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# Concept learning in mathematics : from straight lines to geodesics

Charlotte Barot

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Charlotte Barot. Concept learning in mathematics : from straight lines to geodesics. Cognitive Sciences. Sorbonne Université, 2022. English. NNT : 2022SORUS405 . tel-03968487

**HAL Id: tel-03968487**

**<https://theses.hal.science/tel-03968487v1>**

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SORBONNE UNIVERSITÉ

# De l'apprentissage des nouveaux concepts en mathématiques, le cas des géodésiques

## Concept learning in mathematics: from straight lines to geodesics

Charlotte Barot

Thèse de doctorat de Sorbonne Université

Spécialité: Sciences cognitives

École Doctorale Cerveau Cognition Comportement (Ed3C) n°158

Centre de Neurosciences Intégratives et Cognition (INCC) UMR 8002

Dirigée par Véronique Izard

Soutenue publiquement le 30/09/2022 à Paris

Devant un jury composé de :

<b>Pr. Emmanuel Sander</b>	Université de Genève	<b>Président, rapporteur</b>
<b>Dr. Arnaud Viarouge</b>	Université Paris Cité, CNRS	<b>Rapporteur, examinateur</b>
<b>Pr. Andrew Shtulman</b>	Occidental College	<b>Examinateur</b>
<b>Pr. Manuela Piazza</b>	University of Trento	<b>Examinatrice</b>
<b>Dr. Valeria Giardino</b>	Université Paris Sciences et Lettres, EHESS, ENS, CNRS	<b>Examinatrice</b>
<b>Dr. Véronique Izard</b>	Université Paris Cité, CNRS	<b>Directrice de thèse</b>

# **Concept learning in mathematics: from straight lines to geodesics.**

Charlotte Barot

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«Judicious reviewers must presuppose that nothing genuinely new can be written, for only on that assumption are they in a position to judge, and in no danger of being judged by, the book they are reviewing.»

---

Vladimir Nabokov, *Pale Fire*. (1962)



## Remerciements

Ce travail de thèse s'est trouvé au croisement de questionnements et de recherches qui prennent leurs racines dans différents domaines de la psychologie -évidemment-, mais aussi en épistémologie et en mathématiques. Un des enjeux de cette thèse était donc de réussir à tirer des enseignements de chaque discipline, sans trop les trahir. Il n'aurait pas été possible de mener ce projet sans Véronique Izard.

Merci, Véronique, de m'avoir fait confiance depuis mon entrée dans la psychologie cognitive, d'abord en stage, et d'avoir accepté ensuite de m'encadrer dans une aventure un peu plus longue, et légèrement plus sinueuse. Merci de m'avoir accompagnée sans faillir depuis le début, et toujours encouragée. Mais surtout, d'avoir été un exemple de grande rigueur, de ténacité, d'honnêteté intellectuelle, et d'avoir montré un sens farouche du détail et de la précision, doublés toujours d'une créativité à toute épreuve. Merci aussi d'avoir été une contradictrice bienveillante, menant la réflexion scientifique dans ses retranchements, à tous les étages de la recherche.

Merci à Arnaud Viarouge et Emmanuel Sander d'avoir accepté d'être rapporteurs de ce jury, et pour le travail inspirant qu'ils mènent dans le domaine de la cognition mathématique. I Thank Andrew Shtulman, for showing the "Shtulman effect", one of the main source of this thesis, and who, could have left it at that, because he would have already contributed significantly to this work, but in addition accepted to be an examiner of this thesis. Thanks to Manuela Piazza, for her work on mathematical cognition, and for agreeing to be an examiner of this thesis. Merci à Valeria Giardino d'avoir accepté de découvrir ce travail de recherche depuis ses débuts, pour ses retours et ses conseils, et d'avoir accepté d'être examinatrice de ce travail de thèse.

Je remercie mes collaboratrices et collaborateurs: sur les épiphanies, Louise Chevalier et Lucie Martin, sur les définitions des droites, Annahité Sarré, et sur la géométrie sphérique, Cyril Falcon.

Je remercie l'équipe Langage et cognition de m'avoir ouvert ses portes, Thierry Nazzi, Laurianne Cabrera, Olivier Mascaro, et tous les autres, pour leurs remarques et leurs retours; les membres et administrateurs du Baby Lab de l'INCC -où il est vrai, je n'ai emmené que de vieux enfants, et d'encore plus vieux bébés-, Maxine, Anthony, Viviane.

Je remercie mes camarades de thèse, mes copains et copines de l'INCC, Klara, Hamdi, Lucie, Diane, Marie-Victorine, Garance, Élodie, l'autre Charlotte, Cécile, Axel, Martina, Camille, Laurie, Judith, Théo.

Je remercie mes participants, mathématiciens ou non, professeurs ou non, heureux ou non de s'entendre prodiguer des explications sur les trajectoires qui vont tout droit, sans avoir rien demandé à personne.

Je remercie mes amis et ma famille, mes parents, mon père, soutien chevronné et patient de mes longues études, mes soeurs, pour avoir été elles aussi des cobayes de la première heure, du temps révolu des premiers pilotes, et bien sûr pour leur soutien durant cette thèse.

Je remercie enfin Cyril, pour tout ce qui ne peut être dit -et comme l'a dit un jour un mauvais logicien et très bon philosophe, ce dont on ne peut parler, il faut le taire.

## Contents

<b>Introduction (en français)</b>	1
Contexte	1
Objectifs de la thèse	2
Organisation de la thèse	3
<b>Introduction</b>	5
From mathematical intuitions to academic mathematics	8
Concept learning through bayesian models	13
General context of the thesis: from Euclidean to non-Euclidean geometry.	17
Description of the concept of geodesic.	17
Overview of the thesis	22
<b>An experimental paradigm to study concept learning in the lab</b>	23
1. Experimental design and participants	24
2. Procedure	24
2.1. Inclusion phase	24
2.2. Teaching phase	25
2.3. Test phase	26
2.4. Final phase	30
<b>Chapter 1. Content analysis of the definitions of straight lines</b>	33
1. Introduction	34
2. Methods	36
2.1. Identification of the cited properties	37
2.2. Coding the definition according to the properties	37
3. Results	39
3.1. Exhaustive list of the properties	39
3.2. Representative properties of the first definitions	44
3.3. Evolution of the definitions during the experiment	45



4. Discussion	48
<b>Chapter 2. Insight and mathematical learning</b>	<b>51</b>
1. Introduction	52
2. Methods	55
2.1. Participants	55
2.2. Material and procedure	56
3. Analyses	59
3.1. Effect of the number of lessons on accuracy	60
3.2. Insight Experiences	60
3.3. Relation between Insight Experiences and Learning	60
3.4. Relation between insight and introspection	61
4. Results	61
4.1. Effect of the number of lessons on performance	61
4.2. Effect of the number of lessons on insight reports	64
4.3. Relation between insight report and performance	64
4.4. Relation between insight experiences, confidence and performance	65
4.5. Discussion	67
5. Supplementary analyses including participants without teaching phase	76
5.1. Effect of the number of lessons on performance	76
5.2. Effect of the number of lessons on insight reports	76
5.3. Relation between insight report and performance	77
5.4. Relation between insight experiences, confidence and performance	77
<b>Chapter 3. Dynamic of learning</b>	<b>81</b>
1. Introduction	82
2. Study 1	86
2.1. Introduction.	86
3. Study 2	93
3.1. Introduction.	93
3.2. Power analysis.	94
3.3. Results and discussion.	97
4. General discussion	99

<b>Chapter 4. Planar bias shapes identification of straight lines on curved surfaces</b>	<b>103</b>
1. Introduction	104
2. Results	110
2.1. Study 1	110
2.2. Study 2	117
2.3. Main analyses.	117
2.4. Exploratory analyses.	119
3. Discussion	119
4. Methods	121
4.1. Data availability.	121
4.2. Study 1	121
4.3. Study 2	122
<b>Conclusion</b>	<b>131</b>
Chapter 1	132
Chapter 2	135
Chapter 3	136
Chapter 4	137
Further perspectives	138
4.4. Mathematical concepts as proptotypes	139
4.5. Physical foundations of intuitive mathematics	141
<b>References</b>	<b>143</b>
References	143



## Introduction (en français)

**Contexte.** En dépit du temps et des efforts déployés par les enseignants, de nombreux élèves connaissent des difficultés dans l'apprentissage des concepts scientifiques, au point d'échouer à les appliquer, même après la formation académique - e.g (Vosniadou, 2019). En mathématiques, le cas des fractions, enseignées dès le premier cycle, est un exemple canonique : selon des rapports d'enseignement (Sander et al., 2022; Behr et al., 1984), les enfants peuvent reproduire pendant des années des erreurs dans la manipulation de fractions, et ces confusions imprègnent encore le raisonnement des adultes à propos des pourcentages.

Pourtant, les humains montrent des compétences mathématiques extrêmement précoces constituant une base de mathématiques naïves -e.g, (Dehaene, 2011; Feigenson et al., 2004; Piazza et al., 2007; Dehaene et al., 1998; Spelke, 2017; Mehler & Bever, 1967; Nieder, 2005; Meck & Church, 1983). Ces compétences ont été extensivement étudiées dans le cas de la cognition numérique, montrant que les bébés et jeunes enfants possèdent un sens de l'arithmétique. Agés de quelques mois seulement, les bébés manifestent une sensibilité aux quantités (Xu & Spelke, 2000); d'autre part, avant l'apprentissage du langage, les enfants sont capables d'opération arithmétiques simples comme l'addition, la soustraction, et les ratios (Brannon et al., 2004; J. S. Lipton & Spelke, 2004; J. Lipton & Spelke, 2003; Mccrink & Wynn, 2004; Wood & Spelke, 2005; Gilmore et al., 2007; Xu et al., 2005). Enfin, les performances de primates non humains dans des tâches d'addition et de soustraction montrent que ceux-ci possèdent également des compétences avoisinant celle des adultes humains, (Cantlon & Brannon, 2007), renforçant l'hypothèse d'un noyau de connaissances fondamentales des mathématiques hérité de l'évolution.

Comprendre l'apprentissage des concepts académiques implique donc de comprendre comment ces intuitions interagissent avec l'apprentissage académique, et comment les concepts enseignés sont intégrés dans les réseaux de représentation pré-existants.

Dans les études sur la cognition numérique, l'apprentissage conceptuel est abordé sous l'angle de paradigmes transversaux ou longitudinaux, comparant des enfants différents groupes d'âges ou les mêmes enfants à différents stades de leur développement. Dans le premier cas, on compare des enfants différents potentiellement exposés à différentes sources d'éducation, dans le second cas, on compare des performances espacées par de longues périodes, le temps entre chaque session pouvant aller jusqu'à plusieurs années. Ceci représente un risque d'introduire dans les études de nombreux paramètres confondants (mémoire, différences éducatives, travail consacré au concept, etc.). Il est donc particulièrement difficile de comprendre quelles interactions sont possibles entre les concepts intuitifs et les concepts appris sur des périodes aussi longues, et ainsi des paradigmes d'apprentissage continu, c'est-à-dire à session unique, semblent nécessaires pour suivre la courbe d'apprentissage de manière plus fine.

Il existe, en sciences cognitives, de tels paradigmes : les modèles bayésiens appliqués à la cognition. Ces modèles décrivent l'apprentissage de manière plus continue, sur une période éventuellement très courte, et expriment les interactions entre différentes théories concurrentes au cours de l'apprentissage. Ils sont donc particulièrement pertinents pour étudier l'apprentissage conceptuel. En l'état, cependant, ces modèles décrivent des cas d'apprentissage inductif, procédant par généralisation d'observations directes, et ne s'appliquent pas à l'apprentissage par les règles ou par définitions, qui est le format des concepts scientifiques appris à l'école, et plus spécifiquement des mathématiques.

En conséquence, les mécanismes d'apprentissage conceptuels recrutés pour les concepts mathématiques académiques demeurent mal connus. Par ailleurs, si la cognition numérique a été l'objet de nombreuses études, d'autres domaines des mathématiques, *a priori* distincts des compétences arithmétiques, ont été moins étudiés, notamment sous l'angle de l'apprentissage conceptuel. Cette thèse propose donc d'étudier les mécanismes d'apprentissage dans le cas d'un concept géométrique : les géodésiques.

**Objectifs de la thèse.** Dans cette thèse, nous étudions l'apprentissage conceptuel dans le cas des géodésiques, à trois étapes de du processus d'apprentissage conceptuel : avant, pendant et après une session d'enseignement.

La géodésique est un concept fondamental de la géométrie non-Euclidienne : elle est la généralisation de la ligne droite Euclidienne à toutes les surfaces. Sur une surface quelconque, la

géodésique est définie comme une trajectoire de *direction constante*, c'est-à-dire qui va tout droit, sans jamais tourner. Sur la sphère, par exemple, si l'on suit un chemin toujours dans la même direction, sans dévier, on va tracer un cercle de même rayon que la sphère. Ce cercle croisera n'importe quelle trajectoire droite tracée sur la sphère : ainsi, il n'existe pas de ligne parallèle sur la sphère, contrairement à ce que décrit la géométrie Euclidienne sur le plan. Le modèle de la géométrie sphérique illustre les conséquences étranges - au regard de la géométrie Euclidienne - de l'application des géodésiques sur les surfaces courbes. Pourtant, les géodésiques sont simplement le passage de lignes droites à des trajectoires droites, et peuvent être expliquées à l'aide de modèles physiques simples : un élastique tendu sur une surface, un morceau de scotch plaqué sur une surface, ou un véhicule dont les roues ne peuvent tourner. Pourtant, il y a des raisons de penser que seul le cas particulier des géodésiques dans le plan, c'est-à-dire la ligne droite, est intuitif (Izard et al., 2011; Dehaene et al., 2006), et que l'apprentissage ne pourra se faire qu'au prix d'une révision des intuitions bloquantes. Les géodésiques sont donc un candidat idéal pour l'étude de l'apprentissage conceptuel en mathématiques, car elles peuvent être comprises par une population non-mathématicienne, mais permettent toutefois de faire des inférences complexes.

Un premier objectif de cette thèse est donc le développement d'une situation d'apprentissage-type pour étudier comment les adultes intègrent ce concept de géodésiques à partir de leur connaissance de la ligne droite. Un second objectif est de décrire les changements qualitatifs du concept de ligne droite au fur et à mesure d'une session expérimentale où est introduit le nouveau concept. Ensuite, un troisième objectif est de décrire différents signes de la compréhension durant cet apprentissage : premièrement, trouver une mesure de la compréhension qui permette de tracer une courbe de du progrès au long d'une session, deuxièmement, utiliser des mesures introspectives et objectives de performance pour mieux comprendre les mécanismes d'apprentissage, et notamment pour interroger la possibilité d'un progrès inconscient dans l'apprentissage explicite. Enfin, un dernier objectif est de décrire plus systématiquement les fondements cognitifs du concept géométrique étudié, la ligne droite, et la capacité à généraliser ce concept.

**Organisation de la thèse.** Dans la première partie de cette thèse, je présente les résultats d'un nouveau paradigme qui permet d'observer l'apprentissage dynamique lors d'une seule session en laboratoire. Ce paradigme à session unique permet de suivre la dynamique de l'apprentissage, ainsi que de mesurer les aspects de la performance subjective et objective.

Le [Chapitre 1](#) explore le contenu des intuitions sur les lignes droites à travers une étude qualitative des définitions fournies par les participants avant, pendant et après l'apprentissage.

Les [Chapitre 2](#) et [Chapitre 3](#) explorent la dynamique de l'apprentissage conceptuel : l'apprentissage est-il fait d'étapes conceptuelles discrètes, ou d'ajustements progressifs ? Pour tenter de répondre à cette question, dans le [Chapitre 2](#), nous utilisons différentes mesures d'introspection. Je présente une étude portant sur des modèles alternatifs de l'apprentissage où les mécanismes d'apprentissage ne sont pas nécessairement conscients, et où la conscience n'a accès qu'à des sauts discrets, en mesurant les "insights" survenus pendant une session d'apprentissage.

Dans le [Chapitre 3](#), je présente deux études qui tentent de décrire la courbe d'apprentissage en termes de périodes de progression et de récession et essaient de développer une mesure de la compréhension pour suivre la progression, et ainsi estimer la forme de la courbe d'apprentissage.

Dans la deuxième partie, j'explore les racines intuitives du concept non-euclidien de géodésique chez les non-mathématiciens et les mathématiciens. Dans le [Chapitre 4](#), je demandais si les humains ont tendance à concevoir les trajectoires droites en termes euclidiens. L'étude dans [Chapter 1](#) a établi certaines propriétés intuitives auxquelles les gens pensent lorsqu'ils définissent la notion de ligne droite. Le [Chapitre 4](#) présente deux études systématiques qui testent une hypothèse plus précise : si les gens ont tendance à identifier les sections planes (ligne résultant de l'intersection d'un plan avec des surfaces quelconques) comme des géodésiques (sur la sphère, tous les cercles sont des sections planes, cependant, seuls les grands cercles sont des géodésiques). Cette étude contribue à préciser quelle géométrie serait la plus proche de la façon dont l'esprit humain conçoit l'espace.

## Introduction

Before we learn words, we know objects. From birth, we learn about the world around us and extract knowledge from that environment. In the course of this effortless, empirical learning, we build and enrich intuitive knowledge. Then, with the help of language, we explicitly learn rules or names given to categories of objects that refine this knowledge. Thus, we build a conceptual repertoire from two types of inputs: core concepts, and cultural concepts (Carey, 2009).

Evolutionarily ancient concepts constitute the *core knowledge*, that is, the fundamental knowledge that provides us with cognitive tools to understand the structure of the world (Spelke & Kinzler, 2007; Spelke, 2000; Carey & Spelke, 1996), such as agent (Liu et al., 2019; Liu & Spelke, 2017; Saxe et al., 2005; Gopnik & Wellman, 1992), goal (Liu et al., 2019; Liu & Spelke, 2017; Gergely & Csibra, 2003), object (Vallortigara, 2012; Spelke, 1990; Baillargeon et al., 1985), number (Carey, 2009; Spelke, 2017; Mehler & Bever, 1967), space (Landau et al., 1981). Humans possess them very early or at least develop them naturally through a form of inductive learning resulting from interaction with the environment and cognitive maturation. Such concepts are intuitive in the sense that we can represent them and reason with them effortlessly: infants are immediately and spontaneously surprised by violations of simple physical facts, for instance, that the same object cannot be in two different places at the same time (Spelke, 1994). Other concepts, such as *measure*, *ion*, *chromosome*, or *state*, come from a collective body of knowledge acquired over the course of human history, transmitted through sociocultural processes (Carey, 2009). These concepts are learned explicitly from peers: from family, at school, or from other cultural elements such as books and films. Scientific concepts are among them. Indeed, bodies of scientific knowledge follow a thread that is modified, corrected, and transmitted over centuries, with school and academy reflecting the content of this state of knowledge, in varying degrees of complexity. These concepts are taught in the form of statements or definitions, which are rules to be applied. Unlike intuitive concepts learned through observation, these rules are given explicitly: they are shortcuts to a collective set of inferences from human history.



Concepts taught explicitly, as in school, can be very difficult to grasp, and learning can fail, even if all the effort and time has been devoted to the task (Vosniadou, 2019). Mathematics, for instance, raises a lot of difficulties. The case of rational numbers is a striking example: according to teaching reports and cognitive science studies (Sander et al., 2022; Behr et al., 1984), children can spend years attempting to understand how to manipulate and compare fractions, while confusions in percentage addition still pervade adult reasoning. More generally, mathematics often poses significant challenges, not only to students but also to teachers, because little is known about the mechanisms of learning in this area. These difficulties seem at odds with the fact that humans are not without early and robust intuitions in mathematics. The cognitive sciences have extensively studied early mathematical knowledge in the domain of arithmetics and have identified a sense of number and quantities (Dehaene, 2011; Dehaene et al., 2003). Research has also investigated other mathematical domains, such as geometrical intuitions: some specific intuitions develop from the age of three and are also present in populations without formal mathematical knowledge (Izard et al., 2011; Izard & Spelke, 2009; Spelke et al., 2010; Dillon et al., 2013; Dehaene et al., 2006). This suggests that the human mind has a base of naive mathematics (Sander, 2008; Dehaene, 2011) that contains a lot of intuitive concepts. Mathematics is perhaps as difficult as it is intuitive: cases, where the naive conceptions hinder scholar concept acquisition, were highlighted in the context of mathematical concept learning (Krohn et al., 2021; Gvozdic & Sander, 2018). The rigidity and predominance of some intuitive conceptions may resist enrichment or modification of priors in favor of academic concepts. This internal struggle between intuitive concepts and concepts learned at school is also observed in the context of other scientific fields: even after academic training, participants in physics and biology show traces of intuitive theories as they take longer to respond accurately to questions when the answer violates the naive conception (Shtulman & Lombrozo, 2016; Shtulman & Valcarcel, 2012; Allaire-Duquette et al., 2021).

To better understand human learning capacity, it is, therefore, crucial to understand what intuitive theories humans use in the first place and whether new concepts can be integrated into this intuitive framework. Schools and universities exist because humans believe that explicit instruction can influence and modify knowledge, but are our concepts ever modified? Is new information ever integrated, and if so, in what way?

Let us take the example of gravity. Intuitive gravity is known very early on, as an empirical phenomenon: four-month-old babies know that objects fall if the support on which they are placed is released: the object does not remain at the same point in space but falls down (Needham & Baillargeon, 1993; Baillargeon et al., 1992). It soon becomes obvious that fragile objects held in one's hand should not be dropped. Then, in school, children learn that scientists call this phenomenon "gravity", due to a force, the "Earth's attraction", attracting objects towards the Earth. Later, in high school, a student may learn that behind this notion of gravity is the Newtonian theory of gravitation, and thus a formal counterpart to this theory. Student may also learn notions that predict new observable facts related to gravity, such that two objects of different weights fall at the same speed. And later, in college, this conceptual knowledge can be extended with the theory of relativity, which gives a more general model of the attraction of bodies: celestial bodies appear to be subject to a force that attracts and moves them, but these bodies actually follow straight trajectories (geodesics) in curved space-time. The force is a manifestation of the curvature of this space, which explains why we have the impression, when looking at them from the outside, that they "fall". Thus, if thought of in a naive way, all that the student seems to follow in the different stages of education is only an enrichment of the original concept that "objects fall without support". It is tempting to postulate that the intuitive content is modified and enriched as the student learns, over the years.

First thing culture provides is a name for a concept, and then eventually adds new facts about that concept, which do not always contradict the previous one. This scenario is thus easy to conceive when the concept learned's properties merely extend those of the intuitive concept.

However, this scenario does not explain what happens when the intuitive concept does not match the new concept. We possess conflicting concepts inherited from phenomenological intuitions, and it is not clear how these concepts behave when new information is added.

Many scenarios can describe the interaction between naive theories and learning.

The ideal scenario is that of the blank sheet: the new concept erases the previous one. However, this scenario does not explain the barriers to learning, as it does not seem to match the observed learning difficulties, nor does it explain the persistence of errors after academic training. Moreover, it would not make sense to erase intuitive and otherwise useful content in everyday

naive physics, even if it does not fit the target academic concept. Should we suddenly start trying to drop fragile objects?

In contrast to this ideal scenario, there is also the worst case, the lazy one, where nothing interacts, and the previous concept prevails no matter what. In this case, schooling and teachers have little hope.<sup>1</sup> However, this last scenario does not seem realistic insofar as it does not allow for even superficial learning effects.

There are also many intermediate coexistence scenarios (Shtulman & Lombrozo, 2016). For example, the "incomplete replacement", where the previous concept is not deleted, not even modified. A new concept is added, and one or the other will be used depending on the context. Here, the two concepts are not connected, there is no merging at all.

Finally, merging, the best-of-both-worlds scenario: a new concept is built from the combination of the consistent propositions of the two concepts. Here, the more coherent concept resulting from this merging wins. This view is supported by (Ohlsson & Cosejo, 2014) and is consistent with a Bayesian approach, that will be presented in more detail later in this introduction.

These situations are not necessarily exclusive, and, as put in (Carey, 2009), there is no reason to assume that the scenario is the same in all situations. To understand conceptual learning, it is, therefore, crucial to isolate a concept and evaluate its content before, during, and after learning.

### **From mathematical intuitions to academic mathematics.**

This thesis will focus on mathematical concepts. Cognitive science has extensively described the content of mathematical intuitions and has highlighted many concepts that are easily manipulated by humans and sometimes present from birth. The content of intuitive mathematical theories includes a sense of discrete quantities, laying the foundation for an intuitive concept of natural numbers (Dehaene, 2011). Humans also possess intuitions about geometric principles (Dehaene et al., 2006; Izard et al., 2011; Dillon et al., 2013), combinatorial principles and discrete probabilities (Téglás et al., 2011; Kushnir & Gopnik, 2005), simple syllogistic inferences that set the basis for boolean concepts (Mody & Carey, 2016; Cesana-Arlotti et al., 2018). Thus, there is little doubt that humans have a strong foundation of core mathematical skills, but this does not tell us how these intuitions interact in the context of learning, nor how flexible they are. These intuitive concepts are part of a larger mathematical landscape: how the mind forms concepts for

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<sup>1</sup>Note that this "no learning scenario" is rather transversal to Fodor's view of learning, that is, nothing is ever really learned (Fodor, 1979). In Fodor's sense, that simply means that knowledge is combinatorial : no primitive element is ever added to the conceptual framework. We could imagine that even in this sense, new combinations could be more or less easy to make, which would fit accordingly any of these scenarios about learning.

fundamental mathematical notions, such as function, operator, structure, etc. seems difficult to extract from this basis alone. How, then, do early intuitions extend to academic mathematics?

The numerical cognition study is a fruitful case of connecting intuitive mathematical foundations with the learning of academic concepts. In mathematics, the concept of "number" covers six varieties.<sup>2</sup> The construction of the natural integers is computationally simple compared with other sets of numbers: natural numbers are the set obtained by adding elements one after another to a list, *ad infinitum*. The consensus thesis on numerical learning is that humans possess approximate magnitude intuitions and that these intuitions are recruited to ground the meaning of number symbols.

Many studies have probed the existence of intuitions dedicated to numbers (e.g., (Dehaene, 2011; Feigenson et al., 2004; Piazza et al., 2007; Dehaene et al., 1998; Spelke, 2017; Mehler & Bever, 1967; Nieder, 2005; Meck & Church, 1983)). We apprehend numbers through three different mechanisms of quantification: subitization, which allows the immediate apprehension of small numbers (Starkey & Cooper, 1995), estimation (Izard et al., 2008), which allows the approximate estimation of the cardinality of arbitrarily large sets, and enumeration (Gallistel & Gelman, 1990; Rochel Gelman, 1978), which allows the precise enumeration of any set, using symbols. Two types of evidence support the thesis that humans possess basic arithmetic knowledge, and that this system is evolutionarily ancient. First, number estimation skills, which support simple arithmetic operations, are present from the first months of life in infants (Xu & Spelke, 2000) as well as in children before language development (Brannon et al., 2004; J. S. Lipton & Spelke, 2004; J. Lipton & Spelke, 2003; Mccrink & Wynn, 2004; Wood & Spelke, 2005; Gilmore et al., 2007; Xu et al., 2005). These pre-verbal operations include multiplication and division, as well as ratio computation. Second, other animal species also exhibit these abilities, including macaques, which share most of the pre-verbal arithmetic skills of human adults (Cantlon & Brannon, 2007).

This "intuitive sense of number" (Dehaene, 2011), is supported by a system of perception of quantities (the approximate number system, ANS), and a system of individuation, sensitive to the size of sets of objects. An essential signature of ANS is that the estimation of sets of numbers is less accurate when the numbers are large and comparison of two non-symbolic numbers become more difficult as the difference between them reduces (distance effect). Symbolic manipulation

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<sup>2</sup>Integers, relatives, rationals, reals, complexes, imaginary numbers.

of exact numbers as learned in school is developed on these initial numerical intuitions: several lines of evidence show that we recruit these intuitive number skills. First, comparison operations on symbolic numbers follow the same principles and constraints as magnitude estimates: when people have to compare two symbolic numbers to say which is larger, the error rate and response time increase inversely with the distance between them -distance effect, (Moyer & Landauer, 1967; Dehaene et al., 1990). This suggests that when operating on symbolic numbers, participants transcribe symbolic numbers into approximate quantities, which recruits operations on magnitudes. Second, early quantity representation abilities predict acuity on symbolic numerical tasks: in particular, variations in the distance effect in children predict later competence on arithmetic tasks in school (Holloway & Ansari, 2009). More generally, many studies show a link between these pre-verbal abilities and arithmetic skills in school (Feigenson et al., 2013; Gilmore et al., 2010; Starr et al., 2013; Piazza et al., 2010; Halberda et al., 2008; Elliott et al., 2019; Libertus, 2015; Mussolin et al., 2014). Similarly, training tasks establish the influence of artificial reinforcement of these intuitions on arithmetic performance in children (Hyde et al., 2014) and adults (Park & Brannon, 2014).

The intuitive mapping between magnitude representations and symbolic numbers is crucial, as it lays the foundation for a naive theory of number. This association lends itself to the learning and manipulation of natural integers, but interferes with the learning of rationals written in symbolic form.

Fractions are rational numbers, written as  $a/b$ , where  $a$  and  $b$  are integer coefficients. Rational numbers do not correspond directly to quantities but to *ratios* of whole quantities. The correspondence between symbolic number and magnitude can therefore be misleading here, in several ways. First, a single ratio may be indexed with several natural numbers that do not depend on the quantity denoted by ratio. Hence, a denominator can correspond to different quantities:  $1/3$  can be  $1/3$  of a glass of water, a set of candies, and the density of oxygen in a room, and there is no trivial way to immediately map a fraction to a quantity. Second, the same ratio can be written with several pairs of symbolic numbers. As  $1/3$  is only the representative of an infinite class of 1:3 ratios, of which it is the simplest version, fractions written with integers far apart on the number line can therefore denote rational numbers relatively close together on that same line. This does not coincide with counting operation: translating a fraction into a number accurately requires a division whereas counting requires addition with a constant, 1. There is therefore reason to believe that the naive number theory will be misleading when it comes to learning fractions.

Consistent with these expected difficulties, learning fractions in school poses many problems for children (Sander et al., 2022). Moreover, these errors sometimes persist after school: even educated adults have difficulty with fractions and percentage operations, indicating a tendency to reason in fixed quantities with symbolic numbers (Vamvakoussi et al., 2012). Specifically, studies have identified a source of difficulty with fraction operations: participants tend to overestimate the size of a ratio if written with large numbers: the "whole number bias." People tend to reason with fractions as if they denote direct quantities and view fractions written with large numbers as larger magnitudes. This bias persists after school: Indeed, children (Hoof et al., 2015; DeWolf & Vosniadou, 2015; Vamvakoussi & Vosniadou, 2004), adults (Vamvakoussi et al., 2012, 2013), and even expert mathematicians (Obersteiner et al., 2013) show a tendency to overestimate the size of a rational number if written with large numbers.

Despite this bias, humans prove able of acquiring knowledge of numbers through education. In the case of fractions, some studies show that adult participants succeed in accurately answering and may even build accurate intuitions (based on response time) for rationals presented as a ratio of symbolic numbers, sets of dots, or circles of different sizes (Matthews & Chesney, 2015; Fazio et al., 2016). In the case of natural numbers, some studies show that the perception of numbers is refined over time, with manipulation of symbolic operations (Piazza et al., 2013; Nys et al., 2013).

These biases shaping mathematical reasoning vary between people depending on their overall mathematical competence: in tasks showing interferences with intuitive theories, participants with better "mathematical abilities" or global mathematical ability scores showed less sensitivity to these biases than participants with lower global math ability scores, which suggests they have better flexibility with respect of these biases (Krohn et al., 2021). More precisely, overcoming these natural cognitive constraints may rely on a form of inhibition (Viarouge et al., 2019) as suggested notably by the case of rationals (Roell et al., 2019, 2017; Rossi et al., 2019). (Bugden & Ansari, 2011) shows that mathematical competencies in children probed through arithmetical tasks link to a propensity to use intentional rather than automatic processes. Such inhibition may come with certain expertise: In (Krohn et al., 2021), depending on the mathematical competence, the signature of intuitive biases transfers from accuracy to response time: the less flexible participants make errors, and while in more flexible participants, the errors disappear but time gaps appear in the responses. More generally, in literature probing interferences in other sciences (Allaire-Duquette et al., 2021), interferences impact the response times of experts or educated

adults rather than their accuracy, suggesting that people integrate the new concept but do not erase earlier intuitions. They have to "think twice" and bypass their initial intuitions, which are still present.

Much work has been done in the context of numbers, showing the nature of intuitive tools for manipulating numbers, the relationship between the concept of exact number and these intuitions, the difficulties raised by these same intuitions when the concept needs to be generalized and extended, and how these intuitions evolve over the life course. These studies seem to point to a coexistence-type scenario: intuitions seem to be present throughout life, yet learning, even if delayed, is visible.<sup>3</sup>

However, due to logistical constraints, some research questions cannot be asked under these experimental conditions. To understand the content of concepts and how they change over the life course, these studies compare the results obtained in different age groups. Thus, conceptual learning and changes in skills and representations are addressed from the perspective of cross-sectional or longitudinal paradigms. These constraints are necessary to study children's learning of numbers, because learning sometimes involves very long maturation times, either because the pivotal skills allowing the acquisition of the concept are not yet acquired, or because the concept is difficult in itself. Indeed, some concepts are difficult to acquire or consolidate even in adults, at the end of their brain maturation. Consequently, the time elapsed between each session can be up to several years, which represents a risk of introducing into the studies many confounding parameters (memory, educational differences, work devoted to the concept, etc.) Above all, it makes it difficult to isolate the qualitative factors at play, such as the differences in the presentation of the concepts, but also the raw dynamics of the evolution of the concepts' learning. In particular, it is difficult to understand what interactions are possible between intuitive concepts and learned concepts over such long periods. Single-session learning paradigms appear necessary to follow the learning curve and understand how concepts are integrated.

A whole literature attempts to describe these fine-grained learning dynamics which apply to short periods, such as a single learning session: Bayesian models.

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<sup>3</sup>It is interesting to note that in the case of numbers, two crucial properties must be relaxed to allow for the concept of rationals: (i) a number is directly associated with a quantity, (ii) two different symbolic numbers refer to two different magnitudes. To our knowledge, there has been no qualitative study that could explore whether, when performance increases on the rational, these constraints are consciously overridden by the participants, or at least would they disagree with these more general facts.

### Concept learning through bayesian models.

To explain how children learn so quickly, different theories have been proposed. One of these, Theory theory, proposes that children's learning mechanisms are similar to those of scientific theory formation e.g. (Carey, 1985; Gopnik & Meltzoff, 1997; Wellman & Gelman, 1992). The Bayesian model has provided a computational framework to explain how children "compute" new theories and learn new concepts spontaneously, during development (Gopnik & Tenenbaum, 2007; Tenenbaum et al., 2011; Dehaene, 2020), given some prior intuitions. The main idea behind application of the Bayesian framework to learning is that children produce hypotheses (models) about the world around them and constantly evaluate and re-evaluate them, based on what they observe. This view integrates natural intuitions or core knowledge inside learning models. Thus, this framework postulates both inference mechanisms and innate conceptual content (sense of probability, physical mechanisms), but also great cognitive flexibility and a capacity to change theory when presented with data. It thus reconciles the nativist view of concepts with the empiricist view. It also allows us to understand how the brain generates such a rich body of knowledge during development from a few observations, without necessarily being already present at birth. From this learning procedure would arise many concepts, such as that of "causality" (Goodman et al., 2008).

The learning procedure is based on the Bayes formula:

$$P(h|d) = \frac{P(d|h)P(h)}{\sum_{h' \in H} P(d|h')P(h')}$$

The interpretation of this formula for learning is that the learner has a space of hypotheses  $H$ , which are all possible models. Given some newly observed data  $d$ , and  $h$  in  $H$ , the learner computes a posterior probability  $P(h|d)$ , that is the degree of belief that  $h$  is a relevant model for  $d$  or, in other words, that  $h$  can explain  $d$ . This probability depends on the priors for  $h$ ,  $P(h)$ , i.e. its plausibility independently of the data, and the likelihood  $P(d|h)$ , which expresses the probability of observing  $d$  given  $h$  -the causal relation from  $h$  to  $d$ . The formula takes into account alternative causal models, as the product of priors and likelihoods for  $h$  is normalized by the sum of the same product for each alternate  $h'$  in  $H$ . To modify their theories and conceptual categories, the children must therefore process new data: the Bayesian model is by definition a model of inductive learning.

This framework makes it possible to account for the fine dynamics of learning with respect to the intuitions present (the priors), and thus to test which are the intuitive models of the child.



According to the observed inferences, for example, the surprise of babies or children, one can deduce the intuitive models that the child has, i.e. the content of  $T$ , and what are the weights attributed to the elements of  $H$ . These applications are not specific to development, as they also predict learning situations in adults (Xu & Tenenbaum, 2007).

From a few examples to which the same label is associated, children and adults are able to infer a new category (Tenenbaum et al., 2011). This category is chosen according to a criterion of relevance and economy: a concept is the smallest set of objects that falls into the designated category. If we represent the similarities of objects in tree form, where each branch represents a hypothesis of  $H$ , and each node a conceptual difference, categorizing a concept from a label amounts to selecting the smallest branch that unites all the examples encountered (Tenenbaum et al., 2011). From this perspective, a concept is thus defined extensionally (Frege, 1948), as a category inferred from observed data.

More crucially for our topic of interest, Bayesian models allow us to account for conflicting learning, i.e. situations of conceptual change (Ullman et al., 2012). Such situations correspond to cases of acquisition of new theories, where the basis of intuition is at odds with the new concept to be learned. This model was designed after a laboratory learning situation on magnetism (Bonawitz et al., 2012). This provides a framework for modeling some of the merging-type scenarios discussed above: when new concepts emerge in learning, they come from a consistent combination of new and previous facts. However, such learning mechanisms need fresh data to function: the Bayesian model bayesian models describe inferences over observations, that is, inductive learning (Tenenbaum et al., 2006). It does not deal with learning according to given rules, *a fortiori* explicitly.

We can therefore assume certain limits to the applicability of the Bayesian model to the learning of school and academic concepts: this type of learning is not inductive, and most of the time, it is not empirical, i.e. it is not based on observations. Scholar knowledge is given backward, in the form of definitions that are rules to apply, not examples from which to deduce the rule. School, university, and even scientific communication do not expect the child or researcher to reproduce all the deductions from direct observations: they are provided as shortcuts. This is especially true for mathematics, which is the very example of deductive science, where concepts are given intensionally (Frege, 1948), i.e. by a definition, which is a rule for recognizing examples of the

concept. This does not mean that one learns only by the rules, and that the school does not provide any examples: of course, we all have in our memory striking examples seen in class. Under these conditions, how can we account for explicit learning, by definition, in a Bayesian model?

A natural way to proceed would be to select the hypothesis that corresponds to the one described by the teacher, and to assume that its "posterior" (probability after learning) is equal to 1. In the Bayesian model, learning by rules would thus consist in forcing one of the hypotheses to take precedence over the others, without prior confrontation with new data. The child -or the adult- would be forced to change his theory by a "top-down" process which is a kind of argument of authority or inference to the best explanation (because coming from the teacher or the parents). It is therefore not the data added to the child's - or the scientist's - theory that can overturn it, or modify it, but a competing theory, without data. According to this perspective, however, explicit learning would be trivial, and should work the first time. However, this is not what is observed: learning is not instantaneous, but rather long and delayed.

Crucially, learning by rules is explicit: by giving a rule, one gives in principle all the material necessary to use the new knowledge. In principle, learning by rules is knowledge given on a plate, there is no more work to do. However, the reality seems different: when a rule is given, one must first accept it, then learn to use it and apply it to new examples. We can therefore postulate that the learning mechanisms described by Bayesian learning will not be completely the same as those involved in conceptual academic learning. And if human cognition does indeed proceed in a Bayesian manner, this could explain in part why academic learning is so difficult. Indeed, one can imagine several reasons why learning by rules, under the prism of bayesianism, is more difficult than learning by observing examples.

First, the status of the authority assumption may not update with sufficient priors, namely a probability of 1: the associated probability may still be very low. It may not be enough that the rule is given in a school setting for children to accept it as the only possible rule.

Second, internal models may not be updated explicitly, but only in response to observations. In the case of rules, observations are not directly associated with the rule. A rule is compressed information from which consequences must be extracted. One then has to think of different consequences that are not necessarily obvious from the rule, including consequences that are irrelevant to the applications. One can imagine that inferring the concept of "measure" from examples would be much more complicated though. Some concepts are not directly observable, for example, the atom.

Thirdly, it is also possible that the model updates itself according to the given rule, but is further updated as new observations keep coming in, and that the progress does not persist over time. Indeed there is no reason to think that academic learning acts as a stopping rule and makes the model impervious to intuitive observations and inferences.

One can therefore postulate that some concepts are not learned inductively, or at least require slight adjustments to Bayesian models of learning as they stand. Bayesian models are thus very useful for understanding the overall and fine-grained dynamics of inductive conceptual learning but provide little context for concepts learned by rules: learning mechanisms in deductive or explicit contexts, or abstract concepts that cannot be imagined only in terms of examples.

So we have two branches. On the one hand, studies that describe the content of intuitions and representations of mathematical concepts, such as number. These studies are carried out on the scale of human development, and therefore do not access, for logistical reasons, the fine details of the mechanisms of learning. On the other hand, we have paradigms that describe learning more thinly, in a possibly very short period, and which express the interactions between different competing theories during learning, but these models, as they stand, do not apply to rule-based learning. How can we try to develop fine-grained learning paradigms for complex academic concepts? One of the goals of this thesis is to find a paradigm and a concept allowing the study of learning mechanisms in a single learning session. One of the crucial issues, then, is to find a behavioral measure of learning, or at least of understanding.

To understand concept learning in mathematics, there is a real need to map the mathematical foundations of intuitions to the content of academic mathematics. The very first step then is to set up a one-session laboratory learning situation with a mathematical concept.

Here, we will investigate how humans are equipped for understanding mathematical concepts, in the context of geometry. A long way has been made in the case of numerical intuitions, but geometric intuitions have been less explored so far, and their learning mechanisms remain poorly understood. We have chosen a concept that lends itself to simple physical analogies and does not require the introduction of sophisticated formal concepts - at least when one does not go into the details of the theory. This concept is geodesics, which is the generalization of straight lines to curved surfaces. With this model, we intend to study how the information is integrated,

comparing the intuitive content of the concept with the post-teaching content. One of the goals of this thesis is thus the development of a "model" learning situation to study how adults integrate this more general concept based on their knowledge of the straight line.

**General context of the thesis: from Euclidean to non-Euclidean geometry.** Like the number for arithmetic, the line is a basic concept in geometry, and although not primitive, because it is defined, it is fundamental for the construction of a system of geometry. The straight line has many possible representations, depending on the support on which it is drawn, ranging from a straight line on the plane to a helix on a cylinder. In its most general sense, that of a straight line, it is called a "geodesic". Conceptualization of non-Euclidean geometries gave the concept its most general meaning, applicable to any surface.

The setup of non-Euclidean geometries was a major scientific change that spreads far beyond the scope of mathematics. The conception by Bolyai, Lobatchevski, and Gauss is the first stone to construct a more general geometry and lays the theoretical foundations for relativity in physics (Greenberg, 1993). Formally, a geometry is non-Euclidean if it contradicts the fifth postulate of Euclid, stating that given a straight line and a point outside of it, there exists a unique straight line that is parallel to the given one and going through the given point Table 1, from (Euclid, 300 BCE, 2007). This axiom has been taken as a logical consequence of the rest of the first four postulates for centuries (Greenberg, 1993), and despite the absence of proof, it was taken for granted by almost all mathematicians and philosophers. Arguing by contradiction, mathematicians finally discovered that the system of Euclidean geometry (minus the fifth postulate) and the negation of the fifth axiom gave consistent systems. Geodesics, which are the extension of straight lines to all surfaces, are the cornerstone of the extension from Euclidean to non-Euclidean geometry. Indeed, geodesics are the generalization of the definition of straight lines that allow defining models of geometry in which there is no parallel line (spherical geometry) or infinitely many (hyperbolic geometry).

**Description of the concept of geodesic.** In this thesis, we will focus on the concept of geodesic. A geodesic is a curve of constant direction, i.e. a curve that goes straight on any surface. Geodesics in the plane are very familiar to us: they are straight lines, which we have been manipulating since elementary school. However, we are less used to considering straight trajectories on other surfaces. As an example, consider geodesics on the sphere (Figure 1). Let us imagine I start

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).

TABLE 1. The five Euclid's postulates.

a journey across the earth, starting at the North Pole and going straight South until the equator, following a geodesic for 10,000 km. When I arrive in Amapà, Brazil, I turn left, following another portion of the geodesic, heading straight East to Kampala, Uganda, still following the equator line for 10,000 km. There, I return to my starting point, the North Pole, by a geodesic path transverse to the one I took to start my journey. The triangle drawn by my path on the surface of the earth corresponds to an equilateral triangle with three right angles, an impossible property for a Euclidean triangle (Rouvière, 2016).

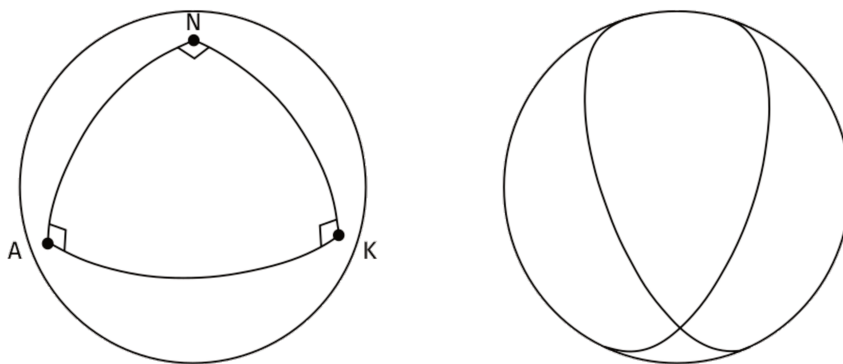


FIGURE 1. Two journeys on the sphere. On the left, an equilateral triangle from North Pole (N) to Amapà (A) and Kampala (K) with three right angles. On the right, two geodesic paths from North Pole crossing at South Pole.

Now let us imagine that someone starts a journey from the North Pole at the same time as I do, and heads directly to Kampala, while I start from the North Pole and head directly to Amapà. If we continue our journey after passing through both cities, without turning, both walking in a straight line, we will meet at a certain point: exactly half the circumference of the earth, at the South Pole.

These two immediate consequences of the extension of straight lines to spheres equally violate Euclidean geometry: On the sphere, there are no parallel lines, and equivalently, the sum of the triangles is not always equal to  $180^\circ$ , as the triangle with three right angles drawn during my trip attests. Otherwise stated, on the sphere, Euclid's fifth axiom is false. We say that the sphere is a model of non-Euclidean geometry, that is to say that it realizes the negation of the fifth axiom while maintaining the other four.

Let us now see what any straight path on the sphere looks like. If I continue my path past the South Pole, I will return exactly to the North Pole. The extension of a straight path on a sphere will always follow a closed curve. In fact, on the sphere, we can give a very simple criterion which is a necessary and sufficient condition to be a geodesic: it will be a circle of the same radius as the sphere, i.e. a "great circle". This is equivalent to saying that any other curve on the sphere, whether it is a circle or not, is not a geodesic (Figure 2).

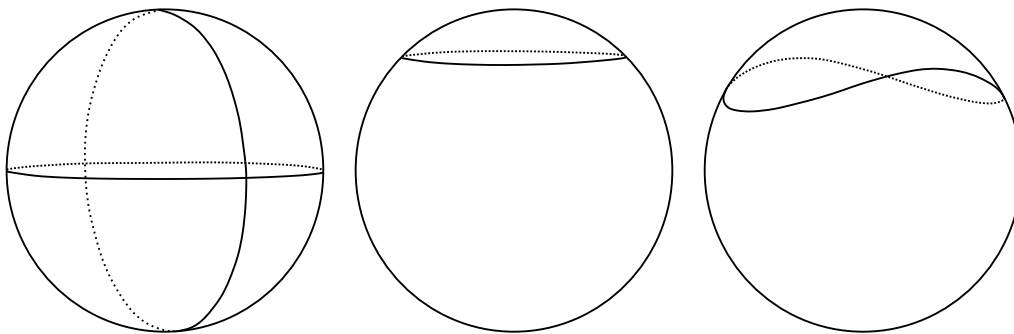


FIGURE 2. Four curves on the sphere. From right to left, two geodesics, a circle non-geodesic, and a non-circle non-geodesic curve.

As the model of spherical geometry illustrates, the seemingly intuitive facet of geodesics - which are simply the switch from straight lines to straight paths - as well as the odd consequences of their applications to curved surfaces regarding Euclidean geometry, make geodesics an ideal candidate for the study of conceptual learning in mathematics. Indeed, geodesics can be explained

without any mathematical background, using only simple physical models. Yet there are reasons to think that only the special case of geodesics in the plane, i.e. the straight line, is intuitive.

The first reason is a historical argument: the model of straight lines on the plane was the first geometric model in history, and it was the only system for nearly two millennia. Interestingly, geometry on the sphere did not appear with non-Euclidean geometry: the ancient Greeks already reasoned, for example, about the properties of circles and triangles on the sphere ([Theodosius et al., 1st century BC, 2010](#)). The crucial difference is that they did not consider the sphere and great circles as a model for geometry, at the cost of the fifth axiom. The geometrical object of the sphere is therefore not a novelty discovered by Bolyai and Lobachevski, but interpretation of this object has changed. In other words, we can reason with geometric system on non-planar surfaces, like spheres. This remark is very important to underline the idea that non-Euclidean geometry brings the crucial idea of the change of referential. There is no longer an absolute referential: concepts are defined from the point of view of the surface, not that of space. Thus, rather than considering the sphere -and surfaces- as a geometric object immersed in space ( $\mathbf{R}^3$ ), and considering directions of space as absolute, in non-Euclidean geometry, one uses relative directions of the surface to define a geometrical model. Constant direction, in  $\mathbf{R}^3$ , amounts to a fixed vector. However, on the sphere, constant direction does not correspond anymore to the constant directions of  $\mathbf{R}^3$ : from the point of view of  $\mathbf{R}^3$ , following a great circle, the directions change as we turn around the sphere. In  $\mathbf{R}^3$ , if we want to verify that a direction is straight, the criterion is the identity of the direction vectors: if the vectors differ during the trajectory, then the direction has changed. On curved surfaces, however, we check that the direction is constant with the parallel transport of the vectors: if we drag the vectors along the trajectory without making them skid, and at the end, they are superimposed, then we have indeed kept a "straight" direction, a constant direction ([Rouvière, 2016](#); [Spivak, 1999](#))<sup>4</sup>.

Moreover, reasoning about straight lines is learned without difficulty in school, and groups of seven to fifteen y.o. children perform above chance on reasoning tasks about the properties of straight lines and the fifth axiom ([Izard et al., 2011](#); [Dillon et al., 2013](#)). Populations with no formal mathematics education also reason with ease about Euclidean properties of straight lines in these same reasoning tasks ([Izard et al., 2011](#); [Dehaene et al., 2006](#)).

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<sup>4</sup>Mathematically, this is stated by the following definition: a geodesic is a trajectory of constant speed whose acceleration is orthogonal to the surface. This definition required specific technical terms, but the main idea of the definition of geodesics can be stated without any specific mathematical concept: it is sufficient to understand the idea of a trajectory going straight ahead.

If the straight line is indeed a very simple concept for humans, what about their ability to generalize it to curved surfaces? Geodesics may seem simple to conceive, but it requires abandoning some criteria that are special cases of the plane. The essential criterion of geodesic is the straight direction, or the constant direction. Geodesics, however, do not satisfy other criteria that may seem obvious, such as being the shortest path globally. This is only true locally, i.e. whenever two points are sufficiently close to each other, any geodesic between them will be minimizing for the distance. On the sphere, critical distance is between two antipodal points. The smallest portion between two points on the sphere is a geodesic, but the complementary of this portion is a geodesic path as well, and is obviously not the shortest path! This criterion of minimal distance path is therefore only local. Other properties differ, as illustrated by the case of the sphere: a geodesic does not always extend to infinity, unlike straight lines. Crucially, a geodesic can have non-zero curvature. Indeed, straight lines in the plane correspond to straight directions in  $\mathbf{R}^3$ , but on curved surfaces, straight lines do not correspond to straight directions in  $\mathbf{R}^3$ .

Straight lines' properties on the plane would thus be an example of an intuitive prior which is contradicted by the generalization of a concept. Importantly, this extension does not require technical vocabulary nor formal notions.

Like early intuitions about quantities, children have intuitions about what an optimal trajectory is, which underlies their inferences in the context of social cognition. Recent evidence from social studies suggests that humans have a coarse sensitivity to optimal trajectories at an early age, which allows them to understand agents who achieve goals (Liu & Spelke, 2017). When three-month-old infants watch someone reach out to press a button, and that hand does not follow the shortest trajectory but instead makes a small detour as if to avoid an invisible wall, the infants tend to be surprised, compared to the case when the hand makes a direct gesture toward the target. This ability to identify goals suggests an early calculation of optimal trajectories and shortest paths. However, this may be too broad to be called in favor of an Euclidean or non-Euclidean conception: *a priori* these paths are computed in  $\mathbf{R}^3$  but they may not correspond exactly to straight lines. Thus, the results say nothing about geometric reasoning abilities *per se*: this knowledge could as well be encapsulated in reasoning dedicated to social cognition. Nevertheless, this indicates that trajectory optimization calculation skills are present at an extremely early stage.



## Overview of the thesis.

Although conceptual learning is at the heart of cognitive science, the ability to track signs of progress by studying the fine dynamics of learning has not been achieved, especially for a complex mathematical concept. This thesis aims to study the development of a geometric concept, geodesics, in the mind, and to describe the state of knowledge at three stages of learning: before, during, and after learning.

In the first part of this thesis, I present results from a new paradigm that allows dynamic learning to be observed in a single session in the laboratory. This single-session paradigm allows for tracking the dynamics of learning, as well as measuring aspects of subjective and objective performance. [Chapter 1](#) explores the content of intuitions about straight lines through a qualitative study of definitions provided by participants before, during, and after learning.

[Chapter 2](#) and [Chapter 3](#) explore the dynamics of conceptual learning: Is learning made of discrete conceptual steps, or of progressive adjustments? To try to answer this question, in [Chapter 2](#) we go through different measures of introspection. I present a study addressing alternative models where learning mechanisms are not necessarily conscious, and where consciousness has access only to discrete jumps.

In [Chapter 3](#), I present two studies that attempt to describe the learning curve in terms of periods of progress and recession and try to develop a measure of understanding to track progress, and thus estimate the shape of the learning curve.

In the second part, I further explore the intuitive roots of the non-Euclidean concept of geodesics in non-mathematicians and mathematicians. In [Chapter 4](#), I ask whether humans are shaped to think about straight trajectories in Euclidean terms. The study in [Chapter 1](#) set up some intuitive properties that people think of when defining the notion of a straight line. [Chapter 4](#) presents two systematic studies that test a more precise hypothesis about the content of the notion of a line: if people tend to identify planar sections (line resulting from the intersection of a plane with any surfaces) as geodesics (on the sphere, all circles are planar sections, however, only the great circles are geodesics). This study contributes to specifying which geometry would be the closest to the way the human mind conceive space.

## **An experimental paradigm to study concept learning in the lab**

Our experimental paradigm was designed as a teaching intervention in the lab that could induce and modulate learning in a single session. This experimental paradigm was conceived to allow observing the dynamic of concept acquisition in the lab and included three research questions. First, it aimed to describe the dynamic of concept acquisition through a new behavioral measure of understanding (Analyse of the learning phase, in [Chapter 3](#)), second, it aimed to describe subjective aspects of performance, such as Eureka moments, in the context of learning, and their relation with performance (Analyse of introspection measure, test performances, and insight reports, in [Chapter 2](#)), and third, it aimed to describe the qualitative content of geometrical intuitions about straight lines before and after learning (Analyses of Definitions 1, 2 and 3, in [Chapter 1](#)).

## 1. Experimental design and participants

This experimental paradigm is a teaching situation where participants have to learn the concept of geodesics. First, participants have to perform a planar geometry test to assess their basic understanding of straight lines on the plane, and a spherical geometry test to ensure that they do not already know the concept taught (Subsection 2.1). Second, participants are given 0, 1, 3, 5 or 7 lessons to learn about the properties of geodesics on the sphere (Subsection 2.2). Third, participants performed three tests (Subsection 2.3). Lastly, participants have to report potential experiences of "insight" during the experiment (Subsection 2.4).

## 2. Procedure

The four phases of the paradigm are comprehensively described below: Inclusion phase, Teaching phase, Test phase, and Final phase.

Inclusion phase	Teaching phase	Test phase	Final phase
Definition of straight lines (1)	Introduction: great circles	Confidence rating (1)	<i>Retrospective ratings of understanding</i>
Planar geometry	Lessons about straight lines on spheres	Subjective efficiency of the teaching phase	Insight report
Straight lines on spheres		Straight lines on spheres	
		Definition of straight lines (2)	
		Straight lines on various surfaces	
		Confidence rating (2)	
		Reasoning about straight lines on spheres and other surfaces	
		Confidence rating (3)	
		Definition of straight lines (3)	

TABLE 2. Comprehensive list of all the tasks administered to the participants in the four phases of the experimental paradigm. The tasks highlighted in blue are analysed in Chapter 1, in green, in Chapter 3, and other tasks in Chapter 2, apart from the task in italics that was not included in the analyses.

### 2.1. Inclusion phase.

**2.1.1. Definition of straight lines.** Participants were first asked to produce a written definition for the notion of straight line.

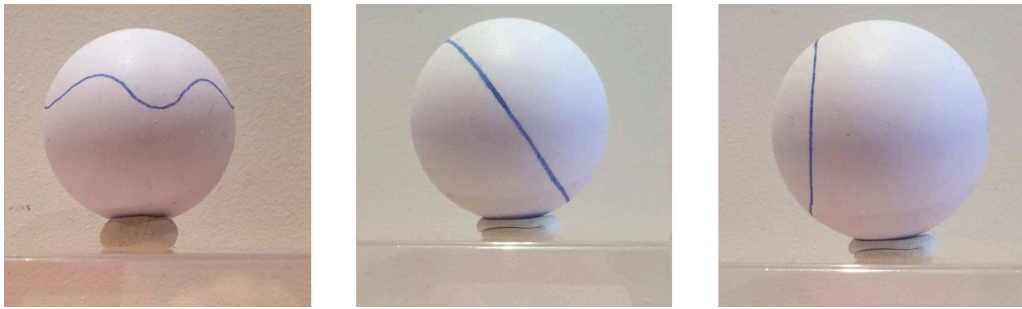


FIGURE 3. Example stimuli from the three conditions of the sphere straight lines test: (left) non-circle line, (middle) great circle, (right) small circle.

**2.1.2. Planar geometry.** This test was adapted from (Izard et al., 2011). In a short introduction, participants were described an infinite plane on which points and straight lines could be drawn. Straight lines were described as lines that never turn, neither on the left nor on the right, and that continue straight ahead indefinitely. After this introduction, participants were asked a series of twenty illustrated questions about the properties of straight lines. Questions were presented both in writing and orally through an audio recording, and participants ticked their answer (yes or no) on a response sheet. Participants were excluded if they made more than 3 errors pilot work indicated that more than 90% of geometry-educated adults should pass this criterion.

**2.1.3. Straight lines on spheres.** In each trial, participants were presented with a photograph of a sphere (a table tennis ball) with a line drawn on it, and asked to indicate whether the line was "straight" or not. Three types of trials were presented: great circles (straight), small circles (non-straight, but typically judged to be straight by most adults), and non-circles (e.g. wavy line, line looping and crossing itself to form an 8 figure). Each category counted 4 trials, for a total of 12 trials presented in a randomized order. Participants responded by pressing the 'o' key for yes ('oui') or the 'n' key for no ('non'). They were included if they made at least two mistakes on the small circle trials (i.e. they incorrectly judged small circles to be straight lines) or if they made at least two mistakes on the great circle trials (i.e. they incorrectly judged that great circles were not straight).

## 2.2. Teaching phase.

**2.2.1. Introduction to great circles.** Participants were given a one-page document describing great circles as circles of the same radius as the sphere, and illustrating great circles traced on spheres at various orientations. A translated version of this document is accessible on the Github repository of the project ([https://github.com/charlusb/Analyses\\_Eurekamaths](https://github.com/charlusb/Analyses_Eurekamaths)).

**2.2.2. Lessons.** Participants were given 1 to 7 lessons to learn about straight lines in spherical geometry. All the lessons explained that straight lines on the sphere correspond to great circles, and not to small circles (synthesized version in [Table 3](#)). Translated version of the 7 different lessons are also accessible on the Github repository of the project ([https://github.com/charlusb/Analyses\\_Eurekamaths](https://github.com/charlusb/Analyses_Eurekamaths)). To encourage the participants to study the lessons in depth, we asked them to write a summary of each lesson just after reading it. We also asked them to rate whether they found the lesson convincing, on a scale graded from 0 to 10. These judgments were collected twice: a first time after reading each lesson, and a second time for all lessons together (except for the group who received only one lesson to read, as this would have resulted in asking the exact same question twice in a row).

This and other judgments on the teaching phase (see Subjective efficiency of the teaching phase below) were included as an effort to develop a measure of the dynamics of the learning process.

### **2.3. Test phase.**

**2.3.1. Confidence rating.** Participants were asked to rate how much they felt they understood the notion of straight line, using a scale graded from 0 to 10 (first confidence rating).

**2.3.2. Subjective efficiency of the teaching phase.** Participants from the 3-, 5- and 7-lesson groups were given two questions to answer. First, they were asked whether they felt that the elements presented had helped them improve their understanding of straight lines (yes or no). Second, participants ranked the different lessons they had studied by mapping them on an oriented line, from the least convincing to the most convincing. This last task was not given to the 1-lesson group because they would have had only 1 lesson to rank. The first question about the subjective impact of the lessons on participants understanding was missing for this group by mistake. At this point, the first experimenter left and was replaced by a second experimenter who was blind to the teaching condition assigned to the participants.

**2.3.3. Straight lines on spheres.** This test was identical to the spherical geometry inclusion test [Figure 3](#).

**2.3.4. Definition of straight lines.** Participants were asked to produce a written definition for the notion of straight line (second definition).

**2.3.5. Straight lines on various surfaces.** Participants were presented with lines drawn on four different surfaces: cone (8 trials), cylinder (6 trials), cube (8 trials) and torus (4 trials). Each



### Synthesis

Imagine a small car whose wheels cannot turn: this car can therefore only go "straight-ahead".

While the car drives along the great circle, it cannot drive along the small circle.

An elastic rubber band tries to minimize its tension, aligning naturally on a straight line. Attached on two points of a small circle, it will follow a different path, but attached on two points of a great circle, the elastic aligns itself on this circle.

To apply a scotch tape piece on a straight line, one needs not twist it. Applied without constraints, the tape will follow a straight line on any surface. On a sphere, it can be applied on a great circle without twisting, but on a small circle, the scotch tape will twist.

On a planisphere, the path taken by planes from Paris to L.A. (in red) may not seem straight. Yet, looking at it on the earth seen from above, it is obvious that the path, which is a portion of a great circle, goes straight, whereas other trajectories (arcs of small circles, in blue), do not.

Imagine a small car whose wheels cannot turn: this car can therefore only go "straight-ahead". When applied on a sphere, the car follows a great circle.

If the two ends of a rubber band are attached to two points on a surface, the rubber band aligns itself on a straight line. On a sphere, if two elastic bands are placed end to end, the complete path forms a great circle.

A piece of scotch tape, if applied without twisting it on a surface, follows a straight line. On a sphere, if applied without twisting, the piece of scotch tape will naturally draw a great circle.

TABLE 3. The seven lessons presented during the learning phase (translated from French).

trial displayed two photographs showing a front and a back view of a surface on which a line had been drawn. Participants were asked to judge whether the line presented was straight or not and indicated their answer by keypress. The task crossed the two variables of straightness (straight, not straight), and intersection (planar, non-planar): 4 trials presented non-straight lines that did not correspond to planar intersections (like non-circle lines on the sphere), 3 trials presented

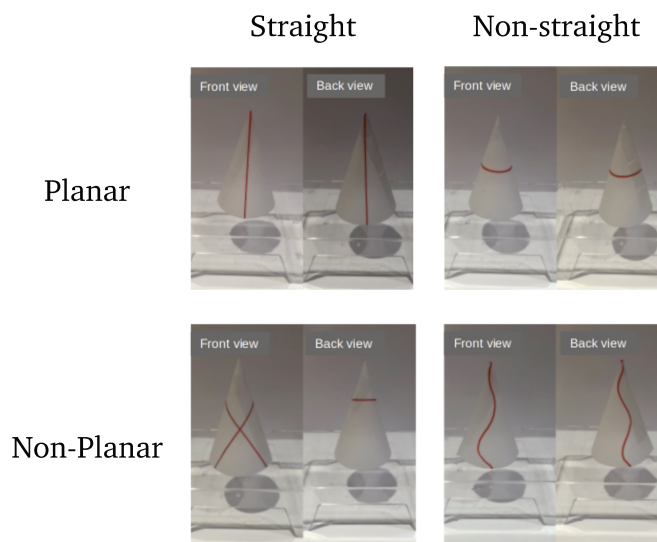


TABLE 4. Example stimuli from the four conditions in the various surfaces straight lines test. Each trial presented two photographs showing a front and a back view of a surface and labeled as such. Three of the conditions correspond to lines that can be realized on the sphere (corresponding to great circles, small circles, and non-circles) while the last one (straight non-planar) cannot be realized on the sphere.

straight lines that corresponded to planar intersections (like great circles on the sphere), 10 trials presented non-straight lines that corresponded to planar intersections (like small circles on the sphere), and 9 trials presented straight lines that did not correspond to planar intersections (there are no such lines on the sphere, but these can exist on other surfaces). Trials were presented in a random order.

One of the torus trials showed a line that is ambiguous in mathematics: a circle resulting from the intersection of the torus with the tangent plane that just touches it on its summit. This curve is considered a geodesic under some mathematical descriptions - the "flat torus" (Borrelli et al., 2013) -, but it does not correspond to a straight line according to the criteria given in our lessons. This trial was thus analyzed as a planar non-straight line. The results presented in Chapter 2 do not change if this trial is analyzed as showing a planar straight line, or if it is removed from the analyses.

**2.3.6. Confidence rating.** Participants were asked to indicate how much they felt they understood the notion of generalized straight line on a 0-10 graduated scale (second confidence rating).

**2.3.7. Reasoning about straight lines on spheres and other surfaces.** This test consisted in a list of mathematical assertions, which participants judged to be true or false: eight assertions

Assertion	Correct answer
1. On a sphere, given two straight lines, one can draw a straight line that intersects the first one but not the second one.	False
2. On a sphere, one can draw two straight lines that get closer to each other.	True
3. On a sphere, there is an infinity of lines perpendicular to a given line (not necessarily at the same point).	True
4. On a sphere, one can draw two straight lines that never intersect.	False
5. On a sphere, it is possible to draw two straight lines that are perpendicular.	True
6. On a sphere, two distinct straight lines always have two points of intersection.	True
7. On a sphere, it is possible to draw a straight line that is parallel to a first straight lines and goes through a given point.	False
8. On a sphere, two straight lines can be drawn at a constant distance from each other.	False
9. There is a surface on which there is always one single straight line that is parallel to a first straight line and that goes through a given point.	True
10. There is a surface on which it is never possible to draw a straight line that is parallel to a first straight line and that goes through a given point.	True
11. There is a surface on which a straight line can go several times through the same point (intersecting itself).	True
12. There is a surface on which two straight lines can be drawn at a constant distance from each other.	True
14. There is a surface on which two straight lines are always intersecting.	True
15. There is a surface on which it is not possible to draw two perpendicular lines.	False
16. There is a surface on which it is not possible to draw two straight lines that intersect.	False

TABLE 5. Assertions presented to participants in the reasoning test about straight lines on spheres (assertions 1-8) and other surfaces (assertions 9-16). Participants were provided with the following definitions on the top of the page: "Two straight lines are parallel if they never intersect. Two straight lines are perpendicular if they intersect at a right angle". All the material presented here is translated from French.

about straight lines drawn on spheres, followed by eight assertions about straight lines drawn on various surfaces. The assertions were presented in a fixed order, on paper. Participants were given written definitions for the terms "parallel" and "perpendicular", which appeared in some of the assertions. They answered by ticking one of four response options for each assertion: 'yes - certain', 'yes - uncertain', 'no - uncertain', and 'no - certain'.



**2.3.8. Confidence rating.** Participants indicated how much they felt they understood the notion of generalized straight line on a 0-10 graduated scale (third confidence rating).

**2.3.9. Definition of straight lines.** Participants were asked to produce a written definition for the notion of straight line (third definition).

**2.3.10. Retrospective ratings of confidence.** Participants were asked to evaluate retrospectively their understanding of straight lines at three time points: before the teaching phase, at the end of the teaching phase and at the end of the test phase.

This task was originally included as another measure of participants' introspection about their own learning: we intended to measure how much people thought that they had progressed in their understanding of straight lines during the course of the experiment. However, inspection of the data of the first two groups tested (1 and 7 lessons) revealed that participants often misunderstood this task. Indeed, if they had responded as intended, i.e. by indicating how much they thought they understood the notion of generalized straight line at several stages of the experiment in the light of the understanding they had gained at the end of the experiment, their ratings should progressively increase, or perhaps remain stable - but they cannot decrease. Quite the contrary, we found that the ratings produced by 19 out of 28 participants expressed a decrease in understanding at some point during the experiment. This suggests that many participants misinterpreted our question as referring to their feeling of understanding as they experienced it at different stages of the experiment - perhaps they thought we wanted to study whether they could faithfully remember these feelings retrospectively. We thus chose to discard this measure from our analyses, yet the task was included for all the participants, for the sake of consistency between groups.

**2.4. Final phase.** The final phase was administered by the experimenter who had been in charge of administering the teaching phase.

**2.4.1. Insight reports.** The final part aimed at measuring whether people experienced insights in our experiment. Participants were first given a description of the sensations associated with insight experiences (adapted from (Danek & Wiley, 2017): a feeling of sudden understanding that comes unexpectedly, and is associated with feelings of relief and certainty. Participants indicated whether they experienced such episodes at all during the course of the experiment (yes or no) - this answer was used as a measure of insight report. They were then presented with vignettes illustrating the different phases of the experiment (one vignette for each lesson, pre-

and post-teaching test, confidence rating, and definition of straight lines), and were asked to indicate when in the experiment they had experienced insights. At the end of the experiment, a few participants in the 3- and 5-lesson groups were asked about the contents of the insights reported.



# 1.

## Content analysis of the definitions of straight lines

This first chapter explores the intuitive content of the concept of a straight line. In [Paradigm 1](#), participants had to define the notion of "straight line" three times during the experiment: first at the beginning, before reading any experimental material, second after a learning sequence and a test, and third, after a test phase. This chapter presents a descriptive analysis of the qualitative content of participants' concept of straight lines, and how it evolves in reaction to learning material. To these ends, I analyze definitions collected and thus attempts to assess the initial geometric intuitions about straight lines.

## 1. Introduction

At first glance, a straight line seems like a very trivial notion, so simple that the question "*What is a straight line?*" may seem like a trick. The reason this question seems easy is that "straight line" seems to be an atomic notion, though it is defined: a straight line is a line that goes straight. What does it mean, "*straight*"?

In ancient Greek literature, there are three physical ways of defining the straight line, before any abstract definition -we rely here on Mugler's article and translations (Mugler, 1957, 1956): First, a definition in terms of tension, where the straight line is a wire taut between two extremities (notably found in Plato's Meno in 380 BCE, the diagonal of the square is described as a segment of the straight line stretched from vertex to vertex (Plato, 380 BCEa), and in Proclus, Commentary on Euclid (Proclus, 5 CE,1970). Second, a kinematic definition where the straight line is the set of fixed points of a spatial rotation: if we think of an axis, the straight line is the line whose points remain fixed during the duration of a rotation - notably in Plato's Republic IV, 436E (Plato, n.d.-b), or Plato's Republic X 616C (Plato, n.d.-a). Third, an optical description where a straight line is a line such that any point on it eclipses its poles (Plato's Parmenides in 137E, (Plato, 380 BCEb) translation in (Mugler, 1956). This optical notion is constructed by analogy with the rectilinear trajectory of the light ray. The idea of the taut wire would come from the stretched rope used to conceive optimal and direct paths in masonry and architecture. This latter definition reduces to saying that the straight line is economical, in matter, or in time. The abstract notion that thus naturally extend these physical pictures is that of minimal distance, as formulated by Archimede "the shortest of the lines having the same ends" (Archimedes, 225 BCE, 2010; Mugler, 1957) . This definition amounts to saying that the straight line is the shortest path between two points and is partly in line with the foundations of the modern definition, in which a geodesic is a trajectory that locally minimizes energy and distance, (Spivak, 1999; Rouvière, 2016). The geodesic is indeed the "straightest" trajectory possible (Rouvière, 2016). Nevertheless, the referential of ancient Greeks is space, as the geodesic is thought of as an axis or a ray, a wire stretched in space. Neither do these definitions contain the idea that straight line preserves direction: they do not yet contain the idea of a possible generalization to other surfaces.

Apart from these physical intuitions, the first mathematical attempt to define the straight line in purely abstract terms is given in Euclid's Elements, as a line "*which lies evenly with points on itself*" (Euclid, 300 BCE, 2007). Although ambiguous, the property of "*lying evenly*" is specified by the addition of other axioms, notably the fifth. Since the plane is the referential of Euclid's geometry, an implicit but crucial assumption of this definition is that a straight line, like any element of this two-dimensional geometric system, is planar. A portion of straight line, in Euclid's geometry, can always be extended, and if extended to infinity, is unbounded. Moreover, seen from the point of view of  $\mathbf{R}^3$ , a straight line is always of zero curvature, does not twist, does not turn, and thus can never cross itself, nor go backward.

One may ask whether these early intuitions from the point of view of human civilization reflect the actual intuitive content of the concept: what is a straight line in people's minds? Are they able to define it by exhibiting more primitive properties, or synonyms?

Here, I present a qualitative content analysis of the responses to the question "What is a straight line?", asked three times in [Paradigm 1](#). The first time, participants had just started the experiment, and had not been given any information or instructions other than this single question. The second time, they had received one or more short lessons explaining what a straight line on the sphere is, and performed a test where they had to identify geodesics on spheres. The third time, they were given two additional tests where they had to identify geodesics on various surfaces and reason on properties of geodesics.

This first description has two purposes. First, to account for the intuitive content of the concept of a straight line as reported by our participants, by analyzing the content of the very first definition collected. Specifically, by exploring the main constituents of the concept as mentioned by the participants, we seek to find out whether the qualitative intuitive content of straight lines is close to a Euclidean version or whether it easily lends itself to a generalization to geodesics. Second, to analyze the evolution of this conception during the learning session by analyzing the differences between definitions 1, 2, and 3. In particular, are the concepts employed by the participants in the first place rigid, or do their definitions show signs of generalization? To this end,

all the properties mentioned have been surveyed and classified to represent an exhaustive list of all the properties encountered in the definitions.

## 2. Methods

In total, 247 definitions were collected, for 84 participants (three definitions for each participant, five missing). As there were different experimental conditions, among participants included in the experiment, some received at least one lesson on geodesics (56), and two baseline groups received no learning material (38). For the analysis of the intuitive content of the concept, I analyzed the first definitions of each participant: 82 definitions, (two definitions were missing). Since the purpose of the analysis of the second and third definitions was to detect changes in the mentioned properties during learning, for these analyses, we discarded the definitions of participants who did not get the learning phase (38 participants), which yielded 54 (two definitions were missing) and 55 definitions (one was missing), respectively.

The collection of reflexive reports on concepts, written or drawn (Vosniadou, 1992; Subramaniam & Padalkar, 2009), provides important information about conceptual content, especially in the field of learning and education (Ercikan & Roth, 2006; Roth, 2005; Madill & Gough, 2008; Erickson, 2012), as they give clues about the different representations used by learners. However, qualitative analyses have obvious limitations: the possible lack of quantification associated with the content captured in an interview - representativeness -, and the room for experimenter's interpretation of the content collected - reliability. We describe below the procedure for identification and quantification over the cited properties, as well as our attempts to address these methodological problems.

Analyses of the definitions followed the following procedure. First, we set up a coding sheet to extract the content of the definitions, which is a set of categories summarizing the mentioned properties. We also classified the different referents mentioned in the definitions, i.e. the surface on which the line has the mentioned property. For example, in the definition "a line on the plane is a line that is infinite and always goes in the same direction", we wanted to be able to point out that the referential mentioned is *the plane*, and that a line on the plane is a line of *constant direction* and *infinite*. The second step was the classification of the definitions into categories (definition coding) according to this coding sheet.

**2.1. Identification of the cited properties.** Identification of the cited properties is the joint work of three people.<sup>1</sup> This allows to overcome, at least partially, the reliability problem. A document was produced that described each of the categories, which the coders had to agree upon. To produce this document, a training coding was done jointly by two of the coders. The procedure was as follows: a first coder initialized a first coding sheet that contained about 20 categories. A second coder joined her at this stage and a strategy was put in place: to consider at each iteration twelve of the 247 definitions, which corresponds to four participants' definitions - about 5% of the data -, to discuss potential conflicts in order to converge towards a new description of the different coding categories, by adding, deleting, renaming categories. Coders then tried to apply these to the next twelve definitions, and counted the number of definitions on which there were conflicts. The process was iterated until there were no more conflicts on all the definitions considered, three times in a row. This was achieved after twenty-six iterations of twelve data points. Next, the third coder reread the categories and made slight changes to the coding conditions.

**2.2. Coding the definition according to the properties.** It then remained to classify each definition according to the observed properties, i.e. to code each definition by describing which properties it mentioned. Because of some natural linguistic ambiguities in the definitions, this procedure could not be computationalized. First, the assignment of properties to a referent under certain terms was usually explained with necessary grammatical vagueness. Therefore, the coding of the definition:

"[Une ligne droite est] une ligne passant par au moins deux points et qui se prolonge tout droit sans se tordre. Sur une surface sphérique une ligne droite formera un grand cercle passant par deux points du diamètre de ce dernier", i.e

"[A straight line is] a line passing through at least two points and extending straight ahead without bending. On a spherical surface a straight line will form a large circle passing through two points of the diameter of the spherical surface"

assumed to understand that the straight line generally has property of passing through two points and extending straight without twisting independently of any referential, and on the sphere, it can also have form a large circle. Yet, a choice could be made here, to only attribute property of being a great circle to a sphere, without reporting the previous properties. This understanding of

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<sup>1</sup>Véronique Izard, Annahita Sarré, Charlotte Barot.



language seemed too subtle for an automatized procedure, along with the correct choice, if any, in this situation.

As with the categories, some of the words used in the definition are not the mathematically correct words to denote a referential, but they were classified according to the concept they were likely to denote. For example, in the expression "*spherical plane*", the word "*plane*" is likely to denote "*surface*" instead of "*plane*", -because a surface can be seen as being made of a sheet of paper, a plane that can form another surface- and so the whole expression denotes the sphere.

Second, words used in the definitions were not always necessary and sufficient conditions for classification in one category, but required an account of the context. For example "*[La ligne droite] peut être infinie ou définie*", i.e. "*[Straight line] can be defined or infinite*" was classified in "infinite or finite" whereas "*a straight line is not always defined depending on the surface*" was coded in "may be found on various surfaces". These interactions with context were a great obstacle to an automatized procedure of coding.

More problematic, definitions were written with common life terms, and most common language terms are underspecified regarding mathematical concepts: they do not naturally map one mathematical concept, which adds some more indeterminism of the coding. For example, the definition

"Une ligne droite est une ligne fixe qui est soit dans la longueur soit en hauteur mais qui ne fléchit pas."

"A straight line is a fixed line that may or may not join two points but does not bend."

could be classified either in "without turns", "null curvature", or "constant direction".

Then, I coded the definitions using the property sheet and the referents and modalities. To avoid the influence of knowing the conditions and the position of the definition in the experiment, I used a blind program that presented each participant's definitions in random order without any mention of the definition's number or participant's identifier. To facilitate understanding of the definitions, however, definitions from the same participant were presented together. All the properties mentioned in the definitions are reported in the first part of the results [Results 3.1](#).

Nevertheless, some mentions are marginal and do not allow to draw interpretations from these results. In an attempt to isolate definitional criteria in our participants, we chose to quantify these

properties in a second step, to limit the problem of representativeness. We defined a percentage of occurrences threshold to describe properties that appeared significantly, based on the observation of the frequency of occurrence of each property (Figure 4). We describe the percentage of occurrences and evolution for these specific properties in Results 3.2 and Results 3.3.

### 3. Results

**3.1. Exhaustive list of the properties.** I present below an exhaustive list of all the properties mentioned in definitions 1, 2 and 3.

#### 3.1.1. Cited properties.

- Points
  - Points: A straight line contains points, or is made of points.
  - Between points: The line is drawn between several points.
  - Connected points: The definition mentions two or more points that are "connected" by the line.
  - Two points: The line is constructed with two points.
  - Two points necessary: Two points are necessary to define a straight line.
  - Two points sufficient: Two points are sufficient to define a straight line.
  - Three points, negative: A straight line cannot go through any three points.
  - Unique: There is only one straight line between points.
  - Multitude: A straight line is made by an infinity or a multitude of points.
  - One dimension: A straight line is a one-dimension object. The property has not to be stated explicitly, but several objects were coded into this category: line, trait, layout, path, trajectory, segment, curve.
- Infinity
  - Infinite: A straight line is infinite, without further precisions.
  - Finite: A straight line is finite.
  - Unboundedness: A straight line is infinite in the specific sense of "without bounds".
  - Bounded: A straight line is finite in the specific sense of "with bounds".
  - Without beginning and end: A straight line is infinite in the sense that it has no apparent limiters -beginning and end- or in the sense that it goes over and over on itself.
- Evenness

- Extension: A straight line extends, or can be extended.
- Continuity: A straight line is continuous, it has no break.
- Constance: A straight line is "constant", without further precisions.
- Straightness and directions
  - Without turns: A straight line is a trajectory that always goes straight ahead, without turning.
  - Straight ahead: A straight line goes straight, follows a straight direction.
  - Direct: The straight line does not make a detour. The category may denote "without turning" or "shortest path", or an intermediate idea, without further specification.
  - Constant direction: A straight line is a trajectory that always goes in the same direction, that follow a constant direction.
  - Fixed vector: A straight line follows a straight direction according to three-dimensional space or the plane as referentials.
  - Negative of fixed vector: A straight line does not always follow the same direction from  $\mathbf{R}^3$ , a straight line on a surface does not necessarily correspond to a straight line from the point of view of 3D space.
  - Relative direction: The definition mentions that the directions should be taken relatively to the surface on which the line is drawn -contrasting with  $\mathbf{R}^3$  directions.
  - Specified direction: The definition mentions horizontal or vertical lines. In sum, what is described here is a special case of what is described by the category "fixed vector".
- Distance
  - Shortest distance: A straight line is a path of minimal distance between points -the points between which the distance is measured are not necessarily explicitated.
- Planarity
  - Planar: A straight line is planar or parallel to a plane.
  - Planar intersection: A straight line is an intersection of a plane with a surface
  - Planar intermediary: The line drawn on a surface becomes straight when we go to the plane, for example unfolding the surface.
  - Curvature: A straight line have a non zero curvature degree, or may appear as so.
  - Negative of Curvature: A straight line is necessary of zero curvature degree.
  - Smoothness: A straight line is a smooth curve, without singularities such as corners.

- Constraint shape: A straight line fits the surface on which it is drawn. It can therefore be curved, but its shape (in particular its curvature) is constrained by the surface.
- Same curvature: A straight line have the same curvature as the surface it is drawn on (thus, on the sphere the great circle has the same curvature as the sphere).
- Intersections
  - Auto-intersection: The line can intersect itself.
  - Closed curve: The line may join itself, or is a closed curve.
- Others
  - Circle: A straight line is a circle.
  - Parallel to the equator: The line is parallel to the equator.
  - Great circle: the line is a great circle.
  - Small circle: the line is a small circle.
  - Half: The line cut the surface in half.
  - Relations between lines: the definition mentions some relations between lines (parallel, perpendicular, cross another line..)
  - Same radius: the line is a circle of the same radius or diameter than the sphere.
  - Counter-intuitive: preception of the line can be counter-intuitive, or is not reliable to identify a straight line.
  - Formal definition for fixed vector.
  - Formal definition for geodesic.
  - Straight: The term straight is mentionned but not defined, e.g. "a straight line is straight".
  - Reference to learning phase: the definition mentions explicitly some elements included in learning phase.

All referentials presented bellow encode the referential mentionned in the definitions.

**3.1.2. Referentials.** Any property is stated according a referential which are the surface or family of surface on which the straight lines are described. They can be:

- a specific surface (sphere, plane, cube...);
- a family of surfaces (non-planar surfaces, compact surfaces);
- omitted in the definition (undefined);

- Singular surfaces The singular surfaces are, except the plane, the surfaces presented in the learning phase (the sphere) and in the second test (Torus, Cylinder, Cube, and Cone). Such surfaces are canonical, and likely to be mentioned. This list is not exhaustive in principle, as it could be extended by any specific surface mentioned.
  - Plane: the plane and synonyms denoting the plane: "planar surface", "planar support", "non-curved surface";
  - Sphere: Sphere and synonyms denoting the sphere: "ball" - the ball is not a surface, but it was assumed that this is likely to designate the sphere, and the confusion between  $\mathbf{R}^2$  and  $\mathbf{R}^3$  is unimportant here-, "oval", "spherical surface", "spherical plane";
  - Cylinder;
  - Cone;
  - Cube;
  - Torus;
  - Three-dimensional space: all references to  $\mathbf{R}^3$ : three-dimensional space, 3D, space, 3D space. Generally, when the definition mentions space, it uses "in" rather than "on": "in the air", "in 3-D space". (Note that the properties of the straight line in  $\mathbf{R}^3$  are the same than properties of the straight line on the plane.)
- Families. Families are referentials which denote a set of surfaces. The most general is Surface, but some specific subset can also be denoted. This list is not exhaustive in principle, and any specific family of surfaces can be added according to the definition.
  - Surfaces: The definition mentions properties of straight lines on any or some surfaces;
  - Bounded Surfaces: corresponds to "*Compact surfaces*" in geometry. These are surfaces which are delimited in space (the plane is not bounded, but the sphere, torus, cube, cone and cylinder are);
  - Curved Surfaces: the complementary of the plane in the surfaces;
  - Smooth surfaces: the surfaces without singularities -vertices, corners.
- Undefined: when the referential is omitted or unexplicited.

In the [Paradigm 1](#), three definitions of straight line were collected. At the beginning of the experiment, before reading any learning materials, tests, or experimenter's remarks, eighty-four participants were asked to briefly write down their definition of straight line. Two more definitions were collected during the experiment, once after reading lessons on straight lines on the sphere

and perform a first test of straight lines on spheres, and once after two additional tests about straight lines on various surfaces.

I describe below which properties mentioned in the first definitions (Analysis 3.2), that are the straight line's properties people spontaneously mention, to explore the content of their concept of the straight line before any explanation and learning materials. Then, I describe the evolutions of the concept, comparing the content of definitions 1, 2, and 3 (Analysis 3.3).

I coded the definitions using the property sheet and the referents and modalities. To avoid the influence of knowledge of the conditions and position of the definition in the experiment, I used a blind program that presented each participant's definitions in a random order without any mention of that participant's number or identifier. To facilitate understanding of the definitions, however, definitions from the same participant were presented together.

All categories appearing at least once in the definitions have been described above, but for the analysis of the frequency of properties, I chose a frequency threshold, to deal only with properties that appear enough times, or evolve in a significant way, and discard the most anecdotal properties. The selection of this criterion was made based on the frequency histogram of the properties (Figure 4). I chose the third gap of the histogram of definitions 1 selecting only the categories that had at least 10 % of occurrences in the first definition. I chose the first gap that appeared in the histogram of evolutions, choosing to mention only the categories that evolved by at least 8% during the session.

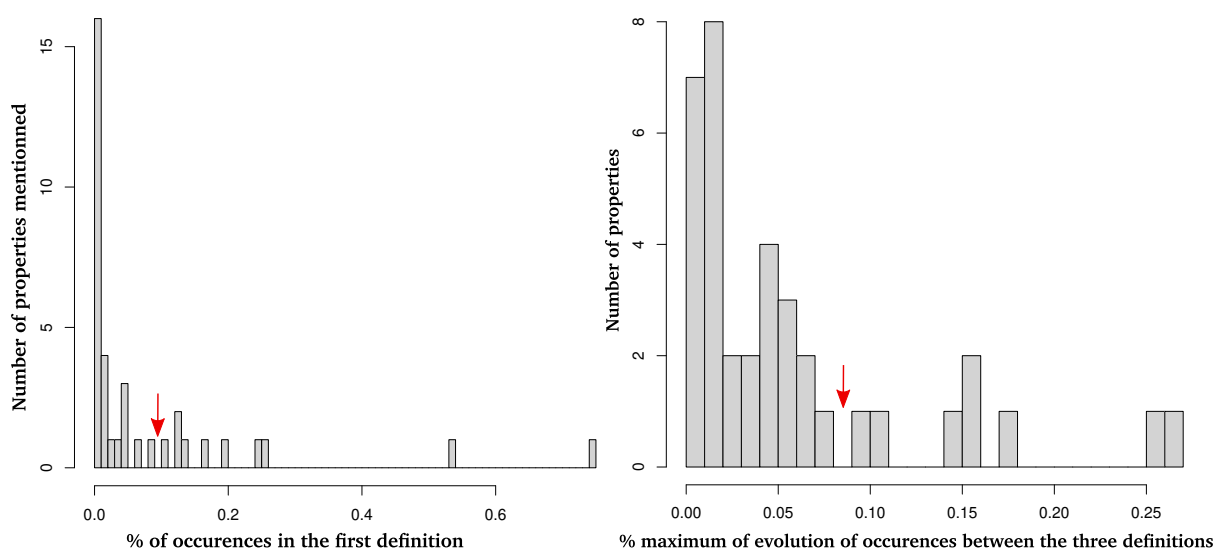


FIGURE 4. On the left, frequency of the percentage of properties' occurrences in the first definition (chosen criterion in red, at least 10% of mentions). On the right, frequency of the percentage of maximal evolution of properties (chosen criterion in red, at least 8% of mentions).

**3.2. Representative properties of the first definitions.**

I calculated the percentage of occurrence of each category across all definitions 1 given by the subjects, a total of 82 definitions. In the first definition, ten properties emerged (Figure 5). The first four are not specific to the straight line and describe properties of the line: that it is a one-dimensional object (75%), that the line can have points (53%) or can be constructed from two points (24%), and that it is a continuous object (17%). Three properties are more descriptive of the straight line but testify to a planar/Euclidean notion of the straight line: the straight line is described as generally infinite (25%), but sometimes as finite (3.6%), as uncurved (20%), and akin to a fixed vector (11%). Finally, two of the properties mentioned are more generalizable to geodesics, though ambiguous: a straight line is a line of minimum distance (13%) and constant direction (12%). The category that generalizes to geodesics, without turning, has been mentioned 12% of times. Interestingly, the criterion planar is absent of this list, as it was only marginally mentioned (4%). Accordingly, very few participants mentioned the plane as a referential (5%), or space (2%). However, as no other referential was mentioned since the first definition, participants may use it only implicitly, because they do not even think of constraining situations.

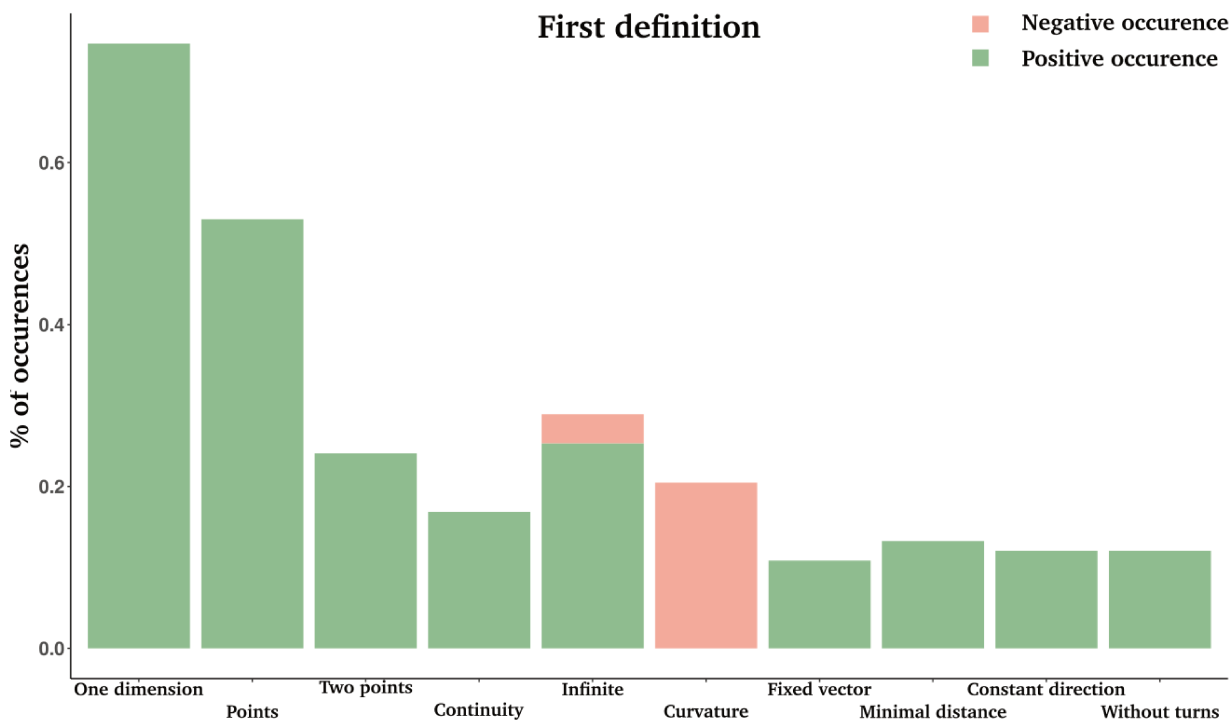


FIGURE 5. Percentage of occurrence of the properties in the first definition (selected properties reach at least 10% of mentions in the first definition).

### 3.3. Evolution of the definitions during the experiment.

First, participants generalized at the level of referentials: the sphere is mentioned (definition 2 24%), and surfaces (3%). After tests, when they are presented with various surfaces, they are more prone to generalize even the referentials (23% surfaces, including finite 5%), with a slight decrease of sphere's mentions (12% of occurrences). The cone anecdotally appears (1%). The plane is mentioned more often in definitions 2 (11% of mentions) and 3 (10%), along with the possibility of imagining other referentials, and space remains marginal 1% and 2% in resp definitions 2 and 3), accordingly with the scenario where participants would need contrasting possibilities to figure out they think of the plane as a support for a straight line.

Participants showed signs of generalization in cited properties: One crucial property has been modified during the experiment: the fact that a straight line has a null curvature. In the first definition, participants tended to write that a straight line is uncurved (20%), thus of zero-curvature. However, during the experiment, participants challenged this idea, mentioning the opposite, i.e., that a straight line can have a non-zero curvature (+11% between definition 1 and definition 2). Moreover, they tend to mention more often the feature "without turning" (+18% between definition 1 and definition 2) more readily than the other criteria to express constant direction. Indeed, it is the way of expressing constant direction that is more prone to generalization, as it lends itself to the idea of a dynamic trajectory that adapts to the surface.

This may seem at odds with the fact that the occurrences of infinity have increased, though this property does not apply to any surface (total +26% of evolution between definition 1 and definition 3, +2% of negative occurrences, and +25% of positive occurrences). Yet, infinity occurrences do not only count bounded infinity: some definitions mention lines "*having no beginning and no end*" or "*which infinitely pass over themselves*". This suggests that the participants are particularly keen on this criterion of infinity to characterize the straight line, but they try to make sense of both their previous model and the examples shown, by suggesting that a closed line can be infinite in some specific sense. Participants also mentioned that a straight line cannot intersect itself (+9% between the definitions 1 and definition 3), a property that was not mentioned in the first definitions, but overlaps with the idea of unbounded infinity. If we think of a straight line



as a fixed vector that extends to infinity on both sides, then the line cannot retrace its steps and intersect itself.

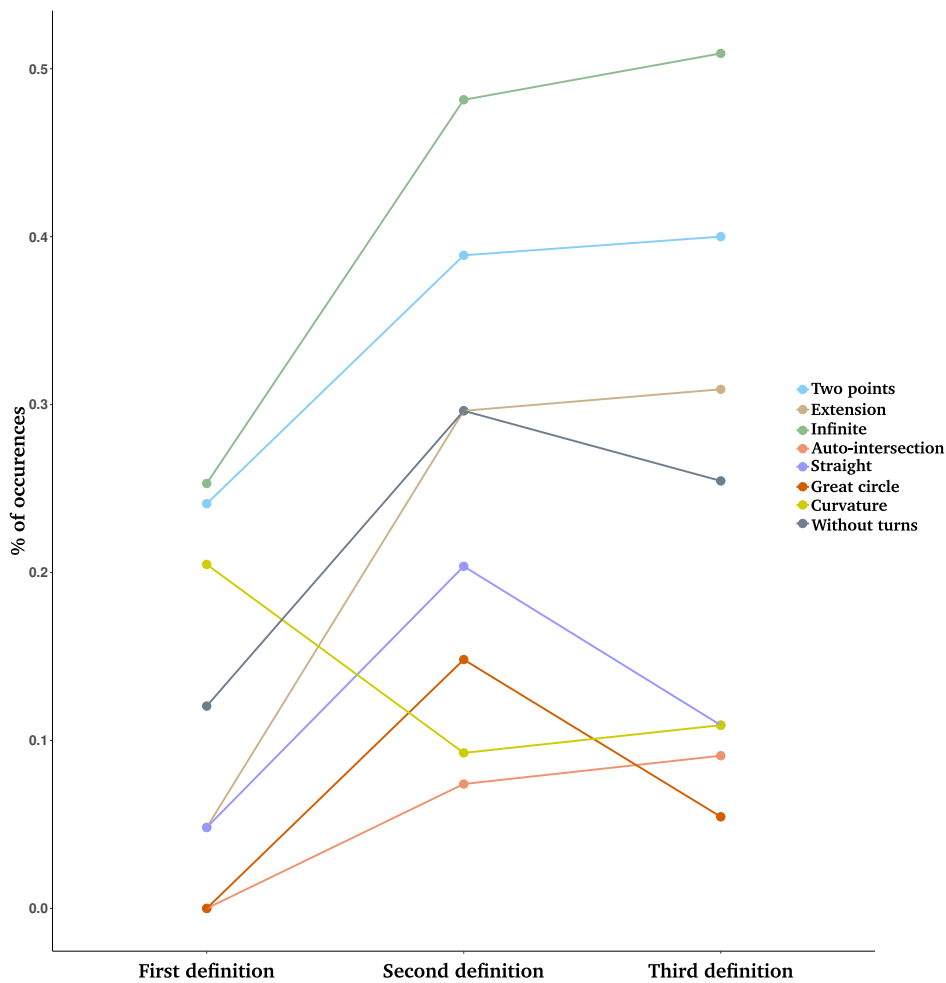


FIGURE 6. Evolution of the mention of categories between the three definitions (selected properties reach at least 8% of difference of mentions' percentage either between definition 1 and definition 2, definition 2 and definition 3, or definition 1 and definition 3).

The notion of "straight" was also mentioned more often (+16% between definition 1 and definition 2). This property is too ambiguous to conclude either recession or generalization: it is possible that participants, to cope with a conceptual difficulty, clung to the least engaging term to describe the straight line, by simply picking up one of the terms of the concept. Moreover, 4% are negative mentions, showing that participants are confused, and accept counterintuitive consequences that they think contradict the concept. The great circle category increased in frequency (+15% between definition 1 and definition 2), but this is probably just an artifact of the learning situation: it is a special case of geodesic that is sometimes mentioned without being included in the definition. It decreases after participants perform the two last tests (between

definition 2 and 3, -10%), and participants are shown other surfaces than the sphere. Finally, the neutral categories "extension" and "two points" have notably increased (resp 26% between definition 1 and 2, and 16% between definitions 1 and 3). Since these categories are not decisive either in favor of a Euclidean concept or in favor of a more general concept, and since they do not correspond to anything that was shown in the learning phase, it is difficult to interpret this increase. To see if these categories are artifacts of increases in other categories, we looked to see if they were paired with other more descriptive categories, and it is the case of extension, that is generally mentioned with infinity, for example, some definitions mentioned a line which "*s'étend à l'infini*", i.e "*extends to infinity*". More generally, we interpret the rise of these neutral categories like the augmentation of "straight": an attempt in the face of confusion to revert to more neutral terms so as not to risk getting the definition wrong.

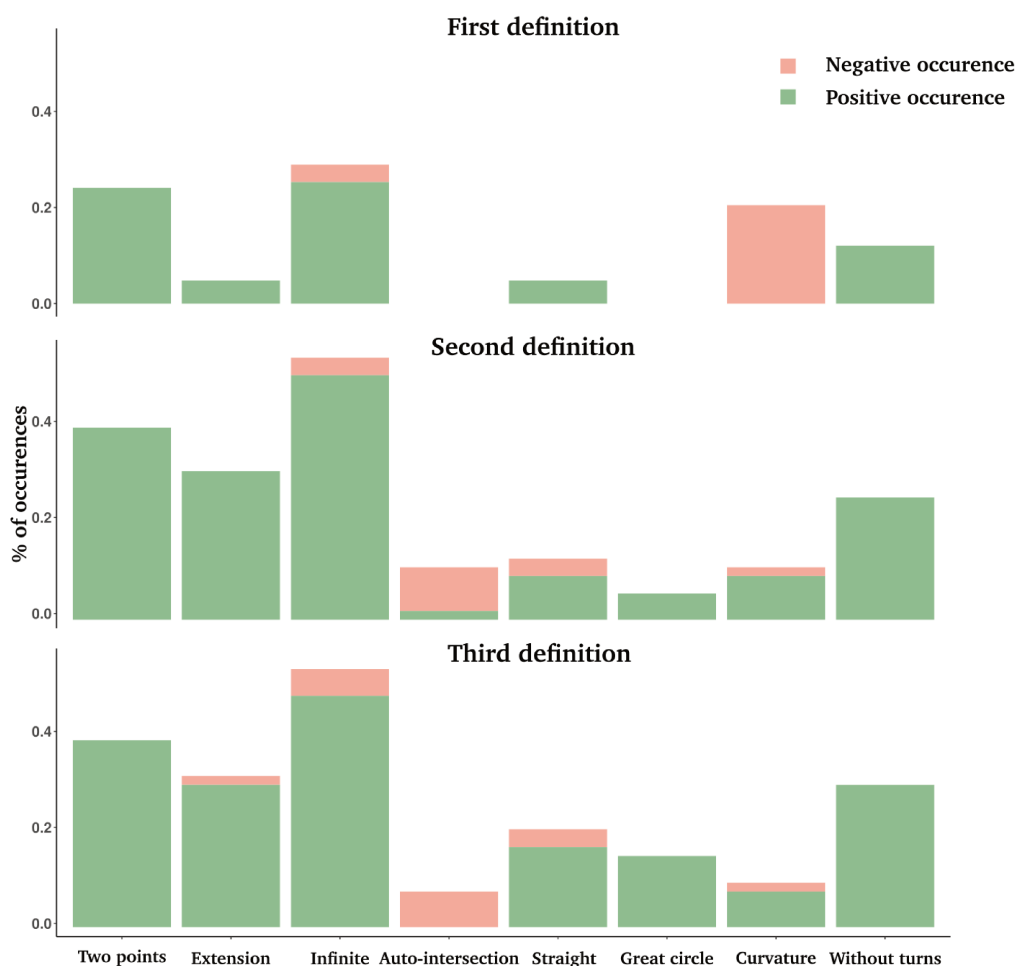


FIGURE 7. Percentage of occurrence of categories in the three definitions (same criterion as in the previous figure).

#### 4. Discussion

The purpose of this first chapter was to provide a preliminary description of people's intuitions about the straight line through the qualitative analysis of the definitions of the straight line. Through the properties mentioned in the first spontaneous definition, participants expressed criteria defining the straight line. Through the evolution of the properties mentioned in definitions two and three, participants tried to reconcile, with varying degrees of flexibility, the theory given during the learning process with their preliminary conception. The evolution of the categories has allowed them to isolate the criteria that they are willing to modify or even reject.

Straight line is difficult to define, with few participants arriving at a precise definition. While participants mentioned a very wide variety of properties, few of them recur regularly, and the description of the properties at a threshold of 10% finally retains only ten. The analysis of the first definitions indicates that overall, participants mention few properties characteristic of the straight line: the categories with the most occurrences mention properties of "line" rather than "straight": one dimension, two points, points. Thus, the exercise of defining the term is probably not easy for them, which suggests that they tend to perceive it as a primitive term: it is difficult for them to break it down.

Moreover, at first, few definitions are generalizable to geodesics. Among the more specific properties appearing in the first definition, these categories apply globally to the Euclidean line: the line is described as infinite, of zero curvature, and corresponding to a fixed vector. A notable portion of the participants identify an essential criterion of the definition of the line: constant direction, but their way of expressing it is not always generalizable: some think only of the Euclidean line (fixed vector), others give no indication of their understanding of the criterion (constant direction), and few give a more generalizable definition (no turning). This last category does not exceed the threshold of significant mentions.

A first crucial change between the first and the last definitions is that the participants admit the possibility that straight lines can be defined on various surfaces. They expand the mentions of reference frames, showing that they admit various surfaces and in particular finite (compact) surfaces as possible reference frames.

Properties compatible with geodesics become more frequent after reading the lessons, suggesting that learning has occurred. Crucially, the valence of some categories was reversed, such as zero curvature, and participants mentioned more appropriate nuances of the "constant direction" criterion, using the "no turn" category in preference to more rigid or ambiguous categories such as fixed vector, for example.

Among the properties that become more frequent, some testify that the generalization is not complete, and show signs of resistance to the new concept. Non-intersection increases, perhaps as a limitation to the possible curvature that a straight line can take: it can bend, but not cross itself. The planar frame of reference also emerged in response to the learning material. Interestingly, some participants anecdotally mentioned properties that are incompatible with the concept: they admitted that straight lines, depending on the reference frame, might not appear straight or even be straight. This probably does not mean that participants think that the concept of straightness includes non-straight objects, but rather that the definition they hold to does not intersect with some of the examples they saw: it may be zero curvature, for example, which would be a direct equivalent of "straightness" for them, or the absence of rotation in three-dimensional space. Infinity also increases, while this property is not generalizable. However, it is possible that this property is accompanied by an intermediate representation that reconciles the idea of infinity with the examples of the circle on the sphere (and thus a closed curve): a closed curve can be perceived as infinite if it traces infinitely the same trajectory.

Some neutral categories have been reinforced, such as extension and two points. They seem to be there for lack of anything better, in the absence of a stable criterion to identify. A tautological category also appears, "straight", to define the straight line.

Have participants progressed in their understanding of the straight line? One might think so, given the signs of generalization, although we cannot totally rule out that these are artifacts of our learning material. that they simply repeated what they were told ("without turning", possibility of straight lines on multiple surfaces)... These properties provide clues that suggest that, *prima facie*, participants have a representation of the straight line that seems more consistent with the Euclidean straight line. Nevertheless, they only marginally mention the plane as a reference

frame or the fact that the straight line must be parallel to a plane. It may be that the participants presuppose this criterion so much that they do not think to make it explicit, but one can assume that they use it, because various properties hold only on the condition that the line is on the plane, e.g., zero curvature, or the vector is fixed-which is another way of saying that a line is plane. Also, the plane as a reference frame is mentioned more in the later definitions, which shows that participants may not even consider at first that straight lines can be defined on non-planar surfaces. Our analysis is descriptive and obviously does not provide statistical criteria for reasoning about straight lines, but it does provide several insights into the content of our participants' representation of the straight line. In [Chapter 4](#), we will present a systematic study of a possible planar bias, which prepares us more for a Euclidean conception of the straight line than for a conception of geodesics.

## 2.

### **Insight and mathematical learning**

Learning new scientific concepts is a difficult task that can fail even with explicit instruction. Crucially, the dynamics of concept learning are not well understood: what stages of progress do students go through? Do they have introspective access to these stages? To better understand the mechanisms of learning in mathematics, this chapter investigates one aspect of people's subjective experience of understanding: insight, or *eureka moment* - episodes when people experience that they suddenly understand a concept, in a flash: "Now I've got it!". Through this study, we want to understand whether mathematical learning has a particular signature: whether people experience "Eureka" moments while learning a new mathematical concept, and whether these experiences signal pivotal stages in learning. We therefore examine the relationship of this subjective sensation to progress in learning. This study also aims to explore whether unconscious mechanisms may be involved in learning, by comparing these subjective feelings of epiphany with measures of confidence.

This chapter has been written as a publication that is currently in review in Open Mind, in collaboration with Louise Chevalier, Lucie Martin and Véronique Izard.

## 1. Introduction

Learning new concepts is difficult and protracted (Carey, 2009), especially in science (Asmuth & Rips, 2006; Vosniadou, 2019; Weber, 2002). Often, students fail to learn the concepts taught, and this even when they have been given all the relevant information, and a long interval of time has been devoted to learning (Caramazza et al., 1981a; Clement, 1982; McCloskey, 1983; D. A. Muller et al., 2007; Shtulman, 2006; Wiser, 1986). Hence, for example, some students continue to have difficulties with the mathematical concepts of function or rational number, several years after these notions have been introduced to them (Breidenbach et al., 1992; Clement, 1982; C. L. Smith et al., 2005; Vamvakoussi & Vosniadou, 2004; Weber, 2002).

Why is concept learning so fallible, and what happens during these long periods of time? In particular, do learners progress towards a better understanding step by step, in a slow and progressive manner, or are new concepts acquired in discrete, fast, but rare events? Currently, the literature provides little information to answer these questions perhaps because addressing them raises difficult experimental challenges. Hence, many studies of conceptual learning have been conducted in classrooms, either in cross sectional designs or in longitudinal designs with long delays between test sessions, leaving the fine dynamics of learning inaccessible to researchers, e.g. (Behr et al., 1984; Moss & Case, 1999; Rittle-Johnson & Alibali, 1999; Schauble, 1996; Siegler, 1995; C. L. Smith et al., 2005; Steinle & Stacey, 1998). Other studies, in contrast, have developed paradigms to probe conceptual learning in lab-based experiments, where participants progress can be tracked in real time e.g. (Feldman, 2000; Goodman et al., 2008; Martí et al., 2018; Ohlsson & Cosejo, 2014; Shepard, 1961). These studies suggest that concept learning can be viewed as a progressive process that weighs different alternative interpretations against the evidence observed, and eventually converges on the interpretation that fits best with this evidence. In all these lab-based studies however, the target concept was not described explicitly to the participants, but instead needed to be inferred from examples. As such, it is unclear whether the learning processes involved in these situations are the same as those involved in the acquisition of mathematics or science concepts: in mathematics and science students can sometimes struggle for years with a concept, and this even after their teachers have provided them with explicit definitions and demonstrations.

To try and gain information about the long process of acquisition of mathematical and science concepts, we turned to History of Science. Interestingly, many famous scientists have reported episodes where they felt that a new understanding occurred to them suddenly, in one flash. For instance, the mathematician Henri Poincaré described his astounding discovery of the Fuchsian functions in these terms: "At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it (...). I did not verify the idea; I should not have had time, (...) but I felt a perfect certainty" (Poincaré, 1946). To cite but a few, Newton, Keukulé, and Helmholtz also reported such "insight experiences" while making scientific discoveries in physics, astrophysics, or chemistry (Hadamard, 1954; Horvitz, 2002).

Importantly, insight experiences are not reserved to privileged minds. Research on problem solving has identified several puzzles known as "insight problems" that typically trigger this kind of experience e.g. the famous six matches problem, (Katona, 1940); two-string problem, (Maier, 1931); candle problem, (Duncker, 1945); and nine dots problem, (Lung & Dominowski, 1985). Specifically, when people find the solution of these puzzles, they report that the solution came suddenly, without awareness that it was about to come (Metcalf, 1986; Metcalf & Wiebe, 1987; R. W. Smith & Kounios, 1996; Bowden & Jung-beeman, 2003), and the solution found is immediately perceived as correct and relevant (Ohlsson, 1984; Danek & Wiley, 2017; Jung-Beeman et al., 2004; Kounios & Beeman, 2014; Laukkonen et al., 2020; van Steenburgh et al., 2012). Thus, these subjective experiences thus comprise all and the same components as those described by Poincaré when he related his discovery of Fuchsian functions. While originally described on a small set of problems (Bowden et al., 2005; R. W. Smith & Kounios, 1996), insight experiences have now been observed in a variety of experimental contexts, as people were tasked with deciphering magic tricks, solving anagrams, understanding jokes and metaphors, or perceiving ambiguous images (Bowden, 1997; Danek et al., 2014; Laukkonen et al., 2020; Laukkonen & Tangen, 2017; Tian et al., 2017). Still, to our knowledge there is currently no empirical evidence that insight experiences arise when people are learning a new concept (for suggestive evidence, see (Liljedahl, 2005)). Anecdotally though, just like Poincaré, many of us will have experienced episodes where we felt that we suddenly understood a notion: "Now, I get it!". Our study was undertaken to address three questions. First, we tested whether people generally experience insights when learning a new scientific concept. Second, we asked whether these insight experiences are reliable. Indeed, insight experiences may reflect genuine learning progress, in



accordance with the learners introspection, but it is also possible that the subjective sensation of progress experienced in an insight episode is illusory. Third, we hoped to try and leverage the insight experiences reported by our participants in order to gain information about the processes involved in concept learning. The occurrence of insight experiences could indicate that concept learning proceeds in discrete steps, each marked by a new experience of insight; or, alternatively, it could be that the learning processes giving rise to insight experiences operate covertly, except in rare episodes where an insight is triggered. Interestingly, and in line with this second hypothesis, several findings converge to indicate that the processes triggering insights in problem solving tasks are largely inaccessible to introspection. Indeed, people cannot predict when an insight is about to occur (Metcalfe, 1986), and moreover, they have no access to the reasoning that led them to the solution after an insight has occurred (Jameson et al., 1990; Miner & Reder, 1994; Schooler et al., 1993; Metcalfe, 1986).

To address these questions, we developed a paradigm where participants were taught the mathematical concept of geodesic, which generalizes the common notion of a straight line to straight trajectories drawn on curved surfaces (Spivak, 1999). Participants were given 1 to 7 lessons to learn about the properties of geodesics on the sphere. They were then tested on their ability to recognize geodesics on the sphere as well as on various surfaces, and to reason about the properties of geodesics traced on these surfaces. In addition, they assessed their confidence in their own understanding three times during the experiment, and were asked at the end of the session whether they had experienced any insight episode. The study was conducted in a single session, in the lab, allowing us to access participants' objective and subjective progress throughout the whole experiment. We made four predictions. First and foremost, if our paradigm is effective in producing learning, we should find that participants perform better in the post-teaching tests when they study more lessons. Second, if concept learning gives rise to insight experiences, participants should report experiencing insights. Importantly, these reports should be modulated by our experimental manipulation (number of lessons studied): an indication that the experiences reported are induced by the teaching phase, not by the general context of the experiment. Third, if insight experiences reflect key computations in learning processes, those participants who report insights should achieve better levels of objective understanding. Fourth, by studying how insight reports and judgments of confidence relate to performance in various post-tests, we can assess

whether insight experiences and introspective feelings of confidence rely on similar or dissociated mechanisms. Specifically, we predict that some learning achievements may be uniquely related to experiencing insights, after factoring out variations in participants judgments of confidence. The reverse relation may also be true: perhaps judgments of confidence also relate to learning performance, independently from the occurrence of insight experiences. Observing such a pattern of double dissociation would indicate that the mechanisms triggering insight experiences and the mechanisms informing confidence are at least partially different, suggesting the existence of conceptual learning mechanisms that operate covertly and are inaccessible to introspection.ă

## 2. Methods

**2.1. Participants.** Fifty-six adults from the greater Paris area were included in the experiment (40 females, age 18-43 years,  $M = 25.5$  years, exact age missing for 10 participants). All participants had attended high school. In France, students can choose to specialize in the humanities and quit studying mathematics after completing 10th grade; accordingly, some of our participants had received education in mathematics only until 10th grade, while others had received up to 7 additional years of mathematics education (average number of years of education in mathematics after 10th grade:  $M = 3.9$  years, Median = 4 years). Thirteen other persons participated in the experiment but were excluded from the analyses because of their performance on inclusion tests (5 participants for poor performance in planar geometry, 3 participants for good performance in spherical geometry) and/or because of an experimenter error (6).

Two comparison groups (38 adults) also participated in the experiment: in one group, participants received no lessons, in the other, they read only the one-page handout showing large circle figures that served as an introduction to the lessons in the teaching groups. Both groups were included in the initial experimental design to control for consistency of performance across experimental conditions. Indeed, while the analysis of conditions 1, 3, 5, and 7-lessons on performance allows us to see whether our paradigm induces learning, we wanted to ensure that reading at least one lesson constitutes an initialization of that learning. These groups are not included in the main analyses because they do not allow us to study the progression of learning, since these participants did not receive the teaching phase. Nevertheless, the results including these two baseline groups remain virtually unchanged, and are presented in the [Section 5](#).

Inclusion phase	Teaching phase	Test phase	Final phase
Planar geometry	Introduction: great circles	Confidence rating (1)	Insight report
Straight lines on spheres		Subjective efficiency of the teaching phase	
		Straight lines on spheres	
		Straight lines on various surfaces	
		Confidence rating (2)	
		Reasoning about straight lines on spheres and other surfaces	
		Confidence rating (3)	

TABLE 6. List of the tasks administered to the participants in the four phases of the experiment.

**2.2. Material and procedure.** The experiment consisted of four phases: i) an inclusion phase, ii) a teaching phase, iii) a testing phase, and iv) a final phase in which participants were asked to share their experiences of insight. Descriptions bellow are also found in [Section 1](#).

### 2.2.1. Inclusion phase.

**Planar geometry.** This test was adapted from ([Izard et al., 2011](#)). In a short introduction, participants were described an infinite plane on which points and straight lines could be drawn. Straight lines were described as lines that never turn, neither on the left nor on the right, and that continue straight ahead indefinitely. After this introduction, participants were asked a series of twenty illustrated questions about the properties of straight lines. Questions were presented both in writing and orally through an audio recording, and participants ticked their answer (yes or no) on a response sheet. Participants were excluded if they made more than 3 errors pilot work indicated that more than 90% of geometry-educated adults should pass this criterion.

**Straight lines on spheres.** In each trial, participants were presented with a photograph of a sphere (a table tennis ball) with a line drawn on it, and asked to indicate whether the line was "straight" or not. Three types of trials were presented: great circles (straight), small circles (non-straight, but typically judged to be straight by most adults), and non-circles (e.g. wavy line, line looping and crossing itself to form an 8 figure). Each category counted 4 trials, for a total of

12 trials presented in a randomized order. Participants responded by pressing the 'o' key for yes ('oui') or the 'n' key for no ('non'). They were included if they made at least two mistakes on the small circle trials (i.e. they incorrectly judged small circles to be straight lines) or if they made at least two mistakes on the great circle trials (i.e. they incorrectly judged that great circles were not straight).

### 2.2.2. Teaching phase.

**Introduction to great circles.** Participants were given a one-page document describing great circles as circles of the same radius as the sphere, and illustrating great circles traced on spheres at various orientations. A translated version of this document is accessible on the Github repository of the project ([https://github.com/charlusb/Analyses\\_Eurekamaths](https://github.com/charlusb/Analyses_Eurekamaths)).

**Lessons.** Participants were given 1 to 7 lessons to learn about straight lines in spherical geometry. All the lessons explained that straight lines on the sphere correspond to great circles, and not to small circles (synthesized version in [Table 3](#)). Translated version of the 7 different lessons are also accessible on the Github repository of the project ([https://github.com/charlusb/Analyses\\_Eurekamaths](https://github.com/charlusb/Analyses_Eurekamaths)). To encourage the participants to study the lessons in depth, we asked them to write a summary of each lesson just after reading it. We also asked them to rate whether they found the lesson convincing, on a scale graded from 0 to 10. These judgments were collected twice: a first time after reading each lesson, and a second time for all lessons together (except for the group who received only one lesson to read, as this would have resulted in asking the exact same question twice in a row).

This and other judgments on the teaching phase (see Subjective efficiency of the teaching phase below) were included as an effort to develop a measure of the dynamics of the learning process.

### 2.2.3. Test phase.

**Confidence rating.** Participants were asked to rate how much they felt they understood the notion of straight line, using a scale graded from 0 to 10 (first confidence rating).

**Straight lines on spheres.** This test was identical to the spherical geometry inclusion test [Figure 3](#).

**Straight lines on various surfaces.** Participants were presented with lines drawn on four different surfaces: cone (8 trials), cylinder (6 trials), cube (8 trials) and torus (4 trials). Each trial displayed two photographs showing a front and a back view of a surface on which a line had been drawn. Participants were asked to judge whether the line presented was straight or not and

indicated their answer by keypress. The task crossed the two variables of straightness (straight, not straight), and intersection (planar, non-planar): 4 trials presented non-straight lines that did not correspond to planar intersections (like non-circle lines on the sphere), 3 trials presented straight lines that corresponded to planar intersections (like great circles on the sphere), 10 trials presented non-straight lines that corresponded to planar intersections (like small circles on the sphere), and 9 trials presented straight lines that did not correspond to planar intersections (there are no such lines on the sphere, but these can exist on other surfaces). Trials were presented in a random order.

One of the torus trials showed a line that is ambiguous in mathematics: a circle resulting from the intersection of the torus with the tangent plane that just touches it on its summit. This curve is considered a geodesic under some mathematical descriptions - the "flat torus" (Borrelli et al., 2013) -, but it does not correspond to a straight line according to the criteria given in our lessons. This trial was thus analyzed as a planar non-straight line. The results presented here do not change if this trial is analyzed as showing a planar straight line, or if it is removed from the analyses.

**Confidence rating.** Participants were asked to indicate how much they felt they understood the notion of generalized straight line on a 0-10 graduated scale (second confidence rating).

**Reasoning about straight lines on spheres and other surfaces.** This test consisted in a list of mathematical assertions, which participants judged to be true or false: eight assertions about straight lines drawn on spheres, followed by eight assertions about straight lines drawn on various surfaces. The assertions were presented in a fixed order, on paper. Participants were given written definitions for the terms "parallel" and "perpendicular", which appeared in some of the assertions. They answered by ticking one of four response options for each assertion: 'yes - certain', 'yes - uncertain', 'no - uncertain', and 'no - certain'.

**Confidence rating.** Participants indicated how much they felt they understood the notion of generalized straight line on a 0-10 graduated scale (third confidence rating).

**2.2.4. Final phase.** The final phase was administered by the experimenter who had been in charge of administering the teaching phase.

**Insight reports.** The final part aimed at measuring whether people experienced insights in our experiment. Participants were first given a description of the sensations associated with insight experiences (adapted from (Danek & Wiley, 2017): a feeling of sudden understanding that comes unexpectedly, and is associated with feelings of relief and certainty. Participants indicated

whether they experienced such episodes at all during the course of the experiment (yes or no) - this answer was used as a measure of insight report. They were then presented with vignettes illustrating the different phases of the experiment (one vignette for each lesson, pre- and post-teaching test, confidence rating, and definition of straight lines), and were asked to indicate when in the experiment they had experienced insights. At the end of the experiment, a few participants in the 3- and 5-lesson groups were asked about the contents of the insights reported.

### 3. Analyses

We conducted analyses to address four main questions. First, we tested whether participants' objective performance varied as a function of the number of lessons studied. Observing effects of the teaching condition would show that our manipulation was successful in inducing learning, a *sine-qua-non* requisite for our study. Second, we looked for evidence that participants may have experienced insights, and tested whether these experiences were related to the number of lessons studied. Finding an effect of the number of lessons on insight reports would ensure that the insight experiences reported are triggered by the teaching phase, not by e.g. the general context of the experiment. Third, we tested whether reports of insight experiences were associated with a better level of objective understanding. This analysis allowed us to test whether experiences of insights are genuinely related to learning progress, or whether the experience of sudden understanding that is associated with insights is better thought as an illusion. Fourth and lastly, we tested whether insight reports were related to participants ratings of confidence, and analyzed how insight reports and ratings of confidence related to performance in the different subtests. Finding that insight reports and ratings of confidence relate to performance in different subtests would indicate that the mechanisms giving rise to insights are at least partially dissociated from the mechanisms informing peoples introspective judgments, supporting the hypothesis that some conceptual learning processes operate covertly and remain inaccessible to introspection. Analyses were conducted in R using the packages *afex* and *emmeans*. Scripts are available on the Github repository of the project ([https://github.com/charlusb/Analyses\\_Eurekamaths](https://github.com/charlusb/Analyses_Eurekamaths)). Unless otherwise stated, all analyses included a numerical variable to account for participants' education in mathematics, corresponding to the number of years studying mathematics after 10th grade. For repeated measures models we included a random effect of participant (function *mixed* in *afex*). The  $\alpha$  level was set at .05. Significant interactions involving a numerical variable (e.g. number of lessons) were explored by computing linear trends by condition (function *emtrends*

in emmeans). Interactions between categorical variables were explored by computing contrasts within each condition (function emmeans). Holms procedure was used to control for multiple comparisons when exploring interactions.

**3.1. Effect of the number of lessons on accuracy.** For the sake of simplicity, and to reduce the number of analyses performed, all three tests of objective understanding were analyzed together. Accuracy was entered in a logistic mixed model analysis, with a random effect for participant, a categorical variable for test condition (total of 9 test conditions corresponding to 3 conditions in the sphere straight line test: non-circle lines, great circles, small circles; 4 conditions in the surfaces straight line test: non-planar non-straight, planar straight, planar non-straight, non-planar straight; and 2 conditions in the reasoning test: sphere, surfaces), numerical variables for teaching condition (corresponding to the number of lessons studied, from 1 to 7) and education in mathematics (number of years studying mathematics after 10th grade, from 0 to 7), as well as interactions between test condition and number of lessons, and between test condition and education in mathematics. If our manipulation was successful, we expected this analysis to yield a significant interaction between test condition and number of lessons, as participants should show different amounts of progress across test conditions. For example, little progress was expected in judging that wavy lines on the sphere are not straight since participants performed very well on this task already before the teaching phase. Moreover, progress was expected to be larger on test conditions that could be solved by direct application of the content of the teaching phase (e.g. judging that small circles drawn on a sphere are not straight) than on the conditions that required further inferences (e.g. reasoning about straight lines on arbitrary surfaces).

**3.2. Insight Experiences.** To analyze whether concept learning triggers insight experiences, we tested whether the occurrence of insights was modulated by the teaching condition in a logistic regression with two numerical variables for the number of lessons studied and participants' years of education in mathematics. An effect of the number of lessons in this analysis would indicate that insight reports are modulated by our experimental manipulation, and do not simply reflect differences in the personality of our participants, or in their level in mathematics.

**3.3. Relation between Insight Experiences and Learning.** We next tested whether the participants who experienced insights reached a better level of understanding, as indicated by better performance in the post-teaching tests. To do so, we used a logistic mixed model on accuracy with a random effect for participant and fixed effects for test condition (categorical variable with

9 levels as above), insight report (dummy variable indicating whether the participant reported experiencing insights or not), and an interaction between test condition and insight report. We then conducted a second version of this analysis with additional variables for number of lessons and years of education in mathematics and their interaction with test condition. This second analysis is more conservative and detects relations between insight and performance that cannot be explained through the influence of the teaching condition or participants education in mathematics. However, the first analysis without covariates is potentially informative as well: if teaching condition and education in mathematics constitute the main source of variance between participants, introducing these variables as covariates may drastically reduce variability and render relations between insight and performance impossible to detect.

**3.4. Relation between insight and introspection.** The last series of analyses aimed at testing whether confidence to have apprehended a notion and experiences of insights reflect similar or different learning processes. To approach this question, we first tested whether our different subjective reports were correlated to each other. Pairwise comparisons were conducted between four measures: the three ratings of confidence collected throughout the test phase, and the insight reports collected at the end of the experiment. For each comparison, we conducted two Spearman correlation analyses, first without covariates, and second with number of lessons and years of education in mathematics as covariates. In each version of the analysis, p-values were corrected for multiple comparisons using Holms procedure. Second, we tested whether insight reports and confidence ratings were related to learning the same aspects of the target mathematical concept. To do so, we conducted a logistic mixed model analysis on post-teaching test accuracy with a random effect for participant and fixed effects for test condition (categorical variable with 9 levels as above), insight report, confidence ratings, as well as interactions of insight report and confidence ratings with test condition. Again, two versions of this analysis were conducted: once without variables accounting for number of lessons and years of education in mathematics, and once including these variables and their interaction with test condition as fixed effects. Since the correlation analysis found all three ratings of confidence to be highly correlated, here we used the mean of participants' three ratings.

## 4. Results

**4.1. Effect of the number of lessons on performance.** Performance varied across test conditions (main effect of test condition,  $p < .001$  (Table 7), ranging from  $M = 33.1\%$  (straight line



judgments: straight non-planar lines on various surfaces) to  $M = 97.3\%$  (straight line judgments: great circles on spheres). In line with our expectations, the teaching phase manipulation had an impact on participants' objective performance on post-teaching tests, as attested by a significant interaction between test condition and number of lessons. To explore the differential effect of the number of lessons across test conditions, we computed linear trends by number of lessons for each test condition (Figure 8). We found a positive effect of the number of lessons on participants' ability to identify small circles on the sphere as non-straight lines ( $\beta = 0.51$ , 95% CI = [0.27, 0.75],  $p < .001$ ), as well as, more generally, on their ability to identify planar non-straight lines on various surfaces (same type of line as small circles on the sphere,  $\beta = 0.16$ , 95% CI = [0.01, 0.32],  $p = .017$ ). In addition, participants who studied more lessons also performed better when asked to draw inferences about the properties of straight lines on the sphere in the reasoning test ( $\beta = 0.18$ , 95% CI = [0.02, 0.35],  $p = .014$ ). Linear trends were non-significant in the other test conditions ( $p_s > .27$ ).

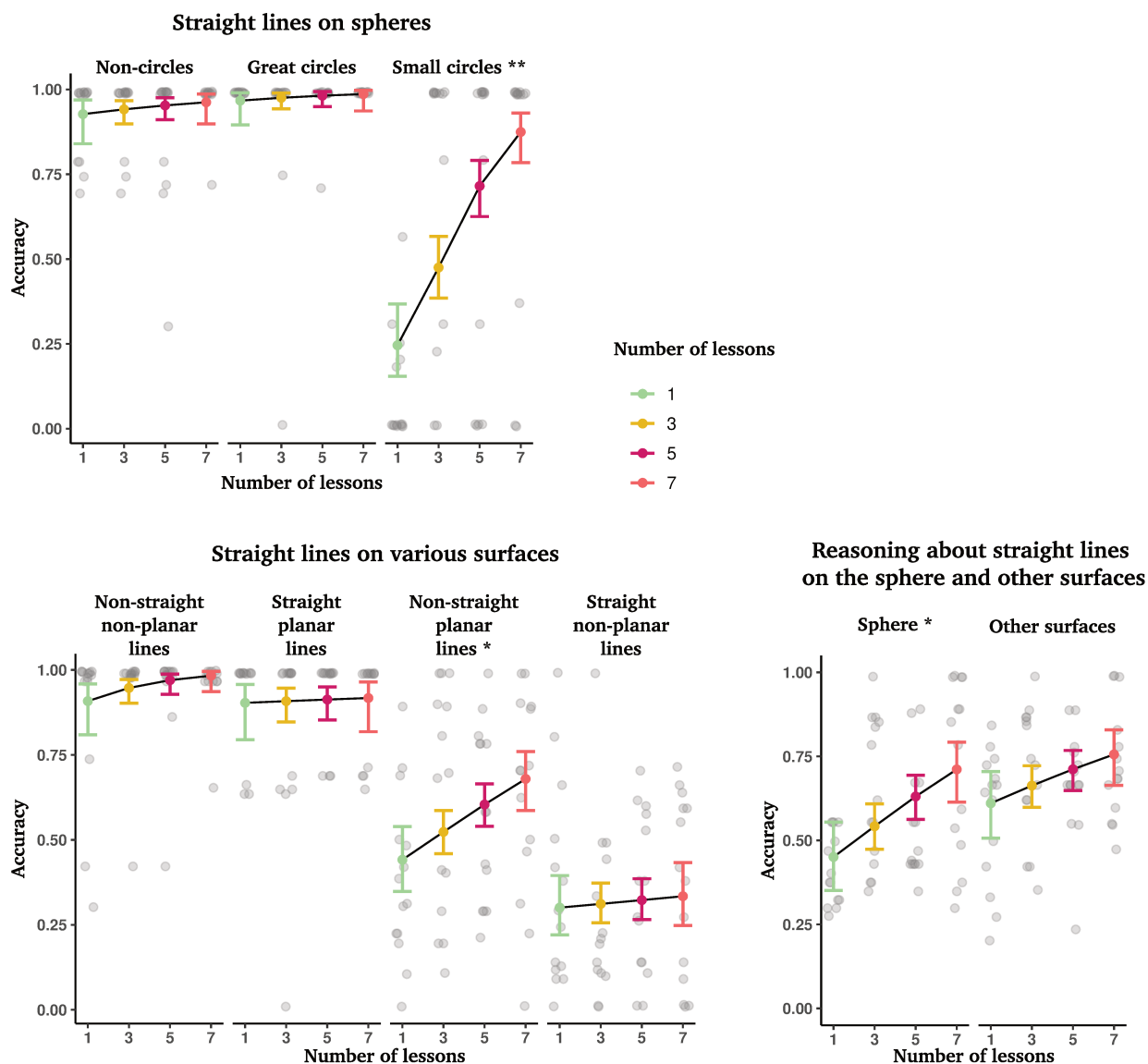


FIGURE 8. Effect of the number of lessons on performance for each test condition. Colored dots and plain lines display the predictions of the logistic mixed model. Error bars represent 95% CI. Transparent dots show the performance of individual participants, corrected for years of mathematics education. P-values from a post-hoc exploration of the linear trends by test condition, holm-corrected for multiple comparisons. \*  $p < .05$ , \*\*  $p < .01$ .

	df	$\chi^2$	p
<b>Test condition</b>	<b>8</b>	<b>60.5</b>	<b>&lt;.001</b>
Number of lessons	1	0.8	.38
Education in Mathematics	1	0.6	.43
<b>Test condition*Number of lessons</b>	<b>6</b>	<b>36.2</b>	<b>&lt;.001</b>
Test condition*Education in mathematics	8	11	.20

TABLE 7. Mixed model analysis of the effects of number of lessons on accuracy. LogLik = -1536.2, Random effect (participant): variance = 0.36.

**4.2. Effect of the number of lessons on insight reports.** Thirty-four of our fifty-six participants reported experiencing an insight (61%). A logistic regression analysis (Table 8, Figure 9) revealed that insights reports were influenced by the number of lessons the participants were given to read but not by their education in mathematics.

	df	$\chi^2$	p
<b>Number of lessons</b>	<b>7.8</b>	<b>1</b>	<b>.005</b>
Education in mathematics	0.7	1	.42

TABLE 8. Effects of the number of lessons on insight reports, as assessed by a logistic regression.

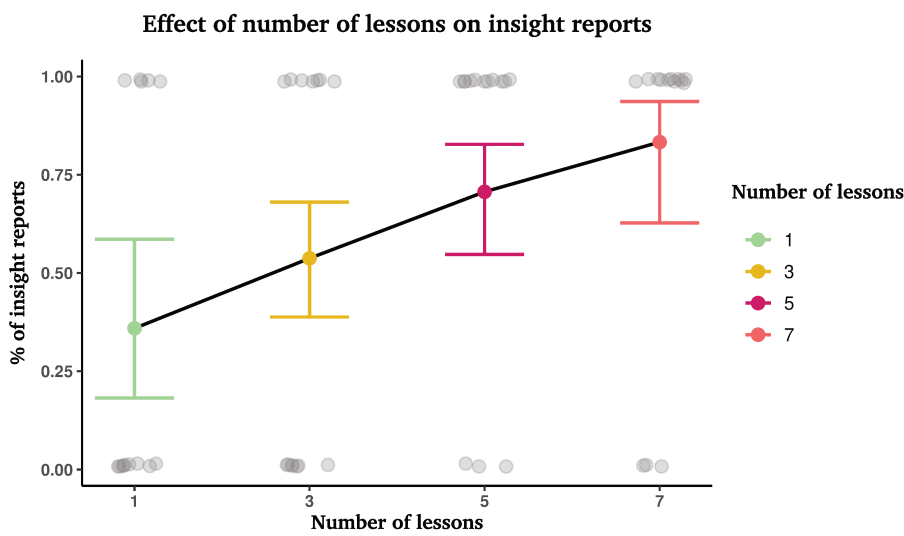


FIGURE 9. Effect of the number of lessons on insight reports. Colored dots and plain lines display the predictions of the logistic regression model. Error bars represent 95% CI. Transparent grey dots show the responses of individual participants, corrected for years of mathematics education.

**4.3. Relation between insight report and performance.** The relation between insight report and performance was assessed in two mixed model analyses with and without variables accounting for number of lessons and years of education in mathematics. Both analyses identified a significant interaction between a variable measuring whether participants reported insights and the test condition variable (Table 9), indicating that participants who did vs. did not report experiencing insights differed in their performance on some of the test conditions. Exploring these interactions revealed that participants who experienced insights were more likely to identify lines that are straight despite being not planar in the various surfaces straight line testã the type of line

that does not exist on the sphere (simple model:  $\beta = 1.06$ , 95% CI = [0.27, 1.86],  $p = .002$ ; model with covariates:  $\beta = 1.22$ , 95% CI = [0.39, 2.05],  $p < .001$ ). In addition, there was a significant effect of insight experiences on participants' accuracy at rejecting small circles as not straight on the sphere, but only in the model that did not account for number of lessons or years of education in mathematics (simple model:  $\beta = 0.96$ , 95% CI = [0, 1.92],  $p = .044$ ; model with covariates:  $\beta = 0.39$ , 95% CI = [-0.67, 1.45],  $p = 1$ ). In all the other test conditions, the linear trends did not reach significance (simple model  $ps = 1.0$ ; model with covariates:  $ps > .62$ ).

	df	$\chi^2$	p	df	$\chi^2$	p
<b>Test condition</b>	<b>8</b>	<b>406.6</b>	<b>&lt;.001</b>	<b>8</b>	<b>78.5</b>	<b>&lt;.001</b>
Insight report	1	0.1	.75	1	0	.9
Number of lessons				1	0.6	.45
Education in mathematics				1	0.6	.43
<b>Test condition*Insight report</b>	<b>8</b>	<b>27.1</b>	<b>&lt;.001</b>	<b>8</b>	<b>34.4</b>	<b>&lt;.001</b>
<b>Test condition*Number of lessons</b>				<b>8</b>	<b>43</b>	<b>&lt;.001</b>
Test condition*Education in mathematics				8	12	.15

TABLE 9. The two mixed models analyzing the relation between insight experiences and learning performance. *On the right, in italics: model accounting for years of education in mathematics and number of lessons.* Model without covariates: Loglik = -1551.5, random effect (participant): variance = 0.45. *Model with covariates: LogLik -1518.5, Random effect (participant): variance = 0.36.*

#### 4.4. Relation between insight experiences, confidence and performance.

**4.4.1. Correlation tests.** Tests of the correlations between participants' three ratings of confidence and their report of insight experiences were conducted twice, once without covariates, and once with number of lessons and years of education in mathematics as covariates. In the two versions of the analysis, the three ratings of confidence were strongly correlated to each other (Table 10). In contrast, ratings of confidence did not correlate with insight reports.

	Confidence rating 1	Confidence rating 2	Confidence rating 3	Insight report
Confidence rating 1	X	$\rho(54) = .6$ $p < .001$	$\rho(53) = .6$ $p < .001$	$\rho(54) = .27$ $p = .13$
Confidence rating 2	$\rho(52) = .57$ $p < .001$	X	$\rho(53) = .85$ $p < .001$	$\rho(54) = .19$ $p = .33$
Confidence rating 3	$\rho(51) = .6$ $p < .001$	$\rho(51) = .85$ $p < .001$	X	$\rho(53) = .17$ $p = .33$
Insight	$\rho(52) = .15$ $p = .81$	$\rho(52) = .14$ $p = .81$	$\rho(51) = .15$ $p = .81$	X

TABLE 10. Spearman's  $\rho$  coefficients and p-values for pairwise correlation tests. Above diagonal: zero-order correlation, below: with number of lessons and years of education in mathematics as covariates. Significant correlations are highlighted in bold. All p-values were corrected for multiple comparisons using Holms method (applied separately to the analyses with and without covariates). Note that the third rating of confidence was missing for one participant in the 1-lesson group, hence the difference in degrees of freedom. Confidence rating 1: measured just after participants completed the teaching phase; Confidence rating 2: measured after the various surfaces straight lines test; Confidence rating 3: measured after the reasoning test.

**4.4.2. Relation between insight experiences, confidence and performance.** Lastly, we tested whether insight experiences and confidence in one's understanding were related to learning the same aspects of the notion of generalized straight lines. Again, we conducted two mixed model analyses, one with and one without variables accounting for number of lessons and years of education in mathematics. The two versions of the model yielded significant interactions between insight report and test condition, as well as between confidence and test condition (Table 11). Exploring the interaction between insight and test condition revealed again a positive relation between insight report and performance in the condition from the various surfaces test that has no equivalent on the sphere (non-planar straight lines, Figure 10), indicating that this relation arises independently of participants' confidence in their own understanding (simple model:  $\beta=1.15$ , 95% CI = [0.33, 1.97],  $p = .001$ ; model with covariates  $\beta = 1.26$ , 95% CI = [0.43, 2.10],  $p < .001$ , Table 11). None of the other test conditions showed a significant association of performance with insight report (model without covariates:  $p_s > .21$ ; model with covariates:  $p_s > .66$ ). Exploring the interaction between test condition and confidence failed to reveal a specific relation between confidence and performance in any of our single test conditions (model without covariates:  $p_s > .10$ ; model with covariates:  $p_s > .30$ ).

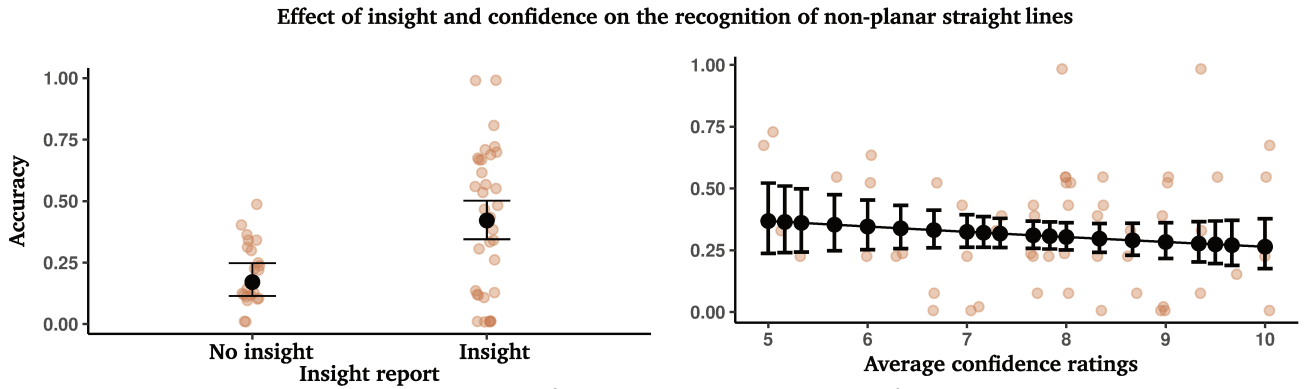


FIGURE 10. Relation between performance, insight and confidence in the condition that has no equivalent on the sphere (straight non-planar lines). On the left, predicted performance for participants who did vs. did not report insight experiences, and individual participants' performance corrected for years of mathematics education, number of lessons, and confidence level. On the right, predicted performance for participants who reported different levels of confidence, and individual participants' performance corrected for years of mathematics education, number of lessons and insight report. Error bars are 95% confidence intervals.

	df	$\chi^2$	p	df	$\chi^2$	p
<b>Test Condition</b>	<b>8</b>	<b>53.9</b>	<b>&lt;.001</b>	<b>8</b>	<b>55.6</b>	<b>&lt;.001</b>
Insight report	1	0	.90	1	0	.93
Confidence in understanding	1	2.7	.10	1	2.2	.14
Number of lessons				1	0.3	.60
Education in mathematics				1	0.6	.44
<b>Test condition*Insight report</b>	<b>8</b>	<b>26.1</b>	<b>.001</b>	<b>8</b>	<b>34.2</b>	<b>&lt;.001</b>
<b>Test condition*Confidence in understanding</b>	<b>8</b>	<b>33</b>	<b>&lt;.001</b>	<b>8</b>	<b>25.9</b>	<b>.001</b>
<b>Test condition*Number of lessons</b>				<b>8</b>	<b>36.1</b>	<b>&lt;.001</b>
Test condition*Education in mathematics				8	10.6	.22

TABLE 11. The two mixed models jointly analyzing the effect of insight experiences and confidence on performance. In italics, on the right: model with covariates for number of lessons and education in mathematics. Model without covariates: LogLik = -1534.9, Random effect (participant): variance = 0.46. Model with covariates: LogLik = -1505.5, Random effect (participant): variance = 0.36.

**4.5. Discussion.** Our study had four goals. First, we aimed to conceive a short, one-session paradigm to study concept learning in the lab. Second, we asked whether learning the mathematical concept introduced would trigger experiences of insights in our participants. Third, we tested whether insight experiences were reliable, in the sense that they were associated with better levels of learning. Fourth and lastly, we studied the relation between learning performance,

insight experiences, and participants introspective assessment of their own understanding. More specifically, we analyzed whether reports of insight experiences and confidence ratings reflected the acquisition of the same or different aspects of the target concept, as a way to assess whether these reports are informed by similar or dissociated learning processes. Below we present our findings and conclusions for each of these four questions.

**4.5.1. Learning a new mathematical concept in the lab.** Participants were invited to study one to seven lessons introducing a mathematical concept generalizing the common notion of a planar straight line to straight lines drawn on curved surfaces (concept of a geodesic). All lessons focused on the sphere and appealed to everyday experiences to explain that great circles drawn on spheres correspond to straight lines, while small circles do not which is counterintuitive. We then assessed whether participants had learned the target concept in three tests requiring an increasing amount of generalization from the teaching phase. In the first post-teaching test, participants were asked to judge whether lines drawn on a sphere were straight or not—so they simply needed to recall the information taught in the lessons. The second post-teaching test requested participants to identify straight lines on various non-sphere surfaces: to solve this task, they needed to consider whether the lines presented fitted the criteria introduced in the lessons. In particular, this second test presented a kind of line (non-planar curves tracing straight trajectories) that does not exist on the sphere, along with other types of lines that can be realized on the sphere: non-planar non-straight lines (corresponding to non-circles on the sphere), planar straight lines (corresponding to great circles), and planar non-straight lines (corresponding to small circles). Lastly, in the third test participants were invited to think about the properties of straight lines either when drawn on a sphere, or on any arbitrary surface. The first part of this last test thus required participants to draw inferences from the information presented in the lessons and reason about the properties of great circles on the sphere, and the second part to reason more broadly about the possible configurations for straight lines as the underlying surface varies. Our results provide evidence that our teaching phase was effective, and participants learned: reading more lessons led to better performance in several of our post-teaching test conditions. Furthermore, participants post-test performance showed two characteristic signatures of conceptual learning. First, learning was difficult, as indicated by the positive linear effects of the number of lessons on test performance. These effects show that learning was not completed after studying the first lesson, and as all the lessons had the same mathematical content (great circles are straight, small circles are not straight), they show that participants benefited from repeated presentations of the

same information. Strikingly, repeating information proved beneficial even in a test where participants simply had to recall the information presented in lessons, i.e. when judging that small circles drawn on spheres are not straight. As a second signature of conceptual learning, the content learned was inferentially rich: participants did not only memorize the information presented in the lessons but were also able to draw inferences from this information. Indeed, participants who received more lessons to study had a better performance in test conditions where they either needed to generalize the notion of straight line to various surfaces, beyond the example of the sphere (condition presenting non-straight planar lines drawn on non-sphere surfaces, equivalent to small circles drawn on spheres), or needed to reason about the properties of straight lines on the sphere.

In this reasoning test, in particular, we found that our teaching phase enabled participants to draw non trivial inferences about the properties of straight lines in spherical geometry, and notably to realize that two straight lines drawn on a sphere can never be parallel: they necessarily cross. This property is highly counterintuitive for geometry-educated people as well as for people without any formal education in geometry (Izard et al., 2011). When tested on a questionnaire that inspired our planar geometry inclusion test, pretty much all U.S. adults, French teenagers, or Mundurucu people from the Amazon judged that it was possible to find parallel lines on the plane and also on the sphere (agreement on the existence of parallel lines on the sphere was above 90% in all groups). Such strongly entrenched intuitions about parallel lines perhaps even contributed to shaping the History of Mathematics. Hence, during nearly 2000 years, mathematicians attempted to demonstrate Euclid's Fifth Postulate on parallel lines from the other simpler postulates and axioms; until Gauss, Bolyai and Lobachevsky finally realized that this enterprise had started on wrong premises, as it is possible to define geometries that are perfectly coherent with the simpler axioms but nonetheless violate the postulate on parallels (Greenberg, 1993). The geometry defined by great circles on the sphere is an example of one such coherent "non-Euclidean" geometry. In that context, it is impressive to observe that a considerable proportion of our participants questioned or even refuted the existence of parallel lines on the sphere (45.5% of negative responses for two assertions of the reasoning test claiming that parallel lines exist on the sphere). Again, the teaching phase played a role in this understanding (linear trend of number of



lessons for these two assertions,  $\beta = 0.53$ , 95% CI = [0.17, 1.23],  $p = .02$ , controlling for years of education in mathematics).

In other test conditions, we did not observe any impact of the teaching phase on performance. Some of these test conditions were easy and appeared compatible with peoples spontaneous intuitions. In our pre-teaching inclusion test, participants generally agreed that non-circle lines drawn on a sphere are not straight (average performance 94%), or that great circles are straight (average performance 84%). Accordingly, in post-teaching tests, we found that all participants reached a good performance in these two conditions, independently of the number of lessons studied (average performance: non-circle lines 94%, great circles 97%). Performance was also good and was not affected by the number of lessons for non-planar non-straight lines (94%) and planar straight lines (90%) drawn on various surfaces—the equivalent of respectively non-circles and great circles drawn on a sphere. In the two last test conditions, the absence of an effect of the teaching phase was associated with lower performance, suggesting that the teaching we provided was not sufficient to solve these tasks. These conditions require a high level of generalization with respect to the information presented in the lessons. For example, on some surfaces it is possible to find straight lines that are not planar; however this is not possible on the sphere, and consequently, this type of line was not exemplified in the lessons. Accordingly, participants generally failed to recognize this type of straight line when tested on various surfaces (average performance 33%), and the presentation of several lessons did not seem to help. Second, the number of lessons presented had little impact on participants reasoning about straight lines on arbitrary surfaces (average performance 67%). This last test condition included questions that required a high level of generalization (e.g. thinking about a cone to find an example of a straight line that intersects itself), as well as very intuitive questions that could be solved by thinking about the plane. Both these very easy and very hard trials presumably contributed to reducing the effects of the teaching condition in this test condition. In summary, analyses of performance indicated that our participants benefited from the lessons presented and learned at least some aspects of the concept of generalized straight line. Our paradigm thus succeeded in creating conditions for studying concept learning in a lab experiment.

**4.5.2. Concept learning triggers insight experiences.** As a second conclusion, our experiment provides evidence that learning a new concept gives rise to insight experiences. At the end of

the session, participants were asked whether they had experienced any insights: episodes where an idea had come to them suddenly, accompanied with a sensation of certainty. The description provided to our participants was taken from a study assessing the insights that arise as people try to decipher magic tricks (Danek et al., 2014), a problem closer in structure to the problem solving tasks classically used in the insight literature. After reading this description, a little over half of our participants reported experiencing insight episodes during the course of our experiment (61%). Together with recent studies (Bowden et al., 2005; Danek et al., 2014; Laukkonen et al., 2020; Laukkonen & Tangen, 2017; Tian et al., 2017; Webb et al., 2016), our findings thus contribute to extending the range of situations known to give rise to insight experiences. Our findings are in line with a recent study, which already suggested that students can experience insights when learning mathematics (Liljedahl, 2005). This previous study however suffered from several shortcomings, raising doubts about the reliability of its results. First, reports of insights were delayed in time until months after the class had finished. Second, and most importantly, students wrote little narratives describing their insights in exchange of credits and writing about an insight episode constituted an alternative to solving a math problem to earn these credits. Given that the population of students involved in the study (education majors) reported high levels of math anxiety, this setting may have constituted a strong incentive for students to amplify or even fabricate false memories of insight episodes.

Importantly, in our study we incorporated several checks to ensure that the insights were not fabricated by participants. First, participants were not rewarded for reporting insights, and our instructions acknowledged that it is fully possible to learn a mathematical notion without experiencing any insight. Most crucially, several of our analyses provide evidence that the insights reported were not fabricated. First, we found evidence that participants' reports of insight episodes were modulated by our experimental manipulation, in that the groups receiving more lessons were more likely to report experiencing insights. This finding attests that reports did not simply reflect variations in the personalities of our participants, or in their taste for mathematics. Second, insight experiences were related to performance in the post-teaching tests; and this relation was modulated across test conditions, in a pattern that held even when controlling for the number of lessons studied, years of education in mathematics, and participants' confidence

ratings of their own understanding. These findings allow us to exclude several deflationary explanations. For example, it is not the case that participants confabulated and reported imaginary insights simply because they had studied many lessons and thought they ought to understand the notion taught very well, or because they had solved the post-teaching tests easily and felt confident about their own understanding. In summary, our findings thus constitute the first robust empirical evidence that learning mathematical concepts can trigger experiences of insights.

**4.5.3. Insight experiences reliably signal learning progresses.** The relation between insight experiences and participants performance in specific subtests, described above, provide evidence that the insights experienced by our participants reflect the functioning of processes integrating the information received in the lessons: insights tended to be triggered when these processes reached a specific state, characterized by a profile of performance across the test conditions; and the probability to reach this state increased when participants received more lessons to study. More specifically, we found that insights predicted performance in one type of generalization test: participants who experienced insights improved in their ability to identify straight lines drawn on new surfaces. In contrast, insights were not related to performance in a task testing a different kind of generalization: drawing inferences from the information taught to reason about the properties of straight lines on the sphere. This pattern suggests that insights were triggered as participants progressed in their understanding of the definitional properties of straight lines identifying the core properties of this concept. Insights were not triggered however when participants reflected on the consequences of adopting this new definition identifying the inferential role of the concept of straight line. In summary, our findings thus provide evidence that learning a new mathematical concept gives rise to insight experiences; and that these insights reflect the functioning of concept learning processes, and especially processes identifying the definitional properties of a concept.

**4.5.4. Dissociations between insight experiences and confidence.** Our experiment also aimed at evaluating whether some concept learning processes (the processes susceptible to trigger insight experiences) may operate covertly and remain inaccessible to introspection. To that avail, besides asking participants to report about their insight experiences, we also asked them to introspect and evaluate their own understanding of the concept of straight line several times during the course of the experiment. Confidence ratings were dissociated from reports of insight experiences in our findings, at two levels. First, there was no correlation between insight reports and confidence ratings. Second, and most importantly, insight reports and confidence ratings

were related to different patterns of performance in the post-teaching tests. In particular, the relation between insights and generalization described above held even after factoring out variations in participants confidence in their understanding: learning abstract definitional properties of generalized straight lines thus involved processes that triggered experiences of insights, yet did not inform participants introspective evaluation of their own understanding. Conversely, we also found that confidence ratings were uniquely associated with a change in performance, independently of insight reports. In detail, however, none of our test conditions showed a specific relation to the confidence ratings, such that it is hard to know whether the significant relation between confidence and performance reflects conceptual learning processes or processes implementing more superficial changes, such as changes in response strategies.

In summary, our findings establish the existence of dissociated processes respectively triggering insight experiences or informing peoples introspective judgments. However, only the processes associated with insights were clearly implicated in conceptual learning. The absence of a correlation between reports of insight experiences and confidence ratings may seem surprising, as this finding apparently contradicts the established relation between insight experiences and confidence, e.g. (Laukkonen et al., 2020). Our own instructions indeed emphasized confidence (feeling "certain") as one key dimension of insight experiences. It is however important to note that in our experiment, participants could sometimes experience an insight and feel certain about an idea without feeling any more confident about their understanding of straight lines. For example, in the informal debriefing that followed the experiment, one of our participants described an insight episode where she had suddenly realized that she did not understand straight lines. This episode was experienced as an insight because the participant felt suddenly certain of her own ignorance, but she certainly did not feel any more confident about her own understanding of straight lines.

This example stands as an exception, however: the positive relation observed between insight reports and performance indeed suggests that most insight episodes did contain information to advance peoples understanding of straight lines. It is thus possible that most insights led participants to feel that they had progressed in their understanding of the concept of straight line. Perhaps we failed to observe a correlation between insights and confidence because the sensation

of certainty associated with insights was transient, and we did not measure participants confidence at the very moment where insights occurred. Suggestively, we found that confidence was highest when measured just after the teaching phase (linear mixed model on confidence, main effect of measurement time,  $F(2,120)=4.1$ ,  $p=.019$ ), i.e. when most insights had just occurred (when asked to report when exactly they had experienced insights, participants identified a total of 65 episodes, 44 of which occurred during the inclusion or teaching phase) and participants had not yet confronted their understanding to the generalization tests. Knowledge that has been gained by insight may be particularly susceptible to interference from further testing: just like participants in problem solving tasks cannot describe their reasoning after having experienced an insight (Jameson et al., 1990; Miner & Reder, 1994; Schooler et al., 1993; Schwartz & Metcalfe, 1992), in concept learning tasks insights may convey knowledge without conscious access to any epistemic justification. Hence, while learners may be able to generalize their knowledge to new situations after an insight, doing so may lead them to realize that they cannot explain why these inferences are founded, and thus lose confidence in their understanding.

In terms of mechanisms, our findings suggest that conceptual learning may involve an interplay between progressive learning processes operating outside the scope of consciousness, and consciousness acting as a discrete filter for access to learned information (for a model presenting consciousness as a discrete filter on perceived information, see (Dehaene et al., 2003; Dehaene, 2014)). Several theories have described conceptual learning as a progressive process, e.g. (Bonawitz et al., 2012; Gopnik & Wellman, 2012; Ohlsson, 2009): according to these theories, we constantly evaluate the coherence of our concepts in the face of the information we receive, and we engage in conceptual change when a competing concept overcomes the current concept in coherence. Insights could correspond to key computational steps in these models: they could be triggered for example when a competing representation reaches a certain threshold (insight about a new idea), or when the current representation drops to a floor level (insight about one's ignorance). Under this view, while covert learning may be progressive, important steps of conceptual change would necessarily be accompanied by the experience of an insight. Interestingly, performance in our most difficult test of straight line categorization (non-planar straight lines on non-sphere surfaces) appears coherent with this suggestion: experiencing an insight seemed to be a necessary precondition to succeed in this condition (see Figure 10).

**4.5.5. Conclusion.** We created a novel paradigm to study conceptual learning in a single laboratory session and analyzed participants reports of insight experiences. More than half of our participants reported experiencing "Eureka" moments during the experimental session; and these episodes participated in the learning process, in the sense that participants who experienced insights hold a more accurate and more generalizable representation of the target concept than those who did not experience insights. Our findings thus provide evidence that insight experiences reflect the functioning of the processes at play when learning difficult mathematical concepts. It should be noted that concept learning differs fundamentally from the situations traditionally associated with insights: here, just like when pupils learn mathematics at school, participants were provided with all the necessary information about the target concept in an explicit form. Hence, a priori, participants did not need to engage in search and should not fall in an impasse in our task, two aspects thought to be crucial for insights to occur (Ansburg & Dominowski, 2000; Knoblich et al., 1999; Ohlsson, 1992; Mayer, 1992; Schooler et al., 1993). Our findings raise several questions. First, are the insight experiences observed in the contexts of concept learning or problem solving qualitatively different, or do they reflect similar psychological processes? Second, if all insight experiences turn out to indicate the termination of a search process, what is the nature of the search involved in conceptual learning, when all the necessary information has been provided explicitly by a teacher? Third, are insights experienced when learning other kinds of material, besides mathematical or scientific concepts, and what kind of learning material do or do not give rise to insights? Answering these questions could advance our understanding of learning, and of the nature of insight experiences.

## 5. Supplementary analyses including participants without teaching phase

### 5.1. Effect of the number of lessons on performance.

Factor	df	$\chi^2$	p
<b>Test condition</b>	<b>8</b>	<b>134.7</b>	<b>&lt; .001</b>
Number of lessons	1	0.8	.37
Education in Mathematics	1	0.3	.56
<b>Test condition*Number of lessons</b>	<b>8</b>	<b>51.9</b>	<b>&lt; .001</b>
<b>Test condition*Education in mathematics</b>	<b>8</b>	<b>19</b>	<b>.015</b>

TABLE 12. Logistic mixed model analysis of the effect of number of lessons on accuracy. LogLik = -2395.5, Random effect (participant): variance = 0.27. Significant effects are highlighted in bold.

To explore the differential effect of the number of lessons across test conditions, we computed linear trends by number of lessons for each test condition. We found a positive effect of the number of lessons on participants' ability to identify small circles on the sphere as non-straight lines ( $\beta = 0.4$ , 95% CI = [0.24, 0.56],  $p < .001$ ), as well as, more generally, on their ability to identify planar non-straight lines on various surfaces (same type of line as small circles on the sphere,  $\beta = 0.09$ , 95% CI = [0, 0.19],  $p = .017$ ). In addition, participants who studied more lessons also performed better in identifying great circles on spheres,  $\beta = 0.4$ , 95% CI = [0.01, 0.78],  $p = .031$ ), straight planar lines on various surfaces,  $\beta = 0.23$ , 95% CI = [0.01, 0.45],  $p = .031$ ), and when asked to draw inferences about the properties of straight lines on the sphere in the reasoning test ( $\beta = 0.12$ , 95% CI = [0.01, 0.22],  $p = .02$ ). Linear trends were non-significant in the other test conditions ( $p$ s > .27).

### 5.2. Effect of the number of lessons on insight reports.

	$\chi^2$	df	p
<b>Number of lessons</b>	<b>7.7</b>	<b>1</b>	<b>.005</b>
Education in mathematics	0.2	1	.63

TABLE 13. Logistic regression analysis of the effect of number of lessons on insight reports. Significant effects are highlighted in bold.

### 5.3. Relation between insight report and performance.

	df	$\chi^2$	p	df	$\chi^2$	p
<b>Test condition</b>	<b>8</b>	<b>508.2</b>	<b>&lt;.001</b>	<b>8</b>	<b>128.3</b>	<b>&lt;.001</b>
Insight report	1	0.5	.49	1	0.2	.64
Number of lessons				1	0.5	.47
Education in mathematics				1	0.4	.55
<b>Test condition*Insight report</b>	<b>8</b>	<b>16.5</b>	<b>.036</b>	<b>8</b>	<b>16.1</b>	<b>.041</b>
<b>Test condition*Number of lessons</b>				<b>8</b>	<b>51.9</b>	<b>&lt;.001</b>
<b>Test condition*Education in mathematics</b>				<b>8</b>	<b>18.8</b>	<b>.017</b>

TABLE 14. Results of the two mixed models analyzing the effect of Insight on performance. On the right, in italics: model accounting for years of education in mathematics and number of lessons. Model without covariates: Loglik = 2430.6, Random effect (participant): variance = 0.33. Model with covariates: LogLik = -2386.6, Random effect (participant): variance = 0.26. Significant effects are highlighted in bold.

Estimated contrast in accuracy between participants who did vs. did not report an insight in each test condition -computed with linear trends of each analyses, with and without covariates- revealed that participants who experienced insights were more likely to identify lines that are straight despite being not planar in the various surfaces straight line test the type of line that does not exist on the sphere (simple model:  $\beta = 0.61$ , 95% CI = [0.03, 1.19],  $p = .032$ , model with covariates:  $\beta = 0.64$ , 95% CI = [0.06, 1.23],  $p < .021$ ). In addition, there was a significant effect of insight experiences on participants' accuracy at accepting great circles as straight on the sphere, but only in the model that did not account for number of lessons or years of education in mathematics (simple model:  $\beta = 1.57$ , 95% CI = [0.07, 3.08],  $p = .032$ ). In all the other test conditions, the linear trends did not reach significance (simple model  $ps = 0.22$ ; model with covariates:  $ps > .26$ ).

### 5.4. Relation between insight experiences, confidence and performance.

#### 5.4.1. Correlation tests.



	Confidence rating 1	Confidence rating 2	Confidence rating 3	Insight report
Confidence rating 1	X	$\rho(84) = .67$ $p < .001$	$\rho(83) = .64$ $p < .001$	$\rho(84) = .2$ $p = .17$
Confidence rating 2	$\rho(82) = .64$ $p < .001$	X	$\rho(83) = .86$ $p < .001$	$\rho(84) = .19$ $p = .17$
Confidence rating 3	$\rho(81) = .65$ $p < .001$	$\rho(81) = .87$ $p < .001$	X	$\rho(83) = .21$ $p = .17$
Insight	$\rho(82) = .12$ $p = .39$	$\rho(82) = .15$ $p = .39$	$\rho(81) = .21$ $p = .19$	X

TABLE 15. Spearman's  $\rho$  coefficients and p-values for pairwise correlation tests. Above diagonal: zero-order correlation, below: with number of lessons and years of education in mathematics as covariates. Significant correlations are highlighted in bold. All p-values were corrected for multiple comparisons using Holms method (applied separately to the analyses with and without covariates). Note that the third rating of confidence was missing for one participant in the 1-lesson group, hence the difference in degrees of freedom. Confidence rating 1: measured just after participants completed the teaching phase; Confidence rating 2: measured after the various surfaces straight lines test; Confidence rating 3: measured after the reasoning test.

#### 5.4.2. Relation between insight experiences, confidence and performance.

	df	$\chi^2$	p	df	$\chi^2$	p
<b>Test Condition</b>	<b>8</b>	<b>53.4</b>	<b>&lt;.001</b>	<b>8</b>	<b>56.8</b>	<b>&lt;.001</b>
Insight report	1	0.3	.58	1	0.2	.67
Confidence in understanding	1	1.4	.23	1	1.1	.31
Number of lessons				1	0.3	.61
Education in mathematics				1	0.3	.57
<b>Test condition*Insight report</b>	<b>8</b>	<b>19.2</b>	<b>.014</b>	<b>8</b>	<b>18.6</b>	<b>&lt;.017</b>
<b>Test condition*Confidence in understanding</b>	<b>8</b>	<b>17.6</b>	<b>.024</b>	<b>8</b>	<b>14.1</b>	<b>.08</b>
<b>Test condition*Number of lessons</b>				<b>8</b>	<b>48.1</b>	<b>&lt;.001</b>
Test condition*Education in mathematics				8	19.2	.015

TABLE 16. The two mixed models jointly analyzing the effect of insight and confidence on accuracy. On the right, in italics: model accounting for years of education in mathematics and number of lessons. Model without covariates: LogLik = -2421.7, Random effect (participant): variance = 0.33, Model with covariates: LogLik = -2378.6, Random effect (participant): variance = 0.25. Significant effects are highlighted in bold.

Estimated contrast in accuracy between participants who did vs. did not report an insight in each test condition, controlling for confidence, revealed that participants who experienced insights were more likely to accept straight non planar lines in the various surfaces straight line test (simple model:  $\beta = 0.67$ , 95% CI = [0.08, 1.27],  $p = .014$ , model with covariates:  $\beta = 0.68$ , 95% CI = [0.09, 1.27],  $p < .012$ ). There was also a significant effect of insight experiences on participants' accuracy at accepting great circles as straight on the sphere, but only in the model without covariates (simple model:  $\beta = 1.38$ , 95% CI = [0.27, 3.39],  $p = .01$ ). In all the other test conditions, the linear trends did not reach significance (simple model  $p$ s = .51; model with covariates:  $p$ s > .08).

Exploring the interaction between test condition and confidence reveal a negative relation between confidence and performance in the accuracy in identifying great circles as straight on the sphere, but only in the model without covariates (simple model:  $\beta = -0.57$ , 95% CI = [-1.08, -0.06],  $p = .019$ ). No other test condition was significant, whether in the simple ( $p$ s > .071), or in the model with covariates ( $p$ s > 1).



# 3.

## Dynamic of learning

In addition to the problem of being able to observe learning as it occurs, experimenters typically face another problem: designing a measure that is appropriate for recording the progress of people's learning. This chapter presents a study of the dynamics of conceptual learning. In order to describe the learning curve, we attempt to find a behavioral measure of understanding that allows us to track progress live, throughout the learning process.

This chapter is being written for publication in collaboration with Véronique Izard.

## 1. Introduction

Learning mathematics can be difficult and slow (Asmuth & Rips, 2006; Carey, 2009; Weber, 2002; Vamvakoussi & Vosniadou, 2004), to the point that some concepts may continue to elude students after several years of training (Caramazza et al., 1981b; Clement, 1982; McCloskey, 1983; D. Muller et al., 2008; Shtulman, 2006; Wisner, 1986). For example, many adults have difficulty with fractions or decimal numbers, despite having studied these concepts throughout elementary and middle school (Vamvakoussi et al., 2012, 2013; DeWolf & Vosniadou, 2015; Hoof et al., 2015; Vamvakoussi & Vosniadou, 2004). What does the concept learning curve look like? Do people progress in a purely incremental fashion, or is learning nonmonotonous, interleaving periods of progress and recession?

To grasp the possible shapes of the learning curve, picture an ideal learner. When this ideal learner encounters new information relevant to her concepts, she should immediately integrate this information into her current knowledge, and she should continue to accumulate information and make progress until the learning is complete. The more information this learner receives, the more he should learn, and his learning curve should therefore be smooth and monotonous. Compared to this ideal description, however, many factors can limit the progression of learning and alter the shape of the learning curve.

First, even an ideal learner may not start from a blank slate. Instead, she may possess knowledge or biases that lead her to initially favor inaccurate assumptions about the concept being taught (Vosniadou, 2019; Carey, 2009; Martí et al., 2018; Bonawitz et al., 2019). Therefore, her learning curve may be non-monotonous, as she will have to reject her initial misconception before accepting the correct one (Ohlsson, 2009; Feldman, 2000). This process can take time, especially if the information presented does not systematically contradict the learner's prior hypothesis - thus explaining why learning can be protracted.

Second, the learner may not be ideal, as she only has limited resources to process the information, which may also affect her learning curve. For example, while the ideal learner model assumes that each new piece of information received is used to update their knowledge, natural learners may not systematically re-evaluate their hypotheses with each new piece of information

received. As a result, the learning curve could show one or more plateaus, rather than following a continuous progression.

In addition, learners may also possess limited memory capacities. Hence, they may progressively forget the information presented earlier, as new information comes in. Memory limitations could cause all kinds of patterns in the learning curve, with both plateaus and even recession periods, depending on the nature of the information that progressively leaks out of memory. For example, a learner may forget a crucial piece of information that was only taught earlier in the process, leading her to consider anew some erroneous hypotheses she had previously dismissed.

As these considerations show, one could expect learning curves for concepts to take all kinds of shapes; and measuring the shape of learning curves will thus be crucial to better understand the process of conceptual change. Doing so, however, raises several experimental issues. First, because of the very protracted nature of concept learning, it is difficult for experimenters to have access to the whole process of learning. Hence, studies of conceptual learning are often conducted over very long periods, with days or months between each testing session, leaving the fine dynamic of the learning process inaccessible to researchers (Behr et al., 1984; Moss & Case, 1999; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001; Schauble, 1996; Siegler, 1995; C. L. Smith et al., 2005; Steinle & Stacey, 1998).

To try and circumvent this issue, other studies have developed paradigms to probe conceptual learning in the lab (Feldman, 2000; Goodman et al., 2008; Martí et al., 2018; Ohlsson & Cosejo, 2014; Shepard, 1961; Bonawitz et al., 2019). These studies found that participants' progress could nicely be captured by formal models where beliefs are updated rationally according to Bayesian inference rules, in line with the ideal learner description sketched above. However, most of these studies have used simple learning targets, and most importantly, the concepts participants needed to learn were never introduced explicitly: rather, participants needed to try and infer the rules for categorizing items. It is not clear whether the learning processes involved in these simple situations are the same as those involved in the acquisition of complex mathematical or science concepts: if participants were given the rule explicitly instead of having to discover it by themselves, they probably would be able to apply it instantly. Such is not the case in science

or mathematics, where students can struggle for years, and this even when concepts have been presented to them in an explicit form. It is thus unclear whether and how the models of rational learning through Bayesian updating can be extended to apply to situations where students learn mathematical or science concepts through formal, explicit instruction.

As a second issue, experimenters face the problem of measuring participants' learning progress across time. At first view, one may be tempted to measure progress by means of tasks directly probing people's understanding, which participants would take repeatedly throughout the learning period. The progression of participants' scores would then directly reflect their learning curve. However, in practice, this solution is difficult to implement, for several reasons.

First, on certain topics, the set of questions that can ask to probe participants' understanding may be limited. Repeating the same questions to the same participants is problematic, however, because it makes it impossible to separate genuine learning progress from mere familiarity effects. To avoid familiarity confounds, experimenters could try and scatter their different questions throughout the learning period they wish to evaluate; however, if the questions are not matched in difficulty, there will be strong effects of order on participants' performance, hampering the measurement of the learning curve. Perhaps even more problematic, asking questions to evaluate how much a learner understands a concept could interfere with the training intervention one wants to assess, because the questions asked to probe participants' understanding may themselves contain implicit or even explicit information about the concept taught in the experiment. More generally, answering questions about the concept they are studying could lead participants to re-evaluate their beliefs, and consequently improve their levels of understanding (for evidence that testing contributes to improving learning, see Roediger).

As a second option, researchers could try and rely upon introspective measures in order to measure participants' level of learning. For example, in studies assessing the effects of different forms of argumentation, participants are asked to report their level of conviction on a given topic before and after an intervention -brief reviews of paradigms based on these types of experimental manipulation in (Mercier, 2016) and (Swire-Thompson et al., 2020)- likewise, studies of concept learning could ask participants to report how much they feel they have understood the notion

taught. However, this solution is problematic too, because people's introspection on their own learning is often unreliable (Moore & Healy, 2008; D. Muller et al., 2008; Dunlosky & Rawson, 2012; Sanchez & Dunning, 2018; Finn & Metcalfe, 2014; Rozenblit & Keil, 2002). Moreover, in situations involving concept change, interpreting people's answers is difficult, because it is impossible to know a priori whether the participants are assessing their understanding concerning their initial (erroneous) concept, or concerning the new (correct) concept, they are intended to learn.

Here, we assess a third option to try and solve this experimental issue and measure people's learning progress in real-time: to measure participants' reaction to the material presented during a learning intervention. We hypothesized that someone with a low level of understanding would tend to judge that the learning material is poor because they find it too complex, obscure, or badly explained. Some may even feel that they do not understand why the information presented is relevant. On the contrary, someone with a good level of understanding would judge the material as clear and convincing.

We developed a paradigm (Paradigm 1) to teach a novel mathematical concept to adult participants: the concept of geodesic, which generalizes the common notion of a straight line to straight trajectories drawn on curved surfaces. Crucially, this concept is counterintuitive: when tested on their spontaneous intuitions, most people make systematic errors when asked to identify straight lines on spheres. In our teaching intervention, participants were presented with one to seven lessons in a row to learn about geodesics on the sphere. These lessons used simple physical models to explain how straight lines generalize in the case of the sphere. This intervention proved effective: participants performance in post-tests administered after the intervention was linearly related to the number of lessons they had studied. In other words, the participants who had studied more lessons eventually reached a better level of understanding than the participants who had studied fewer lessons.

During the intervention, participants were asked to evaluate whether they found each lesson convincing, on a scale from 0 to 10. Here, we ask whether these scores can be used to measure participants learning progress in real-time. We performed two analyses, where the order of the



lesson was entered either as a numerical or as a categorical variable. Given that our previous study found a better understanding of geodesics in participants who had studied more lessons, we predicted that the lessons presented later in the intervention would receive higher scores. In other words, we would observe a positive trend by numerical position in the analysis by entering lesson order as a numerical variable. The second analysis tested whether the learning curve may be non-monotonous, and in particular whether participants may go through an initial recession phase. To do so, we tested whether some lessons may have been rated lower than all the lessons that preceded them.

## 2. Study 1

### 2.1. Introduction.

The first experiment aims to describe the curve of learning for the concept of geodesics (straight lines) on the sphere. Participants were given 1 to 7 lessons explaining how straight lines behave on the sphere, and they were asked to evaluate the quality of each successive lesson. We hypothesized that these scores would reflect people's level of understanding of the concept of the geodesic. In particular, in a previous study, we found that people's understanding of geodesics (as assessed in post-tests administered after the intervention) increased with the number of lessons they had studied (Chapter 2). We thus predicted that the scores attributed to the lessons would globally increase over the course of the teaching phase. Moreover, we tested whether people's learning curve for geodesics is non-monotonous, with an initial drop followed by a later rebound.

#### 2.1.1. Participants.

Fifty-six adults (40 females, age 18-43 years,  $M = 25.5$  years, age missing for 10 participants) were included in the experiment (cohort of Paradigm 1). Education in mathematics, as number of years after 10th grade, ranged from 0 to 10 (average number of years of education in mathematics after 10th grade:  $M = 3.9$  years, Median = 4 years). Thirteen other persons participated but were excluded because they failed inclusion tests.

#### 2.1.2. Methods.

**Procedure.** The present study analyses the scores of the lessons received in Paradigm 1. First, participants were administered two inclusion tests to ensure that they mastered the concept of straight lines on planes (planar geometry inclusion test) but were not yet able to identify straight lines on spheres (spherical geometry test). Most people indeed tend to think that all circles drawn

on spheres are straight lines but this is wrong: only the great circles (e.g. equator) correspond to trajectories that go straight ahead on a sphere, and the smaller circles do not. Second, participants received a sequence of one to seven lessons introducing the concept of straight lines (geodesics) on spheres. Lastly, participants took three tests assessing their understanding of geodesics on spheres and other surfaces. The detailed presentation of the tests and inclusion phases are found in the description of [Paradigm 1](#), in the first part of this thesis. As the present study focuses on the scores attributed to the lessons, we only detail here the material and procedure used in the teaching phase.

**Teaching phase.** Participants received 1, 3, 5, or 7 lessons explaining that great circles correspond to straight lines on the sphere, while small circles do not. These lessons used different physical models with common life objects. Lessons are translated and reported in [Table 3](#). Before reading each lesson, participants were told that they would have to write a brief summary of it. This summary was not analyzed and was only included to ensure that participants read the lessons carefully. Participants were invited to assess the quality of each lesson, by answering the question: *"How convincing was this lesson?"*. Scores were indicated on a scale graded from 0 (not convincing at all) to 10 (very convincing). Two scores were collected for each lesson: one just after the presentation of that lesson (measurement 1), and the second at the end of the teaching phase (measurement 2 - this second measurement was omitted for the 1-lesson group since including it would have resulted in asking the same question twice in a row). The second score was introduced to control for potential sequential effects, and the first score to control for memory effects. Within each experimental condition, we created 14 different counterbalanced orders to ensure that each lesson appeared equally often in each position, and thus tested 14 participants in each condition. Participants did not know in advance how many lessons they would have to read.

### 2.1.3. Results.

#### Analyses.

Lesson scores were analyzed in mixed models analyses, conducted in R using the `lmer` function of package `lme4`. Depending on the analysis, the position of the lesson in the sequence was entered either as a numerical variable, to assess global progress, or as a categorical variable, to test whether the learning curve may be non-monotonous. The analyses also included a categorical variable for score measurement (during vs. at the end of the learning phase) and a numerical

variable for education in mathematics (number of years of education in mathematics after 10th grade), as well as a random effect of participant. Significant effects and interactions were explored using the package emmeans (functions emmeans or emtrends). The script with the analyses and data can be found at <https://github.com/charlusb/learning-curve>.

### Global progress.

The first analysis tested whether the scores attributed to the lessons increased globally as participants progressed through the teaching sequence. To do so, we assessed whether there was a linear effect of the numerical position of each lesson on the lesson scores. As expected, the linear mixed analysis revealed a positive effect of the lesson position on the lesson scores ( $\beta = 0.35$ ,  $t(398.4) = 2.7$ ,  $p = .007$ ), consistently with the fact that people who received more lessons performed better in the tests that assessed their understanding after the learning phase (ref article). This analysis also yielded a significant interaction between the number of years of education in mathematics and the lesson position: the more participants were educated in mathematics, the lesser the scores they attributed increased with lesson position ( $\beta = 0.06$ ,  $t(396.7) = -2.04$ ,  $p = 0.04$ ). There was no main effect of mathematical education or score measurement, and score measurement did not interact with lesson position or education in mathematics.

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	Estimate	Std. Error	df	t value	p value
Education in mathematics	0.1	0.16	89.43	0.61	0.55
<b>Lesson position</b>	<b>0.35</b>	<b>0.13</b>	<b>398.4</b>	<b>2.7</b>	<b>.007</b>
Score number	0.18	0.46	387	0.387	0.7
<b>Education in mathematics x Position</b>	<b>-0.06</b>	<b>0.03</b>	<b>396.65</b>	<b>-2.03</b>	<b>0.04</b>
Education in mathematics x Score number	0.01	0.1	387.44	0.14	0.89
Lesson position x Score number	-0.08	0.12	381.95	-0.68	0.5
Education in mathematics x Lesson position x Score number	0	0.03	382.33	0.27	0.78

TABLE 17. Analysis assessing linear effects of lesson position on the scores.

### Non-monotonicity.

The second analysis assessed whether the scores attributed to the lessons may have been progressing in a non-monotonous manner. This analysis found a significant effect of the position

of the lesson on the lesson scores when lesson position was entered as a categorical factor (S.S. = 74.7,  $F(6, 359.42) = 3$ ,  $p = .008$ ). This result is convergent with the global effect of lesson position found in the previous analysis, but it may also signal the presence of non-monotonous effects. To evaluate this possibility, we used Helmert contrasts to test whether some lessons were attributed scores that differed from the average scores of all the lessons that preceded them. Hence, finding that a lesson was attributed a lower score than the previous lessons would be indicative of an initial phase of recession, in line with the idea that participants may need to deconstruct their initial concept of straight line before acquiring the new concept. The Helmert contrasts revealed that the lesson presented in the sixth position was scored significantly higher than lessons 1-5 ( $\beta = 5.67$ ,  $p = 0.01$ ), but no other contrast proved significant. In particular, there was no indication of an initial drop in performance. In addition to the main effect of lesson position, the analysis also yielded a significant interaction between mathematics education and lesson position (S.S. = 61.05,  $F(6, 358.8) = 2.42$ ,  $p = 0.03$ ). Exploring this interaction revealed a negative trend on the lesson presented in the 5th position ( $\beta = 0.05$ ,  $p = .04$ ): participants who had received more education in mathematics tended to attribute lower scores to this lesson, compared to participants with fewer years of mathematics instruction (see Table 19). Lastly, there was no main effect of score measurement or mathematics education, and no interaction involving score measurement.

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Helmert contrast	Estimate	S.E.	df	t ratio	p.value
Second vs earlier	-0.32	0.31	369	-1.2	0.3
Third vs earlier	-0.92	0.55	360	-1.68	0.09
fourth vs earlier	0.26	0.94	369	0.28	0.78
fifth vs earlier	1.64	1.22	364	1.34	0.18
<b>Sixth vs earlier</b>	<b>5.65</b>	<b>2.18</b>	<b>367</b>	<b>2.59</b>	<b>0.01</b>
Seventh vs earlier	2.94	2.6	364	1.14	0.26

TABLE 18. Helmert contrasts computed from predictions of analysis of categorical effect of position on scores.

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Lesson position	trend	SE	df	lower.CL	upper.CL	t.ratio	p.value
Position 1	0.07	0.17	103	-0.26	0.398	0.42	0.68
Position 2	0.08	0.18	113	-0.27	0.43	0.46	0.65
Position 3	-0.08	0.18	113	-0.43	0.27	-0.46	0.64
Position 4	-0.2	0.19	135	-0.57	0.17	-1.1	0.28
<b>Position 5</b>	<b>-0.47</b>	<b>0.19</b>	<b>135</b>	<b>-0.84</b>	<b>-0.11</b>	<b>-2.55</b>	<b>0.01</b>
Position 6	-0.27	0.22	226	-0.71	0.17	-1.2	0.23
Position 7	0.01	0.22	226	-0.43	0.45	0.04	0.97

TABLE 19. Explorations of interaction between Education in mathematics and Lesson position.

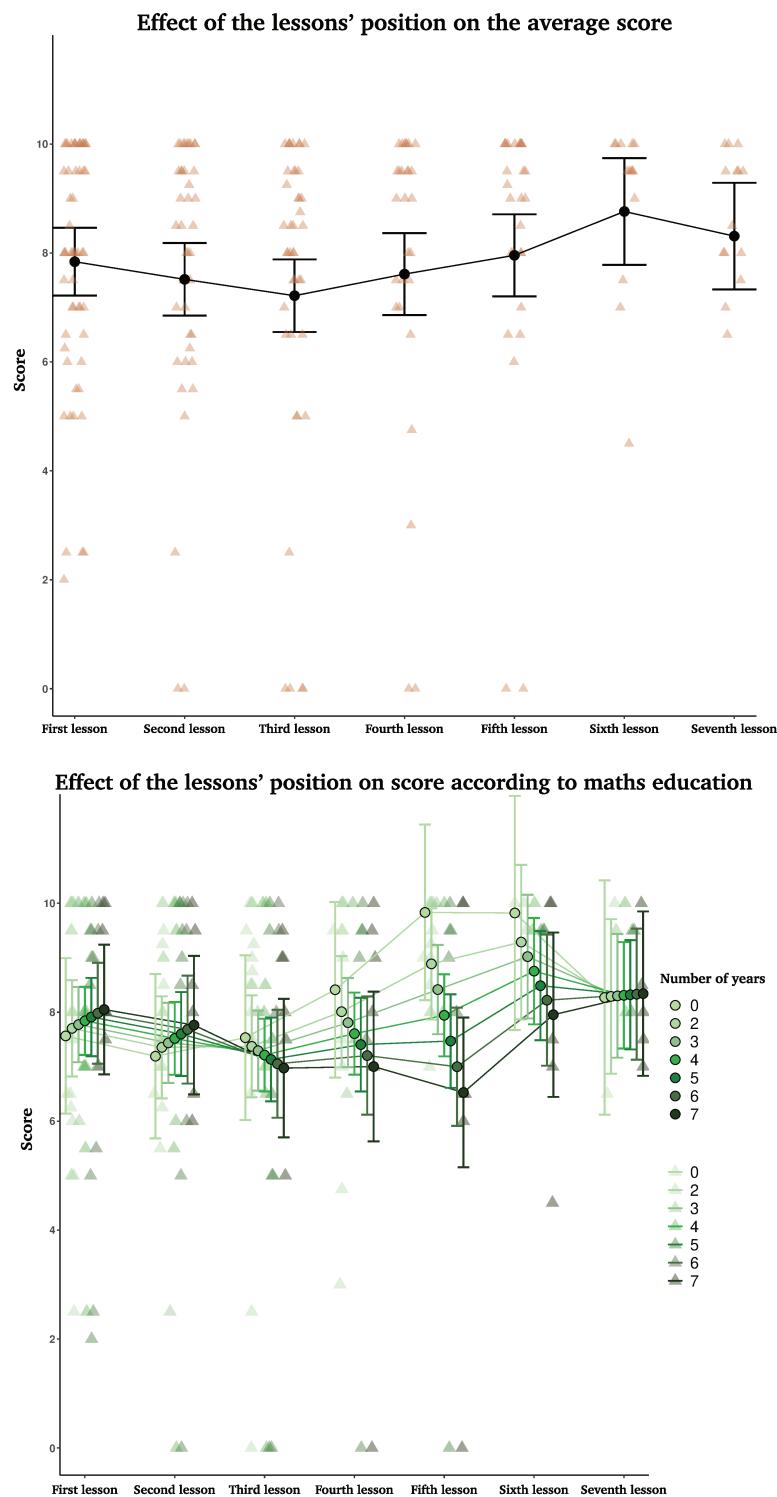


FIGURE 11. Above, effect of categorical position of lesson on the score in Study 1. Plain dots show the scores predicted by the model encoding the position of lesson as a categorical variable. Semi-transparent triangles show the scores attributed by individual participants. Error bars are 95% confidence intervals. Below, interaction between education in math and lesson position on the scores attributed to the lessons in Study 1. Plain dots show the scores predicted by the model according to the position of lesson (categorical variable) and the level of education in mathematics. Semi-transparent triangles show the scores attributed by individual participants, colored by their level of education in mathematics. Error bars are 95% confidence intervals.

#### 2.1.4. Discussion.

The findings of Study 1 suggest that learning may not be purely incremental. First, we found that the scores attributed to the lessons increased globally as participants read more lessons. This finding was attested in two analyses, where lesson position was entered either as a numerical variable (positive effect of the numerical position of lessons on the lesson scores) or as a categorical variable (higher score attributed to the 6th lesson, compared to five first lessons). These effects are in line with the results presented in our previous report, which showed that the participants who had received more lessons performed better in post-tests assessing their understanding of geodesics. As such, this first finding lends credence to the measure we proposed to assess concept learning: asking participants to rate the quality of the teaching material presented to them may provide a good solution to measure participants progress while learning is still occurring, without interfering with the information presented in the teaching phase.

The analysis entering lesson position as a categorical variable was initially undertaken to assess whether progress in concept learning may be non-monotonous. In particular, given that the concept we taught was counterintuitive (participants spontaneously converge on a wrong generalization when asked to identify which lines drawn on a sphere are straight), we intuited that participants may first need to deconstruct their initial misconception, before learning the correct generalization for geodesics. We thus predicted that one of the lessons might be scored lower than the lessons that preceded it. However, our analyses did not yield evidence in this direction.

As one's background in mathematics may affect learning new mathematical concepts, our analyses included a covariate for participants level of education in mathematics. Both our analyses (with lesson position either as a numerical or a categorical variable) identified a significant interaction between education in mathematics and lesson position; and in both cases, the direction of the effect was negative. Specifically, the more the participants were educated in mathematics, the less their scores progressed to the point that the participants who had more education in mathematics attributed lower scores to the 5th lesson, compared to the participants with less education in mathematics. The direction of this effect may be surprising: intuitively, participants with a stronger background in mathematics may be expected to learn faster than the others, not slower. Interestingly, however, the effect observed is compatible with the hypothesis that learning goes

through an initial period of deconstruction: perhaps, this period lasts longer in more educated participants because their initial concept is more strongly entrenched, and more counterevidence is needed to engage the initial deconstruction process.

In summary, the results of Study 1 indicate that measuring people's reaction to teaching material holds promises for tracking learning progresses as they are occurring, i.e. while students are receiving a teaching intervention. However, the design of Study 1 had some limitations. In particular, because the data analyzed in Study 1 stemmed from a larger study, the number of lessons varied across participants. As such, only a smaller number of participants provided scores for the lessons presented towards the end of the teaching sequence. Furthermore, because Study 1 was conducted as an exploratory project, we did not know how large our effects might be, and thus could not assess statistical power a priori. We conducted Study 2 to try and replicate the effects observed in Study 1 in the context of a well-powered experiment.

### 3. Study 2

**3.1. Introduction.** Study 2 was designed to replicate the effects observed in Study 1, in a well-powered study. We used the findings from Study 1 to compute power for our effects of interest: the main effect of lesson position, and the negative interaction between lesson position and education in mathematics. In addition, in this replication Study, we did not vary the number of lessons assigned to each participant, and instead administered seven lessons to all participants.

**3.1.1. Participants.** Due to the design of the teaching phase, which was based on 14 counter-balanced lesson orders, the final sample size had to be a multiple of fourteen. We thus recruited an initial sample of twenty-eight participants, and conducted a power analysis (see below) to check if this sample was sufficient (80% power) to observe our effects of interest: the main effects of lesson position in both analyses -numerical and categorical-, and the interactions between lesson position and education in mathematics. We intended to recruit additional participants in groups of fourteen until 80% power was reached for all these effects. The final group had twenty-eight participants (20 females, age 19-49 years, average 26.19 years, exact age missing for two participants). Participants number of years of education in mathematics after 10th grade ranged from 0 to 10 (average: 3 years, median: 2 years). Four additional persons participated and were excluded because of their performance on the planar geometry inclusion test (more than 3 mistakes).



**3.2. Power analysis.** To estimate the power of Study 2, we used a bootstrapping procedure, based on the models fitted on the data from Study 1 (two models coding lesson position either as a numerical or as a categorical variable). We entered the number of years of education in mathematics of the 28 participants initially recruited in these models and simulated the scores attributed to the lessons (1000 simulations). We then applied the same pipeline of analyses as in Study 1 to these simulated data, and computing power by calculating the percentage of simulations for which the p-value associated with each of our effects of interest was smaller than 0.05. This analysis revealed that we had sufficient power to detect three of our effects of interest in the initial group of twenty-eight participants [Table 20](#). Specifically, the main effect of lesson position was significant in more than 80% of our simulations, whether entered as a numerical or as a categorical variable. In addition, we also reached 80% power for the interaction between education in mathematics and categorical lesson position. Note that power was only 62% when education in mathematics was tested with lesson position as a numerical variable. Given that the powered effects were sufficient to support the conclusions from Study 1, even without this interaction, we decided to stop our recruitment there and analyze the results.

**3.2.1. Methods.** The procedure and all stimuli were identical to the 7-lesson condition of Study 1, except that the test phase was removed. Thus the experiment stopped at the end of the teaching phase.

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	Position effect	Interaction position-math level
Position as a numerical variable	<b>95.6 %</b>	62 %
Position as a categorical variable	<b>99.5 %</b>	<b>84 %</b>

TABLE 20. Power estimates obtained by bootstrapping from the data of Study 1, seeking for effects in two analyses of score according to i) numerical position, ii) categorical position, and interaction with mathematical level in both.

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Helmert contrast	Power
pos2 vs earlier	13 %
pos3 vs earlier <sup>-</sup>	15.4 %
pos4 vs earlier	16 %
pos5 vs earlier	80 %
<b>pos6 vs earlier<sup>+</sup></b>	<b>99 %</b>
pos7 vs earlier	19 %

TABLE 21. Power estimates obtained by bootstrapping from the data of Study 1 (analysis of score according to categorical position factor), by position. <sup>-</sup> Negative trend. <sup>+</sup> Positive trend.

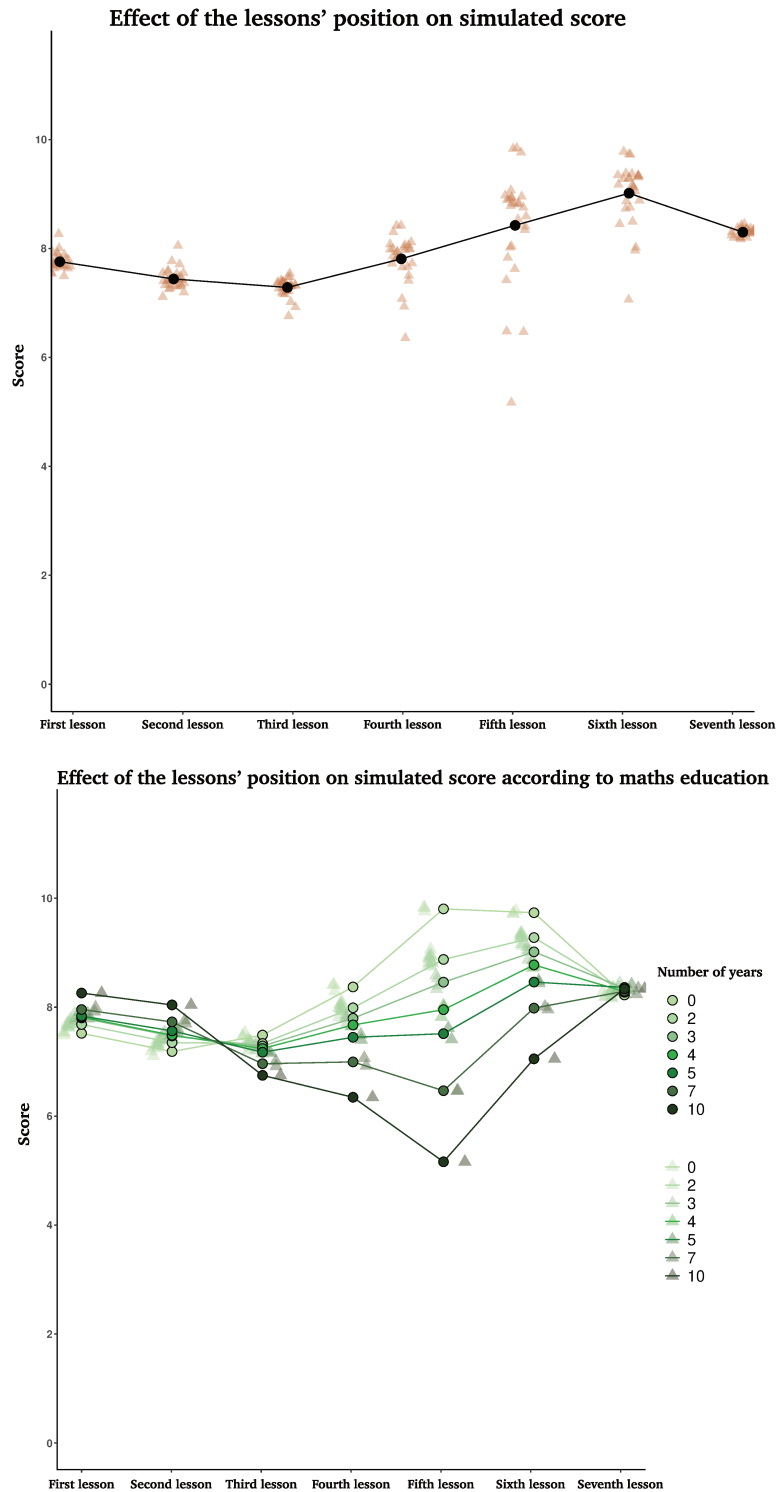


FIGURE 12. Above, average scores of lessons according to position in data simulated by bootstrapping (1000 simulations). Plain dots show the average simulated scores per position, and semi-transparent triangles represent the average simulated scores per (simulated) participant, encoding the position of lesson as a categorical variable. Below, average scores of lessons according to position, by level in mathematics, in data simulated by bootstrapping. Plain dots show the average simulated scores according to the position (categorical variable), by level of education in mathematics. Semi-transparent triangles show the average simulated scores per (simulated) participant, colored by their level of education in mathematics.

### 3.3. Results and discussion.

The analyses revealed no effect of lesson position on the scores attributed to each lesson, neither when lesson position was entered as a numerical ( $p > .3$ ) nor when it was entered as a categorical variable ( $p > .5$ ). There was also no interaction between math education and the position taken as a categorical variable ( $p > .24$ ). However, the interaction between math education and lesson position as a numerical variable was significant with a negative trend, consistent with the results of Study 1 (Results 2.1.3) ( $\beta = -0.041$ ,  $t(358) = -2.15$ ,  $p = .03$ ).

Despite being well-powered, Study 2 thus failed to replicate most of the effects observed in Study 1. In particular, in this second experiment participants did not tend to attribute higher scores to the lessons presented later in the sequence. As one exception, we replicated the interaction between the position of the lessons in the sequence and participants education in mathematics, when lesson position was entered as a numerical variable. This effect was in the same negative direction as in Study 1. Specifically, in Study 2 we found that the lesson scores attributed by the participants with more education in mathematics tended to decrease as they progressed through the learning phase. In contrast, the scores attributed by participants with less education in mathematics tended to remain constant.

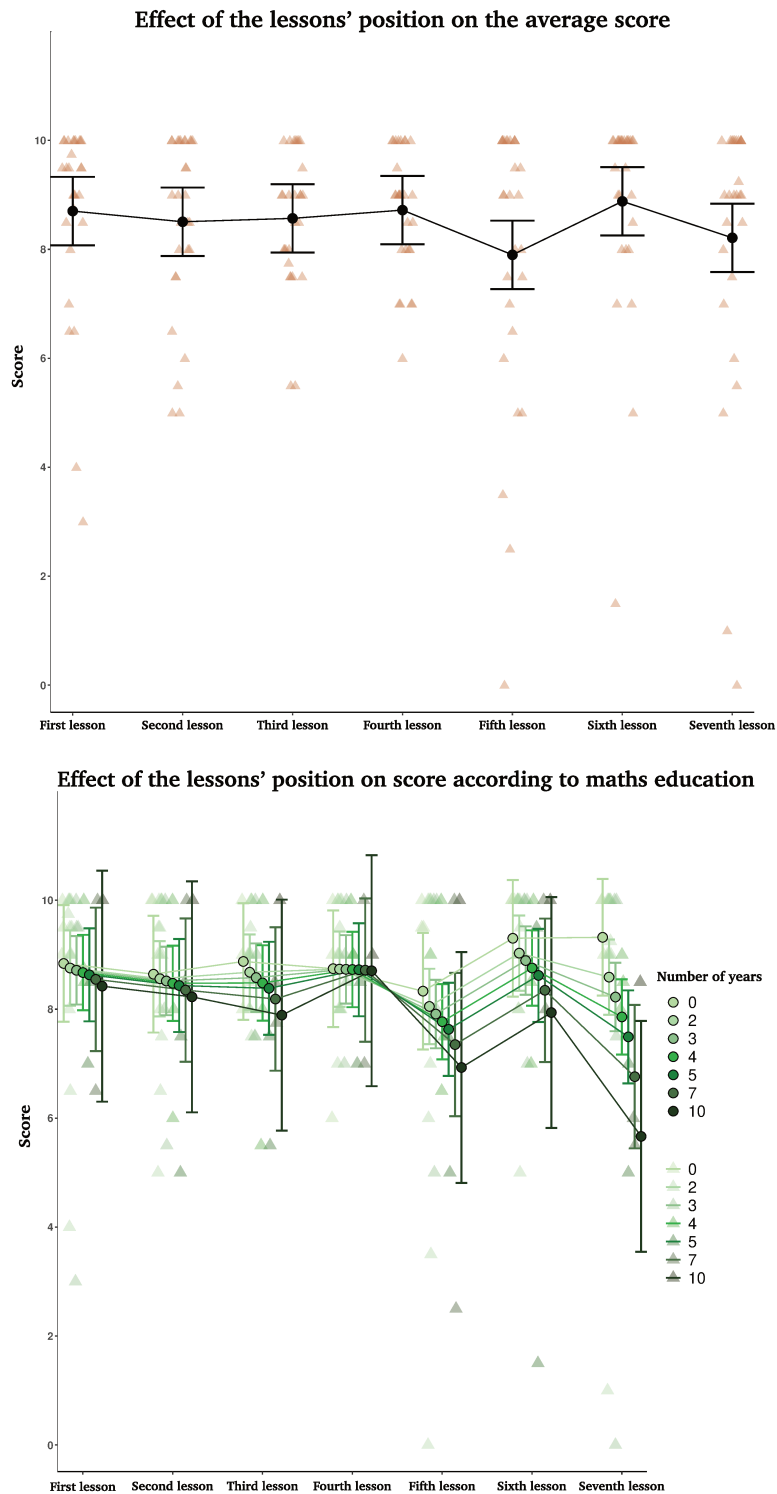


FIGURE 13. Above, effect of categorical position of lesson on the score in Study 2. Plain dots show the scores predicted by the model encoding the position of lesson as a categorical variable. Semi-transparent triangles show the scores attributed by individual participants. Error bars are 95% confidence intervals. Below, interaction between education in math and lesson position on the scores attributed to the lessons in Study 2. Plain dots show the scores predicted by the model according to the position of lesson (categorical variable) and the level of education in mathematics. Semi-transparent triangles show the scores attributed by individual participants, colored by their level of education in mathematics. Error bars are 95% confidence intervals.

#### 4. General discussion

The present study aimed at describing how concept learning unfolds over time, using participants' assessment of the learning material as a measure of their understanding of the target concepts. Participants who were not initially able to identify straight lines on the sphere were given a sequence of up to 7 lessons introducing this notion and were invited to score the quality of each lesson. We hypothesized that the scores attributed to the lessons would reflect participants' understanding of the concept explained in these lessons: a participant with a better understanding of the concept should rate the lesson as convincing, while participants failing to understand the concept would find the lessons unconvincing, or even obscure. Given previous findings that participants receiving more lessons reached a higher level of understanding in our paradigm (as measured by post-tests administered after the teaching phase, see [Chapter 2](#)), we expected that participants should evaluate the lessons more and more positively as the teaching phase progressed. We also suspected that this increase may be non-monotonous.

Study 1, our first exploratory study on this topic, yielded results in line with our main prediction: participants generally attributed higher scores to the lessons presented later in the sequence. However, this finding was not replicated in Study 2, despite a statistical power estimated at more than 95%.

Several interpretations can be given for this non-replication. As a first possibility, perhaps our estimation of the statistical power of Study 2 was inaccurate. The procedure used to estimate Study 2s power rested on the model fitted on the data from Study 1, which rested on several key assumptions, and for example, assumed linear effects of education in mathematics. This or other assumptions made in the model may have been inaccurate, however, and perhaps led to an overestimation of the statistical power of Study 2. If such was the case, the effect observed in Study 1 may still be trusted, despite the absence of a replication in Study 2.

Alternatively, it is also possible that the effect observed in Study 1 is not reliable and that, contrary to our prediction, participants evaluation of the lessons did not increase as the teaching phase progressed. This prediction rested on two key assumptions. First, we hypothesized that people with a better understanding of the target concept should judge the learning material more

positively than people with a poor understanding. Second, in a previous study, we found that the participants who studied more lessons reached a better performance in post-tests testing their understanding of straight lines on curved surfaces. From this, we hypothesized that participants would possess a higher level of understanding when they are reading the last lessons of the sequences. If peoples ratings of the lessons did not vary according to their position in the sequence, however, one or both of these assumptions must have been unwarranted.

On one hand, contrary to our first assumption, perhaps the scores attributed to the lessons did not reflect participants understanding of the target concept. Interestingly, our participants tended to attribute high scores to all lessons, including the very first lesson: 92% of the scores were above the middle of the scale provided. This suggests that they did not realize that the content of lessons was conflicting with their knowledge, and generally failed to integrate this content with their intuitive concept of straight line. Consequently, when they evaluated the lessons, they produced high scores that failed to reflect their poor understanding of the concept of generalized straight line. A similar interpretation was proposed by (D. Muller et al., 2008), in a study where participants watched science outreach videos on counterintuitive phenomena (Newton's First and Second Laws of Motion). After watching the videos, the participants still reasoned in contradiction with the counterintuitive information presented, but nonetheless reported agreeing with the video, and their confidence in their own understanding of Newtonian physics remained unshakably high. Similarly, our lessons were based on simple physical models that involved everyday objects and could thus have seemed quite intuitive to our participants, to the point that they failed to question the implications of these lessons for the concept of a straight line. If this is true, the scores attributed to the lessons cannot be taken to reflect participants understanding at least not at the beginning of the teaching phase.

As a third interpretation, it is possible that our second key assumption was wrong: perhaps participants merely did not progress in their understanding of straight lines during the teaching phase itself, even when the maximum number of lessons were presented (7 lessons). Such could be the case if, for example, participants needed time to fully integrate the information presented during the teaching phase: perhaps the teaching phase was too short to observe any improvement. Perhaps too, learning did not crystallize during the teaching phase, and the post-test administered

in our previous study played an instrumental role in revealing participants' understanding of the concept of generalized straight line. Indeed, and as argued earlier, tests assessing peoples understanding of a concept can potentially induce learning, either because the test questions contain explicit or implicit information that can advance peoples' understanding, or because these questions may lead participants to doubt and re-assess their concepts. If this third interpretation is true, participants' assessment of the teaching material could still constitute measure of their current understanding; yet we failed to find evidence in this direction in the context of our study, because our paradigm failed to generate a better understanding within the limited time of the teaching phase.

Fourth and lastly, it is also possible that we did not observe a global trend on participants' scores because the learning curve for the concept of geodesic is non-monotonous, alternating between phases of progress and of recession. Hence, upon reading the first lessons, perhaps people tried to accommodate the information presented with their intuitive concept of straight line, yielding only partial progress before they realized that the concept presented clashed in important ways with their intuition, and their previous concept needs a thorough revision. If, moreover, the timing of these progress and recession phases is variable across participants, averaging the responses of many participants would result in a flat curve.

Contrary to the main effect of lesson position, which constituted our central prediction, one of the effects observed did replicate across our two experiments: in both experiments, participants who were most educated in mathematics tended to judge the material more poorly towards the end of the learning phase, compared to the participants who were less educated in mathematics. This effect is compatible with the last interpretation offered above, and with one of the hypotheses that motivated our study: that learning a concept may involve an initial phase of recession. Indeed, the concept of (planar) straight line may be more deeply entrenched in mathematically educated people, such that they may need to undergo a longer deconstruction phase than their less mathematically savvy peers. For example, mathematically educated people may be more aware of the necessary relations that exist between different properties of planar straight lines, which no longer exist for straight lines on curved surfaces. On the plane, a straight line is at the same time a trajectory that does not turn, a line of minimal length, the intersection of this plane



with other planes, or a line defined by a vector of constant direction. Many of these properties are not true when straight lines are generalized to geodesics. Also, for those of our participants who were most versed in mathematics, the concept of straight line may have evoked some formal descriptions which may not generalize readily to the case of curved surfaces. In general, thus, our finding of an adverse effect of mathematical education on the scores participants attributed to the lessons is compatible with the idea that concept change involves an initial phase where a whole web of knowledge is unraveled, and this phase takes longer when these knowledge-webs are denser.

Our study presents a new method for tracking progress in concept learning in real time: asking participants to rate the quality of the learning material they are presented with. While further work remains needed to validate this index as a measure of participants' progress, some aspects of our findings are encouraging. In particular, our results are compatible with the idea that learning a new concept involves two phases: a first recession phase where a previous, erroneous conception is being unraveled, followed by a positive phase where a new concept is built. Consequently, our findings suggest that theories presenting concept learning as the result of rational inferences (Bonawitz et al., 2019) may extend to situations where concepts are introduced explicitly, in a formal teaching context. It will be crucial to develop methods to describe the curve of learning for new concepts in order to assess these proposals and, more generally, to understand how information presented by teachers is integrated, and how previous conceptions may interact on this process.

## 4.

### **Planar bias shapes identification of straight lines on curved surfaces**

Chapter 1 examined the reflexive content of the concept of straight line in peoples' mind, and assessed how this reflexive content evolved as participants were given information about straight lines on sphere and other surfaces. The first definitions of straight line, produced by participants before they were presented with any lesson about spheres and surfaces, consistently listed properties specific to planar straight lines, as well as generic properties of lines (a straight line is a line, it is made of points). The present chapter tests more systematically the hypothesis that a planar bias shapes the identification of geodesics, in two groups of participants: mathematicians and non-mathematicians.

This chapter is being written up for publication in collaboration with Véronique Izard and Cyril Falcon.

## 1. Introduction

Our understanding of the world is rooted in spatial representations. The characterization of space was already of interest to ancient scientists: The search for a logical system formalizing the properties of the space of the world gave birth to one of the oldest books of mankind, Euclid's Elements, 300 years B.C (Euclid, 300 BCE, 2007). This very first geometry formalizes the properties of two-dimensional shapes using primitive concepts such as points and lines of the plane, which is the default frame of reference: in this geometry, everything is flat and distances between points are calculated along straight line segments.

Euclidean geometry is based on a list of definitions (for points, straight lines, circles, right angles ...) and a set of five postulates. The first four postulates describe the basic properties of lines and points: (i) one can always draw a straight line between two points, (ii) one can always extend a straight line, (iii) one can always draw a circle from a given point and length (center and radius), and (iv) all right angles are equal. The fifth postulate states that given a straight line  $L$  and a point  $P$  outside  $L$ , there is a single straight line  $L'$  passing through  $P$  and not intersecting  $L$ . This postulate admits several equivalent propositions, notably that the sum of the angles of a triangle is a constant ( $180^\circ$ ).

For nearly 2000 years, the principles of Euclidean geometry were taken as a description of the physical reality (Kant, 1781; Hatfield, 2015). Moreover, because the fifth postulate seemed overly complex compared to the other four, mathematicians believed that it ought to derive from these other, simpler postulates; meaning that the existence of a single parallel line to any straight line is a necessary property of geometry. To prove this, mathematicians attempted to argue by contradiction, that is to show that a system where this principle is denied but the other postulates are maintained results in a contradiction. Such system however turned out to be coherent: we can imagine geometric worlds where it is realized. If the plane is replaced by a curved surface, and the lines by straight paths on the surface, i.e. geodesics, the fifth postulate can be violated in two critical ways: in some geometric worlds there are no parallel lines (spherical geometry), and in other worlds there are an infinity of parallel lines through a given point (hyperbolic geometry). Thus, non-Euclidean geometries arose from a failed attempt to prove that the fifth postulate follows logically from the other principles. This suggests that designing models of non-Euclidean

geometry may be particularly counterintuitive for the human mind, and more generally, that the human mind may be prepared to conceive space according to Euclidean principles.

A large number of experiments have been undertaken to analyse how humans and animals process space (Newcombe & Huttenlocher, 2000; Gallistel & Gelman, 1990). By and large, these experiments show that humans navigate in space using principles that match Euclidean predictions and distances (Gallistel, 1989; Peer et al., 2021; Wang & Spelke, 2002; Landau, 1984), and analyze drawings and objects according to their Euclidean properties e.g (Dehaene et al., 2006; Izard et al., 2022). Nevertheless, humans' representation of space has also be found to present systematic deviations from an Euclidean ideal.

First, the phenomenon of spatial memory distortion suggests that people mentally locate objects in ways that deviate from a pure Euclidean mental map (Hirtle & Jonides, 1985; Sampaio & Wang, 2009; Huttenlocher et al., 1991; Sampaio & Wang, 2017; Stevens & Coupe, 1978). In (Stevens & Coupe, 1978), for example, participants tended to locate cities according to a conceptual hierarchy that subordinates the location of cities on north-south and east-west axes to the location of a (superordinate) state. Sometimes, this hierarchical representation of position leads to errors: for example, if the US is further north than Canada, participants will incorrectly infer that any city in US is further north than any city in Canada, e.g. that Montreal is further north than Seattle. Such errors can constitute local violations to a Euclidean map since they involve changing the actual city position to match the position of the superordinate "state". Thus, these examples have been often cited as proofs that the representation of space in human cognition is not purely Euclidean, leading some authors to propose that a non-Euclidean model might be better suited to predict spatial judgments. These authors however did not exhibit such a model (Hirtle & Jonides, 1985).

While the spatial distortions from memory tasks might seem to inflict only minor tweaks to Euclidean principles, experiments in virtual reality have tested whether human navigation can accommodate more major deviations from Euclidean principles (Warren et al., 2017; Kluss et al., 2015; Zetsche et al., 2009). In one such experiment, (Warren et al., 2017) placed participants in a maze that mostly behaved according to Euclidean geometry, except that a few shortcuts were

introduced, whereby participants, upon entering one corridor in the maze, would find themselves instantly teleported in another place of the maze. The environment at both ends of the shortcut looked exactly the same, such that there was no discontinuity in the stimulation presented to the participants when they walked through the shortcuts. Warren et al.'s environment thus violated one of the most basic and fundamental principles of Euclidean geometry: the continuity of space. Yet, participants proved quite apt to navigate in this environment, taking the shortcuts appropriately in order to minimize the distance traveled between points of interest.

More recently, (Widdowson & Wang, 2022) used VR to assess navigation in curved spaces that mimicked models of non-Euclidean geometry produced by mathematicians. Participants were immersed in planar (Euclidean), spherical or hyperbolic environments (both non-Euclidean), where they had to walk straight along a first segment, turn, walk along a second segment, stop, and then were asked to point in the direction of their starting point. The authors concluded that the responses of the participants globally tended to match the direction of the origin as predicted in a planar (Euclidean) space, irrespective of the nature of the space where they were placed. However, in detail some aspects of their results were not cutting clear. In particular, while the Euclidean response dominated participants responses in the main task, in some cases (when the first leg of the path was twice as long as the second one), responses deviated systematically from the planar (Euclidean) predictions. Again, this was true in all environment, and thus even when people were navigating on a plane.

Thus, while the spatial representations supporting navigation are broadly consistent with Euclidean principles, our perception can also accommodate some major deviations from these principles. Yet, these results do not rule out the possibility that people conceive of space in a purely Euclidean manner, even if their perception is not fully constrained by Euclidean principles. Quite suggestively, in Warren et al.s experiments, participants reported no awareness of the presence of shortcuts, and when asked to draw a map of the environment where they had successively navigated, they invariably portrayed the space on a regular Euclidean, two-dimension map. This all suggest that they never questioned the idea that they were navigating in a usual, Euclidean environment.

The present study aims to test the idea that humans geometric concepts are fundamentally Euclidean, by focusing on one key concept of geometry: the notion of straight line. As alluded before, the notion of planar straight line can be extended to straight trajectories drawn on surfaces: trajectories that we follow when we always go straight ahead on a surface, without turning. In mathematics, this notion has been coined geodesics. Intuitively, a geodesic is a trajectory traced by a particle subjected to no force and which is launched on a surface. Understanding whether people are able to extend straight lines to geodesics is of great significance to understand the shift from Euclidean to non-Euclidean geometries: geodesics on surfaces follow Euclid's first four postulates, but not the last. Thus, for example, the sphere is a model of non-Euclidean geometry, if we accept that the great circles (geodesics) are the "straight lines" of this model.

Recall that (Widdowson & Wang, 2022) tested people as they walked along straight paths in non-Euclidean, curved environments, and found that participants failed to adapt to the curvature of the space they were navigating in. At first view, these findings seem to indicate that humans largely fail to process straight paths in curved spaces, and thus that our mind is not designed to process geodesics. In that study, however, it is conceivable that the participants failed to take into account the curvature of the space in which they were navigating because the nature of the stimulation presented made it too difficult to decode the curvature of the environment. Hence, in this experiment curvature was conveyed by the properties of the visual flow that participants experienced as they moved in the space, and the perceptual processes analyzing visual flow may be automatic and not revisable. As such, it is not surprising that participants always responded in the same way, irrespective of the curvature of the space in which they were immersed.

Could people apprehend geodesics as straight paths, when tested in contexts that do not raise perceptual difficulties? One study suggested that participants may be prone to error even in tasks that do not challenge perception. In (Izard et al., 2011), groups of American adults, French children, and Amazonian adults and children with no formal knowledge of geometry were invited to imagine straight paths on a spherical world, and were asked to answer questions about the properties of these straight lines. Participants from all groups erroneously reported that it is possible to find straight lines that are not intersecting (parallel straight lines) on the sphere. These errors suggest that they considered that all circles traced on the sphere correspond to straight

paths: the great circles of same radius as the sphere (which are geodesics) as well as the smaller circles (which are not geodesics). Therefore, in their view it was fully possible to find two straight paths (circles) that are parallel.

More generally, and based on these findings, we propose that people have a bias to think that straight lines are flat: they tend to identify the straight lines on a surface with the planar sections of this surface. By doing so, they project a property of the Euclidean line (being planar) onto geodesics, a generalization that is not mathematically correct.

One recent study appears to cast doubt on this hypothesis. Huey and colleagues (Huey et al., 2022) assessed participants' ability to distinguish portions of large circles (geodesics) on the sphere from portions of small circles (non-geodesics). In this study, participants had to judge which of two presented paths on a sphere corresponded to the shortest path between two points. The stimuli crossed two variables: the path was either a geodesic (large circle) or a non-geodesic (small circle), and it was either presented in a front view, so that its projection on the screen was perfectly straight (without curvature), or seen at an angle, projecting on a curved segment on the screen. Geodesics and non-geodesics were presented in these different views showing or not showing curvature. In this study, adult participants performed slightly above chance at identifying geodesic paths (arcs from large circles) as shorter paths compared to non-geodesic paths (arcs from small circles).

Despite this finding, Huey et al.'s study does not invalidate our hypothesis, for several reasons. First, while performance was above chance, it was actually quite poor, so it remains possible that other variables would influence people's judgments on lines, more strongly than straightness or non- straightness. In particular, in this study all the paths presented corresponded to planar sections (circles on spheres), so the study could not test whether planarity may drive people's judgments. Second, participants responses were affected by viewpoint, to the point that they were more likely to say that small circles seen from the front are straight, compared to large circles seen from the side. This effect again shows that people's ability to identify straight paths is, at best, very fragile. Also, the interaction with viewpoint again reveals that people do not

clearly distinguish between great circles and small circles. Third, (Huey et al., 2022) used a two-alternative forced choice paradigm where participants had to select one of two paths as shorter, this perhaps enforced distinctions in trials where the participants would have been inclined to consider both the paths presented as examples of straight lines, if these paths had been presented one by one.

The current study assesses whether a planar bias is involved in the identification of straight paths, by evaluating whether participants tend to assimilate straight paths with planar sections. To avoid bias related to viewpoint, and to support accurate perception of the paths presented in our participants, we present lines and surfaces from several viewpoints, with 3D animations showing the lines from all possible viewpoints. In two preregistered experiments, we asked two groups of participants to identify "straight lines" or "geodesics" among four types of curves: geodesic and planar sections, non-planar geodesic sections, non-planar non-geodesic sections. In the first experiment, we tested participants without STEM training after high school, giving them an informal definition of straight lines that applies to any surface. In a second experiment, we tested a group of mathematicians (participants with at least a bachelors degree in mathematics), and provided them with a formal definition of geodesics.



## 2. Results

**2.1. Study 1.** We created 26 trials showing a line drawn on a surface, which could be either a sphere, a cylinder, a cube, or a cone [Table 26](#). Across trials, the lines could be planar or non-planar, and they could be straight (geodesic) or non-straight. Lines were presented one by one, and each trial showed two videos of the surface with its line undergoing full 360° rotations in 3D, as well as a static image showing the line from an informative viewpoint [Figure 2.1](#). Surfaces and lines' visual parameters were harmonized: they were of the same color and the same thickness. The picture showed a semi-transparent surface so to show the other parts of the curve by transparency. The rotating shapes were opaque.

A first group of eleven participants (age 19-36, median 22, mean 25.09, 8 female, sample size estimated from power analysis on pilot data) were tasked to try and identify whether the line corresponded to a planar sections of the surface. Participants were given a short definition of a planar section ('A plane intersection is a plane cut of a surface'), then they were shown two examples, one of a planar section and one of a non-planar section on a mushroom, and then proceeded to test. This group was included to ensure participants are accurate in identifying planar sections, and thus to validate the possibility to rely on such a criterion. A second group of participants (n = 23, age 18-31, median = 21.00, mean = 21.9, 17 female, sample size estimated from power analysis on pilot data) were presented with the same stimuli, but this time they needed to judge whether the line presented was straight or not. Instructions defined straight lines as 'A straight line is a line that always follows the same direction without turning, neither to the left nor to the right, and always goes straight ahead'. Time was unlimited, and participants indicated their answer by pressing either 'o' ('oui') or 'n' ('non').

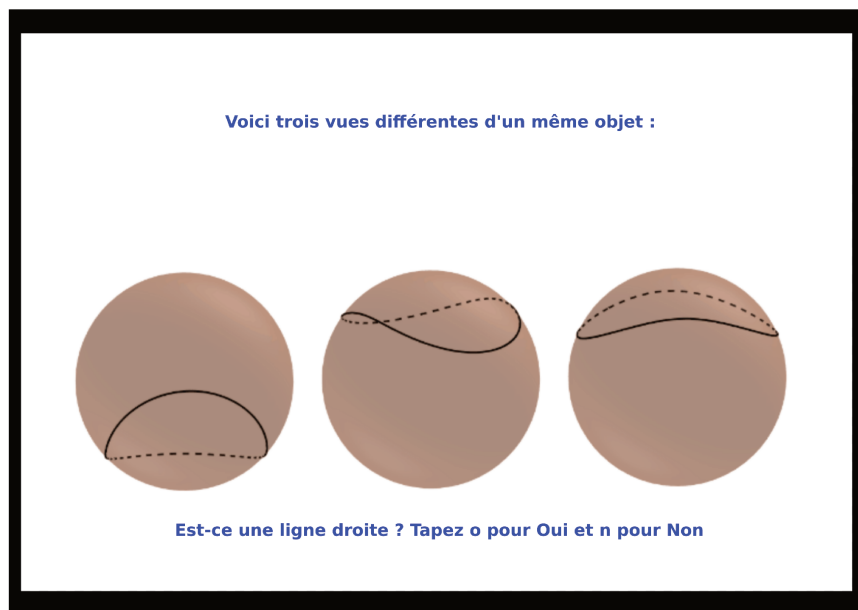


FIGURE 14. Example of a trial of the experiment. On the left, the two figures in rotation around a horizontal axis and a vertical axis, on the right, the image. On the top, "Here are three different views of the same object", on the bottom, "Is it a straight line? Type y for yes and n for no".

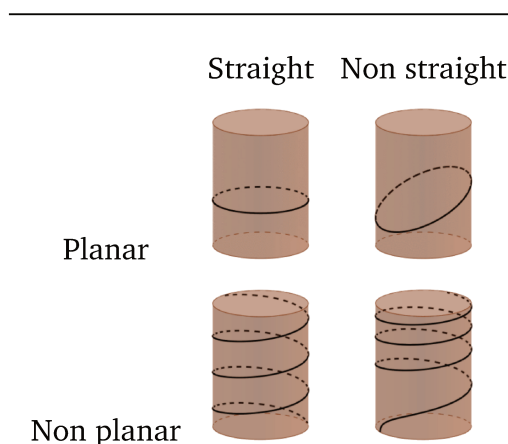


TABLE 22. The four conditions of the stimuli.

**2.1.1. Main analyses.** In both groups, we analyzed the percentage of positive answers according to the straightness and the planarity of the line presented. First, the control group of participants globally succeeded in identifying planar intersections ( $M = 93\%$ ). A 2x2 Anova with 2 within-subject variables for planarity and straightness revealed a significant effect of planarity ( $F(1,10) = 421.06$ ,  $\eta^2 = 0.98$   $p < .001$ ), and no effect of straightness ( $p = .81$ ) nor an interaction between the two variables ( $p = .34$ ). Thus, correctly identifying planar section appears extremely easy.

In the main group, participants were above chance in identifying straight lines in the pool of all stimuli ( $M = 64\%$ ), and As predicted, when asked to identify straight lines, participants tended to answer more positively when the curve was a planar section, as assessed by a significant effect of planarity on the valence of answers ( $F(37,1) = 30.84$ ,  $\eta^2 = 0.71$ ,  $p < .001$ ). The analyses revealed also an effect of straightness, showing sensitivity to geodesics ( $F(37,1) = 65.28$ ,  $\eta^2 = 0.47$ ,  $p < .001$ , and no interaction between the two factors ( $p > .4$ ).

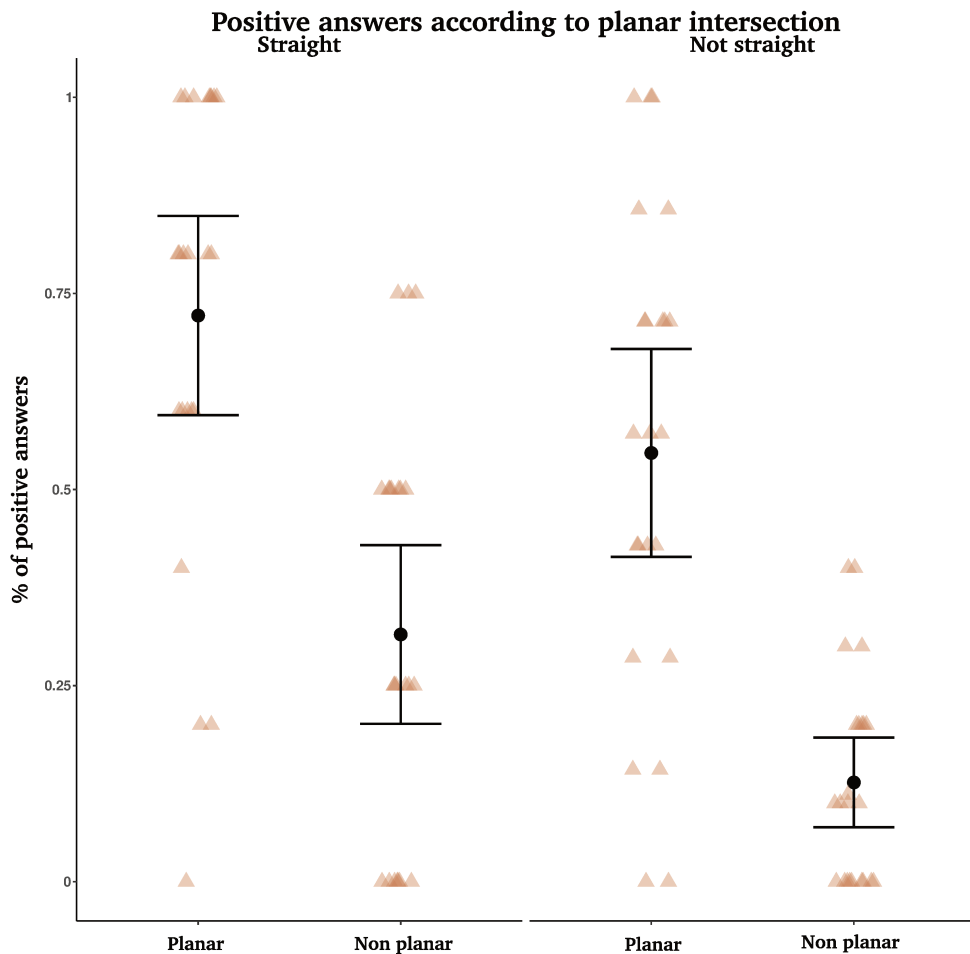


FIGURE 15. Percentage of positive answers to the question "Is this a straight line?" according to planarity for straight and not straight lines. Black dots display the predictions of the anova. Semi-transparent pink triangles show the performance of individual participants.

To check that the responses were not simply driven by other geometric parameters of the lines, such as their curvature or their length, we then conducted further analyses on different subsets of our stimuli. Each subset contained pairs of stimuli matched for the average of the curvature of

the stimulus, seen as a curve drawn in  $\mathbf{R}^3$ <sup>1</sup> -, maximum curvature, and length (within a ratio of 9:10 for mean curvature and length, and 4:5 for maximum curvature). We formed three sets of paired stimuli, that each contrasting along only one variable: First, a set of paired planar lines, one of which was straight and the other was not [Figure 4.3.1](#), second, a set of paired non-planar lines, one of which was straight and the other was not [Figure 4.3.1](#), a set of paired non-straight lines, one of which was planar and the other was not planar [Figure 4.3.1](#). This last subtest was specifically conceived to test for the effect of planarity on peoples responses. A total of thirteen (13) pairs of matched stimuli were thus created.<sup>2</sup>

For each set of paired, we conducted t-tests on the percentage of positive answers by participants. First, these analyses revealed that the responses of the control group were modulated by planarity, as expected: even when the lines were matched for being non-straight, participants were more likely to judge that the line was planar when it actually corresponded to a planar section than when it did not ( $t(10) = 9.12$ ,  $p < .001$ ). Crucially, a similar finding was found for the group judging lines straightness: even when the stimuli were matched for straightness, curvature, and length, participants were more likely to judge that planar lines were straight ( $t(22) = 5.08$ ,  $p < .001$ ). Analyses of the valence of answers on matched pairs of the two other subsets of stimuli (resp. planar, straight vs. not straight, and non-planar, straight vs. not straight), revealed that participants were also sensitive to straightness (Planar, straight vs. non-straight:  $t = 2.45$ ,  $df = 22$ ,  $p\text{-value} = 0.02$ ; Non-planar, straight vs. non-straight  $t = 3.24$ ,  $df = 22$ ,  $p\text{-value} = .004$ ). Detailed results are reported in [Table 24](#).

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<sup>1</sup>Otherwise the curvature of each geodesic stimulus is 0.

<sup>2</sup>Due to geometric constraints, it was not possible to create a fourth set of paired straight lines that planar vs. non-planar. For example, on the sphere, all geodesics are planar sections, so it was not possible to find a twin geodesic for the planar section geodesic.

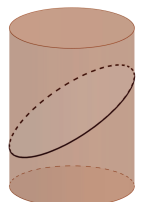
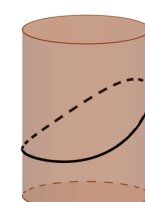
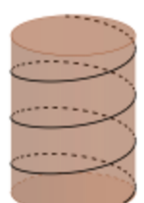
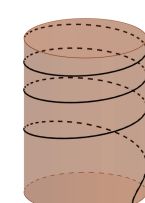
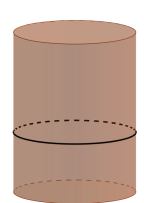
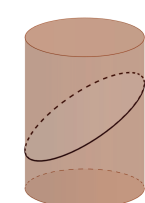
<p>Pairs 1  <b>Not straight</b>            Planar            intersection/            non-planar            intersection</p>	 M = 89%	 M = 19%	$t(10) = 9.12$ $p < .001$
<p>Pairs 2  <b>Non-planar</b>            intersection            Straight/not            straight</p>	 M = 7%	 M = 0%	$t(10) = 1.93$ $p = .08$
<p>Pairs 3  <b>Planar</b>            intersection            Straight/not            straight</p>	 M = 95%	 M = 91%	$t(10) = 0.8$ $p = .44$

TABLE 23. Non mathematicians, planar sections' group. T-tests conducted on pairs matched for maximum and mean curvature, contrasting for planarity (pairs 1, all lines being not straight) or straightness (pairs 2 and 3, all lines being respectively non-planar intersections and planar intersections).

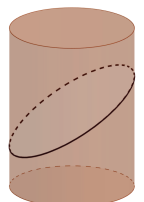
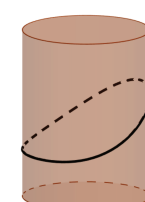
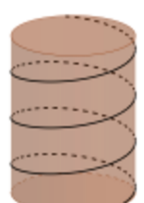
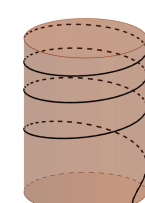
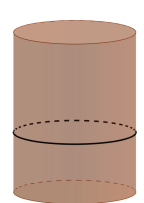
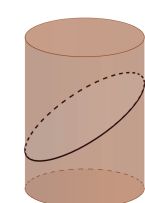
<p>Pairs 1  <b>Not straight</b>            Planar            intersection/            non-planar            intersection</p>	 M = 50%	 M = 13%	$t(22) = 5.0$ $p < .001$
<p>Pairs 2  <b>Non-planar</b>            intersection            Straight/not            straight</p>	 M = 31%	 M = 14%	$t(22) = 2.5$ $p = .02$
<p>Pairs 3  <b>Planar</b>            intersection            Straight/not            straight</p>	 M = 71%	 M = 54%	$t(22)=3.2$ $p < .001$

TABLE 24. Non mathematicians, straight lines group. T-tests conducted on pairs matched for maximum and mean curvature, contrasting for planarity (pairs 1, all lines being not straight) or straightness (pairs 2 and 3, all lines being respectively non-planar intersections and planar intersections).

Our findings indicate that non-mathematicians use planarity as a cue to decide whether a line is straight or not. To test for this idea directly, we conducted a final analysis pooling the responses from the two groups of participants, to see whether the answers of the group that had to identify planar intersections would predict the answers of the group that had to identify straight lines. The analysis revealed that the responses in the two groups are highly correlated, i.e one can predict answers to the question "Is this a straight line?" with the answers to the question "Is this a planar intersection?" ( $t = 6.8635$ ,  $df = 24$ ,  $p\text{-value} < .001$ ,  $r = 0.81$ ).

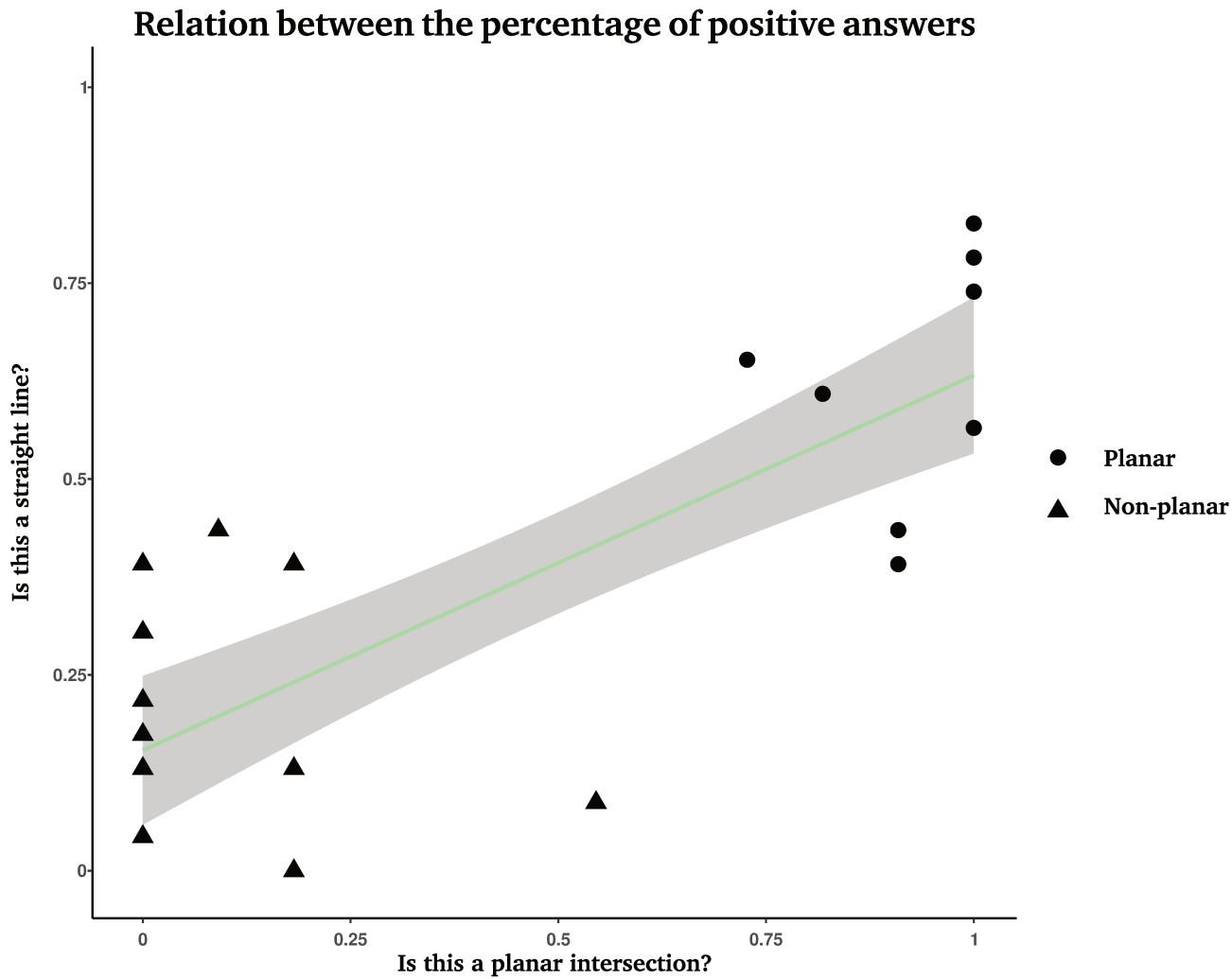


FIGURE 16. Correlation between the responses of two groups asked to judge whether lines are planar or not (x axis), or to judge whether lines are straight or not (y axis).

Finally, we conducted an analysis pooling the main group of participants with the control one, to see if the answers of the group that had to identify planar intersections could predict the answers of the group that had to identify straight lines. The analysis revealed that the answers correlate, i.e. one can predict the answers to the question "Is this a straight line?" with the question "Is this a planar intersection?" ( $t = 6.8635$ ,  $df = 24$ ,  $p\text{-value} < .001$ ,  $r = 0.81$ ).

**2.1.2. Exploratory analyses.** To further explore the effects observed, we conducted two analyses on the responses of the group asked to identify straight lines with additional predictors describing the regularity of the curve, its length and curvature. As curvature is ill-defined on cubes, we conducted an analysis with all the trials testing the effect of regularity, length, and a second analysis without the cubes testing the effect of regularity, maximum curvature, difference

between maximum and minimum curvature, and length. These analyses were conducted using mixed models with random effects for participants, and interaction between planar section and straight factors, as well as a random effect for object.

Both analyses identified significant effects of for planarity (analysis with all stimuli:  $\chi^2 = 13.76$ ,  $p < .001$ , without cubes:  $\chi^2 = 13.29$ ,  $p < .001$ ). None of the other variables tested were significantly related to participants' answers ( $p$ 's  $> 0.5$ ). In particular, there was no effect of straightness in this analysis (with all the stimuli:  $\chi^2 = 3.68$ ,  $p = .055$ , without cubes:  $\chi^2 = 0.82$ ,  $p = .37$ ), suggesting that participants ability to identify geodesic may be rooted on low-level parameters.

**2.1.3. Discussion.** In line with our predictions, we found that adults are biased to identify planar intersections as straight lines. In addition, we also found that participants perform generally above chance in their identification of straight lines, even when planarity was controlled for. Globally, though, participants accuracy was not very high -in line with the results of (Huey et al., 2022). Because our study tested participants without an advanced background in mathematics, we described the notion of straight line informally. Although the participants did not ask for clarification, it is possible that the definition used actually lacked in clarity, and consequently, participants may have reverted to making judgments about planar sections because they did not otherwise understand what was asked of them. To overcome this limitation, we designed a new experiment to test a group of people who could be given a fully unambiguous, formal definition of geodesics: people with an advanced background in mathematics. The definition provided participants with several criteria for identifying geodesics: a formal definition in terms of acceleration, which takes up the intuitive idea that a geodesic is a line that "does not turn" (corresponding to the definition that non-mathematicians were given in Study 1 and a second criterion in terms of local shortest path.

**2.2. Study 2.** Forty-nine mathematicians, tested online, ( $n = 49$ , age = 21-75, mean = 44.43, median = 46, 21 female) were presented with the formal definition of geodesics and had to answer the question "Is this a geodesic?" on 20 different curves drawn on a surface, being either a cone, a cylinder, a sphere or a torus (Table 27).

**2.3. Main analyses.** Participants obtained a good performance in identifying geodesics ( $M = 71\%$ ), but their answers, as predicted, showed a bias of planarity. We used a 2x2 Anova to analyze the effects of planarity and straightness (both within-subjects) on the valence of the answers to



the question "Is this a geodesic?". This analysis identified an effect of planarity on the valence of answers ( $F(48,1) = 51.71$ ,  $\eta^2 = 0.52$ ,  $p < .001$ ), an effect of straightness ( $F(48,1) = 88.04$ ,  $\eta^2 = 0.65$ ,  $p < .001$ ), and no interaction between the two factors ( $p > .8$ ). In line with these first findings, analyses of the valence of answers on matched pairs of stimuli contrasting planar sections and non-planar sections, both non-geodesic, revealed an effect of planarity in this subset of controlled stimuli ( $M = 56\%$  vs  $M = 18\%$ ,  $t(49) = -5.9767$ ,  $p < .001$ ,  $CI = [-0.51, -0.25]$ .)

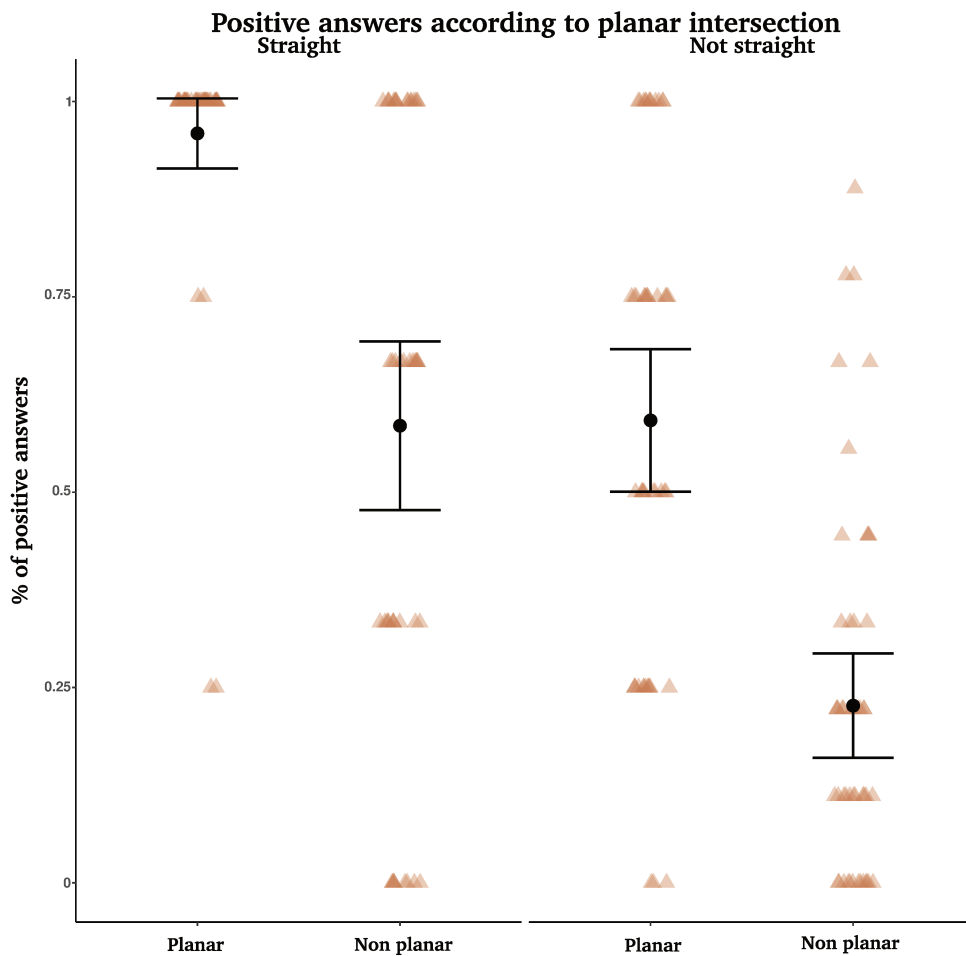


FIGURE 17. Percentage of positive answers to the question "Is this a geodesic?" according to planarity for straight and not straight lines. Black dots display the predictions of the anova. Semi-transparent pink triangles show the performance of individual participants.

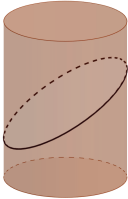
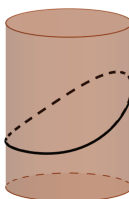
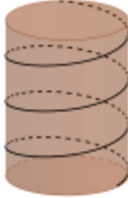
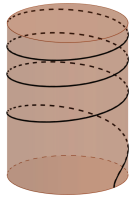
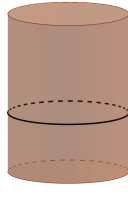
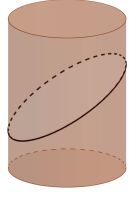
<p>Pairs 1  <b>Not straight</b>            Planar            intersection/            non-planar            intersection</p>	 M = 56%	 M = 18%	<p><math>t(48) = -5.97</math>  <math>p &lt; .001</math></p>
<p>Pairs 2  <b>Non-planar            intersection</b>            Straight/not            straight</p>	 M = 59%	 M = 26%	<p><math>t(48) = 5.36</math>  <math>p &lt; .001</math></p>
<p>Pairs 3  <b>Planar            intersection</b>            Straight/not            straight</p>	 M = 97%	 M = 59%	<p><math>t(48) = 6.79</math>  <math>p &lt; .001</math></p>

TABLE 25. Mathematicians' group. T-tests conducted on pairs matched for maximum and mean curvature, contrasting for planarity (pairs 1, all lines being not straight) or straightness (pairs 2 and 3, all lines being respectively non-planar intersections and planar intersections).

**2.4. Exploratory analyses.** As for the previous experiment, we wanted to assess the role of other properties in the responses of the participants. A mixed model including predictors for the regularity of the curve in  $\mathbf{R}^3$ , the length of the line, its mean curvature, maximum curvature, and difference between maximum and minimum curvature, in addition to planarity and straightness and their interaction, with two random effects for participants and objects, showed an effect of straightness  $\chi^2(1) = 8.93$ ,  $p = .003$  and planarity  $\chi^2(1) = 6.01$ ,  $p = .014$ , but no other significant effect for regularity ( $p > 4$ ), maximum and average of the curvature in  $\mathbf{R}^3$  ( $p > 8$  and  $p > 4$ ), difference between maximum and minimum curvature ( $p > 9$ ), nor length ( $p > 5$ ).

### 3. Discussion

In two experiments, we evaluated the influence of a planarity criterion in identifying geodesics on curved surfaces. We tested two groups of participants: a group of non-mathematicians, with no academic background in mathematics nor other post-highschool STEM fields, and a group of participants who had studied mathematics at least three years after highschool. The first group

received instructions using the common term "straight line", and formulated in everyday language, while the second group was instructed to identify "geodesics", and received a rich formal definition of this notion. Despite these differences, both groups showed a bias to answer positively to the question "Is it a straight line?" (resp. "geodesic") when the line was a planar section.

These findings indicate that participants may have an Euclidean conception of straight trajectories, in that this planar bias is more consistent with a Euclidean conception of straight lines. Indeed, straight lines in the Euclidean plane have several properties (planar, infinite, etc), only some of which remain true on all surfaces. The lines selected by the participants have one crucial property in common with straight lines on the plane: they are planar. It seems that this criterion is an essential property of straight lines for our participants, at the expense of the mathematically valid criterion of geodesics, which is constant direction.

We propose that the participants' planar bias is rooted in a confusion between the directions of space and the directions proper to surfaces. Indeed, to say that a trajectory follows a "constant direction" on a surface requires to compute the direction with respect to the surface itself ("intrinsic direction"). Imagine a bicycle that moves forward without turning the handlebars: it follows the surface, while keeping a constant direction with respect to the surface. However, if we look at the "direction" of the bike in relation to the surrounding space, in which the surface is immersed, its direction changes.<sup>3</sup> We propose that participants use planarity to identify straight lines because doing so minimizes the amount of direction change in  $\mathbf{R}^3$ . Indeed, a planar section always has a coordinate that remains constant in  $\mathbf{R}^3$ , so changed in direction along a planar section are more constrained than along non-planar lines. More generally, this planar bias probably reflects the difficulty of switching from the point of view of space to the point of view of the surface, and thus points out the confusion between intrinsic (directions of the geodesic) and extrinsic (directions of the space in which the surface is immersed) viewpoints.

If this difficulty is at the root of the Euclidean bias we identified, it would be interesting to probe participants intuitions in first-person or action tasks, because performing this type of tasks would force them to adopt an intrinsic viewpoint. Instead of navigation tasks where participants are immersed in wholly different worlds (Widdowson & Wang, 2022) one could ask people to

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<sup>3</sup>Recall that, on curved surfaces, we check that the direction is constant with the parallel transport of the vectors, not identity.

trace straight lines on smaller objects, like a sphere. Perhaps too, this ability to rely on sensory-motor plans to form a first-person view of the surface is what enabled our participants to generally performed slightly above chance for identifying geodesics.

In the end, can we say that Euclidean geometry is more intuitive than other forms of geometry? The straight lines of Euclidean geometry seem to be indeed more intuitive than the straight lines of non-Euclidean geometry. This could perhaps explain why the discovery of non-Euclidean geometries took so much time in the History of mathematics. The Greeks were already quite knowledgeable in spherical geometry, with a focus on spherical triangles made of portions of great circles (Theodosius et al., 1st century BC, 2010). From there, it would have been enough to understand that the great circles of the sphere could play the role of "straight line" to see that the fifth postulate is not a consequence of the other four. Somehow, making this switch appeared too difficult, as it was not possible to relax the absolute referential of space.

While we tested participants with various backgrounds in mathematics, they all had received at least basic education in geometry. Could it be that the bias we observe is a consequence of this education they had received, or more generally to culturally specific factors? The results obtained by (Izard et al., 2011) in a population from the Amazon, the Mundurucu, suggest that the planar bias might be universal, as the Mundurucu also believe that there are parallel lines on the sphere. Conversely, one could try to tackle the universality of our findings by testing whether the bias we have identified can be found even in people who have received a very advanced education in geometry and geodesics. Showing that it is impossible to completely "erase" this bias would provide complementary evidence that the planar bias we identified is deeply entrenched in the human mind.

## 4. Methods

**4.1. Data availability.** Studies are pre-registered at <https://osf.io/4qcf7> (Study 1) and <https://osf.io/7anpu> (Study 2). Data and scripts are available at <https://github.com/charlusb/Straight-Lines>.

**4.2. Study 1.** Based on power analyses conducted on pilot studies, we estimated that we needed to recruit 23 participants in the condition straight-line and 11 participants in the condition planar intersection, participants: group straight 23, 17F, age 18-31, median 21.00, mean 21.9

group inter 11, 8F, age 19-36, median 22, mean 25.09. All participants did the experiment in the lab, were paid and provided written consent. The experiment lasted approximately ten minutes. Inclusion. Based on pilot observations, responses were excluded when the response time was lower than 0.5 ms. Two answers were thus excluded, one in each group. We planed to exclude participants if they had at least three excluded trials, but no participants were excluded for this reason. In the analyses comparing pairs of stimuli, we excluded trials if its match had been excluded.

**4.2.1. Procedure.** Participants were provided instructions by the experimenter and did the experiment on a computer. There was a short introduction giving instructions and a simple definition of "straight line": "a straight line is a line that always goes straight ahead without turning, neither to the right nor to the left. It continues straight ahead indefinitely " (translated from french). Though intuitively stated, this definition corresponds to the mathematical definition of geodesics and gave participants a criterion that generalizes to any surface: a geodesic is a curve of constant direction. Participants performed one trial on an obvious non-straight line (a waving line) and then proceeded to test. At each trial, a 3D- geometrical shape with a line drawn on it was presented on the screen under three perspectives: one video rotating along the y-axis, one video rotating along the x-axis, and one static picture. Participants had to answer the question "Is this a straight line?" by pressing the keyboard for "yes" (o) and "no" (n). Participants were told that could take all the time they wanted to answer. Trial order was random. At the end of the experiment, participants completed a short questionnaire and reported their number of years of education in mathematics.

A second group of participants were presented with the same stimuli, but asked to rate whether the line was a planar intersection or not. The notion of planar section were introduced through a short definition ('A plane intersection is a plane cut of a surface') and two examples that were two pictures, one of a planar cut of a mushroom (example of planar section) and one of a zigzag non-planar cut of a mushroom (example of a non-planar section).

**4.3. Study 2.** According to a power analysis based on the findings of Study 1, we needed 38 participants in Study 2. We posted the task online, until the desired sample size was reached, after exclusions. This procedure resulted in a group of 49 included participants (n = 49, age = 21-75, mean = 44.43 median = 46, 21 female). Their level in mathematics, measured as number

of years of education post highschool, ranged from 3 (Bachelor degree) to 8 (Ph.D.) years, mean 4.24, median 3.

**Inclusion.** It was crucial for the purpose of our hypothesis that participants understood the definition of geodesic, which will be given to them at the beginning of the experiment. Thus, we rejected participants if they declared that they did not understand one of the mathematical terms of the definition, if they declared that they did not understand the definition, or if they stated that the definition is not clear. Furthermore, in this experiment we wished to recruit participants who possessed all the conceptual tools to understand the definition of geodesics, but had no strong familiarity with this notion. We thus excluded participants who declared being familiar with the notion of geodesic to the point that they use it in their work. More specifically, we excluded participants who checked the answers "I know it quite well and have already used it in my work", "I know it well, I use it regularly in my work" or "I know it very well, this notion is central to my work" to the question "Estimate your level of familiarity with this concept [geodesic] based on the criteria below:", among a choice of answers.

#### 4.3.1. Procedure.

**Stimuli.** The stimuli were similar to Study 1. We presented 21 different trials with lines drawn on four geometrical surfaces: cone (5), cylinder (9), sphere (3), and torus (4), using parametrized curves on these surfaces in Geogebra 5.

**Pairs.** Thirteen (10) pairs of matched stimuli were created.

**Definitions.** The mathematicians received the following definition:

Geodesics generalizes to curved surfaces the notion of straight line in the plane.

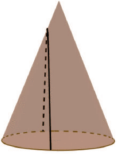
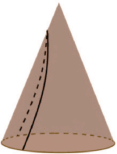
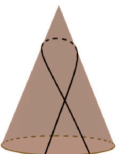
We will consider here the case of a surface immersed in the usual Euclidean space  $\mathbf{R}^3$ . Let  $c(t)$  be a curve drawn on this surface, such that the norm of the velocity  $|c'(t)|$  is constant. The curve  $c(t)$  is a geodesic if and only if its acceleration vector  $c''(t)$  is normal to the surface at any point.

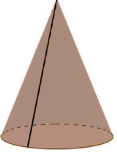
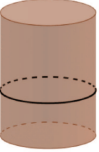
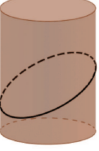
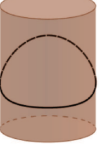
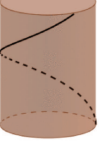

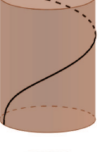
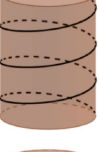
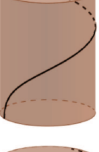
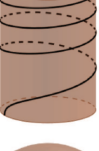
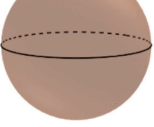
Equivalently, the curve  $c(t)$  is a geodesic if and only if it corresponds locally to the shortest path.

Intuitively, the trajectory of a bicycle moving on a surface describes a geodesic when the cyclist moves forward at constant speed without turning the handlebars.

Measure of regularity. During pilot testing, some of our participants reported using the properties of symmetry of the curve to decide whether it was straight or not. More generally, it is possible that our participants relied on a notion of overall regularity to respond: if the straight line is what goes straight without turning, that limits irregularities - at least on regular surfaces like ours.

To quantify the degree of regularity of our curves, we identified the group of isometries (translations, rotations, symmetries, and combinations thereof) that left the curve unchanged. We then counted how many primitive isometries were necessary and sufficient to reconstruct the whole group of invariant isometries. For example, for a rectangle, the number of primitive invariant transformation is three: 2 symmetries following the two main axes of the rectangle, and 1 symmetry with respect to the plane where the rectangle lies. These elementary symmetries are sufficient to reconstruct all other invariant transformation: for example, combining the 2 symmetries along the axes of the rectangle yields a  $180^\circ$  rotation around the center of the rectangle. In some cases the number of primitives was infinite: for example, for a circle all the rotation around the center of the circle qualify, the angle of the rotation can vary in a continuous manner. The numbers thus obtained for each stimulus were then rank-transformed. Regularity scaled from 0 to 6.

Stimulus	Straight	Planar	Average curv.	Max. Curv.	Diff. max-min	Length	Reg. $\mathbf{R}^3$
	No	Yes	1	1	0	12.57	6
	No	Yes	0.46	5	5	8.9	2
	No	No	0.52	5	4.97	13.44	1
	No	No	0.97	1.64	1.57	11.16	1
	Yes	No	0.89	1.72	1.7	10.34	1

	Yes	Yes	Undefined	Undefined	Undefined	9.94	2
	Yes	Yes	0.5	0.5	0	25.13	6
	No	Yes	0.47	0.56	0.16	26.64	3
	No	No	0.49	0.71	0.36	26.97	1
	Yes	No	0.4	0.4	0	11.86	6
	No	No	0.39	0.44	0.29	9.44	0
	No	No	0.44	0.61	0.38	13.1	0
	Yes	No	0.49	0.49	0	42.69	6
	No	No	0.47	0.82	0.48	18.14	0
	No	No	0.5	1.29	0.83	45.83	0
	Yes	Yes	0.4	0.4	0.4	31.42	6



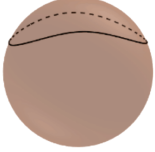
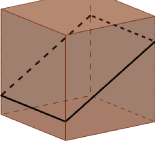
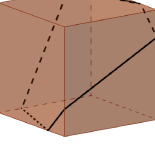
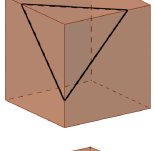
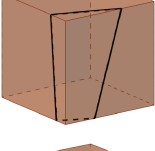
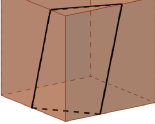
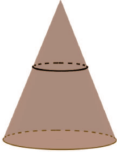
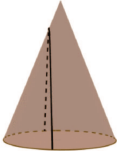
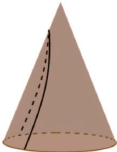
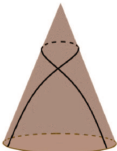
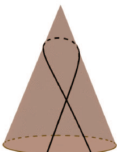
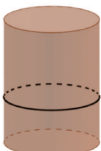
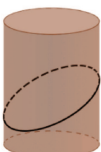
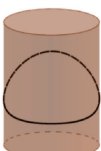
	No	Yes	0.57	0.57	0	11.11	6
	No	No	0.5	0.63	0.23	13.63	2
	Yes	Yes	Undefined	Undefined	Undefined	16	5
	No	Yes	Undefined	Undefined	Undefined	17.26	3
	Yes	No	Undefined	Undefined	Undefined	18	0
	No	No	Undefined	Undefined	Undefined	18	0
	Yes	Yes	Undefined	Undefined	Undefined	16.97	4
	No	Yes	Undefined	Undefined	Undefined	12.5	4
	No	Yes	Undefined	Undefined	Undefined	13.1	0
	No	Yes	Undefined	Undefined	Undefined	13.86	0

Table 26: Table of stimuli presented to non-mathematicians participants in Study 1. For each stimulus, the average and the maximum of the curvature of the curve (seen as a curve drawn in  $\mathbf{R}^3$ ), length, and regularity measure - on the cube, curvature is not defined.

Stimulus	Straight	Planar	Average curv.	Max. curv.	Diff. max-min	Length	Reg. $\mathbf{R}^3$
	No	Yes	7.54	0.83	0.83	0	6
	No	Yes	11.21	5	0.38	5	2
	No	No	11.28	5	0.56	5	1
	No	No	13.99	1.29	0.82	1.25	1
	Yes	No	12.78	1.48	0.76	1.45	1
	Yes	Yes	25.13	0.5	0.5	0	6
	No	Yes	26.64	0.56	0.47	0.16	3
	No	No	26.97	0.71	0.48	0.36	1


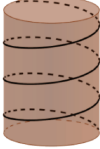

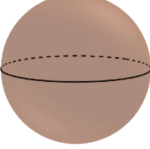
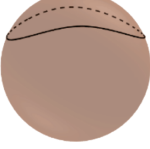
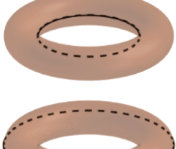
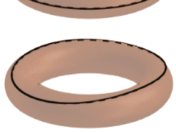
	Yes	No	11.86	0.4	0.4	0	6
	No	No	9.44	0.44	0.39	0.29	0
	No	No	13.1	0.61	0.44	0.38	0
	Yes	No	42.69	0.49	0.49	0	6
	No	No	18.14	0.82	0.47	0.45	0
	No	No	45.83	1.29	0.5	0.83	0
	Yes	Yes	31.42	0.4	0.4	0	6
	No	Yes	11.11	0.57	0.57	0	6
	No	No	13.63	0.63	0.5	0.23	2
	Yes	Yes	9.42	0.67	0.67	0	6
	Yes	Yes	15.71	0.4	0.4	0	6
	No	No	14.31	0.5	0.44	0.17	2

Table 27: Table of stimuli presented to mathematicians participants in Study 2. For each stimulus, the average and the maximum of the curvature of the stimulus (seen as a curve drawn in  $\mathbf{R}^3$ ), length, and regularity measure.

a set of paired planar lines, one of which was straight and the other was not; second, a set of paired non-planar lines, one of which was straight and the other was not; a set of paired non-straight lines, one of which was planar and the other was not planar. This last subtest was specifically conceived to test for the effect of planarity on peoples responses. A total of thirteen (13) pairs of matched stimuli were thus created

paired planar lines, one of which was straight and the other was not

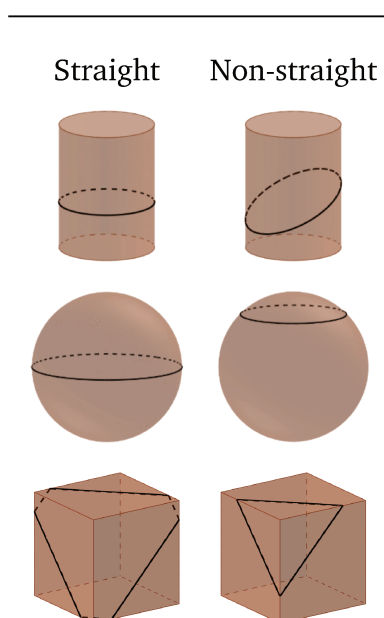


TABLE 28. Set of planar, straight versus non-straight matched pairs of stimuli (cubes were not part of the stimuli in the experiment with mathematicians).

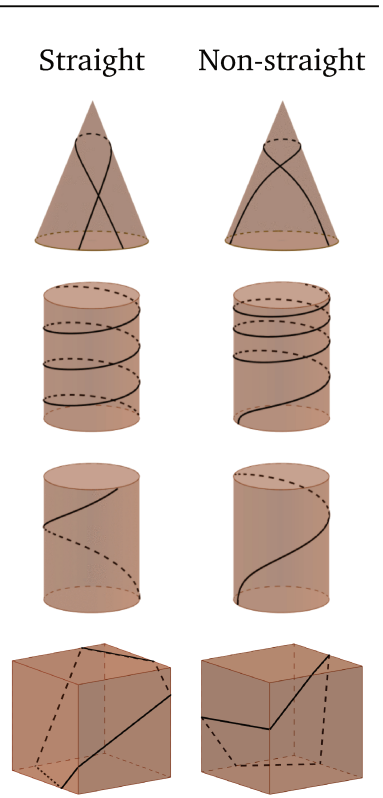


TABLE 29. Set of non-planar, straight versus non-straight matched pairs of stimuli (cubes were not part of the stimuli in the experiment with mathematicians).

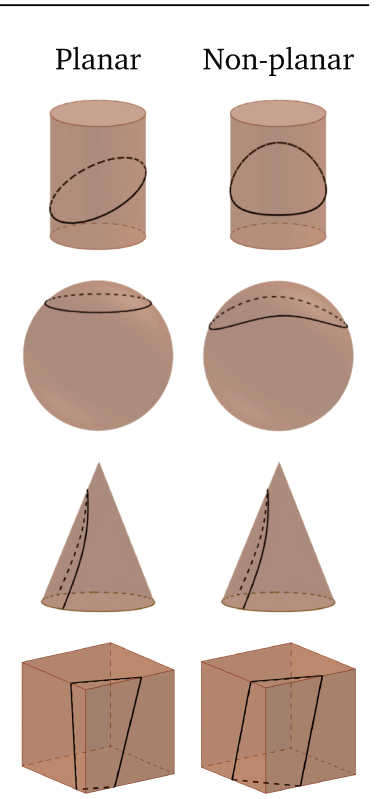


TABLE 30. Set of non-straight, planar versus non-planar matched pairs of stimuli (cubes were not part of the stimuli in the experiment with mathematicians).

## **Conclusion**

The objective of this thesis was to contribute an experimental answer to the question of how new concepts are learned in mathematics, in other words how an academic concept is integrated into a network of intuitive knowledge. We defined three objectives: 1) to set up a learning situation in a laboratory session that would allow us to observe effective learning for a complex concept in a relatively short period, 2) to set up behavioral measures that would allow us to track learning as it unfolds, 3) to describe the conceptual content of mental representations and intuitions specific to our study concept, geodesics, before, during and after learning. Finally, we investigated more systematically the possibility that people's spontaneous concept of straight line is biased towards Euclidean geometry.

I summarize here chapter by chapter the objectives, contributions, and limitations of each study. Finally, I discuss further questions raised by this thesis work.

## Chapter 1

In [Chapter 1](#), I presented a qualitative analysis of the intuitions - collected in the form of definitions written by the participants during the experiment - associated with the concept of straight line before learning, as well as the evolution of this concept during the learning process. The analysis of the first definition, which thus corresponds to the participants' spontaneous intuitions, shows that they tend to define the straight line in terms applying to Euclidean straight lines but not to geodesics: the characteristic properties mentioned most often are that this line is infinite, that it follows a fixed direction in 3-dimensionnal space, and is without curvature. Nevertheless, participants do not explicitly mention that the straight line must be planar, or parallel to the plane (4%). They also rarely mention the fact that straight lines are drawn on a plane (5%).

To explain this lack of explicit mention of the plane as reference frame, one can recall that the straight directions of space correspond to the straight lines of the plane: the participants may therefore reduce themselves to a reference frame that they consider as absolute,  $\mathbf{R}^3$ , and which coincides with Euclidean criteria. Another possibility is that the plane seems so obvious to them that they do not mention it. In fact, the plane is, on the other hand, mentioned more in definitions 2 (11%) and 3 (10%) even if the mention remains rare. The straight line seems to be an atomic notion, as is also shown by the definitions that use the term "straight" (20% mentioned in definition 2 and 11% in definition 3).

Participants correctly identify, for 35% of them, the criterion that generalizes to geodesics: constant direction, but in 11% of the cases define it using only  $\mathbf{R}^3$ 's directions, 12% as an ambiguous criterion ("constant direction"), and 12% of the cases mention the constant direction in a way that really lends itself to generalization: a trajectory that does not deviate, does not turn.

The analysis of definitions 2 and 3 shows that participants have learned: they modify some of the categories by preferentially mentioning "without turns", over the other categories expressing the idea of constant direction, which does not increase throughout the experiment. Crucially, they reverse the value of the property "curvature", admitting that a straight line can eventually have a non-zero curvature.

Despite this, participants also show some rigidity during learning, as some properties remain essential for them, even if they do not lend themselves to generalization: this is the case of infinity, and self-intersection, which does not appear at the beginning but emerges in reaction to the modification of the concept, limiting the range of possible changes to the concept of straight lines. In the case of infinity, this may not be a sign of a complete blockage, but of an attempt to reconcile their previous, "unbounded" concept with the new example of the straight line. Indeed, some participants mentioned in this category "the line that has no beginning or end" or "that passes infinitely over itself." This shows an attempt to apply the category "infinite" to a curve drawn on a compact/fine surface.

The analysis of definition 1 is a first descriptive step for assessing the Euclidean character of geometrical intuitions, and has its natural continuation in the study presented in [Chapter 4](#). Definitions 2 and 3, however, show something different: how participants form intermediate representations composed both of their prior conception, which is not erased, but also of the new information given. Among these intermediate models, there are hybrid models where the different cases are not merged, but merely co-exist: e.g. "*Sur une surface plane, [la ligne droite est une] ligne en forme de droite qui ne tourne ni à gauche ni à droite (pas de rotation) et qui se prolonge à l'infini des deux côtés. Sur une sphere, une ligne droite est un cercle.*", i.e. "*On a planar surface, [a straight line is a] line that does not turn left or right (no rotation) and extends to infinity on both sides. On a sphere, a straight line is a circle.*".

We also find intermediate models that try to reconcile the different representations. The mention of "spherical plane", for example, is an attempt to reconcile the default reference frame of the plane with the new representations given: e.g. "[une ligne droite est] *une infinité de points qui forment le plus court chemin entre deux points d'intersection relativement au plan (qu'il soit sphérique ou "plat")*", i.e. "[a straight line is] *an infinity of points that form the shortest path between two points of intersection relative to the plane (whether spherical or "planar")*". This idea is completed in the notion of developable surface, where a surface is a sheet that unfolds in a plane - note that this is not possible in the case of the sphere: "*Trait infini dont la trajectoire ne varie pas dans un plan de l'espace (si on déplie un cube avec une droite donnée dessus pour obtenir un seul plan, on retrouve une ligne droite sur un plan de l'espace.*", i.e. "*Infinite line whose trajectory does not vary in a plane*".



of space (if we unfold a cube with a given line on it to obtain a single plane, we find a straight line on a plane of space."

To some extent, the planar section may be seen as a criterion resulting from an intermediate model, attempting to reconcile a strictly Euclidean view with curved surfaces: "*Une ligne droite est une ligne qui relie un point A à un point B, par la plus courte distance possible. Lorsqu'elle tranche une forme 3D, les nouvelles surfaces obtenues des deux moitiés sont planes et sans aspérités ou plus d'une variation de niveau.*", i.e "A straight line is a line that connects point A to point B, by the shortest possible distance. When it cuts a 3D shape, the new surfaces obtained from both halves are flat and without any asperities or more than one level variation."

In the most extreme cases, intermediate models are built that accept almost the negation of the concept: "*Une ligne droite peut cependant varier dans un autre plan de l'espace sphère, dans ce cas-là elle ne sera plus droite en vue de tous les plans de l'espace différents.*", i.e "A straight line can however vary in another plane of the sphere space, in this case it will not be straight anymore in view of all the different planes of the space.", "*Une ligne droite est définie par au moins deux points distincts. Elle est courbe sur une sphère si elle passe par le grand angle et droite sur un plan.*", i.e "A straight line is defined by at least two distinct points. It is curved on a sphere if it passes through the large angle and straight on a plane.", "*Une ligne droite, qu'elle soit sur une surface plane ou sphérique, est une ligne qui unit 2 points de l'espace "sans détour". Toutefois, sur une surface sphérique, une ligne droite n'est plus nécessairement la trajectoire tracée par un objet qui va tout droit.*", i.e "A straight line, whether on a flat or spherical surface, is a line that joins 2 points in space "without turning". However, on a spherical surface, a straight line is no longer necessarily the path traced by an object going straight."

4

This suggests that learned data interact at least partially with intuitions and that mental representations reflect this interaction, potentially consisting of different intermediate models. This does not say what happens to these models, which may be too fragile to be accepted and used, but it does show that an effort of integration is made.

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<sup>4</sup>In the same vein, qualitative studies have shown how children compose mental models from contradictory data (Subramaniam & Padalkar, 2009), notably creating the "hollow Earth" that combines both the experience of a flat earth and the information that the earth is round (Vosniadou, 1992).

## Chapter 2

Chapter 2 aimed at presenting a learning situation to analyze learning from objective and subjective perspectives. I designed a paradigm that allows me to observe significant progress in a single session, which shows that the teaching intervention was effective. Moreover, our experimental conditions allowed me to modulate this learning and achieve different levels of generalization: most participants understood that they have to reject some of the planar sections of the sphere, but few of them managed to perform better in a reasoning test. I have shown that this performance follows the experimental condition in which the participants are placed. This chapter has also linked the phenomenon of insight to learning, and shown that insights (i) reflect effective learning progress, are modulated by the experimental condition (number of lessons presented), (ii) are dissociated from introspective measures of confidence, including one taken just before reports of insights, (iii) crucially, insights signal more advanced stages of generalization, showing that they are not mere illusions of understanding. The test positively associated with insights is a test where one has to understand that there are other types of geodesics than planar sections, which is fundamental in reaching an advanced understanding of the concept. Interestingly, the confidence measures are not associated with any particular task. We can therefore conclude that insight experiences are subjective phenomena that occur during learning, and that do not necessarily coincide with confidence taken as a measure of introspection.

This study indicates that insight is linked to effective learning and is not reduced to confrontation with difficulty. This is an important first step toward understanding how insight experiences arise, but we still need to understand the mechanisms by which insight appears if we want to use it as a measure of learning. In particular, understanding whether insight signals learning that takes place outside the field of consciousness could be explored. Indeed, the dissociation with confidence that we observed echoes many observations since it is even one of the characteristics of insight to occur suddenly and unexpectedly. (Metcalf, 1986) had observed that a few seconds before an insight, participants tend to estimate that they are far from solving the problem. Recently, researchers have proposed a computational model of insight, describing it as an error in estimating how long it will take to solve a problem (Dubey et al., 2021). This model gives an algorithmic description of insight as a prediction error of introspection: the solution occurs when one does not expect to find it. Consistently, the intensity and the insight also depend on the difficulty, the time already spent on a problem, and the number of steps to solve a problem, as in

(Danek & Wiley, 2017). In this respect, the relationship between insight and motivation could be explored: insight may also depend on the desire to solve a problem, and on past investment, such that the greater the investment, the greater the reward. It remains, however, to be understood why these prediction errors occur and whether certain learnings give this sensation of an impasse.

In the case of insight, the literature has been built around a very narrow set of problems, such as anagrams, puzzles, and problems traditionally associated with problem-solving. Since then, other contexts have been discovered in which subjective sensations with all the characteristics of insight experiences appear, and indeed it seems more fruitful to define insight phenomena in terms of subjective sensation. This tendency to co-define insight by simply solving this type of problem has been criticized as leading to a circular definition (Bowden et al., 2005). researchers tended to postulate that insight had occurred whenever a participant had solved one of the classic so-called "insight problems", without checking what sensation the participant had experienced when finding the solution. This then raises the question of the scope of the contexts in which insights arise, why these contexts, etc. In the particular case of insight induced by learning contexts, are there intrinsic characteristics of the objects of study that give rise to this type of phenomenon? This raises the question of the relationship between consciousness and learning.

### Chapter 3

Chapter 3 aimed at studying the learning curve for mathematical concepts. The main issue was to design a measure to study understanding at different times of the experimental session, to try to establish what form the dynamics of concept learning takes. The challenge was to avoid measuring it through performance, for two reasons i) because it is difficult to design a task to measure learning that is neither redundant or unbalanced, ii) because questions about the target concepts may influence the participants' response by playing the role of implicit feedback, and thus, interferes with the learning curve one wishes to measure. We also wanted to avoid an introspective measure that requires asking participants if they understood. Instead, we measured participants' reaction to the learning material: we asked participants how compelling the material seemed to them. We hoped to observe an effect of lesson position on the score given, and in particular, what would have validated the measure would have been an overall increase in the scores. We expected that lessons placed at the beginning of the sequence would score lower overall than those placed at the end, that scores would increase throughout the experiment, either linearly or in steps. Unfortunately, while a first experiment suggested that there was an overall

positive effect of position on scores, further work is needed to assess whether the measure could possibly be used as a measure of learning progress.

## Chapter 4

Chapter 4 focused on the content of geometric intuitions about straight lines. This is a study evaluating whether there is a planar bias in the perception of straight trajectories on curved surfaces. In Chapter 1, the descriptive study of the properties mentioned by our participants suggested that, overall, properties specific to planar straight lines were prominent in the definitions produced. This was not a complete coincidence, especially since most of the criteria mentioned were neutral. Moreover, this first study did not allow for a formal test of this planar tendency. In Chapter 4, we systematically test this tendency, assessing whether participants tend to prefer planar to non-planar lines, although this criterion is not correct for identifying straight trajectories. We obtained consistent responses with this hypothesized planar bias, both from naive participants and from non-expert mathematicians from non-expert mathematicians who received a formal definition of geodesics. These findings contribute to extend our knowledge of a long-standing question: whether the human mind may be shaped to think of space in Euclidean terms.

This does not imply, of course, that the human mind is perfectly attuned to Euclidean geometry. One can assume that depending on the context, the reasoning and the spatial principles may vary. If the observed planar tendency is indeed due to an attempt to minimize directional changes from the extrinsic point of view, then one may think that Euclidean geometry is simpler to consider because of its computational simplicity (at least one coordinate remains constant). Do people possess intuitions that would support the identification of straight trajectories? We suspect that using motor responses could elicit such intuitions, as getting people to adopt a first person perspective would help them access a notion of direction defined intrinsically to the surface.

Finally, in order to better understand the evolution of academic concepts, we can also wonder about the persistence of this bias in expert mathematicians. In (Shtulman & Valcarcel, 2012; Allaire-Duquette et al., 2021), interference is observed even in experts, through delayed response times. Is it possible that the accuracy bias observed in mathematician participants also persists in

our paradigm as a response time signature? This would test the robustness of the intuition at the origin of this planar bias, and call in favor of a coexistence scenario in this case.

### Further perspectives

This research was undertaken to better understand the nature of mathematical concepts. We have addressed the question of learning mechanisms, described specific geometrical intuitions, and now suggest two ways to address the question of the format in which these concepts are represented in the mind.

Before presenting these perspectives, let us return to one of the characteristics of mathematics: it is a deductive science, which can be written as a formal language whose grammar is given, to some extent, by first-order logic and a set of axioms. Kant classified mathematics as a formal science describing it as an *analytical science*, i.e. one whose premises are contained in the conclusion, like logic. Under this perspective, no empirical contribution participates in mathematical discovery. This formalist vision of mathematics has had a long posterity: it is found in the foundations of mathematics at the beginning of the 20th century when numerous mathematicians thought that mathematics should be formalized and written in the form of very precise rules of calculation. The reasons however were different: to Hilbert, the goal was to ensure the coherence of mathematical systems (typically Peano's arithmetic), for others, like Kronecker and Brouwer, it was a philosophical and normative position about mathematical practice, to stick to an epistemic realism in reaction to the emergence of Cantor's set theory: is a mathematically valid proposition what is cognitively accessible by human intuition. For others, at last, mathematical statements should be, in principle, calculated explicitly, to make the processes of calculation automatable (Turing, Church) -this is today's computer science.

This vision has conditioned a part of mathematics teaching according to which doing mathematics is above all calculating and writing valid reasoning, however, it is not clear that mathematical skills are reducible to operations of formal logic. Indeed, most evidence in cognitive sciences points in another direction.

First, research on number sense, which represents a considerable part of the research on mathematical cognition, tends to show that the manipulation of numbers recruits above all intuitions

of approximate magnitudes, affected by distance variations. This deviates from the algorithmic description of natural numbers, which are generated by an initial element (0) and the iteration of the successor operation.

Second, deviations from logical rules have been observed, since mathematical, and even logical reasoning suffers from context effects: e.g (Stanovich, 1999; Wason, 1960, 1968; Gros et al., 2021). In our results, several elements seem to indicate conceptions that are not reduced to a formalist conception of mathematics. What, then, is the meaning of mathematics for thinking? In our future research, we would like to study two aspects in particular to explore the foundations of mathematics in the mind.

**4.4. Mathematical concepts as prototypes.** Mathematical concepts are characterized by a definition that is a set of necessary and sufficient conditions, unlike most concepts in everyday life - it is impossible, at first sight, to give a satisfactory definition of the concept of "duck". Common sense concepts are interdefined, to the point that their semantic networks "loop": different dictionary entries refer to each other, sometimes circularly. Common sense concepts sometimes loop. Beyond common sense, this is also the case in science: if we consider for example Newtonian physics, it is difficult to define the concept of force without appealing to the concept of mass, and vice versa. On the contrary, mathematical concepts are well defined in the sense that one can decompose without circularity each concept with the help of rules, going back to the atoms of the definition, the primitives. In fact, the construction of new concepts in mathematics proceeds in a combinatorial way, from concepts already introduced. We can therefore think that the representations we have of mathematical concepts have a particular status, however, the definitions may not be an accurate description of the way mathematical concepts are represented in the mind.

First, it was difficult for our participants to define the concept of straight line. Indeed, they did not only give synonyms, or elementary bricks but sometimes concepts that are neutral with respect to the definition. Perhaps, then, participants failed to give a definition of straight lines because they conceive straight lines as a primitive concept, which cannot be decomposed. Alternatively, it is also possible that straight line is a compositional concept based on simpler notions in their mind, but they fail to consciously access the criteria for identifying straight lines, and the primitives on which these criteria are built. Nevertheless, this does not bode well for the practice of mathematics, which requires the explicitness of criteria. Perhaps, then, these participants are

merely insufficiently experienced in mathematical practice. However, we can see that giving a definition does not eliminate the bias of mathematicians, which indicates that it is not enough to have a valid definition at one's disposal to master a concept.

We can also notice that people tend to form intermediate models rather than coherent propositions. In [Chapter 1](#), we see that participants seem to arrive at representations of the straight line that combine properties of different models, without merging them, as in the [examples mentioned in 4.3.1](#). Yet, these hybrid models present inconsistencies, which are not compatible with a mathematical definition.

Moreover, people tend to reason about examples of a concept instead of the definition. Indeed, some reasoning errors in mathematics education stem from the fact that participants focus on a canonical example, doing so, they reason about a class representative. This is the well-known example of squares and rectangles ([Zazkis & Leikin, 2008](#)): why cannot children easily acknowledge that a square is a particular case of a rectangle? It may be because the concept "rectangle" is usually represented as a non-square rectangle. Yet, if one would ask the question using a definition: "Is a square a quadrilateral with four right angles?" children would probably have no trouble checking the properties, and answering correctly, as an exercise.

As an alternative to a strictly combinatorial description of concepts, one can cite the theory of prototypes, e.g. ([E. H. Rosch, 1973](#); [E. Rosch, 1983](#)). According to this theory, some representatives of a set of concepts are more salient than others, either because they are more central (they share the most properties with the other members of their class), or because they are more familiar. It would be interesting to explore the place of such reasoning in mathematics. According to this theory, concepts are not perceived as a set of elementary bricks that can be decomposed, but as essential cores with layers of more or less superficial characteristics.

If reasoning uses prototypes, one of the difficulties of mathematics would then be to learn how to cut up and decompose concepts that are *a priori* perceived as unbreakable. All the more so since there is not always a single possible decomposition of a concept: let us take the example of the circle. From the mathematical point of view, the circle is the set of points that are at the same distance from a center, but a circle can be described non-mathematically in several ways: it

is what can be drawn with a compass, it is the circumference of the trajectory of a spinning top.<sup>5</sup> In particular, a mathematical concept can receive different pseudo-definitions from naive physics. Do people converge when trying to decompose a concept?

**4.5. Physical foundations of intuitive mathematics.** As a second avenue, we would like to explore the idea that mathematical concepts are rooted in physical knowledge, and that mathematical reasoning arises from the manipulation of these representations.

This idea is in line with many studies that explore the role of analogy in learning and the role of reasoning about concrete problems in learning scientific (Donnelly & McDaniel, 1993; Gentner & Holyoak, 1997) and mathematical concepts (Ngu & Phan, 2020; Alexander et al., 2013). We would like to explore a somewhat stronger hypothesis: that mathematical reasoning does not only benefit from physical representations, but that it is an extension of physical reasoning, that it co-evolves with the development of intuitive physics, and that these take precedence over the use of logic.

The question is even more crucial in the field of geometry, which is a mathematical characterization of shapes and space. Could we not think that naive physics is also a source of intuition in geometry?

More precisely, we would like to test to what extent geometric concepts have emerged as extensions of physical notions, and that the mind associates them with objects subject to the relations predicted by naive physics. This could explain some interferences in mathematics learning: indeed, in (Krohn et al., 2021), the intuitive theories interfering with performance are common sense theories. And these common-sense theories, in the case of geometry, are often rooted in the perception of objects.

Furthermore, geometry can emerge due to the physical constraints of the environment. Humans are not the only animals that do geometry: Geometric patterns and regularity of shapes appear in the behavior of insects, from the webs of spiders to the regularity of hexagonal cells in the honeycombs of bees, to birds, which build nests in circles and ovals. These geometric patterns optimize properties that matter to the animals constructing them, such as strength -interaction between web shape and strength (Du et al., 2006; Cranford et al., 2012)-, the number of cells -hexagons maximize surface area while minimizing the wax used (Tóth, 1964; Klarreich, 2000)-,

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<sup>5</sup>Even in mathematics, concepts can have different definitions from the point of view of geometry, analysis, etc.



volume -protection of offspring, (Mainwaring et al., 2014). These symmetries and regularities testify to animal engineering used as a geometric response to the physical constraints of the environment.

Similarly in the history of the straight line, there are traces of the links between physical representations and geometric concepts: in Greeks' literature, the straight line emerges in analogy with various objects, the taut rope used to trace in construction, the light or visual ray, and the axis of a rotation. These physical intuitions behind geometry's emergence suggest that there are physical properties of these geometric objects that have made them endure in human representations, and that, possibly, both recruit similar intuitions in the human mind.

## References

- Alexander, P. A., White, C. S., & Daugherty, M. (2013, 4). *Analogical reasoning and early mathematics learning*. Taylor and Francis Inc.
- Allaire-Duquette, G., Brault Foisy, L. M., Potvin, P., Riopel, M., Larose, M., & Masson, S. (2021). An fMRI study of scientists with a Ph.D. in physics confronted with naive ideas in science. *npj Science of Learning*, 6(1), 38–41.
- Ansburg, P. I., & Dominowski, R. L. (2000). Promoting insightful problem solving. *Journal of Creative Behavior*, 34, 30-60.
- Archimedes. (225 BCE, 2010). *On the sphere and the cylinder, book i*. Cambridge University Press.
- Asmuth, J. A., & Rips, L. J. (2006). Conceptual change in mathematics. In (pp. 30–35). The Cognitive Science Society.
- Baillargeon, R., Needham, A., & Devos, J. (1992). Intuitions about Support. *Early Development and Parenting*, 1(2), 69–78.
- Baillargeon, R., Spelke, E. S., & Wasserman, S. (1985). Object permanence in five-month-old infants. *Cognition*, 20(3), 191–208.
- Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and Equivalence of Rational Numbers: A Clinical Teaching Experiment. *Journal for Research in Mathematics Education*, 15(5), 323–341.
- Bonawitz, E., Ullman, T., Gopnik, A., & Tenenbaum, J. (2012). Sticking to the evidence? A computational and behavioral case study of micro-theory change in the domain of magnetism. *2012 IEEE International Conference on Development and Learning and Epigenetic Robotics, ICDL 2012*.
- Bonawitz, E., Ullman, T. D., Bridgers, S., Gopnik, A., & Tenenbaum, J. B. (2019, 8). Sticking to the evidence? a behavioral and computational case study of microtheory change in the domain of magnetism. *Cognitive Science*, 43, e12765.
- Borrelli, V., Jabrane, S., Lazarus, F., & Thibert, B. (2013). Isometric embeddings of the square flat torus in ambient space. *Ensaïos Matemáticos*, 24, 1-91.

- Bowden, E. M. (1997). The effect of reportable and unreportable hints on anagram solution and the aha! experience. *Consciousness and cognition*, 6(4), 545–573.
- Bowden, E. M., & Jung-beeman, M. (2003). Normative data for 144 compound remote associate problems. *Behavior Research Methods, Instruments, & Computers*, 35(4), 634–639.
- Bowden, E. M., Jung-Beeman, M., Fleck, J., & Kounios, J. (2005). New approaches to demystifying insight. *Trends in Cognitive Sciences*, 9(7), 322–328.
- Brannon, E. M., Abbott, S., & Lutz, D. J. (2004). Number bias for the discrimination of large visual sets in infancy. *Cognition*, 93(2), 59–68.
- Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics: An International Journal*, 23(3), 247–285.
- Bugden, S., & Ansari, D. (2011). Individual differences in childrens mathematical competence are related to the intentional but not automatic processing of arabic numerals. *Cognition*, 118(1), 32-44.
- Cantlon, J. F., & Brannon, E. M. (2007). Basic math in monkeys and college students. *PLoS Biology*, 5(12), 2912–2919.
- Caramazza, A., McCloskey, M., & Green, B. (1981a). Naive beliefs in "sophisticated" subjects: misconceptions about trajectories of objects. *Cognition*, 9(2), 117–123.
- Caramazza, A., McCloskey, M., & Green, B. (1981b). Naive beliefs in "sophisticated" subjects: misconceptions about trajectories of objects. *Cognition*, 9(2), 117–123.
- Carey, S. (1985). *Conceptual change in childhood*. MIT Press.
- Carey, S. (2009). *The Origin of Concepts*. Oxford University Press.
- Carey, S., & Spelke, E. (1996). Science and core knowledge. *Philosophy of science*, 63(4), 515–533.
- Cesana-Arlotti, N., Martín, A., Téglás, E., Vorobyova, L., Cetnarski, R., & Bonatti, L. L. (2018). Precursors of logical reasoning in preverbal human infants. *Science*, 359, 1263-1266.
- Clement, J. (1982). Students' preconceptions in introductory mechanics. *American Journal of Physics*, 50(1), 66–71.
- Cranford, S. W., Tarakanova, A., Pugno, N. M., & Buehler, M. J. (2012, 2). Nonlinear material behaviour of spider silk yields robust webs. *Nature*, 482, 72-76.
- Danek, A. H., Fraps, T., von Müller, A., Grothe, B., & Öllinger, M. (2014). Working Wonders? Investigating insight with magic tricks. *Cognition*, 130(2), 174–185.

- Danek, A. H., & Wiley, J. (2017). What about false insights? Deconstructing the aha! experience along its multiple dimensions for correct and incorrect solutions separately. *Frontiers in Psychology, 7*, 1–14.
- Dehaene, S. (2011). *The number sense: How the mind creates mathematics, Rev. and updated ed.* Oxford University Press.
- Dehaene, S. (2014). *Le code de la conscience.* O. Jacob.
- Dehaene, S. (2020). *How we learn: Why brains learn better than any machine ... for now.* Penguin Publishing Group, Viking.
- Dehaene, S., Dehaene-Lambertz, G., & Cohen, L. (1998). Abstract representations of numbers in the animal and human brain. *Trends in Neurosciences, 21*(8), 355-361.
- Dehaene, S., Dupoux, E., & Mehler, J. (1990). Is Numerical Comparison Digital? Analogical and Symbolic Effects in Two-Digit Number Comparison. *Journal of Experimental Psychology: Human Perception and Performance, 16*(3), 626–641.
- Dehaene, S., Izard, V., Pica, P., & Spelke, E. (2006). Core Knowledge of geometry in an Amazonian Indigene Group. *Science, 311*, 381–4.
- Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003, 05). Three parietal circuits for number processing. *Cognitive neuropsychology, 20*, 487-506.
- DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction, 37*, 39-49.
- Dillon, M. R., Huang, Y., & Spelke, E. S. (2013). Core foundations of abstract geometry. *Proceedings of the National Academy of Sciences of the United States of America, 110*(35).
- Donnelly, C. M., & McDaniel, M. A. (1993). Use of analogy in learning scientific concepts. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 19*, 975-987.
- Du, N., Xiang, Y. L., Narayanan, J., Li, L., Lim, M. L. M., & Li, D. (2006, 12). Design of superior spider silk: From nanostructure to mechanical properties. *Biophysical Journal, 91*, 4528-4535.
- Dubey, R., Ho, M., Mehta, H., & Griffiths, T. L. (2021). *Aha! moments correspond to meta-cognitive prediction errors.*
- Duncker, K. (1945). *On Problem-Solving* (Psychological Monographs ed., Vol. 58) (No. 5). The American Psychological Association.
- Dunlosky, J., & Rawson, K. A. (2012, 8). Overconfidence produces underachievement: Inaccurate self evaluations undermine students' learning and retention. *Learning and Instruction, 22*,

271-280.

- Elliott, L., Feigenson, L., Halberda, J., & Libertus, M. E. (2019). Bidirectional, Longitudinal Associations Between Math Ability and Approximate Number System Precision in Childhood. *Journal of Cognition and Development, 20*(1), 56–74.
- Ercikan, K., & Roth, W. M. (2006). What Good Is Polarizing Research Into Qualitative and Quantitative? *Educational Researcher, 35*(5), 14–23.
- Erickson, F. (2012, 01). Qualitative research methods for science education. In (pp. 1451–1469).
- Euclid. (300 BCE, 2007). *Elements of Geometry* (J. L. Heiberg & R. Fitzpatrick, Eds.).
- Fazio, L. K., Kennedy, C. A., & Siegler, R. S. (2016, 10). Improving childrens knowledge of fraction magnitudes. *PLOS ONE, 11*, e0165243.
- Feigenson, L., Dehaene, S., & Spelke, E. (2004). Core systems of number. *Trends in Cognitive Sciences, 8*(7), 307–314.
- Feigenson, L., Libertus, M. E., & Halberda, J. (2013). Links between the intuitive sense of number and formal mathematics ability. *Child Development Perspectives, 7*(2), 74-79.
- Feldman, J. (2000). Minimization of Boolean complexity in human concept learning. *Nature, 407*(6804), 630–633.
- Finn, B., & Metcalfe, J. (2014). Overconfidence in children’s multi-trial judgments of learning. *Learning and Instruction, 32*, 1-9.
- Fodor, J. A. (1979). *The language of thought*. Harvard University Press.
- Frege, G. (1948, 5). Sense and reference. *The Philosophical Review, 57*, 209.
- Gallistel, C. R. (1989). Animal cognition: the representation of space, time and number. *Annual review of psychology, 40*, 155–189.
- Gallistel, C. R., & Gelman, R. (1990). The what and how of counting. *Cognition, 34*(2), 197–199.
- Gentner, D., & Holyoak, K. J. (1997). *Reasoning and learning by analogy*. (Vol. 52). doi: 10.1037/0003-066X.52.1.32
- Gergely, G., & Csibra, G. (2003). Teleological reasoning in infancy: the nave theory of rational action. *Trends in Cognitive Sciences, 7*(7), 287-292.
- Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2007). Symbolic arithmetic knowledge without instruction. *Nature, 447*(7144), 589–591.
- Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2010). Non-symbolic arithmetic abilities and mathematics achievement in the first year of formal schooling. *Cognition, 115*(3), 394-406.
- Goodman, N. A., Tenenbaum, J. B., Feldman, J., & Griffiths, T. L. (2008). A rational analysis of

- rule-based concept learning. *Cognitive Science*, 32(1), 108–154.
- Gopnik, A., & Meltzoff, A. N. (1997). *Words, thoughts, and theories (learning, development, and conceptual change)*. The MIT Press.
- Gopnik, A., & Tenenbaum, J. B. (2007). Bayesian networks, Bayesian learning and cognitive development. *Developmental Science*, 10(3), 281–287.
- Gopnik, A., & Wellman, H. M. (1992). Why the child's theory of mind really is a theory. *Mind & Language*, 7(1-2), 145-171.
- Gopnik, A., & Wellman, H. M. (2012). Reconstructing constructivism: Causal models, Bayesian learning mechanisms, and the theory theory. *Psychological Bulletin*, 138(6), 1085–1108.
- Greenberg, M. J. (1993). *Euclidean and non-euclidean geometries: Development and history*. Macmillan.
- Gros, H., Thibaut, J. P., & Sander, E. (2021, 7). What we count dictates how we count: A tale of two encodings. *Cognition*, 212, 104665. doi: 10.1016/j.cognition.2021.104665
- Gvozdic, K., & Sander, E. (2018). When intuitive conceptions overshadow pedagogical content knowledge: Teachers' conceptions of students' arithmetic word problem solving strategies. *Educational Studies in Mathematics*, 98(2), 157–175.
- Hadamard, J. (1954). *The psychology of invention in the mathematical field*. Princeton University Press.
- Halberda, J., Mazocco, M. M., & Feigenson, L. (2008). Individual differences in non-verbal number acuity correlate with maths achievement. *Nature*, 455(7213), 665–668.
- Hatfield, G. (2015, 1). Natural geometry in descartes and kepler. *Res Philosophica*, 92, 117-148.
- Hirtle, S. C., & Jonides, J. (1985). Evidence of hierarchies in cognitive maps. *Memory & Cognition*, 13(3), 208–217.
- Holloway, I. D., & Ansari, D. (2009). Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in childrens mathematics achievement. *Journal of Experimental Child Psychology*, 103(1), 17-29.
- Hoof, J. V., Vandewalle, J., Verschaffel, L., & Dooren, W. V. (2015, 6). In search for the natural number bias in secondary school students' interpretation of the effect of arithmetical operations. *Learning and Instruction*, 37, 30-38.
- Horvitz, L. A. (2002). *Eureka! : scientific breakthroughs that changed the world*. J. Wiley.
- Huey, H., Jordan, M., Hart, Y., & Dillon, M. (2022). *Running head: mind bending geometry*. Retrieved from <https://psyarxiv.com/d4bgc/>

- Huttenlocher, J., Hedges, L. V., & Duncan, S. (1991). Categories and particulars: Prototype effects in estimating spatial location. *Psychological Review*, *98*, 352-376.
- Hyde, D. C., Khanum, S., & Spelke, E. S. (2014). Brief non-symbolic, approximate number practice enhances subsequent exact symbolic arithmetic in children. *Cognition*, *131*(1), 92-107.
- Izard, V., Dehaene-Lambertz, G., & Dehaene, S. (2008). Distinct cerebral pathways for object identity and number in human infants. *PLoS Biology*, *6*(2), 0275–0285.
- Izard, V., Pica, P., & Spelke, E. S. (2022, 8). Visual foundations of euclidean geometry. *Cognitive Psychology*, *136*.
- Izard, V., Pica, P., Spelke, E. S., & Dehaene, S. (2011). Flexible intuitions of Euclidean geometry in an Amazonian indigene group. *Proceedings of the National Academy of Sciences of the United States of America*, *108*(24).
- Izard, V., & Spelke, E. S. (2009). Development of sensitivity to geometry in visual forms. *Human Evolution*, *24*(3), 213–248.
- Jameson, K. A., Narens, L., Goldfarb, K., & Nelson, T. O. (1990). The influence of near-threshold priming on metamemory and recall. *Acta Psychologica*, *73*(1), 55–68.
- Jung-Beeman, M., Bowden, E. M., Haberman, J., Frymiare, J. L., Arambel-Liu, S., Greenblatt, R., ... Kounios, J. (2004, 4). Neural activity when people solve verbal problems with insight. *PLoS Biology*, *2*, e97.
- Kant, I. (1781). *Critique of Pure Reason* (P. Guyer and A. Wood, Trans.) (1999th ed.; P. Guyer & A. Wood, Eds.). Cambridge University Press.
- Katona, G. (1940). *Organizing and memorizing: studies in the psychology of learning and teaching*. Columbia Univ. Press.
- Klarreich, E. G. (2000). Foams and honeycombs: For centuries, the precise architecture of soap foams has been a source of wonder to children and a challenge to mathematicians. *American Scientist*, *88*(2), 152–161.
- Kluss, T., Marsh, W. E., Zetsche, C., & Schill, K. (2015, 9). Representation of impossible worlds in the cognitive map. *Cognitive Processing*, *16*, 271-276.
- Knoblich, G., Ohlsson, S., Haider, H., & Rhenius, D. (1999). Constraint relaxation and chunk decomposition in insight problem solving. *Journal of Experimental Psychology: Learning Memory and Cognition*, *25*, 1534-1555.
- Kounios, J., & Beeman, M. (2014). The cognitive neuroscience of insight. *Annual Review of Psychology*, *65*(July), 71–93.

- Krohn, T., Stricker, J., Vogel, S. E., Schöneburg-Lehnert, S., & Grabner, R. H. (2021). Interference between naïve and scientific theories occurs in mathematics and is related to mathematical achievement. *Cognition*, 214.
- Kushnir, T., & Gopnik, A. (2005, 9). Young children infer causal strength from probabilities and interventions. *Psychological Science*, 16, 678-683. Retrieved from <http://pss.sagepub.com/lookup/doi/10.1111/j.1467-9280.2005.01595.x> doi: 10.1111/j.1467-9280.2005.01595.x
- Landau, B. (1984, 4). Spatial knowledge in a young blind child. *Cognition*, 16, 225-260.
- Landau, B., Gleitman, H., & Spelke, E. (1981). Spatial knowledge and geometric representation in a child blind from birth. , 213(4513), 1275–1278.
- Laukkonen, R. E., Kaveladze, B. T., Tangen, J. M., & Schooler, J. W. (2020). The dark side of Eureka: Artificially induced Aha moments make facts feel true. *Cognition*, 196, 0–5.
- Laukkonen, R. E., & Tangen, J. M. (2017). Can observing a Necker cube make you more insightful? *Consciousness and Cognition*, 48(December), 198–211.
- Libertus, M. E. (2015, 6). The role of intuitive approximation skills for school math abilities. *Mind, Brain, and Education*, 9, 112-120.
- Liljedahl, P. G. (2005). Mathematical discovery and affect: The effect of AHA! experiences on undergraduate mathematics students. *International Journal of Mathematical Education in Science and Technology*, 36(2-3), 219–234.
- Lipton, J., & Spelke, E. (2003, 10). Origins of number sense large-number discrimination in human infants. *Psychological science*, 14, 396-401.
- Lipton, J. S., & Spelke, E. S. (2004). Discrimination of large and small numerosities by human infants. *Infancy*, 5(3), 271-290.
- Liu, S., Brooks, N. B., & Spelke, E. S. (2019). Origins of the concepts cause, cost, and goal in prereaching infants. *Proceedings of the National Academy of Sciences of the United States of America*, 116(36).
- Liu, S., & Spelke, E. S. (2017). Six-month-old infants expect agents to minimize the cost of their actions. *Cognition*, 160, 35–42.
- Lung, C. T., & Dominowski, R. L. (1985). Effects of Strategy Instructions and Practice on Nine-Dot Problem Solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 11(4), 804–811.
- Madill, A., & Gough, B. (2008). Qualitative research and its place in psychological science. ,



13(3), 254–271.

- Maier, N. R. (1931). Reasoning in humans. II. The solution of a problem and its appearance in consciousness. *Journal of Comparative Psychology*, 12(2), 181–194.
- Mainwaring, M. C., Hartley, I. R., Lambrechts, M. M., & Deeming, D. C. (2014, 10). The design and function of birds' nests. *Ecology and Evolution*, 4, 3909-3928.
- Martí, L., Mollica, F., Piantadosi, S., & Kidd, C. (2018). Certainty Is Primarily Determined by Past Performance During Concept Learning. *Open Mind*, 2(2), 47–60.
- Matthews, P. G., & Chesney, D. L. (2015). Fractions as percepts? Exploring cross-format distance effects for fractional magnitudes. *Cognitive Psychology*, 78, 28–56.
- Mayer, R. E. (1992). *Thinking, problem solving, cognition*. WH Freeman/Times Books/Henry Holt & Co.
- McCloskey, M. (1983). Naïve theories of motion. *Mental Models*, 75–98.
- Mccrink, K., & Wynn, K. (2004, 12). Large-number addition and subtraction by 9-month-old infants. *Psychological science*, 15, 776-81.
- Meck, W. H., & Church, R. M. (1983, 7). A mode control model of counting and timing processes. *Journal of Experimental Psychology: Animal Behavior Processes*, 9, 320-334.
- Mehler, J., & Bever, T. (1967, 11). Cognitive capacity of very young children. *Science (New York, N.Y.)*, 158, 141-2.
- Mercier, H. (2016). The argumentative theory: Predictions and empirical evidence. *Trends in Cognitive Sciences*, 20(9), 689–700.
- Metcalf, J. (1986). Feeling of Knowing in Memory and Problem Solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 12(2), 288–294.
- Metcalf, J., & Wiebe, D. (1987). Intuition in insight and noninsight problem solving. *Memory and Cognition*, 15(3), 238–246.
- Miner, A., & Reder, L. M. (1994). A New Look at Feeling of Knowing: Its Metacognitive Role in Regulating Question Answering. *Metacognition*(January 1994).
- Mody, S., & Carey, S. (2016). The emergence of reasoning by the disjunctive syllogism in early childhood. *Cognition*, 154, 40-48.
- Moore, D. A., & Healy, P. J. (2008). The trouble with overconfidence. *Psychological Review*, 115, 502-517.
- Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*,

30(2), 122–147.

- Moyer, R. S., & Landauer, T. K. (1967). Time required for Judgments of Numerical Inequality. *Nature*, 216, 615–616.
- Mugler, C. (1956). Sur deux passages de Platon. *Revue des Études Grecques*, 69(324), 20–34.
- Mugler, C. (1957). Sur l'histoire de quelques définitions de la géométrie grecque et les rapports entre la géométrie et l'optique (première partie). *L'antiquité classique*, 26, 331–345.
- Muller, D., Bewes, J., Sharma, M., & Reimann, P. (2008). Saying the wrong thing: improving learning with multimedia by including misconceptions. *Journal of Computer Assisted Learning*, 24(2), 144-155.
- Muller, D. A., Sharma, M. D., Eklund, J., & Reimann, P. (2007). Conceptual change through vicarious learning in authentic physics settings. *Educational Technology Research and Development*, 57(5), 613–627.
- Mussolin, C., Nys, J., Content, A., & Leybaert, J. (2014, 3). Symbolic number abilities predict later approximate number system acuity in preschool children. *PLoS ONE*, 9, e91839.
- Needham, A., & Baillargeon, R. (1993). Intuitions about support in 4.5-month-old infants. *Cognition*, 47(2), 121–148.
- Newcombe, N. S., & Huttenlocher, J. (2000). *Making space: The development of spatial representation and reasoning, chapter 5: Development of spatial thought*.
- Ngu, B. H., & Phan, H. P. (2020, 9). Learning to solve trigonometry problems that involve algebraic transformation skills via learning by analogy and learning by comparison. *Frontiers in Psychology*, 11, 2590.
- Nieder, A. (2005). Counting on neurons: The neurobiology of numerical competence. *Nature Reviews Neuroscience*, 6, 177-190.
- Nys, J., Ventura, P., Fernandes, T., Querido, L., Leybaert, J., & Content, A. (2013). Does math education modify the approximate number system? a comparison of schooled and unschooled adults. *Trends in Neuroscience and Education*, 2(1), 13-22.
- Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, 28, 64–72.
- Ohlsson, S. (1984). Restructuring revisited. *Scandinavian Journal of Psychology*, 25, 65–78.
- Ohlsson, S. (1992). *Information-processing explanations of insight and related phenomena* (Vol. 1).
- Ohlsson, S. (2009). Resubsumption: A Possible Mechanism for Conceptual Change and Belief

- Revision. *Educational Psychologist*, 44(1), 20–40.
- Ohlsson, S., & Cosejo, D. G. (2014). What Can Be Learned From a Laboratory Model of Conceptual Change? Descriptive Findings and Methodological Issues. *Science and Education*, 23(7), 1485–1504.
- Park, J., & Brannon, E. M. (2014). Improving arithmetic performance with number sense training: An investigation of underlying mechanism. *Cognition*, 133(1), 188–200.
- Peer, M., Brunec, I. K., Newcombe, N. S., & Epstein, R. A. (2021, 1). Structuring knowledge with cognitive maps and cognitive graphs. *Trends in Cognitive Sciences*, 25, 37-54.
- Piazza, M., Facoetti, A., Trussardi, A. N., Berteletti, I., Conte, S., Lucangeli, D., ... Zorzi, M. (2010). Developmental trajectory of number acuity reveals a severe impairment in developmental dyscalculia. *Cognition*, 116(1), 33–41.
- Piazza, M., Izard, V., Pinel, P., Le Bihan, D., & Dehaene, S. (2007). A magnitude code common to numerosities and number symbols in human intraparietal cortex. *Neuron*, 53(2), 293-305.
- Piazza, M., Pica, P., Izard, V., Spelke, E. S., & Dehaene, S. (2013). Education Enhances the Acuity of the Nonverbal Approximate Number System. *Psychological Science*, 24(6), 1037–1043.
- Plato. (n.d.-a). *Republic, book 10*.
- Plato. (n.d.-b). *Republic, book 4*.
- Plato. (380 BCEa). *Meno*. Retrieved from <http://www.perseus.tufts.edu/hopper/text?doc=Perseus%3Atext%3A1999.01.0177%3Atext%3DMeno%3Asection%3D85b>
- Plato. (380 BCEb). *Parmenides*. Retrieved from <https://www.perseus.tufts.edu/hopper/text?doc=Perseus%3Atext%3A1999.01.0173%3Atext%3DParm.%3Asection%3D137e>
- Poincaré, H. (1946). *The foundations of science: Science and hypothesis, the value of science, science and method*. Science press.
- Proclus. (5 CE,1970). *Proclus*. Princeton University Press.
- Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, 91(1), 175–189.
- Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93(2), 346–362.
- Rochel Gelman, C. R. G. (1978). *The child's understanding of number* (1st Edition ed.). Harvard University Press.
- Roell, M., Viarouge, A., Houdé, O., & Borst, G. (2017). Inhibitory control and decimal number

- comparison in school-aged children. *PLoS ONE*, 12(11), 1–17.
- Roell, M., Viarouge, A., Houdé, O., & Borst, G. (2019). Inhibition of the whole number bias in decimal number comparison: A developmental negative priming study. *Journal of Experimental Child Psychology*, 177, 240–247.
- Rosch, E. (1983). Prototype classification and logical classification: The two systems. *New trends in conceptual representation: Challenges to Piagets theory*, 73–86.
- Rosch, E. H. (1973). Natural categories. *Cognitive psychology*, 4(3), 328–350.
- Rossi, S., Vidal, J., Letang, M., Houdé, O., & Borst, G. (2019, Dec.). Adolescents and adults need inhibitory control to compare fractions. *Journal of Numerical Cognition*, 5(3), 314–336.
- Roth, W.-M. (2005). *Doing qualitative research : praxis of method*.
- Rouvière, F. (2016). *Initiation à la géométrie de riemann*. Calvage & Mounet.
- Rozenblit, L., & Keil, F. (2002). The misunderstood limits of folk science: An illusion of explanatory depth. *Cognitive science*, 26(5), 521–562.
- Sampaio, C., & Wang, R. F. (2009, 9). Category-based errors and the accessibility of unbiased spatial memories: A retrieval model. *Journal of Experimental Psychology: Learning Memory and Cognition*, 35, 1331–1337.
- Sampaio, C., & Wang, R. F. (2017, 12). The cause of category-based distortions in spatial memory: A distribution analysis. *Journal of Experimental Psychology: Learning Memory and Cognition*, 43, 1988–1992.
- Sanchez, C., & Dunning, D. (2018). Overconfidence among beginners: Is a little learning a dangerous thing? *Journal of Personality and Social Psychology*, 114, 10–28.
- Sander, E. (2008). Les connaissances naïves en mathématiques. *Les Connaissances Naïves. Paris: Armand Colin*, 57–102.
- Sander, E., Neagoy, M., Rivier, C., Sensevy, G., Thévenot, C., & Scheibling-sève, C. (2022). De la multiplication aux fractions : réconcilier intuition et sens mathématique. *Rapport du Conseil scientifique de l'éducation nationale*.
- Saxe, R., Tenenbaum, J., & Carey, S. (2005). Secret agents: Inferences about hidden causes by 10- and 12-month-old infants. *Psychological Science*, 16(12), 995–1001.
- Schauble, L. (1996). The development of scientific reasoning in knowledge-rich contexts. *Developmental Psychology*, 32(1), 102–119.
- Schooler, J. W., Ohlsson, S., & Brooks, K. (1993). Thoughts Beyond Words: When Language Overshadows Insight. *Journal of Experimental Psychology: General*, 122(2), 166–183.

- Schwartz, B. L., & Metcalfe, J. (1992). Cue Familiarity but not Target Retrievalability Enhances Feeling-of-Knowing Judgments. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 18(5), 1074–1083.
- Shepard, R. N. (1961). Psychological Monographs: General and Applied. *Journal of Chemical Information and Modeling*, 75(9), 1689–1699.
- Shtulman, A. (2006). Qualitative differences between naïve and scientific theories of evolution. *Cognitive Psychology*, 52(2), 170–194.
- Shtulman, A., & Lombrozo, T. (2016). Bundles of Contradiction: A Coexistence View of Conceptual Change. In D. B. . A. S. Baron (Ed.), *Core knowledge and conceptual change* (Oxford University Press, pp. 53–71). Oxford, UK.
- Shtulman, A., & Valcarcel, J. (2012). Scientific knowledge suppresses but does not supplant earlier intuitions. *Cognition*, 124(2), 209–215.
- Siegler, R. S. (1995). How Does Change Occur: A Microgenetic Study of Number Conservation. *Cognitive Psychology*, 28(3), 225–273.
- Smith, C. L., Solomon, G. E., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology*, 51(2), 101–140.
- Smith, R. W., & Kounios, J. (1996). Sudden insight: All-or-none processing revealed by speed-accuracy decomposition. *Journal of Experimental Psychology: Learning Memory and Cognition*, 22(6), 1443–1462.
- Spelke, E. (1990). Principles of Object Perception. *Cognitive Science*, 14(1), 29–56.
- Spelke, E. (1994). Initial knowledge: six suggestions. *Cognition*, 50(1-3), 431–445.
- Spelke, E. (2000). Core knowledge. *American psychologist*, 55(11), 1233.
- Spelke, E. (2017). Core knowledge, language, and number. *Language Learning and Development*, 13(2), 147–170.
- Spelke, E., & Kinzler, K. D. (2007). Core knowledge. *Developmental science*, 10(1), 89–96.
- Spelke, E., Lee, S. A., & Izard, V. (2010). Beyond core knowledge: Natural geometry. *Cognitive Science*, 34(5), 863–884.
- Spivak, M. (1999). *A comprehensive introduction to differential geometry* (No. vol.1). Publish or Perish, Incorporated.
- Stanovich, K. E. (1999). *Who is rational?: Studies of individual differences in reasoning*. Psychology Press.

- Starkey, P., & Cooper, R. G. (1995, 11). The development of subitizing in young children. *British Journal of Developmental Psychology*, 13, 399-420.
- Starr, A., Libertus, M. E., & Brannon, E. M. (2013). Number sense in infancy predicts mathematical abilities in childhood. *Proceedings of the National Academy of Sciences*, 110(45), 18116-18120.
- Steinle, V., & Stacey, K. (1998). The incidence of misconceptions of decimal notation amongst students in grades 5 to 10. *Teaching Mathematics in New Times: Proceedings of the 21st Annual Conference of MERGA*, 2, 548-555.
- Stevens, A., & Coupe, P. (1978). Distortions in judged spatial relations. *Cognitive Psychology*, 10(4), 422-437.
- Subramaniam, K., & Padalkar, S. (2009). Visualisation and reasoning in explaining the phases of the moon. *International Journal of Science Education*, 31(3), 395-417.
- Swire-Thompson, B., DeGutis, J., & Lazer, D. (2020, 9). *Searching for the backfire effect: Measurement and design considerations* (Vol. 9). Elsevier Inc.
- Téglás, E., Vul, E., Girotto, V., Gonzalez, M., Tenenbaum, J. B., & Bonatti, L. L. (2011). Pure reasoning in 12-month-old infants as probabilistic inference. *Science*, 332.
- Tenenbaum, J. B., Griffiths, T. L., & Kemp, C. (2006). Theory-based bayesian models of inductive learning and reasoning. *Trends in Cognitive Sciences*, 10(7), 309-318.
- Tenenbaum, J. B., Kemp, C., Griffiths, T. L., & Goodman, N. D. (2011). How to grow a mind: Statistics, structure, and abstraction. *Science*, 331(6022), 1279-1285.
- Theodosius, Kunitzsch, P., & Lorch, R. (1st century BC, 2010). *Sphaerica : Arabic and medieval latin translations*. Steiner.
- Tian, F., Hou, Y., Zhu, W., Dietrich, A., Zhang, Q., Yang, W., ... Cao, G. (2017). Getting the Joke: Insight during humor comprehension - Evidence from an fMRI study. *Frontiers in Psychology*, 8(OCT), 1-10.
- Tóth, L. F. (1964). What the bees know and what they do not know. *Bulletin of the American Mathematical Society*, 70, 468-481.
- Ullman, T. D., Goodman, N. D., & Tenenbaum, J. B. (2012). Theory learning as stochastic search in the language of thought. *Cognitive Development*, 27(4), 455-480.
- Vallortigara, G. (2012). Core knowledge of object, number, and geometry: A comparative and neural approach. *Cognitive neuropsychology*, 29(1-2), 213-236.
- Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction

- time evidence for a natural number bias in adults. *Journal of Mathematical Behavior*, 31(3), 344–355.
- Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2013). Brief Report. Educated adults are still affected by intuitions about the effect of arithmetical operations: Evidence from a reaction-time study. *Educational Studies in Mathematics*, 82(2), 323–330.
- Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, 14(5 SPEC.ISS.), 453–467.
- van Steenburgh, J. J., Fleck, J. I., Beeman, M., & Kounios, J. (2012, 11). *Insight*. Oxford University Press.
- Viarouge, A., Houdé, O., & Borst, G. (2019). Evidence for the role of inhibition in numerical comparison: A negative priming study in 7- to 8-year-olds and adults. *Journal of Experimental Child Psychology*, 186, 131–141.
- Vosniadou, S. (1992). Mental Models of the Earth: A Study of Conceptual Change in Childhood. *Cognitive Psychology*, 24.
- Vosniadou, S. (2019). The Development of Students' Understanding of Science. *Frontiers in Education*, 4(April), 1–6.
- Wang, R., & Spelke, E. (2002, 10). Human spatial representation: Insights from animals. *Trends in cognitive sciences*, 6, 376.
- Warren, W. H., Rothman, D. B., Schnapp, B. H., & Ericson, J. D. (2017). Wormholes in virtual space: From cognitive maps to cognitive graphs. *Cognition*, 166, 152–163.
- Wason, P. C. (1960). On the failure to eliminate hypotheses in a conceptual task. *Quarterly journal of experimental psychology*, 12(3), 129–140.
- Wason, P. C. (1968). Reasoning about a rule. *Quarterly journal of experimental psychology*, 20(3), 273–281.
- Webb, M. E., Little, D. R., & Cropper, S. J. (2016). Insight is not in the problem: Investigating insight in problem solving across task types. *Frontiers in Psychology*, 7, 1–13.
- Weber, K. (2002). Students' Understanding of Exponential and Logarithmic. *Second international conference on the teaching of mathematics*.
- Wellman, H. M., & Gelman, S. A. (1992). Cognitive development: Foundational theories of core domains. *Annual Review of Psychology*, 43(1), 337–375.
- Widdowson, C., & Wang, R. F. (2022). Human navigation in curved spaces. *Cognition*, 218,

104923.

- Wiser, M. (1986). The differentiaion of heat and temperature: an evaluation of the effect of microcomputer teaching on students' misconceptions. Technical report 87-5. (3).
- Wood, J. N., & Spelke, E. S. (2005). Chronometric studies of numerical cognition in five-month-old infants. *Cognition*, *97*(1), 23–39.
- Xu, F., & Spelke, E. S. (2000). Large number discrimination in 6-month-old infants. *Cognition*, *74*(1), 1–11.
- Xu, F., Spelke, E. S., & Goddard, S. (2005). Number sense in human infants. *Developmental Science*, *8*(1), 88–101.
- Xu, F., & Tenenbaum, J. B. (2007). Word learning as bayesian inference. *Psychological review*, *114*(2), 245.
- Zazkis, R., & Leikin, R. (2008, 10). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics*, *69*, 131-148. doi: 10.1007/s10649-008-9131-7
- Zetzsche, C., Wolter, J., Galbraith, C., & Schill, K. (2009, 9). Representation of space: Image-like or sensorimotor? *Spatial Vision*, *22*, 409-424.