Rational curves on flag varieties. Schubert calculus for incidence varieties

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Courbes rationnelles sur les variétés de drapeaux. Calcul de Schubert pour la variété d'incidence

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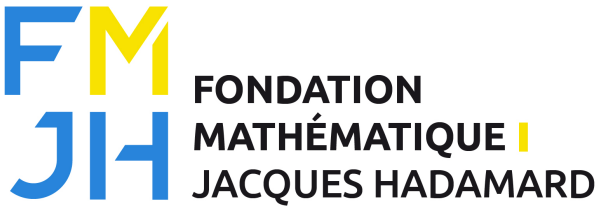
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Titre : Courbes rationnelles sur les variétés de drapeaux. Calcul de Schubert pour la variété d'incidence
Mots clés : Géométrie birationnelle / K-théorie quantique / calcul de Schubert

Résumé : Cette thèse présente trois résultats indépendants sur les variétés de drapeaux. Le premier chapitre est consacré à l'étude de l'espace des courbes rationnelles sur une variété de drapeaux partielle. Le deuxième chapitre procure une formule de comparaison entre corrélateurs de genre 0 en K-théorie quantique T-équivariante de différentes variétés de drapeaux. Le troisième chapitre étudie différentes variantes du calcul de Schubert moderne pour la variété d'incidence X paramétrant les inclusions d'un point dans un hyperplan de l'espace projectif. Nous y calculons une formule fermée décrivant les coefficients de Littlewood-Richardson dans le groupe de Grothendieck $K(X)$ des faisceaux cohérents sur X. Dans le cadre de l'anneau de petite K-théorie quantique de X $QK(X)$ – qui est une déformation de $K(X)$ par des corrélateurs à 3 points marqués – nous procurons une formule de Chevalley, ainsi qu'un algorithme positif calculant les coefficients de Littlewood-Richardson dans $QK(X)$.

Title : Rational curves on flag varieties. Schubert calculus for incidence varieties.
Keywords : Birational geometry / quantum K-theory / Schubert calculus

Abstract : This Phd thesis presents three independent results on flag varieties. In the first chapter, we study the space of rational curves going through points in general position on a partial flag variety. In the second chapter, we provide a comparison formula between T-equivariant genus 0 quantum K-theoretical correlators of different flag varieties. In the third chapter, we study Schubert calculus for the incidence variety X parametrizing points contained in hyperplanes of the projective space. We provide a closed formula for Littlewood-Richardson coefficients in the Groethendieck group $K(X)$ of coherent sheaves on X. We also provide a Chevalley formula in the small quantum K-theory ring $QK(X)$–which is a deformation of $K(X)$ by 3 points correlators– along with a positive algorithm computing all Littlewood-Richardson coefficients in $QK(X)$.

Contents

Introduction

La géométrie énumérative est une branche de la géométrie algébrique considérant le nombre de solutions à un problème géométrique précis. Un exemple marquant de géométrie énumérative est donné par le problème suivant.

Combien de courbes rationnelles de degré d passent par $3d-1$ points en position générale dans $\mathbb{P}^2$?

Ce nombre, que l'on notera N_d, est donné par des invariants de Gromov-Witten. L'associativité du produit dans l'anneau de cohomologie quantique de $\mathbb{P}^2$ (Cf. la Section 0.2 pour des définitions précises des invariants de Gromov-Witten et de l'anneau de cohomologie quantique d'une variété projective lisse) permet de calculer par récurrence tous ces invariants, en fournissant la relation non triviale suivante.

$$N_d = \sum_{d_1+d_2=d,\, d_1,\, d_2>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

On s'intéressera dans cette thèse (cf. Chapitre 3) à une variante particulière de la géométrie énumérative, le calcul de Schubert.

0.1 Calcul de Schubert dans $H^*(G/P, \mathbb{Z})$.

Soit $X = G/P$ une variété de drapeaux généralisée, où G est un groupe algébrique linéaire semi-simple complexe, T un tore maximal de G, B un sous-groupe de Borel de G et P un sous-groupe parabolique de G vérifiant

$$T \subset B \subset P \subset G.$$

Notons $W := N_G(T)/T$ le groupe de Weyl de G. Soit L le sous-groupe de Levi associé à P et B; on note $W_P := N_L(T)/T$ le sous-groupe de W associé à P et $W^P := W/W_P$. Un élément u associé à un élément dans W^P définit une *variété de Schubert*

$$X(u) := \overline{BuP/P}.$$

On note $(X_u)_{u \in W^P}$ les variétés de Schubert de X. Comme $([X_u])_{u \in W^P}$ forme une base de l'homologie à coefficients entiers de $X = G/P$, pour tous éléments u et v dans W^P, il y a une expression unique

$$[X_u] \cup [X_v] = \sum_{w \in I} d^w_{u\,v} [X_w].$$

Les coefficients $d^w_{u\,v}$ sont appelés coefficients de *Littlewood-Richardson*. Une question naturelle se pose alors:

Comment décrire les coefficients $d^w_{u\,v}$?

Soit $Fl_{1,n-1}$ la variété paramétrant les paires (p, h), où p est un point de $\mathbb{P}^{n-1}$ et h est un hyperplan de $\mathbb{P}^{n-1}$ contenant p. Dans le cas où X est la variété d'incidence $Fl_{1,n-1}$, ces coefficients sont calculés dans le chapitre 3. Un algorithme permettant de calculer ces coefficients est appelé *règle de Littlewood-Richardson*. Dans le cadre des Grassmaniennes, des règles de Littlewood-Richardson ont été développées depuis longtemps, par exemple en utilisant la combinatoire des jeux de taquin [Ful13]. Dans le cadre des variétés de drapeaux à deux crans, I. Coskun décrit une règle positive de Littlewood-Richardson [Cos09]. Dans le cadre des variétés minuscules et cominuscules, Thomas et Young ont décrit une règle de Littlewood-Richardson généralisant les jeux de taquin [TY09].

0.2 Cohomologie quantique et K-théorie quantique

On s'intéresse dans le chapitre 2 aux corrélateurs de genre 0 en K-théorie quantique associés à une variété de drapeaux. On s'intéressera dans les parties 3.5 à 3.8 du chapitre 3 à la petite K-théorie quantique de la variété d'incidence $Fl_{1,n-1}$.

Nous rappelons dans cette partie définitions et propriétés classiques des anneaux de grosse cohomologie quantique et de K-théorie quantique d'une variété homogène. Davantage de précisions ainsi que des démonstrations complètes peuvent être trouvées dans [FP96] et [Lee04].

Soit Y une variété projective lisse complexe. Soit γ un élément de $H_2(Y,\mathbb{Z})$ correspondant à une classe effective de courbe. On note $\overline{\mathcal{M}_{0,r}}(Y,\gamma)$ l'espace des application stables de genre 0 à valeurs dans Y représentant la classe γ. Associer à une application stable $(\phi : C \to Y, \{p_1,\dots,p_r\})$ l'image $\phi(p_i)$ de son i-ème point marqué définit un morphisme d'évaluation $ev_i : \overline{\mathcal{M}_{0,r}}(Y,\gamma) \to Y$. Si Y est une variété homogène, $\overline{\mathcal{M}_{0,r}}(Y,\gamma)$ est une variété projective irréductible rationnelle, de dimension la dimension attendue, et à singularités quotient [FP96, Tho98, KP01].

0.2.1 Cohomologie quantique.

Classes effectives de courbes pour $X = G/P$. Notons Δ l'ensemble des racines simples associées à T et positives par rapport à B, et Δ_P l'ensemble des racines simples associées à P. Une racine simple α dans $\Delta^P := \Delta\backslash\Delta_P$ définit une variété de Schubert $X(s_\alpha) \simeq \mathbb{P}^1$ de dimension 1. Les classes $([X(s_\alpha)])_{\alpha\in\Delta\backslash\Delta_P}$ engendrent le groupe abélien libre $H_2(X,\mathbb{Z})$. Notons $E(X)$ le semi-groupe des classes effectives de courbe en homologie singulière. $E(X)$ est l'ensemble des combinaisons linéaires à coefficients positifs de classes $[X(s_\alpha)]$.

Invariants de Gromov-Witten. Soient $\alpha_1, \dots, \alpha_r$ des éléments dans le groupe de Chow $A^*(X) \simeq H^*(X,\mathbb{Z})$, soit γ un élément de $E(X)$. Le *morphisme d'évaluation* $ev_i : \overline{\mathcal{M}_{0,r}}(X,\gamma) \to X$ associe à une application l'image de son i-ème point marqué. Pour X une variété homogène, le morphisme d'évaluation est plat. L'*invariant de Gromov-Witten* associé aux α_i et à γ est défini par:

$$\langle \alpha_1,\dots,\alpha_r\rangle_\gamma^H = \int_{\overline{\mathcal{M}_{0,r}}(X,\gamma)} ev_1^*\alpha_1 \cup \dots \cup ev_r^*\alpha_r. \tag{1}$$

Considérons le cas où les $\alpha_i = [X_i]$ sont les classes de sous-variétés projectives irréductibles de X. Pour $g = (g_1,\dots,g_r)$ en position générale dans G^r, l'invariant $\langle \alpha_1,\dots,\alpha_r\rangle_\gamma$ compte le nombre d'applications stables $(\mathbb{P}^1 \to X, \{p_1,\dots,p_r\})$ représentant la classe γ et envoyant le i-ème point marqué $p_i \in \mathbb{P}^1$ dans la variété $g_i \cdot X_i$.

Anneau de grosse cohomologie quantique. Soit y une classe dans $A^*(X)$. On définit un *potentiel de Gromov-Witten*

$$\mathcal{G}^H(y) = \sum_{n\geqslant 3} \sum_{\beta\in E(X)} \frac{1}{n!} \langle \underbrace{y,\dots,y}_{n}\rangle_\beta^H.$$

Fixons une base $(y_e)_{e\in I}$ du groupe $A^*(X)$. Pour X une variété homogène, les classes de Schubert forment une base naturelle de $A^*(X)$. Notons

$$g_{ef} := \int_X y_e \cup y_f,$$

et g^{ef} la matrice inverse de g_{ef}. Dans la base donnée par les classes de Schubert, pour chaque e il exsiste un unique f tel que g_{ef} est différent de 0; g_{ef} vaut alors 1. On définit le produit entre deux éléments y_i et y_j de la base (y_e) par :

$$y_i * y_j = \sum_{e,f\in I} \frac{\partial^3 \mathcal{G}^H(y)}{\partial y_i \partial y_j \partial y_e} g^{ef} y_f.$$

Ce produit induit par linéarité un produit entre éléments du $\mathbb{Q}[[y]]$-module $A^*(X)\otimes_{\mathbb{Z}}\mathbb{Q}[[y]]$, et en fait une $\mathbb{Q}[[y]]$-algèbre $QH(X)$ commutative, associative, et ayant pour élément neutre $[X]$. L'associativité du produit dans cette algèbre fournit des relations entre invariants de Gromov-Witten de différents degrés.

Anneau de petite cohomologie quantique. Pour X une variété homogène, la plupart des calculs en cohomologie quantique ont lieu dans le cadre de l'anneau de petite cohomologie quantique $QH_s(X)$, plus facile à décrire. En particulier Ciocan-Fontanine donne une présentation des anneaux de petite cohomologie quantique de n'importe quelle variété de drapeaux généralisée [CF95].

Fixons une base $y_1, \ldots, y_p$ de $A^1X \simeq H^2(X,\mathbb{Z})$, et des bases $y_{p+1}, \ldots, y_m$ des autres groupes de cohomologie. On définit le produit en petite cohomologie quantique entre deux classes y_i et y_j par :

$$y_i *_S y_j = \sum_{e,f\in I} \sum_{\beta\in E(X)} \langle y_i, y_j, y_e\rangle_\beta^H q_1^{\int_\beta y_1} \ldots q_p^{\int_\beta y_p} g^{ef} y_f.$$

On obtient ainsi une $\mathbb{Z}[q_1,\ldots,q_p]$-algèbre $QH_s(X) = (A^*(X)\otimes_{\mathbb{Z}} \mathbb{Z}[q_1,\ldots,q_p])$ associative, commutative et ayant pour unité $[X]$. On retrouve A^*X en prenant la limite $q_i \to 0$.

0.2.2 K-théorie quantique.

La K-théorie quantique a été introduite par Givental pour les variétés de drapeaux généralisées puis pour toute variété projective lisse en reformulant le formalisme utilisé dans le cadre de la théorie de Gromov-Witten et en l'adaptant à la K-théorie [Giv00, Lee04].

Corrélateurs en K-théorie quantique. Soit T un groupe algébrique complexe, soit X une variété projective irréductible complexe lisse admettant une action de T. Soient $E_1, \ldots, E_r$ des classes de fibrés vectoriels T-équivariants dans le groupe $K^T(X)$ des fibrés vectoriels T-équivariants de X. Soit β une classe effective de courbe dans $H_2(X,\mathbb{Z})$. On définit le *corrélateur* associé aux E_i et à β par :

$$\langle \phi_1, \ldots, \phi_r\rangle_{T,\beta}^X := \chi_T\left(ev_1^* E_1 \otimes \cdots \otimes ev_r^* E_r \otimes \mathcal{O}^{vir}_{\overline{\mathcal{M}_{0,r}}(X,\beta)}\right),$$

où l'on a noté χ_T la caractéristique d'Euler-Poincaré T-équivariante, et $\mathcal{O}^{vir}_{\overline{\mathcal{M}_{0,r}}(X,\beta)}$ le faisceau structurel virtuel associé à $\overline{\mathcal{M}_{0,r}}(X,\beta)$. Pour X une variété homogène, $\mathcal{O}^{vir}_{\overline{\mathcal{M}_{0,r}}(X,\beta)}$ est le faisceau structurel $\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\beta)}$.

Considérons en particulier le cas où $X = G/P$ est une variété homogène, $T = id$, et les $E_i \otimes [\mathcal{O}_X] = [\mathcal{O}_{X_i}] \in K_\circ(X) \simeq K^{\mathrm{id}}(X)$ sont des classes de faisceaux structuraux au-dessus de sous-variétés X_i de X à singularités rationnelles dans le groupe de Grothendieck $K_\circ(X)$ des faisceaux cohérents sur X. Alors pour $g = (g_1,\ldots,g_r)$ en position générale dans G^r, le corrélateur $\langle E_1,\ldots,E_r\rangle_\beta^X$ est la caractéristique d'Euler-Poincaré de la sous-variété de $\overline{\mathcal{M}_{0,r}}(X,\beta)$ dont les points correspondent à des applications $(C \to X, \{p_1,\ldots,p_r\})$ envoyant leur i-ème point marqué p_i dans $g_i \cdot X_i$. En particulier, si cette variété est de dimension 0, le corrélateur $\langle E_1,\ldots,E_r\rangle_\beta^X$ est égal à l'invariant de Gromov-Witten $\langle [X_1],\ldots,[X_r]\rangle_\beta^H$ défini dans la sous-partie 0.2.1-cf. (1).

Anneau de K-théorie quantique. Soit X une variété projective irréductible complexe lisse. On note $E \subset H_2(X,\mathbb{Z})$ le semi-groupe des classes effectives courbes. Quand X est une variété homogène, ce semi-groupe est engendré par les classes de variétés de Schubert de dimension 1. Soit $\mathbb{C}[E]$ l'anneau de semi-groupe défini par E. On note Q^β l'élément de $\mathbb{C}[E]$ associé à un élément β dans E. On appelle *anneau de Novikov* $N(X)$ l'anneau des séries formelles en Q:

$$N(X) = \left\{ \sum_{\beta\in E} a_\beta Q^\beta \mid a_\beta \in \mathbb{C}. \right\}$$

Soit $(e_i)_{i\in I}$ une base du groupe (algébrique ou topologique) $K(X)$ de K-cohomologie de X. Quand X est une variété homogène, le groupe de Grothendieck des faisceaux cohérents sur X est un groupe abélien libre engendré par les classes de faisceaux structuraux sur les variétés de Schubert de X. On note t_i les coordonnées duales aux e_i. On considère un paramètre formel $t := \sum_{i\in I} t_i e_i$. Le *potentiel de Gromov-Witten* en K-théorie quantique est donné par

$$\mathcal{G}(t) = \sum_{n=0}^{+\infty} \sum_{\beta\in E(X)} \frac{Q^\beta}{n!} \langle \underbrace{t,\ldots,t}_{n}\rangle_\beta.$$

On définit une métrique sur $K(X)$

$$(e_i, e_j) := \frac{\partial^2 \mathcal{G}(t)}{\partial t_i \partial t_j}.$$

Cette métrique est une déformation de la métrique induite par l'accouplement de Poincaré $(e_i, e_j) \to g_{ij} := \chi(e_i \otimes e_j)$ par les corrélateurs de X. On retrouve la métrique g_{ij} en restreignant $\mathcal{G}_{ij} := (e_i, e_j)$ à $Q = 0$. On définit un produit entre éléments e_i et e_j dans $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}[[Q]]$:

$$(e_i * e_j, e_k) = \frac{\partial^3 \mathcal{G}(t)}{\partial t_i \partial t_j \partial t_k}.$$

On obtient par linéarité une algèbre $QK(X) = (K(X) \otimes_{\mathbb{Z}} \mathbb{Q}[[Q, t]], *)$ qui est une algèbre associative et commutative. On retrouve $K(X)$ en prenant la limite $Q \to 0$. La restriction de l'anneau $QK(X)$ à $t = 0$ est l'*anneau de petite K-théorie quantique* $(QK_s(X), \star)$.

0.3 Calcul de Schubert moderne

On rappelle dans cette partie définitions et propriétés classiques du calcul de Schubert pour une variété de drapeaux généralisée, en reprenant les définitions données dans [Bri05] pour le calcul de Schubert en K-théorie et dans [Lee04, BM11] pour le calcul de Schubert en petite K-théorie quantique. Des démonstrations complètes peuvent être trouvées dans [Ful13] pour le calcul de Schubert en homologie à coefficients entiers, [Bri05, Bri02] pour le calcul de Schubert en K-théorie, [FP96] pour le calcul de Schubert en cohomologie quantique, et [Lee04, BM11] pour le calcul de Schubert en petite K-théorie quantique.

DÉFINITIONS. Pour tout u dans W^P on note $Y(u)$ la variété de Schubert opposée à la variété de Schubert $X(u)$.

On s'intéressera ici (cf. chapitre 3) au cas où $G = \mathrm{GL}_n$, T est l'ensemble des matrices diagonales, B est l'ensemble des matrices triangulaires supérieures, B^- est l'ensemble des matrices triangulaires inférieures et P est un sous-groupe parabolique de G associé aux racines $\{\alpha_2, \ldots, \alpha_{n-1}\} \subset \Delta$. La variété $X = G/P$ est alors la variété paramétrant les paires (p, H) où p est un point de $\mathbb{P}^{n-1}$ inclu dans un hyperplan H de $\mathbb{P}^{n-1}$. On étudie dans cette thèse (cf. chapitre 3) différentes variantes du calcul de Schubert moderne dans le cadre de cette variété. Plus précisément, on s'intéresse à la K-théorie et petite K-théorie quantique de la variété d'incidence $Fl_{1,n-1}$.

0.3.1 Calcul de Schubert en K-théorie.

Soit u un élément dans W^P. L'immersion naturelle $i : X(u) \hookrightarrow X$ définit un faisceau $i_*\mathcal{O}_{X(u)}$, qui est un faisceau cohérent de $\mathcal{O}_X$-modules. On note $\mathcal{O}_u := [i_*\mathcal{O}_{X(u)}]$ la classe de $i_*\mathcal{O}_{X(u)}$ dans le groupe de Grothendieck $K_\circ(X)$ des faisceaux cohérents de $\mathcal{O}_X$-modules. Notons $K^\circ(X)$ le groupe de Grothendieck des fibrés vectoriels sur X. Comme X est une variété lisse, l'application naturelle $K^\circ(X) \to K_\circ(X)$ est un isomorphisme. On note $K(X) := K_\circ(X) = K^\circ(X)$. Le produit tensoriel de deux fibrés vectoriels induit une structure d'anneau sur $K(X)$. Les classes de Schubert $(\mathcal{O}_u)_{u \in W^P}$ forment une base de l'anneau $K(X)$. Pour tous éléments u et v dans W^P, il y a donc une expression unique

$$\mathcal{O}_u \cdot \mathcal{O}_v = \sum_{w \in W^P} d^{K\,w}_{u\,v} \mathcal{O}_w.$$

Par analogie avec le cadre classique les coefficients $d^{K\,w}_{u\,v}$ sont appelés *coefficients de Littlewood-Richardson dans* $K(X)$. Notons que le signe de ces coefficients est connu [Bri02].

$$(-1)^{\mathrm{codim}(X(w)) - \mathrm{codim}(X(u)) - \mathrm{codim} X(v)} d^{K\,w}_{u\,v} \geqslant 0.$$

Dans le cas où X est la variété d'incidence, ces coefficients sont calculés dans la partie 0.7.2 du chapitre chapitre 3.

0.3.2 Calcul de Schubert en petite cohomologie quantique.

On note $(QH_s(X), *)$ l'anneau de petite cohomologie quantique de X défini dans la sous-partie 0.2.1. Le semi-groupe $E(X) \subset H_2(X, \mathbb{Z})$ des classes effectives de courbes est engendré par les classes de variétés de Schubert de dimension 1, que l'on a notées $[X(s_\alpha)]$, où $\alpha \in \Delta^P$. On note $m = \#\Delta^P$ le rang de Picard de X. L'anneau $QH_s(X)$ est une déformation de $H^*(X)$ par les variables formelles Q_i, où $i \in \{1, \dots, m\}$. Pour tous éléments u et v dans W^P on a une expression unique

$$[X(u)] * [X(v)] = \sum_{w \in W^P} \sum_{d_1, \dots, d_m \in \mathbb{N}} d_{u,v}^{w, \sum d_i [X(s_{\alpha_i})]} Q_1^{d_1} \dots Q_m^{d_m} [X(w)]$$

où pour $\beta \in E(X)$ le coefficient $d_{u,v}^{w,\beta}$ compte le nombre d'applications $\varphi : \mathbb{P}^1 \to X$ vérifiant :

- $\varphi_*[\mathbb{P}^1] = \beta \in A_* X$.
- Pour un élément $g = (g_1, g_2, g_3)$ général dans G^3, les intersections $\varphi([0:1]) \cap g_1 \cdot X(u)$, $\varphi([1:0]) \cap g_2 \cdot X(v)$ et $\varphi([1:1]) \cap g_3 \cdot Y(w)$ sont non vides.

En particulier les coefficients $d_{u,v}^{w,\beta}$ sont des entiers positifs. On appelle ces coefficients *coefficients de Littlewood-Richardson dans* $QH_s(X)$. Dans le cadre des Grassmaniennes, la petite cohomologie quantique a été étudiée entre autres par Siebert et Tian [ST97], Bertram [Ber97], Bertram-Ciocan-Fontanine-Fulton [BCFF99] et Buch [Buc03]. Dans le cadre des variétés de drapeaux, la petite cohomologie quantique a été étudiée entre autres par Ciocan-Fontanine [CF95]. Kresch et Tamvakis ont étudié la petite cohomologie quantique des Grassmaniennes Lagrangiennes [KT03] et orthogonales [KT04]. Buch-Kresch-Tamvakis ont étudié la petite cohomologie quantique des Grassmaniennes isotropes [BKT12].

0.3.3 Calcul de Schubert en petite K-théorie quantique.

On note $(QK_s(X), \star)$ l'anneau de petite K-théorie quantique de X. D'après la sous-partie 0.2.2, $QK_s(X)$ est une déformation de le l'anneau de K-théorie $K(X)$ par Q^β, pour β dans le semi-groupe $E(X) \subset H_2(X, \mathbb{Z})$ des classes effectives de courbes. Par définition on a donc pour tous éléments u et v dans W^P une expression unique

$$\mathcal{O}_u \star \mathcal{O}_v = \sum_{w \in W^P} P_{u,v}^w(Q) \mathcal{O}_w,$$

où $P_{u,v}^w(Q)$ est à priori une série formelle en Q. Anderson-Chen-Tseng-Iritani ont montré que les coefficients $P_{u,v}^w(Q)$ sont en fait des polynômes en Q [ACT18]. On appelle ces polynômes *coefficients de Littlewood-Richardson dans* $QK_s(X)$. Pour X une Grassmanienne, Buch-Mihalcea donnent un algorithme calculant ces coefficients [BM11]. Buch-Chaput-Perrin-Mihalcea étudient ces coeffcients dans le cadre des variétés cominuscules [BCMP16]. Kato étudie l'anneau de petite K théorie quantique des variétés de drapeaux généralisées G/B dans [Kat18].

On retrouve les coefficients de Littlewood-Richardson dans $K(X)$ en considérant la restriction à $Q = 0$ de ces coefficients. On considère les coefficients $a_{u,v}^{w;\beta}$ des polynômes $P_{u,v}^w(Q)$, i.e.

$$P_{u,v}^w(Q) := \sum_{\beta \in E(X)} a_{u,v}^{w;\beta} Q^\beta.$$

Les coefficients $a_{u,v}^{w,0}$ sont les coefficients de Littlewood-Richardson en K-théorie, et vérifient donc d'après la sous-partie 0.3.1 l'inégalité suivante.

$$(-1)^{\mathrm{codim}(X(w)) - \mathrm{codim}(X(u)) - \mathrm{codim} X(v)} a_{u\,v}^{w,0} \geqslant 0. \tag{2}$$

Définition 0.3.3.1. Soient u et v des éléments dans W^P. Le produit $\mathcal{O}_u \star \mathcal{O}_v = \sum_{w \in I} \sum_{\beta \in E(X)} a_{u,v}^{w;\beta} Q^\beta \mathcal{O}_w$ en petite K-théorie quantique est dit positif si pour tous w dans W^P et β dans $E(X)$

$$(-1)^{\mathrm{codim}(X(w)) - \mathrm{codim}(X(u)) - \mathrm{codim} X(v) + \int_\beta c_1(T_X)} a_{u\,v}^{w,\beta} \geqslant 0.$$

On appelle l'anneau de petite K-théorie quantique $QK_s(X)$ positif si pour tous éléments u, v dans W^P, le produit $\mathcal{O}_u \star \mathcal{O}_v$ est positif.

En particulier, pour $\beta = 0$ on retrouve l'inégalité (2). Notons que Lenart-Maeno ont conjecturé que le produit de deux classes de Schubert est positif dans $QK_s(\mathrm{GL}_n/B)$ [LM06]. Par ailleurs, Buch-Mihalcea ont conjecturé que le produit de deux classes de Schubert est positif dans $QK_s(\mathrm{Gr}(k,n))$, et démontré cette conjecture pour $n < 14$ [BM11].

Soit α un élément dans Δ^P. La variété de Schubert $h_\alpha := Y(\alpha)$ est une variété de codimension 1 dans X. Pour tous α dans Δ^P et v dans W^P, il existe une expression unique

$$[\mathcal{O}_{h_\alpha}] \star \mathcal{O}_v = \sum_{w \in W^P} P^w_{\alpha,v}(Q)\mathcal{O}_w,$$

où les coefficients $P^w_{\alpha,v}(Q)$ sont des polynômes en Q. Une telle formule est appelée *formule de Chevalley dans* $QK_s(X)$.

Une formule de Chevalley est calculée dans la partie 3.8 du chapitre 3, en utilisant des résultats des deux chapitres précédents. On appelle ici un algorithme *positif* si à chaque itération le calcul effectué est une somme de coefficients de mêmes signes. Un algorithme positif permettant de calculer les coefficients de Littlewood-Richardson dans $QK_s(Fl_{1,n-1})$ est donné partie 3.8.1. Le signe de ces coefficients en est déduit dans la partie 3.8.2. Enfin, une formule fermée décrivant ces coefficients est conjecturée dans la partie 3.8.3.

0.4 Résultats

Soit $n > 0$. On considère une collection $I = \{i_1, \ldots, i_m\}$ de m entiers positifs i_k vérifiant $i_0 = 0 < i_1 < \cdots < i_m < i_{m+1} = n$. On note Fl_I la variété de drapeaux paramètrant les drapeaux d'espaces vectoriels du type

$$V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m} \subset \mathbb{C}^n,$$

où V_{i_k} est sous-espace vectoriel de $\mathbb{C}^n$ de dimension i_k.

0.4.1 Variétés de Gromov-Witten des variétés de drapeaux partiels.

Lorsque l'on supprime certains de ses espaces à un drapeau d'espaces vectoriels on obtient un autre drapeau, consitué de moins d'espaces. Ceci induit une application d'oubli $\pi : X \to X'$ entre les variétés de drapeaux associées. Considérons une classe nef $[C] \in H_2(X,\mathbb{Z})$ de courbe de X, où la courbe C est isomorphe à son image par la projection π. Fixons également une classe $\lambda = [l] \in H_2(X,\mathbb{Z})$ de courbe l dont la projection par l'application d'oubli π est un point. On considère dans le chapitre 1 les liens entre la fibre géométrique du morphisme d'évaluation

$$ev_{[C]+\lambda} : \overline{\mathcal{M}_{0,r}}(X, [C] + \lambda) \to X^r$$

et la fibre géométrique du morphisme d'évaluation $ev_{[\pi(C)]}$ à valeur dans $(X')^r$. Pour $r = 3$ et X une variété cominuscule, Chaput-Perrin ont montré que la fibre générale du morphisme d'évaluation est une variété rationnelle [CP11a]. Pour $X = \mathbb{P}^N$, Pandharipande a calculé le genre de la fibre générale de $ev_d : \overline{\mathcal{M}_{0,r}}(\mathbb{P}^N, d) \to (\mathbb{P}^N)^r$ quand cette dernière est de dimension 1 [Pan97].

Définition 0.4.1.1. Soient $X_1, \ldots, X_r$ des variétés de Schubert de la variété de drapeaux X, soit $\mathbf{d}$ un élément de $E(X)$. Pour un élément $g = (g_1, \ldots, g_r)$ général dans GL_n^r, on appelle *variété de Gromov-Witten de degré* $\mathbf{d}$ *associée aux variétés* X_i *et à* g la sous-variété

$$\mathcal{W}^{g,\mathbf{d}}_{X;X_1,\ldots,X_r} := ev_1^{-1}(g_1X_1) \cap \cdots \cap ev_r^{-1}(g_rX_r)$$

de $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$.

Rappelons qu'un morphisme $f : \mathbb{P}^1 \to X$ *représente* un degré $\mathbf{d} \in E(X) \subset H_2(X,\mathbb{Z})$ si f vérifie $f_*[\mathbb{P}^1] = \mathbf{d}$. Notons que la variété de Gromov-Witten $\mathcal{W}^{g,\mathbf{d}}_{X;X_1,\ldots,X_r}$ paramètre les morphismes $\mathbb{P}^1 \to X$ de degré $\mathbf{d}$ dont l'image a une intersection non vide avec les translatés $g_i \cdot X_i$ des variétés de Schubert X_i. Lorsque ces variétés sont de dimension zéro, leur nombre de points est un invariant de Gromov-Witten.

De facon générale, la caractéristique d'Euler-Poincaré de leur adhérence dans l'espace $\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})$ est un invariant en K-théorie quantique [Giv00].

Soit $n \geqslant 0$, soit $I = \{i_1, \ldots, i_m\}$ un ensemble de m entiers vérifiant $0 < i_1 < \cdots < i_m < n$. Considérons la variété de drapeaux $X' = Fl_I$. Pour une classe effective $\mathbf{d} \in H_2(X,\mathbb{Z})$, on note $ev_{\mathbf{d}} : \overline{\mathcal{M}_{0,r}}(Fl_I,\mathbf{d}) \to (Fl_I)^r$ le morphisme d'évaluation. Rappelons que Fl_I paramètre les drapeaux d'espaces vectoriels du type

$$\{0\} \subset V_{i_1} \subset V_{i_2} \cdots \subset V_{i_m} \subset \mathbb{C}^n$$

où $\dim V_{i_k} = i_k$. Soit $J = \{j_1, \ldots, j_M\}$ un ensemble d'entiers vérifiant $0 < j_1 < \cdots < j_M < n$, tel que I est inclu dans J. On note $\pi : X = Fl_J \to X' = Fl_I$ le morphisme d'oubli. Le chapitre 1 démontre l'observation suivante-cf. Théorème 1.1.2.1.

Théorème. *Supposons que $(I, J, [C]+\lambda)$ est une collection stabilisée, au sens de la définition 1.1.1.1. Alors:*

- *Pour toutes variétés de Schubert $X_1, \ldots, X_r$ de X, pour g en position générale dans GL_n^r, chaque variété de Gromov-Witten $\mathcal{W}^{g,[C]+\lambda}_{Fl_J;X_1,\ldots,X_r}$ de Fl_J est une fibration rationnellement connexe au-dessus de variété de Gromov-Witten $\mathcal{W}^{g,[\pi(C)]}_{Fl_I;\pi(X_1),\ldots,\pi(X_r)}$ de Fl_I.*
- *Pour x en position générale dans $(Fl_J)^r$, la fibre $ev^{-1}_{[C]+\lambda}(x)$ est une tour de fibrations unirationnelles au-dessus de la fibre $ev^{-1}_{[\pi(C)]}(\pi(x))$.*

Nous avons appelé une variété W une *fibration rationnellement connexe* (respectivement fibration unirationnelle) au-dessus de W' si il existe un morphisme dominant $W \to W'$ envoyant chaque composante irréductible de W sur une composante irréductible différente de W', et dont la fibre générale est une variété rationnellement connexe (respectivement unirationnelle). Une variété W est une *tour de fibrations unirationnelles* au-dessus d'une variété W' s'il existe une suite de morphismes

$$W_1 = W \to W_2 \to \cdots \to W_l = W'$$

telle que pour $1 \leqslant i < l$, la variété W_i est une fibration unirationnelle au-dessus de W_{i+1}.

0.4.2 Égalités entre corrélateurs de différentes variétés de drapeaux partiels.

Fixons des entiers $1 \leqslant i_1 < \cdots < i_m < n$. On considère la variété de drapeaux X paramétrant les drapeaux d'espaces vectoriels

$$V_{i_1} \subset \cdots \subset V_{i_m} \subset \mathbb{C}^\cdot$$

vérifiant $\dim V_{i_k} = i_k$. La classe $\mathbf{d} = (d_1, \ldots, d_m)$ d'une courbe C de X est déterminée par m entiers d_1, d_2,..., d_m, où d_i est le degré de Plücker de la projection de C sur la Grassmannienne $\mathrm{Gr}(n_i, n)$. Soit T un tore de GL_n agissant sur X par multiplication à gauche. Soient E_1, ..., E_r des fibrés vectoriels de X T-équivariants. Rappelons que les corrélateurs T-équivariants en K-théorie quantique sont donnés par

$$\langle \phi_1, \ldots, \phi_r \rangle^X_{T,(d_1,\ldots,d_m)} := \chi_T \left(ev_1^* E_1 \otimes \cdots \otimes ev_r^* E_r \otimes \mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})} \right), \tag{3}$$

où l'on a noté χ_T la caractéristique d'Euler-Poincaré T-équivariante. Pour un entier $1 \leqslant k \leqslant m$, on note $X_{\hat{k}}$ la variété obtenue à partir de X en oubliant l'espace vectoriel V_{i_k}, et

$$\pi_{\hat{k}} : X \to X_{\hat{k}}$$

le morphisme d'oubli. Afin de faciliter les notations, on note $d_0 = 0 = d_{m+1}$. Le chapitre 2 fournit l'égalité entre corrélateurs suivante-cf. Théorème 2.1.3.1bis.

Théorème. *Supposons:*

- $\forall\, p < k,\ d_k \geqslant n_k\left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$,
- $d_{k-1} \leqslant \lfloor \frac{d_{k+1}}{n_{k+1}} \rfloor$,
- $d_k \geqslant r(n_k - n_{k-1}) + d_{k-1} + (n_k - n_{k-1})(\lfloor \frac{d_{k+1} - d_{k-1}}{n_{k+1} - n_{k-1}} \rfloor + 1)$.

Alors pour tous fibrés vectoriels T-équivariants $E_1, \ldots, E_r$ de $X_{\hat{k}}$, les corrélateurs de X et de $X_{\hat{k}}$ associés au degré $\mathbf{d}$ *et aux fibrés E_i sont égaux:*

$$\langle (\pi_{\hat{k}})^* E_1, \ldots, (\pi_{\hat{k}})^* E_r \rangle^X_{T,(d_1,\ldots,d_m)} = \langle E_1, \ldots, E_r \rangle^{X_{\hat{k}}}_{T,(d_1,\ldots,d_{k-1},d_{k+1},\ldots,d_m)}.$$

Notons que dans le cadre des invariants de Gromov-Witten à trois points marqués, un résultat de Peterson/Woodward permet de déduire tous les invariants à 3 points marqués des variétés de drapeaux généralisées G/P des invariants à trois points marqués de G/B [Woo05].

0.4.3 Calcul de Schubert pour la variété d'incidence $Fl_{1,n-1}$.

On s'intéresse dans le chapitre 3 de cette thèse à différentes variantes du calcul de Schubert moderne pour la variété d'incidence $X = Fl_{1,n-1}$ paramétrant les paires (p, H) où p est un point de $\mathbb{P}^{n-1}$ et H est un hyperplan de $\mathbb{P}^{n-1}$. Notons que X s'écrit sous la forme G/P, où $G = \mathrm{GL}_n$, T est l'ensemble des matrices diagonales, B est l'ensemble des matrices triangulaires supérieures et P est le sous-groupe parabolique de G vérifiant $B \subset P$ associé aux racines $\{\alpha_2, \ldots, \alpha_{n-1}\}$. Le représentant de plus petite longueur w d'un élément dans $W/W_P \simeq \mathfrak{S}_n/\mathfrak{S}_{n-2}$ est une permutation vérifiant $w(1) = i$, $w(n) = j$, et pour $1 < k < n-1$ $w(k) < w(k+1)$. On notera $w_{i,j}$ l'élément associé dans W/W_P. On peut considérer la variété $Fl_{1,n-1}$ comme une sous-variété de bidegré $(1,1)$ de $\mathrm{Gr}(1,n) \times \mathrm{Gr}(n-1,n) \simeq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, définie par la relation d'incidence $L \subset H$. Pour $1 \leqslant i \leqslant n$, on appelle

$$L_i := \{[x_1 : \cdots : x_i : 0 \ldots 0]\},$$

et

$$L^i := \{[0 \ldots 0 : x_i \ldots x_n]\}.$$

Les sous-variétés

$$X(i,j) := (L_i \times L^j) \cap Fl_{1,n-1} \subset Fl_{1,n-1} \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$$

sont les variétés de Schubert $X(w_{i,j})$ de $Fl_{1,n-1}$, pour $1 \leqslant i, j \leqslant n$ et $i \neq j$.

Coefficients de Littlewood-Richardson dans $K(Fl_{1,n-1})$. Pour $1 \leqslant i, j \leqslant n$, $i \neq j$, l'immersion naturelle $i : X(i,j) \hookrightarrow X$ définit un faisceau $i_*\mathcal{O}_{X(i,j)}$, qui est un faisceau cohérent de $\mathcal{O}_{Fl_{1,n-1}}$-modules. On note $\mathcal{O}_{i,j} := [i_*\mathcal{O}_{X(i,j)}]$ la classe de $i_*\mathcal{O}_{X(i,j)}$ dans le groupe de Grothendieck $K(Fl_{1,n-1})$ des faisceaux cohérents de $\mathcal{O}_{Fl_{1,n-1}}$-modules. Rappelons que les classes de Schubert $(\mathcal{O}_{i,j})_{1 \leqslant i,j \leqslant n,\ i \neq j}$ forment une base de $K(Fl_{1,n-1})$, et le produit

$$\mathcal{O}_{i,j} \cdot \mathcal{O}_{k,p} = \sum_{i=0}^{\dim Fl_{1,n-1}} (-1)^i [Tor_i^{Fl_{1,n-1}}\left(\mathcal{O}_{X(i,j)}, \mathcal{O}_{X(k,p)}\right)]$$

définit une structure d'anneau associatif et commutatif $(K(Fl_{1,n-1}), \cdot)$ ayant pour unité $[\mathcal{O}_{Fl_{1,n-1}}]$. On pose $\mathcal{O}_{i,j} = 0$ si $i < 1$ ou $j > n$.

Proposition (Coefficients de Littlewood-Richardson dans $K(Fl_{1,n-1})$-cf. Proposition 3.4.1.1)**.** *Soient $1 \leqslant i, j, k, p \leqslant n$, où $i \neq j$ et $k \neq p$. Alors*

$$\begin{cases} \mathcal{O}_{k,p} \cdot \mathcal{O}_{i,j} = \mathcal{O}_{i+k-n,j+p-1} & \text{si } i+k-n \geqslant j+p \text{ or } i < j \text{ or } k < p; \\ \mathcal{O}_{k,p} \cdot \mathcal{O}_{i,j} = \mathcal{O}_{i+k-n-1,j+p-1} + \mathcal{O}_{i+k-n,j+p} - \mathcal{O}_{i+k-n-1,j+p} & \text{sinon.} \end{cases}$$

Formule de Chevalley et algorithme positif dans $QK_s(Fl_{1,n-1})$. Notons $l_1 := X(2,1)$ et $l_2 := X(1,2)$ les deux variétés de Schubert de $Fl_{1,n-1}$ de dimension 1. Leurs classes $[l_1]$ et $[l_2]$ en homologie singulière de $Fl_{1,n-1}$ engendrent $H_2(Fl_{1,n-1},\mathbb{Z})$. Rappelons que l'anneau de *petite K-théorie quantique* $QK_s(Fl_{1,n-1}) = (K(Fl_{1,n-1}))\otimes\mathbb{Q}[Q_1,Q_2],\star)$ de $Fl_{1,n-1}$ est une déformation de l'anneau $K(Fl_{1,n-1})$ par les variables Q_1 et Q_2. Cette déformation dépend des corrélateurs $\langle \mathcal{O}_{i,j},\mathcal{O}_{k,p},\mathcal{O}_{s,t}\rangle_{d_1[l_1]+d_2[l_2]}$ définis dans la sous-partie 0.2.2. On retrouve l'anneau de K-théorie $K(Fl_{1,n-1})$ en prenant la limite $Q_1,Q_2\to 0$.

Notons $h_1 := X(n-1,1)$ et $h_2 := X(n,2)$ les deux variétés de Schubert de $Fl_{1,n-1}$ de codimension 1.

Proposition (Formule de Chevalley dans $QK_s(Fl_{1,n-1})$-cf. Proposition 3.7.0.1)**.**

$$\mathcal{O}_{h_1}\star\mathcal{O}_{k,p} = \begin{cases} Q_1\mathcal{O}_{n-1,n} + Q_1Q_2\left([\mathcal{O}_X]-\mathcal{O}_{h_1}\right) & \text{si } k=1, p=n \\ Q_1\mathcal{O}_{n,p} & \text{si } k=1,\ p<n \\ \mathcal{O}_{1,2} + Q_1\left([\mathcal{O}_X]-\mathcal{O}_{h_1}\right) & \text{si } k=2, p=1 \\ \mathcal{O}_{k-1,p} & \text{si } k>1,\ k\neq p+1 \\ \mathcal{O}_{p-1,p}+\mathcal{O}_{p,p+1}-\mathcal{O}_{p-1,p+1} & \text{si } 1<p<n-1,\ k=p+1 \end{cases}$$

$$\mathcal{O}_{h_2}\star\mathcal{O}_{k,p} = \begin{cases} Q_2\mathcal{O}_{1,2} + Q_1Q_2\left([\mathcal{O}_X]-\mathcal{O}_{h_2}\right) & \text{si } k=1, p=n \\ Q_2\mathcal{O}_{k,1} & \text{si } k>1,\ p=n \\ \mathcal{O}_{n-1,n} + Q_2\left([\mathcal{O}_X]-\mathcal{O}_{h_2}\right) & \text{si } k=n, p=n-1 \\ \mathcal{O}_{k,p+1} & \text{si } p<n,\ k\neq p+1 \\ \mathcal{O}_{p,p+1}+\mathcal{O}_{p-1,p}-\mathcal{O}_{p-1,p+1} & \text{si } 1<k<n-1,\ k=p+1 \end{cases}$$

Notons que pour X une variété cominuscule, une formule de Chevalley en petite K-théorie quantique est donnée par Buch-Chaput-Mihalcea-Perrin [BCMP16].

Appelons ici un algorithme *positif* si à chaque itération le calcul effectué est une somme de coefficients de mêmes signes.

Proposition. *L'algorithme donné partie 3.8.1 est un algorithme positif calculant les coefficients de Littlewood-Richardson dans $QK_s(Fl_{1,n-1})$.*

Corollaire. *(Cf. Proposition 3.8.2.1) L'anneau de petite K-théorie quantique $QK_s(Fl_{1,n-1})$ est positif au sens de la définition 0.3.3.1.*

Conjecture. *La formule fermée décrite dans la partie 3.8.3 donne les coefficients de Littlewood-Richardson dans $QK_s(Fl_{1,n-1})$.*

Summary

We present here an overview of the different results presented in this thesis. Chapters 2 and 3 use some of the results of Chapter 1, but are otherwise independent and each of them may be read separately.

0.4.4 Chapter 1: Gromov-Witten varieties of partial flag varieties.

Let $n \geqslant 0$, let $J = \{j_1, \ldots, j_M\}$ be a set of M integers satisfying $0 < j_1 < \cdots < j_M < n$. Consider the flag variety $X = Fl_J$ parametrizing the flags of vector spaces of type

$$V_{j_1} \subset V_{j_2} \subset \cdots \subset V_{j_M} \subset \mathbb{C}^n,$$

where V_{j_k} is a vector subspace of $\mathbb{C}^n$ of dimension j_k. For an element $\mathbf{d}$ in the semi-group $E(X) \subset H_2(X, \mathbb{Z})$ of effective classes of curves, we denote by $\mathcal{M}_{0,r}(X, \mathbf{d})$ the space of maps $(f : \mathbb{P}^1 \to X, \{p_1, \ldots, p_r\})$ with r marked points p_i on $\mathbb{P}^1$ satisfying $f_*[\mathbb{P}^1] = \mathbf{d}$. Associating to a map the image of its marked points yields an evaluation morphism $ev_{\mathbf{d}} : \mathcal{M}_{0,r}(X, \mathbf{d}) \to X^r$. For $r = 3$ and X a cominuscule variety, and in particular for X a Grassmannian, Chaput-Perrin proved that the general fiber of the evaluation map is a rational variety [CP11a]. For $X = \mathbb{P}^2$, Pandharipande computed the genus of the general fiber of the evaluation map $ev_d : \mathcal{M}_{0,r}(\mathbb{P}^2, d) \to (\mathbb{P}^2)^r$ when this fiber has dimension 1 [Pan97], genus which turns out to be positive for $d \geqslant 3$. For $X = \mathbb{P}^1$, examples of values of d such that the general fiber of $ev_d : \mathcal{M}_{0,r}(\mathbb{P}^1, d) \to (\mathbb{P}^1)^r$ is not a rationally connected variety have been obtained using techniques from quantum K-theory [IMT14]. One might wonder what happens for a variety of Picard rank greater than one. Chapter 1 provides a step in that direction by studying the geometry of the general fiber of the evaluation map $ev_{\mathbf{d}} : \mathcal{M}_{0,r}(X, \mathbf{d}) \to X^r$ for X a flag variety, and more generally the geometry of Gromov-Witten varieties of flag varieties.

Definition 0.4.4.1. Let $X_1, \ldots, X_r$ be Schubert varieties of the flag variety X, let $\mathbf{d}$ be an element in $E(X)$. For any element $g = (g_1, \ldots, g_r)$ general in GL_n^r, we call *Gromov-Witten variety of degree* $\mathbf{d}$ *associated with the varieties* X_i *and with* g the subscheme

$$\mathcal{W}^{g,\mathbf{d}}_{X;X_1,\ldots,X_r} := ev_1^{-1}(g_1 X_1) \cap \cdots \cap ev_r^{-1}(g_r X_r)$$

of $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$.

Note that the Gromov-Witten variety $\mathcal{W}^{g,\mathbf{d}}_{X;X_1,\ldots,X_r}$ parametrizes morphisms $\mathbb{P}^1 \to X$ representing the class $\mathbf{d}$ whose image has a non empty intersection with the translates $g_i \cdot X_i$ of the Schubert varieties X_i. When these varieties are zero dimensional, their number of points is a Gromov-Witten invariant. More generally, the Euler-Poincaré characteristic of their adherence in the space $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$ is a correlator in quantum K-theory [Giv00].

Let $I = \{i_1, \ldots, i_m\}$ be a set of m integers satisfying $0 < i_1 < \cdots < i_m < n$, such that I is contained in J. We denote by $\pi : X = Fl_J \to X' = Fl_I$ the forgeftul map. Fix a class $\lambda = [l] \in H_2(X, \mathbb{Z})$ of a curve l whose projection by the forgetful map π is a point. We observe in Chapter 1 a relation between the general fiber of the evaluation map

$$ev_{[C]+\lambda} : \overline{\mathcal{M}_{0,r}}(X, [C] + \lambda) \to X^r$$

and the general fiber of the map $ev_{[\pi(C)]} : \overline{\mathcal{M}_{0,r}}(X', [\pi(C)]) \to (X')^r$.

Theorem. *(Theorem 1.1.2.1) Suppose the collection $(I, J, [C] + \lambda)$ is a stabilized collection in the sense of Definition 1.1.1.1. Then:*

- *For any Schubert varieties $X_1, \ldots, X_r$ of X, for g general in GL_n^r, the Gromov-Witten variety $\mathcal{W}^{g,[C]+\lambda}_{Fl_J;X_1,\ldots,X_r}$ of Fl_J is a rationally connected fibration above the Gromov-Witten variety $\mathcal{W}^{g,[\pi(C)]}_{Fl_I;\pi(X_1),\ldots,\pi(X_r)}$ of Fl_I.*
- *For x general in $(Fl_J)^r$, the fiber $ev^{-1}_{[C]+\lambda}(x)$ is a tower of unirational fibrations over the fiber $ev^{-1}_{[\pi(C)]}(\pi(x))$.*

We called a variety W a *rationally connected fibration* (respectively unirational fibration) above W' if there exists a dominant map $W \to W'$ sending each irreducible component of W onto a different irreducible component of W', and whose general fiber is a rationally connected variety (respectively unirational). A variety W is a *tower of unirational fibrations* above a variety W' if there exists a sequence of morphisms

$$W_1 = W \to W_2 \to \cdots \to W_l = W'$$

such that for $1 \leqslant i < l$, the variety W_i is a unirational fibration above W_{i+1}.

0.4.5 Chapter 2: A comparison formula between genus 0 correlators of flag varieties.

Fix integers $1 \leqslant n_1 < \cdots < n_m < n$. We consider the flag variety X parametrizing flags of vector spaces

$$V_{n_1} \subset \cdots \subset V_{n_m} \subset \mathbb{C}^{\cdot}$$

satisfying $\dim V_{n_i} = n_i$. Let T be a torus in GL_n acting on X by left multiplication. Let $\mathbf{d} = (d_1, \ldots, d_m)$ be an effective class of curve in $H_2(X, \mathbb{Z})$. We denote by $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$ the coarse moduli space parametrizing genus 0 stable maps representing the class $\mathbf{d}$. Assocating to a stable map with r marked points the image of its i-th marked point induces an evaluation map $ev_i : \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to X$. Let $E_1, \ldots, E_r$ be T-equivariant fiber bundles over X. In quantum K-theory, T-equivariant correlators are defined by

$$\langle \phi_1, \ldots, \phi_r \rangle^X_{T,(d_1,\ldots,d_m)} := \chi_T \left(ev_1^* E_1 \otimes \cdots \otimes ev_r^* E_r \otimes \mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})} \right), \tag{4}$$

where we denote by χ_T the T-equivariant Euler-Poincaré characteristic [Giv00, Lee04]. For $1 \leqslant k \leqslant m$, we denote by $X_{\widehat{k}}$ the flag variety obtained from X by forgetting the vector space V_{n_k}, and by

$$\pi_{\widehat{k}} : X \to X_{\widehat{k}}$$

the forgetful map. In order to facilitate notations, we set $d_0 = 0 = d_{m+1}$. Chapter 2 yields the following equality between correlators-cf. Theorem 2.1.3.1bis.

Theorem. *Suppose:*

- $\forall\, p < k,\ d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$,
- $d_{k-1} \leqslant \lfloor \frac{d_{k+1}}{n_{k+1}} \rfloor$,
- $d_k \geqslant r(n_k - n_{k-1}) + d_{k-1} + (n_k - n_{k-1})(\lfloor \frac{d_{k+1} - d_{k-1}}{n_{k+1} - n_{k-1}} \rfloor + 1)$.

Then for any T-equivariant fiber bundles $E_1, \ldots, E_r$ over $X_{\widehat{k}}$, the correlators of X and $X_{\widehat{k}}$ associated with the degree $\mathbf{d}$ and the fiber bundles E_i are equal:

$$\langle (\pi_{\widehat{k}})^* E_1, \ldots, (\pi_{\widehat{k}})^* E_r \rangle^X_{T,(d_1,\ldots,d_m)} = \langle E_1, \ldots, E_r \rangle^{X_{\widehat{k}}}_{T,(d_1,\ldots,d_{k-1},d_{k+1},\ldots,d_m)}.$$

Note that for 3-pointed Gromov-Witten invariants, a result of Peterson/Woodward allows to deduce all invariants with 3 marked points of generalized flag varieties G/P from 3-pointed Gromov-Witten invariants of G/B [Woo05].

0.4.6 Chapter 3: Schubert calculus for the incidence variety $Fl_{1,n-1}$.

We study in Chapter 3 different variations of modern Schubert calculus for the incidence variety $X = Fl_{1,n-1}$ parametrizing pairs (p, H) where p is a point in $\mathbb{P}^{n-1}$ and H is a hyperplane in $\mathbb{P}^{n-1}$ containing the point p. Note that X can be written as G/P, where $G = \mathrm{GL}_n$, T is the set of diagonal matrices, B is the set of upper triangular matrices and P is the parabolic subgroup of G satisfying $B \subset P$ associated with the roots $\{\alpha_2, \ldots, \alpha_{n-1}\}$. The smallest length representative w of an element in $W/W_P = \mathfrak{S}_n/\mathfrak{S}_{n-2}$ is a permutation $w_{i,j}$ satisfying $w(1) = i$, $w(n) = j$, and such that for $1 < k < n-1$ $w(k) < w(k+1)$. For $1 \leqslant i, j \leqslant n$ and $i \neq j$, we call $X(i,j) := X(w_{i,j})$ the Schubert variety of $Fl_{1,n-1}$ associated with the element $w_{i,j}$ in $W^P \simeq W/W_P$.

Littlewood-Richardson coefficients in $K(Fl_{1,n-1})$. For $1 \leqslant i, j \leqslant n$, $i \neq j$, the natural embedding $i : X(i,j) \hookrightarrow X$ defines a sheaf $i_*\mathcal{O}_{X(i,j)}$, which is a coherent sheaf of $\mathcal{O}_{Fl_{1,n-1}}$-module. We denote by $\mathcal{O}_{i,j} := [i_*\mathcal{O}_{X(i,j)}]$ the class of $i_*\mathcal{O}_{X(i,j)}$ in the Grothendieck group $K(Fl_{1,n-1})$ of coherent sheaves of $\mathcal{O}_{Fl_{1,n-1}}$-module. Schubert classes $(\mathcal{O}_{i,j})_{1\leqslant i,j\leqslant n,\, i\neq j}$ form a basis of $K(Fl_{1,n-1})$, and the product

$$\mathcal{O}_{i,j} \cdot \mathcal{O}_{k,p} = \sum_{i=0}^{\dim Fl_{1,n-1}} (-1)^i [Tor_i^{Fl_{1,n-1}}\left(\mathcal{O}_{X(i,j)}, \mathcal{O}_{X(k,p)}\right)]$$

defines a structure of associative and commutative ring $(K(Fl_{1,n-1}), \cdot)$ with identity element $[\mathcal{O}_{Fl_{1,n-1}}]$. We set $\mathcal{O}_{i,j} = 0$ if $i < 1$ or $j > n$.

Proposition (Littlewood-Richardson coefficients in $K(Fl_{1,n-1})$-cf. Proposition 3.4.1.1)**.** *Let $1 \leqslant i, j, k, p \leqslant n$, where $i \neq j$ and $k \neq p$. Then*

$$\begin{cases} \mathcal{O}_{k,p} \cdot \mathcal{O}_{i,j} = \mathcal{O}_{i+k-n,j+p-1} & \text{if } i+k-n \geqslant j+p \text{ or } i < j \text{ or } k < p; \\ \mathcal{O}_{k,p} \cdot \mathcal{O}_{i,j} = \mathcal{O}_{i+k-n-1,j+p-1} + \mathcal{O}_{i+k-n,j+p} - \mathcal{O}_{i+k-n-1,j+p} & \text{else.} \end{cases}$$

Chevalley formula and positive algorithm in $QK_s(Fl_{1,n-1})$. We denote by $l_1 := X(2,1)$ et $l_2 := X(1,2)$ the two one-dimensional Schubert varieties of $Fl_{1,n-1}$. Their classes $[l_1]$ and $[l_2]$ in the singular homology of $Fl_{1,n-1}$ generate $H_2(Fl_{1,n-1}, \mathbb{Z})$. The *small quantum K-ring* $QK_s(Fl_{1,n-1}) = (K(Fl_{1,n-1})) \otimes \mathbb{Q}[Q_1, Q_2], \star)$ of $Fl_{1,n-1}$ is a deformation of the ring $K(Fl_{1,n-1})$ by the Novikov variables Q_1 and Q_2 associated with the classes $[l_1]$ and $[l_2]$ [Lee04]. This deformation depends on the correlators $\langle \mathcal{O}_{i,j}, \mathcal{O}_{k,p}, \mathcal{O}_{s,t} \rangle_{d_1[l_1]+d_2[l_2]}$ defined by (4). Considering the limit $Q_1, Q_2 \to 0$ yields the K-ring $K(Fl_{1,n-1})$.

We name $h_1 := X(n-1,1)$ and $h_2 := X(n,2)$ the two Schubert varieties of $Fl_{1,n-1}$ of codimension 1.

Proposition (Chevalley formula in $QK_s(Fl_{1,n-1})$-cf. Proposition 3.7.0.1)**.**

$$\mathcal{O}_{h_1} \star \mathcal{O}_{k,p} = \begin{cases} Q_1\mathcal{O}_{n-1,n} + Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right) & \text{if } k = 1, p = n \\ Q_1\mathcal{O}_{n,p} & \text{if } k = 1,\ p < n \\ \mathcal{O}_{1,2} + Q_1\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right) & \text{if } k = 2, p = 1 \\ \mathcal{O}_{k-1,p} & \text{if } k > 1,\ k \neq p+1 \\ \mathcal{O}_{p-1,p} + \mathcal{O}_{p,p+1} - \mathcal{O}_{p-1,p+1} & \text{if } 1 < p < n-1,\ k = p+1 \end{cases}$$

$$\mathcal{O}_{h_2} \star \mathcal{O}_{k,p} = \begin{cases} Q_2\mathcal{O}_{1,2} + Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_2}\right) & \text{if } k = 1, p = n \\ Q_2\mathcal{O}_{k,1} & \text{if } k > 1,\ p = n \\ \mathcal{O}_{n-1,n} + Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_2}\right) & \text{if } k = n, p = n-1 \\ \mathcal{O}_{k,p+1} & \text{if } p < n,\ k \neq p+1 \\ \mathcal{O}_{p,p+1} + \mathcal{O}_{p-1,p} - \mathcal{O}_{p-1,p+1} & \text{if } 1 < k < n-1,\ k = p+1 \end{cases}$$

Note that for X a cominuscule variety, a Chevalley formula in small quantum K-theory is given by Buch-Chaput-Mihalcea-Perrin [BCMP16].

Since $(\mathcal{O}_w)_{w\in W^P}$ is a basis of $K(Fl_{1,n-1})$, for any elements u and v in W^P, there is a unique decomposition

$$\mathcal{O}_u \star \mathcal{O}_v = \sum_{d_1,\, d_2 \in \mathbb{N}} \sum_{w \in W^P} a_{u,v}^{w;d_1,d_2} Q_1^{d_1} Q_2^{d_2} \mathcal{O}_w.$$

We call the polynomials $\sum_{d_1,\, d_2\in\mathbb{N}} a_{u,v}^{w;d_1,d_2} Q_1^{d_1} Q_2^{d_2}$ *Littlewood-Richardson coefficients in* $QK_s(Fl_{1,n-1})$. We call the ring $QK_s(Fl_{1,n-1})$ *positive* if for any elements u and v in W^P, d_1, d_2 in $\mathbb{N}$ the sign of the coefficient $a_{u,v}^{w;d_1,d_2}$ is given by the following inequality.

$$(-1)^{\mathrm{codim}X(w)-\mathrm{codim}X(u)-\mathrm{codim}X(v)+\int_{d_1[l_1]+d_2[l_2]} c_1(T_X)} a_{u,v}^{w;d_1,d_2} \geqslant 0.$$

Note that for $d_1 = 0 = d_2$, this inequality is satisfied according to M. Brion's result on positivity in the Grothendieck groups of generalized flag varieties [Bri02]. Finally, we call an algorithm *positive* if at each iteration the computation performed is a sum of coefficients having the same sign.

Proposition. *The algorithm given Subsection 3.8.1 is a positive algorithm computing Littlewood-Richardson coefficients in* $QK_s(Fl_{1,n-1})$.

Corollary. *(Cf. Proposition 3.8.2.1) The small quantum K-theory ring* $QK_s(Fl_{1,n-1})$ *is positive.*

Conjecture. *The closed formula described in Subsection 3.8.3 yields Littlewood-Richardson coefficients in* $QK_s(Fl_{1,n-1})$.

Definitions and notations

0.5 Space of stable maps $\overline{\mathcal{M}_{0,r}}(Y,\gamma)$.

Definitions. Let Y be a complex projective smooth variety. Let C be a connecetd reduced nodal curve of genus zero equipped with r distinct non singular marked points. A morphism $C \to Y$ from C to the variety Y is called a *stable map* if each irreducible component of C sent onto a point contains at least three points that are nodal or marked. Let γ be an element in $H_2(Y,\mathbb{Z})$ corresponding to an effective class of curve. We say the morphism $\varphi : C \to Y$ *represents the class* γ if $\varphi_*[C] = \gamma \in H_2(Y,\mathbb{Z})$. Let S be a complex scheme. A family of quasi-stable maps with r marked points above S is given by

- A flat and projective morphism $\pi : \mathcal{C} \to S$ equipped with r sections $p_i : S \to \mathcal{C}$, such that each geometrical fiber is a reduced nodal projective connected curve of genus zero C_s and such that the marked points $p_i(s)$ are distinct and non singular.
- A morphism $\mathcal{C} \to Y$.

The *space* $\overline{\mathcal{M}_{0,r}}(Y,\gamma)$ *of genus* 0 *stable maps to* Y *representing* γ is the coarse moduli space parametrizing families of stable maps with r marked points above S from genus 0 curves to Y representing the class γ, such that the geometric fiber $\mathcal{C}_s$ above any point*s* in S defines a stable map $\mathcal{C}_s \to Y$ [FP96]. For a homogeneous variety Y, the space $\overline{\mathcal{M}_{0,r}}(Y,\gamma)$ is an irreducible rational projective variety, ofe dimension the expected dimension and with quotient singularities [FP96, KP01]. Furthermore, the subvariety $\mathcal{M}_{0,r}(Y,\gamma)$ parametrizing maps $\mathbb{P}^1 \to Y$ is a dense open subset of $\overline{\mathcal{M}_{0,r}}(Y,\gamma)$. A general point in $\overline{\mathcal{M}_{0,r}}(Y,\gamma)$ thus corresponds to a map $\mathbb{P}^1 \to Y$ representing the class γ.

Note that a general point in $\overline{\mathcal{M}_{0,r}}(Y,\gamma)$ is not necessarily associated with an injective map $\mathbb{P}^1 \to Y$. This is immediate if $Y = \mathbb{P}^1$; indeed, a general point in $\overline{\mathcal{M}_{0,r}}(\mathbb{P}^1,d)$ is associated with a degree d map $\mathbb{P}^1 \to \mathbb{P}^1$. This phenomenon may also be observed in higher dimensions. Consider for example the full flag variety $Y = Fl_{1,2} = \mathrm{GL}_3/B$ parametrizing pairs (p,l) where p is a point in $\mathbb{P}^2$ and l is a line in $\mathbb{P}^2$ taht contains p. Fix a line l in $\mathbb{P}^2$. Denote by γ the class of the curve $C \simeq \mathbb{P}^1 \subset Fl_{1,2}$ containing all pairs (p,l), where $p \subset l$ and by $\pi : Fl_{1,2} \to \mathbb{P}^2$ the map forgetting the line l' in a pair (p,l'). Then the restriction of π to the curve C is an isomorphism onto its image, which is the line $l \subset \mathbb{P}^2$. Denote by $\pi' : Fl_{1,2} \to \mathrm{Gr}(2,3)$ the map induced by forgetting the point p in a pair (p,l). An irreducible curve C' in $Fl_{1,2}$ of class 2γ satisfies $\pi_*[C'] = 2$ in $A_*(\mathbb{P}^2)$. Furthermore its image $\pi(C')$ in $\mathbb{P}^2$ is included in the line in $\mathbb{P}^2$ defined by the image $\pi'(C') \in \mathrm{Gr}(2,3)$. Then the projection π induces a degree 2 morphism between the curve C' and its image $\pi(C')$, which contradicts the fact that the restriction of π to C' is an injection. There thus is no irreducible curve of class 2γ in $Fl_{1,2}$, and a general point in $\mathcal{M}_{0,r}(Fl_{1,2},2\gamma) \hookrightarrow \overline{\mathcal{M}_{0,r}}(Fl_{1,2},2\gamma)$ corresponds to a degree 2 map $\mathbb{P}^1 \to Fl_{1,2}$.

G-action on $\overline{\mathcal{M}_{0,r}}(Y,\gamma)$. Let G be a complex algebraic group, Y be a smooth projective G-variety (in this thesis G will be the group GL_n of invertible matrices and Y will be a flag variety parametrizing flags of vector spaces included in $\mathbb{C}^n$). Let γ be an element of $H_2(Y,\mathbb{Z})$ corresponding to an effective class of curve. We consider the natural transformation of functors associating to a morphism $\phi : S \to G$ and to a family of quasi-stable maps $(\pi : \mathcal{C} \to S, \mu : \mathcal{C} \to Y)$ above S with r marked points representing the class γ the following family:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G\cdot\mu} & Y \\ \downarrow{\scriptstyle \pi} & & \\ S & & \end{array}$$

where we denote by $G \cdot \mu : \mathcal{C} \to Y$ the morphism $\mathcal{C} \ni c \to \Phi \circ \pi(c) \cdot \mu(c) \in Y$ translating curves $\mu(\mathcal{C}_s)$ by the element in G image of s by Φ. This natural transformation of functors induces an action $\overline{\mathcal{M}_{0,r}}(Y,\gamma) \times G \to \overline{M_{0,r}(Y,\gamma)}$, associating to an element g in G and to a map $\mu : C \to Y$ the map

$$g \cdot \mu : C \ni p \to g \cdot \mu(p) \in Y.$$

0.6 Flag varieties

We recall here some classical definitions and properties of flag varieties. Let $n > 0$. We consider a collection $I = \{i_1, \ldots, i_m\}$ of m non-negative integers i_k satisfying $i_0 := 0 < i_1 < \cdots < i_m < i_{m+1} := n$. We denote by Fl_I the variety parametrizing flags of vector spaces

$$V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m} \subset \mathbb{C}^n,$$

where V_{i_k} is a vector subspace of $\mathbb{C}^n$ of dimension i_k. Note that the variety Fl_I represents the functor $Sch \to Sets$ associating to a scheme S the following set:

$$\{\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m \subset \mathcal{O}_S^{\oplus n}\},$$

where the fiber bundles over S $\mathcal{F}_k$ have rank i_k and the quotient sheaves $\mathcal{F}_k/\mathcal{F}_{k-1}$ are rank $(i_k - i_{k-1})$ fiber bundles. Furthermore, we consider the complex algebraic group $G = \mathrm{GL}_n$, the maximal torus T of G given by the set of diagonal matrices and the Borel subgroup $B \subset G$ given by the set of upper triangular matrices. We denote by $W \simeq \mathfrak{S}_n$ the Weyl group of $G = \mathrm{GL}_n$, and $\Delta = \alpha_1, \ldots, \alpha_n$ the set of simple roots positive with respect to B. We denote by $\Delta(I) = \{\alpha_1, \ldots \alpha_{i_1 - 1}, \ldots, \alpha_{i_m - 1}\}$ the subset of Δ associated with I, and by $W_I \simeq \prod_{k=1}^{m+1} \mathfrak{S}_{i_k - i_{k-1}}$ the subgroup of W generated by reflexions s_α associated with elements α in $\Delta(I)$. Finally, we denote by $P_I := BW_IB$ the parabolic subgroup associated with $\Delta(I)$. Note that the block diagonal matrix whose k-th block is $GL_{i_k - i_{k-1}}$ is a Levi sugroup L_I associated with P_I. We have the following identification: $Fl_I \simeq G/P_I$.

Let $1 \leqslant k < m$. Composing the map forgetting all vector spaces except from the k-th one with the Plücker embedding $\mathrm{Gr}(i_k, n) \hookrightarrow \mathbb{P}^{\binom{n}{i_k}-1}$ induces a map $\varphi : Fl_I \to \mathbb{P}^{\binom{n}{i_k}-1}$. Let l_i be a curve on Fl_I sent onto a line in $\mathbb{P}^{\binom{n}{i_k}-1}$ by φ, and such that the restriction of φ to l_i is an isomorphism. The classes $[l_i]$ generate the free abelian group $H_2(Fl_I, \mathbb{Z})$, and we may thus identify an effective class of curve $\mathbf{d} = \sum_{k=1}^m d_k[l_k]$ in $E(Fl_I) \subset H_2(Fl_I, \mathbb{Z})$ with m positive integers $(d_1, \ldots, d_m)$.

0.7 Algebraic K-theory

0.7.1 Equivariant algebraic K-theory. We recall here some classical definitions and properties of the equivariant K-theory of a variety; complete proofs can be found in [C$^+$09], chapter 5. Let H be a complex linear algebraic group, let X be an irreducible variety with an H-action. Let E be a vector bundle on X. We call E an *H-equivariant vector bundle* if there exists an H-action $\Phi : H \times E \to E$ satisfying

- The map $E \to X$ commutes with H-actions.
- For any x in X and g in G, the map $\Phi(g, \bullet) : E_x \to E_{g \cdot x}$ is a linear map of vector spaces,

where we denote by E_x the fiber over a point x in X. Let $\mathcal{F}$ be a coherent sheaf on X. Denote by $a : H \times X \to X$ the action of H on X, by $p : H \times X \to X$ the projection map and by $p_{2\,3} : H \times H \times X \to H \times X$ the projection along the first factor H. We call $\mathcal{F}$ an *H-equivariant coherent sheaf* if there exists an H-action $\Phi : H \times \mathcal{F} \to \mathcal{F}$ satisfying

- There exists an isomorphism $I : a^*\mathcal{F} \to p^*\mathcal{F}$.
- ${p_{2\,3}}^* I \circ (\mathrm{id}_H \times a)^* I = (m \times \mathrm{id}_X)^* I$ of sheaves on $H \times X$.

– There is an identification $I_{1\times X} = \mathrm{id} : \mathcal{F} = a^*\mathcal{F}_{|1\times X} \to p^*\mathcal{F}_{|1\times X} = \mathcal{F}$.

Denote by $K_H(X)$ the Grothendieck group of H-equivariant coherent sheaves on X and by $K^H(X)$ the Grothendieck group of H-equivariant vector bundles on X. The tensor product of equivariant vector bundles defines a ring structure on $K^H(X)$. Moreover, the map $K^H(X) \otimes K_H(X) \to K_H(X)$ defined by the tensor product

$$[E] \otimes [\mathcal{F}] \to [E \otimes_{\mathcal{O}_X} \mathcal{F}]$$

makes $K_H(X)$ a $K^H(X)$-module. When X is non singular, the map $[E] \to [E \otimes \mathcal{O}_X]$ yields an isomorphism $K_H(X) \simeq K^H(X)$. Indeed, any equivariant sheaf then has a finite resolution by equivariant vector bundles.
Furthermore, for any equivariant proper morphism of irreducible varieties with an H-action $f : X \to Y$, there is a pushforward map $f_* : K_H(X) \to K_H(Y)$ defined by:

$$[\mathcal{F}] \to \sum_i (-1)^i [R^i f_*(\mathcal{F})].$$

Indeed, if $\mathcal{F}$ is an H-equivariant sheaf on X, the Godement resolution implies that $R^i f_*(\mathcal{F})$ will also be an H-equivariant sheaf on Y.
In particular, if X is an irreducible projective H-variety, the equivariant pushforward $K_H(X) \to K_H(\mathit{point})$ is called *equivariant Euler characteristic*, and denoted by χ_H.
Moreover, an equivariant morphism $f : X \to Y$ induces a pullback map $f^* : K^H(Y) \to K^H(X)$:

$$[E] \to [f^*(E)].$$

Consider an equivariant proper morphism $f : X \to Y$ of irreducible varieties with an H-action. Let E be an equivariant vector bundle on Y, and $\mathcal{F}$ be an equivariant coherent sheaf on X. Note that the following projection formula is satisfied in $K_H(Y)$:

$$f_*(f^*[E] \otimes [\mathcal{F}]) = [E] \otimes f_*[\mathcal{F}].$$

0.7.2 Algebraic K-theory of G/P.

We recall here some classical definitions and properties of the Grothendieck ring of coherent sheaves on a variety. Denote by $K_\circ(X)$ the Grothendieck group of coherent sheaves on X and by $K^\circ(X)$ the Grothendieck group of vector bundles on X. Note that for $H = \mathrm{id}$, the equivariant groups $K_H(X)$ and $K^H(X)$ are equal to $K_\circ(X)$ and $K^\circ(X)$. In particular, the tensor product of vector bundles defines a ring structure on $K^\circ(X)$, and for a non singular variety X we can identify $K(X) = K_\circ(X) \simeq K^\circ(X)$. Furthermore, for any coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X we have

$$[\mathcal{F}] \cdot [\mathcal{G}] = \sum_{i=0}^{\dim X} (-1)^i Tor_i^X(\mathcal{F}, \mathcal{G}).$$

Furthermore, note that for an irreducible projective variety X, the pushforward $K_\circ(X) \to K_\circ(\mathrm{Spec}\mathbb{C})$ is given by:

$$\chi : K_\circ(X) \to \mathbb{Z}, \qquad [\mathcal{F}] \to \chi(\mathcal{F}) = \sum_i (-1)^i h^i(\mathcal{F}),$$

where h^i denote the dimension of the i-th cohomology group of $\mathcal{F}$; i.e. χ is the Euler-Poincaré characteristic of $\mathcal{F}$. A morphism of schemes $f : X \to Y$ is called *perfect* if there is an N such that

$$Tor_i^Y(\mathcal{O}_X, \mathcal{F}) = 0$$

for all $i > N$ and for all coherent sheaves $\mathcal{F}$ on Y. For a perfect morphism $f : X \to Y$, one can define the pull-back

$$f^* : K_\circ(Y) \to K_\circ(X)$$

by

$$f^*[\mathcal{F}] = \sum_{i=0}^{N} [Tor_i^Y(\mathcal{O}_X, \mathcal{F})].$$

Furthermore suppose f is a morphism locally of finite type. Then if f is flat or Y is regular, f is perfect-cf. for example Stacks Project 37.53. We will also be using the following easy results.

Lemma 0.7.2.1. *Let Y be an irreducible projective variety and Φ be an irreducible projective subvariety of $\mathbb{P}^1 \times Y$. Denote by $\pi : \Phi \to \mathbb{P}^1$ the natural projection. Let p_1, p_2 be two points on $\mathbb{P}^1$. Then we have the following equality in $K_\circ(Y)$:*

$$[\mathcal{O}_{\pi^{-1}(p_1)}] = [\mathcal{O}_{\pi^{-1}(p_2)}]$$

Proof. Since the projection map π is flat, we have $\pi^*[\mathcal{O}_{p_i}] = [\mathcal{O}_{\pi^{-1}(p_i)}]$ in $K_\circ(\Phi)$. Denote by $\Pi : \Phi \to Y$ the natural projection. We obtain $\Pi_*\pi^*[\mathcal{O}_{p_i}] = \Pi_*[\mathcal{O}_{\pi^{-1}(p_i)}] = [\mathcal{O}_{\pi^{-1}(p_i)}]$ in $K_\circ(Y)$. Finally, the equality $[\mathcal{O}_{p_1}] = [\mathcal{O}_{p_2}]$ in $K_\circ(\mathbb{P}^1)$ yields the result. □

Lemma 0.7.2.2. *Let X be a projective GL_n-variety. Consider a subvariety Y of X. Then for all g in GL_n, we have the following equality in $K_\circ(X)$:*

$$[\mathcal{O}_{g\cdot Y}] = [\mathcal{O}_Y].$$

Proof. Let g in GL_n. Since GL_n is a dense open subset of $\mathcal{M}_{n,n} \simeq \mathbb{A}^{n^2}$, there exits an irreducible rational curve $f : \mathbb{P}^1 \supset U \to \mathrm{GL}_n$ sending points p_1 and p_2 on $\mathbb{P}^1$ onto the points g and id. Denote by Φ the irreducible projective subvariety of $\mathbb{P}^1 \times X$ obtained as the adherence of the image of the map $U \times Y \to U \times Y \subset \mathbb{P}^1 \times X$ sending (x, y) to $(x, f(x) \cdot y)$. Denote by $\pi : \Phi \to \mathbb{P}^1$ the first projection. Then according to Lemma 0.7.2.1

$$[\mathcal{O}_{\pi^{-1}(p_1)}] = [\mathcal{O}_{\pi^{-1}(p_2)}] \text{ hence } [\mathcal{O}_{g\cdot Y}] = [\mathcal{O}_Y].$$

□

0.8 Rational singularities.

We recall here known definitions and results on rational singularities. The treatment in this section is inspired by [Bri02]. An irreducible complex variety X has *rational singularities* if there exists a desingularization $\psi : \widetilde{X} \to X$ such that:

$$\psi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X \text{ and } \forall i > 0,\ R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0. \tag{5}$$

Note that, according to Zariski's main theorem, the equality $\psi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ is equivalent to the normality of X. The relevance of this notion here comes from the following results.

Theorem 0.8.0.1. *(Viehweg [Vie77]) The quotient of a smooth complex scheme by a finite group has rational singularities.*

Note that, since for X convex the moduli spaces $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$ have quotient singularities [FP96], this implies that for X a generalized flag variety the moduli spaces $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$ have rational singularities.

Let H be a complex linear algebraic group. Recall a variety X is called *unirational* if there exists a dominant map $\mathbb{P}^N \dashrightarrow X$, and *rationally connected* if there exists an irreducible rational curve joining two general points in X. Note that since $\mathbb{P}^N$ is rationally connected, a unirational variety is rationally connected.

Theorem 0.8.0.2. *(Buch-Mihalcea [BM11], Theorem 3.1.) Let $f : X \to Y$ be a surjective equivariant morphism of projective varieties with rational singularities equipped with an H-action. Assume that the general fiber of f is rationally connected. Then $f_*[\mathcal{O}_X] = [\mathcal{O}_Y] \in K_H(Y)$.*

Proof. Buch and Mihalcea prove in [BM11] that this statement is satisfied if a desingularization $\psi : \widetilde{X_y} \to X_y$ of the fiber of f over y general in Y satisfying $H^i(\widetilde{X_y}, \mathcal{O}_{\widetilde{X_y}}) = 0$ for all $i > 0$. Since X_y is rationally connected, and $\widetilde{X_y}$ is birational to X_y, $\widetilde{X_y}$ is also rationally connected. Hence $\widetilde{X_y}$ is a smooth rationally connected variety, hence according to [Deb13], $\widetilde{X_y}$ satisfies $H^i(\widetilde{X_y}, \mathcal{O}_{\widetilde{X_y}}) = 0$ for all $i > 0$. □

Note that for X a full flag variety, Brion gives a nice proof of the rational singularities of Schubert varieties in [Bri05].

Theorem 0.8.0.3. *(Ramanathan, [Ram85]) Schubert varieties have rational singularities.*

Chapter 1

Gromov-Witten varieties of partial flag varieties

1.1 Introduction

Removing subspaces from a partial flag gives another partial flag composed of fewer subspaces. This induces a forgetful map $\pi : X \to X'$ between the corresponding flag varieties. The main result of this chapter is that, for a degree large enough, the variety associated with rational curves of a given degree in X having their marked points within Schubert varieties X_i of X is a rationally connected fibration over its image, which is the variety associated with rational curves of a given degree in X having their marked points within Schubert varieties $\pi(X_i)$ of X'. These subvarieties of the space of genus 0 stable maps to a homogeneous space are called Gromov-Witten varieties. The Euler characteristics of the structure sheaf of these varieties are quantum K-theoretical invariants [Giv00]. When these varieties are zero dimensional, their number of points are Gromov-Witten invariants. Our result in particular implies the equality between (non equivariant) correlators proved in Chapter 2, providing an alternative proof-cf. chapter 2 Section 2.4.

1.1.1 Definitions. Let $n > 0$. Let $I = \{i_1, \ldots, i_m\}$ be a collection of integers satisfying $0 < i_1 < i_2 < \cdots < i_m < n$. Denote by Fl_I the flag variety parametrizing m-tuples of vector spaces ordered by inclusion

$$V_{i_1} \subset V_{i_2} \cdots \subset V_{i_m} \subset \mathbb{C}^n \tag{1.1}$$

such that $\dim V_{i_k} = i_k$. Let B be a Borel subgroup of GL_n. Note that Fl_I is the homogeneous variety $Fl_I \simeq \mathrm{GL}_n/P_I$, where P_I is the parabolic subgroup of GL_n satisfying $B \subset P_I$ with associated reductive subgroup $L_I \simeq \prod_{p=1}^{m+1} \mathrm{GL}(i_p - i_{p-1})$, where we set $i_0 = 0$ and $i_{m+1} = n$. Let $J = \{j_1, \ldots, j_M\}$ be a collection of integers satisfying $0 < j_1 < \cdots < j_M < n$ containing I, i.e. for each k there exists an element s such that $j_s = i_k$. Consider an integer p such that $1 \leqslant p \leqslant M$. We denote by $J_{\geqslant p}$ the collection $J_{\geqslant p} := \{i_1, \ldots, i_{p'}, j_p, \ldots, j_M\}$, where p' is the largest integer such that $i_{p'} < j_p$. Forgetting all vector spaces included in V_{j_p} except those of the form V_{i_k} yields a forgetful morphism

$$\pi_{I/J_{\geqslant p}} : Fl_{J_{\geqslant p}} \to Fl_I.$$

In the same way, for $1 \leqslant p \leqslant M$, we name $J_{\leqslant p} := \{j_1, \ldots, j_p, i_{p'+1} \ldots, i_m\}$, where p' is the smallest integer such that $j_p < i_{p'}$. We name $\pi_{I/J_{\leqslant p}} : Fl_{J_{\leqslant p}} \to Fl_I$ the morphism induced by forgetting all vector spaces containing V_{j_p} except those of the form V_{i_k}.

Let $W = N_{\mathrm{GL}_n}(T)/T \simeq \mathfrak{S}_n$ be the Weyl group of the algebraic group GL_n. We denote by $Fl_J(u) = \overline{BuP_J/P_J}$ the Schubert variety of Fl_J associated with $u \in W$. Note that the image of a Schubert variety $Fl_J(u)$ by the projection $\pi_{I/J}$ is the Schubert variety $\pi_{I/J}(Fl_J(u)) = Fl_I(u)$.

Let $r > 0$. The group $H_2(Fl_I, \mathbb{Z})$ is a free abelian group generated by classes of Schubert varieties of dimension 1. Denote by $E(Fl_I) \subset H_2(Fl_I, \mathbb{Z})$ the semi-group of effective classes of curves. Note

that $H_2(\mathrm{Gr}(k,n),\mathbb{Z}) \simeq \mathbb{Z}$, and the class of a curve on the Grassmannian $\mathrm{Gr}(k,n)$ is given by the degree of its image in $\mathbb{P}^{\binom{n}{k}}$ by the Plücker embedding. The class $\mathbf{d} = (d_1, \ldots, d_m) \in E(Fl_I)$ of a curve C in Fl_I is determined by m non-negative integers d_1, d_2,..., d_m, where d_k is the degree of the projection of C to the Grassmannian $\mathrm{Gr}(i_k, n)$. For a class $\mathbf{d}$ in $E(Fl_I)$, we denote by $\overline{\mathcal{M}_{0,r}}(Fl_I, \mathbf{d})$ the moduli space parametrizing stable maps $C \to Fl_I$ from genus 0 curves C with r marked points $x_i \in C$ to Fl_I representing the class $\mathbf{d}$. We name $ev : \overline{\mathcal{M}_{0,r}}(Fl_I, \mathbf{d}) \to (Fl_I)^r$ the evaluation morphism sending the point p associated with a map $f_p : C \to Fl_I$ to the image $\prod_{i=1}^r f_p(x_I)$ of the r marked points $x_i \in C$. For any class $\mathbf{d}$ in $E(Fl_J) \simeq \mathbb{N}^M$, the morphism $\pi_{I/J} : Fl_J \to Fl_I$ induces a morphism

$$\Pi_{I/J} \times ev : \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_* \mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r. \tag{1.2}$$

Let $1 \leqslant k \leqslant M$ such that the integer j_k is not in I. We denote by

$$\Pi_{J_{\geqslant k}} \times ev : \overline{\mathcal{M}_{0,r}}(Fl_{J_{\geqslant k}}, (\pi_{I/J_{\geqslant k}})_* \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(Fl_{J_{\geqslant k+1}}, (\pi_{I/J_{\geqslant k+1}})_* \mathbf{d}) \times_{(Fl_{J_{\geqslant k+1}})^r} (Fl_{J_{\geqslant k}})^r$$

the morphism induced by forgetting the vector space V_{j_k} and by evaluating the r marked points. Let $1 \leqslant \mu \leqslant k \leqslant M$ such that the element j_k is not in I. We denote by $J_{\mu \leqslant k}$ the collection $\{i_1, \ldots, i_p, j_\mu, \ldots, j_k, i_{p'}, \ldots i_m\}$, where p is the largest element such that $i_p < j_\mu$ and p' is the smallest element such that $i_{p'} > j_k$. We denote by

$$\Pi_{J_{\mu \leqslant k}} \times ev : \overline{\mathcal{M}_{0,r}}(Fl_{J_{\mu \leqslant k}}, (\pi_{J_{\mu \leqslant k}/J})_* \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(Fl_{J_{\mu \leqslant k-1}}, (\pi_{J_{\mu \leqslant k-1}/J})_* \mathbf{d}) \times_{(Fl_{J_{\mu \leqslant k-1}})^r} (Fl_{J_{\mu \leqslant k}})^r$$

the morphism induced by forgetting the vector space V_{j_k} from $Fl_{\mu \leqslant k}$ and evaluating the r marked points.

Definition 1.1.1.1. Consider a collection $\{I, J, \mathbf{d}\}$ of a set $I = \{i_1, \ldots, i_m\}$ of m positive integers satisfying

$$0 < i_1 < \cdots < i_m < n,$$

a set $J = \{j_1, \ldots, j_M\}$ of M positive integers satisfying

$$0 < j_1 < \cdots < j_M < n$$

such that I is contained in J, and a set $\mathbf{d} = (d_1, \ldots, d_M)$ in $\mathbb{N}^M \simeq E(Fl_J)$. We denote by $\mathbf{d}' = (d'_1, \ldots, d'_m) \in \mathbb{N}^m \simeq E(Fl_I)$ the pushforward of $\mathbf{d}$ by the forgetful map $\pi_{I/J}$. Let $1 \leqslant \mu \leqslant M$ such that j_μ is contained in I. We call the collection $\{I, J, \mathbf{d}\}$ *stabilized with respect to* μ if the following conditions are satisfied. First, for all elements $1 \leqslant k < \mu$ such that the integer j_k is not contained in I, the two following conditions are satisfied.

- $$\text{The morphism } \Pi_{J_{\geqslant k}} \times ev \text{ is surjective.} \tag{1.3}$$
- Let k' be the largest integer such that $i_{k'} < j_k$. For any $1 \leqslant p < k'$ we have
$$d_k \geqslant \left\lceil \left(d'_p - d'_{p-1}\right) / (i_p - i_{p-1}) \right\rceil j_k. \tag{1.4}$$

Futhermore, for all $\mu \leqslant k \leqslant M$ such that the integer j_k is not contained in I the two following symmetric conditions are satisfied.

- The morphism $\Pi_{J_{\mu \leqslant k}} \times ev$ is surjective.
- Let k' be the smallest integer such that $i_{k'} > j_k$. For $k' < p \leqslant m$ we have
$$d_k \geqslant \left\lceil \left(d'_p - d'_{p+1}\right) / (i_{p+1} - i_p) \right\rceil (n - j_k).$$

A collection $\{I, J, \mathbf{d}\}$ will be called *stabilized* if there exists an integer μ such that the collection $\{I, J, \mathbf{d}\}$ is stabilized with respect to μ. A collection $\{I, J, \mathbf{d}\}$ will be called *stabilized with respect to* M if for all $1 \leqslant k \leqslant M$ such that j_k is not contained in I the conditions (1.3) and (1.4) are satisfied.

Observe that when $\mu = 1$, the conditions (1.3) and (1.4) are empty. For simplicity we have set $d'_0 = d'_{m+1} = 0 = i_0$ and $i_{m+1} = n$. We have denoted by $\lceil q \rceil$ the ceiling of $q \in \mathbb{Q}$. We will see in Section 1.6 some conditions to ensure that a collection is stabilized. The following interesting cases yield examples of stabilized collections (cf. Subsection 1.6.1).

1. Suppose $I = \{i_1, \dots, i_m\}$ is a collection of integers satisfying $1 < i_1 < \cdots < i_m < n$ and $J = \{j_1, i_1, \dots, i_m\}$ satisfies $0 < j_1 < i_1$. Note that then $Fl_J \to Fl_I$ is the morphism forgetting the first vector space. Then if $d_1 \geqslant j_1 \left(r + 1 + \lfloor d_2/i_1 \rfloor\right)$ the collection $(I, J, (d_1, d_2, \dots d_{m+1}))$ is stabilized.

2. Suppose $J = \{k\}$ and $I = \varnothing$; then Fl_J is the Grassmannian $\mathrm{Gr}(k, n)$ and Fl_I is the point $\mathrm{Spec}(\mathbb{C})$. Then if $d \geqslant rk$ or $d \geqslant r(n-k)$ the collection (I, J, d) is stabilized.

Note that the dependency on the integer r in the examples here above comes from the condition asking the surjectivity of the following morphisms.

$$\Pi_{J_{\mu \leqslant k}} \times ev : \overline{\mathcal{M}_{0,r}}(Fl_{J_{\mu \leqslant k}}, (\pi_{J_{\mu \leqslant k}/J})_* \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(Fl_{J_{\mu \leqslant k-1}}, (\pi_{J_{\mu \leqslant k-1}/J})_* \mathbf{d}) \times_{(Fl_{J_{\mu \leqslant k-1}})^r} (Fl_{J_{\mu \leqslant k}})^r$$

Definition 1.1.1.2. Let $\mathbf{d}$ be an element in $E(Fl_I)$, let $u_1, \dots, u_r$ be elements in W. We call *Gromov-Witten variety of degree* $\mathbf{d}$ *associated with* $u_1, \dots, u_r$ the subvariety

$$\mathcal{W}^{g,\mathbf{d}}_{Fl_I; u_1, \dots, u_r} := ev_1^{-1}(g_1 \cdot Fl_I(u_1)) \cap \cdots \cap ev_r^{-1}(g_r \cdot Fl_I(u_r))$$

of $\overline{\mathcal{M}_{0,r}}(Fl_I, \mathbf{d})$.

Note that for $\mathbf{d}$ in $E(Fl_I)$, $u_1, \dots, u_r$ in W, $\mathcal{W}^{g,\mathbf{d}}_{Fl_I; u_1, \dots, u_r}$ is the variety associated with degree $\mathbf{d}$ rational curves on X having their i-th marked point within $g_i \cdot Fl_I(u_i)$. Denote by 1 the identity in $W \simeq \mathfrak{S}_n$; note that $Fl_I(1) := P_I/P_I$ a point in Fl_I. Furthermore, note that $\mathcal{W}^{g,\mathbf{d}}_{Fl_I; 1, \dots, 1}$ is the variety associated with degree $\mathbf{d}$ rational curves on X whose i-th marked point is the point $g_i P_I/P_I \in Fl_I$.

Remark 1.1.1.1. According to the proof of Kleiman's transversality theorem [Kle74] for a general $g \in \mathrm{GL}_n^r$ the fiber product $overline\mathcal{M}_{0,r}(Fl_I, \mathbf{d}) \underset{(Fl_I)^r}{\times} (g_1 \cdot Fl_I(u_1) \times \cdots \times g_r \cdot Fl_I(u_r))$ is a reduced equidimensional scheme. Hence for g general in GL_n^r, the natural map

$$\overline{\mathcal{M}_{0,r}}(Fl_I, \mathbf{d}) \underset{(Fl_I)^r}{\times} (g_1 \cdot Fl_I(u_1) \times \cdots \times g_r \cdot Fl_I(u_r)) \to \overline{\mathcal{M}_{0,r}}(Fl_I, \mathbf{d}) \supset \mathcal{W}^{g,\mathbf{d}}_{Fl_I; u_1, \dots, u_r}$$

is a closed immersion surjecting into $\mathcal{W}^{g,\mathbf{d}}_{Fl_I; u_1, \dots, u_r}$, and we can identify the two schemes.

1.1.2 Main results.

Definition 1.1.2.1. Let Y, Z be varieties. We call Y a *rationally connected fibration over* Z (respectively *unirational fibration*) if there exists a morphism $Y \to Z$ such that each irreducible component of Y dominates a different irreducible component of Z, Y dominates Z, and the general fiber $Y \to Z$ is a rationally connected (respectively unirational) irreducible variety. We call Y a *tower of unirational fibrations over* Z if there exists a sequence of morphisms

$$Y_1 = Y \to Y_2 \to \cdots \to Y_n = Z$$

such that, for all $1 \leqslant i < n$, Y_i is a unirational fibration over Y_{i+1}.

Theorem 1.1.2.1. *Consider a set $J = \{j_1, \dots, j_M\}$ of M integers satisfying $0 < j_1 < \cdots < j_M < n$, a set $I = \{i_1, \dots, i_m\}$ of m integers satisfying $0 < i_1 < \cdots < i_m < n$ and a degree $\mathbf{d} = (d_1, \dots, d_M) \in E(Fl_J) \simeq \mathbb{N}^M$ such that the collection $\{I, J, \mathbf{d}\}$ is stabilized in the sense of Definition 1.1.1.1.*

i) For x general in $(Fl_J)^r$, the fiber $ev^{-1}(x) \subset \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d})$ is a tower of unirational fibrations over the fiber $ev^{-1}(\pi_{I/J}(x)) \subset \overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_* \mathbf{d})$.

ii) Let $u_1, \ldots, u_r$ be elements in W. For g general in GL_n^r the Gromov-Witten variety $\mathcal{W}^{g,\mathbf{d}}_{Fl_J;u_1,\ldots,u_r}$ is a rationally connected fibration over the Gromov-Witten variety $\mathcal{W}^{g,(\pi_{I/J})_*\mathbf{d}}_{Fl_I;u_1,\ldots,u_r}$.

Let J be a subsrt of I. Denote by $W_J := N_{L_J}(T)/T$ the Weyl group associated with P_J. Denote by

$$p : W/W_J \to W/W_I$$

the natural projection. For u in W/W_I, the inverse image $p^{-1}(u)$ is the set of elements $u \cdot s$, where s is an element in W_I/W_J. Note that the Schubert variety $FL_I(u) := \overline{BuP_I/P_I}$ is independent of the choice of a representative in W of the element u.

Corollary 1.1.2.1. *Let $\mathbf{d} = (d_1, \ldots, d_m)$ be a degree in $E(Fl_I) \simeq \mathbb{N}^m$ and $u_1, \ldots, u_r$ be elements in W such that the following Gromov-Witten invariant is non-zero:*

$$n^{I,\mathbf{d}}_{u_1,\ldots,u_r} := \int_{\overline{\mathcal{M}_{0,r}}(Fl_I,\mathbf{d})} ev_1^*[Fl_I(u_1)] \cup \cdots \cup ev_r^*[Fl_I(u_r)] \neq 0.$$

Let J and I be sets of increasing positive integers and $\mathbf{d}$ be a degree in $E(Fl_J)$ such that the collection $\{I, J, \mathbf{d}\}$ is stabilized in the sense of Definition 1.1.1.1. Then for any elements $v_i \in p^{-1}(u_i)$, for g general in GL_n^r the Gromov-Witten variety $\mathcal{W}^{g,\mathbf{d}}_{Fl_J;v_1,\ldots,v_r}$ has $n^{I,\mathbf{d}}_{u_1,\ldots,u_r}$ connected components, which are irreducible rationally connected varieties.

1.1.3 Previous research.

Gromov-Witten varieties of $\mathbb{P}^1$ and $\mathbb{P}^2$ that are not rationally connected varieties have been found in [IMT14]. For $\mathbb{P}^2$, the locus of $\overline{\mathcal{M}_{0,3d-2}}(\mathbb{P}^2, d)$ corresponding to curves of degree d going through $3d-2$ points is a curve whose genus depends on the degree d, of positive genus as soon as $d > 2$ [Pan97]. A recursive procedure computing the geometric genus of 1-dimensional Gromov-Witten varieties of $\mathbb{P}^r$ was also given in [Ran01]. High degree Gromov-Witten varieties have been shown to be rational varieties for Grassmannians by Buch and Milhacea [BM11], and more generally for cominuscule varieties by Chaput and Perrin [CP11a].

This chapter is organized as follows. We first prove in Section 1.2 that, when the general fiber of the morphism $\Pi_{I/J} \times ev$ defined by (1.2) is rationally connected, Gromov-Witten varieties of Fl_I are rationally connected fibrations over Gromov-Witten varieties of Fl_J obtained by projection. We then recall in Section 1.3 results on flags of vector bundles over $\mathbb{P}^1$. Section 1.4 is dedicated to the study of the general fiber of the map $\Pi_{I/J} \times ev$. We observe that when the flag variety Fl_I is obtained from Fl_J by forgetting one vector space, the general fiber of this map is a unirational variety. This result is central in the derivation of Theorem 1.1.2.1 presented in Section 1.5. Finally, we study the surjectivity of the morphism $\Pi_{I/J} \times ev$ in Section 1.6.

Conventions

Schemes considered in the subsections 1.2.2 and 1.2.3 will be defined over an algebraically closed field. Schemes considered in all other Sections will be defined over $\mathbb{C}$.

We say a property (P) holds for a *general* point in a variety X if (P) is true for points belonging to a dense open subset of X. We say the general fiber of a morphism $f : X \to Y$ satisfies (P) if $f^{-1}(x)$ satisfies (P) for points x in a dense open subset of Y. Let G be an integral algebraic group scheme, let Y be an integral scheme endowed with a transitive G-action, let Z be a subscheme of Y, and $M \to Y$ be a morphism of integral schemes. We say a property (P) holds for a subscheme $M \times_Y g \cdot Z$ of M in *general position* in M if (P) holds for element g of G belonging to a dense open subset of G.

We consider here that a variety is not necessarily irreducible; i.e. a variety is a reduced, finite type scheme defined over an algebraically closed field.

1.2 Restriction to a general subvariety

Let $X \simeq G/P$ and $X' \simeq G/P'$ be (generalized) flag varieties associated with parabolic subgroups $P \subset P'$ of a semi-simple linear algebraic group G. We observe here that some properties of Gromov-Witten varieties of X and X' can be deduced from properties of the general fiber of the morphism $\Pi \times ev : \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d}) \times_{(X')^r} X^r$-cf. Proposition 1.2.4.1. This is one of the key ingredients in the derivation of Theorem 1.2.2.1 we provide in Section 1.5.

More precisely, the property we prove here is the following. Let G be a semi-simple algebraic group, let Z' be a projective variety with a transitive G-action, let Z be a projective variety over Z' such that the morphism $Z \to Z'$ is G-equivariant. Let $\Pi : M \to M'$ be a morphism of projective varieties over Z'. Let $M \to Z$ be a morphism of projective varieties, let X be a subvariety of Z. Then for g general in G the irreducibility (respectively rational connectedness) of the general fiber of the restriction map $\Pi_{|gX} : M_{g\cdot X} \to M'_{g\cdot X}$induced by Π is implied by the irreducibility (respectively rational connectedness) of the general fiber of the morphism $M \to M' \times_{Z'} Z$- cf. Proposition 1.2.3.1.

1.2.1 Rationally connected, unirational and rational schemes.

We call a scheme X *rationally connected* if there is an irreducible rational curve joining two general points on X. Note that a rationally connected scheme is irreducible. By contradiction suppose a scheme Y has several irreducible components Y_i and is rationally connected. We can then find two points y_1 and y_2 that each belongs to an irreducible component Y_i and is not contained in any other irreducible component of Y such that there is an irreducible curve C joining the points y_1 and y_2. Then the irreducible curve C belongs to both Y_1 and Y_2 and is not contained in the intersection of the two irreducible components, which is a contradiction.

We call a variety Y *rational* if there exists $N > 0$ such that Y is birational to $\mathbb{P}^N$. A variety Y is called *unirational* if there exists a dominant rational map $\mathbb{P}^N \dashrightarrow Y$. Note that since $\mathbb{P}^N$ is rationally connected, rational and unirational varieties are rationally connected.

1.2.2 Composition of rationally connected fibrations.

We call a a morphism of schemes $f : Y \to S$ a rationally connected fibration if

- Each irreducible component of Y dominates a different irreducible component of S;
- Y dominates S;
- The general fiber of f is rationally connected.

We call a scheme Y a *rationally connected fibration over a scheme* S if there exists a rationally connected fibration $Y \to S$.

Definition 1.2.2.1. Let $\mathcal{C}$ be a subcategory of the category of schemes Sch. We call a property (P) *stable under composition in* $\mathcal{C}$ if for every morphisms $Y \xrightarrow{f} S$ and $S' \xrightarrow{f'} S$ in $\mathcal{C}$ satisfying (P), the composition $f' \circ f : Y \to S$ also satisfies (P).

Types of morphisms of schemes stable under composition are well known; for example flat, smooth or etale morphisms are stable under composition-cf. for example Stacks Project 62.4. We focus here on composing rationally connected fibrations. Grabber-Harris-Starr's Theorem on rational connectedness implies that the composition of rationally connected fibrations is itself a rationally connected fibration-cf. Theorem 1.2.2.1. This property will be a key ingredient in the derivation of Theorem 1.1.2.1 proposed in Section 1.5.

Theorem 1.2.2.1. *Let $\mathcal{C}$ be the category of irreducible projective schemes.*

(1) The property "THE GENERAL FIBER IS IRREDUCIBLE" *is stable under composition in $\mathcal{C}$.*

(2) Let (P) be a property such that, for all morphisms $Y \to S$ in $\mathcal{C}$, the following statement is verified:

if S and the general fiber of the morphism $Y \to S$ satisfy (P), then Y satisfies (P).

Then the property "THE GENERAL FIBER SATISFIES (P) AND IS IRREDUCIBLE" *is stable under composition in $\mathcal{C}$.*

(3) The property "THE GENERAL FIBER IS RATIONALLY CONNECTED" *is stable under composition in the category of irreducible complex projective schemes.*

(4) Let Y, Z and S be projective complex varieties such that Y is a rationally connected fibration over Z and Z is a rationally connected fibration over S. Then Y is a rationally connected fibration over S.

Lemma 1.2.2.1. *Let $f : Y \to Z$ be a dominant morphism of schemes, of finite presentation, where Y is an equidimensional noetherian scheme. Let V be a dense open subset of Y. Then the intersection of the general fiber of f with V is dense in the fiber.*

Proof. First, since the question is local on Z, we can suppose Z irreducible. Since Y is a noetherian scheme, $Y \backslash V$ can be written as the union of a finite number of schemes, i.e. $Y \backslash V = \cup_{i=1}^{N} F_i$, where the F_i are closed subschemes of Y, of dimension smaller than the pure dimension of Y. Denote by $f_{|F_i}$ the restriction of f to one of the boundary subschemes F_i. If $f_{|F_i}$ does not dominate Z, then the preimage under $f_{|F_i}$ of a general point in Z is empty. On the other side, if $f_{|F_i}$ dominates Z, then the preimage under $f_{|F_i}$ of a general point in Z is a scheme of pure dimension smaller than $\dim Y - \dim Z$. Indeed, since f is a morphism of finite presentation, Chevalley's constructibility theorem implies $f(F_i)$ contains an open subset of Z; hence the subset of points x in $f(F_i)$ such that the fiber $f_{|F_i}^{-1}(x)$ is of expected dimension is an open subset V of Z, verifying $\dim f_{|F_i}^{-1}(x) = \dim F_i - \dim Z$ for all points x in V. Thus, since Y is equidimensional, the general fiber Y_0 of f is a scheme of pure dimension $\dim Y - \dim Z$, whose intersection with F_i is a scheme of dimension smaller than $\dim Y - \dim Z$; this implies the intersection $Y_0 \cap V$ is dense in Y_0. □

Let $h : M \to M'$ and $f : M \to Z$ be morphisms of schemes. We denote by $h \times f : M \to M' \times Z$ the morphism induced by h and f.

Proposition 1.2.2.1. *Let $M \xrightarrow{h} M'$ and $M \xrightarrow{f} Z$ be surjective morphisms of finite presentation of integral noetherian schemes.*

Then, for z general in Z:

i) The image by h of any irreducible component of $f^{-1}(z)$ has dimension equal to $\dim(h \times f)(M) - \dim Z$;

ii) Each irreducible component of $f^{-1}(z)$ dominates one of the irreducible components of $h(f^{-1}(z))$.

Proof. i) Consider a dense open subset U of $(h \times f)(M)$ such that the fiber $(h \times f)^{-1}(u)$ is a scheme of pure dimension $\dim M - \dim(h \times f)(M)$ for all u in U. Consider a point z general in Z, and an irreducible component Z_0 of $f^{-1}(z)$. Denote by Z_0° the variety Z_0 minus the intersection of Z_0 with other irreducible components of $f^{-1}(z)$. We first compute the dimension of the fiber $h^{-1}(h(\mu)) \cap Z_0^\circ$, where μ is a point in $(h \times f)^{-1}(U) \cap Z_0^\circ$, before computing $\dim h(Z_0)$.

Computation of $\dim h^{-1}(h(\mu)) \cap Z_0^\circ$**.** First, we have:

$$\begin{aligned} h^{-1}(h(\mu)) \cap f^{-1}(z) &= (h \times f)^{-1}(h(\mu), z) \\ &= (h \times f)^{-1}(h \times f)(\mu). \end{aligned}$$

Since $(h \times f)(\mu)$ lies in U, this implies $h^{-1}(h(\mu)) \cap f^{-1}(z)$ is a scheme of pure dimension $\dim M - \dim(h \times f)(M)$. Hence the intersection of the irreducible variety Z_0° with $h^{-1}(h(\mu))$ is a scheme of pure dimension $\dim M - \dim(h \times f)(M)$. Indeed, the intersection of the open subvariety Z_0° of $f^{-1}(z)$ with $h^{-1}(h(\mu))$ is a non empty open subvariety of $h^{-1}(h(\mu)) \cap Z_0$ disconnected from the other pure dimensional irreducible components of $h^{-1}(h(\mu)) \cap f^{-1}(z)$. Hence $h^{-1}(h(\mu)) \cap Z_0^\circ$ is a scheme of pure dimension $\dim M - \dim(h \times f)(M)$.

Computation of $\dim h(Z_0)$**.** Note that, since z is general in Z, according to Lemma 1.2.2.1, $(h \times f)^{-1}(U) \cap f^{-1}(z)$ is dense in $f^{-1}(z)$; hence $(h \times f)^{-1}(U) \cap Z_0^\circ$ is a dense open subset of Z_0. Consider a point ν general in $h(Z_0)$; since ν is general and $h\left((h \times f)^{-1}(U) \cap Z_0^\circ\right)$ is dense in the irreducible variety $h(Z_0)$, ν belongs to $h\left((h \times f)^{-1}(U) \cap Z_0^\circ\right)$, and we have:

$$\begin{aligned}\dim Z_0 - \dim h(Z_0) &= \dim h^{-1}(\nu) \cap Z_0^\circ \\ &= \dim M - \dim(h \times f)(M)\end{aligned}$$

where the first equality is deduced from the general position of ν in $h(Z_0)$, and the second equality holds since ν is in $h\left((h \times f)^{-1}(U) \cap Z_0^\circ\right)$. This implies:

$$\begin{aligned}\dim h(Z_0) &= \dim Z_0 - \dim M + \dim(h \times f)(M) \\ &= \dim f^{-1}(z) - \dim M + \dim(h \times f)(M),\end{aligned}$$

where the second equality holds since z is general in Z and thus $f^{-1}(z)$ is equidimensional.

ii) By contradiction, if an irreducible component Z_0 of $f^{-1}(z)$ has an image strictly contained in $h\left(f^{-1}(z)\right)$, then:

$$\dim h(Z_0) < \dim h\left(f^{-1}(z)\right) = \dim(h \times f)(M) - \dim Z,$$

which would contradict i).

□

Proof of Theorem 1.2.2.1. (1) Let $h : Y \to S$ and $S \to Z$ be morphisms in $\mathcal{C}$ whose general fiber is irreducible. Note that since we consider properties of general fibers, we can assume that both morphisms are dominant. Since both morphisms are morphisms of projective schemes, they are then surjective.

Let U be a dense open subset of Z such that the fiber S_z over a point z in U is irreducible. For z general in Z, according to Proposition 1.2.2.1 the image by $h : Y \to S$ of each irreducible component of Y_z dominates one of the irreducible components of S_z, which is irreducible since z is general in Z. Since, according to Lemma 1.2.2.1, for z general in Z the fiber Y_z has a dense intersection with the dense open subset Y_U, the general fiber Y_z is irreducible. Indeed, if we denote by Y_z^1 and Y_z^2 two different irreducible components of Y, and by V the dense open subset of S_z such that the fiber Y_s over a point s in V is irreducible and of exepected dimension, for all v in V the intersection $Y_z^1 \cap Y_v$ is equal to $Y_z^2 \cap Y_v$, since both are closed subschemes of maximal dimension of an irreducible scheme. Hence the intersection of Y_z with the dense open subset $Y_z \cap Y_V$ is irreducible, hence Y_z is irreducible.

(2) Let $Y \xrightarrow{f_1} S$ and $S \xrightarrow{f_2} S'$ be morphisms in $\mathcal{C}$ satisfying (P) and whose general fiber is irreducible. Since we consider properties of the general fiber of $Y \to S'$, we can assume that Y dominates S. Let U be a dense open subset of S such that the fiber of f_1 over a point in U satisfies (P). For x general in S', the following properties are satisfied:

(a) The fiber S_x satisfies (P);

(b) The fiber S_x has a dense open intersection with U;

(c) The morphism $Y_x \to S_x$ is in $\mathcal{C}$;

(d) The general fiber of the morphism $Y_x \to S_x$ satisfies (P);

(e) Y_x satisfies (P).

Note that proving property (e) for x general in S' is our goal here. Property (a) holds since x is general in S'. According to Lemma 1.2.2.1 and since x is general in S', property (b) holds. Property (c) holds since x is general in S' and according to (1) the general fiber of $Y \to S'$ is irreducible. (d) is a direct consequence of (b), the fact that the fiber of $Y_x \to S_x$ over a point y in S_x is isomorphic to the fiber $Y_y = Y \times_{S'} \{y\}$, and the fact that Y_y satisfies (P) for y in U. Finally (e) is implied by (a), (c) and (d).

(3) Note that, according to Grabber-Harris-Starr's theorem on rational connectedness [GHS03], if the general fiber of a morphism $Y \to S$ of irreducible complex projective varieties is rationally connected, and the basis S is rationally connected, then Y is rationally connected. Hence, according to 2., the property "the general fiber is irreducible and rationally connected" is stable under composition in $\mathcal{C}$.

(4) Let Y, Z and S be projective complex schemes, where Y is a rationally connected fibration over Z, and Z is a rationally connected fibration over S. Denote by $f : Y \to Z$ and $g : Z \to S$ the associated morphisms. First, note that since Y dominates Z and Z dominates S, Y dominates S. Furthermore, each irreducible component S_i of S is dominated by exactly one irreducible component Z_i of Z, which itself is dominated by exactly one irreducible component Y_i of Y; hence there is a one-to-one correspondence between irreducible components of Y and S. Finally, since by hypothesis the general fibers of $Y_i \to Z_i$ and $Z_i \to S_i$ are rationally connected, according to (3) the general fiber of $Y_i \to S_i$ is a rationally connected variety. Hence Y is a rationally connected fibration over S.

□

1.2.3 Restriction to a general subvariety.

We study here the restriction of a morphism to a general subvariety. A typical application of the proposition below will be the study of the image of a Gromov-Witten variety. More precisely, consider $\pi : X \to X'$ the projection of a flag variety onto another flag variety, and denote by $\Pi : \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X', (\pi)_*\mathbf{d})$ the morphism induced by π between moduli spaces of genus zero stable maps, where $\mathbf{d}$ is the class of a curve in X. We thus obtain the commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) & \longrightarrow & X^r \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \pi^r} \\ \overline{\mathcal{M}_{0,r}}(X', (\pi)_*\mathbf{d}) & \longrightarrow & (X')^r \end{array}$$

to which we will apply the following proposition.

Proposition 1.2.3.1. *Let G be an integral group scheme, let $f : M \to Z$, $f' : M' \to Z'$, $\pi : Z \to Z'$ be projective morphisms, of integral noetherian schemes, where Z and Z' admit a transitive G-action, and the morphism π is G-equivariant. Let Y be an integral subscheme of Z. Suppose there exists a proper morphism $h : M \to M'$ such that the following diagram commutes :*

$$\begin{array}{ccc} M & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f'} & Z' \end{array}$$

Furthermore, suppose the morphism $h \times f : M \to M' \times_{Z'} Z$ induced by the morphisms $M \xrightarrow{h} M'$ and $M \xrightarrow{f} Z$ is surjective.

Consider a point g general in G. We call $\mathcal{W}_g := f^{-1}(gY)$, $\mathcal{W}'_g := (h \times f)(\mathcal{W}_g)$, and $\mathcal{W}''_g := h(\mathcal{W}_g)$. We denote by $p_1 : M' \times_{Z'} Z \to M'$ the first projection. Then:

i) $\mathcal{W}'_g = M' \times_{Z'} gY$ and $\mathcal{W}''_g = (f')^{-1}(g\pi(Y))$;

ii) $\mathcal{W}_g$, $\mathcal{W}'_g$ and $\mathcal{W}''_g$ are pure dimensional schemes;

iii) The restriction of $h \times f$ to any irreducible component of $\mathcal{W}_g$ is dominant onto an irreducible component of $\mathcal{W}'_g$;

iv) The restriction of p_1 to any irreducible component of $\mathcal{W}'_g$ is dominant onto an irreducible component of $\mathcal{W}''_g$.

Let us introduce the following notation. Let $f : Y \to Z$ and $\varphi : Y' \to Z$ be morphisms of schemes, where an integral group scheme G acts on Z. We denote by $Y \times_Z (gY')$ the fiber product induced by the morphism $\varphi_g : Y' \to Z$ given by $y \to g \cdot \varphi(y)$, i.e. $Y \times_Z (gY')$ is the fiber product associated with the following commutative diagram:

$$\begin{array}{ccc} Y \times_Z (gY') & \longrightarrow & Y' \\ \downarrow & & \downarrow{\scriptstyle \varphi_g} \\ Y & \xrightarrow{f} & Z \end{array}$$

Notice that, if G acts on Y' and φ is G-equivariant, the fiber product $Y \times_Z (gY')$ will simply be the usual fiber product associated with the following commutative diagram:

$$\begin{array}{ccc} Y \times_Z (gY') & \longrightarrow & gY' \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{f} & Z \end{array}$$

Lemma 1.2.3.1. *Let $Y \to Z$ be a morphism of schemes, of finite presentation, where Y is an equidimensional noetherian scheme; let $\varphi : Y' \to Z$ be a morphism of integral schemes. Let V be a dense open subset of Y. Suppose there exists an integral group scheme G such that G acts transitively on Z.*

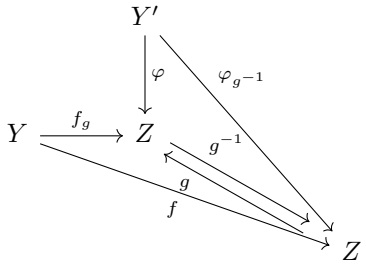
Then, for a point g general in G:

i) The scheme $V \times_Z (gY')$ is a dense subset of $Y \times_Z (gY')$;

ii) The scheme $(gV) \times_Z Y'$ is a dense subset of $(gY) \times_Z Y'$.

Proof. i) Since Y is a noetherian scheme, $Y \backslash V$ can be written as the union of a finite number of irreducible schemes, i.e. $Y \backslash V = \cup_{i=1}^{N} F_i$, where the F_i are irreducible closed subschemes of Y, of dimension smaller than the pure dimension of Y. Consider an element g general in G. Suppose the scheme $Y \times_Z (gY')$ is not empty. Then, according to Kleiman's transversality theorem [Kle74], it has pure dimension given by $\dim Y + \dim Y' - \dim Z'$. Furthermore, for $1 \leqslant i \leqslant N$, the scheme $F_i \times_{Z'} (gY')$ is either empty or of pure dimension smaller than $\dim Y + \dim Y' - \dim Z'$. Hence $V \times_{Z'} (gY') = (Y \backslash \cup_{i=1}^{N} F_i) \times_{Z'} (gY')$ is dense in $Y \times_Z (gY')$.

ii) Consider a point g general in G. Then, according to part *i*)., the scheme $V \times_Z (g^{-1}Y')$ is a dense subset of $Y \times_Z (g^{-1}Y')$. Furthermore, notice that we have an isomorphism $(gY) \times_Z Y' \to Y \times_Z (g^{-1}Y')$. Indeed, if we denote by $g^{-1} : Z \to Z$ the isomorphism of Z induced by the action of g^{-1}, and by $g : Z \to Z$ the isomorphism of Z induced by the action of g, the following commutative diagram



implies that any morphism $T \to (gY) \times_Z Y'$ factors through $Y \times_Z (g^{-1}Y')$, and reciprocally any morphism $T \to Y \times_Z (g^{-1}Y')$ factors through $(gY) \times_Z Y'$.
Finally, in the same tautological way, notice we have an isomorphism $(gV) \times_Z Y' \simeq V \times_Z (g^{-1}Y')$. □

Proof of Proposition 1.2.3.1. i) First, since, by assumption, $h \times f : M \to M' \times_{Z'} Z$ is surjective, $\mathcal{W}'_g = (h \times f)\left(f^{-1}(gY)\right) = (h \times f)(M) \cap (M' \times_{Z'} (gY))$ is equal to $M' \times_{Z'} (gY)$. Furthermore,

$h(f^{-1}(gY))$ is equal to the image of $(h \times f)(f^{-1}(gY))$ by the first projection p_1, where p_1 is defined by the following commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{h\times f} & (h\times f)(M) = M' \times_{Z'} Z & \longrightarrow & Z \\ & \searrow^{h} & \downarrow p_1 & & \downarrow \pi \\ & & M' & \xrightarrow{f'} & Z' \end{array}$$

Meanwhile, $(f')^{-1}(\pi(gY))$ is equal to the image of $M' \times_{Z'} (gY)$ by the first projection p_1, hence the equality between $h(f^{-1}(gY))$ and $(f')^{-1}(\pi(gY))$.
Finally, since π is G-equivariant, this yields: $\mathcal{W}_g'' = (f')^{-1}(g\pi(Y))$.

ii) According to Kleiman's transversality theorem, since g is general in G, the schemes $\mathcal{W}_g = f^{-1}(gY)$, $\mathcal{W}_g' = M' \times_{Z'} gY$ and $\mathcal{W}_g'' = (f')^{-1}(g\pi(Y))$ are either empty or pure dimensional schemes.

iii) We consider the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle h\times f} & & \| \\ M' \times_{Z'} Z & \longrightarrow & Z \end{array}$$

Denote by U a dense open subset of $(h \times f)(M)$ such that, for all z in U, $(h \times f)^{-1}(z)$ has pure dimension equal to $\dim M - \dim(h \times f)(M)$. Fix an irreducible component $\mathcal{W}_0$ of $\mathcal{W}_g$. Denote by $\mathcal{W}_0^\circ$ the variety $\mathcal{W}_0$ minus the intersection of $\mathcal{W}_0$ with other irreducible components o $\mathcal{W}_g$. We first compute the dimension of $(h \times f)^{-1}(x) \cap \mathcal{W}_0^\circ$, where x is a point in $(h \times f)(\mathcal{W}_0^\circ) \cap U$, before computing the dimension of $(h \times f)(\mathcal{W}_0)$.

Computation of $\dim(h \times f))^{-1}(x) \cap \mathcal{W}_0^\circ$**.** First, the fiber of the restriction $(h \times f)_{|\mathcal{W}_g}$ of the morphism $h \times f$ to $\mathcal{W}_g$ is isomorphic to the fiber of $h \times f$. Indeed, for (m, z) in $(h \times f)(\mathcal{W}_g) = M' \times_{Z'} gY$, the fiber ${(h \times f)_{|\mathcal{W}_g}}^{-1}(m, z)$ is equal to $(h\times f)^{-1}(m, z)\cap\mathcal{W}_g = h^{-1}(m)\cap f^{-1}(z)\cap\mathcal{W}_g$, where z lies in gY. Meanwhile, we observe $f^{-1}(z) \cap \mathcal{W}_g = f^{-1}(z) \cap f^{-1}(gY) = f^{-1}(z)$, hence ${(h \times f)_{|\mathcal{W}_g}}^{-1}(m)$ is equal to $(h \times f)^{-1}(m)$.

Since x lies in U, this implies ${(h \times f)}^{-1}(x) \cap \mathcal{W}_g$ is a scheme of pure dimension $\dim M - \dim(h \times f)(M)$. Hence the intersection of the irreducible variety $\mathcal{W}_0^\circ$ with ${(h \times f)}^{-1}(x)$ is a scheme of pure dimension $\dim M - \dim(h \times f)(M)$. Indeed, the intersection of the open subvariety $\mathcal{W}_0^\circ$ of $\mathcal{W}_g$ with $(h\times f)^{-1}(x)$ is a non empty open subvariety of $(h\times f)^{-1}(x)\cap\mathcal{W}_0$ disconnected from the other pure dimensional irreducible components of $(h \times f)^{-1}(x) \cap \mathcal{W}_g$. Hence $(h \times f)^{-1}(x) \cap \mathcal{W}_0^\circ$ is a scheme of pure dimension $\dim M - \dim(h \times f)(M)$.

Computation of $\dim(h \times f)(\mathcal{W}_0)$**.** Note that, since g is general in G, according to Lemma 1.2.3.1, $(h \times f)^{-1}(U) \cap \mathcal{W}_g$ is dense in $\mathcal{W}_g$; hence $(h \times f)^{-1}(U) \cap \mathcal{W}_0^\circ$ is a dense open subset of $\mathcal{W}_0$. Consider a general point ν in $(h\times f)(\mathcal{W}_0)$; since ν is general and $(h\times f)\left((h \times f)^{-1}(U) \cap \mathcal{W}_0^\circ\right)$ is dense in the irreducible variety $(h \times f)(\mathcal{W}_0)$, ν belongs to $(h \times f)\left((h \times f)^{-1}(U) \cap \mathcal{W}_0^\circ\right)$, and we have:

$$\begin{aligned} \dim\mathcal{W}_0 - \dim(h \times f)(\mathcal{W}_0) &= \dim{(h \times f)}^{-1}(\nu) \cap \mathcal{W}_0^\circ \\ &= \dim M - \dim(h \times f)(M) \end{aligned}$$

where the first equality is deduced from the general position of ν in $(h \times f)(\mathcal{W}_0)$, and the second equality holds since ν is in $(h \times f)\left((h \times f)^{-1}(U) \cap \mathcal{W}_0^\circ\right)$. This implies:

$$\begin{aligned} \dim(h \times f)(\mathcal{W}_0) &= \dim\mathcal{W}_0 - \dim M + \dim(h \times f)(M) \\ &= \dim\mathcal{W}_g - \dim M + \dim(h \times f)(M), \end{aligned}$$

where the second equality holds since $\mathcal{W}_g$ is equidimensional. Since $(h \times f)(\mathcal{W}_0)$ is obviously contained in $\mathcal{W}'_g = (h \times f)(\mathcal{W}_g)$, and this dimension does not depend on the irreducible component of $\mathcal{W}_g$ considered, this implies the irreducible variety $(h \times f)(\mathcal{W}_0)$ is an irreducible component of $\mathcal{W}'_g$, which has pure dimension equal to $\dim \mathcal{W}_g - \dim M + \dim(h \times f)(M)$.

iv) First, denote by $\pi_{|Y} : Y \to \pi(Y)$ the restriction of the morphism π to Y. Fix a dense open subset U of $\pi(Y)$ such that, for all x in U, the fiber $\pi_{|Y}^{-1}(x)$ is a scheme of pure dimension equal to $\dim Y - \dim \pi(Y)$.

We consider the following commutative diagram:

$$\begin{array}{ccc} M' \times_{Z'} Z & \xrightarrow{p_2} & Z \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f'} & Z' \end{array}$$

where p_1 and p_2 are respectively the first and second projection morphisms. Consider a general element g in G. We denote by p'_1 the restriction of the morphism p_1 to $\mathcal{W}'_g = M' \times_{Z'} gY$. We want to prove here that the image of any irreducible component of $\mathcal{W}'_g$ by p'_1 is an irreducible component of $\mathcal{W}''_g = (f')^{-1}(g\pi(Y))$. In order to do this, we study the dimension of the fiber $(p'_1)^{-1}(m)$ of p'_1 at a point m in $\mathcal{W}''_g$.

We will use the following observations.

Lemma 1.2.3.2. *Let $f' : M' \to Z'$ and $\pi : Z \to Z'$ be projective morphisms of integral noetherian schemes. Let Y be an integral subscheme of Z. Suppose there exists an integral group scheme G such that G acts transitively on Z, and π is G-equivariant. Consider the following commutative diagram:*

$$\begin{array}{ccc} M' \times_{Z'} Z & \xrightarrow{p_2} & Z \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f'} & Z' \end{array}$$

where p_1 and p_2 are respectively the first and second projection morphisms.

(a) Consider an element g in G. The fiber $p_1^{-1}(m) \cap {p_2}^{-1}(gY)$ of the restriction of the morphism p_1 to $(p_2)^{-1}(gY)$ is isomorphic to $\pi_{|Y}^{-1}(g^{-1}f'(m))$, where we denote by $\pi_{|Y} : Y \to \pi(Y)$ the restriction of the morphism π to Y;

(b) Consider a dense open subset U of Y. Then, for an element g general in G, ${p_2}^{-1}(gU)$ is dense in ${p_2}^{-1}(gY)$.

Proof of Lemma 1.2.3.2. (a) Denote by h' the restriction of the morphism p_1 to ${p_2}^{-1}(gY)$. The fiber $(h')^{-1}(m)$ is equal to $\{m\} \times_{Z'} (gY)$, which is isomorphic to $\pi^{-1}(f'(m)) \cap (gY)$. Furthermore, since π is G-equivariant, the morphism $y \to g^{-1}y$ induces an isomorphism between $\pi^{-1}(f'(m)) \cap (gY)$ and $\pi^{-1}(g^{-1}f'(m)) \cap (Y)$. Finally, notice $\pi^{-1}(g^{-1}f'(m)) \cap (Y)$ is equal to $\pi_{|Y}^{-1}(g^{-1}f'(m))$.

(b) According to Lemma 1.2.3.1 *ii*), since g is general in G, the scheme $M' \times_{Z'} (gU)$ is dense in $M' \times_{Z'} (gY)$. Since the morphism π is G-equivariant, we have here the following equalities: ${p_2}^{-1}(gY) = M' \times_{Z'} (gY)$ and ${p_2}^{-1}(gU) = M' \times_{Z'} (gU)$. Hence the scheme ${p_2}^{-1}(gU)$ is dense in ${p_2}^{-1}(gY)$.

□

Fix an irreducible component $\mathcal{W}'_0$ of $p_2^{-1}(gY) = M' \times_{Z'} gY$. For μ general in $\mathcal{W}'_0$, we have:

$$\begin{aligned} \dim \mathcal{W}'_0 - \dim p_1(\mathcal{W}'_0) &= \dim p_1^{-1}(p_1(\mu)) \cap \mathcal{W}'_0 \\ &= \dim \pi_{|Y}^{-1}\left(g^{-1} \cdot (f' \circ p_1)(\mu)\right) \\ &= \dim Y - \dim \pi(Y), \end{aligned}$$

where the first equality holds since the image of a dense open subset of $\mathcal{W}'_0$ by p_1 is a dense open subset of $p_1(\mathcal{W}'_0)$, the second equality is given by Lemma 1.2.3.2 (a), and the last equality holds since $(f' \circ p_1)(\mu)$ is in $g \cdot \pi(U)$ for μ general in $\mathcal{W}'_0$. Indeed, according to Lemma 1.2.3.2 (b), $p_2^{-1}(g\pi^{-1}(U))$ is dense in $p_2^{-1}(gY)$, hence $p_2^{-1}(g\pi^{-1}(U)) \cap \mathcal{W}'_0$ is dense in $\mathcal{W}'_0$, hence a point in $(p_2 \circ \pi)^{-1}(gU) = (f' \circ p_1)^{-1}(gU)$ is general in $\mathcal{W}'_0$.

Finally, notice that, since g is general in G, any irreducible component $\mathcal{W}'_0$ of $\mathcal{W}'_g$ will be of dimension equal to $\mathrm{dim}M' + \mathrm{dim}Y - \mathrm{dim}Z'$, hence :

$$\mathrm{dim}p_1(\mathcal{W}'_0) = \mathrm{dim}M' + \mathrm{dim}\pi(Y) - \mathrm{dim}Z',$$

which is equal to the pure dimension of $\mathcal{W}''_g = (f')^{-1}(g\pi(Y))$ since g is general in G. Since $p_1(\mathcal{W}'_0)$ is obviously contained in $\mathcal{W}''_g$, this implies that $\mathcal{W}'_0$ dominates an irreducible component of $\mathcal{W}''_g$.

□

1.2.4 Application to Gromov-Witten varieties. Consider a semi-simple algebraic group G, a maximal torus T of G, a Borel subgroup B and parabolic subgroups P and P' satisfying $T \subset B \subset P \subset P'$. Denote by $X := G/P$ and $X' := G/P'$ the corresponding homogeneous varieties, and by

$$\pi : X = G/P \to X' = G/P'$$

the natural projection. For an element u in the Weyl group $W := N_G(T)/T$, denote by $X(u) := \overline{BuP}/P$ the associated Schubert variety. Note that $\pi(X(u)) = X'(u)$. Let $r \geqslant 0$, $u_1, \ldots, u_r$ be elements in W, and $\mathbf{d}$ be a degree in the semi-group $E(G/P)$ of effective classes of curves in $H_2(G/P, \mathbb{Z})$. Note that the morphism $\pi : G/P \to G/P'$ induces a morphism:

$$\Pi \times ev : \overline{\mathcal{M}_{0,r}}(G/P, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(G/P', \pi_*\mathbf{d}) \times_{(G/P')^r} (G/P)^r,$$

where we denote by $\Pi : \overline{\mathcal{M}_{0,r}}(G/P, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(G/P', \pi_*\mathbf{d})$ the morphism induced by π. For an element $g = (g_1, \ldots, g_r)$ in G^r, we will consider the following commutative diagram:

$$\begin{array}{ccccc}
\overline{\mathcal{M}_{0,r}}(G/P, \mathbf{d}) & \xrightarrow{\mathrm{ev}} & (G/P)^r & \hookleftarrow & \prod_i g_i X(u_i) \\
\downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \pi} & & \\
\overline{\mathcal{M}_{0,r}}(G/P', \pi_*\mathbf{d}) & \longrightarrow & (G/P')^r & \hookleftarrow & \prod_i g_i X'(u_i)
\end{array}$$

This commutative diagram allows us to define the following fiber products:

$$\begin{array}{ll}
\mathcal{W}_g := \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \times_{X^r} \prod_i g_i X(u_i) & \hookrightarrow \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \\
\mathcal{W}'_g := \overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d}) \times_{(X')^r} \prod_i g_i X(u_i) & \hookrightarrow \overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d}) \times X^r \\
\mathcal{W}''_g := \overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d}) \times_{(X')^r} \prod_i g_i X'(u_i) & \hookrightarrow \overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d})
\end{array}$$

Note that for $g = (g_1, \ldots, g_r)$ general in G^r, the scheme $\mathcal{W}_g = \mathcal{W}^{g,\mathbf{d}}_{u_1,\ldots,u_r}$ is the variety parametrizing genus zero, degree $\mathbf{d}$, stable maps to X sending their i-th marked point within $g_i \cdot X(u_i)$, and $\mathcal{W}''_g = \mathcal{W}^{g,\pi_*\mathbf{d}}_{u_1,\ldots,u_r}$ is the variety parametrizing genus zero, degree $(\pi_*)\mathbf{d}$ stable maps sending their i-th marked point within $g_i \cdot X'(u_i)$. Indeed, according to the proof of Kleiman's transversality theorem [Kle74], for a point $g = (g_1, \ldots, g_r)$ general in G^r, the natural maps $\mathcal{W}_g \to \overline{\mathcal{M}_{0,r}}(G/P, \mathbf{d})$ and $\mathcal{W}''_g \to \overline{\mathcal{M}_{0,r}}(G/P', \mathbf{d})$ are both closed immersions. This allows us to identify the schemes $\mathcal{W}_g$ and $\mathcal{W}''_g$ with their images in the space of stable maps.

Proposition 1.2.4.1. *Suppose the morphism $\Pi \times$ ev is surjective, and suppose its general fiber is connected.*

1. *For g general in G^r, $(\Pi \times ev)(\mathcal{W}_g) = \mathcal{W}'_g$ and $\Pi(\mathcal{W}_g) = \mathcal{W}''_g$;*

2. *For g general in G^r, each irreducible component of the Gromov-Witten variety $\mathcal{W}_g$ surjects into a different irreducible component of $\Pi(\mathcal{W}_g)$, i.e. there is a 1-to-1 correspondence between irreducible components of $\mathcal{W}_g$ and irreducible components of $\mathcal{W}''_g$;*

3. *Suppose the general fiber of $\Pi \times ev$ is rationally connected. Then for g general in G^r, the Gromov-Witten variety $\mathcal{W}_g$ is a rationally connected fibration over both $\mathcal{W}'_g$ and $\mathcal{W}''_g$, in the sense of Definition 1.1.2.1.*

Lemma 1.2.4.1. *Let S be a normal scheme, and $S \xrightarrow{f} Y$ be a surjective morphism. If the general fiber of f is connected, then each irreducible component of $f(S)$ is dominated by exactly one irreducible component of S.*

Proof. First, observe that, since the general fiber of f is connected, two connected components of S cannot dominate the same irreducible component of $f(S)$. Indeed, if we consider a dense open subset U of $f(S)$ such that the fiber $f^{-1}(s)$ is connected for all s in U, two connected components of S dominating the same irreducible component of $f(S)$ would have a point in common in their image and in U; these two connected components hence would be connected within S. Moreover, since S is normal, its connected components are irreducible. This implies all irreducible components of $f(S)$ are dominated by exactly one irreducible component of S. □

For u in W, we denote by $x(u) := uP/P$ the T-fixed point in $X(u) = \overline{BuP/P}$ associated with u.

Lemma 1.2.4.2. *Let $u \in W$.*

i) Let $f : Z \to X(u)$ be a B-equivariant morphism. Then there exists a surjective map $f^{-1}(BuP/P) \to f^{-1}(x(u))$.

ii) Denote by $\pi_{|u}$ the restriction of the projection π to $X(u)$. The general fiber of the surjective morphism $\pi_{|u} : X(u) \to X'(u)$ is an irreducible unirational variety.

Proof. This is implied by [CMBP13], Proposition 2.3. We include here a proof for completeness. We denote by $U = BuP/P$ the open B-orbit of the T-fixed point $x(u)$.

i) Denote by $s : U \to B$ a section of $B \to BuP/P$, i.e. a morphism such that for all elements x in U, $s(x) \cdot x(u) = x$. Such a section exists according to [CMBP13], Proposition 2.2. Note that for y in $f^{-1}(U)$, the element $(s(f(y)))^{-1} \cdot y$ lies in $f^{-1}(x(u))$. Indeed, since f is B-equivariant, we have $f((s(f(y)))^{-1} \cdot y) = (s(f(y)))^{-1} \cdot f(y) = x(u)$. We obtain a morphism:

$$\begin{aligned} f^{-1}(U) &\to f^{-1}(x(u)) \times U \\ y &\to ((s(f(y)))^{-1} \cdot y; f(y)) \end{aligned}$$

which is an isomorphism; indeed, the morphism $(p, u) \to s(u) \cdot p$ defines its inverse. Composition with the first projection induces a surjective morphism $f^{-1}(U) \to f^{-1}(x(u))$.

ii) According to *i)*, there exists a surjective map $\pi_{|u}^{-1}(U) \to \pi_{|u}^{-1}(x(u))$. Hence, since $\pi_{|u}^{-1}(U)$ is a dense open subvariety of the irreducible rational variety $X(u)$, the fiber $\pi_{|u}^{-1}(x(u))$ is an irreducible unirational variety. Furthermore, since $\pi_{|u}$ is B-equivariant, the fiber $\pi_{|u}^{-1}(x)$ at a point x in U is isomorphic to $\pi_{|u}^{-1}(x(u))$. Indeed, if $x = b \cdot x(u)$, where b lies in B, the morphism $y \to b^{-1} \cdot y$ defines an isomorphism between $\pi^{-1}(x) \cap X(u)$ and $\pi^{-1}(x(u)) \cap X(u)$, since $X(u)$ is B-stable. Hence the general fiber of $\pi_{|u}$ is isomorphic to $\pi_{|u}^{-1}(x(u))$, which is an irreducible unirational variety.

□

We name $(\Pi \times ev)_{|\mathcal{W}_g} : \mathcal{W}_g \to \mathcal{W}'_g$ the restriction of the morphism $\Pi \times ev$ to the Gromov-Witten variety $\mathcal{W}_g$.

Lemma 1.2.4.3. *If the general fiber of $\Pi \times ev$ satisfies a property (P), then for g general in G^r the general fiber of $(\Pi \times ev)_{|\mathcal{W}_g}$ also satisfies (P).*

Proof. Name $M := \overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$. Let U be a dense open subset of $(\Pi \times ev)(M)$ such that the fiber of $\Pi \times ev$ over a point in U satisfies (P). Let (m, x) be a point in U, where m lies in $M' := \overline{\mathcal{M}_{0,r}}(X', \pi_* \mathbf{d})$ and x lies in X^r. Note that, if (m, x) is in the image $(\Pi \times ev)(\mathcal{W}_g)$, then the fiber over (m, x) of $(\Pi \times ev)_{|\mathcal{W}_g}$ satisfies (P). Indeed, we have:

$$\begin{aligned}
(\Pi \times ev)^{-1}_{|\mathcal{W}_g}(m, x) &= (M_m \times_{X^r} \{x\}) \times_{X^r} (g \prod_i X(u_i)) \\
&= M_m \times_{X^r} \left(\{x\} \times_{X^r} (g \prod_i X(u_i)) \right) \\
&= M_m \times_{X^r} \{x\} \\
&= (\Pi \times ev)^{-1}(m, x),
\end{aligned}$$

where the third equality holds since (m, x) is in $(\Pi \times ev)(\mathcal{W}_g)$, and hence x lies in $ev(\mathcal{W}_g) \subset g \prod_i X(u_i)$. Since (m, x) lies in U, the fiber $(\Pi \times ev)^{-1}(m, x)$ satisfies (P); hence the fiber $(\Pi \times ev)^{-1}_{|\mathcal{W}_g}(m, x)$ satisfies (P) for all (m, x) in $(\Pi \times ev)(\mathcal{W}_g) \cap U$

Finally, note that for g general in G^r, according to Lemma 1.2.3.1 $\mathcal{W}_g \cap (\Pi \times ev)^{-1}(U)$ is dense in $\mathcal{W}_g$. Hence $(\Pi \times ev)(\mathcal{W}_g) \cap U$ is dense in $(\Pi \times ev)(\mathcal{W}_g)$. □

Proof of Proposition 1.2.4.1.

1. Since the morphism $\Pi \times ev$ is surjective, this follows directly from Proposition 1.2.3.1 *i*), applied to $h = \Pi : M = \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to M' = \overline{\mathcal{M}_{0,r}}(X', \pi_* \mathbf{d})$, $f = \text{ev} : \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to Z = X^r$, $f' = \text{ev} : \overline{\mathcal{M}_{0,r}}(X', \pi_* \mathbf{d}) \to Z' = (X')^r$, and from considering the integral group G^r.

2. We proceed in two steps; we first prove that each irreducible component of $\mathcal{W}_g$ surjects into a different irreducible component of $\mathcal{W}'_g$, before proving the correspondence with irreducible components of $\mathcal{W}''_g$.

   Correspondence between irreducible components of $\mathcal{W}_g$ and $\mathcal{W}'_g$. First, note that, since the space $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$ and Schubert varieties are normal [FP96, Ram85], according to Kleiman's transversality theorem $\mathcal{W}_g = ev^{-1}(g_1 X(u_1) \times \dots g_r X(u_r))$ is normal for $(g_1, \dots, g_r)$ general in G^r. Furthermore, since the general fiber of $\Pi \times ev$ si connected, according to Lemma 1.2.4.3 for g general in G^r the general fiber of the restriction $(\Pi \times ev)_{|\mathcal{W}_g}$ is connected. Hence according to Lemma 1.2.4.1 all irreducible components of $(\Pi \times ev)(\mathcal{W}_g)$ are dominated by exactly one irreducible component of $\mathcal{W}_g$.

   We then only have to ensure that there is no irreducible component of $\mathcal{W}_g$ whose image is a strict closed subset of an irreducible component of $(\Pi \times ev)(\mathcal{W}_g) = \mathcal{W}'_g$. Proposition 1.2.3.1 *iii*), tells us that, since $\mathcal{W}_g$ is in general position, the image by Π of any of its irreducible components is an irreducible component of $\mathcal{W}'_g$, hence the 1-to-1 correspondence.

   Correspondence between irreducible components of $\mathcal{W}'_g$ and $\mathcal{W}''_g$. According to Lemma 1.2.4.2 the general fiber of the restriction $\pi_u : \prod_i X(u_i) \to \prod_i X'(u_i)$ of π^r to $X(u_1) \times \dots \times X(u_r)$ is irreducible, hence there exists an open subvariety U of $X'(u_1) \times \dots \times X'(u_r)$ such that the fiber of π at x in U is irreducible. Consider the projections p_1 and p_2 defined by the following Cartesian diagram:

$$\begin{array}{ccc}
\mathcal{W}'_g & \xrightarrow{p_2} & \prod_i g_i X(u_i) \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_u} \\
\mathcal{W}''_g & \xrightarrow{ev} & \prod_i g_i X'(u_i)
\end{array}$$

Note that the fiber $p_1^{-1}(m)$ at a point m in $p_2^{-1}(g\pi_u^{-1}(U))$ is irreducible. Indeed, we have:

$$\begin{aligned} p_1^{-1}(m) &= \{m\} \times_{(X')^r} (g_1X(u_1) \times \dots g_rX(u_r)) \\ &\simeq \pi_u^{-1}(ev(m)) \\ &\simeq g \cdot \pi_u^{-1}(g^{-1}ev(m)), \end{aligned}$$

which is irreducible since $ev(m)$ lies in $g\pi_u^{-1}(U)$. Furthermore, according to Lemma 1.2.3.2 *b*), for g general in G^r, $\mathcal{W}'_g \cap p_2^{-1}(g\pi_u^{-1}(U))$ is a dense open subvariety of $\mathcal{W}'_g = p_2^{-1}(\prod_i g_iX(u_i))$. Hence the general fiber of p_1 is irreducible. Furthermore, note that, since $\overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d})$ and Schubert varieties are normal [FP96, Ram85], according to Kleiman's transversality theorem

$$\mathcal{W}'_g = \overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d}) \times_{(X')^r} (g_1X(u_1) \times \dots g_rX(u_r))$$

is normal for $g = (g_1, \dots, g_r)$ general in G^r. According to Lemma 1.2.4.1, all irreducible components of $p_1(\mathcal{W}'_g) = \mathcal{W}''_g$ are thus dominated by exactly one irreducible component of $\mathcal{W}'_g$.

Finally, Proposition 1.2.3.1 *iv*), tells us that, since $\mathcal{W}'_g$ is in general position, the image by p_1 of any of its irreducible components is an irreducible component of $\mathcal{W}''_g$, hence the 1-to-1 correspondence.

To sum up, each irreducible component of $\mathcal{W}_g$ surjects into a different irreducible component of $\mathcal{W}'_g$, and each irreducible component of $\mathcal{W}'_g$ surjects into a different irreducible component of $\mathcal{W}''_g$. Furthermore, $\mathcal{W}_g$ surjects onto $\mathcal{W}''_g$. There thus is a 1-to-1 correspondence between irreducible components of $\mathcal{W}_g$ and irreducible components of $\mathcal{W}''_g$.

3. The only point left to prove is that the general fiber of $\mathcal{W}_g \to \mathcal{W}''_g$ is an irreducible rationally connected variety. According to Lemma 1.2.4.2 there exists a dense open subset U of $\prod_i X'(u_i)$ such that the fiber of $\pi_{|u} : \prod_i X(u_i) \to \prod_i X'(u_i)$ over a point in U is an irreducible rationally connected variety. For g general in G^r, the following properties are satisfied:

   (a) $\left(\overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d})\right)_{g\cdot U}$ is dense in $\mathcal{W}''_g = \left(\overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d})\right)_{g\cdot\prod_i X'(u_i)}$;

   (b) The general fiber of $\mathcal{W}'_g \to \mathcal{W}''_g$ is rationally connected;

   (c) The general fiber of $\mathcal{W}_g \to \mathcal{W}'_g$ is rationally connected;

   (d) The general fiber of $\mathcal{W}_g \to \mathcal{W}''_g$ is rationally connected.

   Note that proving (*d*) is our goal here. (*a*) holds according to Lemma 1.2.3.1. Note that the fiber of the projection $p_1 : \mathcal{W}'_g \to \mathcal{W}''_g$ over a point m in $\left(\overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d})\right)_{g\cdot U}$ is rationally connected; indeed, it is isomorphic to $g \cdot (\pi_{|u})^{-1}(g^{-1} \cdot ev(m))$, where $g^{-1} \cdot ev(m)$ lies in U. Hence, since according to (*a*) $\left(\overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d})\right)_{g\cdot U}$ is dense in $\mathcal{W}''_g = \left(\overline{\mathcal{M}_{0,r}}(X', \pi_*\mathbf{d})\right)_{g\cdot\prod_i X'(u_i)}$, (*b*) holds. (*c*) holds according to Lemma 1.2.4.3. Finally, (*d*) is deduced from (*b*), (*c*) and Theorem 1.2.2.1 (3). □

1.3 Balanced flags of vector bundles

Let X be a flag variety. We describe here the flag of vector bundles associated with a general point in $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$ for a degree $\mathbf{d}$ high enough. All definitions and results presented in this part are contained in I. Coskun's description of rational curves on a flag variety [Cos10], or easily deduced from it.

1.3.1 Notations. Fix two positive integers $m < n$. Let $I = \{i_1, \dots, i_m\}$ be a set of integers satisfying $0 < i_1 < \dots < i_m < n$. Recall Fl_I is the m-step flag variety parametrizing m-tuples $(V_{i_1}, \dots, V_{i_m})$ of vector subspaces of $\mathbb{C}^n$, where the subspaces are ordered by inclusion $0 \subset V_{i_1} \subset \dots \subset V_{i_m} \subset \mathbb{C}^n$, and V_{i_k} is of dimension i_k. The class $\mathbf{d} = (d_1, \dots, d_m)$ of a curve C in X is determined by non-negative integers d_1, d_2,..., d_m, where d_i is the Plücker degree of the projection of C to the Grassmannian $\mathrm{Gr}(n_i, n)$. A point p in $\mathcal{M}_{0,r}(X, \mathbf{d})$ is associated with a morphism $f_p : \mathbb{P}^1 \to X$. By the

functorial definition of flag varieties, the morphism $f_p : \mathbb{P}^1 \to X$ is uniquely determined by the data of a flag of vector bundles over $\mathbb{P}^1$:

$$\mathcal{F}_p = E_1 \subset \cdots \subset E_m \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n},$$

where $\mathrm{rank} E_k = i_k$ and $\deg E_k = -d_k$. Note that we have $f_p^* S_i \simeq E_i$, where S_i is the i-th tautological vector bundle over X.

1.3.2 Admissible sets of sequences.

Recall $I = \{i_1, \ldots, i_m\}$ is a set of m integers $0 < i_1 < \cdots < i_m < n$.

Definition 1.3.2.1. Consider a set $\mathbf{d} = (d_1, \ldots, d_m)$ of m non negative integers. A $(I, \mathbf{d})$-*admissible set of sequences* is a set

$$A_\bullet = \{(a_{1,j})_{j=1}^{i_1}, (a_{2,j})_{j=1}^{i_2}, \ldots, (a_{m,j})_{j=1}^{i_m}\}$$

of m sequences of non-negative integers of lengths $i_1, \ldots, i_m$, respectively, such that:

1. $0 \leqslant a_{k,j} \leqslant a_{k,j+1}$ for every k and every $1 \leqslant j \leqslant i_k - 1$;
2. $a_{k+1,j} \leqslant a_{k,j}$ for every $1 \leqslant k \leqslant m-1$ and $j \leqslant i_k$;
3. $\sum_{j=1}^{i_k} a_{k,j} = d_k$ for every k.

Definition 1.3.2.2. A $(I, \mathbf{d})$-admissible set of sequences $A_\bullet = \{(a_{k,j})\}$ is balanced if $A_\bullet$ minimizes the function

$$\sum_{1 \leqslant k \leqslant m} \sum_{1 \leqslant l < p \leqslant n_i} (a_{k,p} - a_{k,l})$$

among the $(I, \mathbf{d})$-admissible sets of sequences.

Note that, according to [Cos10] (see the proof of Lemma 2.1), for any set $\mathbf{d} = (d_1, \ldots, d_m)$ in $\mathbb{N}^m$ there exists a unique balanced $(I, \mathbf{d})$-admissible set of sequences $(a_{k,j})$, which can be constructed by iteration on k in the following way.

- First, set $a_{1,j} = \lceil d_1/n_1 \rceil - \epsilon_{1,j}$ where $\epsilon_{1,j}$ is in $\{0, 1\}$ and is chosen in such a way that $\sum_{j=1}^{n_1} a_{1,j} = d_1$ and $a_{1,j} \leqslant a_{1,j+1}$ and $\lceil q \rceil$ denote the ceiling of $q \in \mathbb{Q}$;
- Let $1 < k \leqslant m$. Suppose all $a_{k-1,j}$ have been constructed. Denote by r_k the largest index such that the element a_{k-1,r_k} satisfies $a_{k-1,r_k} \leqslant d_k/n_k$. Set:

  $$a_{k,j} = a_{k-1,j} \text{ for } 1 \leqslant j \leqslant r_k. \tag{1.5}$$

  Denote by R_k the largest index $> r_k$ such that the element a_{k-1,R_k} satisfies $a_{k-1,R_k} \leqslant (d_k - \sum_{j=1}^{r_k} a_{k-1,j})/(i_k - r_k)$. Repeat the process replacing r_k with R_k until all elements $a_{k-1,j}$ of index larger than r_k are larger than $(d_k - \sum_{j=1}^{r_k} a_{k-1,j})/(i_k - r_k)$. Then set:

  $$\forall r_k < j \leqslant i_k,\ a_{k,j} = \left\lceil \frac{(d_k - \sum_{j=1}^{r_k} a_{k-1,j})}{(a_k - r_k)} \right\rceil - \epsilon_{k,j}, \tag{1.6}$$

  where $\epsilon_{k,j}$ is in $\{0, 1\}$ and is chosen in such a way that $\sum_{j=1}^{i_k} a_{k,j} = d_k$ and $a_{k,j} \leqslant a_{k,j+1}$.

For simplicity, we set $d_0 := 0$ and $n_0 := 0$.

Proposition 1.3.2.1. *Let $\mathbf{d} = (d_1, \ldots, d_m)$ be a set of nonnegative integers. Let $1 < k \leqslant m$ such that:*

$$\forall\, p < k,\ d_k \geqslant i_k \left\lceil \frac{d_p - d_{p-1}}{i_p - i_{p-1}} \right\rceil. \tag{1.7}$$

Then the balanced $(I, \mathbf{d})$-admissible set of sequences $\{(a_{1,j})_{j=1}^{i_1}, \ldots, (a_{m,j})_{j=1}^{i_m}\}$ satisfies $a_{k,j} = a_{k-1,j}$ for $1 \leqslant j \leqslant i_{k-1}$.

Proof. By iteration on $s < k$, we prove that for $1 \leqslant s < k$ all elements $a_{s,j}$ satisfy $a_{s,j} \leqslant d_k/n_k$. According to (1.5) this will yield the expected result. First, for $s = 1$, since $d_k \geqslant n_k\lceil d_1/i_1\rceil$, $a_{1,j} \leqslant d_k/i_k$. Now let $1 \leqslant s < k$, and suppose $a_{s-1,j} \leqslant d_k/i_k$ for all $1 \leqslant j \leqslant n_{s-1}$. Note that, if $r_s < i_{s-1}$, then there exists elements $a_{s-1,j}$ larger than $(d_s - \sum_{j=1}^{r_s} a_{s-1,j})/(i_s - r_s)$, where we use the notations of (1.6); hence all elements $a_{s,j}$ are smaller than an element $a_{s-1,s'}$ for some s'. Hence all elements $a_{s,j}$ then satisfy $a_{s,j} \leqslant d_k/n_k$. On the other hand, if $r_s = i_{s-1}$, then according to (1.6):

$$\begin{aligned}\forall i_{s-1} < j \leqslant i_s, \quad a_{s,j} &\leqslant \left\lceil \frac{\left(d_s - \sum_{j=1}^{i_{s-1}} a_{s-1,j}\right)}{(i_s - i_{s-1})} \right\rceil \\ &\leqslant \left\lceil \frac{(d_s - d_{s-1})}{(i_s - i_{s-1})} \right\rceil \quad \leqslant \frac{d_k}{n_k}.\end{aligned}$$

□

1.3.3 Balanced flag bundles.

The partial flag variety Fl_I comes equipped with m tautological bundles $S_1, \ldots, S_m$. We describe here the pullback $f_p^* S_i$ by a morphism $f_p : \mathbb{P}^1 \to X$ associated with a point p general in $\overline{\mathcal{M}_{0,r}(X, \mathbf{d})}$.

Let $\mathbf{d} = (d_1, \ldots, d_m)$ be an effective class of curve in $H_2(Fl_I, \mathbb{Z})$, p be a point in $\mathcal{M}_{0,r}(X, \mathbf{d})$, and $f_p : \mathbb{P}^1 \to X$ be its associated morphism. The pullback $f_p^* S_s$ of the s-th tautological bundle S_s by f_p is a vector bundle of degree d_s and rank i_s. According to Grothendieck's decomposition of vector bundles, $f_p^* S_s$ is isomorphic to a direct sum of line bundles

$$f_p^* S_s \simeq \mathcal{O}_{\mathbb{P}^1}(-a_{s,1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{s,i_s}),$$

where $\sum_{j=1}^{i_s - k_s} a_{s,j} = d_s$. Suppose, for a given s, the $a_{s,j}$ are ordered in increasing order. Then, since, for all $1 < s \leqslant m$, $f_p^* S_{s-1}$ is a subbundle of $f_p^* S_s$, we have: $a_{s-1,j} \geqslant a_{s,j}$. Hence the sequence $(a_{s,j})$ forms a $(I, \mathbf{d})$-admissible set of sequence in the sense of Definition 1.3.2.1. We call $a_{s,j}$ the *admissible set of sequences associated with p.*

Proposition 1.3.3.1. *i) The admissible set of sequences associated with a general point in $\mathcal{M}_{0,r}(Fl_I, \mathbf{d})$ is balanced in the sense of Definition 1.3.2.2;*

ii) Suppose $\forall\, p < k$, $d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{i_p - i_{p-1}} \right\rceil$. Denote by $\mathcal{F}_{\mathfrak{p}} = E_1 \hookrightarrow \cdots \hookrightarrow E_m \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ the flag of vector bundles associated with a general element $\mathfrak{p}$ in $\overline{\mathcal{M}_{0,r}}(Fl_I, \mathbf{d})$. Then E_k splits into $E_k \simeq E_{k-1} \oplus_{\sum_{s=n_{k-1}+1}^{i_k}} \mathcal{O}_{\mathbb{P}^1}(-a_{k,s})$.

Proof. i) Cf. proof of [Cos10], Proposition 2.3.

ii) Let $\mathcal{F}_{\mathfrak{p}} = E_1 \subset \cdots \subset E_m \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ be the flag of vector bundles associated with a general element $\mathfrak{p}$ in $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$. Then, according to i), the admissible set of sequences associated with $\mathcal{F}_{\mathfrak{p}}$ is balanced. Since d_k satisfies $\forall\, p < k$, $d_k \geqslant i_k \left\lceil \frac{d_p - d_{p-1}}{i_p - i_{p-1}} \right\rceil$, we have according to Proposition 1.3.2.1 the following sequence:

$$\begin{aligned}0 \to & E_{k-1} \simeq \mathcal{O}_{\mathbb{P}^1}(-a_{k-1,1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{k-1,n_{k-1}}) \to \\ & E_k \simeq \mathcal{O}_{\mathbb{P}^1}(-a_{k-1,1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{k-1,i_{k-1}}) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{k,i_{k-1}+1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{k,n_k})\end{aligned}$$

which splits. Indeed, for all j, l verifying $j \leqslant l$ and $j \leqslant n_{k-1} < l$, we have: $a_{k,j} = a_{k-1,j}$ and $a_{k,j} \leqslant a_{k,l}$, hence $a_{k-1,j} \leqslant a_{k,l}$ for any $l > i_{k-1}$. Hence according to Lemma 1.3.3.1 the morphisms $\mathcal{O}_{\mathbb{P}^1}(-a_{k-1,j}) \to E_k$ factor through $\oplus_{b=1}^{n_{k-1}} \mathcal{O}_{\mathbb{P}^1}(-a_{k,b}) \hookrightarrow E_k$. Finally, since the morphism $E_{k-1} \to E_k$ is injective, E_{k-1} is a rank i_{k-1} subvector bundle of $\oplus_{b=1}^{i_{k-1}} \mathcal{O}_{\mathbb{P}^1}(-a_{k,b})$; hence $E_{k-1} \simeq \oplus_{b=1}^{n_{k-1}} \mathcal{O}_{\mathbb{P}^1}(-a_{k,b})$.

□

Lemma 1.3.3.1. *Let $d_1 \leqslant \cdots \leqslant d_r$ and $d'_1 = d_1 \leqslant \cdots \leqslant d'_r = d_r \leqslant d'_{r+1} \leqslant \cdots \leqslant d'_{r+p}$ be sequences of non negative integers. Then the following sequence splits.*

$$\begin{aligned} 0 \to & E = \mathcal{O}_{\mathbb{P}^1}(-d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-d_r) \to \\ & F = \mathcal{O}_{\mathbb{P}^1}(-d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-d_r) \oplus \mathcal{O}_{\mathbb{P}^1}(-d'_{r+1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-d'_{r+p}) \end{aligned}$$

Proof. Let k be the largest index such that $d_k < d_r$. Note that for $i > k$ all the integers d'_i are greater than d_{k+1} or equal to d_{k+1} and $d_{k+1} = d_r > d_k$. Hence all integers d'_i are greater than d_k for $i > k$. Hence the induced morphism $E_1 := \mathcal{O}_{\mathbb{P}^1}(-d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-d_k) \to F$ factors through a closed immersion $\mathcal{O}_{\mathbb{P}^1}(-d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-d_k) \hookrightarrow F$. Denote by r' the largest index such that $d'_{r'} = d_r$. Note that $r' \geqslant r$. Let us now consider the induced exact sequence

$$\begin{aligned} 0 \to & E/E_1 \simeq \mathcal{O}_{\mathbb{P}^1}(-d_{k+1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-d_r) \simeq \mathcal{O}_{\mathbb{P}^1}(-d_k)^{\oplus r-k} \\ & \to F/E_1 \simeq \mathcal{O}_{\mathbb{P}^1}(-d'_{k+1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-d'_{r+p}) \simeq \mathcal{O}_{\mathbb{P}^1}(-d_r)^{\oplus (r'-k)} \oplus \mathcal{O}_{\mathbb{P}^1}(-d_{r'+1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-d'_{r+p}). \end{aligned}$$

Since for any $i > r'$ the integer d'_i is greater than d_k, the injective morphism $E/E_1 \to F/E_1$ factors through

$$E/E_1 \simeq \mathcal{O}_{\mathbb{P}^1}(-d_k)^{\oplus r-k} \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-d_r)^{\oplus (r'-k)} \hookrightarrow F/E_1$$

Finally observe that the morphism $E/E_1 \simeq \mathcal{O}_{\mathbb{P}^1}(-d_k)^{\oplus r-k} \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-d_r)^{\oplus (r'-k)}$ factors through $E/E_1 \simeq \mathcal{O}_{\mathbb{P}^1}(-d_k)^{\oplus r-k} \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-d_k)^{\oplus r-k} \oplus \mathcal{O}_{\mathbb{P}^1}(-d_k)^{\oplus r'-r} \hookrightarrow F/E_1$. □

1.4 Forgetting one vector space

We denote by $\pi_{\widehat{k}} : X \to X_{\widehat{k}}$ the morphism forgetting the k-th vector space. Let $\mathbf{d} = (d_1, \ldots, d_m)$ be a degree in $E(X)$. In this section, our goal is to prove the following results.

Proposition 1.4.0.1. *Suppose $\forall\, p < k,\ d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$. Then the general fiber of*

$$\Pi_{\widehat{k}} \times ev : \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \left(\Pi_{\widehat{k}} \times ev\right)\left(\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})\right) \subset \overline{\mathcal{M}_{0,r}}(X_{\widehat{k}}, (\pi_{\widehat{k}})_* \mathbf{d}) \times_{(X_{\widehat{k}})^r} X^r$$

is a unirational variety.

Corollary 1.4.0.1. *Suppose $\forall p < k, d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$ and the map $\Pi_{\widehat{k}} \times ev$ is surjective. Then for x general in X^r, the general fiber of the map $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \ni ev^{-1}(x) \to ev^{-1}(\pi_{\widehat{k}}(x)) \in \overline{\mathcal{M}_{0,r}}(X_{\widehat{k}}, (\pi_{\widehat{k}})_* \mathbf{d})$ is a unirational variety.*

We will assume in this section $\forall\ p < k,\ d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$. Let (p, x) be a point in $(\Pi_{\widehat{k}} \times \mathrm{ev})(\mathcal{M}_{0,r}(X, \mathbf{d}))$, where p is in $\mathcal{M}_{0,r}(X_{\widehat{k}}, (\pi_{\widehat{k}}) * \mathbf{d})$, and $x = (x_1, \ldots, x_r)$ is in X^r. Denote by $\Pi_{p,x}$ the fiber of $\Pi_{\widehat{k}} \times \mathrm{ev}$ at (p, x), i.e. :

$$\Pi_{p,x} := \Pi_{\widehat{k}}^{-1}(p) \times_{X^r} x.$$

In order to prove the irreducibility and unirationality of $\Pi_{p,x}$, we define an auxilliary variety $\mathcal{Z}_{p,x}$, which dominates $\Pi_{p,x}$ when (p, x) is general in the image $(\Pi_{\widehat{k}} \times ev)(\mathcal{M}_{0,r}(X, \mathbf{d}))$.

1.4.1 Notations.

The class $\mathbf{d} = (d_1, \ldots, d_m)$ of a curve C in X is determined by non-negative integers d_1, d_2,..., d_m, where d_i is the Plücker degree of the projection of C to the Grassmannian $\mathrm{Gr}(n_i, n)$. The morphism $\pi_{\widehat{k}} : X \to X_{\widehat{k}}$ induces a morphism $\Pi_{\widehat{k}} : \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X_{\widehat{k}}, (\pi_{\widehat{k}})_* \mathbf{d})$. Denote by $\pi_k : X \to \mathrm{Gr}(n_k, n)$ the morphism sending a m-step flag $\{0\} \subset V_1 \subset \cdots \subset V_k \subset \cdots \subset V_m$ to the k-th linear subspace V_k. This morphism induces a morphism $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(\mathrm{Gr}(n_k, n), d_k) \to$

$\overline{\mathcal{M}_{0,3}}(\mathrm{Gr}(n_k,n),d_k)$ into the space of genus zero stable maps with three marked points to the Grassmannian $\mathrm{Gr}(n_k,n)$. We obtain the following commutative diagram:

$$\begin{array}{ccccc}
\overline{\mathcal{M}_{0,r}}(X,\mathbf{d}) & \longrightarrow & \overline{\mathcal{M}_{0,3}}(X,\mathbf{d}) & \longrightarrow & \overline{\mathcal{M}_{0,3}}(\mathrm{Gr}(n_k,n),d_k) \\
\downarrow{\scriptstyle \Pi_{\hat{k}}} & \searrow^{\mathrm{ev}^r} & & & \\
\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},(\pi_{\hat{k}})_*\mathbf{d}) & \longrightarrow & X^r & \xrightarrow{\pi_k\times\cdots\times\pi_k} & \mathrm{Gr}(n_k,n)^r \\
 & & \downarrow{\scriptstyle \pi_{\hat{k}}\times\cdots\times\pi_{\hat{k}}} & & \\
 & & {X_{\hat{k}}}^r & &
\end{array}$$

Furthermore, for p in $\mathcal{M}_{0,r}(X,\mathbf{d})$, we will denote by:

$$f_p : \mathbb{P}^1 \to X$$

the associated morphism. By the functorial definition of flag varieties, f_p is uniquely determined by the data of a flag of vector bundles over $\mathbb{P}^1$:

$$\mathcal{F}_p = E_1 \subset \cdots \subset E_m \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n},$$

where $\mathrm{rank}E_i = n_i$ and $\deg E_i = -d_i$. Note that we have $f_p^* S_i \simeq E_i$, where S_i is the i-th tautological vector bundle over X. Finally, for a point $\mathfrak{p}$ in $\mathbb{P}^1$ we will denote by $(E_i)_{\mathfrak{p}}$ the fiber at $\mathfrak{p}$ of the vector bundle E_i, which has a natural inclusion $(E_i)_{\mathfrak{p}} \subset \mathbb{C}^n$ induced by the morphism $E_i \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$.

1.4.2 A preliminary result.

Denote by $\mathcal{F}_p = E_1 \subset \cdots \subset E_{k-1} \subset E_{k+1} \subset \cdots \subset E_m \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ the flag of vector bundles over $\mathbb{P}^1$ associated with the point p in $\mathcal{M}_{0,r}(X_{\hat{k}},(\pi_{\hat{k}})_*\mathbf{d})$.

Lemma 1.4.2.1. *i) Consider a dense open subset U of $\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})$. Then, for (p,x) general in $(\Pi_{\hat{k}} \times ev)(\overline{\mathcal{M}_{0,r}}(X,\mathbf{d}))$, $\Pi_{p,x} \cap U$ is a dense open subset of $\Pi_{p,x}$;*

ii) For (p,x) general in $(\Pi_{\hat{k}} \times ev)(\overline{\mathcal{M}_{0,r}}(X,\mathbf{d}))$, there exists a dense open subset U of $\Pi_{p,x}$ such that, for all q in U, the flag of vector bundles $\mathcal{F}_q$ associated with q is balanced. Hence there exists an increasing set of integers $a_{n_{k-1}+1} \leqslant \cdots \leqslant a_{n_k}$ such that for all q in U, the flag $\mathcal{F}_q$ is of the form:

$$\mathcal{F}_q = E_1 \subset \cdots \subset E_{k-1} \subset E_k \simeq E_{k-1} \oplus_{i=n_{k-1}+1}^{n_k} \mathcal{O}_{\mathbb{P}^1}(-a_i) \subset E_{k+1} \subset \cdots \subset E_m \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}.$$

Proof. i) Since (p,x) is general in $(\Pi_{\hat{k}} \times ev)(\overline{\mathcal{M}_{0,r}}(X,\mathbf{d}))$, according to Lemma 1.2.2.1 the intersection of $\Pi_{p,x}$ with the dense open subset U is dense in $\Pi_{p,x}$.

ii) According to Proposition 1.3.3.1 *i)*, there exists a dense open subset U of $\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})$ such that a point in U corresponds to a balanced flag of vector bundles over $\mathbb{P}^1$. According to *i)*, $U \cap \Pi_{p,x}$ is dense open subset of $\Pi_{p,x}$. Finally, notice that according to Proposition 1.3.3.1 *ii)*, since $\forall\, p < k$, $d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$, the flag of vector bundles associated with a point in $\Pi_{p,x} \cap U$ splits at E_{k-1}.

□

1.4.3 Definition of $\mathcal{Z}_{p,x}$.

Our goal here is to construct a rational variety $\mathcal{Z}_{p,x}$ dominating $\Pi_{p,x}$. First, consider the integers $a_{n_{k-1}+1}, \ldots, a_{n_k}$ defined in Lemma 1.4.2.1 *ii)* and the following vector space

$$V := Hom(\oplus_{i=n_{k-1}+1}^{n_k} \mathcal{O}_{\mathbb{P}^1}(-a_i), \mathcal{O}_{\mathbb{P}^1}^{\oplus n}).$$

The flag variety X is naturally embedded in $X_{\hat{k}} \times \mathrm{Gr}(n_k,n)$ by Plücker-embedding $X_{\hat{k}} \times \mathrm{Gr}(n_k,n)$ in a product of projective spaces, and cutting out X from this product by considering the relations given by the inclusions of flags. Note that, if we denote by $[f_p : \mathbb{P}^1 \to X_{\hat{k}}, \{p_1,\ldots,p_r\}]$ the stable map

associated with p, points q in $\Pi_{p,x} \cap \mathcal{M}_{0,r}(X, \mathbf{d})$ correspond to stable maps $[f_q : \mathbb{P}^1 \to X, \{p_1, \dots, p_r\}]$ satisfying $f_q(p_i) = x_i$ and such that the following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{P}^1 & \xrightarrow{f_q} & X & \longrightarrow & X_{\hat{k}} \times \mathrm{Gr}(n_k, n) \\ & \searrow^{f_p} & & & \downarrow \\ & & & & X_{\hat{k}} \end{array}$$

Furthermore, by the functorial definition of the Grassmannian, elements u in V corresponding to an injective morphism

$$\Phi_u : E_{k-1} \oplus_i \mathcal{O}_{\mathbb{P}^1}(-a_i) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$$

define a morphism $f_u : \mathbb{P}^1 \to \mathrm{Gr}(n_k, n)$, where we denote by Φ_u the morphism induced by the inclusion $E_{k-1} \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ and the morphism $\mathcal{O}_{\mathbb{P}^1}(-a_i) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ associated with a point u in V. Conversally, any injective morphism $E_{k-1} \oplus_i \mathcal{O}_{\mathbb{P}^1}(-a_i) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ can, up to automorphisms of E_{k-1}, be obtained this way. Points in $\Pi_{p,x}$ are thus associated with elements u in V such that:

1. $\Phi_u : E_{k-1} \oplus_{i=n_{k-1}+1}^{n_k} \mathcal{O}_{\mathbb{P}^1}(-a_i) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ is an injective morphism of vector bundles;
2. For all $1 \leqslant i \leqslant r$, $(F_u)_{p_i} = \pi_k(x_i)$;
3. $f_p \times f_u : \mathbb{P}^1 \to X_{\hat{k}} \times \mathrm{Gr}(n_k, n)$ factors through $X \to X_{\hat{k}} \times \mathrm{Gr}(n_k, n)$,

where we denote by $(F_u)_{p_i}$ the fiber of the vector bundle $F_u := \Phi_u(E_{k-1} \oplus_i \mathcal{O}_{\mathbb{P}^1}(-a_i)) \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ at the point $p_i \in \mathbb{P}^1$, and by f_u the morphism associated with the flag of vector bundles $F_u \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ by functorial definition of the Grassmannian. Note that for u in V the morphism $\Phi_u : E_{k-1} \oplus_i \mathcal{O}_{\mathbb{P}^1}(-a_i) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ might not be an injective morphism, and the fiber $(F_u)_{p_i}$ might be of dimension less than n_k. In order to define a closed subset of V, we define $\mathcal{Z}_{p,x}$ as the subset of elements u in V such that:

$$\forall\, i \in \{1, \dots, r\},\ (F_u)_{p_i} \subset \pi_k(x_i), \text{ where } \pi_k(x_i) \in \mathrm{Gr}(n_k, n); \tag{1.8}$$

$$\forall \mathfrak{p} \in \mathbb{P}^1,\ (F_u)_{\mathfrak{p}} \subset (E_{k+1})_{\mathfrak{p}}, \tag{1.9}$$

where we denote by $\pi_k(x_i)$ the vector subspace of $\mathbb{C}^n$ of dimension n_k associated with the point $\pi_k(x_i)$ in $\mathrm{Gr}(n_k, n)$, and by $(E_{k+1})_{\mathfrak{p}}$ the fiber of the vector bundle E_{k+1} at $\mathfrak{p}$. Note that, when Φ_u defines a morphism $\mathbb{P}^1 \to \mathrm{Gr}(n_k, n)$, Equations (1.8) and (1.9) are equivalent to Equations 2. and 3. here above. Indeed, by construction of Φ_u, $(F_u)_{\mathfrak{p}}$ satisfies $(E_{k-1})_{\mathfrak{p}} \subset (F_u)_{\mathfrak{p}}$ for all $\mathfrak{p}$ in $\mathbb{P}^1$, hence if u satisfies Equation (1.9), u also verifies:

$$\forall \mathfrak{p} \in \mathbb{P}^1,\ (E_1)_{\mathfrak{p}} \subset \cdots \subset (E_{k-1})_{\mathfrak{p}} \subset (F_u)_{\mathfrak{p}} \subset (E_{k+1})_{\mathfrak{p}} \subset \cdots \subset (E_m)_{\mathfrak{p}} \subset \mathbb{C}^n,$$

hence Φ_u defines a morphism to the flag variety X.

1.4.4 Irreducibility and rationality of $\mathcal{Z}_{p,x}$.

We denote by F_μ the subsheaf of $\mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ associated with an element μ in $V \simeq Hom(E_{k-1} \oplus_i \mathcal{O}_{\mathbb{P}^1}(-a_i), \mathcal{O}_{\mathbb{P}^1}^{\oplus r})$, and by $(F_\mu)_{\mathfrak{p}}$ the fiber over a point $\mathfrak{p} \in \mathbb{P}^1$ of F_μ. We consider the usual construction of $\mathbb{P}^1$ by gluing two affine schemes together. Let $U_1 := \mathrm{Spec}(\mathbb{C}[s])$, $U_2 := \mathrm{Spec}(\mathbb{C}[t])$, $U_{12} := \mathrm{Spec}(\mathbb{C}[s, s^{-1}]) = U_1 \backslash \{0\}$, and $U_{21} := \mathrm{Spec}(\mathbb{C}[t, t^{-1}]) = U_2 \backslash \{0\}$. Then $\mathbb{P}^1$ can be obtained by gluing U_1 and U_2 together along $U_{12} \simeq U_{21}$, where we consider the following isomorphism:

$$\mathbb{C}[s, s^{-1}] \simeq \mathbb{C}[t, t^{-1}], \quad s \to t^{-1}.$$

For $d \in \mathbb{N}$, we view $\mathcal{O}_{\mathbb{P}^1}(-d)$ as the line bundle satisfying $\mathcal{O}_{\mathbb{P}^1}(-d)_{|U_i} \simeq U_i \times \mathbb{A}^1$ for $i = 1, 2$, obtained by identifying $U_{12} \times \mathbb{A}^1 = \mathrm{Spec}[s, s^{-1}] \times \mathbb{A}^1$ and $U_{21} = \mathrm{Spec}[t, t^{-1}] \times \mathbb{A}^1$ via the following isomorphism:

$$(s, v) \to (s^{-1}, s^d v).$$

Equivalently we endow $\mathcal{O}_{\mathbb{P}^1}(-d)$ with two trivializations on U_1 and U_2 respectively. We consider here the natural bijection $V \simeq \oplus_j(\mathbb{C}_{a_j}[x])^n$, associating to a section in $(\mathbb{C}_{a_j}[x])^n$ the morphism $\mathcal{O}_{\mathbb{P}^1}(-a_j) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ described by:

$$\begin{array}{ll} \mathrm{Spec}\mathbb{C}[s] \times \mathbb{A}^1 \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n} & (s, v) \to (s, P_j(s)v) \\ \mathrm{Spec}\mathbb{C}[s^{-1}] \times \mathbb{A}^1 \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n} & (s^{-1}, v) \to (s^{-1}, s^{-a_j}P_j(s)v) \end{array}$$

We may hence view a point μ in $\mathcal{Z}_{p,x}$ as a set of $n_k - n_{k-1}$ vectors P_j with coefficients in $\mathbb{C}_{a_j}[x]$. In order to simplify notations, we keep the dependence of the polynomials P_j on the point μ in V associated with them implicit.

Lemma 1.4.4.1. *1. There exist $N \in \mathbb{N}$, $p_1, \ldots, p_N \in \mathbb{P}^1$, such that $\mathcal{Z}_{p,x}$ is the subspace of elements μ in V satisfying the following equations:*

$$\begin{array}{l} \forall i \in \{1, \ldots, r\},\ (F_\mu)_{p_i} \subset \pi_k(x_i), \text{ where } \pi_k(x_i) \in \mathrm{Gr}(n_k, n); \\ \forall \iota \in \{1, \ldots, N\},\ (F_\mu)_{p_\iota} \subset (E_{k+1})_{p_\iota}; \end{array}$$

2. Consider a point $\mathfrak{p}$ in $\mathbb{P}^1$. We denote by $V((E_{k+1})_\mathfrak{p})$ the subset of elements μ in V satisfying $(E_{k+1})_\mathfrak{p} \supset (F_\mu)_\mathfrak{p}$. Then $V((E_{k+1})_\mathfrak{p})$ is a linear subspace of V;

3. $\mathcal{Z}_{p,x}$ is a linear subspace of V.

Proof. 1. By definition, the vectors P_j with coefficients in $\mathbb{C}_{a_j}[x]$ associated with $\mu \in V$ satisfy:

$$\forall \mathfrak{p} \in \mathbb{P}^1,\ (F_\mu)_\mathfrak{p} \simeq \langle (E_{k-1})_\mathfrak{p}, (P_{n_{k-1}+1}(\mathfrak{p}), \ldots, P_{n_k}(\mathfrak{p}))\rangle,$$

where we denote by F_μ the subvector bundle of $\mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ associated with μ. Denote by Q_j the n_{k+1} homogeneous polynomials associated with the inclusion of vector bundles $E_{k+1} \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$. Then there exist $N > 0$, and points $p_1, \ldots, p_N$ in $\mathbb{P}^1$ such that the following statements are equivalent:

$$\begin{array}{l} \forall\, \mathfrak{p} \in \mathbb{P}^1,\ (F_\mu)_\mathfrak{p} \subset (E_{k+1})_\mathfrak{p} \\ \Leftrightarrow \forall\, \mathfrak{p} \in \mathbb{P}^1,\ \langle P_{n_{k-1}+1}(\mathfrak{p}), \ldots, P_{n_k}(\mathfrak{p})\rangle \subset \langle Q_1(\mathfrak{p}), \ldots, Q_{n_{k+1}}(\mathfrak{p})\rangle \\ \Leftrightarrow \forall \iota \in \{1, \ldots, N\},\ \langle P_{n_{k-1}+1}(p_\iota), \ldots, P_{n_k}(p_\iota)\rangle \subset \langle Q_1(p_\iota), \ldots, Q_{n_{k+1}}(p_\iota)\rangle \end{array} \tag{1.10}$$

Indeed, μ satisfies the homogeneous equations (1.10) iff the polynomials P_j associated with μ satisfy (1.10) at N different values of $p_\iota = [x_\iota, y_\iota]$ for N large enough. Notice N only depends on the degrees d_k and d_{k+1}, which are fixed here.

From now on, we fix such an N, and fix points $(p_{r+1}, \ldots, p_N)$ general in $\mathbb{P}^1$. Equation (1.9) is thus equivalent to :

$$\forall \iota \in \{1, \ldots, N\},\ (F_\mu)_{p_\iota} \subset (E_{k+1})_{p_i} \tag{1.11}$$

2. We set $\mathfrak{p} := [x : y]$. We will denote by $\langle E, F\rangle$ the vector subspace of $\mathbb{C}^n$ generated by any two vector subspaces E and F of $\mathbb{C}^n$. The vector subspace $(E_{k+1})_\mathfrak{p}$ is described by linear equations. Hence:

$$\begin{array}{rl} V((E_{k+1})_\mathfrak{p}) &= \{\mu = (P_{i,j}) \in V \mid \langle (E_{k-1})_\mathfrak{p}, (F_\mu)_p\rangle \subset (E_{k+1})_\mathfrak{p}\} \\ &= \left\{\mu = (P_{i,j}) \in V \mid \langle (E_{k-1})_\mathfrak{p}, P_{n_{k-1}+1}(\mathfrak{p}), \ldots, P_{n_k}(\mathfrak{p})\rangle \subset (E_{k+1})_\mathfrak{p}\right\} \end{array}$$

is the intersection of linear equations on the coefficients of P_j, $n_{k-1} + 1 \leqslant j \leqslant n_k$.

3. Fix i in $\{1 \ldots, N\}$. Denote by $\pi_{\widehat{k}}(x_i)$ the image of x_i in the projected flag variety $X_{\widehat{k}}$. The subset V_{p_i} of elements μ in V satisfying $(F_\mu)_{p_i} \subset \pi_k(x_i)$ can be described by:

$$\begin{array}{rl} V_{p_i} :&= \{\mu = (P_{i,j}) \in V \mid \langle (F_\mu)_{p_i}\rangle \subset \pi_k(x_i)\} \\ &= \left\{\mu = (P_{i,j}) \in V \mid \langle P_{n_{k-1}+1}(p_i), \ldots, P_{n_k}(p_i)\rangle \subset \pi_k(x_i)\right\}. \end{array}$$

Since the vector space $\pi_k(x_i)$ is defined by linear equations, V_{p_i} is defined by linear equations on the coefficients of the polynomials $(P_{i,j})$; V_{p_i} is a linear subspace of V.

Finally, according to 1. the variety $\mathcal{Z}_{p,x}$ is the intersection of the linear spaces V_{p_i}, for $1 \leqslant i \leqslant r$, with the linear spaces $V((E_{k+1})_{p_\iota})$, for $r < \iota \leqslant N$. Hence $\mathcal{Z}_{p,x}$ is a linear subspace of V.

□

According to Lemma 1.4.4.1, $\mathcal{Z}_{p,x}$ is a linear subspace of V, and hence is an irreducible rational variety.

1.4.5 Image of $\mathcal{Z}_{p,x} \dashrightarrow \mathcal{M}_{0,r}(X, \mathbf{d})$. First, denote by V^{inj} the subset of elements of V corresponding to injective morphisms $E_{k-1} \oplus_i \mathcal{O}_{\mathbb{P}^1}(-a_i) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$. The complementary of V^{inj} is described by a set of polynomial equations. Indeed, for $u = (P_j)_{n_{k-1}+1 \leqslant j \leqslant n_k}$ in V the associated morphism is injective iff $\langle (E_{k-1})_{\mathfrak{p}}, P_{n_{k-1}+1}(\mathfrak{p}), \ldots, P_{n_k}(\mathfrak{p}) \rangle$ is a vector space of dimension n_k for all $\mathfrak{p} \in \mathbb{P}^1$. Hence V^{inj} is an open subset of V. Call

$$\mathcal{Z}_{p,x}^{inj} := \mathcal{Z}_{p,x} \cap V^{inj}.$$

By the functorial definition of flag varieties, each element u in $\mathcal{Z}_{p,x}^{inj}$ defines a degree $\mathbf{d}$ morphism $f_u : \mathbb{P}^1 \to X$, which by construction of $\mathcal{Z}_{p,x}$ is associated to a point in $\Pi_{p,x}$. Furthermore, by construction of $\mathcal{Z}_{p,x}$, any point q in $\Pi_{p,x}$ whose associated flag of vector bundles is balanced is the image (F_u) of an element u in $\mathcal{Z}_{p,x}^{inj}$. Hence, according to Lemma 1.4.2.1 *ii*), for (p, x) general in $(\Pi_{\hat{k}} \times ev)(\overline{\mathcal{M}_{0,r}}(X, \mathbf{d}))$, the variety $\mathcal{Z}_{p,x}^{inj}$ dominates $\Pi_{p,x}$.

1.4.6 Algebraicity of $\mathcal{Z}_{p,x}^{inj} \to \Pi_{p,x}$. Denote by $\psi : \mathcal{Z}_{p,x}^{inj} \times \mathbb{P}^1 \to X$ the function sending an element $(z, \mathfrak{p})$ in $\mathcal{Z}_{p,x}^{inj} \times \mathbb{P}^1$ to the point of X associated with $(F_z)_{\mathfrak{p}}$. According to Lemma 1.6.1.1 (Appendix A), ψ defines a morphism $\Psi : \mathcal{Z}_{p,x}^{inj} \to \overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$, which dominates $\Pi_{p,x}$ according to here above.

1.4.7 Proof of Proposition 1.4.0.1. Let (p, x) be a general point in $(\Pi_{\hat{k}} \times ev)(\overline{\mathcal{M}_{0,r}}(X, \mathbf{d}))$. We have obtained a dominating morphism $\Psi : \mathcal{Z}_{p,x}^{inj} \to \Pi_{p,x}$. Since $\mathcal{Z}_{p,x}^{inj}$ is a dense open subset of the irreducible rational variety $\mathcal{Z}_{p,x}$, $\mathcal{Z}_{p,x}^{inj}$ is an irreducible rational variety. Hence $\Pi_{p,x}$ is an irreducible unirational variety.

1.4.8 Proof of Corollary 1.4.0.1. For an element x in X^r, we denote by Π_x the restriction of the forgetful map $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, (\pi_{\hat{k}})_* \mathbf{d})$ to $ev^{-1}(x)$. First, note that since the map $\Pi_{\hat{k}} \times ev$ is surjective, according to Proposition 1.2.4.1 1. for x general in X^r we have $\Pi(ev^{-1}(x)) = ev^{-1}(\pi_{\hat{k}}(x))$. Hence for x general in X^r, the map $\Pi_x : ev^{-1}(x) \to ev^{-1}(\pi_{\hat{k}}(x))$ is surjective.

According to Proposition 1.4.0.1, there exists a dense open subset U of $(\Pi_{\hat{k}} \times ev)(\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})) = \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, (\pi_{\hat{k}})_* \mathbf{d}) \times_{X_{\hat{k}}{}^r} X^r$ such that for all elements (m, x) in U, the fiber $(\Pi \times ev)^{-1}(m, x) = \Pi^{-1}(m) \cap ev^{-1}(x)$ is a unirational variety. Denote by $p_1 : \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, (\pi_{\hat{k}})_* \mathbf{d}) \times_{X_{\hat{k}}{}^r} X^r \to \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, (\pi_{\hat{k}})_* \mathbf{d})$ the first projection and by $p_2 : \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, (\pi_{\hat{k}})_* \mathbf{d}) \times_{X_{\hat{k}}{}^r} X^r \to X^r$ the second projection. Note that for all m in $p_1(p_2^{-1}(x) \cap U)$ the fiber $\Pi_x^{-1}(m) = \Pi^{-1}(m) \cap ev^{-1}(x) = (\Pi \times ev)^{-1}(m, x)$ is a unirational variety. Furthermore, note that for x general in X^r, $p_2^{-1}(x) \cap U$ is a dense open subset of $p_2^{-1}(x)$, and hence $p_1(p_2^{-1}(x) \cap U)$ is a dense open subset of $p_1(p_2^{-1}(x)) = \Pi(ev^{-1}(x))$. Hence for x general in X^r the general fiber of Π_x is a unirational variety.

1.5 Proof of Theorem 1.1.2.1.

Let $J = \{j_1, \ldots, j_M\}$ be a set of M integers satisfying $0 < j_1 < \cdots < j_M < n$ such that I is contained in J, let $\mathbf{d} = (d_1, \ldots, d_M)$ be an element in $\mathbb{N}^M$ and $1 \leqslant \mu \leqslant M$ be an integer such that the collection

$\{I, J, \mathbf{d}\}$ is stabilized with respect to μ in the sense of Definition 1.1.1.1. Note that according to Proposition 1.2.4.1 it is enough to prove that the general fiber of

$$\Pi_{I/J} \times ev : \qquad \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_*\mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r \tag{1.12}$$

is rationally connected. We prove this in two times. We first prove that the general fiber of the morphism $\Pi_{J_{\geqslant \mu}/J} \times ev$ is rationally connected, where $\Pi_{J_{\geqslant \mu}/J}$ is the morphism induced by forgetting vector spaces with indices left of μ not associated with elements in I. We then prove that the general fiber of the morphism $\Pi_{I/J_{\geqslant \mu}} \times ev$ is rationally connected. We finally deduce Theorem 1.1.2.1 from these two facts.

1.5.1 Preliminary results on the base change of $\Pi_{I/J} \times ev$.

Let $K = \{k_1, \dots, k_{m'}\}$ be a set of integers satisfying $0 < k_1 < \cdots < k_{m'}$ such that K is contained in J and contains I. Our goal here is to prove that the base change of the morphism $\Pi_{I/J} \times ev$ defined by (1.12) induced by $Fl_K \to Fl_J$ is a morphism of equidimensional projective varieties, whose general fiber satisfies the same properties as the general fiber of $\Pi_{I/J} \times ev$. Since the projection morphism $Fl_K \to Fl_I$ factors through $Fl_J \to Fl_I$, we are considering the following morphism:

$$\begin{array}{ccc} \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_J)^r} (Fl_K)^r & \longrightarrow & \left(\overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_*\mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r\right) \times_{(Fl_J)^r} (Fl_K)^r \\ & & \downarrow \wr \\ & & \overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_*\mathbf{d}) \times_{(Fl_I)^r} (Fl_K)^r \end{array}$$

First, note that $\overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_J)^r} (Fl_K)^r$ is a reduced scheme of pure dimension the expected dimension. Indeed, according to Kleiman's transversality theorem for g general in $(\mathrm{SL}_n)^r$ the projective scheme $\overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_J)^r} g \cdot (Fl_K)^r$ is an equidimensional reduced scheme. Furthermore, since the morphism $(Fl_J)^r \to (Fl_I)^r$ is $G := (SL_n)^r$-equivariant, the following diagram is commutative

$$\begin{array}{ccc} \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_I)^r} (G \times (Fl_J)^r) & \xrightarrow{\sim} & G \times \left(\overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r\right) \\ \downarrow p & & \downarrow \pi \\ G & \xrightarrow{\qquad\qquad \sim \qquad\qquad} & G \end{array}$$

where p and π are the obvious projections. This induces an isomorphism

$$\begin{aligned} p^{-1}(g) = \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_I)^r} (g \cdot (Fl_J)^r) &\simeq \pi^{-1}(g) \\ &\simeq g \cdot \left(\overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r\right) \\ &\simeq \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r \end{aligned}$$

which allows us to deduce the equidimensionality and reducibility of $\overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_J)^r} (Fl_K)^r$ from the equidimensionality and reducibility of $\overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \times_{(Fl_J)^r} g \cdot (Fl_K)^r$.

Finally, Lemma 1.5.1.1 here under ensures that the general fiber of the flat base change of $\Pi_{I/J} \times ev$ satisfies the same properties as the general fiber of $\Pi_{I/J} \times ev$.

Lemma 1.5.1.1. *Let S be a scheme, let $S' \to S$, $M \to S$ and $M' \to S$ be schemes over S, and $M \to M'$ be a morphism of schemes over S. Suppose the general fiber of $M \to M'$ satisfies a property (P), and $S' \to S$ is flat. Then the general fiber of $M_{S'} \to M'_{S'}$ satisfies (P).*

Proof. If the morphism $M \to M'$ is not surjective, it factors through $M \to \mathrm{Im}M$ and then $\mathrm{Im}(M_{S'}) \simeq (\mathrm{Im}M)_{S'}$; hence we can assume $M \to M'$ surjective. Denote by U the dense open subset of M' such that the fiber of $M \to M'$ over $\{m\}$ in U satisfies (P). Note that, since $S' \to S$ is a flat morphism, $U \times_S S'$ is a dense subset of $M' \times_S S'$. Indeed, if we denote by $p_1 : M' \times_S S' \to M'$ the first projection, acccording to [Gro] Proposition 2.3.4, all irreducible components of $p_1^{-1}(M')$ dominate M'; hence the image of each irreducible component of M' by p_1 has a non empty intersection with U,

hence each irreducible component of $M'_{S'}$ has a non empty intersection with $U_{S'}$. Then for (m,s) in $U \times_S S' \subset M' \times_S S'$, we have:

$$\begin{aligned}(M \times_S S')_{(m,s)} &= M \times_S \{s\} \times_{M'} \{m\} \\ &= (M \times_{M'} \{m\}) \times_S \{s\} \simeq M_m,\end{aligned}$$

where M_m satisfies (P) since m lies in U. Hence the general fiber of $M_{S'} \to M'_{S'}$ satisfies (P). □

1.5.2 Forgetting vector spaces left of V_{j_μ}.

Recall a point in Fl_J corresponds to a flag of vector spaces:

$$\{0\} \subset V_{j_1} \subset \cdots \subset V_{j_M} \subset \mathbb{C}^n.$$

Denote by μ' the integer such that $i_{\mu'} = j_\mu$. A point in $Fl_{J_{\geqslant \mu}}$ corresponds to a flag of vector spaces:

$$\{0\} \subset V_{i_1} \subset \cdots \subset V_{i_{\mu'}} = V_{j_\mu} \subset V_{j_{\mu+1}} \subset \cdots \subset V_{j_M} \subset \mathbb{C}^n.$$

For $1 \leqslant k < \mu$, we name $X_k := (Fl_{J_{\geqslant k}})^r$, $\mathbf{d}^k := (\pi_{J_{\geqslant k}/J})_*\mathbf{d}$ and $M_k := \overline{\mathcal{M}_{0,r}}(X_k, \mathbf{d}^k)$. Forgetting all vector spaces left of V_{j_μ} that are not of the type V_{i_k} induces a morphism $\Pi_{J_{\geqslant \mu}/J} \times ev : \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \to M_\mu \times_{X_\mu} (Fl_J)^r$. We consider here the general fiber of this map.

Proposition 1.5.2.1. *Let $\{I, J, \mathbf{d}\}$ be a stabilized collection.*

i) The general fiber of the map $\Pi_{J_{\geqslant \mu}/J} \times ev$ is a unirational variety.

ii) For x general in $(Fl_J)^r$, the fiber $ev^{-1}(x) \in \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d})$ is a tower of unirational fibrations over the fiber $ev^{-1}(\pi_{J_{\geqslant \mu}}(x)) \in M_\mu$.

Proof. We proceed by iteration on k.

i) Let $1 \leqslant k < \mu$ such that the general fiber of $\Pi_{J_{\geqslant k/J}} \times ev : M_J \to M_k \times_{X_k} (Fl_J)^r$ is rationally connected. Suppose j_{k+1} is not contained in I. Recall

$$\mathbf{d} = (d_1, \ldots, d_M).$$

We denote by

$$(\pi_{I/J})_*\mathbf{d} = (d'_1, \ldots, d'_m).$$

Denote by k' the largest integers such that $i_{k'} < j_k$. Note that

- The degree $\mathbf{d}^k$ is given by the set

$$\mathbf{d}^k = (d'_1, \ldots, d'_{k'}, d^k_{k''}, \ldots, d^k_{M'}),$$

obtained from $\mathbf{d}$ by removing all integers d_s such that $s < \mu$ and there exists no integer s' satisfying $i_{s'} = j_s$.

- The degree $\mathbf{d}^{k+1}$ is given by the sequence

$$\mathbf{d}^{k+1} = (d'_1, \ldots, d'_{k'}, d^k_{k''+1} \ldots d^k_{M'}),$$

which is obtained from the set $\mathbf{d}^k$ by forgetting the integer $d^k_{k''}$.

Note that according to Definition 1.1.1.1 for all $p < k$,

$$d^k_{k''} \geqslant \lceil (d'_p - d'_{p-1})/(i_p - i_{p-1}) \rceil j_k.$$

Then according to Proposition 1.4.0.1 the general fiber of

$$\Pi_{J_{\geqslant k+1}/J_{\geqslant k}} \times ev : M_k \to (\Pi_{J_{\geqslant k+1}/J_{\geqslant k}} \times ev)(M_k) = M_{k+1} \times_{X_{k+1}} X_k$$

is unirational, and hence rationally connected. Furthermore, according to Definition 1.1.1.1 the morphism surjects onto $M_{k+1} \times_{X_{k+1}} X_k$. Then according to Subsection 1.5.1 the general fiber of the following morphism

$$M_k \times_{X_k} (Fl_J)^r \to M_{k+1} \times_{X_{k+1}} (Fl_J)^r$$

is also rationally connected. Hence according to Theorem 1.2.2.1 by composition the general fiber of $\Pi_{J_{\geqslant k+1}/J} \times ev : M_J \to M_{k+1} \times_{X_{k+1}} (Fl_J)^r$ is rationally connected. In the same way, if j_{k+1} is contained in I, the general fiber of $\Pi_{J_{\geqslant k+2}/J} \times ev : M_J \to M_{k+2} \times_{X_{k+2}} (Fl_J)^r$ is rationally connected.

ii) Let $1 \leqslant k < \mu$ such that for x general in $(Fl_J)^r$, the fiber $ev^{-1}(x) \in \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d})$ is a tower of unirational fibrations over the fiber $ev^{-1}(\pi_{J_{\geqslant k}/J}(x)) \in M_k$ is unirational. Suppose j_{k+1} is not contained in I. Corollary 1.4.0.1 and Proposition 1.2.4.1 imply that for x general in $(Fl_J)^r$, the fiber $ev^{-1}(x) \in \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d})$ is a tower of unirational fibrations over the fiber $ev^{-1}((\pi_{J_{\geqslant k+1}/J}(x)) \in M_{k+1}$.

□

1.5.3 Forgetting vector spaces right of V_{j_μ}.

We deduce here the rational connectedness of the general fiber of the morphism $\Pi_{I/J_{\geqslant \mu}} \times ev$ from Subsection 1.5.2 by duality. For a set $K = \{k_1, \ldots, k_m\}$ of m integers satisfying $0 < k_1 < \cdots < k_m < n$, we call *dual partition* of K the set

$$K^* := \{n - k_m, \ldots, n - k_1\}.$$

Consider the natural transformation of functors sending a flag $\mathcal{F}_S := E_1 \subset \cdots \subset E_m \subset \mathcal{O}_S^{\oplus n}$ of vector bundles over a scheme S to the dual flag $\mathcal{F}_S^* := (\mathcal{O}_S^{\oplus n}/E_m)^* \subset \cdots \subset (\mathcal{O}_S^{\oplus n}/E_1)^* \subset (\mathcal{O}_S^{\oplus n})^* \simeq \mathcal{O}_S^{\oplus n}$, where we have chosen an isomorphism $(\mathbb{C}^n)^* \simeq \mathbb{C}^n$. This natural transformation of functors induces an isomorphism $Fl_K \simeq Fl_{K^*}$, and we will call Fl_{K^*} the flag variety *dual to* Fl_K. Note that this induces here an isomorphism between the flag variety $Fl_{J_{\geqslant \mu}}$ parametrizing flags of vector spaces

$$V_{i_1} \subset \cdots \subset V_{i_{\mu'}} \subset V_{j_{\mu+1}} \subset \cdots \subset V_{j_M} \subset \mathbb{C}^n.$$

and the dual flag variety $Fl_{(J_{\geqslant \mu})^*}$ parametrizing flags of vector spaces

$$V_{n-j_M} \subset \cdots \subset V_{n-j_{\mu+1}} \subset V_{n-i_{\mu'}} \subset \cdots \subset V_{n-i_1} \subset \mathbb{C}^n.$$

This isomorphism sends a curve of class $\mathbf{d} = (d_1, d_2 \ldots, d_m)$ in $E(Fl_K) \simeq \mathbb{N}^m$ onto a curve of class $\mathbf{d}^* = (d_m, \ldots, d_2, d_1)$ in $E(Fl_{K^*}) \simeq \mathbb{N}^m$. Furthermore, note that, since $\{I, J, \mathbf{d}\}$ is stabilized with respect to μ, according to Lemma 1.5.3.1 the collection $\left\{I, (J_{\geqslant \mu})^*, \left(\left(\pi_{I/J_{\geqslant \mu}}\right)_* \mathbf{d}\right)^*\right\}$ is stabilized with respect to n in the sense of Definition 1.1.1.1. Hence, according to Subsection 1.5.2 here above, the general fiber of the morphism $\Pi_{I^*/(J_{\geqslant \mu})^*} \times ev$ is rationally connected. Finally, the following commutative diagram allows us to deduce the rational connectedness of the general fiber of $\Pi_{I/J_{\geqslant \mu}} \times ev$ from the rational connectedness of the general fiber of the morphism $\Pi_{I^*/(J_{\geqslant \mu})^*} \times ev$.

$$\begin{array}{ccc}
\overline{\mathcal{M}_{0,r}}(Fl_{J_{\geqslant \mu}}, (\pi_{J_{\geqslant \mu}/J})_* \mathbf{d}) & \xrightarrow{\Pi_{I/J_{\geqslant \mu}} \times ev} & \overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_* \mathbf{d}) \underset{(Fl_I)^r}{\times} (Fl_{J_{\geqslant \mu}})^r \\
\updownarrow & & \updownarrow \\
\overline{\mathcal{M}_{0,r}}(Fl_{(J_{\geqslant \mu})^*}, ((\pi_{J_{\geqslant \mu}/J})_* \mathbf{d})^*) & \xrightarrow{\Pi_{I^*/(J_{\geqslant \mu})^*} \times ev} & \overline{\mathcal{M}_{0,r}}(Fl_{I^*}, ((\pi_{I/J})_* \mathbf{d})^*) \underset{(Fl_{I^*})^r}{\times} (Fl_{(J_{\geqslant \mu})^*})^r
\end{array}$$

Note that the isomorphism $\overline{\mathcal{M}_{0,r}}(Fl_{J_{\geqslant \mu}}, (\pi_{J_{\geqslant \mu}/J})_* \mathbf{d}) \simeq \overline{\mathcal{M}_{0,r}}(Fl_{(J_{\geqslant \mu})^*}, ((\pi_{J_{\geqslant \mu}/J})_* \mathbf{d})^*)$ is induced by the isomorphism $Fl_{J_{\geqslant \mu}} \simeq Fl_{(J_{\geqslant \mu})^*}$ and the isomorphism

$$\overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_* \mathbf{d}) \underset{(Fl_I)^r}{\times} (Fl_{J_{\geqslant \mu}})^r \simeq \overline{\mathcal{M}_{0,r}}(Fl_{I^*}, ((\pi_{I/J})_* \mathbf{d})^*) \underset{(Fl_{I^*})^r}{\times} (Fl_{(J_{\geqslant \mu})^*})^r$$

is induced by the isomorphisms $Fl_I \simeq Fl_{I^*}$ and $Fl_{J_{\geqslant \mu}} \simeq Fl_{(J_{\geqslant \mu})^*}$.

Lemma 1.5.3.1. *Let $J = \{j_1, \dots, j_M\}$ be a set of M integers satisfying $0 < j_1 < \cdots < j_M < n$ and let $I = \{i_1, \dots, i_m\}$ be a subset of J. Let $\mathbf{d} = (d_1, \dots, d_M)$ be an element in $\mathbb{N}^M$ and let $1 \leqslant \mu \leqslant M$ such that the collection $\{I, J, \mathbf{d}\}$ is stabilized with respect to μ. Then the collection $\left\{I^*, (J_{\geqslant \mu})^*, \left((\pi_{J_{\geqslant \mu}/J})_* \mathbf{d}\right)^*\right\}$ is stabilized with respect to n.*

Proof. We number by $I^* := \{i_1^*, \dots, i_m^*\}$ and $J^* = \{j_1^*, \dots, j_M^*\}$ the dual partitions of I and J, where for any $1 \leqslant k \leqslant m$, $i_k^* = n - i_{m-k+1}$ and for any $1 \leqslant k \leqslant M$, $j_k^* = n - j_{M-k+1}$. Recall Fl_{I^*} is the flag variety parametrizing flags of vector spaces

$$V_{n-i_m} \subset \cdots \subset V_{n-i_1} \subset \mathbb{C}^n$$

and $Fl_{(J \geqslant \mu)^*}$ is the flag variety parametrizing flags of vector spaces

$$V_{n-j_M} \subset \cdots \subset V_{n-j_{\mu+1}} \subset V_{n-i_{\mu'}} \subset \cdots \subset V_{n-i_1} \subset \mathbb{C}^n.$$

We call $\mathbf{d}' := (\pi_{I/J})_* \mathbf{d} = (d'_1, \dots, d'_m)$. We call $\mathbf{d}^* = (d_1^*, \dots, d_M^*)$ and $\mathbf{d'}^* = ({d'}^*_1, \dots, {d'}^*_m)$ the dual classes, where for $1 \leqslant k \leqslant M$, $d_k^* = d_{M-k+1}$ and for $1 \leqslant k \leqslant m$, ${d'}_k^* = d'_{m-k+1}$. Let $\mu \leqslant k \leqslant M$ such that $j_{M-k+1}^* = n - j_k$ is not contained in I^*, i.e. let $\mu < k \leqslant M$ such that j_k is not contained in the set $I^* = I$. Since the collection $\{I, J, \mathbf{d}\}$ is stabilized with respect to μ, the following properties are satisfied.

- Let k' be the smallest integer such that $i_{k'} > j_k$. Note that $m - k' + 1$ is the largest integer such that $i_{m-k'+1}^* = n - i_{k'} < j_{M-k+1}^* = n - j_k$. Then for any $k' < p \leqslant m$, i.e. for any $1 \leqslant m - p + 1 \leqslant m - k' + 1$, we have

  $$d_{m-k+1}^* = d_k \geqslant \lceil (d'_p - d'_{p+1})/(i_{p+1} - i_p) \rceil (n - j_k) = \lceil ({d'}^*_{m-p+1} - {d'}^*_{m-p})/(i^*_{m-p+1} - i^*_{m-p}) \rceil j^*_{M-k+1}.$$

- Since the morphism $\mathit{\Pi}_{J_{\mu \leqslant k}} \times ev$ is surjective, the morphism $\mathit{\Pi}_{(J_{\geqslant \mu})^*_{\geqslant M-k+1}} \times ev$ is surjective.

Note that the surjectivity of $\mathit{\Pi}_{(J_{\geqslant \mu})^*_{\geqslant M-k+1}} \times ev$ is implied by the following commutative diagram

$$\begin{array}{ccc}
\overline{\mathcal{M}_{0,r}}(Fl_{J_{\mu \leqslant k}}, (\pi_{J_{\mu \leqslant k}/J})_* \mathbf{d}) & \xrightarrow{\mathit{\Pi}_{J_{\mu \leqslant k}} \times ev} & \overline{\mathcal{M}_{0,r}}(Fl_{J_{\mu \leqslant k-1}}, (\pi_{J_{\mu \leqslant k-1}/J})_* \mathbf{d}) \underset{(Fl_{J_{\mu \leqslant k-1}})^r}{\times} (Fl_{J_{\mu \leqslant k-1}})^r \\
\updownarrow & & \updownarrow \\
\overline{\mathcal{M}_{0,r}}(Fl_{(J_{\geqslant \mu})^*_{\geqslant M-k+1}}, ((\pi_{J_{\mu \leqslant k}/J})_* \mathbf{d})^*) & \xrightarrow[\mathit{\Pi}_{(J_{\geqslant \mu})^*_{\geqslant M-k+1}} \times ev]{} & \overline{\mathcal{M}_{0,r}}(Fl_{(J_{\geqslant \mu})^*_{\geqslant M-k+2}}, ((\pi_{J_{\mu \leqslant k-1}/J})_* \mathbf{d})^*) \underset{(Fl_{(J_{\geqslant \mu})^*_{\geqslant M-k+2}})^r}{\times} (Fl_{(J_{\geqslant \mu})^*_{\geqslant M-k+1}})^r
\end{array}$$

where the isomorphism $\overline{\mathcal{M}_{0,r}}(Fl_{J_{\mu \leqslant k}}, (\pi_{J_{\mu \leqslant k}/J})_* \mathbf{d}) \simeq \overline{\mathcal{M}_{0,r}}(Fl_{(J_{\geqslant \mu})^*_{\geqslant M-k+1}}, ((\pi_{J_{\mu \leqslant k}/J})_* \mathbf{d})^*)$ is induced by the isomorphism $Fl_{J_{\mu \leqslant k}} \simeq (Fl_{J_{\mu \leqslant k}})^* = Fl_{(J_{\geqslant \mu})^*_{\geqslant M-k+1}}$. Hence the collection $\left\{I^*, (J_{\geqslant \mu})^*, \left((\pi_{J_{\geqslant \mu}/J})_* \mathbf{d}\right)^*\right\}$ is stabilized with respect to n. □

1.5.4 Proof of Theorem 1.1.2.1.

Let $J_1 := J_{\geqslant (\mu; n_\mu)}$. Consider the morphism

$$\begin{array}{ccc}
\overline{\mathcal{M}_{0,r}}(Fl_{J_1}, (\pi_{J_1/J})_* \mathbf{d}) \times_{(Fl_{J_1})^r} (Fl_J)^r & \longrightarrow & \left(\overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J_1})_* (\pi_{J_1/J})_* \mathbf{d}) \times_{(Fl_I)^r} (Fl_{J_1})^r\right) \times_{(Fl_{J_1})^r} (Fl_J)^r \\
& & \downarrow \wr \\
& & \overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_* \mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r
\end{array}$$

where the first arrow is the morphism obtained by flat base change of $\mathit{\Pi}_{I/J_1} \times ev$ over $(Fl_J)^r \to (Fl_{J_1})^r$. According to Subsection 1.5.3 the general fiber of $\mathit{\Pi}_{I/J_1} \times ev$ is rationally connected, hence according to Lemma 1.5.1.1 the general fiber of $\overline{\mathcal{M}_{0,r}}(Fl_{J_1}, (\pi_{J_1/J})_* \mathbf{d}) \times_{(Fl_{J_1})^r} (Fl_J)^r \to \overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_* \mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r$ is rationally connected. Furthermore, according to Subsection 1.5.2 the general fiber of

$\overline{\mathcal{M}_{0,r}}(Fl_J,\mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(Fl_{J_1},(\pi_{J_1/J})_*\mathbf{d}) \times_{(Fl_{J_1})^r} (Fl_J)^r$ is also rationally connected. Theorem 1.1.2.1 *ii*) then follows from the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}_{0,r}}(Fl_J,\mathbf{d}) & \xrightarrow{\Pi_{J_1/J}\times ev} & \overline{\mathcal{M}_{0,r}}(Fl_{J_1},(\pi_{J_1/J})_*\mathbf{d}) \times_{(Fl_{J_1})^r} (Fl_J)^r \\ & \searrow^{\Pi_{I/J}\times ev} & \downarrow \\ & & \overline{\mathcal{M}_{0,r}}(Fl_I,(\pi_{I/J})_*\mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r \end{array}$$

Indeed, according to Theorem 1.2.2.1 the composition of morphisms of projective complex schemes whose general fiber is rationally connected is a morphism whose general fiber is also rationally connected.

In the same way, since the composition of unirational fibrations is a unirational fibration, Theorem 1.1.2.1 *ii*) is easily deduced from Subsections 1.5.3 and 1.5.2.

1.6 On the surjectivity of $\Pi_{I/J}$.

Let us recall our notations. We consider the flag variety X parametrizing flags $V_{n_1} \subset \cdots \subset V_{n_m} \subset \mathbb{C}^n$, where V_{n_i} is a vector subspace of $\mathbb{C}^n$ of dimension n_i. For $1 \leqslant k \leqslant m$, we denote by $X_{\hat{k}}$ the flag variety obtained from X by forgetting the k-th vector space, and call $\pi_{\hat{k}} : X \to X_{\hat{k}}$ the forgetful morphism. Let $\mathbf{d} = (d_1,\ldots,d_m)$ be an element in $E(X) \simeq \mathbb{N}^m$. We denote by $\pi_{\hat{k}*}\mathbf{d} = (d_1,\ldots,d_{k-1},d_{k+1},\ldots,d_m)$ its pushforward to $E(X_{\hat{k}})$. In order to ease notations, we set $d_0 = 0 = d_{m+1}$. Finally, we denote by $\Pi_{\hat{k}} : \overline{\mathcal{M}_{0,r}}(X,\mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X_{\hat{k}},\pi_{\hat{k}*}\mathbf{d})$ the morphism induced by $\pi_{\hat{k}} : X \to X_{\hat{k}}$.

Our goal here is to give a condition on the degree $\mathbf{d}$, under which the morphism $\overline{\mathcal{M}_{0,r}}(X,\mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X_{\hat{k}},\pi_{\hat{k}*}\mathbf{d}) \times_{X_{\hat{k}}{}^r} X^r$ is surjective. We deduce from it examples of stabilized collections in Subsection 1.6.1.

Lemma 1.6.0.1. *Consider an element p in general position in $\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},\pi_{\hat{k}*}\mathbf{d})$. Denote by $\mathcal{F}_p = E_1 \subset \cdots \subset E_{k-1} \subset E_{k+1} \subset \cdots \subset E_m \subset \mathbb{C}^n$ the flag of vector bundles associated with p. Suppose for all $p < k$, $d_{k+1} \geqslant n_{k+1}\lceil\frac{d_p-d_{p-1}}{n_p-n_{p-1}}\rceil$. Then:*

i) $E_{k+1} = E_{k-1} \oplus E_{k+1}/E_{k-1}$;

ii) $E_{k+1}/E_{k-1} \simeq \mathcal{O}_{\mathbb{P}^1}(-\lfloor\frac{d_{k+1}-d_{k-1}}{n_{k+1}-n_{k-1}}\rfloor)^{\oplus n-r_2} \oplus \mathcal{O}_{\mathbb{P}^1}(-\lfloor\frac{d_{k+1}-d_{k-1}}{n_{k+1}-n_{k-1}}\rfloor+1)^{\oplus r_2})$, *where r_2 is the integer defined by $r_2 = (d_{k+1}-d_{k-1}) - (n_{k+1}-n_{k-1})\lfloor\frac{d_{k+1}-d_{k-1}}{n_{k+1}-n_{k-1}}\rfloor$;*

iii) There exists a $\mathbb{P}^1$-vector bundle F of degree $-d$ where $d \leqslant d_{k-1} + (n_k - n_{k-1})(\left\lfloor\frac{d_{k+1}-d_{k-1}}{n_{k+1}-n_{k-1}}\right\rfloor + 1)$ and of rank n_k such that:

$$E_{k-1} \subset F \subset E_{k+1} \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}.$$

Proof. i) Cf. Proposition 1.3.3.1.

ii) Cf. the description of balanced flags of vector bundles in Section 1.3.3.

iii) Call $d := d_{k+1} - d_{k-1}$ and $n := n_{k+1} - n_{k-1}$. Consider two non negative integers p_1 and p_2 such that $p_1 < n - r_2$, $p_2 < r_2$, and $p_1 + p_2 = n_k - n_{k-1}$.
Denote by F_0 the rank $n_k - n_{k-1}$ subbundle of E_{k+1} associated with $\mathcal{O}_{\mathbb{P}^1}(-\lfloor\frac{d}{n}\rfloor)^{\oplus p_1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\lfloor\frac{d}{n}\rfloor+1)^{\oplus p_2})$ by the isomorphism $E_{k+1}/E_{k-1} \simeq \mathcal{O}_{\mathbb{P}^1}(-\lfloor\frac{d}{n}\rfloor)^{\oplus n-r_2} \oplus \mathcal{O}_{\mathbb{P}^1}(-\lfloor\frac{d}{n}\rfloor+1)^{\oplus r_2})$ of *ii*).
Then $F := E_{k-1} \oplus F_0$ is a vector bundle of rank n_k, of degree the opposite of

$$d_{k-1} + p_1(\lfloor\frac{d}{n}\rfloor) + p_2(\lfloor\frac{d}{n}\rfloor+1) \leqslant d_{k-1} + (n_k - n_{k-1})(\lfloor\frac{d}{n}\rfloor+1).$$

Furthermore, F satisfies $E_{k-1} \subset F \subset E_{k+1} \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$.

□

Proposition 1.6.0.1. *i) Suppose* $d_{k-1} \leqslant \lfloor \frac{d_{k+1}}{n_{k+1}} \rfloor$ *and* $d_k \geqslant d_{k-1} + (n_k - n_{k-1})(\lfloor \frac{d_{k+1}-d_{k-1}}{n_{k+1}-n_{k-1}} \rfloor + 1)$. *Then the morphism*

$$\Pi_{\hat{k}} : \overline{\mathcal{M}_{0,3}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,3}}(X_{\hat{k}}, \pi_{\hat{k}*}\mathbf{d})$$

is surjective;

ii) Consider an integer τ *such that, for all* $d_k \geqslant \tau$, *the morphism* $\Pi_{\hat{k}} : \overline{\mathcal{M}_{0,3}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,3}}(X_{\hat{k}}, \pi_{\hat{k}*}\mathbf{d})$ *is surjective.*
Suppose $d_k \geqslant \tau + r(n_k - n_{k-1})$. *Then the morphism*

$$\Pi_{\hat{k}} \times \mathrm{ev}^r : \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, \pi_{\hat{k}*}\mathbf{d}) \times_{X_{\hat{k}}{}^r} X^r$$

is surjective.

Proof. i) Call $\mathbf{d}_{\hat{k}} := \pi_{\hat{k}*}\mathbf{d}$. Consider an element p in general position in $\overline{\mathcal{M}_{0,3}}(X_{\hat{k}}, \mathbf{d}_{\hat{k}})$. Since X is convex, $\mathcal{M}_{0,3}(X_{\hat{k}}, \mathbf{d}_{\hat{k}})$ is a dense open subset in $\overline{\mathcal{M}_{0,3}}(X_{\hat{k}}, \mathbf{d}_{\hat{k}})$; hence, according to Lemma 1.2.2.1, p is in $\mathcal{M}_{0,3}(X_{\hat{k}}, \mathbf{d}_{\hat{k}})$. We will construct an antecedent of p by $\Pi_{\hat{k}}$ as a concatenation of two morphisms $f : \mathbb{P}^1 \to X$ and $g : \mathbb{P}^1 \to X$.
Construction of f**.** Denote by $\mathcal{F}_p := E_1 \subset \cdots \subset E_{k-1} \subset E_{k+1} \subset \cdots \subset E_m \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ the flag of vector bundles associated with p. According to Lemma 1.6.0.1 *iii*), there exists a $\mathbb{P}^1$-vector bundle F of degree $d \leqslant d_{k-1} + (n_k - n_{k-1})(\lfloor \frac{d_{k+1}-d_{k-1}}{n_{k+1}-n_{k-1}} \rfloor + 1)$ and of rank n_k defining the following flag of vector bundles:

$$\mathcal{F} := E_1 \subset \cdots \subset E_{k-1} \subset F \subset E_{k+1} \subset \cdots \subset E_m \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}.$$

By the functorial definition of flag varieties, $\mathcal{F}$ defines a degree $(d_1, \ldots, d_{k-1}, d, d_{k+1}, \ldots, d_m)$ morphism $f : \mathbb{P}^1 \to X$, such that $f_p : \mathbb{P}^1 \to X_{\hat{k}}$, factors through f. We have denoted by f_p the morphism associated with p.
Construction of g**.** Denote by $g : \mathbb{P}^1 \to X$ a rational curve of class $(d_k - d)\sigma_k$, where σ_k is the degree one Schubert class whose pushforward to $\mathrm{Gr}(n_k, n)$ is not 0. Up to multiplication by an element in SL_n, we can assume $g(\mathbb{P}^1)$ intersects $f(\mathbb{P}^1)$ on a point that is not marked. Notice the image of $\pi_{\hat{k}} \circ g : \mathbb{P}^1 \to X_{\hat{k}}$ is a point.
Finally, the stable map $\mathbb{P}^1 \cup \mathbb{P}^1 \to X$ induced by f and g defines an element in $\overline{\mathcal{M}_{0,3}}(X, \mathbf{d})$, whose projection by $\Pi_{\hat{k}}$ is p.

ii) Call $\mathcal{M}' := \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, \pi_{\hat{k}*}\mathbf{d})$. Fix a genus zero stable map $p := (g_p : \mathbb{P}^1 \to \mathcal{M}', \{p_1, \ldots, p_r\})$ corresponding to a general point in $\mathcal{M}'$. According to Lemma 1.2.3.1, for any dense open subset U of $\mathcal{M}'$, the equidimensional variety $X^r \times_{(X_{\hat{k}})^r} U$ is a dense open subset of $X^r \times_{(X_{\hat{k}})^r} \mathcal{M}'$; hence it is enough to prove that $\Pi_{\hat{k}}^{-1}(\nu)$ dominates $X^r \times_{(X_{\hat{k}})^r} \{p\}$ for a general point p in $\mathcal{M}'$.
Fix an element (x, p) in $X^r \times_{(X_{\hat{k}})^r} \{p\}$, i.e. fix r points $x_1, \ldots, x_r$ on X such that each point x_i gets projected by $\pi_{\hat{k}} : X \to X_{\hat{k}}$ to $g_p(p_i)$. Our goal is to construct an antecedent of (x, p) by $\Pi_{\hat{k}} : \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \mathcal{M}'$.
We proceed in three steps. First, notice there is a degree $(\mathbf{d} - r(n_k - n_{k-1})\sigma_k)$ morphism $f : \mathbb{P}^1 \to X$ getting projected to p, i.e. such that $g_p = \pi_{\hat{k}} \circ f$. Furthermore, we construct rational curves $C_i \to X$ of degree $n_k - n_{k-1}$ joining the points $f(p_i)$ and x_i. Finally, by concatenation, this gives a degree $\mathbf{d}$ stable map $[\mathbb{P}^1 \cup_i C_i \to X, \{p_1, \ldots, p_r\}]$ which is an antecedent of (x, g), as researched.
Existence of f**.** We call:

$$\boldsymbol{\delta} := (d_1, \ldots, d_{k-1}, d_k - r(n_k - n_{k-1}), d_{k+1}, \ldots, d_m) \in A_1(X).$$

Notice that, since g is in general position in $\mathcal{M}_{0,3}(X, \mathbf{d})$ and the morphism $\Pi_{\hat{k}} : \overline{\mathcal{M}_{0,3}}(X, \boldsymbol{\delta}) \to \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, \pi_{\hat{k}*}\mathbf{d})$ is surjective, its reciprocal image $\Pi_{\hat{k}}^{-1}(p)$ has a non empty intersection with the dense open subset $\mathcal{M}_{0,3}(X, \boldsymbol{\delta})$ of $\overline{\mathcal{M}_{0,r}}(X, \boldsymbol{\delta})$. We can thus find an element $[f : \mathbb{P}^1 \to X, \{p_1, \ldots, p_r\}]$ in $\Pi_{\hat{k}}^{-1}(\nu) \cap \mathcal{M}_{0,r}(X, \boldsymbol{\delta})$.
Construction of $C_i \to X$**.** Fix an integer i in $\{1, \ldots, r\}$. Denote by $\mathcal{F}_{p_i} = \{ W_1^i \cdots \subset W_m^i \subset \mathbb{C}^n\}$

the germ at p_i of the flag of $\mathbb{P}^1$-vector bundles associated with f. Since $f : \mathbb{P}^1 \to X$ projects to g_p, the flag of vector spaces associated with $g_p(p_i)$ is:

$$g_p(p_i) = \{W_1^i \subset \cdots \subset W_{k-1}^i \subset W_{k+1}^i \subset \cdots \subset W_m^i \subset \mathbb{C}^n\}.$$

Denote by $< u_{n_{k-1}+1}, \ldots, u_{n_k} >$ a basis of W_k^i/W_{k-1}^i.
Recall x_i is a point in X getting projected to $g_p(p_i)$, i.e. corresponds to a flag of vector spaces

$$W_1^i \subset \cdots \subset W_{k-1}^i \subset V \subset W_{k+1}^i \subset \cdots \subset W_m^i \subset \mathbb{C}^n.$$

Since $\mathrm{ev} : \overline{\mathcal{M}_{0,2}}(\mathrm{Gr}(n_k - n_{k-1}, n_{k+1} - n_{k-1}), d) \to \mathrm{Gr}(n_k - n_{k-1}, n_{k+1} - n_{k-1})^2$ is surjective for $d \geqslant (n_k - n_{k-1})$ (cf. for example [BM11] Corollary 2.2), there exists a degree $(n_k - n_{k-1})$ genus zero stable map

$$h_i : C_i = \bigcup_{j=1}^{N} \mathbb{P}^1 \to \mathrm{Gr}(n_k - n_{k-1}, n_{k+1} - n_{k-1})$$

joining the rank $(n_k - n_{k-1})$ vector spaces V/W_{k-1}^i and W_k^i/W_{k-1}^i within $W_{k+1}^i/W_{k-1}^i \simeq \mathbb{C}^{n_{k+1}-n_{k-1}}$.
By the functorial definition of Grassmannians, each morphism $h_{i|\mathbb{P}^1} : \mathbb{P}^1 \to \mathrm{Gr}(n_k - n_{k-1}, n_{k+1} - n_{k-1})$ defines a flag of $\mathbb{P}^1$-vector bundles

$$E_j^i \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n_{k+1}-n_{k-1}},$$

which has one fiber given by the flag of vector spaces $V_j/W_{k-1}^i \subset W_{k+1}^i/W_{k-1}^i \simeq \mathbb{C}^{n_{k+1}-n_{k-1}}$, and another fiber given by $V_{j+1}/W_{k-1}^i \subset W_{k+1}^i/W_{k-1}^i \simeq \mathbb{C}^{n_{k+1}-n_{k-1}}$, where $V_1 = V$ and $V_N = W_k$. Now consider the following flag of $\mathbb{P}^1$-vector bundles:

$$\mathcal{F}_{i,j} := W_1^i \subset \cdots \subset W_{k-1}^i \subset W_{k-1}^i \oplus E_j^i \subset W_{k+1}^i \subset \cdots \subset W_m^i \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}.$$

$\mathcal{F}_{i,j}$ defines a morphism $f_{i,j} : \mathbb{P}^1 \to X$ which is projected to $g_p(p_i)$. Furthermore, the morphisms $f_{i,j}$ define a degree $(n_k - n_{k-1})$ morphism from the tree of $\mathbb{P}^1$'s C_i to X which joins by construction x_i and $g_p(p_i)$. We will denote by $f_i : C_i \to X$ the morphism thus obtained.
Construction of an antecedent of p. Finally, consider the stable map $q = [\mathbb{P}^1 \cup_i C_i \to X, \{p_1, \ldots, p_r\}]$ associated with $[f : \mathbb{P}^1 \to X]$ and $f_i : C_i \to X$, where we place each of the marked points p_i on an antecedent $f_i^{-1}(x_i)$ on C_i. Since each of the map $f_i : C_i \to X$ projects to $g_p(p_i)$ in $X_{\widehat{k}}$, the morphism $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \mathcal{M}'$ collapses each component C_i and sends q to p. □

1.6.1 Examples of stabilized collections.

1. Forgetting the first vector space. Suppose $I = \{i_1, \ldots, i_m\}$ is a collection of integers satisfying $1 < i_1 < \cdots < i_m < n$ and $J = \{j_1, i_1, \ldots, i_m\}$ satisfies $0 < j_1 < i_1$. Note that then $Fl_J \to Fl_I$ is the morphism forgetting the first vector space. By definition, if the following condition is satisfied, the collection $\{I, J, \mathbf{d}\}$ is stabilized with respect to $\mu = 2$.

   - The morphism $\Pi_{J \geqslant 1} \times ev : \overline{\mathcal{M}_{0,r}}(Fl_J, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(Fl_I, (\pi_{I/J})_* \mathbf{d}) \times_{(Fl_I)^r} (Fl_J)^r$ is surjective.

   Note that since there is no integer $i_{k'}$ such that $i_{k'} < j_k$, the collection $\{I, J, \mathbf{d}\}$ does not have to satisfy any other condition. Furthermore, according to Proposition 1.6.0.1 *i*), if $d_0 = 0 \leqslant \lfloor \frac{d_2}{i_1} \rfloor$ and $d_1 \geqslant d_0 + (j_1 - 0)(\lfloor \frac{d_2 - d_0}{i_1 - 0} \rfloor + 1) = j_1(\lfloor \frac{d_2}{i_1} \rfloor + 1)$, then the morphism $\overline{\mathcal{M}_{0,3}}(Fl_J, \mathbf{d}) \to \overline{\mathcal{M}_{0,3}}(Fl_I, (\pi_{I/J})_* \mathbf{d})$ is surjective. Note that the first condition is always satisfied. Hence according to Proposition 1.6.0.1 *ii*), if

   $$d_1 \geqslant j_1(\lfloor \frac{d_2}{i_1} \rfloor + 1) + r(j_1 - 0) = j_1(\lfloor \frac{d_2}{i_1} \rfloor + r + 1)$$

   then the morphism $\Pi_{J \geqslant 1} \times ev$ is surjective.

   To conclude, we observe the following property:

if $d_1 \geqslant j_1\,(r+1+\lfloor d_2/i_1 \rfloor)$ the collection $(I, J, (d_1, d_2, \dots d_{m+1}))$ is stabilized.

2. Projection from a Grassmannian to the point. Suppose $J = \{k\}$ and $I = \varnothing$. Then Fl_J is the Grassmannian $\mathrm{Gr}(k,n)$ and $Fl_J = \mathrm{Gr}(k,n) \to Fl_I = \mathrm{Spec}\mathbb{C}$ is the projection to the point. By definition, the collection $\{I, J, d\}$ is stabilized iff the morphism

$$\Pi_{J \geqslant 1} \times ev : \overline{\mathcal{M}_{0,r}}(\mathrm{Gr}(k,n), d) \to \overline{M_{0,r}} \times (\mathrm{Gr}(k,n))^r$$

is surjective. Since the projection $\overline{\mathcal{M}_{0,3}}(\mathrm{Gr}(k,n), d) \to \overline{M_{0,3}} = \mathrm{Spec}\mathbb{C}$ is always surjective, according to Proposition 1.6.0.1 *ii*), if

$$d \geqslant r(k-0) = rk,$$

then $\Pi_{J \geqslant 1} \times ev$ is surjective. Hence:

$$\text{if } d \geqslant rk \text{ the collection } \{I, J, d\} \text{ is stabilized.}$$

Finally, since the collection $\{I, J, d\}$ is stabilized, according to Lemma 1.5.3.1, the dual collection $\{I^*, J^*, d^* = d\}$ is also stabilized. Note that $Fl_{J^*} = \mathrm{Gr}(n-k, n)$. Hence:

$$\text{if } d \geqslant r(n-k) \text{ the collection } \{I, J, d\} \text{ is also stabilized.}$$

Appendix A

Let Y be an irreducible projective complex variety, let $\gamma \in H_2(Y, \mathbb{Z})$, and $r \geqslant 0$. We first recall the usual definitions associated with the space of stable maps $\overline{\mathcal{M}_{0,r}}(Y, \gamma)$. The data of a morphism $\varphi : C \to Y$ from a genus zero projective connected reduced and nodal curve C to Y along with r distinct non singular points $\{c_1, \ldots, c_r\}$ on C is called a stable map if any irreducible component of C getting sent to a point contains at least three points which are nodal or marked.

Let S be a complex scheme. A family of r-pointed quasi-stable maps over S consists of the data of morphisms $\mathcal{C} \to Y$ and $\pi : \mathcal{C} \to S$ along with r sections $p_i : S \to \mathcal{C}$, such that each geometric fiber is a genus zero projective connected reduced and nodal curve C_s and the marked points $p_i(s)$ are distinct and non singular. By construction, $\overline{\mathcal{M}_{0,r}}(Y, \gamma)$ is the coarse moduli space parametrizing isomorphism classes of families over S of r-pointed, quasi-stable maps from genus zero curves to Y representing the class γ, and such that each geometric fiber $\mathcal{C}_s$ over a point in S defines a stable map $\mathcal{C}_s \to Y$ [FP96].

Lemma 1.6.1.1. *Fix distinct points $p_1, \ldots, p_r$ in $\mathbb{P}^1$. Let S be a complex scheme, and let $\mu : S \times \mathbb{P}^1 \to Y$ be a morphism such that, for all closed points $s \in S$, the restriction μ_s of μ to $s \times \mathbb{P}^1$ verifies $\mu_{s*}[\mathbb{P}^1] = \gamma \in A_1(Y)$. Let pr_1 be the natural projection from $S \times \mathbb{P}^1$ to S, and $p_i : S \to S \times \mathbb{P}^1$, $1 \leqslant i \leqslant r$, be the morphisms respectively defined by $p_i : s \to (s, p_i)$. Then the family*

$$\begin{array}{ccc} S \times \mathbb{P}^1 & \xrightarrow{\mu} & Y \\ \downarrow{\scriptstyle pr_1} & & \\ S & & \end{array}$$

together with the sections $p_i : S \to S \times \mathbb{P}^1$ is a stable family on S of maps from r-pointed genus 0 curves to Y representing the class γ.

Furthermore, the morphism $f : S \to \overline{\mathcal{M}_{0,r}(Y, \gamma)}$ associated with this family sends a point $s \in S$ into the point of $\mathcal{M}_{0,r}(Y, \gamma)$ associated with $[\mu_s : \mathbb{P}^1 \to Y, \{p_1, \ldots, p_r\}]$.

Proof. Let s be a closed point of S. Then the geometric fiber in s verifies $pr_1^{-1}(s) = (S \times \mathbb{P}^1) \times_S \mathrm{Spec}(\mathbb{C}(s)) \simeq \mathrm{Spec}(\mathbb{C}) \times \mathbb{P}^1$, i.e. is an irreducible rational projective curve. Hence $pr_1^{-1}(s)$ is a projective, connected, smooth, reduced, rational curve with r distinct non singular points.

Since $\mathrm{char}(\mathbb{C}) = 0$, $\mathbb{P}^1$ is flat on $\mathrm{Spec}\mathbb{Z}$, and $pr_1 : S \times \mathbb{P}^1 \to S$ is a flat morphism. Furthermore, pr_1 may be written down as $S \times \mathbb{P}^1 \simeq \mathbb{P}^1_{\mathbb{Z}} \times \mathrm{Spec}\mathbb{C} \times S \to \mathrm{Spec}\mathbb{C} \times S \simeq S$, hence is a projective morphism. The family $(pr_1 : S \times \mathbb{P}^1 \to S, \{p_1, \ldots, p_r\})$ is thus a family of r-pointed genus 0 quasi stable curves on S.

For s in S, denote by $\mu_s : C_s \to Y$ the restriction of μ to the geometric fiber $C_s := pr_1^{-1}(s)$. Note that μ_s is a map from the r-pointed quasi-stable map $(C_s, \{p_1, \ldots, p_r\})$ to Y. The map μ_s is stable, and by hypothesis represents the class γ. The family $(pr_1 : S \times \mathbb{P}^1 \to S, \{p_1, \ldots, p_r\}, \mu : S \times \mathbb{P}^1 \to Y)$ hence is a stable family on S of maps from r-pointed genus 0 curves to Y representing the class γ, i.e. $o = (pr_1 : S \times \mathbb{P}^1 \to S, \{p_1, \ldots, p_r\}, \mu : S \times \mathbb{P}^1 \to Y)/\sim$ is an element of $\overline{\mathcal{M}_{0,r}}(Y, \gamma)(S)$.

Let $s : \mathrm{Spec}(\mathbb{C}) \to S$ be a point of S. Since $\overline{\mathcal{M}_{0,r}}(Y, \gamma)$ is a coarse moduli space, the functor $\overline{\mathcal{M}_{0,r}}(Y, \gamma)$ associates to s the map from $\overline{\mathcal{M}_{0,r}}(Y, \gamma)(S) \to \overline{\mathcal{M}_{0,r}}(Y, \gamma)(\mathrm{Spec}(\mathbb{C}))$ associating to a stable family $(\pi : \mathcal{C} \to S, \{p_1, \ldots, p_r\}, \mu : \mathcal{C} \to Y)/\sim$ corresponding to an element in $\overline{\mathcal{M}}_{0,r}(Y, \gamma)(S)$ the isomorphism class of:

$$\begin{array}{ccccc} \mathrm{Spec}(\mathbb{C}) \times_S \mathcal{C} & \longrightarrow & \mathcal{C} & \xrightarrow{\mu} & Y \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{s} & S & & \end{array}$$

We observe $\overline{\mathcal{M}}_{0,r}(Y, \gamma)(s)$ sends o to the element in $\overline{\mathcal{M}_{0,r}}(Y, \gamma)(\mathrm{Spec}(\mathbb{C}))$ associated with $(C_s, \{p_1, \ldots, p_r\}, \mu_s : C_s \to Y)$.

Since $\overline{M_{0,r}}(Y, \gamma)$ is a coarse moduli space, there exists a natural transformation of functors :

$$\Phi : \overline{\mathcal{M}_{0,r}}(Y, \gamma) \to \mathcal{H}om_{Sch}(S, \overline{\mathcal{M}_{0,r}(Y, \gamma)})$$

Let $f : S \to \overline{M_{0,r}(Y,\gamma)}$ be the morphism associated by Φ with $o = (pr_1 : S \times \mathbb{P}^1 \to S, \{p_1, \ldots, p_r\}, \mu : S \times \mathbb{P}^1 \to Y)/\sim$, i.e. $f = \Phi(o)$. We obtain :

$$\Phi(S)(o) \circ s = \Phi(\mathrm{Spec}(\mathbb{C}))(\overline{\mathcal{M}_{0,r}}(Y,\gamma)(s)),$$

i.e. f sends s to the point of $\overline{\mathcal{M}_{0,r}(Y,\gamma)}$ associated with $(C_s, \{p_1, \ldots, p_r\}, \mu_s)$. □

Chapter 2

A comparison formula between genus 0 correlators of partial flag varieties

2.1 Introduction

Removing a subspace from a partial flag gives another partial flag composed of fewer subspaces. This induces a forgetful map between the corresponding flag varieties. The main result of this chapter is that quantum K-theoretical genus 0 correlators of these two flag varieties are equal to each other, provided their degree is high enough.

2.1.1 Genus zero correlators of G/P. Quantum K-theory takes its simplest form for homogeneous varieties, and genus zero quantum K-theory actually first introduced and defined in this setting by Givental some twenty years ago [Giv00].

Let X be a homogeneous variety G/P, where G is an algebraic group, and P is a parabolic subgroup of G. Fix a degree d in the semi-group $E(X)$ of effective classes of 1-cycles in $A_1(X) \simeq H_2(X, \mathbb{Z})$. Denote by $\overline{\mathcal{M}_{0,n}}(X, d)$ the moduli space parametrizing stable maps $(\mu : C \to X, \{p_1, \ldots, p_n\})$ from a genus zero curve C to X verifying $\mu_*[C] = d$, with n marked points $p_1, \ldots, p_n$ on C. If X is a homogeneous variety, $\overline{\mathcal{M}_{0,n}}(X, d)$ is an irreducible rational variety [KP01, Tho98], and the subvariety $\mathcal{M}_{0,n}(X, d)$ parametrizing stable maps $(\mu : \mathbb{P}^1 \to X, \{p_1, \ldots, p_n\})$ from the projective line to X is a dense open subset of $\overline{\mathcal{M}_{0,n}}(X, d)$ [FP96]. The *evaluation morphism* $\mathrm{ev}_i : \overline{\mathcal{M}_{0,n}}(X, d) \to X$ assigns to a map the image of its i-th marked point. If X is a homogeneous variety, the evaluation morphism is flat.

Consider an algebraic group H acting on $X = G/P$. Let $\phi_1, \ldots, \phi_r$ be elements in the Grothendieck ring $K_H(X)$ of H-equivariant coherent sheaves on X. The *equivariant correlator* $\langle \phi_1, \ldots, \phi_r \rangle_{H,d}^X$ of elements ϕ_i in $K_\circ(X)$ is defined as:

$$\langle \phi_1, \ldots, \phi_r \rangle_{H,d}^X := \chi_H \left(\mathrm{ev}_1^*(\phi_1) \cdot \ \ldots \ \cdot \mathrm{ev}_r^*(\phi_r) \cdot [\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d)}] \right),$$

where χ_H denotes the H-equivariant sheaf Euler characteristic.

2.1.2 H-equivariant decomposition of the diagonal. Let H be a complex linear algebraic group, let X be an irreducible variety with an H-action. Let $R(H)$ denote the representation ring of H, and $K_H(X)$ denote the Grothendieck ring of H-equivariant coherent sheaves on X. We say X *admits a diagonal decomposition in* $K_H(X)$ if there exists a basis $(\alpha)_{i \in I}$ of the $R(H)$-module $K_H(X)$, and a dual basis $(\alpha^\vee)_{i \in I}$ of $K^H(X)$ such that

$$\Delta_*[\mathcal{O}_X] = \sum_{\alpha \in I} \alpha \boxtimes (\alpha^\vee \otimes [\mathcal{O}_X]) \in K_H(X \times X),$$

where $\Delta : X \to X \times X$ is the diagonal embedding, and $(\alpha^\vee)_{i\in I}$ is the dual basis of $(\alpha)_{i\in I}$ with respect to the pairing given by the H-equivariant sheaf Euler characteristic, i.e. $\chi_H(\alpha^\vee \otimes \beta) = \delta_{\alpha,\beta}$ for all elements α, β in $K_H(X)$.

We study here the following setting. Consider a complex linear algebraic subgroup H of GL_n acting on flag varieties via left multiplication such that flag varieties GL_n/P admit a diagonal decomposition in $K_H(X)$. Note that this is in particular true when considering the non equivariant setting or, according to [GK08], when considering the action of a maximal torus T of GL_n.

2.1.3 Main result. Fix two positive integers $m < n$. Let $0 < n_1 < \cdots < n_m < n$ be a strictly increasing collection of non negative integers. Denote by X the m-step flag variety parametrizing m-tuples of vector spaces ordered by inclusion

$$0 \subset V_{n_1} \subset \cdots \subset V_{n_m} \subset \mathbb{C}^n$$

such that $\dim V_{n_i} = n_i$. Fix a positive integer k, $1 \leqslant k \leqslant m$. Denote by $X_{\widehat{k}}$ the $(m-1)$-step flag variety obtained from X by forgetting the k-th vector subspace. Denote by $\pi_{\widehat{k}} : X \to X_{\widehat{k}}$ the projection. The class $\mathbf{d} = (d_1, \ldots, d_m)$ of a curve C in X is determined by non-negative integers d_1, d_2,..., d_m, where d_i is the Plücker degree of the projection of C to the Grassmannian $\mathrm{Gr}(n_i, n)$. Note that $\pi_{\widehat{k}}$ and the evaluation morphism $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to X^r$ induce a morphism $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X_{\widehat{k}}, (\pi_{\widehat{k}})_*\mathbf{d}) \times_{X_{\widehat{k}}{}^r} X^r$. Denote by H an algebraic subgroup of G acting via left multiplication on X and $X_{\widehat{k}}$, such that $X_{\widehat{k}}$ admits a diagonal decomposition in $K_H(X_{\widehat{k}})$.

Theorem 2.1.3.1. *Let $r > 0$, $k \in \{1, \ldots, m\}$, $\mathbf{d} = (d_1, \ldots, d_m)$ be a nef class of 1-cycles such that $\forall\, p < k$, $d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$ and such that the morphism*

$$\overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X_{\widehat{k}}, (\pi_{\widehat{k}})_*\mathbf{d}) \underset{X_{\widehat{k}}{}^r}{\times} X^r$$

is surjective. Then for all elements ϕ_1, ..., ϕ_r in $K_H(X)$, the correlators of X and $X_{\widehat{k}}$ associated with H, $\mathbf{d}$, and the elements ϕ_i are equal:

$$\langle \phi_1, \ldots, \phi_r \rangle^X_{H,\mathbf{d}} = \langle (\pi_{\widehat{k}})_*(\phi_1), \ldots, (\pi_{\widehat{k}})_*(\phi_r) \rangle^{X_{\widehat{k}}}_{H,(\pi_{\widehat{k}})_*\mathbf{d}}.$$

According to Chapter 1 Proposition 1.6.0.1, Theorem 2.1.3.1 implies the following result.

Theorem 1bis. *Let $r > 0$, $k \in \{1, \ldots, m\}$, $\mathbf{d} = (d_1, \ldots, d_m)$ be a nef class of 1-cycles such that*

- $\forall\, p < k$, $d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$,
- $d_{k-1} \leqslant \lfloor \frac{d_{k+1}}{n_{k+1}} \rfloor$,
- $d_k \geqslant r(n_k - n_{k-1}) + d_{k-1} + (n_k - n_{k-1})(\lfloor \frac{d_{k+1} - d_{k-1}}{n_{k+1} - n_{k-1}} \rfloor + 1)$.

Then for any elements ϕ_1, ..., ϕ_r in $K_H(X)$, the correlators of X and $X_{\widehat{k}}$ associated with H, $\mathbf{d}$, and the elements ϕ_i are equal:

$$\langle \phi_1, \ldots, \phi_r \rangle^X_{H,\mathbf{d}} = \langle (\pi_{\widehat{k}})_*(\phi_1), \ldots, (\pi_{\widehat{k}})_*(\phi_r) \rangle^{X_{\widehat{k}}}_{H,(\pi_{\widehat{k}})_*\mathbf{d}}.$$

This Chapter is organized as follows. Sections 2.2 is devoted to recalling some standard facts about equivariant algebraic K-theory, and fixing the notations we use in the rest of the chapter. A proof of Theorem 2.1.3.1 is presented in Section 2.3, using results of Chapter 1. In the non equivariant setting a more geometric proof of Theorem 2.1.3.1 is provided in part 2.4, relying heavily on results from Chapter 1.

2.2 Preliminaries on equivariant algebraic K-theory

2.2.1 Diagonal decomposition of $K_T(X)$. Recall G is a semi-simple linear complex algebraic group, T is a maximal torus of G, B is a Borel subgroup of G and P is a parabolic subgroup of G such that $T \subset B \subset P$. For any element u in the Weyl group W recall $X(u) := \overline{BuP/P}$ is a Schubert variety of $X = G/P$, and $Y(u) := w_0 \cdot \overline{Bw_0uP/P}$ is the Schubert variety opposite to it. Furthermore, denote by $\partial Y(u) := Y(u)\backslash(w_0Bw_0uP/P)$ the complementary of the dense open Schubert cell in $Y(u)$. According to [GK08] we have the following decomposition in $K_T(X)$.

$$\Delta_*[\mathcal{O}_X] = \sum_{u \in W^P} [\mathcal{O}_{X(u)}] \boxtimes [\mathcal{O}_{Y(u)}(-\partial Y(u))] \in K_T(X \times X),$$

where $\Delta : X \to X \times X$ denotes the diagonal embedding. Note that this was first observed in the non equivariant setting by Brion [Bri02].

2.2.2 Equivariant flat base change. We describe here the class of an equivariant flat base change. The main application of this result will be the following setting. Let H be a linear subgroup of GL_n, let X be the flag variety parametrizing flags of vector spaces $V_{n_1} \subset \cdots \subset V_{n_m} \subset \mathbb{C}^n$, let $X_{\widehat{k}}$ be the flag variety obtained by forgetting the vector V_{n_k}, and denote by $\pi_{\widehat{k}} : X \to X_{\widehat{k}}$ the associated morphism. For a class β of curve on $X_{\widehat{k}}$, we consider the fiber product described by the following H-equivariant Cartesian diagram :

$$\begin{array}{ccc} X^r \underset{(X_{\widehat{k}})^r}{\times} \overline{\mathcal{M}_{0,r}}(X_{\widehat{k}}, \beta) & \longrightarrow & \overline{\mathcal{M}_{0,r}}(X_{\widehat{k}}, \beta) \\ \downarrow & & \downarrow ev \\ X^r & \xrightarrow{(\pi_{\widehat{k}})^r} & (X_{\widehat{k}})^r \end{array}$$

Note that, since the action of $(\mathrm{GL}_n)^r$ on $(X_{\widehat{k}})^r$ is transitive, the morphism $(\pi_{\widehat{k}})^r : X^r \to (X_{\widehat{k}})^r$ is flat.

Consider two morphisms of schemes $f_1 : M_1 \to X_1$ and $f_2 : M_2 \to X_2$. We denote by $f_1 \times f_2 : M_1 \times M_2 \to X_1 \times X_2$ the morphism induced by f_1 and f_2, i.e. the morphism induced by the following commutative diagram:

$$\begin{array}{ccccc} M_1 \times M_2 & \xrightarrow{p_2} & M_2 & & \\ \downarrow{\scriptstyle p_1} & \searrow^{f_1 \times f_2} & & \searrow^{f_2} & \\ M_1 & & X_1 \times X_2 & \xrightarrow{\pi_2} & X_2 \\ & \searrow^{f_1} & \downarrow{\scriptstyle \pi_1} & & \\ & & X_1 & & \end{array}$$

where $p_i : M_1 \times M_2 \to M_i$ and $\pi_i : X_1 \times X_2 \to X_i$ are the projection morphisms. Finally, let H be a complex linear algebraic acting on M_1 and M_2. We denote by

$$\begin{aligned} \boxtimes : \quad K^H(M_1) \times K^H(M_2) &\to K^H(M_1 \times M_2) \\ (\phi, \psi) &\to (p_1^*\phi) \otimes (p_2^*\psi) \end{aligned}$$

the external product.

Proposition 2.2.2.1. *Let H be a linear algebraic group. Consider two equivariant morphisms of projective varieties with an H-action $f_1 : M_1 \to Y$ and $f_2 : M_2 \to Y$, such that $M_1 \to Y$ is flat. Denote by $\Delta : Y \to Y \times Y$ the diagonal morphism, $p_i : M_1 \times_Y M_2 \to M_i$ the projection morphisms, and $i : M_1 \times_Y M_2 \to M_1 \times M_2$ the natural morphism; i.e. we consider the following commutative*

diagram:

$$\begin{array}{ccccc} M_1 \times_Y M_2 & & \xrightarrow{\quad i \quad} & & M_1 \times M_2 \\ \downarrow{\scriptstyle p_1} & \searrow^{p_2} & & & \downarrow{\scriptstyle f_1 \times f_2} \\ & & M_2 & & \\ & & \downarrow{\scriptstyle f_2} & & \\ M_1 & \xrightarrow{f_1} & Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

Suppose Y is non singular. Let $(\phi_e)_{e\in I}$ be a basis of $K_H(Y) \simeq K^H(Y)$. Suppose there exists a decomposition of the diagonal in $K_H(Y)$, i.e. suppose there exists a basis $(\phi_e^\vee)_{e\in I}$ of $K_H(Y)$ such that

$$\Delta_*[\mathcal{O}_Y] = \sum_{e\in I} \phi_e \boxtimes \phi_e^\vee \in K_H(Y \times Y)$$

where we consider the diagonal action of H on $Y \times Y$. We then obtain:

$$i_*[\mathcal{O}_{M_1 \times_Y M_2}] = \sum_{e,f\in I} g^{ef} \big(f_1^* \phi_e \boxtimes f_2^* \phi_f\big) \in K_H(M_1 \times M_2),$$

where we denote by $g^{ef} := g_{ef}^{-1}$ the pairing dual to g_{ef}, and consider the diagonal action of H on $M_1 \times M_2$ and $M_1 \times_Y M_2$.

Lemma 2.2.2.1 (Flat base change in equivariant K-theory)**.** *Let H be a linear algebraic group. Consider a Cartesian diagram of proper, equivariant morphisms of varieties with an H-action:*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & S' \end{array}$$

where $S \to S'$ is a flat morphism. Then, for all α in $K_H(X)$:

$$f'_*(g'^*\alpha) = g^*(f_*\alpha) \in K_H(S).$$

Proof. Cf. [C$^+$09] Proposition 5.3.15. □

Proof of Proposition 2.2.2.1. We call $f := f_1 \circ p_1 = f_2 \circ p_2$. First, notice that, since f_1 and p_1 are flat, f is a flat H-equivariant morphism. We hence have: $[\mathcal{O}_{M_1 \times_Y M_2}] = f^*([\mathcal{O}_Y])$. Now consider the diagonal action of H on $M_1 \times M_2$ and $Y \times Y$. Since $\Delta : Y \to Y \times Y$ and $i : M_1 \times_Y M_2 \to M_1 \times M_2$ are proper H-equivariant morphism, $f_1 \times f_2$ is an H-equivariant morphism, and the morphism f_1 is flat, we can apply Lemma 2.2.2.1 to the following commutative diagram of H-equivariant maps.

$$\begin{array}{ccc} M_1 \times_Y M_2 & \xrightarrow{p_2} & M_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{f_1} & Y \end{array}$$

We obtain:

$$i_*[\mathcal{O}_{M_1 \times_Y M_2}] = i_* f^*[\mathcal{O}_Y] = (f_1 \times f_2)^* \Delta_*[\mathcal{O}_Y] \in K_H(Y)$$

Furthermore, we have supposed: $\Delta_*[\mathcal{O}_Y] = \sum_{e,f\in I} g^{ef} \phi_e \boxtimes \phi_f \in K_H(Y) \simeq K^H(Y)$. We thus obtain:

$$i_*[\mathcal{O}_{M_1 \times_Y M_2}] = (f_1 \times f_2)^* \Big(\sum_{e,f\in I} g^{ef} \phi_e \boxtimes \phi_f \Big) = \sum_{e,f\in I} g^{ef} \big(f_1^* \phi_e \boxtimes f_2^* \phi_f\big) \in K_H(M_1 \times M_2).$$

□

2.3 Proof of Theorem 2.1.3.1.

2.3.1 Properties of $\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},(\pi_{\hat{k}})_*\mathbf{d})\times_{(X_{\hat{k}})^r}X^r$. We use here the same notations as in the preceding parts. Let X be the flag variety parametrizing flags of vector spaces $V_{n_1}\subset\cdots\subset V_{n_m}\subset\mathbb{C}^n$. For $1\leqslant k\leqslant m$, we denote by $X_{\hat{k}}$ the flag variety obtained from X by forgetting the k-th vector space, and call $\pi=\pi_{\hat{k}}:X\to X_{\hat{k}}$ the forgetful morphism. For a class β of curve on $X_{\hat{k}}$, we denote by $M':=\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},\beta)$ the moduli space parametrizing stable maps from genus 0 r-pointed curves to $X_{\hat{k}}$. We observe here that the scheme $M'\times_{X_{\hat{k}}{}^r}X^r$ associated with the following Cartesian diagram is a projective variety with rational singularities, obtained via flat base change of M' over $(X_{\hat{k}})^r$.

$$\begin{array}{ccc} M'\times_{X_{\hat{k}}{}^r}X^r & \longrightarrow & X^r \\ \downarrow & & \downarrow{\scriptstyle (\pi_{\hat{k}})^r} \\ M' & \xrightarrow{ev} & (X_{\hat{k}})^r \end{array}$$

Lemma 2.3.1.1. *$\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},(\pi_{\hat{k}})_*\mathbf{d})\times_{(X_{\hat{k}})^r}X^r$ is a projective variety with rational singularities.*

Proof. For simplicity set $M':=\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},(\pi_{\hat{k}})_*\mathbf{d})$. First, notice that since $M'\times_{(X_{\hat{k}})^r}X^r$ is a projective base change, it is projective over the projective scheme X^r and hence over Spec$\mathbb{C}$. Furthermore, according to Kleiman's transversality theorem, for g general in GL_n^r the scheme $M'\times_{(X_{\hat{k}})^r}g\cdot X^r$ is reduced. Since the morphism $X^r\to(X_{\hat{k}})^r$ is GL_n^r-equivariant, and the action of GL_n^r on $(X_{\hat{k}})^r$ is transitive, the action of g^{-1} induces an isomorphism $M'\times_{(X_{\hat{k}})^r}g\cdot X^r\simeq M'\times_{(X_{\hat{k}})^r}X^r$. Hence $M'\times_{(X_{\hat{k}})^r}X^r$ is reduced. Note that according to Theorem 0.8.0.1, M' has rational singularities. Hence, according to [CMBP13], Theorem 2.5, since all varieties considered here have rational singularities, for g general in GL_n^r the scheme $M'\times_{(X_{\hat{k}})^r}g\cdot X^r$ has rational singularities. Hence $M'\times_{(X_{\hat{k}})^r}X^r$ has rational singularities. □

Finally, note that since $\pi_{\hat{k}}:X\to X_{\hat{k}}$ is a fibration in Grassmanians, it is flat.

2.3.2 Proof of Theorem 2.1.3.1. We denote by $i:\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},(\pi_{\hat{k}})_*\mathbf{d})\underset{(X_{\hat{k}})^r}{\times}X^r\hookrightarrow\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},(\pi_{\hat{k}})_*\mathbf{d})\times X^r$ the natural morphism.

Let $(\phi_e)_{e\in I}$ be a basis of $K_H(X)\simeq K^H(X)$. By hypothesis there exists a basis $(\phi_e^\vee)_{e\in I}$ such that $\phi_e\boxtimes\phi_f$ is a diagonal decomposition of X. Note that X^r admits a diagonal decomposition in $K_H(X^r)$, which is given by $\Delta_*[\mathcal{O}_{X^r}]=\sum_{e_1,\dots,e_r\in I}(\boxtimes_{1\leqslant i\leqslant r}\phi_{e_i})_{e_1,\dots,e_r\in I}(\boxtimes_{1\leqslant i\leqslant r}\phi^\vee_{e_i})_{e_1,\dots,e_r\in I}\in K_H(X^r\times X^r)$.

Lemma 2.3.2.1. *Let $\mathbf{d}=(d_1,\dots,d_m)$ be a degree in $E(X)$ such that $\forall\, p<k$, $d_k\geqslant n_k\left\lceil\frac{d_p-d_{p-1}}{n_p-n_{p-1}}\right\rceil$. Suppose the morphism $\Pi\times ev:\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})\to\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},(\pi_{\hat{k}})_*\mathbf{d})\underset{(X_{\hat{k}})^r}{\times}X^r$ is surjective. Then*

$$i_*(\Pi\times ev)_*\left[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}\right]=\sum_{e_1,\dots,e_r\in I}((ev_1^*\phi_{e_1})\otimes\cdots\otimes(ev_r^*\phi_{e_r}))\boxtimes\left(\pi^*(\phi^\vee_{e_1})\boxtimes\cdots\boxtimes\pi^*(\phi^\vee_{e_r})\right)$$

Proof. For simplicity, set $M':=\overline{\mathcal{M}_{0,r}}(X_{\hat{k}},(\pi_{\hat{k}})_*\mathbf{d})$, $S:=X^r$, and $S':=(X_{\hat{k}})^r$. According to Lemma 2.3.1.1, $M'\times_{S'}S$ is a projective variety with rational singularities. Furthermore, since $\Pi\times ev$ is surjective, $M'\times_{S'}S$ is irreducible. Finally, according to Chapter 1 Proposition 1.4.0.1 since $\forall\, p<k$, $d_k\geqslant n_k\left\lceil\frac{d_p-d_{p-1}}{n_p-n_{p-1}}\right\rceil$ the general fiber of $\Pi\times ev$ is a rationally connected variety. Hence $\Pi\times ev$ is an H-equivariant surjective morphism of irreducible projective varieties with rational singularities with

rationally connected general fiber, hence according to Theorem 0.8.0.2 we have in $K_H(M' \times S')$:

$$\begin{aligned} i_*(\Pi \times ev)_* \left[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}\right] &= i_* \left[\mathcal{O}_{(\Pi \times ev)\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}\right] \\ &= i_* \left[\mathcal{O}_{M' \times_{S'} S}\right] \\ &= \sum_{e_1,\ldots,e_r \in I} ev^*(\phi_{e_1} \boxtimes \ldots \phi_{e_r}) \boxtimes ((\pi)^r)^* \left(\phi_{e_1}^\vee \boxtimes \cdots \boxtimes \phi_{e_r}^\vee\right) \\ &= \sum_{e_1,\ldots,e_r \in I} ((ev_1^* \phi_{e_1}) \otimes \cdots \otimes (ev_r^* \phi_{e_r})) \boxtimes \left(\pi^*(\phi_{e_1}^\vee) \boxtimes \cdots \boxtimes \pi^*(\phi_{e_r}^\vee)\right) \end{aligned}$$

where the third equality holds since the morphism $(\pi)^r$ is a flat H-equivariant morphism-cf. Lemma 2.2.2.1. □

Lemma 2.3.2.2. *Let Y and Z be projective varieties with an H-action. Suppose Z non singular and suppose both Y and Z are irreducible. Then for $\alpha \in K_H(Y)$, $\beta \in K_H(Z) \simeq K^H(Z)$, we have $\chi_H(\alpha \boxtimes \beta) = \chi_H(\alpha)\chi_H(\beta)$.*

Proof. We consider the following Cartesian diagram

$$\begin{array}{ccc} Y \times Z & \xrightarrow{p_2} & Z \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_1} \\ Y & \xrightarrow{\pi_2} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

where we have denoted by $p_1 : Y \times Z \to Y$ and $p_2 : Y \times Z \to Z$ the H-equivariant projections. Since Y is irreducible the morphism $Y \to \mathrm{Spec}(\mathbb{C})$ is flat. Since χ_H is the H-equivariant pushforward to the point, we have:

$$\begin{aligned} \chi_H(\alpha \boxtimes \beta) &= \chi_H(p_1^* \alpha \otimes p_2^* \beta) \\ &= \chi_H(\alpha \otimes (p_1)_* p_2^* \beta) \\ &= (\pi_1)_* (\alpha \otimes \pi_1^*((\pi_2)_* \beta)) \\ &= \chi_H(\alpha)\chi_H(\beta), \end{aligned}$$

where the third equality holds according to Lemma 2.2.2.1 and the last equality holds according to the projection formula. □

Proof of Theorem 2.1.3.1. Let $\alpha_1, \ldots, \alpha_r$ be elements in $K^H(X)$. Note that the following diagram is a commutative diagram of proper morphisms:

$$\begin{array}{ccccc} & & \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) & & \\ & & \downarrow{\scriptstyle \Pi \times ev} & \searrow^{ev} & \\ \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, \pi_* \mathbf{d}) \times_{(X_{\hat{k}})^r} X^r & \xrightarrow{i} & \overline{\mathcal{M}_{0,r}}(X_{\hat{k}}, \pi_* \mathbf{d}) \times X^r & \xrightarrow[p_2]{} & X^r \end{array}$$

We obtain in $K_H(\overline{\mathcal{M}_{0,r}}(X_{\widehat{k}},\pi_*\mathbf{d})\times X^r)$:

$$
\begin{aligned}
&i_*(\Pi\times ev)_*\left(ev_1^*\alpha_1\otimes\cdots\otimes ev_r^*\alpha_r\otimes[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}]\right)\\
&=i_*(\Pi\times ev)_*\left(ev^*(\alpha_1\boxtimes\cdots\boxtimes\alpha_r)\otimes[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}]\right)\\
&=i_*(\Pi\times ev)_*\left((p_2\circ i\circ(\Pi\times ev))^*(\alpha_1\boxtimes\cdots\boxtimes\alpha_r)\otimes[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}]\right)\\
&=((p_2)^*(\alpha_1\boxtimes\cdots\boxtimes\alpha_r))\otimes i_*(\Pi\times ev)_*[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}]\\
&=\left([\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X_{\widehat{k}},\pi_*\mathbf{d})}]\boxtimes(\alpha_1\boxtimes\cdots\boxtimes\alpha_r)\right)\otimes i_*(\Pi\times ev)_*[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}]\\
&=\left([\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X_{\widehat{k}},\pi_*\mathbf{d})}]\boxtimes(\alpha_1\boxtimes\cdots\boxtimes\alpha_r)\right)\otimes\sum_{e_1,\ldots,e_r\in I}((ev_1^*\phi_{e_1})\otimes\cdots\otimes(ev_r^*\phi_{e_r}))\boxtimes\left(\pi^*(\phi_{e_1}^\vee)\boxtimes\cdots\boxtimes\pi^*(\phi_{e_r}^\vee)\right)\\
&=\sum_{e_1,\ldots,e_r\in I}((ev_1^*\phi_{e_1})\otimes\cdots\otimes(ev_r^*\phi_{e_r}))\boxtimes\left((\pi^*(\phi_{e_1}^\vee)\otimes\alpha_1)\boxtimes\cdots\boxtimes(\pi^*(\phi_{e_r}^\vee)\otimes\alpha_r)\right),
\end{aligned}
$$

where the third equality holds according to the projection formula, and the last equality holds according to Lemma 2.3.2.1. Finally, decompose each element $\pi_*\alpha_i$ in the basis $(\phi_e)_{e\in I}$ of $K_H(X_{\widehat{k}})\simeq K^H(X_{\widehat{k}})$: $\pi_*\alpha_i=\sum_{e\in I}a_{i,e}\phi_e$. Note that we have $\chi_H(\pi_*\alpha_i\otimes\phi_f^\vee)=a_{i,f}$. Denote by $p_2:\overline{\mathcal{M}_{0,r}}(X_{\widehat{k}},\pi_*\mathbf{d})\times X^r\to X^r$ the second projection, and by $p:(X_{\widehat{k}})^r\to\mathrm{Spec}\mathbb{C}$ the H-equivariant projection to the point. We obtain:

$$
\begin{aligned}
&\chi_H\left(ev_1^*\alpha_1\otimes\cdots\otimes ev_r^*\alpha_r\otimes[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}]\right)\\
&\quad=p_*(\pi^r)_*(p_2)_*i_*(\Pi\times ev)_*\left(ev_1^*\alpha_1\otimes\cdots\otimes ev_r^*\alpha_r\otimes[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}]\right)\\
&\quad=p_*(\pi^r)_*(p_2)_*\left(\sum_{e_1,\ldots,e_r\in I}((ev_1^*\phi_{e_1})\otimes\cdots\otimes(ev_r^*\phi_{e_r}))\boxtimes\left((\pi^*(\phi_{e_1}^\vee)\otimes\alpha_1)\boxtimes\cdots\boxtimes(\pi^*(\phi_{e_r}^\vee)\otimes\alpha_r)\right)\right)\\
&\quad=\sum_{e_1,\ldots,e_r\in I}\chi_H\left((ev_1^*\phi_{e_1})\otimes\cdots\otimes(ev_r^*\phi_{e_r})\right)p_*(\pi^r)_*\left((\pi^*(\phi_{e_1}^\vee)\otimes\alpha_1)\boxtimes\cdots\boxtimes(\pi^*(\phi_{e_r}^\vee)\otimes\alpha_r)\right)\\
&\quad=\sum_{e_1,\ldots,e_r\in I}\chi_H\left((ev_1^*\phi_{e_1})\otimes\cdots\otimes(ev_r^*\phi_{e_r})\right)\prod_{1\leqslant i\leqslant r}\chi_H(\phi_{e_i}^\vee\otimes\alpha_i)\\
&\quad=\sum_{e_1,\ldots,e_r\in I}\chi_H\left((ev_1^*\phi_{e_1})\otimes\cdots\otimes(ev_r^*\phi_{e_r})\right)\prod_{1\leqslant i\leqslant r}a_{i,e_i}
\end{aligned}
$$

where the before last equality is obtained by applying the projection formula to the flat morphism $(\pi)^r$. Finally, we obtain:

$$
\chi_H\left(ev_1^*\alpha_1\otimes\cdots\otimes ev_r^*\alpha_r\otimes[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})}]\right)\quad=\sum_{e_1,\ldots,e_r\in I}\chi_H\left(a_{1,e_1}(ev_1^*\phi_{e_1})\otimes\cdots\otimes a_{r,e_r}(ev_r^*\phi_{e_r})\right)
$$

which yields the expected result since $\sum_{e\in I}a_{i,e}\phi_e=\pi_*\alpha_i$. □

2.4 Geometric interpretation in the non equivariant setting

Consider the case when H is the identity element of G. We provide here an alternative proof of the equality between correlators provided by Theorem 2.1.3.1, using results of Chapter 1.

We use here the same notations as in Chapter 1. As usual, we say a property (P) holds for a general point in a variety X if (P) is true for points belonging to a dense open subset of X. Recall a flag variety X can be written as G/P, where $G=\mathrm{GL}_n$ and P is a parabolic subgroup of X containing a Borel subgroup $B\subset P\subset G$. For an element u in the Weyl group W of $G=\mathrm{GL}_n$ we denote by $X(u):=\overline{BuP/P}$ the associated Schubert variety, and by $\mathcal{O}_u:=[\mathcal{O}_{X(u)}]$ the associated class in the Grothendieck ring $K(X)$ of coherent sheaves on X. Let $r\geqslant 0$. Let $u_1,\ldots,u_r$ be elements in W. Let

$\mathbf{d} = (d_1, \dots, d_m)$ be a degree in $E(X)$. Recall for an element $(g_1, \dots, g_r)$ in G^r we call *Gromov-Witten variety associated with the* u_i the variety

$$\mathcal{W}^X_{g_1,\dots,g_r} := \overline{\mathcal{M}}_{0,r}(X, \mathbf{d}) \times_{X^r} \prod_{i=1}^{r} g_i X(u_i)$$

parametrizing genus zero degree $\mathbf{d}$ stable maps into X sending their i-th marked point within the variety $g_i X(u_i)$. Note that for an element $(g_1, \dots, g_r)$ general in G^r the correlators $\langle \mathcal{O}_{u_1}, \dots, \mathcal{O}_{u_r} \rangle_{\mathbf{d}}$ satisfy

$$\langle \mathcal{O}_{u_1}, \dots, \mathcal{O}_{u_r} \rangle^X_{\mathbf{d}} = \chi(\mathcal{W}^X_{g_1,\dots,g_r}).$$

The flag variety X parametrizes flags of vector spaces $V_{n_1} \subset \cdots \subset V_{n_m} \subset \mathbb{C}^n$, where $0 < n_1 < \cdots < n_m < n$. Let X' be the flag variety obtained from X by forgetting the k-th vector space. Denote by $\pi : X \to X'$ the forgetful map. Note that for all i we have $\pi_* \mathcal{O}_{u_i} = [\mathcal{O}_{X'(u_i)}] = \mathcal{O}_{u_i} \in K(X')$. We obtain

$$\langle \mathcal{O}_{u_1}, \dots, \mathcal{O}_{u_r} \rangle^{X'}_{\pi_* \mathbf{d}} = \chi(\mathcal{W}^{X'}_{g_1,\dots,g_r}).$$

Now suppose the morphism $\Pi \times ev : \overline{\mathcal{M}_{0,r}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,r}}(X', \pi_* \mathbf{d}) \times_{(X')^r} X^r$ is surjective. Furthermore suppose $\forall\, p < k,\ d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$. Then according to Chapter 1 Theorem 1.1.2.1 for an element $(g_1, \dots, g_r)$ general in G^r each irreducible component of the variety $\mathcal{W}^X_{g_1,\dots,g_r}$ surjects into a different irreducible component of the variety $\mathcal{W}^{X'}_{g_1,\dots,g_r}$, and the general fiber of the projection $\mathcal{W}^X_{g_1,\dots,g_r} \to \mathcal{W}^{X'}_{g_1,\dots,g_r}$ is a rationnally connected variety. Furthermore according to [CMBP13] Theorem 2.5 for an element $(g_1, \dots, g_r)$ general in G^r the variety $\mathcal{W}_{g_1,\dots,g_r}$ has rational singularities. Theorem 0.8.0.2 then yields

$$\langle \mathcal{O}_{u_1}, \dots, \mathcal{O}_{u_r} \rangle^X_{\mathbf{d}} = \langle \mathcal{O}_{u_1}, \dots, \mathcal{O}_{u_r} \rangle^{X'}_{\mathbf{d'}},$$

which is the equality described by Theorem 2.1.3.1.

Chapter 3

Schubert calculus for $Fl_{1,n-1}$

3.1 Introduction

Let $X = Fl_{1,n-1}$ be the incidence variety parametrizing lines contained in hyperplanes of $\mathbb{C}^n$. We study here the product of Schubert classes in the small quantum K-ring of X. Part 3.2 is devoted to recalling some standard facts about incidence varieties, and fixing the notations we use in the rest of the chapter. The aim of parts 3.3 and 3.4 is to prove a closed formula for Littlewood-Richardson coefficients in $K(X)$. Part 3.3 describes one parameter degenerations of Richardson varieties, from which the expression of Littlewood-Richardson coefficients in $K(X)$ is derived in Part 3.4. The aim of Part 3.5.2 is to compute all three points quantum K-theoretical correlators of X. Note that in the context of small quantum cohomology the product of two Schubert classes is given directly by three points Gromov-Witten invariants. In small quantum K-theory, the product of two Schubert classes is an alternate sum involving three points correlators and two points correlators. Part 3.6 is dedicated to finding a boundary on the degree of the product of two classes in $QK_s(X)$. A Chevalley formula in $QK_s(X)$ is proven in Part 3.7. Finally we study Littlewood-Richardson coefficients in $QK_s(X)$ in Part 3.8. We provide an algorithm computing these coefficients, and conjecture a closed formula for them. We additionally prove that they satisfy a "positivity rule".

3.2 Incidence varieties.

Let us begin by fixing down some of the usual notations associated with a generalized flag variety. Let $G := \mathrm{GL}_n$, let T be a maximal torus of G and B be a Borel subgroup of G containing T. A generalized flag variety can be written as $X := \mathrm{GL}_n(\mathbb{C})/P$, where P is a parabolic subgroup of GL_n and satisfying

$$T \subset B \subset P \subset G.$$

Let us name $W = N_G(T)/T \simeq \mathfrak{S}_n$ the Weyl group of G, $W_P := N_L(T)/T$ the subgroup of W associated with P, and $W^P := W/W_P$. For $w \in W^P$, note that the variety $\overline{Bw'P/P}$ does not depend on the choice of an element w' in W representing w; we obtain a *Schubert variety* $X(w) := \overline{BwP/P}$. The homology classes $[X(w)]$ from a basis of the integral homology of X, where w runs over W^P. The classes $\mathcal{O}_w := [\mathcal{O}_{X(w)}]$ form an additive basis of the Grothendieck ring $K_\circ(X)$ of coherent sheaves on X. Furthermore, we denote by $(\mathcal{I}_w)_{w\in W^P}$ the dual basis of $(\mathcal{O}_w)$ for the Euler characteristic. If we denote by $\delta_{u,v}$ the symbol equal to 1 if $u = v$, and else equal to 0, we have for all $u, v \in W^P$:

$$\chi(\mathcal{O}_u, \mathcal{I}_v) = \delta_{u,v}.$$

For an element w in W^P, we denote by $\ell(w)$ the length of a minimal length representant of w. Note that $\ell(w) = \dim X(w)$.

We consider here the incidence variety $X = Fl_{1,n-1}$ parametrizing lines included in hyperplanes of $\mathbb{C}^n$. Forgetting a line or a hyperplane yields a forgetful map $X \to \mathbb{P}^{n-1}$ which is a fibration in

$\mathbb{P}^{n-2}$'s. Fix a basis $< e_1, \ldots, e_n >$ of $\mathbb{C}^n$. In this basis, consider the maximal torus in $G = \mathrm{GL}_n$ consisting of diagonal matrices, the Borel subgroup B consisting of upper triangular matrices of GL_n, and the parabolic subgroup $P \supset B$ associated with the reductive group $\mathrm{GL}_1 \times \mathrm{GL}_{n-2} \times \mathrm{GL}_1$. Note that $W = N_G(T)/T \simeq \mathfrak{S}_n$. We denote by w_0 the element of W of greatest length. Note that w_0 is the permutation of $1 \ldots n$ defined by $w_0(i) = n - i + 1$. Furthermore, an element in $W^P \simeq \mathfrak{S}_n/\mathfrak{S}_{n-2}$ is uniquely determined by the choice of a permutation sending 1 to i and n to j, where $1 \leqslant i, j \leqslant n$ and $i \neq j$. We name $w_{i,j}$ the element in W^P thus defined. Schubert varieties of $X \simeq \mathrm{GL}_n(\mathbb{C})/P$ are described by:

$$\begin{aligned} X(i,j) :&= X(w_{i,j}) = \overline{Bw_{i,j}P/P} \\ &= \left\{(L,H) \in X \middle| L \subset < e_1, \cdots, e_i >, < e_1, \cdots, e_{j-1} > \subset H\right\}. \end{aligned} \tag{3.1}$$

Schubert varieties can be embedded in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ as:

$$X(i,j) = \left\{([x_1 : \cdots : x_i : 0 : \cdots : 0], [0 : \ldots 0 : y_j : \cdots : y_n]) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \middle| \sum x_k y_k = 0 \right\}. \tag{3.2}$$

Note that for $i < j$ the condition $\sum x_k y_k = 0$ is redundant, and $X(i,j)$ can simply be described as:

$$X(i,j) = \left\{([x_1 : \cdots : x_i : 0 : \cdots : 0], [0 : \ldots 0 : y_j : \cdots : y_n]) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}\right\}. \tag{3.3}$$

For $1 \leqslant i, j \leqslant n$, $i \neq j$, we name

$$\mathcal{O}_{i,j} := [\mathcal{O}_{X(w_{i,j})}] \in K_\circ(X)$$

the class of the pushforward of the structure sheaf $\mathcal{O}_{X(w_{i,j})}$ in the Grothendieck group $K_\circ(X)$, and by $\mathcal{I}_{i,j} := \mathcal{I}_{w_{i,j}}$ its dual with respect to the pairing $\chi : \alpha, \beta \to \chi(\alpha \cdot \beta)$. For any $u \in W^P$, the Schubert variety *opposite* to the Schubert variety $X(u)$ is

$$X^u := w_0 X(w_0 u).$$

Applying w_0 to (3.1) yields the following description of opposite Schubert varieties of X:

$$X^{w_0 w_{k,l}} = w_0 X(w_{k,l}) = \left\{(L,H) \in X \middle| L \subset < e_{n-k+1}, \cdots, e_n >, < e_{n-l+2}, \cdots, e_n > \subset H\right\}. \tag{3.4}$$

Given v, w in W^P, the corresponding *Richardson variety* is $X_w^v := X(w) \cap w_0 X(w_0 v)$. Consider $1 \leqslant i, j, k, l \leqslant n$, $i \neq j$ and $k \neq l$. Richardson varieties of X are described by:

$$\begin{aligned} X_{i,j}^{k,l} &:= X_{w_{i,j}}^{w_0 w_{k,l}} = X(w_{i,j}) \cap w_0 X(w_{k,l}) \\ &= \{(L,H) \in X | L \subset < e_{n-k+1}, \ldots, e_i >; < e_1, \ldots e_{j-1}, e_{n-l+2}, \ldots, e_n > \subset H\}, \end{aligned} \tag{3.5}$$

if $n - k + 1 \leqslant i$ and $j - 1 \leqslant n - l + 2$, and $X_{i,j}^{k,l} = \varnothing$ else. The second equality comes from considering the description of Schubert varieties given by (3.1) and of opposite Schubert varieties given by (3.4).

3.3 Degenerating Richardson varieties of $Fl_{1,n-1}$.

We describe here one parameter families of deformations of Richardson varieties. More precisely, we describe irreducible projective varieties Φ of $\mathbb{P}^1 \times X$ such that one fiber of the projection morphism $\Phi \to \mathbb{P}^1$ is a Richardson variety, while another fiber is a union of Schubert varieties, up to multiplication by elements of GL_n.

To facilitate the proof, let us first introduce an auxilliary variety. Set $e_{n+1} = 0 = e_0$. For $1 \leqslant r \leqslant h \leqslant n$, $0 \leqslant p \leqslant n - r - 1$, we will consider the following subvariety of X:

$$Y_{r,p}^h := \{(L,H) \in X | L \subset < e_1, \ldots e_h >, < e_1, \ldots e_r, e_{n-p+1}, \ldots, e_n > \subset H\}. \tag{3.6}$$

The Plücker embedding $X \hookrightarrow Gr(1,n) \times Gr(n-1,n) \simeq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ embeds $Y_{r,p}^h$ as:

$$Y_{r,p}^h := \{[x_1, \ldots, x_h, 0, \ldots, 0]; [0 : \cdots : 0 : y_{r+1} : \cdots : y_{n-p} : 0 : \cdots : 0] \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} | \sum_a x_a y_a = 0\}. \tag{3.7}$$

Lemma 3.3.0.1. *1. Let $1 \leqslant i,j,k,l \leqslant n$, where $i \neq j$ and $k \neq l$. Suppose $n-k+1 \leqslant i$ and $j-1 \leqslant n-l+2$.*
Denote by r the number of shared vectors between $(e_{n-k+1},\dots,e_i)$ and $(e_1,\dots e_{j-1},e_{n-l+2},\dots,e_n)$. Call $h := i+k-n$, and $p := j+l-2-r$.
There exists g in GL_n such that $g \cdot X^{k,l}_{i,j} = Y^h_{r,p}$.

2. Fix $1 \leqslant r \leqslant n$, $0 \leqslant p \leqslant n-r-1$. Then there exists g in GL_n such that $Y^r_{r,p} = g \cdot X(r, r+p+1)$.

3. Fix $1 \leqslant r \leqslant h \leqslant n$. Then $Y^h_{r,0} = X(h, r+1)$.

Proof. 1. Recall we have defined r as the number of shared vectors between $(e_{n-k+1},\dots,e_i)$ and $(e_1,\dots e_{j-1},e_{n-l+2},\dots,e_n)$, and $p = j+l-2-r$. Notice:

$$\begin{aligned} p = j+l-2-r &= j-1+n-(n-l+1)-r \\ &= \#(e_1,\dots e_{j-1},e_{n-l+2},\dots,e_n) - r, \end{aligned}$$

hence p is the number of vectors in $(e_1,\dots e_{j-1},e_{n-l+2},\dots,e_n)$ not belonging to $(e_{n-k+1},\dots,e_i)$. Now call $(\epsilon_1,\dots,\epsilon_p)$ these p vectors, and $(\epsilon_{p+1},\dots,\epsilon_{p+r})$ the r vectors in $(e_1,\dots e_{j-1},e_{n-l+2},\dots,e_n)$ belonging to $(e_{n-k+1},\dots,e_i)$. Finally, call $(\eta_{r+1},\dots,\eta_h)$ the $\#(e_{n-k+1},\dots,e_i) - r = h-r$ vectors of $(e_{n-k+1},\dots,e_i)$ that do not belong to $(e_1,\dots e_{j-1},e_{n-l+2},\dots,e_n)$, and $(\eta_{h+1},\dots,\eta_{n-p})$ the remaining vectors. Denote by g the following permutation of the basis vectors:

$$(e_1,\dots,e_n) \to (\epsilon_{p+1},\dots,\epsilon_{p+r},\eta_{r+1},\dots,\eta_h,\eta_{h+1},\dots,\eta_{n-p},\epsilon_1,\dots,\epsilon_p).$$

Then, according to (3.1), we have:

$$\begin{aligned} g \cdot X^{k,l}_{i,j} &= \{(L,H) \in X | L \subset < e_1,\dots,e_h >, < e_1,\dots e_r, e_{n-p+1},\dots,e_n > \subset H\} \\ &= Y^h_{r,p} \end{aligned}$$

where the first equality follows directly from the definition of g and the definition (3.5) of $X^{k,l}_{i,j}$:

$$X^{k,l}_{i,j} = \{(L,H) \in X | L \subset < e_{n-k+1},\dots,e_i >; < e_1,\dots e_{j-1},e_{n-l+2},\dots,e_n > \subset H\}.$$

2. By definition, we have:

$$Y^r_{r,p} := \{(L,H) \in X | L \subset < e_1,\dots e_r >, < e_1,\dots e_r, e_{n-p+1},\dots,e_n > \subset H\}.$$

If $r = n-p$, set $g := id$. Else, denote by g the permutation of the basis vectors sending:

$$(e_1,\dots,e_r,e_{r+1},\dots,e_{n-p},e_{n-p+1},\dots,e_n) \to (e_1,\dots,e_r,e_{n-p+1},\dots,e_n,e_{r+1},\dots,e_{n-p}).$$

Then, according to (3.1), we have:

$$\begin{aligned} g \cdot Y^r_{r,p} &= \{(L,H) \in X | L \subset < e_1,\dots,e_r >, < e_1,\dots e_r, e_{r+1},\dots,e_{r+p} > \subset H\} \\ &= X(r, r+p+1) \end{aligned}$$

3. We have:

$$\begin{aligned} Y^h_{r,0} :&= \{(L,H) \in X | L \subset < e_1,\dots e_h >, < e_1,\dots e_r > \subset H\} \\ &= X(h, r+1) \end{aligned}$$

where the first equality is the definition (3.6) of $Y^h_{r,0}$ and the third equality is the definition (3.1) of Schubert varieties.

□

Proposition 3.3.0.1. *Let $1 \leqslant i, j, k, l \leqslant n$, where $i \neq j$ and $k \neq l$.*

1. *If $n+1 > i+k$ or $j+l \geqslant n+2$, then $X_{i,j}^{k,l} = \varnothing$;*
2. *Suppose $i+k \geqslant n+1$ and $j+l \leqslant n+1$ and ($i<j$ or $k<l$). Then there exists g in GL_n such that:*
$$X_{i,j}^{k,l} = g \cdot X(i+k-n, j+l-1);$$
3. *Suppose $i+k-n \geqslant j+l$ and $j+l \leqslant n+1$. Then there exists g in GL_n, and an irreducible projective subvariety Φ of $\mathbb{P}^1 \times X$ such that:*
$$\pi^{-1}([1:0]) = \{[1:0]\} \times g \cdot X_{i,j}^{k,l} \quad \text{and} \quad \pi^{-1}([0:1]) = \{[0:1]\} \times X(i+k-n, j+p-1);$$
where $\pi : \Phi \subset \mathbb{P}^1 \times X \to \mathbb{P}^1$ is the natural projection;
4. *Suppose $1 \leqslant i+k-n \leqslant j+l-1 \leqslant n$ and $i>j$ and $k>l$. Then there exists g in GL_n, and an irreducible projective subvariety Φ of $\mathbb{P}^1 \times X$ such that:*
$$\pi^{-1}([1:0]) = \{[1:0]\} \times g \cdot X_{i,j}^{k,l}$$
$$\pi^{-1}([0:1]) = \{[0:1]\} \times Y_{i+k-n-1, j+l-1-i-k+n}^{i+k-n-1} \cup \{[0:1]\} \times Y_{i+k-n, j+l-1-i-k+n}^{i+k-n}$$
where $\pi : \Phi \subset \mathbb{P}^1 \times X \to \mathbb{P}^1$ is the natural projection.

Proof. 1. By definition of $X_{i,j}^{k,l}$.

2. Notice that then all vectors of $E_1 = (e_{n-k+1}, \dots, e_i)$ belong to $E_2 = (e_1, \dots e_{j-1}, e_{n-l+2}, \dots, e_n)$, hence the number r of shared vectors between E_1 and E_2 is equal to $h = \#E_1 = i+k-n$. Hence, according to Lemma 3.3.0.1 (i), there exists g in GL_n such that
$$g \cdot X_{i,j}^{k,l} = Y_{h,p}^{h},$$
where $h = i+k-n$ and $p = j+l-2-h$.
Furthermore, according to Lemma 3.3.0.1 (ii), there exists g' in GL_n such that
$$Y_{h,p}^{h} = g' \cdot X(i+k-n, j+l-1).$$
Hence:
$$X_{i,j}^{k,l} = g^{-1} \cdot Y_{h,p}^{h} = g^{-1}g' \cdot X(i+k-n, j+l-1).$$

3. We use here the same notations as in Lemma 3.3.0.1. Denote by r the number of shared vectors between $(e_{n-k+1}, \dots, e_i)$ and $(e_1, \dots e_{j-1}, e_{n-l+2}, \dots, e_n)$. Call $h := i+k-n$, and $p := j+l-2-r$. Let Φ be the subvariety of $X \times \mathbb{P}^1$ defined by :
$$\Phi = \{([u:v]; [x_1 : \cdots : x_h : 0 : \cdots : 0]; [0 : \cdots : 0 : y_{r+1} : \cdots : y_n]) \in \mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \\ \left| \sum_{a=r+1}^{h} x_a y_a = 0 \text{ and } \forall\, r < a \leqslant r+p,\ v y_a = u y_{n-p+a-r} \right\}$$
As usual, we denote by $D(v)$ (respectively $D(u)$) the subvariety of elements $([u:v]; [x_1 : \cdots : x_n]; [y_1 : \cdots : y_n])$ in $\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ satisfying $v \neq 0$ (resp. $u \neq 0$). On $D(v)$, we have
$$\Phi_{|D(v)} \simeq \{(t; [x_1, \dots, x_h]; [0 : \cdots : 0 : t y_{n-p+1} : \cdots : t y_n : y_{r+p+1} : \cdots : y_n] \in \mathbb{C} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \\ | t \sum_{a=r+1}^{r+p} x_a y_{n-p+a-r} + \sum_{a=r+p+1}^{h} x_a y_a = 0\}.$$
Since $r+p = j+l-2 < i+k-n-1 = h-1 < h$, $\Phi_{|D(v)}$ is irreducible. In the same way, $\Phi_{|D(u)}$ is also irreducible. Hence Φ is an irreducible projective subvariety of $X \times \mathbb{P}^1$.

Denote by $\pi : \Phi \to \mathbb{P}^1$ the natural projection. π verifies :

$$\begin{aligned}
&\pi^{-1}(\{[1:0]\}) \\
&\quad = \{([1:0];[x_1:\cdots:x_h:0:\cdots:0];[0:\cdots:0:y_{r+1}:\cdots:y_{n-p},0,\ldots,0]) \in \mathbb{P}^1\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1} | \\
&\qquad\qquad \sum_{i=r+1}^{h} x_i y_i = 0\} \\
&\quad = \{[1:0]\}\times Y^h_{r,p} \\
&\quad = \{[1:0]\}\times g\cdot X^{k,l}_{i,j}
\end{aligned}$$

where the existence of the element g is ensured by Lemma 3.3.0.1 (ii), and the second equality is the definition (3.7) of $Y^h_{r,p}$.

Finally, π verifies:

$$\begin{aligned}
\pi^{-1}([0:1]) &= \{([0:1];[x_1:\cdots:x_h:0:\cdots:0];[0:\cdots:0:y_{r+p+1}:\cdots:y_n]) \in \mathbb{P}^{n-1}\times\mathbb{P}^{n-1} | \sum_{a=r+1}^{h} x_a y_a = 0\} \\
&= \{[0:1]\}\times\{(L,H)\in X | L \subset < e_1,\ldots e_h >, < e_1,\ldots e_{p+r} > \subset H\} \\
&= \{[0:1]\}\times Y^{i+k-n}_{j+l-2,0} \\
&= \{[0:1]\}\times X(i+k-n, j+l-1)
\end{aligned}$$

where the second equality is the definition (3.7) and the last equality holds according to Lemma 3.3.0.1 (iii).

4. Again, we use the same notations as in Lemma 3.3.0.1. Denote by r the number of shared vectors between $(e_{n-k+1},\ldots,e_i)$ and $(e_1,\ldots e_{j-1},e_{n-l+2},\ldots,e_n)$. Call $h := i+k-n$, and $p := j+l-2-r$. Let Φ be the subvariety of $X\times\mathbb{P}^1$ defined by:

$$\Phi = \left\{ ([u:v];[x_1:\cdots:x_h:0:\cdots:0];[0:\cdots:0:y_{r+1}:\cdots:y_{n-p-r+h-1}:0:\cdots:0]) \in \mathbb{P}^1\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1} \right.$$
$$\left. \Big| \forall\ 1\leqslant a\leqslant h-1-r,\ vy_{r+a} = uy_{n-p+a}, \text{ and } \sum_{a=r+1}^{h} x_a y_a = 0 \right\}.$$

On $D(v)$, we have $\Phi_{|D(u)} \simeq \mathrm{Spec}\mathbb{C}[t]\times\mathbb{P}^{h-1}\times Y$, where $Y = \{[0:\cdots:0:y_{r+1}:\cdots:y_n] \in \mathbb{P}^{n-1}|\}$.

$$\Phi_{|D(v)} \simeq \{(t;[x_1,\ldots,x_h:0:\cdots:0];[0:\cdots:0:ty_{n-p+r+1}:\cdots:ty_{n-p+r+h-1}:y_h:\cdots:y_{n-p+r-h-1}:0\cdots:0]$$
$$\in \mathbb{C}\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1} \left| t\sum_{a=r+1}^{h-1} x_a y_{n-p+a} + x_h y_h = 0 \right\}.$$

Now notice that, since $j < n-l+2$, the number r of shared vectors between $(e_{n-k+1},\ldots,e_i)$ and $(e_1,\ldots e_{j-1},e_{n-l+2},\ldots,e_n)$ satisfies $r < h = \#(e_{n-k+1},\ldots,e_i)$. Hence $\Phi_{|D(u)}$ is irreducible. In the same way, $\Phi_{|D(u)}$ is also irreducible. Hence Φ is an irreducible projective subvariety of $X\times\mathbb{P}^1$.

Denote by $\pi : \Phi \to \mathbb{P}^1$ the natural projection. π satisfies:

$$\begin{aligned}
&\pi^{-1}([1:0]) \\
&\quad = \{([1:0];[x_1:\cdots:x_h:0:\cdots:0];[0:\cdots:0:y_{r+1}:\cdots:y_{n-p},0,\ldots,0]) \in \mathbb{P}^1\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1} \\
&\qquad\qquad | \sum_{a=r+1}^{h} x_a y_a = 0\} \\
&\quad = \{[1:0]\}\times Y^h_{r,p} \\
&\quad = \{[1:0]\}\times g\cdot X^{k,l}_{i,j}
\end{aligned}$$

where the existence of the element g is ensured by Lemma 3.3.0.1 (ii), and the second equality is the definition (3.7) of $Y^h_{r,p}$.
Finally, we observe:

$$\begin{aligned}
&\pi^{-1}([0:1])\\
&\quad= \{([0:1];[x_1:\cdots:x_h:0:\cdots:0];[0:\cdots:0:y_h:\cdots:y_{n-p-r+h-1},0,\ldots,0]) \in \mathbb{P}^1\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1} | x_h y_h = 0\}\\
&\quad= \{([0:1];[x_1:\cdots:x_{h-1}:0:\cdots:0];[0:\cdots:0:y_h:\cdots:y_{n-p-r+h-1},0,\ldots,0]) \in \mathbb{P}^1\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}\}\\
&\qquad\cup \{([0:1];[x_1:\cdots:x_h:0:\cdots:0];[0:\cdots:0:y_{h+1}:\cdots:y_{n-p-r+h-1},0,\ldots,0]) \in \mathbb{P}^1\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}\}\\
&\quad= \{[0:1]\}\times Y^{h-1}_{h-1,p+r-h+1} \cup \{[0:1]\}\times Y^h_{h,p+r-h+1}\\
&\quad= \{[0:1]\}\times Y^{i+k-n-1}_{i+k-n-1,j+l-1-i-k+n} \cup \{[0:1]\}\times Y^{i+k-n}_{i+k-n,j+l-1-i-k+n}
\end{aligned}$$

where the third equality is the definition (3.7) and the last equality is deduced from $h = i+k-n$ and $r+p = j+l-2$.

□

3.4 Littlewood-Richardson coefficients in $K(Fl_{1,n-1})$.

3.4.1 Computing Littlewood-Richardson coefficients. We deduce here Littlewood-Richardson coefficients in $K(Fl_{1,n-1})$ from the results of Part 3.3. For simplicity, we set $\mathcal{O}_{i,j} = 0$ if $i < 1$ or $j > n$.

Proposition 3.4.1.1. *Let $1 \leqslant i,j,k,p \leqslant n$, where $i \neq j$ and $k \neq p$. Then*

$$\begin{cases} \mathcal{O}_{k,p}\cdot\mathcal{O}_{i,j} = \mathcal{O}_{i+k-n,j+p-1} & \text{if } i+k-n \geqslant j+p \text{ or } i<j \text{ or } k<p;\\ \mathcal{O}_{k,p}\cdot\mathcal{O}_{i,j} = \mathcal{O}_{i+k-n-1,j+p-1} + \mathcal{O}_{i+k-n,j+p} - \mathcal{O}_{i+k-n-1,j+p} & \text{otherwise.}\end{cases}$$

where, for all $1 \leqslant k,p \leqslant n$, $k \neq p$, we call $\mathcal{O}_{k,p} := [\mathcal{O}_{X(w_{k,p})}]$ the class in $K_\circ(X)$ associated with the longest length element $w_{k,p}$ in the n-th symetric group sending 1 to k and n to p.

Lemma 3.4.1.1. *Let $1 \leqslant h \leqslant n$, $0 \leqslant p \leqslant n-h-1$.*
Then we have the following equality in $K(Fl_{1,n-1})$:

$$[\mathcal{O}_{Y^{h-1}_{h-1,p}\cup Y^h_{h,p}}] = [\mathcal{O}_{Y^{h-1}_{h-1,p}}] + [\mathcal{O}_{Y^h_{h,p}}] - [\mathcal{O}_{Y^{h-1}_{h+1,p}}]$$

Proof. Denote by I_1 and I_2 the bihomogeneous ideals defining $Y^h_{h,p}$ and $Y^{h-1}_{h-1,p}$ embedded in $\mathrm{Proj}\mathbb{C}[x_1,\ldots,x_n]\times\mathrm{Proj}\mathbb{C}[y_1,\ldots,y_n]$.
According to (3.7) we have:

$$I_1 =< x_{h+1},\ldots,x_n; y_1,\ldots,y_h,y_{n-p+1},\ldots,y_n > \quad\text{and}\quad I_2 =< x_h,\ldots,x_n; y_1,\ldots,y_{h-1},y_{n-p+1},\ldots,y_n > .$$

Call $S = \mathbb{C}[x_1,\ldots,x_n]$ and $T = \mathbb{C}[y_1,\ldots,y_n]$ the graded rings. The following is an exact sequence of $S\otimes T$ bigraded-modules:

$$0 \to S\otimes T_{/<I_1\cap I_2>} \to S\otimes T_{/I_1}\oplus S\otimes T_{/I_2} \to S\otimes T_{/<I_1,I_2>} \to 0.$$

This induces the following exact sequence of $\mathcal{O}_{\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}}$-modules:

$$0 \to \mathcal{O}_{Y^{h-1}_{h-1,p}\cup Y^h_{h,p}} \to \mathcal{O}_{Y^{h-1}_{h-1,p}}\oplus\mathcal{O}_{Y^h_{h,p}} \to \widetilde{S\otimes T}_{/<I_1,I_2>} \to 0,$$

where $\widetilde{S\otimes T}_{/<I_1,I_2>}$ is the $\mathcal{O}_{\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}}$-module associated to the $S\otimes T$ bigraded-module $S\otimes T_{/<x_h,\ldots,x_n;y_1,\ldots,y_h,y_{n-p+1},\ldots,y_n>}$. This implies the following equalities in $K(Fl_{1,n-1})$:

$$\begin{aligned}
[\mathcal{O}_{X(i,j)\cup X(i-1,j-1)}] &= [\mathcal{O}_{Y^h_{h,p}}\oplus\mathcal{O}_{Y^{h-1}_{h-1,p}}] - [\widetilde{S\otimes T}_{/<I_1,I_2>}]\\
&= [\mathcal{O}_{Y^h_{h,p}}] + [\mathcal{O}_{Y^{h-1}_{h-1,p}}] - [\mathcal{O}_{Y^{h-1}_{h+1,p}}].
\end{aligned}$$

The second equality is deduced from $\widetilde{S\otimes T}_{/<I_1,I_2>} = \mathcal{O}_{Y^{h-1}_{h+1,p}}$, which is a direct consequence of (3.7).

□

Finally, Proposition 3.4.1.1 follows directly from Lemmas 0.7.2.2, 0.7.2.1, 3.4.1.1 and from Proposition 3.3.0.1. Indeed, notice for any $1 \leqslant i, j, k, l \leqslant n$, such that $i \neq j$ and $k \neq l$, we have:

$$[\mathcal{O}_{X(i,j)}] \cdot [\mathcal{O}_{X(k,l)}] = [\mathcal{O}_{X_{i,j}^{k,l}}],$$

since the intersection between a Schubert varieties and its opposite variety is transverse, and Schubert varieties have rational singularities-cf. for example [Bri05], Theorem 4.2.1 (i). Furthermore, according to Lemmas 0.7.2.1, 3.4.1.1, translations by g and one parameter deformations do not change the sheaf class. Finally, Lemma 3.4.1.1 and Proposition 3.3.0.1 give the result.

3.4.2 Link with cohomological Littlewood-Richardson coefficients.

Let i, k, j and l be integers between 1 and n such that $i \neq j$ and $k \neq l$. Since the morphism $A_*(X) \to GrK_\circ(X)$ is a morphism of graded abelian groups, we can deduce the intersection product between two classes $[X(i,j)]$ and $[X(k,l)]$ in $A_*(X)$ from the product betweeen $[\mathcal{O}_{i,j}]$ and $[\mathcal{O}_{k,l}]$. Proposition 3.4.1.1 describes Littlewood-Richardson coefficients in $A_*(X) \simeq H^*(X)$:

$$\begin{cases} [X(k,l)] \cup [X(i,j)] = 0 & \text{if } i+k \leqslant n \text{ or } j+l \geqslant n+2; \\ [X(k,l)] \cup [X(i,j)] = [X(i+k-n-1, j+l-1)] + & [X(i+k-n, j+l)] \\ & \text{if } 1 \leqslant i+k-n \leqslant j+l-1 \leqslant n \text{ and } i > j,\ k > l; \\ [X(k,l)] \cup [X(i,j)] = [X(i+k-n, j+l-1)] & \text{otherwise.} \end{cases}$$

Remark 3.4.2.1. Chevalley's formula for generalized flag varieties already gives us the intersection product between a codimension 1 Schubert class and any other Schubert class. Let us check we do recover the same formula.
According to the generalization of Chevalley's formula to general G/P (cf. for example [FW04], lemma 8.1), for all $u \in W/W_P$, for all $\beta \in \Delta \backslash \Delta_P$, we have :

$$[Y(s_\beta)] \cup [Y(u)] = \Sigma h_\alpha(\omega_\beta)[Y(\tilde{u}s_\alpha)],$$

where the sum goes over all postive roots α such that $\ell([\tilde{u}s_\alpha]) = \ell(u) + 1$, where $h_\alpha(\omega_\beta) = n_{\alpha\beta}(\beta,\beta)/(\alpha,\alpha)$ (where $n_{\alpha\beta}$ is the coefficient of β in the expansion of α as a positive linear combination of positive roots), and where we write $\tilde{u}$ a minimal length representative of u in W.

We thus obtain here :

$$[Y(\alpha_1)] \cup [Y(u)] = \Sigma[Y(\tilde{u}s_\alpha)],$$

where the sum goes over the $\alpha \in \{\epsilon_1 - \epsilon_i | 2 \leqslant i \leqslant n\} \subset R^+ - R_P^+$ verifying $\ell([\tilde{u}s_\alpha]) = \ell(u) + 1$.

Let $\tilde{u} = w_{i,j}$ and $s_l = s_{\epsilon_1 - \epsilon_l}$. We observe : $\ell(w_{i,j}) = i - 1 + n - j$ if $i < j$, and $\ell(w_{i,j}) = n + i - j - 2$ else; hence $\ell([\tilde{u}s_l]) = \ell(u) + 1$ iff$(l = i+1$ and $i < j)$ or $(l = n$ and $j = i+1)$ or $(l = i$ and $i > j)$, from which we deduce :

$$\begin{aligned} [Y(\alpha_1)] \cup [Y(i,j)] &= [Y(i+1,j)] && \text{if } j \neq i+1,\ i \neq n; \\ [Y(\alpha_1)] \cup [Y(n,j)] &= 0; \\ [Y(\alpha_1)] \cup [Y(i,i+1)] &= [Y(i+2,i+1)] + [Y(i+1,i)], \end{aligned}$$

which does indeed gives us $[X(n-1,1)] \cup [X(k,p)] = [X(k-1,p)]$ if $k \neq p+1$, $k \neq 1$, $[X(n-1,1)] \cup [X(1,p)] = 0$, and $[X(n-1,1)] \cup [X(k,k-1)] = [X(k-2,k-1)] + [X(k-1,k)]$.

Remark 3.4.2.2. Let $\beta \in \Delta \backslash \Delta_P$. If we use the notations of Remark 3.4.2.1, we can write here:

$$[\mathcal{O}_{Y(s_\beta)}] \cdot [\mathcal{O}_{Y(u)}] = \sum_{\substack{\alpha \in R^+ \backslash R_P^+, \\ \ell([\tilde{u}s_\alpha]) = \ell(u)+1}} h_\alpha(\omega_\beta)[\mathcal{O}_{Y(\tilde{u}s_\alpha)}] - \sum_{\substack{\alpha,\gamma \in R^+ \backslash R_P^+ \\ \ell([\tilde{u}s_\alpha]) = \ell(u)+1 = \ell([\tilde{u}s_\gamma])}} \sum_{\substack{v \in R^+ \backslash R_P^+,\ \ell(v) = \ell(u)+2, \\ v \geqslant \alpha \text{ et } v \geqslant \gamma}} h_\alpha(\omega_\beta) h_\gamma(\omega_\beta)[\mathcal{O}_{Y(v)}].$$

3.5 Three points correlators of $Fl_{1,n-1}$.

3.5.1 Three points correlators of $\mathbb{P}^n$. Let $n > 0$. We compute here genus zero correlators of $\mathbb{P}^n$. Note that Buch-Mihalcea fully described the product of two Schubert classes in the ring $QK_s(\mathbb{P}^n)$ [BM11]; we could compute three points correlators using this description. We give here a different proof, by studying the geometry of Gromov-Witten varieties of $\mathbb{P}^n$. This result plays a key role in our derivation of correlators of X presented Part 3.5.3. For $1 \leqslant i \leqslant n+1$, we denote by L_i the Schubert variety associated with the permutation $w_{1,i}$ permuting 1 and i:

$$L_i := \{[x_1 : \cdots : x_i : 0 : \cdots : 0] \in \mathbb{P}^n \mid x_1, \ldots, x_i \in \mathbb{C}\}.$$

Let d in $E(\mathbb{P}^n) \simeq \mathbb{N}$. We consider here the compactification of morphisms $\mathbb{P}^1 \to \mathbb{P}^n$ given by the space of quasi-maps [Bra06]. A degree d morphism $\mathbb{P}^1 \to \mathbb{P}^n$ is defined by $(n+1)$ homogeneous polynomials of degree d. We may thus parametrize degree d morphisms $\mathbb{P}^1 \to \mathbb{P}^n$ by $P = \mathbb{P}(\mathbb{C}^{(d+1)(n+1)})$, where we view a point in $\mathbb{C}^{(d+1)(n+1)}$ as a set of $(n+1)$ degree d polynomials. For a point p in $P = \mathbb{P}(\mathbb{C}^{(d+1)(n+1)})$, we will denote by

$$f_p : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$$

the associated map. Note that if the polynomials associated with a point p in P have common roots x_i, then f_p is a rational map which is not defined on the points $x_i \in \mathbb{P}^1$. The dense subvariety U of P parametrizing polynomials having no common roots is in bijection with $\mathrm{Hom}(\mathbb{P}^1, d)$. Fix distinct points p_1, p_2, p_3 on $\mathbb{P}^1$. According to the appendix of Chapter 2, sending a point p in U to the point in $\mathcal{M}_{0,3}(\mathbb{P}^n, d)$ associated with $(f_p : \mathbb{P}^1 \to \mathbb{P}^n, \{p_1, p_2, p_3\})$ defines a morphism $\phi : U \to \mathcal{M}_{0,3}(\mathbb{P}^n, d)$. Since $\mathcal{M}_{0,3}(\mathbb{P}^n, d)$ is irreducible [KP01] and ϕ is an injective dominant morphism of quasi-projective normal complex varieties, ϕ is birational. We may hence identify a dense open subset V of $\mathcal{M}_{0,3}(\mathbb{P}^n, d)$ with its reciprocal image by ϕ. Finally, note that for an element p in $V \subset P$, the evaluation morphism ev_i assigns to the $(n+1)$ polynomials associated with p the projectivization of their value at p_i.

Proposition 3.5.1.1. *Let $1 \leqslant i_1, i_2, i_3 \leqslant n+1$. Let $d \in \mathbb{N}^*$.*

i) Let g_1, g_2, g_3 be elements in GL_{n+1}, let z be an element in $\mathbb{P}^n$. The variety $ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2}) \cap ev_3^{-1}(g_3 \cdot L_{i_3}) \cap V$ is either empty or an irreducible rational variety. The same holds for the variety $ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2}) \cap ev_3^{-1}(z) \cap V$.

ii) For (g_1, g_2, g_3) general in GL_{n+1}^3, the Gromov-Witten variety $ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2}) \cap ev_3^{-1}(g_3 \cdot L_{i_3})$ is either empty or an irreducible rational variety. In the same way, for (g_1, g_2) general in G^2, the general fiber of the morphism $ev_3 : ev_1^{-1}(g_1 L_{i_1}) \cap ev_2^{-1}(g_2 L_{i_2}) \to ev_3\left(ev_1^{-1}(g_1 L_{i_1}) \cap ev_2^{-1}(g_2 L_{i_2})\right) \subset \mathbb{P}^{n-1}$ evaluating the third marked point is an irreducible rational variety.

iii)

$$\langle w_{1,i_1}, w_{1,i_2}, w_{1,i_3} \rangle_d = \begin{cases} 0 & \text{if } d = 1 \text{ and } i_1 + i_2 + i_3 < n+2 \\ 1 & \text{else} \end{cases}$$

Proof. i) We denote by $L := ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2}) \cap ev_3^{-1}(g_3 \cdot L_{i_3}) \cap V$, which we see here as a subvariety of $P = \mathbb{P}(\mathbb{C}^{(d+1)(n+1)})$. Note that, if we denote by $(P_1, \ldots, P_{n+1})$ the $(n+1)$ polynomials associated to a point in P, L is defined by:

$$L = \left\{(P_1, \ldots, P_{n+1}) \in V \subset (\mathbb{C}^{d+1})^{n+1} \mid \forall 1 \leqslant k \leqslant 3,\ [P_1(p_k) : \cdots : P_{n+1}(p_k)] \in g_k \cdot L_{i_k}\right\}.$$

Since $g_k \cdot L_{i_k}$ is a linear subspaces of $\mathbb{P}^n$, the compactification of L in $P = (\mathbb{C}^{d+1})^{n+1}$ is defined by linear equations. Hence L is the intersection of the dense open subset V with a linear subspace of P, hence L is an irreducible rational variety.

In the same way, if $L := ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2}) \cap ev_3^{-1}(z) \cap V$, since $g_k \cdot L_{i_k}$ is a linear subspace of $\mathbb{P}^n$ and a point in $\mathbb{P}^n$ can be described as the zero locus of a set of linear equations, L is an irreducible rational variety.

ii) Since $\mathbb{P}^n$ is convex, $\mathcal{M}_{0,3}(\mathbb{P}^n, d)$ is a dense open subset of $\overline{\mathcal{M}_{0,3}}(\mathbb{P}^n, d)$ [FP96], hence V is a dense open subset of $\overline{\mathcal{M}_{0,3}}(\mathbb{P}^n, d)$. According to Chapter 1, for $g = (g_k)$ general in GL_{n+1}^3, $ev^{-1}(g \cdot (L_{i_1} \times L_{i_2} \times L_{i_3}))$ has a dense intersection with the dense open subset V of $\mathcal{M}_{0,3}(\mathbb{P}^n, d)$, hence is an irreducible rational variety according to i).

For (g_1, g_2) in G^2 denote by $\mathcal{W}_{g_1,g_2} := ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2})$. For $g = (g_1, g_2)$ general in G^2 the variety $\mathcal{W}_{g_1,g_2}$ has a dense intersection with the dense open subset V; hence for z general in $ev_3(\mathcal{W}_{g_1,g_2})$, the fiber $ev_3^{-1}(z) \cap \mathcal{W}_{g_1,g_2}$ has a dense intersection with the open subset V. Hence for (g_1, g_2) general in G^2 and z general in $ev_3(\mathcal{W}_{g_1,g_2})$, the variety $ev_3^{-1}(z) \cap \mathcal{W}_{g_1,g_2}$ is an irreducible rational variety.

iii) We name $G := \mathrm{GL}_{n+1}$. Since L_i is the Schubert variety associated with $w_{1,i+1}$, according to (3.17) we have for (g_1, g_2, g_3) general in G^3:

$$\langle w_{1,i_1+1}, w_{1,i_2+1}, w_{1,i_3+1} \rangle_d = \chi(\mathcal{O}_{ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2}) \cap ev_3^{-1}(g_3 \cdot L_{i_3})}).$$

Suppose $i_1 + i_2 + i_3 + 1 < n$ and $d = 1$. Then according to Lemma 3.5.1.1 for (g_1, g_2, g_3) general in G^3, the Gromov-Witten variety $ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2}) \cap ev_3^{-1}(g_3 \cdot L_{i_3})$ is empty, hence $\langle w_{1,i_1+1}, w_{1,i_2+1}, w_{1,i_3+1} \rangle_1 = 0$. Now suppose $d > 1$, or $d = 1$ and $i_1 + i_2 + i_3 + 1 \geqslant n$. Then according to Lemma 3.5.1.1 for (g_1, g_2, g_3) general in G^3, the Gromov-Witten variety $ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2}) \cap ev_3^{-1}(g_3 \cdot L_{i_3})$ is non empty. Furthermore, according to [CMBP13] Theorem 2.5. for (g_1, g_2, g_3) general in G^3 this variety has rational singularities. Hence its sheaf Euler characteristic is equal to the sheaf Euler characteristic of its desingularization, which is also a projective rational variety; hence it is equal to 1.

□

Let $1 \leqslant i_1, i_2, i_3 \leqslant n$.

Lemma 3.5.1.1. *i) For $d \geqslant 2$, there exists a genus zero stable map of degree d joining the varieties $g_1 \cdot L_{i_1}$, $g_2 \cdot L_{i_2}$ and $g_3 \cdot L_{i_3}$ for any element (g_1, g_2, g_3) in GL_{n+1}^3;*

ii) For (g_1, g_2, g_3) general in GL_{n+1}^3, there exists a line joining the varieties $g_1 \cdot L_{i_1}$, $g_2 \cdot L_{i_2}$ and $g_3 \cdot L_{i_3}$ iff $i_1 + i_2 + i_3 \geqslant n + 2$.

Proof. i) We first consider the case when $d_2 = 2$. It is enough to show that for any three points P_1, P_2 and P_3 in $\mathbb{P}^n$ there is a genus zero stable map of degree 2 whose image contains all points P_i. Consider a line joining the points P_1 and P_2; now glue this line at P_2 with a line joining the points P_2 and P_3. We obtain a genus zero degree 2 stable map $\mathbb{P}^1 \cup \mathbb{P}^1 \to \mathbb{P}^n$ joining all three points P_i.

Now suppose $d \geqslant 2$. Consider a genus zero degree 2 stable map $f : C \to \mathbb{P}^n$ joining all three points P_i. Let $\mathbb{P}^1 \to \mathbb{P}^n$ be a genus zero degree $d-2$ map, whose image intersects $f(C)$ at the image $f(p)$ of a non special point $p \in C$. Such a map can ba obtained by translating the image of any degree $d-2$ map by an element in GL_{n+1}. Now glue C and $\mathbb{P}^1$ together at the point p. We obtain a degree d stable map $C \cup \mathbb{P}^1 \to \mathbb{P}^n$ joining all three points P_i.

ii) Name $G := \mathrm{GL}_{n+1}$. Let g_1 and g_2 be elements in G^2. By the functorial definition of $\mathbb{P}^n$, there exist vectors u_i and v_i in $\mathbb{C}^{n+1}$ such that $g_1 \cdot L_{i_1}$ parametrizes lines in $\mathbb{C}^{n+1}$ included in the linear subspace $< u_1, \ldots, u_{i_1} > \subset \mathbb{C}^{n+1}$ and $g_2 \cdot L_{i_2}$ parametrizes lines in $\mathbb{C}^{n+1}$ included in the linear subspace $< v_1, \ldots, v_{i_2} > \subset \mathbb{C}^{n+1}$. For (g_1, g_2) general in G^2, the vector space generated by the vectors u_i and v_i is of dimension $\min(n, i_1 + i_2 + 1)$. Hence for (g_1, g_2) general in G^2 the set of points lying on a line joining $g_1 \cdot L_{i_1}$ and $g_2 \cdot L_{i_2}$ parametrizes lines in $\mathbb{C}^{n+1}$ included in the linear subspace

$$< u_1, \ldots, u_{i_1}, v_1, \ldots, v_{i_2} > \simeq \mathbb{C}^{\min(n+1, i_1+i_2)} \subset \mathbb{C}^{n+1}.$$

This can be rephrased more formally as:

$$ev_3(ev_1^{-1}(g_1 \cdot L_{i_1}) \cap ev_2^{-1}(g_2 \cdot L_{i_2})) = h \cdot L_{\min(n+1, i_1+i_2)},$$

where h lies in G and we denote by $ev_i : \overline{\mathcal{M}_{0,3}}(\mathbb{P}^n, 1)$ the evaluation morphism. Hence for (g_1, g_2, g_3) general in G^3, there exists a line joining the varieties $g_1 \cdot L_{i_1}$, $g_2 \cdot L_{i_2}$ and $g_3 \cdot L_{i_3}$ iff the intersection between $h \cdot L_{\min(n+1, i_1+i_2)}$ and $g_3 \cdot L_{i_3}$ is non empty for g_3 general in G-where h is an element in G depending only on the choice of an element (g_1, g_2) general in G^2. This is equivalent to the following condition in the Chow ring $A_*(\mathbb{P}^n)$:

$$[g_3 \cdot L_{i_3}] \cup [h \cdot L_{\min(n, i_1+i_2)}] = [L_{i_3}] \cup [L_{\min(n+1, i_1+i_2)}] \neq 0.$$

Finally, the equality $[L_i] \cup [L_j] = [L_{\min(n, i+j-n)}]$ in $A_*(\mathbb{P}^n)$ yields the result, where we set $L_i = \varnothing$ for $i < 0$.

□

3.5.2 Computing correlators of X. We list here properties of X that will allow us to compute genus zero correlators of X in Part 3.5.3. Recall X can be described as $X \simeq \mathrm{GL}_n/P$, where $P \supset B \supset T$ is a parabolic subgroup of $G = \mathrm{GL}_n$. Note that P is a parabolic subgroup with associated reductive subgroup the block diagonal matrix $\mathrm{GL}_1 \times \mathrm{GL}_{n-2} \times \mathrm{GL}_1$ with GL_{n-2} as the central block. Recall for $1 \leqslant i, j \leqslant n$, where $i \neq j$, we denote by $w_{i,j}$ the representant in W/W_P of any permutation in $N_G(T)/T \simeq \mathfrak{S}_n$ sending 1 to i and n to j. Let $r \geqslant 0$, let $1 \leqslant i_1, \ldots, i_r, j_1, \ldots, j_r \leqslant n$, where $i_k \neq j_k$. For an element $(g_1, \ldots, g_r)$ in G^r, we denote by

$$\mathcal{W}_{g_k} := ev_1^{-1}(g_1 \cdot X(w_{i_1,j_1})) \cap \cdots \cap ev_r^{-1}(g_r \cdot X(w_{i_r,j_r}))$$

the Gromov-Witten variety parametrizing rational curves of degree $\mathbf{d}$ with marked points in the Schubert varieties $g_k X(w_{i_k,j_k})$ in $\overline{\mathcal{M}_{0,r+1}}(X, \mathbf{d})$.

(a) Computing correlators by considering the image of Gromov-Witten varieties.

Proposition 3.5.2.1. *i) Let $(g_1, \ldots, g_r)$ be an element in G^r such that*

A) The general fiber of the evaluation morphism $ev_{r+1} : \mathcal{W}_{g_k} := \cap_{k=1}^r ev_k^{-1}(g_k \cdot X(w_{i_k,j_k})) \to \mathrm{Im}(ev_{r+1}) \subset X$ is a rationally connected variety;

B) $Tor_i^{X^r}(\mathcal{O}_{\overline{\mathcal{M}_{0,r+1}}(X,\mathbf{d})}, \mathcal{O}_{\prod_{k=1}^r g_k X(w_{i_k,j_k})}) = 0$ for all $i > 0$;

C) The Gromov-Witten variety $\mathcal{W}_{g_k}$ has rational singularities;

D) The projected Gromov-Witten variety $ev_{r+1}(\mathcal{W}_{g_k})$ has rational singularities;

Then for any α in $K^\circ(X)$

$$\langle \mathcal{O}_{i_1,j_1}, \ldots, \mathcal{O}_{i_r,j_r}, \alpha \rangle_{\mathbf{d}} = \chi\left(\alpha \cdot [\mathcal{O}_{ev_{r+1}(\mathcal{W}_{g_k})}]\right).$$

ii) If for $(g_1, \ldots, g_r)$ general in G^r:

A) The general fiber of the evaluation morphism $ev_{r+1} : \mathcal{W}_{g_k} := \cap_{k=1}^r ev_k^{-1}(g_k \cdot X(w_{i_k,j_k})) \to \mathrm{Im}(ev_{r+1}) \subset X$ is a rationally connected variety;

B) The projected Gromov-Witten variety $ev_{r+1}(\mathcal{W}_{g_k})$ has rational singularities

then for any α in $K^\circ(X)$ and for $(g_1, \ldots, g_r)$ general in G^r:

$$\langle \mathcal{O}_{i_1,j_1}, \ldots, \mathcal{O}_{i_r,j_r}, \alpha \rangle_{\mathbf{d}} = \chi\left(\alpha \cdot [\mathcal{O}_{ev_{r+1}(\mathcal{W}_{g_k})}]\right)$$

iii) If

A) For (g_1, g_2) general in G^2, the general fiber of the evaluation morphism $ev_3 : \mathcal{W}_{g_1,g_2} = ev_1^{-1}(g_1 X(i_1, j_1)) \cap ev_2^{-1}(g_2 w_0 X(i_2, j_2)) \to \mathrm{Im}(ev_3) \subset X$ is a rationally connected variety;

B) The projected Gromov-Witten variety $ev_3(\mathcal{W}_{1,w_0})$ has rational singularities,

then

$$\langle \mathcal{O}_{i_1,j_1}, \mathcal{O}_{i_2,j_2}, \alpha \rangle_{\mathbf{d}} = \chi\left(\alpha \cdot [\mathcal{O}_{ev_3(\mathcal{W}_{1,w_0})}]\right)$$

Proof. i) Since $Tor_i^{X^r}(\mathcal{O}_{\overline{\mathcal{M}_{0,r+1}}(X,\mathbf{d})}, \mathcal{O}_{\prod_{k=1}^r g_k X(w_{i_k,j_k})}) = 0$ for all $i > 0$, we have the following identity in $K_\circ(\overline{\mathcal{M}_{0,r+1}}(X,\mathbf{d}))$.

$$\begin{aligned} ev^*\left(\prod_{k=1}^r \mathcal{O}_{i_k,j_k}\right) &= ev^*[\mathcal{O}_{\prod_{k=1}^r g_k X(w_{i_k,j_k})}] \\ &= \sum_{i\geqslant 0}[Tor_i^{X^r}(\mathcal{O}_{\overline{\mathcal{M}_{0,r+1}}(X,\mathbf{d}))}, \mathcal{O}_{\prod_{k=1}^r g_k X(w_{i_k,j_k})})] \\ &= [\mathcal{O}_{\overline{\mathcal{M}_{0,r+1}}(X,\mathbf{d})\times_{X^r}\prod_{k=1}^r g_k X(w_{i_k,j_k})}] \\ &= [\mathcal{O}_{\mathcal{W}_{g_k}}]. \end{aligned} \tag{3.8}$$

By definition for α in $K^\circ(X)$ the correlator associated with i_k, j_k and α is given by

$$\langle \mathcal{O}_{i_1,j_1},\dots,\mathcal{O}_{i_r,j_r},\alpha\rangle_{\mathbf{d}} = \chi\left(ev_1^*\mathcal{O}_{i_1,j_1}\dots ev_r^*\mathcal{O}_{i_r,j_r}\cdot ev_{r+1}^*\alpha\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r+1}}(X,\mathbf{d}))}]\right)$$

Using (3.8) here above and the projection formula in $K_\circ(X)$, we obtain:

$$\begin{aligned} \langle \mathcal{O}_{i_1,j_1},\dots,\mathcal{O}_{i_r,j_r},\alpha\rangle_{\mathbf{d}} &= \chi\left(ev_{r+1}^*\alpha\cdot ev_1^*\mathcal{O}_{i_1,j_1}\dots ev_r^*\mathcal{O}_{i_r,j_r}\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r+1}}(X,\mathbf{d}))}]\right) \\ &= \chi\left(ev_{r+1}^*\alpha\cdot[\mathcal{O}_{\mathcal{W}_{g_k}}]\right) \\ &= \chi\left(\alpha\cdot(ev_{r+1})_*[\mathcal{O}_{\mathcal{W}_{g_k}}]\right). \end{aligned}$$

Moreover, since the projected Gromov-Witten variety $ev_{r+1}(\mathcal{W}_{g_k})$ has rational singularities, according to [BM11] Theorem 3.1 we have in $K_\circ(X)$:

$$(ev_{r+1})_*[\mathcal{O}_{\mathcal{W}_{g_k}}] = [\mathcal{O}_{ev_{r+1}(\mathcal{W}_{g_k})}], \tag{3.9}$$

which yields the expected equality.

ii) First, according to [CMBP13] Theorem 2.5 for $(g_1,\dots,g_r)$ general in G^r the Gromov-Witten variety $\mathcal{W}_{g_k}$ has rational singularities. Furthermore, according to Sierra's homological Kleiman-Bertini theorem [Sie09], for $(g_1,\dots,g_r)$ general in G^r, we have $Tor_i^{X^r}(\mathcal{O}_{\overline{\mathcal{M}_{0,r+1}}(X,\mathbf{d})}, \mathcal{O}_{\prod_{k=1}^r g_k X(w_{i_k,j_k})}) = 0$ for all $i > 0$. Hypothesis A) and B) finally imply the expected equality according to i).

iii) Since for (g_1,g_2) general in G^2, the general fiber of the evaluation morphism $ev_3 : \mathcal{W}_{g_k} \to \mathrm{Im}(ev_3) \subset X$ is a rationally connected variety there exists a dense open subset U of G^2 such that elements in U satisfy conditions A) and B) and C) of part i). Since the evaluation morphism $ev : \overline{\mathcal{M}_{0,3}}(X,\mathbf{d}) \to X^3$ is G-equivariant, U is invariant by the diagonal action of G. Hence according to [FW04] Lemma 7.1, there exists (h_1,h_2) in U such that $h_1X(w_{i_1,j_1}) = X(w_{i_1,j_1})$ and $h_2X(w_{i_2,j_2}) = w_0X(w_{i_2,j_2})$. Since the projected Gromov-Witten variety $ev_3(\mathcal{W}_{h_1,h_2})$ has rational singularities, we obtain according to i):

$$\langle \mathcal{O}_{i_1,j_1},\mathcal{O}_{i_2,j_2},\alpha\rangle_{\mathbf{d}} = \chi\left(\alpha\cdot[\mathcal{O}_{ev_3(\mathcal{W}_{1,w_0})}]\right). \tag{3.10}$$

□

(b) Computing correlators of $Fl_{1,n-1}$ when the natural map $\overline{\mathcal{M}_{0,3}}(X,\mathbf{d}) \to \overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1},d_1)\times_{(\mathbb{P}^{n-1})^3} X^3$ is surjective. Recall X parametrizes lines included in hyperplanes of $\mathbb{C}^n$. Denote by $\pi : X \to \mathbb{P}^{n-1}$ the projection induced by forgetting the hyperplane. Let $\mathbf{d} = (d_1,d_2) := d_1l_1 + d_2l_2$ be an effective class of curve in $E(X)$. Note that $\pi_*(d_1,d_2) = d_1 \in E(\mathbb{P}^{n-1})$. The forgetful morphism π induces a morphism $\Pi : \overline{\mathcal{M}_{0,3}}(X,\mathbf{d}) \to \overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1},d_1)$. Together with the evaluation morphisms ev_i, this induces a morphism:

$$\Pi\times ev : \overline{\mathcal{M}_{0,3}}(X,\mathbf{d}) \to \overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1},d_1)\times_{(\mathbb{P}^{n-1})^3} X^3.$$

Proposition 3.5.2.2. *If*

- *The morphism* $\Pi \times ev : \overline{\mathcal{M}_{0,3}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1}, \pi_*\mathbf{d}) \times_{(\mathbb{P}^{n-1})^3} X^3$ *is surjective,*
- *The projected Gromov-Witten variety* $ev_3(\mathcal{W}_{1,w_0}) = ev_3\left(ev_1^{-1}(X(i_1, j_1)) \cap ev_2^{-1}(w_0X(i_2, j_2))\right)$ *has rational singularities,*

Then

$$\langle \mathcal{O}_{i_1,j_1}, \mathcal{O}_{i_2,j_2}, \alpha \rangle_{\mathbf{d}} = \chi\left(\alpha \cdot [\mathcal{O}_{ev_3^{\mathbf{d}}(\mathcal{W}_{1,w_0})}]\right).$$

Proof. Denote by $P_m \supset B$ the parabolic subgroup associated with $\mathbb{P}^{n-1}$, i.e. P_m is a parabolic subgroup satisfying

$$B \subset P \subset P_m$$

with associated reductive subgroup the block diagonal matrix $\mathrm{GL}_1 \times \mathrm{GL}_{n-1}$ whose first block is GL_1. Note that the image of $w_{i,j}$ by the forgetful morphism $W/W_P \to W/W_{P_m}$ is the element $w_{1,i}$ in W/W_{P_m} associated with any permutation sending 1 to i. Recall the Schubert variety of $\mathbb{P}^{n-1}$ associated with the element $w_{1,i}$ in W/W_{P_m} is $L_i := \{[x_1 : \cdots : x_i : 0 : \cdots : 0]\}$. For an element (g_1, g_2) in G^3, denote by

$$\overline{\mathcal{M}_{0,3}}(X, \mathbf{d}) \supset \mathcal{W}_{g_k} := \prod_{k=1,2} ev_k^{-1}(g_kX(w_{i_k,j_k})) \quad \text{and} \quad \mathcal{W}'_{g_k} := \prod_{k=1,2} ev_k^{-1}(g_kL_{i_k}) \subset \overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1}, d_1)$$

the Gromov-Witten variety of X (respectively $\mathbb{P}^{n-1}$) associated with the translated Schubert varieties $g_kX(w_{i_k,j_k})$ (respectively $g_kL_{i_k}$).

Since the morphism $\Pi \times ev$ is surjective and X is a two step flag variety, according to Chapter 2 (Proposition 1.4.0.1) the general fiber of $\Pi \times ev$ is rational. Hence according to Chapter 1 (Proposition 1.2.4.1) the general fiber of the restriction $\mathcal{W}_{g_1,g_2} \to \mathcal{W}'_{g_1,g_2} \times_{(\mathbb{P}^{n-1})^3} g_1X(i_1, j_1) \times g_2X(i_2, j_2) \times X$ of $\Pi \times ev$ is rationally connected. Furthermore, according to Proposition 3.5.1.1, the general fiber of the morphism

$$ev_3^{d_1} : \mathcal{W}'_{g_1,g_2} \to ev_3(\mathcal{W}'_{g_1,g_2}) \subset \mathbb{P}^{n-1}$$

evaluating the third marked point is an irreducible rational variety. Note that since $X \to \mathbb{P}^{n-1}$ is a fibration in $\mathbb{P}^{n-2}$'s, it is a flat morphism. Recall from Chapter 1 that the flat base change and composition of a rationally connected fibration is a rationally connected fibration (Theorem 1.2.2.1 (3) and Lemma 1.5.1.1); hence the general fiber of the morphism $\mathcal{W}'_{g_1,g_2} \times_{(\mathbb{P}^{n-1})^3} g_1X(i_1, j_1) \times g_2X(i_2, j_2) \times X \to \mathbb{P}^{n-1} \times_{(\mathbb{P}^{n-1})^3} g_1X(i_1, j_1) \times g_2X(i_2, j_2) \times X = X$ is also rationally connected. We obtain the following commutative diagram of rationally connected fibrations.

$$\begin{array}{ccc} \mathcal{W}_{g_1,g_2} & & \\ \downarrow{\scriptstyle \Pi \times ev} & \searrow^{ev_3^{\mathbf{d}}} & \\ \mathcal{W}'_{g_1,g_2} \times_{(\mathbb{P}^{n-1})^3} g_1X(i_1, j_1) \times g_2X(i_2, j_2) \times X & \longrightarrow & X \\ \downarrow{\scriptstyle ev_3^{d_1}} & & \\ \mathbb{P}^{n-1} & & \end{array}$$

By composition the general fiber of the morphism $ev_3^{\mathbf{d}} : \mathcal{W}_{g_1,g_2} \to ev_3^{\mathbf{d}}(\mathcal{W}_{g_1,g_2})$ is rationally connected. Since the projected Gromov-Witten variety $ev_3^{\mathbf{d}}(\mathcal{W}_{1,w_0})$ has rational singularities, then according to Proposition 3.5.2.1 for any α in $K_\circ(X)$ we have $\langle \mathcal{O}_{i_1,j_1}, \mathcal{O}_{i_2,j_2}, \alpha \rangle_{\mathbf{d}} = \chi\left(\alpha \cdot [\mathcal{O}_{ev_3(\mathcal{W}_{1,w_0})}]\right)$. □

(c) Symmetry of correlators of $Fl_{1,n-1}$. The isomorphism $Gr(1, n) \simeq Gr(n-1, n)$ yields an automorphism φ of X, which implies the following symmetry for correlators of X.

Proposition 3.5.2.3.

$$\langle w_{i_1,j_1},\dots,w_{i_r,j_r}\rangle_{d_1l_1+d_2l_2}=\langle w_{n-j_1+1,n-i_1+1},\dots,w_{n-j_r+1,n-i_r+1}\rangle_{d_2l_1+d_1l_2}.$$

Proof. We consider here X as the flag variety parametrizing lines included in hyperplanes of $\mathbb{C}^n$. We identify explicitly a line in $\mathbb{C}^n$ with a hyperplane in $\mathbb{C}^n$ in the following way. Consider a Hermitian inner product $\langle\cdot\mid\cdot\rangle$ on $\mathbb{C}^n$. For vectors $u_1,\dots,u_r$ in $\mathbb{C}^n$, we denote by $<u_1,\dots,u_r>$ the $\mathbb{C}$-linear vector space generated by the u_i. For any linear space $E=<u_1,\dots,u_r>$ in $\mathbb{C}^n$ denote by $E^\perp$ its orthogonal with respect to $\langle\cdot\mid\cdot\rangle$; $E^\perp$ may be defined as the intersection of the kernels of the r linear maps sending a vector v in $\mathbb{C}^n$ to the complex number $\langle\overline{u_i}\mid v\rangle$, where we denote by $\overline{u_i}$ the conjugate of u_i. Then the morphism $[L]\to[L^\perp]$ yields an identification $\mathbb{P}^{n-1}\simeq Gr(n-1,n)$. Furthermore, for any hyperplane H in $\mathbb{C}^n$ consider the one dimensional linear space $H^\perp$ orthogonal to it. Since $(L^\perp)^\perp=L$, we obtain the inverse map $Gr(n-1,n)\to\mathbb{P}^{n-1}$. Now consider the natural embedding $X\hookrightarrow\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$. The automorphism φ may be seen as the restriction to X of the map $\mathbb{P}^{n-1}\times Gr(n-1,n)\to\mathbb{P}^{n-1}\times Gr(n-1,n)$ sending the projectivization of a line and a hyperplane $([L],[H])$ onto the projectivization of the orthogonal spaces $([H^\perp],[L^\perp])$. Note that φ exchanges classes l_1 and l_2, and sends a Schubert variety $X(w_{i,j})$ onto a translate of the Schubert variety $X(w_{n-j+1,n-i+1})$. Indeed, according to (3.1) the image by φ of any Schubert variety $X(w_{i,j})$ may be described in the following way, where we consider a basis $(e_i)_{1\leqslant i\leqslant n}$ of $\mathbb{C}^n$ which is orthogonal with respect to the inner product $\langle\cdot\mid\cdot\rangle$.

$$\begin{aligned}
\varphi(X(w_{i,j}))&=\varphi\left(\left\{(L,H)\in X\middle|L\subset<e_1,\cdots,e_i>,<e_1,\cdots,e_{j-1}>\subset H\right\}\right)\\
&=\left\{(H^\perp,L^\perp)\in X\middle|L^\perp\supset(<e_1,\cdots,e_i>)^\perp,\ (<e_1,\cdots,e_{j-1}>)^\perp\supset H^\perp\right\}\\
&=\left\{(L,H)\in X\middle|H\supset<e_{i+1},\cdots,e_n>,<e_j,\cdots,e_n>)^\perp\supset L\right\}\\
&=w_0\cdot\left\{(L,H)\in X\middle|H\supset<e_1,\cdots,e_{n-i}>,<e_1,\cdots,e_{n-j+1}>)^\perp\supset L\right\}\\
&=w_0\cdot X(w_{n-j+1,n-i+1}),
\end{aligned}$$

where w_0 is the element in $G=\mathrm{GL}_n$ associated with a maximal length permutation of the basis vectors e_i. Furthermore, we have $\varphi_*(l_1)=[\varphi(X(w_{2,n}))]=[w_0\cdot X(w_{1,n-1})]=l_2$, and in the same way $\varphi_*(l_2)=l_1$. The isomorphism $\varphi:X\to X$ induces an isomorphism $\phi:\overline{\mathcal{M}_{0,r}}(X,d_1l_1+d_2l_2)\to\overline{\mathcal{M}_{0,r}}(X,\varphi_*(d_1l_1+d_2l_2))=\overline{\mathcal{M}_{0,r}}(X,d_2l_1+d_1l_2)$. We obtain the following commutative diagram.

$$\begin{array}{ccc}
\overline{\mathcal{M}_{0,r}}(X,d_1l_1+d_2l_2) & \xrightarrow{\phi} & \overline{\mathcal{M}_{0,r}}(X,d_2l_1+d_1l_2)\\
\downarrow{\scriptstyle \epsilon v} & & \downarrow{\scriptstyle ev}\\
X^r & \xrightarrow{\varphi^r} & X^r
\end{array}$$

Since φ and ϕ are isomorphisms, for α in $K^\circ(X)\simeq K_\circ(X)$ the projection formula implies

$$\begin{aligned}
\phi_*\left((\epsilon v_i)^*\alpha\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d_1l_1+d_2l_2)}]\right)&=\phi_*\left((\epsilon v)_i^*\varphi^*\varphi_*\alpha\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d_1l_1+d_2l_2)}]\right)\\
&=\phi_*\left(\phi^*ev_i^*\varphi_*\alpha\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d_1l_1+d_2l_2)}]\right)\\
&=ev_i^*\varphi_*\alpha\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d_2l_1+d_1l_2)}].
\end{aligned}$$

This yields the following equalities, where we denote by $\chi:K_\circ(\overline{\mathcal{M}_{0,r}}(X,\mathbf{d}))\to K_\circ(\mathrm{Spec}(\mathbb{C}))$ the pushforward to the point.

$$\begin{aligned}
&\langle w_{i_1,j_1},\dots,w_{i_r,j_r}\rangle_{d_1l_1+d_2l_2}\\
&=\chi\left((\epsilon v_1)^*\mathcal{O}_{w_{i_1,j_1}}\cdot\dots\cdot(\epsilon v_r)^*\mathcal{O}_{w_{i_r,j_r}}\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d_1l_1+d_2l_2)}]\right)\\
&=\chi\left(\phi_*\left(\phi^*ev_1^*\varphi_*\mathcal{O}_{w_{i_1,j_1}}\cdot\dots\cdot\phi^*ev_r^*\varphi_*\mathcal{O}_{w_{i_r,j_r}}\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d_1l_1+d_2l_2)}]\right)\right)\\
&=\chi\left(ev_1^*(\varphi_*\mathcal{O}_{w_{i_1,j_1}})\cdot\dots\cdot ev_r^*(\varphi_*\mathcal{O}_{w_{i_r,j_r}})\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d_2l_1+d_1l_2)}]\right)\\
&=\chi\left(ev_1^*(\mathcal{O}_{w_{n-i_1+1,n-j_1+1}})\cdot\dots\cdot ev_r^*(\mathcal{O}_{w_{n-i_r+1,n-j_r+1}})\cdot[\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d_2l_1+d_1l_2)}]\right),
\end{aligned}$$

where the last equality holds since $\varphi_*\mathcal{O}_{w_{i,j}} = \varphi_*[\mathcal{O}_{X(w_{i,j})}] = [\mathcal{O}_{\varphi(X(w_{i,j}))}] = [\mathcal{O}_{w_0\cdot X(w_{n-j+1,n-i+1})}] = \mathcal{O}_{n-j+1,n-i+1}$. To sum up, correlators of X exhibit the following symmetry.

$$\langle w_{i_1,j_1},\dots,w_{i_r,j_r}\rangle_{d_1l_1+d_2l_2} = \langle w_{n-j_1+1,n-i_1+1},\dots,w_{n-j_r+1,n-i_r+1}\rangle_{d_2l_1+d_1l_2}.$$

□

Remark 3.5.2.1. Let G be a semi-simple linear algebraic group, let P be a parabolic subgroup of G and H be a subgroup of G acting on G/P by left multiplication. Then point (a) holds for H-equivariant correlators of G/P. If $G = \mathrm{GL}_n$, then point (c) implies that H-equivariant correlators of the flag variety GL_n/P are equal to the corresponding correlators of the dual flag variety. The symmetry observed here comes from the fact that $Fl_{1,n-1}$ is self dual.

3.5.3 Three points correlators of X. We use here the results of the preceding parts to compute all genus zero three points correlators of $X = \mathrm{GL}_n/P$. Let $1 \leqslant i_1, i_2, i_3, j_1, j_2, j_3 \leqslant n$, where $i_k \neq j_k$. Recall for $1 \leqslant i, j \leqslant n$, $i \neq j$, we denote by $w_{i,j}$ the element in W^P representing any permutation sending 1 to i and n to j. For a point x in $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$, we denote by

$$(L_x, H_x) \in \mathbb{C}^n \times \mathbb{C}^n$$

vectors satisfying $x = ([L_x],[H_x])$. We consider here X as a subvariety cut out from $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$ by the incidence relation "line included in hyperplane" in the following way. Consider a Hermitian inner product $\langle\cdot\mid\cdot\rangle$ on $\mathbb{C}^n$. Consider a basis (e_i) of $\mathbb{C}^n$ orthogonal with respect to $\langle\cdot\mid\cdot\rangle$. Denote by $\overline{u}$ the conjugate of a complex number. The isomorphism $\mathbb{P}^{n-1} \simeq Gr(n-1,n)$ identifies $H_x \in \mathbb{C}^n$ with the kernel $H_x^\perp$ of the linear map $\mathbb{C}^n \ni v \to \langle\overline{H_x}\mid v\rangle \in \mathbb{C}$, which is a hyperplane of $\mathbb{C}^n$. For any vectors L_x, H_x in $\mathbb{C}^n$ the associated point $([L_x],[H_x])$ lies in X iff L_x lies within the hyperplane $H_x^\perp \subset \mathbb{C}^n$, i.e. iff $\langle L_x \mid \overline{H_x}\rangle = 0$. For an element g in $G = \mathrm{GL}_n$, we denote by $g^\perp$ the adjoint of $\overline{g}$ for the inner product $\langle\cdot\mid\cdot\rangle$, i.e. $\langle gL \mid \overline{g^\perp L'}\rangle = \langle L \mid \overline{L'}\rangle$ for any L, L' in $\mathbb{C}^n$. We consider here the following action of G.

$$\begin{aligned} G\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1} &\to \mathbb{P}^{n-1}\times\mathbb{P}^{n-1} \\ (g,([L_x],[H_x])) &\to ([g\cdot L_x],[g^\perp\cdot H_x]) \end{aligned}$$

The action of G on $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$ we just described induces by restriction the natural action of G by left multilication on $X \simeq G/P$. For $1 \leqslant i \leqslant n$, we set

$$L_i := \{[x_1 : \cdots : x_i : 0 \dots, 0]\} \subset \mathbb{P}^{n-1}.$$

Recall we denote by w_0 the permutation of $1 \dots n$ of largest length. Note that we have

$$\begin{aligned} w_0 L_i &= \{[0 : \dots ; 0 : x_{n-i+1} : \cdots : x_n]\} \subset \mathbb{P}^{n-1} \\ L_i^\perp &= \{[0 : \dots ; 0 : x_{i+1} : \cdots : x_n]\} = w_0 L_{n-i}. \end{aligned}$$

Recall for $1 \leqslant i, j \leqslant n$ and $i \neq j$, $w_{i,j}$ denotes the element in W^P representing any permutation sending 1 to i and n to j. Using (3.2) Schubert varieties of X can be described in the following way.

$$\begin{aligned} X(w_{i,j}) &= \{x = ([L_x],[H_x]) \in X \mid L_x \subset < e_1,\cdots,e_i >,\ H_x \subset < e_j,\cdots,e_n >\} \\ &= (L_i \times w_0 L_{n-j+1}) \cap X. \end{aligned} \tag{3.11}$$

Let $\mathbf{d}$ be a degree in $E(X)$. For (g_1,g_2) in G^2, we denote by

$$X^{\mathbf{d}}_{g_1,g_2} := ev_1^{-1}(g_1\cdot X(w_{i_1,j_1})) \cap ev_2^{-1}(g_2\cdot X(w_{i_2,j_2})) \subset \overline{\mathcal{M}_{0,3}}(X,\mathbf{d})$$

the Gromov-Witten variety parametrizing genus zero degree $\mathbf{d}$ stable maps sending their marked points into the translated Schubert varieties $g_k\cdot X(w_{i_k,j_k})$.

(a) WHEN $\mathbf{d} = l_2$. Denote by $\pi_1 : X \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ (respectively π_2) the first (respectively second) projection. Note that since l_2 is a generating element in the Chow ring $A_*(X)$, curves of class l_2 are irreducible. Furthermore, since $\pi_{1*} l_2 = 0 \in A_*(\mathbb{P}^{n-1})$, a curve C of class l_2 in X projects to the point, and since $\pi_{2*} l_2 = l_2 \in A_*(\mathbb{P}^{n-1})$, a curve C of class l_2 projects to a line in $\mathbb{P}^{n-1}$. Hence curves of class l_2 in X are curves of the form $\{x\} \times L$, where x is a point in $\mathbb{P}^{n-1}$ and L is a line in $\mathbb{P}^{n-1}$. Since lines are rational, any such curves are rational. In particular we observe that there is either a unique rational curve of class l_2 joining two distinct points in X, or none.

We denote by $ev_{3|g_1,g_2}$ the restriction of the morphism evaluating the third marked point to the Gromov-Witten variety $X^{l_2}_{g_1,g_2}$. Denote by $U \subset X^3$ the dense open subset parametrizing distinct triplets of points in X^2.

Lemma 3.5.3.1. *i) For (g_1, g_2) general in G^2, the variety $X^{l_2}_{g_1,g_2}$ has a dense intersection with $(ev_1 \times ev_3)^{-1}(U)$.*

ii) For (g_1, g_2) general in G^2, the general fiber of $ev_{3|g_1,g_2}$ is a rational variety.

iii) Suppose $j_1 + j_2 \leqslant n + 2$. Then there exists an element h in G such that $ev_3(X^{l_2}_{1,w_0}) = h \cdot X(w_{i_1+i_2-n,1})$ if $i_1 + i_2 > n + 1$, and $ev_3(X^{l_2}_{1,w_0}) = h \cdot X(w_{1,2})$ if $i_1 + i_2 = n + 1$, and is empty else.

Proof. We set $X_{g_1,g_2} := X^{l_2}_{g_1,g_2}$.

i) Note that for (g_1, g_2) general in G^2, the intersection of $X^{l_2}_{g_1,g_2}$ with $(ev_1 \times ev_3)^{-1}(U)$ is a dense open subset of $X^{l_2}_{g_1,g_2}$. Indeed, since X is convex, the subset $\mathcal{M}_{0,3}(X, l_2) \subset \overline{\mathcal{M}_{0,3}}(X, l_2)$ parametrizing degree l_2 stable maps $\mathbb{P}^1 \to X$ is a dense open subset of $\overline{\mathcal{M}_{0,3}}(X, l_2)$ [FP96], whose image is contained in U. Hence $(ev_1 \times ev_3)^{-1}(U)$ is a dense open subset of $\overline{\mathcal{M}_{0,3}}(X, l_2)$. Then for (g_1, g_2) general in G^2 the variety $X^{l_2}_{g_1,g_2}$ has a dense intersection with $(ev_1 \times ev_3)^{-1}(U)$.

ii) Let (g_1, g_2) be an element in G^2, let $z = (z_1, z_2)$ be a point in $ev_3(X_{g_1,g_2})$. According to (3.11), we have:

$$\begin{aligned} p_2\left(g_1 X(w_{i_1,j_1}) \times g_2 X(w_{i_2,j_2})\right) &= g_1^{\perp} \cdot p_2(X(w_{i_1,j_1})) \times g_2^{\perp} \cdot p_2(X(w_{i_2,j_2})) \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \\ &= (g_1^{\perp} \cdot w_0 L_{n-j_1+1}) \times (g_2^{\perp} \cdot w_0 L_{n-j_2+1}) \\ &= L_{g_1} \times L_{g_2}, \end{aligned}$$

where we denote by $L_{g_1} := g_1^{\perp} \cdot w_0 L_{n-j_1+1}$ and $L_{g_2} := g_2^{\perp} \cdot w_0 L_{n-j_2+1}$. The point z_1 in $\mathbb{P}^{n-1}$ defines by duality a codimension one linear subspace H_1 in $\mathbb{P}^{n-1} \simeq Gr(n-1, n)$. We consider the set of points in $(L_{g_1} \times L_{g_2}) \cap H_1$ distinct from z that lie in a line containing $z_2 = [z^1 : \cdots : z^n]$; this can be seen as the intersection of the image of the following map with $(L_{g_1} \times L_{g_2}) \cap H_1$.

$$\begin{aligned} & & \mathbb{A}^1 \times L_{g_1} \backslash \{z_2\} &\to \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \\ f : & & (t, x = [x^1 : \cdots : x^n]) &\to (x, [x^1 + t z^1 : \cdots : x^n + t z^n]). \end{aligned}$$

Note that since L_{g_2} and H_1 are described by linear equations, $\mathrm{Im} f \cap (L_{g_1} \times L_{g_2})$ is described by linear equations on the x^i and t, and hence is a rational variety. We now consider the morphism $ev_{3|g_1,g_2}^{-1}(z) \to X^2 \subset (\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})^2$ evaluating the first two marked points. Note that this morphism factors through $ev_{3|g_1,g_2}^{-1}(z) \to (\{z_1\} \times \mathbb{P}^{n-1})^2 \simeq (\mathbb{P}^{n-1})^2$. We denote by f_U the restriction of this morphism to the subset $ev_{3|g_1,g_2}^{-1}(z) \cap (ev_1 \times ev_2)^{-1}(U)$. By definition, $ev_{3|g_1,g_2}^{-1}(z) \cap (ev_1 \times ev_2)^{-1}(U)$ parametrizes degree l_2 stable maps $(h : \mathbb{P}^1 \to \{z_1\} \times \mathbb{P}^{n-1} \simeq \mathbb{P}^{n-1}, \{p_1, p_2, p_3\})$ satisfying the following conditions:

- h sends its first two marked points into the projection $\pi_1(g_k X(w_{i_k,j_k}))$ of the Schubert varieties considered here, i.e. $h(p_1) \in L_{g_1}$, $h(p_2) \in L_{g_2}$;

- h sends its third marked point into z, i.e. $h(p_3) = z_2$;
- $h(p_1) \neq z_2$ and $h(p_2) \neq z_2$;
- The image of h is contained in X, i.e. $h(p_1) \in H_1$ and $h(p_2) \in H_1$.

Since a degree l_2 map $(\mathbb{P}^1 \to \mathbb{P}^{n-1}, \{p_1, p_2, p_3\})$ is uniquely determined by the data of three distinct points in $\mathbb{P}^{n-1}$ joined by a line, points in the fiber $ev_{3|g_1,g_2}^{-1}(z) \cap (ev_1 \times ev_2)^{-1}(U)$ are in one-to-one correspondence with distinct points in $\mathrm{Im}f \cap (L_{g_1} \times L_{g_2}) \cap H_1$. Hence f_U is an injective morphism of quasi-projective complex varieties, dominating the rational variety $\mathrm{Im}f \cap (L_{g_1} \times L_{g_2}) \cap H_1$. Hence $ev_{3|g_1,g_2}^{-1}(z) \cap (ev_1 \times ev_3)^{-1}(U)$ is a rational variety.

Finally, since according to $i)$ for (g_1, g_2) general in G^2, X_{g_1,g_2} has a dense intersection with $(ev_1 \times ev_3)^{-1}(U)$, for (g_1, g_2) general in G^2, the general fiber $ev_{3|g_1,g_2}^{-1}(z)$ has a dense intersection with $(ev_1 \times ev_3)^{-1}(U)$, and hence is also a rational variety.

iii) $ev_3(X_{1,w_0})$ parametrizes elements (L, H) in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ such that:
- There exists a line in $\mathbb{P}^{n-1}$ joining H with elements $L_1 \subset < e_{j_1}, \ldots, e_n >$ and $L_2 \subset < e_1, \ldots, e_{n-j_2+1} >$;
- $L \subset < e_1, \ldots, e_{i_1} > \cap < e_{n-i_2+1}, \ldots, e_n >$;
- $L \subset H^\perp$.

First note that the subset of elements H satisfying the first condition is the vector subspace generated by $< e_1, \ldots, e_{n-j_2+1}, e_{j_1}, \ldots, e_n > \simeq \mathbb{C}^{\min(n, 2n-j_1-j_2+2)}$. Note that the second condition is impossible iff $i_1 < n - i_2 + 1$, and else describes elements L in $< e_{n-i_2+1}, \ldots, e_{i_1} > \simeq \mathbb{C}^{i_1+i_2-n+1}$. Hence $ev_3(X_{1,w_0})$ is the set of elements (L, H) in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ such that
- $H \subset < e_1, \ldots, e_{n-j_2+1}, e_{j_1}, \ldots, e_n >$;
- $L \subset < e_{n-i_2+1}, \ldots, e_{i_1} >$;
- $L \subset H^\perp$.

Note that for $j_1 + j_2 \leqslant n+2$, the linear subspace $< e_1, \ldots, e_{n-j_2+1}, e_{j_1}, \ldots, e_n >$ is $\mathbb{C}^n$, and hence the first condition is allways satsified. Then

$$ev_3(X_{1,w_0}) = \{([0 : \cdots : 0 : x_{n-i_2+1} \ldots x_{i_1} : 0 \ldots 0], [y_1 \ldots y_n]) \in X\}.$$

Translation by a permutation h of the basis vectors yields

$$ev_3(X_{1,w_0}) = h \cdot \{([x_1 \ldots x_{i_1+i_2-n} : 0 : \cdots : 0], [y_1 \ldots y_n]) \in X\},$$

which is equal to $h \cdot X(w_{i_1+i_2-n,1})$ if $i_1 + i_2 - n \neq 1$, and is equal to $h \cdot X(w_{1,2})$ if $i_1 + i_2 = n+1$.

□

Suppose $j_1 + j_2 \leqslant n+2$ and $i_1 + i_2 \geqslant n+1$. According to Lemma 3.5.3.1 $ii)$ the general fiber of the map $ev_3 : X_{g_1,g_2}^{l_2} \to X$ induced by evaluating the third point is a rational variety. Furthermore, according to Lemma 3.5.3.1 $iii)$ its image $ev_3(\mathcal{X}_{1,w_0})$ is a translate of a Schubert variety by an element h in G, and hence has rational singularities according to Theorem 0.8.0.3. According to Part 3.5.2 (b), we have for $i_1 + i_2 > n + 1$:

$$\begin{aligned} \langle w_{i_1,j_1}, w_{i_2,j_2}, \mathcal{I}_{i_3,j_3} \rangle_{l_2} &= \chi\left([\mathcal{O}_{h \cdot X(w_{i_1+i_2-n,1})}] \cdot \mathcal{I}_{i_3,j_3}\right) \\ &= \chi\left([\mathcal{O}_{X(w_{i_1+i_2-n,1})}] \cdot \mathcal{I}_{i_3,j_3}\right). \end{aligned} \tag{3.12}$$

and if $i_1 + i_2 = n + 1$

$$\begin{aligned} \langle w_{i_1,j_1}, w_{i_2,j_2}, \mathcal{I}_{i_3,j_3} \rangle_{l_2} &= \chi\left([\mathcal{O}_{X(w_{1,2})}] \cdot \mathcal{I}_{i_3,j_3}\right) \\ &= \delta_{i_3,1}\delta_{j_3,2} \end{aligned} \tag{3.13}$$

Furthermore, according to Lemma 3.5.3.1 $iii)$ if $j_1 + j_2 \leqslant n + 2$ and $i_1 + i_2 < n + 1$, then

$$\langle w_{i_1,j_1}, w_{i_2,j_2}, \mathcal{I}_{i_3,j_3} \rangle_{l_2} = 0 \tag{3.14}$$

(b) WHEN $\mathbf{d} = l_1 + l_2$. An irreducible rational curve of degree $(d_1 l_1 + d_2 l_2)$ on X may be parametrized as

$$\mathbb{P}^1 \ni [u:v] \to ([P_1(u:v)], [P_2(u:v)]) \in X,$$

where P_1 and P_2 are homogeneous polynomials of degree d_1 and d_2 respectively. Furthermore, for an element (g_1, g_2, g_3) general in G^3, we denote by

$$\mathcal{M}^{\mathbf{d}}_{g_k, w_{i_k, j_k}} := ev_1^{-1}(g_1 \cdot X(w_{i_1,j_1})) \cap ev_2^{-1}(g_2 \cdot X(w_{i_2,j_2})) \cap ev_3^{-1}(g_3 \cdot X(w_{i_3,j_3}))$$

the Gromov-Witten variety associated with the i_k, j_k, where $ev_k : \overline{\mathcal{M}_{0,3}}(X, \mathbf{d}) \to X$ denotes evaluation at the k-th marked point. For a point (x, y) general in X^2 we denote by $C_{x,y}$ the line joining the points $[L_x]$ and $[L_y]$ in $\mathbb{P}^{n-1}$, and by $C'_{x,y}$ the line joining the points $[H_x]$ and $[H_y]$ in $\mathbb{P}^{n-1}$. Recall the varieties h_i are the classes of the Schubert varieties of codimension 1 of X. Note that the class $[h_1]$ (respectively $[h_2]$) in the Chow ring A_*X is the pullbacks of the Cartier divisor $c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))$ by the first (respectively second) projection $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$.

Lemma 3.5.3.2. *i) For a point (x, y) in X^2 one of the two following properties is verified:*

- *There exists a unique connected rational curve of class $(l_1 + l_2)$ joining the points x and y. This curve is the intersection of the variety $C_{x,y} \times C'_{x,y} \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ with X.*
- *There are infinitely many rational curves of class $(l_1 + l_2)$ joining the points x and y. These curves cover the intersection of the variety $C_{x,y} \times C'_{x,y}$ with X.*

ii) For a point (x, y) general in X^2, the rational curve in X of class $(l_1 + l_2)$ joining the points x and y is unique and irreducible.

iii) For (g_1, g_2, g_3) general in G^3, the Gromov-Witten variety $\mathcal{M}^{l_1+l_2}_{g_k, w_{i_k,j_k}}$ is birational to its image in X^3 by the evaluation morphism.

iv) Let (g_1, g_2) be a general element in G^2. Then

$$ev_3\left(ev_1^{-1}(g_1 \cdot h_i) \cap ev_2^{-1}(g_2 \cdot X(w_{i_2,j_2}))\right) = X.$$

Proof. i) We consider X as a divisor of class $[h_1] + [h_2]$ in $A^1(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$. Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ be points on X. If $x_1 = x_2$, we can construct infinitely many genus zero stable map of class $(l_1 + l_2)$ whose image contains the points p_i by glueing together the degree l_2 map $\mathbb{P}^1 \to X$ joining p_1 and p_2 with any map $\mathbb{P}^1 \to X$ of degree l_1 intersecting this curve at another point than the p_i. The same holds if $y_1 = y_2$. Now suppose $x_1 \neq x_2$ and $y_1 \neq y_2$. Denote by L_1 a line in $\mathbb{P}^{n-1}$ joining the points x_1 and x_2, and by L_2 a line in $\mathbb{P}^{n-1}$ joining the points y_1 and y_2. Note that any rational curve of class $(l_1 + l_2)$ in X joining the points p_i projects to L_k, hence any such rational curve is contained in the intersection $(L_1 \times L_2) \cap X$. Since the projective variety $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ is smooth, the intersection of X with $L_1 \times L_2$ has dimension at least 1. If the variety $(L_1 \times L_2) \cap X$ has dimension 2, then any parametrization $\mathbb{P}^1 \to L_1$ and $\mathbb{P}^1 \to L_2$ sending the points $[1:0]$ and $[0:1]$ in $\mathbb{P}^1$ onto the points p_i yield a degree $(l_1 + l_2)$ map $\mathbb{P}^1 \to X$ joining the points p_i. We thus obtain infinitely many irreducible rational curves of class $(l_1 + l_2)$ joining the points p_i. Indeed, since the classes l_i generate $A_1(X)$, the map $\mathbb{P}^1 \to X$ cannot be a multiple covering onto its image, which is then a curve of class $(l_1 + l_2)$. Now suppose the variety $(L_1 \times L_2) \cap X$ has dimension 1. Note that we have the following equality in the Chow ring $A_*(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$:

$$\begin{aligned}
[C'] &= [X] \cup [L_1 \times L_2] \\
&= ([h_1] + [h_2]) \cup [L_1] \boxtimes [L_2] \\
&= ([h_1] + [h_2]) \cup (l_1 + l_2) \\
&= l_1 + l_2
\end{aligned}$$

where we also denote by l_i the pushforward to $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ of the classes $l_i \in H_*(X, \mathbb{Z}) \simeq A_*(X)$. Hence C' is a curve of class $(l_1 + l_2)$ in X. We now only have to consider the case where C' has several irreducible components. Since C' is a curve of class $(l_1 + l_2)$

contained in $L_1 \times L_2$, C' can be described as $C' = (L_1 \times \{y\}) \cup (\{x\} \times L_2)$ where x and y are points in $\mathbb{P}^{n-1}$. Note that since $x_1 \neq x_2$ and $y_1 \neq y_2$, both points p_i cannot belong to the same irreducible component. For example suppose p_1 lies on C_1 and p_2 lies on C_2. Then $C' = (L_1 \times \{y_1\}) \cup (\{x_2\} \times L_2)$. Since L_1 joins x_1 and x_2, we obtain that (x_2, y_1) belongs to $L_1 \times \{y_1\}$. In the same way, (x_2, y_1) belongs to $\{x_2\} \times L_2$; hence the two irreducible components of C' come together at (x_2, y_1). We obtain a genus zero degree $(l_1 + l_2)$ stable map $\mathbb{P}^1 \cup \mathbb{P}^1 \simeq (L_1 \times \{y_1\}) \cup (\{x_2\} \times L_2) \to X$ whose image contains the points p_i.

ii) According to here above, the morphism of irreducible projective varieties

$$ev : \overline{\mathcal{M}_{0,2}}(X, l_1 + l_2) \to X^2$$

is dominant, and hence is surjective. Hence for (x, y) general in X^2, the general fiber $ev^{-1}(x, y)$ has dimension:

$$\begin{aligned} \dim ev^{-1}(x,y) &= \dim \overline{\mathcal{M}_{0,2}}(X, l_1 + l_2) - 2\dim X \\ &= \dim X + \int_{l_1+l_2} c_1(T_X) - 1 - 2\dim X \\ &= \int_{l_1+l_2} ((n-1)[h_1] + (n-1)[h_2]) - \dim X - 1 \\ &= 0. \end{aligned}$$

The general fiber $ev^{-1}(x, y)$ thus contains finitely many points; hence according to i) it contains only one point. The connected rational curve of class $(l_1 + l_2)$ joining two points of X in general position thus is unique. Finally, according to here above such a curve is irreducible.

iii) According to ii), there exists a dense open subset $U \subset X^2$ such that there exists a unique irreducible rational curve of class $(l_1 + l_2)$ joining any two points in U. Since the evaluation map $ev_1 \times ev_2 : \overline{\mathcal{M}_{0,3}}(X, l_1 + l_2) \to X^2$ is surjective, according to Chapter 1 Lemma 1.2.3.1 for (g_1, g_2) general in G^2 the variety $ev_1^{-1}(g_1 \cdot X(w_{i_1,j_1})) \cap ev_2^{-1}(g_2 \cdot X(w_{i_2,j_2}))$ has a dense intersection with the dense open subset $(ev_1 \times ev_2)^{-1}(U)$. Now denote by V the dense open subset of $U \times X$ containing distinct points x, y and z in $U \times X$. Note that the restriction of the evaluation morphism

$$ev_{g_1,g_2} : \big(ev_1^{-1}(g_1 \cdot X(w_{i_1,j_1})) \cap ev_2^{-1}(g_2 \cdot X(w_{i_2,j_2}))\big) \cap ev^{-1}(V) \to X^3$$

is an injective map. Indeed, if we specify two points x and y in U, by definition there is a unique irreducible curve C of class $(l_1 + l_2)$ joining x and y. Consider a third point z lying on C, distinct from x and y. There exists a unique parametrization $\mathbb{P}^1 \to C$ sending the points $[1 : 0]$, $[0 : 1]$ and $[1 : 1]$ onto the points x, y and z; hence the data of three distinct points x, y and z in $U \times X$ defines a unique point in $\overline{\mathcal{M}_{0,3}}(X, l_1 + l_2)$. The restriction of the evaluation morphism to $\mathcal{M}^{l_1+l_2}_{g_k, w_{i_k,j_k}} \cap ev^{-1}(V) = {ev_{g_1,g_2}}^{-1}(g_3 \cdot X(w_{i_3,j_3}))$ thus is an injective morphism of quasi-projective complex varieties, hence is birational onto its image. Finally, note that for (g_1, g_2, g_3) general in G^3 according to Chapter 1 Lemma 1.2.3.1 the variety $\mathcal{M}^{l_1+l_2}_{g_k, w_{i_k,j_k}}$ has a dense open intersection with $ev^{-1}(V)$, and hence is birational onto its image by the evaluation morphism.

iv) Consider points p_1 and p_2 respectively in $g_2 \cdot X(w_{i_2,j_2})$ and X. According to i) there exists a connected rational curve C in X of class $(l_1 + l_2)$ joining p_1 and p_2. Denote by $i : X \hookrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ the natural embedding. We have the following equality in $A_*(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$:

$$[h_i] \cup [C] = [h_i] \cup (l_1 + l_2) = [pt],$$

hence the intersection between h_i and C is non empty.

□

According to Lemma 3.5.3.2 *iii*) for (g_1, g_2) general in G^2, the Gromov-Witten variety $\mathcal{W}_{g_1,g_2} := ev_1^{-1}(g_1 \cdot h_1) \cap ev_2^{-1}(g_2 X(w_{i_2,j_2}))$ is birational onto its image in X^3 via the evaluation map. Then the map $ev_3 : \mathcal{W}_{g_1,g_2} \to X$ induced by evaluating the third point is a degree 1 map of projective complex varieties. Furthermore, according to Lemma 3.5.3.2 *iv*) its image $ev_3(\mathcal{W}_{w_0})$ is the smooth variety X. According to Part 3.5.2 (b), we have:

$$\begin{aligned}\langle w_{i_1,j_1}, w_{i_2,j_2}, \mathcal{I}_{i_3,j_3}\rangle_{l_1+l_2} &= \chi\left([\mathcal{O}_X] \cdot \mathcal{I}_{i_3,j_3}\right) \\ &= \chi\left([\mathcal{O}_{X(w_{n,1})}] \cdot \mathcal{I}_{i_3,j_3}\right) \\ &= \delta_{n,i_3}\delta_{1,j_3}.\end{aligned}$$

(c) When $\mathbf{d} = d_1 l_1 + d_2 l_2$, where $0 \leqslant d_1 \leqslant d_2$ and $d_2 \geqslant 2$. Recall X is the variety parametrizing lines included in hyperplanes of $\mathbb{C}^n$. A point in X corresponds to the inclusion of a line in a hyperplane of $\mathbb{C}^n$. Denote by $\pi : X \to \mathbb{P}^{n-1}$ the projection induced by forgetting the hyperplane. Let $\mathbf{d} = (d_1, d_2) := d_1 l_1 + d_2 l_2 \in E(X)$. Note that $\pi_*(d_1, d_2) = d_1 \in E(\mathbb{P}^{n-1})$. The forgetful morphism π induces a morphism $\Pi : \overline{\mathcal{M}_{0,3}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1}, d_1)$. Together with the evaluation morphisms ev_i, this induces a morphism:

$$\Pi \times ev : \overline{\mathcal{M}_{0,3}}(X, \mathbf{d}) \to \overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1}, d_1) \times_{(\mathbb{P}^{n-1})^3} X^3.$$

We begin by studying when $\Pi \times ev$ is surjective, before computing correlators of X.

Proposition 3.5.3.1. *i) If $\mathbf{d} = (0, d_2)$, where $d_2 \geqslant 2$, then $\Pi \times ev$ is surjective;*

ii) Suppose $\mathbf{d} = (1, d_2)$, where $d_2 \geqslant 2$. The morphism $\Pi \times ev$ is surjective;

iii) Suppose $\mathbf{d} = (d_1, d_2)$, where $d_1 \geqslant 2$ and $d_2 \geqslant d_1$. Then $\Pi \times ev$ is surjective.

iv) Suppose $\mathbf{d} = (d_1, d_2)$, where $d_1 \geqslant 2$ and $d_2 \geqslant 2$. Then $ev^{\mathbf{d}} : \overline{\mathcal{M}_{0,3}}(X, \mathbf{d}) \to X^3$ is surjective.

v) Suppose $\mathbf{d} = (1, d_2)$, where $d_2 \geqslant 2$. Then $ev_3(X_{1,w_0}^{l_1+d_2 l_2}) = h \cdot X(w_{i_1+i_2,1})$, where h is an element in G.

Lemma 3.5.3.3. *Let $f : Y \to Z$ be a morphism of projective irreducible varieties. Let Z_0 be a subvariety of Z of codimension 1 in Z. Suppose the image of f contains Z_0. Furthermore, suppose the inverse image by f of Z_0 is a strict subvariety of Y, i.e. suppose $f^{-1}(Z_0) \subsetneq Y$. Then f is surjective.*

Proof. By contradiction, suppose f is not surjective. Then $f(Y)$ has dimension smaller than $\dim f(f^{-1}(Z_0))$. Indeed, we have $\dim f(f^{-1}(Z_0)) = \dim Z_0 = \dim Z - 1$, hence $\dim f(f^{-1}(Z_0)) \geqslant \dim f(Y)$ since $f(Y)$ is a strict closed subvariety of the irreducible variety Z. Furthermore, note that the irreducible projective variety $f(Y)$ contains the projective variety $f(f^{-1}(Z_0))$. Hence $f(f^{-1}(Z_0))$ is a projective subvariety of the irreducible variety $f(Y)$ of dimension larger than $f(Y)$; hence $f(Y) = f(f^{-1}(Z_0))$ and $f(Y)$ is contained in Z_0. Thus $Y = f^{-1}(Z_0)$, which contradicts the fact that $f^{-1}(Z_0)$ is a strict closed subvariety of Y. □

Proof of Proposition 3.5.3.1. We consider here X as a subvariety of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, where X is cut out from $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ by the relation "line included in hyperplane" in the following way. Consider a point (x, y) in $P^{n-1} \times \mathbb{P}^{n-1}$. By the functorial definition of $\mathbb{P}^{n-1}$, x defines a line $L_x \subset \mathbb{C}^n$ and, if we identify $\mathbb{P}^{n-1}$ with its dual, y defines an hyperplane $H_y \subset \mathbb{C}^n$. Then (x, y) lies in X iff $L_x \subset H_y$. We call *curve* a projective variety of dimension 1. We will denote by pi_1 (respectively π_2) the first (respectively second) projection $X \to \mathbb{P}^{n-1}$.

i) Consider a point p on $\mathbb{P}^{n-1}$; denote by $L_p \subset \mathbb{C}^n$ the line defined by p. Let (p, y_i) be three points on X, where the point y_i in $\mathbb{P}^{n-1}$ correspond to a hyperplane H_i in $\mathbb{C}^n$ containing the line L_p. Our goal here is to construct a genus zero stable map $C \to X$ of degree $d_2 l_2$ whose three marked points are the points (p, y_i). We first construct a curve of degree l_2 joining the points (p, y_1) and (p, y_2), before constructing a genus zero stable map of degree $2l_2$, and finally a genus zero stable map of degree $d_2 l_2$ joining all three points (p, y_i).

Suppose the points y_1 and y_2 are distinct. Denote by C a line in $\mathbb{P}^{n-1}$ joining the points y_1 and y_2; C can be described as the subset of points in $\mathbb{P}^{n-1}$ associated with hyperplanes $uH_1 + vH_2$, where $[u : v]$ runs over $\mathbb{P}^1$. Since the line L_p is included in H_1 and H_2, for all $[u : v]$ in $\mathbb{P}^1$ it is also included in $uH_1 + vH_2$. Hence the curve $C_0 := \{p\} \times C$ in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ induced by C is a curve of class l_2 lying in X joining the points (p, y_1) and (p, y_2). In the same way, if the points y_2 and y_3 are distinct, we may construct a rational curve $C' \subset X$ of class l_2 joining the points (p, y_2) and (p, y_3). Consider parametrizations $f_i : \mathbb{P}^1 \to C_i$, where we fix points q_1, q_2, q_3 and q_4 on $\mathbb{P}^1$ such that f_1 sends q_1 and q_2 to (p, y_1) and (p, y_2), and f_2 sends q_3 and q_4 to (p, y_2) and (p, y_3). Consider the tree of $\mathbb{P}^1$'s $\mathbb{P}^1 \cup \mathbb{P}^1$ with 3 marked points obtained by glueing $(\mathbb{P}^1, \{q_1, q_2\})$ with $(\mathbb{P}^1, \{q_3, q_4\})$ by identifying q_2 and q_3. The morphisms f_i induce a genus zero, degree $2l_2$ stable map $(f : \mathbb{P}^1 \cup \mathbb{P}^1 \to X, \{q_1, q_2, q_4\})$ joining all three points (p, y_i). Finally, notice that if two points y_1 and y_2 come together, a degree $2l_2$ stable map $\mathbb{P}^1 \cup \mathbb{P}^1 \to X$ collapsing an irreducible component $\mathbb{P}^1$ to the point (p, y_1) joins all three points (p, y_i).

Now suppose $d_2 \geqslant 2$. Consider the degree $2l_2$ stable map $(f : \mathbb{P}^1 \cup \mathbb{P}^1 \to X, \{q_1, q_2, q_4\})$ described here above. Now add an irreducible component $\mathbb{P}^1$ to $\mathbb{P}^1 \cup \mathbb{P}^1$, which we glue to $\mathbb{P}^1 \cup \mathbb{P}^1$ at a non marked point, and send this irreducible component to X via a degree $(d_2 - 2)l_2$ map. We obtain a degree $d_2 l_2$ stable map $(\mathbb{P}^1 \cup \mathbb{P}^1 \cup \mathbb{P}^1 \to X, \{q_1, q_2, q_4\})$ joining all three points (p, y_i).

ii) Let L be a line in $\mathbb{P}^{n-1}$, let p_1, p_2, p_3 be three points on L. Denote by (p_i, y_i) three points on X satisfying $\pi(p_i, y_i) = p_i$, i.e. the point y_i in $\mathbb{P}^{n-1}$ corresponds to a hyperplane in $\mathbb{C}^n$ containing the line L_i associated with the point p_i. Our goal here is to prove that there exists a genus zero stable map of degree $l_1 + d_2 l_2$ projecting to L and whose image contains the points (p_i, y_i). Consider a genus zero stable map $(\mathbb{P}^1 \to C \hookrightarrow \mathbb{P}^{n-1}, \{n_1, n_2\})$ of class $d_2 l_2$ in $\mathbb{P}^{n-1}$ sending its marked points n_i onto the points y_1 and y_2. If $d_2 \geqslant 2$, we also consider the case where $\mathbb{P}^1 \to X$ is a genus zero stable map of class $d_2 l_2$ in $\mathbb{P}^{n-1}$ sending its marked points onto the three points y_i. Since $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ is a smooth variety and the intersection $(L \times C) \cap X$ is non empty, $(L \times C) \cap X$ is of dimension at least 1. If it has dimension 2, then $L \times C$ is contained in X. Then identify the line L with $\mathbb{P}^1$; by considering the product of the closed immersion $L \hookrightarrow \mathbb{P}^{n-1}$ with the map $\mathbb{P}^1 \to \mathbb{P}^{n-1}$, we obtain a genus zero stable map $\mathbb{P}^1 \to X$ of degree $(l_1 + d_2 l_2)$ whose image contains all points (p_i, y_i).

Now consider the case where the dimension of the intersection $(L \times C) \cap X$ is 1. Denote by h_1 (respectively h_2) the pullbacks of the Cartier divisor $c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))$ by the first (respectively second) projection. Note that we have the following equality in $A_*(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$: $[X] = h_1 + h_2$. Furthermore, denote by δ the degree of C; note that $\mathbb{P}^1 \to C$ is a degree d_2/δ covering. If the variety $(L \times C) \cap X$ has dimension 1, it is a curve C' in X satisfying the following equality in the Chow ring $A_*(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$:

$$\begin{aligned}[C'] &= [L \times C] \cup [X] \\ &= (l_1 \boxtimes \delta l_2) \cup (h_1 + h_2) \\ &= l_1 + \delta l_2,\end{aligned}$$

where we also denote by l_i the pushforward to $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ of the classes $l_i \in H_*(X, \mathbb{Z}) \simeq A_*(X)$. Hence C' is a curve of class $(l_1 + \delta l_2)$ in X. Note that, since the points (p_i, y_i) belong to both $L \times C$ and X, they lie on C'. Furthermore, the image of C' by the first projection $\pi_1 : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is L. Indeed, C' is a curve of class $(l_1 + \delta l_2)$ contained in $L \times C$; hence its image by the projection π_1 is a curve of class $(\pi_1)_*(l_1 + \delta l_2) = 1 \in A_*(\mathbb{P}^{n-1})$ contained in the irreducible degree one curve $L \subset \mathbb{P}^{n-1}$. Hence C' surjects onto L.

We now only have to prove that C' can be descibed as the image of a stable map. Note that the curve C' is either the union of two curves C_i satisfying $(\pi_2)_*[C_1] + (\pi_2)_*[C_1] = \delta = \pi_2(C')$, or is an irreducible curve of class $(l_1 + \delta l_2)$. First consider the case where C' is the union of two curves C_i, where the first curve C_1 surjects onto L. Then C_2 is obtained as $\{Q_2\} \times C$, where Q_2 is a point on $\mathbb{P}^{n-1}$. Indeed, C_2 is a curve satisfying $(\pi_2)_*[C_2] = \pi_2(C_2)$ projecting to the irreducible degree δ curve C, hence the restriction

$\pi_{2|C_2}$ of the projection map to C_2 is a degree 1 map onto C, hence is an isomorphism. In particular C_2 is a curve of class δl_2. In the same way, C_1 is a curve of class l_1 projecting to L, hence is obtained as $L \times \{Q_1\}$, where Q_1 is a point in $\mathbb{P}^{n-1}$. Then either the points (p_i, y_i) all lie on the irreducible rational curve C_2-which yields a rational curve of class δl_2 containg all points (p_i, y_i)- or C_1 contains one or two point (p_i, y_i). We will consider here the case where C_1 contains one point (p_i, y_i), the same proof holds when C_1 contains two points. If $C_1 = L \times \{Q_1\}$ contains a point (p_i, yi), for example the point (p_1, y_1) then $Q_1 = y_1$ and both (p_1, y_1) and (p_2, y_1) lie on C_1. We obtain a genus zero degree l_1 stable map $f_1 := (\mathbb{P}^1 \simeq L \hookrightarrow X, \{(p_1, y_1), (p_2, y_1)\})$. Denote by $f_2 := (C_2' \to X, \{c_1, c_2, c_3\})$ a genus zero stable map of class $d_2 l_2$ sending its marked points c_i onto the points (p_2, y_1) and (p_2, y_2) and (p_2, y_3); such a map exists according to i). Now glue $\mathbb{P}^1$ and C_2' together by identifying the points (p_2, y_1) and c_1; we obtain a genus zero stable map $\mathbb{P}^1 \cup C_2' \to X$ of class $(l_1 + \delta l_2)$ joining all points (p_i, y_i).

Finally consider the case where C' is irreducible. Denote by $\pi_1 : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ the first projection. Since C' is irreducible and $(\pi_1)_*[C'] = 1 = [\pi_1(C')]$ in $A_*(\mathbb{P}^{n-1})$, the projection $(\pi_1)_{|C'}$ of the curve C' to $\mathbb{P}^{n-1}$ is a degree 1 map; hence C' is birational to its image L in $\mathbb{P}^{n-1}$, hence C' is an irreducible rational curve.

iii) We proceed by iteration on (d_1, d_2). Let us name $D(d_1 - 1, 1)$ the boundary locus of $\overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1}, d_1)$ which is naturally isomorphic to $\overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1}, d_1 - 1) \times_{\mathbb{P}^{n-1}} \overline{\mathcal{M}_{0,2}}(\mathbb{P}^{n-1}, 1)$, and by $D(\mathbf{d} - (l_1 + l_2), l_1 + l_2)$ the boundary locus of $\overline{\mathcal{M}_{0,3}}(X, \mathbf{d} - (l_1 + l_2))$ ismorphic to $\overline{\mathcal{M}_{0,3}}(X, \mathbf{d} - (l_1 + l_2)) \times_X \overline{\mathcal{M}_{0,2}}(X, l_1 + l_2)$. Note that, since $D(\mathbf{d} - (l_1 + l_2), l_1 + l_2)$ is a strict closed subvariety of the irredudible projective variety $\overline{\mathcal{M}_{0,3}}(X, \mathbf{d})$, according to Lemma 3.5.3.3 it is enough to prove that $D(\mathbf{d} - (l_1 + l_2), l_1 + l_2)$ surjects into $D(d_1 - 1, 1)$. Consider an element p in $D(d_1 - 1, 1)$. Denote by $(f_p : C_p \to \mathbb{P}^{n-1}, \{p_1, p_2, p_3\})$ and $(h_p : \mathbb{P}^1 \to \mathbb{P}^{n-1}, \{q_1, q_2\})$ the genus zero stable maps associated with p; f_p is a degree $(d_1 - 1)$ map and h_p is a degree 1 map defining a line $h_p(\mathbb{P}^1)$ intersecting the curve $f_p(C)$ at $h_p(q_1) = f_p(p_3) = x_3$. The induction hypothesis, or if $d_1 = 2$ Lemma 3.5.3.2, ensure that there is a genus zero degree $(\mathbf{d} - (l_1 + l_2))$ stable map $(f_p' : C_p' \to X, \{p_1, p_2, p_3\})$ such that

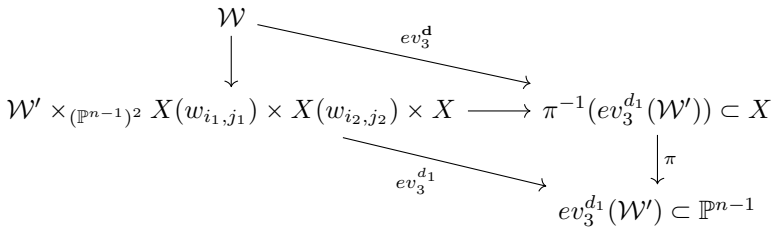
- The image by Π of f_p' is f_p;
- The points $f_p'(p_1)$ and $f_p'(p_2)$ in X are projected to the points $f_p(p_i)$ in $\mathbb{P}^{n-1}$ by the forgetful map $X \to \mathbb{P}^{n-1}$.

If $d_1 > 2$, this is a direct consequence of the induction hypothesis. If $d_1 = 2$ note that according to Lemma 3.5.3.2 there exists a genus zero degree $(l_1 + l_2)$ stable map projecting to f_p. We can then choose marked points p_i projecting to the points $f_p(p_i)$. Denote by p_3' a non singular point in $f_p'^{-1}(p_3)$. Denote by (x_3, y_3) a point in X such that y_3 is contained in $f_p'(f_p^{-1}(p_3))$. According to Lemma 3.5.3.2 there exists a genus zero degree $(l_1 + l_2)$ stable map $(h_q' : C_q \to X, \{q_1', q_2'\})$ sending the point q_1' to the point (x_3, y_3) and the point q_2' to a point of the form $(h_p(q_2), y')$ in X. Now glue the genus zero curves C_p' and C_q together by identifying the point p_3' on C_p' with the point q_1' on C_q. We obtain a genus zero degree $\mathbf{d}$ stable map $(C_p' \cup C_q \to X, \{p_1, p_2, q_1', q_2'\})$ whose projection by Π is p.

iv) Let p_1, p_2, p_3 be three points on X. According to ii), there exist genus zero degree $(l_1 + l_2)$ stable maps $f_1 := (C_1 \to X, \{x_1, x_2\})$ and $f_2 := (C_2 \to X, \{y_1, y_2\})$ sending the marked points x_i onto the points p_1 and p_2 and the marked points y_i onto the points p_2 and p_3. Glue C_1 and C_2 together by identifying x_2 and y_1, and furthermore glue C_2 with $\mathbb{P}^1$ at a non special point q of C_2. Consider a degree $\mathbf{d} - 2(l_1 + l_2)$ map $\mathbb{P}^1 \to X$ sending q onto $f_2(q)$. We obtain a genus zero degree $\mathbf{d}$ stable map $(C_1 \cup C_2 \cup \mathbb{P}^1 \to X, \{x_1, x_2, y_2\})$ sending its marked points onto the points p_i.

v) Let $\mathcal{W} := X_{1,w_0}^{l_1 + d_2 l_2}$ be the Gromov-Witten variety of X associated with the i_k, j_k, and $\mathcal{W}' := \overline{\mathcal{M}_{0,3}}(\mathbb{P}^{n-1}, 1) \times_{(\mathbb{P}^{n-1})^2} (L_{i_1} \times w_0 L_{i_2})$ be the Gromov-Witten variety of $\mathbb{P}^{n-1}$ associated with the i_k. We consider the map $\mathcal{W}' \times_{(\mathbb{P}^{n-1})^3} X^3 \to X$ induced by the third projection

$X^3 \to X$. We obtain the following commutative diagram.



Since according to ii) the map $\Pi \times ev$ is surjective, according to Chapter 1 the restriction $\mathcal{W} \to \mathcal{W}' \times_{(\mathbb{P}^{n-1})^2} X(w_{i_1,j_1}) \times X(w_{i_2,j_2}) \times X$ of $\Pi \times ev$ to $\mathcal{W}$ is also surjective. Furthermore note that since $\pi : X \to \mathbb{P}^{n-1}$ is surjective, the projection $\mathcal{W}' \times_{(\mathbb{P}^{n-1})^3} X(w_{i_1,j_1}) \times X(w_{i_2,j_2}) \to \pi^{-1}(ev_3^{d_1}(\mathcal{W}'))$ is also surjective. Hence by composition $ev_3^{\mathbf{d}} : \mathcal{W} \to \pi^{-1}(ev_3^{d_1}(\mathcal{W}')) \cap (X(w_{i_3,j_3}))$ is also surjective. Note that $ev_3^{d_1}(\mathcal{W}')$ is the locus of points p in $\mathbb{P}^{n-1}$ such that

- p lies on a line $L \subset \mathbb{P}^{n-1}$;
- a point in $L_{i_1} = \{[x_1, \dots, x_{i_1} : 0 : \cdots : 0]\}$ lies on L;
- a point in $w_0 L_{i_2} = \{[0 : \cdots : 0 : x_{n-i_2+1} : \cdots : x_n]\}$ lies on L.

p satisfies these three conditions iff p lies in $\{[x_1 \dots x_{i_1} : 0 \dots 0 : x_{n-i_2+1} \dots x_n]\} = h \cdot L_{\min(n,i_1+i_2)}$, where h is a permutation in $\mathfrak{S}_n$. We obtain

$$\begin{aligned} ev_3(\mathcal{W}) &= \pi^{-1}(h \cdot L_{\min(n,i_1+i_2)}) \\ &= h \cdot \pi^{-1}(L_{\min(n,i_1+i_2)}) \\ &= h \cdot \{([x_1 : \cdots : x_{\min(n,i_1+i_2)} : 0 \cdots : 0], [y_1 : \cdots : y_n]) \in X\} \\ &= h \cdot X(w_{\min(n,i_1+i_2),1}). \end{aligned}$$

□

Suppose $d_2 \geqslant 2$. According to Proposition 3.5.3.1 v) the projected Gromov-Witten variety $ev_3(X_{1,w_0}^{l_1+d_2 l_2})$ is the translate of a Schubert variety and hence according to Theorem 0.8.0.3 has rational singularities. Furthermore there exists an element h in G such that $ev_3(X_{1,w_0}^{l_1+d_2 l_2}) = h \cdot X(w_{i_1+i_2,1})$. Part 3.5.2 (b) then implies the following identity.

$$\begin{aligned} \forall\ d_2 \geqslant 2,\ \langle w_{i_1,j_1}, w_{i_2,j_2}, \mathcal{I}_{i_3,j_3} \rangle_{l_1+d_2 l_2} &= \chi\left([\mathcal{O}_{ev_3^{l_1+d_2 l_2}(X_{1,w_0})}] \cdot \mathcal{I}_{i_3,j_3}\right) \\ &= \chi\left([\mathcal{O}_{h \cdot X(w_{\min(n,i_1+i_2),1})}] \cdot \mathcal{I}_{i_3,j_3}\right) \\ &= \chi\left(\mathcal{O}_{\min(n,i_1+i_2),1} \cdot \mathcal{I}_{i_3,j_3}\right). \end{aligned} \tag{3.15}$$

Consider the partial order $\geqslant$ on $E(X)$ defined by:

$$\mathbf{d} \geqslant \mathbf{d}' \text{ iff } \mathbf{d} - \mathbf{d}' \in E(X).$$

Note that classes $\mathbf{d} = (d_1, d_2)$ and $\mathbf{d}' = (d_1', d_2')$ in $E(X)$ satisfy $\mathbf{d} \geqslant \mathbf{d}'$ iff $d_1 \geqslant d_1'$ and $d_2 \geqslant d_2'$. According to Proposition 3.5.3.1 *iv*) the image of $ev_1^{-1}(X(w_{i_1,j_1}) \cap ev_2^{-1}(w_0 X(w_{i_2,j_2}))$ by the morphism $ev_3^{\mathbf{d}}$ evaluating the third marked point is X for $\mathbf{d} \geqslant 2(l_1 + l_2)$. Since X is smooth, Proposition 3.5.3.1 *iii*) and Part 3.5.2 (b) yield the following implication.

$$\mathbf{d} \geqslant (2l_1 + 2l_2) \ \Rightarrow \quad \langle w_{i_1,j_1}, w_{i_2,j_2}, \mathcal{I}_{i_3,j_3} \rangle_{\mathbf{d}} = \chi\left([\mathcal{O}_X] \cdot \mathcal{I}_{i_3,j_3}\right). \tag{3.16}$$

(d) When $\mathbf{d} = d_1 l_1 + d_2 l_2$, where $0 \leqslant d_2 \leqslant d_1$. Then according to Part 3.5.2 (c) we can dedude degree $\mathbf{d}$ correlators from the preceding points using the symmetry between corelators of degree $d_1 l_1 + d_2 l_2$ and correlators of degree $d_2 l_1 + d_1 l_2$.

3.6 Degree of $\mathcal{O}_{h_i} \star \mathcal{O}_v$.

3.6.1 Definitions. Consider a degree $\mathbf{d}$ in the semi-group $E(X) \simeq \mathbb{N}^2$ of effective classes of curves in $H_2(X,\mathbb{Z}) \simeq \mathbb{Z}^2$. For $r \geqslant 0$, denote by $\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})$ the variety parametrizing degree $\mathbf{d}$ genus zero stable maps to X. The *evaluation morphism* $ev_i : \overline{\mathcal{M}_{0,r}}(X,\mathbf{d}) \to X$ assigns to a map the image of its i-th marked point. For X a homogeneous variety, the evaluation morphism is flat. Recall for u in W^P, we denote by $[\mathcal{O}_u] := [\mathcal{O}_{X(u)}]$ the associated Schubert class in the Grothendieck group $K_\circ(X)$ of coherent sheaves. Since X is non singular, an element $\mathcal{O}_u$ in $K_\circ(X)$ is assimilated with an element in the Grothendieck group $K^\circ(X) \simeq K_\circ(X)$ of locally free sheaves on X. Let $u_1, \ldots, u_r$ be elements in the Weyl group W; the *correlator* $\langle u_1, \ldots, u_r \rangle_{\mathbf{d}}$ associated with the elements u_i is defined by:

$$\langle u_1, \ldots, u_r \rangle_{\mathbf{d}} := \chi\left(ev_1^*(\mathcal{O}_{u_1}) \cdot \ \ldots \ \cdot ev_r^*(\mathcal{O}_{u_r}) \cdot [\mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,d)}] \right).$$

Note that for X a homogeneous variety, this correlator is equal to the sheaf Euler characteristic of the Gromov-Witten variety associated with the u_i (cf. for example [BM11], Part 4.1); i.e. for $(g_1, \ldots, g_r)$ general in G^r, we have:

$$\langle u_1, \ldots, u_r \rangle_{\mathbf{d}} := \chi\left(\mathcal{O}_{ev_1^{-1}(g_1 \cdot X(u_1)) \cap \cdots \cap ev_r^{-1}(g_r \cdot X(u_r))} \right). \tag{3.17}$$

We consider the basis $(\mathcal{O}_u)_{u \in W^P}$ of $K(X) := K^\circ(X) \simeq K_\circ(X)$. Let $t = \sum_{\alpha \in W^P} t^\alpha \mathcal{O}_\alpha$. Classes $l_1 := [X(w_{2,n})]$ and $l_2 := [X(w_{1,n-1})]$ of one dimensional Schubert varieties form a nef basis of $H_2(X,\mathbb{Z})$. Denote by Q_1 and Q_2 the Novikov variables associated with l_1 and l_2. For a degree $\mathbf{d} = (d_1, d_2) := d_1 l_1 + d_2 l_2$ in $E(X)$, we write $Q^{\mathbf{d}} := Q_1^{d_1} Q_2^{d_2}$. Introduce the Gromov-Witten potential

$$\mathcal{G}(t) := \sum_{\mathbf{d} \in E(X)} \sum_{r=0}^{+\infty} \frac{Q^{\mathbf{d}}}{r!} \langle \underbrace{t, \ldots, t}_{r} \rangle_{\mathbf{d}},$$

We define a *deformed metric* by $(\mathcal{O}_\alpha, \mathcal{O}_\beta) = \delta^\alpha \delta^\beta \mathcal{G}(0)$, where $\delta^\alpha = \delta / \delta t^\alpha$. The *genus zero big quantum product* $\mathcal{O}_\alpha \bullet \mathcal{O}_\beta$ of two basis elements of $K^\circ(X)$ is defined by:

$$(\mathcal{O}_\alpha \bullet \mathcal{O}_\beta, \mathcal{O}_\gamma) := \delta^\alpha \delta^\beta \delta^\gamma \mathcal{G}(t).$$

It is then extended by bilinearity over $K^\circ(X) \otimes \mathbb{Q}[[Q,t]]$ and defines the *big quantum K-ring* $(K^\circ(X) \otimes \mathbb{Q}[[Q,t]], \bullet)$. This ring is associative and commutative [Lee04], and we recover the ring $K^\circ(X)$ by taking the classical limit $Q \to 0$. The restriction of the big quantum K-ring to $t = 0$ defines the *small* quantum K-ring $(QK_s(X), \star)$. Note that for X a homogeneous variety, the product of two basis elements in $QK_s(X)$ is a polynomial in Q [ACT18]. The pairing defined by

$$\mathcal{G}(\mathcal{O}_\alpha, \mathcal{O}_\beta) = \mathcal{G}_{\alpha,\beta} = (\mathcal{O}_\alpha, \mathcal{O}_\beta)$$

is extended to a $\mathbb{Q}[[Q,t]]$-bilinear pairing $\mathcal{G}$ on $K(X) \otimes \mathbb{Q}[[Q,t]]$. Denote by $\mathcal{G}^{\alpha,\beta}$ the inverse matrix. By definition, we have:

$$\mathcal{O}_u \star \mathcal{O}_v = \sum_{\gamma, w \in W^P} \delta^u \delta^v \delta^\gamma \mathcal{G}(0)\, \mathcal{G}^{\gamma,w} \mathcal{O}_w.$$

3.6.2 A combinatorial lemma. We prove here a purely combinatorial lemma that will imply that the product of two elements in $K(X)$ is a polynomial in the Novikov variable Q.

Lemma 3.6.2.1. *Let n, m and N be positive integers.*

$$\sum_{\substack{(k,p) \in [\![0;n]\!] \times [\![0;m]\!], \\ k+p=N}} \binom{n}{k} \binom{m}{p} = \binom{n+m}{N}.$$

Hence for $0 \leqslant N \leqslant n, m$ *:*

$$\sum_{k=0}^{N} \binom{n}{k}\binom{m}{N-k} = \binom{m+n}{N},$$

and for $N \geqslant n$, $N \leqslant m$ *:*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{m}{N-k} = \binom{m+n}{N}.$$

Proof. $(x+y)^n(x+y)^m = (x+y)^{n+m}$. □

Lemma 3.6.2.2. *Let* n *and* k *be positive integers, with* $n \geqslant 2$.

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k k = 0.$$

Proof. Let $f : x \to (x-1)^n = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}x^k$. Then $f'(1) = 0$. □

Lemma 3.6.2.3. *Let* $(d, \delta) \in \mathbb{N}^2$*, let* $(d_0, \delta_0) \in \mathbb{N}^2$ *such that* $\delta_0 < \delta - 1$ *or* $d_0 < d - 1$*. Then*

$$\sum_{(\mathbf{d},\boldsymbol{\delta})=((d_0,\delta_0);\cdots;(d_r,\delta_r))} (-1)^r = 0,$$

where we consider the sum over all integers r *and all possible decompositions of a pair* (d, δ) *in* $\mathbb{N}^2$ *on a* $(r+1)$*-tuple* $((d_0, \delta_0); \cdots ; (d_r, \delta_r))$ *satisfying* $\sum_{i=0}^{r} d_i = d$*,* $\sum_{i=0}^{r} \delta_i = \delta$*, and* $d_i > 0$ *or* $\delta_i > 0$ *for all* $i > 0$.

Proof. Let $0 \leqslant d_0 < d$ and $0 \leqslant \delta_0 < \delta - 1$. We set $S = \sum_{(\mathbf{d},\boldsymbol{\delta})=((d_0,\delta_0);\cdots;(d_r,\delta_r))}(-1)^r$. Let r_1 be the elements d_i other than d_0 that satisfy $d_i \neq 0$. Note that $1 \leqslant r_1 \leqslant d - d_0$. For a given r_1 the number r varies from r_1 to $\delta - \delta_0 + r_1$. Denote by r_2 the number of elements $\delta_i \neq 0$ where $i \neq 0$; for given r_1 and r, r_2 varies from $\max(1, r - r_1)$ to $\min(\delta - \delta_0, r)$. For given r_1, r and r_2, we have $\binom{r}{r_1}$ possible choice of an i satisfying $d_i \neq 0$, $\binom{r_1}{r_2-r+r_1}$ possible choice of an i satisfying $\delta_i \neq 0$, $\binom{d-d_0-1}{r_1-1}$ possible choice of the d_i, and $\binom{\delta-\delta_0-1}{r_2-1}$ choice for the integers δ_i. Hence:

$$S = \sum_{r_1=1}^{d-d_0} \sum_{r=r_1}^{\delta-\delta_0+r_1} \sum_{r_2=\max(1,r-r_1)}^{\min(\delta-\delta_0,r)} (-1)^r \binom{r}{r_1}\binom{d-d_0-1}{r_1-1}\binom{r_1}{r_2-r+r_1}\binom{\delta-\delta_0-1}{r_2-1}.$$

Let us now consider $S(r_1, r) = \sum_{r_2=\max(1,r-r_1)}^{\min(\delta-\delta_0,r)} (-1)^r \binom{r}{r_1}\binom{d-d_0-1}{r_1-1}\binom{r_1}{r_2-r+r_1}\binom{\delta-\delta_0-1}{r_2-1}$; alors

$$S = \sum_{r_1=1}^{d-d_0} \sum_{r=r_1}^{\delta-\delta_0+r_1} S(r_1, r).$$

If $r \geqslant \delta - \delta_0$**.** Then according to Lemma 3.6.2.1, if we set $k = r_2 - 1$, $n = \delta - \delta_0 - 1$, $m = r$, $N = r - 1$, we obtain:

$$S(r, r) = (-1)^r \binom{\delta - \delta_0 - 1 + r}{r-1}\binom{d-d_0-1}{r_1-1}.$$

Let $r > r_1$. Set $k = r_2 - r + r_1$, $n = r_1$, $m = \delta - \delta_0 - 1$, $N = \delta - \delta_0 - r + r_1$. Note that $\delta - \delta_0 - r + r_1 \leqslant r_1$ and $\delta - \delta_0 - r + r_1 \leqslant \delta - \delta_0 - 1$. Then according to Lemma 3.6.2.1 we have

$$S(r_1, r) = (-1)^r \binom{r}{r_1}\binom{\delta - \delta_0 - 1 + r_1}{\delta - \delta_0 - r + r_1}\binom{d-d_0-1}{r_1-1}.$$

If $r \leqslant \delta - \delta_0$**.** Set $k = r_2 - 1$, $n = \delta - \delta_0 - 1$, $m = r$, $N = r - 1$. Then according to Lemma 3.6.2.1 we have

$$S(r, r) = (-1)^r \binom{\delta - \delta_0 - 1 + r}{r-1}\binom{d-d_0-1}{r_1-1}.$$

Let $r_1 < r$. Set $k = r_2 - r + r_1$, $N = r_1$, $n = \delta - \delta_0 + r_1 - r$, $m = r - 1$. Note that $\delta - \delta_0 - r + r_1 \geqslant r_1$ et $r_1 \leqslant r - 1$. Then according to Lemma 3.6.2.1 we have

$$\begin{aligned} S(r_1, r) &= (-1)^r \binom{r}{r_1} \binom{\delta - \delta_0 - 1 + r_1}{r_1} \frac{r_1!(\delta - \delta_0 - 1)!}{(r-1)!(\delta - \delta_0 + r_1 - r)!} \binom{d - d_0 - 1}{r_1 - 1}; \\ &= (-1)^r \binom{r}{r_1} \binom{\delta - \delta_0 - 1 + r_1}{r - 1} \binom{d - d_0 - 1}{r_1 - 1}; \\ &= (-1)^r \binom{r}{r_1} \binom{\delta - \delta_0 - 1 + r_1}{\delta - \delta_0 - r + r_1} \binom{d - d_0 - 1}{r_1 - 1}. \end{aligned}$$

Note that $\delta - \delta_0 \geqslant 2$. According to Lemma 3.6.2.2, $\sum_{r=r_1+1}^{\delta-\delta_0+r_1} (-1)^r r \binom{\delta-\delta_0}{r-r_1} = (-1)^{r_1+1}$, which yields

$$\begin{aligned} \sum_{r=r_1+1}^{\delta-\delta_0+r_1} S(r_1, r) &= \sum_{r=r_1+1}^{\delta-\delta_0+r_1} (-1)^r r \frac{(\delta - \delta_0 + r_1 - 1)!}{r_1!(r - r_1)!(\delta - \delta_0 + r_1 - r)!} \binom{d - d_0 - 1}{r_1 - 1}; \\ &= (-1)^{r_1+1} \binom{\delta - \delta_0 + r_1 - 1}{r_1 - 1} \binom{d - d_0 - 1}{r_1 - 1}. \end{aligned}$$

Hence $\sum_{r=r_1}^{\delta-\delta_0+r_1} S(r_1, r) = 0$, hence $S = 0$.

Finally, since this problem is symmetric relatively to d and δ, the same result holds for $d_0 < d - 1$ and $\delta_0 < \delta$. We now only have left to consider the case whare $d_0 = d$ and $\delta_0 < \delta - 1$, or $\delta_0 = \delta$ and $d_0 < d - 1$. For example suppose $\delta_0 = d$. Then, since $\delta_0 < \delta - 1$

$$S = \sum_{r=1}^{\delta-\delta_0} (-1)^r \binom{\delta - \delta_0 - 1}{r - 1} = -(1-1)^{\delta-\delta_0-1} = 0.$$

□

3.6.3 Geometric interpretation of elements in $\mathcal{O}_{h_i} \star \mathcal{O}_{k,p}$ for $k \neq p+1$. From now on we fix elements u and v in W^P such that $X(u) = h_i$ and $v = w_{k,p}$, where $k \neq p+1$. Recall $h_1 := X(w_{n-1,1})$ and $h_2 := X(w_{n,2})$. We fix a degree $\mathbf{d} \in E(X)\backslash\{0\}$. We suppose $\mathbf{d} \geqslant l_1, l_2$, i.e. we suppose $\mathbf{d} = d_1 l_1 + d_2 l_2$ where $d_1 \geqslant 1$ or $d_2 \geqslant 1$.

Let $\mathbf{d} = \mathbf{d}_0 + \sum_{1 \leqslant i \leqslant k} \mathbf{d}_i$ be a decomposition of $\mathbf{d}$ into a sum of degrees $\mathbf{d}_i$. We denote by $M_{\mathbf{d}_0,\ldots,\mathbf{d}_k}$ the following scheme

$$M_{\mathbf{d}_0,\ldots,\mathbf{d}_k} = \overline{\mathcal{M}_{0,3}}(X, \mathbf{d}_0) \times_X \overline{\mathcal{M}_{0,2}}(X, \mathbf{d}_1) \times_X \cdots \times_X \overline{\mathcal{M}_{0,2}}(X, \mathbf{d}_k).$$

Note that $M_{\mathbf{d}_0,\ldots,\mathbf{d}_k}$ is in bijection with the boundary stratum of $\overline{\mathcal{M}_{0,3}}(X, \mathbf{d})$ parametrizing genus zero stable maps $(f : C_0 \cup C_1 \cdots \cup C_k \to X, \{p_1, p_2, p_3\})$ such that

- The curve $\cup_{0 \leqslant i \leqslant k} C_i$ is a union of k genus zero quasi-stable curves C_i meeting in a point.
- The first two marked points p_j lie on C_0 and p_3 lies on C_k
- For all $0 \leqslant i \leqslant k$, the map $f_{|C_i}$ represents the degree $\mathbf{d}_i$.

Furthermore, note that $M_{\mathbf{d}_0,\ldots,\mathbf{d}_k}$ is equipped with evaluation maps ev_i, where $1 \leqslant i \leqslant 3$. We call *boundary Gromov-Witten variety associated with u and v* the inverse image

$$M_{\mathbf{d}_0,\ldots,\mathbf{d}_k}(u, v) := ev_1^{-1}(X(u)) \cap ev_2^{-1}(w_0 X(v)) \subset M_{\mathbf{d}_0,\ldots,\mathbf{d}_k}.$$

Note that $M_{\mathbf{d}_0,\ldots,\mathbf{d}_k}(u, v)$ parametrizes stable maps $(f : C_0 \cup C_1 \cdots \cup C_k \to X, \{p_1, p_2, p_3\})$ satisfying the three conditions here above and sending the first two marked points p_j within the varieties $X(u)$ and $w_0 X(v)$ respectively. Finally, we denote by $\Gamma_{\mathbf{d}_0,\ldots,\mathbf{d}_k}(u, v)$ the projection of the boundary Gromov-Witten variety $M_{\mathbf{d}_0,\ldots,\mathbf{d}_k}(u, v)$ by the third evaluation morphism, i.e.

$$\Gamma_{\mathbf{d}_0,\ldots,\mathbf{d}_k}(u, v) := ev_3(M_{\mathbf{d}_0,\ldots,\mathbf{d}_k}(u, v)) \subset X.$$

Note that $\Gamma_{\mathbf{d}_0,\dots,\mathbf{d}_k}(u,v)$ is the set of points x in X such that there exists a stable map $(\varphi : C \to X, \{p_1,p_2,p_3\})$ associated with an element in $M_{\mathbf{d}_0,\dots,\mathbf{d}_k}$ such that the image $\varphi(p_1)$ of the first marked point belongs to the Schubert variety $X(u)$, the image $\varphi(p_2)$ of the second marked point belongs to the opposite Schubert variety $w_0X(v)$ and the image of the third marked point $\varphi(p_3)$ is the point x.

Lemma 3.6.3.1. *i) Let $f : M \to X$ and $g_i : M' \to X$ be morphisms of schemes, where $i = 1,2$. Let Y be a subvariety of X. Denote by $M \times_X M'$ the fiber product defined by f and g_1, and by $g : M \times_X M' \to X$ the morphism induced by g_2. Note that the morphim f induces a map $f' : M \times_X Y \to X$, and the morphisms g_i induce maps $g_i' : M' \times_X Y \to X$. Then*

$$g(M \times_X \times M' \times_X Y) = g_2' \left({g_1'}^{-1}(f'(M \times_X Y)) \right).$$

ii) Let $k \geqslant 1$. Set $Y := ev_3(M_{\mathbf{d}_0,\dots,\mathbf{d}_{k-1}}(u,v))$. Then $ev_3(M_{\mathbf{d}_0,\dots,\mathbf{d}_k}(u,v)) = ev_2^{\mathbf{d}_k}(ev_1^{-1}(Y))$.

Proof. i) This is a direct consequence of $M \times_X \times M' \times_X Y = (M \times_X Y) \times_X M'$.

ii) Apply $i)$ to $M := M_{\mathbf{d}_0,\dots,\mathbf{d}_{k-1}}$, $M' = \overline{\mathcal{M}_{0,2}}(X,\mathbf{d}_k)$ and $Y = X(u)$.

□

Lemma 3.6.3.2. *Let $\mathbf{d}$ be a degree in $E(X)$. Then $ev_2^{\mathbf{d}}(ev_1^{-1}(X(u)))$ is a Schubert variety, and for any α in $K(X)$, $\langle \mathcal{O}_u, \alpha \rangle_{\mathbf{d}} = \chi(\alpha \cdot [\mathcal{O}_{ev_2^{\mathbf{d}}(ev_1^{-1}(X(u)))}])$.*

Proof. We consider the Gromov-Witten variety $ev_1^{-1}(X(u)) \subset \overline{\mathcal{M}_{0,2}}(X,\mathbf{d})$. Let us consider the morphism $ev_2^{\mathbf{d}} : \overline{\mathcal{M}_{0,2}}(X,\mathbf{d}) \supset ev_1^{-1}(X(u)) \to X$ evaluating the second marked point. According to [BM11] Proposition 3.2., the general fiber of ev_1 is an irreducible rational variety and the image $ev_2^{\mathbf{d}}(ev_1^{-1}(X(u)))$ is a Schubert variety, and hence has rational singularities. Note that since ev_1 and ev_2 are G-equivariant, for any element g_1 in G the variety $ev_2^{\mathbf{d}}(ev_1^{-1}(g_1 \cdot X(u))) = g_1 \cdot ev_2^{\mathbf{d}}(ev_1^{-1}(X(u)))$ is also a Schubert variety. Finally Part 3.5.2 (a) yields the result. □

Proposition 3.6.3.1. *Let u be an element in W^P such that $X(u)$ is a codimension 1 Schubert variety h_i, and $v = w_{k,p}$ be an element in W^P satisfying $k \neq p+1$. Let $\mathbf{d} = \sum_{k=1}^{k} \mathbf{d}_i$ be an element in $E(X)$, where $k > 0$.*

1. *The projected Gromov-Witten variety $\Gamma_{\mathbf{d}_0,\dots,\mathbf{d}_k}(u,v) := ev_3^{\mathbf{d}_0+\dots+\mathbf{d}_r}(M_{\mathbf{d}_0,\dots,\mathbf{d}_k}(u,v))$ is the translate of a Schubert variety of X.*

2. $$\sum_{\alpha_i \in W^P} \langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{I}_{\alpha_1} \rangle_{\mathbf{d}_0} \langle \mathcal{O}_{\alpha_1}, \mathcal{I}_{\alpha_1} \rangle_{\mathbf{d}_1} \dots \langle \mathcal{O}_{\alpha_k}, \mathcal{I}_w \rangle_{\mathbf{d}_k} = \chi([\mathcal{O}_{\Gamma_{\mathbf{d}_0,\dots,\mathbf{d}_k}(u,v)}], \mathcal{I}_w).$$

Proof. 1. By iteration on k. First note that for $\mathbf{d}_0 > 0$ according to Lemma 3.5.3.2 and Proposition 3.5.3.1 there exists an element h in G such that $ev_3^{\mathbf{d}_0}(M_{\mathbf{d}_0}(u,v)) = ev_3^{\mathbf{d}_0}(\overline{\mathcal{M}_{0,3}}(X,\mathbf{d}_0) \times_{X^2} (X(u) \times w_0X(v))$ is given by the translation by h of a Schubert variety. For $\mathbf{d}_0 = 0$, since $k \neq p+1$, according to Proposition 3.3.0.1 $ev_3^0(\overline{\mathcal{M}_{0,3}}(X,\mathbf{d}_0) \times_{X^2} (X(u) \times w_0X(v)) = X(u) \cap w_0X(v)$ is the translation by ane element h in G of a Schubert variety. Now suppose the induction hypothesis verified for all $i < k$. Then the projected Gromov-Witten variety $ev_3(M_{\mathbf{d}_0,\dots,\mathbf{d}_{k-1}}(u,v))$ is the translation by an element h in G of a Schubert variety $X(w)$. According to Lemma 3.6.3.1 we have

$$ev_3(M_{\mathbf{d}_0,\dots,\mathbf{d}_k}(u,v)) = ev_2(ev_1^{-1}(h \cdot X(w))) = h \cdot ev_2(ev_1^{-1}(X(w))),$$

where we consider the maps $ev_i : \overline{\mathcal{M}_{0,2}}(X,\mathbf{d}_k) \to X$, $i = 1,2$. Finally according to [CMBP13] Proposition 3.2 the variety $ev_2(ev_1^{-1}(X(w)))$ is a Schubert variety.

2. By iteration on k. According to Lemma 3.5.3.2 and Proposition 3.5.3.1 since $\mathbf{d}_0 > l_1, l_2$ we have

$$\langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{I}_{\alpha_1} \rangle_{\mathbf{d}_0} = \chi\left(\mathcal{O}_\beta \cdot \mathcal{I}_{\alpha_1}\right),$$

where h is an element in G and β is an element in W^P such that $\Gamma_{\mathbf{d}_0}(u,v) = h \cdot X(\beta)$. Finally notice that $\chi([\mathcal{O}_{h\cdot X(\beta)}], \mathcal{I}_{\alpha_1}) = \chi(\mathcal{O}_\beta, \mathcal{I}_{\alpha_1})$.

Let us now suppose the induction hypothesis satisfied for $i = k-1$. According to i) there exists h in G and β in W^P such that $\Gamma_{\mathbf{d}_0,\ldots,\mathbf{d}_k}(u,v) = h \cdot X(\beta)$. Hence the induction hypothesis yields

$$\sum_{\alpha_i \in W^P, 1 \leqslant i \leqslant k-1} \langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{I}_{\alpha_1}\rangle_{\mathbf{d}_0} \langle \mathcal{O}_{\alpha_1}, \mathcal{I}_{\alpha_1}\rangle_{\mathbf{d}_1} \ldots \langle \mathcal{O}_{\alpha_{k-1}}, \mathcal{I}_{\alpha_{k-1}}\rangle_{\mathbf{d}_k} = \chi([\mathcal{O}_{h\cdot X(\beta)}], \mathcal{I}_{\alpha_{k-1}}) = \delta_{\beta,\alpha_{k-1}},$$

where the symbol $\delta_{\beta,\alpha_{k-1}}$ is one if $\beta = \alpha_{k-1}$, 0 else. We now only have to compute $\sum_{\alpha_{k-1}\in W^P} \delta_{\beta,\alpha_{k-1}} \langle \mathcal{O}_w, \mathcal{I}_w\rangle_{\mathbf{d}_k}$. Let u' be the element in W^P such that $ev_2^{\mathbf{d}_k}(ev_1^{-1}(X(\beta)) = X(u')$. We obtain

$$\sum_{\alpha_{k-1}\in W^P} \delta_{\beta,\alpha_{k-1}} \langle \mathcal{O}_w, \mathcal{I}_w\rangle_{\mathbf{d}_k} = \langle \mathcal{O}_\beta, \mathcal{I}_\beta\rangle_{\mathbf{d}_k}$$

where $\mathcal{O}_\beta = [\mathcal{O}_{ev_2^{\mathbf{d}_k}(ev_1^{-1}(X(\beta)))}] = [\mathcal{O}_{h\cdot ev_2^{\mathbf{d}_k}(ev_1^{-1}(X(\beta)))}] = [\mathcal{O}_{ev_2^{\mathbf{d}_k}(ev_1^{-1}(\Gamma_{\mathbf{d}_0,\ldots,\mathbf{d}_{k-1}}(u,v)))}] = [\mathcal{O}_{\Gamma_{\mathbf{d}_0,\ldots,\mathbf{d}_k}(u,v)}]$ according to Lemma 3.6.3.1.

□

Hence for $v = w_{k,p}$, where $k \neq p+1$, the product $h_i \star \mathcal{O}_v$ can be rewritten as

$$\begin{aligned}
\mathcal{O}_{h_i} \star \mathcal{O}_v = \mathcal{O}_{h_i} \cdot \mathcal{O}_v + &+ Q_1 \left([\mathcal{O}_{\Gamma_{l_1}(h_i,v)}] - [\mathcal{O}_{\Gamma_{0,l_1}(h_i,v)}] \right) \\
&+ Q_2 \left([\mathcal{O}_{\Gamma_{l_2}(h_i,v)}] - \mathcal{O}_{\Gamma_{0,l_2}(h_i,v)}] \right) \\
&+ Q_1Q_2 \Big([\mathcal{O}_{\Gamma_{l_1+l_2}(h_i,v)}] - [\mathcal{O}_{\Gamma_{l_1,l_2}(h_i,v)}] - [\mathcal{O}_{\Gamma_{l_2,l_1}(h_i,v)}] - [\mathcal{O}_{\Gamma_{0,l_1+l_2}(h_i,v)}] \\
&\qquad + [\mathcal{O}_{\Gamma_{0,l_1,l_2}(h_i,v)}] + [\mathcal{O}_{\Gamma_{0,l_2,l_1}(h_i,v)}] \Big)
\end{aligned}$$

3.6.4 Computing the degree of $\mathcal{O}_{h_i} \star \mathcal{O}_v$.

Let $1 \leqslant i_2, j_2 \leqslant n$ where $i_2 \neq j_2$. We note $v = w_{i_2,j_2}$. We compute here the degree of $\mathcal{O}_{h_i} \star \mathcal{O}_v := \mathcal{O}_{h_i} \star \mathcal{O}_{i_2,j_2}$. More precisely, we prove the following result.

Proposition 3.6.4.1. *For any element v in W^P, the product $\mathcal{O}_{h_i} \star \mathcal{O}_{i_2,j_2}$ only includes terms of degree Q_1, Q_2, and Q_1Q_2, i.e.*

$$\mathcal{O}_{h_i} \star \mathcal{O}_{i_2,j_2} = \mathcal{O}_{h_i} \cdot \mathcal{O}_v + Q_1 P_{l_1}(i,v) + Q_2 P_{l_2}(i,v) + Q_1Q_2 P_{l_1+l_2}(i,v),$$

where for $\mathbf{d}$ in $E(X)$, $P_{\mathbf{d}}(i,v)$ is an element in $K(X)$.

Our proof occurs in two times. We begin by rewriting the quantum product between two classes $\mathcal{O}_{h_i}$ and $\mathcal{O}_v$ in $K(X)$. We then note that most of these terms cancel together.

Rewriting the quantum product. Denote by g the bilinear pairing on $K(X)$ induced by the sheaf Euler characteristic; we write $g_{\alpha,\beta} := \chi(\mathcal{O}_\alpha \cdot \mathcal{O}_\beta)$. We have

$$\begin{aligned}
\mathcal{G}_{\alpha,\beta} &= \sum_{\mathbf{d}\in E(X)} Q^{\mathbf{d}} \langle \mathcal{O}_\alpha, \mathcal{O}_\beta\rangle_{\mathbf{d}} \\
&= g_{\alpha,\beta} + \sum_{\mathbf{d}\in E(X)\setminus\{0\}} Q^{\mathbf{d}} \langle \mathcal{O}_\alpha, O_\beta\rangle_{\mathbf{d}}.
\end{aligned}$$

The identity $(g+A)^{-1} = (g(\mathrm{id} + g^{-1}A))^{-1} = \sum(-1)^k (g^{-1}A)^k g^{-1}$ yields the following equality:

$$\begin{aligned}
&\mathcal{G}^{\alpha,\beta} \\
&= \sum_{\mathbf{d}\in E(X)} Q^{\mathbf{d}} \sum_{k=0}^{+\infty} (-1)^k \sum_{\substack{\mathbf{d}_i\in E(X)\setminus\{0\}\\ \mathbf{d}_1+\cdots+\mathbf{d}_k=\mathbf{d}}} \sum_{\alpha_i\in W^P} g^{\alpha,\alpha_1}\langle \mathcal{O}_{\alpha_1}, O_{\alpha_2}\rangle_{\mathbf{d}_1} g^{\alpha_2,\alpha_3}\ldots g^{\alpha_{2k-2},\alpha_{2k-1}}\langle \mathcal{O}_{\alpha_{2k-1}}, O_{\alpha_{2k}}\rangle_{\mathbf{d}_k} g^{\alpha_{2k},\beta} \\
&= g^{\alpha,\beta} + \sum_{k=0}^{+\infty} (-1)^k \sum_{\mathbf{d}_i\in E(X)\setminus\{0\}} Q^{\mathbf{d}_1+\cdots+\mathbf{d}_k} \sum_{\alpha_i\in W^P} g^{\alpha,\alpha_1}\langle \mathcal{O}_{\alpha_1}, \mathcal{I}_{\alpha_2}\rangle_{\mathbf{d}_1}\ldots\langle \mathcal{O}_{\alpha_k}, \mathcal{I}_v\rangle_{\mathbf{d}_k}
\end{aligned}$$

since $\mathcal{I}_\alpha$ is the element in $K(X)$ dual to $\mathcal{O}_\alpha$ for the pairing g. This implies for any elements u and v in W^P

$$\begin{aligned}
&\chi\left((\mathcal{O}_u \star \mathcal{O}_v)\cdot \mathcal{I}_w\right) \\
&= \sum_{\alpha\in W^P}\sum_{\mathbf{d}\in E(X)} Q^{\mathbf{d}}\langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{O}_\alpha\rangle_{\mathbf{d}} \left(g^{\alpha,v} + \sum_{k=0}^{+\infty}(-1)^k \sum_{\mathbf{d}_i\in E(X)\setminus\{0\}} Q^{\mathbf{d}_1+\cdots+\mathbf{d}_k} \sum_{\alpha_i\in W^P} g^{\alpha,\alpha_1}\langle \mathcal{O}_{\alpha_1}, \mathcal{I}_{\alpha_2}\rangle_{\mathbf{d}_1}\ldots\langle \mathcal{O}_{\alpha_k}, \mathcal{I}_w\rangle_{\mathbf{d}_k}\right) \\
&= \sum_{k=0}^{+\infty}(-1)^k \sum_{\mathbf{d}_i\in E(X)\setminus\{0\},\, \mathbf{d}_0\in E(X)} Q^{\mathbf{d}_0+\mathbf{d}_1+\cdots+\mathbf{d}_k} \sum_{\alpha_i\in W^P} \langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{I}_{\alpha_1}\rangle_{\mathbf{d}_0}\langle \mathcal{O}_{\alpha_1}, \mathcal{I}_{\alpha_1}\rangle_{\mathbf{d}_1}\ldots\langle \mathcal{O}_{\alpha_k}, \mathcal{I}_w\rangle_{\mathbf{d}_k}
\end{aligned}$$

Cancellation between different terms of $\mathcal{O}_{h_i} \star \mathcal{O}_v$.

Lemma 3.6.4.1. *Let $1 \leqslant i, j \leqslant n$, $i \neq j$.*

i) For any $1 \leqslant s, t \leqslant n$, $s \neq t$, $\langle \mathcal{O}_{i,j}, \mathcal{I}_{s,t}\rangle_{l_1} = \delta_{s,n}\delta_{t,j}(1-\delta_{j,n}) + \delta_{j,n}\delta_{s,n-1}\delta_{t,n}$.

ii) For any $1 \leqslant s, t \leqslant n$, $s \neq t$, $\langle \mathcal{O}_{i,j}, \mathcal{I}_{s,t}\rangle_{l_2} = \delta_{s,i}\delta_{t,1}(1-\delta_{i,1}) + \delta_{i,1}\delta_{t,2}\delta_{s,1}$.

iii) For any $1 \leqslant i_2, j_2 \leqslant n$, $i_2 \neq j_2$,

$$\begin{aligned}
\sum_{1\leqslant i_1, j_1\leqslant n,\ i_1\neq j_1} &\langle \mathcal{O}_{i,j}, \mathcal{I}_{i_1,j_1}\rangle_{l_1}\langle \mathcal{O}_{i_1,j_1}, \mathcal{I}_{i_2,j_2}\rangle_{l_2} \\
&= (1-\delta_{j,n})\delta_{i_2,n}\delta_{j_2,1} + \delta_{j,n}\delta_{i_2,n-1}\delta_{j_2,1}.
\end{aligned}$$

iv) For any $1 \leqslant i_2, j_2 \leqslant n$, $i_2 \neq j_2$,

$$\begin{aligned}
\sum_{1\leqslant i_1, j_1\leqslant n,\ i_1\neq j_1} &\langle \mathcal{O}_{i,j}, \mathcal{I}_{i_1,j_1}\rangle_{l_2}\langle \mathcal{O}_{i_1,j_1}, \mathcal{I}_{i_2,j_2}\rangle_{l_1} \\
&= (1-\delta_{i,1})\delta_{i_2,n}\delta_{j_2,1} + \delta_{i,1}\delta_{i_2,n}\delta_{j_2,2}.
\end{aligned}$$

v) For any $1 \leqslant i_2, j_2 \leqslant n$, $i_2 \neq j_2$,

$$\sum_{u,v\in W^P} \langle \mathcal{O}_{i,j}, \mathcal{I}_u\rangle_{l_1}\langle \mathcal{O}_u, \mathcal{I}_v\rangle_{l_2}\langle \mathcal{O}_v, \mathcal{I}_{i_2,j_2}\rangle_{l_1} = \delta_{i_2,n}\delta_{j_2,1}.$$

vi) Let $1 \leqslant i_1, j_1, i_2, j_2 \leqslant n$, $i_k \neq j_k$, such that $j_1 + j_2 \leqslant n+2$. Then for any $1 \leqslant i, j \leqslant n$, $i \neq j$, we have

$$\sum_{w\in W^P} \langle \mathcal{O}_{w_{i_1,j_1}}, \mathcal{O}_{w_{i_2,j_2}}, \mathcal{I}_w\rangle_{l_2}\langle \mathcal{O}_w, \mathcal{I}_{i,j}\rangle_{l_2} = \langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{I}_{i,j}\rangle_{l_2}.$$

Proof. i) According to (3.12) we have

$$\langle \mathcal{O}_{w_{i,j}}, \mathcal{I}_w\rangle_{l_1} = \chi\left([\mathcal{O}_{ev_2^{l_1}(ev_1^{-1}(X(w_{i,j})))}], \mathcal{I}_w\right).$$

Note that $ev_2^{l_1}(ev_1^{-1}(X(w_{i,j}))$ parametrizes points (x, y) in X such that

– $y = [0 : \cdots : 0 : x_{n-j+1} \dots x_n]$;
– x lies on a line $L \subset \mathbb{P}^{n-1}$;
– $L_i = \{[x_1 \dots x_i : 0 \dots 0]\}$ intersects L.

Since there is a line joining any two points in $\mathbb{P}^{n-1}$, $ev_2^{l_1}(ev_1^{-1}(X(w_{i,j}))$ is the set of points (x, y) in X such that $y = [0 : \cdots : 0 : x_{n-j+1} \dots x_n]$; this is the Schubert variety $X(w_{n,j})$ if $j \neq n$ and $X(w_{n-1,n})$ if $j = n$.

ii) By symmetry, cf. Part 3.5.2 (c).

iii) This is obtained by multiplying the expressions found in *i*) and *ii*).

iv) This is obtained by multiplying the expressions found in *i*) and *ii*).

v) This is obtained by multiplying the expressions found in *i*) and *iii*).

vi) This obtained by multiplying the expression in *i*) with the expression from (3.12).

□

Recall from Subsection 3.6.3 the following definition. For classes $\mathbf{d}_0, \dots, \mathbf{d}_k$ in $E(X)$, for elements u and v in W^P, we set

$$\Gamma_{\mathbf{d}_0,\dots,\mathbf{d}_k}(u, v) := ev_3\left(ev_1^{-1}(X(u)) \cap ev_2^{-1}(w_0 X(v))\right) \subset X$$

the projected boundary Gromov-Witten variety associated with the classes $\mathbf{d}_i$ and the Schubert varieties $X(u)$ and $w_0X(v)$. In particular when $v = w_0$ satisfies $X(w_0) = X$, we name

$$\Gamma_{\mathbf{d}_0,\dots,\mathbf{d}_k}(u) := \Gamma_{\mathbf{d}_0,\dots,\mathbf{d}_k}(w_0, u) := ev_3\left(ev_2^{-1}(X(u))\right),$$

and when u satisfies $X(u) = h_i$, we name

$$\Gamma_{\mathbf{d}_0,\dots,\mathbf{d}_k}(i, v) := ev_3\left(ev_1^{-1}(h_i) \cap ev_2^{-1}(X(v))\right).$$

Lemma 3.6.4.2. *i) Let $\mathbf{d} = \sum_{i=1}^k \mathbf{d}_i$ be a degree in $E(X)$.*

$$\sum_{\alpha_i \in W^P} \langle \mathcal{O}_u, \mathcal{I}_{\alpha_2} \rangle_{\mathbf{d}_1} \dots \langle \mathcal{O}_{\alpha_k}, \mathcal{I}_\beta \rangle_{\mathbf{d}_k} = \chi\left([\mathcal{O}_{\Gamma_{\mathbf{d}_1,\dots,\mathbf{d}_k}(u)}], \mathcal{I}_\beta\right).$$

ii) $\Gamma_{l_1,l_2,l_1}(u) = X = \Gamma_{l_2,l_1,l_2}(u) = \Gamma_{l_1+l_2,l_1}(u) = \Gamma_{l_1,l_1+l_2}(u)$.

iii) $\Gamma_{dl_1}(u)$ and $\Gamma_{dl_2}(u)$ do not depend on the choice of $d \geqslant 2$.

iv) Consider a degree $\mathbf{d} = d_1 l_1 + d_2 l_2$ in $E(X)$ such that $d_2 \geqslant 2$. Then

$$\sum_{k=0}^{+\infty} (-1)^k \sum_{\substack{\mathbf{d}_i \in E(X)\setminus\{0\} \\ \sum_{1 \leqslant i \leqslant k} \mathbf{d}_i = \mathbf{d}}} \sum_{\alpha_i \in W^P} \langle \mathcal{O}_{\alpha_1}, \mathcal{I}_{\alpha_1} \rangle_{\mathbf{d}_1} \dots \langle \mathcal{O}_\beta, \mathcal{I}_\beta \rangle_{\mathbf{d}_k} = 0.$$

Proof. i) This is a direct consequence of the equality $\langle [\mathcal{O}_X], \alpha, \beta \rangle_{\mathbf{d}} = \langle \alpha, \beta \rangle_{\mathbf{d}}$ for all α, β in $K(X)$ and of Proposition 3.6.3.1.

ii) Cf. Lemma 3.6.4.1.

iii) Cf. Lemma 3.6.4.1.

iv) By applying *i*), *ii*) and *iii*) we obtain

$$\begin{aligned}
&\sum_{k=0}^{+\infty}(-1)^k \sum_{\substack{\mathbf{d}_i\in E(X)\backslash\{0\}\\ \sum_{1\leqslant i\leqslant k}\mathbf{d}_i=\mathbf{d}}} \sum_{\alpha_i\in W^P} \langle \mathcal{O}_{\alpha_1},\mathcal{I}_{\alpha_2}\rangle_{\mathbf{d}_1}\dots\langle \mathcal{O}_{\alpha_k},\mathcal{I}_{\beta}\rangle_{\mathbf{d}_k}\\
&\quad= \sum_{k=0}^{+\infty}(-1)^k \sum_{\substack{\mathbf{d}_i\in E(X)\backslash\{0\}\\ \sum_{1\leqslant i\leqslant k}\mathbf{d}_i=\mathbf{d}}} \chi\left([\mathcal{O}_{\Gamma_{\mathbf{d}_1,\dots,\mathbf{d}_k}(u)}],\mathcal{I}_\beta\right)\\
&\quad= \chi\left([\mathcal{O}_X],\mathcal{I}_\beta\right)\sum_{k=0}^{+\infty}(-1)^k \#\left\{\mathbf{d}_i\in E(X)\backslash\{0\},\ \sum_{1\leqslant i\leqslant k}\mathbf{d}_i=\mathbf{d}\right\}\\
&\quad+ \left(\chi([\mathcal{O}_{\Gamma_{d_1 l_1,d_2 l_2}}],\mathcal{I}_w)-\chi\left([\mathcal{O}_X],\mathcal{I}_\beta\right)\right)\sum_{k=0}^{+\infty}(-1)^k \#\left\{(d_1^i,d_2^j)\in\mathbb{N}^k,\ d_1=\sum_{1\leqslant i\leqslant k}d_1^i,\ d_2=\sum_{1\leqslant i\leqslant k}d_2^i\right\}\\
&\quad+ \left(\chi([\mathcal{O}_{\Gamma_{d_2 l_2,d_1 l_1}}],\mathcal{I}_w)-\chi\left([\mathcal{O}_X],\mathcal{I}_\beta\right)\right)\sum_{k=0}^{+\infty}(-1)^k \#\left\{(d_1^i,d_2^j)\in\mathbb{N}^k,\ d_1=\sum_{1\leqslant i\leqslant k}d_1^i,\ d_2=\sum_{1\leqslant i\leqslant k}d_2^i\right\}
\end{aligned}$$

Finally, since $d_1\geqslant 2$ or $d_2\geqslant 2$, Lemma 3.6.2.3 yields the annulation of the three components of the sum. The annulation of the first component comes from applying the lemma to $d_0=0=\delta_0$ and $\mathbf{d}_i=(d_i,\delta_i)$. The annulation of the second and third part of the sum comes from applying the lemma to $d_0=0=\delta_0$ and $d_i=d_1^i$ and $\delta_i=0$ if $d_1\geqslant 2$, and to $d_0=0=\delta_0$ and $d_i=0$ and $\delta_i=d_2^i$ if $d_2\geqslant 2$.

□

We denote by

$$\begin{aligned}
&P_{\mathbf{d}}(i,v)\\
&:=\sum_{k=0}^{+\infty}(-1)^k \sum_{\substack{\mathbf{d}_i\in E(X)\backslash\{0\},\mathbf{d}_0\in E(X)\\ \mathbf{d}_0+\sum_{1\leqslant i\leqslant k}\mathbf{d}_i=\mathbf{d}}} Q^{\mathbf{d}_0+\mathbf{d}_1+\dots+\mathbf{d}_k} \sum_{\alpha_i\in W^P}\langle \mathcal{O}_{h_i},\mathcal{O}_v,\mathcal{I}_{\alpha_1}\rangle_{\mathbf{d}_0}\langle \mathcal{O}_{\alpha_1},\mathcal{I}_{\alpha_1}\rangle_{\mathbf{d}_1}\dots\langle \mathcal{O}_{\alpha_k},\mathcal{I}_w\rangle_{\mathbf{d}_k}\mathcal{O}_w \quad\in K(X)\otimes\mathbb{Q}_{\mathbf{d}}[Q
\end{aligned}$$

the term of $\mathcal{O}_{h_i}\star\mathcal{O}_v$ of degree $\mathbf{d}$. Let $\mathbf{d}$ be a degree in $E(X)$ such that $\mathbf{d}\geqslant 2l_1$ or $\mathbf{d}\geqslant 2l_2$. According to Lemma 3.6.4.2 most of the elements in $P_{\mathbf{d}}(u,v)$ cancel. More precisely, for a given $\mathbf{d}_0$ satisfying $\mathbf{d}_0\leqslant\mathbf{d}-2l_1$ or $\mathbf{d}_0\leqslant\mathbf{d}-2l_2$, all terms associated with decompositions $\mathbf{d}=\mathbf{d}_0+\sum_{1\leqslant i\leqslant k}\mathbf{d}_i$ cancel together. Suppose $d_1\geqslant 2$ and $d_2\geqslant 2$. The only terms left are then the ones associated with $\mathbf{d}_0=\mathbf{d}$, $\mathbf{d}_0=\mathbf{d}-l_1$, $\mathbf{d}_0=\mathbf{d}-l_2$ and $\mathbf{d}_0=\mathbf{d}-l_1-l_2$. Using Proposition 3.6.3.1 we obtain

$$\begin{aligned}
P_{\mathbf{d}}(u,v)=Q^{\mathbf{d}}\Big(&[\mathcal{O}_{ev_3(\Gamma_{\mathbf{d}}(u,v))}]-[\mathcal{O}_{ev_3(\Gamma_{\mathbf{d}-l_1,l_1}(u,v))}]-[\mathcal{O}_{ev_3(\Gamma_{\mathbf{d}-l_2,l_2}(u,v))}]\\
&-[\mathcal{O}_{ev_3(\Gamma_{d-l_1-l_2,l_1+l_2}(u,v))}]+[\mathcal{O}_{ev_3(\Gamma_{d-l_1-l_2,l_1,l_2}(u,v))}]+[\mathcal{O}_{ev_3(\Gamma_{d-l_1-l_2,l_2,l_1}(u,v))}]\Big),
\end{aligned}$$

which is equal to 0 since according to Lemma 3.6.4.2 for $d_1\geqslant 2$ and $d_2\geqslant 2$ the projected varieties are equal to X. Now suppose $d_2\geqslant 2$ and $d_1=0,1$. The only terms left in $P_{\mathbf{d}}(i,v)$ are those associated with $\mathbf{d}_0=\mathbf{d}$ and $\mathbf{d}_0=\mathbf{d}-l_2$. We obtain

$$P_{\mathbf{d}}(i,v)=Q^{\mathbf{d}}\left([\mathcal{O}_{ev_3(\Gamma_{\mathbf{d}}(i,v))}]-[\mathcal{O}_{ev_3(\Gamma_{\mathbf{d}-l_2,l_2}(i,v))}]\right),$$

which is equal to zero since $1+j_2\leqslant n+2$ according to Lemma 3.5.3.1. Furthermore by symmetry if $d_1\geqslant 2$, since $n+i_2\geqslant n+2$ we have $P_{\mathbf{d}}(i,v)$ is equal to 0.

Hence for any element v in W/W_P all terms of degree $\mathbf{d}\geqslant l_1$ and of degree $\mathbf{d}\geqslant l_2$ cancel in the product $\mathcal{O}_{h_i}\star\mathcal{O}_{i_2,j_2}$, which finally yields

$$\mathcal{O}_{h_i}\star\mathcal{O}_{i_2,j_2}=\mathcal{O}_{h_i}\cdot\mathcal{O}_v+Q_1P_{l_1}(i,v)+Q_2P_{l_2}(i,v)+Q_1Q_2P_{l_1+l_2}(i,v).$$

3.7 A Chevalley formula in $QK_s(X)$

Recall $h_1 := X(w_{n-1,1})$ and $h_2 := X(w_{n,2})$ are the two Schubert varieties whose classes generate $A^1(X) \simeq \mathbb{Z}^2$. We name $\mathcal{O}_{h_i} := [\mathcal{O}_{h_i}]$ the two Schubert classes associated. Let $1 \leqslant k, p \leqslant n$, where $k \neq p$. This part is dedicated to proving the following expressions.

Proposition 3.7.0.1.

$$\mathcal{O}_{h_1} \star \mathcal{O}_{k,p} = \begin{cases} Q_1\mathcal{O}_{n-1,n} + Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right) & \text{if } k=1, p=n \\ Q_1\mathcal{O}_{n,p} & \text{if } k=1,\ p<n \\ \mathcal{O}_{1,2} + Q_1\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right) & \text{if } k=2, p=1 \\ \mathcal{O}_{k-1,p} & \text{if } k>1,\ k\neq p+1 \\ \mathcal{O}_{p-1,p} + \mathcal{O}_{p,p+1} - \mathcal{O}_{p-1,p+1} & \text{if } 1<p<n-1,\ k=p+1 \end{cases} \tag{3.18}$$

$$\mathcal{O}_{h_2} \star \mathcal{O}_{k,p} = \begin{cases} Q_2\mathcal{O}_{1,2} + Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_2}\right) & \text{if } k=1, p=n \\ Q_2\mathcal{O}_{k,1} & \text{if } k>1,\ p=n \\ \mathcal{O}_{n-1,n} + Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_2}\right) & \text{if } k=n, p=n-1 \\ \mathcal{O}_{k,p+1} & \text{if } p<n\ k\neq p+1 \\ \mathcal{O}_{p,p+1} + \mathcal{O}_{p-1,p} - \mathcal{O}_{p-1,p+1} & \text{if } 1<k<n-1,\ k=p+1 \end{cases}$$

3.7.1 Computing terms of degree l_2. Computing $\mathcal{O}_{h_1} \star \mathcal{O}_v$. According to (3.14) if $n-1+k < n+1$, i.e. if $k=1$, we have

$$\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_{l_2} = 0$$

According to (3.13) if $n-1+k = n+1$, i.e. if $k=2$ we have

$$\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_{l_2} = \delta_{i,1}\delta_{j,2}.$$

According to (3.12) if $k>2$ we have

$$\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_{l_2} = \delta_{i,k-1}\delta_{j,1}.$$

Furthermore, the product in K-theory is given by

$$\mathcal{O}_u \cdot \mathcal{O}_v = \sum_{w \in W^P} \chi(\mathcal{O}_u \cdot \mathcal{O}_v \cdot \mathcal{I}_w)\mathcal{O}_w = \sum_{w \in W^P} \langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{I}_w \rangle_0 \mathcal{O}_w.$$

Hence Proposition 3.4.1.1 directly yields the invariants $\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_0$.

$$\begin{aligned} &\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_0 = 0 && \text{if } k=1 \\ &\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_0 = \delta_{i,k-1}\delta_{j,p} && \text{if } k\neq p+1,\ k\neq 1 \\ &\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_0 = \delta_{i,k-2}\delta_{j,p} + \delta_{i,k-1}\delta_{j,p+1} - \delta_{i,k-2}\delta_{j,p+1} && \text{if } k=p+1. \end{aligned}$$

Recall from Lemma 3.6.4.1 that for any $1 \leqslant s,t \leqslant n$, $s \neq t$, $\langle \mathcal{O}_{i,j}, \mathcal{I}_{s,t} \rangle_{l_2} = \delta_{s,i}\delta_{t,1}(1-\delta_{i,1}) + \delta_{i,1}\delta_{s,1}\delta_{t,2}$. We deduce

$$\sum_{w \in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_w \rangle_0 \langle \mathcal{O}_w, \mathcal{I}_{w_{i,j}} \rangle_{l_2} = \begin{cases} 0 & \text{if } k=1 \\ \delta_{i,1}\delta_{j,2} & \text{if } k=2 \\ \delta_{i,k-1}\delta_{j,1} & \text{if } k>2 \end{cases}$$

Moreover recall that the term of degree l_i in $\mathcal{O}_u \star \mathcal{O}_v$ is given by

$$P_{l_i}(u,v) = \sum_{w \in W^P} \langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{I}_w \rangle_{l_i} \mathcal{O}_w - \sum_{w,w' \in W^P} \langle \mathcal{O}_u, \mathcal{O}_v, \mathcal{I}_w \rangle_0 \langle \mathcal{O}_w, \mathcal{I}_{w'} \rangle_{l_i} \mathcal{O}_{w'}.$$

We obtain for all v in W^P

$$P_{l_2}(h_1, v) = 0. \tag{3.19}$$

COMPUTING $\mathcal{O}_{h_2} \star \mathcal{O}_v$. According to (3.13) if $n+k=n+1$, i.e. if $k=1$ we have

$$\langle \mathcal{O}_{h_2}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_{l_2} = \delta_{i,1}\delta_{j,2}.$$

According to (3.12) if $k>1$ we have

$$\langle \mathcal{O}_{h_2}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_{l_2} = \delta_{i,k}\delta_{j,1}.$$

We now obtain the invariants $\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_0$ from Proposition 3.4.1.1.

$$\begin{array}{ll}
\langle \mathcal{O}_{h_2}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_0 = 0 & \text{if } p=n \\
\langle \mathcal{O}_{h_2}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_0 = \delta_{i,k}\delta_{j,p+1} & \text{if } k \neq p+1,\ k \neq 1 \\
\langle \mathcal{O}_{h_2}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_0 = \delta_{i,p}\delta_{j,p+1} + \delta_{i,p+1}\delta_{j,p+2} - \delta_{i,p}\delta_{j,p+2} & \text{if } k=p+1.
\end{array}$$

Recall from Lemma 3.6.4.1 that for any $1 \leqslant s,t \leqslant n$, $s \neq t$, $\langle \mathcal{O}_{i,j}, \mathcal{I}_{s,t} \rangle_{l_2} = \delta_{s,i}\delta_{t,1}(1-\delta_{i,1}) + \delta_{i,1}\delta_{s,1}\delta_{t,2}$. We deduce for $p<n$

$$\sum_{w \in W^P,\ 1 \leqslant i,j \leqslant n,\ i \neq j} \langle \mathcal{O}_{h_2}, \mathcal{O}_{k,p}, \mathcal{I}_w \rangle_0 \langle \mathcal{O}_w, \mathcal{I}_{w_{i,j}} \rangle_{l_2} = \begin{cases}
\delta_{i,1}\delta_{j,2} & \text{if } k=1 \\
\delta_{i,k}\delta_{j,1} & \text{if } k>1 \text{ and } k \neq p+1 \\
\delta_{i,p+1}\delta_{j,1} & \text{if } 2<k<n \text{ and } k=p+1 \\
\delta_{i,2}\delta_{j,1} & \text{if } k=2 \text{ and } p=1 \\
\delta_{i,n-1}\delta_{j,1} & \text{if } k=n \text{ and } p=n-1
\end{cases}$$

We obtain for $2 \leqslant k,p \leqslant n$, $k \neq p$

$$\begin{cases}
P_{l_2}(h_2, w_{1,p}) = \delta_{p,n} Q_2 \mathcal{O}_{1,2} & \\
P_{l_2}(h_2, w_{k,n}) = Q_2 \mathcal{O}_{k,1} & \text{if } k>1 \\
P_{l_2}(h_2, w_{n,n-1}) = Q_2(\mathcal{O}_{n,1} - \mathcal{O}_{n-1,1}) = Q_2([O_X] - \mathcal{O}_{h_1}) & \\
P_{l_2}(h_2, w_{k,p}) = 0 & \text{if } (1<k<n \text{ and } p<n) \text{ or } (k=n \text{ and } p<n-1)
\end{cases} \tag{3.20}$$

3.7.2 Computing terms of degree l_1.

Recall from Part 3.5.2 (c) that correlators of degree l_1 associated with h_1 and $w_{k,p}$ are equal to correlators of degree l_2 associated with h_2 and $w_{n-p+1,n-k+1}$. We deduce from (3.19)

$$P_{l_1}(h_2, w_{k,p}) = 0, \tag{3.21}$$

and we deduce from (3.20)

$$\begin{cases}
P_{l_1}(h_1, w_{k,n}) = \delta_{k,1} Q_1 \mathcal{O}_{n-1,n} & \\
P_{l_1}(h_1, w_{1,p}) = Q_1 \mathcal{O}_{n,p} & \text{if } p<n \\
P_{l_1}(h_1, w_{2,1}) = Q_1(\mathcal{O}_{n,1} - \mathcal{O}_{n,2}) = Q_1([O_X] - \mathcal{O}_{h_2}) & \\
P_{l_1}(h_1, w_{k,p}) = 0 & \text{if } (2<k \text{ and } p<n) \text{ or } (k=2 \text{ and } p>1)
\end{cases} \tag{3.22}$$

3.7.3 Computing terms of degree l_1+l_2.

According to (3.15), for all $1 \leqslant i,j \leqslant n$, $i \neq j$

$$\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_{l_1+l_2} = \delta_{\min(n,k+n-1),i}\delta_{n-\min(2n+1-p,n)+1,j} = \delta_{n,i}\delta_{1,j}.$$

According to (3.13) if $n+k=n+1$, i.e. if $k=1$ we have

$$\langle \mathcal{O}_{h_2}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_{l_2} = \delta_{i,1}\delta_{j,2}.$$

According to (3.12) if $k>1$ we have

$$\langle \mathcal{O}_{h_2}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j} \rangle_{l_2} = \delta_{i,k}\delta_{j,1}.$$

By symmetry we have

$$\begin{aligned}\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j}\rangle_{l_1} &= \delta_{j,n}\delta_{i,n-1} && \text{if } p = n\\ \langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j}\rangle_{l_1} &= \delta_{j,p}\delta_{i,n} && \text{if } p < n\end{aligned}$$

Recall from Lemma 3.6.4.1 that for any $1 \leqslant s,t \leqslant n$, $s \neq t$, $\langle \mathcal{O}_{i,j}, \mathcal{I}_{s,t}\rangle_{l_2} = \delta_{s,i}\delta_{t,1}(1-\delta_{i,1}) + \delta_{i,1}\delta_{s,1}\delta_{t,2}$. We deduce

$$\sum_{w\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_w\rangle_{l_1}\langle \mathcal{O}_w, \mathcal{I}_{w_{i,j}}\rangle_{l_2} = \begin{cases} \delta_{i,n-1}\delta_{j,1} & \text{if } p = n\\ \delta_{i,n}\delta_{j,1} & \text{if } p < n\end{cases}$$

If $k = 1$, we have

$$\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j}\rangle_{l_2} = 0$$

If $k = 2$ we have

$$\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j}\rangle_{l_2} = \delta_{i,1}\delta_{j,2}.$$

If $k > 2$ we have

$$\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j}\rangle_{l_2} = \delta_{i,k-1}\delta_{j,1}.$$

Recall from Lemma 3.6.4.1 that for any $1 \leqslant s,t \leqslant n$, $s \neq t$, $\langle \mathcal{O}_{i,j}, \mathcal{I}_{s,t}\rangle_{l_1} = \delta_{s,n}\delta_{t,j}(1-\delta_{j,n}) + \delta_{j,n}\delta_{s,n-1}\delta_{t,n}$. We deduce

$$\sum_{w\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_w\rangle_{l_2}\langle \mathcal{O}_w, \mathcal{I}_{w_{i,j}}\rangle_{l_1} = \begin{cases} 0 & \text{if } k = 1\\ \delta_{i,n}\delta_{j,2} & \text{if } k = 2\\ \delta_{i,n}\delta_{j,1} & \text{if } k > 2\end{cases}$$

According to Proposition 3.4.1.1 we have

$$\begin{aligned}\langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j}\rangle_0 &= 0 && \text{if } k = 1\\ \langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j}\rangle_0 &= \delta_{i,k-1}\delta_{j,p} && \text{if } k \neq p+1,\ k \neq 1\\ \langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_{i,j}\rangle_0 &= \delta_{i,k-2}\delta_{j,p} + \delta_{i,k-1}\delta_{j,p+1} - \delta_{i,k-2}\delta_{j,p+1} && \text{if } k = p+1.\end{aligned}$$

Furthermore, according to Lemma 3.6.4.1 for any $1 \leqslant i_2, j_2 \leqslant n$, $i_2 \neq j_2$,

$$\sum_{1\leqslant i_1,j_1\leqslant n,\ i_1\neq j_1} \langle \mathcal{O}_{i,j}, \mathcal{I}_{i_1,j_1}\rangle_{l_1}\langle \mathcal{O}_{i_1,j_1}, \mathcal{I}_{i_2,j_2}\rangle_{l_2} = (1-\delta_{j,n})\delta_{i_2,n}\delta_{j_2,1} + \delta_{j,n}\delta_{i_2,n-1}\delta_{j_2,1}.$$

Hence

$$\sum_{w,w'\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_w\rangle_0\langle \mathcal{O}_w, \mathcal{I}_{w'}\rangle_{l_1}\langle \mathcal{O}_{w'}, \mathcal{I}_{i,j}\rangle_{l_2} = \begin{cases} 0 & \text{if } k = 1\\ \delta_{i,n-1}\delta_{j,1} & \text{if } p = n,\ k > 1\\ \delta_{i,n}\delta_{j,1} & \text{if } k > 1 \text{ and } p < n\end{cases}$$

Moreover according to Lemma 3.6.4.1 for any $1 \leqslant i_2, j_2 \leqslant n$, $i_2 \neq j_2$,

$$\sum_{1\leqslant i_1,j_1\leqslant n,\ i_1\neq j_1} \langle \mathcal{O}_{i,j}, \mathcal{I}_{i_1,j_1}\rangle_{l_2}\langle \mathcal{O}_{i_1,j_1}, \mathcal{I}_{i_2,j_2}\rangle_{l_1} = (1-\delta_{i,1})\delta_{i_2,n}\delta_{j_2,1} + \delta_{i,1}\delta_{i_2,n}\delta_{j_2,2}.$$

Hence

$$\sum_{w,w'\in W^P,\ 1\leqslant i,j\leqslant n,\ i\neq j} \langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_w\rangle_0\langle \mathcal{O}_w, \mathcal{I}_{w'}\rangle_{l_2}\langle \mathcal{O}_{w'}, \mathcal{I}_{i,j}\rangle_{l_1} = \begin{cases} 0 & \text{if } k = 1\\ \delta_{i,n}\delta_{j,2} & \text{if } k = 2\\ \delta_{i,n}\delta_{j,1} & \text{if } k > 2\end{cases}$$

Finally according to Lemma 3.6.3.2 and Lemma 3.5.3.2 we have $\langle \mathcal{O}_u, \mathcal{I}_w\rangle_{l_1+l_2} = \chi\left([\mathcal{O}_X]\cdot \mathcal{I}_w\right)$ hence

$$\sum_{u\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_{k,p}, \mathcal{I}_u\rangle_0\langle \mathcal{O}_u, \mathcal{I}_{i,j}\rangle_{l_1+l_2} = \begin{cases} 0 & \text{if } k = 1\\ \delta_{i,n}\delta_{j,1} & \text{else}\end{cases}$$

We are now ready to compute $P_{l_1+l_2}(h_1, W_{k,p})$. Moreover recall that the term of degree (l_1+l_2) in $\mathcal{O}_{h_1} \star \mathcal{O}_v$ is given by

$$\begin{aligned}
&P_{l_1+l_2}(h_1, v)\\
&\quad= \sum_{w\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_{l_1+l_2}\mathcal{O}_w - \sum_{w,w'\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_{l_1}\langle \mathcal{O}_w, \mathcal{I}_{w'}\rangle_{l_2}\mathcal{O}_{w'} - \sum_{w,w'\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_{l2}\langle \mathcal{O}_w, \mathcal{I}_{w'}\rangle_{l_1}\mathcal{O}_{w'}\\
&\quad+ \sum_{w,w',w''\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_0\langle \mathcal{O}_w, \mathcal{I}_{w'}\rangle_{l_1}\langle \mathcal{O}_{w'}, \mathcal{I}_{w''}\rangle_{l_2}\mathcal{O}_{w''} + \sum_{w,w',w''\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_0\langle \mathcal{O}_w, \mathcal{I}_{w'}\rangle_{l_2}\langle \mathcal{O}_{w'}, \mathcal{I}_{w''}\rangle_{l_1}\mathcal{O}_{w''}\\
&\quad- \sum_{w,w'\in W^P} \langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_0\langle \mathcal{O}_w, \mathcal{I}_{w'}\rangle_{l_1+l_2}\mathcal{O}_{w'}.
\end{aligned}$$

a) **If $v = w_{k,p}$, where $k \neq 1$.** Note that then the second and fourth terms of the sum are equal, and the third and fifth terms are equal, and the first and last terms are equal. We obtain

$$\begin{aligned}
P_{l_1+l_2}(h_1, w_{k,p}) &= Q_1Q_2\left(\mathcal{O}_{n,1} - \mathcal{O}_{n,1} - \mathcal{O}_{n,1} + \mathcal{O}_{n,1} + \mathcal{O}_{n,1} - \mathcal{O}_{n,1}\right)\\
&= 0 \qquad\qquad \text{if } k \neq 1.
\end{aligned}$$

b) **If $v = w_{1,p}$, where $p < n$.** Note that then all terms of the sum are equal to zero, except from the first term $\sum_{w\in W^P}\langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_{l_1+l_2}\mathcal{O}_w$, which is non zero iff $w = w_{n,1}$, and $\sum_{w,w'\in W^P}\langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_{l_1}\langle \mathcal{O}_w, \mathcal{I}_{w'}\rangle_{l_2}\mathcal{O}_{w'}$, which is also non zero iff $w = w_{n,1}$. We obtain

$$P_{l_1+l_2}(h_1, w_{1,p}) = 0 \qquad\qquad \text{if } p < n$$

c) **if $v = w_{1,n}$.** Note that then $\mathcal{O}_v = [\mathcal{O}_{X(w_{1,n})}] = [\mathcal{O}_{point}]$ in $K(X)$. Then all terms of the sum are zero, except from the first term $\sum_{w\in W^P}\langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_{l_1+l_2}\mathcal{O}_w$, which is non zero iff $w = w_{n,1}$, and $\sum_{w,w'\in W^P}\langle \mathcal{O}_{h_1}, \mathcal{O}_v, \mathcal{I}_w\rangle_{l_1}\langle \mathcal{O}_w, \mathcal{I}_{w'}\rangle_{l_2}\mathcal{O}_{w'}$, which is non zero iff $w = w_{n-1,1}$. We obtain

$$P_{l_1+l_2}(h_1, w_{1,n}) = Q_1Q_2(\mathcal{O}_{n,1} - \mathcal{O}_{n-1,1}) = Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right).$$

To sum up, we have

$$P_{l_1+l_2}(h_1, w_{k,p}) = \begin{cases} 0 & \text{if } (k,p) \neq (1,n)\\ Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right) & \text{if } (k,p) = (1,n)\end{cases} \tag{3.23}$$

Finally, by symmetry we deduce

$$P_{l_1+l_2}(h_2, w_{k,p}) = \begin{cases} 0 & \text{if } (k,p) \neq (1,n)\\ Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_2}\right) & \text{if } (k,p) = (1,n)\end{cases} \tag{3.24}$$

3.7.4 Computing the Chevalley formula.

[Proof of Proposition 3.7.0.1] According to Part 3.6 the only terms appearing in $\mathcal{O}_{h_i} \star \mathcal{O}_v$ are those of degree l_i and of degree (l_1+l_2). We now only have to sum the classical part $\mathcal{O}_{h_i} \cdot \mathcal{O}_v$ computed Proposition 3.4.1.1 with the part of degree l_1 given by (3.22) and (3.21), the part of degree l_2 given by (3.19) and (3.20), and finally the part of degree (l_1+l_2) given by (3.23) and (3.24).

3.7.5 Comparison with small quantum cohomology.

Fulton and Woodward provide a Chevalley formula in in the small quantum cohomology ring of any generalized flag variety [FW04]. In the case of the incidence variety, this formula yields

$$\begin{cases} h_1 \star_S [X(i,j)] = h_1 \cdot [X(i,j)] + q_1[X]\delta_{i,2}\delta_{j,1} & \text{if } i \neq 1;\\ h_1 \star_S [X(1,j)] = q_1[X(n,j)] & \text{if } j < n;\\ h_1 \star_S [X(1,n)] = q_1[X(n-1,j)] + q_1q_2[X] & \text{if } j = n,\end{cases}$$

and

$$\begin{cases} h_2 \star_S [X(i,j)] = h_2 \cdot [X(i,j)] + q_2[X]\delta_{n,i}\delta_{n-1,j} & \text{si } j \neq n; \\ h_2 \star_S [X(i,n)] = q_2[X(i,1)] & \text{si } i > 1; \\ \qquad\qquad = q_2[X(1,2)] + q_1q_2[X] & \text{si } i = 1, \end{cases}$$

where the product $\star_S$ is the product in the small quantum cohomology ring of X. If we compare the product in small quantum cohomology to the product in small quantum K theory, we observe (in blue the terms which are present in small quantum K-theory but not in small quantum cohomology):

$$\mathcal{O}_{h_1} \star \mathcal{O}_{k,p} = \begin{cases} Q_1\mathcal{O}_{n-1,n} + Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right) & \text{if } k = 1, p = n \\ Q_1\mathcal{O}_{n,p} & \text{if } k = 1,\ p < n \\ \mathcal{O}_{1,2} + Q_1\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right) & \text{if } k = 2, p = 1 \\ \mathcal{O}_{k-1,p} & \text{if } k > 1,\ k \neq p+1 \\ \mathcal{O}_{p-1,p} + \mathcal{O}_{p,p+1} - \mathcal{O}_{p-1,p+1} & \text{if } 1 < p < n-1,\ k = p+1 \end{cases}$$

3.7.6 Generalization to other adjoint varieties ? In [CP11b] Chaput-Perrin interpret Fulton and Woodward's Chevalley formula in quantum cohomology in terms of roots systems. Note that their construction can be generalized to describe the Chevally formula in quantum K-theory described in Proposition 3.7.0.1. One might wonder if this construction holds for other types of adjoint varieties.

3.8 On Littlewood-Richardson coefficients in $QK_s(X)$.

Recall that for any elements u and v in W^P there is a unique expression

$$\mathcal{O}_u \star \mathcal{O}_v = \sum_{w \in W^P} P^w_{u,v}(Q_1, Q_2)\mathcal{O}_w$$

where $P^w_{u,v}(Q_1, Q_2)$ is a polynomial in the Novikov variables Q_1, Q_2. Note that taking the limit $Q_1, Q_2 \to 0$ of $P^w_{u,v}(Q_1, Q_2)$ yields the Littlewood-Richardson coefficients in $K(X)$. We provide here an algorithm computing all coefficients $P^w_{u,v}(Q_1, Q_2)$, and provide a closed formula that matches our computations for small values of n. Furthermore, we prove that the signs of the coefficients of the polynomials $P^w_{u,v}(Q_1, Q_2)$ satisfy a positivity rule.

3.8.1 An algorithm to compute Littlewood-Richardson coefficients in $QK_s(X)$. Recall from Part 3.7

$$\mathcal{O}_{h_1} \star \mathcal{O}_{k,p} = \begin{cases} Q_1\mathcal{O}_{n-1,n} + Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right) & \text{if } k = 1, p = n \\ Q_1\mathcal{O}_{n,p} & \text{if } k = 1,\ p < n \\ \mathcal{O}_{1,2} + Q_1\left([\mathcal{O}_X] - \mathcal{O}_{h_1}\right) & \text{if } k = 2, p = 1 \\ \mathcal{O}_{k-1,p} & \text{if } k > 1,\ k \neq p+1 \\ \mathcal{O}_{p-1,p} + \mathcal{O}_{p,p+1} - \mathcal{O}_{p-1,p+1} & \text{if } 1 < p < n-1,\ k = p+1 \end{cases}$$

$$\mathcal{O}_{h_2} \star \mathcal{O}_{k,p} = \begin{cases} Q_2\mathcal{O}_{1,2} + Q_1Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_2}\right) & \text{if } k = 1, p = n \\ Q_2\mathcal{O}_{k,1} & \text{if } k > 1,\ p = n \\ \mathcal{O}_{n-1,n} + Q_2\left([\mathcal{O}_X] - \mathcal{O}_{h_2}\right) & \text{if } k = n, p = n-1 \\ \mathcal{O}_{k,p+1} & \text{if } p < n\ k \neq p+1 \\ \mathcal{O}_{p,p+1} + \mathcal{O}_{p-1,p} - \mathcal{O}_{p-1,p+1} & \text{if } 1 < k < n-1,\ k = p+1 \end{cases}$$

This Chevalley formula allows us to write down an algorithm computing the product between any two Schubert classes in $QK(X)$ (cf. appendix C for a python code implementing this algorithm); we indeed can compute for all $1 \leqslant k, p \leqslant n$, where $k \neq p$, $\forall u \in W^P$, the product $\mathcal{O}_{w_{k,p}} \star \mathcal{O}_u$ by following the here under steps :

- For all $1 < k \leqslant n$, compute $\mathcal{O}_{w_{k,1}} \star \mathcal{O}_u$ iteratively by noticing $\mathcal{O}_{w_{k,1}} = \mathcal{O}_{h_1} \star \mathcal{O}_{w_{k+1,1}}$ and initializing with $\mathcal{O}_{w_{n,1}} \star \mathcal{O}_u = [\mathcal{O}_X] \star \mathcal{O}_u = \mathcal{O}_u$;
- for all $1 \leqslant p < k \leqslant n$, compute $\mathcal{O}_{w_{k,p}} \star \mathcal{O}_u$ iteratively by noticing $\mathcal{O}_{w_{k,p}} = \mathcal{O}_{h_2} \star \mathcal{O}_{w_{k,p-1}}$;
- compute $\mathcal{O}_{w_{1,2}} \star \mathcal{O}_u$ by noticing $\mathcal{O}_{w_{1,2}} = \mathcal{O}_{h_1} \star \mathcal{O}_{w_{2,1}} - Q_1([\mathcal{O}_X] - \mathcal{O}_{h_1})$;
- for all $1 \leqslant p < n$, compute $\mathcal{O}_{w_{p,p+1}} \star \mathcal{O}_u$ iteratively by noticing $\mathcal{O}_{w_{p,p+1}} = \mathcal{O}_{h_1} \star \mathcal{O}_{w_{p+1,p}} + (\mathcal{O}_{h_2} - id) \star \mathcal{O}_{w_{p-1,p}}$;
- for all $1 \leqslant k < p \leqslant n$, where $k \neq p-1$, compute $\mathcal{O}_{w_{k,p}} \star \mathcal{O}_u$ iteratively by noticing $\mathcal{O}_{w_{k,p}} = \mathcal{O}_{h_1} \star \mathcal{O}_{w_{k+1,p}}$.

3.8.2 Positivity in $QK_s(X)$.

Let $X \simeq G/P$ be a generalized flag variety. For u, v in W^P decompose the quantum product $\mathcal{O}_u \star \mathcal{O}_v = \sum_{w \in W^P, \beta \in E(X)} N_{u,v}^{w,\beta} Q^\beta \mathcal{O}_w$. For any elements u and v in W^P, we call the product $\mathcal{O}_u \star \mathcal{O}_v$ *positive* if for all $w \in W^P$, $\beta \in E(X)$, we have :

$$(-1)^{\mathrm{codim}(X(w))-\mathrm{codim}(X(u))-\mathrm{codim}(X(v))+\int_\beta c_1(T_X)} N_{u,v}^{w,\beta} \geqslant 0. \tag{3.25}$$

Note that this is equivalent to

$$(-1)^{\dim(X(v))-\mathrm{codim}(X(u))-\dim(X(w))+\int_\beta c_1(T_X)} N_{u,v}^{w,\beta} \geqslant 0.$$

Buch-Chaput-Mihalcea-Perrin proved that the product of any two Schubert classes is positive for X a cominuscule variety [BCMP16]. Denote by $\mathcal{O}_u \cdot \mathcal{O}_v = \sum_{w \in W^P} N_{u,v}^w \mathcal{O}_w$ the decomposition of the product of two Schubert classes in $K(X)$. Note that for $\beta = 0$ (3.25) yields

$$(-1)^{\mathrm{codim}(X(w))-\mathrm{codim}(X(u))-\mathrm{codim}(X(v))} N_{u,v}^w \geqslant 0.$$

Brion proved that, for X any generalized flag variety, the product of any two Schubert classes satisfies this positivity rule.

Let us go back now to our setting, i.e. let $X = Fl_{1,n-1}$.

Proposition 3.8.2.1. *Let u, v be elements in W^P. The product $\mathcal{O}_u \star \mathcal{O}_v$ is positive.*

Let $u = w_{k,p}$ in W^P. We will prove here that the product of $\mathcal{O}_u$ with any Schubert class is positive by induction on (k,p), running over the steps of the algorithm described here above. We will use the following formulas. According to (3.3)

$$X(i,j) = \left\{([x_1 : \cdots : x_i : 0 : \cdots : 0], [0 : \ldots 0 : y_j : \cdots : y_n]) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}\right\},$$

we have

$$\dim X(k,p) = \begin{cases} k+n-p-1 & \text{if } k < p \\ k+n-p-2 & \text{if } k > p \end{cases} = (k+n-p-2) + \Gamma_{p-k,0}$$

where we denote by $\Gamma_{n,0}$ the symbol of value 1 if $n \geqslant 0$, 0 else. Furthermore note that since X is a degree $(1,1)$ hypersurface of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ the adjunction formula yields for any $d_1, d_2 \geqslant 0$

$$\int_{d_1 l_1 + d_2 l_2} c_1(T_X) = d_1(n-1) + d_2(n-1)$$

Let us our now begin our induction.

- **Let $1 < k \leqslant n$.** Then $\mathcal{O}_{k,1} = \mathcal{O}_{h_1} \star \mathcal{O}_{k+1,1}$. Suppose the product $\mathcal{O}_{k+1,1} \star \mathcal{O}_v$ is positive for all $v \in W^P$. Let v be an element in W^P. According to Lemma 3.8.2.1 *i)* $\mathcal{O}_{h_1} \star \mathcal{O}_v$ is positive, Hence according to Lemma 3.8.2.1 *iii)* $\mathcal{O}_{k,1} \star \mathcal{O}_v$ is positive. Indeed, for $k > 1$ we have $\dim X(k,1) = k+n-3 = \dim X(k+1,1) - 1$.

- **Let** $1 \leqslant p < k \leqslant n$**.** Suppose the product $\mathcal{O}_{k,p-1} \star \mathcal{O}_v$ is positive for all $v \in W^P$. Note that $\mathcal{O}_{k,p} = \mathcal{O}_{h_2} \star \mathcal{O}_{k,p-1}$. Let v be an element in W^P. According to Lemma 3.8.2.1 *i)* $\mathcal{O}_{h_2} \star \mathcal{O}_v$ is positive, Hence according to Lemma 3.8.2.1 *iii)* $\mathcal{O}_{k,p} \star \mathcal{O}_v$ is positive. Indeed, for $1 \leqslant p < k \leqslant n$ we have $p-1 < k$ and $\dim X(k,p) = k+n-p-1 = \dim X(k,p-1) - 1$.

- **For** $k = 1$ **and** $p = 2$**.** Let v be an element in W^P. Note that $\mathcal{O}_{w_{1,2}} = \mathcal{O}_{h_1} \star \mathcal{O}_{w_{2,1}} - Q_1([\mathcal{O}_X] - \mathcal{O}_{h_1})$. Since $\dim X(1,2) = n-2 = \dim X(2,1) - 1$, according to Lemma 3.8.2.1 *ii)* and *iii)* the term $mathcal O_{h_1} \star \mathcal{O}_{w_{2,1}} \star \mathcal{O}_v$ has the expected sign. Furthermore, since $\dim X - 1 - (n-1) = 2n - 4 - n + 1 = \dim X(1,2)$, according to Lemma 3.8.2.1 *iv*), the terms $-Q_1[\mathcal{O}_X] \star \mathcal{O}_v$ and $Q_1 \mathcal{O}_{h_1} \star \mathcal{O}_v$ have the expected sign. Hence the product $\mathcal{O}_{k,p} \star \mathcal{O}_v$ is positive for all $w \in W^P$.

- **Let** $1 \leqslant p < n$**.** Let v be an element in W^P. Suppose the product $\mathcal{O}_{p+1,p} \star \mathcal{O}_v$ is positive. Note that $\mathcal{O}_{w_{p,p+1}} = \mathcal{O}_{h_1} \star \mathcal{O}_{w_{p+1,p}} + (\mathcal{O}_{h_2} - id) \star \mathcal{O}_{w_{p-1,p}}$. Note that $\dim X(p,p+1) = n-2 = \dim X(p+1,p) - 1$. Hence, since $\mathcal{O}_{p+1,p} \star \mathcal{O}_v$ is positive $-\mathcal{O}_{p+1,p} \star \mathcal{O}_v$ has the same sign as the expected sign for $\mathcal{O}_{p,p+1} \star \mathcal{O}_v$. Furthermore, according to Lemma 3.8.2.1 both $\mathcal{O}_{h_i} \star \mathcal{O}_{p,p+1} \star \mathcal{O}_v$ have the same sign as the expected sign for $\mathcal{O}_{p,p+1} \star \mathcal{O}_v$. Hence $\mathcal{O}_{p,p+1} \star \mathcal{O}_v$ is positive.

- **Let** $1 \leqslant k < p \leqslant n$**, where** $k \neq p-1$**.** Let v be an element in W^P. Suppose the product $\mathcal{O}_{k+1,p} \star \mathcal{O}_v$ is positive. Note that $\mathcal{O}_{w_{k,p}} = \mathcal{O}_{h_1} \star \mathcal{O}_{w_{k+1,p}}$.Since $k < p-1$, $\dim X(k,p) = k+n-p-1 = \dim X(k+1,p)$. Hence according to Lemma 3.8.2.1 *iii*) the product $\mathcal{O}_{k,p} \star \mathcal{O}_v$ is positive.

Finally by induction $\mathcal{O}_{k,p} \star \mathcal{O}_v$ is positive for all v in W^P.

Lemma 3.8.2.1. *i) For any element v in W^P the product $\mathcal{O}_{h_1} \star \mathcal{O}_v$ is positive.*

ii) For any element v in W^P the product $\mathcal{O}_{h_2} \star \mathcal{O}_v$ is positive.

iii) Let u be an element in W^P. Suppose there exists an element u' in W^P such that $\mathcal{O}_u = \mathcal{O}_{h_i} \star \mathcal{O}_{u'}$, and $\mathcal{O}_{u'} \star \mathcal{O}_v$ is positive for all v in W^P. Furthermore suppose $\dim X(u) = \dim X(u') - 1$. Then for any element v in W^P the product $\mathcal{O}_u \star \mathcal{O}_v$ is positive.

iv) Let u be an element in W^P. Suppose there exists an element u' in W^P such that $\mathcal{O}_u = Q_j \mathcal{O}_{u'}$, where $j \in \{1,2\}$, and $\mathcal{O}_{u'} \star \mathcal{O}_v$ is positive for all v in W^P. Furthermore suppose $\dim X(u) = \dim X(u') - (n-1)$. Then for any element v in W^P the product $\mathcal{O}_u \star \mathcal{O}_v$ is positive.

Let $1 \leqslant k, p \leqslant n$.

Proof. i) We compute explicitly $(-1)^{\dim(X(v)) - \mathrm{codim}(h_i) - \dim(X(w)) + \int_\beta c_1(T_X)} N_{u,v}^{w,\beta}$ using (3.18). Note that according to [Bri02] $N_{u,v}^{w,0}$ has the expected sign, hence we only have to consider terms $N_{u,v}^{w,\beta}$ with $\beta > 0$.

(a) Recall we have $\mathcal{O}_{h_1} \star \mathcal{O}_{1,n} = Q_1 \mathcal{O}_{n-1,n} + Q_1 Q_2 ([\mathcal{O}_X] - \mathcal{O}_{h_1})$. We have

$$(-1)^{dim X(n-1,n) - \dim X(1,n) - \mathrm{codim} h_1 + (n-1)} = (-1)^{(n-1)-0-1+(n-2)+(n-1)} = (-1)^{2n-4} \geqslant 0.$$

$$(-1)^{\dim X - \dim X(1,n) - \mathrm{codim} h_1 + 2(n-1)} = (-1)^{2n-3-1} \geqslant 0.$$

Hence $\mathcal{O}_{h_1} \star \mathcal{O}_{1,n}$ is positive.

(b) Let $1 < p < n$. Recall we have $\mathcal{O}_{h_1} \star \mathcal{O}_{1,p} = Q_1 \mathcal{O}_{n,p}$. We have

$$(-1)^{dim X(1,p) - \dim X(n,p) - \mathrm{codim} h_i + (n-1)} = (-1)^{(n-p-1)-(2n-p-2)-1+(n-1)} \geqslant 0.$$

(c) Recall we have $\mathcal{O}_{h_1} \star \mathcal{O}_{2,1} = \mathcal{O}_{1,2} + Q_1 ([\mathcal{O}_X] - \mathcal{O}_{h_1})$. We have

$$(-1)^{dim X - \dim X(2,1) - \mathrm{codim} h_i + (n-1)} = (-1)^{(2n-3)-(n-2)-1+(n-1)} \geqslant 0.$$

ii) Note that by symmetry if there is a term in $Q_2\mathcal{O}_{k,p}$ in $\mathcal{O}_{h_2}\star\mathcal{O}_{i,j}$, we have a term in $Q_1\mathcal{O}_{n-p+1,n-k+1}$ in $\mathcal{O}_{h_1}\star\mathcal{O}_{n-i+1,n-j+1}$, and if there is a term in $Q_1Q_2\mathcal{O}_{k,p}$ in $\mathcal{O}_{h_2}\star\mathcal{O}_{i,j}$, we have a term in $Q_1Q_2\mathcal{O}_{n-p+1,n-k+1}$ in $\mathcal{O}_{h_1}\star\mathcal{O}_{n-i+1,n-j+1}$. Since

$$\begin{aligned}\dim X(n-j+1,n-i+1) &= n-j+1+n-(n-i+1)-1+\Gamma_{n-i+1-(n-j+1),0}\\ &= n+i-j-1+\Gamma_{j-i,0}\\ &= \dim X(i,j),\end{aligned}$$

part *i*) yields the positivity of $\mathcal{O}_{h_2}\star\mathcal{O}_v$ for all v in W^P.

iii) Let v be an element in W^P. Consider the decomposition $\mathcal{O}_{u'}\star\mathcal{O}_v=\sum_{w\in W^P,\beta\in E(X)}N_{u',v}^{w,\beta}Q^\beta\mathcal{O}_w$ of $\mathcal{O}_{u'}\star\mathcal{O}_v$ in the Schubert basis. Since the quantum product $\star$ is associative we have

$$\begin{aligned}\mathcal{O}_u\star\mathcal{O}_v=\mathcal{O}_{h_i}\star\mathcal{O}_{u'}\star\mathcal{O}_v &= \sum_{w\in W^P,\beta_1\in E(X)}N_{u',v}^{w,\beta}Q^{\beta_1}\mathcal{O}_{h_i}\star\mathcal{O}_w\\ &= \sum_{w,\,w'\in W^P,\,\beta_1,\beta_2\in E(X)}N_{u',v}^{w,\beta_1}Q^{\beta_1}N_{h_i,w}^{w',\beta_2}Q^{\beta_2}\mathcal{O}_{w'}\end{aligned}$$

hence $N_{u,v}^{w',\beta}=\sum_{w\in W^P}\sum_{\beta_1,\,\beta_2\in E(X),\,\beta=\beta_1+\beta_2}N_{u',v}^{w,\beta_1}N_{h_i,w}^{w',\beta_2}$. Since $\dim X(u)=\dim X(u')-1$ and the products $\mathcal{O}_{u'}\star\mathcal{O}_v$ and $\mathcal{O}_{h_i}\star\mathcal{O}_w$ are positive we have

$$\begin{aligned}(-1)^{\dim X(u)-\dim X(w')-\mathrm{codim}X(v)+\int_\beta c_1(T_X)}&N_{u,v}^{w',\beta}\\ &= \sum_{w\in W^P}(-1)^{\dim X(u')-1-\dim X(w')-\mathrm{codim}X(v)+\int_\beta c_1(T_X)}N_{u',v}^{w,\beta_1}N_{h_i,w}^{w',\beta_2}\\ &= \sum_{w\in W^P}(-1)^{\dim X(u')-\dim X(w)-\mathrm{codim}X(v)+\int_{\beta_1}c_1(T_X)}N_{u',v}^{w,\beta_1}\\ &\qquad\times(-1)^{\dim X(w)-\dim X(w')-\mathrm{codim}h_i+\int_{\beta_2}c_1(T_X)}N_{h_i,w}^{w',\beta_2}\\ &\geqslant 0\end{aligned}$$

iv) Let v be an element in W^P. Consider the decomposition $\mathcal{O}_{u'}\star\mathcal{O}_v=\sum_{w\in W^P,\beta\in E(X)}N_{u',v}^{w,\beta}Q^\beta\mathcal{O}_w$ of $\mathcal{O}_{u'}\star\mathcal{O}_v$ in the Schubert basis. We have

$$\begin{aligned}\mathcal{O}_u\star\mathcal{O}_v=Q_j\mathcal{O}_{u'}\star\mathcal{O}_v &= \sum_{w\in W^P,\beta\in E(X)}N_{u',v}^{w,\beta+l_j}Q^\beta\mathcal{O}_w\\ &= \sum_{w\in W^P,\,\beta\in E(X)}N_{u',v}^{w,\beta+l_j}Q^\beta\end{aligned}$$

hence $N_{u,v}^{w',\beta}=\sum_{w\in W^P}\sum_{\beta\in E(X),\,\beta\geqslant l_j}N_{u',v}^{w,\beta-l_j}$. Since $\dim X(u)=\dim X(u')-(n-1)$ and the product $\mathcal{O}_{u'}\star\mathcal{O}_v$ is positive we have

$$\begin{aligned}(-1)^{\dim X(u)-\dim X(w')-\mathrm{codim}X(v)+\int_\beta c_1(T_X)}&N_{u,v}^{w',\beta}\\ &= \sum_{w\in W^P}(-1)^{\dim X(u')-(n-1)-\dim X(w')-\mathrm{codim}X(v)+\int_\beta c_1(T_X)}N_{u',v}^{w,\beta-l_j}\\ &= \sum_{w\in W^P}(-1)^{\dim X(u')-\dim X(w)-\mathrm{codim}X(v)+\int_{\beta-l_j}c_1(T_X)}N_{u',v}^{w,\beta-l_j}\\ &\geqslant 0\end{aligned}$$

□

3.8.3 A conjectural formula for Littlewood-Richardson coefficients in $QK_s(X)$.

We computed for various values of n Littlewood-Richardson coefficients in $QK_s(X)$ using the code in Appendix B. The results allowed us to guess and write down the following formula. This formula has been verified for small values of n. Let us now introduce some specific notations we need to express our result. First, we define "degree operators" d_i on $(W^P)^3$ by :

$$\forall 1 \leqslant i,j,k,p,s,t \leqslant n,\ i \neq j,\ k \neq p,\ s \neq t, \quad d_1(w_{i,j},w_{k,p},w_{s,t}) = 1 - \lfloor \frac{i+k-s}{n} \rfloor;$$
$$d_2(w_{i,j},w_{k,p},w_{s,t}) = \lfloor \frac{j+p-t}{n} \rfloor,$$

where $\lfloor x \rfloor$ is the integer part of x. Furthermore, let us define maps $t_i : (W^P)^2 \to W^P$ by :

$$\begin{aligned} \forall 1 \leqslant i,j,k \leqslant n,\ i \neq j,\ k \neq p, \quad t_0(w_{i,j},w_{k,p}) &= w_{(i+k-1)mod(n)+1,(j+p-2)mod(n)+1} \\ t_1(w_{i,j},w_{k,p}) &= w_{(i+k-2)mod(n)+1,(j+p-2)mod(n)+1} \\ t_2(w_{i,j},w_{k,p}) &= w_{(i+k-1)mod(n)+1,(j+p-1)mod(n)+1} \\ t_3(w_{i,j},w_{k,p}) &= w_{(i+k-2)mod(n)+1,(j+p-1)mod(n)+1}. \end{aligned}$$

Finally, we define an operator Δ on $(W^P)^3$ by setting :

$$\begin{aligned} \forall u,\ v,\ w\ \in W^P,\ \Delta(u,v,w) = 1 \qquad &\text{if } (-1)^{\ell(w_0w)-\ell(w_0u)-\ell(w_0v)+\int_{d_1(u,v,w)l_1+d_2(u,v,w)l_2} c_1(T_X)} \geqslant 0 \\ = 0 \qquad &\text{if } (-1)^{\ell(w_0w)-\ell(w_0u)-\ell(w_0v)+\int_{d_1(u,v,w)l_1+d_2(u,v,w)l_2} c_1(T_X)} < 0 \end{aligned}$$

Conjecture 3.8.3.1. *The product between two Schubert classes in $QK_s(X)$ is given by the following formula :*

$$\begin{aligned} \forall\ u,\ v \in W^P,\ \mathcal{O}_u \star \mathcal{O}_v = \quad & \Delta(u,v,t_0(u,v)) && q_1^{d_1(u,v,t_0(u,v))} q_2^{d_2(u,v,t_0(u,v))} \mathcal{O}_{t_0(u,v)} \\ & + (1-\Delta(u,v,t_1(u,v)))\Big(&& q_1^{d_1(u,v,t_1(u,v))} q_2^{d_2(u,v,t_1(u,v))} \mathcal{O}_{t_1(u,v)} \\ & && + q_1^{d_1(u,v,t_2(u,v))} q_2^{d_2(u,v,t_2(u,v))} \mathcal{O}_{t_2(u,v)} \\ & && - q_1^{d_1(u,v,t_3(u,v))} q_2^{d_2(u,v,t_3(u,v))} \mathcal{O}_{t_3(u,v)} \Big) \end{aligned}$$

Appendix B

Here under is the python code used to compute the product between any two Schubert classes in $QK(Fl(1,n-1)$; these computations helped us write down the Littlewood-Richardson formula we show in chapter 4.

```
# Computing product in the small quantum K–theory QKs of incidence varieties Fl(1,n-1;n)
# Date : 18/04/2019
# Author : Sybille Rosset
import math
from sympy import *
from sympy import Matrix
import numpy

# Parametrizing n
n=5 #i.e. we look at lines in hyperplanes of C^n

## This code computes the product in small quantum K theory of Fl(1,n-1;n) between
#Schubert varieties and writes it down within a file named "Output.txt"

#-------------------------------------------------------
#Indexing Schubert varieties X(i,j) by a 1D array !
#-------------------------------------------------------
def coeffij(k,n): #Function sending  0<k<=n(n-1)) to the corresponding pair (i,j)
i=math.ceil(k/(n-1))
if (k-(i-1)*(n-1) < i):
j=k-(i-1)*(n-1)
else:
j=k+1-(i-1)*(n-1)
return([i,j])

def invcoeffij(i,j,n): #Function sending  (i,j) to the corresponding 0<k<=n(n-1))
if(j>i):
t=(i-1)*(n-1)+j-1
else:
t=(i-1)*(n-1)+j
return(t-1)

#for k in range(1,n*(n-1)):
#[i,j]=coeffij(k)
#t=invcoeffij(i,j)
#print(i,j,k,t)
print(invcoeffij(1,2,3))

#-------------------------------------------------------
#Defining the action of O_h1 and O_h2 on QKs           !
#-------------------------------------------------------
q1 = Symbol('q1')
q2 = Symbol('q2')

# !!.I.!! Defining the action of O_h1
def h1(n):
h1 = zeros(n*(n-1))
# defining Oh1*O1,n
h1[invcoeffij(n-1,n,n),invcoeffij(1,n,n)]=q1
```

```
h1[invcoeffij(n,1,n),invcoeffij(1,n,n)]=q1*q2
h1[invcoeffij(n-1,1,n),invcoeffij(1,n,n)]=-q1*q2
#defining Oh1*O1,p
for p in range(2,n):
h1[invcoeffij(n,p,n),invcoeffij(1,p,n)]=q1
#defining Oh1*O2,1
h1[invcoeffij(1,2,n),invcoeffij(2,1,n)]=1
h1[invcoeffij(n,1,n),invcoeffij(2,1,n)]=q1
h1[invcoeffij(n,2,n),invcoeffij(2,1,n)]=-q1
#defining Oh1*Op+1,p
for p in range(2,n):
h1[invcoeffij(p-1,p,n),invcoeffij(p+1,p,n)]=1
h1[invcoeffij(p,p+1,n),invcoeffij(p+1,p,n)]=1
h1[invcoeffij(p-1,p+1,n),invcoeffij(p+1,p,n)]=-1
#defining Oh1*Ok,p
for k in range(2,n+1):
for p in range(1,n+1):
#print(k,p)
if(k!=p+1 and k!=p):
h1[invcoeffij(k-1,p,n),invcoeffij(k,p,n)]=1
return(h1)

# !!.II.!! Defining the action of O_h2
def h2(n):
h2 = zeros(n*(n-1))
# defining Oh2*O1,n
h2[invcoeffij(1,2,n),invcoeffij(1,n,n)]=q2
h2[invcoeffij(n,1,n),invcoeffij(1,n,n)]=q1*q2
h2[invcoeffij(n,2,n),invcoeffij(1,n,n)]=-q1*q2
#defining Oh2*Ok,n
for k in range(2,n):
h2[invcoeffij(k,1,n),invcoeffij(k,n,n)]=q2
#defining Oh2*On,n-1
h2[invcoeffij(n-1,n,n),invcoeffij(n,n-1,n)]=1
h2[invcoeffij(n,1,n),invcoeffij(n,n-1,n)]=q2
h2[invcoeffij(n-1,1,n),invcoeffij(n,n-1,n)]=-q2
#defining Oh2*Op+1,p
for p in range(1,n-1):
h2[invcoeffij(p,p+1,n),invcoeffij(p+1,p,n)]=1
h2[invcoeffij(p+1,p+2,n),invcoeffij(p+1,p,n)]=1
h2[invcoeffij(p,p+2,n),invcoeffij(p+1,p,n)]=-1
#defining Oh2*Ok,p
for p in range(1,n):
for k in range(1,n+1):
if(k!=p+1 and k!=p):
h2[invcoeffij(k,p+1,n),invcoeffij(k,p,n)]=1
return(h2)

#-----------------------------------------------
#Defining the action of O_k,p on any O_i,j  !
#-----------------------------------------------
def O(n):
O=[None]*n*(n-1)            #we create a list O, where we will stock O_k,p as the invcoef
O[invcoeffij(n,1,n)]=eye(n*(n-1))  #initializing OX as the identity matrix
```

```
a = h1(n)
b=h2(n)
J=b-eye(n*(n-1))
#We first define O_k,1
for k in range(1,n-1):
O[invcoeffij(n-k,1,n)]=a*O[invcoeffij(n-k+1,1,n)]
#We then define O_k,p, where k>p
for k in range(2,n+1):
for p in range(2,k):
#print(k,p)
O[invcoeffij(k,p,n)]=b*O[invcoeffij(k,p-1,n)]
#We now define O_p,p+1
O[invcoeffij(1,2,n)]=a*O[invcoeffij(2,1,n)]+q1*J
for p in range(2,n):
#print("p",p)
O[invcoeffij(p,p+1,n)]=a*O[invcoeffij(p+1,p,n)]+J*O[invcoeffij(p-1,p,n)]
#We finally define O_k,p for k<p and k!=p-1
for p in range(3,n+1):
for k in range(2,p):
print(k,p)
O[invcoeffij(p-k,p,n)]=a*O[invcoeffij(p-k+1,p,n)]
return(O)


#-----------------------------------------------------------------
# MAIN TASK : computing O_i,j*O_k,p                              !
#-----------------------------------------------------------------
#Computing QKs(Fl(1,n-1))
L=O(n)

#Writing down small quantum K theory product in a text file named "Output.txt"
Text=[]
for k in range(0,n*(n-1)):
A=L[k]
[l,p]= coeffij(k+1,n)
for r in range(0,n*(n-1)):
[i,j]=coeffij(r+1,n)
R=''
R=R+"O_"+str(l)+','+str(p)+" "+"*O_"+str(i)+','+str(j)+"= "
if(l+i<n+1 and j+p<n+1 and ((i>j and l>p) or (i+l<j+p and i>j and l<p) or (i+l<j+p and i
for s in range(0,n*(n-1)):
if(A[s,r]==0):
R=R
else:
[u,v]=coeffij(s+1,n)
R=R+'+('+str(A[s,r])+')*'+'O_'+str(u)+","+str(v)+" "
#print(R)
Text.append(R+';')

#print(Text)
numpy.savetxt("Output.txt",Text,delimiter=";", fmt='%s')
print("Ok")
```

Annexe C: résumé

Nous présentons ici les différents résultats exposés dans ce manuscrit.

Soit $n > 0$. On considère une collection $I = \{i_1, \ldots, i_m\}$ de m entiers positifs i_k vérifiant $i_0 = 0 < i_1 < \cdots < i_m < i_{m+1} = n$. On note Fl_I la variété de drapeaux paramètrant les drapeaux d'espaces vectoriels du type

$$V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_m} \subset \mathbb{C}^n,$$

où V_{i_k} est sous-espace vectoriel de $\mathbb{C}^n$ de dimension i_k.

3.8.4 Variétés de Gromov-Witten des variétés de drapeaux partiels. Lorsque l'on supprime certains de ses espaces à un drapeau d'espaces vectoriels on obtient un autre drapeau, consitué de moins d'espaces. Ceci induit une application d'oubli $\pi : X \to X'$ entre les variétés de drapeaux associées. Considérons une classe nef $[C] \in H_2(X, \mathbb{Z})$ de courbe de X, où la courbe C est isomorphe à son image par la projection π. Fixons également une classe $\lambda = [l] \in H_2(X, \mathbb{Z})$ de courbe l dont la projection par l'application d'oubli π est un point. On considère dans le chapitre 1 les liens entre la fibre géométrique du morphisme d'évaluation

$$ev_{[C]+\lambda} : \overline{\mathcal{M}_{0,r}}(X, [C] + \lambda) \to X^r$$

et la fibre géométrique du morphisme d'évaluation $ev_{[\pi(C)]}$ à valeur dans $(X')^r$. Pour $r = 3$ et X une variété cominuscule, Chaput-Perrin ont montré que la fibre générale du morphisme d'évaluation est une variété rationnelle [CP11a]. Pour $X = \mathbb{P}^N$, Pandharipande a calculé le genre de la fibre générale de $ev_d : \overline{\mathcal{M}_{0,r}}(\mathbb{P}^N, d) \to (\mathbb{P}^N)^r$ quand cette dernière est de dimension 1 [Pan97].

Définition 3.8.4.1. Soient $X_1, \ldots, X_r$ des variétés de Schubert de la variété de drapeaux X, soit $\mathbf{d}$ un élément de $E(X)$. Pour un élément $g = (g_1, \ldots, g_r)$ général dans GL_n^r, on appelle *variété de Gromov-Witten de degré* $\mathbf{d}$ *associée aux variétés* X_i *et à* g la sous-variété

$$\mathcal{W}^{g,\mathbf{d}}_{X;X_1,\ldots,X_r} := ev_1^{-1}(g_1 X_1) \cap \cdots \cap ev_r^{-1}(g_r X_r)$$

de $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$.

Rappelons qu'un morphisme $f : \mathbb{P}^1 \to X$ *représente* un degré $\mathbf{d} \in E(X) \subset H_2(X, \mathbb{Z})$ si f vérifie $f_*[\mathbb{P}^1] = \mathbf{d}$. Notons que la variété de Gromov-Witten $\mathcal{W}^{g,\mathbf{d}}_{X;X_1,\ldots,X_r}$ paramètre les morphismes $\mathbb{P}^1 \to X$ de degré $\mathbf{d}$ dont l'image a une intersection non vide avec les translatés $g_i \cdot X_i$ des variétés de Schubert X_i. Lorsque ces variétés sont de dimension zéro, leur nombre de points est un invariant de Gromov-Witten. De facon générale, la caractéristique d'Euler-Poincaré de leur adhérence dans l'espace $\overline{\mathcal{M}_{0,r}}(X, \mathbf{d})$ est un invariant en K-théorie quantique [Giv00].

Soit $n \geqslant 0$, soit $I = \{i_1, \ldots, i_m\}$ un ensemble de m entiers vérifiant $0 < i_1 < \cdots < i_m < n$. Considérons la variété de drapeaux $X' = Fl_I$. Pour une classe effective $\mathbf{d} \in H_2(X, \mathbb{Z})$, on note $ev_{\mathbf{d}} : \overline{\mathcal{M}_{0,r}}(Fl_I, \mathbf{d}) \to (Fl_I)^r$ le morphisme d'évaluation. Rappelons que Fl_I paramètre les drapeaux d'espaces vectoriels du type

$$\{0\} \subset V_{i_1} \subset V_{i_2} \cdots \subset V_{i_m} \subset \mathbb{C}^n$$

où $\dim V_{i_k} = i_k$. Soit $J = \{j_1, \ldots, j_M\}$ un ensemble d'entiers vérifiant $0 < j_1 < \cdots < j_M < n$, tel que I est inclu dans J. On note $\pi : X = Fl_J \to X' = Fl_I$ le morphisme d'oubli. Le chapitre 1 démontre l'observation suivante-cf. Théorème 1.1.2.1.

Théorème. *Supposons que* $(I, J, [C] + \lambda)$ *est une collection stabilisée, au sens de la définition 1.1.1.1. Alors:*

- *Pour toutes variétés de Schubert* $X_1, \ldots, X_r$ *de* X*, pour* g *en position générale dans* GL_n^r*, chaque variété de Gromov-Witten* $\mathcal{W}^{g,[C]+\lambda}_{Fl_J;X_1,\ldots,X_r}$ *de* Fl_J *est une fibration rationnellement connexe au-dessus de variété de Gromov-Witten* $\mathcal{W}^{g,[\pi(C)]}_{Fl_I;\pi(X_1),\ldots,\pi(X_r)}$ *de* Fl_I*.*

- *Pour x en position générale dans $(Fl_J)^r$, la fibre $ev^{-1}_{[C]+\lambda}(x)$ est une tour de fibrations unirationnelles au-dessus de la fibre $ev^{-1}_{[\pi(C)]}(\pi(x))$.*

Nous avons appelé une variété W une *fibration rationnellement connexe* (respectivement fibration unirationnelle) au-dessus de W' si il existe un morphisme dominant $W \to W'$ envoyant chaque composante irréductible de W sur une composante irréductible différente de W', et dont la fibre générale est une variété rationnellement connexe (respectivement unirationnelle). Une variété W est une *tour de fibrations unirationnelles* au-dessus d'une variété W' s'il existe une suite de morphismes

$$W_1 = W \to W_2 \to \cdots \to W_l = W'$$

telle que pour $1 \leqslant i < l$, la variété W_i est une fibration unirationnelle au-dessus de W_{i+1}.

3.8.5 Égalités entre corrélateurs de différentes variétés de drapeaux partiels.

Fixons des entiers $1 \leqslant i_1 < \cdots < i_m < n$. On considère la variété de drapeaux X paramétrant les drapeaux d'espaces vectoriels

$$V_{i_1} \subset \cdots \subset V_{i_m} \subset \mathbb{C}^\cdot$$

vérifiant $\dim V_{i_k} = i_k$. La classe $\mathbf{d} = (d_1, \ldots, d_m)$ d'une courbe C de X est déterminée par m entiers d_1, d_2,..., d_m, où d_i est le degré de Plücker de la projection de C sur la Grassmannienne $\mathrm{Gr}(n_i, n)$. Soit T un tore de GL_n agissant sur X par multiplication à gauche. Soient E_1, ..., E_r des fibrés vectoriels de X T-équivariants. Rappelons que les corrélateurs T-équivariants en K-théorie quantique sont donnés par

$$\langle \phi_1, \ldots, \phi_r \rangle^X_{T,(d_1,\ldots,d_m)} := \chi_T \left(ev_1^* E_1 \otimes \cdots \otimes ev_r^* E_r \otimes \mathcal{O}_{\overline{\mathcal{M}_{0,r}}(X,\mathbf{d})} \right), \tag{3.26}$$

où l'on a noté χ_T la caractéristique d'Euler-Poincaré T-équivariante. Pour un entier $1 \leqslant k \leqslant m$, on note $X_{\hat{k}}$ la variété obtenue à partir de X en oubliant l'espace vectoriel V_{i_k}, et

$$\pi_{\hat{k}} : X \to X_{\hat{k}}$$

le morphisme d'oubli. Afin de faciliter les notations, on note $d_0 = 0 = d_{m+1}$. Le chapitre 2 fournit l'égalité entre corrélateurs suivante-cf. Théorème 2.1.3.1bis.

Théorème. *Supposons:*

- $\forall\, p < k,\ d_k \geqslant n_k \left\lceil \frac{d_p - d_{p-1}}{n_p - n_{p-1}} \right\rceil$,
- $d_{k-1} \leqslant \lfloor \frac{d_{k+1}}{n_{k+1}} \rfloor$,
- $d_k \geqslant r(n_k - n_{k-1}) + d_{k-1} + (n_k - n_{k-1})(\lfloor \frac{d_{k+1} - d_{k-1}}{n_{k+1} - n_{k-1}} \rfloor + 1)$.

Alors pour tous fibrés vectoriels T-équivariants $E_1, \ldots, E_r$ de $X_{\hat{k}}$, les corrélateurs de X et de $X_{\hat{k}}$ associés au degré $\mathbf{d}$ *et aux fibrés E_i sont égaux:*

$$\langle (\pi_{\hat{k}})^* E_1, \ldots, (\pi_{\hat{k}})^* E_r \rangle^X_{T,(d_1,\ldots,d_m)} = \langle E_1, \ldots, E_r \rangle^{X_{\hat{k}}}_{T,(d_1,\ldots,d_{k-1},d_{k+1},\ldots,d_m)}.$$

Notons que dans le cadre des invariants de Gromov-Witten à trois points marqués, un résultat de Peterson/Woodward permet de déduire tous les invariants à 3 points marqués des variétés de drapeaux généralisées G/P des invariants à trois points marqués de G/B [Woo05].

3.8.6 Calcul de Schubert pour la variété d'incidence $Fl_{1,n-1}$. On s'intéresse dans le chapitre 3 de cette thèse à différentes variantes du calcul de Schubert moderne pour la variété d'incidence $X = Fl_{1,n-1}$ paramétrant les paires (p, H) où p est un point de $\mathbb{P}^{n-1}$ et H est un hyperplan de $\mathbb{P}^{n-1}$. Notons que X s'écrit sous la forme G/P, où $G = \mathrm{GL}_n$, T est l'ensemble des matrices diagonales, B est l'ensemble des matrices triangulaires supérieures et P est le sous-groupe parabolique de G vérifiant $B \subset P$ associé aux racines $\{\alpha_2, \ldots, \alpha_{n-1}\}$. Le représentant de plus petite longueur w d'un élément dans $W/W_P \simeq \mathfrak{S}_n/\mathfrak{S}_{n-2}$ est une permutation vérifiant $w(1) = i$, $w(n) = j$, et pour $1 < k < n-1$ $w(k) < w(k+1)$. On notera $w_{i,j}$ l'élément associé dans W/W_P. On peut considérer la variété $Fl_{1,n-1}$ comme une sous-variété de bidegré $(1,1)$ de $\mathrm{Gr}(1,n) \times \mathrm{Gr}(n-1,n) \simeq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, définie par la relation d'incidence $L \subset H$. Pour $1 \leqslant i \leqslant n$, on appelle

$$L_i := \{[x_1 : \cdots : x_i : 0 \ldots 0]\},$$

et

$$L^i := \{[0 \ldots 0 : x_i \ldots x_n]\}.$$

Les sous-variétés

$$X(i,j) := (L_i \times L^j) \cap Fl_{1,n-1} \subset Fl_{1,n-1} \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$$

sont les variétés de Schubert $X(w_{i,j})$ de $Fl_{1,n-1}$, pour $1 \leqslant i, j \leqslant n$ et $i \neq j$.

Coefficients de Littlewood-Richardson dans $K(Fl_{1,n-1})$. Pour $1 \leqslant i, j \leqslant n$, $i \neq j$, l'immersion naturelle $i : X(i,j) \hookrightarrow X$ définit un faisceau $i_* \mathcal{O}_{X(i,j)}$, qui est un faisceau cohérent de $\mathcal{O}_{Fl_{1,n-1}}$-modules. On note $\mathcal{O}_{i,j} := [i_* \mathcal{O}_{X(i,j)}]$ la classe de $i_* \mathcal{O}_{X(i,j)}$ dans le groupe de Grothendieck $K(Fl_{1,n-1})$ des faisceaux cohérents de $\mathcal{O}_{Fl_{1,n-1}}$-modules. Rappelons que les classes de Schubert $(\mathcal{O}_{i,j})_{1 \leqslant i,j \leqslant n,\ i \neq j}$ forment une base de $K(Fl_{1,n-1})$, et le produit

$$\mathcal{O}_{i,j} \cdot \mathcal{O}_{k,p} = \sum_{i=0}^{\dim Fl_{1,n-1}} (-1)^i [Tor_i^{Fl_{1,n-1}} \left(\mathcal{O}_{X(i,j)}, \mathcal{O}_{X(k,p)} \right)]$$

définit une structure d'anneau associatif et commutatif $(K(Fl_{1,n-1}), \cdot)$ ayant pour unité $[\mathcal{O}_{Fl_{1,n-1}}]$. On pose $\mathcal{O}_{i,j} = 0$ si $i < 1$ ou $j > n$.

Proposition (Coefficients de Littlewood-Richardson dans $K(Fl_{1,n-1})$-cf. Proposition 3.4.1.1)**.** *Soient $1 \leqslant i, j, k, p \leqslant n$, où $i \neq j$ et $k \neq p$. Alors*

$$\begin{cases} \mathcal{O}_{k,p} \cdot \mathcal{O}_{i,j} = \mathcal{O}_{i+k-n,j+p-1} & \text{si } i+k-n \geqslant j+p \text{ or } i < j \text{ or } k < p; \\ \mathcal{O}_{k,p} \cdot \mathcal{O}_{i,j} = \mathcal{O}_{i+k-n-1,j+p-1} + \mathcal{O}_{i+k-n,j+p} - \mathcal{O}_{i+k-n-1,j+p} & \text{sinon.} \end{cases}$$

Formule de Chevalley et algorithme positif dans $QK_s(Fl_{1,n-1})$. Notons $l_1 := X(2,1)$ et $l_2 := X(1,2)$ les deux variétés de Schubert de $Fl_{1,n-1}$ de dimension 1. Leurs classes $[l_1]$ et $[l_2]$ en homologie singulière de $Fl_{1,n-1}$ engendrent $H_2(Fl_{1,n-1},\mathbb{Z})$. Rappelons que l'anneau de *petite K-théorie quantique* $QK_s(Fl_{1,n-1}) = (K(Fl_{1,n-1}))\otimes\mathbb{Q}[Q_1,Q_2],\star)$ de $Fl_{1,n-1}$ est une déformation de l'anneau $K(Fl_{1,n-1})$ par les variables Q_1 et Q_2. Cette déformation dépend des corrélateurs $\langle \mathcal{O}_{i,j},\mathcal{O}_{k,p},\mathcal{O}_{s,t}\rangle_{d_1[l_1]+d_2[l_2]}$ définis dans la sous-partie 0.2.2. On retrouve l'anneau de K-théorie $K(Fl_{1,n-1})$ en prenant la limite $Q_1,Q_2\to 0$.

Notons $h_1 := X(n-1,1)$ et $h_2 := X(n,2)$ les deux variétés de Schubert de $Fl_{1,n-1}$ de codimension 1.

Proposition (Formule de Chevalley dans $QK_s(Fl_{1,n-1})$-cf. Proposition 3.7.0.1)**.**

$$\mathcal{O}_{h_1}\star\mathcal{O}_{k,p} = \begin{cases} Q_1\mathcal{O}_{n-1,n} + Q_1Q_2\left([\mathcal{O}_X]-\mathcal{O}_{h_1}\right) & \text{si } k=1, p=n \\ Q_1\mathcal{O}_{n,p} & \text{si } k=1,\ p<n \\ \mathcal{O}_{1,2} + Q_1\left([\mathcal{O}_X]-\mathcal{O}_{h_1}\right) & \text{si } k=2, p=1 \\ \mathcal{O}_{k-1,p} & \text{si } k>1,\ k\neq p+1 \\ \mathcal{O}_{p-1,p}+\mathcal{O}_{p,p+1}-\mathcal{O}_{p-1,p+1} & \text{si } 1<p<n-1,\ k=p+1 \end{cases}$$

$$\mathcal{O}_{h_2}\star\mathcal{O}_{k,p} = \begin{cases} Q_2\mathcal{O}_{1,2} + Q_1Q_2\left([\mathcal{O}_X]-\mathcal{O}_{h_2}\right) & \text{si } k=1, p=n \\ Q_2\mathcal{O}_{k,1} & \text{si } k>1,\ p=n \\ \mathcal{O}_{n-1,n} + Q_2\left([\mathcal{O}_X]-\mathcal{O}_{h_2}\right) & \text{si } k=n, p=n-1 \\ \mathcal{O}_{k,p+1} & \text{si } p<n,\ k\neq p+1 \\ \mathcal{O}_{p,p+1}+\mathcal{O}_{p-1,p}-\mathcal{O}_{p-1,p+1} & \text{si } 1<k<n-1,\ k=p+1 \end{cases}$$

Notons que pour X une variété cominuscule, une formule de Chevalley en petite K-théorie quantique est donnée par Buch-Chaput-Mihalcea-Perrin [BCMP16].

Appelons ici un algorithme *positif* si à chaque itération le calcul effectué est une somme de coefficients de mêmes signes.

Proposition. *L'algorithme donné partie 3.8.1 est un algorithme positif calculant les coefficients de Littlewood-Richardson dans $QK_s(Fl_{1,n-1})$.*

Corollaire. *(Cf. Proposition 3.8.2.1) L'anneau de petite K-théorie quantique $QK_s(Fl_{1,n-1})$ est positif au sens de la définition 0.3.3.1.*

Conjecture. *La formule fermée décrite dans la partie 3.8.3 donne les coefficients de Littlewood-Richardson dans $QK_s(Fl_{1,n-1})$.*

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