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$$\rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \nabla \cdot T + f$$

$$e^{i\pi} + 1 =$$

THÈSE DE DOCTORAT

INVESTIGATION DES CATÉGORIES FINIES

Najwa GHANNOUM

Laboratoire de Mathématiques J. A. Dieudonné (LJAD)

Présentée en vue de l'obtention du grade de:

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L'Université Libanaise, L'École Doctorale des Sciences et Technologie

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12 Décembre 2022

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Je dédie cette thèse à mes parents.

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L'objectif principal de cette thèse est de classer les catégories finies en fonction de la structure algébrique de leurs monoïdes d'endomorphismes. À l'aide d'un logiciel appelé `PROVER9/MACE4`, nous dénombrons les catégories avec deux objets tel que chaque ensemble de morphismes a un cardinal égal à 3. Les données nous incitent à étudier la structure des monoïdes afin de mieux comprendre le problème d'énumération. Dans cette thèse, nous étudions quatre types de monoïdes : monoïdes *de type groupe*, *semigroupes simples*, *bandes rectangulaires* et monoïdes avec un élément 0 .

Les *monoïdes de type groupe* sont des monoïdes construits à partir d'un groupe auquel on ajoute un ensemble ordonné d'idempotents. Nous étudions l'action des groupes et l'interaction entre eux. Nous donnons également quelques propriétés sur les ensembles d'idempotents.

Les *semigroupes simples* sont des semigroupes dont le seul idéal est le semigroupe lui-même. Nous prouvons que si nous avons un monoïde alors il contient toujours un unique semigroupe simple et nous pouvons construire une catégorie à deux objets, telle que l'un des objets soit le semigroupe simple et l'autre un groupe.

Les *bandes rectangulaires* sont des semigroupes idempotents avec la propriété $xyz = xz$. Nous prouvons que la matrice d'une catégorie à bandes rectangulaires comme monoïdes d'endomorphisme a certaines restrictions sur ses coefficients.

Les *monoïdes avec un élément 0* sont des monoïdes qui ont un élément absorbant. Nous prouvons que l'existence d'un élément 0 dans les monoïdes induit l'existence d'un élément 0 dans chaque ensemble de morphismes.

Les résultats sur ces monoïdes clarifient les données obtenues à partir de l'énumération et nous aident à en donner une explication.

Mots clés : catégories finies, classification, associativité, `prover9/mace4`, semigroupes, monoïdes de type groupe, catégories de type groupe, semigroupes simples, bandes rectangulaires, catégories avec un 0 .

The main objective of this thesis is to classify finite categories in terms of the algebraic structure of their endomorphism monoids. Using a program called `PROVER9/MACE4`, we give a count to the number of categories with two objects such that every set of morphisms has cardinal 3. The data inspires us to study the structure of the monoids in order to understand the enumeration problem more. In this thesis, we study four types of monoids: *grouplike*, *simple semigroups*, *rectangular bands* and *zero monoids*.

Grouplike monoids are monoids such that they contain a subgroup and an ordered set of idempotents. We study the action of the groups and the interaction between each others. We also give some properties about the idempotent sets.

Simple semigroups are semigroups whose only ideal is the semigroup itself. We prove that if we have a monoid then it always contains a unique simple semigroup and we can construct a category with two objects, such that one of the objects is the simple semigroup and the other is a group.

Rectangular bands are idempotent semigroups with the property $xyz = xz$. We prove that the matrix of a category with rectangular bands as endomorphism monoids has some restrictions on its coefficients.

Zero monoids are monoids that have a zero element. We prove that the existence of a zero element in the monoids induces the existence of a zero element in each set of morphisms. The results about these monoids clarify the data obtained from the enumeration and help us give an explanation to it.

Keywords: finite categories, classification, associativity, prover9/mace4, semigroups, grouplike monoids, grouplike categories, simple semigroups, rectangular bands, zero categories.

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CHAPTER 1

Introduction

1.1 Overview and motivation

Over the years, we have seen many different definitions of a *category*. Steve Awodey defined category theory as the mathematical study of (abstract) algebras of functions. Just as group theory is the abstraction of the idea of a system of permutations of a set of symmetries of a geometric objects, category theory arises from the idea of a system of functions among some objects.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \downarrow g \\ & f \cdot g & C \end{array}$$

A *category* is an algebra, consisting of objects A, B, C, \dots and arrows $f : A \rightarrow B$, $g : B \rightarrow C, \dots$, that are closed under composition and satisfy certain conditions typical of the composition of functions: associativity and identity axioms [7].

In another definition, a *category* is a system of related objects. The objects do not live in isolation: there is some notion of map between objects, binding them together [25].

We can also see categories *themselves* as mathematical objects, and consider maps between them, called *functors*. Along with the same idea, we can go to a higher level and talk about maps between functors, which are called *natural transformations*. In fact, the definition of a category was presented by Samuel Eilenberg and Saunders Mac Lane after the notion of *natural transformation* was used, in the purpose of generalizing the notion [18].

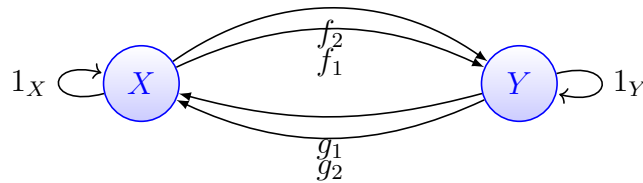
The use of *category theory* has been extended to other areas than in algebraic topology. It actually takes an essential place in pure mathematics. Moreover, we can see some great applications of category theory in computer science and in theoretical physics.

In our work, we focus on *finite categories*, which are categories with a *finite* set of objects and a *finite* set of morphisms. Berger and Leinster represented finite categories as graphs

and associated matrices to these graphs [9], where the entries of the matrix is the number of morphisms between two objects. Going from a category towards a matrix can be done in a unique way, but inversely that's not true. In another point of view, we can represent the relation between finite categories and matrices as a function. This function is not injective, as, for example, a matrix of size 1 could correspond to more than one category, since for instance, there are two distinct monoids with 2 elements. The function is also not surjective, even if we restrict to matrices with non zero diagonal coefficients, because the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

which corresponds to the graph



admits no category, i.e. no pre-image. The reason for that is the following:

$$g_1 = g_1 \circ (f_1 \circ g_2) = (g_1 \circ f_1) \circ g_2 = g_2$$

which highlights the importance of associativity in such structures. The question is then: *what are the matrices that admit a category?*

From the previous example, in 2008, Berger and Leinster proved that if the diagonal coefficients of a matrix are at least 2, then it always admits a category [9]. But Allouch and Simpson wanted to know more, for example what if we want some diagonal coefficients to be 1? In 2014, Allouch and Simpson have shown that a category is associated to a square matrix under certain conditions on the coefficients of the matrices in terms of their determinants [2].

- For example, if we take a matrix M of size 2 with strictly positive integers, and if one of the diagonal coefficients is 1, then $\det(M) \geq 1$, where $\det(M)$ is the determinant of M .
- More generally if we a matrix M of size n having a single 1 on the diagonal coefficients, then we apply the above condition to every sub-matrix of of M of order 2.
- If there is more than one 1 on the diagonal coefficients of a strictly positive matrix,

then the matrix admits no reduced category (in the non-reduced case it would just mean that the coefficients in the corresponding block are 1).

From here, the purpose of the thesis is to find a count of finite categories associated to certain matrices. We don't actually expect an exact number of categories for any square matrix, but we expect that if we fix specific endomorphism monoids of the objects of the category, we could obtain some properties of the category and eventually the exact number according the coefficients of the matrix.

Counting associative structures has been a problem for years. We introduce some of the previous work in this area:

- (2009) A. Distler and T. Kelsey counted the monoids of orders eight, nine and ten. They weren't able to achieve more than that as the number of semigroups of order 10 was unknown [17].
- (2012) the number of semigroups of order 10 was shown by A. Distler, C. Jefferson, T. Kelsey, and L. Kotthoff [16].
- (2014) G. Cruttwell and R. Leblanc introduced the question: *How many categories are there with n morphisms?* It means with a total number of morphisms distributed between objects randomly. They compared the numbers obtained with the number of monoids of order n , which lead to almost the same numbers up to order 10 [15].
- (2017) S. Allouch and C. Simpson counted the categories whose number of morphisms between each two objects is 2. They were able to get an exact count up to order 3, and bounds for a general size [4].

Now, each of the problems stated above was solved differently than the other. Some relied on the use of computer algebra systems to obtain the count, and some did the count by hand. But all of them were blocked at a certain point because of some obstacles, mostly that the used techniques are limited to very small sizes of structures.

The enumeration of associative structures, together with their classification, presents a deep challenge for algebraic combinatorics. The use of computer algebra systems and tools from artificial intelligence have been applied to great success for finite monoids [17], but these techniques have not been applied in the setting of finite categories. Each of the latter consists of a finite set of objects and a finite set of morphisms between them. As associative algebraic structures, these present an avenue for deepening our understanding of enumeration and classification problems in associative algebra in general.

Our method of analyzing finite categories is to use computational resources to generate statistics regarding finite categories of a certain type, and then analyze this data in terms of certain features of the objects. This yields, *inter alia*, an exact count of the number of categories with two non-isomorphic objects and such that all hom-sets have size 3. The objects of such finite categories may be identified with their endomorphism monoids, and the structure of the categories may be analyzed by reference to properties of these endomorphism monoids.

This method relies on the use of McCune’s `PROVER9/MACE4` [27, 26] to construct models. We give a representation of categories as semigroups with a zero (Chapter 4), then using a program written in `PYTHON` (Appendix 1) to generate optimized semigroup equations as an input for `MACE4`ⁱ, we generate non-isomorphic algebraic models satisfying those equations. This provides a count of the corresponding finite categories.

This work follows the line of research began in [3, 9] and continued in [4], viewing categories as associated to certain square matrices. For example, the matrix

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

corresponds to categories with two objects and exactly 3 morphisms between any pair of objects. Viewed from this perspective, our motivation is to study the structure of finite categories for a fixed matrix type. The coefficients on the diagonals constrain the structure and nature of the objects by giving restrictions on their endomorphism monoids, which, when considered as fixed parameters, give insight into the enumeration and classification problem. In particular, we obtain information about the number of categories that can be constructed when the endomorphism monoids of the objects are fixed. Our data shows that some types of endomorphism monoids, such as semilattices, give more options to build categories. This way we also discover which structural properties of finite monoids allow their combinations when realized as objects in categories of a given type.

Allouch and Simpson inaugurated the counting problem in [4], where they count the number of categories associated to matrices whose coefficients are all 2. The calculations in this work are performed by hand, up to matrices of size 3. Using our new methods, we extend the Allouch-Simpson count to the matrices of size 2 whose coefficients are all 3.

In this thesis, we present, using `MACE4`, the number of categories with two objects, where each set of morphisms has cardinal 3. We obtain that there exist 362 non-isomorphic categories of this form. The counting results inspired us to study the categories obtained

ⁱ`MACE4` is a program that searches for finite models of first-order theories.

in terms of their endomorphism monoids. We noticed that the monoids structure plays a very important in the classification, because the number of categories with two specific monoids is not distributed randomly, and sometimes there does not exist any category between two (specific) monoids.

In particular, there are some interesting cases that appear where the categories behave in certain ways, which we study in a general form. In order to study categories in terms of their endomorphism monoids, we represent the matrix of a category as a matrix of monoids and bimodules, we call it *the algebraic matrix*. It means that when we fix the monoids of a category, we can see the sets of morphisms as bimodules with actions on the left and on the right of the respective monoids, such that the actions commute (Chapter 5).

Starting from this idea, we can now discuss finite categories with specific endomorphism monoids. In this thesis, we discuss four types of monoids (or semigroups): *grouplike monoids*, *simple semigroups*, *rectangular bands* and *zero semigroups*. We mainly give a complete classification for grouplike monoids and simple semigroups in finite categories. A *grouplike monoid* is a group G to which we add an ordered set of idempotents (or identities) I , and we denote them by G^{*k} , where k is the cardinal of I . Then a *grouplike category* is a category whose endomorphism monoids are grouplike.

We start with a grouplike G^{*i} and we construct a category with two objects and four copies of G^{*i} . Then we construct a grouplike category from the previous one and we prove that every grouplike category comes from this construction. We also conclude that the number of grouplike categories will depend mostly on the number of bimodules obtained.

A *simple semigroup* is a semigroup S whose only ideal is S itself. By Rees-Sushkevich theorem, we have that a finite semigroup is simple if and only if it is isomorphic to some Rees matrix semigroup [28]. We prove that if we have a simple semigroup S , then we can construct a category with two objects such that one of the objects is S and the other object is a group. Inversely, if we have a category with two objects such that one of the object is a group, then the other monoid contains a simple semigroup (that has the structure of a Rees matrix semigroup).

We can see from the previous two paragraphs the importance of the structure of monoids inside a category, because it imposes some properties and conditions on the whole category structure.

We study other types of monoids as well, for example monoids that have a zero element, and monoids that are rectangular bands. In the case of monoids with a zero, such that all the sets of morphisms are not empty, we prove that every set of morphisms contain a

zero element and it's unique.

For the case of rectangular bands, we prove that there are some restrictions on the cardinal of sets of morphisms.

1.2 Organization of the thesis

Chapter 2: Finite categories and matrices

We present some preliminaries about categories and finite categories in general. We define finite categories and functors. We then give a representation as matrices and study the relation between them. The matrix are the number of morphisms between each two objects. We are mainly interested in categories with non isomorphic objects, we call them *reduced categories*. because otherwise, we can always reduce the category to a reduced one. The results obtained show that a matrix M could have more than one category associated to it and sometimes it could have no categories at all. Which raised the questions: “what is the reason that some matrices do not have a category?” and “what are the matrices that admit a category?” In some cases where there does not exist a category, the reason depends on the determinant of the matrix. For example if we have one set of endomorphisms of cardinal 1, this means it contains only the identity, the determinant of the matrix of size two should be greater or equal to 1.

Chapter 3: Monoids of orders 2, 3 and 4

We classify monoids of orders 2, 3 and 4. There are 2 monoids of order 2, 7 monoids of order 3, and 35 monoids of order 4. The reason that we classify monoids in the first place is that their algebraic structure seems to be very important in the classification problem of finite categories viewing the endomorphism monoids as the objects of a category. We divide the monoids obtained in each case according to their algebraic structure. Which allows us to obtain algebraic properties about categories with specific endomorphism monoids.

We mainly divide the monoids in the following way:

- Monoids that are groups.
- Monoids that contain a group.
- Monoids that have a zero (including left and right rectangular bands).

The number and the structure of these monoids are given in [1] and [21].

Chapter 4: Representation of categories as semigroups with a zero

We represent categories as semigroups with a zero. There are several cases to discuss

here. First case, if the category doesn't have zero elements, second case, if the category has zero elements and they go to different zeros in the semigroup, third case, if the all the zeros go to the same zero in the semigroup. In the last case, it would be difficult to retrieve all the zeros in the category from a semigroup.

Also, in this chapter, we study categories with a zero, these are categories where every set of morphisms has a unique zero element. We define a set \mathcal{Z} of zeros, and we say that the multiplication by an element of \mathcal{Z} is an element of \mathcal{Z} . We prove the following

Proposition 1.1. *(It is Proposition 4.20) Let \mathcal{C} be a finite category whose objects are X and Y , such that $\mathcal{C}(X, Y)$ and $\mathcal{C}(Y, X)$ are not empty. If $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$ have zeros 0_X and 0_Y , then there exists a unique zero $0_{XY} \in \mathcal{C}(X, Y)$ and a unique zero $0_{YX} \in \mathcal{C}(Y, X)$ where 0_{XY} and 0_{YX} are of the form*

$$0_{XY} = 0_X \cdot f \cdot 0_Y \text{ for all } f \in \mathcal{C}(X, Y)$$

and

$$0_{YX} = 0_Y \cdot g \cdot 0_X \text{ for all } g \in \mathcal{C}(Y, X).$$

Chapter 5: Bimodules

In this chapter we represent categories in terms of their bimodules. Suppose we have a category \mathcal{C} associated to the matrix

$$M = \begin{pmatrix} A & L \\ R & B \end{pmatrix}$$

then a bimodule of \mathcal{C} is a category \mathcal{C}_1 associated to the matrix

$$N_1 = \begin{pmatrix} A & L \\ \emptyset & B \end{pmatrix}$$

and the other bimodule is a category \mathcal{C}_2 associated to the matrix

$$N_2 = \begin{pmatrix} A & \emptyset \\ R & B \end{pmatrix}.$$

We call M the algebraic matrix of \mathcal{C} .

Chapter 6: Grouplike categories

This chapter illustrates what we've been saying all along about the importance of the nature of monoids of a category. We study in particular a specific kind of monoids that we call *grouplike monoids*, those are monoids of the form $G \cup I$, where G is a group and

$I = \{e_1, \dots, e_k\}$ such that

$$(e_i \cdot e_j = e_i \text{ and } e_j \cdot e_i = e_i) \text{ if and only if } i \leq j.$$

Then, a *grouplike category* is a category whose endomorphism monoids are grouplike.

Theorem 1.2. *(It is Theorem 6.4) Let G be a group and G^{*i} a grouplike monoid of the form $G \cup I$ such that $I = \{e_1, \dots, e_i\}$. Let $M^{(1)}$ be a matrix of the form*

$$\begin{pmatrix} G^{*i} & G^{*i} \\ G^{*i} & G^{*i} \end{pmatrix}$$

*of similar copies of G^{*i} , let $\mathcal{C}^{(1)}$ be the groupoid-like category associated to this matrix. Then we can extend the endomorphism sets G^{*i} and we obtain a category $\mathcal{C}^{(2)}$ associated to the matrix $M^{(2)}$*

$$\begin{pmatrix} G^{*k_1} & G^{*i} \\ G^{*i} & G^{*k_2} \end{pmatrix}$$

*such that for all $x \in G^{*i}, y \in G^{*k_1}$, we have $x \cdot y = y \cdot x \in G^{*i}$. Same for G^{*k_2} . And there exists $\mathcal{C}^{(2)}$ a finite category associated to $M^{(2)}$.*

Now let $k_1, k_2 \geq i$, suppose that we have the matrix $M^{(3)}$

$$\begin{pmatrix} G^{*k_1} & L \\ R & G^{*k_2} \end{pmatrix}$$

such that L and R are strongly i -unigen (Definition 6.1). Then

$$M^{(1)} \subseteq M^{(2)} \subseteq M^{(3)}.$$

And $M^{(3)}$ is a matrix of a unique grouplike category, denote it by $\mathcal{C}^{(3)}$. We have

$$\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \mathcal{C}^{(3)}.$$

We prove that every grouplike category comes from the construction described above (Theorem 6.5).

Chapter 7: Rectangular bands

In this chapter we treat rectangular bands seen as endomorphism monoids of a category. A *rectangular band* S is an *idempotent* semigroup having the property that $xyz = xz$ for all $x, y, z \in S$. It can also be seen as a set $I \times J$ such that

$$(i, j) \cdot (i', j') = (i, j')$$

In that case we say that S is a rectangular band of size $m \times n$ where $|I| = m$ and $|J| = n$.

Thus, a *left rectangular band* is a rectangular band of size $m \times 1$ and a *right rectangular band* is a rectangular band of size $1 \times n$.

We let

$$M = \begin{pmatrix} I \times J & X \\ Y & A \times B \end{pmatrix}$$

and \mathcal{C} be a category associated to M whose endomorphism monoids are rectangular bands $I \times J$ and $A \times B$ of size mn and pq respectively.

Theorem 1.3. (*It is Theorem 7.8*) *The matrix M admits a category if and only if*

$$|X| \geq mp \text{ and } |Y| \geq nq$$

The idea is to find a representation of the elements that are in X and Y which satisfy the equalities in the above condition. And to understand the role of the extra elements in this case.

Chapter 8: Simple semigroups in finite categories

A *simple semigroup* is a semigroup whose only ideals are \emptyset and itself. By Rees-Sushkevich theorem, we have that a finite semigroup is simple if and only if it is isomorphic to some Rees matrix semigroup [28].

In this chapter, we make a correspondence between simple semigroups and finite categories with two objects where one of the objects is a simple semigroup and the other is a group. (**From a finite monoid to a category**). We prove that every finite monoid M (that is not a group) contains a unique simple semigroup S (different from M), and a group G by taking the intersection of a minimal left ideal L and a minimal right ideal R of M . Then we can construct a category with two objects where one of the objects is S and the other one is G such that

$$|S| = \frac{|L| \cdot |R|}{|G|}.$$

(**From a category to a simple semigroup**). In this section, we start with a finite category with two objects such that one of the objects is a group. We prove that the other monoid contains a simple semigroup, and that the two constructions are isomorphic.

Chapter 9: Matrices of order 2 and coefficients 3

In this chapter, using MACE4 we give a count of categories associated to the matrix

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

which is 362 categories. We divide this number and specify the number of categories

having two specific monoids. We apply all of the results in the previous chapters to analyze the data obtained and to understand the importance of the algebraic structure of each monoid.

CHAPTER 2

Finite categories and matrices

In this chapter, we introduce some properties about finite categories and we present other perspectives to see them. We also introduce some of the results obtained by S. Allouch and C. Simpson in their work in [3]. Let's start with defining a category.

2.1 Categories and Functors

Definition 2.1. A *category* \mathbf{C} consists of:

- a class $\mathbf{Ob}(\mathbf{C})$ of objects,
- a class $\mathbf{Hom}_{\mathbf{C}}(\mathbf{X}, \mathbf{Y})$ of morphisms, or arrows, or maps between the objects, for $X, Y \in \mathbf{Ob}(\mathbf{C})$,
- a domain, or source object class function $\mathbf{dom} : \mathbf{Hom}_{\mathbf{C}}(X, Y) \rightarrow \mathbf{Ob}(\mathbf{C})$,
- a codomain, or target object class function $\mathbf{cod} : \mathbf{Hom}_{\mathbf{C}}(X, Y) \rightarrow \mathbf{Ob}(\mathbf{C})$,
- for every three objects X, Y and Z , a binary operation $\mathbf{Hom}_{\mathbf{C}}(X, Y) \times \mathbf{Hom}_{\mathbf{C}}(Y, Z) \rightarrow \mathbf{Hom}_{\mathbf{C}}(X, Z)$ called *composition of morphisms*; the composition of $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is written as $g \circ f$.

such that the following axioms hold:

- **(associativity)** if $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : Z \rightarrow T$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
- **(identity)** for every object X , there exists a morphism $1_X : X \rightarrow X$ (also denoted by id_X) called the identity morphism of X , such that every morphism $f : A \rightarrow X$ satisfies $1_X \circ f = f$, and every morphism $g : X \rightarrow B$ satisfies $g \circ 1_X = g$.

Notation 2.2. In our work, composing morphisms in the other direction is more practical. Then we denote $g \circ f = f \cdot g = fg$.

Example 2.3. $\mathcal{T}op$: category of topological spaces and continuous applications.

- (i) Objects: the topological spaces (X, \mathcal{T}) ,
- (ii) Morphisms: $Hom_{\mathcal{T}op}(X, Y) = \{f : X \rightarrow Y \text{ continuous}\}$,
- (iii) Composition: \circ of continuous applications,
- (iv) \circ is associative.

Definition 2.4. Let \mathcal{C} be a category. \mathcal{C}^{op} is the *opposite* category to \mathcal{C} , such that:

- 1. $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$,
- 2. $Hom_{\mathcal{C}^{op}}(X, Y) = Hom_{\mathcal{C}}(Y, X)$ for all $X, Y \in Ob(\mathcal{C}^{op})$,
- 3. $Hom_{\mathcal{C}^{op}}(X, X) \ni 1_X = 1_X \in Hom_{\mathcal{C}}(X, X)$ for all $X \in Ob(\mathcal{C}^{op})$,
- 4. $Hom_{\mathcal{C}^{op}}(X, Z) \ni f \circ_{\mathcal{C}^{op}} g = g \circ_{\mathcal{C}} f \in Hom_{\mathcal{C}}(Z, X)$ for all $f \in Hom_{\mathcal{C}^{op}}(X, Y)$ and $g \in Hom_{\mathcal{C}^{op}}(Y, Z)$.

Definition 2.5. A category \mathcal{B} is said to be *sub-category* of \mathcal{C} and denoted by $\mathcal{B} \subset \mathcal{C}$ iff:

- 1. The objects of \mathcal{B} are objects of \mathcal{C} ,
 - 2. the identities of \mathcal{C} are in \mathcal{B} ,
 - 3. for all X, Y objects of \mathcal{B} we have $Hom_{\mathcal{B}}(X, Y) \subset Hom_{\mathcal{C}}(X, Y)$,
 - 4. if $g \circ f : X \rightarrow Y \rightarrow Z$ in \mathcal{B} , then $g \circ_{\mathcal{B}} f = g \circ_{\mathcal{C}} f$.
- \mathcal{B} is said to be *full* sub-category of \mathcal{C} if there exists equality in (2).

Definition 2.6. Let \mathcal{C} be a category, we say that \mathcal{C} is a *finite* category of order n if and only if the sets $Ob(\mathcal{C})$ and $Hom(X, Y)$ are finite for all $X, Y \in Ob(\mathcal{C})$, i.e.:

- there exists a bijection $\delta : Ob(\mathcal{C}) \rightarrow \{1, \dots, n\}$

$$a \mapsto \delta(a) = i$$

we denote a by x_i and the set of objects becomes $Ob(\mathcal{C}) = \{x_1, \dots, x_n\}$.
- for all $X, Y \in Ob(\mathcal{C})$, there exists $m \in \mathbb{N}$ and a bijection,

$$\begin{aligned} \lambda : Hom(X, Y) &\rightarrow \{1, \dots, m\} \\ p &\mapsto \lambda(p) = 1 \end{aligned}$$

we denote p by f_i , and the set of morphisms from X to Y becomes $Hom(X, Y) = \{f_1, \dots, f_m\}$.

In other words, \mathcal{C} is a finite category whose objects are $\{x_1, \dots, x_n\}$, and for all $x_i, x_j \in Ob(\mathcal{C})$ we have $Hom_{\mathcal{C}}(x_i, x_j) = \{f_1, \dots, f_{m_{ij}}\}$.

Definition 2.7. A (totally) *ordered* finite category \mathbf{C} is a finite category, associated with a total order relation over the set of objects. If \mathbf{C} is a finite ordered category of order n , then there exists a unique numbering $\text{Ob}(\mathbf{C}) = \{x_1, \dots, x_n\}$ compatible with the order, i.e. $x_i < x_j \Leftrightarrow i < j$.

Definition 2.8. Let \mathbf{C} and \mathbf{C}' be two categories. A functor $F : \mathbf{C} \rightarrow \mathbf{C}'$ consists of a map $F : \text{Ob}(\mathbf{C}) \rightarrow \text{Ob}(\mathbf{C}')$ and for all $X, Y \in \text{Ob}(\mathbf{C})$, of a map still denoted by $F : \text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{C}'}(F(X), F(Y))$ such that

$$F(id_X) = id_{F(X)}, \quad F(f \circ g) = F(f) \circ F(g).$$

A contravariant functor from \mathbf{C} to \mathbf{C}' is a functor from \mathbf{C}^{op} to \mathbf{C}' . In other words, it satisfies $F(f \circ g) = F(g) \circ F(f)$. If it is not contravariant, then it is also called a covariant functor.

Example 2.9. The forgetful functor $for : \mathcal{T}op \rightarrow \mathbf{Set}$ associates to a topological space X the set X , and to a continuous map f the map f .

Definition 2.10. Let \mathbf{C} be a category of order n with objects x_1, \dots, x_n , we say that x_i and x_j are isomorphic if there exists $f \in \mathbf{C}(x_i, x_j)$ and $g \in \mathbf{C}(x_j, x_i)$ such that $g \circ f = 1_{x_i}$ and $f \circ g = 1_{x_j}$.

Definition 2.11. Let $F : \mathbf{C} \rightarrow \mathbf{C}'$ be a functor.

- (i) F is called *faithful* (resp. *full*, resp. *fully faithful*) if for $X, Y \in \text{Ob}(\mathbf{C})$, $\text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{C}'}(F(X), F(Y))$ is injective (resp. surjective, resp. bijective).
- (ii) F is called *essentially surjective* if for each $Y \in \text{Ob}(\mathbf{C}')$ there exists $X \in \text{Ob}(\mathbf{C})$ and an isomorphism $F(X) \simeq Y$.

Notation 2.12. We denote $\text{Hom}_{\mathbf{C}}(x_i, x_j)$ by $\mathbf{C}(x_i, x_j)$.

Definition 2.13. The notions of *semi-category* is obtained from the definition of a category by removing the condition of the existence of an identity for each object.

If we have a non empty set $U \subset \text{Ob}(\mathbf{C})$ such that there exists id_x for $x \in U$, but not necessarily for $x \notin U$, then we say that (\mathbf{C}, U) is a partially unitary semi-category.

The formal addition of identities allows us to pass from a partially unitary semi-category to a category.

2.2 Finite categories and matrices correspondence

The goal of this section is to study the correspondence between finite categories of order n and square matrices of size n .

We denote by $\mathcal{M}_n(\mathbb{N})$ the set of square matrices of size n , whose entries are natural integers. If \mathbf{C} is a finite category with $Ob(\mathbf{C}) = \{x_1, \dots, x_n\}$, we define its matrix $M_{\mathbf{C}} = (m_{i,j}) \in \mathcal{M}_n(\mathbb{N})$ by

$$m_{ij} = m_{i,j} := |Hom_{\mathbf{C}}(x_i, x_j)|$$

which is the cardinal of the set of morphisms whose source is x_i and target x_j .

Definition 2.14. For $M \in \mathcal{M}_n(\mathbb{N})$, we say that M is a *categorical matrix* if there exists a finite category \mathbf{C} of order n such that $M = M_{\mathbf{C}}$.

We denote $Cat(M) = \{\mathbf{C} \text{ finite category of order } n \mid M = M_{\mathbf{C}}\}$.

Then M is a categorical matrix if and only if $Cat(M) \neq \emptyset$.

If $M \in \mathcal{M}_n(\mathbb{N})$ and $N \in \mathcal{M}_k(\mathbb{N})$, we say that $N = (n_{uv})$ is a *regular sub-matrix* of $M = (m_{ij})$ if there exists an ordered subset

$$\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$$

such that $n_{uv} = m_{i_u i_v}$.

If \mathbf{C} is a finite category of order n , and if $\mathbf{B} \subset \mathbf{C}$ is a full sub-category, with an order that could be different than the order of \mathbf{C} , then $M_{\mathbf{B}}$ is a regular sub-matrix of $M_{\mathbf{C}}$.

2.3 Reduced categories and matrices

Definition 2.15. We say that a category \mathbf{C} is reduced if every two distinct objects are not isomorphic.

Remark 2.16. In this thesis, we will most often be supposing that the categories are reduced.

Definition 2.17. We say that a matrix M is not reduced if there exists $i \neq j$ such that

$$\forall k, M_{ki} = M_{kj}$$

and

$$\forall k, M_{ik} = M_{jk}.$$

Remark 2.18. From the previous two definitions, we can conclude that if \mathbf{C} is not reduced then M is not reduced. Then by contraposition, if M is reduced so is \mathbf{C} . The converse is not always true: we can have a reduced category \mathbf{C} such that $M_{\mathbf{C}}$ is not reduced. For example, we have a reduced category \mathbf{C} with the following non-reduced matrix of order:

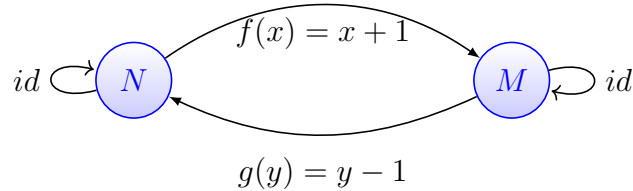
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Theorem 2.19. *If M is a non-reduced matrix, there is a reduced sub-matrix N of M such that M has a category if and only if N has a category.*

Definition 2.20. Let $M \in \mathcal{M}(\mathbb{N})$. Define

$$Cat^{red}(M) := \{C \text{ finite reduced category of order } n \mid M = M_C\}.$$

Remark 2.21. We define $Cat^{red}(M)$ because some matrices could admit a non-reduced category but can't admit a reduced one. For example, let $n \in \mathbb{N}^*$ and let C be a category whose objects are $\{N, M\}$ such that $N = \{1, \dots, n\}$ and $M = \{2, \dots, n + 1\}$, and the morphisms are defined by the following diagram



then C is associated to the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and C is not reduced.

Lemma 2.22. *Let M be a reduced matrix with $M_{i,j} > 0$ for all i, j . If there exists $i \neq j$ such that $M_{i,i} = M_{j,j} = 1$ then $Cat^{red}(M) = \emptyset$.*

2.4 Strictly positive matrices

Let $M = (a_{ij})$ be a square matrix of order n with positive coefficients.

Theorem 2.23 (Berger and Leinster [9]). *Let $M = (m_{ij})$ be a square matrix whose coefficients are positive integers such that for all i , $m_{ii} \geq 2$, then $Cat^{red}(M) \neq \emptyset$.*

C. Simpson and S. Allouch presented the proof of the existence of finite categories associated to matrices taking many cases of matrices of different orders. We state the most important results in their work [3]:

Matrices of order 2

Theorem 2.24. *Let M be a square matrix of order 2 defined by*

$$\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$$

with $b, c, d > 1$. $Cat^{red}(M) \neq \emptyset \iff d \geq bc + 1$.

Corollary 2.25. Let M be a square matrix of order 2 defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with a, b, c, d strictly positive. Then $Cat^{red}(M) \neq \emptyset$ iff

$$\begin{cases} a = b = c = d = 1 \\ \text{or } a = 1 \text{ and } d \geq bc + 1 \\ \text{or } d = 1 \text{ and } a \geq bc + 1 \\ \text{or } a > 1 \text{ and } d > 1. \end{cases}$$

Matrices of order 3

Theorem 2.26. Let M be a matrix of order 3 with strictly positive integers, defined by

$$\begin{pmatrix} 1 & a & b \\ c & n & m \\ p & q & r \end{pmatrix}.$$

Then

$$\begin{cases} n = ac + 1 \\ r = bp + 1 \\ m = bc \\ q = ap \end{cases} \implies Cat^{red}(M) \neq \emptyset.$$

Corollary 2.27. Let M be a matrix of order 3 with strictly positive integers, defined by

$$\begin{pmatrix} z & a & b \\ c & n & m \\ p & q & r \end{pmatrix}.$$

Then $Cat^{red}(M) \neq \emptyset$ if and only if

$$z, n, r > 1$$

or

$$z = 1 \text{ and } n \geq ac + 1, r \geq bp + 1, m \geq bc, q \geq ap$$

or

$$n = 1 \text{ and } z \geq ac + 1, r \geq mq + 1, b \geq am, p \geq cq$$

or

$$r = 1 \text{ and } z \geq bp + 1, n \geq mq + 1, a \geq bq, c \geq mp.$$

Matrices of order n

Theorem 2.28. *Let $M = ((m_{ij}))_{1 \leq i, j \leq n}$ be a strictly positive matrix of order n , such that $m_{11} = 1$ and $m_{ii} > 1$ for all $2 \leq i \leq n$, then*

$$Cat^{red}(M) \neq \emptyset \text{ if and only if } \begin{cases} m_{ii} > m_{i1}m_{1i} & \forall i > 1 \\ m_{ij} > m_{i1}m_{1j} & \forall i > j \end{cases}$$

Remark 2.29. If there exists $j \neq 1$ such that $m_{jj} = 1$, then there is no reduced category.

CHAPTER 3

Monoids of orders 2, 3 and 4

In this chapter, We present the monoids of order 2 and 3, and we give some examples of the monoids of order 4. we will talk about their algebraic structures, that seem to have a huge impact on the classification and counting problem. The importance of the monoids structure is explained later on in the chapters.

We start with some preliminary definitions which will help us classify each monoid according to its algebraic structure.

Definition 3.1. A *monoid* A is a set equipped with a binary operation $\cdot : A \times A \rightarrow A$ such that \cdot is associative and there exists an identity element e such that for every element $a \in A$, the equations $e \cdot a = a$ and $a \cdot e = a$ hold.

Definition 3.2. A *semigroup* S is a set equipped with a binary operation $\cdot : S \times S \rightarrow S$ such that \cdot is associative.

Definition 3.3. A *band* S is an idempotent semigroup, i.e. for all $x \in S$, $x^2 = x$. A *semilattice* is a commutative band.

Definition 3.4. A *left zero semigroup* S is a semigroup that has a left zero element, denote it by s , this means that

$$s \cdot x = s \text{ for all } x \in S.$$

A *right zero semigroup* S is a semigroup that has a right zero element, denote it by t , this means that

$$x \cdot t = t \text{ for all } x \in S.$$

A *zero semigroup* S is a semigroup that is both left and right zero semigroup, denote it by z the left and right zero element, this means that

$$z \cdot x = x \cdot z = z \text{ for all } x \in S.$$

Definition 3.5. A *rectangular band* is a band S that satisfies:

1. $xyx = x$ for all $x, y \in S$, or equivalently,
2. $xyz = xz$ for all $x, y, z \in S$.

Rectangular bands are all of the form $I \times J$, where

$$(i, j) \cdot (i', j') = (i, j') \text{ for all } i \in I, j \in J.$$

3.1 Monoids of order 2

The number of monoids of order 2 is 2.

Suppose $M = \{1, 2\}$, then the possible models are:

$$\begin{array}{c|cc} \cdot & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 1 \end{array} \quad \text{or} \quad \begin{array}{c|cc} \cdot & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 2 \end{array}$$

Interpretation: the first model is the group \mathbb{Z}_2 .

In the case of monoids of order 2, it's not evident to see how much the structure is important here because we don't have a lot of monoids. But for example, in some cases, when we consider categories where all sets of endomorphisms are monoids of order 2, we need to eliminate the case of the group because it contradicts with the fact that we're studying reduced categories (in particular categories whose all sets of morphisms have the same cardinal, see Lemma 3.6).

3.2 Monoids of order 3

Lemma 3.6. Let C be a category such that $Ob(C) = \{x, y\}$ and $|C(x, x)| = |C(x, y)| = |C(y, x)| = |C(y, y)| = n$.

If $f \in C(x, y)$ and $g \in C(y, x)$ are such that $f \cdot g = 1_x$, then $g \cdot f = 1_y$.

Proof. Suppose $f \cdot g = 1_x$.

Define the right composition with $f : x \rightarrow y$, for any object z , by

$$L_{z,f} : C(z, x) \rightarrow C(z, y); L_{z,f}(h) = h \cdot f$$

similarly

$$L_{z,g} : C(z, y) \rightarrow C(z, x); L_{z,g}(h) = h \cdot g.$$

As $f \cdot g = 1$, we obtain that

$$L_{z,g} \cdot L_{z,f} = id \text{ on } \mathbb{C}(z, x).$$

It follows that $L_{z,f}$ is injective and $L_{z,g}$ is surjective. Since the number of morphisms is the same between any objects, these conditions imply that $L_{z,f}$ and $L_{z,g}$ are isomorphisms.

In particular take $\mathbf{z}=\mathbf{y}$.

Since

$$L_{z,f} : \mathbb{C}(y, x) \rightarrow \mathbb{C}(y, y)$$

is an isomorphism, 1_y has a preimage, i.e. there is $h \in \mathbb{C}(y, x)$ such that $L_{z,f}(h) = 1_y$ i.e. $h \cdot f = 1_y$.

But the right inverse g and the left inverse h of f coincide, so that $g \cdot f = h \cdot f = 1_y$. \square

Remark 3.7. Lemma 3.6 can be generalized to any number of objects.

Let \mathbb{C} be a category associated to $M = (3)$ so that $\text{Ob}(\mathbb{C}) = \{x\}$ and $\mathbb{C}(x, x) = \{1, e, f\}$.

We shall now prove that there are seven categories in $\text{Cat}(M)$ given by the compositions in the following table:

<i>comp</i>		\mathbb{C}_1	\mathbb{C}_2	\mathbb{C}_3	\mathbb{C}_4	\mathbb{C}_5	\mathbb{C}_6	\mathbb{C}_7
e^2	=	1	e	e	e	e	e	f
f^2	=	f	e	f	f	e	f	e
$e \cdot f$	=	f	e	e	f	f	e	1
$f \cdot e$	=	f	e	f	e	f	e	1

Table 3.1: Categories associated to $M = (3)$

Lemma 3.8. *If $e^2 = 1$, then*

$$y \cdot f = f \cdot y = f \text{ for all } y \in \mathbb{C}(x, x).$$

Proof. Suppose that there exists $y \in \{e, f\}$ such that $y \cdot f \neq f$ or $f \cdot y \neq f$. There are two cases:

1. **If $y \cdot f = 1$:** we can easily rule out the case where $y = 1$ or $y = e$. Indeed, if $y = 1 \Rightarrow f = 1$ Contradiction! And if $y = e \Rightarrow e \cdot f = 1 \Rightarrow e \cdot (e \cdot f) = e \Rightarrow (e \cdot e) \cdot f = e \Rightarrow f = e$. Contradiction! Same for $f \cdot y = 1$.

Then it is sufficient to study the case $f^2 = 1$.

$$f^2 = 1 \Rightarrow e \cdot f \cdot f = e \Rightarrow (e \cdot f) \cdot f \cdot e = 1$$

- (a) $e \cdot f = 1 \Rightarrow f \cdot e = 1$ contradiction.
- (b) $e \cdot f = e \Rightarrow (e \cdot f) \cdot e = 1 \Rightarrow e \cdot (e \cdot f) = 1$ (Lemma 3.6) $\Rightarrow f = 1$ contradiction.
- (c) $e \cdot f = f \Rightarrow (f \cdot f) \cdot e = 1 \Rightarrow e = 1$ ($f^2 = 1$) contradiction. Then we get $f^2 \neq 1$.

2. **If $y \cdot f = e$ (or $f \cdot y = e$):** There are two cases here:

- (a) $y = e \Rightarrow e \cdot f = e \Rightarrow (e \cdot e) \cdot f = 1 \Rightarrow f = 1$ contradiction.
- (b) $y = f \Rightarrow f^2 = e \Rightarrow (e \cdot f) \cdot f = 1 \Rightarrow f \cdot (e \cdot f) = 1$ (Lemma 3.6)
 - i. $e \cdot f = 1 \Rightarrow f = 1$ contradiction.
 - ii. $e \cdot f = e \Rightarrow f \cdot e = 1 \Rightarrow e \cdot f = 1$ contradiction.
 - iii. $e \cdot f = f \Rightarrow f^2 = 1$ contradiction since $f^2 = e$.

Hence for all $y \in \{e, f\}$, if $e^2 = 1$ then $y \cdot f = f \cdot y = f$.

□

Lemma 3.9. *If $e \cdot f \neq f \cdot e$ then $e^2 = e$ and $f^2 = f$.*

Proof. Note here that if $e \cdot f = 1 \Rightarrow f \cdot e = 1$. Then in this case they can't be different.

Then there is only two cases:

First if $e \cdot f = e$ and $f \cdot e = f$, then

$$f \cdot e = f \Rightarrow f \cdot (e \cdot f) = f^2 \Rightarrow f \cdot e = f^2 \quad (e \cdot f = e) \Rightarrow f^2 = f$$

and

$$e \cdot f = e \Rightarrow e \cdot (f \cdot e) = e^2 \Rightarrow e \cdot f = e^2 \Rightarrow e^2 = e.$$

Second is the opposite case.

□

Lemma 3.10. *If $f^2 = e$, then $e \cdot f = f \cdot e$.*

Proof. Indeed, $e \cdot f = (f \cdot f) \cdot f = f \cdot (f \cdot f) = f \cdot e$. We still need to check e^2 . we have 3 cases:

- 1. If $e \cdot f = f \cdot e = 1$, then

$$e^2 = e \cdot (f \cdot f) = (e \cdot f) \cdot f = f.$$

- 2. If $e \cdot f = f \cdot e = e$, then

$$e^2 = e \cdot (f \cdot f) = (e \cdot f) \cdot f = e \cdot f = e.$$

3. If $e \cdot f = f \cdot e = f$, then

$$f = e \cdot f \Rightarrow f^2 = e \cdot (f \cdot f) = e^2 \text{ and } e^2 = e.$$

□

Remark 3.11. Referring to all of the work above, we get the following results:

1. If $f^2 = 1$, then $e^2 = e \cdot f = f \cdot e = e$, and $M_3^1 \simeq C_1$ (Table 3.1).

2. If $e^2 = f$, then $e \cdot f = f \cdot e$. In this case there 3 possibilities:

(i) If $e \cdot f = f \cdot e = 1$, then $f^2 = e$, and $M_3^1 \simeq C_7$ (Table 3.1).

(ii) If $e \cdot f = f \cdot e = f$, then $f^2 = f$, and $M_3^1 \simeq C_2$ (Table 3.1).

(iii) If $e \cdot f = f \cdot e = e$, then $f^2 = f$, and $M_3^1 \simeq C_5$ (Table 3.1).

3. If $e^2 = e$ and $f^2 = f$, then we have 3 possibilities:

(i) $e \cdot f = f \cdot e = f$ which is isomorphic to C_6 (Table 3.1).

(ii) $e \cdot f = f$ and $f \cdot e = e$ which is isomorphic to C_4 (Table 3.1).

(iii) $f \cdot e = f$ and $e \cdot f = e$ which is isomorphic to C_3 (Table 3.1).

Hence the following proposition.

Proposition 3.12. $Card(Cat(M = (3))) = 7$.

After classifying all of the categories associated to the matrix (3), we interpret each of the semigroups that we have obtained in terms of their algebraic structures, and we remark the following:

1. C_1 and C_2 are zero semigroups.
2. C_3 and C_4 are respectively left and right rectangular bands.
3. C_5 is isomorphic to $\mathbb{Z}_2 +$ an extra identity element.
4. C_6 is a zero semigroup and a semilattice.
5. C_7 is a isomorphic to \mathbb{Z}_3 .

3.3 Monoids of order 4

The number of monoids with 4 elements is given on *The On-line Encyclopedia of Integer Sequences* website [1], and the structure of each of these monoids is presented on Peter Jipsen's webpage [21].

Let M be a monoid of order 4 with $M = \{1, 2, 3, 4\}$. The number of monoids of order 4 is 35. We present some of these monoids according to their algebraic properties.

In the following sections, we write the multiplication tables of elements of M , so that the first row and column are the multiplication by the identity 1.

3.3.1 Monoids that contain subgroups

$$1. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 3 \end{pmatrix}$$

where $\{3, 4\} \simeq \mathbb{Z}_2$

$$2. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{pmatrix}$$

where $\{3, 4\} \simeq \mathbb{Z}_2$

$$3. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 2 & 2 & 4 \\ 4 & 4 & 4 & 2 \end{pmatrix}$$

where $\{2, 4\} \simeq \mathbb{Z}_2$

$$4. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \\ 3 & 2 & 3 & 4 \\ 4 & 2 & 4 & 3 \end{pmatrix}$$

where $\{3, 4\} \simeq \mathbb{Z}_2$

$$5. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 2 & 3 & 4 \\ 4 & 4 & 4 & 2 \end{pmatrix}$$

where $\{2, 4\} \simeq \mathbb{Z}_2$

$$6. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \\ 3 & 2 & 4 & 1 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

where $\{1, 3, 4\} \simeq \mathbb{Z}_3$

$$7. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \\ 4 & 3 & 2 & 2 \end{pmatrix}$$

where $\{2, 3\} \simeq \mathbb{Z}_2$

$$8. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 4 & 2 \\ 4 & 4 & 2 & 3 \end{pmatrix}$$

where $\{2, 3, 4\} \simeq \mathbb{Z}_3$.

3.3.2 Monoids that are groups

$$1. \left(\begin{array}{c|cccc} 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array} \right) \simeq \mathbb{Z}_4$$

$$2. \left(\begin{array}{c|cccc} 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 4 & 2 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{array} \right) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

3.3.3 Monoids that have zeros

All the remaining monoids contain zero elements. We will mention some of them which are more interesting.

$$1. \left(\begin{array}{c|cccc} 1 & 2 & 3 & 4 \\ \hline 2 & 2 & 2 & 2 \\ 3 & 2 & 3 & 2 \\ 4 & 2 & 2 & 4 \end{array} \right)$$

$$2. \left(\begin{array}{c|cccc} 1 & 2 & 3 & 4 \\ \hline 2 & 2 & 2 & 2 \\ 3 & 2 & 3 & 3 \\ 4 & 2 & 3 & 4 \end{array} \right)$$

Both have a zero element 2, and contain semilattices of order 3.

$$3. \left(\begin{array}{c|cccc} 1 & 2 & 3 & 4 \\ \hline 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{array} \right)$$

$$4. \left(\begin{array}{c|cccc} 1 & 2 & 3 & 4 \\ \hline 2 & 2 & 3 & 4 \\ 3 & 2 & 3 & 4 \\ 4 & 2 & 3 & 4 \end{array} \right)$$

Both contain a left (resp. right) band of order 3 which means that every element is a left (resp. right) zero.

The reason we want to point out these kinds of monoids (or semigroups) is that their algebraic structure is very special in the classification. Studying such structures leads to giving general information about categories having these types of monoids of higher orders, which eventually gives a classification of these categories.

CHAPTER 4

Representation of categories as semigroups with a zero

The main goal of this chapter is to illustrate categories as semigroups. The representation between them allows us to write equations of categories as equations of semigroups which makes it easier for later to input them into computer programs to obtain a count.

We offer a representation of categories as semigroups equipped with zero and an additional unary predicate, which encodes the identity morphisms. Our representation allows us to restate the enumeration problem for finite categories in terms of these expanded semigroups.

4.1 Categories as semigroups with a zero

In chapter 2, we gave the definition of a category. Here again we give a slightly modified version of the definition for the purpose of the work in this chapter, but both definitions are the same.

Definition 4.1. A category $\mathcal{C} = (O, \mathbf{m}, Dom, Cod, 1_{(-)}, \cdot)$ consists of the following data:

1. A set of objects O .
2. A set of morphisms \mathbf{m} .
3. Functions $Dom, Cod : \mathbf{m} \rightarrow O$ and $1_{(-)} : O \rightarrow \mathbf{m}$.
4. A partial function $\cdot : \mathbf{m} \times \mathbf{m} \rightarrow \mathbf{m}$ with $domain = \{(f, g) \in \mathbf{m} \times \mathbf{m} \mid Cod(f) = Dom(g)\}$.

The above is stipulated to satisfy:

1. For all $f, g \in \mathbf{m}$, $Dom(f \cdot g) = Dom(f)$ and $Cod(f \cdot g) = Cod(g)$.
2. For all $f \in \mathbf{m}$, $f \cdot 1_{Cod(f)} = f$ and $1_{Dom(f)} \cdot f = f$.

3. For all $f, g, h \in \mathbf{m}$ with $Cod(f) = Dom(g)$ and $Cod(g) = Dom(h)$, $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

Now we need to define the semigroup with a zero element. We will call them *compositional semigroups*. It will be a semigroup with a partial binary operation, the zero element is to make the operation total, and at the end we only need the identity elements, hence we add a unary predicate \mathbf{u} .

Definition 4.2. A *compositional semigroup* $Comp\mathcal{S} = (S, \cdot, 0, \mathbf{u})$ is the following:

1. $0 \notin \mathbf{u}$.
2. For all $e \in \mathbf{u}$, $e \cdot e = e$.
3. For all $x \in S$, there exist unique $e, e' \in \mathbf{u}$ such that $e \cdot x \neq 0$ and $x \cdot e' \neq 0$.
4. There exists $e \in \mathbf{u}$ such that $f \cdot e \neq 0$ and $e \cdot g \neq 0$ if and only if $f \cdot g \neq 0$.
5. $e_1 \cdot f \cdot e_2 \neq 0$ and $e_2 \cdot g \cdot e_3 \neq 0$ if and only if $e_1 \cdot (f \cdot g) \cdot e_3 \neq 0$.
6. $Dom(f) = e$ implies that $e \cdot f = f$ and $Cod(f) = e$ implies that $f \cdot e = f$.

Definition 4.3. The category of categories denoted by Cat_{inj} is the following:

1. Objects: categories.
2. Morphisms: functors that are injective on the objects.

Definition 4.4. The category of compositional semigroups denoted by \mathcal{CSem} consists of the following data:

1. Objects: compositional semigroups.
2. Morphisms: $h : (S_1, \cdot_1, 0_1, \mathbf{u}_1) \rightarrow (S_2, \cdot_2, 0_2, \mathbf{u}_2)$, where:
 - i) $h(x \cdot_1 y) = h(x) \cdot_2 h(y)$.
 - ii) $h[\mathbf{u}_1] \subseteq \mathbf{u}_2$.
 - iii) $h(x) = 0 \iff x = 0$.

An *equivalence of categories* consists of a functor between the involved categories, which is required to have an "inverse" functor. However, in contrast to the situation common for isomorphisms in an algebraic setting, the composition of the functor and its "inverse" is not necessarily the identity mapping. Instead it is sufficient that each object be naturally isomorphic to its image under this composition. Thus one may describe the functors as

being "inverse up to isomorphism". There is indeed a concept of isomorphism of categories where a strict form of inverse functor is required, but this is of much less practical use than the equivalence concept.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ yields an equivalence of categories if and only if it is simultaneously:

- **full**, i.e. for any two objects C_1 and C_2 of \mathcal{C} , the map $Hom_{\mathcal{C}}(C_1, C_2) \rightarrow Hom_{\mathcal{D}}(FC_1, FC_2)$ induced by F is surjective;
- **faithful**, i.e. for any two objects C_1 and C_2 of \mathcal{C} , the map $Hom_{\mathcal{C}}(C_1, C_2) \rightarrow Hom_{\mathcal{D}}(FC_1, FC_2)$ induced by F is injective;
- **essentially surjective**, i.e. each object D in \mathcal{D} is isomorphic to an object of the form FC , for C in \mathcal{C} .

Definition 4.5. Let

$$\begin{aligned} cat : \mathcal{CSem} &\rightarrow \mathcal{Cat}_{inj} \\ (S, \cdot, 0, \mathbf{u}) &\mapsto \mathcal{C}_S \\ S_{\mathcal{C}} &\mapsto \mathcal{C} \end{aligned}$$

be a functor where $\mathbf{u} \subseteq S$, $0 \cdot x = 0$ and $x \cdot 0 = 0$. The functor cat takes morphisms of \mathcal{CSem} to morphisms in \mathcal{Cat}_{inj} , i.e. if $h : S_1 \rightarrow S_2$ and $e, e' \in \mathbf{u}_1$ with $e \neq e'$ then $h(e) \neq h(e')$. Indeed, if $e \cdot e' = 0$ then $h(e) = h(e') = 0$ and this wouldn't be possible if $h(e) = h(e')$.

$cat(S) = \mathcal{C}_S$ consists of:

- $O = \mathbf{u}$.
- $\mathbf{m} = S \setminus \{0\}$.
- $1_{(-)} : \mathbf{u} \subseteq S \setminus \{0\}$.
- $Dom, Cod : S \setminus \{0\} \rightarrow \mathbf{u}$; $e \in \mathbf{u}$ unique such that $ef \neq 0$ ($Dom(e) = e$ and $e \cdot e = e$).

and $cat(\mathcal{C}) = S_{\mathcal{C}}$ consists of:

- $S_{\mathcal{C}} = \mathbf{m} \cup \{0\}$.
- $\mathbf{u} = \{1_a \mid a \in O\}$.
- the operation:

$$\begin{aligned} \hat{\cdot} : S_{\mathcal{C}} \times S_{\mathcal{C}} &\rightarrow S_{\mathcal{C}} \\ (0, x) &\mapsto 0 \\ (x, 0) &\mapsto 0 \\ (f, g) &\mapsto \begin{cases} 0 & \text{if } \text{Cod}(f) \neq \text{Dom}(g) \\ f \cdot g & \text{otherwise} \end{cases} \end{aligned}$$

Lemma 4.6. $S_{\mathcal{C}}$ satisfies the conditions in Definition 4.2.

Proof.

1. Trivial.
2. Trivial.
3. For all f , let $a = \text{Dom}(f)$ then $1_a \cdot f = f \neq 0$.
Suppose a is not unique, let $b \in \mathbf{u}$ such that $b \neq a$, then $\text{Cod}(1_b) \neq \text{Dom}(f)$, hence $1_b \cdot f$ is not defined, then $1_b \hat{\cdot} f = 0$. Same for codomain.
4. $f \cdot e \neq 0$ and $e \cdot g \neq 0 \iff \text{Cod}(f) = \text{Dom}(g) \iff f \cdot g$ defined $\iff f \hat{\cdot} g \neq 0$.
5. $e_1 \hat{\cdot} f \hat{\cdot} e_2 \neq 0 \implies \text{Dom}(f) = e_1$ and $\text{Cod}(g) = e_3$.
 $\text{Dom}(f \cdot g) = \text{Dom}(f) = e_1 \implies e_1 \cdot f \cdot g$ is defined $\implies e_1 \hat{\cdot} f \hat{\cdot} g \neq 0$.

□

Lemma 4.7. Let \mathcal{C} be any category, the following are equivalent:

1. $\text{Cod}(f) = a$.
2. $f \cdot 1_a$ is defined.
3. $f \hat{\cdot} 1_a \neq 0$.

Proof.

- (1) \implies (2) : trivial.
 (2) \implies (3) : $f \hat{\cdot} 1_a$ is defined, i.e. $\text{Cod}(f) = \text{Dom}(1_a)$ i.e. $f \hat{\cdot} 1_a = f \cdot 1_a \neq 0$.
 (3) \implies (1) : $f \hat{\cdot} 1_a \neq 0$ i.e. $f \hat{\cdot} 1_a = f \cdot 1_a$ i.e. $\text{Cod}(f) = \text{Dom}(1_a) = a$.

□

Theorem 4.8. \mathcal{CSem} and Cat_{inj} are equivalent.

Proof. 1. $\text{cat} : \mathcal{CSem} \rightarrow \text{Cat}_{inj}$ is essentially surjective:

We want to prove that for all \mathcal{C} , $\text{cat}(S_{\mathcal{C}}) \simeq \mathcal{C}$.

Let $F : \text{cat}(S_{\mathcal{C}}) \rightarrow \mathcal{C}$ and $G : \mathcal{C} \rightarrow \text{cat}(S_{\mathcal{C}})$ where:

$$\begin{array}{l}
 F_O : 1_a \in \mathbf{u} \mapsto a \quad \text{and} \quad G_O : a \in \mathbf{u} \mapsto 1_a \\
 F_m : f \in \mathbf{m} \mapsto f \quad \text{and} \quad G_m : f \in \mathbf{m} \mapsto f
 \end{array}$$

Check these are functors:

- $Dom(Ff) = F(Dom(f))$?
- $Dom(Ff) = Dom(f) = a$ and $F(Dom(f)) = F(1_a) = a$ for $1_a \hat{\cdot} f \neq 0$.
- $F_m(1_a) = 1_a = 1_{F_O(1_a)} = a = F_O(1_a)$.
- $f \cdot g \neq 0 \implies F_m(f \hat{\cdot} g) = F_m(f \cdot g) = f \cdot g$ and $F_m(f) \cdot F_m(g) = f \cdot g$.

Conclusion: $FG = Id_C$ and $GF = Id_{S_C}$.

2. $cat : \mathcal{CSem} \rightarrow \mathcal{Cat}_{inj}$ is **faithful**:

Let $h, h' : S_1 \rightarrow S_2$ such that $cat(h) = cat(h') \implies h = h'$?

- If $f \in \mathbf{m} \setminus \{0\}$ then $h(f) = h'(f)$.
- If $f = 0$ then $h(0) = 0 = h'(0)$.

3. $cat : \mathcal{CSem} \rightarrow \mathcal{Cat}_{inj}$ is **full**:

$F : cat(S_1) \rightarrow cat(S_2)$, we want to prove that:

$$\begin{array}{l}
 \exists h : S_1 \rightarrow S_2 \\
 x \in S_1 \mapsto \begin{cases} 0 & \text{if } x = 0 \\ F_m(x) \neq 0 & \text{otherwise} \end{cases} ;
 \end{array}$$

such that $F = cat(h)$.

h is semigroups morphism:

- i) $e \in \mathbf{u}_1, F_m(1_e) = 1_{F_m(e)} \in \mathbf{u}_2$.
- ii) $h(x) = 0 \iff x = 0$.
- iii) $x \cdot y \neq 0 \implies h(x \cdot y) = hxhy \implies \exists e \in \mathbf{u}_1; x \cdot e \neq 0$ and $e \cdot y \neq 0 \implies Cod(x) = Dom(y)$ in $cat(S_1) \implies h(x \cdot y) = F(x \cdot y) = Fx Fy = hxhy$.
- iv) if $x \cdot y = 0$ then in this case it means that $Dom(y) \neq Cod(x)$. Then, since F is injective on objects it implies that $Dom(F(y)) \neq Cod(F(x))$. Thus $F(x)F(y) = 0$.

□

Lemma 4.9. $S_1, S_2 \in \mathcal{CompS}$, the following are equivalent:

1. $S_1 \simeq S_2$ in CompS .

2. $S_1 \simeq S_2$ by homomorphisms in signature $(\circ, 0, \mathbf{u})$.

Proof. (2) \implies (1):

$i : S_1 \rightarrow S_2$ then

$$i(x) = 0 \implies x = 0.$$

i is injective and $i(0) = 0$.

(1) \implies (2):

- $x \cdot y \neq 0 \implies i(x \cdot y) = i(x)i(y)$

- $x \cdot y = 0 \implies i(x \cdot y) = 0$, assume that $i(x)i(y) \neq 0$. By (d), $\exists e \in \mathbf{u}_2$ such that $i(x)e \neq 0$ and $ei(y) \neq 0$. Then $i^{-1}(i(x)e) = i^{-1}i(x)i^{-1}(e) = xi^{-1}(e) \neq 0$ the same for $ei(y)$ we obtain $i^{-1}y \neq 0$, by (d) $x \cdot y \neq 0$. Contradiction! \square

Lemma 4.10. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G \simeq \text{Id}$ and $G \circ F \simeq \text{Id}$, then:*

$$\begin{aligned} F/ \simeq: \mathcal{C}/ \simeq &\rightarrow \mathcal{D}/ \simeq \\ [A] &\mapsto [FA] \end{aligned}$$

is a bijection between isomorphism classes.

Proof. • Injective:

$$[FA] = [FB] \implies [A] = [B]?$$

$$FA \simeq FB \implies GFA \simeq GFB \implies A \simeq B.$$

• Surjective:

$$\text{Let } [B] \in \mathcal{D}/ \simeq \text{ then } [GB] \in \mathcal{C}/ \simeq \text{ then } F/ \simeq ([GB]) = [FG(B)] = [B].$$

\square

Remark 4.11. The representation between categories and semigroups is not unique. For example, F. Bonnin [10] calls compositional semigroups, categories without objects, and in his context, the definition of a category consists of a full multiplication and not partial. Also, he doesn't include a zero element in the signature of compositional semigroups. But at the end, the axioms coincide with ours.

4.2 Zero semigroups and zero objects

Definition 4.12. A *left zero semigroup* S is a semigroup that has a left zero element, denote it by s , this means that

$$s \cdot x = s \text{ for all } x \in S.$$

A *right zero semigroup* S is a semigroup that has a right zero element, denote it by t , this means that

$$x \cdot t = t \text{ for all } x \in S.$$

A *zero semigroup* S is a semigroup that is both left and right zero semigroup, denote by z the left and right zero element, this means that

$$z \cdot x = x \cdot z = z \text{ for all } x \in S.$$

Lemma 4.13. *If a semigroup S has a zero, denoted by 0 , then it's unique.*

Proof. Let $0'$ be another zero, then

$$\begin{aligned} 0 \cdot 0' &= 0 \text{ (0 is a zero)} \\ &= 0' \text{ (0' is a zero)}. \end{aligned}$$

□

Definition 4.14. In a category, an object is called a *zero object*, *null object*, or *biterminal* if it is both an initial object and a terminal object.

A category with a zero object is sometimes called a *pointed category*.

Remark 4.15. $0 \in \mathbf{C}$ is a zero object, if for every other object A there is a unique morphism $A \rightarrow 0$ as well as a unique morphism $0 \rightarrow A$.

Remark 4.16. If \mathbf{C} is a pointed category, then an object A of \mathbf{C} is a zero object when the only endomorphism of A is the identity morphism.

Examples 4.17.

- The category of pointed sets has a zero object, namely any one-element set.
- The trivial group is a zero object in the category $\mathcal{G}rp$ of groups and in the category \mathbf{Ab} of abelian groups.

- For \mathcal{R} a ring, the trivial \mathcal{R} -module (that whose underlying abelian group is the trivial group) is the zero object in $\mathcal{R}Mod$.

In particular for $R = \mathbb{K}$ a field, the \mathbb{K} -vector space of dimension 0 is the zero object in \mathbf{Vect} .

From zero objects, we are able to define zero morphisms. In other words, these are going to be the morphisms that absorb the multiplication. What does that mean in terms of the classification? It means that we can reduce the category by eliminating all the zero morphisms, i.e. crushing them into one element, which will facilitate the enumeration.

4.3 Zero semigroups in categories

Definition 4.18. Let \mathcal{C} be a category with a zero object 0 , the *zero morphism* $0_{XY} : X \rightarrow Y$ between two objects $X, Y \in \mathcal{C}$ is the unique morphism that factors through 0 :

$$0_{XY} : X \rightarrow 0 \rightarrow Y.$$

Definition 4.19. A category \mathcal{C} is called a *zero category* if every set of morphisms has a zero element. In this case we define the set \mathcal{Z} of zeros in the following way:

$$\mathcal{Z} = \{f \in \mathcal{C}(X, Y); f \text{ is a zero from } X \text{ to } Y\}.$$

The idea is to prove that whenever we multiply by an element of \mathcal{Z} , the multiplication is an element of \mathcal{Z} . In fact it will be a unique zero element, because every set of morphisms has a unique zero.

Proposition 4.20. *Let \mathcal{C} be a finite category whose objects are X and Y , such that $\mathcal{C}(X, Y)$ and $\mathcal{C}(Y, X)$ are not empty. If $\mathcal{C}(X, X)$ and $\mathcal{C}(Y, Y)$ have zeros 0_X and 0_Y , then there exists a unique zero $0_{XY} \in \mathcal{C}(X, Y)$ and a unique zero $0_{YX} \in \mathcal{C}(Y, X)$ where 0_{XY} and 0_{YX} are of the form*

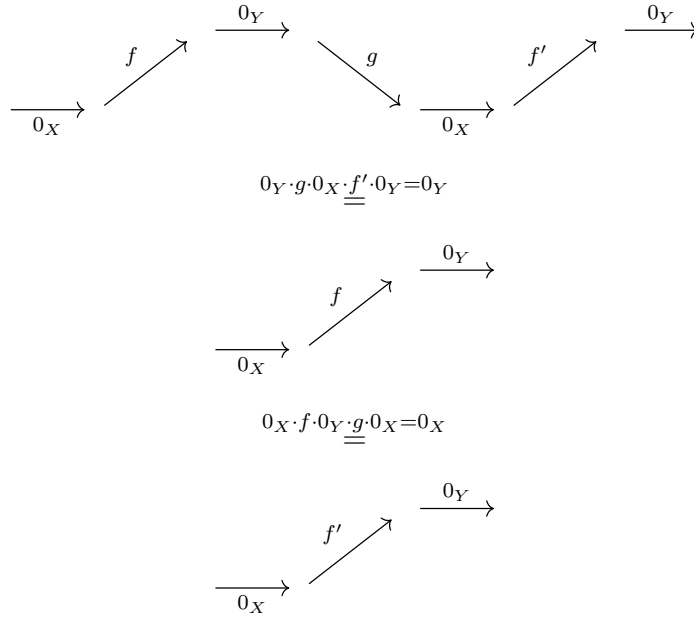
$$0_{XY} = 0_X \cdot f \cdot 0_Y \text{ for all } f \in \mathcal{C}(X, Y)$$

and

$$0_{YX} = 0_Y \cdot g \cdot 0_X \text{ for all } g \in \mathcal{C}(Y, X).$$

Proof. **0_{XY} and 0_{YX} are unique:**

Let $f, f' \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, X)$, then

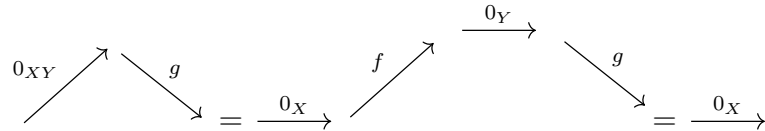


Hence $0_X \cdot f \cdot 0_Y = 0_X \cdot f' \cdot 0_Y$ for all $f, f' \in C(X, Y)$.

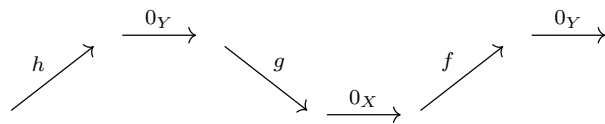
Similarly we prove that 0_{YX} is unique.

0_{XY} and 0_{YX} are zeros:

Let $g \in C(Y, X)$,



Let $h \in C(X, Y)$,



where $f \in C(X, Y)$ and $g \in C(Y, X)$. This diagram is equal to $h \cdot 0_Y$ and to $0_X \cdot f \cdot 0_Y$ which is equal to 0_{XY} . □

Remark 4.21. The condition

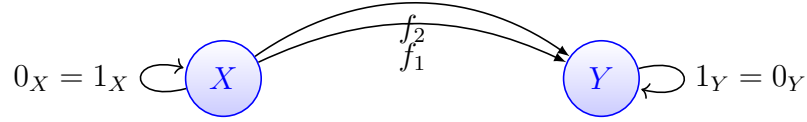
$$C(X, Y) \neq \emptyset \text{ and } C(Y, X) \neq \emptyset$$

is very important, because otherwise the zeros of $C(X, Y)$ and $C(Y, X)$ are not necessarily unique.

For example, let \mathbf{C} be a finite category whose objects are X and Y , associated to the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

which corresponds to the graph



We have 1_X is a zero of $\mathbf{C}(X, Y)$ and 1_Y is a zero of $\mathbf{C}(Y, Y)$ but both f_1 and f_2 are zeros of $\mathbf{C}(X, Y)$ because

$$\begin{aligned} 0_X \cdot f_1 &= f_1 \text{ and } 0_X \cdot f_2 = f_2 \\ f_1 \cdot 0_Y &= f_1 \text{ and } f_2 \cdot 0_Y = f_2. \end{aligned}$$

Conclusion: If we have a zero category, we can shrink all the zeros into one zero, and we obtain a compositional semigroup with a zero with less number of elements. And thus we can classify these compositional semigroups with a zero to obtain a classification of zero categories.

4.4 Categories with a zero and semigroups with a zero

In Section 4.1 we have presented categories as semigroups with a zero, but another question comes to mind that if the category itself has zero elements, what happens to these zeros? Two cases can be discussed: either all the zero elements go to the same zero in the semigroup or they form a set of zeros inside the semigroup.

4.4.1 Zero elements do not collide

A category with a zero is the same as a category but with a little extra conditions.

Definition 4.22. We denote a category with a zero as $\mathcal{C}_0 = (O, \mathbf{m}, Dom, Cod, 1_{(-)}, 0_{(-,-)})$. It is the same definition but we add:

3. Functions $Dom, Cod : \mathbf{m} \rightarrow O$, $1_{(-)} : O \rightarrow \mathbf{m}$ and $0_{(-,-)} : O \times O \rightarrow \mathbf{m}$.

The above is stipulated to satisfy:

4. For all $f \in \mathbf{m}$; $f \cdot 0_{(Cod(f), Cod(f))} = 0_{(Dom(f), Cod(f))}$ and $0_{(Dom(f), Dom(f))} \cdot f = 0_{(Dom(f), Cod(f))}$.

Definition 4.23. A *compositional semigroup with a zero* $\mathcal{CompS}_0 = (S, \circ, 0, \mathbf{u}, \mathfrak{z})$ is the following:

1. $0 \notin \mathbf{u}$ and $0 \notin \mathfrak{z}$.
7. For all $z \in \mathfrak{z}$, $z \cdot z = z$.
8. For all $x \in S, z \in \mathfrak{z}$, we have:

$$z_{(y, \text{Dom}(x))} \cdot x = z_{(y, \text{Cod}(x))} \quad \text{and} \quad x \cdot z_{(y, \text{Cod}(x))} = z_{(y, \text{Dom}(x))}$$

Now, the category of categories with a zero denoted by \mathcal{Cat}_{0inj} consists of objects, and morphisms: categories functors that are injective on objects. And the category of compositional semigroups with a zero denoted by \mathcal{CSem}_0 consists of objects: compositional semigroups with a zero, and morphisms: $h : (S_1, \cdot_1, 0_1, \mathbf{u}_1, \mathfrak{z}_1) \rightarrow (S_2, \cdot_2, 0_2, \mathbf{u}_2, \mathfrak{z}_2)$, where:

iv) $h[\mathfrak{z}_1] \subseteq \mathfrak{z}_2$.

Similarly, the functor between these two categories is going to be:

$$\begin{aligned} \text{cat}_0 : \quad \mathcal{CSem}_0 &\rightarrow \mathcal{Cat}_{0inj} \\ (S, \cdot, 0, \mathbf{u}, \mathfrak{z}) &\mapsto \mathcal{C}_S \\ S_{\mathcal{C}} &\mapsto \mathcal{C} \end{aligned}$$

where $\mathbf{u}, \mathfrak{z} \subseteq S$, $0 \cdot x = 0$ and $x \cdot 0 = 0$. Such that $\text{cat}_0(S) = \mathcal{C}_S$ consists of:

- $0_{(\cdot, \cdot)} : \mathfrak{z} \subseteq S \setminus \{0\}$.

and $\text{cat}_0(\mathcal{C}) = S_{\mathcal{C}}$ consists of:

- $\mathfrak{z} = \{0_{(a,b)} \mid a, b \in O\}$.

- the operation:

$$\begin{aligned} \hat{\cdot} : S_{\mathcal{C}} \times S_{\mathcal{C}} &\rightarrow S_{\mathcal{C}} \\ (0, x) &\mapsto 0 \\ (x, 0) &\mapsto 0 \\ (f, g) &\mapsto \begin{cases} 0 & \text{if } \text{Cod}(f) \neq \text{Dom}(g) \\ f & \text{if } f \in \mathfrak{z} \\ g & \text{if } g \in \mathfrak{z} \\ f \cdot g & \text{otherwise} \end{cases} \end{aligned}$$

Remark 4.24. When we say $f \cdot g = f$ when f is a zero, we don't exactly mean that it should be equal to the same f , but it just means that it's equal to the zero element in the resulting set of morphisms.

Lemma 4.25. $S_{\mathcal{C}}$ satisfies the conditions in Definition 4.23.

Proof.

1. Trivial.
7. Trivial.
8. Let $z \in \mathfrak{z}$, if $z \hat{\cdot} f \neq 0$ then it is equal to some $z' \in \mathfrak{z}$. Same for $f \hat{\cdot} z$.

□

Lemma 4.26. Let \mathcal{C} be any category, the following are equivalent:

1. $\text{Cod}(f) = a$.
2. $f \cdot 1_a$ is defined.
3. $f \hat{\cdot} 1_a \neq 0$.

Theorem 4.27. $\mathcal{C}\text{Sem}_0$ and $\text{Cat}_{0\text{inj}}$ are equivalent.

Proof. 1. $\text{cat}_0 : \mathcal{C}\text{Sem}_0 \rightarrow \text{Cat}_{0\text{inj}}$ is essentially surjective:

We want to prove that for all \mathcal{C} , $\text{cat}(S_{\mathcal{C}}) \simeq \mathcal{C}$.

Let $F : \text{cat}_0(S_{\mathcal{C}}) \rightarrow \mathcal{C}$ and $G : \mathcal{C} \rightarrow \text{cat}_0(S_{\mathcal{C}})$ where:

$$\begin{array}{l} F_O : 1_a \in \mathbf{u} \mapsto a \quad \text{and} \quad G_O : a \in \mathbf{u} \mapsto 1_a \\ F_m : f \in \mathbf{m} \mapsto f \quad \text{and} \quad G_m : f \in \mathbf{m} \mapsto f \end{array}$$

with $F_m(0_{(a,b)}) = 0_{(a,b)}$ for $(a,b) \in \mathbf{u}^2$

Check these are functors:

- $\text{Dom}(Ff) = F(\text{Dom}(f))$?
 $\text{Dom}(Ff) = \text{Dom}(f) = a$ and $F(\text{Dom}(f)) = F(1_a) = a$ for $1_a \hat{\cdot} f \neq 0$.
- $F_m(1_a) = 1_a = 1_{F_O(1_a)} = a = F_O(1_a)$.
- $fg \neq 0 \implies F_m(f \hat{\cdot} g) = F_m(f \cdot g) = f \cdot g$ and $F_m(f) \cdot F_m(g) = f \cdot g$.

Conclusion: $FG = \text{Id}_{\mathcal{C}}$ and $GF = \text{Id}_{S_{\mathcal{C}}}$.

2. $cat_0 : \mathcal{CSem}_0 \rightarrow \mathcal{Cat}_{0inj}$ **is faithful:**

Let $h, h' : S_1 \rightarrow S_2$ such that $cat(h) = cat(h') \implies h = h'$?

- If $f \in \mathbf{m} \setminus \{0\}$ then $h(f) = h'(f)$.
- If $f = 0$ then $h(0) = 0 = h'(0)$.

3. $cat_0 : \mathcal{CSem}_0 \rightarrow \mathcal{FinCat}_{0inj}$ **is full:**

$F : cat_0(S_1) \rightarrow cat_0(S_2)$, we want to prove that:

$$\begin{aligned} \exists h : S_1 &\rightarrow S_2 \\ x \in S_1 &\mapsto \begin{cases} 0 & \text{if } x = 0 \\ F_{\mathbf{m}}(x) \neq 0 & \text{otherwise} \end{cases} ; \end{aligned}$$

such that $F = cat_0(h)$.

h is semigroups morphism:

- i) $e \in \mathbf{u}_1, F_{\mathbf{m}}(1_e) = 1_{F_{\mathbf{m}}(e)} \in \mathbf{u}_2$.
- ii) $h(x) = 0 \iff x = 0$.
- iii) $x \cdot y \neq 0 \implies h(x \cdot y) = h(x)h(y) \implies \exists e \in \mathbf{u}_1; x \cdot e \neq 0 \text{ and } ey \neq 0 \implies$
 $Cod(x) = Dom(y) \text{ in } cat_{0inj}(S_1) \implies h(xy) = F(xy) = F(x)F(y) = h(x)h(y)$.

□

Lemma 4.28. $S_1, S_2 \in \mathcal{CompS}_0$, the following are equivalent:

1. $S_1 \simeq S_2$ in \mathcal{CompS}_0 .
2. $S_1 \simeq S_2$ by homomorphisms in signature $(\cdot, 0, \mathbf{u}, \mathfrak{z})$.

Proof. (2) \implies (1):

$i : S_1 \rightarrow S_2$ then

$$i(x) = 0 \implies x = 0.$$

i is injective and $i(0) = 0$.

(1) \implies (2):

- $x \cdot y \neq 0 \implies i(x \cdot y) = i(x)i(y)$

- $x \cdot y = 0 \implies i(x \cdot y) = 0$, assume that $i(x)i(y) \neq 0$. By (d), $\exists e \in \mathbf{u}_2$ such that $i(x)e \neq 0$ and $ei(y) \neq 0$. Then $i^{-1}(i(x)e) = i^{-1}i(x)i^{-1}(e) = xi^{-1}(e) \neq 0$ the same for $ei(y)$ we obtain $i^{-1}y \neq 0$, by (d) $x \cdot y \neq 0$. Contradiction! □

Lemma 4.29. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G \simeq Id$ and $G \circ F \simeq Id$, then:*

$$\begin{aligned} F/ \simeq: \mathcal{C}/ \simeq &\rightarrow \mathcal{D}/ \simeq \\ [A] &\mapsto [FA] \end{aligned}$$

is a bijection between isomorphism classes.

Proof. • Injective:

$$[FA] = [FB] \implies [A] = [B]?$$

$$FA \simeq FB \implies GFA \simeq GFB \implies A \simeq B.$$

• Surjective:

$$\text{Let } [B] \in \mathcal{D}/ \simeq \text{ then } [GB] \in \mathcal{C}/ \simeq \text{ then } F/ \simeq ([GB]) = [FG(B)] = [B].$$

□

Remark 4.30. Note that the zero elements in the semigroup obtained by the zero elements of the category are not the same as the zero element of the semigroup where the composition is not defined. In this case we are able to retrieve all of the zero elements of the category by the semigroup. But in the next section we will discuss the case where all the zero elements go to the same zero.

4.4.2 Zero elements collide

In this case, all the zero elements in the category collide into one zero.

Let \mathcal{C} be a finite zero category. This means that every set of morphisms has a zero. Let

$$\mathcal{Z} = \{z \in \mathcal{C} \mid z \text{ is a zero of } \mathcal{C}\}.$$

Using Section 4.3, we can view \mathcal{Z} as a zero object of \mathcal{C} . Then the corresponding compositional semigroup to \mathcal{C} is the semigroup quotient by the ideal that has all the zero elements (including the zero when the composition is not defined). The compositional semigroup is denoted by \mathcal{S}/\mathcal{I} where

$$x \cdot y \in \mathcal{I} \iff x \cdot y \in \mathcal{Z} \text{ or } x \cdot y \text{ is not defined.}$$

5.1 Bimodules in finite categories

In this section we present a different point of view of finite categories. In order to make the counting problem easier, we divide the classification of a category into several steps. We take the matrix of a category with positive integers, and we associate specific monoids on the diagonals. It means that the diagonal elements are monoids and the off diagonal elements are just sets. Then we can see the off diagonal sets as bimodules with actions on the left and on the right of the respective monoids. We obtain now a matrix of a category with specific monoids and specific bimodules and it remains to know the multiplication of the chosen bimodules together that goes into the monoids. Therefore, we have two ways of making the classification easier:

1. Specify the monoids and find all the possible categories between them.
2. Specify the monoids and the bimodules and find all the possible categories between them. In our work, this method helps reduce the calculation time and it is used in Chapter 9.

We give the definition of a bimodule and we present some properties and examples to clarify the idea.

Definition 5.1. A *bimodule* is a set with actions on the left and the right of the respective monoids, such that the actions commute i.e. $(g \cdot x) \cdot h = g \cdot (x \cdot h)$. It can be seen as a category such that one of the sets of morphisms is empty, it's called an *upper triangulated category*.

Let

$$N^{(1)} = \begin{pmatrix} A & L \\ \emptyset & B \end{pmatrix}$$

be a matrix of bimodule L , and let $\mathcal{C}^{(1)}$ be a category associated to $N^{(1)}$.

Similarly let

$$N^{(2)} = \begin{pmatrix} A & \emptyset \\ R & B \end{pmatrix}$$

be a matrix of bimodule R , and let $\mathcal{C}^{(2)}$ be a category associated to $N^{(2)}$.

We denote by $N = N^{(1)} \cup N^{(2)}$ the matrix of the form

$$\begin{pmatrix} A & L \\ R & B \end{pmatrix}$$

of monoids A, B and bimodules L, R in $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$.

From this perspective we can see parts of the category as bimodules, taking only the multiplication tables of $A \cdot L \cdot B$ and $B \cdot R \cdot A$. In some cases, the number of bimodules obtained specifies the number of categories that could be obtained by two bimodules.

Example 5.2. In this example, we chose 2 monoids and 2 bimodules and we try to find a category having these specific monoids and bimodules. Let \mathcal{C} be a category with two objects X, Y such that all the sets of morphisms have cardinal 3, then the entries of such a matrix are

$$A = \mathcal{C}(X, X) , L = \mathcal{C}(X, Y) , R = \mathcal{C}(Y, X) \text{ and } B = \mathcal{C}(Y, Y)$$

and we denote the elements of A, L, R and B in the following way

$$A = \{1, 2, 3\} , L = \{4, 5, 6\} , R = \{7, 8, 9\} \text{ and } B = \{10, 11, 12\}.$$

We are looking for a category such that $A = \mathcal{C}(X, X)$, $L = \mathcal{C}(X, Y)$, $R = \mathcal{C}(Y, X)$ and $B = \mathcal{C}(Y, Y)$ (all specified). Then the table of multiplication of elements of such a category looks like the following

	1 2 3	4 5 6	7 8 9	10 11 12
1	1 2 3	4 5 6		
2	2 1 3	4 5 6		
3	3 3 3	4 4 4		
4			3 3 3	4 4 4
5			3 3 3	5 4 4
6			3 3 3	6 4 4
7	7 7 7	11 11 11		
8	8 8 8	12 12 12		
9	9 9 7	11 11 11		
10			7 8 9	10 11 12
11			7 7 7	11 11 11
12			8 8 8	12 12 12

What we do is that we forget the multiplication blocks of elements of L by elements of R and vice versa, we end up with the matrix of monoids and bimodules. In this case, the table of multiplication will look like the following

	1 2 3	4 5 6	7 8 9	10 11 12
1	1 2 3	4 5 6		
2	2 1 3	4 5 6		
3	3 3 3	4 4 4		
4			- - -	4 4 4
5			- - -	5 4 4
6			- - -	6 4 4
7	7 7 7	- - -		
8	8 8 8	- - -		
9	9 9 7	- - -		
10			7 8 9	10 11 12
11			7 7 7	11 11 11
12			8 8 8	12 12 12

Then the question is: how many possibilities are there to fill in $L \times R$ and $R \times L$, respecting the associativity property.

In this thesis, we mostly deal with categories having two objects, but we give the general definition of bimodules in categories.

Definition 5.3. An *ordered category* \mathbf{C} is a category with a linear order on the set of objects. It means if $Ob(\mathbf{C}) = \{x_1, x_2, \dots, x_i, \dots\}$ then there is an order relation on $Ob(\mathbf{C})$ defined by

$$x_i \leq x_j \iff i \leq j.$$

A *finite ordered category* \mathbf{C} of order n is a finite category with a linear order relation on the set of objects $Ob(\mathbf{C}) = \{x_1, \dots, x_n\}$.

Definition 5.4. Let $M = (M_{ij})$ be a matrix of monoids and bimodules where

$$M_{ij} \text{ is a } \begin{cases} \text{monoid} & i = j \\ \text{bimodule} & i \neq j \end{cases}$$

Let \mathbf{C} be a finite ordered category such that $Ob(\mathbf{C}) = \{O_i\}$. An association between \mathbf{C} and the matrix M consists of a linearly ordered set O of objects and a collection of isomorphisms Λ such that

$$\Lambda_{ij} : \mathbf{C}(O_i, O_j) \rightarrow M_{ij}$$

where M_{ii} are monoids and M_{ij} are (M_{ii}, M_{jj}) -bimodules, and the isomorphisms are required to respect these structures. We denote \mathbf{C} by (\mathbf{C}, O, Λ) and we often say that \mathbf{C} is associated to M if it is of the form (\mathbf{C}, O, Λ) .

If there exists a category \mathbf{C}' isomorphic to \mathbf{C} , then there exist an ordering O' of $Ob(\mathbf{C}')$ and isomorphisms

$$\Lambda'_{ij} : \mathbf{C}'(O_i, O_j) \rightarrow M_{ij}$$

such that $(\mathbf{C}', O', \Lambda')$ is also associated to M .

In this case

$$\{(\mathbf{C}, O, \Lambda)\} /_{iso} \simeq \left(\prod_{i,j,k} (Map(M_{ij} \times M_{jk}, M_{ik})) \right)^{assoc} \subset \prod_{i,j,k} (Map(M_{ij} \times M_{jk}, M_{ik})).$$

Then the cardinal of $\{(\mathbf{C}, O, \Lambda)\} /_{iso}$ is going to depend on the cardinal of $\left(\prod_{i,j,k} (Map(M_{ij} \times M_{jk}, M_{ik})) \right)^{assoc}$, which is the multiplication of bimodules by each others respecting the associativity property.

Remark 5.5. In the case of categories with two objects, Definition 5.4 will be interpreted in the following way.

Let $M = (M_{ij})_{1 \leq i,j \leq 2}$ be a matrix of monoids M_{11}, M_{22} and bimodules M_{12}, M_{21} . Then a category \mathbf{C} is associated to M if $Ob(\mathbf{C}) = \{O_1, O_2\}$ and there are isomorphisms of monoids

and bimodules respectively

$$\begin{aligned} \Lambda_{11} : \mathbb{C}(O_1, O_1) &\rightarrow M_{11} & \Lambda_{12} : \mathbb{C}(O_1, O_2) &\rightarrow M_{12} \\ \Lambda_{21} : \mathbb{C}(O_2, O_1) &\rightarrow M_{21} & \Lambda_{22} : \mathbb{C}(O_2, O_2) &\rightarrow M_{22} \end{aligned}$$

where M_{12} is an (M_{11}, M_{22}) -bimodule and M_{21} is an (M_{11}, M_{22}) -bimodule. Then \mathbb{C} is a category associated to the matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

and the number of categories associated to M up to an isomorphism is going to depend on the number of possible multiplications of

$$M_{12} \times M_{21} \rightarrow M_{11}$$

and

$$M_{21} \times M_{12} \rightarrow M_{22}.$$

We can finally define the matrix of monoids and bimodules in the following way: let \mathbb{C} be a finite category. Then we get a matrix where the diagonal entries are monoids and the off diagonal entries are bimodules. This matrix is called *algebraic matrix* (this definition is explained more in Sections 5.3 and 5.4).

Remark 5.6. Let \mathbb{C} and \mathbb{D} be two bimodule categories associated to matrices

$$M = \begin{pmatrix} A & L \\ \emptyset & B \end{pmatrix} \text{ and } N = \begin{pmatrix} A & \emptyset \\ R & B \end{pmatrix}$$

respectively. Define A^{op} , B^{op} and L^{op} to be the opposite monoids and bimodule of A , B and L by

$$x \cdot_{op} y := y \cdot x$$

for all x, y . In particular, if $a \in A$, $b \in B$ and $x \in L$, then

$$x \cdot_{op} a := a \cdot x \text{ and } b \cdot_{op} x := x \cdot b.$$

Then if we apply op to M we get a category \mathbb{C}^{op} associated to the matrix

$$M^{op} = \begin{pmatrix} A^{op} & \emptyset \\ L^{op} & B^{op} \end{pmatrix}$$

such that $\mathbb{C}^{op} \simeq \mathbb{C}$ and \mathbb{C}^{op} is not the same as \mathbb{D} .

Remark 5.7. A matrix of monoids and bimodules does not necessarily admit a category, see example below.

Example 5.8. (Rectangular bands, more explained in Chapter 7)

Let $N^{(1)}$ be matrix of the form

$$\begin{pmatrix} I \times J \cup \{1\} & L \\ \emptyset & A \times B \cup \{1\} \end{pmatrix}$$

And let $N^{(2)}$ be matrix of the form

$$\begin{pmatrix} I \times J \cup \{1\} & \emptyset \\ R & A \times B \cup \{1\} \end{pmatrix}$$

where $I \times J$ is a left rectangular band and $A \times B$ is a right rectangular band, of sizes 2×1 and 1×2 respectively (i.e $|I| = 2$, $|J| = 1$, $|A| = 1$ and $|B| = 2$), and $|L| = |R| = 3$. $N^{(1)}$ admits 20 categories and $N^{(2)}$ admits 16 categories (calculated by MACE4). But the matrix

$$\begin{pmatrix} I \times J \cup \{1\} & L \\ R & A \times B \cup \{1\} \end{pmatrix}$$

does not admit a category, because by Theorem 7.8 there are restrictions on the cardinality of sets of morphisms when we have rectangular bands on the diagonals, and here in this case $|L|$ should be at least 4 in order to obtain a category.

Remark 5.9. The number of categories obtained in the example above for $N^{(1)}$ and $N^{(2)}$ is not the same, because we have a left rectangular band on one side and a right rectangular band on the other side. By Chapter 7 we see that rectangular bands impose some coefficients restrictions on the matrix of a category, which affects the number of bimodules obtained. Also, by Remark 5.6 we see that if we apply op on monoids and bimodules, it's not the same when we interchange monoids, and that explains the different number of bimodules obtained in the example.

5.2 Tensor products

Let B be a monoid, and let L be a right B -module and let R be a left B -module. Then there exists a set

$$L \otimes_B R := L \times R / \sim_B$$

\sim_B is the smallest equivalence relation such that

$$(xb, y) \sim_B (x, by) \quad \forall x \in L, b \in B, y \in R.$$

Then there's a map

$$L \times R \rightarrow L \otimes_B R := L \times R / \sim_B \rightarrow LR; \quad (x, y) \mapsto x \otimes y \mapsto x \cdot y.$$

Lemma 5.10. *Let A, B, A' be monoids.*

1. *If L is an (A, B) -bimodule then $L \otimes_B R$ is a left A -module.*
2. *If R is a (B, A') -bimodule then $L \otimes_B R$ is a right A' -module.*
3. *If L, R are (A, B) and (B, A) bimodules, then $L \otimes_B R$ is an (A, A) -bimodule.*

Proposition 5.11. *Let $Cat(M) := \{(C, \beta); \beta : Mat(C) \simeq M\}$. Then*

$$Cat(M) \subseteq \{(\phi, \psi); \phi : L \times R \rightarrow B, \psi : R \times L \rightarrow B\}.$$

Proof.

$$\begin{aligned} Cat(M) &\subseteq Hom_{A-A \text{ bim}}(L \otimes_B R, A) \times Hom_{B-B \text{ bim}}(R \otimes_A L, B) \\ &\subseteq Hom_{set}(L \times R, A) \times Hom_{set}(R \times L, B). \end{aligned}$$

□

In sections 5.3 and 5.4, we present a more theoretical approach to the categories of bimodules. We discuss two methods of viewing these categories, one through the Grothendieck-Bénabou correspondence and the other through fc-multicategories.

5.3 Grothendieck-Bénabou correspondence

The following is a result described in R. Street's paper [29].

Definition 5.12. [29] Let \mathbf{Bim} be the bicategory whose objects are (small) categories and $Mor_{\mathbf{Bim}}(A, B)$ is the functor category $[B^{op} \times A, \mathbf{Set}]$. The morphisms of \mathbf{Bim} are called *modules* and the 2-cells are called *module morphisms*. Composition of modules is given by a coend formula. For example, if A and B are monoids viewed as categories with one object, then $Mor_{\mathbf{Bim}}(A, B)$ is the category of A, B -bimodules and the composition is given by the tensor product

$$L, L' \mapsto L \otimes_B L'.$$

Then we can identify \mathbf{Cat} as a sub-2-category of the bicategory \mathbf{Bim} by defining a functor $f : A \rightarrow B$ as the module defined by taking $f(b, a)$ to be $B(b, f(a))$.

For any functor $p : E \rightarrow B$, the *fiber* over an object b of B is the subcategory E_b of E given by the pullback

$$\begin{array}{ccc} E_b & \longrightarrow & E \\ \downarrow & & \downarrow p \\ \mathbf{1} & \xrightarrow{b} & B \end{array}$$

Each $\beta : b \rightarrow b'$ determines a module $m_E(\beta) : E_{b'} \rightarrow E_b$ defined by

$$m_E(\beta)(e, e') = \{\xi : e \rightarrow e' \mid p(\xi) = \beta\}$$

for objects e of E_b and e' of $E_{b'}$. We define a lax functor that preserves identities

$$m_E : B^{op} \rightarrow \mathbf{Bim} ; b \mapsto E_b \in Ob(\mathbf{Bim}).$$

Write \mathbf{Cat}/B for the slice category of objects $p : E \rightarrow B$ of \mathbf{Cat} over B in which the morphisms $f : (E, p) \rightarrow (F, q)$ are commutative triangles over B .

We enrich \mathbf{Cat}/B to become a 2-category by accepting those 2-cells $\theta : f \Rightarrow g : (E, p) \rightarrow (F, q)$ satisfying $q\theta = p$. Write $\mathbf{Bicat}(B^{op}, \mathbf{Bim})$ for the bicategory of lax functors $B^{op} \rightarrow \mathbf{Bim}$, lax transformations, and modifications.

Proposition 5.13 (Bénabou, [8] [29]). *The slice 2-category \mathbf{Cat}/B is equivalent to the sub-2-category of the bicategory $\mathbf{Bicat}(B^{op}, \mathbf{Bim})$ whose objects are the lax functors that preserve identities, whose morphisms are the lax transformations with components at objects b of B being actual functors, and whose 2-cells are all the modifications.*

5.3.1 Application to small categories

(I would like to thank Paul-André Melliès for suggesting this point of view).

Let X be a set, then there is a category $\mathbf{Coarse}(X)$ such that $Ob(\mathbf{Coarse}(X)) = X$ and $\mathbf{Coarse}(X)(x, y) := \{*\}$ a singleton.

If \mathbf{C} a category, then there exists $p : \mathbf{C} \rightarrow \mathbf{Coarse}(Ob(\mathbf{C}))$ a unique functor inducing $Id_{\mathbf{C}}$ on $Ob(\mathbf{C})$.

If we apply the Grothendieck-Bénabou construction to this functor we obtain the following:

1. $x \in Ob(\mathbf{C}) \iff x \in \mathbf{Coarse}(Ob(\mathbf{C}))$ and $p^{-1}(x) = \{\mathbf{C}(x, x), x\}$ is the monoid

$\mathbf{C}(x, x)$.

2. An arrow $(x, y) \in \text{Mor}_{\mathbf{Coarse}(\text{Ob}(\mathbf{C}))}$ corresponds to $\mathbf{C}(x, y)$ as a $\mathbf{C}(x, x) - \mathbf{C}(y, y)$ -bimodule.
3. The lax transformation of composition corresponds to

$$\mathbf{C}(x, y) \otimes_{\mathbf{C}(y, y)} \mathbf{C}(y, z) \rightarrow \mathbf{C}(x, z).$$

In the case $X = \{x, y\}$, the category $\text{Coarse}(X)$ has two objects x and y and four morphisms

$$a : x \rightarrow x, \quad b : y \rightarrow y, \quad l : x \rightarrow y, \quad r : y \rightarrow x$$

which gives us two monoids $A = p(a)$ and $B = p(b)$, an (A, B) -bimodule $L = p(l)$ and a (B, A) -bimodule $R = p(r)$. With an (A, A) -bimodule morphism

$$L \otimes_B R \rightarrow A \tag{5.1}$$

and a (B, B) -bimodule morphism

$$R \otimes_A L \rightarrow B. \tag{5.2}$$

Remark 5.14. In condition (3), these are the compositions between the sets of morphisms, which we forget in the case of a bimodule category.

Remark 5.15. To verify associativity, the morphisms 5.1 and 5.2 should satisfy the conditions that the following diagrams are commutative.

$$\begin{array}{ccc} L \otimes_B R \otimes_A L & \longrightarrow & A \otimes_A L \\ \downarrow & & \downarrow \\ L \otimes_B B & \longrightarrow & L \end{array} \quad \text{and} \quad \begin{array}{ccc} R \otimes_A L \otimes_B R & \longrightarrow & B \otimes_B R \\ \downarrow & & \downarrow \\ R \otimes_A A & \longrightarrow & R \end{array}$$

5.4 fc-multicategories

In this section, we present results proved by T. Leinster in [23], thanks to Angel Toledo for giving a talk about this in the *Category Theory Groupe de Travail* in Nice.

In abstract algebra, a *bimodule* is an abelian group that is both a left and a right module, such that the left and right multiplications are compatible. In monoidal category, there is a generalization of the classical notion of modules over rings, which is the notion of modules over monoid objects.

Idea: The classical notion of monoid can be generalized by definition of a monoid (or a *monoid object*) in any monoidal category (\mathbf{C}, \otimes, I) . Classical monoids are just monoids in \mathbf{Set} with the cartesian product.

Definition 5.16. Let \mathbf{C} be a monoidal category. A *monoid* in \mathbf{C} is an object M equipped with a multiplication $\mu : M \otimes M \rightarrow M$ and a unit $\eta : I \rightarrow M$ satisfying the associative law:

$$\begin{array}{ccccc} (M \otimes M) \otimes M & \xrightarrow{\alpha} & M \otimes (M \otimes M) & \xrightarrow{1 \otimes \mu} & M \otimes M \\ & \searrow^{\mu \otimes 1} & & & \swarrow^{\mu} \\ & & M \otimes M & \xrightarrow{\mu} & M \end{array}$$

and the left and right unit laws

$$\begin{array}{ccccc} I \otimes M & \xrightarrow{\eta \otimes 1} & M \otimes M & \xleftarrow{1 \otimes \eta} & M \otimes I \\ & \searrow^{\lambda} & \downarrow^{\mu} & & \swarrow^{\rho} \\ & & M & & \end{array}$$

where α is the associator in \mathbf{C} , λ and ρ are the left and right unitors.

Examples 5.17.

- (i) A monoid object in \mathbf{Ab} (with the usual tensor product of \mathbb{Z} -modules as the tensor product) is a ring. A monoid object in the category of vector spaces over a field k (with the usual tensor product of vector spaces) is an algebra over k .
- (ii) A monoid object in \mathcal{Top} (with cartesian product as the tensor product) is a topological monoid.
- (iii) A monoid object in the category of complete join-semilattices (with its tensor product that represents maps preserving joins in each variable separately) is a unital quantale.
- (iv) The category of pointed sets has a monoidal structure given by the smash product. A monoid object in this monoidal category is an absorption monoid.

Definition 5.18. Let V be a closed monoidal category. Recall that for \mathbf{C} a category enriched over V , a \mathbf{C} -module is a V -functor $\rho : \mathbf{C} \rightarrow V$. We think of the objects $\rho(a)$ for $a \in \text{Ob}(\mathbf{C})$ as the objects on which \mathbf{C} act, and $\rho(\mathbf{C}(a, b))$ as the action of \mathbf{C} on these objects.

In this language a \mathbf{C} - \mathbf{D} *bimodule* for V -categories \mathbf{C} and \mathbf{D} is a V -functor

$$\mathbf{C}^{op} \otimes \mathbf{D} \rightarrow V.$$

Examples 5.19.

- (i) Let $V = \mathbf{Set}$ and let $\mathbf{C} = \mathbf{D}$. Then the *hom* functor $\mathbf{C}(-, -) : \mathbf{C}^{op} \times \mathbf{C} \rightarrow \mathbf{Set}$ is a bimodule. Bimodules can be seen as a kind of generalized *hom*, giving a set of morphisms (or objects of V) between an object of \mathbf{C} and an object of \mathbf{D} .
- (ii) Let $V = \mathbf{Vect}$ and let $\mathbf{C} = \mathbf{BA}_1$ and $\mathbf{D} = \mathbf{BA}_2$ be two one-object \mathbf{Vect} -enriched categories, whose endomorphism vector spaces are hence algebras. Then a \mathbf{C} - \mathbf{D} bimodule is a vector space V with an action of A_1 on the left and an action of A_2 on the right.
- (iii) (**fc-multicategories** [23] [24]) Let V be an **fc**-multicategory. The **fc**-multicategory $\mathbf{Bim}(V)$ is defined as follows:

0-cells A 0-cell of $\mathbf{Bim}(V)$ is a monad in V . That is, it is a 0-cell x of V together with a horizontal 1-cell $x \xrightarrow{t} x$ and 2-cells

$$\begin{array}{ccc} x & \xrightarrow{t} & x & \xrightarrow{t} & x \\ \downarrow 1 & & \Downarrow \mu & & \downarrow 1 \\ x & \xrightarrow{t} & x & & x \end{array} \quad \begin{array}{ccc} x & \xlongequal{\quad} & x \\ \downarrow & & \Downarrow \eta \\ x & \xrightarrow{t} & x \end{array}$$

satisfying the usual axioms, $\mu \circ (\mu, 1_t) = \mu \circ (1_t, \mu)$ and $\mu \circ (\eta, 1_t) = 1 = \mu \circ (1_t, \eta)$.

Horizontal 1-cells A horizontal 1-cell $(x, t, \eta, \mu) \rightarrow (x', t', \eta', \mu')$ consists of a horizontal 1-cell $x \xrightarrow{f} x'$ in V together with 2-cells

$$\begin{array}{ccc} x & \xrightarrow{f} & x' & \xrightarrow{t'} & x' \\ \downarrow 1 & & \Downarrow \theta & & \downarrow 1 \\ x & \xrightarrow{f} & x' & & x' \end{array} \quad \begin{array}{ccc} x & \xrightarrow{t} & x & \xrightarrow{f} & x' \\ \downarrow 1 & & \Downarrow \theta' & & \downarrow 1 \\ x & \xrightarrow{f} & x' & & x' \end{array}$$

satisfying the usual algebra axioms $\theta \circ (\eta, 1_f) = 1$, $\theta \circ (\mu, 1_f) = \theta \circ (1_t, \theta)$, and dually for θ' , and the “commuting action” axiom $\theta' \circ (\theta, 1_{t'}) = \theta \circ (1_t, \theta')$.

Vertical 1-cells $(x, t, \eta, \mu) \downarrow (\hat{x}, \hat{t}, \hat{\eta}, \hat{\mu})$ in $\mathbf{Bim}(V)$ is a vertical 1-cell $\begin{array}{c} x \\ \downarrow p \\ \hat{x} \end{array}$ in V together with

a 2-cell

$$\begin{array}{ccc} x & \xrightarrow{t} & x \\ p \downarrow & & \Downarrow w \\ \hat{x} & \xrightarrow{\hat{t}} & \hat{x} \end{array}$$

such that $w \circ \mu = \hat{\mu} \circ (w, w)$ and $w \circ \eta = \hat{\eta} \circ p$.

2-cells A 2-cell

$$\begin{array}{ccccccc}
 t_0 & \xrightarrow{f_1} & t_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & t_n \\
 \downarrow p & & & & \Downarrow & & \downarrow p' \\
 t & \xrightarrow{f} & & & & & t'
 \end{array}$$

in $\mathbf{Bim}(V)$, where t stands for (x, t, η, μ) , f for (f, θ, θ') , p for (p, w) , and so on, consists of a 2-cell

$$\begin{array}{ccccccc}
 x_0 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & x_n \\
 \downarrow p & & & & \Downarrow \alpha & & \downarrow \\
 x & \xrightarrow{f} & & & & & x'
 \end{array}$$

in V , satisfying the “external equivariance” axioms

$$\alpha \circ (\theta_1, 1_{f_2}, \dots, 1_{f_n}) = \theta \circ (w, \alpha)$$

$$\alpha \circ (1_{f_1}, \dots, 1_{f_{n-1}}, \theta'_n) = \theta' \circ (\alpha, w')$$

and the “internal equivariance” axioms

$$\alpha \circ (1_{f_1}, \dots, 1_{f_{i-2}}, \theta'_{i-1}, 1_{f_i}, 1_{f_{i+1}}, \dots, 1_{f_n}) = \alpha \circ (1_{f_1}, \dots, 1_{f_{i-2}}, \theta_i, 1_{f_{i+1}}, \dots, 1_{f_n})$$

for $2 \leq i \leq n$.

Composition and identities For both 2-cells and vertical 1-cells in $\mathbf{Bim}(V)$, composition is defined directly from the composition in V , and identities similarly.

Theorem 5.20 (Leinster [24]). *If $G : W \rightarrow W'$ is a functor between (S', T') -multicategories then any W -enriched multicategory becomes a W' -enriched multicategory just by composition with G : thus there’s a functor*

$$G_* : (S, T)_W\text{-Multicat} \rightarrow (S, T)_{W'}\text{-Multicat}.$$

In particular, consider for any \mathbf{fc} -multicategory V the forgetful functor $U : \mathbf{Bim}(V) \rightarrow V$. This gives the functor

$$U_* : \mathbf{Cat}_{\mathbf{Bim}(V)} \rightarrow \mathbf{Cat}_V$$

where \mathbf{Cat}_V is the category of V -enriched categories.

The conclusion of Leinster's paper is that if we have an enriched category in V , this gives rise to a category enriched in $\mathbf{Bim}(V)$. In our work, V is the monoidal category \mathbf{Set} , and when we apply Leinster's theorem, we get a bimodule category in which we forget some multiplications. This construction eventually coincides with Grothendieck-Bénabou construction.

Now in Example 5.19 ((iii)), we apply Leinster's theorem to finite categories, and if V is the monoidal category (\mathbf{set}, \times) . Then a 0-cell in $\mathbf{Bim}(V)$ is a monoid, and a horizontal 1-cell is a bimodule (i.e. with monoids actions on the left and on the right that commute). This construction eventually coincides with Grothendieck-Bénabou construction.

Bimodules are explained more in [12] and [22].

5.5 Connectivity of monoids

Definition 5.21. We say that two monoids A and B are *connected* if there exists a category with two objects such that its endomorphism monoids are A and B and all morphism sets are not empty.

Lemma 5.22 (Jérémy Marquès). *If A, B are connected, and B, C are connected then A, C are connected.*

Proof. A, B are connected, then there exists a category \mathbf{C}_1 associated to the algebraic matrix

$$M_1 = \begin{pmatrix} A & L \\ R & B \end{pmatrix}$$

such that L and R are not empty.

Similarly, B, C are connected, then there exists a category \mathbf{C}_2 associated to the algebraic matrix

$$M_2 = \begin{pmatrix} B & L' \\ R' & C \end{pmatrix}$$

such that L' and R' are not empty.

Let

$$M = \begin{pmatrix} A & L \otimes_B L' \\ R' \otimes_B R & C \end{pmatrix}$$

then M is a matrix of a category \mathbf{C} . Indeed, A acts to the left on L and C acts to the right on L' , then A and C act on $L \otimes_B L'$. Same for R and R' . It remains to prove

$(L \otimes_B L') \otimes_C (R' \otimes_B R) \rightarrow A$ and $(R' \otimes_B R) \otimes_A (L \otimes_B L') \rightarrow B$. Indeed,

$$\begin{aligned}
 (L \otimes_B L') \otimes_C (R' \otimes_B R) &\simeq L \otimes_B (L' \otimes_C R') \otimes_B R \\
 &\rightarrow L \otimes_B B \otimes_B R \\
 &\simeq L \otimes_B R \\
 &\rightarrow A.
 \end{aligned}$$

Same for $(R' \otimes_B R) \otimes_A (L \otimes_B L')$. To verify the associativity property, one may refer to Remark 5.15. \square

CHAPTER 6

Grouplike categories

The aim of this chapter is to introduce some categories with certain properties on the objects. The classification of finite categories depends dramatically on the classification of their monoids. Therefore, it's very useful to study some types of semigroups since monoids are semigroups with an identity element. The results in this chapter are published in [19].

6.1 Introduction to the problem

From Chapter 5 (Section 5.1), we can represent matrices of categories in a different way. Instead of number of morphisms between objects, we represent them in terms of the endomorphism monoids of the category.

We study a specific type of finite categories called *grouplike* categories. To define grouplike categories, we need to define grouplike monoids first. A *grouplike monoid* is a monoid of the form $G \cup I$ where G is a group and $I = \{e_1, \dots, e_k\}$ such that

$$(e_i \cdot e_j = e_i \text{ and } e_j \cdot e_i = e_i) \text{ if and only if } i \leq j. \quad (\text{Ord})$$

We denote it by G^{*k} . A *grouplike category* is a category where its endomorphism monoids are grouplike.

In this chapter we present a construction theorem for grouplike categories, and we prove that every grouplike category should come from this construction.

Definition 6.1. Let L be a $(G_1^{*k_1}, G_2^{*k_2})$ -bimodule. We say that L is *i-unigen* if

$$L \cdot f_i = e_i \cdot L := L_i$$

and $L_i \simeq G_1^{*i}$ as a left module and $L_i \simeq G_2^{*i}$ as a right module.

Lemma 6.2. Suppose L is an *i-unigen* $(G_1^{*k_1}, G_2^{*k_2})$ -bimodule with L_i as above, then there exists an isomorphism $G_1 \simeq G_2$ such that $L_i = G_1^{*i}$ as $(G_1^{*k_1}, G_2^{*k_2})$ -bimodule. If $i > 0$ the

isomorphism is unique. If $i = 0$ the isomorphism is well defined up to inner automorphisms.

Because of this lemma we can assume that $G_1 = G_2 = G$. In this case we say that a (G^{*k_1}, G^{*k_2}) -bimodule L is *strongly i -unigen* if it is i -unigen and the isomorphism of Lemma 6.2 can be taken as the identity.

Definition 6.3. Let

$$M = \begin{pmatrix} G^{*k_1} & L \\ R & G^{*k_2} \end{pmatrix}$$

be the algebraic matrix (Definition 5.4) of a grouplike category \mathcal{C} , such that $I_1 = \{e_1, \dots, e_{k_1}\}$ and $I_2 = \{f_1, \dots, f_{k_2}\}$. We define $i^{\max}(\mathcal{C})$ to be the maximum index i such that there exist x, y such that $x \cdot y = e_i$ and $y \cdot x = f_i$.

Let \mathcal{C} be a category with one object $\{\alpha\}$ and endomorphism set G^{*i} . Let x, y be two elements, then there exists a function

$$f : \{x, y\} \rightarrow \alpha$$

and we obtain a new category \mathcal{C}' such that $\mathcal{C}' = f^\alpha(\mathcal{C})$, $Ob(\mathcal{C}') = \{x, y\}$ and $\mathcal{C}'(x, y) = \mathcal{C}(f(x), f(y))$.

If we apply this construction to a grouplike monoid G^{*i} , we get a category associated to the matrix

$$\begin{pmatrix} G^{*i} & G^{*i} \\ G^{*i} & G^{*i} \end{pmatrix} \quad (M^{(1)})$$

of similar copies of G^{*i} , such a category will be called *groupoid-like*.

Theorem 6.4. Let G be a group and G^{*i} a grouplike monoid of the form $G \cup I$ such that $I = \{e_1, \dots, e_i\}$. Let $M^{(1)}$ be a matrix of the form

$$\begin{pmatrix} G^{*i} & G^{*i} \\ G^{*i} & G^{*i} \end{pmatrix} \quad (M^{(1)})$$

of similar copies of G^{*i} , let $\mathcal{C}^{(1)}$ be the groupoid-like category associated to this matrix. Then we can extend the endomorphism sets G^{*i} and we obtain a category $\mathcal{C}^{(2)}$ associated to the matrix

$$\begin{pmatrix} G^{*k_1} & G^{*i} \\ G^{*i} & G^{*k_2} \end{pmatrix} \quad (M^{(2)})$$

such that for all $x \in G^{*i}, y \in G^{*k_1}$, we have $x \cdot y = y \cdot x \in G^{*i}$. Same for G^{*k_2} . And there exists $\mathcal{C}^{(2)}$ a finite category associated to $M^{(2)}$.

Now let $k_1, k_2 \geq i$, suppose that we have the matrix

$$\begin{pmatrix} G^{*k_1} & L \\ R & G^{*k_2} \end{pmatrix} \quad (M^{(3)})$$

such that L and R are strongly i -unigen. Then

$$(M^{(1)}) \subseteq (M^{(2)}) \subseteq (M^{(3)}).$$

And $(M^{(3)})$ is a matrix of a unique grouplike category, denote it by $\mathcal{C}^{(3)}$, such that $i^{\max}(\mathcal{C}^{(3)}) = i$ and for all $x \in L, y \in R, x \cdot y = e_i \cdot x \cdot y \cdot e_i$.

We have

$$\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(2)} \subseteq \mathcal{C}^{(3)}.$$

Theorem 6.5. *Every grouplike category comes from the construction described in Theorem 6.4.*

6.2 Grouplike categories

In this section we introduce grouplike categories. These are groupoid-like having some specific properties, to their set of morphisms. The goal is to study the structure of these categories in order to make the classification problem easier and clearer.

We recall the following definitions from Chapter 3 and Chapter 5:

Definition 3.1. A *monoid* A is a set equipped with a binary operation $\cdot : A \times A \rightarrow A$ such that \cdot is associative and there exists an identity element e such that for every element $a \in A$, the equations $e \cdot a = a$ and $a \cdot e = a$ hold.

Definition 5.1. A *bimodule* is a set with actions on the left and the right of the respective monoids, such that the actions commute i.e. $(g \cdot x) \cdot h = g \cdot (x \cdot h)$. It can be seen as a category such that one of the sets of morphisms is empty, it's called an *upper triangulated category*.

Definition 6.6. A *semicategory* is a category without identity morphisms.

Lemma 6.7. *Let \mathcal{C} be a semicategory, let $u \in \mathcal{C}(x, y)$ with $x, y \in \text{Ob}(\mathcal{C})$, then we can add a morphism u' to $\mathcal{C}(x, y)$ such that $u' \neq u$ and u' duplicates u for all composition operations. Then we get a new semicategory \mathcal{C}' with $\text{Ob}(\mathcal{C}') = \text{Ob}(\mathcal{C})$ and $\text{Hom}(\mathcal{C}') = \text{Hom}(\mathcal{C}) \cup \{u'\}$. On the other hand, we can also add the missing identities in $\text{Ob}(\mathcal{C})$ to obtain a category B .*

The previous lemma shows that we can add morphisms consecutively to a category and obtain a new category (provided that we add identities too). In our case, we define

a category whose objects have grouplike endomorphism monoids, we only add elements consecutively to the monoids. The elements are idempotents and identities to the elements of the monoids. This means that each time we add an identity element, the previous identity is no longer an identity.

We need the notion of bimodules in order to make the classification and counting problem easier.

Definition 6.8. Let A be a semigroup. Whether or not A has a multiplicative identity element, we let e be a fresh element and

$$A^* := A \cup \{e\}.$$

Then A^* is a semigroup if the multiplication of A is extended by stipulating $xe = ex = x$ for all $x \in A^*$. More generally, if A is a semigroup, we recursively define semigroups A^* for $n \in \mathbb{N}$ by:

$$\begin{aligned} A^{*0} &= A \\ A^{*(n+1)} &= (A^{*n})^*. \end{aligned}$$

If A is a group, we say that the semigroups A^{*n} , $n \in \mathbb{N}$, are *grouplike*.

Definition 6.9. We say that a category is called a *grouplike category* with groups G_i if its endomorphism monoids are *grouplike monoids* of the form $G_i^{*k_i}$; $k_i \in \mathbb{N}$.

Definition 6.10. A *band* S is an idempotent semigroup, i.e. for all $x \in S$, $x^2 = x$.

A *semilattice* is a commutative band.

Remark 6.11. The set of idempotents along with the group identity form a semilattice.

Proof. Let $I = \{e_0, \dots, e_k\}$ be the set of idempotents where e_0 is a group identity. I is a semilattice:

- commutative: for $i \leq j$, $e_i \cdot e_j = e_j \cdot e_i = e_i$.
- idempotent: $e_i \cdot e_i = e_i$.

□

6.3 Group action and the orbits of the sets of morphisms

Let \mathbf{C} be a category with n objects X_1, \dots, X_n . For each object X_i, X_j there exists a monoid $A_i = \mathbf{C}(X_i, X_i)$ and a monoid $A_j = \mathbf{C}(X_j, X_j)$ and two operations

$$l : A_i \times \mathbf{C}(X_i, X_j) \rightarrow \mathbf{C}(X_i, X_j)$$

and

$$r : \mathbf{C}(X_i, X_j) \times A_j \rightarrow \mathbf{C}(X_i, X_j)$$

which represent the left monoid action of A_i and the right monoid action of A_j on the set of morphisms from X_i to X_j . Note here that these are the entries of an algebraic matrix.

In our work here, we take monoids of the form A^{*n} , specifically grouplike, which means each monoid contains a subgroup. This subgroup does not act on the whole set of morphisms from X_i to X_j , but it acts on a subset of the previous set (it's important to note here that when we say group action we mean that the identity condition holds). In the following, we introduce how the group action works, and what are exactly the subsets that the group acts on. We will be considering categories with two objects.

Notation 6.12. We denote by $M(G_1^{*k_1}, G_2^{*k_2}, L, R)$ the matrix

$$M = \begin{pmatrix} G_1^{*k_1} & L \\ R & G_2^{*k_2} \end{pmatrix}$$

of monoids and bimodules such that $G_1^{*k_1} = G_1 \cup I_1$ and $G_2^{*k_2} = G_2 \cup I_2$, where G_1 and G_2 are groups and $I_1 = \{e_1, \dots, e_{k_1}\}$ and $I_2 = \{f_1, \dots, f_{k_2}\}$ such that the elements of I_1 and I_2 satisfy (Ord). We denote by e_0 and f_0 the identities of the groups G_1 and G_2 . Let $Cat(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ be the set of grouplike categories associated to M whose objects are X, Y such that the monoids and the bimodules are not empty. If $\mathbf{C} \in Cat(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ then $\mathbf{C}(X, X) = G_1^{*k_1}$, $\mathbf{C}(X, Y) = L$, $\mathbf{C}(Y, X) = R$ and $\mathbf{C}(Y, Y) = G_2^{*k_2}$.

Remark 6.13. More precisely if M is an algebraic matrix i.e. a matrix of monoids and bimodules, then $Cat(M)$ is the set of pairs (\mathbf{C}, β) where \mathbf{C} is a category and β is an isomorphism between the algebraic matrix of \mathbf{C} and M . We usually don't include this notation of β in our discussion.

Remark 6.14. Denote by n_i the order of the group G_i . When we write G^{*k_1} and G'^{*k_2} , this means that the groups G and G' have the same order n .

Lemma 6.15. *Let $C \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. Then:*

1. G_1 acts on $G_1 \cdot L$ and $R \cdot G_1$.
2. G_2 acts on $L \cdot G_2$ and $G_2 \cdot R$.

Proof. Let

$$l : G_1 \times (G_1 \cdot L) \rightarrow G_1 \cdot L.$$

- For all $x \in L$ and $g \in G_1$ we have $e_0 \cdot (g \cdot x) = g \cdot x$.
- For all $g_1, g_2, g_3 \in G_1$ and $x \in L$ we have $(g_2 \cdot g_3) \cdot (g_1 \cdot x) = g_2 \cdot (g_3 \cdot g_1 \cdot x)$.

Same for the other sets.

Moreover, the orbit of the set $G_1 \cdot L$ is itself. Indeed, the orbit of $G_1 \cdot L$ is a subset $G_1 \cdot (G_1 \cdot L)$. It remains to prove the other direction. Let $g \cdot x \in G_1 \cdot L$, then

$$g \cdot x = e_0 \cdot g \cdot x \in G_1 \cdot (G_1 \cdot L).$$

□

Similarly to the case where we have groups as objects, we can conclude that the group action on these sets is free.

Proposition 6.16. *Let $C \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. The actions of the group G_1 on $G_1 \cdot L$ and $R \cdot G_1$ and the group G_2 on $L \cdot G_2$ and $G_2 \cdot R$ are all free.*

Proof. Let $g_1, g_2, g \in G_1$ and $x \in L$. Suppose $g_1 \cdot (g \cdot x) = g_2 \cdot (g \cdot x)$. If we multiply both sides by $y \in R$, we obtain:

$$g_1 \cdot g \cdot x \cdot y = g_2 \cdot g \cdot x \cdot y$$

where $x \cdot y$ has 2 possibilities:

1. If $x \cdot y = e_i$ where $e_i \notin G_1$, then it's sufficient to multiply by g^{-1} on both sides to prove that $g_1 = g_2$.
2. If $x \cdot y = g_3$ where $g_3 \in G_1$, then if we multiply by $g_3^{-1} \cdot g^{-1}$ on both sides, we get $g_1 = g_2$.

□

Corollary 6.17. *Let $C \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. The cardinal of each orbit in the set of morphisms is equal to the order of the group acting.*

Proof. Use Proposition 6.16. □

This corollary could greatly help the enumeration problem here. Now that we know the number of possibilities in some blocks inside a category, then we can compute how many times the multiplication of morphisms appear to obtain non isomorphic copies of blocks.

Lemma 6.18. *Let $C \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. For all $x : X \rightarrow Y$ there exists at least one $y : Y \rightarrow X$ such that $x \cdot y = id_{G_1}$ and vice versa.*

Proof. Let $x \in C(X, Y)$ and $y \in C(Y, X)$. Suppose $x \cdot y \neq id_{G_1}$:

1. If $x \cdot y = g \in G_1$, then

$$x \cdot y \cdot g^{-1} = id_{G_1}.$$

2. If $x \cdot y$ is equal to an idempotent e in $C(X, X)$ then

$$\begin{aligned} x \cdot y &= e \\ x \cdot y \cdot g &= g \ ; \ g \in G_1 \\ x \cdot y \cdot g \cdot g^{-1} &= id_{G_1}. \end{aligned}$$

□

Now since we have two group actions on the sets of morphisms, we want to understand the relation between these actions over the morphisms sets.

Lemma 6.19. *Let $C \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. Then for all $x \in L$, we have*

$$G_1 \cdot x \cdot G_2 \subseteq G_1 \cdot x$$

and

$$G_1 \cdot x \cdot G_2 \subseteq x \cdot G_2.$$

Similarly for all $y \in R$.

Proof. Let $g \cdot x \cdot h \in G_1 \cdot x \cdot G_2$,

$$\begin{aligned} g \cdot x \cdot h &= g \cdot x \cdot h \cdot e_2 \ ; \ e_2 = id_{G_2} \\ &= g \cdot x \cdot h \cdot y \cdot x \ ; \ y \in \mathbb{C}(Y, X) \text{ (Lemma 6.18)} \\ &\in G_1 \cdot x. \end{aligned}$$

Same for the second inequality. □

Lemma 6.20. *Let $\mathcal{C} \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. Let $x \in L$, the orbit of x by the action of G_1 is the same orbit of x by the action of G_2 , i.e. $G_1 \cdot x = x \cdot G_2$.*

Proof. Suppose $n_1 \leq n_2$. From Lemma 6.15, there is a group action by G_1 on $G_1 \cdot L$, and by Corollary 6.17

$$|G_1 \cdot x| = n_1.$$

Similarly, there is a group action by G_2 on $G_1 \cdot L \cdot G_2 \subseteq G_1 \cdot L$, then again by Corollary 6.17 we have

$$|g_1 \cdot x \cdot G_2| = n_2$$

but

$$g_1 \cdot x \cdot G_2 \subseteq G_1 \cdot x \cdot G_2 \subseteq G_1 \cdot x.$$

Therefore, $n_2 \leq n_1$, but we have $n_1 \leq n_2$, then we obtain that n_1 should be equal to n_2 and $G_1 \cdot x \cdot G_2 = G_1 \cdot x$ and $G_1 \cdot x \cdot G_2 = x \cdot G_2$, hence

$$G_1 \cdot x = x \cdot G_2 \text{ for all } x \in L.$$

Similarly, for all $y \in R$, we have $G_2 \cdot y = y \cdot G_1$. □

Proposition 6.21. *Let $\mathcal{C} \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. For all $x, y \in L$ we have*

$$G_1 \cdot x = G_1 \cdot y.$$

Proof. Let $g \cdot x \in G_1 \cdot x$,

$$\begin{aligned} g \cdot x &= (g \cdot e_1) \cdot x \ ; \ e_1 = id_{G_1} \\ &= g \cdot (e_1 \cdot x) \\ &= g \cdot e_1 \cdot x \cdot e_2 \text{ (Lemma 6.20)} \\ &= g \cdot e_1 \cdot x \cdot z \cdot y \in G_1 \cdot y \ ; \ \text{for } z \in R \text{ (Lemma 6.18)} \end{aligned}$$

and since $|G_1 \cdot x| = |G_1 \cdot y|$, hence equality. \square

Proposition 6.22. *Let $C \in \text{Cat}(M(G^{*k_1}, G'^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. Then the multiplication of the elements in the orbit of L and the elements in the orbit of R is the group G , i.e.*

$$G \cdot C(X, Y) \cdot C(Y, X) \cdot G = G$$

and

$$G' \cdot C(Y, X) \cdot C(X, Y) \cdot G' = G'.$$

Proof. $\bullet \subseteq$: evident.

$\bullet \supseteq$: Let $g \in G$

$$g = g \cdot e_0 \cdot e_0 = g \cdot x \cdot y \cdot e_0 \text{ (Lemma 6.18)}$$

where $x : X \rightarrow Y$ and $y : Y \rightarrow X$.

\square

Definition 6.23. Let C be a category with objects $\{X, Y\}$. Suppose that $g \cdot f$ is never equal to an identity for all $f, g \in \text{Hom}(C)$, then we can always reduce the category C to a new semi-category C' such that $\text{Ob}(C') = \text{Ob}(C)$ and $\text{Hom}(C') = \text{Hom}(C) \setminus \{id_C\}$.

Similarly, we can eliminate morphisms other than identities and still obtain a new semi-category.

The reason we want to eliminate some morphisms is because when we take a category whose objects have grouplike endomorphism monoids, then we could restrict the category to a smaller category with only groups as objects and the orbit of the set of morphisms. Following this technique leads to proving some matrix properties about the coefficients on the off-diagonals. It will also clarify how such categories are built.

From the above lemmas and propositions, we conclude that we can divide each set of morphisms into 2 sets: the orbit of the set and the other elements that are not inside the orbit.

6.3.1 Groupoids

Definition 6.24. In category theory, a *groupoid* generalizes the notion of group in several equivalent ways. A groupoid can be seen as a:

- \bullet group with a partial function replacing the binary operation;

- category in which every morphism is invertible. A groupoid with only one object is a usual group.

In the previous sections, we have seen the structure of a grouplike category with two objects. From this these categories, we can extract a subcategory, which is exactly a groupoid.

Theorem 6.25. *In every grouplike category with two objects, there is a sub-semicategory with two objects that is a groupoid, whose groups are the groups of the grouplike monoids.*

Proof. Let \mathcal{C} be a grouplike category with objects X, Y . Let $\mathcal{C}(X, X) = G^{*k_1}$, $\mathcal{C}(Y, Y) = G'^{*k_2}$, $\mathcal{C}(X, Y) = L$ and $\mathcal{C}(Y, X) = R$. Let \mathcal{G} the sub-semicategory of \mathcal{C} of the form:

- $Ob(\mathcal{G}) = \{G, G'\}$;
- $Mor(\mathcal{G}) = \{G, G', G \cdot L = L \cdot G', G' \cdot R = R \cdot G\}$;
- Identities of \mathcal{G} are 1_G and $1_{G'}$;
- Let $x, y \in \mathcal{G}$, $x \circ y = x \cdot y$ with \circ associative.

□

Corollary 6.26. *Let $\mathcal{C} \in Cat(M(G^{*k_1}, G'^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. The category \mathcal{C} determines an isomorphism between G and G' . The isomorphism is well defined up to inner automorphisms.*

Proof. Evident.

□

6.3.2 The sets of idempotents

In this section we study the role of the idempotent elements in the monoids, the idea is to interpret their action on the sets of morphisms.

Let $\mathcal{C} \in Cat(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. Let

$$K_1 = \{e_0, e_1, \dots, e_{k_1}\}$$

and

$$K_2 = \{f_0, f_1, \dots, f_{k_2}\}$$

be the sets of idempotents of $G_1^{*k_1}$ and $G_2^{*k_2}$ where $e_0 = 1_{G_1}$ and $f_0 = 1_{G_2}$. And let:

$$L_{ij} := \{x \in L \mid e_i \cdot x = x \text{ and } x \cdot f_j = x\} = e_i \cdot L \cdot f_j$$

and

$$R_{ji} := \{y \in R \mid y \cdot e_i = y \text{ and } f_j \cdot y = y\} = f_j \cdot R \cdot e_i$$

be the sets of morphisms that are fixed by e_i and f_j .

Notation 6.27. $L_i := L_{ii}$ and $R_j := R_{jj}$.

In general we have

Lemma 6.28. $L_{ij} = e_i \cdot L \cap L \cdot f_j$ and $R_{ji} = f_j \cdot R \cap R \cdot e_i$.

Proof. We always have

$$L_{ij} \subseteq e_i \cdot L \text{ and } L_{ij} \subseteq L \cdot f_j$$

then

$$L_{ij} \subseteq e_i \cdot L \cap L \cdot f_j.$$

Now let $x \in e_i \cdot L \cap L \cdot f_j$ then $e_i \cdot x = x$ and $x \cdot f_j = x$ hence $x \in L_{ij}$. \square

In the following lemma, we present some properties of the multiplication of the sets of morphisms by idempotent elements.

Lemma 6.29. Let $C \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category, we will denote by x, x' elements of L and by y, y' elements of R .

1. $e_i \cdot x = x$ and $x \cdot f_j = x$ for all $e_i \in K_1, f_j \in K_2$ and $x \in L_0$.
2. $e_j \cdot (e_i \cdot x) = e_i \cdot x$ and $(x \cdot f_i) \cdot f_j = x \cdot f_i$ for all $x \in L$ and $i \leq j$.
3. $x \cdot y \in G_1$ and $y \cdot x \in G_2$ for all $x \in L_0$ or $y \in R_0$.
4. If $x \cdot y \in K_1 \setminus \{e_0\}$ then $y \cdot x \in K_2 \setminus \{f_0\}$ and vice versa.
5. If $x \cdot y = e_0$ then $y \cdot x = f_0$. (This result is also proved in Lemma 6.30).
6. If $x \cdot y = e_i$ and $y \cdot x = f_j$ then $x \cdot f_j = e_i \cdot x$ and $y \cdot e_i = f_j \cdot y$.

In this case we can always assume that $x \cdot f_j = e_i \cdot x = x$ and $y \cdot e_i = f_j \cdot y = y$ (because x and y could be replaced with $e_i \cdot x = x \cdot f_j$ and $y \cdot e_i = f_j \cdot y$ respectively).

Proof. 1. Let $x \in L_0$ and $e_i \in K_1$, we have:

$$e_i \cdot x = e_i \cdot (e_0 \cdot x) = (e_i \cdot e_0) \cdot x = e_0 \cdot x = x.$$

Same for $x \cdot f_j$.

2. $e_i \cdot e_j = e_j \cdot e_i = e_i$ for all $i \leq j$ then $e_j \cdot (e_i \cdot x) = x$.
3. Let $g \in G_1$ such that $g \cdot x = g \cdot x'$ then $g^{-1} \cdot g \cdot x = g^{-1} \cdot g \cdot x'$ then $g' \cdot x = g' \cdot x'$.

4. Suppose that $x \cdot y = e_i$ then $x \cdot y \cdot e_0 = e_0$ then $x \cdot y = e_0$ (because $y \cdot e_0 = y$) contradiction.
5. Let $x \in L$ and $y \in R$ such that $x \cdot y = e_i \in K_1 \setminus \{e_0\}$. Suppose that $y \cdot x = g \in G_2$. Then

$$\begin{aligned}(x \cdot y) \cdot (x \cdot y) &= e_i \\ x \cdot g \cdot y &= e_i\end{aligned}$$

but $x \cdot g \cdot y \in G_1$ and $e_i \in K_1 \setminus \{e_0\}$, it means that $e_i \in G_1$, hence contradiction. Therefore, $y \cdot x \in K_2 \setminus \{f_0\}$.

6. From part (4), we can see that if $x \cdot y \in G_1$ then $y \cdot x \in G_2$. Suppose that $x \cdot y = e_0$ and $y \cdot x = g \in G_2$. Proving that $g = f_0$. We have

$$\begin{aligned}y \cdot x &= g \\ \Rightarrow x \cdot y \cdot x &= x \cdot g \\ \Rightarrow y \cdot e_0 &= g \cdot y \\ \Rightarrow f_0 \cdot (y \cdot e_0) &= g \cdot (y \cdot e_0) \quad (\text{because } y \cdot e_0 \text{ and } g \cdot y \text{ are in } G_2 \cdot R).\end{aligned}$$

By the free action of G_2 on $G_2 \cdot R$, we obtain that $g = f_0$.

7. $x \cdot y \cdot x = e_i \cdot x$ thus $x \cdot f_j = e_i \cdot x$
 $y \cdot x \cdot y = f_j \cdot y$ thus $y \cdot e_i = f_j \cdot y$.

In addition, by part (b) $(x \cdot f_j) \cdot f_j = x \cdot f_j$ and $e_i \cdot (e_i \cdot x) = e_i \cdot x$, then we can assume that $x \cdot f_j = e_i \cdot x = x$.

□

From Lemma 6.29 (4) (5), we see that if one of the multiplications is an idempotent then the other way around should be an idempotent as well. That's why in the following, we study the structure of the category whenever we have two elements such that their multiplications are idempotents.

Lemma 6.30. *Let $C \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. Let i be the maximum index such that there exist $x \in L$ and $y \in R$; $x \cdot y = e_i$. And let j' be the maximum index such that there exist $x' \in L$ and $y' \in R$; $y' \cdot x' = f_{j'}$. We can assume following Lemma 6.29 (6) that $e_i \cdot x = x$, $y \cdot e_i = y$, $x' \cdot f_{j'} = x'$ and $f_{j'} \cdot y' = y'$.*

1. If $i = 0$ then $j' = 0$ and vice versa.

2. If $i, j' \geq 1$ then $x \cdot y' = e_i$ and $y' \cdot x = f_{j'}$. In addition, $e_i \cdot x = x \cdot f_{j'} = x$ and $y' \cdot e_i = f_{j'} \cdot y' = y'$.

Proof. For part (1), suppose $i = 0$, in this case by Lemma 6.29 (4) we have $x' \cdot y'$ is an idempotent, and by maximality of i it has to be e_0 . Therefore

$$f_{j'} = f_{j'}^2 = y' \cdot x' \cdot y' \cdot x' = y' \cdot e_0 \cdot x' \in G_1.$$

Therefore, $j' = 0$.

For part (2), suppose $i', j' \geq 1$, then from Lemma 6.29 (4), suppose that $y \cdot x = f_j$ and $x' \cdot y' = e_{i'}$. From our assumption following (Lemma 6.29 (6)), we get

$$f_j \cdot y = y \cdot e_i = y \text{ and } x \cdot f_j = e_i \cdot x = x$$

and

$$f_{j'} \cdot y' = y' \cdot e_{i'} = y' \text{ and } x' \cdot f_{j'} = e_{i'} \cdot x' = x'.$$

By maximality we have $i \geq i'$ and $j' \geq j$. We have

$$x \cdot f_{j'} = (x \cdot f_j) \cdot f_{j'} = x \cdot (f_j \cdot f_{j'}) = x \cdot f_j = x.$$

We obtain that

$$x = x \cdot f_j = x \cdot f_{j'} = x \cdot (y' \cdot x').$$

Then

$$e_i = x \cdot y = (x \cdot y' \cdot x') \cdot y = (x \cdot y') \cdot (x' \cdot y).$$

Therefore $x \cdot y' \notin G_1$ since $i \geq 1$, then there exists $m \leq i$ (by maximality of i) such that $x \cdot y' = e_m$. Then $e_i = e_m \cdot (x' \cdot y')$.

Which implies that $e_m = e_m \cdot e_i = e_m^2 \cdot (x' \cdot y) = e_m \cdot (x' \cdot y) = e_i$. Hence $e_m = e_i$ and $x \cdot y' = e_i$.

Similarly we can prove that $y' \cdot x = f_{j'}$ and $y' \cdot e_i = y'$. □

Corollary 6.31. *In the case of Lemma 6.30 (2), we have $x = x'$ and $y = y'$.*

Proof. $x = e_i \cdot x = x \cdot y' \cdot x = x \cdot f_{j'} \cdot x \cdot y' \cdot x' = e_i \cdot x'$.

Where $x' = e_{i'} \cdot x' = e_{i'} \cdot e_i \cdot x' = e_i \cdot e_{i'} \cdot x' = e_i \cdot x'$. Then $x = x'$.

Similarly we prove that $y = y'$. □

Proposition 6.32. *Let $C \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category. Let i and j be the maximum elements such that there exist $x \in L$ and $y \in R$;*

$x \cdot y = e_i$ and $y \cdot x = f_j$. Then

$$L_{ij} = e_i \cdot L = L \cdot f_j$$

and

$$R_{ji} = f_j \cdot R = R \cdot e_i.$$

Proof. We want to prove that $e_i \cdot L \cdot f_j = e_i \cdot L = L \cdot f_j$.

We always have that $e_i \cdot L \cdot f_j \subseteq e_i \cdot L$ and $e_i \cdot L \cdot f_j \subseteq L \cdot f_j$. We prove the other direction.

Let $x' \in L$

$$\begin{aligned} e_i x' &= e_i e_i x' \\ &= e_i (x y) x' \\ &= e_i x (y x') \\ &= e_i x f_m \\ &= e_i \underbrace{x f_m}_{\in L} f_j \quad (f_m f_j = f_m \text{ because } j \text{ is the maximum}) \\ &\in e_i \cdot L \cdot f_j. \end{aligned}$$

Similarly we prove the others. □

Theorem 6.33. Let $\mathcal{C} \in \text{Cat}(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ (Notation 6.12) be a grouplike category.

Suppose that i and j are the maximum elements (in the sense of Lemma 6.30) such that $x \cdot y = e_i$ and $y \cdot x = f_j$. By Lemma 6.29 (6) we can assume that

$$f_j \cdot y = y \cdot e_i = y \text{ and } x \cdot f_j = e_i \cdot x = x.$$

Then we can construct a sub-semicategory \mathcal{C}' of the form

$$\begin{pmatrix} G_1^{*i} & L_{ij} \\ R_{ji} & G_2^{*j} \end{pmatrix}$$

where $L_{ij} = e_i \cdot L \cdot f_j$ and $R_{ji} = f_j \cdot R \cdot e_i$, such that \mathcal{C}' is the maximum sub-semicategory of this form.

Proof. • **\mathcal{C}' is a category:**

- Objects: X and Y .
- Morphisms: G_1^{*i}, L_{ij}, R_{ji} and G_2^{*j} .
- Composition:

* Let $z \in L_{ij}$ and $e' \in K_i$ then we can write $z = e_i \cdot \tilde{z} \cdot f_j$, then

$$e' \cdot z = e' \cdot (e_i \cdot \tilde{z} \cdot f_j) = (e' \cdot e_i) \cdot \tilde{z} \cdot f_j = e_i \cdot (e' \cdot \tilde{z}) \cdot f_j \in L_{ij}.$$

* Let $z \in L_{ij}$ and $w \in R_{ji}$ then $z = e_i \cdot \tilde{z} \cdot f_j$ and $w = f_j \cdot \tilde{w} \cdot e_i$, then

$$\begin{aligned} z \cdot w &= (e_i \cdot \tilde{z} \cdot f_j) \cdot (f_j \cdot \tilde{w} \cdot e_i) \\ &= e_i \cdot (\tilde{z} \cdot f_j \cdot f_j \cdot \tilde{w}) \cdot e_i \\ &= e_i \cdot (\tilde{z} \cdot f_j \cdot \tilde{w}) \cdot e_i \in G_1^{*i}. \end{aligned}$$

– Identities: e_i and f_j .

• **\mathbf{C}' is the maximum category of this form:**

If we have two elements x' and y' outside of L_{ij} and R_{ji} such that $x' \cdot y' = e_i$ and $y' \cdot x' = f_j$, then by Lemma 6.30 we have $x \cdot y' = e_i$ and $y' \cdot x = f_j$. This is a contradiction with the maximality of i and j . Hence \mathbf{C}' is maximum. □

Corollary 6.34. *The monoids G_1^{*i} and G_2^{*j} are isomorphic. We call \mathbf{C} a groupoid-like category.*

Proof. There exist x, y such that $x \cdot y = e_i$ and $y \cdot x = f_j$ the identities. It follows that $i = j$ because the groups G_1 and G_2 are isomorphic. □

Proposition 6.35. *Let*

$$M = \begin{pmatrix} G_1^{*k_1} & L \\ R & G_2^{*k_2} \end{pmatrix}$$

be a matrix of a grouplike category \mathbf{C} . By Theorem 6.33, we can construct a sub-semicategory \mathbf{C} associated to the matrix

$$M' = \begin{pmatrix} G_1^{*i} & L_i \\ R_i & G_2^{*i} \end{pmatrix}$$

of monoids and bimodules. For all $x \in L$ and all $y \in R$ we have

$$x \cdot y = (e_i \cdot x) \cdot (y \cdot e_i) \in G_1^{*i} \text{ and } y \cdot x = (f_i \cdot y) \cdot (x \cdot f_i) \in G_2^{*i}.$$

Proof. Suppose that $x \cdot y \in G_1^{*k_1} \setminus G_1^{*i}$ then there exists $e_k \notin G_1^{*i}$ such that $x \cdot y = e_k$. This is a contradiction with the maximality of i . □

Conclusion: Let

$$M = \begin{pmatrix} G_1^{*k_1} & L \\ R & G_2^{*k_2} \end{pmatrix}$$

be a matrix of a grouplike category \mathbb{C} whose objects X and Y . Then

1. $G_1 \simeq G_2$.
2. There exists $i^{max} = \max(i)$ such that there exist x, y and $e_{i^{max}} = x \cdot y$.
There exists $j^{max} = \max(j)$ such that there exist x, y and $f_{j^{max}} = y \cdot x$.
3. $i^{max} = j^{max}$, and we obtain the following matrix

$$\begin{pmatrix} G_1^{*i^{max}} & L_{i^{max}j^{max}} \\ R_{j^{max}i^{max}} & G_2^{*j^{max}} \end{pmatrix}$$

of a sub-semicategory \mathbb{C}' of \mathbb{C} whose objects X and Y are isomorphic. Thus

$$G_1^{*i^{max}} \simeq G_2^{*j^{max}} \text{ as monoids}$$

and

$$G_1^{*i^{max}} \simeq L_{i^{max}j^{max}} \simeq R_{j^{max}i^{max}} \text{ as bimodules.}$$

4. The bimodule L has the property

$$L_i = L \cdot f_i = e_i \cdot L \simeq G_1^{*i} \simeq G_2^{*i}$$

i.e. L is i -unigen.

and the bimodule R has the property

$$R_i = f_i \cdot R = R \cdot e_i \simeq G_2^{*i} \simeq G_1^{*i}$$

i.e. R is i -unigen. Where $i = i^{max} = j^{max}$.

5. For all $x \in L$, $e_i \cdot x = x \cdot f_i$.
6. The isomorphisms in 4 are inverses.
7. The multiplications of the elements of L by the elements of R are determined once we fix x and y .

Theorem 6.36. *Let G_1, G_2 be two groups, k_1, k_2 , L, R be $(G_1^{*k_1}, G_2^{*k_2})$ -bimodules, and i ; $i \leq k_1, i \leq k_2$ such that*

$$L_i = L \cdot f_i = e_i \cdot L \text{ and } R_i = R \cdot e_i = f_i \cdot R \tag{P(i)}$$

and such that there is $x_i \in L_i$ such that x_i determines an isomorphism

$$\begin{aligned} G_1^{*i} &\rightarrow L_i \\ g &\mapsto gx_i \end{aligned}$$

and an isomorphism

$$\begin{aligned} G_2^{*i} &\rightarrow L_i \\ h &\mapsto x_i h \end{aligned}$$

Similarly for $y_i \in R_i$, determines an isomorphism $G_2^{*i} \simeq R_i \simeq G_1^{*i}$.

The isomorphisms $G_1^{*i} \simeq L_i \simeq G_2^{*i}$ and $G_2^{*i} \simeq R_i \simeq G_1^{*i}$ are assumed to be inverses for $P(i)$.

Then we get a category \mathcal{C} with algebraic matrix

$$\begin{pmatrix} G_1^{*k_1} & L \\ R & G_2^{*k_2} \end{pmatrix}$$

such that $i = i^{\max} = j^{\max}$, $x_i \cdot y_i = e_i$ and $y_i \cdot x_i = f_i$ (L and R are i -unigen).

For $i \geq 1$, then the choice of x_i is unique and hence the category is unique.

For $i = 0$, then the choice of x_0 is not unique but the category is unique once x_0 and y_0 are fixed.

Proof. The multiplications $G_1^{*k_1} \times L \rightarrow L$ and $L \times G_2^{*k_2} \rightarrow L$ are given by the bimodule structure of L . Similarly for R .

Let $x' \in L$, $y' \in R$,

$$x'y' := \underbrace{(e_i x')}_{\in L_i} \underbrace{(y' e_i)}_{\in R_i} \in G_1^{*i}.$$

Similarly for $y'x'$. We can check that the multiplication is associative.

If we fix the matrix

$$M = \begin{pmatrix} G_1^{*k_1} & L \\ R & G_2^{*k_2} \end{pmatrix}$$

and we take $i^{\text{biggest}} = \max\{i \mid P(i) \text{ holds}\}$, then for the cases $i \geq 1$

$$\text{Cat}_{i^{\max} \geq 1}(M) = \{1 \leq i \leq i^{\text{biggest}}\}.$$

For $i = 0$, the choices of x_0 and y_0 are not unique, see Remark 6.38. □

Remark 6.37. The condition $P(i)$ is what we call i -unigen in Definition 6.1 (Thanks to Carlos Simpson for suggesting the terminology i -unigen).

Remark 6.38. When $i = 0$, once we fix x_0 , there are maybe several choices for y_0 that lead to inverse isomorphisms. The set of choices of y_0 is given by the center of the group. The cardinal of the set of pairs (C, β) in $Cat(M(G_1^{*k_1}, G_2^{*k_2}, L, R))$ with $i^{max} = 0$ is equal to the cardinal of the center of the group.

Proof of Lemma 6.2. Let L be an i -unigen bimodule. Let $x \in L_i$ such that $G_1^{*i} \cdot x = L_i$, where $G_1^{*i} \cdot x \simeq G_1^{*i}$. Then $x \cdot G_2^{*i} = L_i$:

Suppose $x \cdot g = x \cdot g'$, $g \neq g'$ in G_2^{*i} . Then since $G_1^{*i} \cdot x = L_i$, we get $y \cdot g = y \cdot g'$ for all $y \in L_i$, then $g = g'$. Then

$$\{x\} \times G_2^{*i} \hookrightarrow L_i$$

but they have the same cardinal, then

$$\{x\} \times G_2^{*i} \simeq L_i.$$

We define an isomorphism

$$\phi : G_1^{*i} \rightarrow G_2^{*i}$$

as follows

If $g \in G_1^{*i}$ and $g \cdot x \in L_i$ then $g \cdot x = x \cdot h$; $h \in G_2^{*i}$. Set

$$\phi(g) := h \text{ and } g \cdot x = x \cdot \phi(g).$$

Then

$$\begin{aligned} (gg') \cdot x &= g \cdot (g' \cdot x) \\ &= g \cdot (x \cdot \phi(g')) \\ &= (g \cdot x) \cdot \phi(g') \\ &= (x \cdot \phi(g)) \cdot \phi(g') \\ &= x \cdot (\phi(g)\phi(g')) \\ &= x \cdot (\phi(gg')). \end{aligned}$$

By uniqueness of h , we have $\phi(gg') = \phi(g)\phi(g')$, and Lemma 6.2 is proved. □

Proof of Theorem 6.4. Using Theorem 6.36 we prove Theorem 6.4, as we have the same construction of a category. In Theorem 6.36 we start by choosing $i^{max} = j^{max}$ to get to the algebraic matrix $(M^{(3)})$.

For the multiplication table of $C^{(3)}$, the multiplication of G^{*k_1} on L and R is given by the

bimodule structure. It remains to find the maps

$$L \times R \rightarrow G^{*k_1} \text{ and } R \times L \rightarrow G^{*k_2}.$$

Let $x \in L$ and $y \in R$,

$$x \cdot y = e_i \cdot x \cdot y \cdot e_i$$

where $e_i \cdot x$ and $y \cdot e_i$ are in L_i and R_i and these compositions are given by $C^{(1)}$. And as in the conclusion part (7), we get the uniqueness of the category. □

Proof of Theorem 6.5. From Theorem 6.33 and Proposition 6.35 we can prove Theorem 6.5. As in Corollary 6.34 we prove that i and j should be the same, then the algebraic matrix obtained is

$$M = \begin{pmatrix} G^{*i} & L_i \\ R_i & G^{*i} \end{pmatrix}$$

where L_i and R_i are isomorphic to G^{*i} (conclusion part (3)). Then M is the matrix ($M^{(1)}$).

We conclude that if we have a grouplike category then the two bimodules L and R are i -unigen and the resulting isomorphisms between these two bimodules are inverses. If we then identify the groups via these isomorphisms, we can say that L and R become strictly i -unigen. Then we get the structure described in Theorem 6.4 and Theorem 6.5 is proved. □

Notation 6.39. Let

$$N^{(1)} = \begin{pmatrix} A & L \\ \emptyset & B \end{pmatrix}$$

be a matrix of bimodule L , and let $C^{(1)}$ be a category associated to $N^{(1)}$.

Similarly let

$$N^{(2)} = \begin{pmatrix} A & \emptyset \\ R & B \end{pmatrix}$$

be a matrix of bimodule R , and let $C^{(2)}$ be a category associated to $N^{(2)}$.

We denote by $N = N^{(1)} \cup N^{(2)}$ the matrix of the form

$$\begin{pmatrix} A & L \\ R & B \end{pmatrix}$$

of monoids and bimodules L and R in $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$. We denote by \mathcal{C} a category associated to N .

The condition that the isomorphisms should be inverses in Theorem 6.36 is very important to obtain the grouplike category, see example below.

Example 6.40. Consider the matrix

$$N^{(1)} = \begin{pmatrix} \mathbb{Z}_3 & L \\ \emptyset & \mathbb{Z}_3 \end{pmatrix}$$

where

$$\mathbb{Z}_3 = \{1, 2, 3\} \quad L = \{4, 5, 6\} \quad \mathbb{Z}_3 = \{7, 8, 9\}$$

such that

$$\mathbb{Z}_3 \cdot L = L \cdot \mathbb{Z}_3 = \begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{pmatrix}$$

then $N^{(1)}$ is matrix of the bimodule L . Let $\mathcal{C}^{(1)}$ be a bimodule category associated to $N^{(1)}$ with the above table of multiplication.

Consider the matrix

$$N^{(2)} = \begin{pmatrix} \mathbb{Z}_3 & \emptyset \\ R & \mathbb{Z}_3 \end{pmatrix}$$

where

$$\mathbb{Z}_3 = \{1, 2, 3\} \quad R = \{4, 5, 6\} \quad \mathbb{Z}_3 = \{7, 8, 9\}$$

such that

$$\mathbb{Z}_3 \cdot R = \begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \\ 6 & 4 & 5 \end{pmatrix}$$

and

$$R \cdot \mathbb{Z}_3 = \begin{pmatrix} 4 & 6 & 5 \\ 5 & 4 & 6 \\ 6 & 5 & 4 \end{pmatrix}.$$

Similarly, $N^{(2)}$ is a matrix of the bimodule R . Notice that we changed the multiplication table of $R \cdot \mathbb{Z}_3$ by the involution of the group \mathbb{Z}_3 . Let $\mathcal{C}^{(2)}$ be a bimodule category associated to $N^{(2)}$ with the above table of multiplication.

But the matrix

$$N = \begin{pmatrix} \mathbb{Z}_3 & L \\ R & \mathbb{Z}_3 \end{pmatrix}$$

with the above bimodules $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ doesn't admit a grouplike category. This was shown by calculating using MACE4. It also follows from Conclusion part (6) since the isomorphisms given by L and R are not inverses.

We can conclude now the general structure of grouplike categories with only 2 objects X_1, X_2 .

	$X_1 \rightarrow X_1$	$X_1 \rightarrow X_2$	$X_2 \rightarrow X_1$	$X_2 \rightarrow X_2$
X_1	G^{*k_1}	L		
\downarrow				
X_1			$G^{*i^{max}}$	L
\downarrow				
X_2	R	$G^{*i^{max}}$		
\downarrow				
X_1			R	G^{*k_2}
\downarrow				
X_2				

CHAPTER 7

Rectangular bands

In this chapter, we discuss rectangular bands in finite categories. The existence of rectangular bands as objects imposes some restrictions on the sets of morphisms of the category, providing that these sets are not empty. We also discuss a particular case, where we have a left and a right rectangular bands as objects.

7.1 Definition of rectangular bands

We recall some definitions from Chapter 3.

Definition 3.3. A *band* is a semigroup in which every element is idempotent. In other words

$$x^2 = x \quad \forall x.$$

Definition 3.5. A *rectangular band* is a band S that satisfies:

1. $xyx = x$ for all $x, y \in S$, or equivalently,
2. $xyz = xz$ for all $x, y, z \in S$.

Definition 7.1. A *left zero band* is a band satisfying the equation:

$$xy = x.$$

A *right zero band* is a band satisfying the equation:

$$xy = y.$$

There is a complete classification of rectangular bands. Given arbitrary sets I and J , one can define a semigroup operation by setting:

$$(i, j) \cdot (k, l) = (i, l).$$

The resulting semigroup is a rectangular band because:

1. For any pair (i, j) we have $(i, j) \cdot (i, j) = (i, j)$.
2. For any two pairs $(i_x, j_x), (i_y, j_y)$ we have

$$(i_x, j_x) \cdot (i_y, j_y) \cdot (i_x, j_x) = (i_x, j_x).$$

In fact, any rectangular band is isomorphic to one of the above form. Left zero and right zero bands are rectangular bands, and in fact every rectangular band is isomorphic to a direct product of a left zero band and a right zero band. All rectangular bands of prime order are zero bands, either left or right. A rectangular band is said to be purely rectangular if it is not a left or right zero band [13].

In categorical language, one can say that the category of nonempty rectangular bands is equivalent to $\text{Set}_{\neq\emptyset} \times \text{Set}_{\neq\emptyset}$, where $\text{Set}_{\neq\emptyset}$ is the category with nonempty sets as objects and functions as morphisms. This implies that not only that every nonempty rectangular band is isomorphic to one coming from a pair of sets, but also these sets are uniquely determined up to a canonical isomorphism, and all homomorphisms between bands come from pairs of functions between sets [14]. Note that if the set I is empty in the above result, the rectangular band $I \times J$ is independent of J , and vice versa. This is why the above result only gives an equivalence between nonempty rectangular bands and pairs of nonempty sets.

7.2 Rectangular bands in categories

Notation 7.2. let $I \times J$ be the semigroup in one corner and $A \times B$ in the other, with sets X and Y of maps. The elements of $I \times J$ are denoted e_{ij} and the elements of $A \times B$ are denoted f_{ab} . The directions go so that we can define compositions

$$e_{ij}x, \quad xf_{ab}$$

$$f_{ab}y, \quad ye_{ij}.$$

Fix elements denoted 0 in the sets I, J, A, B . We first define some canonical elements denoted $x_{ib} \in X$ and $y_{aj} \in Y$. Choose elements $\tilde{x} \in X$ and $\tilde{y} \in Y$ and set

$$x_{ib} := e_{i0}\tilde{x}f_{0b}$$

and

$$y_{aj} := f_{a0}\tilde{y}e_{0j}.$$

Lemma 7.3. *Let C be a category associated to the matrix*

$$\begin{pmatrix} I \times J^{*1} & X \\ Y & A \times B^{*1} \end{pmatrix}.$$

Then for any $x \in X$ or $y \in Y$ and any i, j, a, b , we have

$$e_{ij}x f_{ab} = x_{ib}$$

and

$$f_{ab}y e_{ij} = y_{aj}.$$

Furthermore,

$$e_{ij}x_{kb} = x_{ib}, \quad x_{ib}f_{ac} = x_{ic}$$

$$f_{ab}y_{cj} = y_{aj}, \quad y_{aj}e_{kl} = y_{al}$$

$$x_{ib}y_{aj} = e_{ij}, \quad y_{aj}x_{ib} = f_{ab}.$$

Proof. Let $x \in X$,

$$\begin{aligned} e_{ij}x f_{ab} &= e_{i0}(e_{0j}x f_{a0})f_{0b} \\ &= e_{i0}\tilde{x}f_{0b} \\ &= x_{ib}. \end{aligned}$$

Same for y_{aj} .

$$\begin{aligned} e_{ij}x_{kb} &= e_{ij}e_{k0}\tilde{x}f_{0b} \\ &= e_{i0}\tilde{x}f_{0b} \\ &= x_{ib}. \end{aligned}$$

Same for the others. □

Proposition 7.4. x_{ib} and y_{aj} are independent of the choice of x_0 and y_0 .

Proof. We have $e_{ij} \cdot (x \cdot f_{ab} \cdot y) \cdot e_{ij} \cdot x_0 \cdot f_{ab} = e_{ij} \cdot e_{kl} \cdot e_{ij} \cdot x_0 \cdot f_{ab} = e_{ij} \cdot x_0 \cdot f_{ab}$.

And $e_{ij} \cdot x \cdot f_{ab} \cdot (y \cdot e_{ij} \cdot x_0) \cdot f_{ab} = e_{ij} \cdot x \cdot f_{ab} \cdot f_{kl} \cdot f_{ab} = e_{ij} \cdot x \cdot f_{ab}$. □

Lemma 7.5. Let C be a reduced category associated to the matrix

$$\begin{pmatrix} M_1 & X \\ Y & M_2 \end{pmatrix}.$$

If there exists $x \in X$ and $y \in Y$ such that $y \cdot x = 1_{M_2}$, then $|M_2| < |M_1|$ and M_2 is a sub-monoid of M_1 disjoint from $\{1_{M_1}\}$.

Proof. Let

$$\begin{aligned} \phi : M_2 &\rightarrow M_1 \\ b &\mapsto x \cdot f \cdot y \end{aligned}$$

We have

$$\phi(ff') = x \cdot ff' \cdot y = x \cdot f \cdot 1_{M_2} \cdot f' \cdot y = (x \cdot f \cdot y) \cdot (x \cdot f' \cdot y) = \phi(f)\phi(f').$$

ϕ is injective, indeed, suppose that $\phi(f) = 1_{M_1}$, then

$$\begin{aligned} x \cdot f \cdot y &= 1_{M_1} \\ \Rightarrow x \cdot f \cdot y \cdot x &= x \\ \Rightarrow x \cdot f &= x \\ \Rightarrow y \cdot x \cdot f &= y \cdot x \\ \Rightarrow f &= 1_{M_2}. \end{aligned}$$

□

Remark 7.6. Lemma 7.5 is the same as Lemma 3.6 but seen in the algebraic matrix point of view. We note that if $|M_1| = |M_2|$ then $x \cdot y = 1_{M_1}$ and C is not reduced.

Proposition 7.7. Let C be a finite category associated to the matrix

$$\begin{pmatrix} I \times J^{*1} & X \\ Y & A \times B^{*1} \end{pmatrix}$$

then $x \cdot y \neq 1_{I \times J^{*1}}$ and $y \cdot x \neq 1_{A \times B^{*1}}$ for all $x \in L$ and $y \in R$.

Proof. Suppose that there exists $y \in Y$ and $x \in X$ such that $y \cdot x = 1_{A \times B^{*1}}$ then by Lemma 7.5 there exists an injection

$$\begin{aligned} \phi : A \times B^{*1} &\rightarrow I \times J \\ 1 &\mapsto e_{ij} \\ f_{ab} &\mapsto e_{kl}. \end{aligned}$$

We have that

$$1 \cdot f_{ab} = f_{ab} \text{ and } f_{ab} \cdot 1 = f_{ab}$$

then

$$\phi(1 \cdot f_{ab}) = \phi(f_{ab}) \text{ and } \phi(f_{ab} \cdot 1) = \phi(f_{ab})$$

then

$$\begin{aligned}\phi(1) \cdot \phi(f_{ab}) &= \phi(f_{ab}) \text{ and } \phi(f_{ab}) \cdot \phi(1) = \phi(f_{ab}) \\ e_{ij} \cdot e_{kl} &= e_{kl} \text{ and } e_{kl} \cdot e_{ij} = e_{kl}\end{aligned}$$

therefore

$$e_{il} = e_{kl} \text{ and } e_{kj} = e_{kl}$$

thus

$$i = k \text{ and } j = l.$$

Hence

$$e_{ij} = e_{kl}$$

this is a contradiction. □

Theorem 7.8. *Let C be a finite category associated to the matrix*

$$\begin{pmatrix} I \times J^{*1} & X \\ Y & A \times B^{*1} \end{pmatrix}$$

where $I \times J$ and $A \times B$ are rectangular bands of size mn and pq respectively such that X and Y are two non empty sets of morphisms. Then $|X| \geq mq$ and $|Y| \geq np$.

Proof. We have $e_{ij} \cdot e_{i'j'} = e_{ij'}$ and $f_{ab} \cdot f_{a'b'} = f_{ab'}$. And for any $x \in X, y \in Y$

$$e_{ij} \cdot x \cdot f_{ab} = x_{ib} \quad \text{and} \quad f_{ab} \cdot y \cdot e_{ij} = y_{aj}.$$

Suppose that there exists $(i, b) \neq (i', b')$ such that $x_{ib} = x_{i'b'}$, i.e. $e_{ij} \cdot x \cdot f_{ab} = e_{i'j'} \cdot x \cdot f_{a'b'}$, then we have two cases:

1. If $i \neq i'$:

$$\begin{aligned}e_{ij} \cdot (x \cdot f_{ab} \cdot y) &= e_{i'j'} \cdot (x \cdot f_{a'b'} \cdot y) \\ e_{ij} \cdot e_{kl} &= e_{i'j'} \cdot e_{k'l'} \\ e_{il} &= e_{i'l'} \text{ contradiction.}\end{aligned}$$

2. If $b \neq b'$:

$$\begin{aligned}(y \cdot e_{ij} \cdot x) \cdot f_{ab} &= (y \cdot e_{i'j'} \cdot x) \cdot f_{a'b'} \\ f_{kl} \cdot f_{ab} &= f_{k'l'} \cdot f_{a'b'} \\ f_{kb} &= f_{k'b'} \text{ contradiction.}\end{aligned}$$

Then $e_{ij} \cdot x \cdot f_{ab} \neq e_{i'j'} \cdot x \cdot f_{a'b'}$. And similarly, we can prove that $y_{aj} = f_{ab} \cdot y \cdot e_{ij} \neq f_{a'b'} \cdot y \cdot e_{i'j'} = y_{a'j'}$. Hence $|X| \geq mq$ and $|Y| \geq np$.

□

Remark 7.9. The multiplication of elements x_{ib} by $y_{a'j'}$ is equal to e_{ij} . And the multiplication of elements $y_{a'j'}$ by x_{ib} is equal to $f_{a'b}$.

Remark 7.10. We use Theorem 7.8 in Chapter 9 to prove why some monoids that contain rectangular bands are not connected in the case of matrix

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

Definition 7.11. Let C be a category associated to the matrix

$$\begin{pmatrix} M_1 & X \\ Y & M_2 \end{pmatrix}.$$

Let $x, x' \in X$, we say that an element x' duplicates x if $e \cdot x' = e \cdot x$ for all $e \in M_1$ and $x' \cdot f = x \cdot f$ for all $f \in M_2$.

Proposition 7.12. *Let C be a category of the form*

$$\begin{pmatrix} I \times J^{*1} & X \\ Y & A \times B^{*1} \end{pmatrix}$$

where $I \times J$ is a left rectangular band of size $m \times 1$ and $A \times B$ is a right rectangular band of size $1 \times q$. Then we obtain a sub-category C' of the form

$$\begin{pmatrix} I \times J^{*1} & I \times B \\ 1 & A \times B^{*1} \end{pmatrix}$$

such that for all $x \in X \setminus I \times B$ there exists a unique element $x' \in I \times B$ such that x duplicates x' .

Proof. Denote by $X_{I,B} \subset X$ the subset of x_{ib} and $Y_{A,J} \subset Y$ the subset of y_{aj} .

Let $X^* = X - X_{I,B}$ and $Y^* = Y - Y_{A,J}$.

Suppose $x \in X^*$. We claim that there is a unique element $\phi = \phi(x, a) \in I$, depending on x and $a \in A$, such that

$$x \cdot y_{aj} = e_{\phi,j}.$$

Indeed we note that changing j corresponds to multiplying on the right by some e_{ij} and this doesn't affect the first coordinate in $I \times J$, and one can see that the second coordinate of the answer must be j the same as the second coordinate of the y_{aj} .

Similarly, there is a unique element $\psi = \psi(x, j) \in B$ such that

$$y_{aj} \cdot x = f_{a,\psi}.$$

We obtain functions $\phi(x) : A \rightarrow I$ and $\psi(x) : J \rightarrow B$.

Furthermore, we claim that

$$x \cdot f_{ab} = x_{\phi(x,a),b}$$

and

$$e_{ij} \cdot x = x_{i,\psi(x,j)}.$$

Indeed,

$$\begin{aligned} x_{\phi(x,a),b} &= e_{\phi(x,a),0} x f_{0b} \\ &= x y_{a0} x f_{0b} \\ &= x \cdot f_{ab}. \end{aligned}$$

Similarly for $e_{ij} \cdot x = x_{i,\psi(x,j)}$.

This gives a “duplication profile” (ϕ, ψ) for the element $x \in X^*$.

Now because we have $I \times J$ and $A \times B$ are left and right bands, and J, A are single-element sets, all functions in question will be constant.

This means that the functions become

$$x \cdot y_{11} = e_{\phi,1} \quad y_{11} \cdot x = f_{1,\psi}$$

and

$$x \cdot f_{1b} = x_{\phi(x,1),b} = x_{c,b} \quad e_{i1} \cdot x = x_{i,\psi(x,1)} = x_{i,c'}.$$

And (ϕ, ψ) is unique. □

Remark 7.13. In the case where $I \times J$ and $A \times B$ are not left and right rectangular bands, the functions $\phi(x)$ and $\psi(x)$ are not necessarily constant. Then in this case, we don’t really know what’s happening.

Remark 7.14. Proposition 7.12 means that if we want to count the categories associated to matrix

$$\begin{pmatrix} I \times J^{*1} & X \\ Y & A \times B^{*1} \end{pmatrix}$$

where $I \times J$ is a left rectangular band of size $m \times 1$ and $A \times B$ is a right rectangular band of size $1 \times q$. It is sufficient to count the sub-categories of the form

$$\begin{pmatrix} I \times J^{*1} & I \times B \\ 1 & A \times B^{*1} \end{pmatrix}.$$

Proposition 7.15. *The multiplication blocks of the set Y are independent of the elements of Y .*

Proof. We need to check the multiplication of Y by the bands and by X .

1. $f_{ab} \cdot y \cdot e_{ij} \cdot e_{i'j'} \cdot x \cdot f_{1b} \cdot f_{kl} \cdot y' \cdot e_{k'l'} = f_{al} \cdot y' \cdot e_{k'l'} = f_{ab} \cdot y \cdot e_{ik'}$.
2. $y_{aj} \cdot x_{ib} = f_{ab'} \cdot y \cdot e_{i'j} \cdot e_{ij'} \cdot f_{a'b} = f_{ab'} \cdot f_{kl} \cdot f_{a'b} = f_{ab}$ (independent of f_{kl}).

□

CHAPTER 8

Simple semigroups in finite categories

The results in this chapter are published in [20].

8.1 Rees matrix semigroups

The following is a preliminaries about Rees matrix semigroups, the results presented in this section are discussed in J.-É. Pin's book [28].

8.1.1 Green's relations

Let S be a semigroup. We define on S four preorder relations $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$, $\leq_{\mathcal{J}}$ and $\leq_{\mathcal{H}}$ as follows

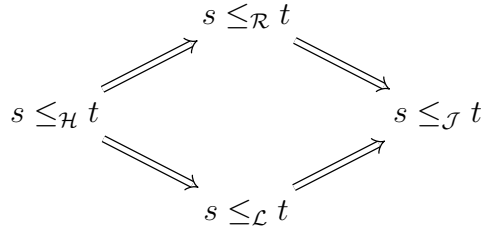
$$\begin{aligned} s \leq_{\mathcal{R}} t & \text{ if and only if } s = tu \text{ for some } u \in S^1 \\ s \leq_{\mathcal{L}} t & \text{ if and only if } s = ut \text{ for some } u \in S^1 \\ s \leq_{\mathcal{J}} t & \text{ if and only if } s = utv \text{ for some } u, v \in S^1 \\ s \leq_{\mathcal{H}} t & \text{ if and only if } s \leq_{\mathcal{R}} t \text{ and } s \leq_{\mathcal{L}} t \end{aligned}$$

These relations can be considered as a non-commutative generalization of the notion of multiple over the integers. For instance $s \leq_{\mathcal{R}} t$ if s is a right multiple of t , in the sense that one can pass from t to s by right multiplication by some element of S^1 . These definitions can be reformulated in terms of ideals as follows

$$\begin{aligned} s \leq_{\mathcal{R}} t & \iff sS^1 \subseteq tS^1 \\ s \leq_{\mathcal{L}} t & \iff S^1s \subseteq S^1t \\ s \leq_{\mathcal{J}} t & \iff S^1sS^1 \subseteq S^1tS^1 \\ s \leq_{\mathcal{H}} t & \iff s \leq_{\mathcal{R}} t \text{ and } s \leq_{\mathcal{L}} t \end{aligned}$$

Thus $s \leq_{\mathcal{J}} t$ [$s \leq_{\mathcal{R}} t$, $s \leq_{\mathcal{L}} t$] if the ideal [right ideal, left ideal] generated by s is contained in the ideal [right ideal, left ideal] generated by t . The following diagram summarizes the

connections between these four preorders.



The equivalences associated with these four preorder relations are denoted by \mathcal{R} , \mathcal{L} , \mathcal{J} and \mathcal{H} , respectively. Therefore

$$\begin{aligned}
 s\mathcal{R}t &\iff sS^1 = tS^1 \\
 s\mathcal{L}t &\iff S^1s = S^1t \\
 s\mathcal{J}t &\iff S^1sS^1 = S^1tS^1 \\
 s\mathcal{H}t &\iff s\mathcal{R}t \text{ and } s\mathcal{L}t
 \end{aligned}$$

We now introduce the fifth Green's relation \mathcal{D} , which is the preorder generated by \mathcal{R} and \mathcal{L} .

Proposition 8.1. *The relation \mathcal{D} is equal to $\mathcal{L} \circ \mathcal{R}$ and $\mathcal{R} \circ \mathcal{L}$.*

One can therefore give the following definition of \mathcal{D} :

$$\begin{aligned}
 s \mathcal{D} t &\iff \text{there exists } u \in S \text{ such that } s \mathcal{R} u \text{ and } u \mathcal{L} t \\
 &\iff \text{there exists } v \in S \text{ such that } s \mathcal{L} v \text{ and } v \mathcal{R} t.
 \end{aligned}$$

The equivalence classes of \mathcal{D} are called the \mathcal{D} -classes of \mathcal{D} , and the \mathcal{D} -class of an element s is denoted by $D(s)$.

8.1.2 Rees matrix semigroups

The Location Theorem indicates that the product of two elements s and t of the same \mathcal{D} -class D either falls out of D or belongs to the intersection of the \mathcal{R} -class of s and of the \mathcal{L} -class of t . In the latter case, the location of st in the egg-box picture is precisely known and the intersection of the \mathcal{R} -class of t and of the \mathcal{L} -class of s is a group. This suggests that the structure of a regular \mathcal{D} -class depends primarily on the coordinates of its elements in the egg-box picture and on its maximal groups. This motivates the following definition.

Let I and J be two nonempty sets, G be a group and $P = (p_{j,i})_{j \in J, i \in I}$ be a $J \times I$ -matrix with entries in G . The Rees matrix semigroup with G as structure group, P as sandwich matrix and I and J as indexing sets, is the semigroup $M(G, I, J, P)$ defined on the set $I \times G \times J$ by the operation

$$(i, g, j)(i', g', j') = (i, gp_{j,i'}g', j')$$

More generally, if $P = (p_{j,i})_{j \in J, i \in I}$ is a $J \times I$ -matrix with entries in G^0 , we let $M^0(G, I, J, P)$ denote the semigroup, called a *Rees matrix semigroup with zero*, defined on the set $(I \times G \times J) \cup \{0\}$ by the operation

$$(i, g, j)(i', g', j') = \begin{cases} (i, gp_{j,i'}g', j') & \text{if } p_{j,i'} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 8.2. *A Rees matrix semigroup with zero is regular if and only if every row and every column of its sandwich matrix has a nonzero entry.*

Theorem 8.3. (Rees-Sushkevich theorem)

1. *A finite semigroup is simple if and only if it is isomorphic to some Rees matrix semigroup.*
2. *A finite semigroup is 0-simple if and only if it is isomorphic to some regular Rees matrix semigroup with zero.*

Corollary 8.4. *Let S be a non empty finite aperiodic semigroup. The following conditions are equivalent:*

1. *S is simple,*
2. *S is idempotent and for all $e, f, s \in S, esf = ef,$*
3. *S is isomorphic to a rectangular band.*

*	*	*	*	*	*
*	*	*	*	*	*
*	*	*	*	*	*
*	*	*	*	*	*

Figure 8.1: An aperiodic simple semigroup: the rectangular band $B(4, 6)$.

Furthermore, if $I[J]$ is a singleton and then $M(I, J, G, P)$ is a right [left] zero semigroup. Conversely, any right [left] zero semigroup is isomorphic to such a Rees matrix semigroup.

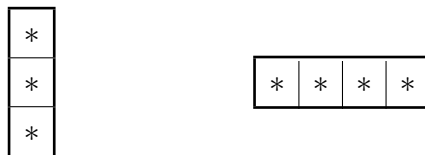


Figure 8.2: A left zero and a right zero semigroup.

8.2 Goals of this chapter

Definition 8.5. A semigroup S is *simple* if its only ideals are \emptyset and S . The Rees-Sushkevich structure theorem says that a simple semigroup is isomorphic to a *Rees matrix semigroup*. In the following, we proceed without using this result, but some of the lemmas and constructions can also be viewed using this structure.

Remark 8.6. In our work we exclude the case where the ideals are empty. This means that whenever we say an ideal, then it's not empty, unless needed.

There are two directions:

Theorem 8.7 (Monoids to categories, Theorem 8.29). *Let A be a finite monoid, and not a group, and let S be its minimal ideal. Then S is a simple semigroup, and we can construct a category \mathcal{C} with two objects associated to*

$$\begin{pmatrix} S^{*1} & L \\ R & G \end{pmatrix}; S^{*1} = I \cup \{1\} \quad (8.1)$$

where L and R are left and right minimal ideals of S respectively and $G = L \cap R$ is a group. And

$$|S| = \frac{|L| \cdot |R|}{|G|}.$$

Then there is an inclusion $\mathcal{C} \subseteq \mathcal{C}$ where \mathcal{C} is a category associated to the matrix

$$\begin{pmatrix} A & L \\ R & G \end{pmatrix}. \quad (8.2)$$

Remark 8.8. If A is a group, then $S = A$ and in this case the identity of S is the identity of A , and we obtain a groupoid [19]. In particular, we get $A \simeq G$ and $|L| = |R| = |G|$. Therefore, for this reason, we generally suppose in the following that A is not a group.

Theorem 8.9 (Categories to monoids). *Let \mathcal{C} be a category associated to*

$$\begin{pmatrix} A & U \\ V & G \end{pmatrix} \quad (8.3)$$

where G is a group and U, V not empty. Then $UV = S$ is the minimal ideal of A and the construction of the category associated to (8.3) is isomorphic to the construction of the category associated to (8.2). And if A is not a group then we get

$$|A| \geq \frac{|L| \cdot |R|}{|G|} + 1.$$

8.3 Categories with groups endomorphism monoids

In this section, we talk about categories with two objects where one of the endomorphism monoids is a group. The existence of a group on one of the diagonals of the matrix of a category imposes some properties over the sets of morphisms. These properties eventually characterize the nature of the other monoid.

Lemma 8.10. *Let C be a category associated to the algebraic matrix*

$$\begin{pmatrix} A & L \\ R & G \end{pmatrix}$$

such that L and R are not empty and G is a group. Then G acts freely on L and R . Therefore $|L|$ and $|R|$ are multiples of $|G|$.

Proof. Let $g_1, g_2 \in G$ and $x \in L$ such that

$$x \cdot g_1 = x \cdot g_2$$

then

$$y \cdot x \cdot g_1 = y \cdot x \cdot g_2 ; y \in R.$$

We have that

$$y \cdot x = z \in G$$

then

$$z^{-1} \cdot z \cdot g_1 = z^{-1} \cdot z \cdot g_2$$

then $g_1 = g_2$.

Same for any $y \in R$. □

Corollary 8.11. *Let C be a category associated to the algebraic matrix*

$$\begin{pmatrix} A & L \\ R & G \end{pmatrix}$$

such that L and R are not empty and G is a group. Then G acts freely on $L \times R$.

Proof. Let $g_1, g_2 \in G$ and let $(x, y) \in L \times R$ such that

$$\begin{aligned} g_1 \cdot (x, y) &= g_2 \cdot (x, y) \\ (x \cdot g_1, g_1 \cdot y) &= (x \cdot g_2, g_2 \cdot y) \end{aligned}$$

then

$$x \cdot g_1 = x \cdot g_2 \text{ and } g_1 \cdot y = g_2 \cdot y$$

and by the free action of G on L and R we get

$$g_1 = g_2.$$

□

Remark 8.12. The condition that L and R are not empty is important to obtain the results. Therefore, in the following we always assume that the bimodules L and R are not empty.

We recall the following from Chapter 5.

Let B be a monoid, and let L be a right B -module and let R be a left B -module. Then there exists a set

$$L \otimes_B R := L \times R / \sim_B.$$

\sim_B is the smallest equivalence relation such that

$$(xb, y) \sim_B (x, by) \quad \forall x \in L, b \in B, y \in R.$$

Then there's a map

$$L \times R \rightarrow L \otimes_B R := L \times R / \sim_B \rightarrow LR; (x, y) \mapsto x \otimes y \mapsto x \cdot y.$$

Lemma 8.13. *Let A, B, A' be monoids.*

1. *If L is an (A, B) -bimodule then $L \otimes_B R$ is a left A -module.*
2. *If R is a (B, A') -bimodule then $L \otimes_B R$ is a right A' -module.*
3. *If L, R are (A, B) and (B, A) bimodules, then $L \otimes_B R$ is an (A, A) -bimodule.*

Lemma 8.14. *If $B = G$ is a group, then G acts on $L \times R$ and*

$$(x, y) \sim (x', y') \iff \exists g \in G \text{ such that } x' = xg^{-1} \text{ and } y' = gy.$$

In this case

$$g \cdot (x, y) := (xg^{-1}, gy) \text{ and } L \otimes_G R = L \times R / G.$$

Remark 8.15. If $B = G$ is a group, we can transform a left action to a right action and vice-versa by multiplying with inverses.

Proposition 8.16. *Let C be a finite category associated to the matrix*

$$\begin{pmatrix} A & L \\ R & G \end{pmatrix}.$$

Consider the quotient $L \times R/G$ with the equivalence relation

$$(x, y) \sim (x', y') \iff \exists g \in G \text{ such that } x' = xg^{-1} \text{ and } y' = gy.$$

Then the function

$$f : L \times R/G \rightarrow LR ; f(x, y) = xy$$

is bijective. In this case $|LR| = |L \times R/G| = \frac{|L||R|}{|G|}$.

Proof. 1. **f is injective:** Let $(x, y), (x', y') \in L \times R/G$ such that $f(x, y) = f(x', y')$ then $xy = x'y'$. There exist $c \in L$ and $d \in R$ such that $yc = 1_G$ and $dx = 1_G$. This gives us

$$x = x \cdot (yc) = x'(y'c)$$

and

$$y = (dx) \cdot y = (dx')y'$$

where $y'c$ and dx' are in G and they are inverses, indeed

$$(dx')(y'c) = d(x'y')c = d(xy)c = (dx)(yc) = 1_G.$$

Then $(x, y) \sim (x', y')$.

2. **f is surjective:** Evident.

3. Since G acts freely on L and on R (Lemma 8.10) then it also acts freely on $L \times R$ (Corollary 8.11). Then $|L \times R/G| = \frac{|L||R|}{|G|}$.

□

Conclusion 8.17. Let C be a category associated to the matrix

$$\begin{pmatrix} A & L \\ R & G \end{pmatrix}$$

where G is a group. Then

1. G acts freely on L and R .
2. $|L|$ and $|R|$ are multiples of $|G|$.
3. If A is not a group then $|A| \geq \frac{|L||R|}{|G|} + 1$.

8.4 From simple semigroups to categories

We aim in this section to construct a finite category with two objects where one of the endomorphism monoids is a group. We start with a simple semigroup S , to which we add an identity element, then we take the minimal left and right ideals of S , denote them by L and R respectively. The intersection of L and R is a group, which will be the second endomorphism monoid of the category.

Definition 8.18. Given a monoid (or semigroup) S , a *left ideal* in S is a subset A of S such that SA is contained in A . Similarly, a *right ideal* is a subset A such that AS is contained in A . Finally, a *two-sided ideal*, or simply *ideal*, in S is a subset A that is both a left and a right ideal.

Proposition 8.19. *Let S be a simple semigroup, L be a minimal left ideal and R be a minimal right ideal of S . Then $LR = S$.*

Proof. We have $L \subset S$ and $R \subset S$ then $LR \subset S$. In addition, LR is a two sided ideal of S . Indeed, let $x \in S$ and $ab \in LR$ such that $a \in L$ and $b \in R$, then

$$x(ab) = \underbrace{(xa)}_{\in L} b \in LR$$

and

$$(ab)x = a \underbrace{(bx)}_{\in R} \in LR.$$

Hence $LR \subset S$ is a two sided ideal, but S is simple, then $LR = S$. \square

Theorem 8.20. *Let S be a simple semigroup, L be a minimal left ideal and R be a minimal right ideal of S . Then:*

1. $G = L \cap R$ is a group.
2. $RL = L \cap R$.

Proof. For part (1), it is sufficient to prove that G is right and left cancellative.

Let $z \in L \cap R$ and zR be the right ideal contained in R , $zR = \{zr_1, \dots, zr_n\}$, let r_i, r_j such that $zr_i = zr_j$, if $r_i \neq r_j$ then $zR \subsetneq R$. This is a contradiction with the minimality of R .

Let Lz be the left ideal contained in L , $Lz = \{l_1z, \dots, l_mz\}$, let l_i, l_j such that $l_i z = l_j z$, if $l_i \neq l_j$ then $Lz \subsetneq L$. This is a contradiction with the minimality of L .

Hence, G is right and left cancellative. Proving now the existing of an identity element.

Since G is right cancellative, then the map $x \mapsto x \cdot z$ is injective, as G is finite then it is also surjective. Then for all w , there exists $e; ez = w$. Take $w = z$.

Also, G is left cancellative, then the map $x \mapsto z \cdot x$ is bijective, and for all y there exists e' such that $ze' = y$. Take $y = z$.

Then:

$$- ey = e(zx) = (ez)x = zx = y.$$

$$- we' = (ez)e' = ez = w.$$

Hence the existence of an identity element.

For part (2), we always have $RL \subseteq L \cap R$.

Let $x \in L \cap R = G$, $x = e \cdot x \in RL$ where e is the identity of G . □

Theorem 8.21. *Let S be a simple semigroup. Let L and R be minimal left and right ideals of S respectively. Then we can construct a category \mathcal{C} associated to the matrix*

$$\begin{pmatrix} S^{*1} & L \\ R & G \end{pmatrix}; S^{*1} = S \cup \{1\}$$

such that $G = RL$ is a group and $LR = S$ where

- $|L|$ and $|R|$ are multiples of $|G|$.
- $|S| = |LR| = \frac{|L||R|}{|G|}$.

Proof. We want to construct a category with two objects starting with a semigroup. The construction we use is called *Karoubi envelope* or *idempotent completion*. It was first introduced in 1963 by M. Artin, A. Grothendieck and J.L. Verdier [6], then the definition also appears in [11] and [5]. The idea is that if \mathcal{C} is a category, then its idempotent envelope \mathcal{C}^{idem} (also denoted $\tilde{\mathcal{C}}$) is the category whose objects are pairs (X, p) where $X \in Ob(\mathcal{C})$ and $p : X \rightarrow X$ is an idempotent. The set of morphisms from X to X' is

$$\mathcal{C}^{idem}((X, p), (X', p')) = p \cdot \mathcal{C}(X, X') \cdot p' \subseteq \mathcal{C}(X, X').$$

Then in the case of a monoid A , we view it as a category with one object $\{*\}$ and A is the set of endomorphisms of $\{*\}$, we denote the category by $(\{*\}, A)$. The Karoubi envelope will be the category

$$Idem(A) := (\{*\}, A)^{idem}$$

where the objects of $Idem(A)$ are the idempotents of A .

If we apply this construction to our work here, we get the following: if S is a simple

semigroup, L minimal left ideal of S , R minimal right ideal of S and G the group obtained by $L \cap R$. Then we take the two idempotents $e_1 = 1_{S^{*1}}$ and $e_2 = 1_G$, and we want to construct a category $Idem_{e_1, e_2}(S^{*1})$, the sets of morphisms are the following

$$\begin{aligned} S^{*1} &= e_1 \cdot S^{*1} \cdot e_1 & L &= e_1 \cdot S^{*1} \cdot e_2 \\ R &= e_2 \cdot S^{*1} \cdot e_1 & G &= e_2 \cdot S^{*1} \cdot e_2. \end{aligned}$$

Note that L and R here are the same of Theorem 8.20, and they are left and right minimal ideals of S respectively.

Then $Idem_{e_1, e_2}(S^{*1})$ is a full subcategory of $Idem(S^{*1}) = \widetilde{S^{*1}}$ the Karoubi envelope of S^{*1} . The multiplication of morphisms is defined by the multiplication of elements of S . \square

Conclusion 8.22. In this section, we started with a simple semigroup S and we proved that we can construct a category with two objects such that one of the objects is S and the other one is a group.

8.5 From finite monoids to categories

In this section, we study how we can construct a category with two objects such that one of the objects is a group by starting with a finite monoid instead of a simple semigroup.

Proposition 8.23. *Let A be a finite monoid, then A has a unique non-empty minimal ideal denoted by S_0 .*

Proof. Let $S = \{S_i \subset A \mid S_i \text{ ideal of } A\} = \{S_1, S_2, \dots, S_k\}$ be the set of all ideals of M , set $S_0 = \bigcap_{S_i \in S} S_i = S_1 \cap S_2 \cap \dots \cap S_k$. S_0 is not empty because it contains at least $S_1 \cdot S_2 \cdot \dots \cdot S_k$ and it is minimal because it doesn't contain any other ideal. S_0 is unique. Indeed, suppose that S' is another minimal ideal, then $S_0 \subseteq S'$ because S_0 is the intersection of all ideals, but S' is minimal then $S_0 = S'$. \square

Remark 8.24. Let A be a finite monoid and S_0 be the minimal ideal of A , then S_0 is a sub-semigroup of A .

Lemma 8.25. *Let A be a finite monoid. If $M \subseteq A$ is a left ideal, then M is a sub-monoid of A and if $L \subseteq M$ is a minimal left ideal of M then L is a minimal left ideal of A .*

Proof. We have $M \cdot L \subseteq L$ and $M \cdot L \subseteq M$ is a left ideal of M contained in L , by the minimal of L in M we have $M \cdot L = L$, i.e. for all $l \in L$ there exists $m \in M, l' \in L$ such that $l = ml'$.

Now let $a \in A$ and $l \in L$ then

$$\begin{aligned}
 a \cdot l &= a \cdot (ml') \\
 &= (am) \cdot l' \\
 &\in M \cdot L \text{ (because } M \text{ is left ideal of } A) \\
 &\in L \text{ (because } L \text{ is left ideal of } M).
 \end{aligned}$$

Therefore, L is a left ideal of A .

M is minimal in A . Indeed, suppose that L' is a left ideal of A such that $L' \subseteq L$, then

$$M \cdot L' \subseteq L' \subseteq L$$

is a left ideal of M , but L is minimal in M , then $M \cdot L' = L$. Therefore, $L' = L$. \square

Lemma 8.26. *Let A be a finite monoid. If $M \subseteq A$ is a right ideal, then M is a submonoid of A and if $R \subseteq M$ is a minimal right ideal of M then R is a minimal right ideal of A .*

Proof. Similar to the proof of Lemma 8.25. \square

Lemma 8.27. *Let A be a finite monoid and let S_0 be its minimal ideal. Then L (resp. R) is a minimal left (resp. minimal right) ideal of A if and only if L (resp. R) is a minimal left (resp. minimal right) ideal of S_0 .*

Proof. We have $S_0 \cdot L \subseteq S_0$ and $S_0 \cdot L \subseteq L$ is a left ideal of A , by the minimality of L in A we obtain that $S_0 \cdot L = L$. Then $L \subseteq S_0$. Also, L is a minimal ideal of S_0 . Indeed, if $L' \subseteq L$ is a left ideal of S_0 then $S_0 \cdot L' \subseteq L' \subseteq L$, but $S_0 \cdot L'$ is a left ideal of A and L is minimal of A then $S_0 \cdot L' = L$ hence $L' = L$ and L is a minimal left ideal of S_0 .

For the other direction, use Lemma 8.25 and Lemma 8.26. \square

Lemma 8.28. *Let A be a finite monoid and S_0 the minimal ideal of A , then S_0 is a simple semigroup. In particular, S_0 has the structure of a Rees matrix semigroup.*

Proof. If $J \subseteq S_0$ is an ideal of S_0 then $S_0 \cdot J \cdot S_0 \subseteq J \subseteq S_0$ and $S_0 \cdot J \cdot S_0$ is an ideal of A , then by the minimality of S_0 in A , we obtain $S_0 \cdot J \cdot S_0 = S_0$ hence $J = S_0$ and S_0 is simple. \square

Theorem 8.29. *Let A be a finite monoid that's not a group, and let S_0 be its minimal ideal. Then by Section 8.4, we obtain a category \mathcal{C}' associated to the matrix*

$$\begin{pmatrix} S_0^{*1} & L \\ R & G \end{pmatrix}$$

where L and R are minimal left and right ideals of S_0 respectively, and G is a group. Then there exists a category \mathcal{C} associated to the matrix

$$\begin{pmatrix} A & L \\ R & G \end{pmatrix}$$

such that $\mathcal{C}' \subseteq \mathcal{C}$.

Proof. We use the same construction of Theorem 8.21. Then if we apply this construction to our work here, we get the following: if A is a monoid, L minimal left ideal of S , R minimal right ideal of A and G the group obtained by $L \cap R$. Then we take the two idempotents $e_1 = 1_A$ and $e_2 = 1_G$, and we want to construct a category $Idem_{e_1, e_2}(A)$, the sets of morphisms are the following

$$\begin{aligned} A &= e_1 \cdot A \cdot e_1 & L &= e_1 \cdot A \cdot e_2 \\ R &= e_2 \cdot A \cdot e_1 & G &= e_2 \cdot A \cdot e_2. \end{aligned}$$

L is a left minimal ideal of A and R is a minimal right ideal of A .

Then $Idem_{e_1, e_2}(A)$ is a full subcategory of $Idem(A) = \tilde{A}$ the Karoubi envelope of A . The multiplication of morphisms is defined by the multiplication of elements of A .

Now since A is not a group, then $1_A \notin S_0$ (Remark 8.8), hence

$$S_0^{*1} \rightarrow A$$

is injective.

L and R are minimal left and right ideals of S_0 respectively. By Lemma 8.27, L and R are also minimal left and right ideals of A respectively, i.e. for all $x \in L, y \in R$ and $a \in A$, we have $a \cdot x \in L$ and $y \cdot a \in R$. By Section 8.4 we have $LR = S_0$. Therefore $LR = S_0 \subset A$. In addition, for all $x \in L$ and $y \in R$, $x \cdot y \neq 1_A$, because G is a group and in this case we get isomorphic objects, which gives us that A is a group and this is a contradiction with the hypothesis. \square

Conclusion 8.30. For any finite monoid A , we can construct a category with two objects such that one of the objects is a group. This result means that every finite monoid A is

connected to a group, which is the intersection of a minimal left and a minimal right ideal of A . Therefore, Theorem 8.7 is proved.

8.6 From categories to simple semigroups

In this section, we give a proof of the direction that starting with a category \mathbf{C} with two objects such that one of the objects is a group, then we can obtain a simple semigroup with cardinality as shown in the previous sections.

Theorem 8.31. *Let \mathbf{C} be a finite category associated to the matrix*

$$\begin{pmatrix} A & U \\ V & G \end{pmatrix}$$

where G is a group and A is not a group, then $S = UV \subseteq A$ is a simple ideal such that $|S| = \frac{|U||V|}{|G|}$ and there exists a sub-category \mathbf{C}' of \mathbf{C} associated to the matrix

$$\begin{pmatrix} S^{*1} & U \\ V & G \end{pmatrix} \tag{8.4}$$

where $S^{*1} = S \cup \{1\}$ and S is a simple semigroup.

Proof. \mathbf{C}' is a sub-category of \mathbf{C} :

- $Ob(\mathbf{C}') = Ob(\mathbf{C})$.
- Morphisms = $\{S^{*1}, U, V, G\}$.

Suppose $S' \subset S$ is a two sided ideal, $S \subset S'$?

Let $x \in U, y \in V$ and $a \in S'$. We have $y \cdot a \cdot x \in G$, then there exists $g \in G$ such that

$$g \cdot (y \cdot a \cdot x) = 1_G$$

then

$$\begin{aligned} x \cdot y &= x \cdot 1_G \cdot y \\ &= x \cdot (g \cdot y \cdot a \cdot x) \cdot y \\ &= \underbrace{(x \cdot g \cdot y)}_{\in S} \cdot \underbrace{a}_{\in S'} \cdot \underbrace{(x \cdot y)}_{\in S} \in S'. \end{aligned}$$

□

Definition 8.32. Let \mathbf{C} be a finite category associated to the matrix

$$\begin{pmatrix} A & U \\ V & G \end{pmatrix}$$

where G is a group. Let $x \in U$ and $y \in V$, define L_y and R_x in the following way

$$L_y := U \cdot y \subseteq A$$

and

$$R_x := x \cdot V \subseteq A.$$

Lemma 8.33. *L_y is a minimal left ideal of A and R_x is a minimal right ideal of A .*

Proof. It is clear that L_y is a left ideal of A . Proving it's minimal. Suppose $L' \subseteq L_y$ a left ideal of A . Let $u \in U, a \in L'$, we have $y \cdot a \cdot u \in G$ then there exists $g \in G$ such that

$$g \cdot (y \cdot a \cdot u) = 1_G$$

then

$$\begin{aligned} u \cdot y &= u \cdot 1_G \cdot y \\ &= u \cdot (g \cdot y \cdot a \cdot u) \cdot y \\ &= \underbrace{(u \cdot g \cdot y)}_{\in L_y} \cdot \underbrace{a}_{\in L'} \cdot \underbrace{(u \cdot y)}_{\in L_y} \in L'. \end{aligned}$$

Hence L_y is minimal. Similarly for R_x . □

Remark 8.34. For all $g \in G$, if we replace y by gy we obtain

$$L_{gy} = U \cdot g \cdot y = U \cdot y = L_y \text{ (because } U \cdot g = U \text{)}.$$

Proposition 8.35. *Let \mathcal{C} be a category associated to the matrix*

$$\begin{pmatrix} A & U \\ V & G \end{pmatrix}.$$

Let $\mathcal{L}(A) = \{L \subseteq A \mid L \text{ is a minimal left ideal of } A\}$ and $\mathcal{R}(A) = \{R \subseteq A \mid R \text{ is a minimal right ideal of } A\}$.

Define the sets

$$\{y\} / y \sim gy := G \setminus^V \quad \text{and} \quad \{x\} / x \sim xg := U /_G.$$

Then

$$G \setminus^V = \mathcal{L}(A) \quad \text{and} \quad U /_G = \mathcal{R}(A).$$

This means that for every minimal left ideal L of A , there exists $y \in V$ such that $L = L_y$. Similarly, for every minimal right ideal R of A , there exists $x \in U$ such that $R = R_x$.

Lemma 8.36. *Let C be a finite category associated to the matrix*

$$\begin{pmatrix} A & U \\ V & G \end{pmatrix}$$

where G is a group and $S = UV$ a simple ideal of A . For $x \in U$ and $y \in V$ such that $y \cdot x = 1_G$, we have

$$L_y R_x = U \cdot y \cdot x \cdot V = UV = S.$$

Proposition 8.37. *Let A be a finite monoid that is not a group, let L be a minimal left ideal and R be a minimal right ideal of A , let $x, y \in A$, then*

$$(xR) \cap (Ly) = xRLy.$$

Proof. Theorem 8.20. □

Notation 8.38. $(xR) \cap (Ly) = xRLy = G_{xy}$.

Theorem 8.39. *Let C be a finite category associated to the matrix*

$$\begin{pmatrix} A & U \\ V & G \end{pmatrix}$$

where G is a group. Let $x \in U, y \in V$ such that $y \cdot x = 1_G$, then by the construction described in Theorem 8.29 there exists a category C associated to the matrix

$$\begin{pmatrix} A & L_y \\ R_x & G_{xy} \end{pmatrix}$$

such that $L_y R_x = S$ the simple ideal of A where L_y is a minimal left ideal of A and R_x is a minimal right ideal of A and $G_{xy} = x \cdot G \cdot y$ is a group. Then $C \simeq C$.

Proof. Consider the maps

$$\phi : G \rightarrow G_{xy} ; g \mapsto xgy$$

$$\psi : U \rightarrow L_y ; u \mapsto uy$$

and

$$\psi' : V \rightarrow R_x ; v \mapsto xv.$$

ϕ, ψ and ψ' provide the isomorphisms needed to prove the theorem. □

Proposition 8.40. *The category obtained is unique.*

Proof. Let A be a monoid, choose L, R such that $G = RL$. Choose another L', R' such that $G' = R'L'$.

Since we have $G = RL$ then we can choose $L_y, R_x \subseteq A$ such that

$$\mathbb{C}(A, L, R, G) \simeq \mathbb{C}(A, L_y, R_x, G_{xy}).$$

Take $L_y = L'$ and $R_x = R'$, then

$$\mathbb{C}(A, L', R', G) = \mathbb{C}(A, L_y, R_x, G_{xy}).$$

Hence $\mathbb{C}(A, L, R, G) \simeq \mathbb{C}(A, L', R', G')$. □

Conclusion 8.41. Finally in this section, we prove that if we have a category \mathbb{C} associated to the matrix

$$\begin{pmatrix} A & U \\ V & G \end{pmatrix}$$

then under a choice of an element $x \in U$ and an element $y \in V$, we obtain a left minimal ideal L_y and a right minimal ideal R_x of A such that $L_y R_x$ is a simple semigroup and $R_x L_y$ is a group. This data gives us a unique category \mathbb{C}' associated to the matrix

$$\begin{pmatrix} A & L_y \\ R_x & G_{xy} \end{pmatrix}$$

such that $\mathbb{C}' \simeq \mathbb{C}$ and $|L_y R_x| = \frac{|L_y||R_x|}{|G_{xy}|}$. If A is not a group, then we get

$$|A| \geq \frac{|U| \cdot |V|}{|G|} + 1.$$

Therefore, Theorem 8.9 is proved.

8.7 Connectivity of monoids

We recall the following definition and lemma introduced in Chapter 5.

Definition 5.21. We say that two monoids A and B are *connected* if there exists a category with two objects such that its endomorphism monoids are A and B and all morphism sets are nonempty.

Lemma 5.22. If A, B are connected, and B, C are connected then A, C are connected. From the above sections, we can define the notion "a monoid has a group".

Definition 8.42. We say that a monoid A has a group G if its minimal ideal is connected to G .

Remark 8.43. Let A be a finite monoid, by the discussion in the previous sections, there is always a group connected to A through the minimal ideal S of A . The group is of the form $x \cdot G \cdot y$ where $x, y \in S$ and $G = RL$ where L and R are minimal left and right ideals of S (eventually A) respectively. In addition, S is a simple semigroup, then by Rees-Sushkevich theorem, it is a Rees matrix semigroup, and the group G is the group that is involved in the definition of the Rees matrix semigroup.

Theorem 8.44. *Two monoids are connected if and only if they have the same group.*

Proof. For the first direction, let A be a monoid that has a group G , then the minimal ideal S_0 of A is a Rees matrix semigroup and connected to G .

Similarly, Let B be a monoid that has a group H , then the minimal ideal T_0 of B is a Rees matrix semigroup and connected to H .

If A is connected to B , then by transitivity (Lemma 5.22), G and H are connected, then G and H are isomorphic (Remark 8.8).

For the second direction, let A and B be two monoids that have the same group G , then A is connected to G and B is connected to G , by transitivity (Lemma 5.22), A and B are connected. □

CHAPTER 9

Matrices of order 2 and coefficients 3

9.1 Using MACE4 for counting

In this chapter, we explain how we use MACE4 in our work in order to obtain the count of categories with two objects and 3 morphisms between each two objects.

We have seen in Chapter 4 that we can represent categories as semigroups with a zero element, where the zero element stands for the compositions that are not defined. We use the PYTHON code described in Appendix 1 to generate semigroups equations that we feed to MACE4. The input of the PYTHON code is a matrix that could be of any size and have any coefficients. This method saves a lot of time because sometimes we could have hundreds of equations that we need to write. Then MACE4 generates models of categories with a given size, which is the number of morphisms in total including the zero element.

Example 9.1. Let C be a category with two objects X, Y associated to the matrix

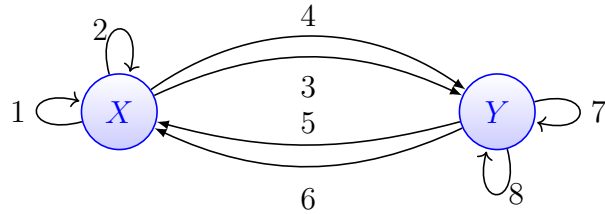
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

We recall that the monoids of order 2 with element $\{1, 2\}$ are

$$\begin{array}{c|cc} \cdot & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 1 \end{array} \quad \text{or} \quad \begin{array}{c|cc} \cdot & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 2 \end{array}$$

The first case is the group \mathbb{Z}_2 , which we exclude because of the reduced property and Lemma 7.5.

We associate numbers to the morphisms of the category, and 0 is for compositions that are not defined. The graph of C is the following



The table of multiplication of such a category will be as follows

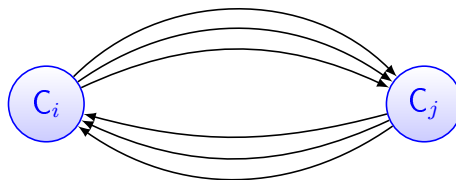
	1	2	3	4	5	6	7	8
1	1	2	3	4	0	0	0	0
2	2	2	–	–	0	0	0	0
3	0	0	0	0	2	2	3	–
4	0	0	0	0	2	2	4	–
5	5	–	8	8	0	0	0	0
6	6	–	8	8	0	0	0	0
7	0	0	0	0	5	6	7	8
8	0	0	0	0	–	–	8	8

We know for sure that the multiplications of $\{3, 4\}$ by $\{5, 6\}$ and vice versa are 2 and 8 respectively because they can't be identities. MACE4 fills all the possibilities left in the table and generates all the models of categories up to an isomorphism.

9.2 Matrix with coefficients 3 and size 2

Recall the 7 monoids of order 3, the elements are $\{1, 2, 3\}$:

<i>comp</i>	C_1	C_2	C_3	C_4	C_5	C_6	C_7
2^2	= 1	2	2	2	2	2	3
3^2	= 3	2	3	3	2	3	2
$2 \cdot 3$	= 3	2	2	3	3	2	1
$3 \cdot 2$	= 3	2	3	2	3	2	1



We combine monoids of 3 elements together in one category, two monoids C_i, C_j are called

connected if there exists a category where the monoids C_i and C_j are the endomorphism monoids of the two objects. Viewing them as objects, each object is one of the monoid of endomorphisms listed before, and this graph of a category is associated to the matrix:

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

The case of C_7 is eliminated as it is a group, and under the assumption of being reduced, groups can't show up as objects in such categories (Lemma 3.6). By computing using MACE4 we obtain the following:

	C_1	C_2	C_3	C_4	C_5	C_6
C_1	10	21	2	2	0	36
C_2	—	62	4	4	0	76
C_3	—	—	2	0	0	8
C_4	—	—	—	2	0	8
C_5	—	—	—	—	2	0
C_6	—	—	—	—	—	123

The entry in the row i and column j indicates the number of categories that can be constructed when C_i and C_j are paired together as objects. The blank cases is to say that the categories obtained from the pair (C_i, C_j) are isomorphic to the categories obtained from the pair (C_j, C_i) , as here we don't care about the order of objects.

We remark that certain kinds of monoids can affect the classification, as we notice that where we have zero elements, as in C_1, C_2 and C_6 , we obtain a high number of categories. Especially in the case of C_6 where it is a *semilattice* (it can be seen also as a grouplike monoid with the trivial group $\{2\}$ union two identities 1 and 3), we notice that more than half of the categories are obtained when C_6 is an object. But whenever we have rectangular bands such as C_3 and C_4 , we obtain less categories.

The most interesting case is C_5 , as when we eliminate (or simply ignore) the identity element, it becomes a group, it is a grouplike with the group \mathbb{Z}_2 union an identity $\{1\}$. And we notice that it is only connected to itself. The properties of this structure obtained in Chapter 6 will help us prove the number of categories between C_5 and C_5 and the reason why it doesn't have a category with other monoids.

Theorem 9.2. $Card(M_3^2) = 362$.

As we noticed from the table above, the algebraic nature of objects affect dramatically the classification problem. This data was a big inspiration that allowed us to make lots of

conjectures and obtain mathematical results. Almost everything that is presented in the thesis is inspired by this table which lead us to prove things about groups inside categories in particular.

9.3 Proof of some entries in the table

We fix the following labels for the tables:

Green: multiplied by identities

Black: known

Red: choices

Orange: consequences.

9.3.1 The number of categories between C_5 and itself

C_5 is a very specific case among these categories, as it is not connected to any other monoid except itself. Since C_5 contains the group \mathbb{Z}_2 , it is logical to study the group action on the sets of morphisms. Clearly that the whole monoid C_5 acts on the sets of morphisms, denote them by L and R , but \mathbb{Z}_2 acts on the subsets $\mathbb{Z}_2 \cdot L$, $L \cdot \mathbb{Z}_2$, $R \cdot \mathbb{Z}_2$ and $\mathbb{Z}_2 \cdot R$ (there is a two-sided multiplication because the group is the same on both sides). From Chapter 6 we conclude that the action of \mathbb{Z}_2 is free and that the orbit of each element in L and R has cardinal $2 = \mathcal{O}(\mathbb{Z}_2)$. Also we proved that the orbit of each element is the same when multiplying by \mathbb{Z}_2 on the left or on the right, which leads to say that all the elements have the same orbit of cardinal 2. What is left to talk about is the multiplication of the orbit elements together, it is exactly the group \mathbb{Z}_2 , i.e. $(\mathbb{Z}_2 \cdot L) \cdot (R \cdot \mathbb{Z}_2) = \mathbb{Z}_2$. Indeed,

- \subseteq : evident.
- \supseteq : Let $g \in \mathbb{Z}_2$

$$g = g \cdot 1_{\mathbb{Z}_2} \cdot 1_{\mathbb{Z}_2} = g \cdot x \cdot y \cdot 1_{\mathbb{Z}_2}$$

where $x \in L$ and $y \in R$.

We are trying to construct a subcategory taking the group part in the monoids as new objects. In this case, we fill in as many numbers as we can in the multiplication table of $C_5 \times C_5$.

Remark 9.3. In the following we use $*$ instead of \cdot for multiplication with the same direction of composition as \cdot ($*$ is the symbol used in MACE4).

Now let's fix

$$\begin{aligned} C_5 &= \{1, 2, 3\} & L &= \{4, 5, 6\} \\ R &= \{7, 8, 9\} & C_5 &= \{10, 11, 12\}. \end{aligned}$$

Where 1 and 10 are the category identities. Let's discuss first the multiplication block of the elements of C_5 by L .

	4	5	6
1	4	5	6
2			
3			

We said that the orbit of each element in L has 2 elements and they are all the same for all the elements. Let's say $Orbit(L) = \{4, 5\}$ (choosing other elements is isomorphic), and since these are the elements of the orbit, then $2 * 4 = 4$ and $2 * 5 = 5$ (because $2 = 1_{\mathbb{Z}_2}$). We conclude that $3 * 4 = 4$ and $4 * 5 = 4$ because otherwise if $3 * x = 2 * x$, then $3 * (x * y) = 2 * (x * y)$ such that $x * y = 1_{\mathbb{Z}_2}$ which gives us $3 = 2$, contradiction! So far, we have the following block:

	4	5	6
1	4	5	6
2	4	5	4
3	5	4	5

Now since $6 \notin Orbit(L)$ then $2 * 6 \neq 6$ and $2 * 6 \in \{4, 5\}$. Here, we are obliged to make a choice and we suppose that $2 * 6 = 4$. The consequences of this choice are:

$$\begin{aligned} & & & & 3 * 6 = 5 \\ 6 * 7 = 2 & & 6 * 8 = 3 & & 6 * 9 = 2 \\ & 6 * 11 = 4 & & 6 * 12 = 5 & \\ 7 * 6 = 11 & & 8 * 6 = 12 & & 9 * 6 = 11. \end{aligned}$$

Now we discuss the multiplication block of R and C_5 :

	1	2	3
7	7	7	8
8	8	8	7
9	9	7	8

For the same reasons, we choose $Orbit(R) = \{7, 8\}$ and we suppose that $9 * 2 = 7$, the consequences are:

$$\begin{array}{ccc}
 & 9 * 3 = 8 & \\
 9 * 4 = 11 & 9 * 5 = 12 & 9 * 6 = 11 \\
 & 11 * 9 = 7 & 12 * 9 = 8 \\
 4 * 9 = 2 & 5 * 9 = 3 & 6 * 9 = 2.
 \end{array}$$

From the choices we made above, we get the following model of a category:

0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	0	0	0	0	0	0
2	2	2	3	4	5	4	0	0	0	0	0	0
3	3	3	2	5	4	5	0	0	0	0	0	0
4	0	0	0	0	0	0	2	3	2	4	4	5
5	0	0	0	0	0	0	3	2	3	5	5	4
6	0	0	0	0	0	0	2	3	2	6	4	5
7	7	7	8	11	12	11	0	0	0	0	0	0
8	8	8	7	12	11	12	0	0	0	0	0	0
9	9	7	8	11	12	11	0	0	0	0	0	0
10	0	0	0	0	0	0	7	8	9	10	11	12
11	0	0	0	0	0	0	7	8	7	11	11	12
12	0	0	0	0	0	0	8	7	8	12	12	11

Since the only choices we made are $2 * 4$ and $9 * 2$, then we can permute them in the following ways:

$$2 * 4 = 4 \quad \text{and} \quad 9 * 2 = 7 \quad (A)$$

$$2 * 4 = 4 \quad \text{and} \quad 9 * 2 = 8 \quad (B)$$

$$2 * 4 = 5 \quad \text{and} \quad 9 * 2 = 7 \quad (C)$$

$$2 * 4 = 5 \quad \text{and} \quad 9 * 2 = 8 \quad (D)$$

It is clear that (C) is isomorphic to (B) and (D) is isomorphic to (A) , and in this case we only have 2 models of categories based on either the choice (A) or the choice (B) . Which proves the number of categories obtained by C_5 with itself is equal to 2.

Remark 9.4. The number of categories between C_5 and itself can also be found in another method using bimodules. Let B be a category with two objects X and Y associated to the matrix

$$M_1 = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$$

such that $C(X, X) = C_5$, $C(Y, Y) = C_5$, $C(X, Y) = L$ and $C(Y, X) = \emptyset$.

And similarly let \mathbf{D} be a category with two objects X and Y associated to the matrix

$$M_2 = \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix}$$

such that $\mathbf{C}(X, X) = \mathbf{C}_5$, $\mathbf{C}(Y, Y) = \mathbf{C}_5$, $\mathbf{C}(X, Y) = \emptyset$ and $\mathbf{C}(Y, X) = R$.

Then there exists a category \mathbf{C} with two objects associated to the matrix

$$M = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

such that $\mathbf{C}(X, X) = \mathbf{C}_5$, $\mathbf{C}(Y, Y) = \mathbf{C}_5$, $\mathbf{C}(X, Y) = L$ and $\mathbf{C}(Y, X) = R$. Where \mathbf{B} is a left bimodule and \mathbf{D} is a right bimodule.

Since there is 1 left and 1 right bimodule and since $i^{max} = 0$, then the number of possible x_0 is equal to the order of the group \mathbb{Z}_2 which is 2 (Theorem 6.36). Hence there are 2 categories between \mathbf{C}_5 and itself.

9.3.2 The number of categories between \mathbf{C}_1 and \mathbf{C}_3

Fix

$$\begin{aligned} \mathbf{C}_1 &= \{1, 2, 3\} & L &= \{4, 5, 6\} \\ R &= \{7, 8, 9\} & \mathbf{C}_3 &= \{10, 11, 12\}. \end{aligned}$$

Since 2 is an invertible element, we can easily conclude, because of the reduced property and Lemma 3.6, that $x * y = 3$ for all $x \in L$ and $y \in R$, and that:

$$2 * x \neq 2 * y \text{ for all } x, y \in L \tag{9.1}$$

and

$$x' * 2 \neq y' * 2 \text{ for all } x', y' \in R. \tag{9.2}$$

Indeed, if $2 * x = 2 * y \implies 2^2 * x = 2^2 * y \implies x = y$ contradiction.

We have:

$$\text{if } 3 * x = y \text{ then } 2 * y = y \text{ and } 3 * y = y$$

and

$$3 * x = 3 * y \text{ for all } x, y \in L. \tag{9.3}$$

The proof of 9.3 is not difficult but very technical, done by PROVER9 (Appendix 2).

Now let's try to fill in the following block:

	4	5	6
1	4	5	6
2	4	5	6
3	4	4	4

We supposed that $3 * 4 = 4$ and $2 * 5 = 5$. Everything else in the table is a consequence of our suppositions. We notice that the choice of $3 * 4 = 4$ is equivalent to when it's equal 5 or 6, but if we change our supposition for $2 * 5$, the category changes.

From this block, we can prove that $11 * x' = 11 * y'$ and $12 * x' = 12 * y'$ for all $x', y' \in R$. Also, we have $11 * x' \neq 12 * x'$ for all $x' \in R$. Indeed, if $11 * x' = 12 * x'$ then $11 * (x' * y) = 12 * (x' * y)$ then $11 = 12$, contradiction! Hence, by supposing that $11 * x' = 7$ and $12 * x' = 8$ we get the following block:

	7	8	9
10	7	8	9
11	7	7	7
12	8	8	8

We notice also that 2 fixes $7 * 3$, i.e. $(7 * 3) * 2 = 7 * 3$ and in our case, $|7 * 3| = 2$. Indeed, suppose $7 * 3 = 9$ then $(11 * 7) * 3 = 11 * 9$ then $7 * 3 = 11 * 9 = 9$ but $11 * 9 = 7$. This means that $7 * 3 \in \{7, 8\}$, $7 * 2 = 7$ and $8 * 2 = 8$ (because if $7 * 2 = 8$ then $8 * 2 = 8$ and $7 * 2 * 2 = 8 * 2 \implies 7 = 8$). And by using 9.2 we obtain that $9 * 2 = 9$. And we have the block:

	1	2	3
7	7	7	7
8	8	8	8
9	9	9	7

Now we can try to obtain some properties about the multiplication table of $\{7, 8, 9\}$ by $\{4, 5, 6\}$. We have $7 * 4 = 7 * (3 * 5) = (7 * 3) * 5 = 7 * 5 = 7 * (3 * 6) = (7 * 3) * 6 = 7 * 6$. Similarly, we can prove that $8 * 4 = 8 * 5 = 8 * 6$. And since $9 * 3 = 7$ then $9 * 4 = 9 * 5 = 9 * 6 = 7 * 4$. We can also easily conclude their values, for example, $7 * 4 = (11 * 7) * 4 = 11 * (7 * 4) = 11$ (since C_3 is a left rectangular band). So basically it depends on the block of $\{11, 12\}$ by $\{7, 8, 9\}$. And we get the following block:

	4	5	6
7	11	11	11
8	12	12	12
9	11	11	11

It's left to fill in the block of $\{4, 5, 6\}$ by $\{10, 11, 12\}$, and it actually depends on the choice of $3 * 4$ because we can always transform the equation into this answer. For example, $4 * 11 = 4 * (7 * 4) = (4 * 7) * 4 = 3 * 4$. Then all the entries are going to be 4, except when it's multiplied by the identity 10. And we obtain the following table of a category:

0	1 2 3	4 5 6	7 8 9	10 11 12
1	1 2 3	4 5 6	0 0 0	0 0 0
2	2 1 3	4 5 6	0 0 0	0 0 0
3	3 3 3	4 4 4	0 0 0	0 0 0
4	0 0 0	0 0 0	3 3 3	4 4 4
5	0 0 0	0 0 0	3 3 3	5 4 4
6	0 0 0	0 0 0	3 3 3	6 4 4
7	7 7 7	11 11 11	0 0 0	0 0 0
8	8 8 8	12 12 12	0 0 0	0 0 0
9	9 9 7	11 11 11	0 0 0	0 0 0
10	0 0 0	0 0 0	7 8 9	10 11 12
11	0 0 0	0 0 0	7 7 7	11 11 11
12	0 0 0	0 0 0	8 8 8	12 12 12

Then the choices made are: $2 * 5$, $7 * 3$ and $11 * 7$. We can permute them in the following way:

$$2 * 5 = 5 \quad 7 * 3 = 7 \quad \text{and} \quad 11 * 7 = 7 \quad (A)$$

$$2 * 5 = 5 \quad 7 * 3 = 7 \quad \text{and} \quad 11 * 7 = 8 \quad (B)$$

$$2 * 5 = 5 \quad 7 * 3 = 8 \quad \text{and} \quad 11 * 7 = 7 \quad (C)$$

$$2 * 5 = 5 \quad 7 * 3 = 8 \quad \text{and} \quad 11 * 7 = 8 \quad (D)$$

$$2 * 5 = 6 \quad 7 * 3 = 7 \quad \text{and} \quad 11 * 7 = 7 \quad (E)$$

$$2 * 5 = 6 \quad 7 * 3 = 7 \quad \text{and} \quad 11 * 7 = 8 \quad (F)$$

$$2 * 5 = 6 \quad 7 * 3 = 8 \quad \text{and} \quad 11 * 7 = 7 \quad (G)$$

$$2 * 5 = 6 \quad 7 * 3 = 8 \quad \text{and} \quad 11 * 7 = 8 \quad (H)$$

We notice that (B) , (C) and (D) are isomorphic to (A) and (F) , (G) and (H) are isomorphic to (E) . Then the number of categories between C_1 and C_3 is 2.

9.3.3 The number of categories between \mathbf{C}_6 and itself

Let \mathbf{B} be a category with two objects X and Y associated to the matrix

$$M_1 = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$$

such that $\mathbf{C}(X, X) = \mathbf{C}_6$, $\mathbf{C}(Y, Y) = \mathbf{C}_6$, $\mathbf{C}(X, Y) = L$ and $\mathbf{C}(Y, X) = \emptyset$. Fix

$$\mathbf{C}_6 = \{1, 2, 3\} \quad L = \{4, 5, 6\} \quad \mathbf{C}_6 = \{7, 8, 9\}.$$

1. **i = 0:** the number of categories associated to M_1 is 64. But Since \mathbf{C}_6 is a grouplike category containing the trivial group $\{2\}$, then the orbit of L has size 1 and it's unique (Chapter 6). This means that $2 * x$ is unique for all $x \in L$. Suppose it's equal to 4. This reduces the number of possibilities to 15.

Similarly, let \mathbf{D} be a category with two objects X and Y associated to the matrix

$$M_2 = \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix}$$

such that $\mathbf{C}(X, X) = \mathbf{C}_6$, $\mathbf{C}(Y, Y) = \mathbf{C}_6$, $\mathbf{C}(X, Y) = \emptyset$ and $\mathbf{C}(Y, X) = R$.

For the same reason, the number of possible categories associated to M_2 with the orbit condition is 15.

Then a category \mathbf{C} associated to the matrix

$$M = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

such that $\mathbf{C}(X, X) = \mathbf{C}_6$, $\mathbf{C}(Y, Y) = \mathbf{C}_6$, $\mathbf{C}(X, Y) = L$ and $\mathbf{C}(Y, X) = R$, has 15 left bimodules A_1, \dots, A_{15} and 15 right bimodules B_1, \dots, B_{15} (Chapter 5).

We have two cases:

- (a) The bimodules are the same

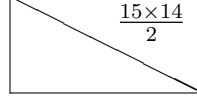
$$\begin{pmatrix} G^{*2} & A_i \\ B_i & G^{*2} \end{pmatrix}.$$

By Theorem 6.36, there are exactly 15 possibilities for this matrix (up to isomorphism).

(b) The bimodules are different

$$\begin{pmatrix} G^{*2} & A_i \\ B_j & G^{*2} \end{pmatrix}$$

such that $1 \leq i < j \leq 15$.



There are $\frac{15 \times 14}{2} = 105$ possibilities.

Hence for $i = 0$, we have 120 categories.

2. **i = 1:** there are 2 left bimodules A_1, A_2 and 2 right bimodules B_1, B_2 . Then the possibilities are:

(a) The bimodules are the same

$$\begin{pmatrix} G^{*2} & A_i \\ B_i & G^{*2} \end{pmatrix}$$

and we have 2 categories that could be associated to this matrix.

(b) The bimodules are different

$$\begin{pmatrix} G^{*2} & A_i \\ B_j & G^{*2} \end{pmatrix}$$

and we have 1 category (up to isomorphism).

Hence in total there are $120 + 3 = 123$ categories between C_6 and itself.

9.3.4 Proof of why C_5 is only connected to itself

Proposition 9.5. Let $M = \begin{pmatrix} 3 & a \\ b & 3 \end{pmatrix}$ and C be a category associated to M whose objects are X and Y . Then

$$C(Y, Y) = C_5 \iff C(X, X) = C_5.$$

Proof. Suppose that $C(X, X) = A, C(X, Y) = L, C(Y, X) = R$ and $C(Y, Y) = C_5 = \mathbb{Z}_2 \cup \{1\}$. Consider the algebraic matrix of C

$$M^{alg} = \begin{pmatrix} A & L \\ R & C_5 \end{pmatrix}$$

such that $|A| = 3, |L| = a, |R| = b$ and $|C_5| = 3$.

Let \mathbf{C}' be a subcategory of \mathbf{C} defined in the following way

1. **(Objects of \mathbf{C}'):** X and Y .
2. **(Morphisms of \mathbf{C}'):** $A, L \cdot \mathbb{Z}_2, \mathbb{Z}_2 \cdot R$ and \mathbb{Z}_2 . (It means that we take the orbits of L and R by the action of \mathbb{Z}_2).
3. **(Composition):** by Proposition 6.22 we have that the composition of orbits goes into the group \mathbb{Z}_2 .
4. **(Identities):** 1_A and $1_{\mathbb{Z}_2}$.

We obtain that \mathbf{C}' is associated to the following algebraic matrix

$$M'^{alg} = \begin{pmatrix} A & L \cdot \mathbb{Z}_2 \\ \mathbb{Z}_2 \cdot R & \mathbb{Z}_2 \end{pmatrix}.$$

Since there exist $x \in L \cdot \mathbb{Z}_2$ and $y \in \mathbb{Z}_2 \cdot R$ such that $x \cdot y = 1_{\mathbb{Z}_2}$, then from Lemma 7.5 we have that \mathbb{Z}_2 is a sub-monoid of A disjoint from $\{1_A\}$. Therefore, $A = \mathbb{Z}_2 \cup \{1\} = \mathbf{C}_5$. \square

Theorem 9.6. Let $M = \begin{pmatrix} 3 & a_{12} & \dots & a_{1n} \\ a_{21} & 3 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & \dots & 3 \end{pmatrix}$ be a strictly positive matrix and let \mathbf{C} be

a reduced category associated to M With $Ob(\mathbf{C}) = \{1, \dots, n\}$.

If there exists $i \in Ob(\mathbf{C})$ such that $\mathbf{C}(i, i) = \mathbf{C}_5$ then $\mathbf{C}(j, j) = \mathbf{C}_5 \quad \forall j \in Ob(\mathbf{C})$.

Proof. If \mathbf{C} is a category associated to M then every regular sub-matrix of M of size 2 is associated to a full sub-category of \mathbf{C} [3] with two objects, then we apply Proposition 9.5 to get the result. \square

9.3.5 Proof of why \mathbf{C}_3 and \mathbf{C}_4 are not connected

\mathbf{C}_3 is a left rectangular band of dimension 2×1 and \mathbf{C}_4 is a right rectangular band of dimension 1×2 . Then by Proposition 7.8 the set of morphisms $\mathbf{C}(X, Y)$ should have a cardinal at least $4 = 2 \times 2$, but it is 3.

\mathbf{C}_3 and \mathbf{C}_4 are connected to themselves because $|\mathbf{C}(X, Y)| = 3 \geq 2 \times 1$ and $|\mathbf{C}(Y, X)| = 3 \geq 2 \times 1$.

Appendices

1

Python script generating semigroups equations

```
testing = False

# input = tuple( [ tuple([3]) ]) # this is how you write matrix 1*1

input = ( (3, 3)
          , (3, 3) )

inputModuleTable = {}

# inputModuleTable['aaa'] = {(1, 2, 3)
#                             , (2, 2, 2)
#                             , (3, 3, 3) }

# input = ( (3, 3, 3)
#           , (3, 3, 3)
#           , (3, 3, 3) )

# Should be only set to True if all the values in the matrix are the same:
forbidCompositionsEqualToUnits = True

#####

obj = tuple( str(chr(97+i)) for i in range(0,len(input)))

def o(i): return obj.index(i)

hom_values_obj = {} # Table with lists of names of morphisms
hash_ = {} # Hash table to map names to numbers
count = 1 # For counting how many morphisms we have

def p_help(i,j,h):
    return i + j + h

for i in obj:
    hom_values_obj[i] = {}
    for j in obj:
        hom_values_obj[i][j] = \
            [str(k) for k in range(1, input[o(i)][o(j)]+1)]
        for h in hom_values_obj[i][j]:
            hash_[p_help(i,j,h)] = str(count)
            if not testing:
                print ("%hash[" + p_help(i,j,h) + "]= " + str(count))
            count = count + 1

print

def hom_values(i,j):
    return hom_values_obj[i][j]

if testing:
    p = p_help
else:
    def p(i,j,h):
        return hash_[p_help(i,j,h)]

#####

unit = '1' # the name of the unit for every object

alphabet = [str(chr(i)) for i in range(ord('a'),ord('z')+1)]

# We now define those constants
unitList = [ "x_" + p(i,i,unit) for i in obj]
unitStr = "".join(unitList)
print ("all_x_(u(x)_{<->_(" + unitStr + ")).")
```

```

# The usual monoid equations
print ("""
x * (y * z) = (x * y) * z.
0 * x = 0.
x * 0 = 0.
""")

# Units, from the left
for i in obj:
    for j in obj:
        for h in hom_values(i, j):
            print (p(i, i, unit) + "*" + p(i, j, h) + "=" + p(i, j, h) + ".")

print

# Units, from the right
for j in obj:
    for i in obj:
        for h in hom_values(j, i):
            print (p(j, i, h) + "*" + p(i, i, unit) + "=" + p(j, i, h) + ".")

print

if forbidCompositionsEqualToUnits:
    # The law that forbids certain compositions to be equal to the units, from the left
    for i in obj:
        for j in obj:
            if i != j:
                for h in hom_values(i, j):
                    for g in hom_values(j, i):
                        print (p(i, j, h) + "*" + p(j, i, g) + "!=" + p(i, i, unit) + ".")

    print

    # The law that forbids certain compositions to be equal to the units, from the right
    for i in obj:
        for j in obj:
            if i != j:
                for h in hom_values(j, i):
                    for g in hom_values(i, j):
                        print (p(j, i, h) + "*" + p(i, j, g) + "!=" + p(j, j, unit) + ".")

    print

# Forcing composition to have the right type
for i in obj:
    for j in obj:
        for k in obj:
            for h in hom_values(i, j):
                for g in hom_values(j, k):
                    if (i == j and h == unit) or (j == k and g == unit):
                        continue

                    options = [ p(i, j, h) + "*" + p(j, k, g) + "=" + p(i, k, l) for l in hom_values(i, k)]
                    print ("|".join(options) + ".")

    print

print

# If the composition doesn't typecheck, make it fail

for i in obj:
    for j in obj:
        for k in obj:
            if j == k: continue

            for l in obj:
                for h in hom_values(i, j):
                    for g in hom_values(k, l):

```

```

        print (p(i,j,h) + "*" + p(k,l,g) + "=0.")

#####

# The tables of multiplication

for i in obj:
    for j in obj:
        for k in obj:
            try:
                t = inputModuleTable[p_help(i,j,k)]
                for f in hom_values(i,j):
                    for g in hom_values(j,k):
                        print (p(i,j,f) + "*_" + p(j,k,g) + "_=" + p(i,k,t[int(f)-1][int(g)-1]) + ".")
            break
        except KeyError:
            continue

# import sys
# sys.exit()

```

2

Proof of Equation 9.3 using PROVER9

```

1 3 * 4 = 3 * 5 & 3 * 4 = 3 * 6 # label(non_clause) # label(goal). [].
2 x * (y * z) = (x * y) * z. [].
3 (x * y) * z = x * (y * z). [2].
4 4 * 7 != 1. [].
5 4 * 8 != 1. [].
6 4 * 9 != 1. [].
7 5 * 7 != 1. [].
8 5 * 8 != 1. [].
9 5 * 9 != 1. [].
10 6 * 7 != 1. [].
11 6 * 8 != 1. [].
12 6 * 9 != 1. [].
13 7 * 4 != 10. [].
14 7 * 5 != 10. [].
15 7 * 6 != 10. [].
16 4 * 7 = 1 | 4 * 7 = 2 | 4 * 7 = 3. [].
17 4 * 7 = 2 | 4 * 7 = 3. [4,16].
18 5 * 7 = 1 | 5 * 7 = 2 | 5 * 7 = 3. [].
19 5 * 7 = 2 | 5 * 7 = 3. [7,18].
20 6 * 7 = 1 | 6 * 7 = 2 | 6 * 7 = 3. [].
21 6 * 7 = 2 | 6 * 7 = 3. [10,20].
22 7 * 2 = 7 | 7 * 2 = 8 | 7 * 2 = 9. [].
23 7 * 4 = 10 | 7 * 4 = 11 | 7 * 4 = 12. [].
24 7 * 4 = 11 | 7 * 4 = 12. [13,23].
25 7 * 5 = 10 | 7 * 5 = 11 | 7 * 5 = 12. [].
26 7 * 5 = 11 | 7 * 5 = 12. [14,25].
27 7 * 6 = 10 | 7 * 6 = 11 | 7 * 6 = 12. [].
28 7 * 6 = 11 | 7 * 6 = 12. [15,27].
29 2 * 2 = 1. [].
30 1 = 2 * 2. [29].
31 3 * 3 = 3. [].
32 12 * 11 = 12. [].
33 3 * 5 != 3 * 4 | 3 * 6 != 3 * 4. [1].
34 6 * 9 != 2 * 2. [30,12].
35 6 * 8 != 2 * 2. [30,11].
36 6 * 7 != 2 * 2. [30,10].
37 5 * 9 != 2 * 2. [30,9].
38 5 * 8 != 2 * 2. [30,8].
39 5 * 7 != 2 * 2. [30,7].
40 4 * 9 != 2 * 2. [30,6].
41 4 * 8 != 2 * 2. [30,5].
42 4 * 7 != 2 * 2. [30,4].
43 4 * 7 = 3 | 2 * x = 4 * (7 * x). [17,3].
44 4 * 7 = 3 | 4 * (7 * x) = 2 * x. [43].
45 5 * 7 = 3 | 2 * x = 5 * (7 * x). [19,3].
46 5 * 7 = 3 | 5 * (7 * x) = 2 * x. [45].
47 6 * 7 = 3 | 2 * x = 6 * (7 * x). [21,3].
48 6 * 7 = 3 | 6 * (7 * x) = 2 * x. [47].

```

49 $7 * 4 = 11 \mid 12 * x = 7 * (4 * x)$. [24,3].
50 $7 * 4 = 11 \mid 7 * (4 * x) = 12 * x$. [49].
51 $7 * 5 = 11 \mid 12 * x = 7 * (5 * x)$. [26,3].
52 $7 * 5 = 11 \mid 7 * (5 * x) = 12 * x$. [51].
53 $7 * 5 = 11 \mid (x * 7) * 5 = x * 12$. [26,3].
54 $7 * 5 = 11 \mid x * (7 * 5) = x * 12$. [3,53].
55 $7 * 6 = 11 \mid 12 * x = 7 * (6 * x)$. [28,3].
56 $7 * 6 = 11 \mid 7 * (6 * x) = 12 * x$. [55].
57 $3 * x = 3 * (3 * x)$. [31,3].
58 $3 * (3 * x) = 3 * x$. [57].
59 $7 * 2 = 7 \mid 7 * 2 = 8 \mid 4 * 7 = 3 \mid 4 * 9 = 2 * 2$. [22,44].
60 $7 * 2 = 7 \mid 7 * 2 = 8 \mid 4 * 7 = 3$. [40,59].
61 $7 * 2 = 7 \mid 7 * 2 = 8 \mid 5 * 7 = 3 \mid 5 * 9 = 2 * 2$. [22,46].
62 $7 * 2 = 7 \mid 7 * 2 = 8 \mid 5 * 7 = 3$. [37,61].
63 $7 * 2 = 7 \mid 7 * 2 = 8 \mid 6 * 7 = 3 \mid 6 * 9 = 2 * 2$. [22,48].
64 $7 * 2 = 7 \mid 7 * 2 = 8 \mid 6 * 7 = 3$. [34,63].
65 $7 * 4 = 11 \mid 7 * (4 * 11) = 12$. [32,50].
66 $7 * 2 = 7 \mid 4 * 7 = 3 \mid 4 * 7 = 3 \mid 4 * 8 = 2 * 2$. [60,44].
67 $7 * 2 = 7 \mid 4 * 7 = 3 \mid 4 * 8 = 2 * 2$. [66].
68 $7 * 2 = 7 \mid 4 * 7 = 3$. [41,67].
69 $4 * 7 = 3 \mid 4 * 7 = 3 \mid 4 * 7 = 2 * 2$. [68,44].
70 $4 * 7 = 3 \mid 4 * 7 = 2 * 2$. [69].
71 $4 * 7 = 3$. [42,70].
72 $3 * x = 4 * (7 * x)$. [71,3].
73 $4 * (7 * x) = 3 * x$. [72].
74 $7 * 4 = 11 \mid 4 * 12 = 3 * 4$. [24,73].
75 $7 * 5 = 11 \mid 4 * 12 = 3 * 5$. [26,73].
76 $7 * 6 = 11 \mid 4 * 12 = 3 * 6$. [28,73].
77 $7 * 5 = 11 \mid 4 * (12 * x) = 3 * (5 * x)$. [52,73].
78 $7 * 5 = 11 \mid 3 * (5 * x) = 4 * (12 * x)$. [77].
79 $7 * 6 = 11 \mid 4 * (12 * x) = 3 * (6 * x)$. [56,73].
80 $7 * 6 = 11 \mid 3 * (6 * x) = 4 * (12 * x)$. [79].
81 $4 * 12 = 3 * 5 \mid 4 * 11 = 3 * 5$. [75,73].
82 $4 * 12 = 3 * 6 \mid 4 * 11 = 3 * 6$. [76,73].
83 $7 * 2 = 7 \mid 5 * 7 = 3 \mid 5 * 7 = 3 \mid 5 * 8 = 2 * 2$. [62,46].
84 $7 * 2 = 7 \mid 5 * 7 = 3 \mid 5 * 8 = 2 * 2$. [83].
85 $7 * 2 = 7 \mid 5 * 7 = 3$. [38,84].
86 $5 * 7 = 3 \mid 5 * 7 = 3 \mid 5 * 7 = 2 * 2$. [85,46].
87 $5 * 7 = 3 \mid 5 * 7 = 2 * 2$. [86].
88 $5 * 7 = 3$. [39,87].
89 $3 * x = 5 * (7 * x)$. [88,3].
90 $5 * (7 * x) = 3 * x$. [89].
91 $7 * 4 = 11 \mid 5 * 12 = 3 * 4$. [24,90].
92 $7 * 4 = 11 \mid 5 * 12 = 3 * (4 * 11)$. [65,90].
93 $4 * 12 = 3 * 4 \mid 5 * 11 = 3 * 4$. [74,90].
94 $4 * 12 = 3 * 6 \mid 5 * 11 = 3 * 6$. [76,90].
95 $7 * 2 = 7 \mid 6 * 7 = 3 \mid 6 * 7 = 3 \mid 6 * 8 = 2 * 2$. [64,48].
96 $7 * 2 = 7 \mid 6 * 7 = 3 \mid 6 * 8 = 2 * 2$. [95].
97 $7 * 2 = 7 \mid 6 * 7 = 3$. [35,96].
98 $6 * 7 = 3 \mid 6 * 7 = 3 \mid 6 * 7 = 2 * 2$. [97,48].
99 $6 * 7 = 3 \mid 6 * 7 = 2 * 2$. [98].
100 $6 * 7 = 3$. [36,99].
101 $3 * x = 6 * (7 * x)$. [100,3].
102 $6 * (7 * x) = 3 * x$. [101].
103 $4 * 12 = 3 * 5 \mid 6 * 11 = 3 * 5$. [75,102].
104 $5 * 12 = 3 * 4 \mid 6 * 11 = 3 * 4$. [91,102].
105 $5 * 12 = 3 * (4 * 11) \mid 6 * 11 = 3 * 4$. [92,102].
106 $4 * 12 = 3 * 4 \mid 7 * 5 = 11 \mid 3 * (3 * 4) = 4 * (12 * 11)$. [93,78].
107 $4 * 12 = 3 * 4 \mid 7 * 5 = 11 \mid 3 * 4 = 4 * (12 * 11)$. [58,106].
108 $4 * 12 = 3 * 4 \mid 7 * 5 = 11 \mid 3 * 4 = 4 * 12$. [32,107].
109 $4 * 12 = 3 * 4 \mid 7 * 5 = 11 \mid 4 * 12 = 3 * 4$. [108].
110 $4 * 12 = 3 * 4 \mid 7 * 5 = 11$. [109].
111 $4 * 12 = 3 * 6 \mid 7 * 5 = 11 \mid 3 * (3 * 6) = 4 * (12 * 11)$. [94,78].
112 $4 * 12 = 3 * 6 \mid 7 * 5 = 11 \mid 3 * 6 = 4 * (12 * 11)$. [58,111].
113 $4 * 12 = 3 * 6 \mid 7 * 5 = 11 \mid 3 * 6 = 4 * 12$. [32,112].
114 $4 * 12 = 3 * 6 \mid 7 * 5 = 11 \mid 4 * 12 = 3 * 6$. [113].
115 $4 * 12 = 3 * 6 \mid 7 * 5 = 11$. [114].
116 $4 * 12 = 3 * 4 \mid 4 * 11 = 3 * 5$. [110,73].
117 $4 * 12 = 3 * 4 \mid 5 * 11 = 3 * 5$. [110,90].
118 $4 * 12 = 3 * 6 \mid 4 * 11 = 3 * 5$. [115,73].
119 $4 * 11 = 3 * 5 \mid 3 * 4 = 3 * 5 \mid 4 * 11 = 3 * 5$. [116,81].
120 $4 * 11 = 3 * 5 \mid 3 * 5 = 3 * 4 \mid 4 * 11 = 3 * 5$. [119].
121 $4 * 11 = 3 * 5 \mid 3 * 5 = 3 * 4$. [120].
122 $4 * 12 = 3 * 4 \mid 4 * 12 = 3 * 4 \mid 3 * 5 = 3 * 4$. [117,93].
123 $4 * 12 = 3 * 4 \mid 3 * 5 = 3 * 4$. [122].
124 $4 * 11 = 3 * 5 \mid 3 * 6 = 3 * 5 \mid 4 * 11 = 3 * 5$. [118,81].
125 $4 * 11 = 3 * 5 \mid 3 * 6 = 3 * 5$. [124].

126 $3 * 5 = 3 * 4$ | $7 * 5 = 11$ | $4 * (7 * 5) = 3 * 4$. [123,54].
127 $3 * 5 = 3 * 4$ | $7 * 5 = 11$ | $3 * 5 = 3 * 4$. [73,126].
128 $3 * 5 = 3 * 4$ | $7 * 5 = 11$. [127].
129 $3 * 5 = 3 * 4$ | $6 * 11 = 3 * 5$. [128,102].
130 $3 * 5 = 3 * 4$ | $5 * 12 = 3 * 4$ | $3 * 5 = 3 * 4$. [129,104].
131 $3 * 5 = 3 * 4$ | $5 * 12 = 3 * 4$. [130].
132 $4 * 12 = 3 * 5$ | $7 * 6 = 11$ | $3 * (3 * 5) = 4 * (12 * 11)$. [103,80].
133 $4 * 12 = 3 * 5$ | $7 * 6 = 11$ | $3 * 5 = 4 * (12 * 11)$. [58,132].
134 $4 * 12 = 3 * 5$ | $7 * 6 = 11$ | $3 * 5 = 4 * 12$. [32,133].
135 $4 * 12 = 3 * 5$ | $7 * 6 = 11$ | $4 * 12 = 3 * 5$. [134].
136 $4 * 12 = 3 * 5$ | $7 * 6 = 11$. [135].
137 $4 * 12 = 3 * 5$ | $4 * 11 = 3 * 6$. [136,73].
138 $4 * 11 = 3 * 6$ | $3 * 5 = 3 * 6$ | $4 * 11 = 3 * 6$. [137,82].
139 $4 * 11 = 3 * 6$ | $3 * 6 = 3 * 5$ | $4 * 11 = 3 * 6$. [138].
140 $4 * 11 = 3 * 6$ | $3 * 6 = 3 * 5$. [139].
141 $3 * 6 = 3 * 5$ | $3 * 6 = 3 * 5$ | $3 * 6 = 3 * 5$. [140,125].
142 $3 * 6 = 3 * 5$ | $3 * 6 = 3 * 5$. [141].
143 $3 * 6 = 3 * 5$. [142].
144 $3 * 5 != 3 * 4$ | $3 * 5 != 3 * 4$. [143,33].
145 $3 * 5 != 3 * 4$. [144].
146 $5 * 12 = 3 * 4$. [145,131].
147 $6 * 11 = 3 * 5$. [145,129].
148 $4 * 11 = 3 * 5$. [145,121].
149 $3 * 4 = 3 * (4 * 11)$ | $6 * 11 = 3 * 4$. [146,105].
150 $3 * 4 = 3 * (3 * 5)$ | $6 * 11 = 3 * 4$. [148,149].
151 $3 * 4 = 3 * 5$ | $6 * 11 = 3 * 4$. [58,150].
152 $3 * 4 = 3 * 5$ | $3 * 5 = 3 * 4$. [147,151].
153 $3 * 5 = 3 * 4$ | $3 * 5 = 3 * 4$. [152].
154 $3 * 5 = 3 * 4$. [153].
155 \$F\$. [145,154].

end of proof

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L'objectif principal de cette thèse est de classer les catégories finies en fonction de la structure algébrique de leurs monoïdes d'endomorphismes. À l'aide d'un logiciel appelé `PROVER9/MACE4`, nous dénombrons les catégories avec deux objets tel que chaque ensemble de morphismes a un cardinal égal à 3. Les données nous incitent à étudier la structure des monoïdes afin de mieux comprendre le problème d'énumération. Dans cette thèse, nous étudions trois types de monoïdes : monoïdes *de type groupe*, *semigroupes simples*, *bandes rectangulaires* et monoïdes avec un élément 0.

Les *monoïdes de type groupe* sont des monoïdes construits à partir d'un groupe auquel on ajoute un ensemble ordonné d'idempotents. Nous étudions l'action des groupes et l'interaction entre eux. Nous donnons également quelques propriétés sur les ensembles d'idempotents.

Les *semigroupes simples* sont des semigroupes dont le seul idéal est le semigroupe lui-même. Nous prouvons que si nous avons un monoïde alors il contient toujours un unique semigroupe simple et nous pouvons construire une catégorie à deux objets, telle que l'un des objets soit le semigroupe simple et l'autre un groupe.

Les *bandes rectangulaires* sont des semigroupes idempotents avec la propriété $xyz = xz$. Nous prouvons que la matrice d'une catégorie à bandes rectangulaires comme monoïdes d'endomorphisme a certaines restrictions sur ses coefficients.

Les *monoïdes avec un élément 0* sont des monoïdes qui ont un élément absorbant. Nous prouvons que l'existence d'un élément 0 dans les monoïdes induit l'existence d'un élément 0 dans chaque ensemble de morphismes.

Les résultats sur ces monoïdes clarifient les données obtenues à partir de l'énumération et nous aident à en donner une explication.

Mots clés : catégories finies, classification, associativité, `prover9/mace4`, semigroupes, monoïdes de type groupe, catégories de type groupe, semigroupes simples, bandes rectangulaires, catégories avec un 0.