Metric properties of the group of Hamiltonian diffeomorphisms

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Contents

Acknowledgments

"By the way", said Vincent. "It might be nice if you go talk to Sobhan. He just came back from IAS and he will be happy to tell you about his research". I was in the office of Vincent Humilière asking him about his research. The previous week, I had decided to work with Claude Viterbo for the masters thesis. I came to see Vincent to have a better perspective of the field I chose to work in – symplectic topology. Just then, a man came into the room. "Salut!" he said. In an energetic voice. A voice that would never come out from a bad person. Vincent and I looked up towards the door. "Oh, there he is!" said Vincent, "We were talking about you!". That was Sobhan. Vincent explained to him what we were discussing. "Okay, you could come to my office next week and we can discuss."

This is how I started to work with Sobhan and Claude which has continued until now. While in masters, I heard many negative stories about life as a PhD student. How tough it is, how stressful it is and so on. Despite all those terrifying stories, I personally ended up having a great time from the beginning. Obviously there were stressful times where I felt creatively empty, like the first semesters of the first year and the second year. But overall, I felt extremely good and fulfilled with the activity of doing mathematical research and this was not possible without my advisors. For me, it was ideal to have both Claude and Sobhan as my advisors from whom I learned different lessons. In particular, I feel grateful and extremely lucky to have seen how these two mathematicians behave when they encounter mathematical difficulties. The process of trying to build a path between yourself and something that seems far away. A thing you cannot really learn in books and yet one of the most important – in my opinion – skills to do research. I thank them both for spending a lot of time with me so that I could get an idea of how to acquire that attitude.

I very much enjoyed and was impressed by the way Claude does not cut mathematics into tiny pieces of sub-fields. He thinks about topology, differential geometry, PDE or probability just as he thinks about symplectic topology. It took away the fear from me to step into topics that I'm less familiar with or I'm totally ignorant about. It was through interactions with him that I learned how to form a project by playing with concrete examples. He also convinced me that the "fun" element is crucial in research. He subconsciously encouraged me to only pursue the directions that make me feel good and excited. I also enjoyed a lot hearing his experiences and stories about his career.

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Chapter 1

Introduction

1.1 Abstract and organization of the thesis

The main theme of this dissertation is the metric properties of the group of Hamiltonian diffeomorphisms. The group of Hamiltonian diffeomorphisms is known to possess two remarkable bi-invariant metrics discovered by Hofer and Viterbo which are referred to as the Hofer metric and the spectral metric, respectively. A third metric on $\mathrm{Ham}(M,\omega)$ which has been of interest in symplectic topology is the C^0 metric. Roughly speaking, my research explores the deep and mysterious relation between these three metrics. It might appear somewhat strange at first glance to consider C^0 metric in symplectic topology which is a realm of *differential* topology.

The dissertation is structured as follows: In Section 1.2, I give a motivation to study C^0-symplectic topology by reviewing its development. In Section 1.3, I summarize the main results of this thesis. In Section 3, I will explain results concerning the comparison between the spectral metric and the C^0 metric. Materials in this section are taken from [Kaw21]. In Section 4, I explain a construction of quasimorphisms that have some new types of continuity properties which are related to a famous question of Entov-Polterovich-Py. Materials in this section are mostly taken from [Kaw20]. In Section 5.1, I explain some new properties of the spectral norm. Materials in this section are based on a work in progress with Egor Shelukhin [KawSh].

Each section is designed to be readable more or less independently from the other sections and therefore there are some overlaps between different sections.

1.2 A biased introduction to C^0-symplectic topology

Since the birth of modern symplectic topology, the study of algebraic and topological properties of the group of Hamiltonian diffeomorphisms of a symplectic manifold (M,ω), denoted by $\mathrm{Ham}(M,\omega)$, has been one of the cornerstones of the field. This direction of research has been strongly influenced by the discovery of two remarkable bi-invariant metrics on $\mathrm{Ham}(M,\omega)$ in the seminal works of Hofer and Viterbo which are referred to as the Hofer distance and the spectral[1] distance, respectively. A third metric on $\mathrm{Ham}(M,\omega)$ which has been of interest in symplectic topology is the C^0 distance. Roughly speaking, my research explores the deep and mysterious relation

[1] The spectral distance is sometimes referred to as the γ distance.

between these three metrics. My results outlined in Section 1.3.1 provide direct comparisons between the C^0 distance and the spectral distance. I have also obtained results on the existence of certain homogeneous quasimorphisms on $\mathrm{Ham}(M,\omega)$; see Section 1.3.3.

Recall that these three norms which induce the three distances mentioned above are defined as follows: The C^0-norm of a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M,\omega)$ is defined by

$$||\phi||_{C^0} := \max_{x\in M} d_M(x,\phi(x))$$

where d_M denotes the distance induced by any choice of a Riemannian metric; any two such distances are equivalent. The Hofer norm is the least "energy" required to produce a given Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M,\omega)$:

$$||\phi||_{\mathrm{Hof}} := \inf\{\int_0^1 (\sup_{x\in M} H_t(x) - \inf_{x\in M} H_t(x))dt : \phi = \phi_H\}.$$

The definition of the spectral norm $\gamma : \mathrm{Ham}(M,\omega) \to \mathbb{R}_{\geqslant 0}$ is more involved than the previous two: it utilizes the sophisticated machinery of Hamiltonian Floer homology. Since its introduction in [Vit92], the spectral norm has been used extensively and has found various applications.

The relation between the Hofer norm and the spectral norm is explained by the following well-known inequality which follows easily from the very definition of the spectral norm:

$$\gamma(\phi) \leqslant ||\phi||_{\mathrm{Hof}}.$$

On the other hand, the relation between the C^0-norm and the other two norms has been very poorly understood even though the significance of role of the C^0-topology in symplectic topology was recognized at a very early stage of modern symplectic topology. But, before going into the story, why do we are we interested in the C^0-topology in the first place? We answer this is the next section.

1.2.1 Why C^0?

All the basic notions one encounters in symplectic geometry are smooth objects e.g. a symplectic manifold, a Hamiltonian, a Lagrangian submanifold and so on. Thus, it might sound strange to talk about C^0-topology at first sight. However, actually the mysterious relationship between symplectic phenomenon and the C^0-topology had been discovered at the very first stage of modern symplectic toplogy, even before the advent of pseudo-holomorphic curve theory of Gromov.

Theorem 1. *(Eliashberg-Gromov)*
Let (M,ω) be any symplectic manifold. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence of symplectic diffeomorphisms i.e. $(\phi_n)^*\omega = \omega$ for all $n \in \mathbb{N}$. Assume

$$\phi_n \xrightarrow{C^0} \phi \in \mathrm{Diff}(M).$$

Then, ϕ is symplectic i.e. $\phi^*\omega = \omega$.

Notice that being symplectic is a condition that involves differentials (pulling-back a differential form!) but here, we do not have any control on the differential as the convergence is given only in the C^0-topology. This C^0-rigidity phenomenon

opened up a new sub-field: C^0-symplectic topology. The main aim of the sub-field is to determine the border between the world of symplectic topology and the world of (general) topology. More precisely and realistically, C^0-symplectic topologists study whether or not symplectic phenomena persist under C^0-limits. With the C^0-rigidity of Eliashberg–Gromov, one might naively think that there is a better chance to expect a rigidity, in other words, many of the symplectic phenomena do persist under C^0-limits. However, it turned out that the story is not that simple. Over the past decade, C^0-symplectic topology has seen a spectacular progress and it was revealed that C^0-rigidity appears often but not any more often than C^0-flexibility. In the next two sections, I will explain some of the main features of the two aspects through my personal selection of important discoveries.

1.2.2 C^0-flexibility

The first instance of the C^0-flexibility was found by Buhovsky–Opshtein along with their invention of the "quantitative h-principle". They started from the following question.

Question 2. *Let (M,ω) be a symplectic manifold. Let S be a symplectic (resp. isotropic, coisotropic, Lagrangian) submanifold. If a symplectic homeomorphism $\phi \in \overline{\mathrm{Symp}}(M,\omega)$ maps S to a smooth submanifold i.e. if $\phi(S)$ is smooth, then will $\phi(S)$ be symplectic (resp. isotropic, coisotropic, Lagrangian)?*

Theorem 3. *([BO16])*
Let $d \geqslant m+2$. In $\mathbb{C}^d$. For any open neighborhood U of $(D^2)^{d-m} \times \{0\} \times \{0\}$, there exists a symplectic homeomorphism $\phi \in \overline{\mathrm{Symp}}(\mathbb{C}^d)$ such that

- *$Supp(\phi) \subset U$.*
- *$\phi|_{(D^2)^{d-m}\times\{0\}\times\{0\}} = \frac{1}{2}\mathrm{id}_{(D^2)^{d-m}\times\{0\}\times\{0\}}$.*

This clearly answers Question 2 in the negative. By applying the "quantitative h-principle", Buhovsky–Humilière–Seyfaddini proved that the analogue of the Arnold conjecture (a typical example of a symplectic rigidity) no longer holds in the C^0 setting which is another instance of the the C^0-flexibility. Let us first recall the Arnold conjecture before stating the theorem of Buhovsky–Humilière–Seyfaddini.

Conjecture 4. (*The Arnold conjecture*)
Let (M^{2n},ω) be a closed symplectic manifold.

1. *For a non-degenerate $\phi \in \mathrm{Ham}(M,\omega)$,*
$$\#\mathrm{Fix}(\phi) \geqslant \sum_j \dim_{\mathbb{C}} H_j(M;\mathbb{C}).$$
2. *For $\phi \in \mathrm{Ham}(M,\omega)$,*
$$\#\mathrm{Fix}(\phi) \geqslant cl(M)$$
where
$$cl(M) := \#\max\{k+1 : \exists a_1, a_2, \cdots, a_k \in H_{*<2n}(M) \text{ s.t. } a_1 \cap a_2 \cap \cdots \cap a_k \neq 0\}$$
and $\cap$ denotes the intersection product.

Since the advent of Floer homology, there has been a huge progress in the two versions of the Arnold conjecture: (1) is now completely settled [FO99], [LT98] and (2) has been confirmed for symplectically aspherical manifolds [Fl89], $\mathbb{C}P^n$ [For85] [ForW85] and negative monotone symplectic manifolds with $N_M \geqslant n$ [LO94]. Although the second verison of the conjecture has not yet been solved in full generality, one can see that Hamiltonian diffeomorphisms tend to possess many fixed points. What Buhovsky–Humilière–Seyfaddini observed is that, all these fixed points can be smashed into a single fixed point which contrasts the case of Hamiltonian diffeomorphisms.

Theorem 5. *([BHS18])*
On any symplectic manifold (M, ω) of dimension $\geqslant 4$, there exists a Hamiltonian homeomorphism ϕ which possess only one fixed point: $\#\mathrm{Fix}(\phi) = 1$.

Remark 6. *On the other hand, the case that is not covered by this Theorem, namely the 2 dimensional case was studied by Matsumoto [Mat00] and the turn out is that the Arnold conjecture for Hamiltonian homeomorphisms do hold! In general, in dimension 2, symplectic features appear less often than in the higher dimensional case e.g. the Gromov embedding theorem becomes a non-symplectic statement in dimension 2.*

1.2.3 C^0-rigidity

There are C^0-rigidity results other than the Eliashberg–Gromov theorem. Leclercq–Humilière–Seyfaddini answered Question 2 for cosiotropic submanifolds and proved the C^0-rigidity.

Theorem 7. *([HLS15])*
Let (M, ω) be a symplectic manifold. Let C be a coisotropic submanifold. If a symplectic homeomorphism $\phi \in \overline{\mathrm{Symp}}(M, \omega)$ maps C to a smooth submanifold i.e. if $\phi(C)$ is smooth, then $\phi(S)$ is coisotropic.

In the previous section, we have seen that the Arnold conjecture no longer holds for Hamiltonian homeomorphisms but using the C^0-continuity of the spectral norm, Buhovsky–Humilière–Seyfaddini [BHS21].

Theorem 8. *([BHS21])*
Let (M, ω) be a symplectically aspherical manifold. If the number of distinct spectral invariants of a Hamiltonian homeomorphism ϕ is less than the cup-length $cl(M)$, then the set of fixed points $\mathrm{Fix}(\phi)$ is homologically non-trivial hence is an infinite set.

Note that this statement is not rigorous as one cannot define spectral invariants for a Hamiltonian homeomorphism ϕ. Yet, one can count the number of distinct "spectral invariants" of a Hamiltonian homeomorphism on a symplectically aspherical manifold by using the theory of barcodes, a notion coming from TDA (topological analysis). Ever since Polterovich–Shelukhin [PS16] brought barcodes into symplectic topology, the application of barcode theory has been a hot topic. According to Polterovich–Shelukhin, given a Hamiltonian diffeomorphism ϕ, by looking at its filtered Floer homology, one can construct a barcode $B(\phi)$. One nice feature of barcodes is the notion of distance– the so-called bottle-neck distance d_{bot} measures how far two barcodes are. This allows us to talk about how far two filtered Floer homologies are. In fact, this distance was estimated Kislev-Shelukhin [KiSh18] as follows:

$$d_{bot}(B(\phi_1), B(\phi_2)) \leqslant \frac{1}{2}\gamma(\phi_1^{-1}\phi_2).$$

This implies that the C^0-continuity of the spectral norm γ will imply the C^0-continuity of barcodes which will even allow us to define barcodes for Hamiltonian homeomorphisms as limits of barcodes of Hamiltonian diffeomorphisms. Now, we can count the number of half-infinite intervals in the barcode of a Hamiltonian homeomorphism with distance starting point and interpret the count as the number of distinct spectral invariants for a Hamiltonian homeomorphism. The above theorem of Buhovsky–Humilière–Seyfaddini should be understood in this way.

1.2.4 Practical application: 2D conservative dynamics

As mentioned earlier, 2 dimensional spaces are not the ideal place to capture symplectic features: they appear less often than in the higher dimensional spaces. Nevertheless, from a practical point of view, C^0-symplectic topology provides a powerful tool to attack questions in 2 dimensional conservative dynamics namely questions concerning area-preserving diffeomorphisms or homeomorphisms. This is because for a 2-sphere S^2 or a 2-disk D^2, Hamiltonian homeomorphisms are precisely the elements of the identity component of the group of area-preserving homeomorphisms.

The very first success in this direction was a result of Seyfaddini which answers the so-called "displaced disks problem"– a question in 2D dynamics – as a direct consequence of a result in C^0-symplectic topology.

The "displaced disks problem" posed by Béguin–Crovisier–Le Roux asks whether or not a C^0-small area-preserving homoemorphism can displace a disk of large area.

Question 9. (*Béguin-Crovisier-Le Roux*)
Let

$$\mathcal{G}_r := \{\phi \in \mathrm{Homeo}(S^2, \sigma_{area}) : \phi(f(B_r)) \cap f(B_r) = \emptyset\}$$

for any $r > 0$ where $f : B_r \to (M, \omega)$ is a symplectic embedding. Does the C^0-closure of $\mathcal{G}_r$ contain the identity element for some $r > 0$?

It turned out that this is a direct consequence of the C^0-continuity of the spectral norm. More recently, Cristofaro-Gradiner–Humilière–Seyfaddini [CGHS20]. [CGHS21] have settled the longstanding "simplicity conjecture" in a similar spirit – their crucial step was to prove a C^0-continuity of a relevant spectral invariant (more precisely, spectral invariants of Hutchings' periodic Floer homology).

Theorem 10. *(Simplicity conjecture [CGHS20], [CGHS21])*
The group of area-preserving homeomorphisms of a 2-disk/sphere is not simple.

1.2.5 Sum-up

The study of C^0-symplectic topology had a significant progress during the past decade and it seems fair to say that the need for studying the C^0-topology in symplectic topology is becoming more and more prominent. I present three pieces of evidence.

1. Recent developments in astrophysics claim that a notion of C^0-Hamiltonian is needed to study the N-body problem [Her19].

2. The C^0-continuity of a certain Floer theoritic invariant played a crucial role in the recent resolution of the historically famous "simplicity conjecture" [CGHS20] which was a long standing conjecture that was studied by many symplectic geometers and dynamicists.

3. There were important discoveries in symplectic dynamics about the C^0 behavior of pseudo-rotations which are important objects in dynamical systems [Bra15a], [Bra15b], [GG18].

1.3 Overview of the results in this thesis

1.3.1 Spectral metric and C^0–metric

As mentioned in the introduction, not much was known about the relation between the C^0-metric and the Hofer or the spectral metrics apart from the C^0-continuity of the spectral metric for $\mathbb{R}^{2n}$ [Vit92]. After this discovery, the question of whether or not the C^0-continuity of the spectral metric holds for every symplectic manifold caught attention within the symplectic community. However, there was very little progress on this question apart from the case of closed surfaces by Seyfaddini [Sey13a] which was based on an argument specific to 2-dimensional manifolds. The case of higher dimensions saw a significant progress in [BHS18] where Buhovsky–Humilière–Seyfaddini proved the C^0–continuity of the spectral norm for symplectically aspherical manifolds. Later, Shelukhin confirmed the case of $\mathbb{C}P^n$ in [Sh18]. In [Kaw21], by following the approach of [BHS18] and adding an analysis of the pseudo-holomorphic spheres which does not appear in the symplectically aspherical setting, I obtained the following results (where my method for $\mathbb{C}P^n$ is different from the approach of Shelukhin).

Theorem 11. ([Kaw21])
The spectral norm $\gamma : \mathrm{Ham}(M,\omega) \to \mathbb{R}$ is C^0–continuous when (M,ω) is either a negative monotone symplectic manifold or $(\mathbb{C}P^n, \omega_{FS})$.

Although I could not achieve the C^0–continuity for other classes of symplectic manifolds, I have obtained the following partial results. Note that C^0–continuity of the spectral norm γ is equivalent to γ being small around the C^0-neighborhood of $\mathrm{id} \in \mathrm{Ham}(M,\omega)$.

Theorem 12. ([Kaw21])

1. *Let (M,ω) be a monotone symplectic manifold. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\phi \in \mathrm{Ham}(M,\omega)$ satisfies $d_{C^0}(\mathrm{id},\phi) < \delta$, then*

$$\gamma(\phi) \leqslant \frac{\dim(M)}{N_M} + \varepsilon$$

where N_M denotes the minimal Chern number i.e. $\langle c_1(TM), \pi_2(M)\rangle = N_M\mathbb{Z}$, $N_M > 0$.

2. *Let (M,ω) be a rational symplectic manifold i.e. $\langle \omega, \pi_2(M)\rangle = \lambda_0\mathbb{Z}$ for some $\lambda_0 > 0$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\phi \in \mathrm{Ham}(M,\omega)$ satisfies $d_{C^0}(\mathrm{id},\phi) < \delta$, then*

$$|\gamma(\phi) - l \cdot \lambda_0| < \varepsilon$$

for some $l \in \mathbb{Z}$.

The first result provides a uniform bound of the spectral norm for monotone symplectic manifolds on a C^0-neighborhood of $\mathrm{id} \in \mathrm{Ham}(M,\omega)$ which unexpectedly played a crucial role in the proof of Theorem 78 which will appear later. The second result gives candidates of values of the spectral norm of Hamiltonian diffeomorphisms on a C^0-neighborhood of $\mathrm{id} \in \mathrm{Ham}(M,\omega)$ for rational symplectic manifolds. The cases considered above i.e. (negative) monotone and rational form important classes of symplectic manifolds.

1.3.2 Persistent homology

Persistent homology (or barcode) is a tool that is broadly used and studied in TDA (topological data analysis). Recently the language of barcodes was brought into symplectic topology by Polterovich–Shelukhin [PS16] (see also [UZ16]) which allowed one to define barcodes from (filtered) Floer homology groups under the assumption that the ambient symplectic manifold is aspherical i.e. $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$. One remarkable feature about barcodes is the notion of distance called the bottleneck distance denoted by d_{bot}. With the bottleneck distance, one can measure how "far" two Floer homology groups are. Barcodes contain a lot of important information of Hamiltonian diffeomorphisms; for example, spectral invariants are the half-infinite intervals in a barcode. It also reads the boundary depth à la Usher [Ush11], [Ush13] which corresponds to the longest finite interval in a barcode. In addition to all these, barcodes contain also "short" finite intervals. Recently, Shelukhin has successfully used the barcode theory to answer several important open problems; see [Sh18], [Sh19a] for his results on the Viterbo conjecture on spectral capacity and [Sh19b] for his result on the Hofer–Zehnder conjecture.

In fact, it turns out that, the C^0-continuity of the spectral norm allows us to define barcodes for Hamiltonian homeomorphisms. Let's briefly see the mechanism. Denote the barcode corresponding to a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M,\omega)$ by $B(\phi)$. The bottleneck distance was estimated by using the spectral norm as follows [KiSh18]

$$\phi_1, \phi_2 \in \mathrm{Ham}(M,\omega),\ d_{bot}(B(\phi_1), B(\phi_2)) \leqslant \frac{1}{2}\gamma(\phi_1^{-1}\phi_2).$$

This inequality and the C^0-continuity of the spectral norm imply the C^0-continuity of barcodes i.e. (filtered) Floer homology only changes a little by a C^0-perturbation of a Hamiltonian diffeomorphism. Now, for a Hamiltonian homeomorphism $\phi \in \overline{\mathrm{Ham}}(M,\omega)$, one can define its barcode $B(\phi)$ by

$$B(\phi) := \lim_{k\to+\infty} B(\phi_k)$$

for any sequence of Hamiltonian diffeomorphisms $\{\phi_k\}$ such that $\phi_k \xrightarrow{C^0} \phi$. The precise meaning of the limit is given in Section 3.3.2. Barcode $B(\phi)$ could be interpreted as the Floer homology of a Hamiltonian homeomorphism ϕ.

Barcodes for Hamiltonian homeomorphisms give us a hope to overcome the fundamental issue in C^0-sympelctic topology, namely the lack of tools. For example, for Hamiltonian diffeomorphisms, Floer theory provides a very rich tool to study dynamics of them. However, Floer homology cannot be defined for Hamiltonian homeomorphisms and therefore it becomes more difficult to study Hamiltonian homeomorphisms. By interpreting barcodes for Hamiltonian homeomorphisms as the Floer

homology for Hamiltonian homeomorphisms, Buhovsky–Humilière–Seyfaddini suggested a way to reformulate the Arnold conjecture for Hamiltonian homeomorphisms in [BHS21]. They proved that, even though it was shown in their previous work that a Hamiltonian homeomorphism might possess only one single fixed point (see Theorem 5), the number of distinct half-infinite intervals should be more than the cup-length of the ambient manifold provided that a Hamiltonian homeomorphism possess finitely many fixed points and the ambient symplectic manifold is aspherical.

In Section 3.2.5 and Section 3.2.6, I extend the work of Buhovsky–Humilière–Seyfaddini to some non-aspherical symplectic manifolds and discuss the C^0-Arnold conjecture.

1.3.3 Quasimorphisms

A celebrated result of Banyaga [Ban78] which states that $\mathrm{Ham}(M,\omega)$ is a simple group when (M,ω) is a closed symplectic manifold implies that there exists no non-trivial homomorphism on $\mathrm{Ham}(M,\omega)$ (as if it did, then the kernel of the homomorphism will be a non-trivial normal subgroup). Nevertheless, Entov–Polterovich proved in the seminal paper [EP03] that there exist "almost homomorphisms" on $\mathrm{Ham}(M,\omega)$ when (M,ω) satisfies a certain condition. More formally, "almost homomorphisms" are called homogeneous quasimorphisms and are defined as follows.

Definition 13. *A homogeneous quasimorphism on a group G is a map*

$$\mu : G \to \mathbb{R}$$

which satisfies

1. $\exists C > 0$ *s.t.* $\forall f, g \in G$,
$$|\mu(f \cdot g) - \mu(f) - \mu(g)| \leqslant C.$$
2. $\forall k \in \mathbb{Z}, \forall f \in G$,
$$\mu(f^k) = k \cdot \mu(f).$$

Quasimorphisms are used to study algebraic and topological (if G is a topological group) properties of G. After the discovery of Entov–Polterovich, an extensive research on the construction of other quasimorphisms as well as their applications to the study of geometric, algebraic and topological properties of $\mathrm{Ham}(M,\omega)$ has been conducted by not only symplectic geometers but also dynamicists.

Entov–Polterovich's homogeneous quasimorphisms are constructed via spectral invariants. It follows easily from a basic property of spectral invariants that Entov–Polterovich's homogeneous quasimorphisms are continuous with respect to the Hofer metric. However, a remarkable property of Entov–Polterovich's homogeneous quasimorphisms called the Calabi property obstructs the C^0–continuity.

The following question was then posed by Entov–Polterovich–Py in [EPP12] which became an important open problem in this direction. This question appears under the name of "Problem 23 Quasimorphism question" in the famous Chapter "Open problems" in the monograph of McDuff-Salamon [MS98].

Question 14. (*"Quasimorphism question" [EPP12], [MS98]*)

1. *Does there exist a nonzero homogeneous quasimorphism*

$$\mu : \mathrm{Ham}(S^2) \to \mathbb{R}$$

that is continuous with respect to the C^0-topology on $\mathrm{Ham}(S^2)$*?*

2. *If yes, can it be made Lipschitz with respect to the Hofer metric?*

Instead of this original question, I considered the following variant.

Question 15. (*[Kaw20]*)

1. *Does there exist a closed symplectic manifold (M, ω) which admits a non-trivial homogeneous quasimorphism*

$$\mu : \mathrm{Ham}(M, \omega) \to \mathbb{R}$$

that is continuous with respect to the C^0-topology?

2. *If yes, can it be made Lipschitz with respect to the Hofer metric?*

In the non-compact setting, there are examples (e.g. closed disks $D^{2n}(1) \subset \mathbb{R}^{2n}$) due to Entov-Polterovich-Py that answer both of the questions in the affirmative. In the closed setting, there is an example due to Gambaudo-Ghys [GG97], [GG04] where they construct a homogeneous quasimorphism $\mu : \mathrm{Ham}(\Sigma) \to \mathbb{R}$ (Σ denotes a closed surface of positive genus), that is C^0-continuous but not Hofer continuous. However, no example of a closed symplectic manifold (M, ω) that answers two questions in the affirmative was known. In [Kaw20], I constructed following homogeneous quasimorphisms which give the first examples of such closed symplectic manifolds.

Theorem 16. (*[Kaw20]*)
Let Q^n denote the complex n-dimensional quadric hypersurface:

$$Q^n := \{(z_0 : z_1 : \cdots : z_n) \in \mathbb{C}P^{n+1} : z_0^2 + z_1^2 + \cdots + z_n^2 = 0\}.$$

There exist non-trivial homogeneous quasimorphisms

$$\mu : \mathrm{Ham}(Q^n) \to \mathbb{R}$$

for $n = 2, 4$ that satisfy the following three properties:

1. *μ is Lipschitz continuous with respect to the Hofer metric.*

2. *μ is C^0-continuous i.e.*

$$\mu : (\mathrm{Ham}(Q^n), d_{C^0}) \to \mathbb{R}$$

is continuous.

3. *We have*

$$|\mu| = \overline{\gamma}$$

where $\overline{\gamma}$ denotes the asymptotic spectral norm:

$$\overline{\gamma} : \mathrm{Ham}(M, \omega) \to \mathbb{R}$$

$$\overline{\gamma}(\phi) := \lim_{k \to +\infty} \frac{\gamma(\phi^k)}{k}.$$

An unexpected consequence of properties 2 and 3 in Theorem 16 is that the asymptotic spectral norm $\overline{\gamma}$ is C^0-continuous for the 2- and 4- quadric hypersurfaces even though the C^0-continuity of the spectral norm γ for these manifolds is not confirmed at the time of writing. In fact, our argument can be generalized as follows.

Theorem 17. *Let (M,ω) be a monotone symplectic manifold such that $QH^*(M;\mathbb{C})$ is semi-simple. Then,*

$$\overline{\gamma} : \mathrm{Ham}(M,\omega) \to \mathbb{R}$$

is C^0-continuous.

Remark 18. *Under the condition of Theorem 17, if $\overline{\gamma} \neq 0$, then it implies the existence of a non-trivial C^0 and Hofer Lipschitz continuous homogeneous quasimorphism on $\mathrm{Ham}(M,\omega)$. However, we can show $\overline{\gamma} \neq 0$ only for the 2- and 4- quadric hypersurfaces at the time of writing.*

As a direct consequence of Theorem 16, we can answer the following question posed by Le Roux on the topology of the group of Hamiltonian diffeomorphisms for Q^2 and Q^4 in the affirmative where, according to [EPP12], Le Roux's question "for closed simply connected manifolds (and already for the case of the 2-sphere) the question is wide open".

Question 19. (*Le Roux [LeR10]*)
Let (M,ω) be any closed symplectic manifold. For any $R>0$, does

$$\mathrm{Ham}_{\geqslant R} := \{\phi \in \mathrm{Ham}(M,\omega) : d_{\mathrm{Hof}}(\mathrm{id},\phi) \geqslant R\}$$

have a non-empty C^0-interior?

The main idea in the proof of Theorem 16 was to use two different quantum cohomology rings: The first quantum cohomology that I considered is

$$QH^*(M;\mathbb{C}) := H^*(M;\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1},t|]$$

where $\mathbb{C}[t^{-1},t|]$ is the field of Laurent series

$$\mathbb{C}[t^{-1},t|] := \{\sum_{k\geqslant k_0} a_k t^k : k_0 \in \mathbb{Z}, a_k \in \mathbb{C}\}$$

and t represents a sphere that satisfies $\omega(t)=\lambda_0$, $c_1(t)=N_M$ (Recall that $\langle\omega,\pi_2(M)\rangle = \lambda_0\mathbb{Z}$ and $\langle c_1,\pi_2(M)\rangle = N_M\mathbb{Z}$). The second quantum cohomology that I used is

$$QH^*(M;\Lambda) := H^*(M;\mathbb{C}) \otimes_{\mathbb{C}} \Lambda$$

where Λ is the universal Novikov field

$$\Lambda := \{\sum_{k=1}^{\infty} b_k T^{\lambda_k} : b_k \in \mathbb{C}, \lambda_k \in \mathbb{R}, \lim_{k\to+\infty} \lambda_k = +\infty\}.$$

These two quantum cohomology rings have different advantages: The former $QH^*(M;\mathbb{C})$ carries a $\mathbb{Z}$-grading which makes it convenient to use spectral invariants as we can study the actions and the indices of theirs. On the other hand, the latter $QH^*(M;\Lambda)$ is commonly used in the mirror symmetry community and many techniques involving

Lagrangian Floer cohomology have been developed in this setting. In particular, I use some non-vanishing results of Lagrangian Floer cohomology which comes from superpotential techniques developed in the latter setting. A more detailed explanation about the difference of $QH^*(M;\mathbb{C})$ and $QH^*(M;\Lambda)$ is explained in Section 4.5.3 with some instructive examples.

Realizing the different advantages of these two quantum cohomology rings led me to answer a question of Polterovich and Wu posed in [Wu15]:

Question 20. (*Question of Polterovich–Wu; Remark 5.2 in [Wu15]*)
Is it possible to distinguish the three Entov-Polterovich type quasimorphisms on $\mathrm{Ham}(\mathbb{C}P^2)$*?*

We answer this in the negative.

Theorem 21. (*[Kaw20]*)
The three Entov-Polterovich type quasimorphisms on $\mathrm{Ham}(\mathbb{C}P^2)$ *all coincide.*

Remark 22. *For the precise statement of the above result, see Section 4.5.6 and Theorem 82. Other examples where we can identify homogeneous quasimorphisms are also discussed in Section 4.5.6.*

1.3.4 Poincaré duality and spectral invariants

In Section 1.3.3, I used two quantum cohomology rings that have (the same ground ring $\mathbb{C}$ but) different coefficient fields, namely the field of Laurent series and the universal Novikov field. This led me to the idea of comparing Floer homology groups defined with different ground rings. In collaboration with Egor Shelukhin, I have been studying the difference between $\gamma_{\mathbb{Z}}$ and $\gamma_{\mathbb{K}}$ where γ_R denotes the spectral norm of the Floer homology of ground ring R and $\mathbb{K}$ is a field. We have observed the following unexpected phenomenon.

Theorem 23. (*[KawSh]*)
For $n = 2, 3$ *we have*

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})} \gamma_{\mathbb{Z}}(\phi) = +\infty$$

while

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})} \gamma_{\mathbb{K}}(\phi) \leqslant \frac{n}{n+1}$$

for any field $\mathbb{K}$*.*

We also answer the generalized question of Entov–Polterovich–Py (see Question 15) for $\mathbb{C}P^2, \mathbb{C}P^3$.

Theorem 24. (*[KawSh]*)
There exist non-trivial homogeneous quasimorphisms

$$\mu : \mathrm{Ham}(\mathbb{C}P^n) \to \mathbb{R}$$

for $n = 2, 3$ *that satisfy the following two properties:*

1. μ *is Lipschitz continuous with respect to the Hofer metric.*

2. μ is C^0-continuous i.e.

$$\mu : (\mathrm{Ham}(\mathbb{C}P^n), d_{C^0}) \to \mathbb{R}$$

is continuous.

The phenomenon observed in Theorem 23 brought us to the following two directions of research.

1.3.5 Algebraic viewpoint

The finiteness of spectral norm for $\mathbb{K}$-coefficients

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})} \gamma_{\mathbb{K}}(\phi) \leqslant \frac{n}{n+1}$$

is a consequence of the so-called "Poincaré duality formula for spectral invariants". Thus,

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})} \gamma_{\mathbb{Z}}(\phi) = +\infty$$

implies that the well-known "Poincaré duality formula for spectral invariants" no longer holds for spectral invariants with $\mathbb{Z}$-coefficients. By considering the universal coefficient theorem for filtered Floer homology, Shelukhin and I have found that for some Hamiltonian H, the filtered $\mathbb{Z}$-coefficient Floer homology $HF^{\tau}(H;\mathbb{Z})$ possesses a torsion element at some filtration level $\tau \in \mathbb{R}$ which disappears when the filtration level τ is sufficiently large.

Theorem 25. ([KawSh])
On $\mathbb{C}P^n, n = 2, 3$, there exists a Hamiltonian H such that for some filtration level $\tau \in \mathbb{R}$, we have

$$Tor(HF^{\tau}(H;\mathbb{Z})) \neq 0.$$

This is a phenomenon that was not previously noticed and I plan to study this torsion part of the filtered Floer homology by using the language of barcodes that I explained in Section 3.3.2.

1.3.6 Dynamical viewpoint

We consider the following question.

Question 26. *What kind of dynamical features does a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})$ satisfying*

$$\sup_{k \in \mathbb{Z}} \gamma_{\mathbb{Z}}(\phi^k) < +\infty$$

possess?

We have obtained the following partial result in this direction.

Theorem 27. ([KawSh])
If $\phi \in \mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})$ is a pseudo-rotation, we have

$$\gamma_{\mathbb{Z}}(\phi) \leqslant \frac{n}{n+1}.$$

Recall that pseudo-rotations are Hamiltonian diffeomorphisms that possess the "minimal" number of periodic points. Pseudo-rotations are very important objects in dynamical systems and their relation to symplectic topology is becoming an active topic of research after the seminal works of Bramham [Bra15a], [Bra15b]. We conjecture that the result of Theorem 137 holds in a broader generality.

Conjecture 28. *Let (M, ω) be a monotone symplectic manifold admitting a pseudo-rotation. Any pseudo-rotation $\phi \in \mathrm{Ham}(M, \omega)$ satisfies*

$$\sup_{k \in \mathbb{Z}} \gamma_{\mathbb{Z}}(\phi^k) < +\infty.$$

Chapter 2

Preliminaries

2.1 Setting

In this section, we briefly review the notions and basic propositions needed later in the proof. For further details, we refer to [MS04].

Let (M, ω) be a symplectic manifold. A Hamiltonian H on M is a smooth time dependent function $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$. We define its Hamiltonian vector field X_{H_t} by $-dH_t = \omega(X_{H_t}, \cdot)$. The Hamiltonian flow of H, denoted by ϕ_H^t, is by definition the flow of X_H. A Hamiltonian diffeomorphism is a diffeomorphism which arises as the time-one map of a Hamiltonian flow. The set of all Hamiltonian diffeomorphisms is denoted by $\mathrm{Ham}(M, \omega)$.

Denote the set of smooth contractible loops in M by $\mathcal{L}M$ and consider its universal cover. Two elements in the universal cover, say $[z_1, w_1]$ and $[z_2, w_2]$, are called equivalent if their boundary sum $w_1 \# \overline{w_2}$ i.e. the sphere obtained by gluing w_1 and w_2 along their common boundary with the orientation on w_2 reversed, satisfies

$$\omega(w_1 \# \overline{w_2}) = 0, \ c_1(w_1 \# \overline{w_2}) = 0.$$

We denote by $\widetilde{\mathcal{L}_0 M}$ the space of equivalence classes. For a Hamiltonian H, define the action functional $\mathcal{A}_H : \widetilde{\mathcal{L}M} \to \mathbb{R}$ by

$$\mathcal{A}_H([z, w]) := \int_0^1 H(t, z(t))dt - \int_{D^2} w^* \omega$$

where $w : D^2 \to M$ is a capping of $z : \mathbb{R}/\mathbb{Z} \to M$. Note that in general, the action functional depends on the capping and not only on the loop. Critical points of this functional are precisely the capped 1-periodic Hamiltonian orbits of H which will be denoted by $\widetilde{\mathcal{P}}(H)$. The set of critical values of $\mathcal{A}_H$ is called the action spectrum and is denoted by $\mathrm{Spec}(H)$:

$$\mathrm{Spec}(H) := \{\mathcal{A}_H(\widetilde{z}) : \widetilde{z} \in \widetilde{\mathcal{P}}(H)\}.$$

We briefly explain some notions of indices used later to construct Floer homology. The Maslov index

$$\mu : \pi_1(Sp(2n)) \to \mathbb{Z}$$

maps a loop of symplectic matrices to an integer. For a capped periodic orbit of a loop of Hamiltonian diffeomorphims $\psi \in \pi_1(\mathrm{Ham}(M,\omega))$ denoted by $[\psi^t(x), w]$, we define its Maslov index $\mu([\psi^t(x), w])$ via the trivialization of w^*TM and the loop of symplectic matrices $d\psi^t(x) : T_xM \to T_{\psi^t(x)}M$. The definition of Maslov indices cannot be directly applied to periodic orbits of a Hamiltonian H since given a periodic orbit $[\phi_H^t(x), w]$, $d\phi_H^t(x) : T_xM \to T_{\phi_H^t(x)}M$ might not define a loop. To overcome this difficulty, Conley-Zehnder modified the definition of the Maslov index and introduced the Conley-Zehnder index

$$\mu_{CZ} : \{A : [0,1] \to Sp(2n) | \det(A(1) - \mathrm{id}) \neq 0\} \to \mathbb{Z}$$

which maps paths of symplectic matrices to integers. Thus, as in the case of Maslov indices, we define the Conley-Zehnder index of a non-degenerate periodic orbit of a Hamiltonian H $[\phi_H^t(x), w]$ (i.e. $d\phi_H(x) : T_xM \to T_xM$ is non-degenerate), denoted by $\mu_{CZ}([\phi_H^t(x), w])$, via the trivialization of w^*TM and the path of symplectic matrices $d\phi_H^t(x) : T_xM \to T_{\phi_H^t(x)}M$.

Let (M, ω) be a closed monotone symplectic manifold i.e.

$$\omega|_{\pi_2(M)} = \kappa \cdot c_1|_{\pi_2(M)}$$

for some monotonicity constant $\kappa > 0$ where $c_1 = c_1(TM)$ denotes the first Chern class. In this paper, we only consider monotone symplectic manifolds unless mentioned otherwise. The positive generators of $\langle \omega, \pi_2(M) \rangle$ and $\langle c_1, \pi_2(M) \rangle \subset \mathbb{Z}$ are respectively called the rationality constant and the minimal Chern number and will be respectively denoted by λ_0 and N_M.

A Hamiltonian H on M is a smooth time dependent function $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$. A Hamiltonian H is called mean-normalized if the following holds:

$$\forall t \in \mathbb{R}/\mathbb{Z}, \ \int_M H_t(x)\omega^n = 0.$$

We define its Hamiltonian vector field X_{H_t} by

$$-dH_t = \omega(X_{H_t}, \ \cdot \ ).$$

The Hamiltonian flow of H, denoted by ϕ_H^t, is by definition the flow of X_H. A Hamiltonian diffeomorphism of H is a diffeomorphism which arises as the time-one map of a Hamiltonian flow and will be denoted by ϕ_H. It is well-known that the set of Hamiltonian diffeomorphisms forms a group and will be denoted by $\mathrm{Ham}(M,\omega)$. We denote its universal cover by $\widetilde{\mathrm{Ham}}(M,\omega)$.

Denote the set of smooth contractible loops in M by $\mathcal{L}_0M$ and consider its universal cover. Two elements in the universal cover, say $[z_1, w_1]$ and $[z_2, w_2]$, are equivalent if $z_1 = z_2$ and their boundary sum $w_1 \# \overline{w_2}$ i.e. the sphere obtained by gluing w_1 and w_2 along their common boundary with the orientation on w_2 reversed, satisfies

$$\omega(w_1 \# \overline{w_2}) = 0, \ c_1(w_1 \# \overline{w_2}) = 0.$$

We denote by $\widetilde{\mathcal{L}_0M}$ the space of equivalence classes.

For a Hamiltonian H, define the action functional $\mathcal{A}_H : \widetilde{\mathcal{L}_0M} \to \mathbb{R}$ by

$$\mathcal{A}_H([z,w]) := \int_0^1 H(t, z(t))dt - \int_{D^2} w^*\omega$$

where $w : D^2 \to M$ is a capping of $z : \mathbb{R}/\mathbb{Z} \to M$. Critical points of this functional are precisely the capped 1-periodic Hamiltonian orbits of H which will be denoted by $\widetilde{\mathcal{P}}(H)$. The set of critical values of $\mathcal{A}_H$ is called the action spectrum and is denoted by $\mathrm{Spec}(H)$:

$$\mathrm{Spec}(H) := \{\mathcal{A}_H(\widetilde{z}) : \widetilde{z} \in \widetilde{\mathcal{P}}(H)\}.$$

2.2 Hofer and C^0 topology on $\mathrm{Ham}(M,\omega)$

Studying the topology of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega)$ is an important topic in symplectic topology. In this section we recall two topologies of $\mathrm{Ham}(M,\omega)$.

Hofer metric (or distance) is defined by

$$d_{Hof}(\phi,\psi) := \inf\{\int_0^1 (\sup_x H_t(x) - \inf_x H_t(x))dt : \phi_H = \psi^{-1}\circ\phi\}$$

for $\phi,\psi \in \mathrm{Ham}(M,\omega)$. The Hofer-topology is the topology induced by the Hofer metric.

The C^0-distance of Hamiltonian diffeomorphisms $\phi,\psi \in \mathrm{Ham}(M,\omega)$ is defined by

$$d_{C^0}(\phi,\psi) := \max_{x\in M} d(\phi(x),\psi(x))$$

where d denotes the distance on M induced by the fixed Riemannian metric on M. The C^0-topology is the topology induced by the C^0-distance. Note that the C^0-topology is independent of the choice of the Riemannian metric.

2.3 Hamiltonian Floer homology

We work with the ground field $\mathbb{C}$ in this paper. We say that a Hamiltonian H is non-degenerate if the diagonal set $\Delta := \{(x,x) \in M\times M\}$ and $\Gamma_{\phi_H} := \{(x,\phi_H(x)) \in M \times M\}$ intersects transversally. We define the Floer chain complex of a non-degenerate Hamiltonian H, denoted by $CF_*(H)$ as follows:

$$CF_*(H) := \{\sum_{\widetilde{z}\in\widetilde{\mathcal{P}}(H)} a_{\widetilde{z}}\cdot\widetilde{z} : \forall\tau\in\mathbb{R},\ \#\{\widetilde{z} : a_{\widetilde{z}}\neq 0, \mathcal{A}_H(\widetilde{z}) \leqslant \tau\} < +\infty\}.$$

Floer chain complex $CF_*(H)$ is $\mathbb{Z}$-graded by the so-called Conley-Zehnder index μ_{CZ}. The differential map counts certain solutions of a perturbed Cauchy-Riemann equation for a chosen ω-compatible almost complex structure J on TM, which can be viewed as isolated negative gradient flow lines of $\mathcal{A}_H$. This defines a chain complex $(CF_*(H),\partial)$ called the Floer chain complex whose homology is called the Floer homology of (H,J) and is denoted by $HF_*(H,J)$. Often it is abbreviated to $HF_*(H)$ as Floer homology does not depend on the choice of an almost complex structure. Note that our convention of the Conley-Zehnder index is as follows:

Let f denote a C^2-small Morse function. For every critical point x of f, we require that

$$\mu_{CZ}([x,w_x]) = i(x)$$

where i denotes the Morse index and w_x is the trivial capping.

Recapping a capped orbit $\widetilde{z} = [z,w]$ by gluing $A \in \pi_2(M)$ changes the action and the Conley-Zehnder index as follows:

- $\mathcal{A}_H([z, w\#A]) = \mathcal{A}_H([z, w]) - \omega(A)$.
- $\mu_{CZ}([z, w\#A]) = \mu_{CZ}([z, w]) - 2c_1(A)$.

We extend the action functional $\mathcal{A}_H$ as follows:

$$\mathcal{A}_H : CF_*(H) \to \mathbb{R}$$

$$\mathcal{A}_H(\sum_{\widetilde{z}\in\widetilde{\mathcal{P}}(H)} a_{\widetilde{z}} \cdot \widetilde{z}) := \max_{a_{\widetilde{z}}\neq 0} \mathcal{A}_H(\widetilde{z}).$$

We then define the $\mathbb{R}$-filtered Floer chain complex of H by the filtration of $\mathcal{A}_H$:

$$CF_*^\tau(H) := \{z \in CF_*(H) : \mathcal{A}_H(z) < \tau\} = \{\sum_{\widetilde{z}\in\widetilde{\mathcal{P}}(H), \mathcal{A}_H(\widetilde{z})<\tau} a_{\widetilde{z}} \cdot \widetilde{z} \in CF_*(H)\}.$$

As the Floer differential map decreases the action, $(CF_*^\tau(H), \partial)$ defines a chain complex whose homology is called the filtered Floer homology of H and is denoted by $HF_*^\tau(H)$.

2.4 Quantum (co)homology and semi-simplicity

Consider a monotone symplectic manifold (M, ω). Let the following denote the field of Laurent series of a formal variable s:

$$\mathbb{C}[|s^{-1}, s] := \{\sum_{k\leqslant k_0} a_k s^k : k_0 \in \mathbb{Z}, a_k \in \mathbb{C}\}.$$

By identifying the variable s with the generator of $\Gamma := \pi_2(M)/\sim$ where the equivalence relation is defined by $A, B \in \pi_2(M)$,

$$A \sim B \iff \omega(A) = \omega(B)$$

satisfying

$$\omega(s) = \lambda_0,\ c_1(s) = N_M,$$

one can define the quantum homology ring $QH_*(M; \mathbb{C})$ as

$$QH_*(M; \mathbb{C}) := H_*(M; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[|s^{-1}, s].$$

Quantum homology ring has the following valuation:

$$\nu : QH_*(M; \mathbb{C}) \to \mathbb{R}$$

$$\nu(\sum_{k\leqslant k_0} a_k s^k) := \max\{k \cdot \omega(s) = k \cdot \lambda_0 : a_k \neq 0, k \leqslant k_0\}.$$

Similarly, for a formal variable t, one can define the quantum cohomology ring $QH^*(M; \mathbb{C})$ as

$$QH^*(M; \mathbb{C}) := H^*(M; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1}, t|]$$

where

$$\mathbb{C}[t^{-1}, t|] := \{\sum_{k\geqslant k_0} b_k t^k : k_0 \in \mathbb{Z}, b_k \in \mathbb{C}\}.$$

Quantum homology ring and quantum cohomology ring has the following isomorphism called the Poincaré duality:

$$\mathrm{PD} : QH^*(M;\mathbb{C}) \xrightarrow{\sim} QH_{2n-*}(M;\mathbb{C})$$

$$a := \sum_{k \geqslant k_0} A_k t^k \mapsto \mathrm{PD}(a) := \sum_{k \geqslant k_0} A_k^{\#} s^{-k}$$

where $\#$ denotes the usual Poincaré duality between singular homology and singular cohomology. Note that t satisfies

$$\omega(t) = \lambda_0, \ c_1(t) = N_M.$$

Quantum cohomology ring has the following valuation:

$$\nu : QH^*(M;\mathbb{C}) \to \mathbb{R}$$

$$\nu(\sum_{k \geqslant k_0} a_k t^k) := \min\{k \cdot \omega(t) = k\lambda_0 : a_k \neq 0, k \geqslant k_0\}.$$

The ring structure of $QH_*(M;\mathbb{C})$ (and of $QH^*(M;\mathbb{C})$) is given by the quantum product which is denoted by $*$. It is defined by a certain count of pseudo-homolorphic spheres. More precisely, in the case of $QH_*(M;\mathbb{C})$,

$$\forall a, b, c \in H_*(M), \ (a * b) \circ c := \sum_{k \in \mathbb{Z}} GW_{3,s^k}(a, b, c) \otimes s^k$$

where $\circ$ denotes the usual intersection index in homology and $GW_{3,s^k}(a, b, c)$ denotes the 3-pointed Gromov-Witten invariant for $a, b, c \in H_*(M)$ in the class $A \in \pi_2(M)$ where $[A] = s^k \in \Gamma$ i.e. the count of pseudo-holomorphic spheres in the homotopy class A passing through cycles representing $a, b, c \in H_*(M)$. See [MS98] for details.

It is known that Floer homology defined in Section 2.3 is canonically isomorphic to the quantum homology ring via the PSS-map:

$$PSS_H : QH_*(M;\mathbb{C}) \xrightarrow{\sim} HF_*(H).$$

Note that the PSS-map preserves the ring structure.

2.5 Hamiltonian Floer theory

We fix a ground field $\mathbb{K}$ of zero characteristic (see Remark 32). We say that a Hamiltonian H is non-degenerate if the diagonal set $\Delta := \{(x, x) \in M \times M\}$ intersects transversally the graph of ϕ, $\Gamma_\phi := \{(x, \phi(x)) \in M \times M)\}$. For a non-degenerate H, we define the Floer chain complex $CF_*(H)$ as follows:

$$CF_*(H) := \{ \sum_{\widetilde{z} \in \widetilde{\mathcal{P}}(H)} a_{\widetilde{z}} \cdot \widetilde{z} : a_{\widetilde{z}} \in R, (\forall \tau \in \mathbb{R}, \ \#\{\widetilde{z} : a_{\widetilde{z}} \neq 0, \mathcal{A}_H(\widetilde{z}) \leqslant \tau\} < +\infty)\}.$$

The Floer chain complex has a $\mathbb{Z}$-grading by the Conley-Zehnder index μ_{CZ}. The boundary map counts certain solutions of a perturbed Cauchy-Riemann equation for a chosen ω-compatible almost complex structure J on TM, which can be viewed as isolated negative gradient flow lines of $\mathcal{A}_H$. This gives us a chain complex $(CF_*(H), \partial)$

called the Floer chain complex. Its homology is called the Floer homology of (H, J) and is denoted by $HF_*(H, J)$. Often it is abbreviated to $HF_*(H)$ as Floer homology does not depend on the choice of an almost complex structure.

Recapping of a capped orbit by $A \in \pi_2(M)$ changes the action and the Conley-Zehnder index as follows:

- $\mathcal{A}_H([z, w\#A]) = \mathcal{A}_H([z, w]) - \omega(A)$.
- $\mu_{CZ}([z, w\#A]) = \mu_{CZ}([z, w]) - 2c_1(A)$.

We define the filtered Floer complex of H by

$$CF_*^\tau(H) := \{\sum a_z z \in CF_*(H) : \mathcal{A}_H(z) < \tau\}.$$

Since the Floer boundary map decreases the action, $(CF_*^\tau(H), \partial)$ forms a chain complex. The filtered Floer homology of H which is denoted by $HF_*^\tau(H)$ is the homology defined by the chain complex $(CF_*^\tau(H), \partial)$.

It is useful to clarify our convention of the Conley-Zehnder index since conventions change according to literature. We fix our convention by requiring that for a C^2-small Morse function $f : M \to \mathbb{R}$, each critical point x of f satisfy

$$\mu_{CZ}([x, w_x]) = i(x)$$

where i denotes the Morse index and w_x is the trivial capping.

2.6 Quantum homology and Seidel elements

We sketch some basic definitions and properties concerning the quantum homology. Once again, we fix a ground field $\mathbb{K}$. For further details of the concepts sketched in this section, we refer to [MS04].

Let (M, ω) be a closed symplectic manifold. Define

$$\Gamma := \pi_2(M)/(\mathrm{Ker}(\omega) \cap \mathrm{Ker}(c_1)).$$

The Novikov ring Λ is defined by

$$\Lambda := \{\sum_{A \in \Gamma} a_A \otimes e^A : a_A \in \mathbb{K}, (\forall \tau \in \mathbb{R}, \#\{a_A \neq 0, \omega(A) < \tau\} < \infty)\}.$$

The quantum homology of (M, ω) is defined by

$$QH_*(M; \mathbb{K}) := H_*(M; \mathbb{K}) \otimes_{\mathbb{K}} \Lambda.$$

The quantum homology has a ring structure with respect to the quantum product denoted by $*$. It is defined as follows:

$$\forall a, b, c \in H_*(M; \mathbb{K}),\ (a * b) \circ c := \sum_{A \in \Gamma} GW_{3,A}(a, b, c) \otimes e^A$$

where $\circ$ denotes the intersection index and $GW_{3,A}$ denotes the 3-pointed Gromov-Witten invariant in the class A. See [MS04] for details.

When (M,ω) is either monotone or negative monotone, then the quantum homology ring $QH_*(M;\mathbb{K})$ can be expressed in a simple way using the field of Laurent series. We first explain the case of monotone symplectic manifolds. In this case, $\Gamma \simeq \mathbb{Z}$ with a generator A such that

$$\omega(A) = \lambda_0,\ c_1(A) = N_M.$$

Thus the Novikov ring Λ is the ring of formal Laurent series

$$\mathbb{K}[|s^{-1}, s] := \{\sum_{k \geqslant k_0} a_k s^k : a_j \in \mathbb{K}, k_0 \in \mathbb{Z}\}$$

where $s := e^A$ and the quantum homology ring $QH_*(M;\mathbb{K})$ is

$$QH_*(M;\mathbb{K}) = H_*(M;\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[|s^{-1}, s].$$

The quantum product is expressed by

$$\forall a, b \in H_*(M;\mathbb{K}),\ a * b = a \cap b + \sum_{k>0} (a*b)_k \cdot s^k.$$

The series on the right hand side runs over only non-positive powers since the elements of Γ appearing in the sum represents pseudo-holomophic spheres and pseudo-holomophic spheres has non-negative ω-area (remember that s represents a sphere A such that $\omega(A) = \lambda_0$).

When (M,ω) is negative monotone, by denoting the generator A of Γ which satisfies

$$\omega(A) = +\lambda_0,\ c_1(A) = -N_M$$

and by denoting $s := e^A$, we have

$$QH_*(M;\mathbb{K}) = H_*(M;\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[|s^{-1}, s]$$

just as in the monotone case.

Example 29. *The quantum homology ring of* $(\mathbb{C}P^n, \omega_{FS})$ *is expressed as follows:*

$$QH_*(\mathbb{C}P^n;\mathbb{K}) = \frac{\mathbb{K}[|s^{-1}, s][h]}{\langle h^{*(n+1)} = [\mathbb{C}P^n] \cdot s^{-1} \rangle}$$

where $h \in H_{2n-2}(\mathbb{C}P^n;\mathbb{K})$ *denotes the projective hyperplane class,* s *denotes the generator of the Novikov ring and* $h^{*(n+1)} := \underbrace{u * u * \cdots * u}_{n+1-times}$.

There is a canonical isomorphism called the PSS-isomorphism between Floer homology and quantum homology which will be denoted by Φ:

$$\Phi_{PSS,H;\mathbb{K}} : QH_*(M;\mathbb{K}) \xrightarrow{\sim} HF_*(H).$$

PSS-isomorphism preserves the ring structure: for $a, b \in QH_*(M;\mathbb{K})$,

$$\Phi_{PSS,H;\mathbb{K}}(a) *_{pp} \Phi_{PSS,H;\mathbb{K}}(b) = \Phi_{PSS,H;\mathbb{K}}(a * b)$$

where $*_{pp}$ denotes the pair-of-pants product.

Next, we briefly recall the definition of the Seidel element. The idea goes back to Seidel [Sei97]. For a Hamiltonian loop $\psi \in \pi_1(\mathrm{Ham}(M,\omega))$, we can define a Hamiltonian fiber bundle

$$(M,\omega) \hookrightarrow (M_\psi, \Omega_\psi) \to (S^2, \omega_{area})$$

where, unit disks $D^2_j,\ j=1,2$,

$$M_\psi := (D^2_1 \times M) \amalg (D^2_2 \times M)/\sim,$$

$$(z_1,x) \sim (z_2,y) \Longleftrightarrow z_1 = z_2 = e^{2\pi i t}, y = \psi^t(x).$$

The form Ω_ψ is a family of symplectic form on $TM^{vert}_\psi = \mathrm{Ker}(d\pi)$ parametrized by points of S^2. We fix almost complex structures j on S^2 and J on M_ψ such that $d\pi$ is pseudo-holomorphic i.e. $j \circ d\pi = d\pi \circ J$ and for every $z \in S^2$, $J|_{\pi^{-1}(z)}$ defines a Ω_ψ-compatible almost complex structure on M_ψ. For a section class $\sigma \in \pi_2(M_\psi)$, we denote the set of (j,J)-pseudo-holomorphic spheres in the class σ by $SecCl(j,J,\sigma)$. The image of $SecCl(j,J,\sigma)$ by the evaluation map $ev : S^2 \to M$ at $z_0 \in S^2$ defines a homology class $[ev_{z_0}(SecCl(j,J,\sigma))]$ of M. We thus define the Seidel element $\mathbb{S}_{\psi,\sigma} \in QH_*(M;\mathbb{K})$ by

$$\mathbb{S}_{\psi,\sigma} := \sum_{A \in \Gamma} [ev_{z_0}(SecCl(j,J,\sigma + A))] \otimes e^A.$$

2.7 Hamiltonian spectral invariants

2.7.1 Homology version

Let $i^\tau : CF^\tau_*(H) \to CF_*(H)$ be the natural inclusion map and denote by $i^\tau_* : HF^\tau_*(H) \to HF_*(H)$ the induced map on homology. For a quantum homology class $a \in QH_*(M;\mathbb{K})$, define the spectral invariant by

$$c(H,a) = c(H,a;\mathbb{K}) := \inf\{\tau \in \mathbb{R} : \Phi_{PSS,H;\mathbb{K}}(a) \in \mathrm{Im}(i^\tau_*)\}.$$

Spectral invariants were introduced by Viterbo [Vit92] in terms of generating functions and later their counterparts in Floer theory were studied by Schwarz for aspherical symplectic manifolds [Sch00] and Oh for closed symplectic manifolds [Oh05]. We list some basic properties of spectral invariants.

Proposition 30. *Spectral invariants satisfy the following properties where H, G are Hamiltonians:*

1. *For any $a \in QH_*(M;\mathbb{K})\backslash\{0\}$,*

$$\mathcal{E}^-(H-G) \leqslant c(H,a) - c(G,a) \leqslant \mathcal{E}^+(H-G)$$

where

$$\mathcal{E}^-(H) := \int_{t=0}^1 \inf_x H_t(x)dt,\ \mathcal{E}^+(H) := \int_{t=0}^1 \sup_x H_t(x)dt,$$
$$\mathcal{E}(H) := \mathcal{E}^+(H) - \mathcal{E}^-(H).$$

2. *For any $a \in QH_*(M;\mathbb{K})\backslash\{0\}$,*

$$c(H,a) \in \mathrm{Spec}(H)$$

- *for any Hamiltonian H when (M,ω) is rational.*
- *for any non-degenerate Hamiltonian H when (M,ω) is a general closed symplectic manifold.*

3. *For any $a, b \in QH_*(M;\mathbb{K})\backslash\{0\}$,*

$$c(H\#G, a*b) \leqslant c(H,a) + c(G,b).$$

4. *Let U be a non-empty subset of M.*

$$c(H,[M]) \leqslant e(\mathrm{Supp}(H)) := \inf\{\mathcal{E}(G) : \phi_G(\mathrm{Supp}(H)) \cap \mathrm{Supp}(H) = \emptyset\}.$$

5. *Let $f : M \to \mathbb{R}$ be an autonomous Hamiltonian and $a \in H_*(M;\mathbb{K})$. For a sufficiently small $\varepsilon > 0$, we have*

$$c(\varepsilon f, a) = c_{LS}(\varepsilon f, a) = \varepsilon \cdot c_{LS}(f,a)$$

where $c_{LS}(f,a)$ is the topological quantity defined by

$$c_{LS}(f,a) := \inf\{\tau : a \in \mathrm{Im}(H_*(\{f \leqslant \tau\}) \to H_*(M))\}.$$

6. *For $\psi \in \pi_1(\mathrm{Ham}(M,\omega))$, a section class σ of the Hamiltonian fiber bundle $M_\psi \to S^2$, and $a \in QH_*(M;\mathbb{K})\backslash\{0\}$ we have*

$$c(\psi^* H, a) = c(H, \mathcal{S}_{\psi,\sigma} * a) + const(\psi,\sigma)$$

where

$$(\psi^* H)_t := (H_t - K_t)\circ\psi^t,\ \psi^t := \phi_K^t,\ \phi_{\psi^* H}^t = (\psi^t)^{-1}\circ\phi_H^t$$

and $const(\psi,\sigma)$ denotes a constant depending on σ and K.

7. *For any $\psi \in \mathrm{Symp}_0(M,\omega)$ and $a \in QH_*(M;\mathbb{K})\backslash\{0\}$,*

$$c(H\circ\psi, a) = c(H,a).$$

Remark 31. *1. For a set A, $e(A) := \inf\{\mathcal{E}(G) : \phi_G(A)\cap A = \emptyset\}$ is called the displacement energy of A.*

2. Strictly speaking, spectral invariants $c(H,\cdot)$ can be defined only if H is non-degenerate since they are defined via Floer homology of H. However, by Proposition 30 (1), one can define $c(H,\cdot)$ for a degenerate Hamiltonian H by considering an approximation of H by non-degenerate Hamiltonians.

The spectral norm of H is defined by

$$\gamma(H) := c(H,[M]) + c(\overline{H},[M])$$

where $[M]$ denotes the fundamental class. We also define a spectral norm for Hamiltonian diffeomorphisms by

$$\gamma : \mathrm{Ham}(M,\omega) \to \mathbb{R}$$
$$\gamma(\phi) := \inf_{\phi_H = \phi} \gamma(H).$$

Remark 32. *In this paper, we work with a fixed ground field $\mathbb{K}$ of zero characteristic but some results hold for other ground rings. More precisely, Theorem 37 and 39 hold for spectral norms respectively with any ground field $\mathbb{K}$ and with any ground ring R that is commutative and unital e.g. $\mathbb{Z}$. In fact, Usher proved in [Ush08] that whenever one can define a Floer chain complex with a ground ring that is Noetherian, spectral invariants can be defined as above and satisfy properties listed in Proposition 30. For weakly-monotone symplectic manifolds, one can define Floer chain complexes with any ground field $\mathbb{K}$ [MS04]. Moreover, for monotone symplectic manifolds, one can define Floer chain complexes with any ground ring R that is commutative and unital [LZ18], [Zap15]. For general closed symplectic manifolds where one needs to use virtual cycle techniques in order to build Floer chain complexes [FO99], [FOOO09], [LT98], the ground field $\mathbb{K}$ should have zero characteristic.*

2.7.2 Cohomology version

In this subsection, we review spectral invariants defined for a quantum cohomology class. Everything is essentially the same as the previous section and the aim of this section is simply to avoid confusion of the notation.

For a non-degenerate Hamiltonian H, let

$$i_*^\tau : CF_*^\tau(H) \hookrightarrow CF_*(H)$$

be the natural inclusion map and denote its map induced on homology by

$$i_*^\tau : HF_*^\tau(H) \to HF_*(H).$$

For a quantum cohomology class $a \in QH^*(M;\mathbb{C})\backslash\{0\}$, define its spectral invariant by

$$\rho(H,a) := \inf\{\tau \in \mathbb{R} : PSS_H \circ \mathrm{PD}(a) \in \mathrm{Im}(i_*^\tau)\}.$$

Spectral invariants are invariant under homotopy rel. end points i.e. if $t \mapsto \phi_H^t$ and $t \mapsto \phi_G^t$ are homotopic paths in $\mathrm{Ham}(M,\omega)$ where H and G are both mean-normalized Hamiltonians, then $\rho(H,\cdot) = \rho(G,\cdot)$. Thus, we can see spectral invariants as follows:

$$\rho : \widetilde{\mathrm{Ham}}(M,\omega) \times QH^*(M) \to \mathbb{R},$$
$$\rho(\tilde{\phi}, a) := \rho(H,a)$$

for any mean-normalized H such that the Hamiltonian path $t \mapsto \phi_H^t$ represents the homotopy class $\tilde{\phi}$.

The obvious relation to the spectral invariants defined in the previous section for quantum homology classes is the following: For $a \in QH_*(M;\mathbb{C})\backslash\{0\}$ and $wt\phi \in \widetilde{\mathrm{Ham}}(M,\omega)$, we have

$$c(\widetilde{\phi}, a) = \rho(\widetilde{\phi}, \mathrm{PD}(a)).$$

We list further properties of spectral invariants.

Proposition 33. *Spectral invariants satisfy the following properties where H, G are Hamiltonians:*

1. *For any $a \in QH^*(M;\mathbb{C})\backslash\{0\}$,*

$$\mathcal{E}^-(H-G) \leqslant \rho(H,a) - \rho(G,a) \leqslant \mathcal{E}^+(H-G)$$

where

- $\mathcal{E}^-(H) := \int_0^1 \inf_{x\in M} H_t(x)dt,$
- $\mathcal{E}^+(H) := \int_0^1 \sup_{x\in M} H_t(x)dt,$
- $\mathcal{E}(H) := \mathcal{E}^+(H) - \mathcal{E}^-(H) = \int_0^1 \{\sup_{x\in M} H_t(x) - \inf_{x\in M} H_t(x)\}dt$

2. *If H is non-degenerate, then for any $a \in QH^*(M;\mathbb{C})\backslash\{0\}$,*
$$\rho(H,a) \in \mathrm{Spec}(H).$$
Moreover, if $a \in QH^{\deg(a)}(M;\mathbb{C})$, then there exists $\tilde{z} \in CF_{2n-\deg(a)}(H)$ such that
$$\rho(H,a) = \mathcal{A}_H(\tilde{z}).$$

3. *For any $a \in QH^*(M;\mathbb{C})\backslash\{0\}$,*
$$\rho(0,a) = \nu(\mathrm{PD}(a))$$
where 0 is the zero-function.

4. *For any $a, b \in QH^*(M;\mathbb{C})\backslash\{0\}$,*
$$\rho(H\#G, a*b) \leqslant \rho(H,a) + \rho(G,b)$$
where
$$(H\#G)(t,x) := H(t,x) + G(t,(\phi_H^t)^{-1}(x))$$
and satisfies $\phi^t_{H\#G} = \phi^t_H\phi^t_G$.

Remark 34. *A priori spectral invariants $\rho(H,\ \cdot\ )$ can be defined only if H is non-degenerate as they are defined via Floer homology of H. However, by the continuity property i.e. Proposition 30 (1), one can define $\rho(H,\ \cdot\ )$ for any $H \in C^0(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R})$ by considering an approximation of H with non-degenerate Hamiltonians.*

The spectral norm γ for Hamiltonians is defined as follows:
$$\gamma : C^\infty(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R}) \to \mathbb{R}_{\geqslant 0}$$
$$\gamma(H) := \rho(H,1) + \rho(\overline{H},1)$$
where $1 \in QH^0(M;\mathbb{C})$ denotes the identity element of $QH^*(M;\mathbb{C})$. We can see the spectral norm as a function on $\widetilde{\mathrm{Ham}}(M,\omega)$ as well:
$$\gamma : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}_{\geqslant 0}$$
$$\gamma(\widetilde{\phi}) := \rho(\widetilde{\phi},1) + \rho(\widetilde{\phi}^{-1},1).$$
We also define the spectral norm for Hamiltonian diffeomorphisms by
$$\gamma : \mathrm{Ham}(M,\omega) \to \mathbb{R}_{\geqslant 0}$$
$$\gamma(\phi) := \inf_{\phi_H=\phi} \gamma(H).$$

Spectral invariants for Floer homology and quantum cohomology with Λ-coefficients were defined in a similar fashion in [FOOO19] and it was proven to satisfy analogous properties listed in Proposition 30. We refer to [FOOO19] for details.

Chapter 3

Comparing C^0-distance and spectral distance

3.1 Abstract of the chapter

The purpose of this paper is to study the relation between the C^0-topology and the topology induced by the spectral norm on the group of Hamiltonian diffeomorphisms of a closed symplectic manifold. Following the approach of Buhovsky-Humilière-Seyfaddini, we prove the C^0-continuity of the spectral norm for complex projective spaces and negative monotone symplectic manifolds. The case of complex projective spaces provides an alternative approach to the C^0-continuity of the spectral norm proven by Shelukhin. We also prove a partial C^0-continuity of the spectral norm for rational symplectic manifolds. Some applications such as the Arnold conjecture in the context of C^0-symplectic topology are also discussed. All the materials in this section are contained in [Kaw21].

3.2 Introduction

The study of topological properties of the group of Hamiltonian diffeomorphisms of a symplectic manifold has been one of the central topics in symplectic topology. The group of Hamiltonian diffeomorphisms is known to carry different metrics such as the Hofer metric, the spectral metric and the C^0-metric and their relations have been studied extensively. This paper studies the relation between the C^0-topology and the topology induced by the spectral metric. More precisely, we study the C^0-continuity of the spectral norm which has been already verified for certain cases: for $\mathbb{R}^{2n}$ by Viterbo [Vit92], for closed surfaces by Seyfaddini [Sey13a], for symplectically aspherical manifolds by Buhovsky-Humilière-Seyfaddini [BHS21] and for complex projective spaces by Shelukhin [Sh18]. In this paper, we push the method developed by Buhovsky-Humilière-Seyfaddini [BHS21] forward to the symplectically non-aspherical setting and confirm the C^0-continuity of the spectral norm for negative monotone symplectic manifolds. We also obtain a partial C^0-continuity of the spectral norm for rational symplectic manifolds and an alternative proof of the C^0-continuity of the spectral norm for complex projective spaces.

3.2.1 Set-up

Throughout this paper, (M, ω) will denote a closed symplectic manifold. A symplectic manifold (M, ω) is called

- rational if $\langle \omega, \pi_2(M) \rangle = \lambda_0 \mathbb{Z}$ for some constant $\lambda_0 > 0$. We refer to the constant λ_0 as the rationality constant.
- monotone (resp. negative monotone) if $\omega|_{\pi_2(M)} = \lambda \cdot c_1|_{\pi_2(M)}$ for some positive (resp. negative) constant λ where $c_1 := c_1(TM)$ denotes the first Chern class of TM. We refer to the constant λ as the monotonicity constant.
- symplectically aspherical when $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$.

The positive generator of $\langle c_1, \pi_2(M) \rangle \subset \mathbb{Z}$ is called the minimal Chern number N_M i.e.

$$\langle c_1, \pi_2(M) \rangle = N_M \mathbb{Z}, \ N_M > 0.$$

Example 35.
- *The complex projective space equipped with the standard Fubini-Study form $(\mathbb{C}P^n, \omega_{FS})$ is monotone and its minimal Chern number $N_{\mathbb{C}P^n}$ is $n+1$.*
- *The degree k Fermat hypersurfaces of $\mathbb{C}P^{n+1}$*

$$M := \{(z_0 : z_1 : \cdots : z_n) \in \mathbb{C}P^{n+1} : z_0^k + z_1^k + z_2^k + \cdots + z_n^k = 0\}$$

is negative monotone for $k > n+1$. The minimal Chern number N_M is $|k - (n+2)|$ if $k \neq n+2$ and $+\infty$ otherwise.

A Hamiltonian H on M is a smooth time dependent function $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$. We define its Hamiltonian vector field X_{H_t} by $-dH_t = \omega(X_{H_t}, \cdot)$. The Hamiltonian flow of H, denoted by $(\phi_H^t)_{t \in \mathbb{R}}$, is by definition the flow of X_H. Its time-one map ϕ_H^1 is called the Hamiltonian diffeomorphism of H and will be denoted by ϕ_H. The set of Hamiltonian diffeomorphisms and its universal cover will be denoted respectively by $\mathrm{Ham}(M, \omega)$ and $\widetilde{\mathrm{Ham}}(M, \omega)$.

3.2.2 C^0-topology

We define the C^0-metric by

$$\phi, \psi \in \mathrm{Ham}(M, \omega), \ d_{C^0}(\phi, \psi) := \max_{x \in M} d(\phi(x), \psi(x))$$

where d denotes the distance on M induced by the Riemannian metric on M. Note that the topology induced by the C^0-distance is independent of the choice of a Riemannian metric. We denote the C^0-closure of $\mathrm{Ham}(M, \omega)$ in the group of homeomorphisms of M by $\overline{\mathrm{Ham}}(M, \omega)$. Their elements are called the Hamiltonian homeomorphisms. Hamiltonian homeomorphisms are central objects in C^0-symplectic topology.

3.2.3 Spectral norms

We roughly outline the notion of the spectral norm. For precise definitions, we refer to Section 2.7. First of all, a Hamiltonian $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})$ is called non-degenerate if for each $x \in \mathrm{Fix}(\phi_H)$, the set of eigenvalues of $d\phi_H(x) : T_xM \to T_xM$

does not include 1. For a non-degenerate Hamiltonian H and a fixed ground field $\mathbb{K}$ (see Remark 32 for more information about the choice of a ground field), one can define the Floer homology group $HF(H) = HF(H;\mathbb{K})$ as well as their filtration with respect to the action functional which will be denoted by $\{HF^\tau(H)\}_{\tau\in\mathbb{R}}$. For each $\tau \in \mathbb{R}$, we denote the natural map induced by the inclusion map of the chain complex by i_*^τ:

$$i_*^\tau : HF^\tau(H) \to HF(H).$$

The quantum homology ring is defined by

$$QH_*(M;\mathbb{K}) := H_*(M;\mathbb{K}) \otimes_{\mathbb{K}} \Lambda$$

where

$$\Gamma := \pi_2(M)/\mathrm{Ker}(\omega) \cap \mathrm{Ker}(c_1),$$

$$\Lambda := \{\sum_{A\in\Gamma} a_A \otimes e^A : a_A \in \mathbb{K}, (\forall \tau \in \mathbb{R}, \#\{a_A \neq 0 : \omega(A) < \tau\} < +\infty)\}.$$

The ring structure of $QH_*(M;\mathbb{K})$ is given by the quantum product $*$: for its definition, see Section 2.6). Floer homology group is ring isomorphic to the quantum homology ring $QH_*(M;\mathbb{K})$ by the PSS-isomorphism

$$\Phi_{PSS,H;\mathbb{K}} : QH_*(M;\mathbb{K}) \to HF(H).$$

For a Hamiltonian H and $a \in QH_*(M;\mathbb{K})\backslash\{0\}$, the spectral invariant of H and a is defined by

$$c(H,a) := \inf\{\tau : \Phi_{PSS,H;\mathbb{K}}(a) \in \mathrm{Im}(i_*^\tau)\}.$$

The spectral norm of a Hamiltonian H is defined by

$$\gamma(H) := c(H,[M]) + c(\overline{H},[M])$$

where $\overline{H}(t,x) := -H(t,\phi_H^t(x))$ which is a Hamiltonian that generates the Hamiltonian flow

$$t \mapsto (\phi_H^t)^{-1}.$$

Since γ is invariant under homotopy i.e. if $\phi_H^t \sim \phi_G^t$ rel. endpoints, then $\gamma(H) = \gamma(G)$, it can be seen as a map defined on the universal cover of $\mathrm{Ham}(M,\omega)$, namely

$$\gamma : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}.$$

We define spectral norms for Hamiltonian diffeomorphisms by

$$\gamma : \mathrm{Ham}(M,\omega) \to \mathbb{R},$$

$$\gamma(\phi) := \inf_{\phi=\phi_H} \gamma(H).$$

3.2.4 Main results

Throughout the paper $\lambda_0 > 0$ denotes the rationality constant i.e. $\langle\omega, \pi_2(M)\rangle = \lambda_0\mathbb{Z}$. We first state our result for rational symplectic manifolds.

Theorem 36. *Let (M,ω) be a rational symplectic manifold. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_{C^0}(\mathrm{id}, \phi_H) < \delta$, then*

$$|\gamma(H) - l \cdot \lambda_0| < \varepsilon$$

for some integer $l \in \mathbb{Z}$ depending on H.

Theorem 36 gives us the candidates of the value of the spectral norm of a C^0-small ϕ_H. When the values of spectral norms are bounded by a number strictly smaller than λ_0, then Theorem 36 implies the C^0-continuity. The complex projective space $\mathbb{C}P^n$ meets this condition.

Theorem 37. *Let $(\mathbb{C}P^n, \omega_{FS})$ be the complex projective space equipped with the Fubini-Study form.*

1. *For any $\phi \in \mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})$,*

$$\gamma(\phi) \leqslant \frac{n}{n+1} \cdot \lambda_0$$

where λ_0 denotes the rationality constant.

2. *The spectral norm is C^0-continuous i.e.*

$$\gamma : (\mathrm{Ham}(\mathbb{C}P^n, \omega_{FS}), d_{C^0}) \to \mathbb{R}$$

is continuous. Moreover, γ extends continuously to $\overline{\mathrm{Ham}}(\mathbb{C}P^n, \omega_{FS})$.

Remark 38. 1. *Theorem 37 is already proven in other papers: (1) appears as Theorem G (2) in [KiSh18] and (2) appears as Theorem C in [Sh18]. Shelukhin's argument in [Sh18], which is different from ours, is based on barcode techniques and is specific for $\mathbb{C}P^n$.*

2. *We will prove an a priori more general result in Section 3.4.3.*

For general monotone symplectic manifolds, instead of the C^0-continuity, we only obtain the following C^0-control of the spectral norm.

Theorem 39. *Let (M,ω) be a monotone symplectic manifold.*

1. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_{C^0}(\mathrm{id}, \phi_H) < \delta$, then*

$$\gamma(H) < \frac{\dim(M)}{N_M} \cdot \lambda_0 + \varepsilon.$$

2. *If $N_M > \dim(M)$, then the spectral norm is C^0-continuous i.e.*

$$\gamma : (\mathrm{Ham}(M, \omega), d_{C^0}) \to \mathbb{R}$$

is continuous. Moreover, γ extends continuously to $\overline{\mathrm{Ham}}(M, \omega)$.

Remark 40. 1. *The author does not know any example satisfying the assumptions in Theorem 39 (2). Note that Theorem 39 (2) follows immediately from Theorem 36 and Theorem 39 (1).*

2. *Theorem 39 applies to spectral norms of any ground ring R (i.e. a commutative ring with unit). See Remark 32 for a comment on the choice of ground rings/fields.*

We now consider the case of negative monotone symplectic manifolds.

Theorem 41. *Let (M,ω) be a negative monotone symplectic manifold.*

1. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_{C^0}(\mathrm{id}, \phi_H) < \delta$, then*

$$\gamma(H) < \varepsilon.$$

In particular, if $\phi_H = \phi_G$ for $H, G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})$, then

$$\gamma(H) = \gamma(G)$$

i.e. $\gamma : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ descends to

$$\gamma : \mathrm{Ham}(M,\omega) \to \mathbb{R}.$$

2. *The spectral norm is C^0-continuous i.e.*

$$\gamma : (\mathrm{Ham}(M,\omega), d_{C^0}) \to \mathbb{R}$$

is continuous. Moreover, γ extends continuously to $\overline{\mathrm{Ham}}(M,\omega)$.

Remark 42. *The independence of the spectral norm of the choice of Hamiltonian follows also from Lemma 3.2.(iv) in [McD10].*

3.2.5 Application 1: C^0-continuity of barcodes

Barcodes are roughly speaking finite sets of intervals which are bounded from below but can be unbounded from above. The set of barcodes carries a metric called the bottleneck distance denoted by d_{bot}. Barcodes have been a common tool in topological data analysis. Polterovich-Shelukhin brought barcodes into symplectic topology in [PS16] where they defined barcodes of (non-degenerate) Hamiltonian diffeomorphisms on symplectically aspherical manifolds and found applications to Hofer geometry. Later, as we will explain in Section 3.3.2, the definition of barcodes was extended to Hamiltonian diffeomorphisms on (negative) monotone symplectic manifolds [LSV18], [PSS17] after considering a completion of the set of barcodes with respect to the bottleneck distance which we will denote by $\widehat{Barcodes}$. An estimate of the bottleneck distance due to Kislev–Shelukhin [KiSh18] (see the inequality **??** in Section 3.3.2) combined with the C^0-continuity of the spectral norm implies the C^0-continuity of barcodes for negative monotone symplectic manifolds.

Corollary 43. *Let (M,ω) be a negative monotone symplectic manifold. The barcode map is C^0-continuous i.e.*

$$B : (\mathrm{Ham}(M,\omega), d_{C^0}) \to (\widehat{Barcodes}, d_{bot})$$

is continuous. Moreover, B extends continuously to $\overline{\mathrm{Ham}}(M,\omega)$.

Remark 44. *Of course, Theorem 37 (2) directly implies the C^0-continuity of barcodes in the case of $(\mathbb{C}P^n, \omega_{FS})$. This is proven by Shelukhin in Corollary 6 in [Sh18].*

3.2.6 Application 2: The C^0-Arnold conjecture

The Arnold conjecture has been historically one of the central topics in symplectic geometry.

Conjecture 45. (*The Arnold conjecture*)
Let (M^{2n}, ω) be a closed symplectic manifold.

1. *For a non-degenerate $\phi \in \mathrm{Ham}(M, \omega)$,*

$$\#\mathrm{Fix}(\phi) \geqslant \sum_j \dim_{\mathbb{C}} H_j(M; \mathbb{C}).$$

2. *For $\phi \in \mathrm{Ham}(M, \omega)$,*

$$\#\mathrm{Fix}(\phi) \geqslant cl(M)$$

where

$$cl(M) := \#\max\{k+1 : \exists a_1, a_2, \cdots, a_k \in H_{*<2n}(M) \text{ s.t. } a_1 \cap a_2 \cap \cdots \cap a_k \neq 0\}$$

and $\cap$ denotes the intersection product.

Since the advent of Floer homology, there has been a huge progress in the two versions of the Arnold conjecture: (1) is now completely settled [FO99], [LT98] and (2) has been confirmed for symplectically aspherical manifolds [Fl89], $\mathbb{C}P^n$ [For85] [ForW85] and negative monotone symplectic manifolds with $N_M \geqslant n$ [LO94].

It caught attention whether or not the Arnold conjecture is C^0-robust i.e. if Hamiltonian homeomorphisms satisfy similar properties. For closed surfaces, this question was answered in the positive by Matsumoto [**?**]. However, Buhovsky-Humilière-Seyfaddini [BHS18] discovered that in higher dimension, this turns out not to be the case.

Theorem 46. (*[BHS18]*)
Let (M, ω) be any closed symplectic manifold of dimension $\geqslant 4$. There exists a Hamiltonian homeomorphism $\phi \in \overline{\mathrm{Ham}}(M, \omega)$ such that

$$\#\mathrm{Fix}(\phi) = 1.$$

In their subsequent paper [BHS21], Buhovsky-Humilière-Seyfaddini have reformulated the Arnold conjecture in a way that is more suited to study the rigidity of Hamiltonian homeomorphisms when the ambient manifold is symplectically aspherical. We will follow their idea to obtain similar results for symplectic manifolds that are not symplectically aspherical by using the quantum product $*$ of $QH_*(M; \mathbb{K})$ (for its definition, see Section 2.6).

Definition 47. *Let (M^{2n}, ω) be a symplectic manifold. Let $a, b \in H_*(M; \mathbb{K}) \backslash \{0\}$. For a Hamiltonian H, define*

$$\sigma_{a,a*b}(H) := c(H, a) - c(H, a * b)$$

and for a Hamiltonian diffeomorphism ϕ, define

$$\sigma_{a,a*b}(\phi) := \inf_{\phi_H = \phi} \sigma_{a,a*b}(H).$$

For (M,ω) for which γ is C^0-continuous (e.g. negative monotone symplectic manifolds and $(\mathbb{C}P^n,\omega_{FS})$), $\sigma_{a,a*b}$ turns out to be C^0-continuous and it extends continuously to $\overline{\mathrm{Ham}}(M,\omega)$: see Section 3.5.2 for details.

Theorem 48. *Let (M^{2n},ω) be either a negative monotone symplectic manifold or $(\mathbb{C}P^n,\omega_{FS})$. For $\phi\in\overline{\mathrm{Ham}}(M,\omega)$, if there exist homology classes $a,b\in H_*(M;\mathbb{K})\backslash\{0\}, b\neq[M]$ such that*

$$\sigma_{a,a*b}(\phi)=0,$$

then $\mathrm{Fix}(\phi)$ is homologically non-trivial, hence it is an infinite set.

Remark 49. *1. Recall that, a subset $A\subset M$ is homologically non-trivial if for every open neighborhood U of A the map $i_*: H_j(U;\mathbb{K})\to H_j(M;\mathbb{K})$, induced by the inclusion $i: U\to M$, is non-trivial for some $j>0$. Homologically non-trivial sets are infinite sets.*

2. In [How12], Howard considers the smooth version of this statement.

3.2.7 Application 3: The displaced disks problem

A topological group G is a Rokhlin group if it possesses a dense conjugacy class i.e. for some $\phi\in G$, $Conj(\phi):=\{\psi^{-1}\phi\psi:\psi\in G\}$ is dense. Béguin-Crovisier-Le Roux formulated the following question so-called the "displaced disks problem", in order to study whether or not the group of area-preserving homeomorphisms on a sphere is a Rokhlin group.

Question 50. (*Béguin-Crovisier-Le Roux*)
Let

$$\mathcal{G}_r:=\{\phi\in\overline{\mathrm{Ham}}(M,\omega):\phi(f(B_r))\cap f(B_r)=\emptyset\}$$

for any $r>0$ where $f: B_r\to(M,\omega)$ is a symplectic embedding. Does the C^0-closure of $\mathcal{G}_r$ contain the identity element for some $r>0$?

This original question which was for $(M,\omega)=(S^2,\omega_{area})$ was solved by Seyfaddini in [?] as a consequence of his earlier result on C^0-continuity of spectral norms for closed surfaces [?]. Other cases, also deduced by C^0-continuity of spectral norms, has also been considered: [BHS21] deals with symplectically aspherical manifolds and [Sh18] treats $\mathbb{C}P^n$. Here we add the case of negative monotone symplectic manifolds.

Theorem 51. *Let (M,ω) be a negative monotone symplectic manifold. For any $r>0$, there exists $\delta>0$ such that if $\phi\in\overline{\mathrm{Ham}}(M,\omega)$ displaces a symplectically embedded ball of radius r, then $d_{C^0}(\mathrm{id},\phi)>\delta$.*

We obtain the following as a direct consequence.

Corollary 52. *Let (M,ω) be a negative monotone symplectic manifold. The group $\overline{\mathrm{Ham}}(M,\omega)$ seen as a topological group with respect to the C^0-topology is not a Rokhlin group.*

Remark 53. *The case of $(\mathbb{C}P^n,\omega_{FS})$ was considered by Shelukhin in [Sh18].*

3.3 Additional preliminaries

In this section, we provide some results that are relevant for the proofs of the results in this chapter but were not covered in Section 4.4. The results in this section will not be used in other chapters.

3.3.1 Some properties of capped orbits

The following elementary properties are often used to calculate the action.

Proposition 54. *Let (M,ω) be a symplectic manifold. Assume the Hamiltonian paths generated by H and G are homotopic rel. end points*
i.e. there exists $W : [0,1]\times[0,1] \to \mathrm{Ham}(M,\omega)$ such that

1. $W(0,t) = \phi_H^t,\ u(1,t) = \phi_G^t$.
2. $W(s,0) = \mathrm{id},\ u(s,1) = \phi_H = \phi_G$.

Let $x \in \mathrm{Fix}(\phi_H) = \mathrm{Fix}(\phi_G)$ and w be a capping of the orbit $\phi_H^t(x)$. Then the action of the capped orbit $[\phi_G^t(x), w']$ where $w' := w\#W$ (W glued to x along $\phi_H^t(x)$) coincides with the action of $[\phi_H^t(x), w]$:

$$\mathcal{A}_H([\phi_H^t(x), w]) = \mathcal{A}_G([\phi_G^t(x), w'])$$

Proposition 55. *Let (M,ω) be a symplectic manifold.*

1. *For any Hamiltonian $H \in C^\infty(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R})$,*

$$\overline{H}(t,x) := -H(t,\phi_H^t(x))$$

generates the Hamiltonian flow

$$t \mapsto (\phi_H^t)^{-1}$$

and has the time-1 map ϕ_H^{-1}.

2. *For any Hamiltonian $H \in C^\infty(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R})$,*

$$\tilde{H}(t,x) := -H(-t,x)$$

generates the Hamiltonian flow

$$t \mapsto \phi_H^{-t}$$

and has the time-1 map ϕ_H^{-1}.

3. *Hamiltonian paths generated by $\overline{H}$ and $\tilde{H}$ are homotopic rel. end points.*

Proposition 56. *Let (M,ω) be a symplectic manifold.*

1. *For any Hamiltonians $H, G \in C^\infty(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R})$,*

$$H\#G(t,x) := H(t,x) + G(t,(\phi_H^t)^{-1}(x))$$

generates the Hamiltonian flow

$$t \mapsto \phi_H^t \circ \phi_G^t$$

and has the time-1 map $\phi_H \circ \phi_G$.

2. *For any Hamiltonians* $H, G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})$,

$$H \wedge G(t,x) := \begin{cases} G(2t,x) \ (0 \leqslant t \leqslant 1/2) \\ H(2t-1, \phi_G(x)) \ (1/2 \leqslant t \leqslant 1) \end{cases}$$

generates the Hamiltonian flow

$$t \mapsto \begin{cases} \phi_G^{2t} \ (0 \leqslant t \leqslant 1/2) \\ \phi_H^{2t-1} \circ \phi_G \ (1/2 \leqslant t \leqslant 1) \end{cases}$$

and has the time-1 map $\phi_H \circ \phi_G$.

3. *Hamiltonian paths generated by* $H\#G$ *and* $H \wedge G$ *are homotopic rel. end points.*

The following two propositions will be used in Section 3.4.1. Proofs will be omitted as they follow from elementary arguments.

Proposition 57. *Let* (M,ω) *be a symplectic manifold,* U *a simply connected non-empty open set and* H *a Hamiltonian such that* $\phi_H(p) = p$ *for all* $p \in U$. *Take any* $x_0 \in U$ *and a capping* $w_0 : D^2 \to M$ *of the orbit* $\phi_H^t(x_0)$ *and fix them.*

For any $x \in U$, *define a capping* $w_x : D^2 \to M$ *of the orbit* $\phi_H^t(x)$ *by*

$$w_x(se^{2\pi it}) := \phi_H^t(c(s))\#w_0$$

where $c : [0,1] \to M$ *is a smooth path from* x_0 *to* x *and* $\phi_H^t(c(s))\#w_0$ *denotes the gluing of* $\phi_H^t(c(s))$ *and* w_0 *along* $\phi_H^t(x_0)$. *Then we have the following:*

1. $\mathcal{A}_H([\phi_H^t(x), w_x]) = \mathcal{A}_H([\phi_H^t(x_0), w_0])$.
2. $\mu([\phi_H^t(x), w_x]) = \mu([\phi_H^t(x_0), w_0])$.

Proposition 58. *Let* (M,ω) *be a symplectic manifold,* H *a Hamiltonian and* $[\phi_H^t(x), w]$ *any capped 1-periodic orbit of* H. *Then*

1. $\overline{w} : D^2 \to M$, $\overline{w}(se^{2\pi it}) := w(se^{2\pi i(-t)})$ *is a capping of the orbit* $\phi_H^{-t}(x)$
2. $\mu([\phi_H^t(x), w]) = -\mu([\phi_H^{-t}(x), \overline{w}])$
3. $\mathcal{A}_H([\phi_H^t(x), w]) = -\mathcal{A}_{\tilde{H}}([\phi_{\tilde{H}}^t(x), \overline{w}])$ *where* $\tilde{H}(t,x) = -H(-t,x)$.

3.3.2 Barcodes

In this subsection, we roughly explain how to define barcodes for Hamiltonian diffeomorphisms on (negative) monotone symplectic manifolds following [LSV18]. We also refer to [PSS17] and [UZ16] for constructions of barcodes in symplectic topology.

A finite barcode is a finite set of intervals

$$B = \{I_j = (a_j, b_j] : a_j \in \mathbb{R}, b_j \in \mathbb{R} \cup \{+\infty\}\}_{1 \leqslant j \leqslant N}.$$

Two finite barcodes B, B' are said to be δ-matched if, after deleting some intervals of length less than 2δ, there exists a bijective matching between the intervals of B and B' such that the endpoints of the matched intervals are less than δ of each other. The bottleneck distance of B, B' is defined as follows:

$$d_{bot}(B, B') := \inf\{\delta > 0 : B \text{ and } B' \text{ are } \delta - \text{matched}\}.$$

Barcodes of non-degenerate Hamiltonian diffeomorphisms were first defined in [PS16] for symplectic manifolds that are symplectically aspherical via filtered Floer homology. For symplectically non-aspherical manifolds, (filtered) Floer homology groups do not satisfy the "finiteness" condition and in order to overcome this issue, [PSS17] defines barcodes for non-degenerate Hamiltonian diffeomorphisms on monotone symplectic manifolds by fixing a degree. Later, in order to define barcodes of degenerate Hamiltonian diffeomorphisms on spheres (their method applies more generally to (negative) monotone symplectic manifolds), in [LSV18] Le Roux-Seyfaddini-Viterbo considered a completion of the set of finite barcodes with respect to the bottleneck distance in the following way:

Let $Barcodes$ denote the set of a collection of intervals $B = \{I_j\}_{j\in\mathbb{N}}$ such that for any $\delta > 0$ only finitely many of the intervals I_j have lengths greater than δ. The bottleneck distance d_{bot} extends to $Barcodes$. The space $(Barcodes, d_{bot})$ is indeed the completion of the space of finite barcodes. Given a barcode $B = \{I_j\}_{j\in\mathbb{N}}$ and $c \in \mathbb{R}$, define $B + c = \{I_j + c\}_{j\in\mathbb{N}}$, where $I_j + c$ is the interval obtained by adding c to the endpoints of I_j. Define an equivalence relation $\sim$ by $B \sim B'$ if $B' = B + c$ for some $c \in \mathbb{R}$. We will denote the quotient space of $Barcodes$ with the relation $\sim$ by $\widehat{Barcodes}$.

We explain briefly how to map a (possibly degenerate) Hamiltonian diffeomorphism on a (negative) monotone symplectic manifold to a barcode following [LSV18]. Given a non-degenerate Hamiltonian H and an integer $k \in \mathbb{Z}$, the filtered k-th Floer homology group $\{HF_k^\tau(H)\}_{\tau\in\mathbb{R}}$ forms a persistence module. For this filtered vector spaces, one can define a barcode in the same way as in [PS16] and we denote the barcode by $B_k(H)$. We define the barcode of H by

$$B(H) := \sqcup_k B_k(H) \in Barcodes.$$

For two Hamiltonians H, G such that $\phi_H = \phi_G$, their Floer homology groups coincide up to shifts of index and action filtration i.e. $HF_*^\tau(H) \simeq HF_{*+k_0}^{\tau+\tau_0}(G)$ for some $k_0 \in \mathbb{Z}, \tau_0 \in \mathbb{R}$. Thus $B(H) = B(G)$ and therefore, we define the barcode map B as follows:

$$B : \mathrm{Ham}(M, \omega) \to \widehat{Barcodes}.$$
$$B(\phi) := B(H)$$

for any H such that $\phi_H = \phi$.

Kislev-Shelukhin [KiSh18] proved the following inequality to estimate the bottleneck distance between barcodes of $\phi, \psi \in \mathrm{Ham}(M, \omega)$:

$$d_{bot}(B(\phi), B(\psi)) \leqslant \frac{1}{2}\gamma(\phi^{-1}\psi).$$

This implies that once we obtain the C^0-continuity of γ, the map

$$B : (\mathrm{Ham}(M, \omega), d_{C^0}) \to (\widehat{Barcodes}, d_{bot})$$

is continuous. Thus, Corollary 43 is a direct consequence of Theorem 41.

3.4 Proofs

In this section, we prove the results claimed in the introduction. We start from the case of negative monotone symplectic manifolds since the proof is based on a similar idea to the case of rational symplectic manifolds but it is simpler.

3.4.1 Proofs of Theorem 39 and 41

We prove Theorem 39 (1) and Theorem 41. It is achieved by combining the following Propositions 59 and 60.

Proposition 59. *Let (M,ω) be a monotone or negative monotone symplectic manifold, U be a simply connected open subset of M. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})$ satisfies $d_{C^0}(\mathrm{id}, \phi_H) < \delta$ and $\phi_H(x) = x$ for all $x \in U$, then*

- *when (M,ω) is monotone,*

$$\gamma(H) < \frac{\dim(M)}{N_M}\lambda_0 + \varepsilon.$$

- *when (M,ω) is negative monotone,*

$$\gamma(H) < \varepsilon.$$

Proposition 60. (*[BHS21] Lemma 4.2*)
Let (M,ω) be any closed symplectic manifold. For any $\varepsilon > 0$, there exists a non-empty open ball $B \subset M$ satisfying the following properties: its displacement energy is estimated by $e(B) < \varepsilon$ and for any $\varepsilon' > 0$, there exists $\delta' > 0$ such that if $\phi_H \in \mathrm{Ham}(M,\omega), d_{C^0}(\mathrm{id}_M, \phi_H) < \delta'$, then there exist $G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M \times M, \mathbb{R})$ such that

1. $\gamma(G) < \varepsilon$
2. $d_{C^0}(\mathrm{id}_{M\times M}, \phi_G) < \varepsilon'$
3. $(\phi_H \times \phi_H^{-1}) \circ \phi_G|_{B\times B} = \mathrm{id}_{B\times B}$

Proposition 60, proven by Buhovsky-Humilière-Seyfaddini [BHS21], claims that given a Hamiltonian diffeomorphism ϕ on M, one can always deform the Hamiltonian diffeomorphism $\phi \times \phi^{-1}$ to a Hamiltonian diffeomorphism on $M \times M$ that does not move any point on a certain open set by composing with a both C^0- and γ-small Hamiltonian diffeomorphism on $M \times M$.

We postpone the proof of Proposition 59 and first briefly review the proof of Proposition 60 due to [BHS21] as we will need some parts of the proof in the proof of Claim 67.

Proof. (of Proposition 60 by [BHS21])
Let $\varepsilon > 0$ and fix any non-empty open ball B' whose displacement energy satisfies $e(B') < \varepsilon/4$.

Claim 61. (*Claim 4.3 [BHS21]*)
There exists a Hamiltonian Q on $M \times M$ and an open ball B'' in M such that

- *$Supp(Q) \subset B' \times B'$.*
- *$\forall (x,y) \in B'' \times B'',\ \phi_Q(x,y) = (y,x)$.*
- *Denote the origin point of the ball B' by x_0. The point (x_0,x_0) is fixed by the flow of Q: $\forall t, \phi_Q^t((x_0,x_0)) = (x_0,x_0)$.*

Now, define

$$G := (0 \oplus \overline{H})\#\overline{Q}\#(0 \oplus H)\#Q. \tag{3.1}$$

This Hamiltonian G satisfies the following:

$$\begin{aligned} c(G, [M \times M]) &= c((0 \oplus \overline{H})\#\overline{Q}\#(0 \oplus H)\#Q, [M \times M]) \\ &\leqslant c((\overline{0 \oplus H})\#\overline{Q}\#(0 \oplus H), [M \times M]) + c(Q, [M \times M]) \\ &= c(\overline{Q}, [M \times M]) + c(Q, [M \times M]) \\ &\leqslant e(Supp(\overline{Q})) + e(Supp(Q)) \leqslant 2e(B' \times B') \end{aligned}$$

and as the ball B' was chosen so that $e(B') < \varepsilon/4$, we get

$$c(G, [M \times M]) < \varepsilon/2. \tag{3.2}$$

We can estimate $c(\overline{G}, [M \times M])$ in the same way and we get

$$\gamma(G) < \varepsilon.$$

Now, let B be a ball whose closure is included in B'' and make sure that the origin of B is the same as the origin of B', namely x_0. If we require ϕ_H to be C^0-close enough to id so that $\phi_H(B) \subset B''$, then for all $(x, y) \in B \times B$, we have

$$(\phi_H \times \phi_H^{-1}) \circ \phi_G(x, y) = (x, y).$$

This finishes the proof of Proposition 60. □

Now, before proving Proposition 59, we prove Theorem 39 (1) and Theorem 41.

Proof. (of Theorem 39 (1) and Theorem 41)

Note that if (M, ω) is (negative) monotone, then so is $(M \times M, \omega \oplus \omega)$. Given any $\varepsilon > 0$, we can take a ball B in M as in Proposition 60. By Proposition 60, for any $\varepsilon' > 0$, there exists $\delta' > 0$ such that if $d_{C^0}(\mathrm{id}_M, \phi) < \delta'$, then there exist $G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M \times M, \mathbb{R})$ such that

1. $\gamma(G) < \varepsilon$
2. $d_{C^0}(\mathrm{id}_{M \times M}, \phi_G) < \varepsilon'$
3. $[(\phi \times \phi^{-1}) \circ \phi_G]|_{B \times B} = \mathrm{id}_{B \times B}$

We take $\varepsilon' > 0$ small enough so that

$$d_{C^0}(\mathrm{id}_{M \times M}, (\phi \times \phi^{-1}) \circ \phi_G) < \delta$$

is satisfied where $\delta > 0$ is a positive number as in Proposition 59 which is determined by $B \times B$ and $\varepsilon > 0$. This is achievable as

$$\begin{aligned} d_{C^0}(\mathrm{id}_{M \times M}, (\phi \times \phi^{-1}) \circ \phi_G) &\leqslant d_{C^0}(\mathrm{id}_{M \times M}, \phi_G) + d_{C^0}(\phi_G, (\phi \times \phi^{-1}) \circ \phi_G) \\ &= d_{C^0}(\mathrm{id}_{M \times M}, \phi_G) + d_{C^0}(\mathrm{id}_{M \times M}, \phi \times \phi^{-1}) \leqslant \varepsilon' + 2\delta'. \end{aligned}$$

Now, take any Hamiltonian H generating ϕ: $\phi_H = \phi$. Then $(H \oplus \overline{H})\#G$ generates $(\phi \times \phi^{-1}) \circ \phi_G$ so if by Proposition 59, we have

- if (M,ω) is monotone, then

$$\gamma((H\oplus\overline{H})\#G) < \frac{\dim(M\times M)}{N_{M\times M}}\lambda_0+\varepsilon = 2\cdot\frac{\dim(M)}{N_M}\lambda_0+\varepsilon.$$

- if (M,ω) is negative monotone,

$$\gamma((H\oplus\overline{H})\#G) < \varepsilon.$$

As $\gamma(H\oplus\overline{H}) = 2\gamma(H)$ (by Theorem 5.1. in [EP09]), we have

$$2\gamma(H) = \gamma(H\oplus\overline{H}) \leqslant \gamma((H\oplus\overline{H})\#G) + \gamma(\overline{G})$$

$$= \gamma((H\oplus\overline{H})\#G) + \gamma(G) < \gamma((H\oplus\overline{H})\#G) + \varepsilon.$$

Therefore,

- if (M,ω) is monotone, then

$$2\gamma(H) < 2\cdot\frac{\dim(M)}{N_M}\lambda_0+\varepsilon+\varepsilon,$$

thus

$$\gamma(H) < \frac{\dim(M)}{N_M}\lambda_0+\varepsilon.$$

This proves Theorem 39 (1).

- if (M,ω) is negative monotone, then $2\gamma(H) < 2\varepsilon$, thus

$$\gamma(H) < \varepsilon.$$

This proves Theorem 41.

□

Proof. (of Theorem 41 (2))

Once we know that spectral norms are well-defined on $\mathrm{Ham}(M,\omega)$, the C^0-continuity at id follows directly from Theorem 41 (1). The C^0-continuity at $\phi\in\mathrm{Ham}(M,\omega)$ is a consequence of the triangle inequality: for any $\varepsilon>0$, if we take $d_{C^0}(\phi,\psi)$ small enough so that $d_{C^0}(\mathrm{id},\phi^{-1}\circ\psi)<\delta$ where δ is taken as in Theorem 41 (1). Then,

$$|\gamma(\psi)-\gamma(\phi)| \leqslant \gamma(\phi^{-1}\circ\psi) < \varepsilon.$$

By using the C^0-continuity, we can define the spectral norm for Hamiltonian homeomorphisms in the following way: for $\phi\in\overline{\mathrm{Ham}}(M,\omega)$, take a sequence $\phi_k\in\mathrm{Ham}(M,\omega)$ that C^0-converges to ϕ. Define $\gamma(\phi) := \lim_{k\to+\infty}\gamma(\phi_k)$. Note that any approximating sequence will give the same limit. This completes the proof of Theorem 41 (2). □

We now prove Proposition 59.

Proof. (of Proposition 59)

Take a Morse function $f : M \to \mathbb{R}$ whose critical points are located in U. We assume that f is C^2-small enough so that its Hamiltonian flow does not admit any non-constant periodic points and that $osc(f) := \max f - \min f < \varepsilon$. Since ϕ_f has no fixed points in $M \backslash U$, there exists $\delta > 0$ such that

$$\forall x \in M \backslash U,\ d(x, \phi_f(x)) > \delta.$$

We will now see that if ϕ_H is C^0-close enough to id, then

$$\mathrm{Crit}(f) = \mathrm{Fix}(\phi_H \circ \phi_f).$$

First, $\mathrm{Crit}(f) \subset \mathrm{Fix}(\phi_H \circ \phi_f)$ follows from $\forall x \in U, \phi_H(x) = x$. Next, we will see $\mathrm{Fix}(\phi_H \circ \phi_f) \subset \mathrm{Crit}(f)$ if ϕ_H is C^0-close enough to id. Let $x \in \mathrm{Fix}(\phi_H \circ \phi_f)$.

1. Assume $x \in U$. Then, $\phi_f(x) = \phi_H \circ \phi_f(x) = x$ and since $\mathrm{Crit}(f) = \mathrm{Fix}(\phi_f)$, we have $x \in \mathrm{Crit}(f)$.

2. Assume $x \notin U$. Then, $\phi_H(x) \notin U$ and

$$\begin{aligned} d_{C^0}(x, \phi_H \circ \phi_f(x)) &\geqslant d_{C^0}(\phi_f(x), x) - d_{C^0}(\phi_f(x), \phi_H \circ \phi_f(x)) \\ &\geqslant \delta - d_{C^0}(\mathrm{id}, \phi_H). \end{aligned}$$

   If we take ϕ_H to be C^0-close enough to id so that the last equation become positive, then $x \notin \mathrm{Fix}(\phi_H \circ \phi_f)$. Thus $x \in \mathrm{Fix}(\phi_H \circ \phi_f)$ implies $x \in U$ and $x = \phi_H \circ \phi_f(x) = \phi_f(x)$. Thus $x \in \mathrm{Crit}(f)$.

We have proven that if ϕ_H is C^0-close enough to id, then

$$\mathrm{Crit}(f) = \mathrm{Fix}(\phi_H \circ \phi_f).$$

Thus, for such ϕ_H and for any $x \in \mathrm{Crit}(f) = \mathrm{Fix}(\phi_H \circ \phi_f)$, its orbit is $\phi^t_{H\#f}(x) = \phi^t_H(x)$ and thus,

$$\mathrm{Spec}(H\#f) = \{f(x) + \mathcal{A}_H([\phi^t_H(x), w]) : x \in \mathrm{Crit}(f), [\phi^t_H(x), w] \in \mathrm{Crit}(\mathcal{A}_H)\}.$$

Take any $x_0 \in \mathrm{Crit}(f)$ and a capping $w_0 : D^2 \to M$ of the orbit $\phi^t_H(x_0)$ i.e. $w_0(e^{2\pi i t}) = \phi^t_H(x_0)$. We fix this capped orbit $[\phi^t_H(x_0), w_0]$ in the sequel.

For any $x \in \mathrm{Crit}(f)$, define a capping $w_x : D^2 \to M$ of the orbit $\phi^t_H(x)$ by

$$w_x(se^{2\pi i t}) := \phi^t_H(c(s))\#w_0$$

where $c : [0,1] \to U$ is a smooth path from x_0 to x and $\phi^t_H(c(s))\#w_0$ denotes the gluing of $\phi^t_H(c(s))$ and w_0 along $\phi^t_H(x_0)$.

Recall that $\gamma(H) = c(H, [M]) + c(\overline{H}, [M])$ and we will estimate $c(H, [M])$ and $c(\overline{H}, [M])$ separately.

By the triangle inequality,

$$c(H, [M]) \leqslant c(H\#f, [M]) + c(\overline{f}, [M]).$$

For the second term we know that

$$c(\overline{f}, [M]) = c(-f, [M]) \leqslant \varepsilon$$

as f is C^2-small and $osc(f) < \varepsilon$.

For the first term,

$$c(H\#f, \cdot) \in \mathrm{Spec}(H\#f)$$

so there exists a point $x \in \mathrm{Crit}(f)$ and a sphere $A : S^2 \to M$ such that

- $\mathcal{A}_{H\#f}([\phi_H^t(x), w_x\#A]) = c(H\#f, [M])$.
- $\mu_{CZ}([\phi_H^t(x), w_x\#A]) = \deg([M]) = 2n$.

The sphere A plays the role of correcting the capping of the capped orbit $[\phi_H^t(x), w_x]$ to achieve the appropriate capped orbit which realizes the spectral invariant $c(H\#f, [M])$.

The action and the index can be rewritten in the following way where i denotes the Morse index:

- $\mathcal{A}_{H\#f}([\phi_H^t(x), w_x\#A]) = f(x) + \mathcal{A}_H([\phi_H^t(x), w_x]) - \omega(A)$.
- $\mu_{CZ}([\phi_H^t(x), w_x\#A]) = i(x) + 2\mu([\phi_H^t(x), w_x]) - 2c_1(A)$.

Thus we get the following two equations.

$$c(H\#f, [M]) = f(x) + \mathcal{A}_H([\phi_H^t(x), w_x]) - \omega(A). \tag{3.3a}$$
$$2n = i(x) + 2\mu([\phi_H^t(x), w_x]) - 2c_1(A). \tag{3.3b}$$

In the same way, there exist a point $y \in \mathrm{Crit}(f)$ and a sphere $B : S^2 \to M$ such that

$$c(\overline{H}\#f, [M]) = f(y) + \mathcal{A}_{\overline{H}}([\phi_{\overline{H}}^t(y), \overline{w_y}]) - \omega(B). \tag{3.4a}$$
$$2n = i(y) + 2\mu([\phi_{\overline{H}}^t(y), \overline{w_y}]) - 2c_1(B). \tag{3.4b}$$

Here, the capping $\overline{w_y}$ is

$$\overline{w_y}(se^{2\pi it}) := w_y(se^{2\pi i(-t)}).$$

Thus, by adding the equations 3.3a and 3.4a, we obtain

$$\gamma(H) \leqslant 2c(-f, [M]) + c(H\#f, [M]) + c(\overline{H}\#f, [M])$$
$$= 2c(-f, [M]) + f(x) + f(y) + \mathcal{A}_H([\phi_H^t(x), w_x]) + \mathcal{A}_{\overline{H}}([\phi_{\overline{H}}^t(y), \overline{w_y}]) - \omega(A + B)$$
$$\leqslant 4\varepsilon - \omega(A + B)$$

where Proposition 57 and 58 were used in the last line. In the same way, by adding the equalities 3.3b and 3.4b, we obtain

$$4n = i(x) + i(y) + 2\mu([\phi_H^t(x), w_x]) + 2\mu([\phi_{\overline{H}}^t(y), \overline{w_y}]) - 2c_1(A + B)$$
$$= i(x) + i(y) - 2c_1(A + B).$$

Now, since $i(x), i(y)$ are Morse indices, we have

$$0 \leqslant i(x), i(y) \leqslant 2n = \dim(M)$$

and thus,

$$0 \leqslant 4n + 2c_1(A + B) \leqslant 4n.$$

Thus,

$$-2n \leqslant c_1(A + B) \leqslant 0.$$

Note that up to now, we have not used the (negative) monotonicity of (M, ω). Now,

- if (M,ω) is negative monotone, then

$$-\omega(A+B) = -\lambda \cdot c_1(A+B) \leqslant 0.$$

- if (M,ω) is monotone, then

$$-\omega(A+B) = -\lambda \cdot c_1(A+B) \leqslant 2n\lambda = \frac{2n}{N_M}\lambda_0.$$

Therefore,

- if (M,ω) is negative monotone, then

$$\gamma(H) \leqslant 4\varepsilon.$$

- if (M,ω) is monotone, then

$$\gamma(H) \leqslant \frac{2n}{N_M}\lambda_0 + 4\varepsilon.$$

This completes the proof of Proposition 59. □

3.4.2 Proof of Theorem 64

The goal of this subsection is to prove Theorem 64 which includes Theorem 36 as a special case. The argument is similar to the negative monotone case. We start by some additional definitions.

Definition 62. *Let (M,ω) be any closed symplectic manifold and $a,b \in H_*(M;\mathbb{K})\backslash\{0\}$. We define the following:*

$$\gamma_{a,b} : C^\infty(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R}) \to \mathbb{R},$$

$$\gamma_{a,b}(H) := c(H,a) + c(\overline{H},b).$$

Remark 63. *Of course, $\gamma_{[M],[M]} = \gamma$ where γ is the usual spectral norm.*

Theorem 64. *Let (M,ω) be a rational symplectic manifold and $a,b \in H_*(M;\mathbb{K})\backslash\{0\}$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_{C^0}(\mathrm{id},\phi_H) < \delta$, then*

$$|\gamma_{a,b}(H) - l\cdot\lambda_0| < \varepsilon$$

for some integer $l \in \mathbb{Z}$ depending on $a,b \in H_*(M;\mathbb{K})\backslash\{0\}$ and H.

Before proving Theorem 64, we will see the following consequence on the C^0-continuity of the spectral norm.

Corollary 65. *Let (M,ω) be a rational symplectic manifold. Assume that there exist constants $0 < \kappa < 1$ and $\delta' > 0$ such that if $\phi \in \mathrm{Ham}(M,\omega)$, $d_{C^0}(\mathrm{id},\phi) \leqslant \delta'$, then $\gamma(\phi) \leqslant \kappa\cdot\lambda_0$. Then, $\gamma : \mathrm{Ham}(M,\omega) \to \mathbb{R}$ is C^0-continuous.*

Corollary 65 will be used to obtain the C^0-continuity of the spectral norm for $\mathbb{C}P^n$ in Theorem 37.

Proof. (of Corollary 65)
It is enough to prove the continuity at id since $|\gamma(\phi) - \gamma(\psi)| \leqslant \gamma(\psi^{-1}\phi)$. For a given $\varepsilon \in (0, \frac{1}{2}(1-\kappa)\lambda_0)$, take $\delta > 0$ as in Theorem 36. Let

$$\phi \in \mathrm{Ham}(M, \omega),\ d_{C^0}(\mathrm{id}, \phi) < \min\{\delta, \delta'\}.$$

There exists a Hamiltonian H such that $\phi_H = \phi$ and

$$\gamma(H) < \gamma(\phi) + \varepsilon < \kappa \cdot \lambda_0 + \frac{1}{2}(1-\kappa)\lambda_0$$

$$= \frac{1}{2}(1+\kappa)\lambda_0 < \lambda_0 - \varepsilon.$$

Thus, by Theorem 36,

$$\gamma(H) < \varepsilon.$$

Thus,

$$\gamma(\phi) \leqslant \gamma(H) < \varepsilon.$$

This implies the continuity of γ at id and hence completes the proof of Corollary 65. □

Now, we move to the proof of Theorem 64. The following Proposition will be needed.

Proposition 66. *Let (M, ω) be a closed symplectic manifold. Fix an arbitrary point $x_0 \in M$. There exists a constant $C > 0$ satisfying the following property: For any point $x \in M$, there exists $\psi \in \mathrm{Ham}(M, \omega)$ such that*

1. $\psi(x) = x_0$
2. $\|d\psi^{-1}\| \leqslant C$

The proof is elementary and thus will be omitted.

Proof. (of Theorem 64)
The proof is similar to the proof of Theorem 41. For a given $\varepsilon > 0$, we take a ball B as in Proposition 60. We will denote the origin of the ball B by x_0. For the open set $B \times B$, consider a Morse function $F : M \times M \to \mathbb{R}$ such that

- $\mathrm{Crit}(F) \subset B \times B$.
- F is C^2-small enough so that $\mathrm{Fix}(\phi_F) = \mathrm{Crit}(F)$ and that $osc(F) := \max F - \min F < \varepsilon$.

As ϕ_F has no fixed points in $M \backslash (B \times B)$, there exists $\delta > 0$ such that for any $x \in M \times M \backslash (B \times B)$, $d(x, \phi_F(x)) > \delta$.

For any $\varepsilon' > 0$, we can take $\delta' > 0$ as in Proposition 60. By Proposition 66, for x_0, there exists a constant $C > 0$ such that for any $x \in M$, there exists $\psi \in \mathrm{Ham}(M, \omega)$ such that

- $\psi(x) = x_0$
- $\|d\psi^{-1}\| \leqslant C$

We consider ϕ_H so that $d_{C^0}(\mathrm{id}, \phi_H) < \delta'/C$. For any $x_* \in \mathrm{Fix}(\phi_H)$, we can take $\psi \in \mathrm{Ham}(M, \omega)$ such that $\psi(x_*) = x_0$ and $\|d\psi^{-1}\| \leqslant C$.

Let $H' := H \circ \psi^{-1}$. We have

$$d_{C^0}(\mathrm{id}, \phi_{H'}) = d_{C^0}(\mathrm{id}, \psi^{-1}\phi_H\psi) = d_{C^0}(\psi^{-1}, \psi^{-1}\phi_H)$$

$$\leqslant \|d\psi^{-1}\| d_{C^0}(\mathrm{id}, \phi_{H'}) \leqslant C \cdot \delta'/C = \delta'.$$

By Proposition 60, there exists $G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M \times M)$ such that

- $\gamma(G) < \varepsilon$.
- $d_{C^0}(\mathrm{id}_{M\times M}, \phi_G) < \varepsilon'$.
- $(\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G|_{B\times B} = \mathrm{id}_{B\times B}$.

In addition, we have seen in the proof of Proposition 60 that G is defined by $G = (0 \oplus \overline{H'})\#\overline{Q}\#(0 \oplus H')\#Q$ where Q is an autonomous Hamiltonian on $M \times M$ whose flow fixes the point (x_0, x_0) for all time t: $\phi_Q^t((x_0, x_0)) = (x_0, x_0)$. The spectral invariant of G was estimated as

$$c(G, [M \times M]) < \frac{1}{2}\varepsilon.$$

All these properties of G and Q will be used in the following.

We will now split the proof into four steps.

• Step 1: The aim of this step is to prove the following:

Claim 67.

$$|c(\overline{H'} \oplus H', a \otimes b) - c((\overline{H'} \oplus H')\#G\#F, a \otimes b)| < \frac{3}{2}\varepsilon.$$

Proof. By the triangle inequality, we have

$$c((\overline{H'} \oplus H')\#G\#F, a \otimes b) - c(\overline{H'} \oplus H', a \otimes b)$$

$$\leqslant c(G\#F, [M \times M]) \leqslant c(G, [M \times M]) + c(F, [M \times M]) < \frac{3}{2}\varepsilon.$$

Note that the final inequality uses,

$$c(F, [M \times M]) \leqslant \max(F) < \varepsilon$$

and the estimate

$$c(G, [M \times M]) < \frac{1}{2}\varepsilon.$$

The other side of the inequality follows from a similar estimate. □

• Step 2: The aim of this step is to prove the following:

Claim 68.

$$c((\overline{H'} \oplus H')\#G\#F, a \otimes b)$$

$$= F(x, y) + \mathcal{A}_{(\overline{H'}\oplus H')\#G}([\phi^t_{(\overline{H'}\oplus H')\#G}((x, y)), w_{x,y}]) + (\omega \oplus \omega)(A_1)$$

for some critical point (x, y) of F, some capping $w_{x,y}$ and some $A_1 \in \pi_2(M \times M)$.

Proof. As

$$d_{C^0}(\mathrm{id}, (\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G) \leqslant d_{C^0}(\mathrm{id}, \phi_G) + d_{C^0}(\phi_G, (\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G)$$
$$= d_{C^0}(\mathrm{id}, \phi_G) + d_{C^0}(\mathrm{id}, \phi_{H'}^{-1} \times \phi_{H'}) \leqslant \varepsilon' + \delta',$$

we can take $\varepsilon' > 0$ small enough so that

$$d_{C^0}(\mathrm{id}, (\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G) \leqslant \delta.$$

Therefore, as

- for all $x \notin B \times B$, $d_{C^0}(x, \phi_F(x)) > \delta$,
- $d_{C^0}(\mathrm{id}, (\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G) \leqslant \delta$,
- $(\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G|_{B\times B} = \mathrm{id}_{B\times B}$,

we have $\mathrm{Fix}((\phi_{H'}^{-1} \times \phi_{H'}) \circ \phi_G \circ \phi_F) = \mathrm{Crit}(F)$. Thus the spectral invariant $c((\overline{H'} \oplus H')\#G\#F, a \otimes b)$ can be expressed as follows:

$$c((\overline{H'} \oplus H')\#G\#F, a \otimes b)$$
$$= F(x, y) + \mathcal{A}_{(\overline{H'}\oplus H')\#G}([\phi^t_{(\overline{H'}\oplus H')\#G}((x, y)), w_{x,y}]) + (\omega \oplus \omega)(A_1)$$

where

- (x, y) is a certain critical point of F which is located in $B \times B$.
- $w_{x,y}$ denotes an arbitrary chosen capping of the orbit $\phi^t_{(\overline{H'}\oplus H')\#G}((x, y))$. We fix this capping in the sequel.
- A_1 denotes the sphere which plays the role of correcting the capping $w_{x,y}$

□

- Step 3: The aim of this step is to prove the following:

Claim 69.

$$\mathcal{A}_{(\overline{H'}\oplus H')\#G}([\phi^t_{(\overline{H'}\oplus H')\#G}((x, y)), w_{x,y}]) = (\omega \oplus \omega)(A_2)$$

for some $A_2 \in \pi_2(M \times M)$.

Proof. By Proposition 57 (2), we obtain

$$\mathcal{A}_{(\overline{H'}\oplus H')\#G}([\phi^t_{(\overline{H'}\oplus H')\#G}((x, y)), w_{x,y}]) = \mathcal{A}_{(\overline{H'}\oplus H')\#G}([\phi^t_{(\overline{H'}\oplus H')\#G}((x_0, x_0)), w_{x_0,x_0}])$$

where w_{x_0,x_0} is the capping of the orbit $\phi^t_{(\overline{H'}\oplus H')\#G}((x_0, x_0))$ corresponding to the capping $w_{x,y}$ in the sense of Proposition 57 (2). As Q is a Hamitonian which generates a time-1 map that switches the coordinate i.e. $(p, q) \mapsto (q, p)$ in $B \times B$ and satisfies $\forall t, \phi^t_Q((x_0, x_0)) = (x_0, x_0)$, we have

$$\mathcal{A}_{(\overline{H'}\oplus H')\#G}([\phi^t_{(\overline{H'}\oplus H')\#G}((x_0, x_0)), w_{x_0,x_0}])$$
$$= \int Q(\phi^t_Q(x_0, x_0))dt + \int (0 \oplus H')(t, x_0, \phi^t_{H'}(x_0))dt - \omega(\overline{\phi^t_{H'}(x_0)})+$$
$$\int \overline{Q}(\phi^t_{\overline{Q}}(x_0, x_0)dt + \int (0 \oplus \overline{H'})(t, x_0, \phi^t_{\overline{H'}}(x_0))dt - \omega(\overline{\phi^t_{\overline{H'}}(x_0)}) + (\omega \oplus \omega)(A_2)$$

where

- $\overline{\phi^t_{H'}(x_0)}$ denotes the capped orbit of $\phi^t_{H'}(x_0)$ whose capping is chosen arbitrarily.
- $\overline{\phi^t_{\overline{H'}}(x_0)}$ denotes the capped orbit of $\phi^t_{\overline{H'}}(x_0)$ whose capping is the same as the the capping of $\phi^t_{H'}(x_0)$ chosen above.
- A_2 denotes the sphere to which corrects the capping of the RHS so that it will meet the capping on the LHS.

Thus, by employing Proposition 58 (3) for $\int H'_t(\phi^t_{H'}(x_0))dt$ and $\int \overline{H'}_t(\phi^t_{\overline{H'}}(x_0))dt$, we obtain,

$$\mathcal{A}_{(\overline{H'}\oplus H')\#G}([\phi^t_{(\overline{H'}\oplus H')\#G}((x_0,x_0)), w_{x_0,x_0}]) = (\omega\oplus\omega)(A_2).$$

□

• Step 4: The aim of this step is to complete the proof.
By Step 2 and 3, we have

$$c((\overline{H'}\oplus H')\#G\#F, a\otimes b) = F(x,y) + (\omega\oplus\omega)(A_2) + (\omega\oplus\omega)(A_1)$$

$$= F(x,y) + l\cdot\lambda_0$$

for some integer $l\in\mathbb{Z}$ such that $(\omega\oplus\omega)(A_1+A_2) = l\cdot\lambda_0$ and

$$c(\overline{H'}\oplus H', a\otimes b) = \gamma_{a,b}(H') = \gamma_{a,b}(H\circ\psi) = \gamma_{a,b}(H)$$

where the last equality uses Proposition 30 (7).

By Step 1, we conclude that

$$|\gamma_{a,b}(H) - l\cdot\lambda_0| \leqslant \frac{5}{2}\varepsilon.$$

Hence we complete the proof.

□

3.4.3 Proof of Theorem 37

The aim of this section is to prove Theorem 37. We prove the following a priori more general result.

Theorem 70. *Let (M^{2n},ω) be a monotone symplectic manifold with a minimal Chern number $N_M > n$. Assume that there exist $\psi\in\pi_1(\mathrm{Ham}(M,\omega))$ and a section class σ of the Hamiltonian fibration $M_\psi\to S^2$, such that its Seidel element $\mathcal{S}_{\psi,\sigma}\in QH_*(M;\mathbb{K})$ satisfies the following:*

- *$(\mathcal{S}_{\psi,\sigma})^{*k} = a_1\cdot[pt]$ for some $a_1\in\mathbb{K}\backslash\{0\}$ and $k\in\mathbb{N}$ where $[pt]$ denotes the point class in $H_0(M;\mathbb{K})$.*
- *$(\mathcal{S}_{\psi,\sigma})^{*k'} = a_2\cdot[M]\cdot s^{-l'}$ for some $a_2\in\mathbb{K}\backslash\{0\}$ and $k',l'\in\mathbb{N}$ where $[M]$ denotes the fundamental class and s denotes the generator of the Novikov ring of (M,ω).*

Then the spectral norm satisfies the following.

1. *For any $\phi\in\mathrm{Ham}(M,\omega)$,*

$$\gamma(\phi) \leqslant \frac{n}{N_M}\cdot\lambda_0.$$

2. *The spectral norm is C^0-continuous i.e.*

$$\gamma : (\mathrm{Ham}(M,\omega), d_{C^0}) \to \mathbb{R}$$

is continuous. Moreover, γ extends continuously to $\overline{\mathrm{Ham}}(M,\omega)$.

Remark 71. 1. *Theorem 70 (1) is essentially contained in Proposition 15 in [KiSh18] where Kislev-Shelukhin considers Lagrangian spectral invariants instead of Hamiltonian ones.*

2. *So far, $(\mathbb{C}P^n, \omega_{FS})$ seems to be the only example that satisfies the assumptions in Theorem 70.*

Proof. (of Theorem 70)

Let $\phi \in \mathrm{Ham}(M,\omega)$ and take any Hamiltonian H such that $\phi_H = \phi$. Let $\psi \in \pi_1(\mathrm{Ham}(M,\omega))$ and σ be as in the statement. Denote

$$a := \mathcal{S}_{\psi,\sigma} \in QH_*(M;\mathbb{K}),\ a^{*k} := \underbrace{a * a * \cdots * a}_{k-times}.$$

By looking at the degree, we have

- $\deg(a^{*k}) = \deg([pt]) = 0$,
- $\deg(a^{*k'}) = \deg([M]\cdot s^{-l'}) = 2n - 2Nl'$,
- For any $m \in \mathbb{N}$, $\deg(a^{*m}) = m\cdot\deg(a) - (m-1)\cdot 2n$.

These equations will give us the following:

$$\frac{k'}{k} = \frac{Nl'}{n} \tag{3.5}$$

and our assumption $N > n$ implies $k' > k$. As $N_M > n$ and $\mathbb{K}$ is a field, the formula in [EP03] Section 2.7 gives us

$$c(\overline{H}, [M]) = -c(H, [pt]),$$

and by Proposition 30, we get the following.

- $\gamma(H) = c(H,[M]) - c(H,[pt]) = c(H,[M]) - c(H,a^{*k})$,
- $\gamma(\psi^*H) = c(H, \mathcal{S}_{\psi,\sigma} * [M]) - c(H, \mathcal{S}_{\psi,\sigma} * a^{*k})) = c(H,a) - c(H,a^{*(k+1)})$.
- $\gamma((\psi^2)^*H) = c(H,a^{*2}) - c(H,a^{*(k+2)})$.

  ...

- $\gamma((\psi^{k'-k})^*H) = c(H,a^{*(k'-k)}) - c(H,a^{*k'})$

  $= c(H,a^{*(k'-k)}) - c(H,[M]) + l'\lambda_0$.

- $\gamma((\psi^{k'-k+1})^*H) = c(H, a^{*(k'-k+1)}) - c(H,a) + l'\lambda_0$.

...

- $\gamma((\psi^{k'-1})^*H) = c(H, a^{*(k'-1)}) - c(H, a^{*(k-1)}) + l'\lambda_0$.

We used that for $j \in \mathbb{Z}$,

$$c(H, a^{*(j+k')}) = c(H, a^{*j}) - l'\lambda_0.$$

Adding up these k'-equations will give us the following.

$$\sum_{0 \leqslant j \leqslant k'-1} \gamma((\psi^j)^*H) = kl' \cdot \lambda_0.$$

As $\gamma(\phi) \leqslant \gamma((\psi^j)^*H)$ for all $0 \leqslant j \leqslant k'-1$,

$$k' \cdot \gamma(\phi) \leqslant kl' \cdot \lambda_0.$$

By equation 3.5, we conclude

$$\gamma(\phi) \leqslant \frac{kl'}{k'} \cdot \lambda_0 = \frac{n}{N} \cdot \lambda_0.$$

The continuity of γ is a direct consequence of Corollary 65. □

Theorem 37 is a direct consequence of Theorem 70.

Proof. (of Theorem 37)
We explain briefly that $\mathbb{C}P^n$ meets the assumptions in Theorem 70. Consider a loop of Hamiltonian diffeomorphism of $\mathbb{C}P^n$ defined by

$$\psi^t([z_0 : z_1 : \cdots : z_{n-1} : z_n]) := [z_0 : e^{2\pi i t} z_1 : e^{2\pi i t} z_2 : \cdots : e^{2\pi i t} z_{n-1} : e^{2\pi i t} z_n].$$

It is known that there exists a section class σ such that $\mathcal{S}_{\psi,\sigma} = [\mathbb{C}P^{n-1}]$ where $[\mathbb{C}P^{n-1}]$ denotes the generator of $H_{2n-2}(\mathbb{C}P^n; \mathbb{K})$. See Example 9.6.1 and Proposition 9.6.4 in [MS04]. This shows that $\mathbb{C}P^n$ satisfies the assumptions in Theorem 70. □

3.5 Proofs of applications

3.5.1 The displaced disks problem

We prove Theorem 51. We use the following energy-capacity inequality proven by Usher in [Ush10].

Proposition 72. ([Ush10])
Let $B := B(r)$ be an open ball in $(\mathbb{R}^{2n}, \omega_{std})$. If $B(r)$ is symplectically embedded to (M, ω)

$$f : B(r) \hookrightarrow (M, \omega)$$

and $\phi(f(B)) \cap f(B) = \emptyset$ for $\phi \in \mathrm{Ham}(M, \omega)$, then

$$\pi r^2 \leqslant \gamma(\phi).$$

Notice that for (M,ω) for which the spectral norm is C^0-continuous, Proposition 72 holds for Hamiltonian homeomorphisms as well.

Proof. (of Theorem 51)
By Theorem 41, we can apply Proposition 72 for Hamiltonian homeomorphisms. Let $r>0$ and take $\delta>0$ so that if $\phi\in\overline{\mathrm{Ham}}(M,\omega), \gamma(\phi)\geqslant \pi r^2$, then $d_{C^0}(\mathrm{id},\phi)>\delta$. Now, we will prove that if $\phi\in\overline{\mathrm{Ham}}(M,\omega)$ displaces an embedded ball of radius r, then $d_{C^0}(\mathrm{id},\phi)>\delta$. By Proposition 72, we have $\gamma(\phi)\geqslant\pi r^2$ and from our choice of δ, this implies $d_{C^0}(\mathrm{id},\phi)>\delta$. □

3.5.2 The C^0-Arnold conjecture

We start by looking at properties of $\sigma_{a,a*b}$ defined earlier in Section 3.2.6.

Proposition 73. *Let (M^{2n},ω) be a symplectic manifold and $a,b\in H_*(M;\mathbb{K})\backslash\{0\}$. For Hamiltonians H,G, we have the following triangle inequality:*

$$|\sigma_{a,a*b}(H)-\sigma_{a,a*b}(G)|\leqslant\gamma(\overline{H}\#G).$$

Proof.

$$\sigma_{a,a*b}(H)-\sigma_{a,a*b}(G)=c(H,a)-c(H,a*b)-(c(G,a)-c(G,a*b))$$
$$\leqslant c(\overline{G}\#H,[M])+c(\overline{H}\#G,[M])=\gamma(\overline{H}\#G).$$

By changing the role of H and G, we get $\sigma_{a,a*b}(G)-\sigma_{a,a*b}(H)\leqslant\gamma(\overline{H}\#G)$ too. This completes the proof. □

Proposition 73 allows us to define the following: Let (M^{2n},ω) be a negative monotone symplectic manifold and $a,b\in H_*(M;\mathbb{K})$.

$$\sigma_{a,a*b}:\mathrm{Ham}(M,\omega)\to\mathbb{R}$$

$$\sigma_{a,a*b}(\phi):=\sigma_{a,a*b}(H)$$

for any H such that $\phi_H=\phi$. Note that the well-definedness is due to Theorem 41. Similarly, we define the following for $\mathbb{C}P^n$: Let $h:=[\mathbb{C}P^{n-1}]$ and $l_1,l_2\in\mathbb{N}, l_1<l_2$.

$$\sigma_{h^{l_1},h^{l_2}}:\mathrm{Ham}(\mathbb{C}P^n,\omega)\to\mathbb{R}$$

$$\sigma_{h^{l_1},h^{l_2}}(\phi):=\inf_{\phi_H=\phi}\sigma_{h^{l_1},h^{l_2}}(H).$$

Corollary 74. *Let (M^{2n},ω) be either a negative monotone symplectic manifold or $(\mathbb{C}P^n,\omega_{FS})$. For $a,b\in H_*(M;\mathbb{K})$, we have the following triangle inequality: For $\phi,\psi\in\mathrm{Ham}(M,\omega)$,*

$$|\sigma_{a,a*b}(\phi)-\sigma_{a,a*b}(\psi)|\leqslant\gamma(\phi^{-1}\psi).$$

Proof. We only explain the case of $(\mathbb{C}P^n,\omega_{FS})$ since the other is simpler. By Proposition 73,

$$\sigma_{h^{l_1},h^{l_2}}(H\#G)\leqslant\sigma_{h^{l_1},h^{l_2}}(H)+\gamma(G).$$

Take an infimum on both sides as in the definition.

$$\sigma_{h^{l_1},h^{l_2}}(\phi\psi)\leqslant\inf_{\phi_H=\phi,\phi_G=\psi}\sigma_{h^{l_1},h^{l_2}}(H\#G)\leqslant\sigma_{h^{l_1},h^{l_2}}(\phi)+\gamma(\psi).$$

Since $\sigma_{h^{l_1},h^{l_2}}$ are finite,

$$\sigma_{h^{l_1},h^{l_2}}(\phi\psi) - \sigma_{h^{l_1},h^{l_2}}(\phi) \leqslant \gamma(\psi).$$

This implies the triangle inequality

$$|\sigma_{h^{l_1},h^{l_2}}(\phi) - \sigma_{h^{l_1},h^{l_2}}(\psi)| \leqslant \gamma(\phi^{-1}\psi)$$

where $\phi, \psi \in \mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})$. □

This corollary and the C^0-continuity of γ implies the C^0-continuity of $\sigma_{a,a*b}$. This allows us to define $\sigma_{a,a*b}$ for Hamiltonian homeomorphisms i.e. for a Hamiltonian homeomorphism ϕ, define $\sigma_{a,a*b}(\phi) := \lim_{n\to\infty} \sigma_{a,a*b}(\phi_n)$ where $\phi_n \in \mathrm{Ham}(M,\omega)$, $\phi_n \xrightarrow{C^0} \phi$.

We are now ready to prove Theorem 48.

Proof. (of Theorem 48)

Since the negative monotone case is simpler than the case of $(\mathbb{C}P^n, \omega_{FS})$, we only prove the latter. We assume that for $\phi \in \overline{\mathrm{Ham}}(\mathbb{C}P^n, \omega_{FS})$ and $l_1 < l_2$, we have

$$\sigma_{h^{l_1},h^{l_2}}(\phi) = 0.$$

It is enough to prove that an arbitrary open neighborhood U of $\mathrm{Fix}(\phi)$ is homologically non-trivial. Let $f : M \to \mathbb{R}$ be a sufficiently C^2-small smooth function such that $f < 0$ on $M\backslash\overline{U}$, $f|_U = 0$ and $c_{LS}(f,\cdot) = c(f,\cdot)$. (See Proposition 30 (5) for the definition of c_{LS}.)

First of all, take a sequence $\phi_j \in \mathrm{Ham}(M,\omega)$, $j \in \mathbb{N}$ such that

$$d_{C^0}(\phi, \phi_j) \leqslant 1/j.$$

The C^0-continuity of γ allows us to take a subsequence $\{j_k\}_{k\in\mathbb{N}}$ so that for each k,

$$\gamma(\phi^{-1}\phi_{j_k}) < 1/k.$$

Next, for each k, take a Hamiltonian H_k which generates ϕ_{j_k} and

$$\sigma_{h^{l_1},h^{l_2}}(H_k) \leqslant \sigma_{h^{l_1},h^{l_2}}(\phi_{j_k}) + 1/k.$$

We borrow the following claim proved in [BHS21].

Claim 75. (*Claim 5.3 in [BHS21]*) *Assume $\phi_{H_k} \xrightarrow{C^0} \phi$. For any $a \in H_*(M;\mathbb{K})\backslash\{0\}$, there exists $0 < \varepsilon_0 < 1$ and an integer k_0 such that for any $k \geqslant k_0$, we have $c(H_k\#\varepsilon_0 f, a) = c(H_k, a)$.*

From this Claim, there exist $\varepsilon_0 > 0$ and $k_0 \in \mathbb{N}$ such that if $k \geqslant k_0$, then

$$c(H_k\#\varepsilon_0 f, a) = c(H_k, a)$$

for all $a \in H_*(\mathbb{C}P^n;\mathbb{K})$. For $k \geqslant k_0$,

$$c(H_k, h^{l_2}) = c(H_k\#\varepsilon_0 f, h^{l_2}) \leqslant c(H_k, h^{l_1}) + c(\varepsilon_0 f, h^{l_2-l_1})$$

and thus,

$$-\sigma_{h^{l_1},h^{l_2}}(H_k) \leqslant c(\varepsilon_0 f, h^{l_2-l_1}) \leqslant c(f, h^{l_2-l_1}).$$

By our choices of ϕ_{j_k} and H_k, we have the following.

$$\sigma_{h^{l_1},h^{l_2}}(H_k) \leqslant \sigma_{h^{l_1},h^{l_2}}(\phi_{j_k}) + 1/k \leqslant \sigma_{h^{l_1},h^{l_2}}(\phi) + \gamma(\phi^{-1}\phi_{j_k}) + 1/k$$

$$\leqslant \sigma_{h^{l_1},h^{l_2}}(\phi) + 2/k = 2/k.$$

Thus,

$$-2/k \leqslant -\sigma_{h^{l_1},h^{l_2}}(H_k) \leqslant c(f, h^{l_2-l_1}).$$

By taking a limit $k \to +\infty$, we obtain

$$0 \leqslant c(f, h^{l_2-l_1}).$$

Thus,

$$0 \leqslant c(f, h^{l_2-l_1}) \leqslant c(f, [M]) \leqslant 0.$$

The last inequality follows from $f \leqslant 0$. Since f was taken to satisfy $c_{LS}(f, \cdot) = c(f, \cdot)$, we have

$$c_{LS}(f, h^{l_2-l_1}) = c_{LS}(f, [M])(= 0).$$

This implies that $\overline{U}$ is homologically non-trivial. □

Chapter 4

Hofer Lipschitz and C^0 continuous quasimorphisms

4.1 Abstarct of the chapter

We construct an example of a non-trivial homogeneous quasimorphism on the group of Hamiltonian diffeomorphisms of the two and four dimensional quadric hypersurfaces which is continuous with respect to both the C^0-metric and the Hofer metric. This answers a variant of a question of Entov–Polterovich–Py which is one of the open problems listed in the monograph of McDuff–Salamon. Throughout the proof, we make extensive use of the idea of working with different coefficient fields in quantum cohomology rings. As a by-product of the arguments in the paper, we answer a question of Polterovich–Wu regarding homogeneous quasimorphisms on the group of Hamiltonian diffeomorphisms of the complex projective plane and prove some intersection results about Lagrangians in the four dimensional quadric hypersurface. Most of the materials in this section are contained in [Kaw20].

4.2 Introduction

A (real-valued) homogeneous quasimorphism on a group G is a map

$$\mu : G \to \mathbb{R}$$

which satisfies

1. $\exists C > 0$ s.t. $\forall f, g \in G,\ |\mu(f \cdot g) - \mu(f) - \mu(g)| \leqslant C$,
2. $\forall k \in \mathbb{Z}, \forall f \in G,\ \mu(f^k) = k \cdot \mu(f)$.

The study of homogeneous quasimorphisms is a very rich topic with numerous connections to other mathematical domains. For example, homogeneous quasimorphisms naturally appear in the theory of bounded cohomology, they play a crucial role in the study of the commutator length and they also have many applications in the study of algebraic and topological properties (in case G is a topological group) of G.

In the context of symplectic topology, the study of algebraic and topological properties of the group of symplectomorphisms and Hamiltonian diffeomorphisms has been an important subject. For a closed symplectic manifold (M,ω), denote the group of Hamiltonian diffeomorphisms by $\mathrm{Ham}(M,\omega)$ and its universal cover by $\widetilde{\mathrm{Ham}}(M,\omega)$. One of the first groundbreaking results in this direction is due to Banyaga [Ban78] which states that $\mathrm{Ham}(M,\omega)$ is a simple group and $\widetilde{\mathrm{Ham}}(M,\omega)$ is a perfect group. This implies that there exist no non-trivial homomorphisms on $\mathrm{Ham}(M,\omega)$ and $\widetilde{\mathrm{Ham}}(M,\omega)$. However, it was discovered that non-trivial (real-valued) homogeneous quasimorphisms on $\mathrm{Ham}(M,\omega)$ and $\widetilde{\mathrm{Ham}}(M,\omega)$ do exist for some symplectic manifolds. Various constructions have been studied extensively as well as their applications to Hamiltonian dynamics. Just to mention a few, there are constructions by Barge–Ghys [BG92], Borman [Bor12], Entov [Ent04], Entov–Polterovich [EP03], Gambaudo–Ghys [GG04], Givental [Giv90], McDuff [McD10], Ostrover [Ost06], Py [Py06] and Shelukhin [Sh14]. Contact counterparts are also considered by Givental [Giv90], Borman–Zapolsky [BorZap15] and Granja–Karshon–Pabiniak–Sandon [GKPS20]. In particular, Entov–Polterovich [EP03] introduced a Floer theoretic method to construct homogeneous quasimorphisms on $\widetilde{\mathrm{Ham}}(M,\omega)$

$$\zeta_e : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$$

where (M,ω) is a closed monotone symplectic manifold which satisfies some property. Recall that a closed symplectic manifold (M,ω) is called monotone if there exists a constant $\kappa > 0$, which is referred to as the monotonicity constant, such that

$$\omega|_{\pi_2(M)} = \kappa \cdot c_1|_{\pi_2(M)}$$

where $c_1 = c_1(TM)$ denotes the first Chern class. In this paper, we only consider monotone symplectic manifolds unless mentioned otherwise. The precise construction of ζ_e is explained in Section 4.4.4. Moreover, (a certain normalization of) ζ_e satisfies the so-called Calabi property which means, roughly speaking, that "locally" it coincides with the Calabi homomorphism: we refer to [EP03] for the precise definition and its proof. In some cases, it is known that this homogeneous quasimorphism descends to $\mathrm{Ham}(M,\omega)$. For an excellent survey of the theory of quasimorphisms in the symplectic context and their relations to other topics, we refer to [Ent14].

4.3 Main results

4.3.1 Homogeneous quasimorphisms

The following question concerning the continuity of quasimorphisms was posed by Entov–Polterovich–Py in [EPP12]. This question appears also in the list of open problems in the monograph of McDuff–Salamon.

Question 76. *([EPP12], [MS98, Chapter 14, Problem 23])*

1. *Does there exist a nonzero homogeneous quasimorphism*

$$\mu : \mathrm{Ham}(S^2) \to \mathbb{R}$$

that is continuous with respect to the C^0-topology on $\mathrm{Ham}(S^2)$*?*

2. If yes, can it be made Lipschitz with respect to the Hofer metric?

Recall that the C^0-topology on $\mathrm{Ham}(M,\omega)$ is induced by the C^0-metric of Hamiltonian diffeomorphisms $\phi,\psi \in \mathrm{Ham}(M,\omega)$, which is defined by

$$d_{C^0}(\phi,\psi) := \max_{x\in M} d(\phi(x),\psi(x)),$$

where d denotes the distance on M induced by a fixed Riemannian metric on M. See Section 2.2 for further remarks on C^0-topology as well as the Hofer metric.

We provide some background and motivation concerning this question of Entov–Polterovich–Py.

• *Hofer metric vs. C^0-metric*: The relation between C^0-topology and the Hofer metric is very subtle. For example, C^0-topology is not continuous with respect to the Hofer metric. Conversely, Entov–Polterovich–Py point out that on $\mathrm{Ham}(D^{2n}(1))$, the group of compactly supported Hamiltonian diffeomorphisms of the closed unit ball $D^{2n}(1)$ in $\mathbb{R}^{2n}$, the Hofer metric is not C^0-continuous. For some striking results that demonstrate rigidity and flexibility of symplectic objects with respect to C^0-topology, see [BHS18], [BO16] and [HLS15].

In fact, for closed surfaces of positive genus Σ, there are examples of homogeneous quasimorphisms defined on $\mathrm{Ham}(\Sigma)$ which are C^0-continuous but not Hofer Lipschitz continuous: for their construction, see Gambaudo–Ghys [GG97], [GG04] and for their discontinuity with respect to the Hofer metric, see [Kha19]. On the other hand, the aforementioned Entov–Polterovich type homogeneous quasimorphisms are Hofer Lipschitz continuous but are not C^0-continuous: in fact, it is known that homogeneous quasimorphisms which have the Calabi property are not C^0-continuous: for a proof, see [EPP12].

• *Homogeneous quasimorphisms on the group of Hamiltonian homeomorphisms*: Given a symplectic manifold (M,ω), consider the C^0-closure of $\mathrm{Ham}(M,\omega)$ inside the group of homeomorphims of M. We denote it by $\overline{\mathrm{Ham}}(M,\omega)$ and call its elements Hamiltonian homeomorphisms. Hamiltonian homeomorphisms are central objects in C^0-symplectic topology. A C^0-continuous homogeneous quasimorphism defined on $\overline{\mathrm{Ham}}(M,\omega)$ will be useful to obtain information about the algebraic and topological properties of $\overline{\mathrm{Ham}}(M,\omega)$. In particular, when (M,ω) is either a 2-sphere S^2 or a 2-disk D^2, $\overline{\mathrm{Ham}}(M,\omega)$ is the identity component of the group of area-preserving homeomorphisms. A (non-trivial) homogeneous quasimorphism on $\overline{\mathrm{Ham}}(M,\omega)$ can be naturally obtained as an extension of a C^0-continuous (non-trivial) homogeneous quasimorphism on $\mathrm{Ham}(M,\omega)$ (see [EPP12, Proposition 1.4]). Therefore, the existence of a non-trivial C^0-continuous homogeneous quasimorphism on $\mathrm{Ham}(S^2)$ and $\mathrm{Ham}(D^2)$ has a strong relation to a question concerning the simplicity of groups $\overline{\mathrm{Ham}}(S^2)$ and $\overline{\mathrm{Ham}}(D^2)$ where the standard area-forms are considered as symplectic forms. The latter was known under the name of the simplicity conjecture ([MS98, Chapter 14, Problem 42]) and has caught the attention of many mathematicians over the years. It has been recently settled by Cristofaro-Gardiner–Humilière–Seyfaddini [CGHS20].

• *Uniqueness of homogeneous quasimorphisms on* $\mathrm{Ham}(S^2)$: Another motivation is the uniqueness of homogeneous quasimorphism on $\mathrm{Ham}(S^2)$. For example, an affirmative answer to the first question will imply the non-uniqueness of such maps, since Entov–Polterovich type homogeneous quasimorphisms are not C^0-continuous.

For more background on this question, see [EPP12].

In this paper, we consider a generalized version of the question of Entov–Polterovich–Py:

Question 77. *Does there exist a closed symplectic manifold (M,ω) which admits a non-trivial homogeneous quasimorphism on $\mathrm{Ham}(M,\omega)$ which is C^0-continuous? If yes, can it be Hofer Lipschitz continuous?*

Entov–Polterovich–Py proved that the vector space consisting of non-trivial homogeneous quasimorphisms on $\mathrm{Ham}(D^{2n}(1))$ that are both C^0 and Hofer Lipschitz continuous is infinite dimensional [EPP12, Proposition 1.9]. However, no example of a closed symplectic manifold (M,ω) which admits a homogeneous quasimorphism on $\mathrm{Ham}(M,\omega)$ that is both Hofer continuous and C^0-continuous is known by the time of writing. In fact, for closed symplectic manifolds, according to [Ent14], constructions of Givental, Entov–Polterovich and Borman are so far the only known examples of homogeneous quasimorphisms (on $\widetilde{\mathrm{Ham}}(M,\omega)$) that are Hofer continuous. The Hofer continuity of Givental's homogeneous quasimorphisms was proven by Borman–Zapolsky [BorZap15]. These examples all possess the Calabi property which implies that, in the case they descend to $\mathrm{Ham}(M,\omega)$, they are not C^0-continuousThe Calabi property of Givental's homogeneous quasimorphisms was proven by Ben Simon [BS07].

Our main result provides such examples for the monotone n-quadric (Q^n,ω) for $n=2,4$. Throughout the paper, we consider the standard monotone symplectic form ω of Q^n with the normalization $\int_{Q^n}\omega^n = 2$ so that the monotonicity constant κ is $1/N_{Q^n} = 1/n$. Note that (Q^2,ω) is symplectomorphic to the monotone product $(S^2\times S^2, \sigma\oplus\sigma)$ where σ is the area-form of S^2 with $\int_{S^2}\sigma = 1$ and (Q^4,ω) is symplectomorphic to $Gr_{\mathbb{C}}(2,4)$ equipped with the standard monotone symplectic form with a certain normalization.

Precisely, we prove the following.

Theorem 78. *(Theorem 16)*
There exist non-trivial homogeneous quasimorphisms

$$\mu : \mathrm{Ham}(Q^n) \to \mathbb{R}$$

where $n=2,4$ that satisfy the following three properties:

1. *μ is Lipschitz continuous with respect to the Hofer metric.*
2. *μ is C^0-continuous i.e.*

   $$\mu : (\mathrm{Ham}(Q^n), d_{C^0}) \to \mathbb{R}$$

   is continuous.
3. *We have*

   $$|\mu| = \overline{\gamma}$$

   where $\overline{\gamma}$ denotes the asymptotic spectral norm:

   $$\overline{\gamma} : \mathrm{Ham}(M,\omega) \to \mathbb{R}$$

   $$\overline{\gamma}(\phi) := \lim_{k\to+\infty} \frac{\gamma(\phi^k)}{k}.$$

Obviously, properties 2 and 3 in Theorem 78 imply that the asymptotic spectral norm $\overline{\gamma}$ is C^0-continuous for the 2- and 4- quadric hypersurfaces even though the C^0-continuity of the spectral norm γ for these manifolds is not confirmed at the time of writing. In fact, our argument can be generalized as follows.

Theorem 79. *(Theorem 17) Let (M,ω) be a monotone symplectic manifold such that $QH^*(M;\mathbb{C})$ is semi-simple. Then,*

$$\overline{\gamma} : \mathrm{Ham}(M,\omega) \to \mathbb{R}$$

is C^0-continuous.

Remark 80.

1. *If $QH^*(M;\mathbb{C})$ is semi-simple and $\overline{\gamma} \neq 0$, then it implies the existence of a non-trivial C^0 and Hofer Lipschitz continuous homogeneous quasimorphism on $\mathrm{Ham}(M,\omega)$. However, we can show $\overline{\gamma} \neq 0$ only for the 2- and 4-quadric hypersurfaces.*

2. *Although it is not explicitly stated, the existence of a homogeneous quasimorphism on $\widetilde{\mathrm{Ham}}(Q^n), \forall n \in \mathbb{N}$ was essentially known since [EP03]. The descent of Entov–Polterovich type homogeneous quasimorphisms to $\mathrm{Ham}(Q^2)$ and $\mathrm{Ham}(Q^4)$ was proven in [EP03] and [Br11], respectively. The homogeneous quasimorphisms in Theorem 78 are different from the Entov–Polterovich type homogeneous quasimorphisms as they are defined as differences of two Entov–Polterovich type homogeneous quasimorphisms.*

3. *In the case of $n = 2$, if we compose $\mu : \mathrm{Ham}(S^2 \times S^2) \to \mathbb{R}$ with*

$$\mathrm{Ham}(S^2) \to \mathrm{Ham}(S^2 \times S^2)$$
$$\phi \mapsto \phi \times \phi,$$

we obtain a C^0-continuous and Hofer Lipschitz continuous homogeneous quasimorphism on $\mathrm{Ham}(S^2)$ but this turns out to be trivial and thus does not answer the question of Entov–Polterovich–Py. See Remark 105 for further explanation.

4. *In Section 4.5.5, we will discuss a generalization of Theorem 78.*

4.3.2 Question of Polterovich–Wu

One of the key ideas in proving Theorem 78 and 117 is to work with quantum cohomology rings with different coefficient fields, namely the field of Laurent series and the universal Novikov field. The advantage of this idea in our context is explained in Section 4.5.3. As another application of this idea, we answer a question of Polterovich–Wu which was posed in [Wu15, Remark 5.2].

We briefly review the question. Details of the question are postponed to Section 4.5.6. In [Wu15], Wu found three homogeneous quasimorphisms $\{\zeta_j\}_{j=1,2,3}$ on $\widetilde{\mathrm{Ham}}(\mathbb{C}P^2)$ via the Entov–Polterovich construction for the quantum cohomology ring with the universal Novikov field. Polterovich posed the following question.

Question 81. *([Wu15, Remark 5.2], see also Question 121)*
Is it possible to distinguish the three homogeneous quasimorphisms $\{\zeta_j\}_{j=1,2,3}$?

We answer this in the negative.

Theorem 82. *The three homogeneous quasimorphisms $\{\zeta_j\}_{j=1,2,3}$ coincide i.e.*

$$\zeta_1 = \zeta_2 = \zeta_3.$$

4.3.3 Application: Le Roux's question

The relation between the Hofer-topology and the C^0-topology on the group of Hamiltonian diffeomorphisms on closed symplectic manifolds still remains a mystery. In [LeR10], Le Roux posed the following question.

Question 83. ([LeR10])
Let (M,ω) be any closed symplectic manifold. For any $R>0$, does

$$\mathrm{Ham}_{\geqslant R} := \{\phi \in \mathrm{Ham}(M,\omega) : d_{\mathrm{Hof}}(\mathrm{id},\phi) \geqslant R\}$$

have a non-empty C^0-interior?

We answer to this question affirmatively for the quadric hypersurface $Q^n \times M$ where $n = 2, 4$.

Theorem 84. *For any $R>0$,*

$$\mathrm{Ham}_{\geqslant R} := \{\phi \in \mathrm{Ham}(Q^n) : d_{\mathrm{Hof}}(\mathrm{id},\phi) \geqslant R\}$$

has a non-empty C^0-interior where $n = 2, 4$.

Theorem 84 seems to be the first case where the question of Le Roux was verified for closed simply connected manifolds. In fact, according to [EPP12, Section 1.4], "for closed simply connected manifolds (and already for the case of the 2-sphere) the question is wide open".

Remark 85. *Our proof applies to any closed monotone symplectic manifold for which the spectral norm can be arbitrarily large: see Theorem 131. See also Theorem 130 for a slightly generalized statement.*

4.3.4 Strategy of the proof and structure of the paper

The strategy of the proof of Theorem 78, which divides into two parts, is as follows:

We first prove that a homogeneous quasimorphism on $\widetilde{\mathrm{Ham}}(M,\omega)$ which is obtained as the difference of any two Entov–Polterovich type homogeneous quasimorphisms descends to $\mathrm{Ham}(M,\omega)$ and is bounded by the spectral norm γ. Next we show that it is C^0-continuous by using a result on the C^0-control of the spectral norm obtained by the author in [Kaw21] (Theorem 101). This is the first part of the proof. Note that this part applies to any monotone symplectic manifold.

In the second part of the proof, we will see that in order to prove that the resulting homogeneous quasimorphism is non-trivial, it suffices to find two disjoint Lagrangian submanifolds with non-vanishing Floer cohomology. We use examples found by Fukaya–Oh–Ohta–Ono and Eliashberg–Polterovich for the case of Q^2 and by Nishinou–Nohara–Ueda and Nohara–Ueda for the case of Q^4 where the Floer cohomology of Lagrangian fibers of a Gelfand–Cetlin system was studied via superpotential techniques.

The crucial idea of the proof is to work with different quantum cohomology rings in Part 1 and 2. The differences of the two quantum cohomology rings as well as their advantages are explained in Section 4.5.3. In Section 4.5.6, we answer a question of Polterovich–Wu also by applying this idea. In Section 4.5.7, we discuss some consequences of the argument to Lagrangian intersections.

4.4 Additional preliminaries

Let (M, ω) be a closed monotone symplectic manifold i.e.

$$\omega|_{\pi_2(M)} = \kappa \cdot c_1|_{\pi_2(M)}$$

for some monotonicity constant $\kappa > 0$ where $c_1 = c_1(TM)$ denotes the first Chern class. In this paper, we only consider monotone symplectic manifolds unless mentioned otherwise. The positive generators of $\langle \omega, \pi_2(M) \rangle$ and $\langle c_1, \pi_2(M) \rangle \subset \mathbb{Z}$ are respectively called the rationality constant and the minimal Chern number and will be respectively denoted by λ_0 and N_M.

A Hamiltonian H on M is a smooth time dependent function $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$. A Hamiltonian H is called mean-normalized if the following holds:

$$\forall t \in \mathbb{R}/\mathbb{Z}, \ \int_M H_t(x)\omega^n = 0.$$

We define its Hamiltonian vector field X_{H_t} by

$$-dH_t = \omega(X_{H_t}, \ \cdot \ ).$$

The Hamiltonian flow of H, denoted by ϕ_H^t, is by definition the flow of X_{H_t}. A Hamiltonian diffeomorphism of H is a diffeomorphism which arises as the time-one map of a Hamiltonian flow and will be denoted by ϕ_H. It is well-known that the set of Hamiltonian diffeomorphisms forms a group and will be denoted by $\mathrm{Ham}(M, \omega)$. We denote its universal cover by $\widetilde{\mathrm{Ham}}(M, \omega)$.

Denote the set of smooth contractible loops in M by $\mathcal{L}_0 M$ and consider its universal cover. Two elements in the universal cover, say $[z_1, w_1]$ and $[z_2, w_2]$, are equivalent if $z_1 = z_2$ and their boundary sum $w_1 \# \overline{w_2}$ i.e. the sphere obtained by gluing w_1 and w_2 along their common boundary with the orientation on w_2 reversed, satisfies

$$\omega(w_1 \# \overline{w_2}) = 0, \ c_1(w_1 \# \overline{w_2}) = 0.$$

We denote by $\widetilde{\mathcal{L}_0 M}$ the space of equivalence classes.

For a Hamiltonian H, define the action functional $\mathcal{A}_H : \widetilde{\mathcal{L}_0 M} \to \mathbb{R}$ by

$$\mathcal{A}_H([z, w]) := \int_0^1 H(t, z(t))dt - \int_{D^2} w^* \omega$$

where $w : D^2 \to M$ is a capping of $z : \mathbb{R}/\mathbb{Z} \to M$. Critical points of this functional are precisely the capped 1-periodic Hamiltonian orbits of H which will be denoted by $\widetilde{\mathcal{P}}(H)$. The set of critical values of $\mathcal{A}_H$ is called the action spectrum and is denoted by $\mathrm{Spec}(H)$:

$$\mathrm{Spec}(H) := \{\mathcal{A}_H(\widetilde{z}) : \widetilde{z} \in \widetilde{\mathcal{P}}(H)\}.$$

4.4.1 Quantum (co)homology and semi-simplicity

Consider a monotone symplectic manifold (M, ω). Let the following denote the field of Laurent series of a formal variable s:

$$\mathbb{C}[[s^{-1}, s] := \{\sum_{k \leqslant k_0} a_k s^k : k_0 \in \mathbb{Z}, a_k \in \mathbb{C}\}.$$

By identifying the variable s with the generator of $\Gamma := \pi_2(M)/\sim$ where the equivalence relation is defined by $A, B \in \pi_2(M)$,

$$A \sim B \Longleftrightarrow \omega(A) = \omega(B)$$

satisfying

$$\omega(s) = \lambda_0,\ c_1(s) = N_M,$$

one can define the quantum homology ring $QH_*(M;\mathbb{C})$ as

$$QH_*(M;\mathbb{C}) := H_*(M;\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[s^{-1}, s].$$

The quantum homology ring has the following valuation:

$$\nu_{QH_*} : QH_*(M;\mathbb{C}) \to \mathbb{R}$$

$$\nu_{QH_*}(\sum_{k \leqslant k_0} a_k s^k) := \max\{k \cdot \omega(s) = k \cdot \lambda_0 : a_k \neq 0\}.$$

Similarly, for a formal variable t, one can define the quantum cohomology ring $QH^*(M;\mathbb{C})$ as

$$QH^*(M;\mathbb{C}) := H^*(M;\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1}, t]]$$

where

$$\mathbb{C}[t^{-1}, t]] := \{\sum_{k \geqslant k_0} b_k t^k : k_0 \in \mathbb{Z}, b_k \in \mathbb{C}\}.$$

The quantum homology and quantum cohomology rings are isomorphic under the Poincaré duality map:

$$\mathrm{PD} : QH^*(M;\mathbb{C}) \xrightarrow{\sim} QH_{2n-*}(M;\mathbb{C})$$

$$a := \sum_{k \geqslant k_0} A_k t^k \mapsto \mathrm{PD}(a) := \sum_{k \geqslant k_0} A_k^{\#} s^{-k}$$

where $\#$ denotes the usual Poincaré duality between singular homology and singular cohomology. Note that t satisfies

$$\omega(t) = \lambda_0,\ c_1(t) = N_M.$$

The quantum cohomology ring has the following valuation:

$$\nu := \nu_{QH^*} : QH^*(M;\mathbb{C}) \to \mathbb{R}$$

$$\nu(\sum_{k \geqslant k_0} a_k t^k) := \min\{k \cdot \omega(t) = k\lambda_0 : a_k \neq 0\}.$$

The ring structure of $QH_*(M;\mathbb{C})$ (and of $QH^*(M;\mathbb{C})$) is given by the quantum product which is denoted by $*$. It is defined by a certain count of pseudo-homolorphic spheres. More precisely, in the case of $QH_*(M;\mathbb{C})$,

$$\forall a, b, c \in H_*(M),\ (a * b) \circ c := \sum_{k \in \mathbb{Z}} GW_{3,s^k}(a, b, c) \otimes s^k$$

where $\circ$ denotes the usual intersection index in homology and $GW_{3,s^k}(a, b, c)$ denotes the 3-pointed Gromov-Witten invariant for $a, b, c \in H_*(M)$ in the class $A \in \pi_2(M)$

where $[A] = s^k \in \Gamma$ i.e. the count of pseudo-holomorphic spheres in the homotopy class A passing through cycles representing $a, b, c \in H_*(M)$. See [MS98] for details.

It is known that the Floer homology defined in Section 2.3 is canonically isomorphic to the quantum homology ring via the PSS-map:

$$PSS_H : QH_*(M;\mathbb{C}) \xrightarrow{\sim} HF_*(H).$$

Note that the PSS-map preserves the ring structure where the ring structure on RHS is given by the pair-of-pants product. See [MS04] for details.

The quantum cohomology ring $QH^*(M;\mathbb{C})$ is called semi-simple if it splits into a finite direct sum of fields i.e.

$$QH^*(M;\mathbb{C}) = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_l$$

for some $l \in \mathbb{N}$ where each Q_j is a field. The identity $1 \in QH^*(M;\mathbb{C})$ can then be decomposed into a sum of units $e_j \in Q_j$:

$$1 = e_1 + e_2 + \cdots + e_l.$$

Remark 86. *The notion of semi-simplicity depends on the algebraic set-up of the quantum (co)homology. The notion explained above is the same as the one in [EP03] which is not suitable to non-monotone settings as the Novikov ring is no longer a field. A more general notion of semi-simplicity was introduced in [Ost06], [EP08]. [EP08, Theorem 5.1] states that in the monotone case, this generalized notion of semi-simplicity coincides with the one of [EP03].*

Examples of monotone symplectic manifolds whose quantum cohomology rings are semi-simple include $\mathbb{C}P^n$, 1, 2 and 3 point monotone blow-ups of $\mathbb{C}P^2$, complex Grassmanians $Gr_{\mathbb{C}}(2,n)$ and their products: see [EP03], [EP08].

Later, we will consider quantum cohomology with a different coefficient field, namely the universal Novikov field Λ defined by

$$\Lambda := \{\sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j\to+\infty} \lambda_j = +\infty\}.$$

Fukaya–Oh–Ohta–Ono [FOOO09], [FOOO19] study Floer theory with coefficients in Λ rather than in the field of Laurent series and considers the following quantum cohomology:

$$QH^*(M;\Lambda) := H^*(M;\mathbb{C}) \otimes_{\mathbb{C}} \Lambda.$$

It has the following valuation:

$$\nu : QH^*(M;\Lambda) \to \mathbb{R}$$

$$\nu(\sum_{j=1}^{\infty} a_j T^{\lambda_j}) := \min\{\lambda_j : a_j \neq 0\}.$$

By considering

$$t \mapsto T^{+\lambda_0},$$

one can embed $QH^*(M;\mathbb{C})$ into $QH^*(M;\Lambda)$:

$$QH^*(M;\mathbb{C}) \hookrightarrow QH^*(M;\Lambda).$$

4.4.2 Quantum homology of quadrics

In this section, we review some information about the quantum homology ring structure of quadric hypersurfaces. For $n \geqslant 2$, the n-quadric Q^n is defined as a hypersurface in $\mathbb{C}P^{n+1}$ as follows:

$$Q^n := \{(z_0 : z_1 : \cdots : z_{n+1}) \in \mathbb{C}P^{n+1} : z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\}.$$

Recall that the minimal Chern number N_{Q^n} of the n-quadric is n. It is well-known that Q^2 and Q^4 are respectively symplectomorphic to $S^2 \times S^2$ and $Gr_{\mathbb{C}}(2,4)$. The ring structure of (quantum) homology of Q^n can be found in [BC09, Section 6.3]. We just recall that $QH_*(Q^n;\mathbb{C})$ satisfies

$$[pt] * [pt] = [Q^n]s^{-2}$$

where $[pt]$ and $[Q^n]$ denote respectively the point class and the fundamental class. The semi-simplicity of the quantum homology ring of Q^n follows from a result of Beauville [Bea95]. In fact, it is easy to see that $QH_*(Q^n;\mathbb{C})$ splits into a direct sum of two fields by using that the minimal Chern number is $N_{Q^n} = n$.

Proposition 87. *For $n \geqslant 2$, $QH_*(Q^n;\mathbb{C})$ splits into a direct sum of two fields $Q_\pm$:*

$$QH_*(Q^n;\mathbb{C}) = Q_+ \oplus Q_-.$$

4.4.3 Lagrangian Floer cohomology with bounding cochain

In this section, we sketch the construction of Lagrangian Floer cohomology deformed by a bounding cochain due to Fukaya–Oh–Ohta–Ono [FOOO09]. In this paper, we mainly consider monotone Lagrangian submanifolds but it is worth mentioning that the theory of Fukaya–Oh–Ohta–Ono sketched in this section applies to any closed oriented Lagrangian submanifold which is relatively spin. We refer to [FOOO09], especially Chapter 3.1 for a detailed description of the material.

Let L be a closed oriented Lagrangian submanifold with a fixed relatively spin structure. Recall that an oriented Lagrangian submanifold is relatively spin if its second Stiefel-Whitney class $w_2(TL)$ is in the image of the restriction map $H^2(M;\mathbb{Z}/2\mathbb{Z}) \to H^2(L;\mathbb{Z}/2\mathbb{Z})$ ([FOOO09, Definition 3.1.1]). For example, if a Lagrangian is spin, then it is relatively spin and in particular, oriented Lagrangians are always relatively spin if $\dim_{\mathbb{R}} M \leqslant 6$.

Define the universal Novikov ring

$$\Lambda_0 := \{\sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j \geqslant 0, \lim_{j\to+\infty} \lambda_j = +\infty\}.$$

The universal Novikov field is given by

$$\Lambda := \{\sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j\to+\infty} \lambda_j = +\infty\}.$$

Define also

$$\Lambda_+ := \{\sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j > 0, \lim_{j\to+\infty} \lambda_j = +\infty\}.$$

Lagrangian intersection Floer theory equips the Λ_0-valued cochain complex of L with the structure of an A_∞-algebra. By taking the canonical model, one obtains an A_∞-structure $\{\mathfrak{m}_k\}_{0\leqslant k\leqslant\infty}$ on $H^*(L;\Lambda_0)$: we refer to [FOOO09, Section 5.4] for details. An element $b\in H^1(L;\Lambda_+)$ is called a weak bounding cochain (in the sequel, we will simply call them a bounding cochain) if it satisfies the weak Maurer-Cartan equation

$$\sum_{k=0}^{\infty}\mathfrak{m}_k(b,b,\cdots,b)=0 \mod \Lambda_0\cdot \mathrm{PD}([L]). \tag{4.1}$$

The set of (weak) bounding cochains will be denoted by $\widehat{\mathcal{M}}_{weak}(L)$. Note that $\widehat{\mathcal{M}}_{weak}(L)$ might be an empty set. We say that the Lagrangian L is unobstructed if

$$\widehat{\mathcal{M}}_{weak}(L)\neq\emptyset.$$

In the case L is unobstructed, for any $b\in\widehat{\mathcal{M}}_{weak}(L)$, one can twist the Floer differential as

$$\mathfrak{m}_1^b(x):=\sum_{k,l\geqslant 0}\mathfrak{m}_{k+l+1}(b^{\otimes k}\otimes x\otimes b^{\otimes l}).$$

The Maurer-Cartan equation 4.1 implies

$$\mathfrak{m}_1^b\circ\mathfrak{m}_1^b=0$$

and the resulting cohomology group

$$HF((L,b);\Lambda_0):=\frac{\mathrm{Ker}(\mathfrak{m}_1^b:H^*(L;\Lambda_0)\to H^*(L;\Lambda_0))}{\mathrm{Im}(\mathfrak{m}_1^b:H^*(L;\Lambda_0)\to H^*(L;\Lambda_0))}$$

will be called the Floer cohomology deformed by a (weak) bounding cochain $b\in\widehat{\mathcal{M}}_{weak}(L)$. We also define

$$HF((L,b);\Lambda):=HF((L,b);\Lambda_0)\otimes_{\Lambda_0}\Lambda.$$

4.4.4 Quasimorphisms via spectral invariants

In this subsection, we recall the Floer theoretic construction of homogeneous quasimorphisms on $\widetilde{\mathrm{Ham}}(M,\omega)$ and the notion of (super)heaviness both due to Entov–Polterovich which are taken from [EP03], [EP09]. However, unlike their version, we use quantum cohomology instead of quantum homology.

Assume $e\in QH^0(M;\mathbb{C})$ is an idempotent. Then we define the asymptotic spectral invariant

$$\zeta_e:C^\infty(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R})\to\mathbb{R}$$

$$\zeta_e(H):=\lim_{k\to+\infty}\frac{\rho(H^k,e)}{k}$$

where $\rho(\cdot,e)$ denotes the spectral invariant corresponding to $e\in QH^0(M;\mathbb{C})$ and the k-times iterated Hamiltonian

$$H^k:=\underbrace{H\#H\#\cdots\#H}_{k\text{-times}}.$$

Its restriction to $C^\infty(M,\mathbb{R})$ i.e. $\zeta_e|_{C^\infty(M,\mathbb{R})} : C^\infty(M,\mathbb{R}) \to \mathbb{R}$ is often referred to the symplectic quasi-state [EP06].

We can also see ζ_e as a function of homotopy classes of Hamiltonian paths:

$$\zeta_e : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$$

$$\zeta_e(\tilde{\phi}) := \lim_{k\to+\infty} \frac{\rho(\tilde{\phi}^k, e)}{k}.$$

Recall that $\rho(\tilde{\phi}, \ \cdot\ ) = \rho(H, \ \cdot\ )$ where H is the mean-normalized Hamiltonian such that the Hamiltonian path $t \mapsto \phi_H^t$ represents the homotopy class $\tilde{\phi}$. It was first discovered by Entov–Polterovich that when some additional condition is satisfied, $\zeta_e : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ is a homogeneous quasimorphism. We will state their result as well as its variant due to Fukaya–Oh–Ohta–Ono.

We denote the even degree part of $QH^*(M;\mathbb{C})$ as follows:

$$QH^{even}(M;\mathbb{C}) := \bigoplus_{k\in\mathbb{Z}} H^{2k}(M;\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1},t]].$$

Theorem 88. *([EP03, Theorem 1.1], [FOOO19])*

1. *If $e \in QH^0(M;\mathbb{C})$ is an idempotent and $e \cdot QH^{even}(M;\mathbb{C})$ is a field, then*

$$\zeta_e : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$$

   is a homogeneous quasimorphism.

2. *If $e \in QH^*(M;\Lambda)$ is an idempotent and $e \cdot QH^*(M;\Lambda)$ is a field, then*

$$\zeta_e : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$$

   is a homogeneous quasimorphism.

Remark 89. *All the examples that appear in this paper satisfy*

$$QH^{even}(M;\mathbb{C}) = QH^*(M;\mathbb{C}).$$

Definition 90. *Let (M,ω) be any closed symplectic manifold and let $e \in QH^*(M;\mathbb{C})$ be an idempotent. A subset S of M is called e-heavy or ζ_e-heavy (resp. e-superheavy or ζ_e-superheavy) if it satisfies the following:*

$$\inf_{x\in S} H(x) \leqslant \zeta_e(H)$$

$$(\textit{resp. } \zeta_e(H) \leqslant \sup_{x\in S} H(x))$$

for any $H \in C^\infty(M,\mathbb{R})$.

Remark 91. *In general, e-heaviness follows from e-superheaviness but not vice versa. In a special case where $\zeta_e : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ is a homogeneous quasimorphism, e-heaviness and e-superheaviness are equivalent. See [EP09] for discussions in this topic.*

The following is a basic intersection property of (super)heavy sets from [EP09].

Proposition 92. *Let (M,ω) be any closed symplectic manifold and let $e \in QH^*(M;\mathbb{C})$ be an idempotent. Let S_1 and S_2 be two disjoint subsets of M. If S_1 is e-superheavy, then S_2 is not e-heavy.*

Proof. If we assume that S_2 is e-heavy, then by the definitions, we have

$$\inf_{x\in S_2} H(x) \leqslant \zeta_e(H) \leqslant \sup_{x\in S_1} H(x)$$

for any $H \in C^\infty(M,\mathbb{R})$. As $S_1 \cap S_2 = \emptyset$, one can take H to be larger on S_2 than on S_1, which contradicts the inequality. □

4.4.5 Closed-open map and heaviness

In this section, we review some properties of the closed-open map defined by Fukaya–Oh–Ohta–Ono in [FOOO09, Theorem 3.8.62]. Note that they also consider the case where the absolute and the relative Floer cohomology groups are deformed with a bulk. However, as bulk deformations are not relevant to the arguments in this paper, we only state a version without them.

Denote the ring homomorphism called the closed-open map, which is a quantum analogue of the restriction map, by

$$\mathcal{CO}^0_b : QH^*(M;\Lambda) \to HF^*((L,b);\Lambda)$$

where b is a bounding cochain. Note that the original notation used in [FOOO09] for $\mathcal{CO}^0_b$ is $i^*_{qm,b}$.

Fukaya–Oh–Ohta–Ono proved the following in [FOOO19] to detect the heaviness of the Lagrangian L, which generalizes the result of Albers [Alb05] and Entov–Polterovich [EP09, Theorem 1.17].

Theorem 93. (*[FOOO19, Theorem 1.6]*)
Assume

$$HF^*((L,b);\Lambda) \neq 0$$

for a certain bounding cochain b. If

$$\mathcal{CO}^0_b(e) \neq 0$$

for an idempotent $e \in QH^*(M;\Lambda)$, then L is e-heavy.

4.4.6 Flag manifolds and Gelfand–Cetlin systems

In this subsection, we provide a brief description of (partial) flag manifolds and Gelfand–Cetlin systems. Materials discussed in this section are only needed to precisely understand the statement of Theorem 115 and will not be used in other parts of the paper. Thus, readers can skip this section in order to read the other parts.

Fix a sequence

$$0 = n_0 < n_1 < \cdots < n_r < n_{r+1} = n$$

of integers, and set

$$k_i := n_i - n_{i-1}$$

for $i = 1, 2, \cdots, r+1$. The (partial) flag manifold $F = F(n_1, n_2, \cdots, n_r, n)$ is a complex manifold parameterizing nested subspaces

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{C}^n,\ \dim V_i = n_i.$$

The dimension of $F = F(n_1, n_2, \cdots, n_r, n)$ is given by

$$\dim_{\mathbb{C}} F(n_1, n_2, \cdots, n_r, n) = \sum_{i=1}^{r} (n_i - n_{i-1})(n - n_i) = \sum_{i=1}^{r} k_i(n - n_i). \tag{4.2}$$

Let $P = P(n_1, n_2, \cdots, n_r, n) \subset GL(n, \mathbb{C})$ be the isotropy subgroup of the standard flag

$$\mathbb{C}^{n_1} \times \{0\} \subset \mathbb{C}^{n_2} \times \{0\} \subset \cdots \subset \mathbb{C}^{n_r} \times \{0\} \subset \mathbb{C}^n.$$

Then, as

$$U(n) \cap P(n_1, n_2, \cdots, n_r, n) = U(k_1) \times U(k_2) \times \cdots U(k_{r+1}),$$

$F(n_1, n_2, \cdots, n_r, n)$ is written as follows:

$$F(n_1, n_2, \cdots, n_r, n) = GL(n, \mathbb{C})/P(n_1, n_2, \cdots, n_r, n)$$

$$= U(n)/(U(k_1) \times U(k_2) \times \cdots U(k_{r+1})).$$

Remark 94. *Note that this description gives the following different expression of the dimension formula 4.2:*

$$\dim_{\mathbb{C}} F(n_1, n_2, \cdots, n_r, n) = n^2 - \sum_{i=1}^{r+1} k_i^2.$$

In this paper, we identify flag manifolds with (co)adjoint orbits. Using a $U(n)$-invariant inner product on the Lie algebra $\mathfrak{u}(n)$ of $U(n)$, denoted by $\langle -, - \rangle$, we identify the dual $\mathfrak{u}(n)^*$ of $\mathfrak{u}(n)$ with the space $\sqrt{-1} \cdot \mathfrak{u}(n)$ of Hermitian matrices. We fix

$$\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \sqrt{-1} \cdot \mathfrak{u}(n)$$

with

$$\underbrace{\lambda_1 = \cdots = \lambda_{n_1}}_{k_1} > \underbrace{\lambda_{n_1+1} = \cdots = \lambda_{n_2}}_{k_2} > \cdots > \underbrace{\lambda_{n_r+1} = \cdots = \lambda_n}_{k_{r+1}}.$$

Then F is identified with the adjoint orbit $\mathcal{O}_\lambda$ of λ (i.e. a set of Hermitian matrices with fixed eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$) by

$$F = U(n)/(U(k_1) \times \cdots \times U(k_{r+1})) \xrightarrow{\sim} \mathcal{O}_\lambda$$

$$[g] \mapsto g\lambda g^*.$$

$\mathcal{O}_\lambda$ has a standard symplectic form ω_λ called the Kirillov–Kostant–Souriau form. Recall that tangent vectors of $\mathcal{O}_\lambda$ at x can be written as

$$ad_\xi(x) = [x, \xi]$$

for $\xi \in \mathfrak{u}(n)$ where $[-,-]$ denotes the Lie bracket. Then the Kirillov–Kostant–Souriau form ω_λ is defined by

$$\omega_\lambda(ad_\xi(x), ad_\eta(x)) := \frac{1}{2\pi}\langle x, [\xi, \eta]\rangle.$$

The following choice of λ gives us a monotone symplectic form ω_λ on $\mathcal{O}_\lambda$:

$$\lambda = (\underbrace{n-n_1, \cdots}_{k_1}, \underbrace{n-n_1-n_2, \cdots}_{k_2}, \cdots, \underbrace{\cdots, n-n_{r-1}-n_r}_{k_r}, \underbrace{-n_r, \cdots}_{k_{r+1}}) + \underbrace{(m, \cdots, m)}_{n=k_1+\cdots+k_{r+1}}$$

for any $m \in \mathbb{R}$. When λ is of this form, we have

$$c_1(T\mathcal{O}_\lambda) = [\omega_\lambda].$$

For $x \in \mathcal{O}_\lambda$ and $k = 1, 2, \cdots, n-1$ let $x^{(k)}$ denote the upper-left $k \times k$ submatrix of x. Since $x^{(k)}$ is also a Hermitian matrix, it has real eigenvalues

$$\lambda_1^{(k)} \leqslant \lambda_2^{(k)} \leqslant \cdots \leqslant \lambda_k^{(k)}.$$

Let $I = I(n_1 \cdots, n_r, n)$ denote the set of pairs (i,k) such that each $\lambda_i^{(k)}$ is non-constant as a function of x. It follows that the number of such pairs coincides with $\dim_{\mathbb{C}} F$ i.e. $|I| = \dim_{\mathbb{C}} F$. The Gelfand–Cetlin system is defined by

$$\Phi : F \to \mathbb{R}^{\dim_{\mathbb{C}} F}$$
$$\Phi(x) := \{\lambda_i^{(k)}(x)\}_{(i,k)\in I}$$

Theorem 95. (*Guillemin–Sternberg, [GS83]*)
The map Φ defines a completely integrable system on $(F(n_1, n_2, \cdots, n_r, n), \omega)$. The image $\Delta := \Phi(F)$ is a convex polytope. A fiber of each interior point $u \in \mathrm{Int}(\Delta)$ is a Lagrangian torus:

$$\Phi^{-1}(u) \simeq T^n$$

for any $u \in \mathrm{Int}(\Delta)$.

We call the convex polytope $\Delta := \Phi(F)$, the Gelfand–Cetlin polytope. The major difference between Delzant polytopes of toric manifolds and Gelfand–Cetlin polytopes appears at fibers of points at the boundary of polytopes. While for a Delzant polytope, a fiber of a relative interior of a k-dimensional face is never Lagrangian, for a Gelfand–Cetlin polytope, a fiber of a relative interior point of a k-dimensional face can be a (non-torus) Lagrangian submanifold. Differences between the two types of polytopes are listed by Y. Cho–Y. Kim–Y-G. Oh in [CKO18].

4.5 Proofs

4.5.1 Proof of Theorem 78–Part 1

The goal of this subsection is to prove the following result and to see how it leads to Theorem 78.

Theorem 96. *Let (M,ω) be a monotone symplectic manifold. Assume its quantum cohomology ring $QH^*(M;\mathbb{C})$ is semi-simple i.e.*

$$QH^*(M;\mathbb{C}) = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_l$$

for some $l \in \mathbb{N}$ where each Q_j is a field. We decompose the identity $1 \in QH^*(M;\mathbb{C})$ into a sum of idempotents with respect to this split:

$$1 = e_1 + e_2 + \cdots + e_l,\ e_j \in Q_j.$$

Then for any $i,j \in \{1,2,\cdots,l\}$,

$$\mu := \zeta_{e_i} - \zeta_{e_j}$$

defines a homogeneous quasimorphism on $\mathrm{Ham}(M,\omega)$ which is C^0-continuous i.e.

$$\mu : (\mathrm{Ham}(M,\omega), d_{C^0}) \to \mathbb{R}$$

is continuous. Moreover, it is Hofer Lipschitz continuous.

Remark 97.

1. *As we do not know if $\zeta_{e_i} \neq \zeta_{e_j}$, the resulting homogeneous quasimorphism*
$$\mu : \mathrm{Ham}(M,\omega) \to \mathbb{R}$$
might be trivial i.e. $\mu \equiv 0$. Thus the point in proving Theorem 78 is to prove $\zeta_{e_+} \neq \zeta_{e_-}$ for the two idempotents $e_\pm \in QH^*(Q^n;\mathbb{C})$ $(n = 2,4)$.
2. *For examples of monotone symplectic manifolds whose quantum cohomology ring is semi-simple, see Section 4.4.1.*
3. *In the spirit of McDuff [McD10], instead of the semi-simplicity we can pose a weaker assumption that $QH^*(M;\mathbb{C})$ has two fields as a direct summand:*
$$QH^*(M;\mathbb{C}) = Q_1 \oplus Q_2 \oplus A$$
where Q_1, Q_2 are fields and no condition is posed on A.

We first show the following estimate.

Proposition 98. *For any $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$, we have*

$$|\mu(\tilde{\phi})| \leqslant \gamma(\tilde{\phi}).$$

Proof of Proposition 98. By the triangle inequality,

- $\rho(\tilde{\phi}^k, e_1) \leqslant \rho(\tilde{\phi}^k, 1) + \rho(\tilde{\mathrm{id}}, e_1)$,
- $-\rho(\tilde{\phi}^k, e_2) \leqslant \rho((\tilde{\phi}^{-1})^k, 1) - \rho(\tilde{\mathrm{id}}, e_2)$.

By adding these inequalities, we obtain

$$\mu(\tilde{\phi}) = \lim_{k\to+\infty} \frac{\rho(\tilde{\phi}^k, e_1) - \rho(\tilde{\phi}^k, e_2)}{k}$$

$$\leqslant \lim_{k\to+\infty} \frac{\rho(\tilde{\phi}^k, 1) + \rho(\tilde{\mathrm{id}}, e_1) + \rho((\tilde{\phi}^{-1})^k, 1) - \rho(\tilde{\mathrm{id}}, e_2)}{k}$$

$$= \lim_{k\to+\infty} \frac{\gamma(\tilde{\phi}^k) + \nu(e_1) - \nu(e_2)}{k} \leqslant \gamma(\tilde{\phi}).$$

As μ is homogeneous, we have

$$-\mu(\tilde{\phi}) = \mu(\tilde{\phi}^{-1})$$

for any $\tilde{\phi}$ and thus

$$-\mu(\tilde{\phi}) = \mu(\tilde{\phi}^{-1}) \leqslant \gamma(\tilde{\phi}^{-1}) = \gamma(\tilde{\phi}).$$

Thus,

$$|\mu(\tilde{\phi})| \leqslant \gamma(\tilde{\phi}).$$

This completes the proof of Proposition 98. □

One can strengthen the statement as follows.

Proposition 99. *The function*

$$\mu : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$$

descends to $\mathrm{Ham}(M,\omega)$ *i.e. if* $\tilde{\phi}$ *and* $\tilde{\psi}$ *have the same endpoint, then* $\mu(\tilde{\phi}) = \mu(\tilde{\psi})$. *Thus, for any* $\phi \in \mathrm{Ham}(M,\omega)$, *we define*

$$\mu(\phi) := \mu(\tilde{\phi})$$

where $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$ *is any element having* ϕ *as the endpoint. We can thus define a map*

$$\mu : \mathrm{Ham}(M,\omega) \to \mathbb{R}.$$

It satisfies

$$|\mu(\phi)| \leqslant \gamma(\phi)$$

for any $\phi \in \mathrm{Ham}(M,\omega)$.

Remark 100. *The Hofer Lipschitz continuity of* μ *follows automatically from Proposition 99 since* $\gamma(\phi) \leqslant d_{\mathrm{Hof}}(\mathrm{id},\phi)$.

Proof of Proposition 99. It suffices to show $\mu|_{\pi_1(\mathrm{Ham}(M,\omega))} \equiv 0$ where we see $\pi_1(\mathrm{Ham}(M,\omega)) \subset \widetilde{\mathrm{Ham}}(M,\omega)$. This is for the following reason.

Assume $\mu|_{\pi_1(\mathrm{Ham}(M,\omega))} \equiv 0$. Let $\tilde{\phi},\tilde{\psi}$ be two homotopy classes of Hamiltonian paths having the same endpoint. For any $k \in \mathbb{N}$, $(\tilde{\phi}^{-1})^k\tilde{\psi}^k$ defines a homotopy class of a Hamiltonian loop i.e. an element in $\pi_1(\mathrm{Ham}(M,\omega))$. Since μ is a quasimorphism on $\widetilde{\mathrm{Ham}}(M,\omega)$, there exists a constant $C > 0$ such that

$$|\mu((\tilde{\phi}^{-1})^k\tilde{\psi}^k) - \mu(\tilde{\psi}^k) - \mu((\tilde{\phi}^{-1})^k)| \leqslant C$$

for any $k \in \mathbb{N}$. From our assumption, the first term vanishes and

$$\mu(\tilde{\psi}^k) = k \cdot \mu(\tilde{\psi}),$$
$$\mu((\tilde{\phi}^{-1})^k) = -k \cdot \mu(\tilde{\phi}).$$

Thus, we have

$$\forall k \in \mathbb{N},\ k \cdot |\mu(\tilde{\psi}) - \mu(\tilde{\phi})| \leqslant C.$$

Therefore, we attain $\mu(\tilde{\phi}) = \mu(\tilde{\psi})$.

Now, we prove $\mu|_{\pi_1(\mathrm{Ham}(M,\omega))} \equiv 0$. We make use of the following theorem proved in [Kaw21]. We restate it with a special emphasis on a particular case which will be used in our argument:

Theorem 101. *([Kaw21, Theorem 4(1)])*
Let (M,ω) be a monotone symplectic manifold. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_{C^0}(\mathrm{id}, \phi_H) < \delta$, then

$$\gamma(H) < \frac{\dim(M)}{N_M} \cdot \lambda_0 + \varepsilon$$

where N_M denotes the minimal Chern number.
In particular, for any $\psi \in \pi_1(\mathrm{Ham}(M,\omega))$, we have

$$\gamma(\psi) \leqslant \frac{\dim(M)}{N_M} \cdot \lambda_0.$$

Now we continue the proof of Proposition 99. Let $\psi \in \pi_1(\mathrm{Ham}(M,\omega))$. For any $k \in \mathbb{N}$, we have

$$k \cdot |\mu(\psi)| = |\mu(\psi^k)| \leqslant \gamma(\psi^k) \leqslant \frac{\dim(M)}{N_M} \cdot \lambda_0.$$

Thus,

$$|\mu(\psi)| \leqslant \lim_{k \to +\infty} \frac{\dim(M)}{N_M} \cdot \frac{\lambda_0}{k} = 0.$$

This completes the proof of the first assertion. The second follows immediately from Proposition 98. □

Remark 102. *The estimate of the spectral norm for Hamiltonian loops that appear in Theorem 101 can be deduced by using basic facts about the Seidel elements as well.*

We will use the following criterion due to Shtern to detect the C^0-continuity of homogeneous quasimorphisms, see [Sht01] and [EPP12, Proposition 1.3].

Proposition 103. *([Sht01], [EPP12, Proposition 1.3])*
Let G be a topological group and $\mu : G \to \mathbb{R}$ a homogeneous quasimorphism. Then μ is continuous if and only if it is bounded on a neighborhood of the identity.

We now complete the proof of Theorem 96.

Proof of Theorem 96. By Propositions 99 and 103, the C^0-continuity of $\mu : \mathrm{Ham}(M,\omega) \to \mathbb{R}$ is reduced to the boundedness of the spectral norm γ around a C^0-neighborhood of *id*. Theorem 101 implies that the spectral norm is bounded around the identity of $\widetilde{\mathrm{Ham}}(M,\omega)$ (thus on $\mathrm{Ham}(M,\omega)$ as well) with respect to the C^0-topology when (M,ω) is monotone. This completes the proof of Theorem 96.
□

By Proposition 87, $QH^*(Q^n;\mathbb{C})$ is semi-simple and splits into a direct sum of two fields

$$QH^*(Q^n;\mathbb{C}) = Q_+ \oplus Q_-$$

and we decompose the identity element as follows:

$$1 = e_+ + e_-.$$

By the Entov–Polterovich theory, we obtain homogeneous (Calabi) quasimorphisms

$$\zeta_{e_\pm} : \widetilde{\mathrm{Ham}}(Q^n) \to \mathbb{R}$$

$$\zeta_{e_\pm}(\widetilde{\phi}) := \lim_{k\to+\infty} \frac{\rho(\widetilde{\phi}^k, e_\pm)}{k}.$$

In the second part of the proof (Section 4.5.4), we will prove the following.

Theorem 104. *For Q^n $(n = 2, 4)$,*

$$\zeta_{e_+} \neq \zeta_{e_-}.$$

Once we prove this, Theorems 96 and 104 imply that

$$\mu := \zeta_{e_+} - \zeta_{e_-}$$

defines a *non-trivial* homogeneous quasimorphism on $\mathrm{Ham}(Q^n)$ $(n = 2, 4)$ which is both C^0 and Hofer Lipschitz continuous and we complete the proof of Theorem 78.

Remark 105. *As remarked in Remark 80 (2), the composition of $\mu : \mathrm{Ham}(S^2\times S^2) \to \mathbb{R}$ and*

$$\mathrm{Ham}(S^2) \to \mathrm{Ham}(S^2 \times S^2)$$
$$\phi \mapsto \phi \times \phi$$

vanishes. This is because, by Proposition 99, we have

$$|\mu(\phi \times \phi)| \leqslant \gamma(\phi \times \phi) = 2\gamma(\phi)$$

for any $\phi \in \mathrm{Ham}(S^2)$. Note that the first and the second γ both denote the spectral norm but the former is for $\mathrm{Ham}(S^2 \times S^2)$ and the latter is for $\mathrm{Ham}(S^2)$. As remarked in Remark 129, the spectral norm for $\mathrm{Ham}(S^2)$ is bounded and thus the homogeneity of μ implies $\mu(\phi \times \phi) = 0$ for any $\phi \in \mathrm{Ham}(S^2)$.

4.5.2 Proof of Theorem 79

In this section, we prove Theorem 79.

Proof of Theorem 79. Denote the split of the semi-simple quantum homology ring $QH^*(M;\mathbb{C})$ by

$$QH^*(M;\mathbb{C}) = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_l.$$

We prove the following.

Claim 106. *For any Hamiltonian H,*

$$\overline{\rho}_{1_M}(H) = \max_{1\leqslant j\leqslant l} \overline{\rho}_{e_j}(H).$$

Before proving this Claim, we complete the proof of Theorem 79. This Claim implies that for any $\phi \in \mathrm{Ham}(M,\omega)$,

$$\overline{\gamma}(\phi) = \max_{1\leqslant i,j\leqslant l} \mu_{i,j}(\phi)$$

where

$$\mu_{i,j}(\phi_H) = \mu_{i,j}(H) := \overline{\rho}_{e_i}(H) - \overline{\rho}_{e_j}(H).$$

In fact,

$$\overline{\gamma}(\phi_H) = \overline{\rho}_{1_M}(H) + \overline{\rho}_{1_M}(\overline{H}) = \max_{1\leqslant i\leqslant l} \overline{\rho}_{e_i}(H) + \max_{1\leqslant j\leqslant l} \overline{\rho}_{e_j}(\overline{H})$$

$$= \max_{1\leqslant i,j\leqslant l} \overline{\rho}_{e_i}(H) - \overline{\rho}_{e_j}(H) = \max_{1\leqslant i,j\leqslant l} \mu_{i,j}(\phi).$$

By Theorem 96, we know that for each i, j, $\mu_{i,j}$ is C^0-continuous and therefore, $\overline{\gamma}$ is C^0-continuous.

We now prove the Claim. We first prove $\overline{\rho}_{1_M}(H) \geqslant \max_{1\leqslant j\leqslant l} \overline{\rho}_{e_j}(H)$. By the triangle inequality, we get

$$\rho(H, 1_M) + \nu(e_j) \geqslant \rho(H, e_j)$$

for any j and Hamiltonian H and thus

$$\overline{\rho}_{1_M}(H) \geqslant \overline{\rho}_{e_j}(H)$$

for any j and Hamiltonian H. Therefore,

$$\overline{\rho}_{1_M}(H) \geqslant \max_{1\leqslant j\leqslant l} \overline{\rho}_{e_j}(H)$$

for any Hamiltonian H.

Next, we prove $\overline{\rho}_{1_M}(H) \leqslant \max_{1\leqslant j\leqslant l} \overline{\rho}_{e_j}(H)$. A standard property of spectral invariants implies

$$\rho(H, 1_M) \leqslant \max_{1\leqslant j\leqslant l} \rho(H, e_j)$$

as $1_M = e_1 + e_2 + \cdots + e_l$ and thus

$$\overline{\rho}_{1_M}(H) \leqslant \max_{1\leqslant j\leqslant l} \overline{\rho}_{e_j}(H)$$

for any Hamiltonian H. This completes the proof of the Claim.

□

4.5.3 Comparing different quantum cohomology rings

In the first part of the proof of Theorem 78, we have used the quantum cohomology ring denoted by $QH^*(M;\mathbb{C})$ but in the second part of the proof, we work with a different quantum cohomology ring, namely the quantum cohomology ring with the universal Novikov field which is denoted by $QH^*(M;\Lambda)$. In this section, we explain the different advantages of working with $QH^*(M;\mathbb{C})$ and $QH^*(M;\Lambda)$. Working with these two different quantum cohomology rings plays a crucial role not only in the proof of Theorem 78 but also in the proof of Theorem 82. We also compare spectral invariants of a quantum cohomology class in $QH^*(M;\mathbb{C})$ and its embedded quantum cohomology class in $QH^*(M;\Lambda)$. Note that results in this subsection concern not only the n-quadric but any monotone symplectic manifold.

Let (M,ω) be a monotone symplectic manifold. Recall from Section 4.4.1 that $QH^*(M;\mathbb{C})$ was defined by

$$QH^*(M;\mathbb{C}) := H^*(M;\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1}, t|]$$

where the variable t represents an element in $\pi_2(M)$ that satisfies

$$\omega(t) = \lambda_0,\ c_1(t) = N_M.$$

On the other hand $QH^*(M;\Lambda)$ is defined by

$$QH^*(M;\Lambda) := H^*(M;\mathbb{C}) \otimes_{\mathbb{C}} \Lambda$$

and one can embed $QH^*(M;\mathbb{C})$ to $QH^*(M;\Lambda)$ by

$$\iota : QH^*(M;\mathbb{C}) \hookrightarrow QH^*(M;\Lambda)$$
$$t \mapsto T^{+\lambda_0}$$

and ι is a ring homomorphism.

We explain the different advantages of working with $QH^*(M;\mathbb{C})$ and $QH^*(M;\Lambda)$ as well as examples of cases where those advantages are used.

• *The advantage of working with $QH^*(M;\mathbb{C})$:*

1. $QH^*(M;\mathbb{C})$ carries a $\mathbb{Z}$-grading while $QH^*(M;\Lambda)$ does not. Thus, to use spectral invariants it is preferable to work with $QH^*(M;\mathbb{C})$ than $QH^*(M;\Lambda)$ as the $\mathbb{Z}$-grading allows us to study both the action and the index of spectral invariants.

   Example 107. *Theorem 101, which plays a crucial role in the first part of the proof of Theorem 78, is proven by using the information of both the action and the index of spectral invariants and thus, it is proven only in the setting where we have a $\mathbb{Z}$-grading of the quantum cohomology ring.*

2. The algebraic structure of $QH^*(M;\mathbb{C})$ tends to be simpler than that of $QH^*(M;\Lambda)$.

   Example 108. *$QH^*(\mathbb{C}P^2;\mathbb{C})$ is a field and $QH^*(\mathbb{C}P^2;\Lambda)$ splits into a direct sum of three fields. $QH^*(S^2\times S^2;\mathbb{C})$ splits into a direct sum of two fields and $QH^*(S^2\times S^2;\Lambda)$ splits into a direct sum of four fields.*

   The quantum cohomology ring $QH^*(\mathbb{C}P^2;\mathbb{C})$ being a field has important consequences as pointed out in Remark 129 which do not follow only from semi-simplicity. This is precisely what we use in the proof of Theorem 82.

• *The advantage of working with $QH^*(M;\Lambda)$:* With Λ-coefficients, we have a very rich Lagrangian Floer theory developed by Fukaya–Oh–Ohta–Ono. In particular, the superpotential techniques are very useful to detect Lagrangian submanifolds that have non-trivial Floer coholomogy groups.

Example 109. *Finding certain Lagrangian submanifolds that have non-trivial Floer cohomology groups via superpotential techniques is a key step in the second part of the proof of Theorem 78 explained in Section 4.5.4.*

To sum up, in the first part of the proof of Theorem 78 (Section 4.5.1), we need to work with $QH^*(M;\mathbb{C})$ while in the second part of the proof of Theorem 78 (Section 4.5.4), we greatly benefit from the advantage of working with $QH^*(M;\Lambda)$. In order to connect arguments in Part 1 and Part 2 which are done in different algebraic settings, we will need the following comparison between spectral invariants of a quantum cohomology class in $QH^*(M;\mathbb{C})$ and its embedded quantum cohomology class in $QH^*(M;\Lambda)$.

Lemma 110. *Let (M,ω) be a monotone symplectic manifold.*

1. *For any $a\in QH^*(M;\mathbb{C})\backslash\{0\}$, we have*

$$\rho(H,\iota(a)) = \rho(H,a)$$

   for any Hamiltonian H.

2. *Let $e \in QH^0(M;\mathbb{C})$ be an idempotent. Assume that $e \cdot QH^{even}(M;\mathbb{C})$ is a field. Then, we have*
$$\zeta_{\iota(e)}(\widetilde{\phi}) = \zeta_e(\widetilde{\phi})$$
for any $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$. In particular,
$$\zeta_{\iota(e)} : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$$
is a homogenous quasimorphism.

The value $\rho(\cdot, a)$ denotes the spectral invariant of $a \in QH^*(M;\mathbb{C})$ while the value $\rho(\cdot, \iota(a))$ denotes the spectral invariant of its embedded element $\iota(a) \in QH^*(M;\Lambda)$.

Remark 111. *A priori Lemma 110 (2) is not obvious as we do not know if $\iota(e)$ is a unit of a field factor of $QH^*(M;\Lambda)$ i.e. $\iota(e)\cdot QH^*(M;\Lambda)$ is a field (Theorem 88). For example, $QH^*(\mathbb{C}P^2;\mathbb{C})$ is a field but $QH^*(\mathbb{C}P^2;\Lambda)$ splits into a direct sum of three fields and the identity element $1 \in QH^*(\mathbb{C}P^2;\mathbb{C})$ embeds to $1_\Lambda \in QH^*(\mathbb{C}P^2;\Lambda)$ which is not an unit of a field factor.*

Proof of Lemma 110. 1. Essentially the equality follows from [UZ16, Propositions 2.21, 6.6]. See also [BC09, Section 5.4] where they also explain that spectral invariants are preserved under field extension. Nevertheless, we give a brief sketch of the proof.

By the continuity property of spectral invariants (Proposition 30 (1)), it is enough to prove the case where H is non-degenerate. Let $a \in QH^*(M;\mathbb{C})\backslash\{0\}$ and H be a non-degenerate Hamiltonian. Now, consider the natural embedding of the Floer chain complex $CF_*(H)$ into the Fukaya–Oh–Ohta–Ono type Floer chain complex $CF_*(H;\Lambda)$
$$j : CF_*(H) \hookrightarrow CF_*(H;\Lambda)$$
and the induced map on homology
$$j_* : HF_*(H) \to HF_*(H;\Lambda).$$
Note that
$$PSS_{H,\Lambda} \circ PD \circ \iota = j_* \circ PSS_H \circ PD$$
where PSS_H on the right hand side denotes the PSS-isomorphism
$$PSS_H : QH_*(M;\mathbb{C}) \xrightarrow{\sim} HF_*(H)$$
while PSS_H on the left hand side denotes the PSS-isomorphism
$$PSS_{H,\Lambda} : QH_*(M;\Lambda) \xrightarrow{\sim} HF_*(H;\Lambda)$$
and PD denotes the Poincaré duality between quantum homology and quantum cohomology.
$$\begin{array}{ccccc} HF_*^\tau(H) & \xrightarrow{i_*^\tau} & HF_*(H) & \xleftarrow{PSS_H \circ PD} & QH^*(M;\mathbb{C}) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle \iota} \\ HF_*^\tau(H;\Lambda) & \xrightarrow{i_*^\tau} & HF_*(H;\Lambda) & \xleftarrow{PSS_{H,\Lambda} \circ PD} & QH^*(M;\Lambda) \end{array}$$

As j_* preserves the action filtration, the diagram commutes and for tautological reasons, we get

$$\rho(H, \iota(a)) = \rho(H, a).$$

2. This follows immediately from (1).

□

We obtain the following from Lemma 110 (2).

Corollary 112. *Let (M, ω) be a monotone symplectic manifold. Assume that $e \in QH^0(M; \mathbb{C})$ is an idempotent and $e \cdot QH^{even}(M; \mathbb{C})$ is a field. If a subset $S \subset M$ is $\iota(e)$-heavy, then S is e-superheavy.*

Proof of Corollary 112. Lemma 110 (2) implies that S is e-heavy. However, as $e \in QH^0(M; \mathbb{C})$ is a unit of a field factor of $QH^{even}(M; \mathbb{C})$, ζ_e is a homogeneous quasi-morphism so S is e-superheavy. □

4.5.4 Proof of Theorem 78–Part 2

In this subsection, we prove Theorem 104 which was used to complete the proof of Theorem 78 in the end of Section 4.5.1.

Proof of Theorem 104. We argue the cases $n = 2$ and $n = 4$ separately.

• *Case $n = 2$:* In this case, $\zeta_{e_+} \neq \zeta_{e_-}$ was already proven by Eliashberg–Polterovich in [EliP10] by an approach different to what we discuss in this section. In this section, we will prove $\zeta_{e_+} \neq \zeta_{e_-}$ by using the following result of Fukaya–Oh–Ohta–Ono [FOOO12], [FOOO19]. The same argument will be used in the case where $n = 4$.

Theorem 113. (*[FOOO12, Lemma 23.3(1), Lemma 23.5]*)

1. *In $S^2 \times S^2$, there exists a monotone Lagrangian submanifold L_0 diffeomorphic to T^2 such that*
$$HF((L_0, b_0); \Lambda) \neq 0$$
for a certain bounding cochain $b_0 \in H^1(L_0; \Lambda_0)/H^1(L_0; 2\pi i\mathbb{Z})$.

2. *The anti-diagonal in $S^2 \times S^2$ denoted by L_1 is unobstructed and satisfies*
$$HF(L_1; \Lambda) \neq 0.$$

3. *L_0 and L_1 are disjoint:*
$$L_0 \cap L_1 = \emptyset.$$

Now, consider the natural embedding

$$\iota : QH^*(Q^2; \mathbb{C}) \hookrightarrow QH^*(Q^2; \Lambda).$$

As the closed-open map maps the identity element of the quantum cohomology ring to the identity element of the Lagrangian Floer cohomology group, we have

- $\mathcal{CO}^0_b(1) = \mathrm{PD}([L_0]) \neq 0 \in HF^*((L_0, b); \Lambda)$
- $\mathcal{CO}^0(1) = \mathrm{PD}([L_1]) \neq 0 \in HF^*(L_1; \Lambda)$.

Since

$$1 = \iota(e_+) + \iota(e_-),$$

it is

$$\mathcal{CO}^0_b(\iota(e_+)) \neq 0 \quad or \quad \mathcal{CO}^0_b(\iota(e_-)) \neq 0.$$

As $\iota(e_\pm)$ are both idempotents, by Theorem 93, we deduce that L_0 is at least either $\iota(e_+)$-heavy or $\iota(e_-)$-heavy. Corollary 112 implies that L_0 is at least either e_+-superheavy or e_--superheavy. Next, by looking at the second equation, the same argument implies that L_1 is at least either e_+-superheavy or e_--superheavy. As L_0 and L_1 are disjoint, Proposition 92 implies that they cannot be both e_+-superheavy or both e_--superheavy at once. This implies

$$\zeta_{e_+} \neq \zeta_{e_-}.$$

Remark 114. *From this argument, it follows that either*

- *L_0 is e_+-superheavy and L_1 is e_--superheavy*
- *L_0 is e_--superheavy and L_1 is e_+-superheavy*

but it is not clear which one of the two actually holds. Eliashberg–Polterovich's approach shows that the former holds [EliP10].

•*Case $n = 4$*: The key of the proof is to find two disjoint Lagrangian submanifolds in Q^4 having non-vanishing Floer cohomology just as in the previous case. We use results of Nishinou–Nohara–Ueda and Nohara–Ueda which we will now briefly explain.

The relation between the superpotential and Lagrangian Floer cohomology has been studied extensively. After a pioneering work of Cho [Cho04], Fukaya–Oh–Ohta–Ono computed the superpotential for toric symplectic manifolds in [FOOO10]. Later, Nishinou–Nohara–Ueda computed the superpotential for symplectic manifolds admitting a toric degeneration in [NNU10]. This lead Nohara–Ueda to study the Floer cohomology of non-torus fibers in partial flag manifolds in [NU16]. We state some of their results which will be relevant for us.

Theorem 115. (*[NNU10, Theorem 10.1, Section 11], [NU16, Theorem 1.2, Example 3.3]*)

Let $\Phi : Gr_{\mathbb{C}}(2,4) \to \mathbb{R}^4$ be the Gelfand–Cetlin system with the Gelfand–Cetlin polytope $\Delta := \Phi(Gr_{\mathbb{C}}(2,4))$. Denote the fiber of $u \in \Delta$ by $L(u)$:

$$L(u) := \Phi^{-1}(u).$$

We identify $Gr_{\mathbb{C}}(2,4)$ with the adjoint orbit of

$$\lambda = \mathrm{diag}(4,4,0,0)$$

so that it is monotone.

1. *For*

$$u_0 := (2,3,1,2) \in \mathrm{Int}(\Delta),$$

there exists a bounding cochain $b \in H^1(L(u_0);\Lambda)$ such that

$$HF((L(u_0),b);\Lambda) \simeq QH^*(T^4;\Lambda).$$

2. *There exists $u_1 \in \partial\Delta$ such that its fiber $L(u_1)$ is Lagrangian and diffeomorphic to $U(2) \simeq S^1 \times S^3$ with non-trivial Floer cohomology:*

$$HF((L(u_1), \pm\pi i/2 \cdot e_1); \Lambda) \simeq QH^*(S^1 \times S^3; \Lambda)$$

for a bounding cochain $b = \pm\pi i/2 \cdot e_1$ where e_1 is the generator of $H^1(L(u_1); \mathbb{Z})$.

Remark 116. 1. *Q^4 is symplectomorphic to $Gr_{\mathbb{C}}(2,4)$.*

2. *Theorem 115 (1) was proven for any identification of $Gr_{\mathbb{C}}(2,4)$ with $\mathcal{O}_\lambda$ where*

$$\lambda = \mathrm{diag}(2\alpha, 2\alpha, 0, 0),$$

$$u_0 := (\alpha, 3\alpha/2, \alpha/2, \alpha) \in \mathrm{Int}(\Delta)$$

for any $\alpha > 0$. If we choose $\alpha = 2$, the Kirillov-Konstant form ω_λ defines a monotone symplectic form by the monotonicity criterion in Section 4.4.6.

3. *Note that $L(u_0) \simeq T^4$ and $L(u_1) \simeq U(2) \simeq S^1 \times S^3$ are both monotone since they are both located in the center of a Lagrangian facet of the Gelfand–Cetlin polytope. This follows from a result of Yunhyung Cho and Yoosik Kim [?] where they classify monotone fibers of Gelfand–Cetlin polytopes.*

Let $L(u_0), L(u_1)$ be as in Theorem 115. We argue exactly as in the case where $n = 2$. As the closed-open map maps the identity element of the quantum cohomology ring to the identity element of the Lagrangian Floer cohomology group, we have

- $\mathcal{CO}^0_b(1) = \mathrm{PD}([L(u_0)]) \neq 0 \in HF^*((L(u_0), b); \Lambda)$
- $\mathcal{CO}^0_{\pm\pi i/2\cdot e_1}(1) = \mathrm{PD}([L(u_1)]) \neq 0 \in HF^*((L(u_1), \pm\pi i/2 \cdot e_1); \Lambda)$.

Since

$$1 = \iota(e_+) + \iota(e_-),$$

the first equation and Theorem 93 imply that $L(u_0)$ is e_+-superheavy or e_--superheavy. We have used that by Corollary 112, $\iota(e_\pm)$-heaviness is equivalent to $e_\pm$-superheaviness. Next, by looking at the second equation, the same argument implies that $L(u_1)$ is e_+-superheavy or e_--superheavy. As $L(u_0)$ and $L(u_1)$ are disjoint (recall that they are fibers of distinct points in the Gelfand–Cetlin polytope), we conclude that they cannot be both e_+-superheavy or both e_--superheavy at once. This implies

$$\zeta_{e_+} \neq \zeta_{e_-}.$$

□

4.5.5 Generalization of Theorem 78

In this section, we prove the following slight generalization of Theorem 78.

Theorem 117. *Let (M, ω) be a monotone symplectic manifold (with the same monotonicity constant as Q^n, $n = 2, 4$) such that $QH^*(M; \mathbb{C})$ is semi-simple. Assume that there exists a Lagrangian submanifold L of (M, ω) such that $HF((L, b); \Lambda) \neq 0$ for some bounding cochain b. Then, there exists a non-trivial homogeneous quasimorphism*

$$\mu : \mathrm{Ham}(Q^n \times M) \to \mathbb{R}$$

which is both C^0-continuous and Hofer Lipschitz continuous where $Q^n \times M$ $(n = 2, 4)$ denotes the monotone product.

Remark 118. *The existence of a homogeneous quasimorphism on* $\mathrm{Ham}(Q^n \times M)$, $n = 2, 4$ *(instead of on the universal cover) where* (M, ω) *is as in Theorem 117 was not known to the best of our knowledge. Note that examples of* (M, ω) *which satisfy the assumptions in Theorem 117 include* $\mathbb{C}P^n$*, 1, 2 and 3 point monotone blow-ups of* $\mathbb{C}P^2$*,* Q^n *and their monotone products.*

We start with some preliminary results on the product of semi-simple algebras.

Let (M_j, ω_j) $(j = 1, 2)$ be monotone symplectic manifolds. Denote the generators of $\pi_2(M_j)/\mathrm{Ker}(\omega_j)$ by s_j which satisfy

$$\omega_j(s_j) = \lambda_{M_j}, \quad c_1(TM_j)(s_j) = N_{M_j}$$

where λ_{M_j} denotes the rationality constant and N_{M_j} denotes the minimal Chern number of (M_j, ω_j).

In the case where the monotonicity constants of (M_j, ω_j) $(j = 1, 2)$ coincide, one can consider their product $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ which is also a monotone symplectic manifold. It has the same monotonicity constant as (M_j, ω_j) $(j = 1, 2)$ and its minimal Chern number $N_{M_1 \times M_2}$ is the greatest common divisor of N_{M_1} and N_{M_2}. As above, we denote the generator of the $\pi_2(M_1 \times M_2)/\mathrm{Ker}(\omega_1 \oplus \omega_2)$ by s which satisfies

$$(\omega_1 \oplus \omega_2)(s) = \lambda_{M_1 \times M_2}, \quad c_1(T(M_1 \times M_2))(s) = N_{M_1 \times M_2}.$$

Entov–Polterovich proved the following in [EP08].

Theorem 119. *([EP08, Theorem 5.1, Theorem 6.1])*
Let (M_j, ω_j) $(j = 1, 2)$ *be monotone symplectic manifolds. Assume that their quantum homology rings*

$$QH_{even}(M_j; \mathbb{C}) = H_{even}(M_j; \mathbb{C}) \otimes \mathbb{C}[|s_j^{-1}, s_j]$$

are both semi-simple and that at least one of M_j, $j = 1, 2$ *satisfies* $H_{2k-1}(M_j; \mathbb{C}) = 0$ *for all* $k \in \mathbb{Z}$*. Then,*

$$QH_{even}(M_1 \times M_2; \mathbb{C}) = H_{even}(M_1 \times M_2; \mathbb{C}) \otimes \mathbb{C}[|s^{-1}, s]$$

is semi-simple.

One can consider the following embedding:

$$\sigma : QH_*(M_1; \mathbb{C}) \hookrightarrow QH_*(M_1 \times M_2; \mathbb{C})$$

$$a \cdot s_1 \mapsto a \otimes [M_2] \cdot s^{N_{M_1}/N_{M_1 \times M_2}}.$$

Of course, one can consider an analogous embedding for M_2.

We are now ready to prove Theorem 117. We will use the cohomological counterpart of the results above.

Proof of Theorem 117. As $QH^*(M; \mathbb{C})$ is semi-simple, it splits into a direct sum of fields $\{Q_j\}$:

$$QH^*(M; \mathbb{C}) = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_l.$$

We decompose the identity element $1_M \in QH^*(M; \mathbb{C})$ with respect to this decomposition:

$$1_M = e_1 + e_2 + \cdots + e_l$$

where e_j is a unit of Q_j. As we argued in the proof of Theorem 78, as

$$\mathcal{CO}_b^0(1_M) = \mathcal{CO}_b^0(\iota(e_1)) + \mathcal{CO}_b^0(\iota(e_2)) + \cdots + \mathcal{CO}_b^0(\iota(e_l)) \neq 0,$$

Proposition 93) implies that L is $\iota(e_j)$-heavy for some $j \in \{1, 2, \cdots, l\}$. Without loss of generality, we assume $j = 1$. Moreover, Corollary 112 implies that L is actually e_1-superheavy.

Recall that in the proof of Theorem 78, we have seen that in Q^n, $n = 2, 4$, there exist two disjoint Lagrangian submanifolds L_0 and L_1 which satisfy either one of the following:

1. L_0 is e_+-superheavy and L_1 is e_--superheavy.
2. L_0 is e_--superheavy and L_1 is e_+-superheavy.

Without loss of generality, we assume the former. By [EP09, Theorem 1.7], $L_0 \times L$ is $e_+ \otimes e_1$-superheavy and $L_1 \times L$ is $e_- \otimes e_1$-superheavy.

Now, as $QH^*(Q^n \times M; \mathbb{C})$ is also semi-simple, we consider its decomposition into fields and the decomposition of the identity element $1_{Q^n \times M} \in QH^*(Q^n \times M; \mathbb{C})$ with respect to this split:

$$QH^*(Q^n \times M; \mathbb{C}) = Q'_1 \oplus Q'_2 \oplus \cdots \oplus Q'_{l'}$$

$$1_{Q^n \times M} = e'_1 + e'_2 + \cdots + e'_{l'}$$

for some $l' \in \mathbb{N}$. As $e_+ \otimes e_1$ and $e_- \otimes e_1$ are idempotents of $QH^*(Q^n \times M; \mathbb{C})$, by Theorem 1.5 (3) in [EP09], there exist $j_0, j_1 \in \{1, 2, \cdots, l'\}$ such that $L_0 \times L$ is e'_{j_0}-heavy and $L_1 \times L$ is e'_{j_1}-heavy. For $\zeta_{e'_{j_0}}, \zeta_{e'_{j_1}} : \widetilde{\mathrm{Ham}}(Q^n \times M) \to \mathbb{R}$ both being homogeneous quasimorphisms, heaviness and superheaviness are equivalent for e'_{j_0} and e'_{j_1}, thus $L_0 \times L$ is e'_{j_0}-superheavy and $L_1 \times L$ is e'_{j_1}-superheavy. As $L_0 \times L$ and $L_1 \times L$ are disjoint, $L_1 \times L$ is not e'_{j_0}-superheavy. This implies

$$\zeta_{e'_{j_0}} \neq \zeta_{e'_{j_1}}.$$

Thus, it follows from Theorem 96 that

$$\mu := \zeta_{e'_{j_0}} - \zeta_{e'_{j_1}}$$

defines a non-trivial homogeneous quasimorphism

$$\mu : \mathrm{Ham}(Q^n \times M) \to \mathbb{R}$$

which is both C^0-continuous and Hofer Lipschitz continuous.

□

4.5.6 Proof of Theorem 82

In this subsection, we precisely state the question of Polterovich–Wu which appeared in Section 4.3.2 and prove Theorem 82 as an application of Lemma 110.

According to a computation due to Wu [Wu15], $QH^*(\mathbb{C}P^2; \Lambda)$ is semi-simple and splits into a direct sum of three fields:

$$QH^*(\mathbb{C}P^2; \Lambda) = Q_1 \oplus Q_2 \oplus Q_3.$$

We denote the corresponding split of the identity element $1_\Lambda \in QH^*(\mathbb{C}P^2;\Lambda)$ as follows:

$$1_\Lambda = e_1 + e_2 + e_3$$

where $\{e_j\}_{j=1,2,3}$ are

$$e_j := \frac{1}{3}(1_\Lambda + \theta^j u T^{-\frac{1}{3}\lambda_0} + \theta^{2j} u^2 T^{-\frac{2}{3}\lambda_0}),$$

u is the generator of $H^2(\mathbb{C}P^2;\mathbb{C})$ and

$$\lambda_0 := \langle \omega_{FS}, [\mathbb{C}P^1] \rangle,\ \theta := e^{\frac{2\pi i}{3}}.$$

Note that u satisfies

$$u^3 = T^{\lambda_0}.$$

These idempotents give rise to three homogenous quasimorphisms (or symplectic quasi-states) $\{\zeta_{e_j}\}_{j=1,2,3}$:

$$\zeta_{e_j} : \widetilde{\mathrm{Ham}}(\mathbb{C}P^2) \to \mathbb{R}$$

$$\zeta_{e_j}(\widetilde{\phi}) := \lim_{k\to+\infty} \frac{\rho(\widetilde{\phi}^k, e_j)}{k}$$

for each $j = 1, 2, 3$.

Remark 120. *It will not be used in this paper but we point out that ζ_{e_j} descends to* $\mathrm{Ham}(\mathbb{C}P^2)$ *as* $\pi_1(\mathrm{Ham}(\mathbb{C}P^2)) = \mathbb{Z}_3$.

Polterovich posed the following question:

Question 121. (*[Wu15, Remark 5.2]*)
Is it possible to distinguish the symplectic quasi-states/morphisms for the three idempotents of $QH^*(\mathbb{C}P^2;\Lambda)$?

Note that ζ_j which appeared in the statement of this question in Section 4.3.2 is precisely ζ_{e_j} defined above. We now prove Theorem 82 which answers this question in the negative.

Proof of Theorem 82. We will show that

$$\zeta_{e_j} = \zeta_{1_\Lambda}$$

for all $j = 1, 2, 3$ where $1_\Lambda \in QH^*(\mathbb{C}P^2;\Lambda)$. By the triangle inequality, we have

$$\rho(\widetilde{\phi}^k, e_j) \leqslant \rho(\widetilde{\phi}^k, 1_\Lambda) + \nu(e_j)$$

for any $k \in \mathbb{N}$. Thus,

$$\zeta_{e_j} \leqslant \zeta_{1_\Lambda}$$

where

$$\zeta_{1_\Lambda}(\widetilde{\phi}) := \lim_{k\to+\infty} \frac{\rho(\widetilde{\phi}^k, 1_\Lambda)}{k}$$

for $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(\mathbb{C}P^2)$. By Lemma 110, as $QH^*(\mathbb{C}P^2;\mathbb{C})$ is a field, we have

$$\zeta_{1_\Lambda} = \zeta_1$$

where $1 \in QH^*(\mathbb{C}P^2;\mathbb{C})$. Thus, we have

$$\zeta_{e_j} \leqslant \zeta_1$$

and as ζ_{e_j}, ζ_1 are both homogeneous quasimorphisms, it follows that

$$\zeta_{e_j} = \zeta_1.$$

Thus we have proven

$$\zeta_{e_1} = \zeta_{e_2} = \zeta_{e_3} = \zeta_1 = \zeta_{1_\Lambda}.$$

□

Remark 122. *A similar argument is applicable to the case where $M := S^2 \times S^2$: As we have seen in Section 4.4.2, $QH^*(S^2 \times S^2;\mathbb{C})$ splits into a direct sum of two fields. On the other hand, Fukaya–Oh–Ohta–Ono have computed in the proof of [FOOO19, Theorem 23.4], $QH^*(S^2 \times S^2;\Lambda)$ splits into a direct sum of four fields. Denote the two units of field factors of $QH^*(S^2 \times S^2;\mathbb{C})$ by $e_\pm$ which satisfy*

$$\mathrm{PD}(e_\pm) = \frac{[M] \pm [pt \times pt]s}{2}.$$

Denote the four units of the field factors of $QH^*(S^2 \times S^2;\Lambda)$ by $e_{(\pm,\pm)}$ which satisfy

$$\mathrm{PD}(e_{(+,\pm)}) = \frac{[M] + P \cdot T^{\lambda_0}}{4} \pm \frac{(A+B)T^{\lambda_0/2}}{4},$$

$$\mathrm{PD}(e_{(-,\pm)}) = \frac{[M] - P \cdot T^{\lambda_0}}{4} \pm \frac{(A-B)T^{\lambda_0/2}}{4}$$

where

$$[M] := [S^2 \times S^2], \quad P := [pt \times pt],$$
$$A := [S^2 \times pt], \quad B := [pt \times S^2].$$

By using

$$\iota(e_+) = e_{(+,+)} + e_{(+,-)},$$
$$\iota(e_-) = e_{(-,+)} + e_{(-,-)},$$

we obtain

$$\zeta_{e_+} = \zeta_{\iota(e_+)} = \zeta_{e_{(+,+)}} = \zeta_{e_{(+,-)}},$$
$$\zeta_{e_-} = \zeta_{\iota(e_-)} = \zeta_{e_{(-,+)}} = \zeta_{e_{(-,-)}}.$$

4.5.7 Results on Lagrangian intersections

In this section, we discuss consequences of the proof of Theorem 78 for Lagrangian intersections.

In proving Theorem 78, detecting disjoint Lagrangian submanifolds whose Floer cohomology is non-trivial is a crucial step which we discussed in Section 4.5.1. As a by-product, we obtain certain results on Lagrangian intersections.

A closed Lagrangian submanifold L is called monotone if it satisfies

$$\omega|_{\pi_2(M,L)} = \lambda \cdot \mu|_{\pi_2(M,L)}$$

for some $\lambda > 0$ where $\mu = \mu_L$ denotes the Maslov class. The minimal Maslov number N_L is the positive generator of $\mu(\pi_2(M, L))$ i.e. $\mu(\pi_2(M, L)) = N_L\mathbb{Z}$. Recall that Λ denotes the universal Novikov field

$$\Lambda = \{\sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j\to+\infty} \lambda_j = +\infty\}.$$

All the Lagrangian submanifolds concerned in the following are assumed to be oriented and relatively spin (for its definition, see Section 4.4.3). The statements in this section include the notion of deformed Floer cohomology defined by Fukaya–Oh–Ohta–Ono [FOOO09]. For a quick review, see Section 4.4.3.

The main statement for Lagrangian intersection is the following.

Theorem 123. *In Q^n $(n = 2, 4)$, there exist two monotone Lagrangian submanifolds L_0, L_1 that satisfy the following:*

1. *L_0 and L_1 are respectively diffeomorphic to*
   - *T^2 and S^2 when $n = 2$.*
   - *T^4 and $S^1 \times S^3$ when $n = 4$.*
2. *L_0 and L_1 are disjoint.*
3. *Let L be a Lagrangian submanifold in Q^n which is*
   - *oriented when $n = 2$.*
   - *oriented and relatively spin when $n = 4$.*

   If L is disjoint from both L_0 and L_1 i.e. if

   $$L \cap (L_0 \cup L_1) = \emptyset$$

   then

   $$HF((L, b); \Lambda) = 0$$

   for any bounding cochain b.

Remark 124.
1. *Under the symplectomophism between Q^2 and $S^2 \times S^2$, the Lagrangian submanifolds L_0 and L_1 in Theorem 123 correspond respectively to the so-called exotic torus defined by*

   $$\{(x, y) \in S^2 \times S^2 : x_1y_1 + x_2y_2 + x_3y_3 = -1/2, x_3 + y_3 = 0\}$$

   which was studied in [EP09], [FOOO12] and the anti-diagonal

   $$\{(x, y) \in S^2 \times S^2 : x = -y\}.$$

2. *For more information about the two Lagrangian submanifolds in Theorem 123, see Theorems 113, 115 and related references.*

For example, Theorem 123 can be applied to the following two well-known Lagrangians in Q^2 and Q^4. In Q^2, there is a Lagrangian torus T which corresponds to the product of equatorial circles $S^1 \times S^1$ in $S^2 \times S^2$ under the symplectomophism between Q^2 and $S^2 \times S^2$. In Q^4, there is the standard Lagrangian sphere S^4 which appears as the real locus

$$S^4 = \{(x_0 : \cdots : x_4) \in \mathbb{C}P^5 : x_0^2 + \cdots + x_3^2 = x_4^2,\ x_j \in \mathbb{R}, j = 0, \cdots, 4\}.$$

These Lagrangians T and S^4 are known to have non-trivial Floer cohomology groups

$$HF(T; \Lambda) \neq 0,\ HF(S^4; \Lambda) \neq 0.$$

Theorem 123 directly implies the following.

Corollary 125. *Any Hamiltonian deformation of T in Q^2 or the standard Lagrangian sphere S^4 in Q^4 intersects either one of L_0 or L_1 in Theorem 123:*
For any $\phi \in \mathrm{Ham}(Q^2)$,

$$L_0 \cap \phi(T) \neq \emptyset \quad or \quad L_1 \cap \phi(T) \neq \emptyset.$$

For any $\phi \in \mathrm{Ham}(Q^4)$,

$$L_0 \cap \phi(S^4) \neq \emptyset \quad or \quad L_1 \cap \phi(S^4) \neq \emptyset.$$

Remark 126. *In Theorem 123, it is crucial that we consider Floer cohomology without bulk-deformation. As it was studied by Fukaya–Oh–Ohta–Ono [FOOO12] and Cho–Kim–Oh [CKO18], there exist Lagrangians in Q^n $(n = 2, 4)$ intersecting neither L_0 nor L_1 that have non-trivial bulk-deformed Floer cohomology.*

There are several ways to construct monotone Lagrangian submanifolds in Q^n such as the Albers–Frauenfelder-type construction [AF08] and the Biran-type construction [B01], [B06]. Their precise constructions and the relations among them are explained in [OU16]. In particular, Oakley–Usher constructs monotone Lagrangian submanifolds in Q^4 which are diffeomorphic to $S^1 \times S^3$ by these methods in [OU16, Section 1.2] denoted by $L^Q_{0,3}$ and $\mathbb{S}^Q_{0,3}$, which turn out to be Hamiltonian isotopic (see [OU16, Theorem 1.4]). However, the monotone Lagrangian submanifold L_1 in Q^4 which appeared in Theorem 123 is not Hamiltonian isotopic to these examples due to Oakley–Usher as L_1 has minimal Maslov number 4 (see [NU16, Section 4.4]) and Oakley–Usher's Lagrangian submanifold has minimal Maslov number 2. Thus, we have the following.

Proposition 127. *The 4-quadric Q^4 has two monotone Lagrangian submanifolds diffeomorphic to $S^1 \times S^3$ which are not Hamiltonian isotopic.*

Basically, Theorem 123 comes from the fact that the quantum cohomology ring $QH^*(Q^n; \mathbb{C}) = H^*(Q^n; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1}, t]]$ splits into a direct sum of two fields. In the case where the quantum cohomology ring does not split i.e. itself is a field, we have a stronger rigidity result as follows.

Proposition 128. *Let (M, ω) be a closed symplectic manifold for which the spectral pseudo-norm is bounded i.e.*

$$\sup\{\gamma(H) : H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})\} < +\infty.$$

Let L_1 be a Lagrangian submanifold such that

$$HF((L_1, b_1); \Lambda) \neq 0$$

for some bounding cochain b_1. Then, any Lagrangian submanifold L_2 which is disjoint from L_1 has a vanishing Floer cohomology:

$$HF((L_2, b_2); \Lambda) = 0$$

for any bounding cochain b_2.

Remark 129. *1. When (M, ω) is monotone, if $QH^*(M; \mathbb{C})$ is a field, then the spectral norm is bounded. Thus, Proposition 128 applies to $\mathbb{C}P^n$ (see [EP03]).*

2. Proposition 128 is not restricted to monotone symplectic manifolds. Examples of non-monotone symplectic manifolds for which the spectral norm is bounded includes a large one point blow-up of $\mathbb{C}P^2$ and $(S^2 \times S^2, \sigma \oplus \lambda\sigma)$ for $\lambda > 1$ where σ denotes an area form with $\int_{S^2} \sigma = 1$. See Section 4.5.7 for further remarks.

Proof of Theorem 123. We assume that there exists a bounding cochain b such that

$$HF((L, b); \Lambda) \neq 0$$

and show that L must intersect either L_0 or L_1. As the closed-open map maps the identity element of the quantum cohomology ring to the identity element of the Lagrangian Floer cohomology group, we have

$$\mathcal{CO}^0_b(1) = \mathrm{PD}([L]) \neq 0 \in HF^*((L, b); \Lambda).$$

Since

$$QH^*(Q^n; \mathbb{C}) := Q_+ \oplus Q_-$$

$$1 = e_+ + e_-,$$

Theorem 93 implies that L (is either $\iota(e_+)$-heavy or $\iota(e_-)$-heavy so by Corollary 112) is either e_+-superheavy or e_--superheavy where

$$\iota : QH^*(Q^n; \mathbb{C}) \hookrightarrow QH^*(Q^n; \Lambda).$$

(This was explained in more detail in the proof of Theorem 78.) If L intersects neither of L_0, L_1, then we have two disjoint sets which are either both e_+-superheavy or both e_--superheavy, which contradicts Proposition 92. Thus, L must intersect either L_0 or L_1 and this completes the proof. □

We now prove Proposition 128.

Proof of Proposition 128. Assume there exist two Lagrangian submanifolds L_1 and L_2 such that

$$L_1 \cap L_2 = \emptyset$$

and

$$HF((L_1, b_1); \Lambda) \neq 0, \ HF((L_2, b_2); \Lambda) \neq 0.$$

Then by Theorem 93, L_1 and L_2 are both ζ_1-heavy where ζ_1 denotes the asymptotic spectral invariant with respect to the idempotent $1 \in QH^*(M;\Lambda)$. Thus, for any Hamiltonian H we have

$$\gamma(H) = \rho(H,1) + \rho(\overline{H},1) \geqslant \zeta_1(H) + \zeta_1(\overline{H})$$

$$\geqslant \inf_{x\in L_1} H(x) + \inf_{x\in L_2} \overline{H}(x).$$

As $L_1 \cap L_2 = \emptyset$, we can consider a Hamiltonian which is arbitrarily large on L_1 and arbitrarily small on L_2 which contradicts the assumption

$$\sup\{\gamma(H) : H \in C^\infty(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R})\} < +\infty.$$

This completes the proof. □

As we have pointed out in Remark 129, examples of closed symplectic manifolds that satisfy

$$\sup\{\gamma(H) : H \in C^\infty(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R})\} < +\infty$$

include $\mathbb{C}P^n$, a large one point blow-up of $\mathbb{C}P^2$ and $(S^2\times S^2, \sigma\oplus\lambda\sigma)$ with $\lambda > 1$. We provide a brief explanation to these examples.

One can easily check that, for any closed symplectic manifold (M,ω), the condition

$$\sup\{\gamma(H) : H \in C^\infty(\mathbb{R}/\mathbb{Z}\times M,\mathbb{R})\} < +\infty$$

is equivalent to

$$\rho(\,\cdot\,,1) : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$$

being a quasimorphism where $1 \in QH^*(M;\mathbb{C})$. When (M,ω) is monotone, then $\rho(\cdot,1)$ is a quasimorphism when $QH^*(M;\mathbb{C})$ is a field. Thus, the case of $\mathbb{C}P^n$ follows. When (M,ω) is non-monotone, [Ost06, Theorem 1.3] or [EP08, Theorem 3.1] imply that $\rho(\cdot,1)$ is a quasimorphism when "$QH^0(M;\mathbb{C})$" is a field where a different set-up of the quantum cohomology is considered. For a precise definition of this set-up, we refer to [Ost06], [EP06]. As pointed out in [Ost06, Lemma 3.1, Remark 3.4], "$QH^0(M;\mathbb{C})$" is a field when (M,ω) is a large one point blow-up of $\mathbb{C}P^2$ or $(S^2\times S^2,\sigma\oplus\lambda\sigma)$ with $\lambda > 1$.

4.5.8 Proof of Application

In this section, we prove the following Theorem, which includes Theorem 84.

Theorem 130. *Let (M,ω) be a symplectic manifold which is either symplectically aspherical or monotone with the same monotonicity constant as Q^n, $n = 2,4$ (we also allow it to be an empty set). For any $R > 0$,*

$$\mathrm{Ham}_{\geqslant R} := \{\phi \in \mathrm{Ham}(Q^n\times M) : d_{\mathrm{Hof}}(\mathrm{id},\phi) \geqslant R\}$$

has a non-empty C^0-interior.

Theorem 130 follows as a corollary of the following statement.

Theorem 131. *Let (M,ω) be a monotone symplectic manifold. Assume that*

$$\sup\{\gamma(\phi) : \phi \in \mathrm{Ham}(M)\} = +\infty.$$

For any $R > 0$,

$$\mathrm{Ham}_{\geqslant R} := \{\phi \in \mathrm{Ham}(M,\omega) : d_{\mathrm{Hof}}(\mathrm{id},\phi) \geqslant R\}$$

has a non-empty C^0-interior.

Proof of Theorem 131. From the assumption, for any $R > 0$, we can find $\phi \in \mathrm{Ham}(M,\omega)$ such that

$$\gamma(\phi) > R + \frac{2\dim(M)}{N_M}$$

where N_M is the minimal Chern number of M. By Theorem 101, there exists $\delta > 0$ such that if $d_{C^0}(\mathrm{id},\phi') < \delta$, then $\gamma(\phi') < \frac{2\dim(M)}{N_M}$. Thus, for any $\psi \in \mathrm{Ham}(M,\omega)$ such that $d_{C^0}(\phi,\psi) < \delta$, we have

$$\gamma(\psi) > \gamma(\phi) - \gamma(\psi\circ\phi^{-1}) > R + \frac{2\dim(M)}{N_M} - \frac{2\dim(M)}{N_M} = R$$

thus, $\psi \in \mathrm{Ham}_{\geqslant R}$. This completes the proof. □

Now Theorem 130 follows immediately.

Proof of Theorem 130. From the Künneth formula for spectral invariants (see [EP09, Section 5.1]), we have

$$\sup\{\gamma(\phi) : \phi \in \mathrm{Ham}(Q^n \times M)\} \geqslant \sup\{\gamma(\phi_1 \times \mathrm{id}) : \phi_1 \in \mathrm{Ham}(Q^n), \mathrm{id} \in \mathrm{Ham}(M)\}$$

$$= \sup\{\gamma(\phi_1) : \phi_1 \in \mathrm{Ham}(Q^n)\} \geqslant \sup\{\mu(\phi_1) : \phi_1 \in \mathrm{Ham}(Q^n)\} = +\infty$$

for $n = 2, 4$. Note that the last equality uses the non-triviality and the homogeneity from Theorem 78. Therefore, Theorem 130 follows directly from Theorem 131. □

Chapter 5

Spectral norm and ground rings

5.1 Overview of the chapter

Floer homology is a powerful tool in symplectic topology but its construction is very technical and subtle. For example, they depend on the choice of the ground ring (e.g. $\mathbb{R}P^n$ in $\mathbb{C}P^n$ where n is odd has non-vanishing Floer homology with $\mathbb{Z}_2$-coefficients $HF(\mathbb{R}P^n;\mathbb{Z}_2) \neq 0$ whereas with $\mathbb{C}$-coefficients), its Floer homology vanishes $HF(\mathbb{R}P^n;\mathbb{C}) = 0$) and sometimes one needs to twist the boundary map in order to obtain a non-trivial Floer homology (e.g. The Clifford torus T^2_{Clif} in $\mathbb{C}P^2$ has vanishing Floer homology with the usual boundary map $HF(T^2_{Clif},\mathbb{C}) = 0$ but by twisting the boundary map with a certain local system $b \in H^1(T^2_{Clif};\mathbb{C})$, one gets a non-vanishing Floer homology $HF((T^2_{Clif},b);\mathbb{C}) \neq 0)$. There are also several different versions of Floer homology: First of all, there is the traditional one introduced by Floer [Fl88] and developed by Oh [Oh93a]. Later on, Fukaya-Oh-Ohta-Ono built a massive theory that allows one to construct Floer homology on any closed symplectic manifold. The former and the latter use different coefficient fields in the construction of homology. It seems very difficult to decide if this dependence on the algebraic setting is just technical of rather conceptual.

The main results in Chapter 4 (Theorem 78 which answers a variant of a question of Entov-polterovich-Py on the C^0 and Hofer-Lipschitz continuous quasimorphisms and Theorem 82 which answers a question of Polterovich-Wu) were obtained by using Floer homologies with different algebraic setting. This led us to the following question.

Question 132. *How does the behavior of the spectral norm γ_R change according to the choice of the ground ring R?*

Note that the spectral norm is a Floer theoritic quanitty and the subscript R denotes the choice of the ground ring we use in order to define Floer homology. Our main observation is the following.

Theorem 133. ([KawSh])
For $n = 2, 3$ we have

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n)} \gamma_{\mathbb{Z}}(\phi) = +\infty$$

while

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n)} \gamma_{\mathbb{K}}(\phi) \leqslant 1$$

for any field $\mathbb{K}$.

We will study this phenomenon from two different perspectives.

5.1.1 Algebraic viewpoint

The finiteness of spectral norm for $\mathbb{K}$-coefficients

$$\sup_{\phi\in \mathrm{Ham}(\mathbb{C}P^n)} \gamma_{\mathbb{K}}(\phi) \leqslant \frac{n}{n+1}$$

is a consequence of the so-called "Poincaré duality formula for spectral invariants". Thus,

$$\sup_{\phi\in \mathrm{Ham}(\mathbb{C}P^n)} \gamma_{\mathbb{Z}}(\phi) = +\infty$$

implies that the well-known "Poincaré duality formula for spectral invariants" no longer holds for spectral invariants with $\mathbb{Z}$-coefficients. We have found that for some Hamiltonian H, the filtered $\mathbb{Z}$-coefficient Floer homology $HF^{\tau}(H;\mathbb{Z})$ possesses a torsion element at some filtration level $\tau \in \mathbb{R}$ which disappears when the filtration level τ is sufficiently large.

Theorem 134. (*KawSh*)
On $\mathbb{C}P^n$ $(n = 2, 3)$*, there exists a Hamiltonian* H *such that for some filtration level* $\tau \in \mathbb{R}$*, we have*

$$Tor(HF^{\tau}(H;\mathbb{Z})) \neq 0.$$

5.1.2 Dynamical viewpoint

The relation between the asymptotic of the spectral norm and the dynamics of a Hamiltonian diffeomorphism has caught attention of some symplectic geometers. For example, Ginzburg-Gurel proved the folllowing in [GG18].

Theorem 135. (*GG18*)
If $\phi \in \mathrm{Ham}(\mathbb{C}P^n)$ *is a pseudo-rotation, then there exists a sequence* $n_k \nearrow +\infty$ *such that*

$$\lim_{k\to+\infty} \gamma_{\mathbb{C}}(\phi^{n_k}) = 0.$$

This result led us to the following question.

Question 136. *Let* (M,ω) *be a closed monotone symplectic manifold and* $R \in \{\mathbb{C},\mathbb{Z}\}$ *. If* $\phi \in \mathrm{Ham}(M,\omega)$ *is a pseudo-rotation, then do we have*

$$\overline{\gamma}_R(\phi)(= \lim_{k\to\infty} \frac{\gamma_R(\phi^k)}{k}) = 0 \ ?$$

Moreover, can we say that

$$\sup_{k\in\mathbb{Z}} \gamma_R(\phi^k) < +\infty \ ?$$

We have obtained the following partial result in this direction.

Theorem 137. ([KawSh])
For $\mathbb{C}P^n$, we have

$$\sup_{\phi:pseudo-rotation} \gamma_{\mathbb{Z}}(\phi) \leqslant 1$$

even though

$$\sup_{\phi\in \mathrm{Ham}(\mathbb{C}P^n)} \gamma_{\mathbb{Z}}(\phi) = +\infty$$

when $n = 2, 3$.

5.1.3 Set-up and notations

Throughout the chapter, we will denote Floer theoretic quantities with the choice of the ground ring as follows. Note that we only consider a closed monotone symplectic manifold (M, ω) in this chapter.

For a ground ring R, define the Floer homology group as usual and denote it by $HF(H; R)$. Similarly, we denote the quantum homology ring by

$$QH_*(M; R) := H_*(M; R) \otimes_R R[|s^{-1}, s].$$

Denote the PSS-map by

$$PSS_{H,R} : QH_*(M; R) \xrightarrow{\sim} HF(H; R).$$

For any $a \in QH_*(M; R)\backslash\{0\}$ and a Hamiltonian H, we define the spectral invariant with respect to H and a by

$$c(H, a; R) := \inf\{\tau \in \mathbb{R} : PSS_{H;R}(a) \in \mathrm{Im}(i_*^\tau)\}$$

where

$$i_*^\tau : HF^\tau(H; R) \to HF(H; R).$$

We often abbreviate the spectral invariant of the fundamental class $c_R(H, [M])$ by $c(H; R)$.

We denote the asymptotic spectral invariant of an idempotent $e \in QH_*(M; R)$ by

$$\overline{c}_e(H; R) := \lim_{k\to+\infty} \frac{c(H, e; R)}{k}.$$

When the choice of the ground ring is obvious, we sometimes lift it from the notation.

5.2 Proofs

5.2.1 Proof of Theorem 133

The quantum homology ring of $\mathbb{C}P^n$ is a field for any ground field $\mathbb{K}$ so by Theorem **??**,

$$\overline{c}_{\mathbb{K}}(\tilde{\phi}) := \lim_{k\to+\infty} \frac{c(\tilde{\phi}, [\mathbb{C}P^n]; \mathbb{K})}{k}$$

defines a homogeneous quasimorphism on $\widetilde{\mathrm{Ham}}(\mathbb{C}P^2)$ for any ground field $\mathbb{K}$.

Proposition 138. *For $\mathbb{C}P^2$, we have*

$$\overline{c}_{\mathbb{Z}_2} \neq \overline{c}_{\mathbb{C}}$$

Proof. (of Proposition 138)

In $\mathbb{C}P^2$, there are two disjoint monotone Lagrangians with non-vanishing Floer (co)homology, namely $\mathbb{R}P^2$ and the Chekanov torus T^2_{Ch}. These satisfy

- $HF(\mathbb{R}P^2;\mathbb{Z}_2) \neq 0$ ([Oh93b]).
- $HF((T^2_{Ch},b);\Lambda) \neq 0$ where b is a certain bounding cochain ([Wu15]).

Thus, by considering the closed-open map (Theorem 93), we deduce that $\mathbb{R}P^2$ is $\overline{c}_{\mathbb{Z}_2}$-superheavy and T^2_{Ch} is $\overline{c}_{1_\Lambda}$-superheavy. From the answer to the question of Polterovich-Wu (Theorem 82), T^2_{Ch} is $\overline{c}_{\mathbb{C}}$-superheavy. As $\mathbb{R}P^2$ and T^2_{Ch} are disjoint, we conclude that

$$\overline{c}_{\mathbb{Z}_2} \neq \overline{c}_{\mathbb{C}}.$$

□

One might wonder we will obtain a C^0 and Hofer-Lipschitz continuous homogeneous homogeneous quasimorphism by defining

$$\mu := \overline{c}_{\mathbb{C}} - \overline{c}_{\mathbb{Z}_2}$$

as in Chapter 4.

However, there is one issue: Recall that the proof the C^0 continuity of a quasimorphism $\mu : \mathrm{Ham}(Q^n) \to \mathbb{R}$ in Theorem 78 went through the following steps:

1. Define $\mu := \overline{c}_{e_+} - \overline{c}_{e_-}$ where $e_\pm \in QH(M;\mathbb{C})$ are units of field factors.
2. Get the estimate $|\mu| \leqslant \gamma_{\mathbb{C}}$
3. $\gamma_{\mathbb{C}}$ is bounded in a C^0-neiborhood of id in $\mathrm{Ham}(Q^n)$.
4. Apply Shtern's continuity criterion for quasimorphisms.

The issue in the current situation is that, as we work with quasimorphisms with two different ground fields, namely $\mathbb{Z}_2$ and $\mathbb{C}$, we do not get the estimate of the first step in the same way. In order to over come this issue, we will consider the following property of spectral invariants.

Proposition 139. *Let $\mathbb{K}$ be any field. For any $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$, we have*

$$c(\widetilde{\phi},[M];\mathbb{K}) \leqslant c(\widetilde{\phi},[M];\mathbb{Z}).$$

From this basic property, we get

$$\mu(\widetilde{\phi}) = \overline{c}_{\mathbb{C}}(\widetilde{\phi}) - \overline{c}_{\mathbb{Z}_2}(\widetilde{\phi}) = \overline{c}_{\mathbb{C}}(\widetilde{\phi}) + \overline{c}_{\mathbb{Z}_2}(\widetilde{\phi}^{-1}) \leqslant \overline{c}_{\mathbb{Z}}(\widetilde{\phi}) + \overline{c}_{\mathbb{Z}}(\widetilde{\phi}^{-1}) \leqslant \gamma_{\mathbb{Z}}(\widetilde{\phi}).$$

Thus, we obtain

$$|\mu| \leqslant \gamma_{\mathbb{Z}}$$

and we are in the position where we can apply the C^0-control of the spectral norm (Theorem 12): For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_{C^0}(\mathrm{id}, \phi) < \delta$, then

$$\gamma_{\mathbb{Z}}(\phi) \leqslant \frac{\dim_{\mathbb{R}}(M)}{N_M} + \varepsilon.$$

Shtern's continuity criterion for quasimorphisms implies that

$$\mu := \overline{c}_{\mathbb{C}} - \overline{c}_{\mathbb{Z}_2}$$

defines a non-trivial C^0 and Hofer-Lipschitz continuous homogeneous homogeneous quasimorphism. The estimate $|\mu| \leqslant \gamma_{\mathbb{Z}}$ and the homogeniety implies

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^2)} \gamma_{\mathbb{Z}}(\phi) = +\infty.$$

Remark 140. *We can alternatively use the following computations of Floer homologies as well:*

- $HF_{Zap}(T^2_{Ch}; \mathbb{C}) \neq 0$ *for a certain choice of a local system for the Zapolsky-type Floer homology (see [Zap15] and [LZ18]).*
- $HF_{Zap}(\mathbb{R}P^2; \mathbb{Z}_2) \neq 0$ *for the Zapolsky-type Floer homology (see [Zap15]).*

In comparison to Theorem 78, I pose the following question.

Question 141. *Does the homogeneous quasimorphism* $\mu : \mathrm{Ham}(\mathbb{C}P^2) \to \mathbb{R}$, $\mu := \overline{c}_{\mathbb{C}} - \overline{c}_{\mathbb{F}_2}$ *satisfy*

$$\overline{\gamma_{\mathbb{Z}}} = |\mu| \quad ??$$

5.3 Poincaré duality and spectral invariants

In this section, we prove a formula concerning spectral invarinats and Poincaré duality with $\mathbb{Z}$-coefficients. The version with $\mathbb{C}$-coefficients appeared in [EP03].

5.3.1 Basic sequences

First of all, we introduce exact sequences which will be helpful to consider Poincaré duality for spectral invariants.

Proposition 142. *We have the following exact sequences:*

$$\begin{array}{ccc}
0 & & HF^{\tau}_{*}(\overline{H}) \\
\downarrow & & \downarrow{\scriptstyle i_\tau} \\
Ext(HF^{-\tau}_{2n-*-1}(H), \mathbb{Z}) & & HF_*(\overline{H}) \\
\downarrow{\scriptstyle F_\tau} & & \downarrow{\scriptstyle j_\tau} \\
H_{2n-*}(Hom(CF_*(H)/CF^{\geqslant -\tau}_*(H)), \mathbb{Z}) & \xleftarrow[PD]{} & H_*(CF(\overline{H})/CF^{\tau}(\overline{H})) \\
\downarrow{\scriptstyle G_\tau} & & \\
Hom(HF^{-\tau}_{2n-*}(H), \mathbb{Z}) & & \\
\downarrow & & \\
0 & &
\end{array} \tag{5.1}$$

Proof. First of all, we have the following diagram which comes from Poincaré duality:

$$\begin{array}{ccc} CF_k^\tau(\overline{H}) & \longrightarrow & Hom(CF_{2n-k}^{\geqslant -\tau}(H),\mathbb{Z}) \\ \downarrow & & \downarrow \\ CF_k(\overline{H}) & \longrightarrow & Hom(CF_{2n-k}(H),\mathbb{Z}) \\ \downarrow & & \downarrow \\ CF_k(\overline{H})/CF_k^\tau(\overline{H}) & \longrightarrow & Hom(CF_{2n-k}(H),\mathbb{Z})/Hom(CF_{2n-k}^{\geqslant -\tau}(H),\mathbb{Z})) \end{array} \tag{5.2}$$

and by using

$$Hom(CF_*(H),\mathbb{Z})/Hom(CF_*^{\geqslant -\tau}(H),\mathbb{Z}) \simeq Hom(CF_*(H)/CF_*^{\geqslant -\tau}(H),\mathbb{Z})$$

(we do not use $Hom(CF_*(H)/CF_*^{\geqslant -\tau}(H),\mathbb{Z}) \simeq Hom(CF_*^{-\tau}(H),\mathbb{Z})$ as, if we did, then the following homology will not be defined) and passing to homology, we obtain

$$\begin{array}{ccc} HF_k^\tau(\overline{H}) & \longrightarrow & HF_{-\tau}^{2n-k}(H) \\ \downarrow & & \downarrow \\ HF_k(\overline{H}) & \longrightarrow & HF^{2n-k}(H) \\ \downarrow & & \downarrow \\ H_k(CF(\overline{H})/CF^\tau(\overline{H})) & \longrightarrow & H_{2n-k}(Hom(CF_*(H)/CF_*^{\geqslant -\tau}(H)),\mathbb{Z})) \end{array} \tag{5.3}$$

By the universal coefficient theorem for the chain complex $(CF_*(H)/CF_*^{\geqslant -\tau}(H),\partial_{Floer})$, we have

$$0 \longrightarrow Ext(H_{2n-k-1}(CF_*(H)/CF_*^{\geqslant -\tau}(H),\partial_{Floer})),\mathbb{Z}) \longrightarrow H_{2n-k}(Hom(CF_*(H)/CF_*^{\geqslant -\tau}(H),\mathbb{Z})),\delta_{Floer})$$

$$\longrightarrow Hom(H_{2n-k}(CF_*(H)/CF_*^{\geqslant -\tau}(H),\partial_{Floer}),\mathbb{Z}) \longrightarrow 0 \tag{5.4}$$

Note that

$$H_{2n-k}(CF_*(H)/CF_*^{\geqslant -\tau}(H),\partial_{Floer}) = HF_{2n-k}^{-\tau}(H)$$

and thus

$$0 \longrightarrow Ext(HF_{2n-k-1}^{-\tau}(H),\mathbb{Z}) \longrightarrow H_{2n-k}(Hom(CF_*(H)/CF_*^{\geqslant -\tau}(H),\mathbb{Z})),\delta_{Floer})$$

$$\longrightarrow Hom(HF_{2n-k}^{-\tau}(H),\mathbb{Z}) \longrightarrow 0 \tag{5.5}$$

Together with the exact sequence 5.5, we obtain the vertical sequence in Proposition 142.

□

Remark 143. *If we work with a ground field $\mathbb{K}$ instead of $\mathbb{Z}$, then we get a sequence similar to 142 with $\mathbb{Z}$ replaced with $\mathbb{K}$.*

5.3.2 Two extreme cases

Consider the following diagram:

$$\begin{array}{ccccc} 0 & & & & \\ \downarrow & & & & \\ Ext(HF^{-\tau}_{2n-*-1}(H);\mathbb{Z}) & & & & \\ \downarrow{\scriptstyle F_{-\tau}} & & & & \\ H_{2n-*}(Hom(CF_*(H)/CF_*^{\geqslant -\tau}(H)),\mathbb{Z}) & \xleftarrow[PD\circ j_\tau]{} & HF_*(\overline{H}) & \xleftarrow[i_\tau]{} & HF^\tau_*(\overline{H}) \\ \downarrow{\scriptstyle G_{-\tau}} & & \uparrow{\scriptstyle PSS_{\overline{H}}} & & \\ Hom(HF^{-\tau}_{2n-*}(H);\mathbb{Z}) & & QH_*(M;\mathbb{Z}) & & \\ \downarrow & & & & \\ 0 & & & & \end{array} \tag{5.6}$$

There are two extreme cases: given $a \in QH_*(M;\mathbb{Z})$, denote $a^\# := PD \circ j_\tau \circ PSS_{\overline{H}}(a)$.

1. $G_{-\tau}$ is an isomorphism. This happens for example when
   - The ground ring is a field.
   - $Ext(HF^\tau_{*-1}(H);\mathbb{Z}) = 0$ for all $\tau \in \mathbb{R}$ e.g. ϕ_H is a pseudo-rotation.
2. $F_{-\tau}$ is an isomorphism. This happens if and only if
$$a \in Tor(QH_*(M;\mathbb{Z})).$$

In these two cases, what would the PD formulae be?

1. If $G_{-\tau}$ is an isomorphism, then
$$c(\overline{H}, a) = -\inf\{c(H,b) : \langle a, b\rangle \neq 0\}$$
which recovers the formula in [EP03].
2. If $F_{-\tau}$ is an isomorphism, then the formula becomes much more complicated: See Section 5.5.

5.3.3 Results for $\mathbb{C}P^n$

In this subsection, we prove the following Theorem.

Theorem 144. ([KawSh])
On $\mathbb{C}P^n, n = 2, 3$, there exists a Hamiltonian H such that for some filtration level $\tau \in \mathbb{R}$, we have
$$Tor(HF^\tau(H;\mathbb{Z})) \neq 0.$$

Proof. Assume that for any Hamiltonian H, we have $Ext(HF^{\tau}_{*-1}(H);\mathbb{Z}) = 0, \forall \tau \in \mathbb{R}$. Then, Proposition 142 implies

$$c(\overline{H},[M];\mathbb{Z}) = -\inf\{c(H,b;\mathbb{Z}) : \langle [M], b\rangle \neq 0\}$$

$$= -\inf\{c(H,k\cdot[pt];\mathbb{Z}) : k \in \mathbb{Z}\}$$

Note that we have used that $N_{\mathbb{C}P^n} = n+1$ and therefore $QH_0(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}\langle [pt]\rangle$. The following implies that this cannot happen.

Claim 145. *On $M := \mathbb{C}P^n$, there exists a Hamiltonian H such that*

$$c(\overline{H},[M];\mathbb{Z}) > -\inf\{c(H,k\cdot[pt];\mathbb{Z}) : k \in \mathbb{Z}\}.$$

Proof. (of Claim 145) By the Poincaré duality formula, we know that for any H,

$$-c(\overline{H},[M];\mathbb{Z}) \leqslant \inf_{k\in\mathbb{Z}} c(H,k\cdot[pt];\mathbb{Z}).$$

Assume that we have the equality

$$-c(\overline{H},[M];\mathbb{Z}) = \inf_{k\in\mathbb{Z}} c(H,k\cdot[pt];\mathbb{Z})$$

for any H. Then,

$$-c(\overline{H},[M];\mathbb{Z}) = \inf_{k\in\mathbb{Z}} c(H,k\cdot[pt];\mathbb{Z}) \geqslant c(H,[pt];\mathbb{C}) = -c(\overline{H},[M];\mathbb{C}).$$

Note that the last equality is a consequence of the Poincaré duality formula for $\mathbb{C}$-coefficients. Thus,

$$c(\overline{H},[M];\mathbb{Z}) \leqslant c(\overline{H},[M];\mathbb{C}).$$

On the other hand, by the property of the ground ring, we know that

$$c(\overline{H},[M];\mathbb{Z}) \geqslant c(\overline{H},[M];\mathbb{C}).$$

These put together, we obtain

$$c(\overline{H},[M];\mathbb{Z}) = c(\overline{H},[M];\mathbb{C})$$

for any H, which contradicts Proposition 138. This finishes the proof of Claim 145. □

Thus we conclude that there exists a Hamiltonian H such that for some filtration level $\tau \in \mathbb{R}$, $Ext(HF^{\tau}_{*-1}(H);\mathbb{Z}) \neq 0$.

□

Question 146. *Then what is $c_{\mathbb{Z}}(\overline{H},[M])$ in terms of $c_{\mathbb{Z}}(H,\cdot)$?*

5.4 Dynamical results

Corollary 147. *(spectral bound for PR)*
If $\phi \in \mathrm{Ham}(\mathbb{C}P^n)$ is a pseudo-rotation, then

$$\gamma_{\mathbb{Z}}(\phi) \leqslant \lambda_0 = \langle \omega_{FS}, [\mathbb{C}P^1] \rangle.$$

Proof. (of Corollary 147)
For a pseudo-rotation $\phi = \phi_H \in \mathrm{Ham}(\mathbb{C}P^n)$,

$$\gamma_{\mathbb{Z}}(H) = c_{\mathbb{Z}}(H, [M]) + c_{\mathbb{Z}}(\overline{H}, [M])$$

$$= c_{\mathbb{F}_2}(H, [M]) + c_{\mathbb{F}_2}(\overline{H}, [M]) = \gamma_{\mathbb{F}_2}(H) \leqslant \langle \omega_{FS}, [\mathbb{C}P^1] \rangle.$$

□

Corollary 148. *Let $\phi_H \in \mathrm{Ham}(\mathbb{C}P^n)$ be a pseudo-rotation. For any ground ring R, we have*

$$\bar{c}_R(H) = \bar{c}_{\mathbb{Z}}(H).$$

Remark 149. *When we work with a ground field $\mathbb{K}$, the asymptotic spectral invariant $\bar{c}_{\mathbb{K}}$ is this corolarry equals the so-called augmented action. This was shown by Ginzburg-Gürel in [GG09].*

Proof. For any H, we have

$$\bar{c}_R(H) \leqslant \bar{c}_{\mathbb{Z}}(H).$$

Together with

$$\bar{c}_R(H) + \bar{c}_R(H) \geqslant 0,$$

we obtain

$$-\bar{c}_R(H) \leqslant \bar{c}_{\mathbb{Z}}(\overline{H}).$$

Now, for a pseudo-rotation $\phi_H \in \mathrm{Ham}(\mathbb{C}P^n)$, we have

$$\bar{c}_{\mathbb{Z}}(\overline{H}) = \bar{c}_{\mathbb{F}_2}(\overline{H}) = -\bar{c}_{\mathbb{F}_2}(H),$$

so all together,

$$-\bar{c}_R(H) \leqslant -\bar{c}_{\mathbb{F}_2}(H).$$

Thus,

$$\bar{c}_{\mathbb{F}_2}(H) \leqslant \bar{c}_R(H) \leqslant \bar{c}_{\mathbb{Z}}(H) = \bar{c}_{\mathbb{F}_2}(H).$$

This implies

$$\bar{c}_R(H) = \bar{c}_{\mathbb{Z}}(H).$$

□

Proposition 150. *For $\mathbb{C}P^n$, $n = 2, 3$, there exist $l \in \mathbb{Z}$ and a Hamiltonian H such that*

$$c_{\mathbb{Z}}(H, l[M]) < c_{\mathbb{Z}}(H, [M]).$$

Proof. As we have seen in Proposition 138, we have

$$\overline{c}_{\mathbb{F}_2} \neq \overline{c}_{\mathbb{C}}.$$

This implies that there exists H such that

$$c_{\mathbb{F}_2}(H,[M]) < c_{\mathbb{C}}(H,[M]).$$

This implies that there exists $l \notin 2\mathbb{Z}$ such that

$$c_{\mathbb{Z}}(H,l[M]) < c_{\mathbb{Z}}(H,[M]).$$

□

5.5 Closing remarks

In $M := \mathbb{C}P^n$, there might be $b \in Tor(HF^\tau(H))$, which eventually disappears, such that $c(\overline{H},[M])$ corresponds to $-\tau$ where $PSS_H([pt])+b$ appears at $HF^\tau(H)$. In order to understand this torsion class, together with Egor Shelukhin, I plan to study the "filtered quantum linking form" which the quantum counterpart of the linking form in singular homology theory (which is considered as the Poincaré pairing for the torsion classes) by taking the filtration into account. The non-filtered version of this looks like

$$\mathcal{L} : QH_*(M;\mathbb{Z}) \otimes QH_{2n-1-*}(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

which comes from the following operation:

$$\circledast : QH_i(M;\mathbb{Z}) \otimes Tor(QH_j(M;\mathbb{Z})) \to QH_{i+j-2n+1}(M;\mathbb{Q}/\mathbb{Z})$$

$$(a,b) \mapsto a * \beta^{-1}(b)$$

where

$$\beta : QH_j(M;\mathbb{Q}/\mathbb{Z}) \to QH_{j-1}(M;\mathbb{Z})$$

denotes the Bockstein homomorphism.

We have for $a \in QH_i(M;\mathbb{Z})$, $b \in QH_{2n-1-i}(M;\mathbb{Z})$,

$$\mathcal{L}(a,b) = \langle a \circledast b, 1\rangle,\ 1 \in QH^0(M;\mathbb{Q}/\mathbb{Z}).$$

We plan to study properties of this pairing and aim to refine the Poincaré duality formula for spectral invariants.

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RÉSUMÉ

Le groupe des difféomorphismes hamiltoniens possède deux métriques bi-invariantes remarquables, notamment la métrique d'Hofer et la métrique spectrale. La troisième métrique de ce groupe qui a attiré l'intérêt en topologie symplectique est la métrique C^0. Cette thèse étudie les relations mystérieuses entre ces trois métriques. Les résultats incluent la comparaison entre la métrique spectrale et la métrique C^0 dans le cas non-asphérique, une construction de quasimorphismes qui possèdent un nouveau type de continuité inspiré par une question célèbre d'Entov-Polterovich-Py, et quelques nouvelles propriétés de la norme spectrale.

MOTS CLÉS

Homologie de Floer, difféomorphismes hamiltoniens, géométrie d’Hofer, invariants spectraux, quasimorphismes, cohomologie quantique.

ABSTRACT

The group of Hamiltonian diffeomorphisms is known to possess two remarkable bi-invariant metrics called the Hofer metric and the spectral metric. A third metric on the group which has been of interest in symplectic topology is the C^0 metric. Roughly speaking, this thesis explores the deep and mysterious relation between these three metrics. Results contain the comparison between the spectral metric and the C^0 metric, a construction of quasimorphisms that have some new types of continuity properties which are related to a famous question of Entov-Polterovich-Py, and some new properties of the spectral norm.

KEYWORDS

Floer homology, Hamiltonian diffeomorphisms, Hofer geometry, spectral invariants, quasimorphisms, quantum coholomogy.