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Thèse

présentée pour obtenir le grade de

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École Doctorale ED-386

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par

Thibault Alexandre

Cohomologie cohérente et géométrie de la fibre spéciale des variétés de Siegel

Coherent cohomology and geometry of the special fiber of Siegel varieties

Soutenue publiquement le 09/12/2022 devant le jury composé de :

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À ma grand-mère, Monique.

Résumé

Cette thèse porte sur la cohomologie cohérente et la géométrie de certaines variétés de Shimura en caractéristique première p. Plus précisément, nous considérons les variétés de Siegel avec un niveau hyperspécial en p. Nous commençons par établir des résultats de positivité pour certains fibrés automorphes. Ces résultats nécessitent l'introduction d'une nouvelle notion de positivité pour les fibrés vectoriels plus faible que l'amplitude mais plus forte que la nefness et la bigness. Ensuite nous montrons des résultats d'annulations pour la cohomologie cohérente à coefficients dans certains fibrés automorphes. Les poids accessibles avec notre méthode peuvent être calculés à l'aide d'un algorithme implémenté sur SageMath. Puis nous étudions le défaut d'hyperbolicité en essayant de caractériser les sous-variétés de type général. Nous montrons en particulier qu'en dessous d'une certaine codimension, toutes les sous-variétés de la variété de Siegel sont de type général. Enfin, inspiré par un travail récent de Boxer et Pilloni, nous redéfinissons un opérateur de Hecke à l'aide de la théorie des foncteurs de Schur et nous montrons qu'il est auto-dual pour la dualité de Serre.

Summary

This thesis deals with the coherent cohomology and geometry of certain Shimura varieties in prime characteristic p. More precisely, we consider Siegel varieties with a hyperspecial level at p. We start by establishing positivity results for some automorphic bundles. These results require the introduction of a new positivity notion for vector bundles weaker than amplitude but stronger than nefness and bigness. Then we show vanishing results for the coherent cohomology with coefficients in some automorphic bundles. The weights that are accessible with our method can be computed with an algorithm implemented on SageMath. Then we study the failure of hyperbolicity by trying to characterize the subvarieties of general type. In particular, we show that below a certain codimension, all subvarieties of the Siegel variety are of general type. Finally, inspired by a recent work of Boxer and Pilloni, we redefine a Hecke operator with the theory of Schur functors and we show that it is self-dual for Serre duality.

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Je tiens à remercier mes amis Dilshad et Mamad' que j'ai rencontrés aux Mines il y a déjà quelques années... Mais aussi mes amis du master de Jussieu : Archia, Arnab, Dylan, Olivier,

[1] Je ne prétends pas que je sais ce que c'est aujourd'hui ! Et oui je sais que j'ai dit que je ne parlerai pas de maths au début ...

[2] Même si je me suis retrouvé exilé au 4ème étage la dernière année...

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[3]Ma mère dit souvent que tu mérites une médaille...

CHAPTER 1

Introduction

1.1. Motivation générale

Cette thèse en mathématiques fondamentales s'inscrit d'une part dans le cadre du programme de Langlands, et de l'autre dans celui de la géométrie arithmétique. Introduit par Robert Langlands en 1967, ce programme propose de relier la théorie des nombres et l'analyse harmonique. Grâce au langage schématique introduit par Grothendieck et Dieudonné, nous pouvons étudier la géométrie algébrique d'objets se trouvant à mi-chemin entre la théorie des nombres et l'analyse harmonique. Cette thèse s'intéresse plus particulièrement à la géométrie de ces objets lorsqu'on les regarde définis au dessus d'un corps dont la caractéristique est un nombre premier p. Nous allons présenter brièvement deux résultats clés qui ont motivés la recherche actuelle en géométrique arithmétique dans le cadre du programme de Langlands : la conjecture de Ramanujan sur la fonction τ [**Del71**] et la conjecture de Taniyama–Shimura [**Wil95**][**BCDT01**] (et son lien avec le grand théorème de Fermat). Cette partie n'a aucun lien direct avec le reste de cette thèse mais peut apporter quelques motivations intéressantes au lecteur non averti. Cette présentation est très succincte, ne prétend pas servir de référence aux sujets abordés et contient de nombreux abus assumés visant à alléger la lecture.

1.1.1. La fonction tau de Ramanujan. Une forme modulaire, dans sa version la plus simple, est une fonction holomorphe sur le demi-plan de Poincaré

$$\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

qui possède énormément de symétries. Plus précisément, on considère le sous-groupe de $\mathrm{SL}_2(\mathbb{Z})$

$$\Gamma_1(N) = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N,\ c \equiv 0 \bmod N\right\}$$

et on dit que $f : \mathcal{H} \to \mathbb{C}$ est une forme modulaire de niveau N et poids k si

(1) f est holomorphe,

(2) pour toute matrice $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$ et tout nombre complexe z, on a

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \tag{1.1}$$

(3) et $|f(z)|$ est bornée lorsque $\mathrm{Im}(z) \to \infty$.

Comme la matrice $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ appartient à $\Gamma_1(N)$, l'équation fonctionnelle vérifiée par f lui impose la relation

$$f(z+1) = f(z)$$

et cela entraine que f est développable en série de Fourier $f(z) = \sum_{n\in\mathbb{Z}} c_n q^n$ où $q = e^{2i\pi z}$. Par ailleurs, comme $|f(z)|$ est bornée lorsque $\mathrm{Im}(z) \to \infty$, les coefficients de Fourier c_n pour

$n < 0$ sont tous nuls. On appelle le développement de Fourier

$$f = \sum_{n \geq 0} c_n q^n$$

la q-expansion de f et celle-ci caractérise complètement la forme modulaire f. Le premier exemple de forme modulaire lorsque $N = 1$ est une forme de poids 12, plus connue sous le nom de discriminant modulaire[1] et dont la q-expansion[2] est la suivante.

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

où τ désigne la fonction tau de Ramanujan. On trace les valeurs du discriminant modulaire sur la figure 1.1.1 dont le code couleur est expliqué par la remarque 1.1.

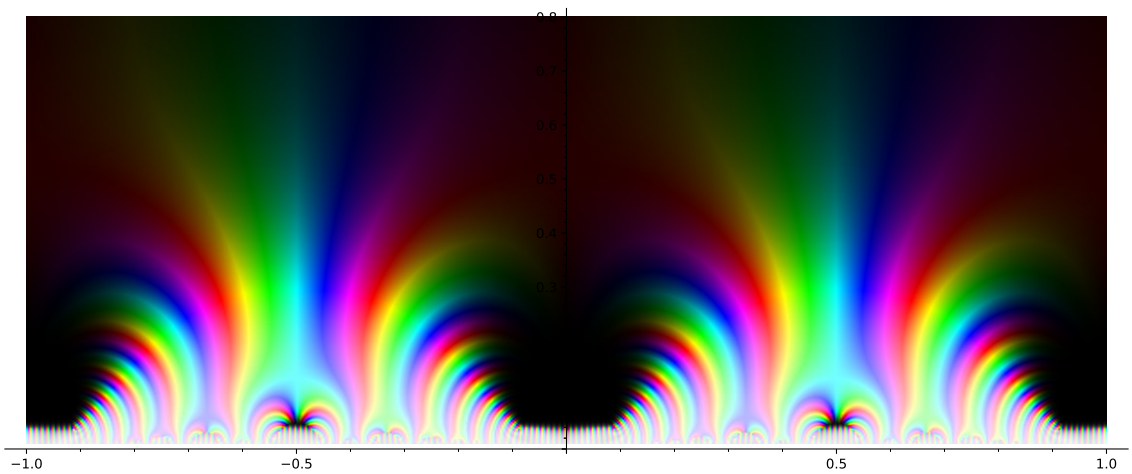


FIGURE 1.1.1. Tracé de $\Delta(z)$.

Remarque 1.1. Pour tracer des fonctions holomorphes sur un plan, on utilise des couleurs à la place des nombres complexes selon le "mapping" de la figure 1.1.2.

Chose remarquable, tous les coefficients de Fourier de Δ sont entiers. Voici ses premières valeurs.

n	1	2	3	4	5	6	7	8	9	10
$\tau(n)$	1	-24	252	-1472	4830	-6048	-16744	84480	-113643	-115920

[1] Label 1.12.a.a sur [**LMF22**]

[2] Il n'est pas du tout évident à partir d'une q-expansion de savoir si cela définit une forme modulaire. Pour le discriminant modulaire, cela résulte du fait que

$$\Delta(z) = \frac{1}{(2\pi)^{12}} \left(g_2^3(z) - 27 g_3^2(z) \right)$$

où

$$\begin{cases} g_2(z) = \dfrac{4}{3}\pi^4 \left[1 + 240 \displaystyle\sum_{k=1}^{\infty} \sigma_3(k) q^{2k} \right], \\ g_3(z) = \dfrac{8}{27}\pi^6 \left[1 - 504 \displaystyle\sum_{k=1}^{\infty} \sigma_5(k) q^{2k} \right] \end{cases}$$

sont des séries d'Eisenstein renormalisées avec $\sigma_\alpha(k) = \sum_{d|k} d^\alpha$.

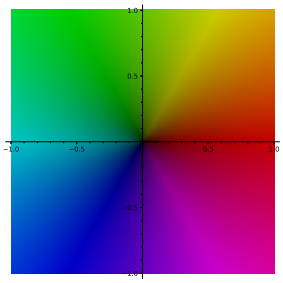


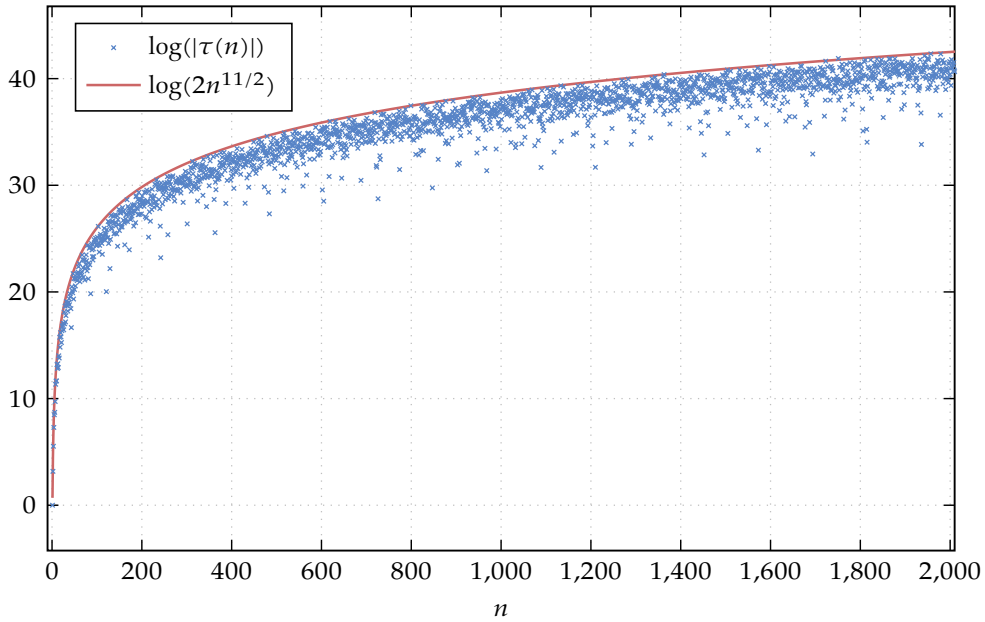
FIGURE 1.1.2. Tracé de $z \mapsto z$.

Notons aussi les propriétés multiplicatives de cette fonction τ.

(1) Si n et m sont deux nombres premiers entre eux, alors $\tau(nm) = \tau(n)\tau(m)$[3].

(2) Si $r \geq 2$ et p est premier alors $\tau(p^r) = \tau(p)\tau(p^{r-1}) - p^{11}\tau(p^{r-2})$.

Mettons de coté le signe de $\tau(n)$ et traçons les valeurs de $|\tau(n)|$ en fonction de n.



On observe que pour tout $n \in [1, 2000] \setminus \{799, 1751\}$, on a

$$|\tau(n)| < 2n^{11/2}$$

et

$$\begin{cases} |\tau(799)| = 18558651348375264 > 2 \times 799^{11/2} \simeq 1.840930028820411\text{e+}16, \\ |\tau(1751)| = 1559050018604597232 > 2 \times 1751^{11/2} \simeq 1.3775386699905252\text{e+}18. \end{cases}$$

Les nombres 799 et 1751 ne sont pas premiers car ils se décomposent $799 = 17 \times 47$ et $1751 = 17 \times 103$. Il semblerait que la borne $|\tau(n)| < 2n^{11/2}$ tient tout de même pour les nombres premiers. Cette borne semble optimale dans le sens où si on prend $\varepsilon > 0$, il parait

[3]Ce résultat est non-trivial et fut démontré initialement par Mordell [**SB17**, p. 117].

possible de trouver p premier tel que $|\tau(p)| > 2p^{11/2-\varepsilon}$. Par exemple, on a

$$\begin{cases} |\tau(47)| = 2687348496 > 2 \times 47^{11/2-0.1} \simeq 2.1397273039381979\text{e+9}, \\ |\tau(103)| = 225755128648 > 2 \times 103^{11/2-0.01} \simeq 2.2464996708384985\text{e+11}. \end{cases}$$

L'existence d'une telle borne est l'objet d'une conjecture formulée en 1916 par Ramanujan mais il a fallu attendre 1973 et les outils de la géométrie arithmétique pour voir une preuve de ce résultat par Deligne. La preuve de Deligne exploite l'action d'un opérateur (dit de Hecke) sur un objet géométrique appelée la courbe modulaire. Voici les grandes lignes de la preuve.

Interprétation géométrique des formes modulaires avec la courbe modulaire. La forme modulaire Δ est une fonction sur le demi-plan de Poincaré vérifiant une équation "d'invariance" par le sous-groupe $\mathrm{SL}_2(\mathbb{Z})$ qui agit par homographie sur $\mathcal{H}$. Pour éviter d'avoir à parler de champs algébriques de Deligne-Mumford, nous allons ajouter du niveau. Soit $N \geq 5$ un entier. On considère le quotient $Y_N := \Gamma_1(N)\backslash\mathcal{H}$ qui est naturellement muni d'une structure de variété complexe de dimension 1. Un domaine fondamental de l'action de $\mathrm{SL}_2(\mathbb{Z})$ sur le demi-plan de Poincaré est représenté par la zone grisée de la figure 1.1.3.

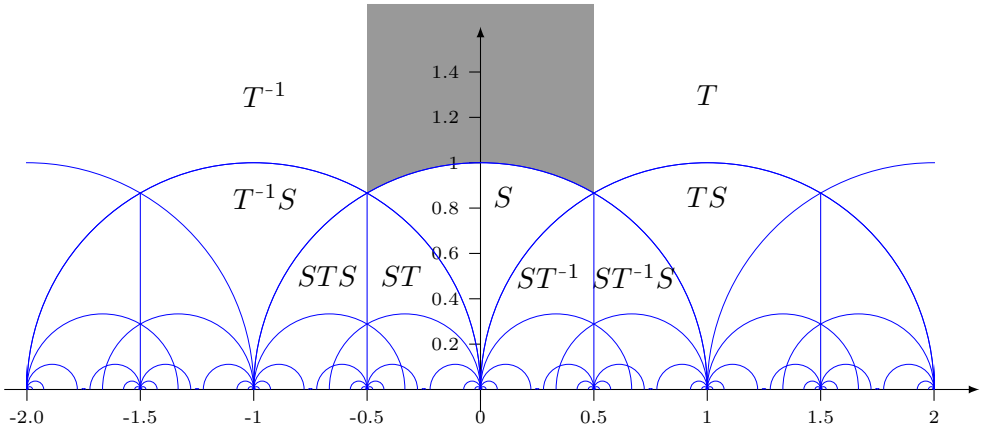


FIGURE 1.1.3. Action de $\mathrm{SL}_2(\mathbb{Z})$ par homographie sur le demi-plan de Poincaré. Le domaine fondamental $\mathcal{D}$ est représenté en gris. Les différents cercles représentent contours des transformées de $\mathcal{D}$ par les matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

On appelle simplement courbe modulaire de niveau N la variété Y_N. Lorsque l'on dispose d'un objet géométrique, il est naturel de vouloir étudier ses formes différentielles. On cherche alors les 1-formes différentielles de la forme

$$\omega(z) = f(z)dz$$

où f est une fonction holomorphe définie sur $\mathcal{H}$. La condition de modularité (1.1) pour le groupe $\Gamma_1(N)$ que doit vérifier ω pour descendre au quotient s'écrit

$$\begin{aligned}\omega\left(\frac{az+b}{cz+d}\right) &= f\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right)\\ &= f\left(\frac{az+b}{cz+d}\right)\frac{ad-bc}{(cz+d)^2}dz\\ &= f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-2}dz \quad = f(z)dz\end{aligned}$$

et on remarque que f doit satisfaire l'équation fonctionnelle de la définition d'une forme modulaire de poids 2 et niveau N. Comme on a la relation

$$2i\pi dz = \frac{dq}{q},$$

on peut réinterpréter les formes modulaires de poids 2 s'annulant lorsque $q \to 0$ comme les sections globales du faisceau des 1-formes différentielles $\Omega^1_{X_N}$ où X_N est la compactification minimale de Y_N. Ainsi, les sections de $(\Omega^1_{X_N})^{\otimes k}$ sont des formes modulaires de poids $2k$ ayant un pôle d'ordre k en $q = 0$. Pour résumer, si on note $D = X_N - Y_N$ le bord de la compactification minimale, la fonction Δ peut être vue comme un élément de

$$H^0(X_N, (\Omega^1_{X_N})^{\otimes k}(-kD)),$$

le groupe de cohomologie de degré 0 de la courbe modulaire compactifiée X_N à coefficients dans $(\Omega^1_{X_N})^{\otimes k}(-kD)$, la k-ième puissance du faisceau des 1-formes différentielles holomorphes sur X_N dont les sections ont au plus un pôle d'ordre k le long du bord D.

Modèle entier des courbes modulaires. La courbe modulaire compactifiée X_N étant lisse, il résulte de la théorie des surfaces de Riemann compactes qu'elle peut être représentée par une équation algébrique définie sur $\mathbb{C}$. Motivé par le fait que les coefficients de la q-expansion de la forme modulaire Δ sont tous entiers, nous nous attendons à pouvoir définir cette équation algébrique au dessus de $\mathbb{Q}$, les nombres rationnels. Une telle construction nécessite de réinterpréter la courbe modulaire Y_N comme un espace de module de courbes elliptiques. Rappelons qu'une courbe elliptique complexe est une surface de Riemann projective (lisse) munie d'une loi de groupe. Toute courbe elliptique complexe admet au moins une description comme un quotient de la forme $\mathbb{C}/\Lambda$ où $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\, z$ est un réseau du plan complexe avec $z \in \mathcal{H}$. Toute courbe elliptique admet également au moins une équation cubique dans $\mathbb{P}^2_{\mathbb{C}}$ de la forme

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

où a_1, a_3, a_2, a_4, a_6 sont des nombres complexes. Construisons une bijection entre le quotient ensembliste $\mathrm{SL}_2(\mathbb{Z})\diagdown\mathcal{H}$ et l'ensemble X des courbes elliptiques à isomorphisme près. Pour cela, on considère l'application

$$\begin{aligned}\varphi\colon \mathcal{H} &\to X\\ z &\mapsto \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\, z\end{aligned}$$

qui est clairement surjective mais pas injective. On montre facilement que $\varphi(z) = \varphi(z')$ est équivalent à ce qu'il existe $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ tel que $z = \gamma \cdot z'$. Cela entraine que l'application passe au quotient en une application bijective

$$\overline{\varphi}\colon\ \mathrm{SL}_2(\mathbb{Z})\diagdown\mathcal{H} \xrightarrow{\sim} X.$$

Ainsi, on est invité à réinterpréter la courbe modulaire comme un espace de module de courbe elliptiques complexes. Or les courbes elliptiques sont algébriques et peuvent donc être définies au dessus d'un anneau quelconque (comme $\mathbb{Q}$). La théorie des schémas permet même de définir une courbe elliptique au dessus d'un schéma quelconque S. Notons Sch_R la catégorie des schémas au dessus d'un anneau R et Set la catégorie des ensembles. On redéfinit Y_N comme le schéma au dessus de $\operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$ qui représente le foncteur

$$Y_N : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}]} \to \mathrm{Set}$$

associant à un schéma S, l'ensemble des couples (E, P) constitués d'une courbe elliptique $E \to S$ avec $P \in E[N](S)$, un S-point d'ordre N. Le foncteur Y_N est représentable dans la catégorie des schémas sur $\operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$ car on a supposé $N \geq 5$. Si $N < 5$, on peut voir Y_N comme un champ algébrique de Deligne-Mumford. Le schéma Y_N est une courbe quasi-projective et lisse au dessus de $\mathbb{Z}[\frac{1}{N}]$ dont la compactification minimale X_N est une courbe projective lisse munie d'un diviseur de Cartier $D = X_N \diagdown Y_N = \sum_{i=1}^{n}[z_i]$ où les z_i sont des points fermés de X_N appelés pointes ou cusps. ll est connu que l'on dispose d'un schéma semi-abélien

$$\begin{tikzcd} E \arrow[r, "f"] & X_N \arrow[l, bend left, "e"] \end{tikzcd}$$

au dessus de X_N qui étend la courbe elliptique universelle au dessus de Y_N. Notons

$$\omega = e^*\Omega^1_{E/X_N}$$

le faisceau localement libre de rang 1. On peut alors redéfinir les formes modulaire de poids k et niveau N au dessus de $\mathbb{Z}[\frac{1}{N}]$ comme les sections globales de $\omega^{\otimes k}$. Le lien avec la discussion précédente fait appel à l'isomorphisme de Kodaira-Spencer

$$\omega^{\otimes 2} \xrightarrow{\sim} \Omega^1_{X_N}(-D)$$

qui permet d'interpréter une forme modulaire de poids 2 comme une 1-forme différentielle sur X_N qui s'annule au bord D. Pour toute $\mathbb{Z}[\frac{1}{N}]$-algèbre R, on note

$$M_{k,R}(N) = H^0(X_N, \omega^{\otimes k} \otimes_{\mathbb{Z}[\frac{1}{N}]} R)$$

les formes modulaires de poids k et niveau N à coefficient dans R et

$$S_{k,R}(N) = H^0(X_N, \omega^{\otimes k}(-D) \otimes_{\mathbb{Z}[\frac{1}{N}]} R)$$

les formes modulaires cuspidales de poids k et niveau N à coefficients dans R.

Opérateur de Hecke T_p sur la courbe modulaire X_N. Nous avons réussi à interpréter le discriminant modulaire Δ comme une section globale d'un fibré en droite sur la courbe modulaire $X_N \to \operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$. Pour mieux comprendre les coefficients $\tau(n)$ de sa q-expansion, nous introduisons des opérateurs T_p pour chaque nombre premier p ne divisant pas N tels que Δ soit une fonction propre pour chaque T_p, avec valeur propre correspondante $\tau(p)$. Comme τ est une fonction multiplicative[4], cela permettra de retrouver toute l'information sur les $\tau(n)$. On définit l'opérateur de Hecke T_p à partir d'une correspondance géométrique sur la courbe

[4]La multiplicativité de τ peut être démontrée en introduisant des opérateurs de Hecke T_n pour tout $n \geq 2$ et en on montrant que $T_n\Delta = \tau(n)\Delta$ et $T_nT_m = T_{nm}$ si $(n, m) = 1$. Voir [**Ser70**, Corollaire 2, p. 166].

modulaire X_N. Pour cela, on introduit le diagramme suivant

$$\begin{tikzcd} & C_N \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[drr, bend left] & & \\ Y_N \arrow[drr, bend right] & & Y_N \arrow[r] & \operatorname{Spec} \mathbb{Z}[\frac{1}{N}] \end{tikzcd}$$

où C_N est l'espace de module sur $\mathbb{Z}[\frac{1}{N}]$ dont les S-points paramètrent les triplets (E, H, P) formés d'une courbe elliptique E sur S, un sous-groupe fini et plat $H \subset E[p]$ d'ordre p et un S-point $P \in E[N](S)$ d'ordre N. La flèche p_1 est la projection qui envoie un triplet (E, H, P) sur un couple (E, P). La flèche p_2 envoie un triplet (E, H, P) sur un couple $(E/H, P_{E/H})$ où E/H est la courbe elliptique quotient de E par le sous-groupe fini et plat $H \subset E[p]$ et $P_{E/H} \subset (E/H)[N](S)$ est l'image de P par la flèche quotient $E \to E/H$. Sur les S-points, T_p associe à un couple (E, P) la somme formelle

$$T_p(E, P) := \sum_{\pi : E \to E'} (E', P_{E'})$$

où chaque $\pi : E \to E'$ est une isogénie d'ordre p. Considérons la courbe elliptique universelle $E \to C_N$ munie de son sous-groupe universel $H \subset C_N$ et de sa section neutre $e : C_N \to E$. Nous admettons que cette correspondance géométrique munie de $\pi : E \to E/H$ peut être étendue aux compactifications minimales de Y_N, C_N et qu'elle permet de définir une action d'un opérateur de Hecke encore noté T_p sur

(1) La cohomologie cohérente cuspidale $H^i(X_N, \omega^k(-D))$ pour tout $k \geq 1$.
(2) La cohomologie étale l-adique (intérieure) $\overline{H}^i_{\text{ét}}((Y_N)_{\overline{\mathbb{Q}}}, \mathcal{V}_{k,l} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ où l est un nombre premier quelconque et $\mathcal{V}_{k,l} := \operatorname{Sym}^k \overline{R}^1 f_{*,\text{ét}} \mathbb{Q}_l$ est la k-ième puissance symétrique de l'image directe supérieure étale l-adique (intérieure) du faisceau constant sur la courbe elliptique universelle $f : E \to Y_N$ changée de base à $\mathbb{Q}$.

Le lien entre Δ et T_p. Dans cette section, nous allons énoncer, sans aucun détail, des résultats qui sont le fruit d'une étude minutieuse de l'action de l'opérateur T_p sur les groupes de cohomologie cohérente et étale l-adiques que l'on vient d'introduire. Le lecteur pourra se référer à [**Del71**] pour tous les détails qui permettent d'obtenir ces résultats. Plaçons nous dans le cas où $k = 12$. Pour tout nombre premier l, la forme modulaire Δ engendre une représentation galoisienne

$$\rho_{\Delta,l} : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{Q}_l)$$

non-ramifiée en dehors de l. Cela signifie pour tout nombre premier $p \neq l$, la représentation $\rho_{\Delta,l}$ se factorise par $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Pour tout $p \neq l$, l'endomorphisme $\rho_{\Delta,l}(\sigma_p)$, où $\sigma_p : x \mapsto x^p$ est le Frobenius arithmétique, a pour polynôme annulateur

$$X^2 - \tau(p)X + p^{11}$$

et on a les égalités

$$\begin{cases} \operatorname{tr}(\rho_{\Delta,l}(\sigma)) = \alpha_p + \beta_p = \tau(p) \\ \det(\rho_{\Delta,l}(\sigma)) = \alpha_p \times \beta_p = p^{12-1} = p^{11} \end{cases}$$

où α_p, β_p sont les valeurs propres de $\rho_{\Delta,l}(\sigma_p)$. La conjecture de Ramanujan énoncée plus haut est équivalente au fait que pour tout p le polynôme $X^2 - \tau(p)X + p^{11}$ ait un discriminant

$\tau(p)^2 - 4p^{11}$ négatif. Il suffit donc de montrer que les valeurs propres α_p et β_p sont complexes conjuguées de même module $p^{11/2}$. Ce dernier résultat est une conséquence des conjectures de Weil démontrées par Deligne [**Del74**][**Del80**].

1.1.2. Le grand théorème de Fermat et la modularité des courbes elliptiques. Dans la section précédente, nous avons introduit une représentation galoisienne

$$\rho_{\Delta,l} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_l)$$

associée à la forme modulaire Δ de niveau 1 et poids 12. Plus généralement, Deligne a associé à toute forme modulaire cuspidale f de poids $k \geq 2$, de niveau N, de Nebentypus trivial[5], propre pour les opérateurs de Hecke T_p pour tout $p \nmid Nl$ et telle que le premier coefficient de sa q-expansion soit égal à 1, une représentation galoisienne $\rho_{f,l} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$. Si on se restreint de plus aux formes modulaires nouvelles[6], cette représentation galoisienne caractérise entièrement la forme modulaire f et on a les propriétés suivantes.

(1) Pour tout p ne divisant pas Nl, $\rho_{f,l}$ est non-ramifiée en p.
(2) Pour tout p ne divisant pas Nl, la trace du Frobenius arithmétique $a_p := \mathrm{tr}(\rho_{f,l}(\sigma_p))$ est égale au p-ième coefficient de Fourier de la q-expansion $f = \sum_{n>0} a_n q^n$ de f.
(3) Pour tout p ne divisant pas Nl, le déterminant du Frobenius arithmétique $\det(\rho_{f,l}(\sigma_p))$ vaut p^{k-1}.

Il existe par ailleurs une source naturelle de représentation galoisienne de dimension 2 qui provient directement de la géométrie des courbes elliptiques. Considérons une courbe elliptique $E \to \mathrm{Spec}\,\mathbb{Q}$ au dessus des nombres rationnels. Les $\overline{\mathbb{Q}}$-points de la l-torsion de E forment un groupe qui admet une trivialisation de la forme

$$E[l](\overline{\mathbb{Q}}) \simeq (\mathbb{Z}/l\,\mathbb{Z})^2.$$

Comme $E[l]$ est un schéma en groupe fini et étale au dessus de $\mathbb{Q}$, ses $\overline{\mathbb{Q}}$-points sont également munis d'une action du groupe de Galois $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. En passant à la limite inverse, on obtient un $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module

$$\varprojlim_n E[l^n](\overline{\mathbb{Q}}) \simeq \mathbb{Z}_l^2$$

de dimension 2 connu sous le nom de module de Tate l-adique de E que l'on notera simplement

$$\rho_{E,l} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_l).$$

Par un résultat de Faltings, le module de Tate l-adique $\rho_{E,l}$ caractérise la courbe elliptique E à isogénie près et il est non ramifié en tout nombre premier $p \neq l$ ne divisant pas le discriminant de la courbe elliptique E. Le module de Tate l-adique se réinterprète également comme le dual de la cohomologie étale l-adique, i.e.

$$\rho_{E,l} = \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(H^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_l), \mathbb{Q}_l(1)^{7})$$

[5]Cela signifie que f vérifie l'équation de modularité (1.1) pour

$$\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}$$

[6]Une forme modulaire de poids k et niveau N est dite nouvelle si elle est cuspidale et elle ne provient pas d'un espace de forme modulaire de poids k et niveau $M \neq N$ avec $M|N$.

[7]La notation $\mathbb{Q}_l(1)$ désigne le twist de Tate.

Soit $p \neq l$ un nombre premier ne divisant pas le discriminant Δ_E de E. Notons α_p et β_p les valeurs propre de $\rho_{E,l}(\sigma_p)$ où σ_p est le Frobenius arithmétique en p. Grâce à la formule des traces de Lefschetz pour la cohomologie étale appliquée au Frobenius, on peut écrire

$$\begin{aligned}\#E(\mathbb{F}_p) &= \mathrm{tr}(\sigma_p|H^0(E_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)) - \mathrm{tr}(\sigma_p|H^1(E_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)) + \mathrm{tr}(\sigma_p|H^2(E_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)) \\ &= 1 - (\alpha_p + \beta_p) + \alpha_p\beta_p \\ &= 1 - a_p + p\end{aligned}$$

où $E_{\overline{\mathbb{F}}_p}$ est la réduction modulo p de E, $a_p := \mathrm{tr}(\rho_{E,l}(\sigma)) = \alpha_p + \beta_p$ et $\det(\rho_{E,l}(\sigma)) = p$. Si on cherche une forme modulaire f telle que $\rho_{f,l} = \rho_{E,l}$, il faut que celle-ci soit de poids 2 et que les coefficients $(a_n)_{n>0}$ de sa q-expansion soient reliés au nombre de $\mathbb{F}_p$-points $\#E(\mathbb{F}_p)$ de la réduction modulo p de la courbe elliptique E par la relation

$$a_p = p + 1 - \#E(\mathbb{F}_p)$$

pour tout p premier ne divisant pas le discriminant de E. Une courbe elliptique dont le module de Tate provient d'une forme modulaire de poids 2 est dite courbe elliptique modulaire. Commençons avec un exemple concret. Soit E_0 la courbe elliptique d'équation

$$E_0 \ : \ y^2 = x^3 + 1$$

et dont les points réels sont représentés sur la figure 1.1.4. Son discriminant est

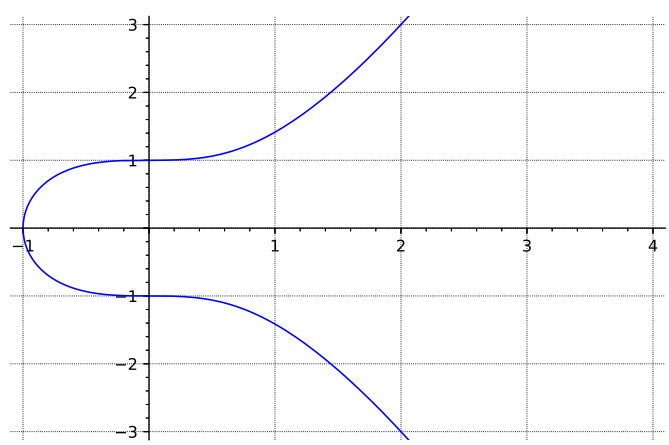


FIGURE 1.1.4. Points réels de la courbe elliptique E_0.

$$\Delta_{E_0} = -432 = -2^4 \cdot 3^3,$$

donc on peut la réduire modulo tout nombre premier $p \geq 5$. Calculons les coefficients a_p de E_0 à partir du nombre de $\mathbb{F}_p$-points de la réduction de E_0 modulo p.

p	$\#E_0(\mathbb{F}_p)$	a_p
5	6	0
7	12	-4
11	12	0
13	12	2
17	18	0
19	12	8
23	24	0
29	30	0
31	36	-4
37	48	-10
41	42	0
43	36	8

p	$\#E_0(\mathbb{F}_p)$	a_p
47	48	0
53	54	0
59	60	0
61	48	14
67	84	-16
71	72	0
73	84	-10
79	84	-4
83	84	0
89	90	0
97	84	14
101	102	0

Considérons maintenant la forme modulaire nouvelle f_0[8] de poids 2 et niveau 36

$$f_0(z) = \eta(6z)^4$$

où $\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ est la fonction êta de Dedekind. Les premiers termes de sa q-expansion sont

$$\begin{aligned} f_0(q) &= q\prod_{n\geq 1}(1-q^{6n})^4 \\ &= q - 4q^7 + 2q^{13} + 8q^{19} - 5q^{25} - 4q^{31} - 10q^{37} + 8q^{43} + 9q^{49} \\ &\qquad + 14q^{61} - 16q^{67} - 10q^{73} - 4q^{79} - 8q^{91} + 14q^{97} + O(q^{100}) \end{aligned}$$

et son graphe est tracé sur la figure 1.1.5.

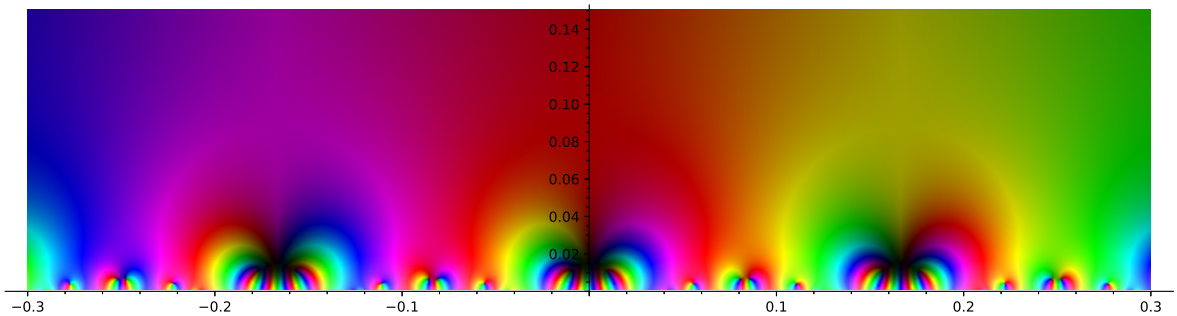


FIGURE 1.1.5. Tracé de $f_0(z)$.

On constate que les coefficients de la q-expansion de f_0 coïncident parfaitement avec la suite des a_p de la courbe elliptique E_0. Il est même possible de montrer qu'il existe un isomorphisme de représentations galoisiennes

$$\rho_{E_0,l} = \rho_{f_0,l}.$$

Voici quelques exemples supplémentaires de courbes elliptiques modulaires.

[8] Label 36.2.a.a sur [**LMF22**]

(1) La courbe elliptique d'équation

$$E_1 \ : \ y^2 + y = x^3 - 331$$

et de discriminant $\Delta_{E_1} = -1 \cdot 3^9 \cdot 7^4$ dont les points réels sont représentés sur la figure 1.1.6, provient d'une forme modulaire nouvelle f_1[9] de poids 2 et niveau 441

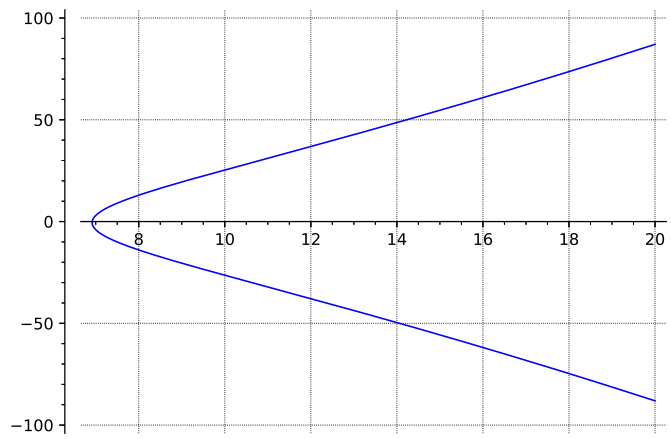


FIGURE 1.1.6. Points réels de la courbe elliptique E_1.

dont la q-expansion s'écrit

$$f_1 = q - 2q^4 - 7q^{13} + 4q^{16} - 7q^{19} - 5q^{25} - 7q^{31} - q^{37} + 5q^{43} + 14q^{52} + 14q^{61} - 8q^{64} + 11q^{67} - 7q^{73} + 14q^{76} - 13q^{79} + 14q^{97} + O(q^{100})$$

et dont le graphe est tracé sur la figure 1.1.7.

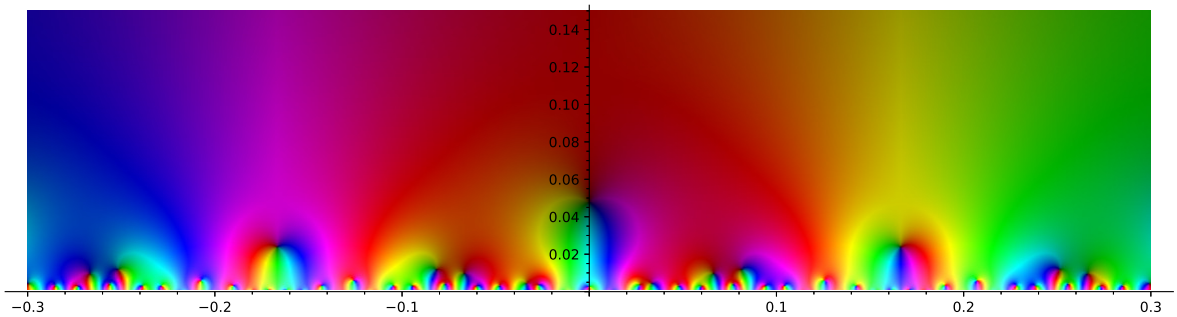


FIGURE 1.1.7. Tracé de $f_1(z)$.

(2) La courbe elliptique d'équation

$$E_2 \ : \ y^2 + y = x^3 - 169$$

et de discriminant $\Delta_{E_2} = -1 \cdot 3^9 \cdot 5^4$ dont les points réels sont représentés sur la figure 1.1.8, provient d'une forme modulaire nouvelle f_2[10] de poids 2 et niveau 675

[9] Label 441.2.a.d sur [**LMF22**]

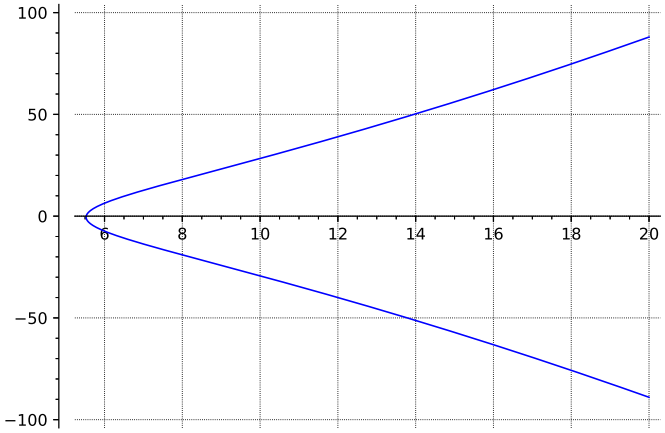


FIGURE 1.1.8. Points réels de la courbe elliptique E_2.

dont la q-expansion s'écrit

$$\begin{aligned} f_2 = q &- 2q^4 - 4q^7 + 5q^{13} + 4q^{16} + 8q^{19} + 8q^{28} + 11q^{31} \\ &- q^{37} - 13q^{43} + 9q^{49} - 10q^{52} + 14q^{61} - 8q^{64} + 5q^{67} \\ &+ 17q^{73} - 16q^{76} - 13q^{79} - 20q^{91} + 14q^{97} + O(q^{100}) \end{aligned}$$

et dont le graphe est tracé sur la figure 1.1.9.

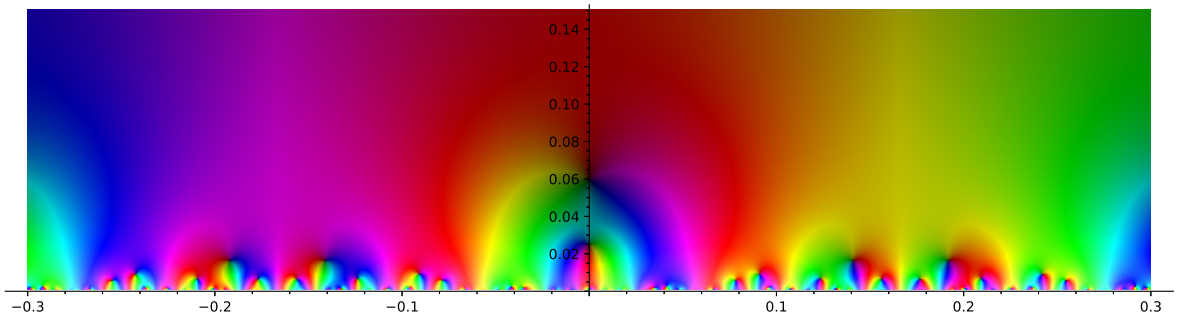


FIGURE 1.1.9. Tracé de $f_2(z)$.

Le théorème de modularité (aussi appelé conjecture de Shimura-Taniyama-Weil) démontré par Andrew Wiles en 1995 [**Wil95**] dans le cas des courbes elliptiques semi-stables vérifiant une hypothèse technique supplémentaire[11] et par Breuil-Conrad-Diamond-Taylor [**BCDT01**] dans le cas général, dit essentiellement que toute courbe elliptique définie sur $\mathbb{Q}$ est modulaire. Cela implique le fameux théorème de Fermat car si on considère par l'absurde une solution entière (a, b, c) non-triviale ($abc \neq 0$)

$$a^p + b^p = c^p$$

[10] Label 675.2.a.d sur [**LMF22**]

[11] Il faut que la restriction à $\mathbb{Q}(\sqrt{3})$ de l'action galoisienne sur $E[3]$ soit absolument irréductible.

avec $p \geq 3$ un nombre premier, alors Frey [**Fre86**] propose de construire la courbe elliptique

$$y^2 = x(x - a^p)(x + b^p)$$

et Ribet [**Rib90**] montre qu'elle ne peut pas être modulaire, ce qui contredit la conjecture de Shimura-Taniyama-Weil.

1.2. Présentation des principaux résultats

Les variétés de Shimura peuvent être vues comme des généralisations en dimension supérieure des courbes modulaires introduites dans la section précédente. Les variétés de Siegel forment une classe de variétés de Shimura associées aux groupes réductifs $\mathrm{Sp}_{2g} / \mathrm{GSp}_{2g}$ où $g \geq 1$ est un nombre entier appelé genre. Elles sont définies comme des espaces de modules fins de variétés abéliennes principalement polarisées avec une base de leur N-torsion où $N \geq 3$ est un entier appelé niveau. Le cas $g = 1$ correspond aux courbes modulaires dont le groupe associé est $\mathrm{SL}_2 / \mathrm{GL}_2$. Dans cette thèse, nous étudions principalement la géométrie et la cohomologie cohérente de la fibre spéciale des variétés de Siegel de genre $g \geq 2$. Nous pensons que la plupart de nos résultats pourraient se généraliser à d'autres variétés de Shimura de type PEL ou type Hodge. Certains des résultats présentés dans cette thèse sont déjà disponible en ligne, voir [**Ale22b**] et [**Ale22a**].

Soit $N \geq 3$ un nombre entier et p un nombre premier ne divisant pas N. On considère une compactification toroïdale projective lisse $\tilde{\mathrm{Sh}}^{\mathrm{tor}}$[12] du modèle entier canonique $\tilde{\mathrm{Sh}}$ sur $\mathbb{Z}_p$ de la variété modulaire de Siegel de niveau N et genre $g \geq 2$. On note D_{red} le diviseur à croisements normaux défini comme le bord $\tilde{\mathrm{Sh}}^{\mathrm{tor}} - \tilde{\mathrm{Sh}}$ de cette compactification toroïdale. À partir du modèle entier $\tilde{\mathrm{Sh}}^{\mathrm{tor}}$, on peut définir les objets géométriques suivants.

(1) Le schéma $\mathrm{Sh}^{\mathrm{tor}}$ sur $\mathbb{F}_p$, obtenu comme tiré en arrière de $\tilde{\mathrm{Sh}}^{\mathrm{tor}}$ à $\mathbb{F}_p$.

(2) Le schéma $\mathrm{Sh}^{\mathrm{tor}}_{\eta}$ sur $\mathbb{Q}_p$, obtenu comme tiré en arrière de $\tilde{\mathrm{Sh}}^{\mathrm{tor}}$ à $\mathbb{Q}_p$.

Sur le modèle entier $\tilde{\mathrm{Sh}}^{\mathrm{tor}}$, on définit le fibré vectoriel de Hodge Ω comme le tiré en arrière

$$\Omega := e^* \Omega^1_{A/\tilde{\mathrm{Sh}}^{\mathrm{tor}}}$$

du fibré des 1-formes différentielles sur le schéma semi-abélien

$$\begin{tikzcd} A \arrow[r, "f"] & \tilde{\mathrm{Sh}}^{\mathrm{tor}} \arrow[l, bend left, "e"] \end{tikzcd}$$

qui étend la variété abélienne universelle sur $\tilde{\mathrm{Sh}}$. Le fibré de Hodge est de rang g et on peut considérer des fibrés vectoriels automorphes définis sur $\tilde{\mathrm{Sh}}^{\mathrm{tor}}$ comme produit contracté de Ω avec une représentation algébrique de GL_g sur $\mathbb{Z}_p$. Au dessus de $\mathbb{Q}_p$, la catégorie des représentations algébriques de GL_g[13] est semi-simple et les objets simples $\nabla(\lambda)$ sont classifiés par leur plus haut poids λ qui est un caractère dominant du tore maximal standard de GL_g. Au dessus de $\mathbb{F}_p$, la catégorie des représentations algébriques de GL_g n'est plus semi-simple et pour chaque caractère dominant λ, on peut introduire deux représentations indécomposables $\nabla(\lambda)$ et $\Delta(\lambda)$ (dites costandard et standard), duale l'une de l'autre, mais en général non-isomorphes. On peut également définir un module autodual (dit tilting ou basculant) $T(\lambda)$ de plus haut poids λ. Ces représentations $\nabla(\lambda), \Delta(\lambda), T(\lambda)$ peuvent être définies au dessus de $\mathbb{Z}_p$ et permettent donc de définir des définir des fibrés automorphes sur $\tilde{\mathrm{Sh}}^{\mathrm{tor}}$ que l'on notera avec les mêmes symboles.

[12] Une telle compactification existe toujours mais n'est pas unique car elle repose sur des choix combinatoires.

[13] Plus généralement d'un groupe réductif connexe déployé sur $\mathbb{Z}_p$.

1.2.1. Amplitude des fibrés automorphes sur la fibre spéciale. À notre connaissance, le seul résultat dont on disposait sur la positivité des fibrés automorphes $\nabla(\lambda)$ sur la fibre spéciale $\mathrm{Sh}^{\mathrm{tor}}$ concernait essentiellement le cas du poids parallèle $\lambda = \text{-}(1, \ldots, 1)$, c'est à dire le déterminant du fibré de Hodge

$$\det \Omega = \nabla(\text{-}1, \ldots, \text{-}1) = \Delta(\text{-}1, \ldots, \text{-}1) = T(\text{-}1, \ldots, \text{-}1)^{14}.$$

Plus précisément, il existe un diviseur effectif D[15] sur $\mathrm{Sh}^{\mathrm{tor}}$ dont le diviseur réduit associé est D_{red} et un entier $n_0 \geq 1$ tel que le fibré en droite

$$(\det \Omega)^{\otimes n} (\text{-}D)$$

soit ample sur $\mathrm{Sh}^{\mathrm{tor}}$ pour tout $n \geq n_0$. Historiquement, ce résultat est une conséquence de la construction des compactifications toroïdales et de la redéfinition de la compactification minimale de la variété de Siegel par Faltings et Chai [**FC90**], et de la semi-amplitude[16] du faisceau $\det \Omega$ due à Moret-Bailly [**MB85**]. On en déduit facilement que $\det \Omega$ descend à la compactification minimale en un fibré en droite ample et que $(\det \Omega)^{\otimes n} (\text{-}D)$ est ample pour tout entier $n \geq 1$ assez grand et un diviseur effectif D supporté sur le bord. Dans cette thèse, nous généralisons ce résultat à d'autres fibrés vectoriels de rang supérieur. Pour cela, nous introduisons un nouvelle notion, dite de (φ, D)-amplitude[17] pour un fibré vectoriel $\mathcal{E}$ sur un schéma X défini sur un corps de caractéristique p, où $\varphi : X \to X^{(p)}$ est le morphisme de Frobenius relatif et D est un diviseur de Cartier effectif sur X.

Définition (Definition 3.43)**.** *Un fibré vectoriel $\mathcal{E}$ au dessus de X est dit (φ, D)-ample s'il existe un entier $r_0 \geq 1$ tel que pour tout entier $r \geq r_0$, le fibré vectoriel $\mathcal{E}^{(p^r)}(-D) := (\varphi^r)^* \mathcal{E} \otimes \mathcal{O}_X(-D)$ soit ample.*

Rappelons qu'un fibré en droite ample est en particulier nef et big et que l'on définit un fibré vectoriel nef (resp. big) comme un fibré $\mathcal{E}$ dont le fibré en droite universel $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ est nef (resp. big). Contrairement à l'amplitude ou la nefness, la bigness (pour les fibrés vectoriels) n'est pas stable par quotient[18]. On montre alors qu'un fibré vectoriel (φ, D)-ample est nef et big mais que la (φ, D)-amplitude se comporte mieux car elle est stable par somme directe, extension[19], quotient, produit tensoriel, racine tensorielle, tiré en arrière par un morphisme fini et qu'elle satisfait la descente le long d'un morphisme fini surjectif. Inspiré par un résultat de Mourougane [**Mou97**, Théorème 1] sur $\mathbb{C}$ à propos de l'amplitude du fibré adjoint $\pi_*(\mathcal{L} \otimes \omega_{Y/X})$ où $\pi : Y \to X$ est un morphisme surjectif, $\omega_{Y/X}$ est le fibré canonique relatif de π et $\mathcal{L}$ est un fibré en droite ample sur Y, on prouve des résultats similaires en caractéristique p quand π est une fibration en variétés de drapeaux. Plus précisément, on considère G un groupe réductif connexe déployé au dessus d'un corps k de caractéristique p. On fixe une paire de Borel (B, T) de G et on note ρ la demi-somme des racines positives de G. Soit X un schéma propre sur k, E un G-torseur sur X et $\pi : Y \to X$ la fibration en variétés de drapeaux G/B qui paramètre les B-réductions de E. On rappelle que l'on peut associer

[14] Le choix de nos conventions est expliqué au chapitre 4.

[15] Ce diviseur dépend d'un choix d'une fonction de polarisation sur la décomposition en cônes polyhédraux intervenant dans la construction des compactifications toroïdales.

[16] Un fibré en droite est semi-ample s'il possède une puissance tensorielle générée par ses sections globales.

[17] On parlera simplement de D-amplitude dans le cas des fibrés en droites.

[18] Un contre-exemple consiste à prendre $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ qui est big par [**Jab07**, p. 24] mais dont le quotient $\mathcal{O}_{\mathbb{P}^1}$ n'est clairement pas big.

[19] Sous une hypothèse de régularité sur X.

un fibré en droite $\mathcal{L}_\lambda$ sur Y à tout caractère λ de T[20] et que si λ est dominant, le faisceau $\pi_* \mathcal{L}_\lambda$ est un fibré vectoriel isomorphe au produit contracté de E avec le G-module costandard $\nabla(\lambda)$. Nous montrons le résultat suivant.

Théorème (Theorem 3.61 and 3.62)**.** *Si $\mathcal{L}_{2\lambda+2\rho}$ est ample (resp. $(\varphi, \pi^{-1}D)$-ample) sur Y, alors $\pi_* \mathcal{L}_\lambda$ est un fibré vectoriel ample (resp. (φ, D)-ample) sur X.*

Comme le fibré canonique relatif $\omega_{Y/X}$ est isomorphe à $\mathcal{L}_{-2\rho}$, notre résultat peut être vu comme une version en caractéristique p du résultat de Mourougane. Pour démontrer ce résultat, nous exploitons les propriétés cohomologiques particulières de la représentation de Steinberg de degré r

$$\mathrm{St}_r := \nabla\left((p^r-1)\rho\right) = \Delta\left((p^r-1)\rho\right) = T\left((p^r-1)\rho\right)$$

où $r \geq 1$ est un entier. La restriction aux fibrations en variétés de drapeaux complètes est reliée à l'utilisation de la représentation St_r dans notre argument.

Notons simplement G le groupe réductif Sp_{2g} sur $\mathbb{F}_p$, $(X^*, \Phi, X_*, \Phi^\vee)$ ses données radicielles, W son groupe de Weyl, $w_0 \in W$ son plus long élément, (B, T) une paire de Borel, $\Delta \subset \Phi^+$ les racines simples et positives correspondantes, $B \subset P \subset G$ le sous-groupe parabolique qui stabilise la filtration de Hodge sur le demi-plan de Siegel $\mathbb{H}_g$ de genre g, $L = \mathrm{GL}_g$ le sous-groupe de Levi associé à P, $\Phi_L \subset \Phi$ (resp. $\Phi_L^+ \subset \Phi^+$) les racines (resp. racines positives) de L, $I \subset \Delta$ le type de P, ρ_L la demi-somme des racines positives de L et $w_{0,L}$ l'élément de longueur maximal du groupe de Weyl de L. Rappelons que le fibré de Hodge intervient comme premier gradué de la filtration de Hodge-de Rham

$$0 \to \Omega \to \mathcal{H}^1_{\mathrm{dR}}(A/\mathrm{Sh}^{\mathrm{tor}}) \to \Omega^\vee \to 0,$$

qui est équivalente à la donnée d'un P-torseur sur la variété de Siegel $\tilde{\mathrm{Sh}}$. Pour tout sous-ensemble $I_0 \subset I$, on note Φ_{I_0} (resp. $\Phi_{I_0}^+$) l'ensemble des racines (resp. racines positives) engendrée par I_0, ρ_{I_0} la demi-somme des racines positives engendrée par I_0, $B \subset P_0 \subset P$ le sous-groupe parabolique intermédiaire de type I_0 et $\pi_{I_0} : Y_{I_0}^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$ la fibration en variété de drapeaux P/P_0 qui paramètre les P_0-réduction du P-torseur correspondant à la filtration de Hodge-de Rham sur la variété de Siegel $\tilde{\mathrm{Sh}}$. En appliquant le théorème précédent à la fibration en variété de drapeaux[21]

$$\pi : Y_\emptyset^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}},$$

on montre que la (φ, D)-amplitude du fibré automorphe

$$\pi_* \mathcal{L}_\lambda = \nabla(\lambda)$$

est une conséquence directe de la (φ, D)-amplitude du fibré en droite $\mathcal{L}_{2\lambda+2\rho_L}$ sur $Y_\emptyset^{\mathrm{tor}}$. D'après un résultat de [**BGKS**], on sait que $\mathcal{L}_\lambda$ est D-ample si ses puissances tensorielles possèdent des invariants de Hasse généralisés, ce qui est assuré si le caractère λ vérifie des inégalités provenant de la théorie des G-Zip introduites dans [**GK19a**]. On obtient finalement le résultat suivant.

Théorème (Theorem 6.7)**.** *Soit λ un caractère dominant du tore T. Si $\gamma := 2\lambda + 2\rho_L$ est*

[20] Dans le cas de GL_n, on écrit le caractère $\lambda = (k_1, \ldots, k_n)$ et on a $\mathcal{L}_\lambda = \mathrm{gr}_1^{\otimes k_1} \otimes \ldots \otimes \mathrm{gr}_n^{\otimes k_n}$ où les gr_i sont les gradués du drapeau universel du fibré vectoriel associé à E sur Y.

[21] Il s'agit d'une fibration en $P/P_0 = L/(P_0 \cap L)$.

(1) orbitalement p-proche, i.e.

$$\max_{\alpha\in\Phi, w\in W, \langle\gamma,\alpha^\vee\rangle\neq 0} |\frac{\langle\gamma, w\alpha^\vee\rangle}{\langle\gamma,\alpha^\vee\rangle}| \leq p-1$$

(2) et $\mathcal{Z}_\emptyset$-ample, i.e.

$$\langle\gamma,\alpha^\vee\rangle > 0 \text{ pour tout } \alpha\in I \text{ et } \langle\gamma,\alpha^\vee\rangle < 0 \text{ pour tout } \alpha\in\Phi^+\backslash\Phi_L^+$$

alors le fibré vectoriel automorphe $\nabla(\lambda)$ est (φ, D)-ample sur $\mathrm{Sh}^{\mathrm{tor}}$.

1.2.2. Annulation de la cohomologie cohérente des fibrés automorphes sur la fibre spéciale. La cohomologie des fibrés vectoriels automorphes sur les variétés de Shimura a joué un rôle important dans l'étude des propriétés arithmétiques des représentations automorphes comme cela est expliqué dans [**Har85**]. Pour mieux comprendre la cohomologie de $\mathrm{Sh}^{\mathrm{tor}}$, il est utile de savoir que certains degrés cohomologiques s'annulent. Il existe déjà plusieurs résultats d'annulation de cohomologie dans ce cadre obtenus par Lan et Suh [**LS12**][**LS13**] et Stroh [**Str13**]. Lan et Suh montrent plus généralement des résultats d'annulation pour la cohomologie cohérente de la fibre spéciale de variétés de Shimura PEL. Dans le cas des variétés modulaires de Siegel, Lan et Suh arrivent à accéder à des fibrés automorphes $\Delta(\lambda)^\vee$ pour des poids λ dans toutes les chambres de Weyl, du moment que λ peut s'écrire

$$\lambda = w\cdot\mu + \underline{k} \tag{1.2}$$

où w est un élément des représentants de longueur minimale du coset à gauche ${}^IW$ de type I, μ est un poids suffisamment régulier[22] qui est p-petit pour Sp_{2g} et tel que $|\mu|_{\mathrm{re},+}$[23] $< p$ et $\underline{k} = (k,\ldots,k)$ est un poids parallèle positif (i.e. $k>0$). Ils utilisent des résultats de [**PT02**] sur les complexes duaux de Bernstein-Gelfand-Gelfand et un pléthysme géométrique qui imposent plusieurs restrictions sur la taille du poids par rapport à la caractéristique p. La méthode de Stroh est similaire à celle de Lan et Suh bien qu'il se soit restreint au cas de la chambre de Weyl anti-dominante en vue de démontrer un résultat de classicité pour la théorie de Hida. Comme il n'y a qu'un nombre de fini de poids p-petit pour Sp_{2g}, leur méthode n'est capable d'accéder qu'à un nombre fini de fibrés automorphes à translation près par les poids parallèles positifs.

Dans cette thèse, nous introduisons une méthode permettant de déduire de nouveaux résultats d'annulation à partir de résultats déjà connus et qui fournit déjà de nouveaux résultats sans aucune initialisation non-triviale. Les poids accessibles avec notre méthode ne sont pas nécessairement réguliers ou p-petits (même à translation près par un poids parallèle) mais ils appartiennent à la chambre de Weyl anti-dominante. Pour tout caractère dominant λ de P_0, on note $\nabla^{\mathrm{sub}}(\lambda) := \nabla(\lambda)(\text{-}D_{\mathrm{red}})$ l'extension sous-canonique du fibré vectoriel automorphe costandard de plus haut poids λ et $\mathcal{L}_\lambda^{\mathrm{sub}}$ le fibré en droite $\mathcal{L}_\lambda(\text{-}\pi^{-1}D_{\mathrm{red}})$ sur la fibration en variété de drapeaux $Y_{I_0}^{\mathrm{tor}}$. On construit alors une fonction croissante

$$g_{I_0,e} : \mathcal{P}(X^*) \to \mathcal{P}(X^*)$$

sur l'ensemble des parties des caractères X^*, ordonné par l'inclusion, qui dépend d'un sous-ensemble $I_0\subset I$ et d'un entier $0\leq e\leq d-1$ où $d = g(g+1)/2$ est la dimension de $\mathrm{Sh}^{\mathrm{tor}}$. Nous reportons la définition de $g_{I_0,e}$ à la fin de cette sous-section car elle repose sur la dégénérescence de plusieurs suites spectrales que nous devons d'abord introduire.

[22] Voir [**LS12**, Def. 7.18].
[23] Voir [**LS12**, 7.22].

Théorème (Theorem 3.61)**.** *Supposons que $p > g^2$. Soit $\mathcal{C}$ un ensemble de caractères λ pour lesquels la cohomologie $H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\lambda))$ est concentrée en degrés $[0, e+1]$. Alors, l'image de $\mathcal{C}$ par la fonction $g_{I_0,e}$ est un ensemble de caractères λ pour lesquels la cohomologie $H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\lambda))$ est concentrée en degrés $[0, e]$.*

Puisque la définition de $g_{I_0,e}$ est difficile à exploiter, nous avons implémenté un algorithme sur SageMath[24] qui calcule les différents résultats d'annulations. Cet algorithme dépend d'un choix du genre $g \geq 2$, de la caractéristique p, d'un entier e compris entre 0 et $d-1$, d'un sous-ensemble $I_0 \subset I$ et d'un ensemble de résultats d'annulations préalablement connu $\mathcal{C}$. Nous insistons sur le fait que même sans aucun résultat d'annulation connu pour initialiser notre algorithme, celui-ci fournit des résultats non-triviaux dans les cas $e = 0$ et $e = d-1$ qui peuvent être réutilisés avec d'autres valeurs de e.

Notre code SageMath définit une classe *SiegelVariety* avec plusieurs méthodes permettant de calculer explicitement des résultats d'annulation. Dans cet algorithme, les valeurs de p et g sont fixées mais le niveau N peut varier. La ligne suivante permet de définir la variété de Siegel de genre 2 au dessus de $\mathbb{F}_7$.

In [1]:
```
X = SiegelVariety(g = 2, p = 7)
```

Si la prochaine ligne renvoie *True*, cela signifie que le fibré en droite automorphe $\mathcal{L}_{(-2,-8)}$ est D-ample sur la fibration en variétés de drapeaux $\pi : Y_{\emptyset}^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$.

In [2]:
```
X.ample([],[-2,-8])
```

Out[2]:
```
True
```

La méthode *compute* calcule des résultats d'annulation pour tous les caractères $\lambda = (k_1, k_2)$ avec $-50 \leq k_2 \leq k_1 \leq 0$ en utilisant la fonction $g_{I_0,e}$ dans le cas où $I_0 = \emptyset$ et $e = 0$. Ces résultats sont enregistrés dans la liste C_{van}. La méthode renvoie True si l'algorithme a trouvé de nouveaux résultats d'annulations.

In [3]:
```
X.compute([], e = 0, kmin = -50, kmax = 0)
```

```
100%|██████████| 1326/1326 [00:02<00:00, 571.40it/s]
```

Out[3]:
```
True
```

La méthode *computeAll* appelle la méthode *compute* pour tous les sous-ensembles $I_0 \subset I$ et tous les entiers $0 \leq e \leq d-1$. Nous n'avons qu'à spécifier la plage de caractères $\lambda = (k_1, k_2)$ que l'on veut considérer. Elle renvoie *True* si l'algorithme a trouvé de nouveaux résultats d'annulations. Il est souhaitable d'appeler cette méthode plusieurs fois jusqu'à ce qu'elle renvoie *False*.

In [4]:
```
X.computeAll(-50, 0)
```

[24] Voir github.com/ThibaultAlexandre/vanishing-results-over-the-siegel-variety

```
Step 1/6 : e = 0, Iprime = [0]
100%|██████████| 1326/1326 [00:02<00:00, 496.87it/s]
Step 2/6 : e = 1, Iprime = [0]
100%|██████████| 1326/1326 [00:01<00:00, 1205.39it/s]
Step 3/6 : e = 2, Iprime = [0]
100%|██████████| 1326/1326 [00:01<00:00, 1293.54it/s]
Step 4/6 : e = 0, Iprime = [1]
100%|██████████| 1326/1326 [00:00<00:00, 2673.82it/s]
Step 5/6 : e = 1, Iprime = [1]
100%|██████████| 1326/1326 [00:00<00:00, 2609.12it/s]
Step 6/6 : e = 2, Iprime = [1]
100%|██████████| 1326/1326 [00:00<00:00, 2994.65it/s]
```

Out[4]:
```
True
```

La prochaine ligne renvoie *True* si nous savons que

$$H^i(X, \nabla^{\mathrm{sub}}(\text{-}4, \text{-}6)) = 0$$

pour tout $i > 1$.

In [5]:
```
X.vanishes(1,(-4,-6))
```

Out[5]:
```
True
```

Si la prochaine ligne renvoie *False*, cela signifie que l'on ne sait pas si

$$H^i(X, \nabla^{\mathrm{sub}}(\text{-}4, \text{-}6)) = 0$$

pour tout $i > 0$.

In [6]:
```
X.vanishes(0,(-4,-6))
```

Out[6]:
```
False
```

Notre code SageMath permet également de tracer lorsque $g = 2$ ou $g = 3$[25] les résultats d'annulations calculés. On illustre alors nos résultats dans le cas spécial $g = 2$, $p = 5$ avec la figure 1.2.2. Remarquons que dans le cas $g = 2$, $p = 5$ de la figure 1.2.2, le seul poids p-petit pour Sp_4 est $(0, 0)$; cela signifie que la méthode de Lan et Suh est seulement capable d'accéder à des poids de la forme $w\rho - \rho + \underline{k}$ pour $w \in W$ là où notre méthode est capable d'accéder à une infinité de poids modulo les poids parallèles. Notre méthode exploite encore la D-amplitude des fibrés en droites $\mathcal{L}_\lambda$ sur la fibration en variété de drapeaux $\pi : Y_{I_0}^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$ obtenue par [**BGKS**]. Comme Lan et Suh, nous utilisons une version logarithmique, due à Esnault et Viehweg [**EV92**], du théorème d'annulation de Kodaira-Nakano en caractéristique p. Sous l'hypothèse $p > d_0 := \dim Y_{I_0}^{\mathrm{tor}}$, nous avons

$$H^i(Y_{I_0}^{\mathrm{tor}}, \Omega^{d_0-e}_{Y_{I_0}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) = 0,$$

[25] Pour $g > 3$, on peut aussi calculer des résultats d'annulations mais c'est plus difficile à tracer et l'ordinateur prend plus de temps à calculer.

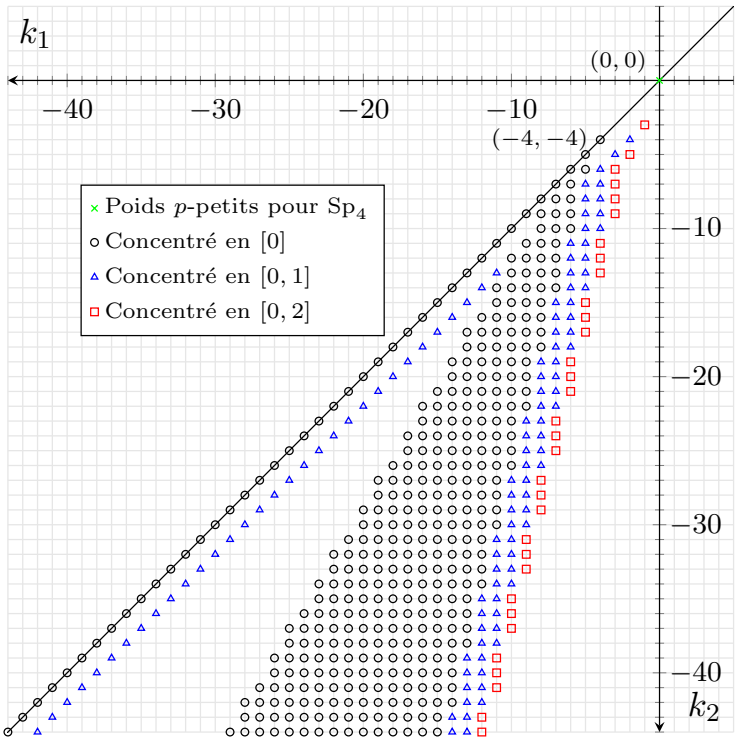


FIGURE 1.2.1. Résultat d'annulations obtenus avec notre méthode (sans initialisation) lorsque $g = 2$, $p = 5$. Chaque élément de la grille correspond à un fibré automorphe $\nabla(k_1, k_2)$.

pour tout $i > e$ et tout caractère λ qui admet des invariants de Hasse généralisés. Ensuite, nous filtrons le fibré

$$\Omega^{d_0-e}_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_{\lambda}$$

avec une filtration croissante $F_\bullet$

$$F_k = \pi^* \Omega^{d_0-e-k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \wedge \Omega^k_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_{\lambda}$$

et nous considérons la suite spectrale associée

$$E^{t,k}_{2,e,\lambda} = H^{t+k}(Y^{\mathrm{tor}}_{I_0}, \mathrm{gr}_k \otimes \mathcal{L}^{\mathrm{sub}}_{\lambda}) \Rightarrow H^{t+k}(Y^{\mathrm{tor}}_{I_0}, \Omega^{d_0-e}_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_{\lambda}). \tag{1.3}$$

C'est une suite spectrale commençant à la seconde page $E_2^{i,j}$ dont la limite est nulle lorsque $i + j > e$ par le théorème d'annulation de Kodaira-Nakano logarithmique. En général, il est difficile d'extraire de l'information sur la seconde page d'une suite spectrale dont la limite est nulle. Cependant, si on arrive à montrer que suffisamment de termes $E_2^{i,j}$ sont déjà nuls (avec les théorèmes d'annulation contenus dans $\mathcal{C}$), alors la suite spectrale dégénèrera partiellement et on en déduira de nouveaux résultats d'annulation de termes $E_2^{i,j}$[26]. Le but est donc de déterminer les résultats d'annulation nécessaires à la dégénérescence de cette suite spectrale. De plus, au cours de la démonstration, nous montrons que la cohomologie de

[26]Dans le cas $e = 0$, la suite spectrale est concentrée sur une ligne, ce qui explique pourquoi nous n'avons pas besoin de résultats d'annulation préliminaires pour en obtenir de nouveaux.

produits tensoriels de fibrés automorphes de la forme

$$\Lambda^k \nabla(\lambda) \otimes \nabla(\mu)$$

s'annule en certains degrés. Pour relier la cohomologie de ce produit tensoriel avec la cohomologie des fibrés automorphes, on considère une suite spectrale associée à une ∇-filtration[27] de ce produit tensoriel, dont l'existence est assurée par [**Mat90**] sous l'hypothèse $p > k$. Là encore, on utilise des résultats d'annulation préliminaires pour faire dégénérer partiellement cette suite spectrale et en déduire de nouveaux résultats d'annulation. Notre méthode est technique parce qu'elle implique récursivement un nombre inconnu de suites spectrales. Voici la définition de la fonction $g_{I_0,e}$ du théorème.

Définition (Definition 7.18)**.** *Notons $(\mu_j^k)_j$ les poids de $\Lambda^k Sym^2 std_{GL_g}$*[28]*, $s_M = \sum_{\alpha \in M} \alpha$ où $M \subset \Phi^+$, $\rho_{I_0} = \frac{1}{2} s_{\Phi_{I_0}^+}$ et $\mathcal{C}_{amp,I_0}$ l'ensemble des poids λ tels que*

(1) $\max_{\alpha \in \Phi, w \in W} \left\{ \frac{|\langle \lambda, w\alpha^\vee \rangle|}{|\langle \lambda, \alpha^\vee \rangle|} \mid w \in W \right\} \leq p - 1$ *(orbitalement p-proche)*

(2) $\forall \alpha \in I \backslash I_0 \ \langle \lambda, \alpha^\vee \rangle > 0$ *et* $\forall \alpha \in \Phi^+ \backslash \Phi_L^+ \ \langle \lambda, \alpha^\vee \rangle < 0$ *($\mathcal{Z}_{I_0}$-ample)*

On définit la fonction $g_{I_0,e} : \mathcal{P}(X^*) \to \mathcal{P}(X^*)$ par

$$g_{I_0,e}(\mathcal{C}) = \mu^{d-e}_{\binom{d}{d-e}} + X^*(P_0)^+ \cap (-2\rho_{I_0} + \mathcal{C}_{amp,I_0}) \cap \bigcap_{k,j,M} (s_M - 2\rho_{I_0} - \mu_j^{d-e+k} + \mathcal{C}),$$

pour tout $\mathcal{C} \subset X^*$ où la dernière intersection porte sur l'ensemble des k, j, M vérifiant $0 \leq k \leq e$, $1 \leq j \leq \binom{d}{d-e+k}$ et $M \subset \Phi_L^+ - \Phi_{I_0}^+$ tel que $|M| = r_0 - k$ à l'exception de $j = \binom{d}{d-e}$ lorsque $k = 0$.

1.2.3. Hyperbolicité des variétés de Siegel en caractéristique p. Pour une variété projective lisse X au dessus des nombres complexes $\mathbb{C}$, on dit qu'elle est algébriquement hyperbolique si toute sous-variété $V \subset X$ est de type général, i.e. admet une désingularisation $\tilde{V}$ telle que son fibré canonique $\omega_{\tilde{V}}$ soit big. Pour une variété quasi-projective, il faut remplacer la condition de type général par la condition de type log-général, i.e. V admet une désingularisation $\tilde{V}$ et une immersion ouverte $\tilde{V} \subset W$ tel que $H = W - \tilde{V}$ soit un diviseur de Cartier effectif tel que $\omega_W(H)$ soit big. Il est connu que les variétés de Siegel au dessus de $\mathbb{C}$ sont algébriquement hyperboliques mais que peut-on dire si l'on remplace le corps des nombres complexe par un corps de caractéristique p. Comme les techniques de désingularisation n'existent pas toujours en caractéristique p, nous nous restreignons aux sous-variétés lisses. Autrement dit, on se pose la question suivante.

Question (Caractéristique p)**.** *Est-ce que $(\mathrm{Sh}^{\mathrm{tor}}, D_{\mathrm{red}})$ est algébriquement hyperbolique ? En d'autres termes, est-ce que toute sous-variété lisse*

$$\iota : V \hookrightarrow \mathrm{Sh}^{tor}$$

telle que $\iota^{-1} D_{\mathrm{red}}$ est bien défini comme diviseur de Cartier effectif est de type log-général ?

Il est connu depuis [**MB81**] que la réponse à cette question est négative lorsque $g = 2$. Moret-Bailly construit une famille non-isotriviale $A \to \mathbb{P}^1$ de surfaces abéliennes principalement polarisée avec une structure de niveau N au dessus de la droite projective sur $\mathbb{F}_p$. Cette famille induit une immersion fermée $\mathbb{P}^1 \hookrightarrow \mathrm{Sh}_{g=2}$ qui contredit l'hyperbolicité de la variété

[27] Voir définition 2.12.

[28] En réalité, pour suivre la convention de la définition 3.43, on doit tordre ces poids par $w_0 w_{0,\mathrm{GL}_g}$ et supposer qu'ils sont ordonnés de telle façon que $w_0 w_{0,\mathrm{GL}_g}(\mu^n_{\binom{d}{n}})$ soit le plus haut poids.

de Siegel lorsque $g = 2$ au dessus de $\mathbb{F}_p$. Il est facile d'utiliser cette famille pour contredire l'hyperbolicité de la variété de Siegel pour $g \geq 2$. Soit k un corps de caractéristique p et notons encore $\mathrm{Sh}^{\mathrm{tor}}$ une compactification toroïdale projective lisse de la variété de Siegel de genre g et niveau N au dessus de k. Un des principaux apports de cette thèse est une méthode reposant sur le pléthysme des foncteurs de Schur pour montrer que les sous-variétés de $\mathrm{Sh}^{\mathrm{tor}}$ ayant une certaine codimension sont nécessairement de type log-général. Cela nous permet de montrer le résultat suivant.

Théorème (Corollary 8.15)**.** *Supposons que $p \geq g^2+3g+1$. Toute sous-variété $\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ de codimension $\leq g-1$ satisfaisant*

(1) V est lisse,

(2) $\iota^{-1}D_{\mathrm{red}}$ est un diviseur à croisements normaux,

est de type log-général avec lieu exceptionnel contenu dans le bord D_{red}.

Motivé par la conjecture de Green-Griffiths-Lang, on peut imaginer qu'il existe un lieu exceptionnel $E \subset \mathrm{Sh}^{\mathrm{tor}}$ tel que pour toute sous-variété lisse V non-contenue dans le bord, V est de type log-général si et seulement si $V \nsubseteq E$. Notre résultat indique que ce lieu exceptionnel hypothétique a une codimension strictement supérieure à $g-1$. Il est peut-être équidimensionnel de codimension exactement g. Afin d'expliquer notre stratégie pour démontrer le corollaire 8.15, nous devons discuter des foncteurs de Schur.

Les foncteurs de Schur sont des endofoncteurs

$$S : \mathrm{FinVect}_k \to \mathrm{FinVect}_k$$

de la catégorie abélienne des k-espaces vectoriels de dimension finie qui généralisent la construction des puissances symétriques et extérieures d'un espace vectoriel. Ces foncteurs sont indexés par des partitions entières ou des diagrammes de Young, et ils peuvent être définis sur la catégorie des modules localement libres de rang fini au dessus d'un schéma. Nous nous intéressons à ces foncteurs parce que si $\lambda = (k_1 \geq \ldots \geq k_g \geq 0)$ est un poids G-dominant, on peut l'identifier avec un diagramme de Young dont la $i^{\text{ième}}$-ligne a k_i colonnes et on obtient un isomorphisme

$$S_\lambda \Omega = \nabla(-w_{0,L}\lambda).$$

La stratégie pour prouver le théorème 8.14 est de montrer que le fibré $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ est (φ, D)-ample pour certains choix spécifiques de λ. Puisque la (φ, D)-amplitude est stable par quotients, tiré en arrière par un morphisme fini et que S_λ respecte les surjections, la (φ, D)-amplitude de $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ implique que le quotient

$$\iota^* S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \twoheadrightarrow S_\lambda \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}})$$

est aussi $(\varphi, \iota^{-1}D)$-ample pour toute sous-variété lisse $\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ telle que $\iota^{-1}D_{\mathrm{red}}$ est bien défini comme diviseur à croisements normaux. Il résulte alors de la théorie générale des foncteurs de Schur que le fibré vectoriel $S_\lambda \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}})$ est non-nul exactement quand la dimension de V est plus grande que le nombre de parties (aussi appelée hauteur) de λ. Dans ce cas, le déterminant de $S_\lambda \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}})$ est une puissance tensorielle positive de $\omega_V(\iota^{-1} D_{\mathrm{red}})$ et la (φ, D)-amplitude de $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ implique que V est de type-général avec lieu exceptionnel contenu dans le bord D_{red}. Par l'isomorphisme de Kodaira-Spencer

$$\rho_{\mathrm{KS}} : \mathrm{Sym}^2 \Omega \xrightarrow{\sim} \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}),$$

nous sommes réduits à étudier la composition de foncteurs de Schur de la forme $S_\lambda \circ \mathrm{Sym}^2$. Au dessus d'un corps de caractéristique 0, la composition de deux foncteurs de Schur $S_\lambda \circ S_\mu$ se décompose en une somme directe

$$S_\lambda \circ S_\mu = \bigoplus_\eta S_\eta^{\oplus c^\eta_{\lambda,\mu}}$$

de foncteurs de Schur et on appelle pléthysme le problème de déterminer les coefficients $c^\eta_{\lambda,\mu}$. Pour étudier le pléthysme au dessus d'un corps de caractéristique p, on se place dans la catégorie Pol des foncteurs polynomiaux stricts qui contient les composés des S_λ. La catégorie Pol n'est pas semi-simple mais on peut au moins espérer que la composition $S_\lambda \circ S_\mu$ admette une filtration par des foncteurs polynomiaux dont les gradués sont des foncteurs de Schur. Malheureusement, un tel résultat est tout simplement faux : Boffi [**Bof91**] et Touzé [**Tou13**, Corollary 6.10.] ont trouvé des obstructions à l'existence de telles filtrations. Par exemple, le pléthysme $\Lambda^2 \circ \Lambda^2$ sur $\mathbb{F}_2$ n'admet pas de telle filtration comme cela est expliqué par la proposition 2.49. On évite ces contre-exemples avec une restriction supplémentaire sur la caractéristique p.

Proposition (Proposition 2.43)**.** *Soit λ et μ des partitions de taille $|\lambda|$ et $|\mu|$. Si $p \geq 2|\lambda|-1$, le foncteur polynomial strict $S_\lambda \circ S_\mu$ admet une filtration finie*

$$0 = T^n \subsetneq T^{n-1} \subsetneq \cdots \subsetneq T^0 = S_\lambda \circ S_\mu$$

par des foncteurs polynomiaux stricts de degré $|\lambda||\mu|$ dont les gradués sont des foncteurs de Schur. Par ailleurs, lorsque S_λ est une puissance extérieure, il suffit de supposer $p > |\lambda|$.

Remarque. On obtient au corollaire 2.48 un résultat analogue sans restriction sur la taille de p si on remplace les foncteurs de Schur S_λ par des foncteurs tilting T_λ et si on suppose que la partition du deuxième foncteur que l'on applique est p-régulière.

Puisque la (φ, D)-amplitude est stable par extension, on peut utiliser la proposition 2.43 pour voir que $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ est (φ, D)-ample si les gradués $\nabla(\eta)$ apparaissant dans le pléthysme $S_\lambda \circ \mathrm{Sym}^2$ sont (φ, D)-amples. Il est important de noter que les calculs directs de pléthysme sont extrêmement complexes et qu'à ce jour, nous ne connaissons pas de règle combinatoire générale permettant de calculer efficacement les coefficients $c^\eta_{\lambda,\mu}$. De plus, déterminer effectivement si un fibré automorphe $\nabla(\eta)$ est (φ, D)-ample est difficile car cela nécessite de vérifier la condition orbitalement p-proche du théorème 6.7. Par exemple pour $g = 3$, un calcul sur SageMath[29] permet de voir que les fibrés

$$\begin{cases} \Lambda^6 \mathrm{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(\text{-}4, \text{-}4, \text{-}4) \\ \Lambda^5 \mathrm{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(\text{-}2, \text{-}4, \text{-}4) \end{cases}$$

sont (φ, D)-ample dès que $p \geq 11$ et que le fibré $S_{(2^4)} \mathrm{Sym}^2 \Omega^{\mathrm{tor}}$ qui est filtré par

$$\nabla(\text{-}2, \text{-}6, \text{-}8), \nabla(\text{-}3, \text{-}6, \text{-}7), \nabla(\text{-}4, \text{-}4, \text{-}8) \text{ et } \nabla(\text{-}4, \text{-}6, \text{-}6)$$

est (φ, D)-ample dès que $p \geq 17$. Cela montre que lorsque $p \geq 17$, les (bonnes) sous-variétés de dimension ≥ 4 sont de type log-général. Pour $g = 2, 3, 4$, les calculs de pléthysme sur SageMath sont accessibles avec la puissance de calcul d'un ordinateur standard. Au delà de $g = 4$, nous avons besoin d'une autre méthode pour démontrer le théorème 8.14. Il est connu depuis [**Wil09**, Lemma 7] que le pléthysme $\Lambda^k \circ \mathrm{Sym}^2$ appartient aux rares cas où une

[29] Voir github.com/ThibaultAlexandre/positivity-of-automorphic-bundles

formule générale est disponible. Avec cette formule et une borne supérieure sur la condition orbitalement p-proche, nous montrons le résultat suivant qui implique le corollaire 8.15.

Théorème (Théorème 8.14)**.** *Supposons $p \geq g^2+3g+1$. Pour tout entier $k \geq g(g-1)/2+1$, le fibré $\Lambda^k \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ est (φ, D)-ample.*

Dans le cas $g = 2, 3$, on arrive également à montrer avec des calculs d'intersections sur les strates d'Ekedahl-Oort que la borne sur k est optimale dans le théorème 8.14, ce qui fournit un argument pour espérer que le lieu exceptionnel hypothétique $E \subset \mathrm{Sh}^{\mathrm{tor}}$ est pur de codimension g.

1.2.4. Opérateur de Hecke cohomologique en niveau hyperspécial. Les modèles entiers des variétés de Siegel avec un niveau parahorique en p permettent de définir intégralement une correspondance de Hecke $T_{p,g}$. Dans [**FP19**], Fakhruddin et Pilloni compactifient cette correspondance de Hecke afin d'obtenir une action sur toute la cohomologie cohérente et pas seulement le H^0. En lisant l'article [**BP20**] qui étudie cet opérateur de Hecke compactifié dans le cas des courbes modulaires, nous avons voulu redémontrer leur résultat d'auto-dualité pour la dualité de Serre et commencer à étudier la situation analogue pour des variétés de Siegel de genre supérieur à 1. Toutes les idées que l'on développe sont dues à Boxer et Pilloni et on peut voir ce chapitre comme un exercice d'étudiant, indépendant du reste de la thèse, où l'on adapte le résultat d'auto-dualité de [**BP20**] au cadre des variétés de Siegel. Une difficulté qui n'apparait que pour les variétés de Siegel de genre $g > 1$, est que les fibrés automorphes proviennent de la catégorie des représentations algébriques du Lévi $L = \mathrm{GL}_g$ qui n'est pas semi-simple en caractéristique p dès que $g > 1$. Ainsi, l'action de l'opérateur $T_{p,g}$ sur la cohomologie d'un fibré automorphe costandard $\nabla(\lambda)$ devient par dualité une action sur la cohomologie d'un fibré automorphe standard $\Delta(-w_{0,L}\lambda + 2\rho^L)$. Nous proposons une (re)définition explicite (voir définition 9.4.3) à l'aide des foncteurs de Schur de l'action de l'opérateur $T_{p,g}$ sur la cohomologie $R\Gamma(\mathrm{Sh}^{\mathrm{tor}}, \mathcal{F}_\lambda)$ où $\mathcal{F}_\lambda = \nabla(\lambda), \Delta(\lambda)$ ou $T(\lambda)$. On définit également une version transposée et cuspidale de l'opérateur $T_{p,g}$. Un des défis techniques de cette définition consiste à établir des résultats de renormalisation par un facteur $p^{-v(\lambda)}$ reposant ultimement sur les propriétés des modules S_k et le fait que les foncteurs de Schur admettent une base indexée par des tableaux de Young. On montre alors le résultat suivant.

Théorème (Théorème 9.42, Remarque 9.43)**.** *Soit λ un caractère GL_g-dominant. Alors on a des identifications*

$$\begin{cases} D(T_{p,g}^{\nabla(\lambda)}) = T_{p,g}^{\Delta(-w_{0,L}\lambda+2\rho^L),t}, \\ D(T_{p,g}^{\Delta(\lambda)}) = T_{p,g}^{\nabla(-w_{0,L}\lambda+2\rho^L),t}, \\ D(T_{p,g}^{T(\lambda)}) = T_{p,g}^{T(-w_{0,L}\lambda+2\rho^L),t} \end{cases}$$

où D désigne la dualité de Serre et t désigne la transposition. Autrement dit, l'opérateur $T_{p,g}$ est auto-dual pour la dualité de Serre.

La preuve du théorème 9.42 n'est pas complètement formelle car elle repose sur la commutativité du diagramme du lemme 9.41 faisant intervenir les morphismes traces des deux projections définissant $T_{p,g}$.

CHAPTER 2

Schur functors and the plethysm operation

Classically, Schur functors are certain endofunctors

$$S : \mathrm{FinVect}_{\mathbb{C}} \to \mathrm{FinVect}_{\mathbb{C}}$$

of the abelian category of finite dimensional complex vector spaces. The first example is given by the nth-symmetric power Sym^n which sends a vector space V to the space of $\mathfrak{S}_n$-coinvariants $(V^{\otimes n})_{\mathfrak{S}_n}$ where $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors. A second example is given by the nth-exterior power Λ^n which sends a vector space V to the space of $\mathfrak{S}_n$-coinvariants $(V^{\otimes n})_{\mathfrak{S}_n}$ where an element $\sigma \in \mathfrak{S}_n$ acts on $V^{\otimes n}$ by antisymmetrization

$$\sigma(v_1 \otimes \cdots \otimes v_n) = \varepsilon(\sigma)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})^{1}.$$

In general, we start with a $\mathbb{S}$-module, which a collection $\pi = (\pi(n))_{n\geq 1}$ where each $\pi(n)$ is a finite dimensional representation of the symmetric group $\mathfrak{S}_n$ and such that only a finite number of $\pi(n)$ are non-zero. We define the Schur functor

$$S_\pi : \mathrm{FinVect}_{\mathbb{C}} \to \mathrm{FinVect}_{\mathbb{C}}$$

associated to π as

$$S_\pi(V) = \bigoplus_{n\geq 1} (V^{\otimes n} \otimes \pi(n))_{\mathfrak{S}_n}$$

where each $\mathfrak{S}_n$ acts via permutation on the first factor. It is well-known that irreducible representations π of $\mathfrak{S}_n$ are in bijection with partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$ of n. This bijection is made explicit by sending a partition λ of n to the Specht module Sp_λ of shape λ [**Spe35**]. We call $S_\lambda = S_{\mathrm{Sp}_\lambda}$, the Schur functor of weight λ.

A key feature of Schur functors is their close relationship to polynomial representations of the general linear group GL_n, i.e. algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$ that can be lifted to a scheme morphism $\hat{\rho} : \mathrm{Mat}_n \to \mathrm{End}(V)$ where Mat_n is the scheme of $n \times n$ matrices and $\mathrm{End}(V)$ is the scheme of endomorphisms of the vector space V. We denote $\mathrm{Rep}_k(\mathrm{GL}_n)^{\mathrm{pol}}$ the polynomial representations for GL_n over a field k. We say that ρ is homogeneous of degree $d \geq 1$ if the center $\mathbb{G}_m$ acts as

$$z \mapsto \begin{pmatrix} z^d & & \\ & \ddots & \\ & & z^d \end{pmatrix} \in \mathrm{GL}(V).$$

Given a Schur S_λ and a polynomial representation V of GL_n, the evaluation $S_\lambda V$ yields by functoriality a new polynomial representation of GL_n. More precisely, if we start with a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0)$ of height n, then the evaluation $S_\lambda(\mathrm{std}_{\mathrm{GL}_n})$ at

[1] This definition is not correct over a field of characteric p if $p \geq n$. We should instead consider a quotient by the ideal generated by tensors of the form $x_1 \otimes \cdots \otimes x_n$ where $x_i = x_j$ for some $i \neq j$.

the standard representation of GL_n is isomorphic to the algebraic representation of GL_n of highest weight

$$\lambda : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \to t_1^{\lambda_1} \cdots t_n^{\lambda_n}.$$

This process being functorial, we would like to describe it as a single functor from a category of Schur functors to a category of polynomial representation for GL_n for any $n \geq 1$. According to Friedlander and Suslin [**FS97**], the correct category to describe Schur functor is the category of strict polynomial functors Pol. In particular, they show that this category Pol can be splitted

$$\mathsf{Pol} = \bigoplus_{d \geq 0} \mathsf{Pol}_d$$

where Pol_d is the category of strict polynomial functors homogeneous of degree d. They also prove that, as long as $n \geq d$, the evaluation functor yields an equivalence of categories

$$\begin{aligned} \mathrm{ev}_d : \mathsf{Pol}_d &\longrightarrow \mathrm{Rep}_k(\mathrm{GL}_n)_d^{\mathrm{pol}} \\ S &\longmapsto S(\mathrm{std}_{\mathrm{GL}_n}) \end{aligned}$$

between the category of strict polynomial functors homogeneous of degree d and the category of polynomial representation of GL_n homogeneous of degree d. This category Pol of strict polynomial functors can be seen as a clever way of stacking all the polynomial representation of GL_n for different n.

Since composing two strict polynomial functors yields a third strict polynomial functor, it is a simple consequence of the semisimplicity of the categories $\mathrm{Rep}_k(\mathrm{GL}_n)$ over $\mathbb{C}$ that for any two partition λ and μ, we have a direct sum decomposition

$$S_\lambda \circ S_\mu \simeq \bigoplus_\eta S_\eta^{\oplus c_{\lambda,\mu}^\eta} \tag{2.1}$$

in the category of stricty polynomial functors. The problem of determining the coefficients $c_{\lambda,\mu}^\eta$ is called plethysm[2].

Example 2.1. There is no known combinatorial rule for computing the coefficients $c_{\lambda,\mu}^\eta$. To illustrate how hard is the plethysm problem, we give the following examples.

(1) $S_{(2,1)} \circ S_{(1,1)} = S_{(2,1,1,1,1)} \oplus S_{(2,2,1,1)} \oplus S_{(3,2,1)}$.
(2) The composition $S_{(4,2)} \circ S_{(3,1)}$ involves $1,238$ different partitions η with a maximum multiplicity $c_{\lambda,\mu}^\eta$ of $8,408$. Counted with multiplicity, there are $958,705$ endofunctors in the direct sum.
(3) The composition $S_{(3,2,1)} \circ S_{(4,2)}$ involves $11,938$ different partitions η with a maximum multiplicity $c_{\lambda,\mu}^\eta$ of $9,496,674$. Counted with multiplicity, there are $4,966,079,903$ endofunctors in the direct sum.

[2]According to [**Sta99**, Appendix 2], the term "plethysm" was suggested to Littlewood by M. L. Clark after the Greek word plethysmos, or $\pi\lambda\eta\theta\upsilon\sigma\mu o\varsigma$, which means "multiplication" in modern Greek (though apparently the meaning goes back to ancient Greek). The related term plethys in Greek means "a big number" or "a throng", and this in turn comes from the Greek verb plethein, which means "to be full", "to increase", "to fill", etc.

In this chapter, we study Schur functors and the plethysm operation over a field of characteristic p. We start by recollecting some results on algebraic representations of reductive of groups.

2.1. Representation of reductive groups in positive characteristic

In this section, we recall some well-known results about algebraic representations of reductive groups over a field of prime characteristic that can be found in [**Jan03**]. Let p be a prime number. Let k be a field of characteristic p and G a geometrically connected split reductive algebraic group over k. We choose a Borel pair (B, T) of G, i.e. a Borel sugroup $B \subset G$ together with a maximal torus $T \subset G$ defined over k. Denote $(X^*, \Phi, X_*, \Phi^\vee)$ the root datum of G where X^* is the group of characters of T, X_* is the group of cocharacters of T, Φ is the set of roots of G, $\Phi^\vee$ is the set of coroots of G and

$$\langle \cdot, \cdot \rangle : X^* \times X_* \to \mathbb{Z}$$

is a perfect pairing between the characters and the cocharacters of T. To any root $\alpha \in \Phi$, there is an associated coroot $\alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. This choice of (B, T) determines a set of positive roots Φ^+ and a set of simple roots $\Delta \subset \Phi^+$. To simplify the statement of the proposition 4.18, we follow a non-standard convention for the positive roots by declaring $\alpha \in \Phi$ to be positive if the root group $U_{-\alpha}$ is contained in B. A character $\lambda \in X^*$ is said to be G-dominant (or simply dominant if there is no ambiguity on the group G) if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Phi^+$. We denote ρ the half-sum of positive roots. We denote W the Weyl group of G, $l : W \to \mathbb{N}$ its length function and w_0 its longest element. Consider a collection $I \subset \Delta$ of simple roots. We denote Φ_I (resp. Φ_I^+) the set of roots (resp. positive roots) obtained as $\mathbb{Z}$-linear combination of roots in I. We denote $W_I \subset W$ the subgroup generated by the reflections s_α where $\alpha \in I$ and ${}^I W \subset W$ the set of minimal length representatives of $W_I \backslash W$. We denote $\varphi : G \to G^{(p)}$ the relative Frobenius morphism of G where $G^{(p)} = G \times_{k,\sigma} k$ is the pullback along the Frobenius map $\sigma : k \to k$ of k. Since any split reductive group is a base change of a split reductive group over $\mathbb{Z}$, the reductive group G is isomorphic to $G^{(p)}$. For any G-module M, we define $M^{(p^r)}$ as the same module M with a G-action twisted by φ^r.

Denote $\mathrm{Rep}_k(G)$ the category of algebraic representations of G on finite dimensional k-vector spaces. We will use interchangeably the term G-module to denote any representation $V \in \mathrm{Rep}_k(G)$. It is well-known that this category is not semi-simple but we can still define some interesting highest weight representations.

Definition 2.2 ([**Jan03**, Part I, sect. 5.8])**.** *Let $\lambda : T \to \mathbb{G}_m$ be a character of T. We define a line bundle $\mathcal{L}_\lambda$ on the flag variety G/B as the B-quotient of the vector bundle $G \times_k \mathbb{A}^1 \to G$, where B acts on $G \times_k \mathbb{A}^1$ by*

$$(g, x)b = (gb^{-1}, \lambda(b^{-1})x),$$

and where λ is naturally extended by 0 on the unipotent part of B. The global section group $H^0(G/B, \mathcal{L}_\lambda)$ is given the structure of a G-module through left translation. As a consequence we get an algebraic representation of G, and we will denote it simply $\nabla(\lambda)$.

Proposition 2.3 ([**Jan03**, Part II, sect. 2.6])**.** *The G-module $\nabla(\lambda)$ is non-zero exactly when λ is dominant. Moreover, its highest T-weight is λ and we call $\nabla(\lambda)$ the induced module or costandard module of highest weight λ.*

Remark 2.4. A different convention can be found in the litterature where we set the dominance to be relative to B.

Definition 2.5 ([**Jan03**, Part II, sect. 2.13])**.** *Let $\lambda \in X^*(T)$ be a character. The standard module of highest weight λ can be defined*

$$\Delta(\lambda) := \nabla(-w_0\lambda)^\vee,$$

where w_0 is the longest element of the Weyl group W of G and $\vee$ denotes the linear dual in $\mathrm{Rep}_k(G)$.

As a consequence from the definitions, $\nabla(\lambda)$ and $\Delta(\lambda)$ must have the same characters but they are usually not simple and not isomorphic. However $L(\lambda)$ is the socle of $\nabla(\lambda)$ and the head of $\Delta(\lambda)$ (see [**Jan03**, Part II, Chap. 2]). We give a condition on the highest weight of a standard/costandard module to be simple.

Proposition 2.6 ([**Jan03**, Part II, sect. 5.6])**.** *If λ is a p-small character, i.e.*

$$\forall \alpha \in \Phi^+ \ \langle \lambda + \rho, \alpha^\vee \rangle \leq p,$$

then we have isomorphisms

$$\nabla(\lambda) = \Delta(\lambda) = L(\lambda).$$

Remark 2.7. If λ is p-small and $\mu \leq \lambda$, then μ is also p-small.

In positive characteristic, there is a very special algebraic representation called the Steinberg representation St_r. The Steinberg representation is a self-dual simple G-module whose highest weight is never p-small.

Definition 2.8 ([**Jan03**, Part II, sect. 3.18])**.** *Assume $p \neq 2$ or $\rho \in X^*(T)$. For each $r \geq 1$, we define the Steinberg module as*

$$\mathrm{St}_r := \nabla((p^r - 1)\rho).$$

Proposition 2.9 ([**Jan03**, Part II, sect. 3.19])**.** *We have isomorphisms*

$$\nabla((p^r - 1)\rho) = \Delta((p^r - 1)\rho) = L((p^r - 1)\rho).$$

In particular, St_r is a simple G-module.

We come to the main proposition that justifies the interest of the Steinberg representation.

Proposition 2.10. *Let λ be a character and $r \geq 1$ an integer. For all $i \geq 0$, we have an isomorphism of G-modules:*

$$H^i(G/B, \mathcal{L}_{(p^r-1)\rho} \otimes \mathcal{L}_{p^r\lambda}) = \mathrm{St}_r \otimes H^i(G/B, \mathcal{L}_\lambda)^{(p^r)}.$$

Proof. See [**Jan03**, Part II, Chap. 3, Sect. 19]. □

We recall Kempf's vanishing theorem.

Proposition 2.11 ([**Jan03**, Part II, sect. 4.5])**.** *Let λ be a dominant character. For each $i > 0$, we have*

$$H^i(G/B, \mathcal{L}_\lambda) = 0.$$

More generally, let P be a standard parabolic of type $I \subset \Delta$ and λ a I-dominant character of P(i.e. $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta \backslash I$ and $\langle \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in I$). There is an associated line bundle $\mathcal{L}_\lambda$ on G/P and we have

$$H^i(G/P, \mathcal{L}_\lambda) = 0,$$

for all $i > 0$.

Proof. We give a sketch of the argument. The first step is to show that $\mathcal{L}_\lambda$ is ample over the flag variety G/B exactly when λ is strictly dominant by reducing to the case $G = \mathrm{SL}_2$ and $G/B = \mathbb{P}^1_k$. Then, in characteristic 0, we can conclude with the Kodaira-Nakano vanishing theorem since the canonical bundle $\omega_{G/B}$ of G/B is anti-ample. Indeed, we have an isomorphism

$$\omega_{G/B} = \mathcal{L}_{-2\rho}$$

and if we consider a dominant character λ, the line bundle

$$\omega_{G/B}^{-1} \otimes_{\mathcal{O}_{G/B}} \mathcal{L}_\lambda = \mathcal{L}_{2\rho+\lambda}$$

is ample since $2\rho + \lambda$ is strictly dominant. The Kodaira-Nakano vanishing theorem applied to $\mathcal{L}_{2\rho+\lambda}$ says that

$$H^i(G/B, \underbrace{\omega_{G/B} \otimes \mathcal{L}_{2\rho+\lambda}}_{=\mathcal{L}_\lambda}) = 0,$$

for all $i > 0$. In positive characteristic, we can conclude with Serre's cohomological criterion for ampleness and the formula in proposition 2.10 with the Steinberg module $\nabla((p^r-1)\rho)$. $\square$

We insist on the fact that the proof in [**Jan03**, Part II, sect. 5.3] of the more general Borel-Weil-Bott theorem which gives information on the higher cohomology groups of $\mathcal{L}_\lambda$ when λ is no longer dominant requires to divide by binomial numbers $\binom{n}{k}$ with $n \geq p$, which is impossible in characteristic p. Actually, one can find counterexamples to the Borel-Weil-Bott theorem in positive characteristic (see [**Jan03**, Part II, sect. 15.8]). In characteristic 0, it is easier to understand tensor product of highest weight representations: we know that $L(\lambda) \otimes L(\mu)$ is a direct sum of $L(\lambda')$ where λ' can be expressed as $\lambda + \mu'$ where $\mu' \leq \mu$ is a weight of $L(\mu)$. Going back to our positive characteristic case, we would like to have a weaker but similar kind of result for $\nabla(\lambda)$'s.

Definition 2.12. *Let V be an algebaic representation of G. We say that*

(1) V admits a ∇-filtration if there is a finite filtration

$$0 = V^n \subsetneq V^{n-1} \subsetneq \cdots \subsetneq V^0 = V$$

with graded pieces

$$V^i/V^{i+1} \simeq \nabla(\nu_i),$$

for some dominant characters ν_i.

(2) V admits a Δ-filtration if there is a finite filtration

$$0 = V^n \subsetneq V^{n-1} \subsetneq \cdots \subsetneq V^0 = V$$

with graded pieces

$$V^i/V^{i+1} \simeq \Delta(\nu_i),$$

for some dominant characters ν_i.

We give a cohomological criterion due to Donkyn for the existence of ∇ and Δ-filtration.

Proposition 2.13. *Let V be an algebraic representation of G. The following assertions are equivalent.*

(1) V admits a ∇-filtration.

(2) For all dominant character λ and $i > 0$, $\mathrm{Ext}^i_G(\Delta(\lambda), V) = 0$.

(3) For all dominant character λ, $\mathrm{Ext}^1_G(\Delta(\lambda), V) = 0$.

Dually, the following assertions are equivalent.

(1) V admits a Δ-filtration.
(2) For all dominant character λ and $i > 0$, $\mathrm{Ext}^i_G(V, \nabla(\lambda)) = 0$.
(3) For all dominant character λ, $\mathrm{Ext}^1_G(V, \nabla(\lambda)) = 0$.

Proof. See [**Jan03**, Part II, sect. 4.16]. □

We are now able to define tilting modules and explain some of their properties.

Definition 2.14 ([**Jan03**, Part II, sect. E.1])**.** *Let V denote a G-module. We say that V is a tilting module if it admits both a ∇-filtration and a Δ-filtration.*

Tilting modules are interesting because they are no extension between them and indecomposable tilting modules are classified by their highest weight.

Proposition 2.15. *Let V be an algebraic representation of G. The following assertions are equivalent.*

(1) V is a tilting module.
(2) For all dominant character λ and $i > 0$, $\mathrm{Ext}^i_G(\Delta(\lambda), V) = \mathrm{Ext}^i_G(V, \nabla(\lambda)) = 0$.
(3) For all dominant character λ, $\mathrm{Ext}^1_G(\Delta(\lambda), V) = \mathrm{Ext}^1_G(V, \nabla(\lambda)) = 0$.

In particular, if V and W are two tilting modules, then

$$\mathrm{Ext}^i_G(V, W) = 0$$

for all $i > 0$.

Proof. Direct consequence of the Donkyn criterion from proposition 2.13. □

Corollary 2.16. *Let V and W be two algebraic representations of G. If V admits a ∇-filtration (resp. Δ-filtration) and W is a direct factor of V, then W admits a ∇-filtration (resp. Δ-filtration).*

Proposition 2.17. *For each dominant character λ, there is a unique indecomposable tilting module $T(\lambda)$ of highest weight λ. Moreover, any tilting module V admits a direct sum decomposition*

$$V = \bigoplus_{\lambda} T(\lambda)^{\oplus n_\lambda}$$

where λ ranges over the set of dominant characters and the $n_\lambda \geq 0$ are almost all 0.

Proof. See [**Jan03**, Part II, sect. E.6]. □

Remark 2.18. The module $T(\lambda)$ is usually not isomorphic to $\nabla(\lambda)$ or $\Delta(\lambda)$ unless we already have an isomorphism $\nabla(\lambda) = \Delta(\lambda)$[3]. The weights of the module $T(\lambda)$ are hard to describe: a formula involving p-Kazhdan–Lusztig polynomials was proven in [**RW22**].

The following proposition states the existence of a ∇-filtration for a tensor product $\nabla(\lambda) \otimes \nabla(\mu)$ and gives some details about its graded pieces. This result is due to Donkyn [**Don85**] when G does not contain any components of type E_7, E_8 or that $p \neq 2$. His approach relies on a case by case analysis of each Dynkin diagram and requires long and difficult calculations. A more general proof, without the technical restrictions, was given later by Mathieu. We first need a lemma.

Lemma 2.19. *Let λ, μ denote T-characters such that $\mathrm{Ext}^1_G(\nabla(\lambda), \nabla(\mu)) \neq 0$. Then, $\lambda \geq \mu$.*

[3] It is the case when λ is p-small for example.

Proof. We have

$$\begin{aligned}\mathrm{Ext}^1_G(\nabla(\lambda),\nabla(\mu)) &= H^1(G,\Delta(-w_0\lambda)\otimes\nabla(\mu))\\ &= H^1(P,\Delta(-w_0\lambda)\otimes\mu) \text{ by [Jan03, Part II, sect. 4.7]}\end{aligned}$$

and by [Jan03, Part II, sect. 4.10 b)], there exists a weight ν of $\Delta(-w_0\lambda)$ such that $-(\nu+\mu)$ is a $\mathbb{N}$-linear combination of positive roots Φ^+. In particular, we have $-\nu\geq\mu$. Since $w_0(-w_0\lambda)=-\lambda$ is the lowest weight of $\Delta(-w_0\lambda)$, we deduce that $\lambda\geq-\nu\geq\mu$. □

Proposition 2.20 ([Mat90])**.** *Let λ,μ be two dominant characters in $X^*(T)$. Then $\nabla(\lambda)\otimes\nabla(\mu)$ admits a ∇-filtration $(V^i)_{i\geq0}$ with graded pieces*

$$V^i/V^{i+1}\simeq\nabla(\lambda+\mu_i),$$

where $(\mu_i)_i$ is a collection of weights of $\nabla(\mu)$ with $\mu_0=\mu$. In particular, the first graded piece is given by $V^0/V^1=\nabla(\lambda+\mu)$.

Proof. We add some details to the result of Mathieu to explain how to get a filtration with the desired properties. The result of Mathieu assumes that G is a connected, simply-connected, semi-simple algebraic group over an algebraically closed field k of characteristic $p>0$ and it is not hard to reduce to this case. By [Mat90, Theorem 1], there exists a filtration

$$0=V^n\subset V^{n-1}\subset\cdots\subset V^1\subset\cdots V^0=\nabla(\lambda)\otimes\nabla(\mu),$$

where for each i the graded piece V^i/V^{i+1} is a costandard module $\nabla(\nu_i)$ for some dominant character ν_i. The character class of $\nabla(\lambda)\otimes\nabla(\mu)$ is

$$\mathrm{ch}(\nabla(\lambda)\otimes\nabla(\mu))=\sum_i\mathrm{ch}\,\nabla(\lambda+\mu_i),$$

where the sum is taken over some weights $(\mu_i)_i$ of $\nabla(\mu)$. As the highest weight of this module, $\lambda+\mu$ contributes to the sum. Note that the non-zero terms are those such that $\lambda+\mu_i$ is dominant. We choose an ordering of the $(\mu_i)_i$ such that whenever $\mu_i<\mu_j$ for some i,j then $i>j$. It implies that there exists a permutation σ on $0,1,\cdots,n-1$ such that

$$V^i/V^{i+1}=\nabla(\lambda+\mu_{\sigma(i)}),$$

for all i between 0 and $n-1$. We remake the argument in [Jan03, Part 11, sect. 4.16, remark 4] to explain how to reorganize the terms. If $\sigma(i)<\sigma(i+1)$ for some i, $0\leq i\leq n-2$, then $\lambda+\mu_{\sigma(i)}\not<\lambda+\mu_{\sigma(i+1)}$ and the exact sequence

$$0\longrightarrow\nabla(\lambda+\mu_{\sigma(i+1)})\longrightarrow V^i/V^{i+2}\longrightarrow\nabla(\lambda+\mu_{\sigma(i)})\longrightarrow0$$

is split because $\mathrm{Ext}^1_G(\nabla(\lambda+\mu_{\sigma(i+1)}),\nabla(\lambda+\mu_{\sigma(i)}))=0$ by lemma 2.19. It shows that

$$V^i/V^{i+2}=\nabla(\lambda+\mu_{\sigma(i+1)})\oplus\nabla(\lambda+\mu_{\sigma(i)})$$

and we can replace V^{i+1} by a submodule $\tilde{V}^{i+1}$ between V^{i+2} and V^i such that $\tilde{V}^{i+1}/V^{i+2}=\nabla(\lambda+\mu_{\sigma(i)})$ and $V^i/\tilde{V}^{i+1}=\nabla(\lambda+\mu_{\sigma(i+1)})$. We iterate this process to produce the desired filration. □

Remark 2.21.
(1) Not all the weights $\mu'\leq\mu$ of $\nabla(\mu)$ such that $\lambda+\mu'$ is dominant will contribute to the filtration.
(2) We note that the dual statement says that tensor products of standard modules $\Delta(\lambda)\otimes\Delta(\mu)$ admits a Δ-filtration.

Corollary 2.22. *Let V and W be two algebraic representations of G that admit a ∇-filtration (resp. Δ-filtration), then $V \otimes W$ admits a ∇-filtration (resp. Δ-filtration). In particular, the tensor product of tilting modules is a tilting module.*

2.2. Strict polynomial functors

In their founder article [**FS97**], Friedlander and Suslin introduced the category of strict polynomial functors Pol over a field k of any characteristic which is a full subcategory of the category of endofunctors of $\mathrm{FinVect}_k$. In particular, when $n \geq d$, they prove an equivalence of categories

$$\mathsf{Pol}_d \simeq S(n,d) - \mathrm{Mod}$$

between the category Pol_d of strict polynomial functors homogeneous of degree d and the category of modules over the Schur algebra $S(n,d)$[4], which is itself equivalent to the category of polynomial representation for GL_n of degree d. If V, W are finite dimensional vector spaces over k, we denote by $\mathrm{Hom}_{\mathrm{pol}}(V,W)$ the abelian group of scheme morphisms over k between $\underline{V}$ and $\underline{W}$. Elements of $\mathrm{Hom}_{\mathrm{pol}}(V,W)$ are called polynomial maps between V and W.

Definition 2.23 ([**FS97**, Definition 2.1])**.** *A strict polynomial functor*

$$T : \mathrm{FinVect}_k \to \mathrm{FinVect}_k$$

is a pair of functions, the first of which assigns to each $V \in \mathrm{FinVect}_k$ a vector space $T(V) \in \mathrm{FinVect}_k$ and the second assigns a polynomial map

$$T_{V,W} \in \mathrm{Hom}_{\mathrm{pol}}(\mathrm{Hom}_k(V,W), \mathrm{Hom}_k(T(V),T(W)))$$

to each V, W. These two functions should satisfy the usual conditions of the definition of a functor

(1) For any vector space $V \in \mathrm{FinVect}_k$, we have $T_{V,V}(\mathrm{id}_V) = \mathrm{id}_{T(V)}$

(2) For any U, V, W, the following diagram of polynomial maps commute

$$\begin{tikzcd}
\mathrm{Hom}_k(V,W) \times \mathrm{Hom}_k(U,V) \arrow[r] \arrow[d, "T_{V,W} \times T_{U,V}"] & \mathrm{Hom}_k(U,W) \arrow[d, "T_{U,W}"] \\
\mathrm{Hom}_k(T(V),T(W)) \times \mathrm{Hom}_k(T(U),T(V)) \arrow[r] & \mathrm{Hom}_k(T(U),T(W))
\end{tikzcd}$$

Let $T : \mathrm{FinVect}_k \to \mathrm{FinVect}_k$ be a strict polynomial functor. We say that T is homogeneous of degree d if for all vector spaces V, W, the polynomial map

$$T_{V,W} \in \mathrm{Hom}_{\mathrm{pol}}(\mathrm{Hom}_k(V,W), \mathrm{Hom}_k(T(V),T(W)))$$

has degree d. We denote Pol the category of strict polynomial functors of finite degree where the morphism are morphism between the underlying functors.

Proposition 2.24 ([**FS97**, Proposition 2.6])**.** *The category of strict polynomial functors decomposes*

$$\mathsf{Pol} = \bigoplus_{d \geq 0} \mathsf{Pol}_d$$

where Pol_d is the full subcategory of Pol consisting of strict polynomial functors homogeneous of degree d. In particular, there are no extension between two strict polynomial functors homogeneous of different degrees.

[4]Recall that $S(n,d)$ is defined as the algebra of endomorphisms of $V^{\otimes d}$ commuting with the usual action of the symmetric group $\mathfrak{S}_d$, where V is any k-vector space of dimension n.

Example 2.25. We give some simple example of strict polynomial functors:

(1) The nth-tensor power $(\cdot)^{\otimes n}$ which sends a k-vector space V to $V^{\otimes n}$ is homogeneous of degree n.
(2) The nth-symmetric power Sym^n which sends a k-vector space V to the space of $\mathfrak{S}_n$-coinvariants $(V^{\otimes n})_{\mathfrak{S}_n}$ where $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors is homogeneous of degree n.
(3) The nth-divided power Γ^n which sends a k-vector space V to the space of $\mathfrak{S}_n$-invariants $(V^{\otimes n})^{\mathfrak{S}_n}$ where $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors is homogeneous of degree n.
(4) The nth-exterior power Λ^n which sends a vector space V to the quotient space $V^{\otimes n}/I$ where I is the ideal generated by elements $x_1 \otimes \cdots \otimes x_n$ such that $x_i = x_j$ for some $i \neq j$ is homogeneous of degree n.
(5) If $\mathrm{char}(k) = p$, the Frobenius twist $(\cdot)^{(p)}$ which sends a k-vector space V to its pullback $V \otimes_{k,\sigma} k$ by the Frobenius map $\sigma : k \to k$ is homogeneous of degree p.

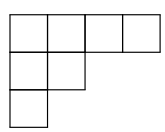
Remark 2.26. The functors Sym^n and Γ^n are isomorphic over a field of characteristic 0 but not over a field of characteristic $p > 0$ when $n \geq p$.

Following [**ABW82**], we now define Schur functors and Weyl functors that are indexed by partitions λ as strict polynomial functors.

Definition 2.27. *Given a partition $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_r > 0)$, we write $|\lambda| = \sum_{i=1}^r k_r$ for its size and $ht(\lambda) = r$ for its height.*

Definition 2.28. *We represent a partition $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_r > 0)$ with a diagram containing r rows and such that for each i, the ith-row contains k_i columns. Such a representation is called a Young diagram.*

Example 2.29. The Young diagram of the partition $\lambda = (4, 2, 1)$ is



Its size is 7 and its height is 3.

Definition 2.30. *Given a partition $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_r > 0)$, we define its conjugate partition $\lambda' = (k'_1 \geq k'_2 \geq \ldots \geq k'_s > 0)$ as the partition where k'_i is the number of terms of k_j that are greater or equal to i. Note that λ and λ' have the same size. Any integer l between 1 and d determines a unique position (i, j) in the Young diagram of λ such that $l = k_1 + \cdots k_{i-1} + j$. Then, we define a permutation on $|\lambda|$-letters $\sigma_\lambda \in \mathfrak{S}_{|\lambda|}$ by setting*

$$\sigma_\lambda(l) = k'_1 + \cdots k'_{i-1} + j.$$

Note that we have $\sigma_{\lambda'} = \sigma_\lambda^{-1}$.

Example 2.31. The conjugate partition of $\lambda = (8, 4, 2)$

$$\lambda = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \cline{1-4} 13 & 14 \\ \cline{1-2}\end{array}$$

is $\lambda' = (3,3,2,2,1,1,1,1)$,

$$\lambda' = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} 9 & 10 \\ \cline{1-2} 11 \\ \cline{1-1} 12 \\ \cline{1-1} 13 \\ \cline{1-1} 14 \\ \cline{1-1}\end{array}$$

and we have

$$\sigma_\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 4 & 7 & 9 & 11 & 12 & 13 & 14 & 2 & 5 & 8 & 10 & 3 & 6 \end{pmatrix}.$$

Definition 2.32. *Let $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_r > 0)$ denote a partition, $\lambda' = (k'_1 \geq k'_2 \geq \ldots \geq k'_s > 0)$ its conjugate partition and V a finite dimensional vector space over k. We define $S_\lambda V$ as the image of the map*

$$\bigotimes_{1\leq j\leq s} \Lambda^{k'_j} V \xrightarrow{\Delta^{\otimes s}} V^{\otimes|\lambda|} \xrightarrow{\sigma_\lambda} V^{\otimes|\lambda|} \xrightarrow{\nabla^{\otimes r}} \bigotimes_{1\leq i\leq r} \mathrm{Sym}^{k_i} V$$

where $\Delta : \Lambda^l V \to V^{\otimes l}$ is the comultiplication given by

$$\Delta(v_1 \wedge \cdots \wedge v_l) = \sum_{\sigma\in\mathfrak{S}_l} \epsilon(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(l)},$$

$\nabla : V^{\otimes l} \to \mathrm{Sym}^l V$ is the multiplication and $\sigma_\lambda : V^{\otimes|\lambda|} \to V^{\otimes|\lambda|}$ is given by

$$\sigma_\lambda(v_1 \otimes \cdots v_{|\lambda|}) = v_{\sigma_\lambda(1)} \otimes \cdots \otimes v_{\sigma_\lambda(|\lambda|)}.$$

We define $W_\lambda V$ as the image of the map

$$\bigotimes_{1\leq i\leq r} \Gamma^{k_i} V \xrightarrow{\Delta^{\otimes r}} V^{\otimes|\lambda|} \xrightarrow{\sigma_{\lambda'}} V^{\otimes|\lambda|} \xrightarrow{\nabla^{\otimes s}} \bigotimes_{1\leq j\leq s} \Lambda^{k'_j} V$$

where $\Delta : \Gamma^l V \to V^{\otimes l}$, $\nabla : V^{\otimes l} \to \Lambda^l V$ are defined with the same formulas. These construction are functorial in V and define strict polynomial functors S_λ and W_λ that are homogeneous of degree $|\lambda|$.

Example 2.33. We give the following examples:

(1) If $\lambda = (n)$, then $S_\lambda = \mathrm{Sym}^n$ and $W_\lambda = \Gamma^n$.
(2) If $\lambda = (1, \cdots, 1)$ is a partition of n, then $S_\lambda = W_\lambda = \Lambda^n$.

Proposition 2.34. *Let λ be a partition of d and $V \in \mathrm{FinVect}_k$. We have an isomorphism*

$$S_\lambda(V)^\vee = W_\lambda(V^\vee)$$

which is functorial in V.

Proof. It follows from the fact that $\mathrm{Sym}^n(V)^\vee = \Gamma^n(V^\vee)$ and $\Lambda^n(V)^\vee = \Lambda^n(V^\vee)$. □

To relate Schur functors with algebraic representations of the general linear group GL_n for different n, we briefly recall the main definitions and properties of highest weight categories.

Definition 2.35. *Let $\mathcal{A}$ denote a k-linear abelian category such that for any $x, y \in \mathcal{A}$, the k-vector space $\mathrm{Hom}_{\mathcal{A}}(x, y)$ has finite dimension. We denote Λ the set of isomorphism classes of irreducible objects. A highest weight structure is the datum of a partial order $\leq$ on Λ satisfying the condition stated in* [**Ric16**, Definition 7.1].

Definition 2.36. *Let $\lambda \in \Lambda$, we denote $L_\lambda, \Delta_\lambda, \nabla_\lambda$ and T_λ the irreducible, standard, costandard and tilting objects indexed by λ.*

Remark 2.37. Note that the highest weight structure is determined by the collections $(\Delta_\lambda)_\lambda$ and $(\nabla_\lambda)_\lambda$.

Definition 2.38. *Let $\mathcal{A}, \mathcal{B}$ denote two highest weight categories. An equivalence of categories $\Phi : \mathcal{A} \to \mathcal{B}$ is an equivalence of highest weight categories if Φ sends the standard (resp. costandard) object of weight λ on the standard (resp. costandard) object of weight $\Phi_\Lambda(\lambda)$ where $\Phi_\Lambda : \Lambda_{\mathcal{A}} \to \Lambda_{\mathcal{B}}$ is the induced application on irreducible classes. In that case, Φ sends the tilting object of weight λ on the tilting object of weight $\Phi_\Lambda(\lambda)$.*

It is well-known that the category of algebraic representation of the general linear group has a highest weight structure.

Proposition 2.39. *Let V be a k-vector space of dimension n. The category $\mathrm{Rep}_k(\mathrm{GL}(V))$ has a highest weight structure where the irreducible are indexed by dominant $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ with the usual order*

$$\lambda \leq \mu \Leftrightarrow \lambda - \mu \text{ is a } \mathbb{N}\text{-linear combination of simple roots.}$$

Recall from previous section that the irreducible, standard, costandard and titling objetcts of $\mathrm{Rep}_k(\mathrm{GL}(V))$ are denoted $L(\lambda), \nabla(\lambda), \Delta(\lambda)$ and $T(\lambda)$ for each dominant character λ.

Proof. See [**Jan03**, Part II, Chap. E]. □

Proposition 2.40 ([**Kra15**])**.** *The category Pol_d has a highest weight structure where the irreducible are indexed by the set Λ_d of integer partition of size d with the dominance order*

$$\lambda \leq \mu \Leftrightarrow \forall k \geq 1 \ \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i.$$

Moreover, for any $\lambda \in \Lambda_d$, the standard object Δ_λ is given by the Weyl functor W_λ and the costandard object ∇_λ is given by the Schur functor S_λ. In particular, we have a tilting functor T_λ and an irreducible functor L_λ.

Proof. See [**Kra15**, Theorem 7.1, Theorem 5.8, Corollary 5.9]. □

We state the main result of this section.

Proposition 2.41 ([**FS97**])**.** *Let $d \geq 1$ be an integer and V a k-vector space of dimension n. If $n \geq d$, the evaluation functor at V*

$$\begin{aligned} \mathrm{ev}_V : \mathsf{Pol} &\longrightarrow \mathrm{Rep}_k(\mathrm{GL}(V))^{pol} \\ T &\longmapsto T(V) \end{aligned}$$

restricts to an equivalence of category between Pol_d and the category $\mathrm{Rep}_k(\mathrm{GL}(V))_d^{pol}$ of polynomial representation of $\mathrm{GL}(V)$ homogeneous of degree d. Moreover, through this equivalence, the Schur functor S_λ maps to the costandard module $\nabla(\lambda)$, the Weyl functor W_λ maps to the

standard module $\Delta(\lambda)$ *and the tilting functor* T_λ *maps to the tilting module* $T(\lambda)$ *where we see* $\lambda = (k_1, \dots, k_r)$ *as a character of the standard maximal torus of* $\mathrm{GL}(V)$

$$\lambda : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{k_1} \cdots t_r^{k_r}.$$

Proof. The equivalence of categories is proven in [**FS97**, Lemma 3.4.]. For the equivalence as highest weight categories see [**Tou13**, Remark 6.1] and use the fact that the orders on the irreducible classes of Pol_d and $\mathrm{Rep}_k(\mathrm{GL}(V))_d^{\mathrm{pol}}$ match when $n \geq d$. □

Remark 2.42. If we replace $\mathrm{FinVect}_k$ with the category $\mathrm{Loc}(\mathcal{O}_X)$ of locally free sheaves of finite rank over a k-scheme X, we are still able to define objects $\Delta_\lambda, \nabla_\lambda$ and T_λ in the categories $\mathsf{Pol}_{\mathcal{O}_X}$ and $\mathrm{Rep}_{\mathcal{O}_X}(\mathrm{GL}_n)$ and our result are still valid by standard reduction to the case $\mathrm{FinVect}_k$. In particular, the proposition 2.41 is still valid.

2.3. The plethysm problem

Take two Schur functors S_λ and S_μ as strict polynomial functors over k and consider the composition $S_\lambda \circ S_\mu$. It is a strict polynomial functor homogeneous of degree $|\lambda||\mu|$. Since the category of algebraic representation of GL_n is not semisimple, we have no reason to hope for a decomposition of $S_\lambda \circ S_\mu$ as a direct sum of Schur functors. One might hope, that there exists at least a filtration of $S_\lambda \circ S_\mu$ where the graded pieces are Schur functors. Unfortunately, Boffi [**Bof91**] and Touzé [**Tou13**, Corollary 6.10.] have found counter-examples to the existence of such filtrations for plethysm of the form $\mathrm{Sym}^k \circ \mathrm{Sym}^d$, $\Lambda^k \circ \mathrm{Sym}^d$ and $\mathrm{Sym}^k \circ \Lambda^d$ with $d \geq 3$ and $p|k$. More precisely, Touzé has found an obstruction to the existence of such filtration that lives in the p-torsion of the homology of the Eilenberg-Mac Lane space $K(\mathbb{Z}, d)$. In this section, we prove the following existence result.

Proposition 2.43. *Let* λ *and* μ *be partitions. If* $p \geq 2|\lambda| - 1$*, the strict polynomial functor* $S_\lambda \circ S_\mu$ *admits a finite filtration*

$$0 = T^n \subsetneq T^{n-1} \subsetneq \cdots \subsetneq T^0 = S_\lambda \circ S_\mu$$

by strict polynomial functors of degree $|\lambda||\mu|$ *where the graded pieces are Schur functors.*

We start with the following lemma.

Lemma 2.44. *If* $p \geq 2|\lambda| - 1$*, then* S_λ *is a direct summand of* $(\cdot)^{\otimes|\lambda|}$ *in* $\mathsf{Pol}_{|\lambda|}$.

Remark 2.45. If $S_\lambda = \Lambda^n$, then it is enough to ask $p > n$.

Proof. Write $\lambda = (k_1 \geq \cdots \geq k_r > 0)$. By proposition 2.41, it is enough to prove that $S_\lambda V$ is a direct summand of $V^{\otimes|\lambda|}$ in the category of $\mathrm{GL}(V)$-module for any k-vector space of dimension greater than $|\lambda|$. Consider a vector space V of dimension n. Notice that the surjection

$$\nabla^{\otimes r} : V^{\otimes|\lambda|} \to \mathrm{Sym}^\lambda V := \bigotimes_{1 \leq i \leq r} \mathrm{Sym}^{k_i} V$$

admits a section when $p > \max_i k_i = k_1$. Indeed, we define it as $s = s_1 \otimes \cdots \otimes s_r$ where

$$s_i(v_1 v_2 \cdots v_r) = \frac{1}{k_i!} \sum_{\sigma \in \mathfrak{S}_{k_i}} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k_i)}.$$

By definition $S_\lambda V$ is a sub-$\mathrm{GL}(V)$-module of $\mathrm{Sym}^\lambda V$ and we would like to find a condition on p that guarantees it is also a direct summand. The following exact sequence of $\mathrm{GL}(V)$-modules

$$0 \longrightarrow S_\lambda V \longrightarrow \mathrm{Sym}^\lambda V \longrightarrow \mathrm{Sym}^\lambda V/S_\lambda V \longrightarrow 0$$

is split if we can show that $\mathrm{Ext}^1_{\mathrm{GL}(V)}(\mathrm{Sym}^\lambda V/S_\lambda V, S_\lambda V)$ vanishes. Let $\tilde{\lambda}$ denote the character $(|\lambda|, 0, \cdots, 0)$ of the standard maximal torus of $\mathrm{GL}(V)$. Since $\mathrm{Sym}^\lambda V$ is of highest weight $\tilde{\lambda}$, the $\mathrm{GL}(V)$-modules $S_\lambda V$ and $\mathrm{Sym}^\lambda V/S_\lambda V$ are filtered by simple modules $L(\nu)$'s with $\nu \leq \tilde{\lambda}$. Under the assumption that $\tilde{\lambda}$ is p-small, proposition 2.6 implies that we have isomorphisms

$$L(\nu) = \nabla(\nu) = \Delta(\nu)$$

for all characters ν satisfying $\nu \leq \tilde{\lambda}$. Since $\mathrm{Ext}^1_{\mathrm{GL}(V)}(\mathrm{Sym}^\lambda V/S_\lambda V, S_\lambda V)$ is the limit of a spectral sequence involving Ext groups

$$\mathrm{Ext}^1_{\mathrm{GL}(V)}(L(\nu), L(\nu')) = \mathrm{Ext}^1_{\mathrm{GL}(V)}(\Delta(\nu), \nabla(\nu')),$$

that vanishes by corollary 2.22, $\mathrm{Ext}^1_{\mathrm{GL}(V)}(\mathrm{Sym}^\lambda V/S_\lambda V, S_\lambda V)$ must vanish. In conclusion, we have the desired splitting, provided that $\tilde{\lambda}$ is p-small, i.e.

$$\begin{aligned} p \geq \max_{\alpha \in \Phi^+} \langle \tilde{\lambda} + \rho, \alpha^\vee \rangle &= \max_{1 \leq i < j \leq n} \langle \tilde{\lambda} + \rho, \varepsilon_i - \varepsilon_j \rangle \\ &= \max_{1 \leq i < j \leq n} \langle (|\lambda| + \frac{n-1}{2}, \frac{n-3}{2}, \cdots, -\frac{n-1}{2}), \varepsilon_i - \varepsilon_j \rangle \\ &= |\lambda| + n - 1. \end{aligned}$$

Since our argument is valid only when $n \geq |\lambda|$, we get the bound $p \geq 2|\lambda| - 1$. □

Proof of proposition 2.43. By proposition 2.41, it is enough to show that $S_\lambda \circ S_\mu(V)$ admits a ∇-filtration as a $\mathrm{GL}(V)$-module where V is any vector space of dimension greater that $|\lambda||\mu|$. Consider a vector space V of dimension $n \geq |\lambda||\mu|$. By lemma 2.44, $S_\lambda(S_\mu V)$ is a direct summand of $(S_\mu V)^{\otimes|\lambda|}$ as $\mathrm{GL}(S_\mu V)$-modules. After restriction to the category of $\mathrm{GL}(V)$-modules through the map $\mathrm{GL}(V) \to \mathrm{GL}(S_\mu V)$ induced by S_μ, $S_\lambda(S_\mu V)$ is again a direct summand of $(S_\mu V)^{\otimes|\lambda|}$. By corollary 2.22, the $\mathrm{GL}(V)$-module $(S_\mu V)^{\otimes|\lambda|}$ admits a ∇-filtration. By corollary 2.16, the $\mathrm{GL}(V)$-module $S_\lambda \circ S_\mu(V)$ admits a ∇-filtration. □

Remark 2.46. Under the assumption of the proposition 2.43, the partitions (counted with multiplicity) of the Schur functors appearing in the graded pieces of the filtration of $S_\lambda \circ S_\mu$ are the same as the one appearing in the decomposition (2.1) over the complex numbers. This is just a consequence of the $\mathbb{Z}$-linearity of the $\nabla(\lambda)$'s in the space $X^*(T)^W$ of W-invariants characters, but we reprove it directly. First note that the weights of the $\mathrm{GL}(V)$-module $S_\lambda \circ S_\mu(V)$ where V is a vector space of dimension $\geq |\lambda||\mu|$ do not depend on the characteristic of the base field of V. Then, we are left to check that a direct sum $M = \bigoplus_\lambda \nabla(\lambda)^{\oplus c_\lambda}$ of costandard modules is uniquely determined by its characters $\mathrm{ch}(M) = \sum_\eta d_\eta \eta$. We prove it with a descending induction on the number of distinct factors of M. Consider the highest weight η_0 appearing in the sum $\mathrm{ch}(M)$. Clearly, $\nabla(\eta_0)$ is a direct factor of the module M because η_0 cannot appear in the weights of a costandard module $\nabla(\lambda)$ with $\lambda < \eta_0$. Moreover, the multiplicity of $\nabla(\eta_0)$ in M is exactly d_{η_0} and we can pursue the induction with $M' = \bigoplus_{\lambda < \eta_0} \nabla(\lambda)^{\oplus c_\lambda}$.

We would like to discuss the existence of direct sum decomposition for composition of tilting polynomial functors $T_\lambda \circ T_\mu$. Basically, we have two cases: the partition λ is p-regular or p-singular.

Proposition 2.47. *Let λ denote a p-regular integer partition of d, i.e. such that there is no i such that*

$$\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1} > 0.$$

Then T_λ is a direct factor of the functor $\otimes^d$ in Pol_d.

Proof. See [**Jan03**, Part II, sect. E.17.] and use proposition 2.41. □

Corollary 2.48. *Let λ denote a p-regular integer partition of d and μ an integer partition of d'. Then there exists a direct sum decomposition*

$$T_\lambda \circ T_\mu = \bigoplus_\eta T_\eta^{\oplus n^\eta_{\lambda,\mu}}$$

in the category $\mathsf{Pol}_{dd'}$.

Proof. Same argument as in the proof of proposition 2.43. □

We have seen that in the case where the first partition is p-regular, we have a direct sum decomposition of a plethysm of tilting functors in characteristic p. In the p-singular case, even the simplest example fails.

Proposition 2.49. *Assume $p = 2$. Then the plethysm $\Lambda^2 \circ \Lambda^2 = S_{(1,1)} \circ S_{(1,1)}$ does not admit a direct sum decomposition (nor a filtration) by tilting functors.*

Proof. Consider a $\mathbb{F}_p$-vector space V of dimension 4. By [**AB88**, Sect. 4], we have

$$\Lambda^2(\Lambda^2 V) = \begin{cases} S_{(2,1,1)}V \text{ if } p \neq 2 \\ \Lambda^4 V \oplus W_{(2,1,1)}V/\Lambda^4 V \text{ if } p = 2 \end{cases}$$

The weight $(2,1,1,0)$ is p-small for $\mathrm{GL}(V)$ if $p \geq 3$ (easy computation), which shows that when $p \neq 2$, $\Lambda^2 \circ \Lambda^2$ is equal to $S_{(2,1,1)} = W_{(2,1,1)} = T_{(2,1,1)}$. When $p = 2$, the $\mathrm{GL}(V)$-module $W_{(2,1,1)}V/\Lambda^4 V$ is of highest weight $(2,1,1,0)$ and has dimension $15 - 1 = 14$, hence it cannot be equal to $T(2,1,1,0)$ (which has a dimension ≥ 15). In particular, it is not a tilting module and neither is $\Lambda^2 \circ \Lambda^2$. □

Remark 2.50. Again, if we replace $\mathrm{FinVect}_k$ with the category $\mathrm{Loc}(\mathcal{O}_X)$ of locally free sheaves of finite rank over a k-scheme X, our plethysm existence results are still valid by standard reduction to the case $\mathrm{FinVect}_k$.

2.4. A canonical basis for Schur and Weyl functors

In this section, we describe canonical basis indexed by Young tableaux for Schur functors and Weyl functors. This allows us to prove a factorization result which will be useful to define a Hecke operator in the chapter 9. Let S be any scheme.

Definition 2.51. *Let $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_r \geq 0)$ denote an integer partition of d. A (semistandard) Young tableau of shape λ is a filling of the boxes in the Young diagram associated to λ with positive integer values such that the entries weakly increase along each row and strictly increase down each column.*

Remark 2.52. We will only consider semistandard Young tableaux in this chapter.

Definition 2.53. *Let $\mathcal{V}$ be a $\mathcal{O}_S$-free module of rank n. Let $v_1, \ldots, v_n$ be a basis of $\mathcal{V}$. Let us consider*

$$\lambda = (k_1 \geq k_2 \geq \ldots \geq k_r \geq 0)$$

a partition of d and

$$\lambda' = (k'_1 \geq k'_2 \geq \ldots \geq k'_s \geq 0)$$

its conjugate partition. To each Young tableau T of shape λ whose values are between 1 and n, we can associate two elements $v_T \in S_\lambda \mathcal{V}$ and $\hat{v}_T \in W_\lambda \mathcal{V}$ by the formulas

$$v_T = v^{k_1} \otimes \cdots \otimes v^{k_r} \in \Gamma^{k_1}\mathcal{V} \otimes \cdots \otimes \Gamma^{k_r}\mathcal{V}$$

where v^{k_i} is the product in $\Gamma\mathcal{V}$ of v_k for k running through the values of the i-th row of T and

$$\hat{v}_T = \hat{v}^{k'_1} \otimes \cdots \otimes \hat{v}^{k'_s} \in \Lambda^{k'_1}\mathcal{V} \otimes \cdots \otimes \Lambda^{k'_s}\mathcal{V}$$

where $\hat{v}^{k'_j}$ is the product in $\Lambda\mathcal{V}$ of the v_k for k running through the values of the j-th column of T.

Remark 2.54. The product in $\Gamma\mathcal{V}$ is given by

$$x.y = \sum_{\sigma \in \mathfrak{S}_{n+d}/\mathfrak{S}_n \times \mathfrak{S}_d} \sigma(x \otimes y) \in \Gamma^{n+d}\mathcal{V}$$

where $x \in \Gamma^n\mathcal{V}$ and $y \in \Gamma^d\mathcal{V}$.

Example 2.55. Let T denote the Young tableau

1	2	2	3
2	3		
3			

of shape $\lambda = (4, 2, 1)$, then

$$\begin{cases} v_T = (v_1.v_2.v_2.v_3) \otimes (v_2.v_3) \otimes (v_3), \\ \hat{v}_T = (v_1 \wedge v_2 \wedge v_3) \otimes (v_2 \wedge v_3) \otimes (v_3). \end{cases}$$

Proposition 2.56. *Let λ be a partition of d. Let $\mathcal{V}$ be a free $\mathcal{O}_S$-module of rank n. Then the collection of v_T (resp. $\hat{v}_T$) for T running through Young tableaux of shape λ with values between 1 and n forms a basis of the $\mathcal{O}_S$-module $W_\lambda\mathcal{V}$ (resp. $S_\lambda\mathcal{V}$).*

Proposition 2.57. *Let $\mathcal{V} = \mathcal{O}_S^g$ the free $\mathcal{O}_S$-module of rank g. Let*

$$\lambda = (k_1 \geq \cdots \geq k_g \geq 0)$$

be a partition of $d := \sum_{j=1}^g k_i$. Let us consider the map $\varphi : \mathcal{V} \to \mathcal{V}$ of multiplication by the matrix

$$\begin{array}{c} i\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ \\ \\ \\ \\ \end{array}\begin{pmatrix} p & & & & & \\ & \ddots & & & & \\ & & p & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}\begin{array}{c} \\ \\ \\ \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} g-i \end{array}$$

then the induced application $S_\lambda\varphi$ factorizes as follows

$$\begin{tikzcd} S_\lambda\mathcal{V} \arrow[r, "S_\lambda\varphi"] \arrow[dr, dotted] & S_\lambda\mathcal{V} \\ & p^{\sum_{j=g-i+1}^{g} k_j} S_\lambda\mathcal{V} \arrow[u] \end{tikzcd}$$

and likewise for $W_\lambda\varphi$

$$\begin{tikzcd} W_\lambda\mathcal{V} \arrow[r, "W_\lambda\varphi"] \arrow[dr, dotted] & W_\lambda\mathcal{V} \\ & p^{\sum_{j=g-i+1}^{g} k_j} W_\lambda\mathcal{V} \arrow[u] \end{tikzcd}.$$

Proof. Thanks to the proposition 2.56, we know that it is enough to count the number of elements v_k with $k \leq i$ which appear in the v_T and the $\hat{v}_T$ when T goes through Young tableaux of shape λ whose values are between 1 and g. This brings us back to an enumeration problem.

Let us treat the case $g = 3$ and $\lambda = (4, 2, 1)$ to visualise what happens. The modules $S_\lambda\mathcal{O}_S^{\oplus 3}$ and $W_\lambda\mathcal{O}_S^{\oplus 3}$ are free of rank 15 because there are exactly 15 Young tableaux T of shape λ with values between 1 and 3.

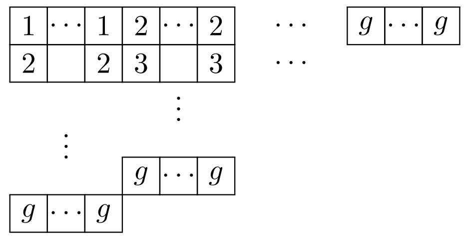
$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&2&3&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}$$

The last Young tableau has $k_3 = 1$ values less than 1, $k_3 + k_2 = 3$ values less than 2 and $k_3 + k_2 + k_1 = 7$ values less than 3. And the same is true for the other Young tableau as the last one has maximum values. To prove the general case, this suggests looking at the Young tableau $T_{\max}$ of shape λ with maximum values between 1 and g. $T_{\max}$ looks like the following



with k_t columns in the t-th row. We then count the number of entries with values less than i in this Young tableau and find

$$\begin{aligned}
\#_{\leq i} T_{\max} &= k_g \times i + (k_{g-1} - k_g) \times (i-1) + \cdots (k_1 - k_2) \times 1 \\
&= k_g i + \sum_{t=1}^{i-1} (k_{g-t} - k_{g-t+1})(i-t) \\
&= k_g i + \sum_{t=1}^{i-2} k_{g-t}((i-t) - (i-t-1)) + k_{g-i+1} - k_g(i-1) \\
&= \sum_{t=g-i+1}^{g} k_t,
\end{aligned}$$

which concludes the proof. □

CHAPTER 3

A new positivity notion for vector bundles called (φ, D)-ampleness

In algebraic geometry, certain line bundles are considered to be positive. The most important positivity notion for line bundles is ampleness. Let X denote a projective scheme over a field k. A standard reference for the classical materials in the introduction of this chapter is [**Laz04a**].

Definition 3.1. *A line bundle $\mathcal{L}$ over X is said to be ample if for any coherent sheaf $\mathcal{F}$, the coherent sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global section for n large enough.*

Remark 3.2. In differential geometry, this notion does not appear because one can prove that all line bundles are ample. Let M denote a paracompact manifold and $\mathcal{C}^\infty$ denote the sheaf of smooth functions on M. Such manifolds always admit smooth partitions of unity, i.e. the sheaf $\mathcal{C}^\infty$ is fine. As a consequence any finite type sheaf in $\mathcal{C}^\infty$-modules (in particular any tensor product $\mathcal{F} \otimes_{\mathcal{C}^\infty} \mathcal{L}$ where $\mathcal{F}$ is a coherent sheaf and $\mathcal{L}$ a smooth line bundle) is also fine. But fine sheaves are soft, hence acyclic. We conclude with the cohomological criterion for ampleness, which is still valid in differential geometry as its proof only involves the long exact sequence in cohomology associated to a short exact sequence of sheaves.

Roughly speaking, it means that $\mathcal{L}$ possesses a lot of global sections. One of the reason to call such an ample line bundle positive is the fact that the tensor product of ample line bundles is ample but there is a lot more to say. One may argue that the historical importance of ample line bundles is that their global sections can be used to define an embedding to a projective space.

Proposition 3.3. *Let $\mathcal{L}$ be a line bundle over X. The following assertions are equivalent.*

(1) $\mathcal{L}$ is ample.

(2) There exists an integer $n \geq 1$ and a closed immersion $\iota : X \hookrightarrow \mathbb{P}^d_k$ such that $\iota^* \mathcal{O}_{\mathbb{P}^d}(1) \simeq \mathcal{L}$ where $\mathcal{O}_{\mathbb{P}^d}(1)$ is the universal line bunde on the projective space $\mathbb{P}^d_k$.

One may also argue that if we think of line bundles on "good"[1] schemes as Weil divisors, i.e. $\mathbb{Z}$-linear combination of the form $D = \sum_{i=1}^r n_i [Z_i]$ modulo rational equivalence where each Z_i is a reduced subvariety of X of codimension 1, then the Nakai-Moishezon criterion says that ample line bundles corresponds to Weil divisors whose intersection product with all the subvariety of X is positive.

Proposition 3.4 (Nakai–Moishezon criterion)**.** *Assume that X is a smooth integral projective scheme over k. Let $\mathcal{L}$ be a line bundle on X and denote D the associated Weil divisor. The following assertions are equivalent.*

(1) $\mathcal{L}$ is ample.

[1] A regular scheme over a field for example.

(2) *For any subvariety V of X of pure dimension, the intersection product $D^{\dim V} \cdot [V]$ is strictly positive.*

One may also argue that ample line bundles are the one that kills the higher cohomology of coherent sheaves.

Proposition 3.5. *Let $\mathcal{L}$ be a line bundle over X. The following assertions are equivalent.*

(1) *$\mathcal{L}$ is ample.*
(2) *For any coherent sheaf $\mathcal{F}$ on X, the cohomology group $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ vanishes for all $i > 0$ and all $n \geq 1$ large enough.*

For the convenience of the reader, we also recall the classical weaker notion of nefness and bigness for line bundles.

Definition 3.6. *A line bundle $\mathcal{L}$ over X is said to be nef if the following equivalent assertions are satisfied.*

(1) *For any subvariety V of X of pure dimension (equivalently, any curve) , we have*
$$c_1(\mathcal{L})^{\dim V} \cdot [V] \geq 0$$
in the Chow ring of X.
(2) *For any subvariety V of X (equivalently, any curve), there is an integer $d \geq 1$ and a non-zero global section s of $\mathcal{L}^{\otimes d}_{|V}$.*

Definition 3.7. *A line bundle $\mathcal{L}$ over X is said to be big if the following there is an ample line bundle (equivalently, any ample line bundle) $\mathcal{A}$ and an integer $n \geq 1$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{A}^{\otimes -1}$ is globally generated.*

Until now we have restricted the discussion to line bundles and it seems natural to expect a generalization to higher dimensional vector bundles. The following definition is due to Hartshorne for ample vector bundles and to Lazarsfeld for nef and L-big vector bundles.

Definition 3.8 ([**Har66a**][**Laz04b**])**.** *A vector bundle $\mathcal{E}$ over X is said to be ample (resp. nef, L-big) if the corresponding universal line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on the projective bundle $\mathbb{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_{\mathcal{O}_X} \mathrm{Sym}^\bullet \mathcal{E}$ is ample (resp. nef, big).*

Over a field of characteristic p, Hartshorne has defined in [**Har66a**] another notion of ampleness for vector bundles called p-ampleness. This notion is strictly stronger than ampleness. Furthermore, Kleiman has defined in [**Kle69**] a third notion, again strictly stronger, called cohomological p-ampleness. We start this chapter with the preliminary section 3.1 on the flag bundle construction and G-torsors because it will be useful for the rest of the chapter. In the section 3.2, we recall the main properties of the different notions of ampleness for vector bundles. In the section 3.3, we consider an effective Cartier divisor D and we develop a new positivity notion for vector bundles over a field of characteristic p, called (φ, D)-ampleness. The letter φ stands the relative Frobenius map $X \to X^{(p)}$. In the case of line bundles, this notion is equivalent to being nef and big with D as exceptional divisor. This new notion is better behaved than nef and L-big vector bundles because it satisfies many stability properties whereas L-bigness is not even stable under quotient[2]. In the section 3.4, we recall some important properties of nef vector bundles as it does not always appear in the literature over

[2] A counter-example is the bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ which is L-big on $\mathbb{P}^1$ by [**Jab07**, p. 24] but whose quotient $\mathcal{O}_{\mathbb{P}^1}$ is definitely not big.

a field of characteristic p. Inspired by a result of Mourougane [**Mou97**, Théorème 1] over $\mathbb{C}$ about the ampleness of the adjoint bundle $\pi_*(\mathcal{L} \otimes \omega_{Y/X})$ where $\pi : Y \to X$ is a surjective morphism and $\mathcal{L}$ is an ample line bundle on Y, we prove in the section 3.5, similar results over a field of characteristic p when π is a flag bundle.

The following table summarizes the different stability properties of ampleness, p-ampleness, cohomological p-ampleness, (φ, D)-ampleness and nefness with the corresponding reference in the text.

	Cohomologically p-ample	p-ample	Ample	(φ, D)-ample	Nef
Stability of direct sum	3.41	3.38	3.29	3.49	3.57
Stability of extension	?	?	3.29	3.50 (X regular)	3.57
Stability of quotient	?	3.38	3.29	3.51	3.57
Stability of tensor product	?	3.38	3.29	3.52	3.57
Stability of tensor roots	?	?	3.30	3.53	3.57
Stability of pullback by finite morphism	3.42	?	3.31	3.54 ($f^{-1}D$ defined)	3.57 (works without finite hypothesis)
Descent along finite surjective morphism	?	?	3.31	3.54	3.57 (works without finite hypothesis)

TABLE 3.0.1. Main properties of the different positivity notions, from the strongest to the weakest one.

3.1. Flag bundle associated to a G-torsor

3.1.1. Higher direct image. In this subsection only, $\pi : Y \to X$ is a general scheme morphism. We recall some generalities about cohomology and higher direct images.

Proposition 3.9. *For any $\mathcal{O}_Y$-module $\mathcal{F}$, there is a spectral sequence starting at page 2*

$$E_2^{i,j} = H^i(X, R^i\pi_*(\mathcal{F})) \Rightarrow H^{i+j}(Y, \mathcal{F}).$$

PROOF. See [**Sta21**, Lemma 01F2]. □

We recall the projection formula.

Proposition 3.10. *Let $\mathcal{F}$ be a $\mathcal{O}_Y$-module, $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank and $i \geq 0$ an integer. The natural map*

$$R^i\pi_* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E} \to R^i\pi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \pi^* \mathcal{E})$$

is an isomorphism.

PROOF. See [**Sta21**, Lemma 01E8]. □

We recall the proper base change theorem for coherent cohomology.

Proposition 3.11 (Proper base change, non reduced case)**.** *Let $f : X \to S$ be a proper morphism between locally noetherian schemes. Let $\mathcal{F}$ be a coherent sheaf over X which is flat over S. Let $p \geq 0$ and $s \in S$. If $\theta_s^p : (R^p f_* \mathcal{F})_s \otimes_{\mathcal{O}_{S,s}} k(s) \to H^p(X_s, \mathcal{F}_{|X_s})$ is surjective, then there is an open neighbourhood U of s such that for all $s' \in U$, $\theta_{s'}^p$ is an isomorphism and the following conditions are equivalent*

(1) θ_s^{p-1} is surjective,

(2) $R^p f_* \mathcal{F}$ is free on U

and under these conditions, the formation of $R^p f_* \mathcal{F}$ commutes under base change. This means that for any $g : S' \to S$, we have $g^* R^p f_* \mathcal{F} \simeq R^p f'_* g'^* \mathcal{F}$ where the maps are defined in the following cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

PROOF. See [**Har77**, Part III, Theorem 12.11]. □

Remark 3.12. (1) We assume θ_s^{-1} to be the zero morphism.

(2) The reference cited in proposition 3.11 gives a base change result only for geometric points of S. To see how it implies the base change for any morphism, see [**Con**, Proposition 2.1].

Lemma 3.13. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Artin stacks and $\pi : \mathcal{Y} \to \mathcal{X}$ a proper representable morphism. Let $\mathcal{L}$ be a coherent sheaf over $\mathcal{Y}$, flat over $\mathcal{X}$, such that for all geometric points $x : \operatorname{Spec} \bar{k} \to \mathcal{X}$ fitting in the cartesian diagram*

$$\begin{tikzcd} \mathcal{Y}_x := \mathcal{Y} \times_{\mathcal{X},x} \operatorname{Spec} \bar{k} \arrow[r, "i"] \arrow[d, "\pi_x"] & \mathcal{Y} \arrow[d, "\pi"] \\ \operatorname{Spec} \bar{k} \arrow[r, "x"] & \mathcal{X} \end{tikzcd}$$

the complex

$$R(\pi_x)_* \mathcal{L}_{|\mathcal{Y}_x}$$

is concentrated in degree 0. Then, the complex

$$R\pi_* \mathcal{L}$$

is also concentrated in degree 0.

PROOF. Consider a presentation $f : X \to \mathcal{X}$ of the Artin stack $\mathcal{X}$ where X is a scheme and f is a surjective and smooth morphism. Consider the double cartesian diagram

$$\begin{tikzcd} Y_x := Y \times_{X,x} \operatorname{Spec} \bar{k} \arrow[r, "i"] \arrow[d, "\pi_x"] & Y := X \times_{\mathcal{X}} \mathcal{Y} \arrow[r, "f'"] \arrow[d, "\tilde{\pi}"] & \mathcal{Y} \arrow[d, "\pi"] \\ \operatorname{Spec} \bar{k} \arrow[r, "x"] & X \arrow[r, "f"] & \mathcal{X} \end{tikzcd}$$

where x is a geometric point of X. For any $i > 0$, we have $H^i(Y_x, \mathcal{L}_{|Y_x}) = 0$ by hypothesis. As a consequence, the base change morphism for the first cartesian diagram

$$\theta_x^i : R^i\tilde{\pi}_*\mathcal{L}_{|Y} \otimes_{\mathcal{O}_{X,x}} k(x) \to H^i(Y_x, \mathcal{L}_{|Y_x})$$

is surjective. By lemma 3.13, we deduce that $\theta_{x'}^i$ is an isomorphism for all x' in a neighborhood of x. We deduce that $R^i\tilde{\pi}_*\mathcal{L}_{|Y}$ is zero for all $i > 0$. Since f is flat, the base change theorem for the second cartesian diagram says that there is an isomorphism

$$f^* \circ R\pi_*\mathcal{L} \to R\tilde{\pi}_* \circ f'^*\mathcal{L}.$$

Since f is faithfully flat, it implies that $R\pi_*\mathcal{L}$ is concentrated in degree 0. □

3.1.2. G-torsors. In this subsection, G is a connected split reductive group over k, $P \subset G$ is a parabolic subgroup and X is a k-scheme. If Y is a scheme, we denote $\mathrm{Mod}(\mathcal{O}_Y)$ the abelian category of $\mathcal{O}_Y$-module on Y and $\mathrm{Loc}(\mathcal{O}_Y) \subset \mathrm{Mod}(\mathcal{O}_Y)$ the fully faithful additive subcategory of locally free $\mathcal{O}_Y$-module of finite rank.

Definition 3.14. *Let E be a G-torsor over X and let $P \subset G$ be a parabolic subgroup of G. We define the flag bundle of type P of E to be the scheme $\mathcal{F}_P(E)$ over X that represents the functor whose S-points are P-reduction of $E \times_X S$ over S.*

Remark 3.15. Note that if $G = \mathrm{GL}_n$ and

$$P = \left(\begin{array}{ccc|c} * & & & * \\ & \ddots & & \vdots \\ & & * & * \\ 0 & \dots & 0 & * \end{array}\right),$$

then the flag bundle $\mathcal{F}_P(E)$ is just the projective bundle $\mathbb{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_{\mathcal{O}_X} \mathrm{Sym}^\bullet \mathcal{E}$ where $\mathcal{E}$ is the vector bundle corresponding to the GL_n-torsor E.

Definition 3.16. *Let V be an algebraic representation of G and E a G-torsor over X. We define the contracted product of E and V over G to be the representable quotient X-scheme*

$$V \times^G E := \underline{V} \times_k E/G$$

where $\underline{V}$ is the k-vector space scheme associated to V and G acts on functorial points by $g(v, e) = (gv, ge)$. Note that the structure of k-vector space on V endows $V \times^G E$ with a structure of vector bundle of rank $\dim_k V$ over X.

Definition 3.17. *Let E be a G-torsor over X. We define a functor*

$$\mathcal{W} : \mathrm{Rep}(G) \to \mathrm{Loc}(\mathcal{O}_X)$$

through the formula

$$\mathcal{W}(V) = V \times^G E$$

where V is an algebraic representation of G.

Definition 3.18. *Let E be a G-torsor over X and $\pi : \mathcal{F}_P(E) \to X$ the flag bundle of type P of E. We define a functor*

$$\mathcal{L} : \mathrm{Rep}(P) \to \mathrm{Loc}(\mathcal{O}_{\mathcal{F}_P(E)})$$

through the formula

$$\mathcal{L}(V) = V \times^P H$$

where V is an algebraic representation of P and H is the universal P-torsor on $\mathcal{F}_P(E)$.

Remark 3.19. If $\lambda \in X^*(P)$ is a character of P, we simply write $\mathcal{L}_\lambda$ for the associated line bundle on $\mathcal{F}_P(E)$. We also write $\mathcal{W}_\lambda$ for the vector bundle associated to the G-representation $H^0(G/P, \mathcal{L}_\lambda) = \nabla(\lambda)$. We simply denote St_r the image of the Steinberg module by $\mathcal{W}$.

Remark 3.20. When $G = \mathrm{GL}_n$ and P is the standard parabolic as in remark 3.15, the universal line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is equal to the associated line bundle $\mathcal{L}_\lambda$ with $\lambda = (1, 0, \cdots, 0)$ through the canonical isomorphism $X^*(T) \simeq \mathbb{Z}^n$ where T is the standard maximal torus of GL_n.

Proposition 3.21. *The functor $\mathcal{W}$ and $\mathcal{L}$ are monoidal and exact.*

Proof. This is a general result on associated sheaves [**Jan03**, Part. 1, Chap. 5]. □

Proposition 3.22. *Let E be a G-torsor over X. Then, the following diagram*

$$\begin{tikzcd}
\mathrm{Rep}(P) \arrow[r, "\mathcal{L}"] \arrow[d, "\mathrm{Ind}_P^G"] & \mathrm{Loc}(\mathcal{O}_{\mathcal{F}_P(E)}) \arrow[d, "\pi_*"] \\
\mathrm{Rep}(G) \arrow[r, "\mathcal{W}"] \arrow[d, "\mathrm{Res}_P^G"] & \mathrm{Loc}(\mathcal{O}_X) \arrow[d, "\pi^*"] \\
\mathrm{Rep}(P) \arrow[r, "\mathcal{L}"] & \mathrm{Loc}(\mathcal{O}_{\mathcal{F}_P(E)})
\end{tikzcd}$$

commutes where $\mathrm{Ind}_P^G : \mathrm{Rep}_k(P) \to \mathrm{Rep}_k(G)$ and $\mathrm{Res}_P^G : \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(P)$ are the induction and restriction functors. Moreover, if λ is a dominant character of P, then $R\pi_*\mathcal{L}_\lambda$ is isomorphic to $\mathcal{W}_\lambda$ concentrated in degree 0.

Proof. The commutativity of the lower square follows directly from the definitions. The commutativity of the upper square is a formal consequence of the definition of induced modules and standard base change theorem. We have a cartesian diagram

$$\begin{tikzcd}
\mathcal{F}_P(E) \arrow[r, "\zeta_P"] \arrow[d, "\pi"] & \lfloor P\backslash * \rfloor \arrow[d, "\tilde{\pi}"] \\
X \arrow[r, "\zeta"] & \lfloor G\backslash * \rfloor
\end{tikzcd}$$

where the horizontal arrows corresponds to the universal G-torsor on X and the universal P-torsor on $\mathcal{F}_P(E)$ and where the vertical arrow $\tilde{\pi}$ between the classifying stacks is induced by the inclusion $P \subset G$. For every representation V of P, we have a vector bundle $\tilde{\mathcal{L}}(V)$ on the classifying stack of P. We denote by $\tilde{\mathcal{W}}(V)$ the vector bundle on the classifying stack of G associated to the G-module $H^0(G/P, \mathcal{L}(V)) = \mathrm{Ind}_P^G(V)$. By definition, we have isomorphisms

$$\begin{cases}
\tilde{\pi}_*\tilde{\mathcal{L}}(V) = \tilde{\mathcal{W}}(V) \\
\zeta_P^*\tilde{\mathcal{L}}(V) = \mathcal{L}(V) \\
\zeta^*\tilde{\mathcal{W}}(V) = \mathcal{W}(V)
\end{cases}$$

on $\lfloor G\backslash * \rfloor$. Since ζ is flat, the base change theorem in the derived category of quasi-coherent sheaves over X says that the natural map

$$\zeta^* \circ R\tilde{\pi}_*\tilde{\mathcal{L}}(V) \to R\pi_* \circ \zeta^*\tilde{\mathcal{L}}(V) \tag{3.1}$$

is an isomorphism. Taking H^0 in equation 3.1, we get an isomorphism

$$\mathcal{W}(V) = \zeta^* \tilde{\pi}_* \tilde{\mathcal{L}}(V) \xrightarrow{\simeq} \pi_* \mathcal{L}(V)$$

over X. If λ is P-dominant, Kempf's vanishing theorem from proposition 2.11 combined with lemma 3.13 implies that

$$\begin{cases} R\pi_* \mathcal{L}_\lambda = \pi_* \mathcal{L}_\lambda \\ R\tilde{\pi}_* \mathcal{L}_\lambda = \tilde{\pi}_* \mathcal{L}_\lambda \end{cases}$$

and we get

$$R\pi_* \mathcal{L}_\lambda \simeq \mathcal{W}_\lambda[0].$$

□

Proposition 3.23. *Let λ be a character. For all $r \geq 1$, we have isomorphisms*

$$\pi_*(\mathcal{L}_{p^r(\lambda+\rho)-\rho}) = \mathrm{St}_r \otimes \mathcal{W}_\lambda^{(p^r)}.$$

Proof. This is a direct consequence of proposition 2.10 and 3.22. □

Proposition 3.24. *Let P_I be a standard parabolic subgroup of G of type I. Let E be a G-torsor over X and $\pi : \mathcal{F}_{P_I}(E) \to X$ the flag bundle ot type P_I of X. We have an isomorphism*

$$\Omega^{top}_{\mathcal{F}_{P_I}(E)/X} \simeq \mathcal{L}_{-2\rho_I}$$

where top denotes the relative dimension of π.

Proof. From the cartesian diagram

$$\begin{tikzcd} \mathcal{F}_{P_I}(E) \arrow[r, "\tilde{\zeta}"] \arrow[d, "\pi"] & \lfloor P_I \backslash * \rfloor \arrow[d, "\tilde{\pi}"] \\ X \arrow[r, "\zeta"] & \lfloor G \backslash * \rfloor \end{tikzcd}$$

we deduce an isomorphism $\tilde{\zeta}^* \Omega^1_{\tilde{\pi}} = \Omega^1_\pi$. We know that

$$\Omega^1_{\tilde{\pi}} \simeq \mathcal{L}(\mathrm{Lie}(G)/\mathrm{Lie}(P_I)^\vee),$$

hence

$$\Omega^{top}_{\tilde{\pi}} \simeq \mathcal{L}(\Lambda^{top} \mathrm{Lie}(G)/\mathrm{Lie}(P_I)^\vee).$$

The weights of the T-action on $\mathrm{Lie}(G)/\mathrm{Lie}(P_I)$ are the roots $-\Phi_I^+$, so $\Lambda^{top}\mathrm{Lie}(G)/\mathrm{Lie}(P_I)$ is a one-dimensional module of weight $-2\rho_I$ and by taking the linear dual, we get an isomorphism $\Omega^{top}_\pi = \tilde{\zeta}^* \Omega^{top}_{\tilde{\pi}} = \mathcal{L}_{-2\rho_I}$. □

3.2. Ampleness in positive characteristic

Let k denote a field of characteristic p. Let X denote a projective scheme over k. We write $\varphi : X \to X^{(p)}$ for the relative geometric Frobenius of X fitting in the following cartesian diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\ & X^{(p)} \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}\, k \arrow[r, "F_k"] & \mathrm{Spec}\, k \end{tikzcd}$$

where F_X is the absolute Frobenius on X and F_k is the absolute Frobenius on k. We denote $\varphi^r : X \to X^{(p^r)}$ and $\sigma^r : X^{(p^r)} \to X$ the corresponding r^{th}-iterates. If $\mathcal{F}$ is a sheaf on X and $r \geq 1$ is an integer, we write $\mathcal{F}^{(p^r)} := (\varphi^r)^*(\sigma^r)^*\mathcal{F} = (F_X^r)^*\mathcal{F}$. We endow the finite dimensional real vector space $A_1(X)$ of 1-cycles on X modulo linear equivalence with a norm $\|\cdot\|$. If C is a projective curve and $\mathcal{E}$ is a vector bundle on C, we denote by $\delta(\mathcal{E})$ the minimum of the degrees of quotient line bundles of $\mathcal{E}$. For the convenience of the reader, we recall the definitions and main properties of globally generated sheaves and relatively ample line bundles.

Definition 3.25. *We say that a coherent sheaf $\mathcal{F}$ is globally generated at $x \in X$ if the canonical map*

$$H^0(X, \mathcal{F}) \otimes_k \mathcal{O}_X \to \mathcal{F}$$

is surjective at $x \in X$. We say $\mathcal{F}$ is globally generated over $U \subset X$ if it is globally generated at x for all $x \in U$.

The following lemma is well-known.

Lemma 3.26. *Let x be a point of X. We have the following assertions.*

(1) The direct sum of two globally generated sheaves at x is globally generated at x.
(2) Let $\mathcal{F} \to \mathcal{F}'$ be a morphism of coherent sheaves wich is sujective at x. If $\mathcal{F}$ is globally generated at x, then $\mathcal{F}'$ is also globally generated at x.
(3) The tensor product of two globally generated sheaves at x is globally generated at x.
(4) The pullback of a globally generated sheaf at x is globally generated at x.

Proof. Assertion (1) follows directly from the definition. For assertion (2), we consider the following commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{F}) \otimes_k \mathcal{O}_X & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{F}') \otimes_k \mathcal{O}_X & \longrightarrow & \mathcal{F}' \end{array}$$

and since the top horizontal and the right vertical maps are surjective at x, the bottom horizontal map must be surjective at x. For (3), we first assume that the natural map

$$\varphi : H^0(X, \mathcal{F}) \otimes_k \mathcal{O}_X \to \mathcal{F}$$

is surjective at x. Since $\mathcal{F}$ is coherent, the k-vector space $H^0(X, \mathcal{F})$ is finite dimensional and we can choose a finite basis

$$H^0(X, \mathcal{F}) \simeq k^{\oplus d}.$$

By tensoring φ with the identity of $\mathcal{F}'$, we get a map of $\mathcal{O}_X$-modules

$$(\mathcal{F}')^{\oplus d} \simeq k^{\oplus d} \otimes_k \mathcal{O}_X \otimes \mathcal{F}' \to \mathcal{F} \otimes \mathcal{F}'$$

which is surjective at x. Since $\mathcal{F}'$ is globally generated at x, so is $(\mathcal{F}')^{\oplus d}$ and we conclude with (1). Assertion (4) is a direct consequence of the right-exactness of the pullback functor. $\square$

From now on, we fix an ample line bundle $\mathcal{O}_X(1)$ on X and we write $\mathcal{F}(m)$ instead of $\mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes m}$ for any coherent sheaf $\mathcal{F}$ on X and integer m. We recall the definition of relative ample line bundles.

Definition 3.27. *Let Y be a projective scheme over a base scheme S. Write $f : Y \to S$ for the structure morphism. We say that a line bundle $\mathcal{L}$ on Y is f-ample if the following equivalent propositions are satisfied.*

(1) For all coherent sheaf $\mathcal{F}$ on Y, there is an integer n_0 such that the adjunction morphism $f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is surjective for all $n \geq n_0$.

(2) For all coherent sheaf $\mathcal{F}$ on Y, there is an integer n_0 such that the higher direct image sheaves $R^i f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ vanishes for all $i > 0$, $n \geq n_0$.

Proof. For the equivalence of the definitions, see [**Laz04b**, Theorem 1.7.6] or [**Sta21**, Lemma 02O1]. □

Proposition 3.28. *Let $\mathcal{E}$ be a vector bundle on X. The following assertions are equivalent.*

(1) $\mathcal{E}$ is ample on X.

(2) For all coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that $\mathcal{F} \otimes \operatorname{Sym}^n \mathcal{E}$ is globally generated for all $n \geq n_0$.

(3) For all coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that the cohomology groups $H^i(X, \mathcal{F} \otimes \operatorname{Sym}^n \mathcal{E})$ vanishes for all $i > 0$, $n \geq n_0$.

(4) There exists a real number $\varepsilon > 0$ such that for all finite morphism $g : C \to X$ where C is a smooth projective curve, we have

$$\delta(g^* \mathcal{E}) \geq \varepsilon \|g_* C\|.$$

Recall that $\delta(g^* \mathcal{E})$ is the minimum of the degrees of quotient line bundles of $g^* \mathcal{E}$ and $\|\cdot\|$ denotes a norm on $A_1(X)$, the finite dimensional real vector space of 1-cycles modulo linear equivalence.

Proof. See [**Har66a**, Proposition 3.2/3.3] for a complete proof of

$$(1) \Leftrightarrow (2) \Leftrightarrow (3).$$

For (1) ⇔ (4), this numerical criterion is due to Barton [**Bar71**]. □

Proposition 3.29. *We have the following assertions*

(1) Let $\mathcal{E}$ and $\mathcal{E}'$ be two ample vector bundles on X. Then $\mathcal{E} \oplus \mathcal{E}'$ is ample.

(2) Consider an extension of vector bundles on X

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are ample. Then $\mathcal{E}$ is ample.

(3) Let $\mathcal{E}$ and $\mathcal{E}'$ be two vector bundles on X such that $\mathcal{E}$ is ample and $\mathcal{E}'$ is globally generated over X. Then the tensor product $\mathcal{E} \otimes \mathcal{E}'$ is an ample vector bundle.

(4) Let $\mathcal{E} \twoheadrightarrow \mathcal{E}'$ be a surjective morphism of $\mathcal{O}_X$-modules between two vector bundles. If $\mathcal{E}$ is ample, then $\mathcal{E}'$ is also ample.

(5) The tensor product of ample vector bundles over X is ample.

Proof. See [**Har66a**, Proposition 2.2/Corollary 2.5] for assertions (1), (3), (4), [**Har66a**, Corollary 3.4] for assertion (4) and [**Bar71**, Theorem. 3.3] for assertion (5). □

Proposition 3.30. *If $\mathcal{E}$ is a vector bundle such that $\mathcal{E}^{\otimes n}$ is ample for some $n \geq 1$, then $\mathcal{E}$ is also ample.*

Proof. Assume that $\mathcal{E}^{\otimes n}$ is ample. As a quotient of $\mathcal{E}^{\otimes n}$, $\operatorname{Sym}^n \mathcal{E}$ is ample and we conclude with [**Har66a**, Proposition 2.4]. □

Proposition 3.31. *Let $f : Y \to X$ be an finite morphism of projective schemes and $\mathcal{E}$ be an ample vector bundle on X. If $\mathcal{E}$ is ample on X, then $f^*\mathcal{E}$ is ample on Y. If furthermore f is assumed surjective, then the converse holds.*

Proof. See [**Laz04a**, Proposition 1.2.9] and [**Laz04a**, Corollary 1.2.24]. □

Corollary 3.32. *Let $\mathcal{E}$ be a vector bundle and $r \geq 1$ an integer. Then $\mathcal{E}$ is ample if and only if $\mathcal{E}^{(p^r)}$ is ample.*

Proof. Since the Frobenius map is finite surjective, it follows from the proposition 3.31. □

Definition 3.33 ([**Har66a**])**.** *We say that a vector bundle $\mathcal{E}$ on X is p-ample if for all coherent sheaf $\mathcal{F}$ on X, there is an integer r_0 such that $\mathcal{F} \otimes \mathcal{E}^{(p^r)}$ is globally generated for all $r \geq r_0$.*

Lemma 3.34. *For any coherent sheaf $\mathcal{F}$ and $m \geq 0$ large enough, we can write $\mathcal{F}$ as a quotient of $\mathcal{O}_X(-m)^{\oplus s}$ for a suitable $s \geq 1$.*

Proof. Choose $m \geq 0$ large enough such that $\mathcal{F}(m)$ is globally generated over X. We get a sujective morphism

$$\mathcal{O}_X^{\oplus s} \to \mathcal{F}(m)$$

for some $s \geq 1$ and then we tensor by $\mathcal{O}_X(-m)$. □

Proposition 3.35. *In the definition 3.33, we can restrict ourselves to coherent sheaves of the form $\mathcal{F} = \mathcal{O}_X(-m)$ for all $m \geq 0$ large enough.*

Proof. We use the lemma 3.34 to write $\mathcal{F}$ as a quotient of $\mathcal{O}_X(-m)^{\oplus s}$ for a suitable $s \geq 1$. Take n large enough such that $\mathcal{O}_X(-m) \otimes \mathcal{E}^{(p^r)}$ is globally generated. Since the quotient of a globally generated sheaf is globally generated, we get that $\mathcal{F} \otimes \mathcal{E}^{(p^r)}$ is globally generated. □

Proposition 3.36. *If $\mathcal{E}$ is p-ample on X, then $\mathcal{E}$ is ample.*

Proof. Choose n large enough such that $\mathcal{E}^{(p^n)}(-1)$ is globally generated. We deduce that $\mathcal{E}^{(p^n)}$ is quotient of $\mathcal{O}_X(1)^{\oplus s}$ for a suitable $s \geq 1$. By the assertion (3) of proposition 3.29, $\mathcal{E}^{(p^n)}$ is ample and by corollary 3.32, $\mathcal{E}$ is ample. □

Remark 3.37. The converse to the previous proposition is false in general (see [**Gie71**] for a counter example). However, in the special case where $\mathcal{E}$ is a line bundle or X is curve, it holds by [**Har66a**, Proposition 7.3].

Proposition 3.38. *We have the following assertions:*

(1) Let $\mathcal{E}$ and $\mathcal{E}'$ be two p-ample vector bundles on X. Then $\mathcal{E} \oplus \mathcal{E}'$ is p-ample.

(2) Let $\mathcal{E}$ and $\mathcal{E}'$ be two vector bundles on X such that $\mathcal{E}$ is p-ample and $\mathcal{E}'$ is globally generated over X. Then, the tensor product $\mathcal{E} \otimes \mathcal{E}'$ is a p-ample vector bundle.

(3) Let $\mathcal{E} \to \mathcal{E}'$ be a surjective morphism of $\mathcal{O}_X$-modules between two vector bundles. If $\mathcal{E}$ is p-ample, then so is $\mathcal{E}'$.

(4) The tensor product of p-ample vector bundles over X is p-ample.

Proof. See [**Har66a**, Proposition 6.4/Corollary 6.7] for assertions (1), (2) and (4). Hartshorne does not state assertion (3), so we give a proof. Let $\mathcal{F}$ be a coherent sheaf and $r_0 \geq 1$ be an

integer such that $\mathcal{F} \otimes \mathcal{E}^{(p^r)}$ is globally generated for all $r \geq r_0$. For all $r \geq r_0$, the surjective morphism $\mathcal{E} \to \mathcal{E}'$ induces a surjective morphism of $\mathcal{O}_X$-modules

$$\mathcal{F} \otimes \mathcal{E}^{(p^r)} \to \mathcal{F} \otimes (\mathcal{E}')^{(p^r)}$$

and from assertion (2) of lemma 3.26, the module $\mathcal{F} \otimes (\mathcal{E}')^{(p^r)}$ is globally generated over X. □

There is no known cohomological criterion for p-ampleness. However, Kleiman has defined in [**Kle69**] the strictly[3] stronger notion of cohomological p-ampleness.

Definition 3.39 ([**Kle69**])**.** *We say that a vector bundle $\mathcal{E}$ on X is cohomologically p-ample if for all coherent sheaves $\mathcal{F}$ on X, there is an integer $r_0 \geq 1$ such that the cohomology groups $H^i(X, \mathcal{F} \otimes \mathcal{E}^{(p^r)})$ vanishes for all $i > 0$, $r \geq r_0$.*

Proposition 3.40. *If $\mathcal{E}$ is cohomologically p-ample on X, then $\mathcal{E}$ is p-ample.*

Proof. See [**Kle69**, Proposition 9]. □

To the best of our knowledge, the following statements do not appear in the litterature so we state them and provide a proof.

Proposition 3.41. *A direct sum of cohomologically p-ample vector bundle is cohomologically p-ample.*

Proof. It follows directly from the isomorphism

$$H^i(X, \mathcal{F} \otimes (\mathcal{E} \oplus \mathcal{E}')^{(p^r)}) = H^i(X, \mathcal{F} \otimes \mathcal{E}^{(p^r)}) \oplus H^i(X, \mathcal{F} \otimes \mathcal{E}'^{(p^r)}).$$

□

Proposition 3.42. *Let $f : Y \to X$ be a finite morphism of projective schemes and $\mathcal{E}$ be a cohomologically p-ample vector bundle on X. Then $f^* \mathcal{E}$ is cohomologically p-ample on Y.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on Y. Since f is finite, the Leray spectral sequence degenerates at page 2 and we have isomorphisms

$$H^i(X, f_*(\mathcal{F} \otimes f^* \mathcal{E}^{(p^r)})) = H^i(Y, \mathcal{F} \otimes f^* \mathcal{E}^{(p^r)})$$

for all $i \geq 0$ and $r \geq 0$. Since f is finite, the pushforward $f_* \mathcal{F}$ is a coherent $\mathcal{O}_X$-module and the projection formula implies that

$$f_*(\mathcal{F} \otimes f^* \mathcal{E}^{(p^r)}) = f_* \mathcal{F} \otimes \mathcal{E}^{(p^r)}.$$

Since $\mathcal{E}$ is cohomologically p-ample on X, there is an integer $r_0 \geq 1$ such that

$$H^i(X, f_* \mathcal{F} \otimes \mathcal{E}^{(p^r)}) = 0 = H^i(Y, \mathcal{F} \otimes (f^* \mathcal{E})^{(p^r)})$$

for all $i > 0$ and $r \geq r_0$. In particular, $f^* \mathcal{E}$ is cohomologically p-ample on Y. □

[3] See again [**Gie71**] for an example of p-ample vector bundle that is not cohomologically p-ample.

3.3. (φ, D)-ample bundles and their properties

If D is a Cartier divisor, we write $\mathcal{O}_X(D)$ for the associated line bundle. If $\mathcal{F}$ is a coherent sheaf on X, then we simply write $\mathcal{F}(D)$ instead of $\mathcal{F} \otimes \mathcal{O}_X(D)$. We consider an *effective* Cartier divisor D on X and we define the notion of (φ, D)-ampleness for vector bundles over X. This new notion is better behaved than nef and L-big vector bundles because it satisfies many stability properties whereas L-bigness is not even stable under quotient[4].

Definition 3.43. *Let $\mathcal{E}$ be a vector bundle over X. We say that $\mathcal{E}$ is (φ, D)-ample if there is an integer $r_0 \geq 1$ such that for all integer $r \geq r_0$, the vector bundle $\mathcal{E}^{(p^r)}(-D)$ is ample.*

In the case of line bundles, (φ, D)-ampleness has the following characterization.

Proposition 3.44. *Let $\mathcal{L}$ be a line bundle over X. Then $\mathcal{L}$ is (φ, D)-ample if and only if $\mathcal{L}$ is nef and there is an integer $n_0 \geq 1$ such that $\mathcal{L}^{\otimes n_0}(-D)$ is ample.*

Proof. Note that $\mathcal{L}^{(p^r)} = \mathcal{L}^{\otimes p^r}$ for all $r \geq 0$. Assume that $r_0 \geq 1$ is an integer such that $\mathcal{L}^{\otimes p^r}(-D)$ is ample for all $r \geq r_0$. If $\mathcal{L}$ was not nef, we could find a subcurve $C \subset X$ such that the intersection product

$$c_1(\mathcal{L}) \cdot [C]$$

is negative. It would imply that the intersection product

$$c_1(\mathcal{L}^{\otimes p^r}(-D)) \cdot [C] = p^r \underbrace{(c_1(\mathcal{L}) \cdot [C])}_{<0} - D \cdot [C]$$

is negative for some $r \geq r_0$ large enough, which contradicts the ampleness of $\mathcal{L}^{\otimes p^r}(-D)$. Inversely, we assume that $\mathcal{L}$ is nef and there exists an integer $n_0 \geq 1$ such that $\mathcal{L}^{\otimes n_0}(-D)$ is ample. Let r be an integer such that

$$r \geq \log_p n_0$$

and consider

$$\mathcal{L}^{(p^r)}(-D) = \mathcal{L}^{\otimes p^r}(-D) = \mathcal{L}^{\otimes n_0}(-D) \otimes \mathcal{L}^{\otimes p^r - n_0}$$

which is ample as the tensor product of an ample line bundle with a nef line bundle. □

Remark 3.45. In the case of line bundles we will drop the φ from the notation and simply say that the line bundle is D-ample.

Proposition 3.46. *Let $\mathcal{L}$ be a line bundle over X. The following assertions are equivalent.*

(1) $\mathcal{L}$ is nef and big.

(2) There exists an effective Cartier divisor H on X, such that $\mathcal{L}$ is H-ample.

Proof. Assume that there exists an effective Cartier divisor H on X such that $\mathcal{L}$ is H-ample. We have seen in proposition 3.44 that $\mathcal{L}$ is nef and there is an integer $n_0 \geq 1$ such that $\mathcal{L}^{\otimes n_0}(-H)$ is ample. Moreover, since we can write $\mathcal{L}^{n_0}$ as a tensor product

$$\mathcal{L}^{n_0} = \mathcal{L}^{n_0}(-H) \otimes \mathcal{O}_X(H)$$

of an ample line bundle with an effective line bundle, $\mathcal{L}$ is big. We are left to show the implication $(i) \Rightarrow (ii)$. Since $\mathcal{L}$ is big, there exists an integer $n_0 \geq 1$ and an ample line bundle $\mathcal{A}$ such that $\mathcal{L}^{n_0} \otimes \mathcal{A}^{-1} = \mathcal{O}_X(H)$ with H an effective divisor. In particular, the line bundle $\mathcal{L}^{n_0}(-H)$ is ample. We conclude with proposition 3.44. □

[4] A counter-example is the bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ which is L-big on $\mathbb{P}^1$ by [**Jab07**, p. 24] but whose quotient $\mathcal{O}_{\mathbb{P}^1}$ is definitely not big.

We prove some stability properties of (φ, D)-ample vector bundles. We first prove the following easy lemma.

Lemma 3.47. *Let C be a projective curve and $\mathcal{E}$ be a vector bundle on C. Recall that $\delta(\mathcal{E})$ denotes the minimum of degrees of quotient line bundles of $\mathcal{E}$. Then, we have*

(1) If $\mathcal{L}$ is a line bundle on C, then $\delta(\mathcal{E} \otimes \mathcal{L}) = \delta(\mathcal{E}) + \deg \mathcal{L}$.

(2) If $f : C' \to C$ is a finite morphism of degree d with C' a projective curve, then $d\delta(\mathcal{E}) \geq \delta(f^* \mathcal{E})$.

Proof of the lemma 3.47. For (1), take a line bundle $\mathcal{E} \twoheadrightarrow \mathcal{L}'$ such that $\delta(\mathcal{E}) = \deg \mathcal{L}'$. If we tensor it by $\mathcal{L}$, we get $\delta(\mathcal{E} \otimes \mathcal{L}) \leq \deg \mathcal{L}' + \deg \mathcal{L} = \delta(\mathcal{E}) + \deg \mathcal{L}$. The same argument applied to $\mathcal{E} \otimes \mathcal{L}^{-1}$ shows the reverse inequality. For (2), take a line bundle $\mathcal{E} \twoheadrightarrow \mathcal{L}'$ such that $\delta(\mathcal{E}) = \deg \mathcal{L}'$. The pullback f^* induces a quotient map $f^* \mathcal{E} \twoheadrightarrow f^* \mathcal{L}' = \mathcal{L}'^{\otimes d}$ which shows that $\delta(f^* \mathcal{E}) \leq d \deg \mathcal{L}' = d\delta(\mathcal{E})$. □

Proposition 3.48. *Let $\mathcal{E}$ be a vector bundle on X and $n \geq 1$ an integer. The following assertions are equivalent.*

(1) $\mathcal{E}$ is (φ, D)-ample.

(2) $\mathcal{E}$ is (φ, nD)-ample.

Proof. Assume that $\mathcal{E}$ is (φ, D)-ample and consider $r_0 \geq 1$ such that

$$\mathcal{E}^{(p^r)}(-D)$$

is ample for all $r \geq r_0$. By Barton's numerical criterion of ampleness recalled in assertion (4) of proposition 3.28, for all $r \geq r_0$, we have a real number $\varepsilon_r > 0$ such that for all finite morphism $g : C \to X$ where C is a smooth projective curve over k, we have

$$\delta(g^* \mathcal{E}^{(p^r)}(-D)) \geq \varepsilon_r \|g_* C\|$$

which is equivalent to

$$\delta(g^* \mathcal{E}^{(p^r)}) - D \cdot C \geq \varepsilon_r \|g_* C\|$$

where $D \cdot C$ is the degree of the line bundle $g^* \mathcal{O}_X(D) = \mathcal{O}_X(D)_{|C}$ (it is also equal to the intersection number of D with C). If $D \cdot C \leq 0$, then

$$\begin{aligned} \delta(g^* \mathcal{E}^{(p^r)}(-nD)) &= \delta(g^* \mathcal{E}^{(p^r)}) - D \cdot C - (n-1) D \cdot C \\ &\geq \delta(g^* \mathcal{E}^{(p^r)}) - D \cdot C \\ &\geq \varepsilon_r \|g_* C\| \end{aligned}$$

for all $r \geq r_0$. If $D \cdot C > 0$, we take $r_1 \geq r_0$ such that $r_1 \geq r_0 + \log_p(n)$ and let $r \geq r_1$ be an integer. Since $\mathcal{E}^{(p^r)}(-D)$ is ample, the bundle

$$(\mathcal{E}^{(p^{r_0})}(-D))^{(p^{r-r_0})} = \mathcal{E}^{(p^r)}(-p^{r-r_0} D)$$

is ample and we have

$$\delta(g^* \mathcal{E}^{(p^r)}(-p^{r-r_0} D)) \geq \varepsilon'_r \|g_* C\|$$

for some real number $\varepsilon'_r > 0$. Thus,

$$\begin{aligned} \delta(g^* \mathcal{E}^{(p^r)}(-nD)) &= \delta(g^* \mathcal{E}^{(p^r)}(-p^{r-r_0} D)) + (p^{r-r_0} - n) D \cdot C \\ &\geq \delta(g^* \mathcal{E}^{(p^r)}(-p^{r-r_0} D)) \\ &\geq \varepsilon'_r \|g_* C\|. \end{aligned}$$

In conclusion, we have

$$\delta(g^* \mathcal{E}^{(p^r)}(-nD)) \geq \min(\varepsilon_r, \varepsilon'_r)\|g_*C\|$$

for all $r \geq r_1$ and all $g : C \to X$, which means that $\mathcal{E}$ is (φ, nD)-ample. Inversely, consider an integer $r_0 \geq 1$ such that for all $r \geq r_0$, we have a real number $\varepsilon_r > 0$ such that for all finite morphism $g : C \to X$ where C is a smooth projective curve over k, we have

$$\delta(g^* \mathcal{E}^{(p^r)}) - nD \cdot C \geq \varepsilon_r \|g_*C\|.$$

If $D \cdot C \geq 0$, we have

$$\begin{aligned}\delta(g^* \mathcal{E}^{(p^r)}(-D)) &= \delta(g^* \mathcal{E}^{(p^r)}(-nD)) + (n-1)D \cdot C \\ &\geq \delta(g^* \mathcal{E}^{(p^r)}(-nD)) \\ &\geq \varepsilon_r \|g_*C\|\end{aligned}$$

for all $r \geq r_0$. Consider an integer $r_1 \geq r_0 + \log_p n$. If $D \cdot C < 0$, we have

$$\begin{aligned}p^{r-r_0}\delta(g^* \mathcal{E}^{(p^{r_0})}(-D)) &\geq \delta(g^* \mathcal{E}^{(p^r)}(-p^{r-r_0}D)) \\ &\geq \delta(g^* \mathcal{E}^{(p^r)}(-nD)) + (n - p^{r-r_0})D \cdot C \\ &\geq \delta(g^* \mathcal{E}^{(p^r)}(-nD)) \\ &\geq \varepsilon_r \|g_*C\|.\end{aligned}$$

for all $r \geq r_1$. In conclusion, we have

$$\delta(g^* \mathcal{E}^{(p^r)}(-D)) \geq \frac{\varepsilon_r}{p^{r-r_0}} \|g_*C\|$$

for all $r \geq r_1$ and all $g : C \to X$, which means that $\mathcal{E}$ is (φ, D)-ample. □

Proposition 3.49. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two (φ, D)-ample vector bundle on X. Then, $\mathcal{E} \oplus \mathcal{E}'$ is (φ, D)-ample.*

Proof. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundles $\mathcal{E}^{(p^r)}(-D)$ and $(\mathcal{E}')^{(p^r)}(-D)$ are ample. For all $r \geq r_0$, we have

$$(\mathcal{E} \oplus \mathcal{E}')^{(p^r)}(-D) = \mathcal{E}^{(p^r)}(-D) \oplus (\mathcal{E}')^{(p^r)}(-D)$$

which is ample by the assertion (1) of proposition 3.29. □

Proposition 3.50. *Consider an extension of vector bundles on X*

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are (φ, D)-ample and assume that X is regular over k. Then $\mathcal{E}$ is (φ, D)-ample.

Proof. On a regular scheme, the Frobenius morphism is flat by [**Kun69**] or [**Sta21**, Lemma 0EC0]. As a consequence, we have an integer $r_0 \geq 1$ and an exact sequence

$$0 \longrightarrow (\mathcal{E}_1)^{(p^r)}(-D) \longrightarrow (\mathcal{E})^{(p^r)}(-D) \longrightarrow (\mathcal{E}_2)^{(p^r)}(-D) \longrightarrow 0$$

of vector bundles on X where $(\mathcal{E}_1)^{(p^r)}(-D)$ and $(\mathcal{E}_2)^{(p^r)}(-D)$ are ample for all $r \geq r_0$. We conclude with assertion (2) of proposition 3.29. □

Proposition 3.51. *Let $\mathcal{E} \to \mathcal{E}'$ be a surjective morphism of $\mathcal{O}_X$-modules between two vector bundles. If $\mathcal{E}$ is (φ, D)-ample, then $\mathcal{E}'$ is also (φ, D)-ample.*

PROOF. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundle $\mathcal{E}^{(p^r)}(-D)$ is ample. For all $r \geq r_0$, the surjective morphism $\mathcal{E} \to \mathcal{E}'$ induces a surjection

$$\mathcal{E}^{(p^r)}(-D) \to (\mathcal{E}')^{(p^r)}(-D)$$

and we conclude with the assertion (3) of proposition 3.29. □

Proposition 3.52. *The tensor product of (φ, D)-ample vector bundles is (φ, D)-ample.*

PROOF. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundles $\mathcal{E}^{(p^r)}(-D)$ and $(\mathcal{E}')^{(p^r)}(-D)$ are ample. For all $r \geq r_0$, we have

$$(\mathcal{E} \otimes \mathcal{E}')^{(p^r)}(-2D) = \mathcal{E}^{(p^r)}(-D) \otimes (\mathcal{E}')^{(p^r)}(-D)$$

which is ample by assertion (5) of proposition 3.29. It shows that $\mathcal{E} \otimes \mathcal{E}'$ is $(\varphi, 2D)$-ample and we conclude with the proposition 3.48. □

Proposition 3.53. *If $\mathcal{E}$ is a vector bundle such that $\mathcal{E}^{\otimes n}$ is (φ, D)-ample for some $n \geq 1$, then $\mathcal{E}$ is also (φ, D)-ample.*

PROOF. Assume that $\mathcal{E}^{\otimes n}$ is (φ, D)-ample. By proposition 3.48, $\mathcal{E}^{\otimes n}$ is also (φ, nD)-ample. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundle $(\mathcal{E}^{\otimes n})^{(p^r)}(-nD)$ is ample. For all $r \geq r_0$, the bundle

$$(\mathcal{E}^{(p^r)}(-D))^{\otimes n} = (\mathcal{E}^{\otimes n})^{(p^r)}(-nD)$$

is ample. Thus, the bundle $\mathcal{E}^{(p^r)}(-D)$ is ample for all $r \geq r_0$ by proposition 3.30. □

Proposition 3.54. *Let $f : Y \to X$ be a finite morphism of projective schemes such that the pullback $f^{-1}D$ is defined as an effective Cartier divisor of Y*[5] *and $\mathcal{E}$ be a (φ, D)-ample bundle on X. Then $f^*\mathcal{E}$ is $(\varphi, f^{-1}D)$-ample on Y. If furthermore, f is assumed surjective, then the converse holds.*

PROOF. Let $r_0 \geq 1$ be an integer such that for all $r \geq r_0$, the bundle $\mathcal{E}^{(p^r)}(-D)$ is ample. For all $r \geq r_0$, the bundle

$$f^*(\mathcal{E}^{(p^r)}(-D)) = (f^*\mathcal{E})^{(p^r)}(-f^{-1}D)$$

is ample by proposition 3.31. If f is assumed surjective, the converse holds by proposition 3.31 again. □

We explain the relationship between (φ, D)-ampleness and other positivity notion.

Proposition 3.55. *Let $\mathcal{E}$ be a vector bundle on X. Then,*

$$\begin{array}{ccccc} & & \mathcal{E} \text{ is } L\text{-big} & & \\ & & \Uparrow & & \\ \mathcal{E} \text{ is ample} & \Longrightarrow & \mathcal{E} \text{ is } (\varphi, D)\text{-ample} & \Longrightarrow & \mathcal{E} \text{ is nef} \\ & & \Downarrow & & \\ & & \mathcal{E}^{(p^r)} \text{ is } V\text{-big for some } r \geq 1 & & \end{array}$$

[5]By [**Sta21**, Lemma 02OO], it is is the case when $f(x) \notin D$ for any weakly associated point x of X or when f is flat.

Proof. The first implication follows directly from [**Bar71**, Proposition 3.1]. Now, assume that $\mathcal{E}$ is (φ, D)-ample and consider the universal line bundle $\mathcal{O}(1)$ on the projective bundle $\mathbb{P}(\mathcal{E})$. We have a surjective map $\pi^* \mathcal{E} \twoheadrightarrow \mathcal{O}(1)$ and since (φ, D)-ampleness is stable under quotient by proposition 3.51, $\mathcal{O}(1)$ is in particular nef and big by proposition 3.46. It shows that $\mathcal{E}$ is nef and L-big. Take $r \geq 1$ such that $\mathcal{E}^{(p^r)}(-D)$ is ample. We deduce that there is an integer $n \geq 1$ such that

$$\mathrm{Sym}^n(\mathcal{E}^{(p^r)}(-D)) \otimes \mathcal{O}_X(-1) = \mathrm{Sym}^n(\mathcal{E}^{(p^r)})(-nD) \otimes \mathcal{O}_X(-1)$$

is globally generated. Since $\mathrm{Sym}^n(\mathcal{E}^{(p^r)}) \otimes \mathcal{O}_X(-1)$ can be expressed as a tensor product of a globally generated vector bundle with $\mathcal{O}_X(nD)$, it is globally generated on the complementary open subset of the support of D. It implies that the augmented base locus of $\mathcal{E}^{(p^r)}$ is not equal to X, i.e. that $\mathcal{E}^{(p^r)}$ is V-big. □

3.4. Nef vector bundles

For the convenience of the reader, we recall some important properties of nef vector bundles.

Proposition 3.56 (Barton-Kleiman criterion)**.** *A vector bundle $\mathcal{E}$ on X is nef if and only if for all finite morphism $g : C \to X$ where C is a smooth projective curve, we have*

$$\delta(g^* \mathcal{E}) \geq 0$$

Recall that $\delta(g^* \mathcal{E})$ is the minimum of the degrees of quotient line bundles of $g^* \mathcal{E}$.

Proof. See [**Laz04b**, Proposition 6.1.16.]. □

All the stability properties for ample vector bundles are also valid for nef vector bundles.

Proposition 3.57 ([**Laz04b**, Proposition 6.2.12.])**.** *We have the following assertions*

(1) Let $\mathcal{E}$ and $\mathcal{E}'$ be two nef vector bundles on X. Then $\mathcal{E} \oplus \mathcal{E}'$ is nef.

(2) Let $\mathcal{E} \twoheadrightarrow \mathcal{E}'$ be a surjective morphism of $\mathcal{O}_X$-modules between two vector bundles. If $\mathcal{E}$ is nef, then $\mathcal{E}'$ is also nef.

(3) Let $\mathcal{E}$ be a nef vector bundle and $\mathcal{L}$ an ample line bundle. Then, the bundle $\mathcal{E} \otimes \mathcal{L}$ is ample.

(4) The tensor product of nef vector bundles over X is nef.

(5) Consider an extension of vector bundles on X

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are nef. Then $\mathcal{E}$ is nef.

(6) Let $f : Y \to X$ be a morphism of projective schemes and $\mathcal{E}$ be an nef vector bundle on X. If $\mathcal{E}$ is nef on X, then $f^* \mathcal{E}$ is nef on Y. If furthermore f is assumed surjective, then the converse holds.

(7) Let $n \geq 1$ be an integer. A vector bundle $\mathcal{E}$ is nef if and only $\mathrm{Sym}^n \mathcal{E}$ is nef.

Remark on the proof. The properties (2) and (6) are direct consequences of the corresponding facts for line bundles. The property (3) can be shown directly with the Barton-Kleiman criterion. The other properties are a consequence of [**Laz04b**, Proposition 6.2.11.] and the analogue properties for ampleness from section 3.2. In [**Laz04b**, Remark 6.2.12.], Hartshorne writes that we do not know if exterior powers of nef bundles are nef in characteristic p but we think it must be true because an exterior power $\Lambda^n \mathcal{E}$ is just a quotient $\mathcal{E}^{\otimes n} / \mathcal{I}$

where $\mathcal{I}$ is the ideal generated by tensors of the form $x_1 \otimes \cdots \otimes x_n$ such that $x_i = x_j$ for some $i \neq j$. However, this argument does not work with divided powers Γ^n as their definition involves taking invariants. $\square$

3.5. Pushforward of positive line bundles

Let D be an effective Cartier divisor over X, E be a G-torsor over X and write $\pi : Y \to X$ for the flag bundle of type B of E. We write also D for the Cartier divisor $\pi^{-1}(D)$ on Y. Recall that we have fixed an ample line bundle $\mathcal{O}_X(1)$ on X and that we write $\mathcal{F}(m)$ instead of $\mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes m}$ for any coherent sheaf $\mathcal{F}$ on X and integer m. We start this section with some preliminary results.

Lemma 3.58. *Consider a finite surjective morphism $g : G' \to G$ of algebraic groups with central kernel. Then there exists a projective scheme X' and a finite surjective morphism $f : X' \to X$ such that the pullback of G-torsor f^*E reduces to a G'-torsor on X'.*

Proof. Let us denote $BG = \lfloor \operatorname{Spec} k/G \rfloor$ and $BG' = \lfloor \operatorname{Spec} k/G' \rfloor$ the classifying stacks of G and G'. The G-torsor E on X corresponds to a map $b_E : X \to BG$ and $g : G' \to G$ induces a map $b_g : BG' \to BG$ on the classifying stacks. Let K denote the kernel of g. We consider the following cartesian product

$$\begin{tikzcd}
X' \arrow[dr, dashed, "h"] \arrow[ddr, dashed, bend right, "\alpha \circ h"'] & & \\
& \mathcal{X} \arrow[r, "\beta"] \arrow[d, "\alpha"] & BG' \arrow[d, "b_g"] \\
& X \arrow[r, "b_E"] & BG
\end{tikzcd}$$

in the category of Artin stacks over k and the objective is now to prove that there exists a scheme X' over k and a morphism $h : X' \to \mathcal{X}$ such that $\alpha \circ h : X' \to X$ is finite surjective. The first step is to show that α is quasi-finite, proper[6] and surjective. By base change along b_E, it is enough to show it for b_g. Since K is central, we have the following cartesian product

$$\begin{tikzcd}
BK \arrow[r] \arrow[d] & BG' \arrow[d, "b_g"] \\
\operatorname{Spec} k \arrow[r, "b_G"] & BG
\end{tikzcd}$$

where b_G is the classifying map of the trivial G-torsor on $\operatorname{Spec} k$. We claim the map $BK \to \operatorname{Spec} k$ is proper, quasi-finite and surjective. The only non-trivial part is to show that $BK \to \operatorname{Spec} k$ is separated, i.e. that its diagonal is proper. We have the following cartesian product

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\
BK \arrow[r] & BK \times_k BK
\end{tikzcd}$$

and since K is finite, the diagonal $BK \to BK \times_k BK$ is also finite by faithfully flat descent. By faithfully flat descent along b_G, it implies that the map b_g is quasi-finite, proper and surjective. The second step is to find a finite surjective morphism $h : X' \to \mathcal{X}$ approximating the Artin stack $\mathcal{X}$. By [**Ryd15**, Theorem B], we have to check that the diagonal of $\alpha : \mathcal{X} \to X$

[6] See [**Ols16**, Sect. 10.1] for a reference on properness for non-representable morphisms of stacks.

is quasi-finite and separated. By base change and faithfully flat descent, it follows from the fact that the diagonal of $BK \to \operatorname{Spec} k$ is finite. Combining the two steps, the composition $\alpha \circ h$ is finite surjective. □

We give a sufficient cohomological condition for a vector bundle to be ample.

Proposition 3.59. *Let $\mathcal{E}$ be a vector bundle over X. Let $\lambda \in X^*$ be a character. If for all coherent sheaf $\mathcal{F}$, there is an integer r_0 such that*

$$H^i(X, \mathcal{F} \otimes \mathrm{St}_r \otimes \mathcal{W}_\lambda^{(p^r)}) = 0$$

for all $i > 0$ and $r \geq r_0$, then $\mathcal{W}_\lambda$ is ample.

Proof. We consider a coherent sheaf $\mathcal{F} = \mathcal{O}_X(-m)$ with $m \geq 0$ and we write $\mathcal{G}_r = \mathrm{St}_r \otimes \mathcal{W}_\lambda^{(p^r)} \otimes \mathcal{O}_X(-m)$. Let $x \in X$ be a closed point. From our hypothesis, there is an integer r_0 such that

$$H^1(X, \mathcal{G}_{r_0} \otimes \mathcal{I}_x) = 0$$

where $\mathcal{I}_x$ is the ideal sheaf defining the closed point x. From the long exact sequence of cohomology associated to the exact sequence

$$0 \longrightarrow \mathcal{G}_{r_0} \otimes \mathcal{I}_x \longrightarrow \mathcal{G}_{r_0} \longrightarrow \mathcal{G}_{r_0} \otimes k(x) \longrightarrow 0$$

we deduce that the map

$$H^0(X, \mathcal{G}_{r_0}) \to H^0(X, \mathcal{G}_{r_0} \otimes k(x))$$

is surjective. In other words, $\mathcal{G}_{r_0}$ is globally generated at x. It implies there exists an open U containing x such that $\mathcal{G}_{r_0}$ is globally generated over U. Since St_{r_0} is self dual, there is a canonical surjective map

$$\mathrm{St}_{r_0}^{\otimes 2} \to \mathcal{O}_X\,.$$

Since the tensor product of globally generated sheaves over U is again globally generated over U, we deduce that

$$\mathcal{G}_{r_0}^{\otimes 2} = \mathrm{St}_{r_0}^{\otimes 2} \otimes (\mathcal{W}_\lambda^{\otimes 2})^{(p^{r_0})} \otimes \mathcal{O}_X(-2m)$$

is a globally generated over U. Since the quotient of a globally generated sheaf over U is globally generated over U, we know that

$$(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r_0})}(-2m)$$

is globally generated sheaf over U. Now, let $r \geq r_0$ be an integer. From the equality

$$(\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-2p^{r-r_0}m) = \Big((\mathcal{W}_\lambda^{\otimes 2})^{(p^{r_0})}(-2m)\Big)^{(p^{r-r_0})}$$

we deduce that $(\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-2p^{r-r_0}m)$ is globally generated over U. Now take r_1 large enough to have $\mathcal{O}_X((2p^{r_1-r_0}-1)m)$ globally generated. We deduce that

$$(\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-2p^{r-r_0}m) \otimes \mathcal{O}_X((2p^{r-r_0}-1)m) = (\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-m)$$

is globally generated over U for all $r \geq r_1$. Since X is quasi-compact, we can find an integer $r_2 \geq r_1$ such that $(\mathcal{W}_\lambda^{\otimes 2})^{(p^r)}(-m)$ is globally generated over X for all $r \geq r_2$. We use the proposition 3.35 to deduce that $\mathcal{W}_\lambda^{\otimes 2}$ is p-ample. By proposition 3.36 and 3.30, it implies that $\mathcal{W}_\lambda$ is ample. □

We give a sufficient cohomological condition for a vector bundle to be (φ, D)-ample.

Proposition 3.60. *Let $\mathcal{E}$ be a vector bundle over X. Let $\lambda \in X^*(T)$ be a character. If there exists an effective Cartier divisor D and an integer $r_0 \geq 1$ such that for all $r \geq r_0$ and all coherent sheaf $\mathcal{F}$ over X, there is an integer $r_1 \geq 1$ such that for all $r' \geq r_1$ and $i > 0$, we have*

$$H^i(X, \mathcal{F} \otimes \mathrm{St}_{r+r'} \otimes \mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}D)) = 0,$$

then $\mathcal{W}_\lambda$ is (φ, D)-ample.

Proof. Using proposition 3.35 and proposition 3.36, it is sufficient to see that there exists integers $n \geq 1$ and $r_0 \geq 1$ such that for all $r \geq r_0$, the bundle $\mathcal{W}_\lambda^{(p^r)}(-nD)$ is p-ample. In other words, it is sufficient to see that there exists integers $n \geq 1$ and $r_0 \geq 1$ such that for all $r \geq r_0$ and all $m \geq 1$, there exists $r_1 \geq 1$, such that for all $r' \geq r_1$, the bundle

$$\mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}nD)(-m)$$

is globally generated over X. By hypothesis, we have a Cartier divisor D and $r_0 \geq 1$ an integer. Consider two integers $r \geq r_0$, $m \geq 0$ and write

$$\mathcal{G}_{r'} = \mathrm{St}_{r+r'} \otimes \mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}D) \otimes \mathcal{O}_X(-m).$$

Let $x \in X$ be a closed point. By hypothesis, we have an integer $r_1 \geq 1$ such that

$$H^1(X, \mathcal{G}_{r_1} \otimes \mathcal{I}_x) = 0$$

where $\mathcal{I}_x$ is the ideal sheaf defining the closed point x. From the long exact sequence of cohomology associated to the exact sequence

$$0 \longrightarrow \mathcal{G}_{r_1} \otimes \mathcal{I}_x \longrightarrow \mathcal{G}_{r_1} \longrightarrow \mathcal{G}_{r_1} \otimes k(x) \longrightarrow 0$$

we deduce that the map

$$H^0(X, \mathcal{G}_{r_1}) \to H^0(X, \mathcal{G}_{r_1} \otimes k(x))$$

is surjective. In other words, $\mathcal{G}_{r_1}$ is globally generated at x. It implies there exists an open U containing x such that $\mathcal{G}_{r_1}$ is globally generated over U. Since the Steinberg module is self dual, there is a canonical surjective map

$$\mathrm{St}_{r+r_1}^{\otimes 2} \to \mathcal{O}_X\,.$$

Since the tensor product of globally generated sheaves over U is again globally generated over U, we deduce that

$$\mathcal{G}_{r_1}^{\otimes 2} = \mathrm{St}_{r+r_1}^{\otimes 2} \otimes (\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r_1})}(-2p^{r_1}D) \otimes \mathcal{O}_X(-2m)$$

is globally generated over U. Since the quotient of a globally generated sheaf over U is globally generated over U, we know that

$$(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r_1})}(-2p^{r_1}D) \otimes \mathcal{O}_X(-2m)$$

is a globally generated sheaf over U. From the equality

$$\Big((\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r_1})}(-2p^{r_1}D)(-2m)\Big)^{(p^{r'-r_1})} = (\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}(-2p^{r'}D)(-2p^{r'-r_1}m)$$

we deduce that $(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}(-2p^{r'}D)(-2p^{r'-r_1}m)$ is globally generated over U. Now take $r_2 \geq r_1$ large enough to have $\mathcal{O}_X((2p^{r'-r_1}-1)m)$ globally generated for all $r' \geq r_2$. We deduce that

$$(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}(-2p^{r'}D)(-2p^{r'-r_1}m) \otimes \mathcal{O}_X((2p^{r'-r_1}-1)m) = (\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}(-2p^{r'}D)(-m)$$

is globally generated over U for all $r' \geq r_2$. Since X is quasi-compact, we can find an integer $r_3 \geq r_2$ such that $(\mathcal{W}_\lambda^{\otimes 2})^{(p^{r+r'})}(-2p^{r'}D)(-m)$ is globally generated over X for all $r' \geq r_3$. In conclusion, we have proven that $\mathcal{W}_\lambda^{\otimes 2}$ is $(\varphi, 2D)$-ample, which is equivalent to (φ, D)-ample by proposition 3.48. Then, we use the proposition 3.53 to deduce that $\mathcal{W}_\lambda$ is (φ, D)-ample. □

Theorem 3.61. *Let $\lambda \in X^*(T)$ be character. If $\mathcal{L}_{2\lambda+2\rho}$ is ample on Y, then $\mathcal{W}_\lambda$ is ample on X.*

Proof. Consider a semi-simple cover of G, i.e. a finite surjective morphism $h : G' \to G$ of reductive groups with central kernel such that $G' = G_{\mathrm{sc}} \times T_1$ is a product of a semisimple simply-connected group G_{sc} with a torus T_1. Since ampleness can be tested after a pullback by a finite surjective morphism by proposition 3.31, we can use lemma 3.58 to assume that ρ is a genuine character. Assume that $\mathcal{L}_{\lambda+\rho}$ is ample and consider a coherent sheaf $\mathcal{F}$ on X. We have a Leray spectral sequence starting at second page

$$E_2^{i,j} = H^i(X, \mathcal{F} \otimes R^j\pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho})) \Rightarrow H^{i+j}(Y, \pi^*\mathcal{F} \otimes \mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho})$$

Since $\mathcal{L}_{\lambda+\rho}$ is ample on Y, it is also π-ample and we have

$$R^j\pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho}) = 0$$

for all $j > 0$ and r large enough. We deduce that the spectral sequence degenerates at page 2 and we get isomorphisms:

$$H^i(X, \mathcal{F} \otimes \pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho})) = H^i(Y, \pi^*\mathcal{F} \otimes \mathcal{L}_{\lambda+\rho}^{\otimes p^r} \otimes \mathcal{L}_{-\rho})$$

for all $i \geq 0$ and r large enough. Moreover, since $\mathcal{L}_{\lambda+\rho}$ is ample, the right hand side vanishes for $i > 0$ and r large enough. From the proposition 2.10, we know that

$$\pi_*(\mathcal{L}_{p^r(\lambda+\rho)-\rho}) = \mathrm{St}_r \otimes \mathcal{W}_\lambda^{(p^r)}$$

and from proposition 3.59, we deduce that $\mathcal{W}_\lambda$ is ample. □

Theorem 3.62. *Let $\lambda \in X^*(T)$ be character. If $\mathcal{L}_{2\lambda+2\rho}$ is D-ample over Y, then $\mathcal{W}_\lambda$ is (φ, D)-ample on X.*

Proof. Since (φ, D)-ampleness can be tested after a pullback by a finite surjective morphism by proposition 3.54, we use the same trick as in theorem 3.61 to assume that ρ is a genuine character. Consider $r_0 \geq 1$ large enough such that $\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D)$ is ample for all $r \geq r_0$. Let $\mathcal{F}$ be a coherent sheaf on X and $r \geq r_0$ integer. For all integer $r' \geq 1$, we have a Leray spectral sequence starting at second page

$$E_2^{i,j} = H^i(X, \mathcal{F} \otimes R^j\pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D) \otimes \mathcal{L}_{-\rho})) \Rightarrow H^{i+j}(Y, \pi^*\mathcal{F} \otimes \mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D) \otimes \mathcal{L}_{-\rho}).$$

Since $\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D)$ is π-ample, there is a $r_1 \geq 1$ large enough such that

$$R^j\pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D) \otimes \mathcal{L}_{-\rho}) = R^j\pi_*((\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D))^{\otimes p^{r'}} \otimes \mathcal{L}_{-\rho}) = 0$$

for all $j > 0$ and $r' \geq r_1$. We deduce that the spectral sequence degenerates at page 2 and we get isomorphisms:

$$H^i(X, \mathcal{F} \otimes \pi_*(\mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D) \otimes \mathcal{L}_{-\rho})) = H^i(Y, \pi^*\mathcal{F} \otimes \mathcal{L}_{\lambda+\rho}^{\otimes p^{r+r'}}(-p^{r'}D) \otimes \mathcal{L}_{-\rho})$$

for all $i \geq 0$ and $r' \geq r_1$. Since $\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D)$ is ample on Y, there exists $r_2 \geq r_1$ such that we have

$$H^i(Y, \pi^* \mathcal{F} \otimes (\mathcal{L}_{\lambda+\rho}^{\otimes p^r}(-D))^{\otimes p^{r'}} \otimes \mathcal{L}_{-\rho}) = 0$$

for all $i > 0$ and $r' \geq r_2$. From proposition 2.10, we know that

$$\pi_*(\mathcal{L}_{p^{r+r'}(\lambda+\rho)-\rho}(-p^{r'}D)) = \mathrm{St}_{r+r'} \otimes \mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}D)$$

which implies that

$$H^i(X, \mathcal{F} \otimes \mathrm{St}_{r+r'} \otimes \mathcal{W}_\lambda^{(p^{r+r'})}(-p^{r'}D)) = 0$$

for all $r' \geq r_2$. We deduce with the technical proposition 3.60 that $\mathcal{W}_\lambda$ is (φ, D)-ample. $\square$

CHAPTER 4

Siegel varieties and their automorphic bundles

Shimura varieties belong to a certain class of locally symmetric Hermitian spaces which admit a canonical model over a number field $E/\mathbb{Q}$. Siegel varieties are exactly the Shimura varieties associated to the symplectic group $G = \mathrm{Sp}_{2g}$ where $g \geq 1$ is an integer called the genus and the first example is the Siegel variety of genus $g = 1$, also known as the modular curve. We can define the modular curve as a quotient space

$$Y_N = \mathcal{H}/\Gamma_1(N)$$

where $N \geq 5$ is an integer and

$$\Gamma_1(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}$$

acts on the Poincaré half plane $\mathcal{H}$ through the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Defined this way, Y_N is only a Riemann surface but since it can be shown that it is also a moduli space of complex elliptic curves with $\Gamma_1(N)$-level structure, it is natural to define the same moduli problem over a more general ring (such as $\mathbb{Z}$ or $\mathbb{Q}$) and it will give us an algebraic model of the complex modular curve. In fact, we have to restrict to a ring over $\mathbb{Z}[\frac{1}{N}]$ to make sense of the $\Gamma_1(N)$-level structure. The modular curve Y_N is now a scheme of relative dimension 1 over $\mathbb{Z}[\frac{1}{N}]$ and it is not proper but it admits a minimal compactification X_N of Y_N by adding a finite number of points named cusps. The boundary $D = X_N - Y_N$ is then a Cartier divisor in X_N. On X_N lives a semi-abelian scheme

$$\begin{tikzcd} E \arrow[r, "\pi"'] & X_N \arrow[l, bend right, "e"'] \end{tikzcd}$$

that extends the universal elliptic curve on Y_N. We define the Hodge bundle

$$\Omega := e^*\Omega^1_{E/X_N}$$

which is a locally free sheaf of rank 1. Let R be a $\mathbb{Z}[\frac{1}{N}]$-module. We define modular forms of weights $k \geq 0$ and level N with coefficients in R to be the sections in

$$M(k, N, R) := H^0(X_N, \Omega^{\otimes k} \otimes_{\mathbb{Z}[\frac{1}{N}]} R)$$

and cusps forms $S(k, N, R) \subset M(k, N, R)$ to be modular forms that vanishes at the cusps. It makes sense to try to compare the abelian groups

$$M(k, N, R) \text{ and } M(k, N, \mathbb{Z}[\frac{1}{N}]) \otimes_{\mathbb{Z}[\frac{1}{N}]} R.$$

Since cohomology commutes with filtered colimits, we are reduced to the case where $R = \mathbb{F}_p$ with $p \nmid N$. Through standard base change theorem for coherent cohomology, the question

can be answered positive if we have

$$H^1((X_N)_{\mathbb{F}_p}, \Omega^{\otimes k}) = 0$$

where $(X_N)_{\mathbb{F}_p} = X_N \times_{\mathbb{Z}[\frac{1}{N}]} \mathbb{F}_p$. In fact Ω is ample on X_N for $N \geq 5$, so we must have the vanishing for k large enough by Serre's vanishing theorem. With a more careful approach, Katz shows the following

Theorem ([**Kat73**, Theorem 1.7.1])**.** *Assume that $N \geq 5$. If either $k \geq 2$ or $k = 1$ and $N \leq 11$, then $H^1((X_N)_{\mathbb{F}_p}, \Omega^{\otimes k}) = 0$.*

Modular curves are Shimura varieties corresponding to the reductive group SL_2 and we already see that a vanishing result of the H^1 over the special fiber has an application to the base change of modular forms. In this chapter, we introduce the general Siegel variety of genus g with their automorphic vector bundles, which is the main object of study of this thesis. For the reader who is familiar with the case of modular curves, we provide the following analogy table.

	Modular curve	Siegel variety
Group	$G = \mathrm{SL}_2$	$G = \mathrm{Sp}_{2g}$
Dimension	1	$g(g+1)/2$
Universal cover	Poincaré upper half-plane	Siegel upper half-space of genus g
Level	$K \subset \mathrm{SL}_2(\mathbb{A}_f)$	$K \subset \mathrm{Sp}_{2g}(\mathbb{A}_f)$
Compactification	Minimal = Toroïdal	Minimal = singular Toroidal = can be chosen smooth but not unique
Moduli space	Parametrizes elliptic curves with level structure	Parametrizes abelian varieties of dimension g with polarization and level structure
Hodge bundle	Rank 1	Rank g
Automorphic vector bundles	$\Omega^{\otimes k}$, $k \in \mathbb{Z}$	$\nabla(\lambda)$, $\Delta(\lambda)$, $T(\lambda)$, $\lambda \in \mathbb{Z}^g$
Positivity of the Hodge bundle	Ω is ample	Ω is never nef when $g \geq 2$

Table 4.0.1. Analogy table between modular curves and Siegel varieties.

4.1. Siegel varieties and their toroidal compactifications

In this section, we follow [**FC90**] for generalities about Siegel varieties. We denote by Sch_R the category of schemes over a ring R and $\mathbb{A}_f$ the finite adeles of $\mathbb{Q}$.

Definition 4.1. *Let A and A' be abelian schemes of relative dimension g over a scheme S. A quasi-isogeny $A \to A'$ is an equivalence class of pairs (α, N) where $\alpha : A \to A'$ is an isogeny*

over S and N is a positive integer with the relation

$$(\alpha, N) \sim (\alpha', N') \text{ if and only if } N'\alpha = N\alpha'.$$

Definition 4.2. *Let V be the $\mathbb{Z}$-module $\mathbb{Z}^{2g}$ endowed with the standard non-degenerate symplectic pairing*

$$\begin{aligned} \psi : V \times V &\longrightarrow \mathbb{Z} \\ (x, y) &\longmapsto {}^t x J y \end{aligned}$$

where

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

We denote Sp_{2g} the algebraic group over $\mathbb{Z}$ of $2g \times 2g$ matrices M that preserve the symplectic pairing ψ, i.e. such that

$${}^t M J M = J.$$

In the following proposition, we define the Siegel variety of level K as a scheme over $\mathbb{Z}_p$ when the level is small enough and p is a prime such that $K_p = \mathrm{Sp}_{2g}(\mathbb{Z}_p)$.

Proposition 4.3 ([FC90]). *Let $K \subset \mathrm{Sp}_{2g}(\mathbb{A}_f)$ be an open compact subgroup and assume that p is prime number such that $K_p = \mathrm{Sp}_{2g}(\mathbb{Z}_p)$ is hyperspecial. Consider the fibered category in groupoids $\mathcal{A}_{g,K}$ on $\mathrm{Sch}_{\mathbb{Z}_p}$ whose S-points are groupoids with*

- *Objects: $(A/S, \lambda, \psi)$ where A/S is abelian scheme over S of relative dimension g, $\lambda : A \to A^\vee$ is a $\mathbb{Z}_{(p)}$-multiple of a principal polarization and for all primes $l \neq p$ and all geometric points $s \in S$, ψ_l is a K_l-orbit of symplectic* isomorphisms *from $H_1(A_s, \mathbb{Q}_l)$ to $V \otimes \mathbb{Q}_l$ which is invariant under $\pi_1(S, s)$. The structure of symplectic $\mathbb{Q}_l$-vector space on the l-adic étale homology group $H_1(A_s, \mathbb{Q}_l)$ (it is also the rational Tate module of A_s) is the one induced by the polarization (which is an isomorphism since we tensor by $\mathbb{Q}_l$) and the Weil paring.*
- *Morphisms: A morphism $(A/S, \lambda, \psi) \to (A'/S, \lambda', \psi')$ is a quasi-isogeny $\alpha : A \to A'$ over S such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\lambda"] & A' \arrow[d, "\lambda'"] \\ A^\vee & A'^\vee \arrow[l, "\alpha^\vee"] \end{tikzcd}$$

  is commutative up to a constant in $\mathbb{Z}_{(p)}$ and the pullback of ψ_l by the quasi-isogeny α is ψ'_l.

If the level away from p, K^p, is small enough[1]*, then $\mathcal{A}_{g,K}$ is representable by a smooth integral quasi-projective scheme over $\mathbb{Z}_p$.*

Remark 4.4. We often prefer the GSp_{2g} group that fits into the PEL framework because it yields a morphism $\mathrm{Sh} \to \operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$ with geometrically connected fibers. That said, we choose the Sp_{2g} group to have the Levi subgroup GL_g (and not $\mathbb{G}_m \times \mathrm{GL}_g$) when defining automorphic vector bundles. As Sp_{2g} allows us to access the geometric connected components of the Siegel variety for GSp_{2g}, this does not change anything for our results on cohomology vanishing, (phi, D)-ampleness of automorphic bundles and hyperbolicity.

[1] It is the case in particular when K is the kernel of the reduction map $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ with $N \geq 3$ such that $p \nmid N$.

Remark 4.5. Without the hypothesis on the smallness of K, $\mathcal{A}_{g,K}$ is only a Deligne-Mumford stack over $\mathbb{Z}_p$.

Notation 4.6. We denote G the base change of the algebraic group Sp_{2g} over $\mathbb{Z}_p$. We fix a genus $g \geq 1$ and a level $N \geq 3$ such that $p \nmid N$. We denote simply Sh the Siegel modular variety $\mathcal{A}_{g,K}$ where K is the kernel of the reduction map $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let μ be the following minuscule cocharacter of G

$$\begin{aligned} \mu : \mathbb{G}_m &\longrightarrow G \\ z &\longmapsto \begin{pmatrix} zI_g & 0 \\ 0 & z^{-1}I_g \end{pmatrix}. \end{aligned}$$

We denote $P^+ := P_\mu$ and $P := P_{-\mu}$ the associated opposite parabolic subgroups with common Levi subgroup $L = \mathrm{GL}_g$ over $\mathbb{Z}_p$. We denote $B \subset P$ the Borel of upper triangular matrices in $G = \mathrm{Sp}_{2g}$ over $\mathbb{Z}_p$. We denote Φ_L (resp. Φ_L^+) the corresponding roots of L (resp. positive roots of L).

Definition 4.7 ([FC90])**.** *As a fine moduli space, the Siegel variety* Sh *is endowed with a universal principally polarized abelian scheme of relative dimension g*

$$\begin{tikzcd} A \arrow[r, "f"] & \mathrm{Sh} \arrow[l, bend left, "e"] \end{tikzcd}$$

where $e : \mathrm{Sh} \to A$ is the neutral section. Recall the following associated objects on Sh*.*

(1) We denote $\mathcal{H}^1_{\mathrm{dR}} := R^1 f_*(\Omega^\bullet_{A/\mathrm{Sh}})$ the de Rham cohomology vector bundle of rank $2g$ over Sh*.*

(2) We denote $\Omega = e^*\Omega^1_{A/\mathrm{Sh}}$ the Hodge vector bundle of rank g over Sh*.*

Note that the Weil paring and the principal polarization on the universal abelian scheme $f : A \to \mathrm{Sh}$ induce a symplectic pairing of the same type as ψ on $\mathcal{H}^1_{\mathrm{dR}}$. In other words, the de Rham cohomology is equivalent to the data of a G-torsor on Sh*.*

Proposition 4.8 ([DI87])**.** *The Hodge-de Rham spectral sequence*

$$E_1^{i,j} = R^j f_*(\Omega^i_{A/\mathrm{Sh}}) \Rightarrow R^{i+j} f_*(\Omega^\bullet_{A/\mathrm{Sh}})$$

degenerates at page 1 which proves the existence of the Hodge-de Rham filtration

$$0 \longrightarrow \Omega \longrightarrow \mathcal{H}^1_{\mathrm{dR}} \longrightarrow R^1 f_* \mathcal{O}_A \longrightarrow 0.$$

Moreover, the Hodge bundle Ω is totally isotropic for the symplectic pairing on $\mathcal{H}^1_{\mathrm{dR}}$ which implies that the Hodge-de Rham filtration is equivalent to the data of a P-reduction of the G-torsor $\mathcal{H}^1_{\mathrm{dR}}$ on the Siegel variety.

Remark on the proof. The Hodge filtration comes from the degeneration at second page of the Hodge-de Rham spectral sequence over $\mathbb{Z}_p$. Over $\mathbb{Q}_p$, it is a consequence of the classical Hodge theory. Over $\mathbb{F}_p$, it is a result of Deligne and Illusie [DI87]. The module Ω is locally free of rank g because $\pi : A \to \mathrm{Sh}$ is smooth. Actually, we also have an isomorphism

$$\Omega \simeq \pi_* \Omega^1_{A/\mathrm{Sh}}.$$

Indeed, as a group scheme π satisfies

$$\Omega^1_{A/\mathrm{Sh}} = \pi^* e^* \Omega^1_{A/\mathrm{Sh}}$$

and for any proper morphism $f : X \to Y$ with geometrically connected fibers, we have

$$f_*\mathcal{O}_X = \mathcal{O}_Y.$$

From the projection formula, we deduce

$$\pi_*\Omega^1_{A/\operatorname{Sh}} = \pi_*(\pi^* e^*\Omega^1_{A/\operatorname{Sh}} \otimes \mathcal{O}_A) = e^*\Omega^1_{A/\operatorname{Sh}} \otimes \pi_*\mathcal{O}_A = \Omega^1_{A/\operatorname{Sh}}. \qquad \square$$

In the next definition, we recall the main properties of toroidal compactifications of Siegel varieties.

Definition 4.9 ([**FC90**, Chapter 4][**Lan12**, Th. 2.15])**.** *Let C be the cone of all positive semi-definite symetric bilinear forms on $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ whose radicals are defined over $\mathbb{Q}$. Let $\Sigma = \{\sigma_\alpha\}_\alpha$ be a smooth $GL(X^*(T))$-admissible decomposition in polyhedral cones of C as defined in* [**FC90**, Chapter 4, Definition 2.2/2.3]*. We assume that Σ admits a $GL(X^*(T))$-equivariant polarization function as defined in* [**FC90**, Chapter 4, Definition 2.4]*. See* [**AMRT10**] *or* [**KKMSD73**] *for a proof of the existence of such polyhedral cone decompositions. We consider the corresponding toroidal compactification $\operatorname{Sh}^{\operatorname{tor},\Sigma}$ of the Siegel variety Sh. It is a smooth and projective scheme over $\mathbb{Z}_p$ which satisfies the following properties.*

(1) The complementary $D_{\operatorname{red}} = \operatorname{Sh}^{\operatorname{tor},\Sigma} - \operatorname{Sh}$, when endowed with its reduced structure, is a Cartier divisor with normal crossings.

(2) The universal abelian scheme $f : A \to \operatorname{Sh}$ extends to a semi-abelian scheme $f^{\operatorname{tor}} : A^{\operatorname{tor}} \to \operatorname{Sh}^{\operatorname{tor}}$ such that

(3) $\Omega^{\operatorname{tor}} := e^*\Omega^1_{A^{\operatorname{tor}}/\operatorname{Sh}^{\operatorname{tor},\Sigma}}$ is a vector bundle that extends the Hodge bundle to $\operatorname{Sh}^{\operatorname{tor},\Sigma}$.

(4) By [**FC90**, Chapter 4] *or* [**Lan12**, Th. 2.15, (2)] *the semi-abelian scheme $f^{\operatorname{tor}} : A^{\operatorname{tor}} \to \operatorname{Sh}^{\operatorname{tor}}$ can be compactified into a proper and log-smooth scheme $\bar{f}^{\operatorname{tor}} : \bar{A}^{\operatorname{tor}} \to \operatorname{Sh}^{\operatorname{tor}}$ which is projective and smooth over $\mathbb{F}_p$*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & A^{\operatorname{tor}} \arrow[r] \arrow[d, "f^{\operatorname{tor}}"] & \bar{A}^{\operatorname{tor}} \arrow[d] \arrow[dl, "\bar{f}^{\operatorname{tor}}"] \\
\operatorname{Sh} \arrow[r] & \operatorname{Sh}^{\operatorname{tor}} \arrow[r] & \operatorname{Spec}\mathbb{F}_p
\end{tikzcd}$$

and we denote again D_{red} the normal crossing divisor $\bar{A}^{\operatorname{tor}} - A$.

(5) Following [**FC90**, Chapter 4] *or* [**Lan12**, Th. 2.15, (3)]*, the log-de Rham complex $\bar{\Omega}^\bullet_{\bar{A}^{\operatorname{tor}}/\operatorname{Sh}^{\operatorname{tor}}}$ is the complex of log-differentials $\bar{\Omega}^i_{\bar{A}^{\operatorname{tor}}/\operatorname{Sh}^{\operatorname{tor}}} := \Lambda^i\bar{\Omega}^1_{\bar{A}^{\operatorname{tor}}/\operatorname{Sh}^{\operatorname{tor}}}$ where*

$$\bar{\Omega}^1_{\bar{A}^{\operatorname{tor}}/\operatorname{Sh}^{\operatorname{tor}}} = \Omega^1_{\bar{A}^{\operatorname{tor}}}(\log D_{\operatorname{red}})/(\bar{f}^{\operatorname{tor}})^*\Omega^1_{\operatorname{Sh}^{\operatorname{tor}}}(\log D_{\operatorname{red}}).$$

(6) and the log-de Rham cohomology

$$\mathcal{H}^1_{\log-\operatorname{dR}} := R^1(\bar{f}^{\operatorname{tor}})_*\bar{\Omega}^\bullet_{\bar{A}^{\operatorname{tor}}/\operatorname{Sh}^{\operatorname{tor}}}$$

is a Sp_{2g}-torsor that extends $\mathcal{H}^1_{\operatorname{dR}}$ on Sh.

(7) and the logarithmic Hodge-de Rham spectral sequence

$$E_1^{i,j} = R^j(\bar{f}^{\operatorname{tor}})_*\bar{\Omega}^i_{\bar{A}^{\operatorname{tor}}/\operatorname{Sh}^{\operatorname{tor}}} \Rightarrow \mathcal{H}^i_{\log-\operatorname{dR}} := R^i(\bar{f}^{\operatorname{tor}})_*\bar{\Omega}^\bullet_{\bar{A}^{\operatorname{tor}}/\operatorname{Sh}^{\operatorname{tor}}}$$

degenerates at page 1. It defines a P-reduction of the Sp_{2g}-torsor $\mathcal{H}^1_{\log-\operatorname{dR}}$ on $\operatorname{Sh}^{\operatorname{tor},\Sigma}$ that extends the Hodge fitlration on Sh.

From now on, we drop the upperscript Σ to denote $\operatorname{Sh}^{\operatorname{tor},\Sigma}$ since the coherent cohomology does not depend on this choice.

Definition 4.10 ([FC90]). *We define the minimal compactification as the projective scheme*

$$\mathrm{Sh}^{\min} := \mathrm{Proj}(\oplus_{n\geq 0} H^0(\mathrm{Sh}^{\mathrm{tor}}, \omega^{\otimes n})),$$

where $\omega = \det \Omega^{\mathrm{tor}}$ *is the Hodge line bundle.*

Proposition 4.11 ([MB85, Chap. IX, Theorem 2.1, p. 208]). *The Hodge line bundle* ω *is semi-ample on* $\mathrm{Sh}^{\mathrm{tor}}$*, i.e. there exists an integer* $m \geq 1$ *such that* $w^{\otimes m}$ *is globally generated over* $\mathrm{Sh}^{\mathrm{tor}}$*. In particular, the Hodge line bundle descends to an ample line bundle on the minimal compactification.*

Proposition 4.12 ([FC90, Chap. V, Theorem 5.8]). *The toroidal compactification* $\mathrm{Sh}^{\mathrm{tor}}$ *is the normalization of the blow-up of* $\mathrm{Sh}^{\min}$

$$\nu : \mathrm{Sh}^{\mathrm{tor}} \to \mathrm{Sh}^{\min}$$

along a coherent sheaf of ideals $\mathcal{I}$ *of* $\mathcal{O}_{\mathrm{Sh}^{\min}}$*.*

In particular, the pullback $\nu^*\mathcal{I}$ is of the form $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}(\text{-}D)$ where D is an effective Cartier divisor whose associated reduced Cartier divisor is the boundary D_{red}. It follows from the ampleness of ω on $\mathrm{Sh}^{\min}$ and the ν-ampleness of $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}(\text{-}D)$ that there exists $\eta_0 > 0$ such that $\omega^{\otimes\eta}(\text{-}D)$ is ample for every $\eta \geq \eta_0$. In other words, we have

Corollary 4.13. *The Hodge line bundle* $\omega = \det \Omega^{\mathrm{tor}}$ *is* D*-ample on the toroidal compactification* $\mathrm{Sh}^{\mathrm{tor}}$*.*

Remark 4.14. The effective Cartier divisor D appearing in the corollary obviously depends on the choice of the $GL(X^*)$-equivariant polarization function on the decomposition in polyhedral cones Σ.

4.2. Automorphic vector bundles

Recall that we consider the Siegel variety $\mathrm{Sh}^{\mathrm{tor}}$ defined over $\mathbb{Z}_p$. We define a collection of automorphic vector bundles over the Siegel variety. We choose an intermediary parabolic subgroup $P_0 \subset P$ of type $I_0 \subset I \subset \Delta$ and we denote $P_{0,L} := P_0 \cap L \subset L$ the parabolic subgroup of $L = \mathrm{GL}_g$. We denote ρ_L the half-sum of positive roots of L, $\rho^L = \rho - \rho_L$, $W_L \subset W$ the Weyl group of L and $w_{0,L}$ the element of maximal length in W_L.

Definition 4.15. *We define the flag bundle* $\pi : Y^{\mathrm{tor}}_{I_0} \to \mathrm{Sh}^{\mathrm{tor}}$ *of type* I_0 *as the flag bundle* $\mathcal{F}_{P_{0,L}}(\Omega^{\mathrm{tor}})$ *(as in definition 3.14) of type* $P_{0,L}$ *of the* L*-torsor* Ω^{tor}*.*

Definition 4.16. *From definitions 3.17 and 3.18, we have functors*

$$\mathcal{W} : \mathrm{Rep}(L) \to \mathrm{Loc}(\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}),$$
$$\mathcal{L} : \mathrm{Rep}(P_{0,L}) \to \mathrm{Loc}(\mathcal{O}_{Y^{\mathrm{tor}}_{I_0}})$$

and we call automorphic bundle any vector bundle in the essential image of these functors. Moreover, if λ *is a character of* P_0*, we denote* $\nabla(\lambda)$ *the automorphic vector bundle* $\mathcal{W}(\mathrm{Ind}^L_{P_{0,L}}\lambda)$ *on* $\mathrm{Sh}^{\mathrm{tor}}$ *and* $\mathcal{L}_\lambda$ *the automorphic line bundle* $\mathcal{L}(\lambda)$ *on* $Y^{\mathrm{tor}}_{I_0}$*. With our conventions the module* $\mathrm{Ind}^L_{P_{0,L}}\lambda$ *is isomorphic to the costandard representation of highest weight* $w_0 w_{0,L}\lambda$*. More generally, we define (costandard, standard and tilting) automorphic vector bundles of highest weight* λ

(1) $\nabla(\lambda) := \mathcal{W}(\nabla(w_0 w_{0,L}\lambda))$*,*
(2) $\Delta(\lambda) := \mathcal{W}(\Delta(w_0 w_{0,L}\lambda))$*,*
(3) $T(\lambda) := \mathcal{W}(T(w_0 w_{0,L}\lambda))$*.*

The following proposition explains the relationship between automorphic vector bundles and strict polynomial functors.

Proposition 4.17. *Let $\lambda = (0 \geq k_1 \geq k_2 \geq \ldots \geq k_g)$ be a dominant character of P_0. Recall from chapter 2 that we can see $w_0 w_{0,L}\lambda$ as an integer partition*

$$w_0 w_{0,L}\lambda = (\text{-}k_g \geq \text{-}k_{g-1} \geq \ldots \geq \text{-}k_1 \geq 0)$$

and that we have a Schur functor $S_{w_0 w_{0,L}\lambda}$, a Weyl functor $W_{w_0 w_{0,L}\lambda}$ and a tilting functor $T_{w_0 w_{0,L}\lambda}$ associated this partition. We have the following isomorphisms of vector bundles

(1) $S_{w_0 w_{0,L}\lambda}\Omega^{\mathrm{tor}} = \nabla(\lambda)$,
(2) $W_{w_0 w_{0,L}\lambda}\Omega^{\mathrm{tor}} = \Delta(\lambda)$,
(3) $T_{w_0 w_{0,L}\lambda}\Omega^{\mathrm{tor}} = T(\lambda)$.

Proof. This is a consequence of proposition 2.41. □

The following proposition explains how the costandard automorphic vector bundle $\nabla(\lambda)$ and the automorphic line bundle $\mathcal{L}_\lambda$ are related to each other.

Proposition 4.18. *Let λ be a dominant character of P_0. We have an isomorphism of (complex of) vector bundles (in the derived category of coherent sheaves)*

$$R\pi_* \mathcal{L}_\lambda = \nabla(\lambda)[0].$$

Proof. It is a direct consequence of proposition 3.22. □

Example 4.19. We have the following special cases.

(1) If $\lambda = (0, \ldots, 0, \text{-}1)$, then $\nabla(\lambda) = \Delta(\lambda) = T(\lambda) = \Omega^{\mathrm{tor}}$.
(2) If $\lambda = (0, \ldots, 0, \text{-}n)$ with $n \geq 1$, then $\nabla(\lambda) = \mathrm{Sym}^n \Omega^{\mathrm{tor}}$ and $\Delta(\lambda) = \Gamma^n \Omega^{\mathrm{tor}}$.
(3) If $\lambda = (\text{-}1, \ldots, \text{-}1)$, then $\nabla(\lambda) = \Delta(\lambda) = T(\lambda) = \Lambda^g \Omega^{\mathrm{tor}} = \omega$.
(4) If $g = 3$ and $\lambda = (\text{-}2, \text{-}3, \text{-}11)$, then $\nabla(\lambda) = S_{(11,3,2)}\Omega^{\mathrm{tor}}$, $\Delta(\lambda) = W_{(11,3,2)}\Omega^{\mathrm{tor}}$ and $T(\lambda) = T_{(11,3,2)}\Omega^{\mathrm{tor}}$. The automorphic bundles $\nabla(\lambda)$ and $\Delta(\lambda)$ always have rank 99. When $p \geq 11$, the weight λ is p-small for GL_3 and we have isomorphisms $\nabla(\lambda) = \Delta(\lambda) = T(\lambda)$. When $p < 11$, the automorphic bundle $T(\lambda)$ can have a rank higher than 99.

We recall the Kodaira-Spencer isomorphism.

Proposition 4.20 ([**FC90**, Chap. 3, sect. 9])**.** *The Kodaira-Spencer map on the toroidal compactification of the Siegel variety*

$$\rho_{KS} : \mathrm{Sym}^2 \Omega^{\mathrm{tor}} \xrightarrow{\sim} \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D)$$

is an isomorphism between the automorphic bundle $\nabla(0, \cdots, 0, \text{-}2)$ and the sheaf of logarithmic 1-differentials $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D)$. Taking the determinant yields an isomorphism of line bundles

$$\Omega^d_{\mathrm{Sh}^{\mathrm{tor}}}(\log D) \simeq \nabla(-2\rho^L)$$

where d is the dimension of $\mathrm{Sh}^{\mathrm{tor}}$ and

$$\rho^L = \frac{1}{2} \sum_{\alpha \in \Phi^+ \setminus \Phi_L^+} \alpha.$$

CHAPTER 5

The theory of G-Zip and G-ZipFlag

In this chapter all objects will be considered over $\mathbb{F}_p$. Consider the Siegel variety Sh over $\mathbb{F}_p$ of genus $g \geq 1$ and neat level $K \subset \mathrm{Sp}_{2g}(\mathbb{A}_f)$ such that $K_p = \mathrm{Sp}_{2g}(\mathbb{Z}_p)$ is hyperspecial. It possesses a stratification called the Ekedahl-Oort stratification which is a genuine new structure which does not exist in characteritic 0. For the modular curve defined over $\mathbb{F}_p$, there are two strata: the ordinary locus and the supersingular locus. The ordinary locus is an open subscheme corresponding to ordinary elliptic curves over $\mathbb{F}_p$ and the supersingular locus is a reduced closed subscheme corresponding to supersingular elliptic curves over $\mathbb{F}_p$. Hence, the closure of the ordinary locus is the whole modular curve. In the series of papers [**Wed99**] [**MW04**] [**PWZ11**] [**PWZ15**], Moonen, Wedhorn, Pink and Ziegler define an Artin stack G-Zip$^{\mu}$ which depends on the reductive group G over $\mathbb{F}_p$ and a cocharacter μ of G. The underlying topological space of this stack is finite and its topology captures the closure relations of the EO stratification. Furthermore, one can construct a morphism

$$\zeta : \mathrm{Sh} \to G\text{-Zip}^{\mu}$$

from the G-torsor $\mathcal{H}^1_{\mathrm{dR}}$ corresponding to de Rham cohomology of the universal abelian scheme. Zhang, in his thesis [**Zha18**], has proven that ζ is a smooth morphism. One can recover the EO stratification on Sh through a pullback of some substack w of G-Zip$^{\mu}$. Recall that P is the parabolic associated to -μ defined in notation 4.6 and $I \subset \Delta$ is its type. The EO stratification on the Shimura variety can be further generalized on the flag bundle Y_{I_0} corresponding to a standard parabolic subgroup $P_0 \subset P$ of type $I_0 \subset I$. In [**GK19b**], Goldring and Koskivirta define a stack G-ZipFlag$^{\mu,I_0}$, a smooth morphism

$$\zeta_{I_0} : Y_{I_0} \to G\text{-ZipFlag}^{\mu,I_0}$$

and a stratification of the flag bundle Y_{I_0} through the pullback of a collection of some substacks $[w]$ of G-ZipFlag$^{\mu,I_0}$. For the convenience of the reader, we recall how [**GK19a**] and [**GK19b**] use the formalism of G-Zips and G-ZipFlags to define and study the stratifications on the Siegel variety Sh and its flag bundle Y_{I_0}. In particular, we recall their result on the existence of generalized Hasse invariants.

5.1. G-Zip and G-ZipFlag

In order to consider the stratification of the stack G-ZipFlag$^{\mu,I_0}$ for all $I_0 \subset I$, it is convenient to use a general zip datum $\mathcal{Z}$ and to define a stack G-Zip$^{\mathcal{Z}}$ for a general reductive group G over $\mathbb{F}_p$.

Definition 5.1. *A zip datum of exposant $n \geq 1$ is a tuple*

$$\mathcal{Z} = (G, P, L, Q, M, \varphi^n),$$

where G is a reductive group over $\mathbb{F}_p$, $\varphi : G \to G$ is the relative Frobenius map and $P, Q \subset G \times_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ are parabolics over $\overline{\mathbb{F}_p}$ with Levi subgroups $L \subset P$, $M \subset Q$ such that $\varphi^n(L) = M$. We write U and V for the unipotent radical of P and Q.

Definition 5.2. *A morphism of zip data of exposant $n \geq 1$*

$$\mathcal{Z} = (G, P, L, Q, M, \varphi^n) \to \mathcal{Z}' = (G', P', L', Q', M', \varphi'^n)$$

is the data of a group morphism $f : G \to G'$ such that $f(\Diamond) \subset \Diamond'$ for $\Diamond = G, P, L, Q, M, U, V$.

Recall from notation 4.6 that for each $g \geq 1$, we have a minuscule cocharacter $\mu : \mathbb{G}_m \to \mathrm{Sp}_{2g}$ defined over $\mathbb{F}_p$. The couple (Sp_{2g}, μ) (Sp_{2g} is defined over $\mathbb{F}_p$) is a cocharacter datum according to the following definition.

Definition 5.3. *A cocharacter datum is a couple (G, μ) where G is a reductive group over $\mathbb{F}_p$ and $\mu : \mathbb{G}_m \to G$ is a cocharacter defined over $\bar{\mathbb{F}}_p$. A morphism of cocharacter data*

$$(G, \mu) \to (G', \mu')$$

is a group morphism $f : G \to G'$ such that $\mu = f \circ \mu'$. A cocharacter data (G, μ) determines a opposite parabolic subgroup $P_\mu, P_{-\mu}$ with common Levi subgroup $L = P_{-\mu} \cap P_\mu$.

From a cocharacter datum (G, μ) we can construct a zip datum of exposant n

$$\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi^n)$$

by setting $P = P_{-\mu}$, $Q = \varphi^n(P_\mu)$, $L = P_{-\mu} \cap P_\mu$, $M = \varphi^n(L)$. We explain how to define a stack $G\text{-Zip}^{\mathcal{Z}}$ from a zip datum $\mathcal{Z}$.

Definition 5.4. *Let $\mathcal{Z}$ be a zip datum and S be a scheme over $\mathbb{F}_p$. A zip of type $\mathcal{Z}$ over S is a tuple*

$$\underline{I} = (\mathcal{I}, \mathcal{I}_P, \mathcal{I}_Q, \psi),$$

where $\mathcal{I}$ is a G-torsor over S, $\mathcal{I}_P \subset \mathcal{I}$ is a P-reduction of $\mathcal{I}$, $\mathcal{I}_Q \subset \mathcal{I}$ is a Q-reduction of $\mathcal{I}$ and

$$\psi : (\varphi^n)^*(\mathcal{I}_P/U) \to \mathcal{I}_Q/V$$

is an isomorphism of M-torsors over S. A morphism of zips of type $\mathcal{Z}$ over S

$$\underline{I} = (\mathcal{I}, \mathcal{I}_P, \mathcal{I}_Q, \psi) \to \underline{I}' = (\mathcal{I}', \mathcal{I}'_P, \mathcal{I}'_Q, \psi')$$

is a morphism of G-torsors $f : \mathcal{I} \to \mathcal{I}'$ over S such that $f(\Diamond) \subset \Diamond'$ for $\Diamond = \mathcal{I}_P, \mathcal{I}_Q$ and such that the following diagram commutes

$$\begin{array}{ccc} (\varphi^n)^*(\mathcal{I}_P/U) & \xrightarrow{\psi} & \mathcal{I}_Q/V \\ \downarrow & & \downarrow \\ ((\varphi')^n)^*(\mathcal{I}'_P/U') & \xrightarrow{\psi'} & \mathcal{I}'_Q/V' \end{array}$$

where the vertical arrows are induced by f.

Proposition 5.5. *Let $\mathcal{Z}$ be a zip datum and S be a scheme over $\mathbb{F}_p$. The category $G\text{-Zip}^{\mathcal{Z}}(S)$ of zips of type $\mathcal{Z}$ over S is a groupoid. The association $S \to G\text{-Zip}^{\mathcal{Z}}(S)$ defines an algebraic stack over $\mathbb{F}_p$ that we simply denote $G\text{-Zip}^{\mathcal{Z}}$.*

Proof. See [**PWZ15**, Proposition 3.2/3.11]. □

Note that the association $\mathcal{Z} \to G\text{-Zip}^{\mathcal{Z}}$ defines a functor from the category of zip data to the category of algebraic stacks over $\mathbb{F}_p$. We simply write $G\text{-Zip}^{\mu}$ instead of $G\text{-Zip}^{\mathcal{Z}_\mu}$ when the zip datum comes from a cocharacter datum (G, μ). Most of the interesting properties of $G\text{-Zip}^{\mathcal{Z}}$ can be deduced from its presentation as a quotient stack. From now on, we fix a zip datum of exposant n

$$\mathcal{Z} = (G, P, L, Q, M, \varphi^n).$$

Proposition 5.6. *G-Zip$^{\mathcal{Z}}$ is a smooth stack of dimension 0 over $\mathbb{F}_p$ and it is presented as a quotient stack*

$$G\text{-Zip}^{\mathcal{Z}} = \lfloor E_{\mathcal{Z}} \backslash G \rfloor$$

where $E_{\mathcal{Z}} = \{(x,y) \in P \times Q \mid \varphi^n(\bar{x}) = \bar{y}\}$, $x \to \bar{x}$ denotes the natural projection $P \to L$, $Q \to M$ and $(x,y) \in E_{\mathcal{Z}}$ acts on $g \in G$ by

$$(x,y)g = xgy^{-1}.$$

Proof. See [**PWZ15**, Proposition 3.2/3.11]. □

Denote by W the Weyl group of G, $I \subset \Delta$ the type of the parabolic P, $J \subset \Delta$ the type of the parabolic Q, $W_I \subset W$ the subgroup generated by the reflexions in I, ${}^I W$ the set of elements w that are of minimal length in $W_I w$, $W_J \subset W$ the subgroup generated by the reflexions in J and W^J the set of elements w that are of minimal length in wW_J. The element of maximal length in W (resp. W_I and W_J) is denoted w_0 (resp. $w_{0,I}$ and $w_{0,J}$). Denote by z the element $w_0 w_{0,J}$.

Proposition 5.7. *If there exists a Borel pair (B,T) of G defined over $\mathbb{F}_p$, then there exists an element $z \in W$ such that the triple (B,T,z) is a W-frame for $\mathcal{Z}$. It means that the following conditions are satisfied*

(1) $B \subset P$,
(2) $zBz^{-1} \subset Q$,
(3) $\varphi(B \cap L) = zBz^{-1} \cap M$.

Proof. See the proof of [**PWZ11**, Proposition 3.7]. □

For each $w \in W$, we choose a lift $\dot{w}$ in $N_G(T)$. The following proposition explains how G decomposes in $E_{\mathcal{Z}}$-orbits.

Proposition 5.8. *The map $w \mapsto G_w := E_{\mathcal{Z}} \dot{w} \dot{z}^{-1}$ restricts to bijections*[1] *between*

(1) ${}^I W$ and the $E_{\mathcal{Z}}$-orbits of G,
(2) W^J and the $E_{\mathcal{Z}}$-orbits of G.

Moreover, we have the following dimension formula for all $w \in {}^I W \cup W^J$

$$\dim G_w = l(w) + \dim(P).$$

Proof. See [**PWZ11**, Theorem 7.5/11.2]. □

Corollary 5.9. *The stack G-Zip$^{\mathcal{Z}}$ decomposes*

$$G\text{-Zip}^{\mathcal{Z}} = \bigsqcup_{w \in {}^I W} \lfloor E_{\mathcal{Z}} \backslash G_w \rfloor.$$

The stack G-Zip$^{\mathcal{Z}}$ has a topology which describes the closure relations between the G_w.

Proposition 5.10. *The underlying topological space of G-Zip$^{\mathcal{Z}}$ is homeomorphic to the finite topological space ${}^I W$ where the topology is given by the partial order:*

$$w \preccurlyeq w' \Leftrightarrow \text{ if and only if there is } w \in W_I \text{ such that } vwxv^{-1}x^{-1} \leq w',$$

where x is the unique element of minimal length in $W_J w_0 W_I$.

[1] In the case $G = \mathrm{Sp}_{2g}$, these two bijections coincide.

Proof. The result follows from the isomorphism

$$\overline{G_w} = \bigsqcup_{w' \preccurlyeq w,\ w' \in {}^I W} G_{w'},$$

for all $w \in {}^I W$ which is proven in [**PWZ11**, Theorem 6.2]. □

We simply write $[w]$ for the locally closed substack $\lfloor E_{\mathcal{Z}} \backslash G_w \rfloor$ of G-Zip$^{\mathcal{Z}}$. Now, we describe how to define a more general stack G-ZipFlag$^{\mathcal{Z},P_0}$ which depends on the zip datum $\mathcal{Z}$ and an auxiliary parabolic subgroup $B \subset P_0 \subset P$.

Definition 5.11. *Let $B \subset P_0 \subset P$ be a parabolic subgroup of P and S be a scheme over $\mathbb{F}_p$. A zip flag of type $(\mathcal{Z}, P_0)$ over S is a tuple*

$$\underline{J} = (\underline{I}, \mathcal{J}),$$

where $\underline{I} = (\mathcal{I}, \mathcal{I}_P, \mathcal{I}_Q, \psi)$ is a zip of type $\mathcal{Z}$ over S and $\mathcal{J} \subset \mathcal{I}_P$ is a P_0-reduction of the P-torsor $\mathcal{I}_P$. A morphism of zip flags of type $(\mathcal{Z}, P_0)$ over S

$$\underline{J} = (\underline{I}, \mathcal{J}) \to \underline{J}' = (\underline{I}', \mathcal{J}')$$

is a morphism of zip $\underline{I} \to \underline{I}'$ of type $\mathcal{Z}$ over S such that the underlying morphism of G-torsor $\mathcal{I} \to \mathcal{I}'$ restricts to a morphism of P_0-torsor $\mathcal{J} \to \mathcal{J}'$ over S.

Proposition 5.12. *Let $B \subset P_0 \subset P$ be a parabolic subgroup of P and S be a scheme over $\mathbb{F}_p$. The category G-ZipFlag$^{\mathcal{Z},P_0}(S)$ of zip flags of type $(\mathcal{Z}, P_0)$ over S is a groupoid. The association $S \to G$-ZipFlag$^{\mathcal{Z},P_0}(S)$ defines an algebraic stack over $\mathbb{F}_p$ that we simply denote G-ZipFlag$^{\mathcal{Z},P_0}$.*

Proof. See [**GK19b**, Theorem 2.1.2]. □

From now on, we fix an auxiliary parabolic subgroup $B \subset P_0 \subset P$.

Proposition 5.13. *The stack G-ZipFlag$^{\mathcal{Z},P_0}$ is a smooth stack of dimension $\dim(P/P_0)$ over $\mathbb{F}_p$ and it can be presented as the quotient stack*

$$\lfloor E_{\mathcal{Z},P_0} \backslash G \rfloor$$

where $E_{\mathcal{Z},P_0} := E_{\mathcal{Z}} \cap (P_0 \times G) \subset P_0 \times Q$ acts on G by restriction of the $E_{\mathcal{Z}}$-action on G. It can also be presented as the quotient stack

$$\lfloor E_{\mathcal{Z}} \times P_0 \backslash G \times P \rfloor$$

where $((x, y), p_0) \in E_{\mathcal{Z}} \times P_0$ acts on $(g, p) \in G \times P$ through the formula

$$((x, y), p_0).(g, p) = (xgy^{-1}, xpp_0^{-1}).$$

Proof. See [**GK19b**, Theorem 2.1.2]. □

Definition 5.14. *The map sending a zip flag $\underline{J} = (\underline{I}, \mathcal{J})$ of type $(\mathcal{Z}, P_0)$ over S to the zip $\underline{I}$ of type $\mathcal{Z}$ over S defines a morphism of algebraic stacks over $\mathbb{F}_p$*

$$\pi : G\text{-ZipFlag}^{\mathcal{Z},P_0} \to G\text{-Zip}^{\mathcal{Z}}.$$

Proposition 5.15. *The inclusion $E_{\mathcal{Z},P_0} \subset E_{\mathcal{Z}}$ induces a morphism*

$$\lfloor E_{\mathcal{Z},P_0} \backslash G \rfloor \to \lfloor E_{\mathcal{Z}} \backslash G \rfloor$$

which corresponds to π through the isomorphisms in proposition 5.13.

Proposition 5.16. *The morphism $\pi : G\text{-ZipFlag}^{\mathcal{Z},P_0} \to G\text{-Zip}^{\mathcal{Z}}$ is proper and smooth with fibers isomorphic to the flag variety P_0/P.*

PROOF. See [**GK19b**, Theorem 2.1.2]. □

It is natural to hope for a stratification of $G\text{-ZipFlag}^{\mathcal{Z},P_0}$ that generalizes the stratification on $G\text{-Zip}^{\mathcal{Z}}$ however the $E_{\mathcal{Z},P_0}$-orbits of G are not as easy to understand as the $E_{\mathcal{Z}}$-orbits. Instead, we define a smooth surjective map

$$G\text{-ZipFlag}^{\mathcal{Z},P_0} \to G\text{-Zip}^{\mathcal{Z}_0}$$

where $\mathcal{Z}_0$ is a zip datum constructed from $\mathcal{Z}$ and P_0 and then pullback the stratification of $G\text{-Zip}^{\mathcal{Z}_0}$.

Definition 5.17. *We denote by $\mathcal{Z}_0$ the zip datum*

$$\mathcal{Z}_0 = (G, P_0, L_0, Q_0, M_0, \varphi^n)$$

where Q_0 is a parabolic subgroup of Q defined by

$$Q_0 = \varphi^n(P_0 \cap L)R_u(Q) \subset Q$$

with $R_u(Q)$ the unipotent radical of Q and where L_0, M_0 are the Levi subgroups of P_0, Q_0.

Proposition 5.18. *We have inclusions*

$$E_{\mathcal{Z},P_0} \subset E_{\mathcal{Z}_0} \subset P_0 \times Q_0$$

and the induced maps

$$G\text{-ZipFlag}^{\mathcal{Z},P_0} \xrightarrow{\psi_1} G\text{-Zip}^{\mathcal{Z}_0} \xrightarrow{\psi_2} \lfloor P_0 \backslash G / Q_0 \rfloor$$

are smooth and surjective.

PROOF. See [**GK19b**, Section 3.1]. □

Definition 5.19. *The fine stratification of $G\text{-ZipFlag}^{\mathcal{Z},P_0}$ is the stratification of $G\text{-Zip}^{\mathcal{Z}_0}$ pulled back by ψ_1 and the coarse stratification of $G\text{-ZipFlag}^{\mathcal{Z},P_0}$ is the stratification of the Bruhat stack $\lfloor P_0 \backslash G / Q_0 \rfloor$ pulled back by $\psi_2 \circ \psi_1$. If $w \in {}^{I_0}W \cup W^{J_0}$, then we write $G\text{-ZipFlag}_w^{\mathcal{Z},P_0}$ for the corresponding fine strata.*

In the special case where $P_0 = B$ is the Borel subgroup, the map ψ_2 is an isomorphism, so the coarse and the fine stratification of $G\text{-ZipFlag}^{\mathcal{Z},P_0}$ coincide. Note that if we have an inclusion of auxiliary parabolics $B \subset P_0 \subset P_1 \subset P$, then there exists natural maps making the following diagram 2-cartesian.

$$\begin{tikzcd} G\text{-ZipFlag}^{\mathcal{Z},P_0} \arrow[r] \arrow[d] & \lfloor P_0 \backslash G / Q_0 \rfloor \arrow[d] \\ G\text{-ZipFlag}^{\mathcal{Z},P_1} \arrow[r] & \lfloor P_1 \backslash G / Q_1 \rfloor \end{tikzcd}$$

However, we don't know if a similar statement holds if we replace the Bruhat stacks with $G\text{-Zip}^{\mathcal{Z}_0}$ and $G\text{-Zip}^{\mathcal{Z}_1}$.

Corollary 5.20. *The stack $G\text{-ZipFlag}^{\mathcal{Z},P_0}$ decomposes*

$$G\text{-ZipFlag}^{\mathcal{Z},P_0} = \bigsqcup_{w \in {}^{I_0}W} G\text{-ZipFlag}_w^{\mathcal{Z},P_0}$$

and for all $w \in {}^{I_0}W$, we have the closure relation

$$\overline{G\text{-ZipFlag}_w^{\mathcal{Z},P_0}} = \bigsqcup_{w' \preccurlyeq w,\ w' \in {}^{I_0}W} G\text{-ZipFlag}_{w'}^{\mathcal{Z},P_0},$$

where the order on ${}^{I_0}W$ is the one introduced in proposition 5.10.

Corollary 5.21. *Let $w \in {}^{I_0}W \cup W^{J_0}$ and G-$\mathrm{ZipFlag}_w^{\mathcal{Z},P_0}$ be the corresponding fine strata. Then G-$\mathrm{ZipFlag}_w^{\mathcal{Z},P_0}$ is a smooth stack over $\mathbb{F}_p$ of pure dimension $l(w) + \dim P - \dim G$.*

Now we want to construct some sections of vector bundles on G-$\mathrm{Zip}^{\mathcal{Z}}$, G-$\mathrm{ZipFlag}^{\mathcal{Z},P_0}$ and relate their non-vanishing loci to the stratification we have introduced. We start by introducing vector bundles on G-$\mathrm{Zip}^{\mathcal{Z}}$.

Definition 5.22. *Let $\rho : L \to \mathrm{GL}(V)$ be a finite dimensional algebraic representation of the Levi L. Consider the map $f : E_{\mathcal{Z}} \to L$ which is the composition of the first projection $E_{\mathcal{Z}} \to P$ with the quotient map $P \to L$. The composition $\rho \circ f$ is an algebraic representation of $E_{\mathcal{Z}}$. It induces a locally free sheaf $\mathcal{W}(V)$ of rank $\dim_{\mathbb{F}_p} V$ on $\lfloor E_{\mathcal{Z}} \backslash G \rfloor$. If $\lambda \in X^*(T)$ is a I-dominant character of T, we simply denote $\nabla(\lambda)$ the locally free sheaf $\mathcal{W}(H^0(L/B_L, \mathcal{L}_\lambda))$. Note that $H^0(L/B_L, \mathcal{L}_\lambda)$ is the costandard L-represention of highest weight $w_0 w_{0,I}\lambda$. More generally, we denote $\Delta(\lambda)$ and $T(\lambda)$ the locally free sheaves $\mathcal{W}(\Delta(w_0 w_{0,I}\lambda))$ and $\mathcal{W}(T(w_0 w_{0,I}\lambda))$.*

More generally, we can define vector bundles on G-$\mathrm{ZipFlag}^{\mathcal{Z},P_0}$.

Definition 5.23. *Let $\rho : P_0 \to \mathrm{GL}(V)$ be a finite dimensional algebraic representation of the parabolic P_0. Consider the first projection map $f : E_{\mathcal{Z},P_0} \to P_0$. The composition $\rho \circ f$ is an algebraic representation of $E_{\mathcal{Z},P_0}$ and it induces a locally free sheaf $\mathcal{L}(V)$ of rank $\dim_{\mathbb{F}_p} V$ on $\lfloor E_{\mathcal{Z},P_0} \backslash G \rfloor$. If $\lambda \in X^*(L_0) \subset X^*(T)$ is a character of L_0, we also denote $\mathcal{L}_\lambda$ the line bundle $\mathcal{L}(\lambda)$ where we see λ as a one dimensional representation of P_0.*

We have defined vector bundles $\nabla(\lambda), \Delta(\lambda)$ and $T(\lambda)$ on G-$\mathrm{Zip}^{\mathcal{Z}}$ and line bundles $\mathcal{L}_\lambda$ on G-$\mathrm{ZipFlag}^{\mathcal{Z},P_0}$ for certain characters $\lambda \in X^*(T)$. The next proposition gives a direct relation between $\nabla(\lambda)$ and $\mathcal{L}_\lambda$.

Proposition 5.24. *Recall that $\pi : G$-$\mathrm{ZipFlag}^{\mathcal{Z},P_0} \to G$-$\mathrm{Zip}^{\mathcal{Z}}$ is the proper and smooth map that forgets the P_0-torsor from a zip flag of type $(\mathcal{Z}, P_0)$. Let $\lambda \in X^*(L_0)$ be a I_0-dominant character of L_0. We have a canonical isomorphism*

$$\pi_* \mathcal{L}_\lambda \simeq \nabla(\lambda).$$

Proof. This is a similar proof to the one for proposition 3.22. Consider the cartesian diagram

$$\begin{tikzcd} G\text{-ZipFlag}^{\mathcal{Z},P_0} \arrow[r, "\tilde{\zeta}"] \arrow[d, "\pi"] & \lfloor */P_0 \rfloor \arrow[d, "\tilde{\pi}"] \\ G\text{-Zip}^{\mathcal{Z}} \arrow[r, "\zeta"] & \lfloor */P \rfloor \end{tikzcd}$$

where the horizontal maps are given by the universal P_0-torsor on G-$\mathrm{ZipFlag}^{\mathcal{Z},P_0}$ and the universal P-torsor on G-$\mathrm{Zip}^{\mathcal{Z}}$. For each character $\lambda \in X^*(P_0)$, we have a line bundle $\mathcal{L}_\lambda$ on $\lfloor */P_0 \rfloor$ and a vector bundle $\nabla(\lambda)$ on $\lfloor */P \rfloor$ (corresponding to the induced P-representation

$H^0(P/P_0, \mathcal{L}_\lambda))$ that satisfies

$$\begin{cases} \tilde{\zeta}^* \mathcal{L}_\lambda = \mathcal{L}_\lambda, \\ \zeta^* \nabla(\lambda) = \nabla(\lambda). \end{cases}$$

It is straightforward from the definitions that

$$\tilde{\pi}_* \mathcal{L}_\lambda = \nabla(\lambda).$$

As the map is fibered in P/P_0 by proposition 5.16, we know by proposition 2.11 and lemma 3.13 that for a I_0-dominant character λ, we have

$$\begin{cases} R\tilde{\pi}_* \mathcal{L}_\lambda = \tilde{\pi}_* \mathcal{L}_\lambda, \\ R\pi_* \mathcal{L}_\lambda = \pi_* \mathcal{L}_\lambda. \end{cases}$$

Since ζ is flat, we conclude as in the end of the proof of proposition 4.18 with the base change theorem in the derived category that says that the natural map

$$\zeta^* \circ R\tilde{\pi}_* \mathcal{L}_\lambda \to R\pi_* \circ \zeta^* \mathcal{L}_\lambda$$

is an isomorphism. □

On the Bruhat stack $\mathrm{Brh} = \lfloor B \backslash G / B \rfloor$, we have the Bruhat stratification

$$\mathrm{Brh} = \bigsqcup_{w \in W} \mathrm{Brh}_w,$$

where $\mathrm{Brh}_w = \lfloor B \backslash BwB / B \rfloor$ and for all $w \in W$ we have the closure relation

$$\overline{\mathrm{Brh}_w} = \bigsqcup_{w' \leq w,\ w' \in W} \mathrm{Brh}_{w'},$$

where $\leq$ is the Bruhat order. We consider the morphism

$$\psi : G\text{-ZipFlag}^{\mathcal{Z},B} \to \mathrm{Brh}$$

defined as the composition of the morphism induced by the inclusion

$$E_{\mathcal{Z},B} \subset B \times zBz^{-1}$$

with the isomorphism

$$\alpha_z : \lfloor B \backslash G / zBz^{-1} \rfloor \to \lfloor B \backslash G / B \rfloor,$$

that sends x to xz. We use this stack to construct some sections on G-ZipFlag$^{\mathcal{Z},P_0}$.

Proposition 5.25. *Given two characters $(\lambda, \eta) \in X^*(T) \times X^*(T)$, the associated line bundle $\mathcal{L}_{\lambda,\eta}$ on* Brh *has the following properties.*

(1) We have a canonical isomorphism $\psi^* \mathcal{L}_{\lambda,\eta} = \mathcal{L}_{\lambda + p^\sigma(z\eta)}$ where $\sigma : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$ is the inverse of the Frobenius.

(2) For all $w \in W$, we have $H^0(\mathrm{Brh}_w, \mathcal{L}_{\lambda,\eta}) \neq 0 \Leftrightarrow \eta = -w^{-1}\lambda$.

(3) $\dim_{\mathbb{F}_p} H^0(\mathrm{Brh}_w, \mathcal{L}_{\lambda,-w^{-1}\lambda}) = 1$.

(4) For any non-zero $s \in H^0(\mathrm{Brh}_w, \mathcal{L}_{\lambda,-w^{-1}\lambda})$ viewed as a rational function on $\overline{\mathrm{Brh}}_w$, one has

$$\mathrm{div}(s) = -\sum_{\alpha \in E_w} \langle \lambda, w\alpha^\vee \rangle \overline{\mathrm{Brh}}_{ws_\alpha}$$

where $E_w = \{\alpha \in \Phi^+ \mid ws_\alpha < w \text{ and } l(ws_\alpha) = l(w) - 1\}$. The set of ws_α for $\alpha \in E_w$ is called the set of lower neighbors of w.

Proof. For (i), see [**GK19a**, Lemma 3.1.1]. For (ii) to (iv), see [**GK19a**, Theorem 2.2.1]. □

Definition 5.26. *Let $w \in W$ and $n \geq 0$. We define by induction on n, the element $w^{(n)}$ by setting*

(1) $w^{(0)} = e$,
(2) $w^{(n)} = {}^{\sigma}(w^{(n-1)}w)$ if $n \geq 1$.

Proposition 5.27. *The function*

$$D_w : \begin{array}{ccc} X^*(T) & \to & X^*(T) \\ \lambda & \mapsto & \lambda - p\,{}^{\sigma}(zw^{-1}\lambda) \end{array}$$

induces a $\mathbb{Q}$-linear automorphism of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. If χ is a character, it's inverse by D_w is given by the $\mathbb{Q}$-character

$$\lambda = \frac{-1}{p^{rn}-1} \sum_{i=0}^{rn-1} p^i (zw^{-1})^{(i)}\, {}^{\sigma^i}\chi,$$

where r is an integer such that $(zw^{-1})^{(r)} = e$ and n is an integer such that χ is defined over $\mathbb{F}_{p^n}$.

Proof. See [**GK19a**, Lemma 3.1.3]. □

Having defined sections on stacks G-Zip$^{\mathcal{Z}}$, we can study their vanishing locus.

Definition 5.28. *Let $\lambda \in X^*(P_0)$ be a L_0-dominant character of P_0 and*

$$s \in H^0(G\text{-ZipFlag}_w^{\mathcal{Z},P_0}, \mathcal{L}_\lambda),$$

a non-zero section. We say that s is a generalized Hasse invariant for G-ZipFlag$_w^{\mathcal{Z},P_0}$ if there exists some $d \geq 1$ such that s^d extends to $\overline{G\text{-ZipFlag}_w^{\mathcal{Z},P_0}}$ with non-vanishing locus G-ZipFlag$_w^{\mathcal{Z},P_0}$. We define the sets

$$\mathcal{C}_{\mathrm{Ha},I_0,w} = \left\{\lambda \in X^*(P_0) \mid \mathcal{L}_\lambda \text{ has a generalized Hasse invariant for } G\text{-ZipFlag}_w^{\mathcal{Z},P_0}\right\}$$

and

$$\mathcal{C}_{\mathrm{Ha},I_0} = \bigcap_{w \in W} \mathcal{C}_{\mathrm{Ha},I_0,w}.$$

Now, we give a strong result for the existence of generalized Hasse invariants on the stack G-ZipFlag$^{\mathcal{Z},P_0}$.

Proposition 5.29. *Let $\lambda \in X^*(P_0)$ be a L_0-dominant character, w be an element of ${}^{I_0}W$ and s be a non zero section of $H^0(G\text{-ZipFlag}_w^{\mathcal{Z},P_0}, \mathcal{L}_\lambda)$. Then, the following statements are equivalent:*

(1) s is a generalized Hasse invariant for G-ZipFlag$_w^{\mathcal{Z},P_0}$.
(2) For all $\alpha \in E_w$, we have

$$\sum_{i=0}^{rn-1} \langle (zw^{-1})^{(i)} ({}^{\sigma^i}\lambda), w\alpha^\vee \rangle p^i > 0,$$

where r is an integer such that $(zw^{-1})^{(r)} = e$ and n is an integer such that λ is defined over $\mathbb{F}_{p^n}$.

Proof. See [**GK19a**, Proposition 3.2.1]. □

Example 5.30. We give more details in the case $G = \mathrm{Sp}_4$ and $\mathcal{Z} = \mathcal{Z}_\mu$ with μ the cocharacter that stabilizes the Hodge filtration of the Siegel datum. The Levi L is GL_2. We denote by s_1 and s_2 the simple reflections associated to the simple roots $(1,-1)$ and $(0,2)$. We represent the elements of W in the diagram

$$\begin{tikzcd}
 & w_0 = s_2s_1s_2s_1 \arrow[dl] \arrow[dr] & \\
w_1 = s_2s_1s_2 \arrow[d] \arrow[drr] & & w_1' = s_1s_2s_1 \arrow[d] \arrow[dll] \\
w_2 = s_2s_1 \arrow[d] \arrow[drr] & & w_2' = s_1s_2 \arrow[d] \arrow[dll] \\
w_3 = s_2 \arrow[dr] & & w_3' = s_1 \arrow[dl] \\
 & e &
\end{tikzcd}$$

where an arrow is drawn from w to w' if $w' \leq w$ and $l(w') = l(w) - 1$. For each $w \in W$ and $\lambda \in X^*$, we denote by $s_{\lambda,w}$ the quasi-section[2] in $H^0(G\text{-ZipFlag}_w^{\mathcal{Z},B}, \mathcal{L}_\lambda)$ obtained via pullback from a non-zero quasi-section of $H^0(\mathrm{Brh}_w, \mathcal{L}_{\chi,-w^{-1}\chi})$ where χ is a $\mathbb{Q}$-character such that $D_w(\chi) = \lambda$. We write $\lambda = (k_1, k_2)$ and we compute $\mathrm{div}(s_{\lambda,w})$ for each w.

$$\begin{cases}
\mathrm{div}(s_{\lambda,w_0}) = \dfrac{1}{p^2-1}\left((p-1)(k_1-k_2)\overline{[w_1]} - (k_2+pk_1)\overline{[w_1']}\right), \\
\mathrm{div}(s_{\lambda,w_1}) = \dfrac{1}{p-1}\left(-k_1\overline{[w_2]} - k_2\overline{[w_2']}\right), \\
\mathrm{div}(s_{\lambda,w_1'}) = \dfrac{1}{p^2+1}\left(-((p-1)k_1+(p+1)k_2)\overline{[w_2]} + ((p+1)k_1-(p-1)k_2)\overline{[w_2']}\right), \\
\mathrm{div}(s_{\lambda,w_2}) = \left(\dfrac{k_1}{p+1} - \dfrac{k_2}{p-1}\right)\overline{[w_3]} + \dfrac{-k_2}{p-1}\overline{[w_3']}, \\
\mathrm{div}(s_{\lambda,w_2'}) = -\left(\dfrac{k_1}{p-1} + \dfrac{k_2}{p+1}\right)\overline{[w_3]} + \dfrac{-k_1}{p-1}\overline{[w_3']}, \\
\mathrm{div}(s_{\lambda,w_3}) = \dfrac{1}{p^2+1}(k_2-pk_1)\overline{[e]}, \\
\mathrm{div}(s_{\lambda,w_3'}) = \dfrac{1}{p+1}(k_1-k_2)\overline{[e]}.
\end{cases}$$

We deduce that the following set of characters

$$\mathcal{C}_1 := \{\lambda = (k_1,k_2) \mid 0 > k_1 > \frac{p-1}{p+1}k_2 \text{ and } k_2 > pk_1\}$$

has generalized Hasse invariant for all strata of $G\text{-ZipFlag}_w^{\mathcal{Z},B}$. Since the character of the Levi are $X^*(L) = \mathbb{Z}(1,1)$ and the minimal length left coset representatives are

$${}^IW = \{e, w_3, w_2, w_1\},$$

[2] It means a section of a certain positive tensorial power of $\mathcal{L}_\lambda$.

we deduce that the following set of characters

$$\mathcal{C}_2 := \{\lambda = (k_1, k_1) \mid k_1 < 0\}$$

has generalized Hasse invariant for all strata of G-Zip$^{\mathcal{Z}}$.

Example 5.31. We give some details in the case $G = \mathrm{Sp}_6$. The Levi L is GL_3 and we denote by s_1, s_2 and s_3 the simple reflections associated to the simple roots $(1, -1, 0), (0, 1, -1)$ and $(0, 0, 2)$. The Weyl group W is isomorphic to $S_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$ (48 elements). The computations can be painful without a computer, so we have implemented an algorithm on SageMath[3] that computes the divisor of all the $s_{w,\lambda}$ for any $g \geq 2$. Take $p = 7$, $w = s_1 s_2 s_3$, and $\lambda = (-1, -3, -5)$. We get

$$\mathrm{div}(s_{w,\lambda}) = \frac{5}{6}\overline{[s_2 s_3]} + \frac{1}{2}\overline{[s_3 s_1]} + \frac{1}{6}\overline{[s_1 s_2]}.$$

Koskivirta and Goldring introduced a notion called orbitally p-closeness that guarantees a character to have generalized Hasse invariants for all strata without having to compute all the $\mathrm{div}(s_{w,\lambda})$. However, this notion is not necessary for a character to have Hasse invariants.

Definition 5.32. *Let λ be a character of T. For every coroot such that $\langle \lambda, \alpha^\vee \rangle \neq 0$, we set*

$$\mathrm{Orb}(\lambda, \alpha^\vee) = \left\{ \frac{|\langle \lambda, w\alpha^\vee \rangle|}{|\langle \lambda, \alpha^\vee \rangle|} \mid w \in W \rtimes Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p) \right\}$$

and we say that λ is

(1) orbitally p-close if $\max_{\alpha \in \Phi} \mathrm{Orb}(\lambda, \alpha^\vee) \leq p - 1$,
(2) $\mathcal{Z}_0$-ample if $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in I \backslash I_0$ and $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Phi^+ \backslash \Phi_L^+$.

Example 5.33. We plot in the figure 5.1.1 the weights $\lambda = (k_1, k_2)$ that are orbitally 7-close when $G = \mathrm{Sp}_4$.

Proposition 5.34. *Let λ be a character of P_0. If λ is orbitally p-close and $\mathcal{Z}_0$-ample then, there exists $d \geq 1$ such that for all $w \in {}^{I_0}W$ and all non-zero section s in*

$$H^0(G\text{-ZipFlag}_w^{\mathcal{Z},P_0}, \mathcal{L}_\lambda),$$

the dth-power s^d extends to $\overline{G\text{-ZipFlag}_w^{\mathcal{Z},P_0}}$ with non-vanishing locus G-ZipFlag$_w^{\mathcal{Z},P_0}$.

Proof. See [**GK19a**, Proposition 3.2.3]. □

5.2. The Ekedahl-Oort stratification and generalized Hasse invariants

In this subsection, we specialize our discussion to the Siegel case. Recall that the Siegel variety Sh is a smooth scheme over $\mathbb{F}_p$. We denote by $\pi : Y_{I_0} \to \mathrm{Sh}$ the Siegel flag bundle of type $I_0 \subset I$. Recall that π extends to the toroidal compactifications $\pi : Y_{I_0}^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$. We also have a minimal compactification $\mathrm{Sh}^{\mathrm{min}}$ for the Shimura variety but not for the flag bundle. The goal of this subsection is to define the maps ζ and ζ_{I_0}. We need some results on the Hodge and filtrations of abelian schemes. We recall a result due to Deligne and Illusie.

Proposition 5.35. *Let S be a scheme of characteristic p. Let $f : A \to S$ be an abelian scheme over S. Consider the Hodge to de Rham spectral sequence*

$$E_2^{i,j} = R^j f_* \Omega^i_{A/S} \Rightarrow H^{i+j}_{\mathrm{dR}}(A/S).$$

[3]See github.com/ThibaultAlexandre/generalized-hasse-invariants to download the algorithm.

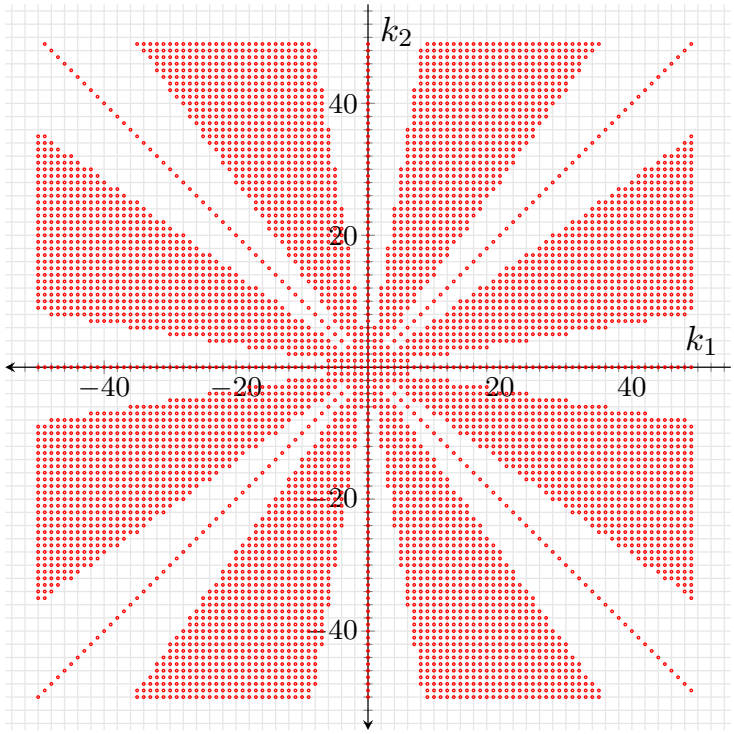


FIGURE 5.1.1. Orbitally 7-close weights when $G = \mathrm{Sp}_4$

and the conjugate spectral sequence

$$E_1'^{i,j} = R^i f_*(\mathcal{H}^j(\Omega^\bullet_{A/S})) \Rightarrow H^{i+j}_{\mathrm{dR}}(A/S).$$

Then

(1) $E_2^{i,j}$ *degenerates at page* 2,

(2) $E_1'^{i,j}$ *degenerates at page* 1.

PROOF. See [**DI87**, Corollaire 2.4 and Remarques 2.6 (iv)]. □

Definition 5.36. *Let S be a scheme of characteristic p. Let $f : A \to S$ be an abelian scheme over S. The Hodge filtration of A over S is the two-step underlying filtration on $H^1_{\mathrm{dR}}(A/S)$ coming from the degeneration of the Hodge to de Rham spectral sequence*

$$0 \to \pi_*\Omega^1_{A/S} \to H^1_{\mathrm{dR}}(A/S) \to R^1\pi_*\mathcal{O}_A \to 0.$$

Definition 5.37. *The conjugate filtration is the two-step underlying filtration on $H^1_{\mathrm{dR}}(A/S)$ coming from the degeneration of the conjugate spectral sequence*

$$0 \to R^1\pi_*\mathcal{H}^0(\Omega^\bullet_{A/S}) \to H^1_{\mathrm{dR}}(A/S) \to \pi_*\mathcal{H}^1(\Omega^\bullet_{A/S}) \to 0.$$

The Hodge and the conjugate filtration are related on their graded pieces by the Cartier isomorphism which we recall the definition.

Definition 5.38. *Let S be a scheme of characteristic p and $f : A \to S$ be a smooth morphism*[4]*. The Cartier morphism is map of graded algebra*

$$C^{-1} : \bigoplus_i \Omega^i_{A^{(p)}/S} \to \bigoplus_i \mathcal{H}^i(F_*\Omega^\bullet_{A/S}),$$

where $F : A \to A^{(p)}$ is the relative geometric Frobenius of A over S. It is enough to define it in degree 0 and 1 and then use the graded algebra structure to extend it. In degree 0, it is the map $F^* : \mathcal{O}_{A^{(p)}} \to F_*\mathcal{O}_A$. In degree 1, it is a map

$$\Omega^1_{A^{(p)}/S} \to \mathcal{H}^1(F_*\Omega^\bullet_{A/S})$$

coming from the S-derivation $\delta : \mathcal{O}_{A^{(p)}} \to \mathcal{H}^1(F_*\Omega^\bullet_{A/S})$ satisfying (we use the isomorphism $\mathcal{O}_{A^{(p)}} = \mathcal{O}_A \otimes_{\mathcal{O}_S, F^*} \mathcal{O}_S$)

(1) $\delta(fs \otimes s') = \delta(f \otimes s^p s')$,

(2) $\delta(fg \otimes s') = f^p\delta(g \otimes s) + g^p\delta(f \otimes s)$,

for all $f, g \in \mathcal{O}_A$ and $s, s' \in \mathcal{O}_S$. If $f \in \mathcal{O}_A$ and $s \in \mathcal{O}_S$, we define $\delta(f \otimes s)$ to be the cohomology class of $sf^{p-1}df$.

Proposition 5.39 ([Car57])**.** *The Cartier morphism C^{-1} is an isomorphism and it satisfies*

(1) $C^{-1}(1) = 1$,

(2) $C^{-1}(w \wedge w') = C^{-1}(w) \wedge C^{-1}(w')$ for all $w \in \Omega^i_{A^{(p)}/S}$, $w' \in \Omega^{i'}_{A^{(p)}/S}$,

(3) $C^{-1}(d(f \otimes 1)) = \left[f^{p-1}df\right]$.

The Hodge filtration and the conjugate filtration of an abelian scheme $f : A \to S$ of relative dimension g can be seen as a P-reduction $\mathcal{I}$ and a Q-reduction $\mathcal{I}$ of the $G = \mathrm{Sp}_{2g}$-torsor $H^1_{\mathrm{dR}}(A/S)$ where P and Q are the maximal parabolic subgroups associated[5] to the cocharacter datum (Sp_{2g}, μ). With the Cartier isomorphism, we can construct an isomorphism

$$\psi : (\varphi^n)^*(\mathcal{I}_P/U) \to \mathcal{I}_Q/V$$

of M-torsors. In other words, we can construct a zip $(H^1_{\mathrm{dR}}(A/S), \mathcal{I}_P, \mathcal{I}_Q, \psi)$ over S of type $\mathcal{Z} = (G, P, L, Q, M, \varphi^n)$.

Definition 5.40. *The morphism*

$$\zeta : \mathrm{Sh} \to G\text{-Zip}^{\mathcal{Z}}$$

is the classifying map of the universal zip $\underline{I} = (\mathcal{H}^1_{\mathrm{dR}}, \mathcal{I}_P, \mathcal{I}_Q, \psi)$ associated to the universal abelian scheme $f : A \to \mathrm{Sh}$ over Sh. For all $w \in {}^I W$, we define the locally closed subscheme $\mathrm{Sh}_w := \zeta^{-1}(G\text{-Zip}^{\mathcal{Z}}_w)$.

Over Y_{I_0}, we have a universal P_{I_0}-reduction $\mathcal{J}$ of the P-torsor $\mathcal{I}_P$ corresponding to the Hodge filtration. The pair $(\underline{I}, \mathcal{J})$ is a zip flag of type $(\mathcal{Z}, I_0)$.

Definition 5.41. *The morphism*

$$\zeta_{I_0} : Y_{I_0} \to G\text{-ZipFlag}^{\mathcal{Z}, I_0}$$

is the classifying map of the universal zip flag $(\underline{I}, \mathcal{J})$ of type $(\mathcal{Z}, I_0)$. For all $w \in {}^{I_0}W$, we define the locally closed subscheme $(Y_{I_0})_w := \zeta_{I_0}^{-1}(G\text{-ZipFlag}^{\mathcal{Z}}_w)$.

[4] Like an abelian scheme A over S.

[5] See the paragraph after definition 5.3.

Proposition 5.42. *The morphisms ζ and ζ_{I_0} are smooth and surjective.*

PROOF. See [**Zha18**, Theorem 3.1.2] for the smoothness. See [**Oor01**] for the surjectivity. □

As generalizations lift along flat morphisms[6], we deduce in particular that we have the following closure relations.

(1) For all $w \in {}^I W$, $\overline{\mathrm{Sh}_w} = \bigsqcup_{w' \preccurlyeq w,\ w' \in {}^I W} \mathrm{Sh}_{w'}$.

(2) For all $w \in {}^{I_0} W$, $\overline{(Y_{I_0})_w} = \bigsqcup_{w' \preccurlyeq w,\ w' \in {}^{I_0} W} (Y_{I_0})_{w'}$.

We give a statement about the extension of these results on the toroidal compactifications.

Proposition 5.43. *The universal zip $\underline{I}$ extends to a zip of type $\mathcal{Z}$ over the toroidal compactification $\mathrm{Sh}^{\mathrm{tor}}$ of Sh. The corresponding classifying morphism ζ^{tor} extends the morphism ζ:*

$$\begin{tikzcd} \mathrm{Sh}^{\mathrm{tor}} \arrow[r, "\zeta^{\mathrm{tor}}"] & G\text{-}\mathrm{Zip}^{\mathcal{Z}} \\ \mathrm{Sh} \arrow[u] \arrow[ur, "\zeta"'] & \end{tikzcd}$$

PROOF. See [**GK19a**, Theorem 6.2.1]. □

Corollary 5.44. *The universal zip flag $(\underline{I}, \mathcal{J})$ extends to a zip flag of type $(\mathcal{Z}, I_0)$ over the toroidal compactification $\mathrm{Sh}^{\mathrm{tor}}$ of Sh. The corresponding classifying morphism $\zeta_{I_0}^{\mathrm{tor}}$ extends the morphism ζ_{I_0}*

$$\begin{tikzcd} Y_{I_0}^{\mathrm{tor}} \arrow[r, "\zeta_{I_0}^{\mathrm{tor}}"] & G\text{-}\mathrm{ZipFlag}^{\mathcal{Z}, I_0} \\ Y_{I_0} \arrow[u] \arrow[ur, "\zeta_{I_0}"'] & \end{tikzcd}$$

Proposition 5.45. *The morphisms ζ^{tor} and $\zeta_{I_0}^{\mathrm{tor}}$ are smooth.*

PROOF. See [**And21**, Theorem 1.2.]. □

We conclude this chapter with an obvious remark.

Remark 5.46. All the bundles $\nabla(\lambda)$, $\Delta(\lambda)$, $T(\lambda)$ and $\mathcal{L}_\lambda$ coincide with the automorphic vector bundles constructed in definition 4.16. It is a consequence of the stability of contracted products under base change.

[6] See [**Sta21**, Tag 03HV]

CHAPTER 6

Ampleness of automorphic vector bundles

In this chapter all objects will be considered over $\mathbb{F}_p$. However, since we obtain positivity results on the special fiber and since (φ, D)-ampleness is an open condition on the base, the results are also valid over $\mathbb{Z}_p$, $\mathbb{Q}_p$ and any field of characteristic p. Let $\mathrm{Sh}^{\mathrm{tor}}$ be a smooth and projective toroidal compactification of the special fiber of the Siegel variety as in definition 4.9. Let $P_0 \subset P$ denote a parabolic subgroup of type $I_0 \subset I$ and $\pi : Y_{I_0}^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$ the associated flag bundle that parametrizes P_0-reduction of the Hodge filtration over $\mathrm{Sh}^{\mathrm{tor}}$. In the chapter 5, we have recalled how [**Wed99**] [**MW04**] [**PWZ11**] [**PWZ15**][**Zha18**] [**GK19a**] [**GK19b**] [**And21**] defined smooth morphisms

$$\zeta^{\mathrm{tor}} : \mathrm{Sh}^{\mathrm{tor}} \to G\text{-Zip}^{\mathcal{Z}_\mu}$$

and

$$\zeta_{I_0}^{\mathrm{tor}} : Y_{I_0}^{\mathrm{tor}} \to G\text{-ZipFlag}^{\mathcal{Z}_\mu, P_0},$$

that allowed Goldring and Koskivirta to construct generalized Hasse invariants on the stratification of $\mathrm{Sh}^{\mathrm{tor}}$ and $Y_{I_0}^{\mathrm{tor}}$. We denote D_{red} the normal crossing Cartier divisors supported on the boundary of $\mathrm{Sh}^{\mathrm{tor}}$. Recall from corollary 4.13 that there exists an effective Cartier divisor D whose associated reduced divisor is D_{red} and an integer $\eta_0 > 0$ such that $\omega^{\otimes \eta}(-D)$ is ample on $\mathrm{Sh}^{\mathrm{tor}}$ for every $\eta \geq \eta_0$. To lighten our notations, we write D, D_{red} instead of $\pi^{-1}D, \pi^{-1}D_{\mathrm{red}}$ when no confusion is possible.

By a result of [**BGKS**], we know that the line bundles $\mathcal{L}_\lambda$ with generalized Hasse invariants are D-ample[1] on the flag bundle $Y_{I_0}^{\mathrm{tor}}$. We state and give a proof of this result in theorem 6.5. Then, we use the pushforward theorem 3.62 from chapter 3 to deduce that certain costandard automorphic bundles $\nabla(\lambda)$ are (φ, D)-ample on the Siegel variety. In the last section we compute some intersection products with the Ekedahl-Oort strata to prove that certain automorphic vector bundles are not nef.

6.1. Positive automorphic line bundles over the flag bundle

Recall the following definition from previous chapter due to Goldring and Koskivirta.

Definition 6.1 (Definition 5.32)**.** *Let λ be a character of T. For every coroot such that $\langle \lambda, \alpha^\vee \rangle \neq 0$, we set*

$$Orb(\lambda, \alpha^\vee) = \left\{ \frac{|\langle \lambda, w\alpha^\vee \rangle|}{|\langle \lambda, \alpha^\vee \rangle|} \mid w \in W \right\}$$

and we say that λ is

(1) orbitally p-close if $\max_{\alpha \in \Phi} Orb(\lambda, \alpha^\vee) \leq p - 1$,

(2) $\mathcal{Z}_\emptyset$-ample if $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in I$ and $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Phi^+ \backslash \Phi_L^+$.

It is now convenient to introduce a relevant subset of characters of P_0.

[1] See definition 3.43.

Definition 6.2. *We set*

$$\mathcal{C}_{\mathrm{ample},I_0} = \left\{\lambda \in X^*(P_0) \mid \mathcal{L}_\lambda \text{ is } D\text{-ample on } Y^{\mathrm{tor}}_{I_0}\right\}.$$

Proposition 6.3. *The subset $\mathcal{C}_{\mathrm{ample},I_0}$ is a saturated cone of $X^*(P_0)$ and we call it the D-ample cone of $Y^{\mathrm{tor}}_{I_0}$.*

Proof. It is a consequence of the proposition 3.52 and 3.53. □

Definition 6.4. *Let $\lambda \in X^*(P_0)$ and $w \in {}^{I_0}W$. We call a generalized Hasse invariant for $\mathcal{L}_\lambda$ any section s of $\mathcal{L}_\lambda^{\otimes d}$ (for some $d \geq 1$) over $\overline{Y}^{\mathrm{tor}}_{I_0,w}$ that vanishes exactly on the boundary $\overline{Y}^{\mathrm{tor}}_{I_0,w} - Y^{\mathrm{tor}}_{I_0,w}$. It follows from definition 5.28 that any $\mathcal{L}_\lambda$ with $\lambda \in \mathcal{C}_{\mathrm{Ha},I_0,w}$ admits a generalized Hasse invariant obtained as a pullback by $\zeta^{\mathrm{tor}}_{I_0}$.*

We can now state and give a proof of the main result of [**BGKS**].

Theorem 6.5 ([**BGKS**])**.** *If $\lambda \in X^*(P_0)$ is a character in $\mathcal{C}_{\mathrm{Ha},I_0}$, then $\mathcal{L}_\lambda$ is D-ample on $Y^{\mathrm{tor}}_{I_0}$. In other words, we have an inclusion*

$$\mathcal{C}_{\mathrm{Ha},I_0} \subset \mathcal{C}_{\mathrm{ample},I_0}.$$

Proof. We start by proving that $\mathcal{L}_\lambda$ is nef on $Y^{\mathrm{tor}}_{I_0}$ for any $\lambda \in \mathcal{C}_{\mathrm{Ha},I_0}$. Let V be a subvariety of $Y^{\mathrm{tor}}_{I_0}$ and consider the minimal element w of ${}^{I_0}W$ such that $V \subset \overline{Y}^{\mathrm{tor}}_{I_0,w}$. Such an element always exists because $\overline{Y}^{\mathrm{tor}}_{I_0,w_0} = Y^{\mathrm{tor}}_{I_0}$. We consider a generalized Hasse invariant $s \in H^0(\overline{Y}^{\mathrm{tor}}_{I_0,w}, \mathcal{L}_\lambda^{\otimes d})$ (for some $d \geq 1$) and we claim that the restriction $s_{|V}$ is not identically zero. If it were, we would have

$$V \subset \overline{Y}^{\mathrm{tor}}_{I_0,w} - Y^{\mathrm{tor}}_{I_0,w} = \bigsqcup_{w' \preccurlyeq w, w' \neq w} Y^{\mathrm{tor}}_{I_0,w'},$$

which would contradict the minimality of w (V is irreducible). In particular, we have shown that $\mathcal{L}_\lambda$ is nef. Let λ be a character in $\mathcal{C}_{\mathrm{Ha},I_0}$. Recall that $\eta_0 \geq 1$ is an integer such that $\omega^{\otimes \eta}(-D)$ is ample for all $\eta \geq \eta_0$. Since $\mathcal{L}_\lambda$ is π-ample, we deduce that

$$\mathcal{L}_\lambda \otimes (\pi^*\omega^{\otimes\eta}(-D))^{\otimes m}$$

is ample on $Y^{\mathrm{tor}}_{I_0}$ for m large enough. Since λ belongs to $\mathcal{C}_{\mathrm{Ha},I_0}$ and the inequalities that define $\mathcal{C}_{\mathrm{Ha},I_0}$ are strict, we know that for all n large enough,

$$\mathcal{L}_\lambda^{\otimes n} \otimes \pi^*\omega^{\otimes -\eta}$$

has generalized Hasse invariants for all strata $Y^{\mathrm{tor}}_{I_0,w}$, so it is nef. Hence, we know

$$\mathcal{L}_\lambda \otimes (\pi^*\omega^{\otimes\eta}(-D))^{\otimes m} \otimes (\mathcal{L}_\lambda^{\otimes n} \otimes \pi^*\omega^{\otimes -\eta})^{\otimes m} = \mathcal{L}_\lambda^{\otimes nm+1}(-mD)$$

is ample on $Y^{\mathrm{tor}}_{I_0}$ for n, m large enough. We consider some integer $n_0, m_0 \geq 1$ such that $\mathcal{L}_\lambda^{\otimes n_0 m_0 + 1}(-m_0 D)$ is ample. Since $\mathcal{L}_\lambda$ is nef, we must have $\mathcal{L}_\lambda^{\otimes\eta}(-m_0 D)$ ample for all $\eta \geq n_0 m_0 + 1$. In particular, $\mathcal{L}_\lambda$ is $m_0 D$-ample, hence D-ample by proposition 3.48. □

Using proposition 5.34 for the existence of generalized Hasse invariants on the stack G-ZipFlag$^{\mathcal{Z}_\mu, P_0}$, we get

Theorem 6.6. *Let λ be a character of P_0. If λ is orbitally p-close and $\mathcal{Z}_0$-ample, then $\mathcal{L}_\lambda$ is D-ample on $Y^{\mathrm{tor}}_{I_0}$.*

6.2. Application to automorphic vector bundles

We can now state and prove our main result of this chapter.

Theorem 6.7. *Let λ be a dominant character of T.*

(1) If λ is a positive parallel weight, i.e. $\lambda = k(1, \cdots, 1)$ with $k < 0$, or

(2) if $2\lambda + 2\rho_L$ is orbitally p-close and $\mathcal{Z}_\emptyset$-ample

then the automorphic vector bundle $\nabla(\lambda)$ is (φ, D)-ample on $\mathrm{Sh}^{\mathrm{tor}}$.

Proof. This a direct consequence from theorem 3.62 and proposition 6.6. □

We represent the weights $\lambda = (k_1, k_2)$ such that the automorphic bundle $\nabla(\lambda)$ is (φ, D)-ample on the Siegel threefold over $\mathbb{F}_p$ for different values of p.

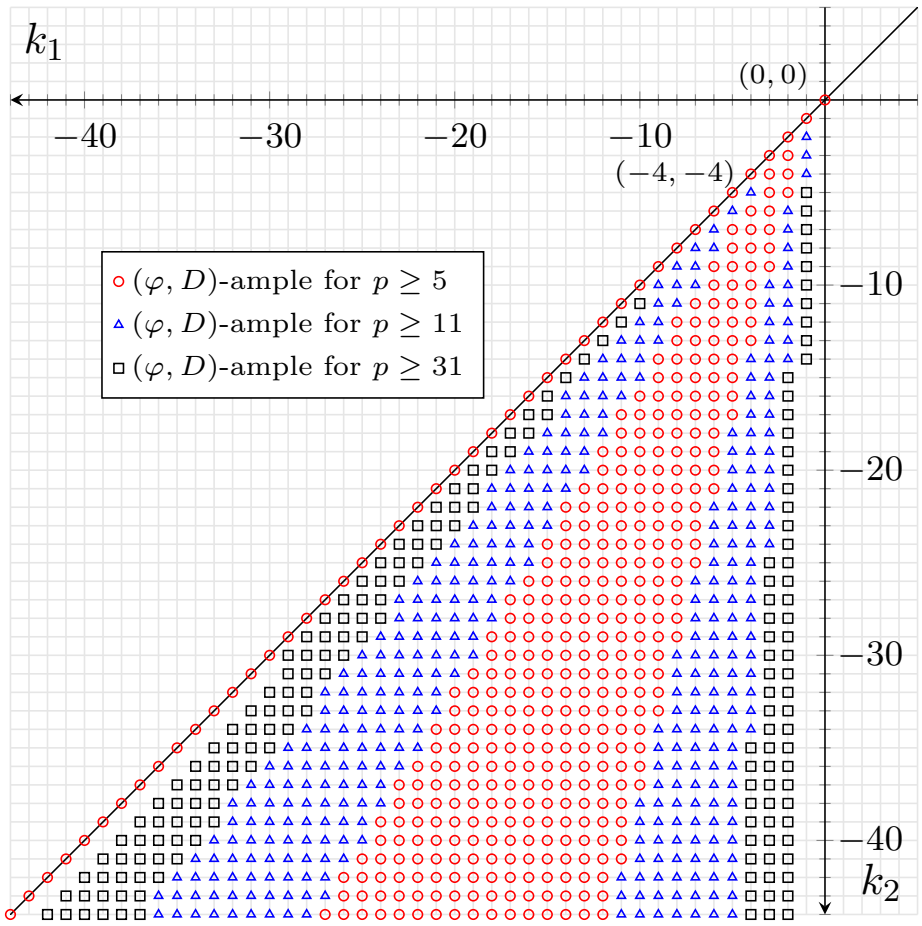


Figure 6.2.1. (φ, D)-ampleness of automorphic bundles $\nabla(\lambda)$ when $g = 2$.

We represent the weights $\lambda = (k_1, k_2, k_3)$ such that the automorphic bundle $\nabla(\lambda)$ is (φ, D)-ample on the Siegel variety of genus 3 over $\mathbb{F}_p$ for different values of p.

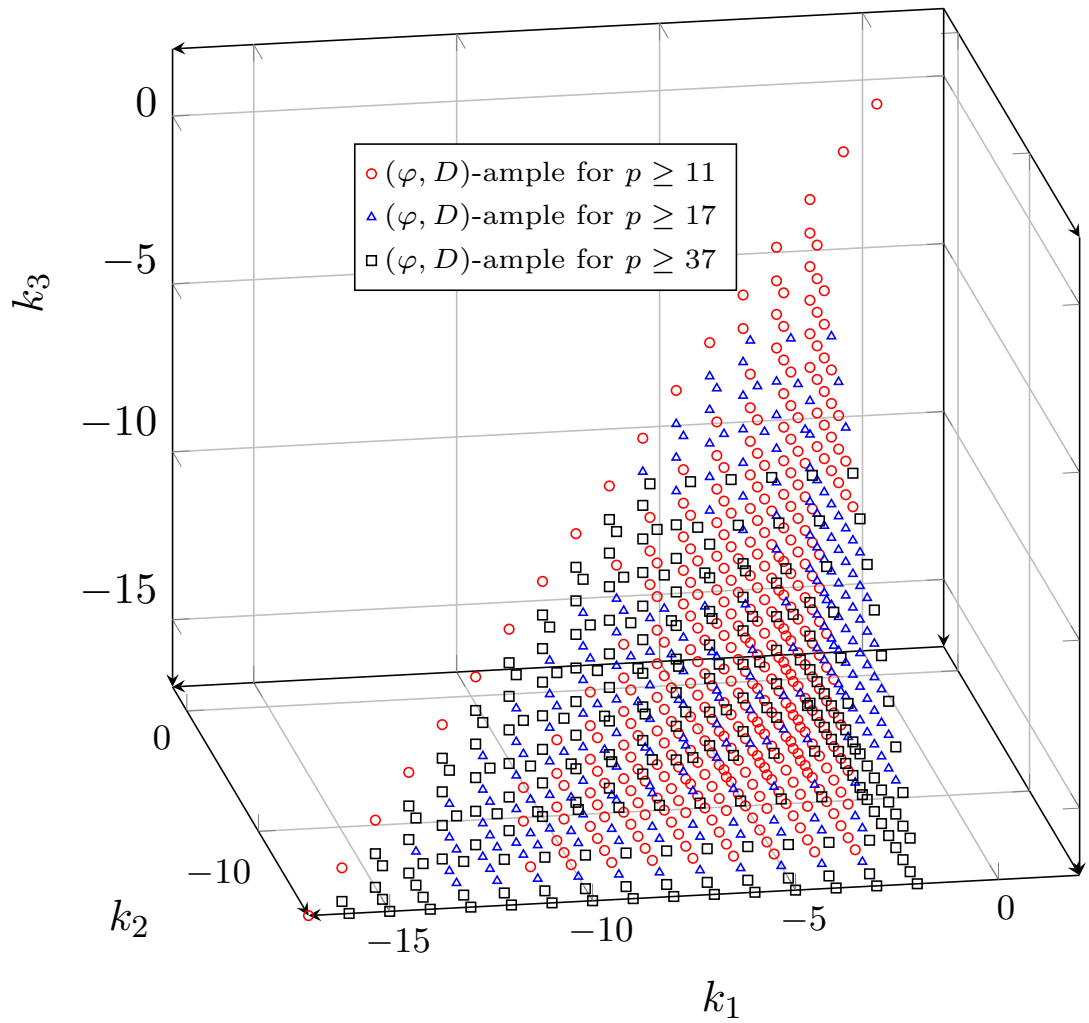


FIGURE 6.2.2. (φ, D)-ampleness of automorphic bundles $\nabla(\lambda)$ when $g = 3$.

We notice that, as p grows, there are more and more weights λ in the anti-dominant chamber such that $\nabla(\lambda)$: this is a consequence of the orbitally p-closeness condition.

6.3. Intersection computations on Ekedahl-Oort strata

We will use the results of [**WZ18**] to do some computation on the Chow $\mathbb{Q}$-algebra of the partial flag bundle $\mathbb{P}(\Omega^{\mathrm{tor}})$. Recall that I denotes the type of the parabolic subgroup $P = P_{-\mu}$ of Sp_{2g}. Let I_0 denote a subset of I and recall the morphisms

$$\zeta : \mathrm{Sh}^{\mathrm{tor}} \to \mathrm{Sp}_{2g}\text{-}\mathrm{Zip}^{\mu}$$

and

$$\zeta_{I_0} : Y_{I_0}^{\mathrm{tor}} \to \mathrm{Sp}_{2g}\text{-}\mathrm{ZipFlag}^{\mu, I_0}$$

as defined in chapter 5. For all $w \in {}^I W$, recall that $\mathrm{Sh}_w^{\mathrm{tor}} := \zeta^{-1}([w])$ is the EO stratum of the Siegel variety where $[w] \subset \mathrm{Sp}_{2g}\text{-}\mathrm{Zip}^{\mu}$ is the corresponding substack. More generally[2], for

[2] If $I_0 = I$, then $Y_{I_0}^{\mathrm{tor}} = \mathrm{Sh}^{\mathrm{tor}}$

all $w \in {}^{I_0}W$, recall that $Y^{\mathrm{tor}}_{I_0,w} := \zeta_{I_0}^{-1}([w])$ is the EO stratum of the partial flag bundle of type $I_0 \subset I$ where $[w] \subset \mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,I_0}$ is the corresponding substack. The morphism ζ_{I_0} induces a pullback map on the corresponding Chow $\mathbb{Q}$-algebra

$$A^\bullet(\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,I_0}) \xrightarrow{\zeta_{I_0}^*} A^\bullet(Y^{\mathrm{tor}}_{I_0})$$

and we call the image of $\zeta_{I_0}^*$, the tautological ring $\mathcal{T}_{I_0}$ of $Y^{\mathrm{tor}}_{I_0}$. Clearly, the Chow $\mathbb{Q}$-algebra of $\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,I_0}$ is generated by the cycle classes of the EO strata $\overline{[w]}$ for $w \in {}^{I_0}W$ but we would like another description relying on chern classes of automorphic bundles. We have a morphism of $\mathbb{Q}$-vector spaces

$$\begin{aligned} c_1 : X^*(T) &\longrightarrow A^1(\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,\emptyset}) \\ \lambda &\longmapsto c_1(\mathcal{L}_\lambda) \end{aligned}$$

which induces a morphism of $\mathbb{Q}$-algebras $S \to A^\bullet(\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,\emptyset})$ where $S = \mathrm{Sym}\, X^*(T)$ is the symmetric algebra of the characters of T. By [**WZ18**, Theorem 3], this map is surjective with kernel generated by the W-invariant elements of degree > 0. We deduce a description of the Chow $\mathbb{Q}$-algebra of $\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,\emptyset}$ as

$$\begin{tikzcd} S \arrow[r] \arrow[d] & A^\bullet(\mathrm{Sp}_{2g}\text{-ZipFlag}^{\mu,\emptyset}) \\ S/\mathcal{I}S \arrow[ur] & \end{tikzcd}$$

where $\mathcal{I}$ is the augmentation ideal of the W-invariant elements of S. This ideal admits an explicit description as the augmentation ideal of a polynomial algebra

$$\mathcal{I} = \mathbb{Q}[f_1, \cdots, f_g]_{\geq 1} \quad f_i = x_1^{2i} + \cdots + x_g^{2i}.$$

In particular, the tautological ring $\mathcal{T}_\emptyset$ is generated as a $\mathbb{Q}$-algebra by the cycle classes of the closed EO strata $\overline{Y^{\mathrm{tor}}_{\emptyset,w}}$ and by the chern classes of the automorphic line bundles $c_1(\mathcal{L}_\lambda)$. The goal is now to express $[\overline{Y^{\mathrm{tor}}_{\emptyset,w}}]$ as an element of $S/\mathcal{I}S$ and to compute intersection products of the form

$$c_1(\mathcal{L}_\lambda)^{l(w)} \cdot [\overline{Y^{\mathrm{tor}}_{\emptyset,w}}].$$

Following the strategy of [**WZ18**], we have implemented on SageMath an algorithm which computes $[\overline{Y^{\mathrm{tor}}_{\emptyset,w}}]$ as an element of $S/\mathcal{I}S$. In order to be more explicit, we choose a system of positive roots in a way to obtain:

$$I = \{e_i - e_{i+1} \mid i = 1, \cdots g-1\} \subset \Delta = \{e_i - e_{i+1} \mid i = 1, \cdots g-1\} \cup \{2e_g\}.$$

The Weyl group $W = S_g \ltimes (\mathbb{Z}/2\,\mathbb{Z})^g$ contains $2^g g!$ elements we can write as a product of the simple reflections $s_1, s_2, \cdots, s_g$ associated to $e_1 - e_2, \cdots, e_{g-1} - e_g, 2e_g$.

6.3.1. The case $g = 2$. We represent the Weyl group of Sp_4 with the following diagram

$$\begin{tikzcd}
 & w_0 = s_2s_1s_2s_1 \arrow[dl] \arrow[dr] & \\
w_1 = s_2s_1s_2 \arrow[d] \arrow[drr] & & w_1' = s_1s_2s_1 \arrow[d] \arrow[dll] \\
w_2 = s_2s_1 \arrow[d] \arrow[drr] & & w_2' = s_1s_2 \arrow[d] \arrow[dll] \\
w_3 = s_2 \arrow[dr] & & w_3' = s_1 \arrow[dl] \\
 & e &
\end{tikzcd}$$

where an arrow is drawn from w to w' if $w' \leq w$ and $l(w') = l(w) - 1$. Consider the line bundle $\mathcal{L}_\lambda$ on $Y_\emptyset^{\mathrm{tor}} = \mathbb{P}(\Omega^{\mathrm{tor}})$ where $\lambda = (k_1, k_2)$ and recall that $\mathcal{L}_{\lambda_\Omega} = \mathcal{L}_{(0,-1)} = \mathcal{O}(1)$. In the graded algebra

$$\mathcal{T}_\emptyset = \mathbb{Q}[x_1, x_2]/(x_1^2 + x_2^2, x_1^2x_2^2),$$

we have the following formulas

$$\begin{cases}
[\overline{Y_{\emptyset,w_0}^{\mathrm{tor}}}] = 1, \\
[\overline{Y_{\emptyset,w_1}^{\mathrm{tor}}}] = x_1 - px_2, \\
[\overline{Y_{\emptyset,w_1'}^{\mathrm{tor}}}] = -(p-1)(x_1 + x_2), \\
[\overline{Y_{\emptyset,w_2}^{\mathrm{tor}}}] = -(p-1)(px_1 + x_2)x_1, \\
[\overline{Y_{\emptyset,w_2'}^{\mathrm{tor}}}] = (p-1)(px_2 - x_1)x_1, \\
[\overline{Y_{\emptyset,w_3}^{\mathrm{tor}}}] = (p^2-1)(px_2 - x_1)x_1^2, \\
[\overline{Y_{\emptyset,w_3'}^{\mathrm{tor}}}] = (p^2+1)(p-1)(x_1^3 + x_2^3), \\
[\overline{Y_{\emptyset,e}^{\mathrm{tor}}}] = (p^4-1)x_1x_2^3.
\end{cases}$$

Since the cycle $x_1x_2^3$ has positive degree and since we are only concerned with the sign of the intersection products, we make the identification $x_1x_2^3 = 1$ and we get the following intersection products.

$$\begin{cases}
c_1(\mathcal{L}_\lambda)^4 \cdot [\overline{Y_{\emptyset,w_0}^{\mathrm{tor}}}] = (k_1x_1 + k_2x_2)^4 = 4(k_1k_2^3 - k_1^3k_2), \\
c_1(\mathcal{L}_\lambda)^3 \cdot [\overline{Y_{\emptyset,w_1}^{\mathrm{tor}}}] = (pk_1^3 - 3pk_1k_2^2 - 3k_1^2k_2 + k_2^3), \\
c_1(\mathcal{L}_\lambda)^3 \cdot [\overline{Y_{\emptyset,w_1'}^{\mathrm{tor}}}] = (p-1)(k_1^3 + 3k_1^2k_2 - 3k_1k_2^2 - k_2^3), \\
c_1(\mathcal{L}_\lambda)^2 \cdot [\overline{Y_{\emptyset,w_2}^{\mathrm{tor}}}] = (p-1)(k_1^2 - k_2^2 + 2pk_1k_2), \\
c_1(\mathcal{L}_\lambda)^2 \cdot [\overline{Y_{\emptyset,w_2'}^{\mathrm{tor}}}] = (p-1)(p(k_2^2 - k_1^2) + 2k_1k_2), \\
c_1(\mathcal{L}_\lambda) \cdot [\overline{Y_{\emptyset,w_3}^{\mathrm{tor}}}] = (p^2-1)(k_2 - pk_1), \\
c_1(\mathcal{L}_\lambda) \cdot [\overline{Y_{\emptyset,w_3'}^{\mathrm{tor}}}] = (p-1)(p^2+1)(k_1 - k_2).
\end{cases}$$

6.3.2. The case $g = 3$. The degree 9 part of the graded algebra $\mathcal{T}_\emptyset = \mathbb{Q}[x_1, x_2, x_3]/\mathcal{I}$ is a $\mathbb{Q}$-vector space of dimension 1 generated by $x_1^5 x_2^3 x_3$. We have

$$\overline{[Y^{\mathrm{tor}}_{\emptyset,e}]} = \underbrace{(p^9 - p^8 + p^7 - p^6 - p^3 + p^2 - p + 1)}_{>0} x_1^5 x_2^3 x_3$$

and since this polynomial in p is always positive, we may identify $x_1^5 x_2^3 x_3$ with 1. Let λ denote the character $(k_1 \geq k_2 \geq k_3)$. We have the following intersection product

$$c_1(\mathcal{L}_\lambda) \cdot \overline{[Y^{\mathrm{tor}}_{\emptyset,s_3}]} = k_1 \underbrace{(-p^8 + p^7 + p^2 - p)}_{<0} + k_3 \underbrace{(p^7 - p^6 - p + 1)}_{>0},$$

which shows that $\mathcal{L}_\lambda$ is not nef when $k_1 = 0$ and $k_3 < 0$.

6.3.3. The case $g = 4$. The degree 16 part of the graded algebra $\mathcal{T}_\emptyset = \mathbb{Q}[x_1, x_2, x_3, x_4]/\mathcal{I}$ is a $\mathbb{Q}$-vector space of dimension 1 generated by $x_1^7 x_2^5 x_3^3 x_4$. We have

$$\overline{[Y^{\mathrm{tor}}_{\emptyset,e}]} = \underbrace{(p^{16} + p^{14} - p^{10} - 2p^8 - p^6 + p^2 + 1)}_{>0} x_1^7 x_2^5 x_3^3 x_4$$

and since this polynomial in p is always positive, we may identify $x_1^7 x_2^5 x_3^3 x_4$ with 1. Let λ denote the character $(k_1 \geq k_2 \geq k_3 \geq k_4)$. We have the following intersection product

$$c_1(\mathcal{L}_\lambda) \cdot \overline{[Y^{\mathrm{tor}}_{\emptyset,s_4}]} = k_1 \underbrace{(-p^{15} + p^9 + p^7 - p)}_{<0} + k_4 \underbrace{(p^{14} - p^8 - p^6 + 1)}_{>0},$$

which shows that $\mathcal{L}_\lambda$ is not nef when $k_1 = 0$ and $k_4 < 0$. We are now able to prove the following results.

Proposition 6.8. *Assume that $g \in \{2, 3, 4\}$. Any non-zero automorphic bundle $\nabla(k_1, \cdots, k_g)$ on $\mathrm{Sh}^{\mathrm{tor}}$ where $k_1 = 0$ is not nef.*

Remark 6.9.
(1) We believe that this result generalizes to every $g \geq 2$.
(2) In particular, we know that the bundle $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \nabla(0, \cdots, 0, \text{-}2)$ is not nef when $g \in \{2, 3, 4\}$. The proposition 6.10 imply that the cotangent bundle is never nef when $g > 1$.

Proof. Consider a dominant character λ of T and write $I_0 \subset I$ for the set of simple roots such that $\langle \lambda, \alpha^\vee \rangle = 0$. As a consequence, the line bundle $\mathcal{L}_\lambda$ on $Y^{\mathrm{tor}}_{I_0}$ is relatively π-ample which implies that we have a surjective map for some $n \geq 1$ large enough

$$\pi^* \pi_* \mathcal{L}_\lambda^{\otimes n} = \pi^* \nabla(n\lambda) \to \mathcal{L}_\lambda^{\otimes n}.$$

In particular, if $\nabla(\lambda)$ was nef, it would imply that $\nabla(\lambda)^{\otimes n}$, hence $\nabla(n\lambda)$ and $\mathcal{L}_\lambda$ would be nef. We are reduced to show the non-nefness of $\mathcal{L}_\lambda$ on $Y^{\mathrm{tor}}_{I_0}$, which can be tested on $Y^{\mathrm{tor}}_\emptyset$. We have seen in the cases $g = 2, 3$ that we can always find a EO stratum $Y^{\mathrm{tor}}_{I_0,w}$ such that the following intersection product is negative

$$c_1(\mathcal{L}_\lambda)^{l(w)} \cdot \overline{[Y^{\mathrm{tor}}_{I_0}]} < 0.$$

□

Proposition 6.10. *For any $g \geq 2$, the bundle $\Omega^{\mathrm{tor}} = \nabla(0, \cdots, 0, \text{-}1)$ is not nef.*

Proof. We add a subscript g to our notations to specify the genus. Recall the Kodaira-Spencer isomorphism

$$\Omega^1_{\mathrm{Sh}_g^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \mathrm{Sym}^2\,\Omega_g^{\mathrm{tor}}$$

from proposition 4.20. We assume that $g \geq 2$. Take an abelian variety A_0 of dimension $g-2$ over $\mathbb{F}_p$ and consider the closed immersion $\iota_{A_0} : \mathrm{Sh}_2^{\mathrm{tor}} \to \mathrm{Sh}_g^{\mathrm{tor}}$ that sends an abelian surface A to the fibre product $A \times_{\mathbb{F}_p} A_0$. The closed immersion ι_{A_0} induces a surjective morphism

$$\iota_{A_0}^* \underbrace{\Omega^1_{\mathrm{Sh}_g^{\mathrm{tor}}}(\log D_{\mathrm{red}})}_{=\mathrm{Sym}^2\,\Omega_g^{\mathrm{tor}}} \twoheadrightarrow \Omega^1_{\mathrm{Sh}_2^{\mathrm{tor}}}(\log D_{\mathrm{red}})$$

and if Ω_g^{tor} was nef, it would imply that $\Omega^1_{\mathrm{Sh}_2^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \mathrm{Sym}^2\,\Omega_2^{\mathrm{tor}} = \nabla(0, \text{-}2)$ is nef, which contradicts the previous proposition. $\square$

CHAPTER 7

Vanishing of coherent cohomology

Let p be a prime number and $N \geq 3$ be an integer such that $p \nmid N$. Consider a smooth and projective toroidal compactification $\mathrm{Sh}^{\mathrm{tor}}$ of the Siegel variety of level N and genus $g \geq 1$ over $\mathbb{F}_p$. Lan and Suh [**LS12**][**LS13**] and Stroh [**Str13**] prove many vanishing results for the coherent cohomology of $\mathrm{Sh}^{\mathrm{tor}}$. We note that Lan and Suh deal more generally with PEL Shimura varieties and, in the Siegel case, they were able to access automorphic bundles $\Delta(\lambda)^\vee$ in all Weyl chambers as long as the weight λ can be written

$$\lambda = w \cdot \mu + \underline{k} \tag{7.1}$$

where w is an element of the minimal left coset representatives ${}^I W$ of type I, μ is a sufficiently regular weight[1] which is p-small for Sp_{2g} and such that $|\mu|_{\mathrm{re},+}$[2] $< p$ and $\underline{k}$ is a positive parallel weight[3]. They use results from [**PT02**] on dual Bernstein-Gelfand-Gelfand complexes and a geometric plethysm that imposes many restriction on the size of the weight compared to p. We note that for such a weight λ, we have $\Delta(\lambda)^\vee = \nabla(-w_0\lambda) = \Delta(-w_0\lambda)$. Stroh's method is similar to Lan and Suh's although he restricted himself to the case of the anti-dominant Weyl chamber in order to prove a classicity result in Hida theory. As there is only a finite number of p-small characters for Sp_{2g}, their method access only a finite number of weights up to positive parallel weights. The main result of this chapter is a general recipe to produce new vanishing results from old ones. Namely, we define a non-decreasing function $g_{I_0,e}$ on the power set of characters

$$g_{I_0,e} : \mathcal{P}(X^*) \to \mathcal{P}(X^*)$$

that depends on a subset $I_0 \subset I$ and an integer $0 \leq e \leq d-1$ where $d = g(g+1)/2$ is the dimension of $\mathrm{Sh}^{\mathrm{tor}}$.

Theorem (Theorem 7.19)**.** *Assume that $p > g^2$. Let $\mathcal{C}$ be a set of characters λ for which the cohomology $H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\lambda))$ is concentrated in degrees $[0, e+1]$. Then the image of $\mathcal{C}$ by the function $g_{I_0,e}$ is a set of characters λ for which the cohomology $H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\lambda))$ is concentrated in degrees $[0, e]$.*

Moreover, in the extreme cases $e = 0$ and $e = d-1$, our method produces new vanishing results without any prior knowledge. These results can then be used in the other cases $0 < e < d-1$. As an example, see the figure 7.5.1 for the vanishing results we are able to obtain in the case $g = 2$, $p = 5$ without initialization. The accessible weights with this method are not necessarily regular and not necessarily p-small (even up to a positive parallel weight) but they belong to the anti-dominant Weyl chamber[4]. Since the definition of the function $g_{I_0,e}$ is hard to grasp, we have implemented on SageMath an algorithm[5] that computes the

[1] See [**LS12**, Def. 7.18].

[2] See [**LS12**, 7.22].

[3] It means of the form $(k, k, \cdots, k)$ for some $k > 0$.

[4] It corresponds to the dominant Weyl chamber in the work of Lan and Suh.

[5] See github.com/ThibaultAlexandre/vanishing-results-over-the-siegel-variety

vanishing results with our method. We were able to access automorphic bundles $\nabla(\lambda)$ where λ is not necessarily of the form $w \cdot \mu + \underline{k}$ as in equation (7.1). In particular, we have no reason to expect that $\Delta(\lambda) = \nabla(\lambda)$. The p-smallness restriction is replaced with the much weaker restriction from the theory of G-Zip called orbitally p-closeness. Before explaining our method to deduce new vanishing results, we start with some preliminary results concerning the spectral sequence associated to the cohomology of a filtered sheaf. In section 7.5, we present our algorithm and plot some of the vanishing results we were able to get without any prior knowledge.

7.1. Spectral sequence associated to a filtered sheaf

We consider a scheme morphism $f : X \to S$ and a sheaf $\mathcal{F}$ on X endowed with an increasing filtration $\mathcal{F}_\bullet$ with graded pieces $\mathrm{gr}_k = \mathcal{F}_k/\mathcal{F}_{k-1}$ where $k \in \mathbb{Z}$.

Proposition 7.1. *There is a spectral sequence starting at page* 2

$$E_2^{t,k} = R^{t+k} f_*(\mathrm{gr}_k) \Rightarrow R^{t+k} f_*(\mathcal{F}).$$

Proof. This result is well-known: see the appendix of [**EV92**] for example. We just recall how the differentials of the second page are defined. For all $k \in \mathbb{Z}$, there is an exact sequence

$$0 \to \mathcal{F}_{k-1}/\mathcal{F}_{k-2} \to \mathcal{F}_k/\mathcal{F}_{k-2} \to \mathcal{F}_k/\mathcal{F}_{k-1} \to 0$$

and the differentials are the connecting morphisms

$$\forall i \geq 0 \ R^i f_*(\mathrm{gr}_k) \to R^{i+1} f_*(\mathrm{gr}_{k-1}).$$

□

From the study of this spectral sequence, we deduce several results.

Lemma 7.2. *Let $i_0 \geq 0$ and assume*

$$\forall k \in \mathbb{Z} \ R^{i_0} f_*(\mathrm{gr}_k) = 0.$$

Then,

$$R^{i_0} f_*(\mathcal{F}) = 0.$$

Proof. We pass from a page of a spectral sequence to the next one by taking cohomology and since $E_2^{i_0-k,k} = R^{i_0} f_*(\mathrm{gr}_k) = 0$ for all $k \in \mathbb{Z}$, we have

$$\forall a \geq 2 \ \forall k \in \mathbb{Z} \ E_a^{i_0-k,k} = 0.$$

Thus,

$$\forall a \geq 2 \ \forall k \in \mathbb{Z} \ E_\infty^{i_0-k,k} = 0$$

and

$$R^{i_0} f_*(\mathcal{F}) = 0.$$

□

Lemma 7.3. *Let $i_0 \geq 0$ and assume that there exists $n \in \mathbb{Z}$ such that for all $k > n$, $\mathrm{gr}_k = 0$. If*

$$\begin{cases} R^{i_0} f_*(\mathcal{F}) = 0, \\ \forall k \leq n-1 \ R^{i_0+1} f_*(\mathrm{gr}_k) = 0, \end{cases}$$

then

$$R^{i_0} f_*(\mathrm{gr}_n) = 0.$$

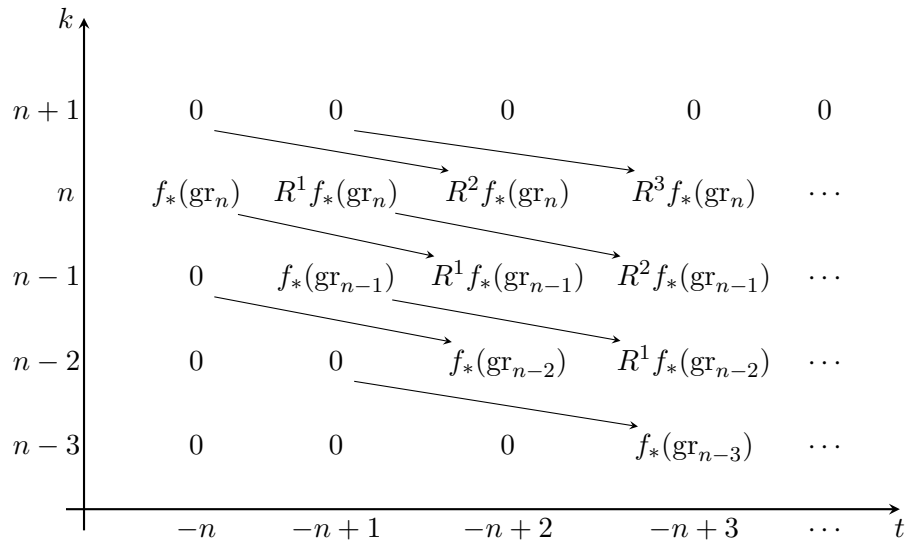


FIGURE 7.1.1. E_2-page of the spectral sequence.

PROOF. For a visual support, see the figure 7.1.1. From the hypothesis on the graded pieces, we know that for all $a \geq 2$, the differential with target $E_a^{-n+i_0,n}$ vanishes. Since for all $k \leq n-1$ we have $R^{i_0+1}f_*(\mathrm{gr}_k) = 0$, then for all $a \geq 2$ the differential with source $E_a^{-n+i_0,n}$ must vanish. Thus, we get $R^{i_0}f_*(\mathrm{gr}_n) = E_2^{-n+i_0,n} = E_\infty^{-n+i_0,n}$ and $E_\infty^{-n+i_0,n} = 0$ as a graded piece of $R^{i_0}f_*(\mathcal{F})$.

□

7.2. A logarithmic Kodaira-Nakano vanishing theorem in positive characteristic

In this section, we review the Kodaira-Nakano vanishing theorem in positive characteristic due to Deligne and Illusie in [**DI87**] and a logarithmic version due to Esnault and Viehweg in [**EV92**]. Let X be a smooth projective variety of dimension n over a perfect field k of characteristic p. Let D_{red} be a normal crossing divisor of X. We have an open immersion $\tau : U := X - D_{\mathrm{red}} \to X$.

Proposition 7.4. *Recall that X is a smooth projective variety over k and let $\mathcal{L}$ be an ample line bundle over X. Denote by d the dimension of X. Assume that $(X, \mathcal{L})$ lifts to the Witt vectors $W_2(k)$ and $p \geq d$, then*

$$\forall i + j > d, \qquad H^i(X, \Omega_X^j \otimes \mathcal{L}) = 0.$$

PROOF. The detailed proof can be found in [**DI87**]. □

Over the flag bundle $Y_{I_0}^{\mathrm{tor}}$ of the toroidal compactification of the Siegel variety, we have seen that certain line bundles $\mathcal{L}_\lambda$ are D-ample for some effective divisor D supported on the boundary. This motivates[6] this refined version of the result of Deligne and Illusie due to Esnault and Viehweg.

Proposition 7.5. *Recall that X is a smooth projective variety over k. Recall that D_{red} denotes a normal crossing divisor on X. Let D be an effective Cartier divisor whose associated*

[6]This result already appears in [**LS13**].

reduced divisor is D_{red} and let $\mathcal{L}$ be a D-ample line bundle on X. Denote by d the dimension of X. Assume that the triple $(X, D_{\mathrm{red}}, \mathcal{L})$ lifts to $W_2(k)$ and $p \geq d$, then

$$\forall i + j > d, \quad H^i(X, \Omega^j_X(\log D_{\mathrm{red}}) \otimes \mathcal{L}(-D_{\mathrm{red}})) = 0.$$

Proof. The proof of [**EV92**, Proposition 11.5] shows that

$$\forall i + j < \min(d, p), \quad H^i(X, \Omega^j_X(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{-1}) = 0,$$

which is equivalent to

$$\forall i + j > \max(2d - p, d), \quad H^i(X, \Omega^j_X(\log D_{\mathrm{red}}) \otimes \mathcal{L}(-D_{\mathrm{red}})) = 0$$

by Serre duality. We use that for all $i + j = n$, the pairing $\Omega^i_X(\log D_{\mathrm{red}}) \otimes \Omega^j_X(\log D_{\mathrm{red}}) \to \Omega^n_X(\log D_{\mathrm{red}})$ mapping $\alpha \otimes \beta$ to $\alpha \wedge \beta$ is perfect. □

Remark 7.6. Motivated by proposition 3.46, one might be tempted to replace the assumption D-ample by nef and big. However, the proposition 7.5 requires a normal crossing divisor.

7.3. The general case

The goal of this subsection is to explain how to deduce new vanishing results for the coherent cohomology from known ones. We recall the notations. The variety $\mathrm{Sh}^{\mathrm{tor}}$ is a smooth projective toroidal compactification of the Siegel variety over $\mathbb{F}_p$ of genus $g \geq 2$ and level N and $\pi : Y^{\mathrm{tor}}_{I_0} \to \mathrm{Sh}^{\mathrm{tor}}$ is the flag bundle in P/P_0 where $P_0 \subset \mathrm{Sp}_{2g}$ is a parabolic subgroup of type $I_0 \subset I \subset \Delta$ which is contained in the parabolic $P \subset \mathrm{Sp}_{2g}$ of type I. We denote D_{red} the normal crossing divisor supported on the boudary of $\mathrm{Sh}^{\mathrm{tor}}$. We use the same notation D_{red} for the normal crossing divisor $\pi^{-1}D_{\mathrm{red}}$ of $Y^{\mathrm{tor}}_{I_0}$ when no confusion is possible. We denote d, d_0 the dimension of $\mathrm{Sh}^{\mathrm{tor}}$, $Y^{\mathrm{tor}}_{I_0}$ and $r_0 = d_0 - d$ the relative dimension of π. We choose a system of positive roots in a way to obtain

$$I = \{e_i - e_{i+1} \mid i = 1, \cdots g-1\} \subset \Delta = \{e_i - e_{i+1} \mid i = 1, \cdots g-1\} \cup \{2e_g\}.$$

The Levi subgroup L of $P \subset \mathrm{Sp}_{2g}$ is GL_g and to each representation V of L, we have an associated vector bundle $\mathcal{W}(V)$ on $\mathrm{Sh}^{\mathrm{tor}}$. With our conventions, the Hodge bundle Ω^{tor} is the vector bundle of rank g associated to the standard representation std_L of L. To each character λ of P_0, we have an associated line bundle $\mathcal{L}_\lambda$ on $Y^{\mathrm{tor}}_{I_0}$. Assuming that $p \geq d_0$, the basic idea is to use the logarithmic Kodaira-Nakano vanishing theorem (see proposition 7.5) on the flag bundle $Y^{\mathrm{tor}}_{I_0}$ with D-ample line bundle $\mathcal{L}_\lambda$. Since the determinant of

$$\Omega^1_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}})$$

is a line bundle over $Y^{\mathrm{tor}}_{I_0}$, it is not hard to express it as an automorphic bundle and it provides vanishing results for cohomology groups H^i with $i > 0$. The accessible weights with this method are regular. To access less regular weights, a natural idea is to use the logarithmic Kodaira-Nakano vanishing theorem for

$$\Omega^m_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}})$$

with $m < d_0$. However, this bundle is not a line bundle and doesn't seem related to automorphic bundles (see remark 7.12). A solution is to filter it by automorphic vector bundles and then use the associated spectral sequence. The following result is well-known but since we haven't found a reference, we give a proof.

Lemma 7.7. *We have an exact sequence of vector bundles*

$$0 \longrightarrow \pi^*\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \longrightarrow \Omega^1_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}) \longrightarrow \Omega^1_{Y^{\mathrm{tor}}_{I_0}/\,\mathrm{Sh}^{\mathrm{tor}}} \longrightarrow 0.$$

Proof. By [Del70, II. §3.], we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}} \arrow[r] \arrow[d, "a"] & \pi^*\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \arrow[r] \arrow[d, "b"] & \pi^*\mathcal{O}_{D_{\mathrm{red}}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_{Y^{\mathrm{tor}}_{I_0}} \arrow[r] & \Omega^1_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}) \arrow[r] & \mathcal{O}_{\pi^{-1}D_{\mathrm{red}}} \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact (use also that π is flat for the first row). Since π is smooth, $\ker a = 0$ and by the snake lemma, the sequence

$$0 \longrightarrow \ker b \longrightarrow 0 \longrightarrow \Omega^1_{Y^{\mathrm{tor}}_{I_0}/\,\mathrm{Sh}^{\mathrm{tor}}} \longrightarrow \Omega^1_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}})/\pi^*\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \longrightarrow 0$$

is exact. The desired exact sequence is obtained from b. □

Definition 7.8. *Let $e \geq 0$ be an integer. We define an increasing filtration $F_\bullet$ of $\Omega^{d_0-e}_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}})$ by*

$$F_k = \pi^*\Omega^{d_0-e-k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \wedge \Omega^k_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}),$$

with graded pieces

$$\mathrm{gr}_k = \pi^*\Omega^{d_0-e-k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^k_{Y^{\mathrm{tor}}_{I_0}/\,\mathrm{Sh}^{\mathrm{tor}}}.$$

From the proposition 7.1, we get an associated spectral sequence starting at page 2 for each $\lambda \in X^*(P_0)$

$$E^{t,k}_{2,e,\lambda} = H^{t+k}(Y^{\mathrm{tor}}_{I_0}, \mathrm{gr}_k \otimes \mathcal{L}_\lambda(-D_{\mathrm{red}})) \Rightarrow H^{t+k}(Y^{\mathrm{tor}}_{I_0}, \Omega^{d_0-e}_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}_\lambda(-D_{\mathrm{red}})). \tag{7.2}$$

This spectral sequence doesn't degenerate in general, so we need to consider weights λ that ensure partial degeneration results. This will allow us to deduce vanishing results for tensor products of the form

$$\Omega^k_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \nabla(\lambda).$$

Another difficulty arises because, in positive characteristic, algebraic representations of reductive groups are not semi-simple, so we can't easily deduce vanishing results for automophic bundles from vanishing results for such tensor products. However, from proposition 7.10, the bundle $\Omega^k_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ admits a ∇-filtration if $p > d$ and we can use the corollary 2.22 to see that the tensor product $\Omega^k_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \nabla(\lambda)$ admits also a ∇-filtration: this allows us to deduce new vanishing results for automophic bundles. Since our method relies heavily on partial degeneration results that requires vanishing results, we can think of it as a way to deduce new vanishing results from known ones. This is why we present them in two steps.

- *Degeneration:* We determine the vanishing results we need to ensure the degeneration of relevant spectral sequences.
- *Propagation:* Given a set of known vanishing results, we determine the new vanishing results we can deduce from them.

To lighten our notations, we will denote the subcanonical automorphic bundle by $\nabla^{\mathrm{sub}}(\lambda)$ instead of $\nabla(\lambda)(-D_{\mathrm{red}})$ and $\mathcal{L}^{\mathrm{sub}}(V)$ instead of $\mathcal{L}(V)(-D_{\mathrm{red}})$. We introduce some notations for the weights of our automorphic bundles.

Definition 7.9. *For all $n \geq 0$, we set*

$$(\mu_j^n)_{1\leq j\leq \binom{d}{n}} = (w_0 w_{0,L}\nu_j^n)_{1\leq j\leq \binom{d}{n}},$$

where the ν_j^n's are the characters of the L-representation

$$\wedge^n \operatorname{Sym}^2 \operatorname{std}_L .$$

We assume that $\nu^n_{\binom{d}{n}}$ is the highest weight.

Proposition 7.10. *If $p > d = g(g+1)/2$, then for any $n \geq 1$ the vector bundle $\Omega^n_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ admits a filtration*

$$0 = \mathcal{V}^s \subsetneq \mathcal{V}^{s-1} \subsetneq \cdots \subsetneq \mathcal{V}^0 = \Omega^n_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}),$$

where the graded pieces are automorphic vector bundles of the form $\nabla(\mu_j^n)$ with μ_j^n dominant.

Proof. It is a consequence of the remark after the theorem 2.43 but we re-explain the argument here. Recall from proposition 4.20 that the Kodaira-Spencer map induces an isomorphism

$$\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \mathcal{W}(\operatorname{Sym}^2 \operatorname{std}_L) = \nabla(0, \cdots, 0, -2).$$

We only need to see that for any $1 \leq n \leq \frac{g(g+1)}{2}$, the GL_g-module $\Lambda^n \operatorname{Sym}^2 \operatorname{std}_L$ admits a ∇-filtration. The module

$$\operatorname{Sym}^2 \operatorname{std}_L = \nabla(2, 0, \cdots, 0)$$

is already a costandard module. From proposition 2.20, the module

$$(\operatorname{Sym}^2 \operatorname{std}_L)^{\otimes n}$$

admits also a ∇-filtration. Since $p > \frac{g(g+1)}{2} \geq n$, p does not divide $n!$ and the surjection of G-modules

$$(\operatorname{Sym}^2 \operatorname{std}_L)^{\otimes n} \to \Lambda^n \operatorname{Sym}^2 \operatorname{std}_L$$

admits a GL_g-equivariant section s defined by the formula

$$s(v_1 \wedge \cdots \wedge v_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

As a direct factor of $(\operatorname{Sym}^2 \operatorname{std}_L)^{\otimes n}$, the corollary 2.16 implies that $\Lambda^n \operatorname{Sym}^2 \operatorname{std}_L$ admits a ∇-filtration. □

Proposition 7.11. *We have an isomorphism*

$$\Omega^1_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} = \mathcal{L}(\operatorname{Lie} L/\operatorname{Lie}(P_0 \cap L))^\vee,$$

and for all $i \geq 0$, the vector bundle $\Omega^i_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}}$ is filtered by line bundles

$$\mathcal{L}_{-s_M} \text{ where } s_M = \sum_{\alpha \in M} \alpha \text{ for all } M \subset \Phi_L^+ - \Phi_{I_0}^+ \text{ such that } |M| = i.$$

In particular, $\Omega^{r_0}_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} \simeq \mathcal{L}_{-2\rho_{I_0}}$ with

$$\rho_{I_0} = \frac{1}{2} \sum_{\alpha \in \Phi_L^+ \setminus \Phi_{I_0}^+} \alpha.$$

PROOF. Consider the cartesian diagram

$$\begin{array}{ccc} Y_{I_0}^{\mathrm{tor}} & \xrightarrow{\tilde{\zeta}} & \lfloor P_0 \backslash * \rfloor \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} \\ \mathrm{Sh}^{\mathrm{tor}} & \xrightarrow{\zeta} & \lfloor P \backslash * \rfloor \end{array}$$

where the horizontal arrows corresponds to the universal P-torsor on $\mathrm{Sh}^{\mathrm{tor}}$ and the universal P_0-torsor on $Y_{I_0}^{\mathrm{tor}}$ and where the vertical arrow $\tilde{\pi}$ between the classifying stacks is induced by the inclusion $P_0 \subset P$. Coherent sheaves on the classifying stack $\lfloor P_0 \backslash * \rfloor$ are algebraic representations of P_0 and clearly, we have

$$\Omega^1_{\tilde{\pi}} = \mathrm{Lie}(P)/\mathrm{Lie}(P_0)^\vee,$$

where the action of P_0 on $\mathrm{Lie}(P)$ is induced by the restriction of the adjoint action of P. From the isomorphism

$$\tilde{\zeta}^* \Omega^1_{\tilde{\pi}} = \Omega^1_{\pi},$$

we deduce that

$$\Omega^1_{Y_{I_0}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} = \mathcal{L}(\mathrm{Lie}(P)/\mathrm{Lie}(P_0)^\vee).$$

Since the T-weights on $\mathrm{Lie}(P)/\mathrm{Lie}(P_0)^\vee$ are the $-\alpha$ wih $\alpha \in \Phi_L^+ - \Phi_{I_0}^+$, the result follows. □

Remark 7.12. The exact sequence

$$0 \to \pi^* \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \to \Omega^1_{Y_{I_0}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \to \Omega^1_{Y_{I_0}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} \to 0$$

doesn't seem to split and we cannot prove the vanishing of the abelian group

$$\mathrm{Ext}^1_{\mathcal{O}_{Y_{I_0}^{\mathrm{tor}}}}(\Omega^1_{Y_{I_0}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}}, \pi^* \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}))$$

using known vanishing results because the vector bundle

$$\pi^* \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes {\Omega^1}^\vee_{Y_{I_0}^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}}$$

is filtered by $\mathcal{L}_\lambda$'s with $\lambda \in X^*(P_0)$ outside the anti-dominant Weyl chamber for which the first cohomology is non-zero in general. Outside the case $I_0 = I$, we don't even know if $\Omega^1_{Y_{I_0}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is automorphic. In other words, we don't know if $\Omega^1_{Y_{I_0}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is of the form $\mathcal{L}(V)$ for an algebraic representation V of P_0.

7.3.1. Degeneration. Mutliple subsets of $X^*(P_0)$ will occur in the formulations of our degeneration results, we gather them in the following definition.

Definition 7.13. *Consider an integer $0 \leq e \leq d-1$. We denote $\mathcal{C}^0_{\mathrm{deg}}$, $\mathcal{C}^1_{\mathrm{deg},e}$ and $\mathcal{C}^2_{\mathrm{deg},e}$ the following set of characters.*

$$\begin{cases} \mathcal{C}^0_{\mathrm{deg}} := \{\lambda \in X^*(P_0) \mid \lambda - 2\rho_{I_0} \in X^*(P_0)^+\}, \\ \mathcal{C}^1_{\mathrm{deg},e} := \{\lambda \in X^*(P_0) \mid \forall i > e+1 \ \forall j \ \forall 1 \leq k \leq e \ \forall M \subset \Phi_L^+ - \Phi_{I_0}^+ \text{ such that} \\ \qquad\qquad |M| = r_0 - k, \ H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\mu_j^{d-e+k} + \lambda - s_M)) = 0\}, \\ \mathcal{C}^2_{\mathrm{deg},e} := \{\lambda \in X^*(P_0) \mid \forall i > e+1 \ \forall j \neq \begin{pmatrix} d \\ d-e \end{pmatrix} \\ \qquad\qquad H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\mu_j^{d-e} + \lambda - 2\rho_{I_0})) = 0\}. \end{cases}$$

Lemma 7.14. *Let $\lambda \in \mathcal{C}^0_{\deg}$ and $\mathcal{F}$ be a coherent sheaf on $\mathrm{Sh}^{\mathrm{tor}}$. For all $0 \leq i \leq r_0$ and $n \geq 0$, we have the following isomorphism*

$$H^n(Y^{\mathrm{tor}}_{I_0}, \pi^*\mathcal{F} \otimes \Omega^i_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda) = H^n(\mathrm{Sh}^{\mathrm{tor}}, \mathcal{F} \otimes \pi_*(\Omega^i_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda)).$$

Proof. Let $i \geq 0$. We know by proposition 7.11 that the vector bundle $\Omega^i_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}}$ is filtered by line bundles

$$\mathcal{L}_{-s_M} \text{ where } s_M = \sum_{\alpha \in M} \alpha \text{ for all } M \subset \Phi^+_L - \Phi^+_{I_0} \text{ such that } |M| = i.$$

From the definition of $\mathcal{C}^0_{\deg}$ and the fact that the roots in $\Phi^+_L - \Phi^+_{I_0}$ are I_0-dominant, we know that all $\lambda - s_M$ are I_0-dominant characters. From Kempf's vanishing theorem (see proposition 2.11 and proposition 3.22), we get

$$\forall M \ \forall k > 0 \ R^k\pi_*(\mathcal{L}_{\lambda - s_M}) = 0$$

and by lemma 7.2 we deduce

$$\forall k > 0 \ R^k\pi_*(\Omega^i_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}_\lambda) = 0.$$

Since $\pi^*\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}(-D_{\mathrm{red}}) = \mathcal{O}_{Y^{\mathrm{tor}}_{I_0}}(-D_{\mathrm{red}})$, the projection formula implies

$$\forall k > 0 \ R^k\pi_*(\Omega^i_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}_\lambda(-D_{\mathrm{red}})) = 0.$$

Using again the projection formula, it implies that the Leray spectral sequence

$$E_2^{t,k} = H^t(\mathrm{Sh}^{\mathrm{tor}}, R^k\pi_*(\pi^*\mathcal{F} \otimes \Omega^i_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda)) \Rightarrow H^{t+k}(Y^{\mathrm{tor}}_{I_0}, \pi^*\mathcal{F} \otimes \Omega^i_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda)$$

is concentrated on one row and we get the desired isomorphisms. □

Proposition 7.15. *Assume that $p > d = g(g+1)/2$. Let $e \geq 0$ and $i > e$ be integers. For any character*

$$\lambda \in \mathcal{C}^0_{\deg} \cap \mathcal{C}^1_{\deg,e} \cap \mathcal{C}^2_{\deg,e},$$

the vanishing

$$H^i(Y^{\mathrm{tor}}_{I_0}, \Omega^{d_0-e}_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_\lambda) = 0$$

implies the vanishing

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\mu^{d-e}_{\binom{d}{d-e}} + \lambda - 2\rho_{I_0})) = 0.$$

Proof. We use lemma 7.2 for the filtration of

$$\pi^*\Omega^{d-e+k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^{r_0-k}_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda$$

obtained from the one defined in proposition 7.11 to see that the vanishing

$$\forall 1 \leq k \leq e \ H^{i+1}(Y^{\mathrm{tor}}_{I_0}, \pi^*\Omega^{d-e+k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^{r_0-k}_{Y^{\mathrm{tor}}_{I_0}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda) = 0,$$

follows from the vanishing

$$H^{i+1}(Y^{\mathrm{tor}}_{I_0}, \pi^*\Omega^{d-e+k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_{\lambda - s_M}) = 0 \tag{7.3}$$

for all $1 \leq k \leq e$ and all $M \subset \Phi_L^+ - \Phi_{I_0}^+$ such that $|M| = r_0 - k$. Since $\lambda \in \mathcal{C}^0_{\mathrm{deg}}$, we know by lemma 7.14 that

$$H^{i+1}(Y^{\mathrm{tor}}_{I_0}, \pi^*\Omega^{d-e+k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_{\lambda - s_M}) \\ = H^{i+1}(\mathrm{Sh}^{\mathrm{tor}}, \Omega^{d-e+k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \nabla^{\mathrm{sub}}(\lambda - s_M)).$$

We use proposition 2.20 and proposition 7.10 to see that the bundle

$$\Omega^{d-e+k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \nabla^{\mathrm{sub}}(\lambda - s_M)$$

admits a filtration where the graded pieces are isomorphic to

$$\nabla(\mu_j^{d-e+k} + \lambda - s_M).$$

By lemma 7.2, we deduce that the vanishing in equality (7.3) follows from $\lambda \in \mathcal{C}^1_{\mathrm{deg},e}$. Since

$$H^i(Y^{\mathrm{tor}}_{I_0}, \Omega^{d_0-e}_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_{\lambda}) = 0$$

by hypothesis, we can apply lemma 7.3 to $E_{2,d_0-e,\lambda}$ to deduce that

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \Omega^{d-e}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \nabla^{\mathrm{sub}}(\lambda - 2\rho_{I_0})) = 0.$$

Combining again the proposition 2.20 and 7.10, we know that

$$\Omega^{d-e}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \nabla^{\mathrm{sub}}(\lambda - 2\rho_{I_0})$$

admits a ∇-filtration. Since $\lambda \in \mathcal{C}^2_{\mathrm{deg},e}$, we use again lemma 7.3 for the ∇-filtration of $\Omega^{d-e}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \nabla^{\mathrm{sub}}(\lambda - 2\rho_{I_0})$ to see that

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\mu^{d-e}_{\binom{d}{d-e}} + \lambda - 2\rho_{I_0})) = 0.$$

□

7.3.2. Propagation. In this section, we construct a non-decreasing function on the power set of characters that gives new vanishing results from known ones. Our main result is theorem 7.19.

Definition 7.16. *For all $k \geq 0$, we define a subset $\mathcal{C}^k_{\mathrm{van}}$ of X^* as*

$$\mathcal{C}^k_{\mathrm{van}} = \left\{ \lambda \in X^* \mid \forall i > k \ H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\lambda)) = 0 \right\}.$$

Remark 7.17. $\mathcal{C}^k_{\mathrm{van}}$ always contains the non-dominant characters.

Definition 7.18. *We define a function $g_{I_0,e} : \mathcal{P}(X^*) \to \mathcal{P}(X^*)$ by*

$$g_{I_0,e}(\mathcal{C}) = \mu^{d-e}_{\binom{d}{d-e}} + X^*(P_0)^+ \cap (-2\rho_{I_0} + \mathcal{C}_{\mathrm{ample},I_0}) \cap \bigcap_{k,j,M} (s_M - 2\rho_{I_0} - \mu_j^{d-e+k} + \mathcal{C}),$$

for all $\mathcal{C} \subset X^*$ where the last intersection is taken over the set of k, j, M where $0 \leq k \leq e$, $1 \leq j \leq \binom{d}{d-e+k}$ and $M \subset \Phi_L^+ - \Phi_{I_0}^+$ such that $|M| = r_0 - k$ with the exception of $j = \binom{d}{d-e}$ when $k = 0$.

Theorem 7.19. *Assume that $p > d_0$. Let $\mathcal{C}$ be a subset of $\mathcal{C}^{e+1}_{\mathrm{van}}$. Then, we have*

$$g_{I_0,e}(\mathcal{C}) \subset \mathcal{C}^e_{\mathrm{van}}.$$

In other words, if we have a set $\mathcal{C}$ of characters λ for which the cohomology

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\lambda))$$

is concentrated in degrees $[0, e+1]$*, then the image of* $\mathcal{C}$ *by the function* $g_{I_0,e}$ *is a set of characters* λ *for which the cohomology* $H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\lambda))$ *is concentrated in degrees* $[0, e]$*.*

Proof. Since $g_{I_0,e}$ is non-decreasing, it suffices to show $g_{I_0,e}(\mathcal{C}^{e+1}_{\mathrm{van}}) \subset \mathcal{C}^{e}_{\mathrm{van}}$. Let $\lambda \in g_{I_0,e}(\mathcal{C}^{e+1}_{\mathrm{van}})$ be a character and define $\lambda' := \lambda + 2\rho_{I_0} - \mu^{d-e}_{\binom{d}{d-e}}$. From the definition of $g_{I_0,e}$, we first deduce that

$$\lambda' \in \mathcal{C}^0_{\mathrm{deg}} \cap \mathcal{C}^1_{\mathrm{deg}} \cap \mathcal{C}^2_{\mathrm{deg}}$$

and

$$\lambda' \in \mathcal{C}_{\mathrm{ample},I_0}.$$

Since the triple $(Y^{\mathrm{tor}}_{I_0}, D_{\mathrm{red}}, \mathcal{L}_{\lambda'})$ lifts to $\mathbb{Z}/p^2\mathbb{Z}$ and $p \geq d_0$, we apply proposition 7.5 to see that

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \Omega^{d_0-e}_{Y^{\mathrm{tor}}_{I_0}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_{\lambda'}) = 0$$

for all $i + d_0 - e > d_0$ (i.e. $i > e$) and we use proposition 7.15 (as $p > d_0 \geq d$) to see that

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\mu^{d-e}_{\binom{d}{d-e}} + \lambda' - 2\rho_{I_0})) = H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\lambda)) = 0$$

for all $i > e$. □

Remark 7.20. The theorem is still valid if we use subsets of $\mathcal{C}_{\mathrm{ample},I_0}$ instead of $\mathcal{C}_{\mathrm{ample},I_0}$. In particular, by theorem 6.6, we can use it with the subset of orbitally p-close and $\mathcal{Z}_0$-ample characters.

7.4. The Siegel threefold case

In this subsection, we give more details in the case $g = 2$ because we believe it already contains some of the idea of the general method and it requires less notations. Assume that p is a prime larger than $g^2 = 4$. The Siegel threefold $\mathrm{Sh}^{\mathrm{tor}}$ is projective variety of dimension $d = 3$ over $\mathbb{F}_p$. From the Kodaira-Spencer isomorphism, we have an identification

$$\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \nabla(0, -2).$$

From proposition 7.10, we know that any exterior power of $\mathrm{Sym}^2 \mathrm{std}_{\mathrm{GL}_2}$ admits a ∇-filtration. It direclty implies that we have

$$\begin{cases} \Omega^2_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \nabla(-1, -3), \\ \Omega^3_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \nabla(-3, -3), \end{cases}$$

and the weights of these three automorphic vector bundles are

$$\begin{cases} (\mu^1_j)_j = \{(-2, 0), (-1, -1), (0, -2)\}, \\ (\mu^2_j)_j = \{(-3, -1), (-2, -2), (-1, -3)\}, \\ (\mu^3_j)_j = \{(-3, -3)\}. \end{cases}$$

We start with the case $I_0 = \emptyset$. The associated complete flag bundle $\pi : Y^{\mathrm{tor}} \to \mathrm{Sh}^{\mathrm{tor}}$ parametrizes quotient line bundles of the rank 2 Hodge bundle Ω^{tor}. It is a $\mathbb{P}^1$-fibration and we have an identification:

$$\Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} = \mathcal{L}_{-2\rho} = \mathcal{L}_{(-1,1)}.$$

For any integer $0 \leq e \leq d-1 = 2$, we have an increasing filtration on the bundle $\Omega^{4-e}_{Y^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ given by

$$F_k = \pi^* \Omega^{4-e-k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \wedge \Omega^k_{Y^{\mathrm{tor}}}(\log D_{\mathrm{red}})$$

$$
\begin{tikzcd}
k & & & & & \\
2 & 0 \arrow[rrr] & 0 \arrow[rrr] & 0 & 0 & 0 \\
1 & H^0(\mathrm{gr}_1 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \arrow[rrr] & H^1(\mathrm{gr}_1 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) & H^2(\mathrm{gr}_1 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) & H^3(\mathrm{gr}_1 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) & 0 \\
0 & 0 \arrow[rrr] & 0 \arrow[rrr] & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
 & -1 & 0 & 1 & 2 & 3 \quad t
\end{tikzcd}
$$

FIGURE 7.4.1. E_2-page of the spectral sequence when $e = 0$

with graded pieces

$$\mathrm{gr}_k = \pi^* \Omega^{4-e-k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^k_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}}.$$

For any character $\lambda = (k_1 \geq k_2)$, we have an associated spectral sequence:

$$E^{t,k}_{2,e,\lambda} = H^{t+k}(Y^{\mathrm{tor}}, \pi^* \Omega^{4-e-k}_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^k_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda) \Rightarrow H^{t+k}(Y^{\mathrm{tor}}, \Omega^{4-e}_{Y^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_\lambda)$$

starting at page 2. We will study this spectral sequence for each e, starting with $e = 0$. In this case (see the corresponding figure), the second page of the spectral sequence is concentrated on one row as the only graded piece is $\mathrm{gr}_1 = \pi^* \Omega^3_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}}$. In particular, the spectral sequence degenerates at page 2 and we get

$$H^i(Y^{\mathrm{tor}}, \pi^* \Omega^3_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda) = H^i(Y^{\mathrm{tor}}, \Omega^4_{Y^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_\lambda)$$

for all $i \geq 0$. Moreover, if we assume that $\lambda - 2\rho$ is dominant (which is equivalent to $k_1 \geq k_2 + 2$), we get

$$\begin{aligned}
&H^i(Y^{\mathrm{tor}}, \pi^* \Omega^3_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda) \\
&= H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla(-3,-3) \otimes \nabla^{\mathrm{sub}}(k_1 - 1, k_2 + 1)) \\
&= H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k_1 - 4, k_2 - 2))
\end{aligned}$$

for all $i \geq 0$. We assume that $\mathcal{L}_{(k_1,k_2)}$ is D-ample on Y^{tor} and we use the logarithmic Kodaira-Nakano vanishing theorem to see that

$$H^i(Y^{\mathrm{tor}}, \Omega^4_{Y^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_\lambda) = 0$$

for all $i > 0$. We summarize this discussion by saying that we have

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k_1 - 4, k_2 - 2)) = 0$$

for all $i > 0$ and (k_1, k_2) such that

- $k_1 \geq k_2 + 2$,
- $(k_1, k_2) \in \mathcal{C}_{\mathrm{ample},\emptyset}$.

$$
\begin{tikzcd}
k & & & & & \\
2 & 0 \arrow[rrrr] & 0 & 0 & 0 & 0 \\
1 & H^0(\mathrm{gr}_1 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \arrow[rrrr] & H^1(\mathrm{gr}_1 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \arrow[rrrr] & H^2(\mathrm{gr}_1 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \arrow[rrrr] & H^3(\mathrm{gr}_1 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \arrow[rrrr] & H^4(\mathrm{gr}_1 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \\
0 & 0 & H^0(\mathrm{gr}_0 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \arrow[rrrr] & H^1(\mathrm{gr}_0 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \arrow[rrrr] & H^2(\mathrm{gr}_0 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) & H^3(\mathrm{gr}_0 \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \\
-1 & 0 & 0 & 0 & 0 & 0 \\
 & -1 & 0 & 1 & 2 & 3 \arrow[r] & t
\end{tikzcd}
$$

FIGURE 7.4.2. E_2-page of the spectral sequence when $e = 1$

Now, we consider the spectral sequence in the case $e = 1$ (see the corresponding figure). The graded pieces are $\mathrm{gr}_1 = \pi^*\Omega^2_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}}$ and $\mathrm{gr}_0 = \pi^*\Omega^3_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$. The limit is $H^i(Y^{\mathrm{tor}}, \Omega^3_{Y^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}_\lambda^{\mathrm{sub}})$ and by the logarithmic Kodaira-Nakano theorem, it vanishes for all $i > 1$ when $(k_1, k_2) \in \mathcal{C}_{\mathrm{ample},\emptyset}$. The critical differential is

$$
\begin{aligned}
d^{1,1} : H^2(Y^{\mathrm{tor}}, \pi^*\Omega^2_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \\
\to H^3(Y^{\mathrm{tor}}, \pi^*\Omega^3_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}_\lambda^{\mathrm{sub}}),
\end{aligned}
$$

because when $d^{1,1} = 0$, we have $E_2^{1,1} = E_\infty^{1,1}$ and $E_2^{2,1} = E_\infty^{2,1}$. Under the additional hypothesis $k_1 \geq k_2 + 2$, we deduce that

$$
\begin{aligned}
H^i(Y^{\mathrm{tor}}, \pi^*\Omega^2_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) \\
= H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla(-1,-3) \otimes \nabla^{\mathrm{sub}}(k_1 - 1, k_2 + 1)) = 0
\end{aligned}
$$

for all $i > 1$ and $(k_1, k_2) \in \mathcal{C}_{\mathrm{ample},\emptyset}$. Consider an integer $i = 2$ or 3. The tensor product of automorphic vector bundles

$$\nabla(-1,-3) \otimes \nabla^{\mathrm{sub}}(k_1 - 1, k_2 + 1)$$

is filtered by the automorphic bundles

$$\nabla^{\mathrm{sub}}(\mu_j^2 + (k_1 - 1, k_2 + 1))$$

where $j = 1, 2, 3$ and if we ask for the vanishing

$$H^{i+1}(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\mu_j^2 + (k_1 - 1, k_2 + 1))) = 0$$

for $j = 1, 2$, it will imply

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}((-1,-3) + (k_1 - 1, k_2 + 1))) = H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k_1 - 2, k_2 - 2)) = 0.$$

To see that the critical differential $d^{1,1}$ is zero, it is sufficient to have

$$H^3(Y^{\mathrm{tor}}, \pi^*\Omega^3_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}_\lambda^{\mathrm{sub}}) = H^3(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k_1 - 3, k_2 - 3)) = 0.$$

We summarize this discussion by saying that we have

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k_1 - 2, k_2 - 2)) = 0$$

for all $i > 1$ and (k_1, k_2) such that

- $k_1 \geq k_2 + 2$,
- $(k_1, k_2) \in \mathcal{C}_{\mathrm{ample},\emptyset}$,
- $H^3(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\mu_j^2 + (k_1 - 1, k_2 + 1))) = 0$ for $j = 1, 2$,
- $H^3(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k_1 - 3, k_2 - 3)) = 0$.

Now, we consider the spectral sequence in the case $e = 2$. The graded pieces are $\mathrm{gr}_1 = \pi^*\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}}$ and $\mathrm{gr}_0 = \pi^*\Omega^2_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$. The limit is

$$H^i(Y^{\mathrm{tor}}, \Omega^2_{Y^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_\lambda)$$

and by the logarithmic Kodaira-Nakano theorem, it vanishes for all $i > 2$ when $(k_1, k_2) \in \mathcal{C}_{\mathrm{ample},\emptyset}$. The differential

$$d^{2,1} : H^3(Y^{\mathrm{tor}}, \pi^*\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda) \to \underbrace{H^4(Y^{\mathrm{tor}}, \pi^*\Omega^2_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \mathcal{L}^{\mathrm{sub}}_\lambda)}_{=0}$$

is already 0 since the Siegel threefold has dimension 3. Under the additional hypothesis $k_1 \geq k_2 + 2$, we deduce that

$$H^i(Y^{\mathrm{tor}}, \pi^*\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \Omega^1_{Y^{\mathrm{tor}}/\mathrm{Sh}^{\mathrm{tor}}} \otimes \mathcal{L}^{\mathrm{sub}}_\lambda) = H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla(0, -2) \otimes \nabla^{\mathrm{sub}}(k_1 - 1, k_2 + 1)) = 0$$

for all $i > 2$ and $(k_1, k_2) \in \mathcal{C}_{\mathrm{ample},\emptyset}$. The tensor product of automorphic vector bundles

$$\nabla(0, -2) \otimes \nabla^{\mathrm{sub}}(k_1 - 1, k_2 + 1)$$

is filtered by the automorphic bundles

$$\nabla^{\mathrm{sub}}(\mu_j^1 + (k_1 - 1, k_2 + 1))$$

where $j = 1, 2, 3$ and since the vanishing

$$H^{i+1}(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(\mu_j^2 + (k_1 - 1, k_2 + 1))) = 0$$

for $j = 1, 2$ is automatic, it implies the vanishing of

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}((0, -2) + (k_1 - 1, k_2 + 1))) = H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k_1 - 1, k_2 - 1)).$$

We summarize this discussion by saying that we have

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k_1 - 1, k_2 - 1)) = 0$$

for all $i > 2$ and (k_1, k_2) such that

- $k_1 \geq k_2 + 2$,
- $(k_1, k_2) \in \mathcal{C}_{\mathrm{ample},\emptyset}$.

We consider the case $I_0 = I = \{(1, -1)\}$. This case corresponds to $Y^{\mathrm{tor}}_{I_0} = \mathrm{Sh}^{\mathrm{tor}}$. In this degenerate case, the spectral sequence is trivial and the D-ample automorphic line bundle are powers of the determinant of the Hodge bundle : $\nabla(k, k)$ for all $k < 0$. By the logarithmic Kodaira-Nakano vanishing theorem, we have

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \Omega^j_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \otimes \nabla^{\mathrm{sub}}(k, k)) = 0$$

for all $i + j > 3$ and $k < 0$. In the case $j = 3$, we get

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k - 3, k - 3)) = 0$$

for all $i > 0$. In the case $j = 2$, we get

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k-1, k-3)) = 0$$

for all $i > 1$. In the case $j = 1$, we get

$$H^i(\mathrm{Sh}^{\mathrm{tor}}, \nabla^{\mathrm{sub}}(k, k-2)) = 0$$

for all $i > 2$.

7.5. An algorithm for the degeneration of spectral sequences

From the description of the degeneration of the different spectral sequences in the case of the Siegel threefold, it is clear that an algorithm implemented on a computer could be useful to make the vanishing results more explicit. We present an algorithm[7] written in SageMath that uses our main result (theorem 7.19) to compute new vanishing results from known ones.

It depends on the following parameters.

(1) The genus $g \geq 2$ (the case $g = 1$ is obvious).
(2) A prime p such that $p > g^2$.
(3) A set of known vanishing result $\mathcal{C}_{\mathrm{van}}$ for each cohomological degree.
(4) The integer e that appears on the spectral sequence 7.2.
(5) A subset $I_0 \subset \Delta$ for the choice of the flag bundle $Y_{I_0}^{\mathrm{tor}}$ over the Siegel variety.

In the special case where $e = 0$, our algorithm does not need any vanishing result for the degeneration process as the spectral sequence 7.2 is concentrated on one row. In the special case where $e = d - 1$, the degeneration is automatic as it is given by the vanishing of the coherent cohomology in degree $i > d$. Then, these results can be used to run the algorithm with $e = 1$ and with differents $I_0 \subset \Delta$ and so on.

Our SageMath code defines a class SiegelVariety with some methods that can be used to compute vanishing results. We create the Siegel threefold X over $\mathbb{F}_7$.

```
In [7]:   X = SiegelVariety(g = 2, p = 7)
```

If the next line returns *True*, it means that the automorphic line bundle $\mathcal{L}_{(-2,-8)}$ is D-ample on the complete flag variety Y over X.

```
In [8]:   X.ample([],[-2,-8])

Out[8]:   True
```

The next line compute vanishing results for characters $\lambda = (k_1, k_2)$ with $-50 \leq k_2 \leq k_1 \leq 0$ using the function $g_{I_0,e}$ in the case where $I_0 = \emptyset$ and $e = 0$. The results are registered in the list C_{van}. It returns *True* if the algorithm has found new vanishing results.

```
In [9]:   X.compute([], e = 0, kmin = -50, kmax = 0)

          100%|██████████| 1326/1326 [00:02<00:00, 571.40it/s]

Out[9]:   True
```

[7] See github.com/ThibaultAlexandre/vanishing-results-over-the-siegel-variety

The next line runs the *compute* method for each $I_0 \subset I$ and $0 \leq e \leq d-1$. We only need to specify the range of characters $\lambda = (k_1, k_2)$ we want to consider. It returns *True* if the algorithm has found new vanishing results. You may want to run this command several times until it returns *False*.

In [10]:
```
X.computeAll(-50, 0)
```

```
Step 1/6 : e = 0, Iprime = [0]
100%|██████████| 1326/1326 [00:02<00:00, 496.87it/s]
Step 2/6 : e = 1, Iprime = [0]
100%|██████████| 1326/1326 [00:01<00:00, 1205.39it/s]
Step 3/6 : e = 2, Iprime = [0]
100%|██████████| 1326/1326 [00:01<00:00, 1293.54it/s]
Step 4/6 : e = 0, Iprime = [1]
100%|██████████| 1326/1326 [00:00<00:00, 2673.82it/s]
Step 5/6 : e = 1, Iprime = [1]
100%|██████████| 1326/1326 [00:00<00:00, 2609.12it/s]
Step 6/6 : e = 2, Iprime = [1]
100%|██████████| 1326/1326 [00:00<00:00, 2994.65it/s]
```

Out[10]:
```
True
```

The next line returns *True* if we know that

$$H^i(X, \nabla^{\mathrm{sub}}(-4,-6)) = 0$$

for all $i > 1$.

In [11]:
```
X.vanishes(1,(-4,-6))
```

Out[11]:
```
True
```

If the next line returns *False*, it means we don't know if

$$H^i(X, \nabla^{\mathrm{sub}}(-4,-6)) = 0$$

for all $i > 0$.

In [12]:
```
X.vanishes(0,(-4,-6))
```

Out[12]:
```
False
```

7.5.1. Explicit vanishing for $G = \mathrm{Sp}_4$. We plot some vanishing results we have obtained for the Siegel threefold with our algorithm. We have also added the p-small characters for Sp_4 with a twist by $-w_0$ to have them in the anti-dominant Weyl chamber.

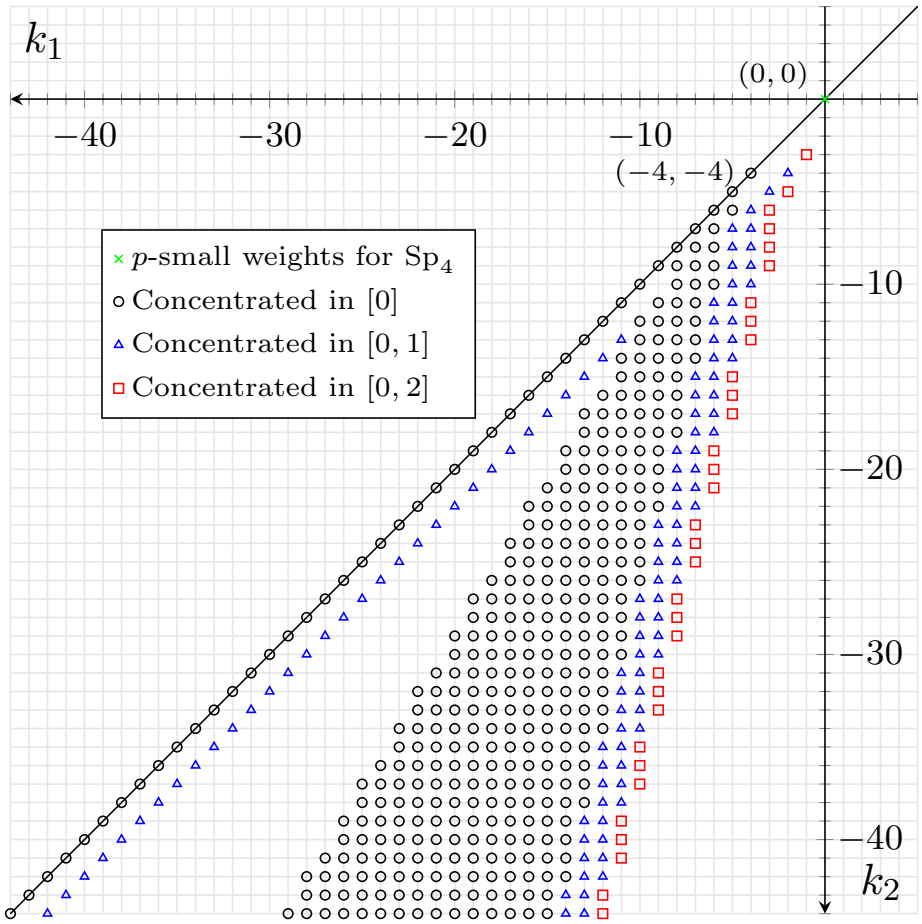


FIGURE 7.5.1. $g = 2, p = 5$. The weights $\lambda = (k_1 \geq k_2)$ such that the cohomology is concentrated in degree 0 contains in particular the positive parallel weights (k, k) below $(-4, -4)$. The vanishing results in the region located near the positive parallel line come from the degeneration with $I = \{(1, -1)\}$ and the rest corresponds to the degeneration with $I = \emptyset$.

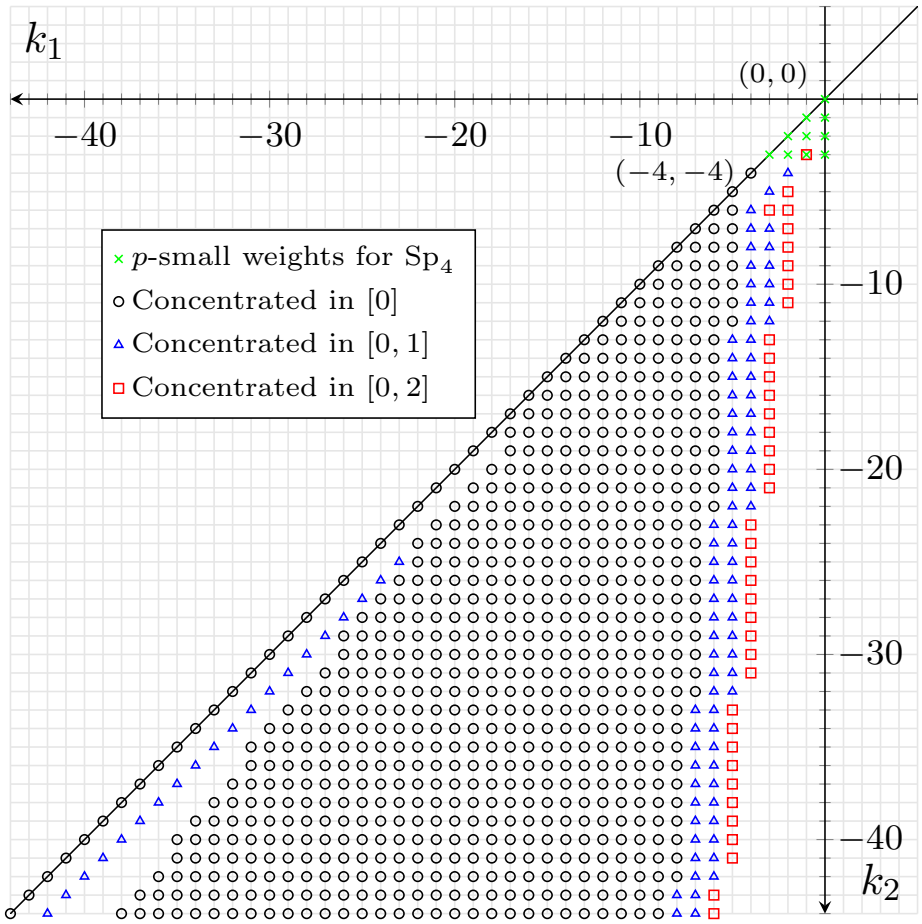


FIGURE 7.5.2. $g = 2, p = 11$. Notice that since the orbitally p-close condition is less restrictive with $p = 11$ than with $p = 5$, we are able to access more weights. However, if we look far enough, we notice the same phenomenon with the two regions corresponding to $I = \{(1, -1)\}$ and $I = \emptyset$.

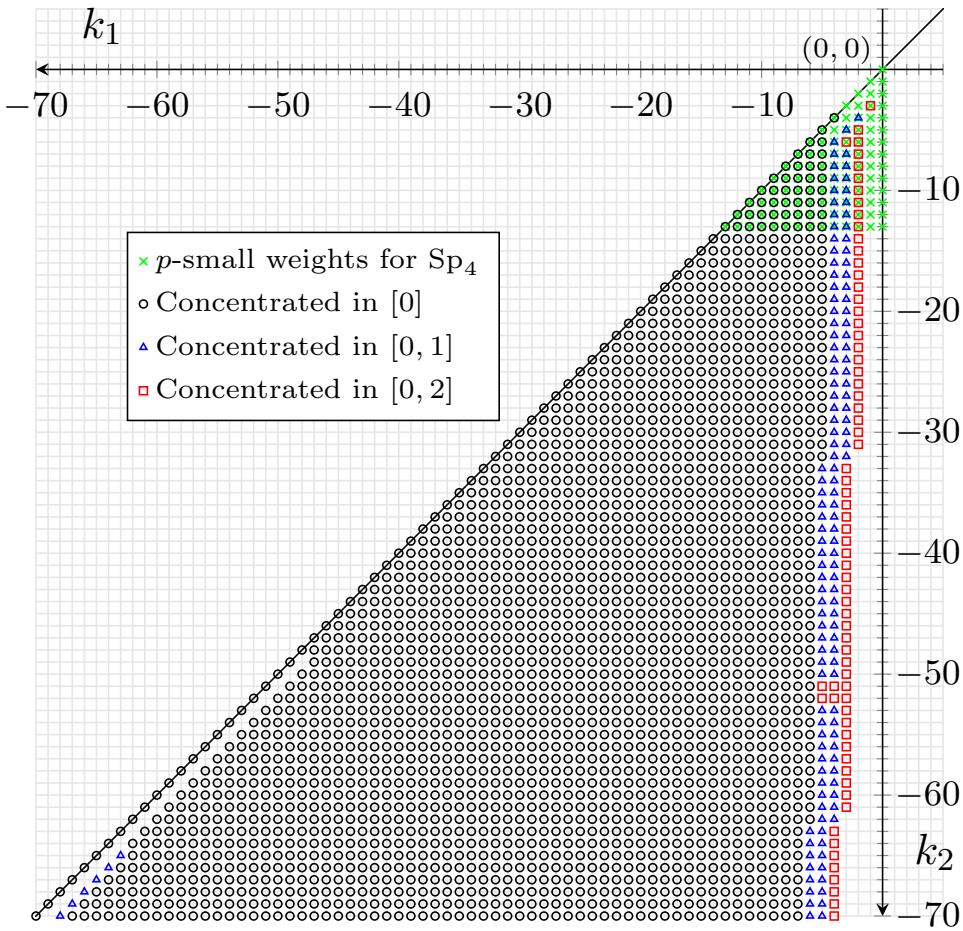


FIGURE 7.5.3. $g = 2, p = 31$.

7.5.2. Explicit vanishing for $G = \mathrm{Sp}_6$. We plot some vanishing results we have obtained in the case $g = 3$ with our algorithm. The weights live in a three-dimensional space and we need 6 different labels.

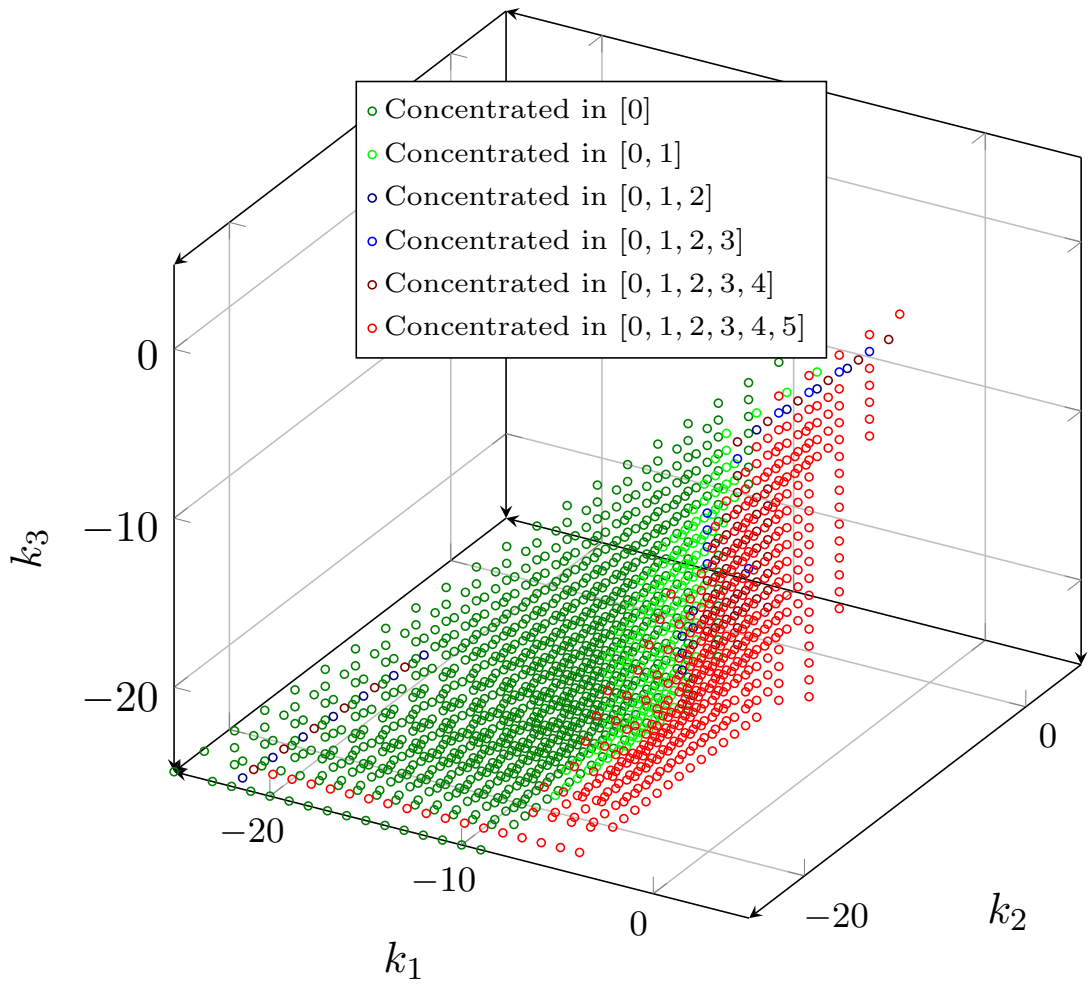


FIGURE 7.5.4. $g = 3$, $p = 11$.

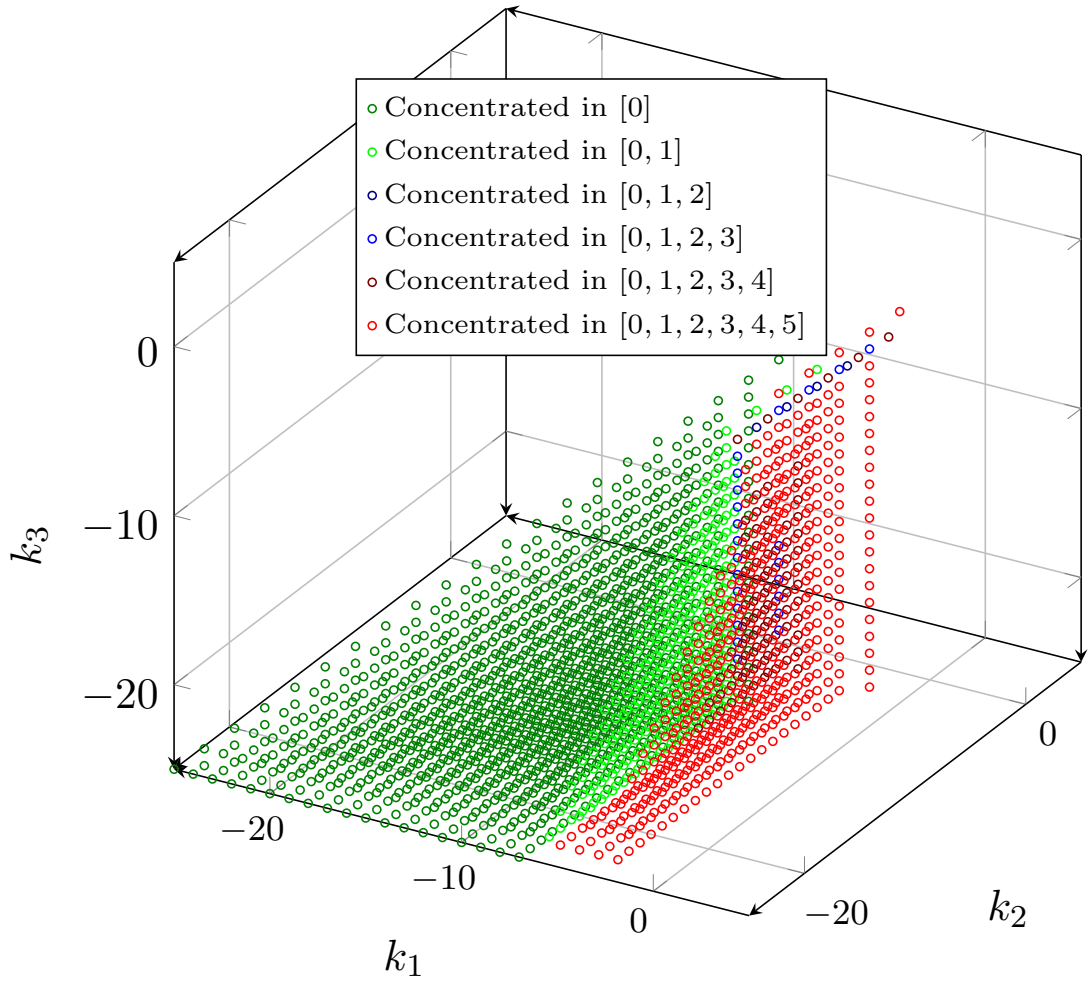


FIGURE 7.5.5. $g = 3$, $p = 691$.

7.5.3. Explicit vanishing for $G = \mathrm{Sp}_8$. We plot the vanishing results we have obtained in the case $g = 4$, $p = 47$ with our algorithm. The weights live in a four-dimensional space and the dimension of the variety $\mathrm{Sh}^{\mathrm{tor}}$ is 10. We have restricted to weights λ such that each k_i satisfies -10 $\leq k_i \leq$ -3.

(1) $H^i = 0$ with $i > 0$ for

(-10, -10, -10, -10), (-9, -10, -10, -10), (-9, -9, -10, -10), (-9, -9, -9, -10),
(-9, -9, -9, -9), (-8, -8, -10, -10), (-8, -8, -9, -10), (-8, -8, -8, -10),
(-8, -8, -9, -9), (-8, -8, -8, -9), (-8, -8, -8, -8), (-7, -7, -7, -10),
(-7, -7, -7, -9), (-7, -7, -7, -8), (-7, -7, -7, -7), (-6, -6, -6, -6).

(2) $H^i = 0$ with $i > 1$ for

(-8, -10, -10, -10), (-7, -9, -9, -9), (-6, -8, -8, -8), (-5, -7, -7, -7), (-4, -6, -6, -6).

(3) $H^i = 0$ with $i > 2$ for

(-7, -9, -10, -10), (-6, -8, -9, -9), (-5, -7, -8, -8), (-4, -6, -7, -7), (-3, -5, -6, -6).

(4) $H^i = 0$ with $i > 3$ for

(-7, -7, -10, -10), (-6, -6, -9, -9), (-5, -5, -8, -8), (-4, -4, -7, -7), (-3, -3, -6, -6).

(5) $H^i = 0$ with $i > 4$ for

(-6, -7, -9, -10), (-5, -6, -8, -9), (-4, -5, -7, -8), (-3, -4, -6, -7).

(6) $H^i = 0$ with $i > 5$ for

(-6, -6, -8, -10), (-5, -5, -7, -9), (-4, -4, -6, -8), (-3, -3, -5, -7).

(7) $H^i = 0$ with $i > 6$ for no new character.

(8) $H^i = 0$ with $i > 7$ for

(-6, -7, -7, -10), (-5, -6, -6, -9), (-4, -5, -5, -8), (-3, -4, -4, -7).

(9) $H^i = 0$ with $i > 8$ for

(-7, -7, -8, -10), (-6, -6, -7, -9), (-5, -5, -6, -8), (-4, -4, -5, -7), (-3, -3, -4, -6).

(10) $H^i = 0$ with $i > 9$ for

(-7, -8, -8, -10), (-6, -8, -8, -10), (-5, -8, -8, -10), (-4, -8, -8, -10), (-6, -7, -8, -10),
(-5, -7, -8, -10), (-4, -7, -8, -10), (-5, -6, -8, -10), (-4, -6, -8, -10), (-5, -5, -8, -10),
(-4, -5, -8, -10), (-4, -4, -8, -10), (-5, -7, -7, -10), (-4, -7, -7, -10), (-6, -6, -7, -10),
(-5, -6, -7, -10), (-4, -6, -7, -10), (-5, -5, -7, -10), (-4, -5, -7, -10), (-4, -4, -7, -10),
(-6, -6, -6, -10), (-5, -6, -6, -10), (-4, -6, -6, -10), (-5, -5, -6, -10), (-4, -5, -6, -10),
(-4, -4, -6, -10), (-5, -5, -5, -10), (-4, -5, -5, -10), (-4, -4, -5, -10), (-4, -4, -4, -10),
(-6, -7, -7, -9), (-5, -7, -7, -9), (-4, -7, -7, -9), (-5, -6, -7, -9), (-4, -6, -7, -9),
(-4, -5, -7, -9), (-4, -4, -7, -9), (-6, -6, -6, -9), (-4, -6, -6, -9), (-5, -5, -6, -9),
(-4, -5, -6, -9), (-4, -4, -6, -9), (-5, -5, -5, -9), (-4, -5, -5, -9), (-4, -4, -5, -9),
(-4, -4, -4, -9), (-6, -6, -6, -8), (-5, -6, -6, -8), (-4, -6, -6, -8), (-4, -5, -6, -8),
(-5, -5, -5, -8), (-4, -4, -5, -8), (-4, -4, -4, -8), (-5, -5, -5, -7), (-4, -5, -5, -7),
(-4, -4, -4, -7), (-4, -4, -4, -6), (-3, -3, -8, -10), (-3, -3, -7, -10), (-3, -3, -6, -10),
(-3, -3, -5, -10), (-3, -3, -4, -10), (-3, -3, -3, -10), (-3, -3, -7, -9), (-3, -3, -6, -9),
(-3, -3, -5, -9), (-3, -3, -4, -9), (-3, -3, -3, -9), (-3, -3, -6, -8), (-3, -3, -5, -8),
(-3, -3, -4, -8), (-3, -3, -3, -8), (-3, -3, -4, -7), (-3, -3, -3, -7), (-3, -3, -3, -6),
(-3, -3, -3, -5).

We get much more weights whose cohomology is concentrated in the range $[0, 9]$ because the vanishing results needed for the degeneration of the corresponding spectral sequences are automatic.

CHAPTER 8

Hyperbolicity of the Siegel variety

One of the most celebrated result in Diophantine geometry is the following.

Theorem ([Fal83]). *Consider a geometrically integral smooth projective curve C over $\mathbb{Q}$. The following three assertions are equivalent.*

(1) For all finite extension $F/\mathbb{Q}$, the set of F-rational points of C is finite.

(2) Every holomorphic map $\mathbb{C} \to C^{an}_{\mathbb{C}}$ is constant.

(3) The canonical bundle w_C is big, i.e. the genus g of C satisfies $g \geq 2$.

A curve satisfying these assertions is called hyperbolic and generalizing hyperbolicity to higher dimensional varieties is an open problem. One might be tempted to consider the following three definitions of hyperbolicity which are conjectured to be equivalent [Lan86, Conjecture 5.6/5.8].

Definition. *Let X denote a geometrically integral projective smooth variety over $\mathbb{Q}$.*

(1) X is arithmetically hyperbolic if for all finite extension $F/\mathbb{Q}$, the set of F-rational points of X is finite.

(2) X is Brody hyperbolic if every holomorphic map $\mathbb{C} \to X^{an}_{\mathbb{C}}$ is constant.

(3) X is algebraically hyperbolic if every integral subvariety V of $X_{\mathbb{C}}$ is of general type, i.e. there exists a desingularization $\tilde{V} \to V$ such that $\omega_{\tilde{V}}$ is big.

These definitions are motivated by two conjectures from Lang which says that Brody hyperbolicity is equivalent to algebraic hyperbolicity [Lan86, Conjecture 5.6] and Brody hyperbolicity implies arithmetic hyperbolicity [Lan86, Conjecture 5.8]. These conjectures from Lang are related to the Green-Griffiths-Lang conjecture and the Bombieri-Lang conjecture.

Conjecture 8.1. *Let X be an irreducible projective complex variety. Denote $Exc(X)$ the Zariski closure of the union of the images of all non-constant holomorphic maps $\mathbb{C} \to X$. Then X is of general type if and only if $Exc(X) \neq X$.*

Conjecture 8.2. *Let X be an irreducible projective complex variety of general type over $\mathbb{Q}$. Then the $\mathbb{Q}$-rational point of X do not form a dense set in the Zariski topology.*

It is well-known that if X is a smooth projective variety over $\mathbb{Q}$ such that the cotangent bundle Ω^1_X is ample and globally generated, X is algebraically hyperbolic. For a non-proper quasi-projective variety X over $\mathbb{Q}$, we should consider a compactification $\bar{X}$ such that the boundary $D_{\text{red}} = \bar{X} - X$ is a normal crossing Cartier divisor over $\bar{X}$. We refer to such a pair $(\bar{X}, D_{\text{red}})$ as a logarithmic pair over $\mathbb{Q}$. These pairs naturally arise in arithmetic when studying integral points. Following [AT20], we give the following definition.

Definition 8.3. *Let $(\bar{X}, D_{\text{red}})$ denote a logarithmic pair consisting of a geometrically integral projective variety $\bar{X}$ over $\mathbb{Q}$ and a reduced effective Cartier divisor D_{red} on $\bar{X}$. We say that $(\bar{X}, D_{\text{red}})$ is algebraically hyperbolic if every integral subvariety V of $\bar{X}_{\mathbb{C}}$ not contained in*

D_{red} is of log-general type, i.e. there exists a desingularization $\tilde{V} \to V$ and a projective embedding $\iota : \tilde{V} \to \bar{X}_{\mathbb{C}}$ where the pullback $\iota^{-1}(D_{\mathrm{red}})$ is a normal crossing divisor and such that $\omega_{\tilde{V}}(\iota^{-1}D_{\mathrm{red}})$ is big.

In [**AT20**], they conjecture that the three different kind of hyperbolicity (where general type is replaced by log general type and rational points by integral points) coincide for quasi-projective varieties X. It has been proven that Siegel varieties over $\mathbb{Q}$ are algebraically hyperbolic [**Zuo00**] [**Bru18**]. Let's replace the base field $\mathbb{Q}$ by the characteristic p analogue $\mathbb{F}_p(T)$. Since desingularization techniques do not always exist in characteristic p, we will restrict ourselves to *smooth* subvarieties. Assume that p does not divide N and consider a smooth projective toroidal compactification $\mathrm{Sh}^{\mathrm{tor}}$ of the Siegel variety of genus $g \geq 2$ over k, a field of characteristic p, and denote D_{red} its reduced boundary as a normal crossing divisor.

Question (Characteristic p)**.** *Is $(\mathrm{Sh}^{\mathrm{tor}}, D_{\mathrm{red}})$ algebraically hyperbolic over k? In other words, does every smooth subvariety*

$$\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$$

such that $\iota^{-1}D$ is well defined is of log general type?

The answer to this question is in fact negative as we will see in the next section.

8.1. The supersingular pencil of Moret-Bailly

From now on, the letter k will denote a field of characteristic p. Denote Sh_g the Siegel variety of genus g and full level $N \geq 3$ (with $p \nmid N$) over k and $\mathrm{Sh}_g^{\mathrm{tor}}$ a smooth toroidal compactification with boundary a normal crossing divisor D_{red}. Recall that D denotes the effective divisor supported on the boundary that appears as the exceptional divisor of the blow-up from $\mathrm{Sh}_g^{\mathrm{tor}}$ to the minimal compactification of Sh_g. In [**MB81**], Moret-Bailly constructs a non-isotrivial family $A \to \mathbb{P}^1_k$ of principally polarized supersingular abelian surfaces over the projective line with a full level N-structure. This family yields a closed immersion $\iota_2 : \mathbb{P}^1_k \hookrightarrow \mathrm{Sh}_2$ whose image belongs to the supersingular locus of the Siegel threefold. In particular, we already know that $\mathrm{Sh}_g^{\mathrm{tor}}$ is not hyperbolic when $g = 2$. This family can be used to contradict the hyperbolicity of the Siegel variety for all $g \geq 2$: Take an abelian variety A_0 of dimension $g-2$ over k and consider the closed immersion $\iota := \iota_{A_0} \circ \iota_2$

$$\begin{tikzcd} \mathbb{P}^1_k \arrow[r, "\iota_2"] \arrow[dr, "\iota"'] & \mathrm{Sh}_2^{\mathrm{tor}} \arrow[d, "\iota_{A_0}"] \\ & \mathrm{Sh}_g^{\mathrm{tor}} \end{tikzcd}$$

where ι_{A_0} sends an abelian surface A to the fibre product $A \times_k A_0$. It also shows that the logarithmic cotangent bundle $\Omega^1_{\mathrm{Sh}_g^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ cannot be nef. Indeed, ι induces a surjective morphism

$$\iota^*\Omega^1_{\mathrm{Sh}_g^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \to \Omega^1_{\mathbb{P}^1}$$

and if $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ was nef, it would imply that $\Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-1)$ is nef.

8.2. Understanding the failure of hyperbolicity in positive characteristic

We have seen that $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ cannot be nef as we can always see $\mathbb{P}^1$ as a closed curve in $\mathrm{Sh}^{\mathrm{tor}}$. Consider a partition λ with height $\mathrm{ht}(\lambda) \leq \dim \mathrm{Sh}^{\mathrm{tor}}$ and denote S_λ the

corresponding Schur functor

$$S_\lambda : \mathrm{Loc}(\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}}) \to \mathrm{Loc}(\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}})$$

as a strict polynomial functor on the category of locally free sheaves of $\mathrm{Sh}^{\mathrm{tor}}$. We start with the following lemma.

Lemma 8.4. *If $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample and $\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ is any subvariety such that*

(1) V is smooth,
(2) $\iota^{-1} D_{\mathrm{red}}$ is a normal crossing divisor,
(3) $\dim V \geq ht(\lambda)$,

then the logarithmic canonical bundle $\omega_V(\iota^{-1} D_{\mathrm{red}})$ is $(\varphi, \iota^{-1} D)$-ample. In particular, it is nef and big and V is of log general type with exceptional locus contained in the boundary.

Proof. The surjective morphism

$$\iota^* \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \to \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}})$$

induces a surjective morphism

$$\iota^* S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) \to S_\lambda \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}})$$

and by proposition 3.54 and 3.51, we deduce that $S_\lambda \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}})$ is $(\varphi, \iota^{-1} D)$-ample. Since $\mathrm{ht}(\lambda) \leq \dim V$, the bundle

$$\det S_\lambda \Omega^1_V(\log \iota^{-1} D_{\mathrm{red}}) = \left(\omega_V(\iota^{-1} D_{\mathrm{red}})\right)^{\otimes \frac{|\lambda| \dim \nabla(\lambda)}{g}}$$

is non-zero and $(\varphi, \iota^{-1} D)$-ample. We conclude with proposition 3.53. □

With this fundamental lemma in mind, the aim is to find partitions λ that ensure the (φ, D)-ampleness of $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$, which is isomorphic to $S_\lambda \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$ by the Kodaira-Spencer isomorphism (proposition 4.20). Recall that under the assumption $p \geq 2|\lambda| - 1$, the plethysm $S_\lambda \circ \operatorname{Sym}^2$ is filtered by Schur functors S_η by proposition 2.43. This allows us to state the following lemma.

Lemma 8.5. *Let λ be a partition and assume that $p \geq 2|\lambda| - 1$. If $S_\lambda \circ \operatorname{Sym}^2$ is filtered by Schur functors S_η such that $S_\eta \Omega^{\mathrm{tor}}$ is (φ, D)-ample or zero, then $S_\lambda \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.*

Proof. Since $\mathrm{Sh}^{\mathrm{tor}}$ is smooth, this is a direct consequence of proposition 3.50. □

Remark 8.6. If η has more than g parts, then $S_\eta \Omega^{\mathrm{tor}} = 0$. Otherwise, $S_\eta \Omega^{\mathrm{tor}} = \nabla(w_0 w_{0,L} \eta)$.

By theorem 6.7, we are reduced to find a partition λ such that all the partition η with at most g parts appearing in the plethysm $S_\lambda \circ \operatorname{Sym}^2$ are such that $2\rho_L + 2w_0 w_{0,L} \eta$ is orbitally p-close and $\mathcal{Z}_\emptyset$-ample. We start with some explicit plethysm computations in the cases $g = 2, 3, 4$ which helped us to build some intuition about the general case.

8.2.1. The low dimensional cases.

8.2.1.1. *The case $g = 2$.* The Hodge bundle Ω^{tor} is locally free of rank 2 and

$$\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$$

is locally free of rank 3. Recall there is no need to assume that $p \geq 2|\lambda| - 1$ when taking the highest exterior power. Under the assumption $p \geq 2 \times 2 - 1 = 3$, we have

$$\begin{cases} \Lambda^3\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(\text{-}3, \text{-}3) \\ \Lambda^2\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(\text{-}1, \text{-}3) \\ \mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(0, \text{-}2). \end{cases}$$

Clearly, the line bundle $\nabla(\text{-}3, \text{-}3)$ is D-ample for any $p > 0$ and $\nabla(0, \text{-}2)$ is never nef (hence never (φ, D)-ample) by proposition 6.8. For $\nabla(\text{-}1, \text{-}3)$, we need to check wether $(\text{-}1, \text{-}7)$ is orbitally p-close and $\mathcal{Z}_\emptyset$-ample. This condition is satisfied as soon as $p \geq 11$. By lemma 8.4, this shows that any (good) subsurface of the Siegel threefold is of log general type when $p \geq 11$. Putting some extra effort, one can show that this result holds with $p = 7$ as well. When $p \geq 2 \times 4 - 1 = 7$, the bundle

$$S_{(2,2)} \circ \mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$$

is filtered by the automorphic vector bundles $\nabla(\text{-}2, \text{-}6)$ and $\nabla(\text{-}4, \text{-}4)$ which are (φ, D)-ample when $p \geq 7$. We get the following result:

Proposition 8.7. *Assume that $g = 2$.*

(1) If $p \geq 11$, then $\Lambda^2\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.

(2) If $p = 7$, then $S_{(2,2)}\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.

Corollary 8.8. *Assume that $g = 2$ and $p \geq 7$. If $\iota : S \to \mathrm{Sh}^{\mathrm{tor}}$ is a subvariety of dimension ≥ 2 such that*

(1) S is smooth,

(2) $\iota^{-1}D_{\mathrm{red}}$ is a normal crossing divisor,

then S is of log general type with exceptional locus contained in the boundary.

8.2.1.2. *The case $g = 3$.* The Hodge bundle Ω^{tor} is locally free of rank 3 and

$$\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$$

is locally free of rank 6. Under the assumption $p \geq 2 \times 5 - 1 = 9$, we have

$$\begin{cases} \Lambda^6\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(\text{-}4, \text{-}4, \text{-}4) \\ \Lambda^5\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(\text{-}2, \text{-}4, \text{-}4) \\ \Lambda^4\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(\text{-}1, \text{-}3, \text{-}4) \\ \Lambda^3\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(\text{-}1, \text{-}1, \text{-}4), \nabla(0, \text{-}3, \text{-}3) \\ \Lambda^2\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(0, \text{-}1, \text{-}3) \\ \mathrm{Sym}^2\,\Omega^{\mathrm{tor}} = \nabla(0, 0, \text{-}2). \end{cases}$$

We deduce that $\Lambda^i\,\mathrm{Sym}^2\,\Omega^{\mathrm{tor}}$ is (φ, D)-ample for $i = 5, 6$ when $p \geq 11$. For $i = 4$, we do not know if the bundle $\nabla(\text{-}1, \text{-}3, \text{-}4)$ is (φ, D)-ample since

$$2(\text{-}1, \text{-}3, \text{-}4) + 2\rho = (0, \text{-}6, \text{-}10)$$

is not $\mathcal{Z}_\emptyset$-ample. This incites us to consider the plethysm

$$S_{(2,2,2,2)} \circ \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$$

which is filtered by $\nabla(-2,-6,-8)$, $\nabla(-3,-6,-7)$, $\nabla(-4,-4,-8)$ and $\nabla(-4,-6,-6)$ when $p \geq 2 \times 8 - 1 = 15$. These automorphic bundles are (φ, D)-ample when $p \geq 17$ by theorem 6.7. It implies that $S_{(2,2,2,2)} \circ \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$ is (φ, D)-ample when $p \geq 17$. We get the following result

Proposition 8.9. *Assume that $g = 3$.*

(1) If $p \geq 11$, then $\Lambda^5 \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.

(2) If $p \geq 17$, then $S_{(2,2,2,2)} \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.

Corollary 8.10. *Assume that $g = 3$ and $p \geq 17$. If $\iota : V \to \mathrm{Sh}^{\mathrm{tor}}$ is a subvariety of dimension ≥ 4 such that:*

(1) V is smooth,

(2) $\iota^{-1} D_{\mathrm{red}}$ is a normal crossing divisor,

then V is of log general type with exceptional locus contained in the boundary.

8.2.1.3. *The case $g = 4$.* The Hodge bundle Ω^{tor} is locally free of rank 4 and

$$\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}}) = \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$$

is locally free of rank 10. Under the assumption $p \geq 2 \times 9 - 1 = 17$, we have

$$\begin{cases} \Lambda^{10} \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(-5,-5,-5,-5) \\ \Lambda^{9} \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(-3,-5,-5,-5) \\ \Lambda^{8} \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(-2,-4,-5,-5) \\ \Lambda^{7} \operatorname{Sym}^2 \Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-1,-4,-4,-5), \nabla(2,-2,-5,-5) \\ \Lambda^{6} \operatorname{Sym}^2 \Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-1,-2,-4,-5), \nabla(0,-4,-4,-4) \\ \Lambda^{5} \operatorname{Sym}^2 \Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-1,-1,-3,-5), \nabla(0,-2,-4,-4) \\ \Lambda^{4} \operatorname{Sym}^2 \Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-1,-1,-1,-5), \nabla(0,-1,-3,-4) \\ \Lambda^{3} \operatorname{Sym}^2 \Omega^{\mathrm{tor}} \text{ is filtered by } \nabla(-0,-1,-1,-4), \nabla(0,-0,-3,-3) \\ \Lambda^{2} \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(-0,-0,-1,-3) \\ \operatorname{Sym}^2 \Omega^{\mathrm{tor}} = \nabla(-0,-0,-0,-2). \end{cases}$$

We deduce that $\Lambda^i \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$ is (φ, D)-ample for $i = 8, 9, 10$ when $p \geq 17$. It does not work for $i = 7$ since the character

$$2(-1,-4,-4,-5) + 2\rho = (1,-7,-9,-13)$$

is not $\mathcal{Z}_\emptyset$-ample. Under the assumption $p \geq 2 \times 14 - 1 = 27$, the plethysm

$$S_{(2^7)} \circ \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$$

is filtered by the following list of automorphic bundles

$$\begin{cases} \nabla(\text{-}2, \text{-}8, \text{-}8, \text{-}10) \\ \nabla(\text{-}3, \text{-}6, \text{-}9, \text{-}10) \\ \nabla(\text{-}3, \text{-}7, \text{-}9, \text{-}9) \\ \nabla(\text{-}3, \text{-}8, \text{-}8, \text{-}9) \\ \nabla(\text{-}4, \text{-}4, \text{-}10, \text{-}10) \\ \nabla(\text{-}4, \text{-}6, \text{-}8, \text{-}10) \\ \nabla(\text{-}4, \text{-}7, \text{-}8, \text{-}9) \\ \nabla(\text{-}4, \text{-}8, \text{-}8, \text{-}8) \\ \nabla(\text{-}5, \text{-}5, \text{-}9, \text{-}9) \\ \nabla(\text{-}5, \text{-}6, \text{-}8, \text{-}9) \\ \nabla(\text{-}5, \text{-}7, \text{-}7, \text{-}9) \\ \nabla(\text{-}6, \text{-}6, \text{-}6, \text{-}10) \\ \nabla(\text{-}6, \text{-}6, \text{-}8, \text{-}8) \\ \nabla(\text{-}7, \text{-}7, \text{-}7, \text{-}7). \end{cases}$$

which are all (φ, D)-ample when $p \geq 31$.

Proposition 8.11. *Assume that $g = 4$.*

(1) If $p \geq 17$, then $\Lambda^i \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample for $i \geq 8$

(2) If $p \geq 31$, then $S_{(2^7)}\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.

Corollary 8.12. *Assume that $g = 4$ and $p \geq 31$. If $\iota : V \to \mathrm{Sh}^{\mathrm{tor}}$ is a subvariety of dimension ≥ 7 such that:*

(1) V is smooth,

(2) $\iota^{-1} D_{\mathrm{red}}$ is a normal crossing divisor,

then V is of log general type with exceptional locus contained in the boundary.

Remark 8.13. For the case $g = 5$, the plethysm computations were too long to conclude and it became clear that we needed a different method.

8.2.2. The general case. Consider the Siegel variety $\mathrm{Sh}^{\mathrm{tor}}$ of genus g over k. Based on the results obtained in the cases $g = 2, 3, 4$, we notice[1] that subvarieties of codimension lower or equal to $g - 1$ seem to be of log-general type.

Siegel variety	Dimension of the global variety	Dimension of the subvarieties of log-general type	Codimension of the subvarieties of log-general type
$g = 2$	3	≥ 2	≤ 1
$g = 3$	6	≥ 4	≤ 2
$g = 4$	10	≥ 7	≤ 3

In this section, we prove the following result.

[1] We ignore the technical restriction on the prime p.

Theorem 8.14. *Assume that $p \geq g^2 + 3g + 1$. For all $k \geq g(g-1)/2 + 1$, the bundle $\Omega^k_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is (φ, D)-ample.*

Corollary 8.15. *Assume that $p \geq g^2 + 3g + 1$. Any subvariety $\iota : V \to \mathrm{Sh}^{\mathrm{tor}}$ of codimension $\leq g - 1$ satisfying*

(1) V is smooth,

(2) $\iota^{-1} D_{\mathrm{red}}$ is a normal crossing divisor,

is of log general type with exceptional locus contained in the boundary.

Motivated by the Green-Griffiths-Lang conjecture 8.1, we can formulate the following conjecture.

Conjecture 8.16. *For p large enough, there is a closed subscheme $E \subset \mathrm{Sh}^{\mathrm{tor}}$ such that for any subvariety $\iota : V \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}$ satisfying*

(1) V is smooth,

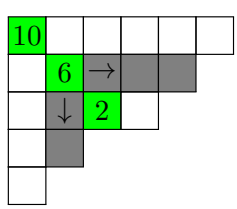
(2) $\iota^{-1} D_{\mathrm{red}}$ is a normal crossing divisor,

V is of log general type if and only if $V \nsubseteq E$.

The theorem 8.14 indicates that such an exceptional locus $E \subset \mathrm{Sh}^{\mathrm{tor}}$ should have codimension $> g - 1$. We believe it has exactly codimension g.

Proof of theorem 8.14. The strategy is to study a ∇-filtration of the kth-exterior power of the bundle $\Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ and check that all the graded pieces are (φ, D)-ample automorphic vector bundles when $p \geq g^2 + 3g + 1$ and $k \geq g(g+1)/2 - (g-1)$. By the Kodaira-Spencer isomorphism of proposition 4.20, the bundle $\Lambda^k \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is isomorphic to $\mathcal{W}(\Lambda^k \operatorname{Sym}^2 \mathrm{std}_{\mathrm{GL}_g})$. By proposition 2.43, the GL_g-module $\Lambda^k \operatorname{Sym}^2 \mathrm{std}_{\mathrm{GL}_g}$ has a ∇-filtration when $p > k$ and it implies that $\Lambda^k \Omega^1_{\mathrm{Sh}^{\mathrm{tor}}}(\log D_{\mathrm{red}})$ is filtered by automorphic bundles $\nabla(w_0 w_{0,L} \lambda)$'s where the λ's are the highest weights of the ∇-filtration of $\Lambda^k \operatorname{Sym}^2 \mathrm{std}_{\mathrm{GL}_g}$. As explained in the example 2.1, determining the Schur functors appearing in a plethysm $S_\lambda \circ S_\mu$ is often a hard task, however the plethysm $\Lambda^k \circ \operatorname{Sym}^2$ belongs to the one of the few cases where a general formula is known. We start with a notation.

Notation 8.17. Let k be a positive integer and λ a partition of k in r distinct parts. We denote $2[\lambda]$ the partition of $2k$ whose main-diagonal hook lengths are $2\lambda_1, \cdots, 2\lambda_r$, and whose i^{th}-part has length $\lambda_i + i$. For example, we have

$2[(5,3,1)] =$ 

$= (6,5,4,2,1)$

where the diagonal hook have lengths $10, 6, 2$.

Lemma 8.18 ([**Wil09**, Lemma 7])**.** *Assume that $p > k$. Then the polynomial functor $\Lambda^k \circ \operatorname{Sym}^2$ has a filtration where the graded pieces are the Schur functors $S_{2[\lambda]}$ where λ range over the set of partition of k in distinct parts.*

Example 8.19. Consider the case $k = 5$. The partitions of 5 in distinct parts are $(5), (4, 1)$ and $(3, 2)$. The plethysm $\Lambda^5 \circ \mathrm{Sym}^2$ is then filtered by the Schur functors $S_{2[(5)]} = S_{(6,1^4)}$, $S_{2[(4,1)]} = S_{(5,3,1^2)}$ and $S_{2[(3,2)]} = S_{(4,4,2)}$.

Since we evaluate this plethysm at the Hodge bundle Ω^{tor} which has rank g, we can discard the partitions $2[\lambda]$ of height strictly greater than g (for such partitions, the evaluation vanishes). Since the height of $2[\lambda]$ is λ_1, we want to study the (φ, D)-ampleness of the automorphic bundles

$$S_{2[\lambda]}\Omega^{\mathrm{tor}} = \mathcal{W}(\nabla(2[\lambda])) = \nabla(w_0 w_{0,L} 2[\lambda])$$

where λ is partition of k in distinct parts with $\lambda_1 \leq g$. By theorem 6.7, we know it is the case when $2w_0w_{0,L}2[\lambda] + 2\rho_L$ is $\mathcal{Z}_\emptyset$-ample and orbitally p-close. Even if the second condition is always satisfied for p large enough, the first condition may not be satisfied as explained in the case $g = 3$. In proposition 6.8, we have seen that automorphic bundles of the form $\nabla(\eta)$ where $\eta = (\eta_1 \geq \cdots \geq \eta_g)$ is a dominant character such that $\eta_1 = 0$ are not nef, hence not (φ, D)-ample. Conversely, we will see that any automorphic bundle $\nabla(\eta)$, where η is a dominant character such that $\eta_1 \leq -1$, is (φ, D)-ample if p is greater than a specific bound which depends on η. We start with the following lemma.

Lemma 8.20. *Consider two* GL_g*-dominant character* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_g \geq 0)$ *and* $\mu = (\mu \geq \cdots \geq \mu \geq 0)$*. The* GL_g*-module* $\nabla(\lambda) \otimes \nabla(\mu)$ *is filtered by costandard modules* $\nabla(\eta)$ *such that* $\eta_g \geq \lambda_g + \mu_g$ *and* $\eta_1 \leq \lambda_1 + \mu_1$*.*

Proof of the lemma. See proposition 2.20 for the existence of the ∇-filtration. The tensor product of two polynomial representation of GL_g is still a polynomial representation. Apply it to $\nabla(\lambda - (\lambda_g^g)) \otimes \nabla(\mu - (\mu_g^g))$ where $(\lambda_g^g) = (\lambda_g, \cdots, \lambda_g)$ and $(\mu_g^g) = (\mu_g, \cdots, \mu_g)$ to get the first inequality. The second inequality follows from the fact that $\lambda + \mu$ is the highest weight of $\nabla(\lambda) \otimes \nabla(\mu)$. □

Proposition 8.21. *Let* $\eta = (\eta_1 \geq \cdots \geq \eta_g)$ *be a dominant character such that* $\eta_1 \leq -1$*. Then the automorphic bundle* $\nabla(\eta)$ *is* (φ, D)*-ample if* $p \geq (g + 1)|\eta_g| + g$*.*

Proof of the proposition. By proposition 3.53, it is enough to show that $\nabla(\eta)^{\otimes n}$ is (φ, D)-ample for some $n \geq 1$. By lemma 8.20, the bundle $\nabla(\eta)^{\otimes n}$ is filtered by automorphic bundles of the form $\nabla(\delta)$ where $\delta_1 \leq n\eta_1$ and $\delta_g \geq n\eta_g$. To apply theorem 6.7, we need to see that each $2\delta + 2\rho_L$ is $\mathcal{Z}_\emptyset$-ample and orbitally p-close. We first focus on the $\mathcal{Z}_\emptyset$-ampleness of $\gamma := 2\delta + 2\rho_L$. In other words, we need to check that

$$\begin{aligned}\gamma = 2\delta + 2\rho_L &= (2\delta_1, \cdots, 2\delta_g) + (g-1, g-3, \cdots, -(g-1)) \\ &= (2\delta_1 + g - 1, \cdots, 2\delta_g - g + 1)\end{aligned}$$

is such that $0 > 2\delta_1 + g - 1 > 2\delta_2 + g - 3 > \cdots > 2\delta_g - g + 1$. The first inequality being the only one non-trivial, it is enough to have $n > (g-1)/2$ as it implies

$$\begin{aligned}2\delta_1 + g - 1 &\leq 2n\eta_1 + g - 1 \\ &\leq -2n + g - 1 \\ &< 0.\end{aligned}$$

For the orbitally p-closeness of $\gamma = 2\delta + 2\rho_L$, we have the following bound

$$\begin{aligned}\max_{\alpha\in\Phi, w\in W, \langle\gamma,\alpha^\vee\rangle\neq 0} |\frac{\langle\gamma, w\alpha^\vee\rangle}{\langle\gamma,\alpha^\vee\rangle}| &\leq \max_{1\leq i\leq j\leq g} \frac{|\gamma_j| + |\gamma_i|}{2} \\ &\leq \frac{2|\gamma_g|}{2} \\ &\leq |2\delta_g - (g-1)| \\ &\leq 2|\delta_g| + (g-1) \\ &\leq 2n|\eta_g| + (g-1) \text{ by lemma 8.20}\end{aligned}$$

and we deduce that it is enough to have $2n|\eta_g| + g \leq p$. Combining it with the restriction $n = \lfloor (g-1)/2 \rfloor + 1 \leq (g+1)/2$ which ensure the $\mathcal{Z}_\emptyset$-ampleness of γ, we get

$$p \geq (g+1)|\eta_g| + g.$$

□

With the proposition 8.21 in mind, recall that we want to prove that the bundle

$$\nabla(w_0 w_{0,L} 2[\lambda])$$

is (φ, D)-ample when λ is a partition of k in distinct parts such that $\mathrm{ht}(2[\lambda]) = \lambda_1 \leq g$. If there exists such a partition λ with $\lambda_1 \leq g-1$, we will not be able to apply the proposition 8.21 to $w_0 w_{0,L} 2[\lambda]$ as the first term will be 0. To avoid these partitions, we prove the following lemma.

Lemma 8.22. *Assume that $p > k$. All the automorphic bundles $\nabla(\eta)$ appearing as graded pieces of the ∇-filtration of $\Lambda^k \operatorname{Sym}^2 \Omega^{\mathrm{tor}}$ satisfy $\eta_1 \leq -1$ if and only if $k \geq g(g-1)/2 + 1$.*

Proof of the lemma. Assume that $k \geq g(g-1)/2 + 1$. We need to check that there exists no partition λ of k in distinct parts such that $\mathrm{ht}(2[\lambda]) = \lambda_1 \leq g-1$. Consider a partition λ of k in r-distinct parts. We have

$$\frac{g(g-1)}{2} + 1 \leq k = \lambda_1 + \lambda_2 + \cdots + \lambda_r \leq \frac{\lambda_1(\lambda_1+1)}{2}$$

which is possible only if $\lambda_1 \geq g$. Conversely, if $k \leq g(g-1)/2$, it is not hard to find a partition λ of k in distinct parts such that $\lambda_1 \leq g-1$. □

Since $(2[\lambda])_1 = \lambda_1 + 1$, we conclude with the proposition 8.21 which says that each automorphic bundle $\nabla(w_0 w_{0,L} 2[\lambda])$ where $\lambda_1 = g$ is (φ, D)-ample when

$$p \geq g^2 + 3g + 1 = (g+1)\underbrace{|(w_0 w_{0,L} 2[\lambda])_g|}_{=g+1} + g$$

□

CHAPTER 9

Opérateur de Hecke entier et auto-dualité

Ce chapitre est indépendant du reste de la thèse et fortement inspiré de l'article [**BP20**]. Les modèles entiers des variétés de Siegel avec un niveau parahorique en p permettent de définir intégralement une correspondance de Hecke $T_{p,g}$. Dans [**FP19**], Fakhruddin et Pilloni compactifient cette correspondance de Hecke afin d'obtenir une action sur toute la cohomologie cohérente et pas seulement le H^0. En lisant [**BP20**] qui étudie cet opérateur de Hecke compactifié dans le cas des courbes modulaires, nous avons voulu redémontrer leur résultat d'auto-dualité pour la dualité de Serre et commencer à étudier la situation analogue pour des variétés de Siegel de genre supérieur à 1. Toutes les idées que l'on développe sont dues à Boxer et Pilloni et on peut voir ce chapitre comme un exercice d'étudiant, indépendant du reste de la thèse, où l'on adapte leur résultat d'auto-dualité au cadre des variétés de Siegel.

Dans la section 9.1, nous rappelons quelques propriétés algébriques et cohomologiques des modules S_k afin de redémontrer le lemme fondamental permettant de renormaliser les opérateurs de Hecke entier. Dans la section 9.2, nous rappelons les principales propriétés du complexe dualisant en cohomologie cohérente. Dans la la section 9.3, nous rappelons comment définir la correspondance de Hecke cohomologique $T_{p,g}$. Dans la section 9.4, nous proposons une nouvelle définition de l'opérateur de Hecke cohomologique $T_{p,g}$ agissant sur les fibrés automorphes $\nabla(\lambda), \Delta(\lambda)$ et $T(\lambda)$. Dans la section 9.5, nous redémontrons le résultat d'auto-dualité de [**BP20**] en utilisant notre définition de $T_{p,g}$. Pour ne pas compliquer la discussion avec les compactifications toroïdales, nous travaillons avec la variété de Siegel non compactifiée Sh et le lecteur intéressé pourra se référer à [**FP19**] pour obtenir un opérateur de Hecke sur toute la cohomologie $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}, \mathcal{F}_\lambda)$ où $\mathcal{F}_\lambda = \nabla(\lambda), \Delta(\lambda)$ ou $T(\lambda)$.

9.1. Modules S_k

Cette section permet de rappeler quelques propriétés des modules au dessus d'un anneau noethérien ou au dessus d'un schéma localement noethérien.

Définition 9.1. *Soit R un anneau noethérien et M un R-module de type fini. On note* $\mathrm{Ass}(M)$ *l'ensemble des idéaux premiers associés de M, c'est à dire l'ensemble des idéaux premiers de la forme*

$$I = \mathrm{Ann}_R(m) = \{r \in R \mid rm = 0\}$$

avec $m \in M$.

Proposition 9.2. *Soit R un anneau noethérien et M un R-module de type fini. On a* $\mathrm{Ass}(M) \subset \mathrm{Supp}(M)$.

Démonstration. Considérons $\mathfrak{p} = \mathrm{Ann}_R(m) \in \mathrm{Ass}(M)$. Alors pour tout $r \in R \backslash \mathfrak{p}$, on a $rm \neq 0$, donc $\frac{m}{1} \in M_{\mathfrak{p}}$ est non nul. □

Proposition 9.3. *Soit R un anneau noethérien et M un R-module de type fini. Les ensembles suivants sont les mêmes*

(1) Les éléments minimaux de $\mathrm{Ass}(M)$*,*
(2) Les éléments minimaux de $\mathrm{Supp}(M)$*.*

Démonstration. Voir [**Sta21**, Tag 02CE]. □

Proposition 9.4. *Soit R un anneau noethérien et M un R-module de type fini. La flèche naturelle*

$$M \to \prod_{\mathfrak{p}\in \mathrm{Ass}(M)} M_{\mathfrak{p}}$$

est injective.

On définit maintenant la profondeur d'un module.

Définition 9.5. *Soit R un anneau noethérien, M un R-module de type fini et $I \subset R$ un idéal tel que $IM \neq M$. On appelle profondeur de M le long de I et on note* $\mathrm{prof}_I(M)$ *le supremum de la longueur des suites M-régulières dans I. Autrement dit,* $\mathrm{prof}(M)$ *est le maximum des entiers $n \geq 0$ tels qu'il existe une suite $f_1, \cdots, f_n \in I$ vérifiant*

$$\forall i \ f_i \text{ est régulier dans } M/(f_1, \cdots, f_{i-1})M.$$

Remarque 9.6.
(1) Lorsque $(R, \mathfrak{m})$ est un anneau local, on note simplement $\mathrm{prof}(M) := \mathrm{prof}_{\mathfrak{m}}(M)$ la profondeur le long de l'idéal maximal $\mathfrak{m}$.
(2) Étant donné un schéma X localement noethérien, Z un sous-schéma fermé de X correspondant à un idéal $\mathcal{I}$ et $\mathcal{F}$ un $\mathcal{O}_X$-module cohérent sur X dont le support n'est pas inclu dans Z, alors on définit de la même façon la profondeur $\mathrm{prof}_Z(\mathcal{F})$ de $\mathcal{F}$ le long de Z comme le maximum des entiers $n \geq 0$ tels qu'il existe une suite de sections $f_1, \cdots, f_n \in \mathcal{I}$ vérifiant

$$\forall i \ f_i \text{ est régulier dans } \mathcal{F}/(f_1, \cdots, f_{i-1})\mathcal{F}.$$

(3) Si $IM = M$, alors on pose $\mathrm{prof}_I(M) = +\infty$.

Proposition 9.7. *Soit $(R, \mathfrak{m})$ un anneau local noethérien et M un R-module de type fini. On a*

(1) $\mathrm{prof}(M) \leq \dim \mathrm{Supp}\, M$*.*
(2) Si $\mathfrak{p} \in \mathrm{Ass}(M)$*, alors* $\mathrm{prof}(M) \leq \dim R/\mathfrak{p}$*.*

Démonstration. Voir [**Sta21**, Tag 00LK] et [**Sta21**, Tag 0BK4] □

Proposition 9.8. *Soit X un schéma localement noethérien, $\mathcal{F}$ un faisceau cohérent sur X et $Z \subset X$ un sous-schéma fermé d'idéal $\mathcal{I}$. Alors, on a*

$$\mathrm{prof}_Z(\mathcal{F}) = \inf_{x\in Z} \mathrm{prof}\, \mathcal{F}_x$$

où $\mathcal{F}_x$ est vu comme $\mathcal{O}_{X,x}$-module.

Il est utile de caractériser la profondeur grâce à la cohomologie locale. Pour cela on suit [**Har67**].

Définition 9.9. *Soit X un schéma localement noethérien, $\iota : Z \hookrightarrow X$ un sous-schéma fermé. On dispose d'un foncteur $\underline{\Gamma}_Z$ qui à un $\mathcal{O}_X$-module cohérent $\mathcal{F}$ associe le $\mathcal{O}_X$-module $\iota_*\iota^!\mathcal{F}$ et d'un foncteur $\Gamma_Z = \Gamma \circ \underline{\Gamma}_Z$ qui à un $\mathcal{O}_X$-module cohérent $\mathcal{F}$ associe le groupe abélien $H^0(X, \iota_*\iota^!\mathcal{F})$. On note $\underline{R\Gamma}_Z(X, \_)$ et $R\Gamma_Z(X, \_)$ les foncteurs dérivés de ces deux foncteurs. C'est la cohomologie à support dans Z. Si $\mathcal{F}$ est un faisceau cohérent, on note pour $i \geq 0$*

(1) $\mathcal{H}^i_Z(X, \mathcal{F}) = \mathcal{H}^i(\underline{R\Gamma}_Z(X, \mathcal{F}))$*,*

(2) $H^i_Z(X,\mathcal{F}) = \mathcal{H}^i(R\Gamma_Z(X,\mathcal{F}))$.

Si $Z = X$, on retrouve la cohomologie usuelle et on enlève l'indice Z des notations.

Proposition 9.10. *Soit X un schéma localement noethérien, $\mathcal{F}$ un faisceau cohérent sur X, $Z \subset X$ un sous-schéma fermé et U l'ouvert complémentaire de Z dans X. On a des triangles distingués*

$$\begin{tikzcd} \underline{R\Gamma}_Z(X,\mathcal{F}) \arrow[rr] & & \underline{R\Gamma}(X,\mathcal{F}) \arrow[dl] \\ & \underline{R\Gamma}(U,\mathcal{F}) \arrow[ul, "+1"] & \end{tikzcd}$$

et

$$\begin{tikzcd} R\Gamma_Z(X,\mathcal{F}) \arrow[rr] & & R\Gamma(X,\mathcal{F}) \arrow[dl] \\ & R\Gamma(U,\mathcal{F}) \arrow[ul, "+1"] & \end{tikzcd}.$$

Démonstration. Voir [**Har67**, Proposition 1.9]. □

Le résultat qui relie ces groupes de cohomologie locale à la profondeur est le suivant.

Proposition 9.11. *Soit X un schéma localement noethérien, $\mathcal{F}$ un faisceau cohérent sur X et $Z \subset X$ un sous-schéma fermé. On a l'égalité*

$$\operatorname{prof}_Z(\mathcal{F}) = \min\{i \geq 0 \mid \mathcal{H}^i_Z(X,\mathcal{F}) \neq 0\}.$$

Démonstration. Voir [**Har67**, Theorem 3.8]. □

On définit maintenant des propriétés faisant intervenir la profondeur.

Définition 9.12. *Soit X un schéma localement noethérien et $\mathcal{F}$ un faisceau cohérent sur X et $k \geq 0$ un entier.*

(1) On dit que $\mathcal{F}$ est (S_k) si pour tout $x \in X$, le $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ vérifie

$$\operatorname{prof}(\mathcal{F}_x) \geq \min(k, \dim \operatorname{Supp} \mathcal{F}_x).$$

(2) On dit que le schéma X est (S_k) si $\mathcal{O}_X$ est (S_k) en tant que $\mathcal{O}_X$-module.

Définition 9.13. *Soit X un schéma localement noethérien et $\mathcal{F}$ un faisceau cohérent sur X.*

(1) On dit que $\mathcal{F}$ est CM si $\mathcal{F}$ est (S_k) pour tout $k \geq 0$.

(2) On dit le schéma X est CM si $\mathcal{O}_X$ est CM en tant que $\mathcal{O}_X$-module.

On a alors une caractérisation plus simple des modules CM.

Proposition 9.14. *Soit $(R,\mathfrak{m})$ un anneau local noethérien et M un R-module de type fini. Alors*

$$M \text{ est CM si et seulement si } \operatorname{prof} M = \dim \operatorname{Supp} M.$$

Démonstration. Découle directement de la définition de CM. □

On détaille plusieurs résultats de ces modules (S_k).

Proposition 9.15. *Soit R un anneau noethérien et M un R-module de type fini. On suppose que M est (S_1). Alors M n'a pas d'idéaux premiers associés non minimaux dans $\operatorname{Supp} M$. C'est à dire, les ensembles suivants sont les mêmes.*

(1) $\operatorname{Ass}(M)$.

(2) Les éléments minimaux de $\mathrm{Ass}(M)$*.*

(3) Les éléments minimaux de $\mathrm{Supp}(M)$*.*

Démonstration. Soit $\mathfrak{p}$ un idéal premier associé de M, alors l'idéal maximal $\mathfrak{p}R_{\mathfrak{p}}$ est un idéal associé de $M_{\mathfrak{p}}$ et cela implique

$$\operatorname{prof} M_{\mathfrak{p}} = 0$$

et donc on a

$$0 = \operatorname{prof} M_{\mathfrak{p}} \geq \min(1, \dim \operatorname{Supp} M_{\mathfrak{p}}),$$

donc $\dim \operatorname{Supp} M_{\mathfrak{p}} = 0$ et $\mathfrak{p}$ est minimal dans $\operatorname{Supp} M$. □

Proposition 9.16. *Soit X un schéma localement noethérien, $\mathcal{F}$ un faisceau cohérent sur X et $Z \subset X$ un sous-schéma fermé de codimension ≥ 1 et $j : U \to X$ l'ouvert complémentaire de Z. On suppose que $\mathcal{F}$ est (S_1) et que $\operatorname{Supp} \mathcal{F} = X$. Alors toute section de $\mathcal{F}$ définie sur X et nulle sur U est nulle.*

Démonstration. On écrit le début de la suite longue du triangle distingué de la proposition 9.10

$$0 \longrightarrow \mathcal{H}^0_Z(X, \mathcal{F}) \longrightarrow \mathcal{F} \xrightarrow{\text{res}} j_* j^* \mathcal{F} \longrightarrow \mathcal{H}^1_Z(X, \mathcal{F})$$

et par hypothèse on a

$$\operatorname{prof}_Z(\mathcal{F}) = \inf_{x \in Z} \operatorname{prof} \mathcal{F}_x \geq \inf_{x \in Z} \min(1, \dim \operatorname{Supp} \mathcal{F}_x) = 1,$$

donc

$$\mathcal{H}^0_Z(X, \mathcal{F}) = 0$$

et la flèche $\mathrm{res} : \mathcal{F} \to j_* j^* \mathcal{F}$ est injective. □

Proposition 9.17. *Soit X un schéma localement noethérien, $\mathcal{F}$ un faisceau cohérent sur X et $Z \subset X$ un sous-schéma fermé de codimension ≥ 2 et $j : U \to X$ l'ouvert complémentaire de Z. On suppose que $\mathcal{F}$ est (S_2) et que $\operatorname{Supp} \mathcal{F} = X$. Alors toute section de $\mathcal{F}$ définie sur U s'étend uniquement à X.*

Démonstration. On écrit le début de la suite longue du triangle distingué de la proposition 9.10

$$0 \longrightarrow \mathcal{H}^0_Z(X, \mathcal{F}) \longrightarrow \mathcal{F} \xrightarrow{\text{res}} j_* j^* \mathcal{F} \longrightarrow \mathcal{H}^1_Z(X, \mathcal{F})$$

et par hypothèse on a

$$\operatorname{prof}_Z(\mathcal{F}) = \inf_{x \in Z} \operatorname{prof} \mathcal{F}_x \geq \inf_{x \in Z} \min(2, \dim \operatorname{Supp} \mathcal{F}_x) = 2$$

donc

$$\mathcal{H}^0_Z(X, \mathcal{F}) = \mathcal{H}^1_Z(X, \mathcal{F}) = 0$$

et la flèche $\mathrm{res} : \mathcal{F} \to j_* j^* \mathcal{F}$ est un isomorphisme. □

9.2. Complexe dualisant

Les résultats de cette section proviennent principalement de [**Har66b**]. On fixe un schéma S affine noethérien.

Définition 9.18. *Soit X un schéma localement noethérien. On note $\mathcal{D}^b_{\mathrm{qcoh}}(\mathcal{O}_X)$ la catégorie dérivée des complexes bornés de $\mathcal{O}_X$-modules quasi-cohérents.*

Définition 9.19. *Soient X, Y deux schémas localement noethériens sur S et $f : X \to Y$ un morphisme de S-schémas. On dit que f est plongeable s'il existe un S-schéma lisse P, un entier $n \geq 1$ et un morphisme fini $\iota : X \to P \times_S Y$ tel que le diagramme suivant soit commutatif*

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "f"] & P \times_S Y \arrow[dl, "p_2"] \\ Y & \end{tikzcd} .$$

Proposition 9.20. *Soient X, Y deux S-schémas localement noethérien. Soit $f : X \to Y$ un morphisme de S-schéma plongeable. Alors le foncteur*

$$Rf_* : \mathcal{D}^b_{\mathrm{qcoh}}(\mathcal{O}_X) \to \mathcal{D}^b_{\mathrm{qcoh}}(\mathcal{O}_Y)$$

possède un adjoint à droite

$$f^! : \mathcal{D}^b_{\mathrm{qcoh}}(\mathcal{O}_Y) \to \mathcal{D}^b_{\mathrm{qcoh}}(\mathcal{O}_X).$$

Les résultats du reste de cette section sont issus de [**FP19**], qui repose principalement sur [**Har66b**][**Con00**].

Proposition 9.21. *Soit X un S-schéma localement noethérien. Soit $f : X \to S$ un morphisme plongeable de dimension relative n. Alors*

$$\omega^\bullet_{X/S} := f^! \mathcal{O}_S$$

est représentable par un complexe concentré en degrés $[-n, 0]$.

(1) Si on suppose que f est lisse, alors $\omega^\bullet_{X/S}$ est représentable par le faisceau inversible $\det \Omega^1_{X/S}$ concentré en degré $-n$.

(2) Si on suppose que f est lci, alors le complexe cotangent $\mathbb{L}_{X/S}$ est parfait et $\omega^\bullet_{X/S}$ est représentable par le faisceau inversible $\det \mathbb{L}_{X/S}$ concentré en degré $-n$.

(3) Si on suppose que f est CM, alors $\omega^\bullet_{X/S}$ est représentable par un faisceau cohérent $\omega_{X/S}$ concentré en degré $-n$. Ce faisceau $\omega_{X/S}$ est plat sur S, il vérifie la propriété (S_2) de Serre et

$$\operatorname{Supp} \omega_{X/S} = X.$$

Et si on suppose de plus que S est un schéma CM, alors $\omega_{X/S}$ est un module CM.

Démonstration. Voir [**FP19**, Lemma 2.2/2.3/2.4]. Pour la propriété (S_2) du faisceau dualisant, voir [**Sta21**, Tag 0AWN]. □

On se sert de cette proposition pour étudier la situation suivante.

Proposition 9.22. *Soient X, Y deux S-schémas localement noethériens. On considère un diagramme commutatif*

$$\begin{tikzcd} Y \arrow[d, "p"] \arrow[dr, "g"] & \\ X \arrow[r, "f"] & S \end{tikzcd}$$

où f,g et p sont plongeables. On suppose que f et g ont la même dimension relative n et que S est CM. Alors

(1) Si f est lci et g est CM alors $p^!\mathcal{O}_X$ est représentable par un faisceau cohérent CM $\omega_{Y/X}$ concentré en degré 0 vérifiant

$$\operatorname{Supp} \omega_{Y/X} = Y.$$

(2) Si de plus il y a des ouverts lisses sur S, $U \subset Y$ et $V \subset X$ tels que $p(U) \subset V$ et $\operatorname{codim} Y \backslash U = 2$, alors on dispose d'une application trace

$$\mathrm{tr} : \mathcal{O}_Y \to \omega_{Y/X}.$$

Démonstration. Voir [**FP19**, Proposition 2.6]. □

9.3. Une correspondance de Hecke sur la variété de Siegel entière

Les résultats que nous présentons ici sont des cas particuliers de [**FP19**] où la question de définir des opérateurs de Hecke sur la cohomologie cohérente de variétés de Shimura plus générale est considérée. Pour ne pas compliquer la discussion avec les compactifications toroïdales, nous travaillons avec la variété de Siegel non compactifiée et le lecteur intéressé pourra se référer à [**FP19**] pour compactifier cette correspondance de Hecke. On considère $X \to \operatorname{Spec} \mathbb{Z}_p$ la variété de Siegel lisse de genre g et de niveau N où $p \nmid N$ (que l'on notait Sh dans les autres chapitres de cette thèse). On a une description comme espace de modules : les S-points de X sont les classes d'isomorphismes de triplets (A, λ, ψ_N) où A/S un schéma abélien de dimension g avec une polarisation principale $\lambda : A \to A^t$ et ψ_N est une similitude symplectique

$$\psi_N : A[N] \to (\underline{\mathbb{Z}/N\mathbb{Z}}_S)^{2g}.$$

Notons $f : A \to X$ le schéma abélien universel et $\omega_A = e^*\Omega^1_{A/S}$ le fibré de de Hodge qui est un faisceau localement libre de rang g sur X. Nous souhaitons définir un opérateur de Hecke T_p correspondant à l'indicatrice

$$\mathbf{1}_{\mathrm{GSp}_{2g}(\mathbb{Z}_p) M \mathrm{GSp}_{2g}(\mathbb{Z}_p)} \in \mathcal{C}_c^\infty(\mathrm{GSp}_{2g}(\mathbb{Z}_p) \backslash \mathrm{GSp}_{2g}(\mathbb{Q}_p) / \mathrm{GSp}_{2g}(\mathbb{Z}_p))$$

où M est la matrice

$$\begin{array}{c} g\left\{\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right. \\ g\left\{\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right. \end{array} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & p^{-1} & & \\ & & & & \ddots & \\ & & & & & p^{-1} \end{pmatrix}$$

dans $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$. Le groupe fini et plat $A[p]$ sur X est muni d'un accouplement symplectique qui provient de l'accouplement de Weil où on utilise la polarisation principale pour identifier $A[p]$ et ${}^tA[p]$.

On considère maintenant la correspondance $C \to \operatorname{Spec} \mathbb{Z}_p$ qui est un espace de module sur $\mathbb{Z}_p$ dont les S-points sont les quadruplets (A, λ, ψ_N, H) où $(A, \lambda, \psi_N) \in X(S)$ et $H \subset A[p]$ est un sous-groupe totalement isotrope de la p-torsion. Comme $H^\perp = H$, H est un sous-groupe d'ordre p^g.

Proposition 9.23. *L'application $g : C \to \operatorname{Spec} \mathbb{Z}_p$ est CM.*

Démonstration. Voir [**He13**, Theorem 1.2]. □

Sur le schéma C, on dispose d'une chaine d'isogénie $A \to A/H \to A$ entre schémas abéliens principalement polarisés où la première flèche est la projection et la deuxième le dual de la projection composée avec la polarisation principale λ. On a des flèches de projections

$$\begin{tikzcd} & C \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

où p_1 oublie A/H et p_2 oublie A. La polarisation sur A/H est celle qui fait commuter le diagramme

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "\lambda"] & A/H \arrow[d, "p\lambda'"] \\ {}^tA & {}^t(A/H) \arrow[l, "{}^t\pi"] \end{tikzcd} .$$

Proposition 9.24. *Les flèches p_1 et p_2 vérifient les hypothèses de la proposition 9.22.*

Démonstration. Les flèches $p_1, p_2 : C \to X$ sont plongeables avec $P = \mathbb{P}^n_{\mathbb{Z}_p}$ pour un $n \geq 1$ car elles sont projectives et admettent un faisceau ample. Pour construire ce faisceau relativement ample, il suffit de construire une immersion fermée vers une grassmannienne. L'anneau des fonctions de $A[p]$, où $A \to X$ est la variété abélienne universelle, est un fibré vectoriel de rang p^{2g} sur X munie d'une structure d'algèbre de Hopf venant de la loi de groupe schématique. Donc $p_1, p_2 : C \to X$ sont des fibrations en variétés de drapeaux qui se plonge vers une grassmannienne $Q \to X$. Le X-morphisme $C \to Q$ paramètre des drapeaux partiels de l'anneau des fonctions sur $A[p]$, compatibles à la loi de Hopf, c'est donc bien une immersion fermée. Le morphisme $C \to \operatorname{Spec} \mathbb{Z}_p$ est plongeable comme composée de deux morphismes plongeables. On prend pour U et V les lieux ordinaires de C et X. □

Si on possède seulement une application $p_2^*\mathcal{F} \to p_1^*\mathcal{F}$, il suffit de la composer par l'application trace

$$\mathrm{tr}_1 \otimes 1_{p_1^*\mathcal{F}} : p_1^*\mathcal{O}_X \otimes p_1^*\mathcal{F} = p_1^*\mathcal{F} \to p_1^!\mathcal{O}_X \otimes p_1^*\mathcal{F} = p_1^!\mathcal{F},$$

pour obtenir une correspondance cohomologique $p_2^*\mathcal{F} \to p_1^!\mathcal{F}$. Nous en profitons pour démontrer le résultat suivant qui sera utilisé plusieurs fois après.

Proposition 9.25. *Considérons une correspondance cohomologique sur C*

$$T : p_2^*\mathcal{F} \to p_1^!\mathcal{F},$$

telle que $\mathcal{F}$ soit un faisceau localement libre. Soit $k \in \mathbb{Z}$ un entier. Soit $C^{ord}_{\mathbb{F}_p}$ l'ouvert de la fibre spéciale de C constitué du lieu ordinaire. On suppose que pour tout point (de façon équivalente, en tout point géométrique) x de $C^{ord}_{\mathbb{F}_p}$, on a la factorisation

$$\begin{tikzcd} (p_2^*\mathcal{F})_x \arrow[r, "T_x"] \arrow[dr, dotted] & (p_1^!\mathcal{F})_x \\ & p^k(p_1^!\mathcal{F})_x \arrow[u] \end{tikzcd}$$

alors, l'image de T est un sous-module de $p^k p_1^!\mathcal{F}$ et l'application $p^{-k}T$ est bien définie sur C.

Démonstration. Premièrement, on suppose $k \geq 1$ sinon il n'y a rien à démontrer. Comme $p_1^!\mathcal{O}_X$ est plat sur $\mathbb{Z}_p$, $p_1^!\mathcal{F}$ est également plat sur $\mathbb{Z}_p$ et donc p n'est pas un diviseur de 0 dans $p_1^!\mathcal{F}$. Si on montre que l'image de T est un sous-module de $p^k p_1^!\mathcal{F}$, alors comme la multiplication par p^k est injective dans $p_1^!\mathcal{F}$, la conclusion de la proposition suit. Notons U l'ouvert de C constitué de la fibre générique et du lieu ordinaire de la fibre spéciale de C et Z son complémentaire dans C. Par platitude de C sur $\mathbb{Z}_p$, l'ensemble Z est un fermé de codimension 2 de C. Soit V un ouvert de C et s une section de $p_2^*\mathcal{F}$ sur V. Par hypothèse, la restriction de $T(s)$ à $U \cap V$ appartient à $p^k p_1^!\mathcal{F}(U \cap V)$, ce qui signifie qu'il existe une section s' de $p_1^!\mathcal{F}$ sur $U \cap V$ telle que

$$T(s)_{|U\cap V} = p^k s'.$$

Comme $p_1^!\mathcal{O}_X$ est un module (S_2) supporté sur tout C, il en est de même pour $p_1^!\mathcal{F}$ et par la proposition 9.17, on sait que s' s'étend de façon unique à V. Ainsi, $T(s) = p^k s'$ avec $s' \in p_1^!\mathcal{F}(V)$. Ceci conclut la preuve. □

9.4. Définition de l'opérateur de Hecke T_p

On souhaite définir une correspondance cohomologique $T_p : p_2^*\mathcal{F} \to p_1^!\mathcal{F}$ où $\mathcal{F}$ sera un fibré automorphe de poids λ qui coïncide avec l'action classique du même opérateur sur $X_{\mathbb{Q}_p}$ en prenant en compte les renormalisations usuelles. La renormalisation dépend du poids et on introduit donc la notation suivante.

Définition 9.26. *Pour tout caractère $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_g)$, on pose*

$$v(\lambda) = \inf_{0\leq i\leq g} \sum_{j=i+1}^{g} k_j + \frac{i(i+1)}{2}$$

Notation 9.27. Dans ce chapitre seulement, on change notre convention pour les fibrés automorphes $\nabla(\lambda), \Delta(\lambda)$ et $T(\lambda)$. Le lecteur est invité à comparer avec la définition 4.16. On définit directement

(1) $\nabla(\lambda) := \mathcal{W}(\nabla(\lambda))$,
(2) $\Delta(\lambda) := \mathcal{W}(\Delta(\lambda))$,
(3) $T(\lambda) := \mathcal{W}(T(\lambda))$,

où $\mathcal{W}$ est le foncteur défini dans la définition 3.17. Ainsi on a simplement

(1) $S_\lambda\Omega = \nabla(\lambda)$,
(2) $W_\lambda\Omega = \Delta(\lambda)$,
(3) $T_\lambda\Omega = T(\lambda)$.

L'idée générale est d'utiliser l'isogénie universelle de schémas abéliens

$$\pi : A \to A/H$$

ou l'isogénie universelle duale

$${}^t\pi : {}^t(A/H) \to {}^tA$$

sur C et de considérer le tiré en arrière par la section neutre e des flèches induites sur les différentielles. En identifiant les variétés abéliennes A et A/H avec leurs variétés abéliennes duales grâce aux polarisations principales, on obtient des applications

$$\varphi : p_2^*\omega_A = \omega_{A/H} \to \omega_A = p_1^*\omega_A$$

et

$${}^t\varphi : p_1^*\omega_A = \omega_A \to \omega_{A/H} = p_2^*\omega_A$$

telles ques $\varphi \circ {}^t\varphi$ et ${}^t\varphi \circ \varphi$ soient les applications de multiplication par p.

9.4.1. Définition sur la fibre générique $C_{\mathbb{Q}_p}$. Commençons par expliquer ce qu'il se passe sur la fibre générique de C.

Proposition 9.28. *Sur $C_{\mathbb{Q}_p}$, l'isogénie universelle $\pi : A \to A/H$ est étale et sa différentielle φ est un isomorphisme. Soit $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_g)$ un poids GL_g-dominant et $n \geq 0$ un entier assez grand pour que $k_g + n \geq 0$. On note λ_n le poids $(k_1 + n \geq k_2 + n \geq \ldots \geq k_g + n)$ et on définit l'opérateur*

$$(T_p)^{\mathrm{naif},\nabla(\lambda)}_{\mathbb{Q}_p} : p_2^*\nabla(\lambda)_{\mathbb{Q}_p} \to p_1^!\nabla(\lambda)_{\mathbb{Q}_p}$$

comme la composée de

$$S_{\lambda_n}\varphi \otimes (\det \varphi^{-1})^{\otimes -n} : p_2^*\nabla(\lambda_n) \otimes p_2^*\nabla(1, \cdots, 1) = p_2^*\nabla(\lambda) \to p_1^*\nabla(\lambda_n) \otimes p_1^*\nabla(1, \cdots, 1) = p_1^*\nabla(\lambda)$$

avec l'application trace $p_1^*\nabla(\lambda) \to p_1^!\nabla(\lambda)$. Et on renormalise,

$$(T_p)^{\nabla(\lambda)}_{\mathbb{Q}_p} := p^{-v(\lambda)}(T_p)^{\mathrm{naif},\nabla(\lambda)}_{\mathbb{Q}_p}.$$

Remarque 9.29. Si on faisait la même chose avec $(T_p)^{\Delta(\lambda)}_{\mathbb{Q}_p}$, on obtiendrait le même opérateur car on a un isomorphisme canonique

$$\nabla(\lambda)_{\mathbb{Q}_p} = \Delta(\lambda)_{\mathbb{Q}_p}.$$

9.4.2. Définition sur C pour un poids Sp_{2g}-dominant.

Définition 9.30. *Soit $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_g)$ un poids Sp_{2g}-dominant, c'est à dire tel que $k_g \geq 0$, alors on peut définir l'opérateur $T_p^{\mathrm{naif},\nabla(\lambda)}$ comme la composée*

$$p_2^*\nabla(\lambda) \xrightarrow{S_\lambda \varphi} p_1^*\nabla(\lambda) \xrightarrow{\mathrm{tr}_1 \otimes 1_{p_1^*\nabla(\lambda)}} p_1^!\nabla(\lambda)$$

et l'opérateur $T_p^{\mathrm{naif},\Delta(\lambda)}$ comme la composée

$$p_2^*\Delta(\lambda) \xrightarrow{W_\lambda \varphi} p_1^*\Delta(\lambda) \xrightarrow{\mathrm{tr}_1 \otimes 1_{p_1^*\Delta(\lambda)}} p_1^!\Delta(\lambda) \ .$$

Proposition 9.31. *On se place dans la situation de la définition précédente. Alors,* $T_p^{\mathrm{naif},\nabla(\lambda)}$ *peut se factoriser*

$$\begin{tikzcd} p_2^*\nabla(\lambda) \arrow[r, "T_p^{\mathrm{naif},\nabla(\lambda)}"] \arrow[dr, dotted] & p_1^!\nabla(\lambda) \\ & p^{v(\lambda)}p_1^!\nabla(\lambda) \arrow[u] \end{tikzcd}$$

et de même $T_p^{\mathrm{naif},\Delta(\lambda)}$ *peut se factoriser*

$$\begin{tikzcd} p_2^*\Delta(\lambda) \arrow[r, "T_p^{\mathrm{naif},\Delta(\lambda)}"] \arrow[dr, dotted] & p_1^!\Delta(\lambda) \\ & p^{v(\lambda)}p_1^!\Delta(\lambda) \arrow[u] \end{tikzcd} .$$

Démonstration. Par la proposition 9.25, il suffit de le montrer en tout point géométrique ξ de $C_{\mathbb{F}_p}^{\mathrm{ord}}$, le lieu ordinaire de la fibre spéciale. On est donc ramené à étudier le comportement de T_p au voisinage de ξ. Le tiré en arrière A_ξ est une variété abélienne ordinaire au dessus de k un corps de caractéristique p algébriquement clos et $H_\xi \subset A_\xi[p]$ a pour rang étale $0 \leq i \leq g$. C'est à dire que l'on a une décomposition de groupes finis et plats

$$H_\xi \simeq \mu_p^{g-i} \times \underline{\mathbb{Z}/p\mathbb{Z}}_k^i.$$

Ainsi, il existe des trivialisation de $(\omega_A)_\xi$ telles que la différentielle $\varphi_\xi : p_2^*(\omega_A)_\xi \to p_1^*(\omega_A)_\xi$ fasse commuter le diagramme suivant

$$\begin{tikzcd} p_2^*(\omega_A)_\xi \arrow[r, "\varphi_\xi"] \arrow[d, "\simeq"] & p_1^*(\omega_A)_\xi \arrow[d, "\simeq"] \\ \mathcal{O}_{C,\xi}^{\oplus g} \arrow[r, "h"] & \mathcal{O}_{C,\xi}^{\oplus g} \end{tikzcd}$$

où h est l'application de multiplication par

$$\begin{array}{c} g-i \left\{ \begin{pmatrix} p & & & & & \\ & \ddots & & & & \\ & & p & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \right\} i \end{array}$$

Par la proposition 2.57, cela entraine que les applications $S_\lambda\varphi_\xi$ et $W_\lambda\varphi_\xi$ se factorisent par $p^{\sum_{j=i+1}^g k_j}$. Enfin, il résulte de la théorie de Serre-Tate [**Kat81**] que l'application trace $p_1^*(\omega_A)_\xi \to p_1^!(\omega_A)_\xi$ se factorise par $p^{\frac{i(i+1)}{2}}$. □

Proposition 9.32. *Soit* $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_g)$ *un poids* Sp_{2g}*-dominant, c'est à dire tel que* $k_g \geq 0$*, alors les opérateurs*

$$p^{-v(\lambda)}T_p^{\mathrm{naif},\nabla(\lambda)}$$

et

$$p^{-v(\lambda)}T_p^{\mathrm{naif},\Delta(\lambda)}$$

sont bien définis et coïncident avec la renormalisation usuelle sur la fibre générique $\mathbb{Q}_p$.

9.4.3. Définition sur C pour un poids GL_g-dominant. On propose maintenant une construction des opérateurs de Hecke sur C pour tous les poids GL_g-dominant.

Définition 9.33 (Objet costandard)**.** *Soit $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_g)$ un poids GL_g-dominant et $n \geq 0$ un entier assez grand pour que $k_g + n \geq 0$. On note λ_n le poids $(k_1 + n \geq k_2 + n \geq \ldots \geq k_g + n)$. Lorsque l'on applique le foncteur S_{λ_n} au morphisme φ on obtient une application*

$$S_{\lambda_n}\varphi : p_2^*\nabla(\lambda_n) \to p_1^*\nabla(\lambda_n).$$

Lorsque l'on applique le foncteur $S_{(1,\cdots,1)} = W_{(1,\cdots,1)} = \det$ au morphisme ${}^t\varphi$, on obtient une application

$$\det {}^t\varphi : p_1^* \det \omega_A \to p_2^* \det \omega_A$$

et en prenant la puissance tensorielle $-n$ de ce morphisme de faisceaux inversibles (qui est contravariante car $-n \leq 0$) on obtient un morphisme

$$(\det {}^t\varphi)^{\otimes -n} : p_2^*(\det \omega_A)^{\otimes -n} \to p_1^*(\det \omega_A)^{\otimes -n}$$

et on définit l'opérateur $T_p^{\mathrm{naif},n,\nabla(\lambda)}$ comme la composée

$$p_2^*\nabla(\lambda) \xrightarrow{S_{\lambda_n}\varphi \otimes (\det^t \varphi)^{\otimes -n}} p_1^*\nabla(\lambda) \xrightarrow{\mathrm{tr}_1 \otimes 1_{p_1^*\nabla(\lambda)}} p_1^!\nabla(\lambda)$$

Définition 9.34 (Objet standard)**.** *Soit $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_g)$ un poids GL_g-dominant et $n \geq 0$ un entier assez grand pour que $k_g + n \geq 0$. On note λ_n le poids $(k_1 + n \geq k_2 + n \geq \ldots \geq k_g + n)$. Lorsque l'on applique le foncteur W_{λ_n} au morphisme φ on obtient une application*

$$W_{\lambda_n}\varphi : p_2^*\Delta(\lambda_n) \to p_1^*\Delta(\lambda_n).$$

Lorsque l'on applique le foncteur $S_{(1,\cdots,1)} = W_{(1,\cdots,1)} = \det$ au morphisme ${}^t\varphi$, on obtient une application

$$\det {}^t\varphi : p_1^* \det \omega_A \to p_2^* \det \omega_A$$

et en prenant la puissance tensorielle $-n$ de ce morphisme de faisceaux inversibles (qui est contravariante car $-n \leq 0$) on obtient un morphisme

$$(\det {}^t\varphi)^{\otimes -n} : p_2^*(\det \omega_A)^{\otimes -n} \to p_1^*(\det \omega_A)^{\otimes -n}$$

et on définit l'opérateur $T_p^{\mathrm{naif},n,\Delta(\lambda)}$ comme la composée

$$p_2^*\Delta(\lambda) \xrightarrow{W_{\lambda_n}\varphi \otimes (\det^t \varphi)^{\otimes -n}} p_1^*\Delta(\lambda) \xrightarrow{\mathrm{tr}_1 \otimes 1_{p_1^*\Delta(\lambda)}} p_1^!\Delta(\lambda)$$

Proposition 9.35. *Soit $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_g)$ un poids GL_g-dominant et $n \geq 0$ un entier assez grand pour que $k_g + n \geq 0$. On note λ_n le poids $(k_1 + n \geq k_2 + n \geq \ldots \geq k_g + n)$. Alors les opérateurs de Hecke $T_p^{\mathrm{naif},n,\nabla(\lambda)}$ et $T_p^{\mathrm{naif},n,\Delta(\lambda)}$ se factorisent*

$$\begin{tikzcd}
p_2^*\nabla(\lambda) \arrow[r, "T_p^{\mathrm{naif},n,\nabla(\lambda)}"] \arrow[dr, dotted] & p_1^!\nabla(\lambda) \\
& p^{v(\lambda)+gn}p_1^!\nabla(\lambda) \arrow[u]
\end{tikzcd}$$

et

$$\begin{tikzcd} p_2^*\Delta(\lambda) \arrow[r, "T_p^{\mathrm{naif},n,\Delta(\lambda)}"] \arrow[dr, dotted] & p_1^!\Delta(\lambda) \\ & p^{v(\lambda)+gn}p_1^!\Delta(\lambda) \arrow[u] \end{tikzcd}.$$

Démonstration. Par la proposition 9.25, il suffit de le montrer en tout point géométrique ξ de $C^{\mathrm{ord}}_{\mathbb{F}_p}$, le lieu ordinaire de la fibre spéciale. On est donc ramené à étudier le comportement de T_p au voisinage de ξ. Le tiré en arrière A_ξ est une variété abélienne ordinaire au dessus de k un corps de caractéristique p algébriquement clos et $H_\xi \subset A_\xi[p]$ a pour rang étale $0 \leq i \leq g$. C'est à dire que l'on a une décomposition de groupes finis et plats

$$H_\xi \simeq \mu_p^{g-i} \times \underline{\mathbb{Z}/p\mathbb{Z}}_k^i.$$

Ainsi, il existe des trivialisation de $(\omega_A)_\xi$ telles que les différentielles $\varphi_\xi : p_2^*(\omega_A)_\xi \to p_1^*(\omega_A)_\xi$ et ${}^t\varphi_\xi : p_1^*(\omega_A)_\xi \to p_2^*(\omega_A)_\xi$ fasse commuter le diagramme suivant

$$\begin{tikzcd} p_2^*(\omega_A)_\xi \arrow[r, "\varphi_\xi"] \arrow[d, "\simeq"] & p_1^*(\omega_A)_\xi \arrow[r, "{}^t\varphi_\xi"] \arrow[d, "\simeq"] & p_2^*(\omega_A)_\xi \arrow[d, "\simeq"] \\ \mathcal{O}_{C,\xi}^{\oplus g} \arrow[r, "h"] & \mathcal{O}_{C,\xi}^{\oplus g} \arrow[r, "h'"] & \mathcal{O}_{C,\xi}^{\oplus g} \end{tikzcd}$$

où h est l'application de multiplication par

$$\begin{array}{c} g-i \left\{ \right. \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} p & & & & & \\ & \ddots & & & & \\ & & p & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \begin{array}{c} \left. \right\} i \\ \\ \\ \\ \\ \end{array}$$

et h' est l'application de multiplication par

$$\begin{array}{c} i \left\{ \right. \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} p & & & & & \\ & \ddots & & & & \\ & & p & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \begin{array}{c} \left. \right\} g-i \\ \\ \\ \\ \\ \end{array}$$

Par la proposition 2.57, cela entraine que $S_{\lambda_n}\varphi_\xi$ et $W_{\lambda_n}\varphi_\xi$ se factorisent par $p^{\sum_{j=i+1}^g (k_j+n)}$ et $(\det {}^t\varphi_\xi)^{\otimes -n}$ se factorise par p^{in}. Enfin, il résulte de la théorie de Serre-Tate que l'application

trace $p_1^*(\omega_A)_\xi \to p_1^!(\omega_A)_\xi$ se factorise par $p^{\frac{i(i+1)}{2}}$. En sommant, on déduit que les opérateurs de Hecke $T_p^{\mathrm{naif},n,\nabla(\lambda)}$ et $T_p^{\mathrm{naif},n,\Delta(\lambda)}$ se factorisent par

$$\begin{aligned} p^{\sum_{j=i+1}^g (k_j+n)} \times p^{in} \times p^{\frac{i(i+1)}{2}} &= p^{\sum_{j=i+1}^g (k_j+n)+in+\frac{i(i+1)}{2}} \\ &= p^{(\sum_{j=i+1}^g k_j)+(g-i)n+in+\frac{i(i+1)}{2}} \\ &= p^{\sum_{j=i+1}^g k_j+\frac{i(i+1)}{2}+gn} \end{aligned}$$

□

Définition 9.36. *Soit $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_g)$ un poids GL_g-dominant et $n \geq 0$ un entier assez grand pour que $k_g + n \geq 0$. Grâce à la proposition 9.35, on peut définir*

$$\begin{cases} T_p^{\nabla(\lambda)} := p^{-(v(\lambda)+gn)} T_p^{\mathrm{naif},n,\nabla(\lambda)} \\ T_p^{\Delta(\lambda)} := p^{-(v(\lambda)+gn)} T_p^{\mathrm{naif},n,\Delta(\lambda)} \end{cases}$$

et il n'est pas difficile de voir que cette définition ne dépend pas de n.

Proposition 9.37. *Soit $\lambda = (k_1 \geq k_2 \geq \ldots \geq k_g)$ un poids GL_g-dominant. Sur la fibre spéciale $C_{\mathbb{Q}_p}$, les opérateurs $T_p^{\nabla(\lambda)}$ et $T_p^{\Delta(\lambda)}$ coïncident avec l'opérateur $(T_p)_{\mathbb{Q}_p}^{\nabla(\lambda)}$ de la proposition 9.28.*

Démonstration. Soit $n \geq 0$ un entier tel que $k_g + n \geq 0$. Sur la fibre générique $C_{\mathbb{Q}_p}$, on a la relation

$$\varphi \circ {}^t\varphi = p \cdot \mathrm{id},$$

ce qui entraine

$$\det \varphi \circ \det {}^t\varphi = p^g$$

puis

$$(\det {}^t\varphi)^{\otimes -n} = p^{gn} (\det \varphi^{-1})^{\otimes -n}.$$

On a donc

$$\begin{aligned} T_p^{\nabla(\lambda)} \otimes \mathbb{Q}_p &= (\mathrm{tr}_1 \otimes 1_{p_1^*\nabla(\lambda)}) \circ p^{-(v(\lambda)+gn)} S_{\lambda_n}\varphi \otimes (\det {}^t\varphi)^{\otimes -n} \\ &= (\mathrm{tr}_1 \otimes 1_{p_1^*\nabla(\lambda)}) \circ p^{-v(\lambda)} S_{\lambda_n}\varphi \otimes p^{-gn} p^{gn} (\det \varphi^{-1})^{\otimes -n} \\ &= (\mathrm{tr}_1 \otimes 1_{p_1^*\nabla(\lambda)}) \circ p^{-v(\lambda)} S_{\lambda_n}\varphi \otimes (\det \varphi^{-1})^{\otimes -n} \\ &= (T_p)_{\mathbb{Q}_p}^{\nabla(\lambda)} \end{aligned}$$

et de même

$$T_p^{\Delta(\lambda)} \otimes \mathbb{Q}_p = (T_p)_{\mathbb{Q}_p}^{\Delta(\lambda)}.$$

□

9.4.4. Correspondance transposée. Sur C, on a l'isogénie universelle $\pi : A \to A/H$ et l'isogénie duale ${}^t\pi : {}^t(A/H) \to {}^tA$ qui fournit des différentielles

$$p_1^*\omega_A \xrightarrow{{}^t\varphi} p_2^*\omega_A \xrightarrow{\varphi} p_1^*\omega_A$$

En appliquant la même construction que précédemment à ${}^t\varphi$ au lieu de φ, on obtient des correspondances transposées

$$\begin{cases} T_p^{\Delta(\lambda),t} : p_1^*\Delta(\lambda) \to p_2^!\Delta(\lambda) \\ T_p^{\nabla(\lambda),t} : p_1^*\nabla(\lambda) \to p_2^!\nabla(\lambda) \end{cases}$$

que l'on normalise similairement.

9.5. Autodualité de T_p

Le but de cette section est d'étudier l'opérateur de Hecke T_p lorsque l'on lui applique le foncteur dualisant de Serre qui est défini dans [**Har66b**].

Définition 9.38. *On rappelle que l'on a un diagramme commutatif*

$$\begin{tikzcd} & C \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "g"] & \\ X \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\ & \operatorname{Spec} \mathbb{Z}_p & \end{tikzcd} \tag{9.1}$$

et on note

(1) $D_X = \underline{\mathrm{RHom}}_{\mathcal{O}_X}(\_, f^! \mathbb{Z}_p)$ *le foncteur dualisant sur* X

(2) $D_C = \underline{\mathrm{RHom}}_{\mathcal{O}_C}(\_, g^! \mathbb{Z}_p)$ *le foncteur dualisant sur* C

(3) $D_{\mathbb{Z}_p} = \mathrm{RHom}_{\mathbb{Z}_p}(\_, \mathbb{Z}_p)$ *le foncteur dualisant sur* $\operatorname{Spec} \mathbb{Z}_p$.

D'après la proposition 9.21, nous avons

$$\begin{cases} f^! \mathbb{Z}_p = \omega_X[d] \\ g^! \mathbb{Z}_p = \omega_C[d] \end{cases}$$

où $d = g(g+1)/2$, $\omega_X = \det \Omega^1_{X/\mathbb{Z}_p}$ est un faisceau inversible et ω_C un $\mathcal{O}_C$-module cohérent. Afin de relier ω_X aux fibrés automorphes, nous rappelons les propriétés du morphisme de Kodaira-Spencer.

Définition 9.39. *Soit* $h : A \to S$ *un schéma abélien avec une polarisation principale* $\lambda : A \to A^t$ *où* S *est un schéma lisse sur* $\operatorname{Spec} \mathbb{Z}_p$. *Nous avons une filtration de Hodge*

$$0 \longrightarrow e^*\Omega^1_{A/S} \xrightarrow{\iota} \mathcal{H}^1_{\mathrm{dR}}(A/S) \xrightarrow{q} R^1 h_* \mathcal{O}_A \simeq (e^*\Omega^1_{A^t/S})^\vee \longrightarrow 0$$

qui provient de la dégénérescence de la suite spectrale de Hodge-de Rham sur S. *On note simplement* $\omega_A = e^*\Omega^1_{A/S} = e^*\Omega^1_{A^t/S}$ *le fibré de Hodge où le deuxième isomorphisme est induit par* λ. *Le fibré* $\mathcal{H}^1_{\mathrm{dR}}(A/S)$ *est muni de la connexion de Gauss-Manin*

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(A/S) \to \mathcal{H}^1_{\mathrm{dR}}(A/S) \otimes \Omega^1_S$$

et on appelle morphisme de Kodaira-Spencer $\mathrm{KS}_{A/S}$ *la composition*

$$\begin{tikzcd} \omega_A \arrow[r, "\iota"] \arrow[drr, "\mathrm{KS}_{A/S}"'] & \mathcal{H}^1_{\mathrm{dR}}(A/S) \arrow[r, "\nabla"] & \mathcal{H}^1_{\mathrm{dR}}(A/S) \otimes \Omega^1_S \arrow[d, "q \otimes 1"] \\ & & \omega_A^\vee \otimes \Omega^1_S \end{tikzcd} .$$

Proposition 9.40. *Soit $h : A \to X$ le schéma abélien universel muni de sa polarisation principale $\lambda : A \to A^t$. Le morphisme de Kodaira-Spencer induit un isomorphisme*

$$\begin{tikzcd} \omega_A \otimes \omega_A \arrow[r, "1_{\omega_A} \otimes \mathrm{KS}_{A/X}"] \arrow[d] & \Omega^1_X \\ \mathrm{Sym}^2\, \omega_A \arrow[ur, "\simeq"] & \end{tikzcd}$$

On considère maintenant l'opérateur

$$T_p^{\nabla(\lambda)} : p_2^* \nabla(\lambda) \to p_1^! \nabla(\lambda)$$

vu comme un morphisme de $\mathcal{O}_C$-modules. Lorsque l'on applique le foncteur dualisant D_C et que l'on fait un décalage par $[-d]$, par [**Con00**], on obtient un morphisme

$$D_C(T_p^{\nabla(\lambda)}) : p_1^*(\Delta(-w_{0,L}\lambda) \otimes \omega_X) \to p_2^!(\Delta(-w_{0,L}\lambda) \otimes \omega_X)$$

où $w_{0,L}$ est l'élément de longueur maximale du groupe de Weyl de $L = \mathrm{GL}_g$. Or la proposition 9.40 entraine que $w_X \simeq \nabla(g+1, \cdots, g+1)$ et on obtient donc un morphisme

$$D_C(T_p^{\nabla(\lambda)}) : p_1^* \Delta(-w_{0,L}\lambda + 2\rho^L) \to p_2^! \Delta(-w_{0,L}\lambda + 2\rho^L)$$

où $2\rho^L = \underline{g+1} = (g+1, \cdots, g+1)$ est la somme des racines non-compactes. Si on utilise la correspondance transposée de C, les résultats de la section 9.4 fournissent des opérateurs

$$\begin{cases} T_p^{\nabla(\lambda),t} : p_1^* \nabla(\lambda) \to p_2^! \nabla(\lambda) \\ T_p^{\Delta(\lambda),t} : p_1^* \nabla(\lambda) \to p_2^! \Delta(\lambda) \end{cases}$$

Avant d'énoncer le résultat principal de cette section, nous démontrons le lemme suivant :

Lemme 9.41. *Considérons le changement de base à $\mathbb{Q}_p$ du diagramme 9.1*

$$\begin{tikzcd} & C_{\mathbb{Q}_p} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "g"] & \\ X_{\mathbb{Q}_p} \arrow[dr, "f"'] & & X_{\mathbb{Q}_p} \arrow[dl, "f"] \\ & \mathrm{Spec}\, \mathbb{Q}_p & \end{tikzcd} .$$

Alors le diagramme suivant

$$\begin{tikzcd} p_1^* \omega_{X_{\mathbb{Q}_p}} \arrow[r, "\mathrm{tr}_1"] & p_1^! \omega_{X_{\mathbb{Q}_p}} \arrow[d, equal] \\ p_2^* \omega_{X_{\mathbb{Q}_p}} \arrow[u, "p^{-\frac{g(g+1)}{2}} S_{2\rho^L}\, {}^t\varphi"] \arrow[r, "\mathrm{tr}_2"] & p_2^! \omega_{X_{\mathbb{Q}_p}} \end{tikzcd}$$

est commutatif.

Démonstration. On oublie d'écrire $\mathbb{Q}_p$ en indice des objets pour alléger les notations et comme les flèches $g : C \to \mathrm{Spec}\, \mathbb{Q}_p$ et $f : X \to \mathrm{Spec}\, \mathbb{Q}_p$ sont lisses, les morphismes de trace $tr_1 : p_1^* \omega_X \to \omega_C$ et $tr_2 : p_2^* \omega_X \to \omega_C$ correspondent aux déterminants des applications de tiré

en arrière sur les différentielles $dp_1 : p_1^*\Omega_X^1 \to \Omega_C^1$ et $dp_2 : p_2^*\Omega_X^1 \to \Omega_C^1$. L'isogénie universelle $\pi : A \to A/H$ au dessus de C induit un morphisme

$$p_2^*(H^1_{\mathrm{dR}}(A/X), \nabla) \to p_1^*(H^1_{\mathrm{dR}}(A/X), \nabla)$$

de fibrés à connexions qui induit un diagramme commutatif

$$\begin{tikzcd}
p_2^*\omega_A \arrow[r, "p_2^*\,\mathrm{KS}"] \arrow[d, "\varphi"] & p_2^*\Omega_X^1 \otimes p_2^*\omega_A^\vee \arrow[r, "dp_1\otimes 1"] & \Omega_C^1 \otimes p_2^*\omega_A^\vee \arrow[d, "1\otimes {}^t\varphi^\vee"] \\
p_1^*\omega_A \arrow[r, "p_1^*\,\mathrm{KS}"] & p_1^*\Omega_X^1 \otimes p_1^*\omega_A^\vee \arrow[r, "dp_2\otimes 1"] & \Omega_C^1 \otimes p_1^*\omega_A^\vee
\end{tikzcd}$$

C'est maintenant un problème d'algèbre linéaire d'établir le résultat souhaité. On se ramène à la situation où l'on dispose d'un diagramme commutatif

$$\begin{tikzcd}
N \arrow[r, "\rho"] \arrow[d, "\varphi"] & M \otimes N^\vee \arrow[d, "1\otimes {}^t\varphi^\vee"] \\
N \arrow[r, "\rho"] & M \otimes N^\vee
\end{tikzcd}$$

où N, M sont des k-espaces vectoriels de dimensions respectives g et $d := g(g+1)/2$ et ρ est supposé symétrique dans le sens suivant : on voit ρ comme une application

$$\begin{array}{rcl} \rho : N & \to & \mathrm{Hom}(N, M) \\ n & \mapsto & \rho_n \end{array}$$

et on a $\rho_n(n') = \rho_{n'}(n)$ pour tout $n, n' \in N$. On obtient alors un diagramme

$$\begin{tikzcd}
\mathrm{Sym}^2 N \arrow[r, "n\otimes n' \mapsto \rho_n(n')"] \arrow[d, "n\otimes n' \mapsto \varphi(n)\otimes n'"'] & M \arrow[d, "\rho_n(n') \mapsto \rho_n({}^t\varphi(n'))"] \\
\mathrm{Sym}^2 N \arrow[r, "\varphi(n)\otimes n' \mapsto \rho_{\varphi(n)}(n')"'] & M
\end{tikzcd} \tag{9.2}$$

dont la commutativité dit que $\rho_{\varphi(n)}(n') = \rho_n({}^t\varphi(n'))$ pour tout $n, n' \in N$. On considère maintenant un diagramme

$$\begin{tikzcd}
\Lambda^d \mathrm{Sym}^2 N = \det N^{\otimes(g+1)} \arrow[r, "h"] \arrow[d, "S_{2\rho^L}\varphi"] & \Lambda^d M \arrow[d, equal] \\
\Lambda^d \mathrm{Sym}^2 N = \det N^{\otimes(g+1)} \arrow[r, dashed, "p^{-d}h"] & \Lambda^d M
\end{tikzcd} \tag{9.3}$$

où l'on a appliqué le foncteur de Schur Λ^d aux objets du diagramme 9.2 mais on impose les flèches sauf celle horizontale en bas que l'on cherche à déterminer pour le rendre commutatif. La première flèche horizontale h envoie $(x_1 \otimes y_1) \wedge \cdots \wedge (x_d \otimes y_d)$ sur $\rho_{x_1}(y_1) \wedge \cdots \wedge \rho_{x_d}(y_d)$. Pour que le diagramme 9.3 soit commutatif il faut que la flèche en pointillé envoie

$$(\varphi(x_1) \otimes \varphi(y_1)) \wedge \cdots \wedge (\varphi(x_d) \otimes \varphi(y_d))$$

sur $\rho_{x_1}(y_1) \wedge \cdots \wedge \rho_{x_d}(y_d)$ or on a

$$\begin{aligned} h((\varphi(x_1) \otimes \varphi(y_1)) \wedge \cdots \wedge (\varphi(x_d) \otimes \varphi(y_d))) &= \rho_{\varphi(x_1)}(\varphi(y_1)) \wedge \cdots \wedge \rho_{\varphi(x_d)}(\varphi(y_d)) \\ &= \rho_{x_1}(t\varphi(\varphi(y_1))) \wedge \cdots \wedge \rho_{x_d}(t\varphi(\varphi(y_d))) \\ &= p^d(\rho_{x_1}(y_1) \wedge \cdots \wedge \rho_{x_d}(y_d)). \end{aligned}$$

Cela montre que c'est $p^{-d}h$ qui fait commuter le diagramme 9.3. □

Théorème 9.42. *Soit λ un poids GL_g-dominant. Alors on a des identifications*

$$D_C(T_p^{\nabla(\lambda)}) = T_p^{\Delta(-w_{0,L}\lambda+2\rho^L),t}$$

et

$$D_C(T_p^{\Delta(\lambda)}) = T_p^{\nabla(-w_{0,L}\lambda+2\rho^L),t}$$

Démonstration. Grâce à la proposition 9.16, il suffit de démontrer l'égalité des deux morphismes après localisation en tout point géométrique ξ de la fibre générique de C. Pour ne pas alourdir les notations, on oublie d'écrire $\mathbb{Q}_p$ en indice des objets. Le fait d'écrire S_λ ou W_λ n'a pas d'importance puisque au dessus de $\mathbb{Q}_p$, les deux foncteurs sont canoniquement isomorphes, néanmoins on distingue les deux notations pour rendre la preuve plus claire. Fixons $\lambda = (k_1, \cdots, k_g)$ un poids GL_g-dominant et n un entier tel que $k_g + n$ soit positif. Le dual de Serre $D(T_p^{\mathrm{naif},n,\nabla(\lambda)})$ de $T_p^{\mathrm{naif},n,\nabla(\lambda)}$ est un morphisme

$$\begin{tikzcd} p_1^*(\nabla(\lambda)^\vee \otimes \omega_X) \arrow[r, "\mathrm{tr}_1 \otimes 1"] \arrow[ddr, "D(T_p^{\mathrm{naif},n,\nabla(\lambda)})"'] & p_1^!(\nabla(\lambda)^\vee \otimes \omega_X) \arrow[d, equal] \\ & p_1^*\Delta(-w_{0,L}\lambda) \otimes \omega_C \arrow[d, "(S_{\lambda_n}\varphi \otimes (\det {}^t\varphi)^{\otimes -n})^\vee \otimes 1_{\omega_C}"] \\ & p_2^*\Delta(-w_{0,L}\lambda) \otimes \omega_C \end{tikzcd}$$

Comme $\varphi \circ {}^t\varphi$ est la multiplication par p, cela permet d'écrire

$$(S_{\lambda_n}\varphi \otimes (\det {}^t\varphi)^{\otimes -n})^\vee = p^{gn}(S_{\lambda_n}\varphi \otimes (\det \varphi^{-1})^{\otimes -n})^\vee.$$

En outre, on sait qu'il existe un entier $m \geq 0$ assez grand tel que

$$\begin{aligned} (S_{\lambda_n}\varphi \otimes (\det \varphi^{-1})^{\otimes -n})^\vee &= W_{(-w_{0,L}\lambda)_m}\varphi^{-1} \otimes (\det \varphi)^{\otimes -m} \\ &= p^{\sum_{i=1}^g (k_i - m)} W_{(-w_{0,L}\lambda)_m}{}^t\varphi \otimes (\det \varphi)^{\otimes -m}. \end{aligned}$$

Par ailleurs, le lemme 9.41 dit que le diagramme suivant est commutatif

$$\begin{tikzcd} p_1^*\omega_X \arrow[r, "\mathrm{tr}_1"] & p_1^!\omega_X \arrow[d, equal] \\ p_2^*\omega_X \arrow[u, "p^{-\frac{g(g+1)}{2}} S_{2\rho^L}{}^t\varphi"] \arrow[r, "\mathrm{tr}_2"] & p_2^!\omega_X \end{tikzcd}$$

ce qui permet d'écrire

$$\begin{aligned} D(T_p^{\mathrm{naif},n,\nabla(\lambda)}) &= (S_{\lambda_n}\varphi \otimes (\det {}^t\varphi)^{\otimes -n})^\vee \otimes 1 \circ \mathrm{tr}_1 \otimes 1 \\ &= \mathrm{tr}_2 \otimes 1 \circ p^{gn} p^{\sum_{i=1}^g (k_i - m)} p^{-\frac{g(g+1)}{2}} W_{(-w_{0,L}\lambda + 2\rho_L)_m} {}^t\varphi \otimes (\det \varphi)^{\otimes -m} \\ &= p^{gn - gm + \sum_{i=1}^g k_i - \frac{g(g+1)}{2}} T_p^{\mathrm{naif},m,\Delta(-w_{0,L}\lambda + 2\rho^L),t}. \end{aligned}$$

Comme

$$T_p^{\nabla(\lambda)} = p^{-v(\lambda) - gn} T_p^{\mathrm{naif},n,\nabla(\lambda)}$$

et

$$T_p^{\Delta(-w_{0,L}\lambda + 2\rho^L),t} = p^{-v(-w_{0,L}\lambda + 2\rho^L) - gm} T_p^{\mathrm{naif},m,\Delta(-w_{0,L}\lambda + 2\rho^L),t},$$

la conclusion du théorème découle de l'égalité

$$v(\lambda) + \frac{g(g+1)}{2} - \sum_{i=1}^g k_i = v(-w_{0,L}\lambda + 2\rho^L).$$

□

Remarque 9.43. Il est possible de définir une correspondance de Hecke cohomologique faisant intervenir les fibrés automorphes tilting

$$T_p^{T(\lambda)} : p_2^* T(\lambda) \to p_1^! T(\lambda)$$

pour tous les poids GL_g-dominants λ. Au lieu d'utiliser les foncteurs S_λ ou W_λ, il faut utiliser le foncteur tilting T_λ introduit au chapitre 2. Rappelons que sur $\mathbb{Q}_p$, on a automatiquement une décomposition de fibrés automorphes

$$T(\lambda) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \bigoplus_\mu \nabla(\mu)_{\mathbb{Q}_p}^{n_\mu^\lambda}$$

et que les coefficients n_μ^λ sont difficiles à exprimer directement car ils font intervenir des polynômes de p-Kazhdan–Lusztig (voir remark 2.18). La renormalisation ne se fait pas avec un facteur $p^{-v(\lambda)}$, mais un facteur $p^{-\underline{v}(\lambda)}$ où

$$\underline{v}(\lambda) = \sum_\mu n_\mu^\lambda v(\mu)$$

et de façon analogue aux cas des fibrés automorphes standards et costandards, on obtient le résultat d'auto-dualité suivant

$$D_C(T_p^{T(\lambda)}) = T_p^{T(-w_{0,L}\lambda + 2\rho^L),t}.$$

Bibliography

[AB88] Kaan Akin and David A. Buchsbaum. Characteristic-free representation theory of the general linear group. II. Homological considerations. *Adv. in Math.*, 72(2):171–210, 1988.

[ABW82] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman. Schur functors and Schur complexes. *Adv. in Math.*, 44(3):207–278, 1982.

[Ale22a] Thibault Alexandre. Positivity, plethysm and hyperbolicity of siegel varieties in positive characteristic, 2022.

[Ale22b] Thibault Alexandre. Vanishing results for the coherent cohomology of automorphic vector bundles over the siegel variety in positive characteristic, 2022.

[AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai. *Smooth compactifications of locally symmetric varieties*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. With the collaboration of Peter Scholze.

[And21] Fabrizio Andreatta. On two mod p period maps: Ekedahl–oort and fine deligne–lusztig stratifications, 2021.

[AT20] Kenneth Ascher and Amos Turchet. Hyperbolicity of varieties of log general type. In *Arithmetic geometry of logarithmic pairs and hyperbolicity of moduli spaces—hyperbolicity in Montréal*, CRM Short Courses, pages 197–247. Springer, Cham, [2020] ©2020.

[Bar71] Charles M. Barton. Tensor products of ample vector bundles in characteristic p. *Amer. J. Math.*, 93:429–438, 1971.

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over $\mathbf{Q}$: wild 3-adic exercises. *J. Amer. Math. Soc.*, 14(4):843–939, 2001.

[BGKS] Yohann Brunebarbe, Wushi Goldring, Jean-Stefan Koskivirta, and Benoit Stroh. Ample automophic bundles on zip-schemes. *In preparation*.

[Bof91] Giandomenico Boffi. On some plethysms. *Adv. Math.*, 89(2):107–126, 1991.

[BP20] George Boxer and Vincent Pilloni. Higher hida and coleman theories on the modular curve, 2020.

[Bru18] Yohan Brunebarbe. Symmetric differentials and variations of Hodge structures. *J. Reine Angew. Math.*, 743:133–161, 2018.

[Car57] Pierre Cartier. Une nouvelle opération sur les formes différentielles. *C. R. Acad. Sci. Paris*, 244:426–428, 1957.

[Con] Brian Conrad. Applications of base change for coherent cohomology. http://virtualmath1.stanford.edu/~conrad/248BPage/handouts/cohom.pdf.

[Con00] Brian Conrad. *Grothendieck duality and base change*, volume 1750 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.

[Del71] Pierre Deligne. Formes modulaires et représentations l-adiques. In *Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363*, volume 175 of *Lecture Notes in Math.*, pages Exp. No. 355, 139–172. Springer, Berlin, 1971.

[Del74] Pierre Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[DI87] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[Don85] Stephen Donkin. *Rational representations of algebraic groups*, volume 1140 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985. Tensor products and filtration.

[EV92] Hélène Esnault and Eckart Viehweg. *Lectures on vanishing theorems*, volume 20 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1992.

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983.

[FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.

[FP19] Najmuddin Fakhruddin and Vincent Pilloni. Hecke operators and the coherent cohomology of shimura varieties, 2019.

[Fre86] Gerhard Frey. Links between stable elliptic curves and certain Diophantine equations. *Ann. Univ. Sarav. Ser. Math.*, 1(1):iv+40, 1986.

[FS97] Eric M. Friedlander and Andrei Suslin. Cohomology of finite group schemes over a field. *Invent. Math.*, 127(2):209–270, 1997.

[Gie71] David Gieseker. p-ample bundles and their Chern classes. *Nagoya Math. J.*, 43:91–116, 1971.

[GK19a] Wushi Goldring and Jean-Stefan Koskivirta. Strata Hasse invariants, Hecke algebras and Galois representations. *Invent. Math.*, 217(3):887–984, 2019.

[GK19b] Wushi Goldring and Jean-Stefan Koskivirta. Stratifications of flag spaces and functoriality. *Int. Math. Res. Not. IMRN*, (12):3646–3682, 2019.

[Har66a] Robin Hartshorne. Ample vector bundles. *Inst. Hautes Études Sci. Publ. Math.*, (29):63–94, 1966.

[Har66b] Robin Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin-New York, 1966.

[Har67] Robin Hartshorne. *Local cohomology*. Lecture Notes in Mathematics, No. 41. Springer-Verlag, Berlin-New York, 1967. A seminar given by A. Grothendieck, Harvard University, Fall, 1961.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Har85] Michael Harris. Arithmetic vector bundles and automorphic forms on Shimura varieties. I. *Invent. Math.*, 82(1):151–189, 1985.

[He13] Xuhua He. Normality and Cohen-Macaulayness of local models of Shimura varieties. *Duke Math. J.*, 162(13):2509–2523, 2013.

[Jab07] Kelly Jabbusch. *Notions of positivity for vector bundles*. ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–University of Washington.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[Kat73] Nicholas M. Katz. p-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 69–190. Lecture Notes in Mathematics, Vol. 350, 1973.

[Kat81] Nicholas Katz. Serre-tate local moduli. In *Surfaces algébriques*, pages 138–202. Springer, 1981.

[KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973.

[Kle69] Steven L. Kleiman. Ample vector bundles on algebraic surfaces. *Proc. Amer. Math. Soc.*, 21:673–676, 1969.

[Kra15] Henning Krause. Highest weight categories and strict polynomial functors, 2015.

[Kun69] Ernst Kunz. Characterizations of regular local rings of characteristic p. *Amer. J. Math.*, 91:772–784, 1969.

[Lan86] Serge Lang. Hyperbolic and Diophantine analysis. *Bull. Amer. Math. Soc. (N.S.)*, 14(2):159–205, 1986.

[Lan12] Kai-Wen Lan. Toroidal compactifications of PEL-type Kuga families. *Algebra Number Theory*, 6(5):885–966, 2012.

[Laz04a] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Laz04b] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[LMF22] The LMFDB Collaboration. The L-functions and modular forms database. http://www.lmfdb.org, 2022. [Online; June 2022].

[LS12] Kai-Wen Lan and Junecue Suh. Vanishing theorems for torsion automorphic sheaves on compact PEL-type Shimura varieties. *Duke Math. J.*, 161(6):1113–1170, 2012.

[LS13] Kai-Wen Lan and Junecue Suh. Vanishing theorems for torsion automorphic sheaves on general PEL-type Shimura varieties. *Adv. Math.*, 242:228–286, 2013.

[Mat90] Olivier Mathieu. Filtrations of G-modules. *Ann. Sci. École Norm. Sup. (4)*, 23(4):625–644, 1990.

[MB81] Laurent Moret-Bailly. Familles de courbes et de variétés abéliennes sur $\mathbb{P}^1$. II. Exemples. Number 86, pages 125–140. 1981. Seminar on Pencils of Curves of Genus at Least Two.

[MB85] Laurent Moret-Bailly. Pinceaux de variétés abéliennes. *Astérisque*, (129):266, 1985.

[Mou97] Christophe Mourougane. Images directes de fibrés en droites adjoints. *Publ. Res. Inst. Math. Sci.*, 33(6):893–916, 1997.

[MW04] Ben Moonen and Torsten Wedhorn. Discrete invariants of varieties in positive characteristic. *Int. Math. Res. Not.*, (72):3855–3903, 2004.

[Ols16] Martin Olsson. *Algebraic spaces and stacks*, volume 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016.

[Oor01] Frans Oort. A stratification of a moduli space of abelian varieties. In *Moduli of abelian varieties (Texel Island, 1999)*, volume 195 of *Progr. Math.*, pages 345–416. Birkhäuser, Basel, 2001.

[PT02] Patrick Polo and Jacques Tilouine. Bernstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups over $\mathbb{Z}_{(p)}$ for representations with p-small weights. Number 280, pages 97–135. 2002. Cohomology of Siegel varieties.

[PWZ11] Richard Pink, Torsten Wedhorn, and Paul Ziegler. Algebraic zip data. *Doc. Math.*, 16:253–300, 2011.

[PWZ15] Richard Pink, Torsten Wedhorn, and Paul Ziegler. F-zips with additional structure. *Pacific J. Math.*, 274(1):183–236, 2015.

[Rib90] K. A. Ribet. On modular representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ arising from modular forms. *Invent. Math.*, 100(2):431–476, 1990.

[Ric16] Simon Riche. *Geometric Representation Theory in positive characteristic*. Habilitation à diriger des recherches, Université Blaise Pascal (Clermont Ferrand 2), December 2016.

[RW22] Simon Riche and Geordie Williamson. Smith–treumann theory and the linkage principle. *Publications mathématiques de l'IHÉS*, pages 1–68, 2022.

[Ryd15] David Rydh. Noetherian approximation of algebraic spaces and stacks. *J. Algebra*, 422:105–147, 2015.

[SB17] Cambridge Philosophical Society. and Niels Henrik David Bohr. *Proceedings - Cambridge Philosophical Society Mathematical and physical sciences*, volume v.19-20 (1917-1921). Cambridge [etc.], Cambridge Philosophical Society [etc.], 1917. https://www.biodiversitylibrary.org/bibliography/39008.

[Ser70] Jean-Pierre Serre. *Cours d'arithmétique*, volume 2 of *Collection SUP: "Le Mathématicien"*. Presses Universitaires de France, Paris, 1970.

[Spe35] Wilhelm Specht. Die irreduziblen darstellungen der symmetrischen gruppe. *Mathematische Zeitschrift*, 39(1):696–711, 1935.

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Sta21] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2021.

[Str13] Benoît Stroh. Classicité en théorie de Hida. *Amer. J. Math.*, 135(4):861–889, 2013.

[Tou13] Antoine Touzé. Ringel duality and derivatives of non-additive functors. *J. Pure Appl. Algebra*, 217(9):1642–1673, 2013.

[Wed99] Torsten Wedhorn. Ordinariness in good reductions of Shimura varieties of PEL-type. *Ann. Sci. École Norm. Sup. (4)*, 32(5):575–618, 1999.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

[Wil09] Mark Wildon. Multiplicity-free representations of symmetric groups. *J. Pure Appl. Algebra*, 213(7):1464–1477, 2009.

[WZ18] Torsten Wedhorn and Paul Ziegler. Tautological rings of shimura varieties and cycle classes of ekedahl-oort strata, 2018.

[Zha18] Chao Zhang. Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type. *Canad. J. Math.*, 70(2):451–480, 2018.

[Zuo00] Kang Zuo. On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. volume 4, pages 279–301. 2000. Kodaira's issue.