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Harriet Walsh

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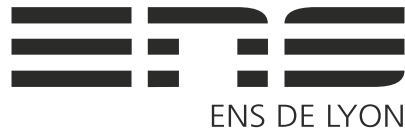
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Nouvelles classes d'universalité pour les partitions aléatoires

New universality classes for random integer partitions

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New universality classes for random integer partitions

Version submitted to the defence

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September 29, 2022

Abstract

Random integer partitions have been particularly useful in explaining the connections between diverse physical and combinatorial models exhibiting the same asymptotic phenomena. Famously, a partition under the Plancherel measure encodes the lengths of monotone subsequences of a uniform random permutation; its parts also correspond to positions of free fermions on a lattice, a connection that allows their statistics to be studied exactly. It has a deterministic limit shape and edge fluctuations with a universal critical exponent of $1/3$, associated with out-of-equilibrium physics. This thesis presents two generalisations of the Plancherel measure with edge behaviour escaping its universality class. First, we introduce measures on partitions corresponding to natural models of free fermions, and show that they give rise to “multicritical” asymptotic edge fluctuations, with new critical exponents. These measures relate multicritical free fermions to random unitary matrices, explaining the appearance of the same asymptotic distributions for both. Second, we introduce a measure related to the enumeration of transposition factorisations on symmetric groups and certain discrete surfaces. We show that, in a regime where the corresponding surfaces are of high genus, it produces a novel twofold limit behaviour where the first part becomes very large. As a consequence, we find an asymptotic estimate for the unconnected Hurwitz numbers at high genus. The laws studied each have integrable structures. In the first case our analysis exploits integrability directly; in the second, we use an entropy method to study an asymptotic regime which is inaccessible by integrability approaches.

Résumé

Les partitions aléatoires d’entiers ont servi à expliquer les connexions entre modèles physiques et combinatoires très différentes. Une partition sous la célèbre mesure de Plancherel donne les longueurs des sous-suites monotones d’une permutation aléatoire uniforme ; ses parties correspondent aussi aux positions de fermions libres sur réseau, ce qui permet d’étudier leurs statistiques de manière exacte. Elle a une forme limite déterministe et des fluctuations de bord avec exposant critique universel $1/3$, associé à la physique hors d’équilibre. Cette thèse porte sur deux généralisations de la mesure de Plancherel, dont le comportement de bord échappe l’universalité. On introduit en premier une famille de mesures correspondant aux fermions libres, et montre qu’elles entament des fluctuations de bord « multicritiques », avec d’autres exposants critiques. Ces mesures relient les fermions multicritiques aux matrices unitaires aléatoires, et expliquent l’apparition des mêmes distributions asymptotiques pour chaque modèle. Ensuite, on introduit une mesure liée à l’énumération des factorisations par transpositions sur les groupes symétriques et de certaines surfaces discrètes. On montre que, dans un régime où les surfaces concernées sont de grand genre, elle produit un comportement limite inédit où la première partie devient très grande. Par conséquent, on obtient une approximation asymptotique pour les nombres d’Hurwitz non connexes à grand genre. Les lois étudiées possèdent des structures intégrables. Dans le premier cas, on les exploite directement ; dans le deuxième, on étudie via une méthode d’entropie un régime asymptotique inaccessible par l’approche d’intégrabilité.

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Index of notation

Here we collect some notation conventions used throughout the text.

Partitions (and Young diagrams) are labelled with Greek letters from the middle of the alphabet (e.g. λ, μ, ν). Boxes in Young diagrams are denoted by squares \square . Random objects are denoted by bold letters (e.g. $\boldsymbol{\lambda}$), fixed ones by light letters (e.g. λ).

$\ell(\lambda)$	the length of a partition λ , i.e. the number of nonzero parts
$ \lambda $	the size of λ , i.e. the sum of its parts $\lambda_1 + \lambda_2 + \dots + \lambda_{\ell(\lambda)}$ or the size of a standard Young tableau of shape λ
λ'	the conjugate partition of λ , with parts $\lambda'_j = \#\{\lambda_i \lambda_i \geq j\}$
λ/μ	the skew partition, whose Young diagram is λ with the boxes of μ removed (defined if $\lambda_i \geq \mu_i$ for all i)
$\mu \nearrow \lambda$	λ is obtained from μ by adding one box
μ^{1+}, λ^{1-}	$\mu^{1+} = (\mu_1 + 1, \mu_2, \dots, \mu_{\ell(\mu)})$ is μ with a box added to the first row, $\lambda^{1-} = (\lambda_1 - 1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ is λ with a box removed from the first row
$\lambda \succ \mu$	λ and μ are interlaced, i.e. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$
\emptyset	any empty set, and the empty partition $\emptyset \vdash 0$
f_λ	the number of standard Young tableaux of shape λ
$\eta_\lambda(\square)$	the length of the hook from \square to the edges of the Young diagram λ
$c(\square), C_\lambda$	the contents of \square in a Young diagram, the content-sum $\sum_{\square \in \lambda} c(\square)$ of λ
$\psi_{\lambda; \alpha}$	the rescaled profile describing the upper edge of the tilted Young diagram of λ in coordinates where the centres of the boxes are $1/\alpha$ apart
$S(\lambda)$	the (charge zero) fermion configuration $\{\lambda_i - i + \frac{1}{2}, i \in \mathbb{Z}_{\geq 1}\}$ associated with the partition λ
Q_0, Q_1, Q_2	the charge, energy and content-sum of a fermion configuration
$h_\lambda; e_\lambda; s_\lambda$	the complete homogeneous; elementary; Schur symmetric function indexed by the partition λ
$s_\lambda(x_1, x_2, \dots)$	a symmetric function (e.g. Schur) of the parameters x_1, x_2, \dots , i.e. a symmetric function specialised to x_1, x_2, \dots
$s_\lambda[t_1, t_2, \dots]$	a symmetric function (e.g. Schur) with Miwa times specialised to t_1, t_2, \dots , i.e. a symmetric function of x_1, x_2, \dots where $\frac{1}{r} \sum_i x_i^r = t_r$
Sym, Sym_n	the algebra of symmetric functions over \mathbb{C} , the vector space of symmetric functions of degree n over \mathbb{C}
\mathfrak{S}_n	the symmetric group of order n
$\mathcal{C}_\lambda, V_\lambda, \chi^\lambda$	the conjugacy class of \mathfrak{S}_n indexed by $\lambda \vdash n$, the irreducible representation of \mathfrak{S}_n , indexed by λ , and the character on V_λ
$\mathbf{1}_E$	indicator of the event E , i.e. 1 if E is true and 0 if E is false
$\mathbb{P}(\mathbf{s}), \mathbb{P}(\mathbf{E})$	law of the statistic \mathbf{s} , probability of the event \mathbf{E}
$\mathbb{E}(\mathbf{s}), \text{Var}(\mathbf{s})$	expectation and variance of the statistic \mathbf{s}
\xrightarrow{p}	convergence in probability

Index of notation

$L^2(R), l^2(R)$	the space of square integrable functions on R , the space of square summable functions on a discrete R
$o(f(n))$	“little o ”: if $g(n) = o(f(n))$ as $n \rightarrow \infty$, then $f(n)/g(n) \rightarrow 0$ in that limit
$O(f(n))$	“big O ”: if $g(n) = O(f(n))$ as $n \rightarrow \infty$, there exist positive numbers G, n_0 such that $g(n) \leq Gf(n)$ for all $n > n_0$
$\Theta(f(n))$	“big Θ ”: if $g(n) = \Theta(f(n))$ as $n \rightarrow \infty$, there exist positive numbers G_1, G_2, n_0 such that $G_1f(n) \leq g(n) \leq G_2f(n)$ for all $n > n_0$
$o_p(f(n))$	“little o in probability”: if $\mathbf{g}(n) = o_p(f(n))$ as $n \rightarrow \infty$, then $\mathbf{g}(n)/f(n) \xrightarrow{p} 0$ in that limit
$O_p(f(n))$	“big O in probability”: if $\mathbf{g}(n) = O_p(f(n))$ as $n \rightarrow \infty$, then for all ε , there exist positive numbers G, n_0 such that $\mathbb{P}(\mathbf{g}(n)/f(n) > G) < \varepsilon$ for all $n > n_0$
$\mathbb{P}_n, \mathbb{P}_\theta$	the Poissonised Plancherel measure $\mathbb{P}_n = f_\lambda^2/n!$ on partitions of n , the Poissonised Plancherel measure \mathbb{P}_θ on all partitions with parameter θ
\mathbb{P}_θ^m	an order m multicritical Schur measure on all partitions with parameter θ
$\mathbb{P}_\theta^{a,m}, \mathbb{P}_\theta^{s,m}$	the order m minimal asymmetric and symmetric multicritical Schur measures with parameter θ (where $\mathbb{P}_\theta = \mathbb{P}_\theta^{a,1} = \mathbb{P}_\theta^{s,1}$)
$\mathbb{P}_{n,\ell}, \mathbb{P}_{n,\ell}^+$	the Plancherel–Hurwitz measure with exponent ℓ on partitions of n (where $\mathbb{P}_n = \mathbb{P}_{n,0}$) and its positive half $\mathbb{P}_{n,\ell}^+(\lambda) = 2\mathbb{P}_{n,\ell}(\lambda)\mathbf{1}_{C_\lambda > 0}$
c_k^\dagger, c_k	the fermionic creation and annihilation operators on site k
a_r^\dagger, a_r	the bosonic creation operator $a_r^\dagger = a_{-r} := \sum_k :c_k^\dagger c_{k-r}:$ at level r , and the corresponding annihilation operator a_r
$F_{\text{GUE}}(s)$	the Tracy–Widom GUE distribution
$F_{2m+1}(s)$	the order m analogue of the Tracy–Widom GUE distribution (the classical distribution is $F_{\text{GUE}}(s) = F_3(s)$)
\mathcal{A}	the zero temperature Airy integral kernel
\mathcal{A}_{2m+1}	the order m analogue of the zero temperature Airy kernel (the classical kernel is $\mathcal{A} = \mathcal{A}_3$)

The following abbreviations are also used throughout:

BDJ	Baik–Deift–Johansson
DPP	determinantal point process
GUE	Gaussian unitary ensemble
GWW	Gross–Witten–Wadia
i.i.d.	independent and identically distributed
KPZ	Kardar–Parisi–Zhang
RSK	Robinson–Schensted–Knuth
(S)SYT	(semi-)standard Young tableau
TW	Tracy–Widom
VKLS	Vershik–Kerov–Logan–Shepp
w.h.p.	with high probability, i.e. with probability tending to 1

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- [2] Guillaume Chapuy, Baptiste Louf, and Harriet Walsh. Random Partitions Under the Plancherel–Hurwitz Measure, High Genus Hurwitz Numbers and Maps. In *33rd International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2022)*, volume 225 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 6:1–6:12, 2022. Extended abstract. Long version submitted for publication, available at arXiv:2206.11315 [math.CO].

Introduction

Combinatorics has a natural role in statistical physics. We might think, for instance, of computing the thermodynamic entropy of an isolated system at equilibrium, which amounts to counting its microscopic states, or consider finding the density of states of Bose and Fermi gases as solving two simple box and ball problems. Among the most notable achievements of combinatorial approaches to statistical physics are the simple computation of the partition function and critical point of the Ising ferromagnet in two dimensions by Kac and Ward in 1952 [KW52] and the six vertex model introduced by Pauling in 1935 [Pau35] to estimate the number of molecular configurations accounting for the residual entropy of ice, which was solved exactly by Lieb in 1967 [Lie67]. Beyond direct applications of enumeration, methods from analytic combinatorics provide important tools for finding the asymptotic behaviour of observables of thermodynamic systems.

Combinatorial approaches have predominantly been developed for models in one and two dimensions which, as in the cases of the Ising and six vertex model, yield exact solutions by algebraic methods. Such models, called *integrable*, have rich symmetries and are extremely rare. But although algebraic approaches can only be applied to systems with very particular microscopic interactions, the macroscopic behaviour of those systems at critical points is typically observed for many others. The behaviour of the Ising ferromagnet near its second order phase transition, as characterised by the critical exponents of intensive quantities, is also observed in physical magnets and in liquid vapour systems, where it has been experimentally observed [Hel67]. The *universality* of critical phenomena means that algebraic methods can be used to make precise predictions for the behaviour of complex systems at criticality, although universality itself is difficult to explain rigorously. For discrete models, we might hope to explain the asymptotic equivalence of statistics of models in the same universality class combinatorially by identifying bijections between those models.

This thesis follows broadly in the tradition of using combinatorial models with integrable structures to study universal physical phenomena. In particular, we introduce new probability laws for integer partitions (that is, weakly decreasing sequences of positive integers) as tools to study universality classes that are not yet well understood. Partitions are natural objects for approaches that are simultaneously combinatorial and algebraic. On the one hand, they play a key role in representation theory, as they index both the irreducible representations and conjugacy classes of the symmetric group. On the other, they encode data about the monotone subsequences of permutations thanks to the Robinson–Schensted correspondence. And, importantly, they are in direct correspondence with particle configurations in certain discrete physical models.

In this chapter we present an informal overview of the context and main results of this thesis, with particular emphasis on the physical models and phenomena motivating the combinatorial ones we study.

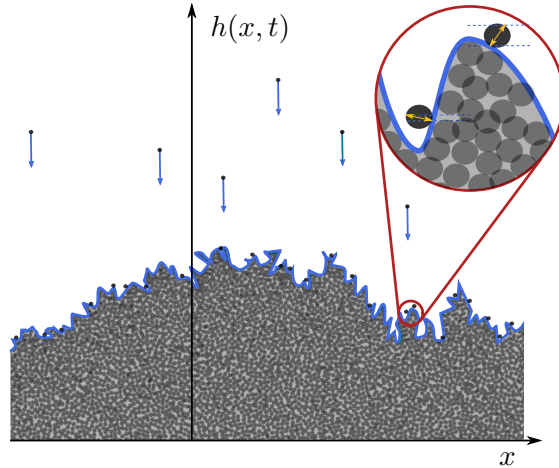


Figure 0.1 A sketch of 2D growth by deposition as modelled by the KPZ equation. The non-linear term in (0.1) is a correction to the average deposition speed to account for the growth being normal to the surface, as illustrated: if the interface at position x grows normally by d between t and $t + \Delta t$, the corresponding change in the height function is $\Delta h(x, t) = d - \frac{1}{2}d(\partial_x h(x, t))^2 + O(d^3)$.

I An out-of-equilibrium universality class, a case for random partitions

While universality has provided a rather complete description of critical phenomena in systems at equilibrium, in recent years the study of critical phenomena has been extended to the new frontier of out-of-equilibrium processes, with the discovery of the universality of the interface statistics of a diverse class of models.

Let us consider two models of generic interface behaviour, in two very different contexts.

Universal growing interfaces In 1986, Kardar, Parisi and Zhang [KPZ86] proposed a stochastic differential equation to govern generic random growth in diffusive media. For a one-dimensional interface whose height (overlooking overhangs) is described by the single valued random function $\mathbf{h}(x, t)$ at time t , the famous Kardar–Parisi–Zhang (KPZ) equation reads

$$\partial_t \mathbf{h} = \frac{\nu}{2} \partial_x^2 \mathbf{h} - \frac{\lambda}{2} (\partial_x \mathbf{h})^2 + \sqrt{D} \boldsymbol{\xi}(x, t) \quad (0.1)$$

where ν , λ and D are positive constants and $\boldsymbol{\xi}(x, t)$ is Gaussian white noise, with mean zero. This describes the growth of the height function under simple assumptions: the growth is local, with quickly decorrelating noise, and the first Laplacian term accounts for the smoothing of the interface, with a surface tension of ν , while the second non-linear term ensures normal growth (see Figure 0.1)¹. Since (0.1) does not satisfy a detailed balance equation, KPZ growth is a nonequilibrium process.

If we remove the non linear term, we recover the Edwards–Wilkinson (EW) random growth equation [EW82]; then, the growth of $h(x, t)$ is independent for each x at long times. Under EW growth, the fluctuations of the height function about its mean have an exponent of $1/2$, and are driven by the Gumbel distribution. For KPZ growth, however, by identifying the scaling regime in which the non-linear term remains positive and no

¹Although (0.1) is ill-defined, since the height function can't be expected to be more regular than a Brownian motion in the presence of white noise, it defines a well-posed stochastic heat equation for the Cole–Hopf transformation $\mathbf{W}(x, t) = \exp(\lambda/2\nu \mathbf{h}(x, t))$ of the height function [BG97].

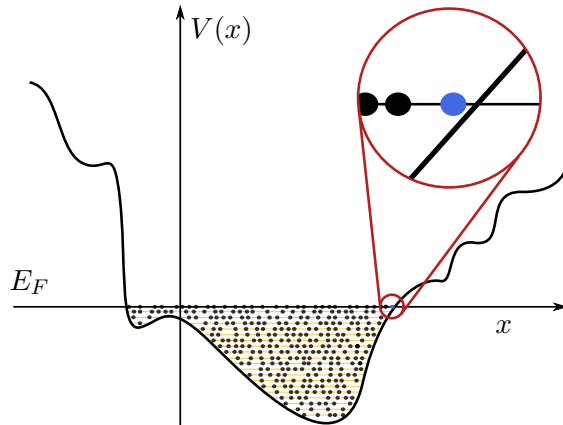


Figure 0.2 A sketch of fermions trapped in an arbitrary potential $V(x)$ in 1D. For fermions near the edge at the Fermi energy E_F , the effective potential is linear.

terms blow up, the authors predicted that the fluctuations of the height function would have an exponent of $1/3$, escaping the Gaussian universality (and this is indeed the most valid scaling regime: renormalisation group analysis of how the normal growth model varies under coarse-graining of time and space shows the non-linear term to be “relevant”, so it remains positive in the continuum limit).

In 2010, Amir, Corwin and Quastel [ACQ11] and, independently, Sasamoto and Spohn [SS10] solved the KPZ equation exactly for the “narrow wedge” initial condition $h(x, 0) = |x|/\delta$ with $\delta \ll 1$, notably proving that, when $\nu = \lambda = D = 1$ (the general case can be extracted by a change of variables) at a fixed position x

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{\mathbf{h}(x, t) - t/24}{(t/2)^{1/3}} > s \right) = F_{\text{GUE}}(-s) := \det(1 - \mathcal{A})_{L^2[-s, \infty)}. \quad (0.2)$$

The distribution F_{GUE} (also denoted F_2) driving the long time limit fluctuations is the *Tracy–Widom GUE (TW-GUE) distribution*, defined as the Fredholm determinant of the classical Airy kernel \mathcal{A} .

Following some experimental evidence for the KPZ scaling exponents, in the earliest case from the growth of bacteria colonies [WIMM97], the TW-GUE fluctuations of a 1D interface were observed in a liquid crystal turbulence experiment by Takeuchi and Sano [TS10] (see Figure 0.3) the same year their existence was proven mathematically.

Universality for edge fermions The same distribution F_{GUE} governs the fluctuations around a rather generic interface in quantum mechanics. Consider N spinless non-interacting fermions – that is to say, indistinguishable point particles which only interact through the Pauli exclusion principle – at temperature zero in one dimension, and suppose they are trapped to a region of space by some smooth potential $V(x)$. Then (in first quantisation; this is detailed in Section 1.3.2), the probability density of fermion in the ground state is given by the square modulus of a Slater determinant of individual particle wavefunctions, which are eigenfunctions of a Hamiltonian operator $H = -d^2/dx^2 + V(x)$ (in dimensionless position space coordinates).

The fermions near the Fermi energy E_F and near the right edge x_{edge} of the trap such that $V(x_{\text{edge}}) = E_F$ and $V'(x_{\text{edge}}) \geq 0$ experience an effective potential that’s generically the first order linear term $V_{\text{eff}}(x) = V'(x_{\text{edge}})(x - x_{\text{edge}})$. If $V'(x_{\text{edge}}) > 0$, in suitably rescaled dimensionless coordinates $\tilde{x} = (x - x_{\text{edge}})/\kappa$ the effective position

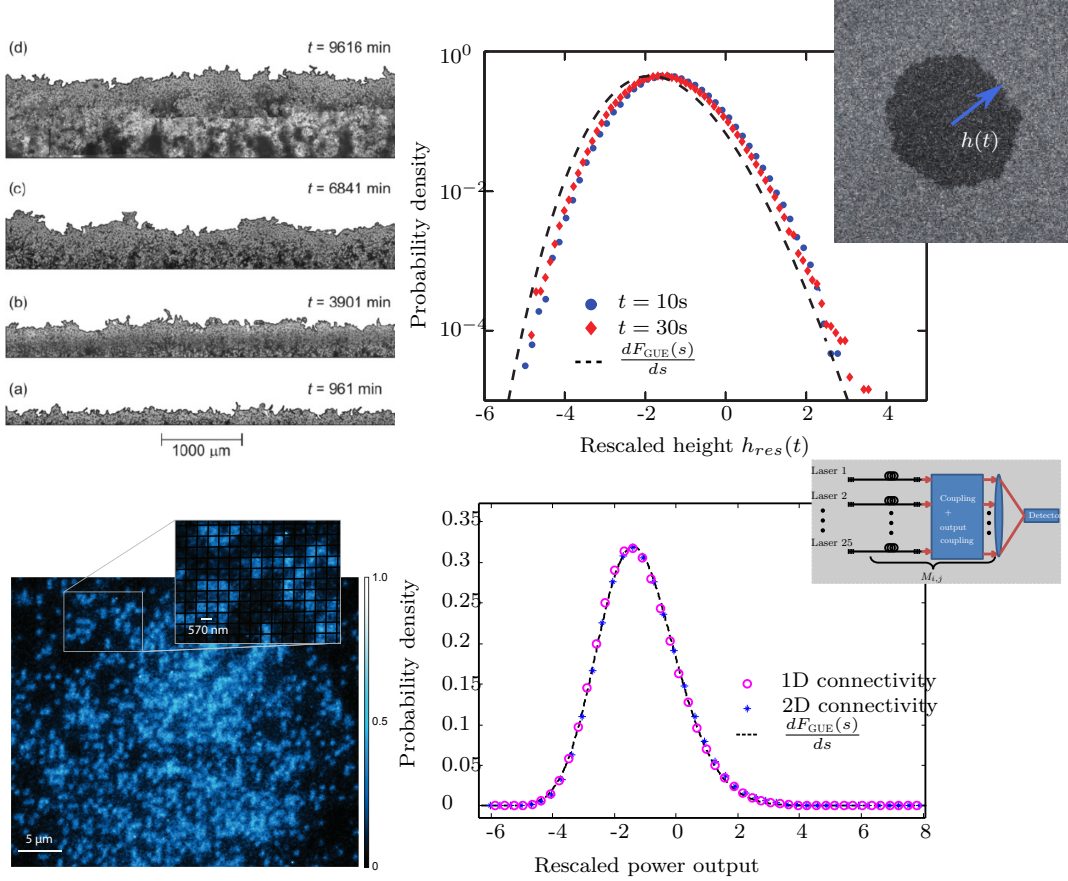


Figure 0.3 Experimental investigations of models the TW-GUE universality class. Top left, cancer cell colonies growing by cell division in a Petri dish, where the KPZ exponents were observed [HPG⁺12]. Top right, observation of F_{GUE} in liquid crystal growth with a single nucleation point corresponding to a narrow wedge initial condition, where the log scale plot shows strong agreement with F_{GUE} for the very rare events in the tails [TS10] (reproduced with modification). Bottom left, an experiment with cold Lithium atoms [PHM⁺15], offering a promising way to study trapped fermions and their edge statistics. Bottom right, observation of F_{GUE} in a coupled laser experiment, where the power output was modelled by random matrices [FPN⁺12] (reproduced with modification).

space Hamiltonian is

$$H_{\text{edge}} = -\frac{d^2}{d\tilde{x}^2} + \tilde{x}, \quad (0.3)$$

whose square integrable eigenfunction is the classical Airy function and whose ground state propagator is the Airy kernel. As shown by Eisler [Eis13], as N tends to infinity the the fluctuations in the counting statistics at the edge are driven by the TW-GUE distribution, just as for 2D random growth: the law of the rightmost fermion is

$$\mathbb{P}\left(\frac{\mathbf{x}_{\max} - x_{\text{edge}}}{\kappa} < s\right) = F_{\text{GUE}}(s) := \det(1 - \mathcal{A})_{L^2[s, \infty)}, \quad (0.4)$$

from precisely the ground state Airy kernel; dimensional analysis shows that κ scales with $x^{1/3}$.

This universality result even extends to some models of interacting fermions, as shown by Stéphan [Sté19]. The non-interacting case was recently proven by rigorous semiclassical analysis by Deleporte and Lambert [DL21]. Meanwhile, recent advances in cold atom technology (see e.g. [PHM⁺15] and Figure 0.3) have lead to increased interest in low dimensional trapped fermion physics as the experimental realisation of such systems is now feasible.

Distributions and phase transitions from random matrix theory The distribution F_{GUE} was first discovered by Tracy and Widom [TW93] in another context deeply related to statistical physics: random matrices. They proved that the law of the largest eigenvalue ζ_{max} of an $N \times N$ random Hermitian matrix \mathbf{M} in the Gaussian unitary ensemble (GUE) (i.e. sampled with probability density proportional to $e^{-\text{tr} \mathbf{M}^2/2}$) is asymptotically

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{\zeta_{\text{max}} - (2N)^{1/2}}{2^{-1/2} N^{-1/6}} < s \right) = F_{\text{GUE}}(s). \quad (0.5)$$

The connection between the edge statistics for the GUE and for trapped fermions in 1D is clear in one special case. In the harmonic potential $V(x) = x^2$ (in dimensionless coordinates), the positions (and indeed momenta) of N fermions coincide exactly with the eigenvalues of an $N \times N$ GUE random matrix. This connection is lost, however, for more general potentials where the same edge behaviour appears. But the matrix model origins of F_{GUE} do give some insight into characteristics of the models it is observed in, which we call the TW-GUE universality class. They notably led to the association of a phase transition to the universality class, as we have for equilibrium universality phenomena, by Majumdar and Schehr [MS14]. From the tails of F_{GUE} , these authors recognised the traces of a *third order strong-to-weak coupling transition* first found in a model of unitary matrices from gauge theory by Gross, Witten [GW80] and, independently, Wadia [Wad80].

In fact, this type of phase transition was identified as early as 1972 by May [May72], who proposed approximating the time evolution of a vector $\vec{p}(t)$ of populations of N different species which have random pairwise interactions (as, for instance, predators or prey) by the linear equation

$$\frac{d}{dt} \vec{p} = -\vec{p} + \mathbf{M} \vec{p} \quad (0.6)$$

where \mathbf{M} is a random real matrix. When N is large, the populations transition between stable and unstable phases as the largest eigenvalue of \mathbf{M} crosses a certain threshold. TW-GUE fluctuations are observed for interfaces between stable (weakly interacting) and metastable (strongly interacting) phases. In the case of KPZ growth, there is a stable cluster below the interface height h and a metastable noisy one above it; for trapped fermions, the almost empty region to the right of x_{edge} is stable while the region occupied by fermions is metastable. This coincides with an asymmetry in F_{GUE} itself: the transition from metastable to stable is fast and the negative tail of F_{GUE} decays rapidly, while the reverse transition is suppressed and the positive tail decays more gradually (the logarithm of dF_{GUE}/ds scales with s^3 as $s \rightarrow -\infty$ and with $-s^{3/2}$ as $s \rightarrow \infty$ [MS14]).

Exploring the TW-GUE class with random partitions With these elements of the TW-GUE universality class in mind, let us finally introduce our main protagonist, random partitions. In the time between the introduction of the KPZ equation and its exact solution, the long time limiting law (0.2) for the height function was predicted from a family of discrete stochastic models sharing the key properties of KPZ growth, called the KPZ universality class² – the proof of (0.2) is effectively the proof that the KPZ equation belongs to its universality class. Most notably, for some of these models every correlation can be computed exactly using algebraic methods, and their discovery has

²We will use the term “TW-GUE class” rather than the more established “KPZ class” to refer to any models with fluctuations driven by F_{GUE} with an exponent of 1/3 in order to include models with no time dependence (the KPZ class is more strictly defined in terms of the 3 : 2 : 1 scaling and the KPZ fixed point conjecture).

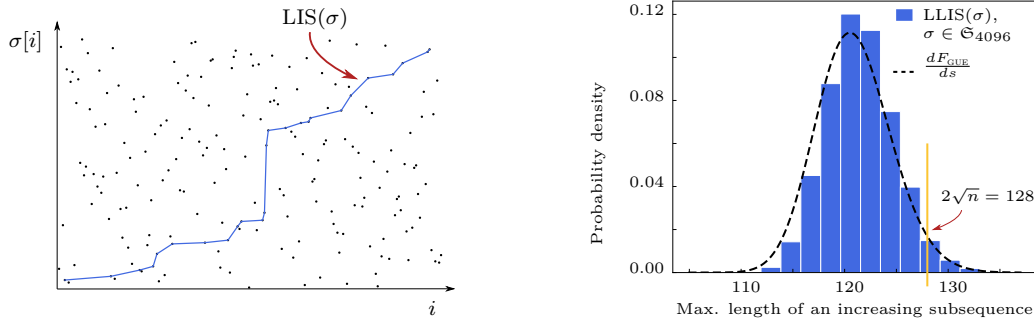


Figure 0.4 Left, a permutation of $1, \dots, 200$ sampled uniformly, with one of its longest increasing subsequences (of length 25) highlighted. Right, a histogram of the lengths of the longest increasing subsequences of 10000 random permutations of $1, \dots, 4096$ compared to the asymptotic probability density $F'_{\text{GUE}}(s)$. The asymmetry of F_{GUE} is visible from the displacement of its peak from its centre.

led to the growth of a new field, *integrable probability* (to which the lecture notes [BG16] by Borodin and Gorin are a thorough guide).

The simplest such model comes from the Ulam–Hammersley *longest increasing subsequence problem* [Ham72]. Consider a uniform random permutation σ of the numbers $1, \dots, n$, and the maximum length of a subsequence $I \subseteq (1, \dots, n)$ such that for all pairs $i < j \in I$, $\sigma[i] < \sigma[j]$, which we denote $\text{LLIS}(\sigma)$. If we plot σ on a grid, $\text{LLIS}(\sigma)$ can be interpreted as the maximum length of a path directed up and right through a uniform random medium (see Figure 0.4. This statistic can be studied by considering a probability law on partitions induced by the *Robinson–Schensted* algorithm, a powerful combinatorial tool which identifies each permutation in the symmetric group of order n (denoted \mathfrak{S}_n) bijectively with a pair of standard Young tableaux, each of which is an arrangement of boxes labelled 1 to n aligned on the left and on the bottom such that the labels increase along rows and columns. Under this bijection, both of the tableaux (P, Q) generated from a given $\sigma \in \mathfrak{S}_n$ have the same shape, where the lengths of the rows, counting up from the bottom, are $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\sigma)})$. This sequence of row lengths is a partition λ , and since its “parts” λ_i sum to n it is a partition of n (denoted $\lambda \vdash n$); removing the labels from the boxes of P or Q produces the Young diagram of λ . Most importantly, the first part λ_1 of λ is exactly equal to $\text{LLIS}(\sigma)$.

The law of $\text{LLIS}(\sigma)$ is thus equal to the law of λ_1 where λ is sampled with the *Plancherel measure*

$$\mathbb{P}_n(\lambda) = \frac{f_\lambda^2}{n!} \tag{0.7}$$

on partitions of n , where f_λ is the number of standard Young tableaux of shape λ . The same measure also arises from “Fourier transforming” the uniform measure on \mathfrak{S}_n to its irreducible representations, and it has particularly elegant properties. A famous result due independently to Vershik and Kerov [VK77] and Logan and Shepp [LS77] is that as n tends to infinity, a random partition λ distributed by \mathbb{P}_n has a deterministic limit shape – more precisely, the piecewise linear function $\psi_{\lambda, \sqrt{n}}$ describing the upper edge of the tilted Young diagram in coordinates rescaled by $1/\sqrt{n}$ converges in probability to a fixed known smooth curve Ω (shown in Figure 0.5).

In a breakthrough that would ultimately relate the TW-GUE distribution to the KPZ class, Baik, Deift and Johansson [BDJ99] proved that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{\text{LLIS}(\sigma) - 2n^{1/2}}{n^{1/6}} < s \right] = \lim_{n \rightarrow \infty} \mathbb{P}_n \left[\frac{\lambda_1 - 2n^{1/2}}{n^{1/6}} < s \right] = F_{\text{GUE}}(s) \tag{0.8}$$

Their proof exploits the exact solvability of the *Poissonised Plancherel measure*, the

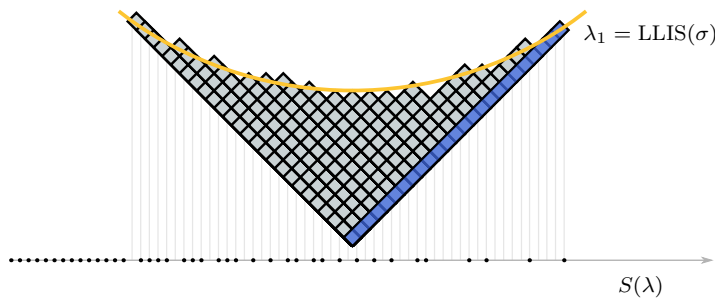


Figure 0.5 The shape of the Young tableaux associated to the uniformly sampled permutation of $1, \dots, 200$ plotted in Figure 0.4 by the Robinson–Schensted algorithm (drawn in the Russian convention). The corresponding fermion configuration is drawn below. The Vershik–Kerov–Logan–Shepp limit shape for the rescaled profile describing the upper edge of a Plancherel random partition is shown in yellow.

measure on all partitions

$$\mathbb{P}_\theta(\lambda) = e^{-\theta^2} \sum_{n=0}^{\infty} \frac{\theta^{2n}}{n!} \mathbb{P}_n(\lambda) \mathbf{1}_{\lambda \vdash n} = e^{-\theta^2} \frac{\theta^{2|\lambda|} f_\lambda^2}{|\lambda|!^2} \quad (0.9)$$

where θ is a positive parameter, which we might think of as a Gibbs measure on a grand canonical ensemble of partitions. Under \mathbb{P}_θ , the size of a random partition is a Poisson random variable with mean θ^2 . The exact solution \mathbb{P}_θ uses a mapping to a model of free fermions on a one dimensional lattice, discovered by Johansson [Joh98] and by Borodin, Okounkov and Olshanski [BOO00].

Partitions are in one-to-one correspondence with what we will call *fermion configurations* at charge zero, which are infinite subsets of the $\mathbb{Z} + \frac{1}{2}$ lattice containing all of the negative integers less than some finite number and none of the positive half integers above some finite point; the configuration $S(\lambda)$ corresponding to a partition λ can be read off the upper edge of its tilted Young diagram by including each half-integer where the slope of the profile is -1 and excluding each one where the slope is $+1$ (see Figure 0.5). $\mathbb{P}_\theta(\lambda)$ defines a *determinantal point process (DPP)* on these configurations: there is a known kernel K such that if λ is distributed by \mathbb{P}_θ , for any finite set of half integers k_1, k_2, \dots, k_N we have

$$\mathbb{P}_\theta(\{k_1, k_2, \dots, k_N\} \subset S(\lambda)) = \det_{1 \leq i, j \leq N} K(k_i, k_j). \quad (0.10)$$

With this exact expression for every correlation function of finite observables, we consider this model to be integrable. Baik, Deift and Johansson also notably related F_{GUE} to classical integrable differential equations, expressing the logarithm of its derivative as a Painlevé II transcendent by way of a Riemann–Hilbert problem.

The general formalism of DPPs was introduced by Macchi [Mac75] to model the experimentally observable spatial distribution of fermions in optical beams. In fact, the DPP defined by \mathbb{P}_θ corresponds to a physically sensible model of free fermions on the $\mathbb{Z} + \frac{1}{2}$ lattice, with a linear potential and nearest neighbour hopping dynamics weighted by the parameter θ . The linear potential could appear unnatural, since it can never saturate, but we can make sense of it by preparing an infinite number of fermions in a *domain wall state* with all negative sites occupied and all positive ones empty. Then, in the ground state, the fermion positions will have precisely the law of $S(\lambda)$ (justifying the name we gave it). The limit $\theta \rightarrow \infty$, from which the $n \rightarrow \infty$ limit of the \mathbb{P}_n is obtained through a de-Poissonisation procedure, is a continuum limit for the model. In this limit the right edge, corresponding to λ_1 , coincides with precisely the edge of a confining trap with the

Hamiltonian (0.3). The appearance of the Gross–Witten–Wadia (GWW) matrix model phase transition can also be explained directly for the edge statistics of the Poissonised Plancherel measure: as proven by Johansson using purely algebraic identities [Joh98], the cumulative distribution λ_1 is exactly equal to the partition function of the unitary matrix model they considered (before the $\theta \rightarrow \infty$ limit).

The utility of random partitions in relating different elements of the TW-GUE class to one another goes further, as the Plancherel measure is also the fixed time distribution for a random growth process. Let $\sigma^{(n)}$ be a uniform random permutation in \mathfrak{S}_n ; then, a uniform random permutation $\sigma^{(n+1)}$ in \mathfrak{S}_{n+1} can be sampled by inserting $n+1$ uniformly into $\sigma^{(n)}$. For the corresponding pair of Young tableaux, the insertion corresponds to the addition of a box, and there is a coupling between the laws \mathbb{P}_n and \mathbb{P}_{n+1} of their shapes.

Beyond the Plancherel measure and beyond the TW-GUE distribution Several natural variations of the Plancherel measure also correspond to models in the TW-GUE universality class. Considering a simple random growth process on a Young diagram in continuous time, where a box is added to each available corner with the same average rate. Then, the upper edge of the tilted Young diagram can be interpreted as the height function for the totally symmetric particle exclusion process (TASEP) prepared in the wedge initial condition; this is a reference model for discrete nonequilibrium dynamics. The longest increasing subsequence problem generalises to a problem of finding the longest directed path through different random medium [Joh00], which, via the more general Robinson–Schensted–Knuth (RSK) correspondence [Knu70], in turn corresponds to a model of random partitions [BR01a]. An interpretation of one such problem as polynuclear growth led to a strong conjecture for the long time asymptotic fluctuations of the KPZ equation by Prähofer and Spohn [PS02].

One significant development in the application of random partitions to nonequilibrium systems has been the introduction of infinite parameter families of integrable measures. In the most famous case, Okounkov’s *Schur measures* [Oko01] are defined as

$$\mathbb{P}(\lambda) = \frac{1}{Z} s_\lambda(x_1, x_2, \dots) s_\lambda(y_1, y_2, \dots) \quad (0.11)$$

where Z is the normalisation and $s_\lambda(x_1, x_2, \dots)$ is the Schur function indexed by λ , which is a particularly interesting symmetric function of the complex valued parameters x_1, x_2, \dots (the choice of which is limited only by the requirement that $\mathbb{P}(\lambda)$ must be non-negative for all λ and that Z is well defined). By definition, these measures have a high level of symmetry, being invariant under exchange of their parameters. Due to a combinatorial interpretation of the Schur function, they encode for instance the directed path problems in inhomogeneous media mentioned above (which has a symmetry under exchange of inhomogeneity parameters; see [BC22] for an application of this); the Schur function is also a determinant of other symmetric functions. Schur measures are integrable, defining DPPs on the corresponding fermion configurations; they are also related to classical integrable equations, as the Schur measure partition function is a τ function of the infinite Kadomtsev–Petviashvili (KP) hierarchy.

The Schur measure admits two important extensions. The *Schur process* introduced by Okounkov and Reshitikhin [OR03] is a measure on plane partitions, which are sequences of partitions satisfying some interlacing conditions (or 3D Young diagrams, called Wulff crystals in physics literature). It is a time dependant generalisation of the Schur measure, with growth model interpretations via the RSK algorithm, and in more combinatorial terms it is a generalisation of to two dimensions, encoding a wide range of tiling problems

(see e.g. [BBC⁺15]). The *periodic Schur process* introduced by Borodin [Bor07] is a measure on cylindric partitions, which are sequences of interlaced partitions with a periodic boundary condition, was shown by Betea and Bouttier [BB19] to encode a positive temperature fermion model.

Recently, these extensions of random partitions have offered a promising path towards explaining a connection between interface statistics for KPZ growth and trapped fermions before their respective long time and zero temperature limits. At a finite time t , the fluctuations in the interface height $\mathbf{h}(x, t)$ evolving by the KPZ equation (0.1) from the narrow wedge initial condition have an exponent of $1/3$ but are driven by the Fredholm determinant of a finite time Airy kernel, which is equivalent to the classical Airy kernel \mathcal{A} in the limit $t \rightarrow \infty$ [ACQ11, SS10]. The same Fredholm determinant was shown by Dean, Le Doussal, Majumdar and Schehr [DLDMS15] to govern the fluctuations in the rightmost of N fermions in a confining trap in 1D at a positive temperature T , where $N^{1/3}/T$ plays the role of the time t . Thanks to a new bijection which relates edge statistics for periodic Schur processes and of q -Whittaker measures discovered by Imamura, Mucciconi and Sasamoto [IMS21], there is now a path towards identifying finite time statistics for asymmetric simple exclusion processes and discrete growth processes bijectively with positive temperature statistics for lattice fermion models [IMS22].

II Multicritical Schur measures and new microscopic edge fluctuations

Let us now give an overview of the first contribution of this thesis, which deals precisely with the edge fluctuations of random partitions. In collaboration with Dan Betea and Jérémie Bouttier [1], we introduced new probability laws for partitions which escape the TW-GUE universality class, both in terms of critical exponents for fluctuations and the asymptotic law itself. Our main result is that

Result 0.1 (see Definition 2.1 and Theorem 2.2). *For each positive integer m , we can tune sequences of real numbers $\gamma_1, \gamma_2, \dots$ with finite support such that a random partition $\lambda = (\lambda_1, \lambda_2, \dots)$ under the Hermitian Schur measure*

$$P_\theta^m = \frac{1}{Z} s_\lambda(x_1, x_2, \dots) s_\lambda(\bar{x}_1, \bar{x}_2, \dots) \quad (0.12)$$

specialised in the Miwa times by

$$\frac{1}{r} \sum_i x_i^r = \frac{1}{r} \sum_i \bar{x}_i^r = \theta \gamma_r, \quad r \geq 1 \quad (0.13)$$

has asymptotic edge fluctuations which we characterise as “order m multicritical”, with

$$\lim_{\theta \rightarrow \infty} \mathbb{P}_\theta^m \left[\frac{\lambda_1 - b\theta}{(d\theta)^{\frac{1}{2m+1}}} < s \right] = F_{2m+1}(s) := \det(1 - \mathcal{A}_{2m+1})_{[s, \infty)} \quad (0.14)$$

for positive constants b, d depending on $\gamma_1, \gamma_2, \dots$, where \mathcal{A}_{2m+1} is the order m analogue of the Airy kernel ($\mathcal{A}_3 = \mathcal{A}$ is the classical Airy kernel and $F_3 = F_{\text{GUE}}$ is the TW-GUE distribution).

Multicritical edge momenta of trapped fermions This work was inspired by the discovery of the same asymptotic distribution for largest fermion momentum in particular models of trapped fermions in 1D by Le Doussal, Majumdar and Schehr [LDMS18]. By considering fermions trapped in “flat potentials” $V(x) = x^{2m}$ (m is again a positive integer) in momentum space, these authors escaped the universality of the position space edge with linearised potential; rather, in suitably rescaled dimensionless coordinates $\tilde{p} = (p - p_{\text{edge}})/\kappa$ near the Fermi edge $p_{\text{edge}}^2 = E_F$, they found a new edge Hamiltonian generalising (0.3),

$$H_{\text{edge}} = (-1)^m \frac{d^{2m}}{d\tilde{p}^{2m}} + \tilde{p}. \quad (0.15)$$

The higher-order Airy kernel \mathcal{A}_{2m+1} whose Fredholm determinant $F_{2m+1}(s)$ drives the multicritical edge fluctuations is the zero temperature ground state propagator of this Hamiltonian.

In [LDMS18], some indications about the nature of the universality class associated with these edge statistics are found by analysis of the distribution $F_{2m+1}(s)$. In particular, the authors derived a differential equation satisfied by the square root of $d^2 \log F_{2m+1}(s)/ds^2$ for each m , finding the order $2m$ equation of the Painlevé II hierarchy. Cafasso, Claeys and Girotti [CCG19] later proved by a rigorous Riemann–Hilbert method that

$$F_{2m+1}(s) = \exp \left[\int_s^\infty (x - s) q_m^2(x) dx \right] \quad (0.16)$$

where q_m is a solution of the order $2m$ Painlevé II equation which is equivalent to the order $2m + 1$ classical Airy function (the eigenfunction of (0.15) from which the order m Airy kernel \mathcal{A}_{2m+1} is built) at positive infinity. These authors found generalisations of the higher-order Airy kernel whose Fredholm determinants satisfy the same relation, which had not been considered in the context of trapped fermions [CCG19]. The appearance of higher order Painlevé transcendents implied a link to certain unitary matrix models studied by Periwal and Shevitz [PS90a, PS90b], which generalise the GWW model. For $\ell \times \ell$ matrices, these models have a partition function of the form

$$Z_\ell = \int_{\mathcal{U}(\ell)} e^{u\ell \text{tr}(V(U)+V(U^*))} \mathcal{D}U. \quad (0.17)$$

The Painlevé II hierarchy was found from these models by using orthogonal polynomials on the unit circle to tune a polynomial potential $V(U)$ such that, in the $\ell \rightarrow \infty$ limit with u finite, the square root of $Z_{\ell-1}Z_{\ell+1}/Z_\ell^2$ (which coincides with the the second derivative of the free energy) would satisfy an order $2m$ differential equation in u . For a random matrix U sampled with probability density $e^{v\ell \text{tr}(V(U)+V(U^*))}/Z_\ell$, this tuning means that in the limit $\ell \rightarrow \infty$ its eigenvalue density on the unit circle will vanish with an exponent of $2m$, in analogy with Kazakov’s multicritical Hermitian matrix model [Kaz89]. Le Doussal, Majumdar and Schehr called the novel edge behaviour they observed for fermion momenta “multicritical” after the name Periwal and Shevitz gave these unitary matrix models.

In order to develop the universality picture for the multicritical edge statistics and allow for combinatorial connections, we found discrete models with the same asymptotic edge distributions.

From Hermitian Schur measures to multicriticality We constructed measures on partitions with order m multicritical edge statistics by tuning the parameters of *polynomial*

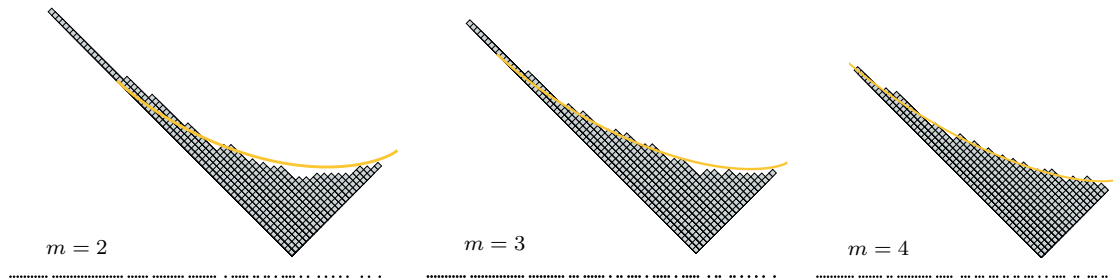


Figure 0.6 The tilted Young diagrams of partitions sampled under the minimal asymmetric multicritical measures $\mathbb{P}_\theta^{a,m}$ at orders $m = 2, 3, 4$, along with their limit shapes. Their right edges vanish more quickly than those of TW-GUE class random partitions.

Hermitian Schur measures. This corresponds to specialising not the direct parameters x_1, x_2, \dots , in which the Schur measure has an exchange symmetry, but rather the Miwa times $t_r = \sum_i x_i^r / r$ ($t_r = p_r / r$ in terms of the power-sum symmetric functions p_r). We will allow only finitely many non-zero Miwa times, so there is an infinite number of nonzero terms in the underlying direct parameter specialisation. Importantly, these specialisations are not Schur positive (that is to say, under these specialisations s_λ is negative for some partitions λ); hence, they cannot be extended to define time dependant processes. However, this atypical construction of a Schur measure corresponds to a natural model of lattice fermions, generalising the linear potential and nearest neighbour hopping dynamics of the Poissonised Plancherel measure to include longer finite range hopping dynamics, with the hopping term for a distance of r sites weighted by t_r .

The multicritical measures \mathbb{P}_θ^m for which we prove the asymptotic edge fluctuation theorem are very general. At order m , the coefficients γ_r need only satisfy $m - 1$ homogeneous linear equations in addition to some positivity conditions (see Definition 2.1). From this, we see immediately that we find the largest space of possible measures at $m = 1$, so TW-GUE edge behaviour is generic for Hermitian Schur measures. Each Miwa time grows linearly with the single parameter θ , so that the mean size of a random partition grows with θ^2 ; heuristically, tuning a Hermitian Schur measure to multicriticality corresponds to tuning its dynamic terms to coincide in the $\theta \rightarrow \infty$ continuum limit with the $d^{2m} / d\tilde{p}^{2m}$ term of the momentum space edge Hamiltonian (0.15). The proof of Result 0.1 involves asymptotic analysis of the discrete Fredholm determinant giving the exact gap probability for the multicritical Schur measure. The higher-order Airy kernel \mathcal{A}_{2m+1} is recovered from the kernel of the discrete DPP in the limit as $\theta \rightarrow \infty$ by saddle point analysis involving a “multicritical point” where $2m$ saddle points coalesce – another way in which these edge statistics are multicritical.

Limit shapes Asymptotic analysis of DPP kernel associated with \mathbb{P}_θ^m also allows us to prove that

Result 0.2 (see Theorem 2.4). *As $\theta \rightarrow \infty$, the rescaled profile $\psi_{\lambda;\theta}(x)$ describing the upper edge of the \mathbb{P}_θ^m -distributed partition λ in coordinates that scale with $1/\theta$ converges in probability to a deterministic limit shape $\Omega(x)$ which depends on the specific choice of coefficients γ_r but which has a universal vanishing exponent on its right edge, with*

$$\Omega'(x) \sim 1 - C(b - x)^{1/2m} \quad (0.18)$$

when x is close to b , where C and b are constants depending on the precise set of parameters (and θb is the expectation value of λ_1).

This generalises the $\frac{1}{2}$ exponent in the case of the $m = 1$ TW-GUE class, which coincides with the edge of the Wigner semi circle law. The existence of a limit shape is a generic property of Hermitian Schur measures in an asymptotic regime where each Miwa time grows linearly in θ .

In addition to the general criteria for multicriticality, we find a pair of one-parameter families of “minimal” multicritical Schur measures, for which we find explicit limit shapes. In the first case, $\mathbb{P}_\theta^{a,m}$ are the unique set of multicritical measures where $\gamma_1 = 1$ and only $\gamma_1, \gamma_2, \dots, \gamma_m$ are nonzero (we have $\mathbb{P}_\theta^{a,1} = \mathbb{P}_\theta$). Partitions sampled with some of these measures and the corresponding limit shapes are shown in Figure 0.6; for $m > 1$, the limit shapes are asymmetric under horizontal reflection of the tilted Young diagram, with the left edge vanishing with a $\frac{1}{2}$ exponent (and TW-GUE fluctuations in the length $\ell(\lambda)$). The second family of measures we find, $\mathbb{P}_\theta^{s,m}$, is constructed to have a symmetry under flipping the tilted Young diagram (and a second multicritical edge on the left for $\ell(\lambda)$), with $\gamma_1 = 1$ and only the odd labelled coefficients $\gamma_1, \gamma_3, \dots, \gamma_{2m-1}$ non-zero.

Connection to random matrices For the multicritical Schur measures, the connection with the Periwal–Shevitz unitary matrix models can be explained exactly (just as the Poissonised Plancherel measure was related to the GWW model in [Joh98]). Using a series of algebraic identities notably employed by Gessel [Ges90], we show that

Result 0.3 (see Theorem 3.1). *Where λ is a random partition under \mathbb{P}_θ^m , the cumulative distribution of its first part satisfies*

$$e^{\sum_r r\theta^2\gamma_r^2} \cdot \mathbb{P}_\theta^m[\lambda_1 \leq \ell] = \int_{\mathcal{U}(\ell)} e^{\theta \operatorname{tr} \sum_r (-1)^{r-1} \gamma_r (U^r + U^{*r})} \mathcal{D}U$$

*where $\mathcal{D}U$ is the Haar measure on the $\ell \times \ell$ unitary matrices. The unitary matrix model with potential $V(U) = \theta \sum_r (-1)^{r-1} \gamma_r (U^r + U^{*r})$ has order $2m$ vanishing in its eigenvalue density as $\theta, \ell \rightarrow \infty$ such that θ/ℓ is finite.*

The equivalence between the cumulative distribution for λ_1 and the unitary matrix integral is exact for all θ , while the condition that the asymptotic edge fluctuations on the partition side are multicritical ensures that the unitary matrix model is asymptotically multicritical too. The sets of coefficients γ_r defining the minimal measures $\mathbb{P}_\theta^{a,m}$ are precisely the ones found by tuning the potential on the unitary matrix side in [PS90a]. These matrix models exhibit a strong-to-weak type phase transition, generalising the one in the GWW model, with a scaling exponent of $2 + 1/m$.

Beyond zero temperature multicritical edge statistics By extending multicritical Schur measures to multicritical periodic Schur processes (measures on cylindrical partitions), we can recover models with asymptotic edge fluctuations driven by Fredholm determinants of positive temperature higher-order Airy kernels found in [LDMS18]. These Fredholm determinants were related to Painlevé II equations by Krajenbrink [Kra20] and then rigorously expressed in terms of higher-order Painlevé transcendents by Bothner, Cafasso and Tarricone [BCT21]. These statistics have yet to be related to matrix models beyond the $m = 1$ TW-GUE case, where the GUE has as a positive temperature analogue the Moshe–Neuberger–Shapiro model [MNS94]. The multicritical periodic Schur process could eventually provide an algebraic means to find a positive temperature unitary matrix model.

From the Hermitian Schur measure construction, we can also find discrete models

whose DPP kernel converges to the generalised higher-order Airy kernel introduced in [CCG19]. This construction also naturally includes measures with a *split Fermi sea*, whose asymptotic analysis presents interesting challenges and for which we find new edge distributions given by modified multicritical Fredholm determinants.

III A universality class from two-dimensional quantum gravity

Let us shift our attention to a universality class for rather different probabilistic models: random maps. Combinatorial maps are discrete surfaces, defined as gluings of polygons, or equivalently as embeddings of graphs in surfaces. As in the case of the TW-GUE universality class, the study of maps has in large part developed alongside random matrix theory (see e.g. [Eyn16]). Map data can generally be encoded by sets of permutations, making random partitions a natural (but thus far less exploited) approach to these objects too.

Combinatorial maps in physics Maps arise in physics in two main contexts (which are highly related to one another). The first is as Feynman diagrams for toy models of quantum field theories with a large gauge symmetry group, namely the special unitary group $SU(N)$ where N is large. Following 't Hooft's identification of the large symmetry group limit with a planar limit for Feynman diagrams [tH74], Brézin, Itzykson, Parisi and Zuber [BIPZ78] proposed zero-dimensional theories whose field is an $N \times N$ matrix M and whose action has a global Lie group invariance, and studied them diagrammatically. They considered models with a partition function (obtained from the path integral by the formal identification of i/\hbar with the inverse temperature β) of the form

$$Z_N = \int_{N \times N} \exp[\beta \operatorname{tr} V(M)] \mathcal{D}M \quad (0.19)$$

where $V(M)$ is a polynomial. Where the field is a Hermitian matrix and the symmetry group is $SU(N)$, the diagrams in question are *ribbon graphs*, whose edges are flat strips, with two sides of opposite orientation (each corresponding to a matrix entry M_{ij} , with an exchange of indices corresponding to flipping the orientation; the coefficients of $V(M)$ weight vertices with the corresponding number of incident edges).

The orientation of the flat strip edges fix a cyclic ordering around each vertex and identifies faces. These data define a discrete orientable surface, in precisely the way gluing together polygons does (these different ways of drawing the same object are shown Figure 0.7). Once the graph is connected, the *genus* g of that surface (the number of "handles" of a closed surface) is given by the famous relation for the Euler characteristic topological invariant

$$\chi = \#\text{vertices} - \#\text{edges} + \#\text{faces} = 2 - 2g. \quad (0.20)$$

In the limit as $N \rightarrow \infty$, as shown for very general field theories in [tH74], the dominant diagrams contributing to a path integral of the form (0.19) have $g = 0$, and can be drawn on a plane with no edges overlapping.

The ribbon graphs generated this way had previously been introduced and called maps by Tutte, who enumerated various families in the planar case by a decomposition approach [Tut63]; asymptotic analysis of matrix integrals led to derivations of Tutte's formulas. Although the matrix integral approach is non-rigorous, typically involving divergent series, other enumerative formulas it has predicted were proven combinatorially by a series of bijections to trees and mobiles initiated by Schaeffer [Sch99] and further

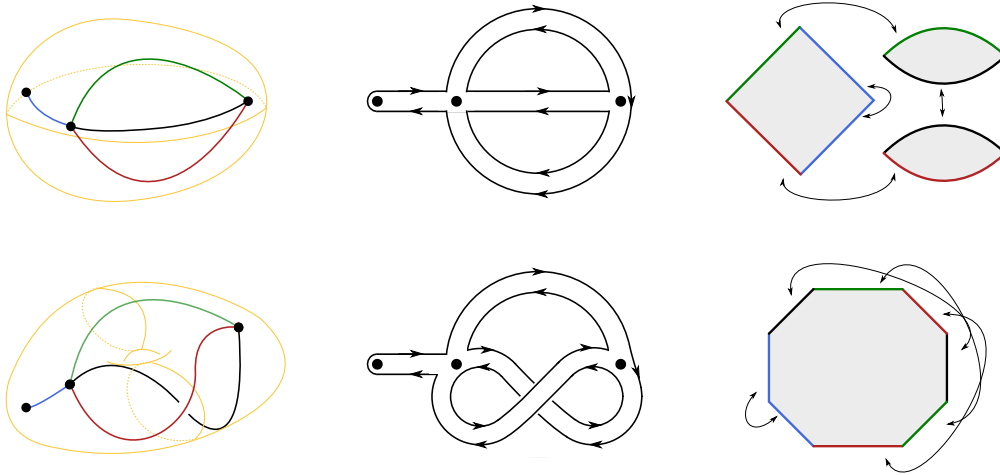


Figure 0.7 Left, two distinct maps from the same graph (note the difference in the ordering of edges around the rightmost vertex). The minimal surfaces they can be drawn on, of genus 0 and 1 respectively, are shown in yellow. Centre, the corresponding ribbon graph Feynman diagrams, both of which contribute to an integral over $\text{tr } M \cdot \text{tr } M^4 \cdot \text{tr } M^3$. Right, the corresponding gluing of polygons (showing the maps have 3 and 1 faces respectively).

developed by Bouttier, Di Francesco and Guitter [BDFG02]. Matrix integral generating functions also led to the development of topological recursion [EO07], which has since been applied to a wide range of geometric objects. The effectiveness of matrix models in map enumeration can be attributed to their integrability. The same algebraic structures that allow these models to be solved exactly have led to recurrence formulas for map enumeration via integrable hierarchies [GJ08, CC15], in some cases corresponding to map bijections [CFF13, Lou19].

The second appearance of maps in physics is as discretisations of space-time in 2D quantum gravity (or indeed discretisations of the worldsheet in string theory; we refer to the book [ADJ97] for a general overview). In 2D, gravity is topological, since by the Gauss–Bonnet theorem the scalar space-time curvature is equal to the Euler characteristic χ ; where μ denotes the metric, R the Ricci scalar, Λ the cosmological constant and \mathcal{N} Newton’s constant, the Einstein–Hilbert action for a 2D space-time Σ with area $A(\Sigma)$ and Euler characteristic $\chi(\Sigma)$ is just

$$S_{\text{EH}}(\Sigma) = \int_{\Sigma} (\Lambda \sqrt{\det \mu} + \mathcal{N} R \sqrt{\det \mu}) d^2 x = \Lambda A(\Sigma) + \mathcal{N} \chi(\Sigma). \quad (0.21)$$

To quantise this action, the path integral is computed by integrating $\exp(iS_{\text{EH}}(\Sigma)/\hbar)$ over surfaces Σ at fixed topology, and summing over topologies. The integration over all surfaces required here is not well defined, but by considering discrete ones in the form of maps, the path integral can be reframed as a well defined sum; identifying i/\hbar with β recasts the quantum mechanical problem in terms of an ensemble of random maps.

Universality of planar maps Asymptotic map enumeration results at fixed genus obtained by the approaches mentioned above or more classical generating function methods (e.g. in [BC86, BMJ06]) have universal exponents: regardless of the precise composition of the maps (whether they are, for instance, glueings of polygons of fixed degree, maps with alternately coloured vertices or faces, or maps of all degree), the number of maps of size (which can be the number of vertices or the number of faces) n with a labelled “root” vertex at genus g is, as $n \rightarrow \infty$,

$$\#\{\text{maps of size } n\} \sim C \gamma^n n^{5/2(g-1)} \quad (0.22)$$

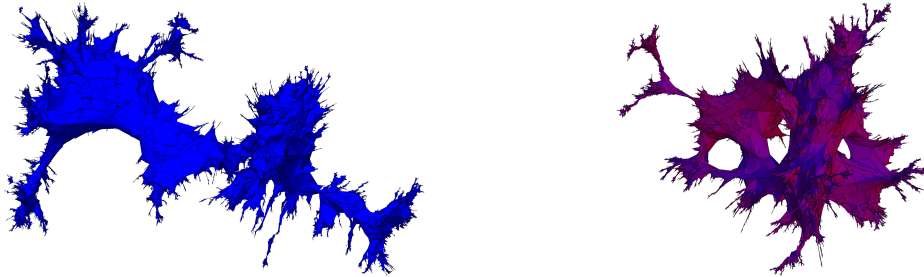


Figure 0.8 Approximations of the Brownian sphere and the Brownian triple torus (genus 3) by uniformly sampled large maps, produced by Jérémie Bettinelli. The algorithm used, introduced in [Cha09b], has been programmed up to genus 4.

where C, γ are model dependent constants (see e.g. [BFSS01] for the $g = 0$ case and [BC86, Gao93, Cha09a] for $g \geq 1$).

For planar maps, the universality extends to the asymptotic limiting objects themselves. Bijections to trees allow maps to be defined as metric spaces [BDFG04]. Le Gall [Gal13] and Miermont [Mie13] independently proved that when the map size tends to infinity while the distance is rescaled to remain finite, there exists a unique universal scaling limit for the metric space of a uniform random planar p -angulation as its size tends to infinity, the *Brownian sphere* (see Figure 0.8). This limiting object can be considered as a canonical model of a continuous random surface, and has important consequences for the study of 2D quantum gravity, allowing for the rigorous discrete construction of conformal field theories predicted for 2D quantum gravity from string theory [Pol81] (see for instance [DKRV16] for the physical models in question and [Gwy20] for a review on probabilistic advances). The growing family of models proven to converge to the Brownian sphere at genus zero make up the universality class of maps.

For maps with fixed positive genus, there is a similar, though less well explored, universality picture: analogous Brownian surfaces were found by Bettinelli [Bet10, Bet12] as scaling limits of particular quadrangulations, using the the Chapuy–Marcus–Schaeffer bijection [CMS09] to define a metric space.

The high genus frontier If instead of keeping the genus fixed, we consider an asymptotic regime where the genus g grows with the size n , established methods fail. In particular, the linear “high genus” regime where $g \sim qn$ for some constant q is an exciting new frontier. In terms of asymptotic enumeration, this regime requires the development of new tools; in terms of the geometry of large random maps (specifically their *local limit*; see Figure 0.9), a neighbourhood of a fixed radius around the root vertex of a map of size n looks the same for any fixed genus as $n \rightarrow \infty$, as the root vertex do not “see” the handles, but if the genus grows with n the local limit will change dramatically. The high genus regime was pioneered in [ACCR13] and [Ray15] in the case of maps with only one face, which do not belong to the map universality class but to a class of generalised trees. A breakthrough in understanding this regime for maps was recently made by Budzinski and Louf [BL21], who proved the Benjamini–Curien conjecture [Cur16] on the local limits of uniform random triangulations in this regime. As a by-product, using a combination of probabilistic and combinatorial methods, they obtained the estimate

$$\#\{\text{triangulations of size } n \text{ and genus } g\} = n^{2g} \exp[C(q)n + o(n)] \quad (0.23)$$

where $C(q)$ is a known function. Subsequent work by the same authors identifying related local limits for a wider class of glueing of p -gons [BL22] led them to conjecture

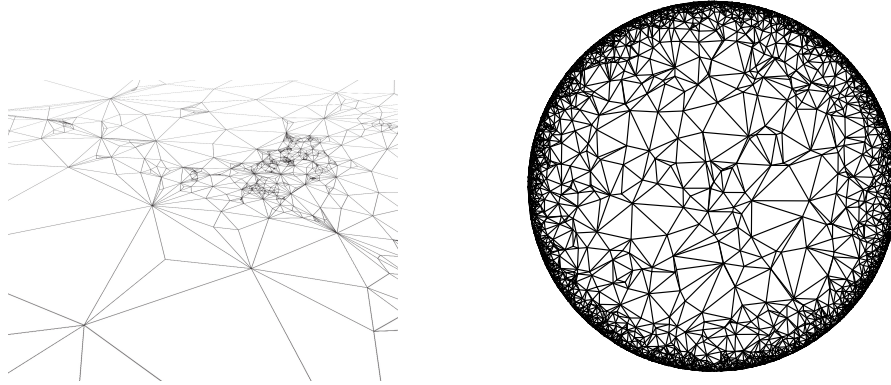


Figure 0.9 Left, a simulation of the neighbourhood of a vertex of a uniform infinite planar triangulation produced by Igor Kortchemski. Right, a simulation of a planar stochastic hyperbolic triangulation embedded in the hyperbolic plane, produced by Nicolas Curien. This is the local limit of uniform triangulations at high genus [BL21].

that this estimate is universal up to the function $C(q)$.

Random partitions and maps Returning to our main objects of interest, random partitions, and to the TW-GUE universality class, map enumeration played a key role in proving a conjecture of Baik, Deift and Johansson. In [BDJ99], the authors inferred from (0.8) that the equivalence in law between the maximum eigenvalue $\xi_1 := \xi_{\max}$ of an $n \times n$ GUE random matrix and the first part λ_1 of a Plancherel random partition $\lambda \vdash n$ as $n \rightarrow \infty$ would extend to equivalence of the joint distributions of the m highest eigenvalues $\xi_1 \geq \xi_2 \geq \dots \geq \xi_m$ and parts $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ where m is fixed. Okounkov [Oko00b] proved this combinatorially, without exploiting the DPPs encoded by the GUE and by the Plancherel measure after Poissonisation (the DPP approach was later used in [BOO00] and [Joh01]). Instead, his proof used the fact that the moments of traces of a GUE random matrix asymptotically count planar maps, then related these discretisations of the sphere to ramified coverings of the Riemann sphere which are in turn counted by moments of parts of a Plancherel random partition via the Jucys–Murphy elements.

Matrix model generating functions of maps have also been directly expressed in terms of characters of the symmetric group, which are expressed in terms of partition data, leading to closed expressions for numbers of not necessarily connected maps [DFI93]. The unitary matrix models of the form (0.3) which are exactly equivalent to gap probabilities for Schur random partitions resemble the Hermitian ones of the form (0.19), and in fact the multicritical matrix models were proposed as a form of map generating function in [PS90a] (more precisely, writing a unitary matrix as the exponential of a Hermitian one, as series of map generating functions). This interpretation has not yet been formulated rigorously, and remains an interesting open question.

IV The Plancherel–Hurwitz measure and new macroscopic limit behaviour

Let us now present the second contribution of this thesis, where, in collaboration with Guillaume Chapuy and Baptiste Louf [2], we tested a new random partition approach to the asymptotic enumeration of maps at high genus. To do so we introduced a new generalisation of the Plancherel measure on partitions whose normalisation counts

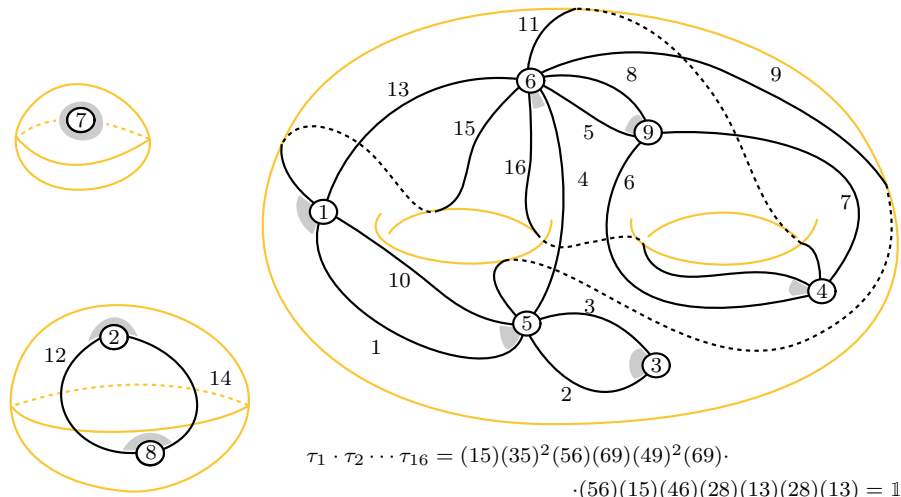


Figure 0.10 The pure Hurwitz map with Euler characteristic $\chi = 2$ corresponding to the transposition factorisation of the identity on \mathfrak{S}_9 shown (it is unconnected as the factorisation is not transitive). Each transposition gives the labels of two vertices which are connected by an edge, the order of the transpositions determines the cyclic ordering of the edges around vertices. Each face has exactly one corner which is an edge-label descent, indicated in grey.

certain natural not necessarily connected maps. In a regime where the negative Euler characteristic of the corresponding maps grows linearly with their size, this measure gave rise to a novel two part limit phenomenon for a random partition, with its first part becoming much larger than all of the others.

Hurwitz numbers and maps We generalise the Plancherel measure \mathbb{P}_n by considering different elements of the symmetric group \mathfrak{S}_n . The *unconnected Hurwitz numbers* $H_{n,\ell}$ count factorisations of the identity by ℓ transpositions on \mathfrak{S}_n which are not necessarily transitive. A purely representation theoretic relation, *Frobenius’ theorem*, expresses these numbers in terms of the characters of transpositions on the irreducible representations of \mathfrak{S}_n , which can be expressed in terms of Young diagram data as

$$H_{n,\ell} := \#\{(\tau_1, \tau_2, \dots, \tau_\ell) \in (\mathfrak{S}_n)^\ell, \tau_1 \cdot \tau_2 \cdots \tau_\ell = \mathbb{1}\} = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 C_\lambda^\ell \quad (0.24)$$

where each τ_i is a transposition, f_λ is the number of standard Young tableaux of shape λ and C_λ is the *content-sum*, which can be written simply as $C_\lambda = \sum_i \lambda_i(\lambda_i - 2i + 1)/2$. For each even ℓ , we can interpret $H_{n,\ell}$ as the partition function normalising what we call the *Plancherel–Hurwitz measure*, with $\mathbb{P}_{n,\ell}(\lambda) := f_\lambda^2 C_\lambda^2 / n! H_{n,\ell}$, which reduces to the classical Plancherel measure when $\ell = 0$. The measure $\mathbb{P}_{n,\ell}$ is invariant under conjugation of the partition (that is, flipping the tilted Young diagram horizontally), and for $\ell > 0$ it is natural to consider only its “positive half” $\mathbb{P}_{n,\ell}^+(\lambda) := 2\mathbb{P}_{n,\ell}(\lambda)\mathbf{1}_{C_\lambda > 0}$, conditioned on the content-sum being positive (so the boxes are predominantly to the right in the tilted Young diagram).

From a high genus map perspective, the Hurwitz numbers are interesting because transposition factorisations of the identity are in one-to-one correspondence with *unconnected pure Hurwitz maps* (they also enumerate ramified coverings of the Riemann sphere [Hur91], an enumerative geometry perspective we will not develop). These maps are special cases of *constellations* [BMS00], and are strongly conjectured to be in the map universality class [DPS14]; from a simple rule for drawing a transposition (see Figure 0.10) we see that $H_{n,\ell}$ counts not necessarily connected maps with n vertices, ℓ

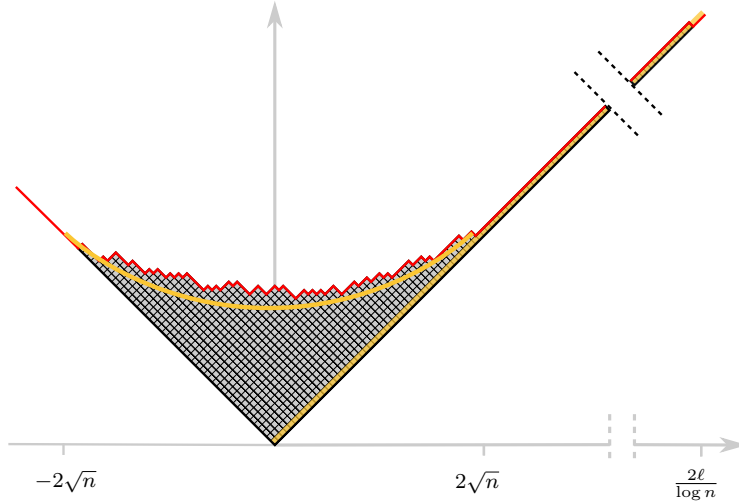


Figure 0.11 A partition sampled under the Plancherel–Hurwitz measure $\mathbb{P}_{n,\ell}$ at high genus. The twofold asymptotic behaviour is shown in yellow: the first part becomes very large and escapes the picture, while the rest of the partition has the limit shape of the Plancherel measure \mathbb{P}_n . The profile of the partition is shown in red; without a DPP for $\mathbb{P}_{n,\ell}$, our proof relies on functional optimisation of this curve.

edges and n faces. The Euler characteristic relation here reads

$$\chi = 2n - \ell = 2\kappa - 2G \tag{0.25}$$

where κ is the number of connected components in the map and G the sum of their genera. Then for a sequence of $\ell := \ell(n) \sim 2qn$, if $q > 1$, the sum of genera G grows linearly with n and the connected components of the corresponding maps are of high genus (although this is not strictly the high genus regime studied for the genus may be divide unevenly across different connected components).

A high genus limit shape for Young diagrams Considering the corresponding random partitions, we found a deterministic limit behaviour associated with this kind of high genus regime, and proved that

Result 0.4 (see Definition 4.1 and Theorem 4.2). *Under the positive half of the Plancherel–Hurwitz measure $\mathbb{P}_{n,\ell}^+$ whose partition function counts not necessarily transitive factorisations of the identity on \mathfrak{S}_n by ℓ transpositions, in the regime given by $\ell = \ell(n) \sim 2qn$ for some $q > 0$, a random partition $\lambda \vdash n$ with positive content-sum has the following limit behaviour as $n \rightarrow \infty$:*

- (i) *the first part λ_1 is equivalent to $2\ell/\log n$ in probability;*
- (ii) *the rest of the partition $\tilde{\lambda} = (\lambda_2, \lambda_3, \dots)$ has a VKLS limit shape. Namely, the rescaled profile $\psi_{\tilde{\lambda};\sqrt{n}}(x)$ converges in probability to the limit shape Ω of the Plancherel measure $\mathbb{P}_{n,0}$ in the supremum norm;*
- (iii) *the second part λ_2 and the number of parts $\ell(\lambda)$ are both equivalent to $2\sqrt{n}$ in probability.*

This phenomenon is illustrated in Figure 0.11. A random partition under the full measure $\mathbb{P}_{n,\ell}$ is a partition distributed by $\mathbb{P}_{n,\ell}^+$ which is then conjugated with probability $\frac{1}{2}$ (or, its tilted Young diagram is flipped horizontally with probability $\frac{1}{2}$).

The “bulk” $\tilde{\lambda} = (\lambda_2, \lambda_3, \dots)$ behaves exactly like a Plancherel random partition, and λ_1 is the only part growing faster than \sqrt{n} ; this twofold structure is quite novel. Our proof, using comparisons of entropies, also follows this twofold structure, with a second layer of refinement required for the precise bounds of $2\sqrt{n}$ on the λ_2 and $\ell(\lambda)$. The limit behaviour can be thought of as resulting the competing “forces” of the content-sum term C_λ^ℓ and Plancherel measure term f_λ^2 : while the former is maximised by a very skewed Young diagram, there is a cost from the latter in deviating from the symmetric VKLS limit shape. Result 0.4 shows that a random partition balances these factors by obtaining a large content-sum only from the first part, and leaving the rest of the partition to maximise the Plancherel part only.

Connections to random walks and integrable hierarchies The Plancherel–Hurwitz measure is a natural extension of the classical Plancherel one, and it previously appeared (without a name) in two seminal works.

The first, by Diaconis and Shahshahani [DS81], considered a random walk on \mathfrak{S}_n generated by choosing a transposition τ_i uniformly ℓ times. They were interested regimes in which the walk is asymptotically *well mixed*, that is in finding $\ell := \ell(n)$ such that the law of the random permutation $\sigma^{(\ell)} = \tau_\ell \cdot \tau_{\ell-1} \cdots \tau_1 \sigma^{(0)} \in \mathfrak{S}_n$ is close to uniform as $n \rightarrow \infty$. They found a mixing threshold for ℓ , showing that if $\ell > (1+\varepsilon)n \log n/2$ for any $\varepsilon > 0$, the total variation distance from the law of $\sigma^{(\ell)}$ to the uniform law tends to zero. This leads directly to an asymptotic estimate for $H_{n,\ell}$ with ℓ above the mixing threshold. The Plancherel–Hurwitz measure appears in the formula for the total variation distance, and in fact the proof uses the trivial limiting behaviour of a partition under $\mathbb{P}_{n,\ell}$ with ℓ above the mixing threshold: then, λ almost surely has only one part or n parts. The limiting behaviour in Result 0.4 can be seen as combining this phenomenon with the Plancherel measure VKLS behaviour.

The second, by Okounkov [Ok00a], used the partition expression for the unconnected Hurwitz numbers $H_{n,\ell}$ to show that their generating function encodes solutions of a hierarchy of integrable differential equations – namely, that it is a τ -function of the Toda lattice hierarchy, which is a generalisation of the KP hierarchy for which the Schur measure partition function is a τ -function. This uses precisely the same correspondence with fermion configurations that allowed the Poissonised Plancherel measure and Schur measures to be solved exactly and asymptotically using DPPs. The Plancherel–Hurwitz measure ostensibly also defines a DPP after Poissonisation, however in this case we do not have a generating function for the integral kernel even at fixed ℓ , and thus cannot use such an approach. The integrability of this measure at the level of classical hierarchies, however, has been used by Dubrovin, Yang and Zagier [DYZ17] to find recurrence relations for the connected Hurwitz numbers $h_{n,\ell}$. These recurrence relations have not so far lead to asymptotics for $h_{n,\ell}$.

An estimate for $H_{n,\ell}$ at high genus and the corresponding map model As a direct consequence of the limit shape behaviour, we obtain an asymptotic estimate for $H_{n,\ell}$ in the same regime, up to a exponential terms.

Result 0.5 (see Theorem 4.3). *Let $H_{n,\ell}$ be the number of (not necessarily transitive) factorisations of the identity on \mathfrak{S}_n by an even number $\ell = \ell(n) \sim 2qn$ of transpositions. As $n \rightarrow \infty$,*

$$H_{n,\ell} = \left(\frac{\ell}{\log n} \right)^{2\ell} \exp [(-2 + \log 2)\ell + o(n)]. \quad (0.26)$$

This should not be directly compared to the Budzinski–Louf estimate (0.23) because there is a fundamental difference between our model and theirs, and most map models considered in the literature: the maps counted by $H_{n,\ell}$ are not necessarily connected. Indeed, we show that a uniform random map in our model is almost surely unconnected, in particular that

Result 0.6 (see Theorem 4.8). *For all $q > 1$, a uniformly random Hurwitz map with n vertices and an even number $\ell = \ell(n) \sim 2qn$ of edges contains a connected component with at least $c(q)\ell$ edges, for some function $c(q) > 0$, and no connected component has more than $O_p(n/\log n)$ vertices.*

Thus, the Plancherel–Hurwitz measure does not provide a direct approach to the high genus map regime that motivated it. Rather, we might look at the asymptotic estimate for $H_{n,\ell}$ as giving a return probability for the Diaconis–Shahshahani random walk [DS81] in a linear regime. Knowing the limit behaviour of partitions under this measure and the limit behaviour of corresponding uniform random transposition factorisations (which for now we only understand to a limited extent) is a promising step towards finding a bijective interpretation of Frobenius’ theorem as stated in (0.24). A natural question in this context is to find an interpretation for the statistic λ_1 , which in the purely Plancherel case corresponds to $\text{LLIS}(\sigma)$ where σ is a uniform random permutation. We can note that under the Plancherel–Hurwitz measure at high genus, λ_1 has a limiting value related to the Diaconis–Shahshahani mixing threshold, which is also related to upper bound on the number on vertices in a connected component of a uniform pure Hurwitz map.

V Organisation of the text

The main text is divided into four chapters.

Chapter 1 (“Integer partitions and integrability”) gathers mathematical and physical preliminaries, starting with classical definitions pertaining to integer partitions. We review some aspects of the representation theory of the symmetric group and of symmetric functions, then discuss some important classical probability laws on partitions. Finally we discuss the integrability of Schur measures, reviewing determinantal point processes and relating them to physical fermion models.

Chapter 2 (“Multicritical Schur measures”) is based on joint work with Dan Betea and Jérémie Bouttier [1], in which we construct multicritical Schur measures on partitions and study their asymptotic behaviour using steepest descent analysis of associated correlation kernels. This chapter includes a broad discussion of the connection between lattice fermion models and Schur measures in the Hermitian case, and a heuristic description of how they asymptotically coincide with trapped fermions on the edge. Three natural extensions of the multicritical measures are presented.

Chapter 3 (“Random matrices and random maps”) presents connections between random partitions and random matrices. We discuss an exact correspondence between Schur measures and unitary matrix models, and present a new contribution from [1] relating multicritical Schur measures to multicritical unitary matrix models. We also review classical results relating the Gaussian unitary ensemble to the Plancherel measure and to free fermions. In connection with Hermitian matrix models we review definitions for combinatorial maps, and discuss how random maps relate to random partitions.

Chapter 4 (“The Plancherel–Hurwitz measure”) is based on joint work with Guillaume Chapuy and Baptiste Louf [2], in which we introduce a deformation of the Plancherel

measure and relate it to a model of not necessarily connected random maps. We study its limiting behaviour in an asymptotic regime related to high genus maps using various entropy optimisation and comparison methods, and study the connectedness of the corresponding random maps. We relate our results to random transposition walks on symmetric groups.

Finally we collect open questions and general conclusions in Chapter 5.

Integer partitions and integrability

In this chapter we gather definitions and classical algebraic and combinatorial results relating to the main objects we study, integer partitions, and discuss natural probability measures on partitions that arise from those results. Of particular interest are measures which are *integrable*, meaning that every correlation function can be computed exactly. We review how certain measures define determinantal point processes, and how these relate to physical discrete fermion models.

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1.1 Integer partitions and elements of algebra

1.1.1 Preliminary definitions

This section collects some basic definitions, first for ways of representing integer partitions (as diagrams, sets of half integers, and functions) then ways of filling their diagrams that provide useful combinatorial data.

Partitions and how to draw them Let us begin by formally defining the central objects of this thesis, integer partitions (and by dropping the word “integer” from their name).

Definition 1.1 (Partition). A *partition* λ is a weakly decreasing finite sequence of positive integers called *parts*

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)}) \subset \mathbb{Z}_{>0}. \quad (1.1)$$

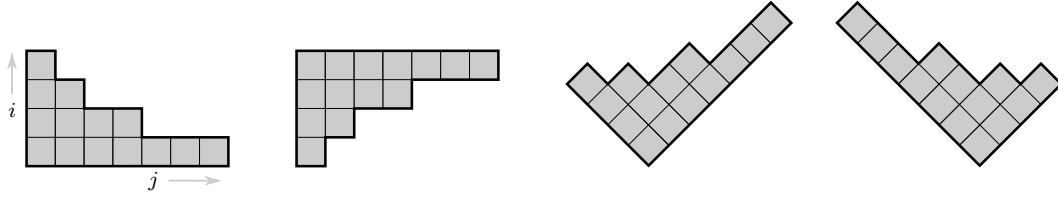


Figure 1.1 From left to right: the Young diagram of the partition $\lambda = (7, 4, 2, 1) \vdash 14$ in our (French) convention; the Young diagram of λ in the English convention; the tilted (Russian) Young diagram of λ ; the tilted Young diagram of the conjugate partition $\lambda' = (4, 3, 2, 2, 1, 1, 1)$.

It is extended to an arbitrarily long vector by employing the convention $\lambda_i := 0$ for all $i > \ell(\lambda)$. The number of positive parts $\ell(\lambda) \geq 0$ is called the *length* of λ , and the sum of the parts, denoted $|\lambda|$, is called its *size*. Where $|\lambda| := \sum_i \lambda_i = n$, λ is said to be a *partition of n* , and this is denoted $\lambda \vdash n$. The unique partition of 0 is the empty partition denoted \emptyset .

A partition is naturally encoded by a second partition of the same size:

Definition 1.2 (Conjugate partition). For each partition $\lambda \vdash n$, its *conjugate* λ' is the unique partition of n with parts

$$\lambda'_j = \#\{i | \lambda_i \geq j\}. \quad (1.2)$$

The conjugation operation $\lambda \mapsto \lambda'$ is an involution. Partitions are represented graphically by the diagrams composed of square boxes defined as follows (we refer to a partition and its diagram interchangeably, using the same symbol for both):

Definition 1.3 (Young diagram). Where λ is a partition, the Young diagram λ is a stack of left aligned rows of boxes with λ_i boxes in the i th row from the bottom.

There are λ'_j boxes in the j th column from the left, so the conjugation operation exchanges rows and columns at the level of Young diagrams. This is defined in the “French convention”. To switch the Young diagram to the “English convention”, which is perhaps the most standard, one flips it vertically. It will prove convenient to switch to a third “Russian” convention, where we consider the *tilted Young diagram*, which is the Young diagram rotated counter-clockwise by 45° . The tilted Young diagram of λ' is just that of λ flipped horizontally. See Figure 1.1.

Partitions as fermion configurations, and how to draw them A partition can also be encoded by an infinite set of distinct half integers, as follows:

Definition 1.4 (Fermion configuration of a partition). For each partition λ , its *fermion configuration* $S(\lambda)$ is the set

$$S(\lambda) = \left\{ \lambda_i - i + \frac{1}{2}, i \in \mathbb{Z}_{>0} \right\} \subset \mathbb{Z} + \frac{1}{2}. \quad (1.3)$$

A generic fermion configuration is a subset S of $\mathbb{Z} + \frac{1}{2}$ with all but finitely many negative half integers included and only finitely many positive elements, and corresponds to a *Maya diagram* (that is, a row of black and white nodes where far enough to the left every node is black and far enough to the right every node is white), where one places a

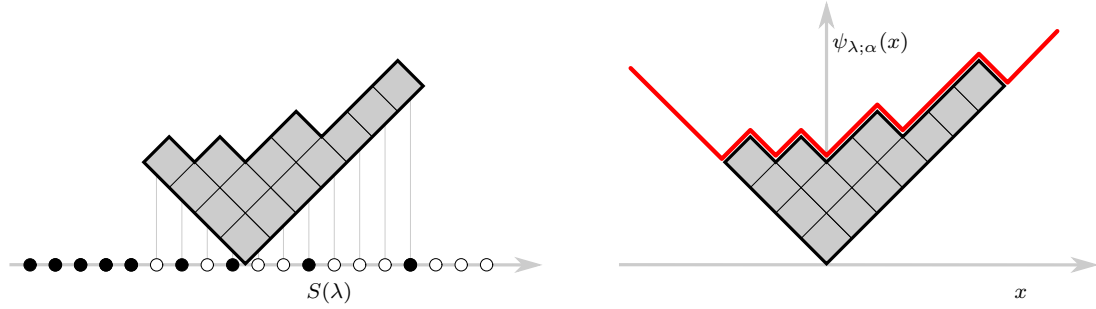


Figure 1.2 The tilted Young diagram of the partition $\lambda = (7, 4, 2, 1) \vdash 14$ again, shown on the left with the Maya diagram of the corresponding fermion configuration $S(\lambda) = \{\frac{13}{2}, \frac{5}{2}, -\frac{1}{2}, -\frac{5}{2}, -\frac{9}{2}, -\frac{11}{2}, \dots\}$ and on the right with a coordinate system and the rescaled profile $\psi_{\lambda; \alpha}(x)$.

black node on each point $k \in S$ of a line labelled with $\mathbb{Z} + \frac{1}{2}$ and a white node on every other site. The charge Q_0 of such a set is the difference

$$Q_0(S) = \#\{k \in S | k > 0\} - \#\{k \notin S | k < 0\}, \quad k \in \mathbb{Z} + \frac{1}{2}. \quad (1.4)$$

The fermion configuration $S(\lambda)$ associated to a partition has charge zero: for an empty partition, $S(\emptyset)$ has charge zero trivially, and each other $S(\lambda)$ be obtained from $S(\emptyset)$ by an operation replacing an element $-i + \frac{1}{2}$ with $\lambda_i - i + \frac{1}{2}$, which preserves Q_0 . More generally, a charge Q_0 fermion configuration can be assigned to λ by shifting each element of $S(\lambda)$ to $\lambda_i - i + \frac{1}{2} + Q_0$. In terms of its fermion configuration, the size of a partition is $|\lambda| = Q_1(S(\lambda))$ where

$$Q_1(S) = \sum_{k > 0, k \in S} k - \sum_{k < 0, k \notin S} k. \quad (1.5)$$

The Maya diagram of $S(\lambda)$ can be drawn from the tilted Young diagram of λ by adding axes centred at the bottom corner, with coordinates on the horizontal x -axis such that the corners of the boxes align with the integers. Consider the curve traced by upper edge of the tilted Young diagram extended to the curve $|x|$ on either side, which we called the *profile* of λ . At each $\mathbb{Z} + \frac{1}{2}$ labelled point on the x -axis we place a black node if the profile has slope -1 , a white node if it has slope $+1$. See Figure 1.2.

Partitions as real functions The profile described above can itself be used to encode a partition; since it is a continuous function, this formulation allows us to recast partition data in analytic terms. It is most convenient if the profile is defined more generally in coordinates that scale with some parameters, as follows:

Definition 1.5 (Rescaled profile of a partition). For each partition λ and $\alpha > 0$ which may depend on λ or other parameters, the *rescaled profile* $\psi_{\lambda; \alpha}(x)$ of λ is the piecewise linear continuous function composed of elements with slope ± 1 tracing the upper edge of the tilted diagram of λ in coordinates in which the centres of the boxes are $1/\alpha$ apart, and extended to a function on all \mathbb{R} by appending a line of slope $+1$ on the right and one of slope -1 on the left. To write formulas for $\psi_{\lambda; \alpha}(x)$, we start from the expressions

$$v = \phi_{\lambda; \alpha}(u) := \frac{1}{\alpha} \lambda_{[\alpha u] + 1}, \quad u \in (0, \infty) \quad \text{and} \quad u = \phi_{\lambda; \alpha}^{-1}(v) := \frac{1}{\alpha} \lambda'_{[\alpha v] + 1}, \quad v \in (0, \infty) \quad (1.6)$$

for the edges of the Young diagram in continuous rescaled coordinates; then, with u and v defined by the union of these curves, the rescaled profile is given implicitly by the change of coordinates

$$\psi_{\lambda; \alpha}(x) = u + v, \quad x = u - v. \quad (1.7)$$

One such profile is illustrated in Figure 1.2. We can see that for all x and x'

$$|\psi_{\lambda;\alpha}(x) - \psi_{\lambda;\alpha}(x')| \leq |x - x'|, \quad (1.8)$$

or in other words that $\psi_{\lambda;\alpha}$ is *1-Lipschitz*, and that $\psi_{\lambda;\alpha}(x) \geq |x|$ everywhere. The size of the partition is

$$|\lambda| = \frac{\alpha^2}{2} \int (\psi_{\lambda;\alpha}(x) - |x|) dx; \quad (1.9)$$

the most natural choices of α are then proportional to $\sqrt{|\lambda|}$, or to $\sqrt{\mathbb{E}(|\lambda|)}$ if we consider the profile of a random partition λ whose size is not fixed. The “unrescaled profile” where each box has area 2 is $\psi_{\lambda;1}$. The profile $\psi_{\lambda;\alpha}$ can also be expressed in terms of the fermion configuration $S(\lambda)$: restricting to integer values of αx , we have

$$\psi_{\lambda;\alpha}(x) = x + \frac{2}{\alpha} \cdot \#\{k \in S \mid k > \alpha x\}, \quad x \in \frac{1}{\alpha}\mathbb{Z}. \quad (1.10)$$

Filled Young diagrams and related objects For what follows, it is useful to define extensions of Young diagrams, which appear notably in combinatorics and representation theory.

Definition 1.6 (Young tableau). A *Young tableau* of shape λ is a filling of the boxes of the Young diagram λ with positive integers. It is said to be *semi-standard* (and abbreviated as *SSYT*) if the entries are strictly increasing bottom to top along columns and weakly increasing left to right along rows, and *standard* (and abbreviated as *SYT*) if they are also strictly increasing along rows and the numbers are $1, \dots, |\lambda|$.

One of the most important quantities relating these objects is the number of SYT of a given shape, which can be computed in terms of another filling of the boxes of the same Young diagram:

Theorem 1.7 (Hook length formula [FRT54]). *The number of SYT of shape $\lambda \vdash n$ is*

$$f_\lambda = \frac{n!}{\prod_{\square \in \lambda} \eta_\lambda(\square)}, \quad \eta_\lambda(\square_{i,j}) = \lambda_i - i + \lambda'_j - j + 1 \quad (1.11)$$

where the product is taken over all boxes in the Young diagram λ and the hook length $\eta_\lambda(\square)$ is number of boxes in the hook from \square to the edges of the diagram (written explicitly for the the box $\square_{i,j}$ in the i th row and j th column).

This formula has been proven in several ways since its discovery by Frame, Robinson and Thrall [FRT54]. We review one simple “probabilistic” proof.

Sketch of Greene, Nijenhuis and Wilf’s “hook walk” proof [GNW79]. Fix $\lambda \vdash n$, and let “ $\mu \nearrow \lambda$ ” denote that $\mu \vdash n - 1$ is a Young diagram obtained from λ by the removal of one box. An SYT of shape λ differs from an SYT of some $\mu \nearrow \lambda$ by the addition of one box filled with the number n to an outer corner; hence, the number f_λ of SYT of shape λ satisfies

$$f_\lambda = \sum_{\mu \nearrow \lambda} f_\mu. \quad (1.12)$$

Now let

$$g_\lambda = \frac{n!}{\prod_{\square \in \lambda} \eta_\lambda(\square)}. \quad (1.13)$$

Trivially, we have $f_\emptyset = g_\emptyset = 1$, so by induction $f_\lambda = g_\lambda$ if and only if $g_\lambda = \sum_{\mu \nearrow \lambda} g_\mu$.

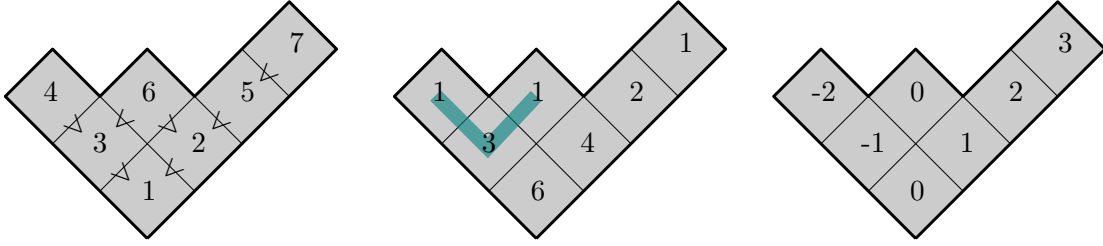


Figure 1.3 The tilted Young diagram of the partition $(4, 2, 1) \vdash 7$. Left, its boxes are filled to produce a SYT of shape $(4, 2, 1)$; center, they are filled with their hook lengths, showing there are $f_{(4,2,1)} = 7!/(6 \cdot 4 \cdot 3 \cdot 2) = 35$ such tableaux; right, each box is filled with its content, and the content-sum is $C_{(4,2,1)} = -2 - 1 + 0 + 0 + 1 + 2 + 3 = 3$.

Now choose a box $\square^{(0)}$ from λ uniformly at random, and start a random “hook walk” from that box as follows: at step i , choose the new box $\square^{(i)}$ uniformly from other boxes in the hook from $\square^{(i-1)}$ to the edges. This walk will move outward toward the edge via shorter and shorter hooks, and terminates at a box $\square^{\text{end}} := \square_{\mathbf{a}, \mathbf{b}}$ where (\mathbf{a}, \mathbf{b}) is the position of some corner box. Removing \square^{end} from λ generates a random partition $\mu \nearrow \lambda$. By construction of all walks out to some $\square_{\mathbf{a}, \mathbf{b}}$ (see e.g. [Rom15, Section 1.9]), one can show that the law of μ is

$$\begin{aligned} \mathbb{P}(\mu = \lambda \setminus \square_{\mathbf{a}, \mathbf{b}}) &= \frac{1}{n} \sum_{\substack{A \subseteq \{1, \dots, a-1\} \\ B \subseteq \{1, \dots, b-1\}}} \prod_{i \in A} \frac{1}{\eta_{\lambda}(\square_{i, b}) - 1} \prod_{j \in B} \frac{1}{\eta_{\lambda}(\square_{\mathbf{a}, j}) - 1} \\ &= \frac{1}{n} \prod_{i=1}^{a-1} \frac{\eta_{\lambda}(\square_{i, b})}{\eta_{\lambda}(\square_{i, b}) - 1} \prod_{j=1}^{b-1} \frac{\eta_{\lambda}(\square_{\mathbf{a}, j})}{\eta_{\lambda}(\square_{\mathbf{a}, j}) - 1} = \frac{g_{\lambda \setminus \square_{\mathbf{a}, \mathbf{b}}}}{g_{\lambda}}. \end{aligned} \quad (1.14)$$

But since this law is well normalised $\sum_{\mu \nearrow \lambda} g_{\mu}/g_{\lambda} = 1$, which is what is required. \blacksquare

A third filling of boxes appears in representation theoretic contexts:

Definition 1.8 (Contents, content-sum of a partition). The *contents* of a box $\square_{i, j}$ in the i th row and j th column of a Young diagram is

$$c(\square_{i, j}) = j - i. \quad (1.15)$$

The *content-sum* of a partition λ is

$$C_{\lambda} = \sum_{\square \in \lambda} c(\square) = \sum_{i=1}^{\ell(\lambda)} \frac{\lambda_i(\lambda_i - 2i + 1)}{2}. \quad (1.16)$$

In the tilted Young diagram, the contents of a box is its horizontal position (in the “unrescaled” coordinates). In terms of its fermion configuration $S(\lambda)$, the content-sum of λ is $C_{\lambda} = Q_2(S(\lambda))$ where

$$Q_2(S) = \sum_{k > 0, k \in S} \frac{k^2}{2} - \sum_{k < 0, k \notin S} \frac{k^2}{2}. \quad (1.17)$$

Generalisation to skew partitions and related sequences Let us introduce the following notion of a “difference” of partitions, which corresponding generalisations of Young diagrams and Young tableaux:

Definition 1.9 (Skew partition, Young diagram, Young tableau). A pair of partitions λ, μ where $\lambda_i \geq \mu_i$ for all i (a partial ordering relation denoted $\lambda \supseteq \mu$) defines the *skew partition* λ/μ with size $|\lambda/\mu| = |\lambda| - |\mu|$. The *skew Young diagram* λ/μ is the array of boxes obtained by removing the boxes of μ from the Young diagram λ . A *skew Young tableau* of shape λ/μ is a filling of λ/μ with positive integers.

The skew partition λ/\emptyset is equivalent to λ . Skew Young tableaux can similarly be standard or semi-standard. A skew Young diagram is not necessarily a connected array of boxes. If the array is connected and does not contain any 2×2 blocks, we call it a *ribbon*. If λ/μ has only one box in each column, λ and μ are interlaced; in terms of the partitions themselves,

Definition 1.10 (Interlaced partitions). The partitions λ, μ are said to be *interlaced* if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \tag{1.18}$$

This is denoted $\lambda \succ \mu$.

If $\lambda \succ \mu$, then if we draw the Maya diagram of $S(\lambda)$ above that of $S(\mu)$ we can connect all of the black nodes with a zigzagging line passing from one line to the other without having to intersect itself. Consider a sequence of interlaced partitions from the empty set to some λ , $\emptyset \prec \lambda^{(1)} \prec \lambda^{(2)} \dots \prec \lambda^{(m)} = \lambda$; if we fill each skew partition $\lambda^{(i)}/\lambda^{(i-1)}$ with i then recombine them to the shape λ , we have a SSYT of shape λ filled with the numbers $1, \dots, m$.

An asymptotic enumeration estimate To conclude this section, let us note a useful bound on the number of partitions of n ,

$$\#\{\lambda \vdash n\} = \exp [O(\sqrt{n})]. \tag{1.19}$$

It follows for instance from Hardy and Ramanujan's precise asymptotic estimate [HR18].

1.1.2 Symmetric groups and partitions

Partitions of n are fundamental to the study of the symmetric group of order n , which we denote \mathfrak{S}_n . This section describes how three pieces of information about permutations are encoded by partitions: their conjugacy classes, the lengths of their monotone subsequences, and their irreducible representations. In the third instance we focus on how factorisations of the identity on \mathfrak{S}_n are enumerated by way of combinatorial data of partitions.

Conjugacy classes of \mathfrak{S}_n Let G be a group and g one of its elements. The *conjugacy class* $\mathcal{C} \subseteq G$ containing g is the set of elements $\{hgh^{-1} | h \in G\}$. Where the group is \mathfrak{S}_n , the conjugacy class of a permutation

$$\sigma = \left(\begin{array}{cccc} 1 & 2 & \dots & n \\ \sigma[1] & \sigma[2] & \dots & \sigma[n] \end{array} \right), \tag{1.20}$$

here shown in two-line notation, includes all permutations obtained by relabelling numbers. If σ contains an *m-cycle*, that is to say there is a sequence $I = (i_1, i_2, \dots, i_m)$ such that $\sigma[i_1] = i_2, \sigma[i_2] = i_3, \dots, \sigma[i_m] = i_1$, then any element $\varpi\sigma\varpi^{-1}$ of its conjugacy class will also contain an *m-cycle*: where

$$\varpi \in \left(\begin{array}{cccc} \dots & i_1 & \dots & i_m & \dots \\ \dots & j_1 & \dots & j_m & \dots \end{array} \right) \in \mathfrak{S}_n \tag{1.21}$$

exchanges each entry of I with the corresponding one of a sequence $J = (j_1, j_2, \dots, j_m)$, J provides the indices of the m -cycle in $\varpi\sigma\varpi^{-1}$. Since any permutation can be written as a product of disjoint cycles, is straightforward to prove that

Proposition 1.11 (Conjugacy classes of symmetric groups). *The conjugacy classes \mathcal{C}_λ of \mathfrak{S}_n are indexed by partitions $\lambda \vdash n$, where a permutation $\sigma \in \mathcal{C}_\lambda$ has cycles of length $\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}$.*

For example, the conjugacy class $\mathcal{C}_{(1^n)}$ (where (1^n) denotes 1 repeated n times) contains only the identity; $\mathcal{C}_{(2, 1^{n-2})}$ is composed of *transpositions*, which exchange two numbers and leave all others fixed, and of which there are $n(n-1)/2$; $\mathcal{C}_{(n)}$ is composed of the $(n-1)!$ *long cycles* which cannot be factorised into smaller disjoint cycles.

Monotone subsequences of $\sigma \in \mathfrak{S}_n$ If we consider a permutation $\sigma \in \mathfrak{S}_n$ as a word rather than a group element, we might overlook its symmetries and interest ourselves in its monotone subsequences rather than by its cycles. Let $I = (i_1 < i_2 < \dots < i_m)$ be a subsequence of $(1, 2, \dots, n)$; if $\sigma[i_1] < \sigma[i_2] < \dots < \sigma[i_m]$, then I indexes an *increasing subsequence* of σ (or a *decreasing* one if $\sigma[i_1] > \sigma[i_2] > \dots > \sigma[i_m]$). The length of the longest I indexing an increasing subsequence can be found by sorting the letters of σ with an algorithm resembling a game of solitaire (which may be very simple, we refer to e.g. [Rom15, Section 1.5] for one such example). A sophisticated recursive algorithm can achieve this without losing information about the original permutation; in particular,

Theorem 1.12 (Robinson–Schensted correspondence [Rob38, Sch61]). *There exists a bijection associating each permutation $\sigma \in \mathfrak{S}_n$ to a pair of standard Young tableaux (P, Q) with the same shape $\lambda \vdash n$.*

This bijection finds the length of the longest increasing subsequences [Sch61], but also much finer information:

Theorem 1.13 (Greene's theorem [Gre74]). *If the shape of the standard Young tableaux associated to a given $\sigma \in \mathfrak{S}_n$ is $\lambda \vdash n$, then for each m from 1 to n the greatest sum of lengths of m disjoint increasing subsequences of σ is $\lambda_1 + \lambda_2 + \dots + \lambda_m$. The greatest sum of lengths of m disjoint decreasing subsequences of σ is $\lambda'_1 + \lambda'_2 + \dots + \lambda'_m$.*

Sketch of proof of Theorems 1.12 and 1.13. The following algorithm generates a unique pair of SYT (P, Q) filled with the numbers $1, \dots, n$ from any permutation $\sigma \in \mathfrak{S}_n$.

At step 1, create two tableaux of just one box: $P^{(1)}$ (the “insertion” tableau) consisting of a box filled with $\sigma[1]$ and $Q^{(1)}$ (the “recording” tableau) consisting of a box filled with 1. Introduce an index a to keep track of the row into which the algorithm is inserting a number, and an index b to keep track of the number being inserted.

At step i for each i from 2 to n , set $a := 1$ and $b := \sigma[i]$, then repeat the following protocol until $P^{(i)}$ is set:

1. If b is greater than all of the entries on the a th row of $P^{(i-1)}$, set $P^{(i)}$ to $P^{(i-1)}$ with a box filled with b appended to the a th row. Otherwise:
2. Let \square^{change} be the leftmost box on the a th row of $P^{(i-1)}$ whose filling c is greater than a .
3. Refill \square^{change} with b , reset $b := c$ and increase a by 1.

Complete the step by setting $Q^{(i)}$ to be $Q^{(i-1)}$ with a box filled with i appended to the i th row to finish step i . After step n , set $P := P^{(n)}$, $Q := Q^{(n)}$.

At each step of this procedure, $P^{(i)}$ and $Q^{(i)}$ have the same shape $\lambda^{(i)} \vdash i$ and their entries are strictly increasing along rows and columns. At step n , each tableau is filled with $1, \dots, n$ and is an SYT of shape $\lambda := \lambda^{(n)} \vdash n$. It is clear that the length $\lambda_1^{(i)}$ of the first row of $P^{(i)}$ or $Q^{(i)}$ is equal to the length of the longest increasing subsequence of the word $\sigma[1]\sigma[2] \dots \sigma[i]$ while the length $\lambda_1^{(i)}$ of the first column is the length of the longest decreasing subsequence; it can further be shown that since the algorithm is recursive the lengths of the next rows and columns encode the maximal sums of lengths of disjoint monotone subsequences.

Since the recording tableau Q records the order in which each entry of σ was inserted into the insertion tableau P , each step of the algorithm can be reversed to find the permutation σ . The reversal can be applied to any SYT (P, Q) with the same shape, so this algorithm defines the required bijection. \blacksquare

The Robinson–Schensted correspondence has other remarkable properties; for instance, the inverse permutation σ^{-1} maps to (Q, P) (i.e. inversion corresponds to exchanging the insertion and recording tableaux). Theorem 1.12 provides a combinatorial proof of the identity

$$\sum_{\lambda \vdash n} f_\lambda^2 = |\mathfrak{S}_n| = n!. \quad (1.22)$$

Monotone subsequences in generalised permutations An important generalisation of this bijection was found by Knuth [Knu70]. The *Robinson–Schensted–Knuth (RSK)* correspondence uniquely associates pairs of SSYT of the same shape $\lambda \vdash m$ to two-line arrays

$$w = \begin{pmatrix} \ell_1 & \ell_2 & \dots & \ell_m \\ w[\ell_1] & w[\ell_2] & \dots & w[\ell_m] \end{pmatrix} \quad (1.23)$$

where the top line generalises $(1, 2, \dots, n)$ to any weakly increasing sequence $L = (\ell_1 \leq \ell_2 \leq \dots \leq \ell_m)$ of integers between 1 and n and the bottom line can be any sequence from the same set with the constraint that if $\ell_i = \ell_{i+1}$ then $w[\ell_i] \leq w[\ell_{i+1}]$ (these objects are called *generalised permutations* or *biwords*). While permutations in \mathfrak{S}_n are represented by $n \times n$ matrices with 1 appearing exactly once in each row and each column and 0 everywhere else, in general a two-line array w is represented (uniquely) by one with entries

$$a_{ij} = \#\{\ell \in L \mid \ell = i, w[\ell] = j\}. \quad (1.24)$$

Hence, using precisely the algorithm of the Robinson–Schensted correspondence where at the i th step $w[\ell_i]$ is inserted into $P^{(i)}$ and ℓ_i is added to $Q^{(i)}$,

Theorem 1.14 (Robinson–Schensted–Knuth correspondence [Knu70]). *There exists a bijection associating each matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ with non-negative integer entries summing to m with a pair of SSYT (P, Q) of the same shape $\lambda \vdash m$ filled with numbers $1, \dots, n$. The number of times j appears in P is $\sum_i a_{ij}$ and the number of times i appears in Q is $\sum_j a_{ij}$.*

For simplicity we will refer to the algorithm with the initials RSK of all three authors, even when it is applied to permutations. Greene’s theorem 1.13 also generalises to monotone subsequences of two-line arrays: the length of the longest subsequence(s) $I \subseteq L$ such that for all $\ell_i, \ell_j \in I$, if $\ell_i < \ell_j$ then $w[\ell_i] < w[\ell_j]$ is equal to the first part λ_1 of the shape λ of P and Q [Gre74].

Irreducible representations of \mathfrak{S}_n We now present elements of the representation theory of \mathfrak{S}_n , starting with some basic terminology. Let G be a finite group. A *finite representation* V_ρ of G is a finite-dimensional complex vector space equipped with a map π_ρ from G to the general linear group of V_ρ

$$\pi_\rho : G \rightarrow \mathrm{GL}(V_\rho), \quad g \mapsto \pi_\rho(g); \quad \pi_\rho(g) : V_\rho \rightarrow V_\rho \quad (1.25)$$

which is a group homomorphism, meaning that

$$\pi_\rho(g)\pi_\rho(h) = \pi_\rho(gh), \quad \text{for all } g, h \in G. \quad (1.26)$$

If $\pi_\rho(g) = \mathbb{1}$ for all $g \in G$, V_ρ is called trivial. If V_ρ cannot be decomposed as $V_\rho = V_{\rho'} \oplus V_{\rho''}$ such that $V_{\rho'}, V_{\rho''}$ are non-trivial representations, V_ρ is called *irreducible* (and there are no subspaces of V invariant under $\pi_\rho(G)$). The *character* χ^ρ of a representation V_ρ is the function

$$\chi^\rho : G \rightarrow \mathbb{C}, \quad \chi^\rho(g) := \mathrm{tr}_{V_\rho} \rho(g) = \chi^\rho(hgh^{-1}) \quad \text{for all } h \in G \quad (1.27)$$

where the final straightforward equality shows that we can write $\chi^\rho(\mathcal{C}) = \chi^\rho(g)$ where \mathcal{C} is the conjugacy class g belongs to. The character of the identity is just $\chi^\rho(\mathbb{1}) = \dim V_\rho$.

One simple representation of any finite group G is the *group algebra* $\mathbb{C}[G]$, which is the algebra of formal sums $\alpha = \alpha_1 g_1 + \alpha_2 g_2 + \dots$ of elements of G and which inherits the multiplication of G ; we have $\pi_{\mathbb{C}[G]}(g)\alpha = g\alpha$ for all $\alpha \in \mathbb{C}[G]$. $\mathbb{C}[G]$ can be decomposed by the algebra isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{\rho \text{ irr. rep.}} \mathrm{End}(V_\rho) \quad (1.28)$$

where $\mathrm{End}(V_\rho)$ denotes the endomorphisms on V_ρ and the index ρ in the direct sum runs over irreducible representations of G (and V_ρ are equivalent up to isomorphism class; we refer to [LZ04, Appendix A.1] for a concise overview). This decomposition has profound consequences; by comparing the dimensions on either side, we find

$$|G| = \sum_{\rho \text{ irr. rep.}} (\dim V_\rho)^2. \quad (1.29)$$

Considering the conjugacy classes \mathcal{C}_α of G , we can see that the elements $K_\alpha = \sum_{g \in \mathcal{C}_\alpha} g \in \mathbb{C}[G]$ are in the centre of $\mathbb{C}[G]$, i.e. $gK_\alpha = K_\alpha g$ for all $g \in G$, and they form a basis for it. Defining characters of elements of $\mathbb{C}[g]$ by linearity, $\chi^\rho(K_\alpha) = |\mathcal{C}_\alpha| \chi^\rho(\mathcal{C}_\alpha)$. Thanks to (1.28), we can compare the dimension of the centre of $\mathbb{C}[G]$ to the one dimensional centres of $\mathrm{End} V_\rho$ to find that

$$\#\{\text{conjugacy classes } \mathcal{C}_\alpha \text{ of } G\} = \#\{\text{irreducible representation } V_\rho \text{ of } G\}. \quad (1.30)$$

In the case of the symmetric group $G = \mathfrak{S}_n$, it is possible to further show (see e.g. [Ful96, Chapter 7]), following Proposition 1.11, that

Proposition 1.15 (Irreducible representations of symmetric groups). *The irreducible representations of \mathfrak{S}_n are indexed by partitions of n . The dimension of the representation V_λ indexed by λ is equal to f_λ , the number of SYT of shape λ .*

For \mathfrak{S}_n , the identity (1.29) is, as already mentioned, also proven by the Robinson–Schensted correspondence. The bijective and representation theoretic approaches are quite independent; the action of an RSK insertion step on an endomorphism of an irreducible representation is not clear.

Characters and factorisations of the identity on \mathfrak{S}_n The characters $\chi^\lambda(\mathcal{C}_\mu)$ on irreducible representations of \mathfrak{S}_n can be computed using combinatorics of Young diagrams by the *Murnaghan–Nakayama rule* [Mur37, Nak40]. The procedure amounts to identifying all sequences of partitions $\mathcal{S}(\lambda, \mu) = \lambda^{(1)} \supseteq \lambda^{(2)} \supseteq \dots \supseteq \lambda^{(\ell(\mu))} \supseteq \emptyset$ such that $\lambda^{(1)} := \lambda$ and each skew partition $\lambda^{(i)}/\lambda^{(i+1)}$ has size μ_i and is a ribbon. Then we have

$$\chi^\lambda(\mathcal{C}_\mu) = \sum_{\mathcal{S}(\lambda, \mu)} (-1)^{h(\mathcal{S}(\lambda, \mu)) - \ell(\mu)} \quad (1.31)$$

where $h(\mathcal{S}(\lambda, \mu))$ is the sum of the heights of the skew partitions $\lambda^{(i)}/\lambda^{(i+1)}$ from $\mathcal{S}(\lambda, \mu)$. Since the characters can be explicitly computed, the following classical result provides a practical way of extracting data about \mathfrak{S}_n :¹

Theorem 1.16 (Frobenius' formula, see e.g. [LZ04, Appendix A.1]). *The number of factorisations of the identity on \mathfrak{S}_n by a product of permutations with given cycle lengths can be expressed in terms of characters on irreducible representations V_λ of \mathfrak{S}_n . For any sequence $(\mu^1, \mu^2, \dots, \mu^\ell)$ of partitions $\mu^i \vdash n$,*

$$\begin{aligned} & \#\{(\sigma_1, \sigma_2, \dots, \sigma_\ell) \in \mathcal{C}_{\mu^1} \times \mathcal{C}_{\mu^2} \times \dots \times \mathcal{C}_{\mu^\ell} \mid \sigma_1 \cdot \sigma_2 \cdots \sigma_\ell = \mathbb{1}\} \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} (\dim V_\lambda)^2 \frac{|\mathcal{C}_{\mu^1}| \chi^\lambda(\mathcal{C}_{\mu^1})}{\dim V_\lambda} \frac{|\mathcal{C}_{\mu^2}| \chi^\lambda(\mathcal{C}_{\mu^2})}{\dim V_\lambda} \dots \frac{|\mathcal{C}_{\mu^\ell}| \chi^\lambda(\mathcal{C}_{\mu^\ell})}{\dim V_\lambda}. \end{aligned} \quad (1.32)$$

Proof. Consider the elements $K_\mu = \sum_{\sigma \in \mathcal{C}_\mu} \sigma \in \mathbb{C}[\mathfrak{S}_n]$, and the product $K_{\mu^1} \cdot K_{\mu^2} \cdots K_{\mu^\ell}$. The trace of a permutation acting on $\mathbb{C}[\mathfrak{S}_n]$ normalised by $n!$ is just $\text{tr}_{\mathbb{C}[\mathfrak{S}_n]} \sigma = \mathbf{1}_{\sigma=1}$, so we have

$$\text{tr}_{\mathbb{C}[\mathfrak{S}_n]} K_{\mu^1} \cdots K_{\mu^\ell} = \#\{(\sigma_1, \dots, \sigma_\ell) \in \mathcal{C}_{\mu^1} \times \dots \times \mathcal{C}_{\mu^\ell} \mid \sigma_1 \cdots \sigma_\ell = \mathbb{1}\}. \quad (1.33)$$

Now consider the action of the same product on an irreducible representation V_λ . By linearity, $\pi_\lambda(K_\mu)$ is well defined for each K_μ , and since K_μ is a central element, by Schur's lemma it acts as multiplication by some scalar $\kappa_\lambda(\mu)$. So, we have

$$\pi_\lambda(K_{\mu^1}) \cdot \pi_\lambda(K_{\mu^2}) \cdots \pi_\lambda(K_{\mu^\ell})v = \kappa_\lambda(\mu^1) \kappa_\lambda(\mu^2) \cdots \kappa_\lambda(\mu^\ell)v \quad \text{for all } v \in V_\lambda. \quad (1.34)$$

By the algebra isomorphism (1.28) and by (1.33), we have an equivalence of traces over either algebra

$$\begin{aligned} \sum_{\lambda \vdash n} \text{tr}_{\text{End}(V_\lambda)} \pi_\lambda(K_{\mu^1}) \cdot \pi_\lambda(K_{\mu^2}) \cdots \pi_\lambda(K_{\mu^\ell}) &= \sum_{\lambda \vdash n} (\dim V_\lambda)^2 \kappa_\lambda(\mu^1) \kappa_\lambda(\mu^2) \cdots \kappa_\lambda(\mu^\ell) \\ &= \#\{(\sigma_1, \sigma_2, \dots, \sigma_\ell) \in \mathcal{C}_{\mu^1} \times \mathcal{C}_{\mu^2} \times \dots \times \mathcal{C}_{\mu^\ell} \mid \sigma_1 \cdot \sigma_2 \cdots \sigma_\ell = \mathbb{1}\} \end{aligned} \quad (1.35)$$

using the fact that $\dim \text{End}(V) = (\dim V)^2$. It remains only to calculate $\kappa_\lambda(\mu)$; since the character $\chi^\lambda(K_\mu)$ is

$$\chi^\lambda(K_\mu) = \text{tr}_{V_\lambda} \pi_\lambda(K_\mu) = \kappa_\lambda(\mu) \dim V_\lambda \quad (1.36)$$

we can insert $\kappa_\lambda(\mu) = \chi^\lambda(K_\mu)/\dim V_\lambda$ to complete the proof. ■

¹We write this only for \mathfrak{S}_n for simplicity, but it holds analogously for any finite group.

Contents and factorisations of the identity on \mathfrak{S}_n The Murnaghan–Nakayama rule is, however, generally quite inefficient to use. Modern approaches to the representation theory of \mathfrak{S}_n , as reviewed in [VO05], have made extensive use the powerful *Jucys–Murphy elements* [Juc74, Mur81] of $\mathbb{C}[\mathfrak{S}_n]$, which are the sums of transpositions

$$J_i = (1, i) + (2, i) + \dots + (i - 1, i) \quad (1.37)$$

for each i from 1 to n . These elements are not in the centre of $\mathbb{C}[\mathfrak{S}_n]$, but they do commute with one another. We can also see that any symmetric polynomial f of the J_i is in the centre, and in fact its action on an irreducible representation can be computed exactly in terms of the contents of the corresponding Young diagram:

Proposition 1.17 (Action of Jucys–Murphy elements on irreducible representations [Juc74, Mur81]). *Let f be a symmetric polynomial and let V_λ be an irreducible representation of \mathfrak{S}_n . Then, denoting the boxes of λ by $\square_1, \dots, \square_n$,*

$$\pi_\lambda(f(J_1, J_2, \dots, J_n))v = f(c(\square_1), \dots, c(\square_n))v \quad \text{for all } v \in V_\lambda. \quad (1.38)$$

These elements notably allow us to prove a reformulation of Theorem 1.16, where we control the number of cycles of some factors rather than their precise conjugacy class².

Theorem 1.18 (Frobenius’ formula, controlling numbers of cycles). *On \mathfrak{S}_n , let $\mathcal{P}_m = \bigcup_{\mu \vdash n, \ell(\mu)=m} \mathcal{C}_\mu$ denote the set of elements of with m cycles and let*

$$N_{\mu, \nu}^{m_1, \dots, m_\ell} = \#\{(\varpi_a, \sigma_1, \dots, \sigma_\ell, \varpi_b) \in \mathcal{C}_\mu \times \mathcal{P}_{m_1} \times \dots \times \mathcal{P}_{m_\ell} \times \mathcal{C}_\nu \mid \varpi_a \cdot \sigma_1 \cdots \sigma_\ell \cdot \varpi_b = \mathbb{1}\}. \quad (1.39)$$

The generating function for these numbers can be expressed in terms of the contents of Young diagrams $\lambda \vdash n$ as

$$\sum_{m_1, \dots, m_\ell} N_{\mu, \nu}^{m_1, \dots, m_\ell} \prod_{i=1}^{\ell} u_i^{m_i} = \frac{1}{n!} \sum_{\lambda \vdash n} |\mathcal{C}_\mu| \chi^\lambda(\mathcal{C}_\mu) |\mathcal{C}_\nu| \chi^\lambda(\mathcal{C}_\nu) \prod_{i=1}^{\ell} \prod_{\square \in \lambda} (u_i + c(\square)) \quad (1.40)$$

where each u_i is a formal variable.

Proof. We can write the left hand side of (1.40) (following the arguments in the proof of Theorem 1.16) as

$$\sum_{m_1, \dots, m_\ell} N_{\mu, \nu}^{m_1, \dots, m_\ell} \prod_{i=1}^{\ell} u_i^{m_i} = \text{tr}_{\mathbb{C}[G]} K_\mu \cdot K_\mu \cdot \prod_{i=1}^{\ell} \sum_{\sigma \in \mathfrak{S}_n} u_i^{m(\sigma)} \sigma \quad (1.41)$$

where $m(\sigma)$ denotes the number of cycles of σ . The final sum is a central element of $\mathbb{C}[\mathfrak{S}_n]$, since it can be decomposed as $\sum_{\lambda \vdash n} u^{\ell(\lambda)} K_\lambda$. Moreover,

Lemma 1.19 (Permutation generating function and Jucys–Murphy elements). *Where J_i are the Jucys–Murphy elements on \mathfrak{S}_n ,*

$$P_n(u) := \sum_{\sigma \in \mathfrak{S}_n} u^{m(\sigma)} \sigma = \prod_{i=1}^n (u + J_i). \quad (1.42)$$

²The formula we write with the cycle structure of two of the factors fixed is a special case that will be useful to us; a similar formula can be written with any number of factors of given structure.

Proof. This holds trivially for $n = 1$. For each $n > 1$, consider the following bijection between \mathfrak{S}_n and $\mathfrak{S}_{n-1} \times \{1, \dots, n\}$: consider each $\sigma \in \mathfrak{S}_{n-1}$ as an element of \mathfrak{S}_n where n is fixed; then, n is factored out of an element $\sigma' \in \mathfrak{S}_n$ as $\sigma' = (i, n) \cdot \sigma$ for some i from 1 to n and some $\sigma \in \mathfrak{S}_{n-1}$, where $(n, n) := \mathbb{1}_{\mathfrak{S}_n}$. If i is less than n , we have $m(\sigma') = m(\sigma)$, since n “joins” the cycle i is in. If $\sigma' = \mathbb{1}_{\mathfrak{S}_n} \cdot \sigma$, n is in a cycle on its own so we have $m(\sigma') = m(\sigma) + 1$. Now considering generating functions of permutations by number of cycles, we have

$$P_n(u) := \sum_{\sigma' \in \mathfrak{S}_n} u^{m(\sigma')} \sigma' = (u + J_n) \sum_{\sigma \in \mathfrak{S}_{n-1}} u^{m(\sigma)} \sigma = (u + J_n) P_{n-1}(u) \quad (1.43)$$

which proves the lemma by induction. \blacktriangleleft

To complete the proof of Theorem 1.18 we write the action of $K_\mu K_\nu \prod_i P_n(u_i)$ on an irreducible representation V_λ . Since $P_n(u)$ is a symmetric polynomial of the J_i , by Proposition 1.17 it acts by scalar multiplication by $\prod_{\square \in \lambda} (u + c(\square))$. Recalling the action of K_μ, K_ν on V_λ , we find that $\text{tr}_{\text{End}(V_\lambda)} \pi_\lambda(K_\mu K_\nu \prod_i P_n(u_i))$ is precisely the right hand side of (1.40); by the algebra isomorphism (1.28) both sides are equivalent as required. \blacksquare

Transposition factorisations and Hurwitz numbers To conclude this discussion, let us mention one case in which Theorem 1.18 allows us to compute numbers of factorisations of the form shown in Theorem 1.16 explicitly in terms of fillings of boxes of Young diagrams.

Definition 1.20 (Hurwitz number). The *unconnected classical Hurwitz number* $H_{n,\ell}$ is the number of factorisations of the identity on \mathfrak{S}_n by ℓ transpositions. The *connected classical Hurwitz number* $h_{n,\ell}$ is the number of *transitive* factorisations of the identity on \mathfrak{S}_n by ℓ transpositions (meaning the group generated by the transpositions in the factorisation is \mathfrak{S}_n).

These numbers have been studied at least since the work of Hurwitz himself [Hur91], who found a geometric interpretation for them: they count degree n coverings of the Riemann sphere with ℓ labelled simple ramification points by an oriented surface (which is connected if the transposition factorisations are required to be transitive). Lando and Zvonkin’s book [LZ04] provides a thorough reference on this connection, and on how ramified coverings relate to maps. The maps interpretation is the most concrete and combinatorial one, and we will discuss it in chapters 3 and 4.

The conjugacy class $\mathcal{C}_{(2,1^{n-2})} \subset \mathfrak{S}_n$ of transpositions is the only one with $n - 1$ cycles; hence, by extracting the coefficient of $\prod_i u_i^{n-1}$, Theorem 1.18 gives (recalling that $\dim V_\lambda = f_\lambda$)

Corollary 1.21 (Frobenius’ formula for transposition factorisations). *The number of factorisations of the identity on \mathfrak{S}_n by ℓ transpositions is*

$$H_{n,\ell} := \#\{(\tau_1, \dots, \tau_\ell) \in (\mathcal{C}_{(2,1^{n-2})})^\ell \mid \tau_1 \cdots \tau_\ell = \mathbb{1}\} = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 C_\lambda^\ell. \quad (1.44)$$

The classical numbers Hurwitz yield several important generalisations; to mention just one, the double Hurwitz numbers (written here in the not necessarily connected case) are defined in [Oko00a] for all $\mu, \nu \vdash n$ as

$$H_{n,\ell}(\mu, \nu) = \#\{(\varpi_a, \tau_1, \dots, \tau_\ell, \varpi_b) \in \mathcal{C}_\mu \times (\mathcal{C}_{(2,1^{n-2})})^\ell \times \mathcal{C}_\nu \mid \varpi_a \cdot \tau_1 \cdots \tau_\ell \cdot \varpi_b = \mathbb{1}\}; \quad (1.45)$$

their enumeration by Theorem 1.18 is a rather simple generalisation of the one in Corollary 1.21. They enumerate a more general class of coverings of the Riemann sphere, with a ramifications of arbitrary type determined by μ and ν at 0 and ∞ and ℓ labelled simple ramification points elsewhere.

1.1.3 Symmetric functions and partitions

Another role played by partitions is as indices for bases of the algebra of symmetric functions. This section gives an overview of some of these bases and how they relate to one another, with particular emphasis on the Schur functions and their remarkable combinatorial and algebraic properties. We refer to [Sta99, Chapter 7] for details of everything mentioned in this section.

The algebra of symmetric functions Let $x = \{x_1, x_2, \dots\}$ be a possibly infinite set of formal parameters. A symmetric function of x is a formal power series of bounded degree (and not, in fact, a function in general) which is invariant under any exchange of the parameters. A degree n homogeneous symmetric function f of x has the form

$$f(x) = \sum_{\alpha: \alpha_1 + \alpha_2 + \dots = n} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \dots \quad (1.46)$$

where the sum is taken over all *weak compositions* of n , which are ordered sequences of non-negative integers $\alpha = (\alpha_1, \alpha_2, \dots)$ whose elements sum to n , and where c_α are some coefficients such that $c_{\sigma(\alpha)} = c_\alpha$ for all permutations $\sigma(\alpha)$ of the composition α . For simplicity we will always take the coefficients c_α to be complex valued, so the set of these series is a complex vector space which we denote \mathbf{Sym}_n . The symmetric functions of all degree make up the graded algebra $\mathbf{Sym} = \bigoplus_n \mathbf{Sym}_n$ (since if $f \in \mathbf{Sym}_n$ and $g \in \mathbf{Sym}_m$, $f \cdot g \in \mathbf{Sym}_{n+m}$).³

Multiplicative bases for Sym Partitions enter the story as a more efficient means of decomposing symmetric functions than weak compositions. Let us construct natural bases for each \mathbf{Sym}_n , starting with multiplicative definitions where in each case the basis element b_λ is just the product $b_{\lambda_1} \cdot b_{\lambda_2} \cdots b_{\lambda_{\ell(\lambda)}}$ of a generating family indexed by positive integers. In the first instance:

Definition 1.22 (Power-sum symmetric function, Miwa times). For each positive integer r , the *power-sum* symmetric function of the degree r of $x = \{x_1, x_2, \dots\}$ is

$$p_r(x) = \sum_i x_i^r. \quad (1.47)$$

For each partition $\lambda \vdash n$, $p_\lambda \in \mathbf{Sym}_n$ is

$$p_\lambda(x) = p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots p_{\lambda_{\ell(\lambda)}}(x) = \sum_{i_1, i_2, \dots, i_{\ell(\lambda)}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}. \quad (1.48)$$

For a set of parameters x , the corresponding *Miwa times* $t = (t_1, t_2, \dots)$ are

$$t_r = \frac{1}{r} p_r(x) = \frac{1}{r} \sum_i x_i^r. \quad (1.49)$$

³More generally, the coefficients c_α in (1.46) can be taken to belong to a commutative ring R , for instance \mathbb{Z} which is the convention used in [Mac95] and many other references. In this case \mathbf{Sym}_n is an R -module. Our discussion of bases of \mathbf{Sym}_n only applies to the case where \mathbf{Sym}_n is a vector space, i.e. the coefficients c_α are in a field.

Defining the Miwa times is superfluous in terms of the present discussion of bases, but having this notation is convenient to recast symmetric functions as formal power series of the power-sums rather than the “direct” parameters x (the new normalisation also proves useful). The name “Miwa times” refers to a connection to integrable systems, discussed in the second half of this chapter.

At this point we also introduce square bracket notation for a symmetric function f of x defined as a function of Miwa times:

$$f[t] = f[p_1(x), \frac{1}{2}p_2(x), \dots] := f(x). \quad (1.50)$$

If $f[t]$ is defined explicitly, $f(x)$ is well defined as a composition of formal series (which is valid because there are no constant terms in the power-sums). We define two more families, this time giving their generation functions in both the direct and Miwa time parameters (their formal equivalence is straightforward).

Definition 1.23 (Elementary and complete homogeneous symmetric functions). The *elementary* symmetric functions of $x = \{x_1, x_2, \dots\}$ are

$$e_m(x) = \sum_{i_1 < i_2 < \dots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \quad e_\lambda(x) = e_{\lambda_1}(x) e_{\lambda_2}(x) \cdots e_{\lambda_{\ell(\lambda)}}(x) \quad (1.51)$$

where we set $e_0 := e_\emptyset = 1$ and $e_m = 0$ if $m < 0$, and are generated by

$$E(x; z) = \sum_m e_m(x) z^m = \prod_i (1 + x_i z) \quad (1.52)$$

or $E[t; z] = \sum_m e_m[t] z^m = \exp[\sum_r (-1)^{r+1} t^r z^r]$. The *complete homogeneous* symmetric functions of x are

$$h_m(x) = \sum_{i_1 \leq i_2 \leq \dots \leq i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \quad h_\lambda(x) = h_{\lambda_1}(x) h_{\lambda_2}(x) \cdots h_{\lambda_{\ell(\lambda)}}(x) \quad (1.53)$$

where we set $h_0 := h_\emptyset = 1$ and $h_m = 0$ if $m < 0$, and are generated by

$$H(x; z) = \sum_m h_m(x) z^m = \prod_i (1 - x_i z)^{-1} \quad (1.54)$$

or $H[t; z] = \sum_m h_m[t] z^m = \exp[\sum_r t^r z^r]$.

The elementary symmetric function $e_m(x)$ is 0 if x has less than m elements, while the complete homogeneous function h_m contains in its sum every degree n monomial. We have $h_{(n)} = e_{(1^n)}$, and these families are in some way “dual” to one another.

Let us mention that these families do fulfil the goal we set out with. Firstly,

Proposition 1.24 (Bases). *Both $\{p_\lambda | \lambda \vdash n\}$ and $\{e_\lambda | \lambda \vdash n\}$ are bases of Sym_n . Moreover, both $\{p_r | r \in \mathbb{Z}_{>0}\}$ and $\{e_m | m \in \mathbb{Z}_{>0}\}$ are algebraically independent and generate Sym .*

Sketch of proof (see [Sta99, Theorem 7.4.4 and Corollary 7.7.2]). We start from a canonical basis for Sym_n , the *monomial* symmetric functions $\{m_\lambda | \lambda \vdash n\}$ where each m_λ is the sum over *distinct* indexings of the monomial $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}$ (such that $i_1 \neq i_2 \neq \dots$). This is precisely the way a symmetric function is decomposed in (1.46), and it is immediately clear that this is a basis; now, we just need to show that there is an invertible change of basis between $\{m_\lambda | \lambda \vdash n\}$ and the other families.

For $\{p_\lambda | \lambda \vdash n\}$, we have

$$p_\lambda = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu \quad (1.55)$$

where $R_{\lambda,\mu}$ accounts for additional monomial terms counted in the power-sums. To give an explicit example, $p_{(1^2)}(x) = \sum_{i,j} x_i x_j$ whereas $m_{(1^2)} = \sum_{i < j} x_i x_j$, so we have $R_{(1^2)(1^2)} = 2$ and $R_{(1^2)(2)} = 1$; then $p_{(2)}(x) = m_{(2)} = \sum_i x_i^2$ so $R_{(2)(1^2)} = 0$ and $R_{(2)(2)} = 1$. Each p_λ contains m_λ at least once, so $R_{\lambda\lambda} \geq 1$ for all λ . Then we can see that $R_{\lambda\mu} = 0$ if $\lambda_1 > \mu_1$, and one can show that $R_{\lambda\mu} = 0$ if the first m parts satisfy $\lambda_1 + \dots + \lambda_m > \mu_1 + \dots + \mu_m$ for any m . This condition defines a partial ordering on $\{\lambda \vdash n\}$, called *dominance order*. If we index the rows and columns of R by a total ordering that's compatible with dominance order, such as *lexicographic order* where partitions are ordered by their first part and then by their second part and so on, then R is a lower triangular matrix with a positive diagonal and it can be inverted, proving that $\{p_\lambda | \lambda \vdash n\}$ is a basis.

For $\{e_\lambda | \lambda \vdash n\}$, we proceed similarly, now setting

$$e_\lambda = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu. \quad (1.56)$$

We have $e_{(n)} = m_{(1^n)}$, and it's straightforward to see that $R_{\lambda\lambda'} \geq 1$ for all $\lambda \vdash n$. Since the largest exponent of an x_i in any term of e_λ is $\ell(\lambda)$, we can see that $R_{\lambda\mu} = 0$ if $\ell(\lambda) < \mu_1$, and it is possible to further show that $R_{\lambda\mu} = 0$ whenever the sum of the first m parts of λ' and μ' satisfy $\lambda'_1 + \dots + \lambda'_m < \mu_1 + \dots + \mu_m$ for any m . Then if we index the columns by the partitions of n in, for example, lexicographic order and the rows with the conjugates of partitions in lexicographic order, R is an upper triangular matrix with positive diagonal, which is invertible as required.

The algebraic independence of $\{p_r | r \in \mathbb{Z}_{>0}\}$, and similarly $\{e_r | r \in \mathbb{Z}_{>0}\}$, follows directly from the grading of the algebra $\mathbf{Sym} = \bigoplus_n \mathbf{Sym}_n$; since the products of these elements form bases for each \mathbf{Sym}_n , they generate the algebra. ■

Following this result we can state the duality between the elementary and complete homogeneous symmetric functions (and show that the latter form a basis in turn):

Proposition 1.25 (Classical involution). *The multiplication-preserving linear map*

$$\omega : \mathbf{Sym} \rightarrow \mathbf{Sym}, \quad \omega(e_m) := h_m, \quad \omega(e_\lambda) := \omega(e_{\lambda_1})\omega(e_{\lambda_2}) \cdots \omega(e_{\lambda_{\ell(\lambda)}}) = h_\lambda \quad (1.57)$$

is an involution. The set $\{h_\lambda | \lambda \vdash n\}$ is a basis for \mathbf{Sym}_n , and $\{h_m | m \in \mathbb{Z}_{>0}\}$ generates \mathbf{Sym} .

Proof. The generating functions of $e_m(x)$ and $h_m(x)$ immediately satisfy $E(x; -z)H(x; z) = 1$, so cancelling all of the coefficients for z^m , $m \geq 1$, we have

$$\sum_{m=1}^n (-1)^m e_m h_{n-m} = 0 \quad \text{for all } n \geq 1. \quad (1.58)$$

Applying ω to this series and then relabelling indices, we have

$$\sum_{m=1}^n (-1)^m h_m \omega(h_{n-m}) = 0 = (-1)^n \sum_{m=1}^n (-1)^m h_{m-n} \omega(h_m) \quad \text{for all } n \geq 1. \quad (1.59)$$

These n linear equations fix $\omega(h_m)$ for the first n h_m ; by (1.58) the solution is $\omega(h_m) = e_m$ as required.

Since there is an invertible map between $\{h_\lambda | \lambda \vdash n\}$ and $\{e_\lambda | \lambda \vdash n\}$, $\{h_\lambda | \lambda \vdash n\}$ is also a basis for \mathbf{Sym}_n . Since it is a multiplicative basis, the elements $\{h_m | m \in \mathbb{Z}_{>0}\}$ generate \mathbf{Sym} . ■

A determinantal basis Now we shift our attention to the main family that interests us, the Schur functions, which have a more subtle definition. In fact they have several definitions, and it will be most convenient to define them in terms of the complete homogenous functions using the Jacobi–Trudi identity, then work backwards to a more classical combinatorial definition.

Definition 1.26 (Schur function, via the Jacobi–Trudi identity). The degree n Schur symmetric function indexed by $\lambda \vdash n$ is

$$s_\lambda = \det_{1 \leq i, j \leq \ell(\lambda)} h_{\lambda_i - i + j} \quad (1.60)$$

where each h_m is a complete homogenous symmetric function. For any $\lambda \supseteq \mu$, the skew Schur function indexed by λ/μ is

$$s_{\lambda/\mu} = \det_{1 \leq i, j \leq \ell(\lambda)} h_{\lambda_i - i - \mu_j + j}. \quad (1.61)$$

The skew Schur function $s_{\lambda/\mu}$ is defined even when $\lambda \not\supseteq \mu$; in that case, we have $s_{\lambda/\mu} = 0$. The most canonical definition of Schur functions is also determinantal, with s_λ defined in terms of the direct parameters $x = (x_1, \dots, x_m)$ as the ratio $s_\lambda(x) = a_\lambda(x)/a_\emptyset(x)$ of the determinants $a_\lambda(x) = \det_{1 \leq i, j \leq m} x_i^{\lambda_j + m - j}$. This is indeed equivalent to our definition, as shown in e.g. [Sta99, Section 7.15]. We will focus instead on a more combinatorial way to define these functions.

Theorem 1.27 (Combinatorial definition of Schur functions, see e.g. [Sta99]). *The Schur function indexed by $\lambda \vdash n$ can equivalently be defined in terms of the direct parameters $x = \{x_1, x_2, \dots\}$ as*

$$s_\lambda(x) = \sum_{SSYT \text{ of } \lambda} x_1^{\#\{1 \in T\}} x_2^{\#\{2 \in T\}} \dots \quad (1.62)$$

where the sum runs over SSYT of shape λ , and the skew Schur function $s_{\lambda/\mu}$ can similarly be defined as a sum over skew SSYT.

This equivalence has a particularly elegant combinatorial proof exploiting the Linstrom–Gessel–Viennot theorem [Lin73, GV85], which identifies determinants with sums over configurations of non-intersecting lattice paths. We present a different one using skew partitions.

Proof. We prove the equivalence for $s_{\lambda/\mu}$ (s_λ is the case $\mu = \emptyset$), and for a set with a fixed number m of parameters (it is quite clear that this holds as $m \rightarrow \infty$). Start from a skew Schur function of a single parameter $x = x_1$,

$$s_{\lambda/\mu}(x) = \det_{1 \leq i, j \leq \ell(\lambda)} (x)^{\lambda_i - i - \mu_j + j} \mathbf{1}_{\lambda_i - i - \mu_j + j \geq 0}. \quad (1.63)$$

If $\lambda \succ \mu$, we immediately see that this is the determinant of an upper triangular matrix, and the product of the diagonal gives $s_{\lambda/\mu}(x) = x^{|\lambda/\mu|}$. If, however, $\lambda \not\succ \mu$, there are two rows or columns proportional to one another. Hence, $s_{\lambda/\mu}(x) = x^{|\lambda/\mu|} \mathbf{1}_{\lambda \not\succ \mu}$.

To produce a skew function in m parameters, we use the following relation:

Lemma 1.28 (Branching for skew Schur functions). *The skew Schur function in $x = (x_1, \dots, x_m)$ satisfies*

$$s_{\lambda/\mu}(x) = \sum_{\nu} s_{\lambda/\nu}(x_1, \dots, x_{m-1}) s_{\nu/\mu}(x_m). \quad (1.64)$$

Proof. Let $x' = (x_1, \dots, x_{m-1})$ and let $x'' = x_m$, and define the following two matrices: let A have $\ell(\lambda)$ rows and N columns where $N > \lambda_1$ is arbitrarily large and entries $h_{\lambda_a - a - b}(x')$, and let B have $\ell(\lambda)$ columns and N rows and entries $h_{a - \mu_b + b}(x'')$. By Definition 1.26, the skew Schur function $s_{\lambda/\nu}(x')$ is the minor of A taken over the columns $b = \nu_j - j$ and all rows $a = i$ where i, j run from 1 to $\ell(\lambda)$. Similarly $s_{\nu/\mu}(x'')$ is the minor of B , taken over rows $a = \nu_i - i$ and columns $b = j$ where i, j run from 1 to $\ell(\nu)$. Equivalently, $s_{\nu/\mu}(x'')$ can be rewritten as an $\ell(\lambda) \times \ell(\lambda)$ minor by allowing i, j run up to $\ell(\lambda)$; the additional rows have 1 in the diagonal and 0 everywhere to the left of it so the determinant is unchanged.

The sum over ν on the right hand side of (1.64) is just a sum over $\ell(\lambda) \times \ell(\lambda)$ minors of A and B . The Cauchy–Binet identity (see e.g. [Ait56, Chapter IV]) then gives

$$\sum_{\nu} s_{\lambda/\nu}(x') s_{\nu/\mu}(x'') = \sum_{\substack{L \subset \{1, \dots, N\} \\ |L| = \ell(\lambda)}} \det A|_L \det B|_L = \det AB \quad (1.65)$$

where $A|_L$ denotes the submatrix of A found by restricting to the set of labels L . So the right hand side is

$$\begin{aligned} \det_{1 \leq i, j \leq \ell(\lambda)} \sum_{a=1}^N h_{\lambda_i - i - a}(x') h_{a - \mu_j + j}(x'') \\ = \det_{1 \leq i, j \leq \ell(\lambda)} \sum_b h_{\lambda_i - i - \mu_j + j - b}(x_1, \dots, x_{m-1}) (x_m)^b \end{aligned} \quad (1.66)$$

which is exactly $s_{\lambda/\mu}(x)$. ◀

Applying the branching relation (1.64), from $s_{\lambda/\mu}(x_1)$ we construct $s_{\lambda/\mu}(x_1, x_2)$ and so on up to $s_{\lambda/\mu}(x = x_1, \dots, x_m)$. At each step only a sum over interlaced partitions contributes, and we have a sum over interlaced sequences

$$s_{\lambda/\mu}(x) = \sum_{\mu \prec \lambda^{(1)} \prec \dots \prec \lambda^{(m-1)} \prec \lambda} x_1^{|\lambda^{(1)}/\mu|} x_2^{|\lambda^{(2)}/\lambda^{(1)}|} \dots x_m^{|\lambda/\lambda^{(m-1)}|}. \quad (1.67)$$

Each sequence of interlaced partitions in the sum corresponds to a skew SSYT of shape λ/μ filled with numbers from $1, \dots, m$, so this completes the proof. ■

Directly following this expression for s_{λ} , we can prove two key results. Firstly,

Corollary 1.29 (Schur basis). *The set $\{s_{\lambda} | \lambda \vdash n\}$ is a basis of Sym_n .*

Proof. We proceed as in Proposition 1.24, by showing that the matrix $R_{\lambda\mu}$ where

$$s_{\lambda} = \sum_{\mu \vdash n} R_{\lambda\mu} m_{\mu} \quad (1.68)$$

give a change of basis from the Schur functions to the monomials. From the combinatorial expression from Theorem 1.27, we see that $R_{\lambda\lambda} = 1$ by considering the weights from a SSYT where in each row every box is filled with the same number. The exponent of x_1 in s_{λ} is at most λ_1 and the sum of the exponents of the first x_1, \dots, x_m is at most $\lambda_1 + \dots + \lambda_m$, so $R_{\lambda\mu} = 0$ if $\lambda_1 + \dots + \lambda_m < \mu_1 + \dots + \mu_m$ for any m . So R indexed by partitions in lexicographic order is a lower triangular matrix with positive diagonal, and is invertible as required. ■

In fact, the Schur functions form an orthonormal basis with respect to a scalar product under which the m_λ and h_λ are dual bases, with $\langle m_\lambda, h_\mu \rangle = \mathbf{1}_{\lambda=\mu}$; they can also be defined as the result of Gram–Schmidt orthogonalisation. The second identity we can prove from the combinatorial definition is the following:

Corollary 1.30 (Cauchy identity for Schur functions). *Let x, x' be two sets of direct parameters and t, t' two sequences of Miwa times. We have*

$$\sum_{\lambda \vdash n} s_\lambda(x) s_\lambda(x') = \prod_{i,j} (1 - x_i x'_j)^{-1} \quad \text{and} \quad \sum_{\lambda \vdash n} s_\lambda[t] s_\lambda[t'] = \exp \left[\sum_r r t_r t'_r \right]. \quad (1.69)$$

Proof. Both versions of the identity are equivalent by composition of formal power series, we prove only the direct parameter version.

The right hand side can be expanded as

$$\prod_{i,j} (1 - x_i x'_j)^{-1} = \prod_{i,j} \sum_{a_{ij} \geq 0} (x_i x'_j)^{a_{ij}}. \quad (1.70)$$

The sum is taken over matrices A with non-negative integer entries a_{ij} . Consider the term in this expansion

$$N_{\alpha,\beta} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_1^{\beta_1} x_2^{\beta_2} \cdots. \quad (1.71)$$

The coefficient $N_{\alpha,\beta}$ counts the number of matrices A whose i th row sums to α_i and whose j th column sums to β_j . But by Theorem 1.14 non-negative integer matrices A are in bijection with pairs of SSYT (P, Q) with the same shape λ , where the sum of the i th row of A is the number of times i appears in Q , and the sum of the j th row is the number of times j appears in P . So, (1.70) is just

$$\sum_{\lambda} \sum_{\text{SSYT } (P,Q) \text{ of } \lambda} x_1^{\#\{1 \in Q\}} x_2^{\#\{2 \in Q\}} \cdots x_1^{\#\{1 \in P\}} x_2^{\#\{2 \in P\}} \cdots \quad (1.72)$$

which, by Theorem 1.27, is precisely $\sum_{\lambda} s_{\lambda}(x) s_{\lambda}(x')$ as required. ■

The classical involution again We can find yet another definition for the Schur functions from the duality of the elementary and complete homogeneous bases.

Theorem 1.31 (Dual Jacobi–Trudi identity, see e.g. [Mac95, Section 1.2]). *The classical involution $\omega : e_m \leftrightarrow h_m$ acts on the Schur functions by conjugating the index, as $\omega(s_\lambda) = s_{\lambda'}$. The Schur function can equivalently be defined by the dual Jacobi–Trudi identity*

$$s_\lambda = \det_{1 \leq i, j \leq \lambda_1} e_{\lambda'_i - i + j}, \quad (1.73)$$

and the skew Schur function similarly by $s_{\lambda/\mu} = \det_{i,j} e_{\lambda'_i - i - \mu'_j + j}$.

Proof. Let $N = \lambda_1 + \ell(\lambda)$. Consider the lower triangular matrices $E = ((-1)^{a-b} e_{a-b})_{0 \leq a, b \leq N}$ and $H = (h_{a-b})_{0 \leq a, b \leq N}$. We have $\det H = \det E = 1$, and by (1.58), E is the inverse of H . To prove that $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$, we consider the minor of H taken over rows $a = \lambda_i - i + \ell(\lambda)$ and columns $b = \mu_j - j + \ell(\lambda)$ for $1 \leq i, j \leq \ell(\lambda)$, which we denote $\det H^{\lambda, \mu}$. Let \tilde{H} be the matrix obtained from H by permuting the rows a to the top and the columns b to the left (and otherwise maintaining their order); the determinant of \tilde{H} is then $(-1)^{|\lambda| - |\mu|}$ and its inverse is the matrix \tilde{E} obtained by moving the rows b to the top and the columns a to the left. The minor $\det \tilde{H}^{\lambda, \mu}$ is the determinant of the upper

left block of \tilde{H} , so Jacobi's determinant identity (see e.g. [Ait56, Chapter IV]) states that

$$\det H^{\lambda, \mu} = \det \tilde{H} \cdot \det E^{(\lambda, \mu)'} \quad (1.74)$$

where $\det E^{(\lambda, \mu)'}$ is the determinant of the lower right block of \tilde{E} in the same decomposition, i.e. the minor of E taken over all rows except b and all columns except a . But we can show that these remaining rows are indexed by $b' = \ell(\lambda) + 1 - \mu'_j + j$ for $1 \leq i, j \leq \lambda_1$ and the remaining columns by $a' = \ell(\lambda) + 1 - \lambda'_i + i$, for $1 \leq i, j \leq \ell(\lambda)$. Hence, we have

$$\det_{1 \leq i, j \leq \ell(\lambda)} h_{\lambda_i - \mu_j - i + j} = (-1)^{|\lambda| - |\mu|} \det_{1 \leq i, j \leq \lambda_1} (-1)^{\lambda'_i - \mu'_j - i + j} e_{\lambda'_i - \mu'_j - i + j} \quad (1.75)$$

and the signs cancel to give the dual Jacobi–Trudi identity (1.73). So, applying ω to both sides we find

$$\omega(s_\lambda) = \det_{1 \leq i, j \leq \lambda_1} h_{\lambda'_i - \mu'_j - i + j} = s_{\lambda'} \quad (1.76)$$

which is what we wanted to show. ■

The involution can be directly applied to the parameters themselves; considering the generating functions of the e_m and the h_m , we can write $\omega(E(x; z)) = H(x; z)$ as

$$\omega_x \prod_i (1 + x_i z) = \prod_i (1 - x_i z)^{-1} \quad (1.77)$$

where the notation ω_x indicates that the involution is acting on the family of parameters x . Then, we can act on the Cauchy identity (1.69) and apply Theorem 1.31 to find a new one

$$\sum_\lambda s_\lambda(x) s_\lambda(x') = \omega_x \prod_{i, j} (1 - x_i x'_j)^{-1} = \prod_{i, j} (1 + x_i x'_j). \quad (1.78)$$

Like the original Cauchy identity, this can be proven bijectively, using a dual version of the RSK correspondence which associates matrices with entries 0, 1 with pairs of SSYT.

Considering $\omega(E[t; z]) = H[t; z]$ at the level of the Miwa times, we find the action of ω on the one remaining basis:

Proposition 1.32 (Involution of power-sums). *The classical involution $\omega : e_m \leftrightarrow h_m$ acts on the Miwa times (and power-sums) by alternating the sign, with $\omega(t_r) = (-1)^{r+1} t_r$ (and $\omega(p_r) = (-1)^{r+1} p_r$).*

Characters of symmetric groups again The Schur functions also bring us back to the representation theory of \mathfrak{S}_n , for instance by way of the following identity, which we present without proof:

Proposition 1.33 (Schur function in the Miwa times). *The Schur function indexed by $\lambda \vdash n$ in the Miwa times $t = (t_1, t_2, \dots)$ is*

$$s_\lambda[t] = \frac{1}{n!} \sum_{\mu \vdash n} |\mathcal{C}_\mu| \chi^\lambda(\mathcal{C}_\mu) \mu_1 \mu_2 \cdots \mu_{\ell(\mu)} t_{\mu_1} t_{\mu_2} \cdots t_{\mu_{\ell(\mu)}}. \quad (1.79)$$

Of course, in terms of the power-sum basis this says that the coefficients of the change of basis to the Schur functions are $\frac{1}{n!} |\mathcal{C}_\lambda| \chi^\mu(\mathcal{C}_\lambda)$.

Specialisations In all of our discussions so far, the symmetric functions have been formal power series. But we will mostly be interested in extracting numbers from them (and using them to write such concrete things as probabilities); to this end, we use algebra homomorphisms

$$\varphi : \text{Sym} \rightarrow \mathbb{C}, \quad f \mapsto \varphi(f) \quad (1.80)$$

called *specialisations*. The action of any given φ may be written by assigning values in \mathbb{C} to an algebraically independent family generating Sym . This could be the direct parameters x , or indeed the elements $\{p_r | r \in \mathbb{Z}_{>0}\}$, or $\{e_m | m \in \mathbb{Z}_{>0}\}$ or $\{h_m | m \in \mathbb{Z}_{>0}\}$. Note that the existence of $\varphi(f)$ for given φ, f needs to be justified, in terms of the convergence of complex series; for specialisations with finite support, this is generally straightforward.

Let us mention two important specialisations. For a real valued parameter q and a positive integer n , the *principal specialisation* of a symmetric function of $x = \{x_1, x_2, \dots\}$ is defined in the direct parameters as

$$\text{ps}_{q,n}(f) = f(1, q, q^2, \dots, q^{n-1}, 0, 0, \dots). \quad (1.81)$$

The elements of the bases we discussed are all well defined in the limit $n \rightarrow \infty$ if $q \in [0, 1)$, and $\text{ps}_{1,n}$ is well defined at fixed n . For a real parameter θ , the *exponential specialisation* is defined as the limit

$$\text{ex}_\theta(f) = \lim_{n \rightarrow \infty} f\left(\underbrace{\frac{\theta}{n}, \dots, \frac{\theta}{n}}_{n \text{ times}}, 0, 0, \dots\right) \quad (1.82)$$

or equivalently as

$$\text{ex}_\theta(f) = f[\theta, 0, 0, \dots]. \quad (1.83)$$

The elements of all the bases we consider are very simple under the exponential specialisation, with

$$\text{ex}_\theta(h_\lambda) = \text{ex}_\theta(e_\lambda) = \frac{\theta^{|\lambda|}}{\lambda_1! \lambda_2! \dots}, \quad \text{ex}_\theta(s_\lambda) = \frac{f_\lambda \theta^{|\lambda|}}{|\lambda|!} \quad (1.84)$$

where the specialisation of the Schur function is found by evaluating of the character of $\mathcal{C}_{(1^{|\lambda|})} = \{\mathbb{1}\}$ in (1.79). We will denote specialised symmetric functions directly as functions of the parameters assigned by the specialisation.

1.2 Measures and processes on partitions

In this section, we review classical probability laws on partitions which are motivated directly by the algebraic and combinatorial results reviewed in Sections 1.1.2 and 1.1.3: in the first instance, the Plancherel measure and its Poissonisation, then the infinite parameter family of Schur measures. We then discuss how Schur measures generalise to processes in discrete time, or measures on sequences of interlaced partitions.

1.2.1 The Plancherel measure

Consider the identity (1.22) for the size of symmetric group. Dividing both sides by $n!$, we can recognise a probability law on partitions:

Definition 1.34 (Plancherel measure). The *Plancherel measure* on partitions $\lambda \vdash n$ is

$$\mathbb{P}_n(\lambda) = \frac{1}{n!} f_\lambda^2. \quad (1.85)$$

Just as (1.22) can be proven either combinatorially by the RSK bijection or by identities from finite group representation theory applied to \mathfrak{S}_n , the Plancherel measure is motivated two different contexts. At this point let us establish the convention that random objects are denoted by bold letters (we use it throughout the text).

Fourier transform on a finite group The name of this measure comes from the representation theory side. Letting G be a finite group and $G^\wedge = \{V_\rho\}$ be (the equivalence classes of) its irreducible representations, the generic Plancherel measure on G^\wedge can be read from (1.29) as

$$\mathbb{P}_{G;\text{Plancherel}}(V_\rho) = \frac{(\dim V_\rho)^2}{|G|}. \quad (1.86)$$

The name of this measure comes from an analogy with classical analysis: the generalised Fourier transform from the functions $\{f : G \rightarrow \mathbb{C}\}$ to the endomorphisms $\{\hat{f}(\rho) : V_\rho \rightarrow V_\rho \mid V_\rho \in G^\wedge\}$ is defined by

$$\hat{f}(\rho) = \sum_{g \in G} f(g) \pi_\rho(g), \quad (1.87)$$

which is the usual discrete Fourier transform if G is a cyclic group \mathbb{Z}_n (in that case, for each irreducible representation we can put $V_\rho = \mathbb{C}$ as each $\pi_\rho(g)$ is an n th roots of unity, and \hat{f} is a function from \mathbb{Z}_n to \mathbb{C} just as f is). For the usual (discrete or continuous) Fourier transform, Plancherel proved [PL10] that if f is a square integrable function, then its Fourier transform \hat{f} is a square integrable function with the same L^2 -norm; in other words, the discrete Fourier transform is an isometry from $L^2(\mathbb{Z}_n)$ to itself. For the generalised Fourier transform on a finite group, we have the Plancherel formula

$$\sum_{g \in G} |f(g)|^2 = \frac{1}{|G|} \sum_{\rho} \dim V_\rho \operatorname{tr}_{V_\rho}(\hat{f}(\rho) \hat{f}^*(\rho)) \quad (1.88)$$

where the left hand side defines the integral of $|f|^2$ on G with respect to the counting measure and the right hand side defines the integral of $\hat{f} \hat{f}^*$ on G^\wedge with respect to the Plancherel measure; it means that the finite group Fourier transform is an isometry from the space $L^2(G)$ with respect to the counting measure on G to $L^2(G^\wedge)$ with respect to the Plancherel measure on G^\wedge .

The longest increasing subsequence problem The Plancherel measure for \mathfrak{S}_n has been subject to a great level of interest due to its combinatorial interpretation. Consider the following question, popularised by Hammersley [Ham72] who credited Ulam [Ula61] with posing it previously: if $\sigma \in \mathfrak{S}_n$ is a uniform random permutation of $1, \dots, n$, then, what is the law of the length of the longest increasing subsequences $\text{LLIS}(\sigma)$ as $n \rightarrow \infty$? The RSK correspondence (detailed at Theorem 1.12 above) and Greene’s theorem 1.13 translate this into a random partition problem: if we let λ be a random partition of n under the Plancherel measure \mathbb{P}_n , then λ_1 is equivalent in law to $\text{LLIS}(\sigma)$. This formulation is powerful, as \mathbb{P}_n behaves particularly well as n tends to infinity. We review two profound asymptotic results for this measure.

Limit shape of a Plancherel random partition The first, proven by Vershik and Kerov [VK77] and independently by Logan and Shepp [LS77], is that \mathbb{P}_n gives rise to a deterministic limit shape for large Young diagrams; to be precise (where “ \xrightarrow{P} ” denotes convergence in probability),

Theorem 1.35 (Limit shape of the Plancherel measure [VK77, LS77]). *Let $\lambda \vdash n$ be a random partition under the Plancherel measure \mathbb{P}_n . Then, as $n \rightarrow \infty$,*

(i) *the rescaled profile⁴ $\psi_{\lambda, \sqrt{n}}(x)$ of λ has a deterministic limit shape, with*

$$\sup_x |\psi_{\lambda, \sqrt{n}}(x) - \Omega(x)| \xrightarrow{p} 0,$$

$$\text{where } \Omega(x) = \begin{cases} \frac{2}{\pi} (\arcsin \frac{x}{2} + \sqrt{4 - x^2}), & |x| \leq 2, \\ |x|, & |x| > 2. \end{cases} \quad (1.89)$$

(ii) *the first part λ_1 and the length $\ell(\lambda)$ are both equivalent to $2\sqrt{n}$ in probability, with*

$$\frac{\lambda_1}{\sqrt{n}} \xrightarrow{p} 2, \quad \frac{\ell(\lambda)}{\sqrt{n}} \xrightarrow{p} 2. \quad (1.90)$$

We call Ω the ‘‘VKLS curve’’, after the authors’ initials. This curve is found and part (i) of the theorem is proven by variational analysis of the Plancherel measure as a functional of the rescaled profile, starting from the fact that

Proposition 1.36 (Asymptotic expression for the Plancherel measure). *As $n \rightarrow \infty$,*

$$\mathbb{P}_n(\lambda) = \exp \left[-n(1 + 2I_{\text{hook}}(\psi_{\lambda; \sqrt{n}})) + O(\sqrt{n} \log n) \right] \quad (1.91)$$

uniformly for all $\lambda \vdash n$, where $I_{\text{hook}}(\psi_{\lambda; \sqrt{n}}) = O(1)$ is a functional expressed in terms of $\phi_{\lambda, \sqrt{n}}, \phi_{\lambda; \sqrt{n}}^{-1}$ defined at (1.6) as

$$I_{\text{hook}}(\psi_{\lambda; \sqrt{n}}) = \int_0^\infty \int_0^{\phi_{\lambda; \sqrt{n}}(u)} \log(\phi_{\lambda; \sqrt{n}}(u) - u + \phi_{\lambda; \sqrt{n}}^{-1}(v) - v) dv du. \quad (1.92)$$

This is shown by applying the hook length formula (Theorem 1.7) to write

$$\mathbb{P}_n(\lambda) = \exp \left[\log n! - 2 \sum_{\square \in \lambda} \log \eta_\lambda(\square) \right] \quad (1.93)$$

$$= \exp \left[-n - 2 \sum_{i, j: \square_{i, j} \in \lambda} \log \left(\frac{\lambda_i - i + \lambda'_j - j + 1}{\sqrt{n}} \right) + O(\log n) \right] \quad (1.94)$$

where the second equality comes from the Stirling approximation $\log n! = n \log n - n + O(\log n)$. Then, the difference between the final summation and its integral approximation $nI_{\text{hook}}(\psi_{\lambda; \sqrt{n}})$ defined at (1.92) can be shown to be $O(\sqrt{n} \log n)$. Following Proposition 1.36, application of the change of coordinates (1.7) re-expresses I_{hook} as a series of integrals in terms of the single continuous and 1-Lipshitz function $\psi_{\lambda; \sqrt{n}}$. In terms of the ‘‘height function’’ $h(x) := \psi_{\lambda; \sqrt{n}}(x) - |x|$, we have

$$I_{\text{hook}}(\psi_{\lambda; \sqrt{n}}) = \log \sqrt{2} - \frac{1}{2} \iint \log |x - y| h'(x) h'(y) dx dy - 2 \int h'(x) (x \log |x| - x) dx. \quad (1.95)$$

I_{hook} is then be minimised; we refer to [Rom15, Sections 1.13–1.17] for a clear presentation of this analysis (we will present an alternative proof of Theorem 1.35 by Borodin, Okounkov and Olshanski [BOO00] as a special case in Section 2.3.2). The main result is that the unique function Ω minimising $I_{\text{hook}}(\Omega)$ under the constraint $\int (\Omega(x) - |x|) dx = 2$

⁴Some references, notably [Rom15] which we cite regularly, define the rescaled profile as what we denote $\psi_{\lambda, \sqrt{2n}}$, so that each box has area $1/\sqrt{n}$.

is the VKLS curve given in (1.89), and that $I_{\text{hook}}(\Omega) = -1/2$. Hence, for any $\lambda \vdash n$ that is “far” from the partition whose shape is described by Ω , $\mathbb{P}_n(\lambda)$ tends to zero exponentially fast as $n \rightarrow \infty$.

Although the support of $\Omega(x) - |x|$ is $[-2, 2]$ as expected, part (i) of Theorem 1.35 is insufficient to prove part (ii) since the supremum norm is insensitive to a finite number of parts at the start of λ or λ' ; one must additionally bound the expectation values of λ_1 and $\ell(\lambda)$ above. One simple way to do so is using the *Plancherel growth process*, which was explicitly introduced by Kerov [Ker98].

Proof of Theorem 1.35 (ii) (see e.g. [Rom15, Section 1.19]). Consider the following random growth process on partitions. For each $n > 1$, let each $\sigma^{(n)} \in \mathfrak{S}_n$ be a random permutation sampled from $\sigma^{(n-1)}$ by choosing m uniformly at random from $1, \dots, n$ and setting

$$\sigma^{(n)}[i] = \begin{cases} \sigma^{(n-1)}[i] & i < n, \sigma^{(n-1)}[i] < m \\ \sigma^{(n-1)}[i] + 1, & i < n, \sigma^{(n-1)}[i] \geq m \\ m, & i = n. \end{cases} \quad (1.96)$$

Starting from the trivial permutation $\sigma^{(1)} = (1) \in \mathfrak{S}_1$, this generates a uniform random permutation of $1, \dots, n$ at the n th step. From this, define a sequence of random partitions $\lambda^{(n)} \vdash n$ as the shape of the pair of SYT associated to $\sigma^{(n)}$ by the RSK correspondence, so that the probability that $\lambda^{(n)} = \lambda$ is $\mathbb{P}_n(\lambda) \mathbf{1}_{\lambda \vdash n}$.

Fix $\lambda \vdash n$ and consider the transition probability $\mathbb{P}(\lambda^{(n)} = \lambda | \lambda^{(n-1)} = \mu)$; this is

$$\frac{\mathbb{P}(\lambda^{(n)} = \lambda \cap \lambda^{(n-1)} = \mu)}{\mathbb{P}(\lambda^{(n-1)} = \mu)} = \frac{f_\lambda f_\mu / n!}{f_\mu^2 / (n-1)!} \mathbf{1}_{\mu \nearrow \lambda} = \frac{f_\lambda}{n f_\mu} \mathbf{1}_{\mu \nearrow \lambda} \quad (1.97)$$

where the numerator is the fraction of permutations in \mathfrak{S}_n such that the RSK algorithm outputs tableaux of shape μ at the penultimate step and tableaux of shape λ at the last step where $\mu \nearrow \lambda$; the only constraint is that in the recording tableau Q , the label n goes on the box λ/μ , so there are f_μ recording tableaux and f_λ insertion tableaux to count.

Now, consider the expectation of $\lambda^{(n)} - \lambda^{(n-1)}$. For any $\mu \vdash n-1$ we let μ^{1+} denote the partition $(\mu_1 + 1, \mu_2, \dots, \mu_{\ell(\lambda)}) \vdash n$; then, this is just the probability of the event $\mathbf{E}_{n,1+} = \{\lambda^{(n)} = \lambda^{(n-1)1+}\}$, which is

$$\mathbb{P}(\mathbf{E}_{n,1+}) = \sum_{\mu \vdash n-1} \mathbb{P}_{n-1}(\mu) \mathbb{P}(\lambda^{(n)} = \mu^{1+} | \lambda^{(n-1)} = \mu) = \sum_{\mu \vdash n-1} \frac{f_\mu^2}{(n+1)!} \frac{f_{\mu^{1+}}}{n f_\mu}. \quad (1.98)$$

Since this is an average of $f_{\mu^{1+}}/n f_\mu$ it can be bounded above by the Cauchy–Schwarz inequality, to find

$$\mathbb{P}(\mathbf{E}_{n,1+}) \leq \left[\sum_{\mu \vdash n-1} \frac{f_\mu^2}{(n+1)!} \frac{f_{\mu^{1+}}^2}{n^2 f_\mu^2} \right]^{1/2} = \frac{1}{\sqrt{n}} \left[\sum_{\mu \vdash n-1} \frac{f_{\mu^{1+}}^2}{n!} \right]^{1/2} \leq \frac{1}{\sqrt{n}} \quad (1.99)$$

where the final inequality comes from the fact that the partitions $\{\mu^{1+} | \mu \vdash n-1\}$ are a subset of the partitions $\{\lambda \vdash n\}$. Then, the expectation of the first part of any $\lambda \vdash n$ under \mathbb{P}_n is bounded as

$$\mathbb{E}(\lambda_1) = \sum_{m=2}^n \mathbb{P}(\mathbf{E}_{m,1+}) \leq \sum_{m=2}^n \frac{1}{\sqrt{m}} \leq 2\sqrt{n} \quad (1.100)$$

Noting that precisely the same arguments can be applied to the conjugate partition λ' , we equally have $\mathbb{E}(\ell(\lambda)) \leq 2\sqrt{n}$. Now, from part (i) of Theorem 1.35, since $\Omega(x) - |x| > 0$ for $|x| < 2$, we have that for all $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_n(\lambda_1/\sqrt{n} \leq 2 - \varepsilon) = \lim_{n \rightarrow \infty} \mathbb{P}_n(\ell(\lambda)/\sqrt{n} \leq 2 - \varepsilon) = 0 \quad (1.101)$$

and each of λ_1/\sqrt{n} and $\ell(\lambda)/\sqrt{n}$ converge in probability to 2 as required. \blacksquare

Of course, by Greene's theorem 1.13, Theorem 1.35 is immediately applicable to the Ulam–Hammersley problem: part (ii) states that if $\sigma \in \mathfrak{S}_n$ is a uniform random permutation then as $n \rightarrow \infty$, we have $\text{LLIS}(\sigma)/\sqrt{n} \xrightarrow{p} 2$, and similarly for the length of the longest decreasing subsequences of σ , denoted $\text{LLDS}(\sigma)$, we have $\text{LLDS}(\sigma)/\sqrt{n} \xrightarrow{p} 2$. In terms of representation theory, the limit shape result was applied by Vershik and Kerov to compute asymptotic characters of symmetric groups [VK81]. For more general measures on partitions induced by decomposition of tensor representations of \mathfrak{S}_n , deterministic limit shapes in coordinates scaling with $1/\sqrt{n}$ were shown by Biane [Bia98, Bia01] to arise wherever characters asymptotically factorise over these representations.

Asymptotic edge fluctuations The second result we present for large Plancherel random partitions is at the much finer scale of $n^{1/6}$, around the limiting value $2\sqrt{n}$ of λ_1 or $\ell(\lambda)$. Baik, Deift and Johansson [BDJ99] found the exact distribution asymptotically driving the fluctuations in each of these statistics:

Theorem 1.37 (Asymptotic edge fluctuations of the Plancherel measure [BDJ99]). *Let $\lambda \vdash n$ be a random partition under the Plancherel measure \mathbb{P}_n . Then, the asymptotic cumulative distribution for its first part λ_1 is*

$$\lim_{n \rightarrow \infty} \mathbb{P}_n\left(\frac{\lambda_1 - 2\sqrt{n}}{n^{1/6}} > s\right) = F_{\text{GUE}}(s) := \det(1 - \mathcal{A})_{L^2([s, \infty))} \quad (1.102)$$

where $F_{\text{GUE}}(s)$ is the Fredholm determinant

$$\det(1 - \mathcal{A})_{L^2([s, \infty))} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_s^{\infty} \cdots \int_s^{\infty} \det_{1 \leq i, j \leq n} \mathcal{A}(x_i, x_j) dx_1 \cdots dx_n \quad (1.103)$$

of the Airy integral kernel

$$\mathcal{A}(x, y) = \int_0^{\infty} \text{Ai}(x+v) \text{Ai}(y+v) dv = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x-y}, \quad (1.104)$$

$$\text{Ai}(x) = \frac{1}{2\pi i} \int_{1+i\mathbb{R}} \exp\left[\frac{\zeta^3}{3} - x\zeta\right] d\zeta. \quad (1.105)$$

In the second expression for \mathcal{A} , the case $x = y$ should be taken as a limit using L'Hôpital's rule. Although we write it only for λ_1 , this result equally applies to $\ell(\lambda)$, or indeed $\text{LLIS}(\sigma)$ and $\text{LLDS}(\sigma)$ for uniform random $\sigma \in \mathfrak{S}_n$. The limiting ‘‘TW-GUE distribution’’ F_{GUE} defined at (1.103) was discovered by Tracy and Widom [TW93], who proved it governs the asymptotic fluctuations of the largest eigenvalue of a random matrix in the Gaussian unitary ensemble (GUE); we will revisit this connection in Section 3.2.1 and treat the asymptotic fluctuations of the Plancherel measure independently until then. We finally present a proof of the Baik–Deift–Johansson (BDJ) theorem 1.37 in Section 2.3.2 as a special case of the more general Theorem 2.2, using methods from ‘‘integrability’’ reviewed in Section 1.3.3. The key step in applying these methods, first used in [Joh98] and [BOO00] is to define a new measure:

Definition 1.38 (Poissonised Plancherel measure). The *Poissonised Plancherel measure* with real parameter θ on all partitions λ is the law

$$\mathbb{P}_\theta(\lambda) = e^{-\theta^2} \sum_{n=1}^{\infty} \frac{\theta^{2n}}{n!} \mathbb{P}_n(\lambda) \mathbf{1}_{\lambda \vdash n} = e^{-\theta^2} \frac{\theta^{2|\lambda|} f_\lambda^2}{|\lambda|!^2}. \quad (1.106)$$

The expectation value of the size of a partition λ under \mathbb{P}_θ is θ^2 , and Theorems 1.35 and 1.37 can be rewritten for \mathbb{P}_θ with θ playing the role of \sqrt{n} . The Poissonisation procedure used is well established in statistical physics: to pass from a canonical ensemble, in which an isolated system composed of a fixed number of particles is in thermodynamic equilibrium, to a grand canonical ensemble in which the same system can exchange particles with a reservoir and is at both thermodynamic and “chemical” equilibrium, one promotes the particle number for the system to a Poisson random variable of parameter $e^{-\mu_{\text{ch}}}$ where μ_{ch} is the *chemical potential*. We revisit this in Section 1.3.2. It also relates the Plancherel measure to symmetric functions: we recognise from (1.84) that

$$\mathbb{P}_\theta(\lambda) = e^{-\theta^2} \text{ex}_\theta(s_\lambda)^2 = e^{-\theta^2} s_\lambda[\theta]^2. \quad (1.107)$$

1.2.2 Schur measures

An important infinite parameter family of measures on partitions was introduced by Okounkov [Ok01], by considering more general specialisations of Schur functions to the complex numbers. Natural measures of this kind arise both from specialisations in which the direct parameters are assigned complex values and, as in the case of the Poissonised Plancherel measure, ones in which the Miwa times are assigned complex values; we give explicit definitions from either perspective.

Definition 1.39 (Schur measure). Let $x = \{x_1, x_2, \dots\}$ and $x' = \{x'_1, x'_2, \dots\}$ be two sets of complex valued parameters, and let $t = (t_1, t_2, \dots)$ and $t' = (t'_1, t'_2, \dots)$ be two sequences of complex valued parameters. Then,

- (i) if for all partitions λ , $s_\lambda(x)s_\lambda(x') \geq 0$ and the partition function $Z := \prod_{i,j} (1 - x_i x'_j)^{-1}$ is finite, the *Schur measure* specialised to x, x' is the probability law on all partitions

$$\mathbb{P}(\lambda) = s_\lambda(x)s_\lambda(x') \prod_{i,j} (1 - x_i x'_j) \quad (1.108)$$

- (ii) if for all partitions λ , $s_\lambda[t]s_\lambda[t'] \geq 0$ and $Z := \exp[\sum_r r t_r t'_r]$ is finite, the Schur measure specialised to t, t' in the Miwa times is

$$\mathbb{P}(\lambda) = s_\lambda[t]s_\lambda[t'] e^{-\sum_r r t_r t'_r}. \quad (1.109)$$

The *partition functions* Z (that is, the normalisation factors for the weights $s_\lambda(x)s_\lambda(x')$ or $s_\lambda[t]s_\lambda[t']$) are computed from the Cauchy identity (1.69). Schur measures are natural generalisations of the Poissonised Plancherel measure, and importantly they are all integrable in the same way.

The size of a partition λ under a Schur measure is random, and its expectation can be found directly from the definition. By homogeneity of the Schur function, for a parameter q we have

$$s_\lambda[qt_1, q^2 t_2, q^3 t_3, \dots] = q^{|\lambda|} s_\lambda[t_1, t_2, t_3, \dots] \quad (1.110)$$

and hence by (1.69),

$$\begin{aligned} \mathbb{E}(|\boldsymbol{\lambda}|) &= \frac{d}{dq} e^{-\sum_r r t_r t'_r} \sum_{\lambda} s_{\lambda}[q t_1, q^2 t_2, q^3 t_3, \dots] s_{\lambda}[t'] \Big|_{q=1} \\ &= \frac{d}{dq} e^{\sum_r r q^r t_r t'_r} e^{-\sum_r r t_r t'_r} \Big|_{q=1} = \sum_{r \geq 1} r^2 t_r t'_r. \end{aligned} \quad (1.111)$$

An inhomogeneous longest increasing subsequence problem Let us motivate this family of measures by returning to the Ulam–Hammersley problem, and considering generalisations of it. First, consider a uniform random permutation $\sigma \in \mathfrak{S}_n$ as a sequence of random points in $\mathbb{Z}_{>0}^2$ with coordinates $(i, \sigma[i])_{1 \leq i \leq n}$. Then, in terms of “up/right paths” P of points (i_k, j_k) where $k = 1, \dots, 2n - 1$ such that $(i_0, j_0) = (1, 1)$, $(i_{2n-1}, j_{2n-1}) = (n, n)$ and at each step $(i_{k+1} - i_k, j_{k+1} - j_k)$ is equal to either $(1, 0)$ or $(0, 1)$, the main statistic of interest is

$$\text{LLIS}(\sigma) = \mathbf{L}_{n,n} := \max_{P:(1,1) \rightarrow (n,n)} \sum_{(i,j) \in P} \mathbf{a}(i,j), \quad \mathbf{a}(i,j) = \mathbf{1}_{j=\sigma[i]}. \quad (1.112)$$

Following this reformulation as a *last passage percolation* model, it is natural to consider the analogous statistic $\mathbf{L}_{n,n}$ for more general random weights $\mathbf{a}(i,j)$; such a model was proposed by Johansson in [Joh00], and generalised and recast in algebraic terms by Baik and Rains [BR01b, BR01a] (and studied by implicitly using Schur measures before their introduction). To model directed paths through an inhomogeneous random medium, fix two sequences of numbers $0 < x_i, y_i < 1$ for i from 1 to n and let each weight be an independent geometrically distributed random non-negative integer, with

$$\mathbb{P}(\mathbf{a}(i,j) = m) = \frac{x_i^m y_j^m}{1 - x_i y_j}. \quad (1.113)$$

The $n \times n$ matrix with entries $\mathbf{a}(i,j)$ can be written as a two-line array (or generalised permutation) \mathbf{w} of $1, \dots, n$, and we can see that, as in the permutation case,

$$\mathbf{L}_{n,n} = \max_{P:(1,1) \rightarrow (n,n)} \sum_{(i,j) \in P} \mathbf{a}(i,j) = \text{LLIS}(\mathbf{w}). \quad (1.114)$$

Now, under the RSK correspondence (Theorem 1.14), $\text{LLIS}(\mathbf{w})$ is the first part $\boldsymbol{\lambda}_1$ of the shape $\boldsymbol{\lambda}$ of the SSYT corresponding to \mathbf{w} . Hence, following the arguments of Corollary 1.30 the law of $\mathbf{L}_{n,n}$ is

$$\begin{aligned} \mathbb{P}(L_{n,n}) &= \sum_{\lambda: \lambda_1 = L_{n,n}} \sum_{\substack{P, Q \\ \text{SSYT of } \lambda}} \prod_{i,j=1}^n (1 - x_i y_j) x_i^{\#i \in P} y_j^{\#j \in Q} \\ &= \prod_{i,j=1}^n (1 - x_i y_j) \sum_{\lambda: \lambda_1 = L_{n,n}} s_{\lambda}(x_1, \dots, x_n) s_{\lambda}(y_1, \dots, y_n) \end{aligned} \quad (1.115)$$

and we recognise the Schur measure in the final sum.

The length $\mathbf{L}_{n,n}$ is equivalent to the height function $\mathbf{h}(x,t)$ at the centre $x = 0$ in a *discrete polynuclear growth* model with wedge initial condition $h(x,0) = |x|$ (see e.g. [Joh03]), and last passage percolation with geometric weights plays an important role in establishing the *TW-GUE universality class* associated with the fluctuation exponents and asymptotic distribution of the BDJ theorem 1.37. For the homogeneous case $x_i =$

$y_i = q$ for all i of last passage percolation with i.i.d. geometric random weights presented in [Joh00], Johansson proved⁵ that as $n \rightarrow \infty$, the expectation of $\mathbf{L}_{n,n}$ scales with n while its fluctuations scale with $n^{1/3}$ and are driven by the TW-GUE distribution, in direct analogy with the BDJ theorem 1.37.

1.2.3 Plane and cylindric partitions, processes and periodic processes

As discussed in part I of the introduction, the TW-GUE universality class notably includes several time-dependant models of out-of-equilibrium physical systems, such as KPZ stochastic growth and asymmetric exclusion processes with particular initial conditions. The integrable algebraic approach to this class of Schur measures was extended to a discrete time dependent setting by Okounkov and Reshetikhin [OR03, OR07], who defined measures on (*skew*) *plane partitions*, which are represented by three-dimensional (*skew*) Young diagrams where stacks of cubes are aligned in a corner (or a series of corners) and correspond to sequences of interlaced partitions $\emptyset \prec \lambda^{(1)} \dots \lambda^{(n-1)} \succ \emptyset$; these measures are defined by way of transition weights between interlaced partitions, which have also been applied to sequences with more general boundary conditions. One notable generalisation by Borodin [Bor07] went beyond the usual hypothesis of a time-dependent process to have periodic boundary conditions, defining measures on *cylindric partitions* introduced in [GK96]. This family may be informally defined as follows⁶:

Definition 1.40 (Schur process). Given a suitable sequence of N pairs of sets of complex parameters x^{i+}, x^{i-} and a non-negative parameter u , the *Schur process* of rank N is a measure on sequences of $2N + 1$ partitions $\Lambda = (\mu^{(0)} \subseteq \lambda^{(1)} \supseteq \mu^{(1)} \subseteq \lambda^{(2)} \supseteq \dots \subseteq \lambda^{(N)} \supseteq \mu^{(N)} = \mu^{(0)})$ defined as the product of transition weights

$$\mathbb{P}(\Lambda) = \frac{1}{Z} u^{|\mu^{(0)}|} \left[\prod_{i=1}^N s_{\lambda^{(i)}/\mu^{(i-1)}}(x^{i+}) s_{\lambda^{(i)}/\mu^{(i)}}(x^{i-}) \right] \quad (1.116)$$

and where Z is the partition function normalising the measure.

Fixing $u = 0$, we have (for sequences with non-zero probability) $\mu^{(0)} = \mu^{(N)} = \emptyset$, and the rank 1 Schur process at $u = 0$ is the usual Schur measure. Fixing $x^{i-} = 0$ for $i = 1, \dots, M$, $x^{i+} = 0$ for $i = M + 1, \dots, N$ we have $\mu^{(i)} = \lambda^{(i)}$, $i \leq M$ and $\mu^{(i-1)} = \lambda^{(i)}$, $i > M$; if moreover each remaining set x^{i+}, x^{i-} contains just one non-zero parameter, the sequences of partitions are interlaced, with $\Lambda = (\emptyset \prec \lambda^{(1)} \prec \dots \prec \lambda^{(M)} \succ \dots \succ \lambda^{(N-1)} \succ \emptyset)$ corresponding to a plane partition, which was the original case considered in [OR03]. Allowing for order of the pairs x^{i+}, x^{i-} to be changed, each Λ is a skew plane partition, as considered in [OR07]. For $u > 0$, Λ is a cylindric partition and $\mathbb{P}(\Lambda)$ is the *periodic Schur process* introduced in [Bor07].

The simplest periodic Schur process is the rank 1 *cylindric Plancherel measure*, where both x^{1+} and x^{1-} correspond to exponential specialisations,

$$\mathbb{P}_{u,\theta}(\mu, \lambda) = \frac{1}{Z} u^{|\mu|} s_{\lambda/\mu}[\theta]^2, \quad Z = \frac{\exp[\frac{\theta^2}{1-u}]}{\prod_{i \geq 1} (1-u^i)}. \quad (1.117)$$

The same processes appeared implicitly in prior works of Johansson [Joh03, Joh05], where in the second case they were applied to domino tilings of the Aztec diamond. The connection between Schur processes (including ones with different boundary conditions) and tiling problems, specifically configurations of dimers on a certain family of graphs, was made explicit and shown to be very general in [BBC⁺15] and [BCC17].

⁵The proof in [Joh00] includes more general up/right paths ending on (m, n) where $m \sim c \cdot n$ for some constant c .

⁶This definition is not given in full generality; see e.g. [BG16] for a concise review including other cases.

Specialisations and Schur positivity It is important to emphasise that the Schur process is only well defined for certain x^{i+}, x^{i-} and parameters u . In the case of the periodic Schur process, the parameter u must be less than 1 for the measure to be normalisable. For $\mathbb{P}(\Lambda)$ to represent a sensible stochastic process, each weight $\mathcal{T}(\lambda, \mu)$ should define a probability for all λ, μ after normalisation, which is ensured if each x^{i+}, x^{i-} defines a *Schur positive* (or more accurately Schur non-negative) specialisation, meaning that $s_\lambda(x^{i+}), s_\lambda(x^{i-}) \geq 0$ for all λ . This in turn ensures that all skew Schur functions are non-negative under the corresponding specialisation; by Definition 1.26, this corresponds to a non-negativity condition for the minors of a lower triangular matrix, and was classified in [AESW51] and [Tho64]:

Proposition 1.41 (Schur positivity condition [AESW51, Tho64]). *A set of complex parameters $x = \{x_1, x_2, \dots\}$ defines a Schur positive specialisation if and only if there exist parameters $\alpha_i, \beta_i, \gamma \geq 0$ such that*

$$H(x; z) = \sum_{m=0}^{\infty} h_m(x) z^m = e^{\gamma z} \prod_i \frac{1 + \beta_i z}{1 - \alpha_i z}. \quad (1.118)$$

A periodic generalisation of the BDJ theorem Analysis of Schur processes has led to a wide range of asymptotic results, such as limit surfaces for random plane partitions [CLP98, OR03] or convergence to the Pearcey process on convex corners of skew plane partitions [OR07]. Let us highlight one such result, which leads to an extension of the TW-GUE universality class. In [BB19], Betea and Bouttier studied the edge behaviour of the larger partition in a random sequence Λ under $\mathbb{P}_{u,\theta}$ in the $\theta \rightarrow \infty$ limit, which, as the case of the Poissonised Plancherel measure \mathbb{P}_θ , is a large partition limit. By simultaneously considering a $u \rightarrow 1$ limit, the smaller partition μ is also large, and they proved the following result:

Theorem 1.42 (Asymptotic edge fluctuations of the cylindric Plancherel measure [BB19]). *Let $\Lambda = (\mu \subseteq \lambda \supseteq \mu)$ be a random cylindric partition under $\mathbb{P}_{u,\theta}$. Then, in a critical scaling regime where $\theta(1-u)^2 \rightarrow \alpha^3 > 0$ as $\theta \rightarrow \infty$ and $u \rightarrow 1$, we have*

$$\lim_{\theta \rightarrow \infty, u \rightarrow 1} \mathbb{P}_{u,\theta}^m \left(\frac{\lambda_1 - b\Theta}{(d\Theta)^{\frac{1}{3}}} < s \right) = F_{\text{GUE}}^\alpha(s) := \det(1 - \mathcal{A}^\alpha)_{L^2([s, \infty))} \quad (1.119)$$

$$\text{where } \Theta := \frac{\theta}{1-u}, \quad \Theta \sim \left(\frac{\theta}{\alpha} \right)^{\frac{3}{2}}$$

where F_{GUE}^α is the Fredholm determinant of the α -Airy integral kernel composed of classical Airy functions,

$$\mathcal{A}^\alpha(x, y) = \int_{-\infty}^{\infty} \frac{e^{\alpha v}}{1 + e^{\alpha v}} \text{Ai}(x+v) \text{Ai}(y+v) dv. \quad (1.120)$$

The distribution F_{GUE}^α was previously found by Johansson [Joh07] and shown by Dean, Le Doussal, Majumdar and Schehr [DLDMS15, LDMS17] to govern edge fluctuations in free fermion models at positive temperature. The factor of $e^{\alpha v}/(1 + e^{\alpha v})$ in the kernel coincides with the Fermi density of states, with α acting as the “limiting inverse temperature” – we see that in the limit $\alpha \rightarrow \infty$, this factor becomes $\mathbf{1}_{v>0}$ and $\mathcal{A}^\alpha \rightarrow \mathcal{A}$, recovering the “zero temperature” classical Airy kernel. Moreover, after some more subtle scaling arguments, in the $\alpha \rightarrow 0$ high temperature limit one recovers the Gumbel distribution $G(s) = e^{e^{-s}}$ associated with cumulative probabilities from independent events, and edge fluctuations of models in the Edwards–Wilkinson universality class.

The asymptotic regime in Theorem 1.42 parametrises a crossover in edge behaviour between two universality classes. Betea and Bouttier’s proof exploits a correspondence between these models and discrete space fermions at positive temperature (partially presented in Section 1.3.3).

The same distribution F_{GUE}^α also governs the finite time asymptotic fluctuations in the height of an interface growing randomly under the KPZ equation with the narrow wedge initial condition [SS10, ACQ11]. The periodic Schur process does not have a direct interpretation as a time dependant process that could explain the origin of F_{GUE}^α in this context combinatorially. However, by using bijective connections to another class of processes defined in terms of q -Whittaker functions, Imamura, Mucciconi and Sasamoto recently presented a periodic Schur process approach to models in the KPZ class [IMS21, IMS22].

1.3 Integrable measures and free fermions

As previously mentioned, the Schur measures and processes introduced in the last section have the rare and very useful distinction of being *integrable*, and this section explains what that means. The notion of integrability comes from physics, in the first instance from classical mechanics, where a system is integrable if its equations of motion can be solved exactly. A probability law can be integrable in the sense that a quantum mechanical model can be, where every *correlation function* can be computed exactly by an algebraic algorithm. Schur measures are *determinantal point processes*, meaning that this algorithm is particularly simple; in physical terms, they correspond to models of free fermions. We will first describe the family of integrable laws that Schur measures belong to, then review the physical models which inspired this categorisation, before finally showing how correlation functions of Schur measures can be computed exactly using fermionic calculus.

1.3.1 Determinantal point processes

The class of integrable probability laws we consider was defined by Macchi [Mac75]:

Definition 1.43 (Determinantal point process). Let χ be a configuration of random points in a metric space R equipped with a reference measure μ_{ref} such that for any compact subspace $B \subset R$, $\#\{\chi \cap B\} < \infty$, and suppose that χ admits an n -point correlation function ρ_n (also called *joint intensity*) with respect to μ_{ref} such that for any finite set $\{k_1, \dots, k_n\} \subset R$ and any symmetric function f ,

$$\begin{aligned} \mathbb{E} \left(\sum_{\{k_1, \dots, k_n\} \subseteq \chi} f(k_1, \dots, k_n) \right) \\ = \int_R f(k_1, \dots, k_n) \rho_n(k_1, \dots, k_n) d\mu_{\text{ref}}(k_1) \cdots d\mu_{\text{ref}}(k_n). \end{aligned} \quad (1.121)$$

The random configuration χ is a *determinantal point process*⁷ (DPP) if there exists a kernel $K : R \times R \rightarrow \mathbb{C}$ of which every correlation function of χ is a minor,

$$\rho_n(k_1, \dots, k_n) = \det_{1 \leq i, j \leq n} K(k_i, k_j). \quad (1.122)$$

⁷The term “process” here does not connote time dependence, rather it borrows the terminology of spatial point processes for locally finite configurations of random points.

The n -point correlation function has a particularly natural definition if R is discrete and μ_{ref} is the counting measure; in this case,

$$\rho_n(k_1, \dots, k_n) = \mathbb{P}(\{k_1, \dots, k_n\} \subseteq \mathcal{X}). \quad (1.123)$$

If R is continuous, we have

$$\rho_n(k_1, \dots, k_n) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\prod_i \mu_{\text{ref}}(B_\varepsilon(k_i))} \mathbb{P}(\mathcal{X} \cap B_\varepsilon(k_i) \neq \emptyset \text{ for all } i = 1, \dots, n) \quad (1.124)$$

where $B_\varepsilon(k_i)$ denotes a ball of radius ε around k_i . DPPs are reviewed in detail in [AGZ09, Section 4.2], and we refer also to [HM19] for a concise and enjoyable introduction. They notably include eigenvalues in a number of random matrix models, such as the Ginibre ensemble of matrices whose entries are i.i.d. complex numbers of mean 0 and variance 1 and the Gaussian unitary ensemble which we revisit in Section 3.2.1. Note that they also include completely uncorrelated processes, for which

$$\rho_n(k_1, \dots, k_n) = \rho_1(k_1)\rho_1(k_2) \cdots \rho_1(k_n); \quad (1.125)$$

in this case the kernel is diagonal, with $K(k, \ell) = 0$ if $k \neq \ell$.

In this text we will only be concerned with DPPs where R is \mathbb{R} or $\mathbb{Z} + \frac{1}{2}$ (the shift of $\frac{1}{2}$ is arbitrary but will prove convenient), for which μ_{ref} is respectively the Lebesgue measure and the counting measure.

Kernels and integral operators The kernel K associated with a DPP must be *locally trace-class*, with

$$\text{tr}_B K = \int_B K(k, k) d\mu_{\text{ref}}(k) < \infty \quad (1.126)$$

for any compact $B \subset R$. From this point let us write the integration with respect to μ_{ref} implicitly, with $dk := d\mu_{\text{ref}}(k)$. One important sufficient condition for a locally trace-class operator to be a valid DPP kernel was stated by Soshnikov [Sos00]:

Theorem 1.44 (Condition for a Hermitian determinantal point process kernel [Mac75, Sos00]). *Let K be a locally trace-class Hermitian kernel on R , with $K(k, \ell) = \bar{K}(\ell, k)$. Then, K defines a DPP if and only if the eigenvalues of the integral operator $K_f|_B : L^2(B) \rightarrow L^2(B)$*

$$K_f(f)|_B = \int_B K(k, \ell) f(\ell) d\ell \quad (1.127)$$

all lie in $[0, 1]$ for $B \subseteq R$.

It is worth noting, however, that a DPP kernel is not necessarily Hermitian.

Most natural DPPs are defined in terms of *projection kernels*, where $(K_f)^2 = K_f$ and

$$\int K(k_i, \ell) K(\ell, k_j) d\ell = K(k_i, k_j). \quad (1.128)$$

The linear statistics of a DPP \mathcal{X} have simple formulas, with

$$\begin{aligned} \mathbb{E}\left(\sum_{k \in \mathcal{X}} f(k)\right) &= \int_R K(k, k) f(k) dk \\ \text{Var}\left(\sum_{k \in \mathcal{X}} f(k)\right) &= \int_R f^2(k) K(k, k) dk - \iint_R K(k, \ell) K(\ell, k) f(k) f(\ell) dk d\ell \end{aligned} \quad (1.129)$$

and in particular, $\mathbb{E}(\#\{\mathcal{X} \cap B\}) = \int_B K(k, k) dk = \text{tr}_B K$ for any compact $B \subset R$.

Gap probabilities and Fredholm determinants If $\chi \subset R$ is a DPP, we can write particularly simple expressions for its *gap probabilities*, that is the probability that there are no elements of χ in a given region $B \subset R$. By the inclusion-exclusion principle, we have

$$\begin{aligned} \mathbb{P}(\{\chi \cap B\} = \emptyset) &= 1 - \int_B \rho_1(k) dk + \frac{1}{2} \iint_B \rho_2(k_1, k_2) dk_1 dk_2 \\ &\quad - \frac{1}{3!} \iiint_B \rho_3(k_1, k_2, k_3) dk_1 dk_2 dk_3 + \dots \end{aligned} \quad (1.130)$$

where the factor of $n!$ on the n -point correlation function corrects the over counting by relabellings of elements of χ . In terms of the DPP kernel, this is

$$\mathbb{P}(\{\chi \cap B\} = \emptyset) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \iint \dots \int_B \det_{1 \leq i, j \leq n} K(k_i, k_j) dk_1 dk_2 \dots dk_n. \quad (1.131)$$

On the right hand side, we can recognise an expression for the determinant of the identity perturbed by an $N \times N$ matrix A as a sum of minors

$$\det(\mathbb{1} + xA) = \sum_{L \subseteq \{1, \dots, N\}} x^{|L|} \det(A)_L \quad (1.132)$$

in the limit $N \rightarrow \infty$. In this sense, (1.131) defines a generalisation of this determinant for the integral operator K_f on $L^2(B)$. This is called a *Fredholm determinant*, and denoted $\det(\mathbb{1} - K_f)_{L^2(B)}$. For simplicity, from now on we will denote the integral operator associated to a kernel with the same symbol.

The BDJ theorem again The distribution F_{GUE} driving the fluctuations in a random partition λ under \mathbb{P}_n as $n \rightarrow \infty$ in the BDJ theorem 1.37 is one such Fredholm determinant. An extension of this theorem due to Okounkov [Ok00b], reproven more directly by Borodin, Okounkov, Olshanski and Johansson [BO00, Joh01], explains its origin: it states that for any finite m , as $n \rightarrow \infty$, the sequence $(\lambda_i - 2\sqrt{n})n^{-1/6}$ for i from 1 to m converges in law to the *Airy ensemble*, which is the DPP whose kernel is the Airy kernel given in (1.104).

1.3.2 Physical fermion models

Macchi originally introduced DPPs in order to model the spatial distribution of fermions in optical beams [Mac75]. The definition of the n -point correlation function is motivated by experimental observations: one can in principle place any finite number n of detectors to measure different points in the space at once, but an infinite number measurements could not be performed at once. DPPs themselves arise directly from the quantum mechanics of free (that is to say, non-interacting except through the Pauli exclusion principle) fermions, which we review in this section.

Quantum mechanics in a nutshell In the formalism of quantum mechanics, any observable \mathcal{O} of a system (for example, the position \mathbf{x} or momentum \mathbf{p} of a particle) is a random variable, whose law is determined by the *state* Φ that the system is in. If the system exists in a space R (parametrising e.g. the positions or momenta of particles), its state Φ lives in a Hilbert space $\mathcal{H} \subset L^2(R)$ of square integrable⁸ complex functions

⁸As in the previous section, R may be discrete; each integral is to be taken with respect to a suitable reference measure. Elsewhere we write $l^2(R)$ rather than $L^2(R)$ if R is discrete.

with norm one, with $\int_R |\Phi(x)|^2 dx = 1$. Then, the expectation of an observable \mathcal{O} is

$$\mathbb{E}(\mathcal{O}) = \int_R \Phi^*(x) \hat{\mathcal{O}} \Phi(x) dx. \quad (1.133)$$

where $\hat{\mathcal{O}} : L^2(R) \rightarrow L^2(R)$ is the corresponding operator.

While one can use either position or momentum space to represent a system, an important element of the fundamental quantum formalism is that both cannot be used at once, as the corresponding operators \hat{x} , \hat{p} do not commute with one another; rather we have

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \quad (1.134)$$

In dimensionless coordinates, we set the reduced Planck constant \hbar to 1. Then, the momentum operator acts as $\hat{p} = -id/dx$ on position space with corresponding coordinate x and the position operator acts as $\hat{x} = id/dp$ on momentum space with coordinate p .

The state Φ itself is an eigenfunction of the system's *Hamiltonian*, which is a Hermitian operator $H : L^2(R) \rightarrow L^2(R)$, with (real valued) eigenvalue equal to the state's energy. We will only consider models with no time dependence.

Bosons and fermions in first quantisation To construct a system in the first quantisation formalism, we consider only the particles themselves to have random positions and momenta; they are assumed to be in a deterministic environment, experiencing a fixed potential. We refer to [DDMS19, Section 3] for a clear example of this set-up.

Consider a system of N indistinguishable particles in one dimension, which do not interact with one another; in position space, its Hamiltonian has the form

$$H = \sum_{i=1}^N -\frac{\partial^2}{\partial x_i^2} + V(x_i) \quad (1.135)$$

for some potential V in dimensionless coordinates, where we set the convention that the mass equals $\frac{1}{2}$ (so that the kinetic energy of the i th particle is just p_i^2). The single particle states $\varphi_i(x)$ are then given by eigenfunctions of $d/dx + V(x)$ with L^2 norm 1 (let us assume the potential $V(x)$ is well chosen so these eigenfunctions exist).

The collective state Φ is determined by the nature of the particles, which come in two flavours: a system of *bosons* is symmetric under exchange of particles so their state function Φ is too, whereas a system of *fermions* is antisymmetric under particle exchange. In the latter case (with the additional condition that the fermions are spinless), by the Pauli exclusion principle each state can only be occupied by one particle. In the ground state, which is occupied at zero temperature, only the N lowest energy states are filled and the collective state is a *Slater determinant* of single particle state functions

$$\Phi_0(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det_{1 \leq i, j \leq N} \varphi_i(x_j). \quad (1.136)$$

Considering the n -point correlation function $\rho_n(k_1, \dots, k_n)$ on this system, giving the probability that the positions x_1, \dots, x_n of n particles coincide with k_1, \dots, k_n , we have

$$\rho_n(k_1, \dots, k_n) = \frac{N!}{(N-n)!} \int \dots \int |\Phi_0(k_1, \dots, k_n, x_{n+1}, \dots, x_N)|^2 dx_{n+1} \dots dx_N. \quad (1.137)$$

We can compute $|\Phi_0(k_1, \dots, k_n, x_{n+1}, \dots, x_N)|^2$ by recalling that $\det A^t \det B = \det AB$ and that each single particle state function φ_i has L^2 norm 1, to find that the random

positions form a DPP with kernel

$$K(k, \ell) = \sum_{i=1}^N \varphi_i^*(k) \varphi_i(\ell). \quad (1.138)$$

$K(k, \ell)$ is manifestly Hermitian, and it meets exactly the conditions of the Macchi-Soshnikov theorem 1.44. Moreover, it is a projection kernel.

Positive temperature fermions in first quantisation At positive temperature, or at finite inverse temperature β (which we treat as dimensionless, setting the Boltzmann constant to 1), fermions are not constrained to occupy the lowest energy states available. Rather, if a state φ_i has energy ϵ_i (that is, if $(d^2/dx^2 + V(x))\varphi_i = \epsilon_i\varphi_i$) it is weighted with a Boltzmann factor of $e^{-\beta\epsilon_i}$ (see e.g. [DDMS19, Section 5]). Then, the collective state of a “canonical ensemble” of N fermions is a weighted sum of Slater determinants

$$\Phi_{\beta^{-1}}(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}Z_\beta} \sum_{i_1 < \dots < i_N} e^{-\frac{\beta}{2}(\epsilon_{i_1} + \dots + \epsilon_{i_N})} \det_{1 \leq k, j \leq N} \varphi_{i_k}(x_j), \quad (1.139)$$

where Z_β is a normalisation factor. Unlike in the zero temperature case, here we can only write ρ_N as a determinant, and cannot integrate over the remaining degrees of freedom to write a determinant for the n -point correlation function $\rho_n(k_1, \dots, k_n)$. However, we restore this integration property by moving the *grand canonical ensemble*, which, at chemical potential μ_{ch} , contains a random number of particles \mathbf{N} with law $\mathbb{P}(N) = e^{\mu_{\text{ch}}N}/N!e^{e^{\mu_{\text{ch}}}}$. In this ensemble at inverse temperature β , the fermion positions form the DPP with Hermitian projection kernel

$$K^\beta(k, \ell) = \sum_{i=1}^{\infty} \frac{1}{e^{\beta(\epsilon_i - \mu_{\text{ch}})} + 1} \varphi_i^*(k) \varphi_i(\ell). \quad (1.140)$$

The fixed particle number N of a canonical ensemble is related to the chemical potential μ_{ch} of the analogous grand canonical ensemble are related by

$$N = \text{tr } K^\beta = \sum_{i=1}^{\infty} \frac{1}{e^{\beta(\epsilon_i - \mu_{\text{ch}})} + 1}. \quad (1.141)$$

At $N \rightarrow \infty$, the canonical ensemble is equivalent to the grand canonical ensemble – this is a universal principle of statistical mechanics, see e.g. [LPS94].

Airy fermions at the edge of confining traps The Airy ensemble found in the large n limit of the Plancherel measure \mathbb{P}_n arises naturally, and universally, for free fermions in one dimension at zero temperature. Consider the case mentioned in part I of the introduction (see Figure 0.2) where the space is continuous, the potential V confines the fermions to some region with a “soft edge”; if the *Fermi energy*, which is the highest energy of an occupied state, is E_F , then the edge of that region is x_{edge} such that $E_F = V(x_{\text{edge}})$, and in the soft edge case the derivative of the potential is $0 < V'(x_{\text{edge}}) < \infty$ at the edge (the “hard edge” case where dV/dx diverges at x_{edge} has different limiting edge behaviour, see [DDMS19]). Then, as shown by Eisler [Eis13] and proven rigorously by Deleporte and Lambert [DL21], the positions of fermions rescaled near the edge to $\tilde{\mathbf{x}}_i = (\mathbf{x}_i - x_{\text{edge}})/\kappa$ are a DPP whose kernel, of the form (1.138), is built up from square integrable eigenfunctions of the effective edge Hamiltonian

$$H_{\text{edge}} = -\frac{d^2}{d\tilde{x}^2} + \kappa^3 V'(x_{\text{edge}})\tilde{x} = \kappa^2 H + O(\kappa^2(x - x_{\text{edge}})^2). \quad (1.142)$$

The critical scaling regime, in which H_{edge} has non-trivial square integrable eigenfunctions, is $\kappa \sim c \cdot V'(x_{\text{edge}})^{-1/3}$ for some constant; for simplicity we set $\kappa := V'(x_{\text{edge}})^{-1/3}$ and replace \tilde{x} with x .

Considering the classical Airy functions $\text{Ai}(x+v)$ defined at (1.105) for a real number v , we find (for instance by differentiating under the integral) that they are real square integrable eigenfunctions of H_{edge} , with

$$H_{\text{edge}} \text{Ai}(x+v) = \left(-\frac{d^2}{dx^2} + x \right) \text{Ai}(x+v) = -v \text{Ai}(x+v); \quad (1.143)$$

so, H_{edge} has a continuous unbounded spectrum, and by taking any $v > -E_F$ we find an infinite number of states $\text{Ai}(x+v)$ which can be occupied in the ground state. Since the ground state cannot saturate, this model is in general not physically sensible; if, however, the number of particles N tends to infinity, we can have a physically meaningful collective ground state where $E_F = 0$ and all of the negative energy single particle states are filled (a ‘‘Dirac sea’’ in quantum field theory terms). Then, the kernel of the DPP on the edge is precisely the Airy kernel \mathcal{A} defined at (1.104).

Further considering a grand canonical ensemble of fermions in the same edge regime, as shown by Dean, Le Doussal, Majumdar and Schehr [DLDMS15], in a regime where $\beta = \alpha v / \mu_{\text{ch}}$ the formula (1.140) gives precisely the kernel \mathcal{A}^α defined at (1.120); in terms of the particle number in the corresponding canonical ensemble given by (1.141), the crossover regime is $\beta = \alpha N^{-1/3}$ as $N \rightarrow \infty$.

Bosons and fermions in second quantisation In the second quantisation formalism, the particles are subject to quantised potentials. Here, rather than considering one Hilbert space for the whole position or momentum space R , we have a different Hilbert \mathcal{H}_x space for each point $x \in R$. A collective state then belongs to the tensor product of these spaces, which is the *Fock space* \mathcal{F} ; we denote a state as a vector $|\Phi\rangle$, which has a conjugate transpose $\langle\Phi|$ defining an inner product $\langle\Psi|\Phi\rangle \in \mathbb{C}$ on \mathcal{F} ; the $L^2(R)$ norm of a physical state is $\langle\Phi|\Phi\rangle = 1$, and an observable \mathcal{O} has expectation $\mathbb{E}(\mathcal{O}) = \langle\Phi|\hat{\mathcal{O}}|\Phi\rangle$.

Let us take R to be an ordered one dimensional space (in practice we will always consider \mathbb{R} or $\mathbb{Z} + \frac{1}{2}$) and consider non-interacting fermions on it. By the Pauli exclusion principle, each Hilbert space \mathcal{H}_x has only two possible states: filled or empty. For a set of points $B \subset R$, let $|B\rangle$ denote the state with a particle in \mathcal{H}_x for each $x \in B$ and every other \mathcal{H}_x empty; let us further insist that there exists a maximum element of B , so that there are finitely many elements of B in any subset of R bounded below. These states form a basis of \mathcal{F} , which we can take to be orthonormal with respect to the inner product, with $\langle B|B'\rangle = \mathbf{1}_{B=B'}$. We refer to [Sté21, Appendix A] for a concise and concrete example of this construction.

Starting from a given $|B\rangle$, another element of this basis be generated using the fermionic *creation and annihilation* operators c_x^\dagger, c_x , which act as

$$\begin{aligned} c_x^\dagger |B\rangle &= \begin{cases} (-1)^{\#\{B>x\}} |B \cup \{x\}\rangle & \text{if } x \notin B \\ 0 & \text{if } x \in B \end{cases} \\ c_x |B\rangle &= \begin{cases} 0 & \text{if } x \notin B \\ (-1)^{\#\{B>x\}} |B \setminus \{x\}\rangle & \text{if } x \in B. \end{cases} \end{aligned} \quad (1.144)$$

It is clear that c_x^\dagger and c_x are adjoint to one another with respect to the inner product (for example, if $|B'\rangle = c_x^\dagger |B\rangle$, then $\langle B'| = \langle B|c_x$). Moreover, from the orthonormality condition, c_x^\dagger, c_x satisfy the *canonical anticommutation relations*

$$c_x^\dagger c_y + c_y c_x^\dagger = \mathbf{1}_{x=y}, \quad c_x c_y + c_y c_x = c_x^\dagger c_y^\dagger + c_y^\dagger c_x^\dagger = 0. \quad (1.145)$$

From them, we construct two other useful operators. Firstly, indicator functions

$$c_x^\dagger c_x |B\rangle = \mathbf{1}_{x \in B} |B\rangle, \quad c_x c_x^\dagger |B\rangle = \mathbf{1}_{x \notin B} |B\rangle \quad (1.146)$$

and secondly the “level” r bosonic creation operator a_r^\dagger

$$a_r^\dagger := a_{-r} := \sum_{x \in R} c_x^\dagger c_{x-r}, \quad r \neq 0. \quad (1.147)$$

Its adjoint is a_r , which is bosonic annihilation operator at level r , and from the fermionic anticommutation relations, we can show they satisfy

$$[a_r, a_s] = a_r a_s - a_s a_r = r \mathbf{1}_{r=-s}. \quad (1.148)$$

These operators are so-called because they play a role analogous to the one of c_x^\dagger, c_x for fermions: if instead we considered a bosonic Fock space, it is made up of Hilbert spaces each of which can contain any non-negative integer number of particles; an operator a_r^\dagger adding a particle to the space \mathcal{H}_r satisfies the commutation relations (1.148).

In this formalism, both the kinetic and potential energy terms of the Hamiltonian act on \mathcal{F} through combinations of creation and annihilation operators c_x^\dagger, c_x ; it is in this sense that the potential is also quantised. We will revisit this kind of quantum mechanical model in Section 2.2.2. In the following section, we present an example using elements of second quantised fermions to construct a DPP, without reference to a physical Hamiltonian.

1.3.3 Integrability of Schur measures

Let us finally return to random partitions, and prove the following result:

Theorem 1.45 (Determinantal point process from the Schur measure [Ok01]). *Fix two sequences $t = (t_1, t_2, \dots)$ and $t' = (t'_1, t'_2, \dots)$ such that $\mathbb{P}(\lambda) := e^{-\sum_r r t_r t'_r} s_\lambda[t] s_\lambda[t']$ is a Schur measure, and let $\boldsymbol{\lambda}$ be a random partition that measure. Then, for each finite set $\{k_1, \dots, k_n\} \subset \mathbb{Z} + \frac{1}{2}$, the n -point correlation function of the fermion configuration $S(\boldsymbol{\lambda})$ is*

$$\rho_n(k_1, \dots, k_n) = \mathbb{P}(\{k_1, \dots, k_n\} \subset S(\boldsymbol{\lambda})) = \det_{1 \leq i, j \leq n} K(k_i, k_j) \quad (1.149)$$

where

$$K(k, \ell) = \sum_{i=0}^{\infty} J_{k+i+1/2}(t, t') J_{\ell+i+1/2}(t, t') \quad (1.150)$$

where $J_n(t, t')$ is the multivariate Bessel function

$$J_n(t, t') = \frac{1}{2\pi i} \oint \exp \left[\sum_r t_r z^r - \sum_r t'_r z^{-r} \right] \frac{dz}{z^{n+1}}. \quad (1.151)$$

The kernel K is generated by

$$\sum_{k, \ell \in \mathbb{Z} + \frac{1}{2}} z^k w^{-\ell} K(k, \ell) = \frac{\exp \left[\sum_r t_r z^r - \sum_r t'_r z^{-r} \right] \sqrt{zw}}{\exp \left[\sum_r t_r w^r - \sum_r t'_r w^{-r} \right] z - w}, \quad |w| < |z|. \quad (1.152)$$

In order to prove this, we consider a Fock space \mathcal{F} over the half integers generated by an orthonormal basis $\{|S\rangle\}$ indexed by the “fermion configurations” $S \subset \mathbb{Z} + \frac{1}{2}$ introduced

in Section 1.1.1, with $\langle S|S' \rangle = \mathbf{1}_{S=S'}$. As their name suggests, these configurations are naturally described in these terms. Once we define the *normal ordering* $::\cdot::$,

$$:c_k^\dagger c_k: := \begin{cases} c_k^\dagger c_k, & k > 0 \\ c_k c_k^\dagger, & k < 0, \end{cases} \quad (1.153)$$

we can note that key data for a fermion configuration S are eigenvalues of normal ordered operators, with

$$\begin{aligned} Q_0(S)|S\rangle &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} :c_k^\dagger c_k: |S\rangle, & Q_1(S)|S\rangle &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} k :c_k^\dagger c_k: |S\rangle \\ Q_2(S)|S\rangle &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{k^2}{2} :c_k^\dagger c_k: |S\rangle. \end{aligned} \quad (1.154)$$

Each of these eigenvalues is finite for any S , and we recall that for a partition λ , its fermion configuration satisfies $Q_0(S(\lambda)) = 0$, $Q_1(S(\lambda)) = |\lambda|$, $Q_2(S(\lambda)) = C_\lambda$.

The bosonic creation operator a_r^\dagger have natural interpretations in terms of partitions; for each $\mu \vdash n$, we have

$$a_r^\dagger |S(\mu)\rangle = \sum_{\substack{\lambda \vdash n+r: \\ \lambda/\mu \text{ is a ribbon}}} |S(\lambda)\rangle \quad (1.155)$$

and in particular $a_1^\dagger |S(\mu)\rangle = \sum_{\mu \nearrow \lambda} |S(\lambda)\rangle$.

Proof of Theorem 1.45. We construct the DPP on $\mathbb{Z} + \frac{1}{2}$ as described by defining vectors $|\Phi\rangle, |\Phi'\rangle$ in the Fock space \mathcal{F} over $\mathbb{Z} + \frac{1}{2}$ such that⁹

$$\rho_n(k_1, \dots, k_n) := \langle \Phi | c_{k_1}^\dagger c_{k_1} \cdots c_{k_n}^\dagger c_{k_n} | \Phi' \rangle = \det_{1 \leq i, j \leq n} K(k_i, k_j), \quad (1.156)$$

then show that if a random configuration \mathbf{S} is a DPP with this kernel, it is equivalent in law to the fermion configuration of a random partition $\boldsymbol{\lambda}$ under the Schur measure.

To start, we let the “domain wall” $|S(\emptyset)\rangle$ be the *vacuum state* and define the *vertex operators*

$$\Gamma_\pm(t) = \exp \left[\sum_{r \geq 1} t_r a_{\pm r} \right] \quad (1.157)$$

in terms of the positive integer indexed bosonic creation and annihilation operators $a_{\pm r}$. Then we set $|\Phi\rangle = Z^{-1/2} \Gamma_-(t) |S(\emptyset)\rangle$ and $|\Phi'\rangle = Z^{-1/2} \Gamma_-(t') |S(\emptyset)\rangle$ where Z is a normalisation factor, such that $\rho_0(\emptyset) = 1$; for now let us assume that these vectors define a probability law, as this will be confirmed at the end of the proof. The adjoint of $\Gamma_-(t)$ is $\Gamma_+(t)$, so the n -point correlation is

$$\rho_n(k_1, \dots, k_n) = \frac{1}{Z} \langle S(\emptyset) | \Gamma_+(t) c_{k_1}^\dagger c_{k_1} \cdots c_{k_n}^\dagger c_{k_n} \Gamma_-(t') | S(\emptyset) \rangle \quad (1.158)$$

From the anticommutation relations (1.145), we find

$$[a_r, c_k^\dagger] = c_{k+r}^\dagger, \quad [a_r, c_k] = -c_{k+r}, \quad (1.159)$$

and for the vertex operators, in terms of the generating functions

$$c^\dagger(z) := \sum_k z^k c_k^\dagger \quad \text{and} \quad c(w) := \sum_\ell w^{-\ell} c_\ell \quad (1.160)$$

⁹This is a more general construction than the quantum mechanical one, which would require $|\Phi'\rangle = |\Phi\rangle$.

we have

$$\Gamma_{\pm}(t)c^{\dagger}(z) = e^{\sum_r t_r z^{\pm r}} c^{\dagger}(z)\Gamma_{\pm}(t), \quad \Gamma_{\pm}(t)c(w) = e^{-\sum_r t_r z^{\pm r}} c(w)\Gamma_{\pm}(t). \quad (1.161)$$

By application of the *Baker–Campbell–Hausdorff* formula $e^A e^B = e^B e^A e^{[A,B]}$ where $[A, [A, B]] = [B, [A, B]] = 0$, we have

$$\Gamma_+(t)\Gamma_-(t') = e^{\sum_r r t_r t'_r} \Gamma_-(t')\Gamma_+(t). \quad (1.162)$$

Noting that $\Gamma_+|S(\emptyset)\rangle = |S(\emptyset)\rangle$, we then compute $\rho_0(\emptyset)$ to find $Z = e^{\sum_r r t_r t'_r}$. Then, since we similarly have $\Gamma_+^{-1}|S(\emptyset)\rangle = |S(\emptyset)\rangle$, the correlation function (1.158) can then be written

$$\begin{aligned} \rho_n(k_1, \dots, k_n) &= \langle S(\emptyset) | \tilde{c}_{k_1}^{\dagger} \tilde{c}_{k_1} \cdots \tilde{c}_{k_n}^{\dagger} \tilde{c}_{k_n} | S(\emptyset) \rangle \\ \tilde{c}_k^{\dagger} &= \Gamma_+(t)\Gamma_-(t')^{-1} c_k^{\dagger} \Gamma_-(t')\Gamma_+(t)^{-1}, \quad \tilde{c}_k = \Gamma_+(t)\Gamma_-(t')^{-1} c_k \Gamma_-(t')\Gamma_+(t)^{-1}. \end{aligned} \quad (1.163)$$

In this form, ρ_n can be expressed as a determinant using a classical identity, which follows directly from the anticommutation relations (1.161). Let $\langle \cdot \rangle = \langle S(\emptyset) | \cdot | S(\emptyset) \rangle$ denote the expectation on the vacuum state; then,

Lemma 1.46 (Wick's lemma [Wic50]). *For any sequence of half integers $(k_1, k_2, \dots, k_{2n})$ we have*

$$\langle c_{k_1}^{\dagger} c_{k_2} c_{k_3}^{\dagger} c_{k_4} \cdots c_{k_{2n-1}}^{\dagger} c_{k_{2n}} \rangle = \det_{1 \leq i, j \leq n} \langle c_{k_{2i-1}}^{\dagger} c_{k_{2j}} \rangle. \quad (1.164)$$

Each \tilde{c}_k^{\dagger} is a linear combination of operators c_{ℓ}^{\dagger} , and similarly \tilde{c}_k is a linear combination of c_{ℓ} s, so by linearity Lemma 1.46 gives

$$\rho_n(k_1, \dots, k_n) = \det_{1 \leq i, j \leq n} \langle \tilde{c}_{k_i}^{\dagger} \tilde{c}_{k_j} \rangle. \quad (1.165)$$

It remains to show that $K(k, \ell) := \langle \tilde{c}_k^{\dagger} \tilde{c}_{\ell} \rangle$ is the kernel given in the statement.

Using the generating functions $c^{\dagger}(z), c(z)$ and the relations (1.161), we have

$$\begin{aligned} \sum_{k, \ell} z^k w^{-\ell} K(k, \ell) &= \langle \Gamma_+(t)\Gamma_-(t')^{-1} c^{\dagger}(z) c(w) \Gamma_-(t')\Gamma_+(t)^{-1} \rangle \\ &= e^{\sum_r t_r z^r - t_r z^{-r}} \langle c^{\dagger}(z) c(w) \rangle e^{\sum_r t'_r w^{-r} - \sum_r t_r w^r}. \end{aligned} \quad (1.166)$$

First, we evaluate the final vacuum state expectation, which is just a geometric series; requiring that $|w| < |z|$ we have

$$\sum_{k, \ell \in \mathbb{Z} + \frac{1}{2}} \frac{z^k}{w^{\ell}} \langle c_k^{\dagger} c_{\ell} \rangle = \sum_{k, \ell \in \mathbb{Z} + \frac{1}{2}} \frac{z^k}{w^{\ell}} \mathbf{1}_{k=\ell, k < 0} = \frac{\sqrt{zw}}{z-w}, \quad (1.167)$$

which gives (1.152) as required. Second, we manipulate the formal series in (1.166) further, in terms of the multivariate Bessel functions defined in (1.151), to get

$$\begin{aligned} \sum_{k, \ell} z^k w^{-\ell} K(k, \ell) &= \sum_{k, \ell} z^k w^{-\ell} \sum_{m \in \mathbb{Z}} z^m J_m(t, t') \sum_{n \in \mathbb{Z}} w^{-n} J_n(t, t') \mathbf{1}_{k=\ell, k < 0} \\ &= \sum_{i=0}^{\infty} \sum_{m, n \in \mathbb{Z}} z^{m-i-\frac{1}{2}} J_m(t, t') w^{i-n+\frac{1}{2}} J_n(t, t') \\ &= \sum_{k, \ell} z^k w^{-\ell} \sum_{i=0}^{\infty} J_{k+i+\frac{1}{2}}(t, t') J_{\ell+i+\frac{1}{2}}(t, t'). \end{aligned} \quad (1.168)$$

This recovers (1.150) as required.

To conclude the proof, let us find an explicit expression for the law of a random configuration under this DPP, in terms of the corresponding partition. We compute this by inserting the projection operation $|S\rangle\langle S|$ on \mathcal{F} , to write

$$\mathbb{P}(S) = \langle \Phi | S \rangle \langle S | \Phi' \rangle \quad (1.169)$$

and evaluate each inner product, letting $S := S(\lambda)$. For the first inner product,

$$\langle \Phi | S(\lambda) \rangle = e^{-\frac{1}{2} \sum_r r t_r t'_r} \langle \Gamma_+(t) c_{\lambda_1 - \frac{1}{2}}^\dagger c_{-\frac{1}{2}}^\dagger c_{\lambda_2 - \frac{3}{2}}^\dagger c_{-\frac{3}{2}}^\dagger \cdots c_{\lambda_\ell(\lambda) - \ell(\lambda) + \frac{1}{2}}^\dagger c_{-\ell(\lambda) + \frac{1}{2}}^\dagger \rangle. \quad (1.170)$$

By (1.161), and recalling that $e^{\sum_r t_r z^r}$ is the generating function of the complete homogeneous symmetric functions as given in (1.54), we have

$$\begin{aligned} \Gamma_+(t) c_k^\dagger &= \sum_{m=0}^{\infty} h_m[t] c_{k-m}^\dagger \Gamma_+(t) =: \tilde{c}_k^\dagger \Gamma_+(t), \\ \Gamma_+(t) c_k &= \sum_{m=0}^{\infty} h_m[t] c_{k+m} \Gamma_+(t) =: \tilde{c}_k \Gamma_+(t). \end{aligned} \quad (1.171)$$

Then by Wick's lemma 1.46, we have

$$\begin{aligned} \langle \Phi | S(\lambda) \rangle &= e^{-\frac{1}{2} \sum_r r t_r t'_r} \langle \tilde{c}_{\lambda_1 - \frac{1}{2}}^\dagger \tilde{c}_{-\frac{1}{2}}^\dagger \cdots \tilde{c}_{\lambda_\ell(\lambda) - \ell(\lambda) + \frac{1}{2}}^\dagger \tilde{c}_{-\ell(\lambda) + \frac{1}{2}}^\dagger \rangle \\ &= e^{-\frac{1}{2} \sum_r r t_r t'_r} \det_{1 \leq i, j \leq \ell(\lambda)} \langle \tilde{c}_{\lambda_i - i + \frac{1}{2}}^\dagger \tilde{c}_{-j + \frac{1}{2}} \rangle. \end{aligned} \quad (1.172)$$

Evaluating the final vacuum expectation value, we find

$$\begin{aligned} \langle \tilde{c}_{\lambda_i - i + \frac{1}{2}}^\dagger \tilde{c}_{-j + \frac{1}{2}} \rangle &= \sum_{m, n} h_m[t] h_n[t] \mathbf{1}_{\lambda_i - i - m = n - j} \\ &= \sum_n h_{\lambda_i - i + j - n}[t] h_n[t] = h_{\lambda_i - i + j}[t]; \end{aligned} \quad (1.173)$$

hence, by Definition 1.26 of the Schur function, we have

$$\langle \Phi | S(\lambda) \rangle = e^{-\frac{1}{2} \sum_r r t_r t'_r} s_\lambda[t] \quad (1.174)$$

and noting the similar expression for $\langle S(\lambda) | \Phi' \rangle$, we have

$$\mathbb{P}(S(\lambda)) = e^{-\sum_r r t_r t'_r} s_\lambda[t] s_\lambda[t'] =: \mathbb{P}(\lambda). \quad (1.175)$$

which completes the proof (and shows that the condition that t, t' define a Schur measure is sufficient for the vectors $\Gamma_-(t) | S(\emptyset) \rangle, \Gamma_-(t') | S(\emptyset) \rangle$ to define a DPP). \blacksquare

The Plancherel measure again As a direct consequence, we can characterise the DPP associated to the Poissonised Plancherel measure:

Corollary 1.47 (Bessel ensemble from the Poissonised Plancherel measure). *Let λ be a random partition under the Poissonised Plancherel measure \mathbb{P}_θ . Then, its fermion configuration $S(\lambda)$ is equivalent in law to the DPP with Bessel kernel*

$$\mathcal{J}_\theta(k, \ell) = \frac{1}{(2\pi i)^2} \iint \frac{e^{\theta(z-1/z)}}{e^{\theta(w-1/w)}} \frac{dz dw}{z^{k+1} w^{1-\ell} (z-w)} = \sum_{i=0}^{\infty} J_{k+i+\frac{1}{2}}(2\theta) J_{\ell+i+\frac{1}{2}}(2\theta) \quad (1.176)$$

where $J_n(2\theta)$ is the classical Bessel function

$$J_n(2\theta) := J_n((\theta, 0, \dots), (\theta, 0, \dots)) = \frac{1}{2\pi i} \oint e^{\theta(z-1/z)} \frac{dz}{z^{n+1}}. \quad (1.177)$$

In this case, the equivalence of laws

$$\mathbb{P}_\theta(\lambda) = e^{-\theta^2} \langle S(\emptyset) | e^{\theta a_1} | S(\lambda) \rangle \langle S(\lambda) | e^{\theta a_{-1}} | S(\emptyset) \rangle \quad (1.178)$$

has an immediate combinatorial interpretation. Consider the action of a_{-1}^n at the level of partitions: we have

$$a_{-1}^n | S(\emptyset) \rangle = \sum_{\emptyset \nearrow \lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \dots \nearrow \lambda^{(n)}} | S(\lambda^{(n)}) \rangle = \sum_{\lambda \vdash n} f_\lambda | S(\lambda) \rangle \quad (1.179)$$

where we use the fact that a sequence of n partitions where each is obtained by adding one box to the last corresponds to a SYT. It follows that the vertex operator $\Gamma_-(\theta, 0, 0, \dots)$ acts on the vacuum by generating states $| S(\lambda) \rangle$ weighted with factors of $\theta^{|\lambda|} / |\lambda|!$.

Both the VKLS theorem 1.35 and the BDJ theorem 1.37 can be proven by analysis of the Bessel kernel $\mathcal{J}_\theta(k, \ell)$ as $\theta \rightarrow \infty$, in regimes where $k, \ell \sim u\theta$ and $k, \ell \sim 2\theta + s\theta^{1/3}$ respectively (see Section 2.3), then by “de-Poissonisation” [Joh01] (see also [Rom15, Lemma 2.31]); in spirit this is a proof the asymptotic equivalence of the canonical and grand canonical ensembles of partitions under the Plancherel measure.

The $\theta \rightarrow \infty$ limit of \mathbb{P}_θ is physically meaningful too. From a first quantisation perspective, the Bessel kernel \mathcal{J}_θ is Hermitian and has the form (1.138); this corresponds to a system of an infinite number of particles with positions on $\mathbb{Z} + \frac{1}{2}$ with single particle wavefunctions given by classical Bessel functions, with $\phi_i(k) = J_{k+i+\frac{1}{2}}(2\theta)$. We can see (for instance by differentiating (1.177) under the integral) that these wavefunctions satisfy (putting $\ell = i + \frac{1}{2}$)

$$-J_{k+\ell+1}(2\theta) - J_{k+\ell-1}(2\theta) + \frac{\ell}{\theta} J_{\ell+k}(2\theta) = -\frac{k}{\theta} J_{\ell+k}(2\theta). \quad (1.180)$$

So, $J_{i+k+\frac{1}{2}}(2\theta)$ are eigenfunctions of a discrete difference operator which we can interpret as a Hamiltonian H_θ , with an unbounded negative linear spectrum. In a regime where $k \sim 2\theta + x\theta^{1/3}$, the $\theta \rightarrow \infty$ limit is a continuum limit in which we can informally consider the rescaled Hamiltonian to converge to a differential operator, with $\theta^{-1/3} H_\theta \rightarrow H_{\text{edge}}$.

Multicritical Schur measures

This chapter presents joint work with Dan Betea and Jérémie Bouttier, adapted from [1]. We construct new “multicritical” probability laws on partitions, tuned to be outside the TW-GUE universality class, with edge fluctuations characterised by critical exponents differing from the generic $1/3$. The asymptotic distribution of the first part of a random partition under these measures is a Fredholm determinant previously encountered by Le Doussal, Majumdar and Schehr [LDMS18] in momentum space models of trapped fermions in one dimension, and is a higher-order analogue of the TW-GUE distribution. The multicritical laws are defined as Schur measures, of a form that arises naturally from physical lattice fermion models, and our analysis directly exploits their integrability.

We also consider extensions of these measures: to multicritical measures on cylindric partitions, which, in a suitable asymptotic regime, have edge statistics interpolating between higher-order TW-GUE and Gumbel distributions, previously observed for positive temperature fermions in [LDMS18]; to measures with asymptotic edge fluctuations driven by more general Fredholm determinant solutions of Painlevé II equations found by Cafasso, Claeys and Girotti [CCG19]; and to measures corresponding to fermion models with split momentum spectra, whose limiting law is not yet fully understood but for which we conjecture new asymptotic edge statistics.

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2.1 New edge fluctuations for random partitions

In this section we introduce the measures on partitions we consider, and characterise their asymptotic edge behaviour. First, we define them by way of a set of criteria for a rather general class of measures, then we give explicit expressions for two one-parameter families of “minimal” measures. In the latter case we also state limit curves for the rescaled profiles of large random partitions.

The TW-GUE class and how to escape it Our motivation comes from the universal appearance of TW-GUE fluctuations for interfaces in random matrices [TW93], random growth [SS10, ACQ11], free and interacting fermion models [Eis13, Sté19, DL21], domino tilings [Joh05] and directed paths through random media [Joh00], to name but a few. This asymptotic behaviour is exemplified by Baik, Deift and Johansson’s [BDJ99] famous result for random partitions under the Plancherel, stated in the last chapter as Theorem 1.37, where we see the characteristic $1/3$ exponent for the scale of the fluctuations relative to the edge distance, and the characteristic distribution F_{GUE} . Random partitions provide a convenient means of investigating the TW-GUE universality class, as several models in the class can be reformulated in their terms – we can cite, for instance, the connection between geometric last passage percolation and Schur measures discussed in Section 1.2.2, the discrete random growth process associated with the Plancherel measure from the proof of Theorem 1.35 (ii), and the asymptotic equivalence of the edges in the Poissonised Plancherel measure and in trapped fermion models discussed in Section 1.3.3, as well as the equivalence of Schur processes and certain tiling models proven in [BBC⁺15]. We were interested in the question of how a model of random partitions can escape this universality class.

Alternative asymptotic edge behaviour has previously been found for elements of random sequences of partitions: as discussed in Section 1.2.3, distributions interpolating between the TW-GUE and Gumbel distributions were found for cylindric partitions [BB19], and internal edges of random skew plane partitions were found to converge to the Pearcey Process [OR07]. To directly construct measures on single partitions with new edge behaviour, we took inspiration from the work of Le Doussal, Majumdar and Schehr [LDMS18], who studied free fermions in continuous one dimensional space confined to a fixed region of position space by “flat trap” potentials $V(x) = x^{2m}$. While the fluctuations of the maximal fermion position in such a model are universal (see Section 1.3.2), these authors showed the maximal fermion *momentum* to be asymptotically distributed by a novel distribution. The distributions they found, indexed by a positive integer m , are Fredholm determinants, and encode solutions of equations in the Painlevé II hierarchy, generalising the connection between the TW-GUE distribution and the Painlevé II equation; this was rigorously proven in [CCG19]. We find measures on partitions in the corresponding universality classes using a correspondence between certain Schur measures and systems of free fermions in one dimension, whose edges asymptotically coincide with the momentum space edges in flat traps.

A further motivation was a seemingly coincidental connection with partition functions of so-called “multicritical” unitary matrix models previously considered by Periwal and Shevitz [PS90b, PS90a] observed in [LDMS18]. In a large matrix limit, derivatives of these partition functions satisfy the same Painlevé II hierarchy equations as the higher-order TW-GUE distributions. In fact, these partition functions are *exactly* equal to certain cumulative distributions under our measures. We will discuss this connection in detail in Section 3.1.1 of the next chapter. For now, following [LDMS18], we borrow the term *multicritical* to refer to this new edge behaviour.

Independent work by Kimura and Zahabi [KZ21b] arguing that the multicritical edge phenomena is found for Schur measures appeared on the arXiv shortly after the extended abstract for [1] did. The authors considered the semiclassical analysis of the multivariate Bessel functions related to Schur measure, and presented results that are consistent with ours. Our approach is somewhat more concrete, we use direct asymptotic analysis of gap probabilities and find explicit measures with multicritical edge behaviour.

2.1.1 Classification and properties of multicritical measures

We consider the following family of measures, each parametrised by a single positive number:

Definition 2.1 (Multicritical measures). For each positive integer m , an order m multicritical measure with parameter θ is a Schur measure $\mathbb{P}_\theta^m := e^{-\theta^2 \sum_r r \gamma_r^2} s_\lambda[\theta \gamma]^2$ with both sets of Miwa times specialised to $\theta \gamma := (\theta \gamma_1, \theta \gamma_2, \dots)$ where $\gamma = (\gamma_1, \gamma_2, \dots)$ is a sequence of real numbers with finite support satisfying

$$\sum_{r \geq 1} r \gamma_r > 0, \quad \sum_{r \geq 1} r^{2m+1} \gamma_r \neq 0, \quad \sum_{r \geq 1} r^2 \gamma_r \sin r \phi \geq 0 \quad \text{for all } \phi \in [0, \pi] \quad (2.1)$$

and, if $m > 1$,

$$\sum_{r \geq 1} r^{2p+1} \gamma_r = 0, \quad p = 1, 2, \dots, m-1. \quad (2.2)$$

If $m = 1$ the measure is called *simply critical*.

The positive constants

$$b := 2 \sum_{r \geq 1} r \gamma_r, \quad \tilde{b} := 2 \sum_{r \geq 1} (-1)^{r+1} r \gamma_r, \quad d := \frac{2(-1)^{m+1}}{(2m)!} \sum_{r \geq 1} r^{2m+1} \gamma_r \quad (2.3)$$

are respectively the right edge, left edge and fluctuation coefficients associated with the measure.

Recalling (1.111), if λ is a random partition under \mathbb{P}_θ^m , $|\lambda|$ has expectation $\theta^2 \sum_r r^2 \gamma_r^2$. The parameter θ thus defines a typical length scale for the parts λ_i, λ'_i .

The $m = 1$ simply critical measures are the most generic, since they do not need to satisfy any “multicriticality conditions” (2.2), and they notably include the Poissonised Plancherel measure \mathbb{P}_θ . The space of order $m+1$ measures is one dimension smaller than that of order m . Of the three conditions (2.1), the first establishes a convenient sign convention (immediately giving $b > 0$) and the second sets the order m along with (2.2), while the third is more subtle. It ensures that there are no cuts in the Fourier transform of the kernel, which is a necessary assumption for the asymptotic analysis, and also ensures that $d > 0$. Physically, this corresponds to having no gap in the quantum number configuration of a system. The situation where there is a gap was studied in an integrable boson model in [FEC14], and called a *split Fermi sea*. Limit shapes for free fermion models with split fermi seas were considered in [BS21]. In Section 2.4.3, we consider the situation where this condition is lifted.

The multicritical Schur measures are *Hermitian*, meaning the two sequences of Miwa times are complex conjugate to one another (for simplicity we consider only real valued Miwa times, but our arguments may be generalised). Along with the condition that the Miwa times have finite support (so the measure may be called *polynomial*), this is

sufficient for a Schur measure to be well defined. However, for $m > 1$, the multicritical measures do not satisfy the Schur positivity condition of Proposition 1.41, so they do not extend to define Schur processes with discrete time evolution. A law corresponding to a $m = 2$ multicritical measure was found from a system of lattice fermions evolving in imaginary time considered by Bocini and Stéphan [BS21], but this used a construction described as “non-probabilistic” by the authors, with negative Boltzmann weights at certain times.

Definition 2.1 amounts to tuning polynomial Hermitian Schur measures to have the following edge behaviour:

Theorem 2.2 (Asymptotic edge fluctuations of multicritical measures). *Let λ be a random partition distributed by an order m multicritical measure $\mathbb{P}_\theta^m(\lambda) = e^{-\theta^2 \sum_r r \gamma_r^2} s_\lambda[\theta \gamma]^2$, with edge position and fluctuation coefficients b, d . Then the asymptotic cumulative distribution of its first part is*

$$\lim_{\theta \rightarrow \infty} \mathbb{P}_\theta^m \left[\frac{\lambda_1 - b\theta}{(d\theta)^{\frac{1}{2m+1}}} \leq s \right] = F_{2m+1}(s) := \det(1 - \mathcal{A}_{2m+1})_{L^2([s, \infty))} \quad (2.4)$$

where \mathcal{A}_{2m+1} is the higher-order Airy kernel defined in [LDMS18]¹

$$\begin{aligned} \mathcal{A}_{2m+1}(x, y) &= \int_0^\infty \text{Ai}_{2m+1}(x+v) \text{Ai}_{2m+1}(y+v) dv \\ &= \sum_{i=0}^{2m-1} \frac{(-1)^{m+i+1} \text{Ai}_{2m+1}^{(i)}(x) \text{Ai}_{2m+1}^{(2m-1-i)}(y)}{x-y}, \end{aligned} \quad (2.5)$$

$$\text{Ai}_{2m+1}(x) = \frac{1}{2\pi i} \int_{1+i\mathbb{R}} \exp \left[(-1)^{m-1} \frac{\zeta^{2m+1}}{2m+1} - x\zeta \right] d\zeta. \quad (2.6)$$

In the second expression in (2.5), we use the notation $f^{(n)}(x) := d^n f/dx^n$, and the $x = y$ case is recovered by a limit using L’Hôpital’s rule i.e. evaluating the derivative of the numerator at $x = y$. Note that the higher-order Airy functions Ai_{2m+1} decay to zero at positive infinity, and that \mathcal{A} has finite trace on any $L^2([t, \infty))$ where t is finite. In the simply critical case $m = 1$, we have $\text{Ai}_3 = \text{Ai}$ and $F_3 = F_{\text{GUE}}$. Then, Theorem 2.2 is the Poissonised version of the BDJ theorem 1.37 generalised to a large class of measures (note that the constants b, d are not universal). For $m > 1$, the theorem defines higher-order analogues of the TW-GUE distribution associated with critical exponents $1/(2m + 1)$.

These distributions were related to classical integrable equations, first in [LDMS18, Appendix G], then by rigorous Riemann–Hilbert analysis in [CCG19]:

Theorem 2.3 (Painlevé II hierarchy and higher-order TW-GUE distributions [CCG19]). *The distribution $F_{2m+1}(s) = \det(1 - \mathcal{A}_{2m+1})_{L^2([s, \infty))}$ encodes a Fredholm determinant solution of the order $2m$ equation of the Painlevé II hierarchy; in particular, this equation has a solution q_m with boundary behaviour*

$$\begin{aligned} q_m((-1)^m s) &= O(e^{-Cs \frac{2m+1}{2m}}) \quad \text{as } s \rightarrow +\infty, \\ q_m((-1)^m s) &\sim \left(\frac{m!^2}{(2m)!} |s| \right)^{\frac{1}{2m}} \quad \text{as } s \rightarrow -\infty \end{aligned} \quad (2.7)$$

¹Our integration convention differs from [LDMS18, Equation 5] which defines the same function. In their expression the integration is taken over a line to the left of the origin for even m , and is recovered from ours by the change of integration variable $\zeta \rightarrow -\zeta$.

for some constant $C > 0$, which satisfies

$$F_{2m+1}(s) = \exp \left[- \int_s^\infty (x-s) q_m^2 ((-1)^{m+1} x) dx \right]. \quad (2.8)$$

The $m = 1$ case is just the Painlevé transcendent expression for F_{GUE} given by Tracy and Widom [TW93]. In [CCG19], the authors showed that Fredholm determinants of more general higher-order Airy kernels satisfy the same relations, see Addendum 2.17.

A trace of multicriticality is also visible at the “macroscopic” scale of the parts of a random partition. A second critical exponent, characterising the vanishing at the right edge of the Young diagram of a large random partition (in terms of its rescaled profile, see Definition 1.5) generalises from the value of $1/2$ characteristic of the TW-GUE class (and Wigner’s semi-circle law for random matrices):

Theorem 2.4 (Limit shapes of multicritical measures). *The rescaled profile $\psi_{\lambda, \theta}$ of a random partition λ under an order m multicritical measure $\mathbb{P}_\theta^m = e^{-\theta^2 \sum_r r \gamma_r^2} s_\lambda[\theta \gamma]^2$ has a deterministic limit curve: as $\theta \rightarrow \infty$,*

$$\sup_x |\psi_{\lambda, \theta}(x) - \Omega(x)| \xrightarrow{p} 0 \quad (2.9)$$

where Ω is the function

$$\Omega(x) = \begin{cases} x + 2\tilde{b} - \frac{2}{\pi} \int_{-\tilde{b}}^x \chi(v) dv, & x \in [-\tilde{b}, b] \\ |x|, & x > b \text{ and } x < -\tilde{b} \end{cases} \quad (2.10)$$

where $\chi(x) \in [0, \pi]$ is a solution of

$$2 \sum_r r \gamma_r \cos r \chi(x) = x, \quad x \in [-\tilde{b}, b]. \quad (2.11)$$

The right edge of the Young diagram vanishes with exponent $1/2m$, with

$$\Omega'(x) \sim 1 - \frac{2}{\pi} \left(\frac{b-x}{d} \right)^{\frac{1}{2m}} \quad \text{as } x \rightarrow b. \quad (2.12)$$

In (2.10) we have a general formula to find the limit shape, but that shape is dependant on the precise coefficients γ ; only the vanishing exponent of $1/2m$ is universal for order m multicritical measures.

Although we only state our main theorems for the right edge at λ_1 , analogous results for the second interface at $\ell(\lambda)$ can be extracted directly, because

Proposition 2.5 (Conjugate partition under a multicritical measure). *If λ is a random partition under $\mathbb{P}_\theta^m = e^{-\theta^2 \sum_r r \gamma_r^2} s_\lambda[\theta \gamma]^2$, then the law of its conjugate λ' is*

$$\mathbb{P}_\theta^m(\lambda') = e^{-\theta^2 \sum_r r \tilde{\gamma}_r^2} s_\lambda[\theta \tilde{\gamma}]^2, \quad \tilde{\gamma}_r = (-1)^{r-1} \gamma_r. \quad (2.13)$$

This follows directly from Theorem 1.31 and Proposition 1.32, since, in terms of the classical involution ω ,

$$s_{\lambda'}[\theta \gamma] = \omega(s_\lambda[\theta \gamma]) = s_\lambda[\omega(\theta \gamma)] = s_\lambda[\theta \tilde{\gamma}]. \quad (2.14)$$

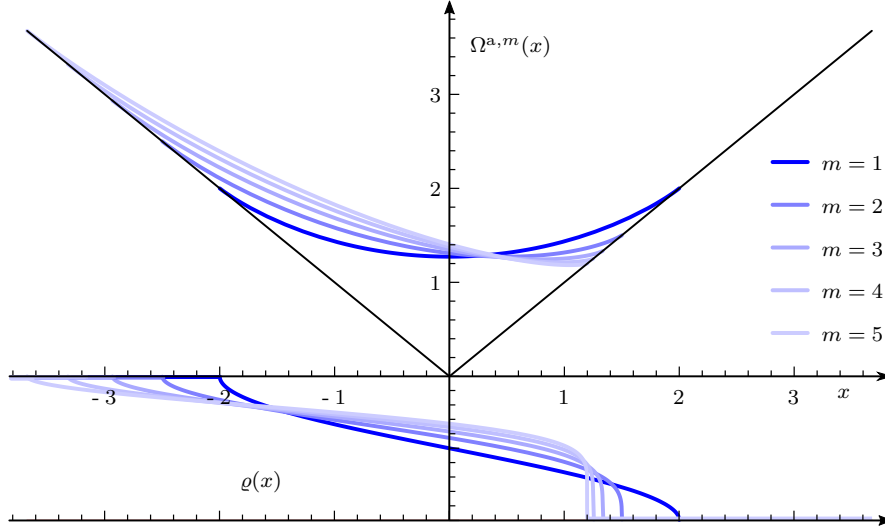


Figure 2.1 Limit curves of the minimal multicritical measures $\mathbb{P}^{a,m}$ for m from 1 to 5, and the limiting density profiles of the corresponding free fermion models. See also Figure 0.6 for Young diagrams and fermion configurations of partitions sampled under these measures.

2.1.2 One parameter families of multicritical measures

To give concrete examples of multicritical measures, let us introduce two families of multicritical measures, with a unique measure for each order m . First, we fix the sequence γ by allowing only the first m coefficients γ_r to be non-zero, and fixing $\gamma_1 = 1$ in each case (so, in particular, we have the Poissonised Plancherel measure \mathbb{P}_θ for $m = 1$). We find the following definition:

Definition 2.6 (Minimal multicritical measures). The order m minimal multicritical measure is $\mathbb{P}_\theta^{a,m}(\lambda) = e^{-\theta^2 \sum_r r \gamma_r^2} s_\lambda[\theta \gamma]^2$ where

$$\gamma_r = \begin{cases} \frac{(-1)^{r+1}}{r} \binom{2m}{m+r} / \binom{2m}{m-1}, & r = 1, 2, \dots, m \\ 0, & r > m. \end{cases} \quad (2.15)$$

Its edge and fluctuation coefficients are

$$b = \frac{m+1}{m}, \quad \tilde{b} = 4^m \binom{2m}{m}^{-1} - \frac{m+1}{m}, \quad d = \binom{2m}{m-1}. \quad (2.16)$$

From Theorem 2.4, we have explicit limit shapes for these measures:

Corollary 2.7 (Limit shapes of minimal multicritical measures). The rescaled profile $\psi_{\lambda,\theta}$ of a random partition under $\mathbb{P}_\theta^{a,m}$ converges in probability to

$$\Omega^{a,m}(x) = \begin{cases} x + 2\tilde{b} - \frac{2}{\pi} \int_{-\tilde{b}}^x \arccos \left[1 - \frac{1}{2} \binom{2m}{m-1}^{\frac{1}{m}} (b-v)^{\frac{1}{m}} \right] dv, & x \in [-\tilde{b}, b] \\ |x|, & x > b \text{ and } x < -\tilde{b}. \end{cases} \quad (2.17)$$

These limit shapes are shown for the first few m in Figure 2.1; for $m = 1$, the curve is the VKLS limit curve given explicitly in (1.89). By Proposition 2.5 and Definition 2.1, if λ is distributed by $\mathbb{P}_\theta^{a,m}$ then, for all m , λ' is distributed by a simply critical measure (the fluctuations in $\ell(\lambda)$ are asymptotically driven by the TW-GUE distribution; the

critical exponents are $1/3$ for the fluctuations and $1/2$ for the vanishing). Proposition 2.5 motivates a definition for another one-parameter family of measures which are manifestly invariant under conjugation of the partition, where we fix γ by letting only the first m odd-indexed coefficients γ_r be non-zero, and again fixing $\gamma_1 = 1$ (so again we have \mathbb{P}_θ for $m = 1$).

Definition 2.8 (Symmetric minimal multicritical measures). The order m symmetric minimal multicritical measure is $\mathbb{P}_\theta^{s,m}(\lambda) = \mathbb{P}_\theta^{s,m}(\lambda') = e^{-\theta^2 \sum_r r \gamma_r^2} s_\lambda[\theta \gamma]^2$ where

$$\gamma_{2r-1} = \begin{cases} \frac{(-1)^{r+1} (2m-1)}{(2r-1)^2} \binom{2m-1}{m-r} / \binom{2m-1}{m-1}, & r = 1, 2, \dots, m \\ 0 & r > m \end{cases} \quad (2.18)$$

and $\gamma_{2r} = 0$ for each positive integer r . Its edge and fluctuation coefficients are

$$b = \tilde{b} = \frac{2^{4m-1} (m!)^4}{m((2m)!)^2}, \quad d = \frac{(2m-2)!!}{(2m-1)!!}. \quad (2.19)$$

The symmetric limit shape for this measure can again be found from Theorem 2.4, and are shown for the first few m in Figure 2.2 (note that here again we have the VKLS curve at $m = 1$):

Corollary 2.9 (Limit shapes of symmetric minimal multicritical measures). *The rescaled profile $\psi_{\lambda,\theta}$ of a random partition under $\mathbb{P}_\theta^{s,m}$ converges in probability to*

$$\Omega^{s,m}(x) = \begin{cases} x + 2b - \frac{2}{\pi} \int_{-b}^x \chi(v) dv, & x \in [-b, b] \\ |x|, & |x| > b \end{cases} \quad (2.20)$$

where $\chi(x)$ satisfies

$$\int_0^{\chi(x)} \sin^{2m-1} \phi d\phi = \frac{(-1)^{m+1}}{2^{2m-1}} \binom{2m-1}{m} x, \quad x \in [-b, b]. \quad (2.21)$$

2.1.3 Plan of the chapter

The following Section 2.2 explains the connection with free fermions: first, we discuss the models of fermions on a line originally considered by Le Doussal, Majumdar and Schehr in [LDMS18], then we discuss models of fermions on a 1D lattice which correspond to Hermitian Schur measures, and finally present heuristic arguments for the equivalence of the edges in either case in a continuum limit in a critical scaling regime. In Section 2.3, we prove Theorems 2.2 and 2.4, by analysing the DPPs associated with the multicritical Schur measures (in several respects we follow the approach of [BOO00]), and justify the expressions for the minimal and symmetric minimal measures introduced in this section.

The final section discusses extensions of the multicritical measures which exhibit more general edge behaviour. Section 2.4.1 introduces multicritical measures on pairs of partitions (or simple cylindric partitions) using the periodic Schur processes of Borodin [Bor07] with same specialisations of Miwa times. We generalise a theorem of Betea and Bouttier [BB19] to the multicritical case, or equivalently generalise Theorem 2.2 to “positive temperature”. Section 2.4.2 introduces generalised multicritical measures, whose asymptotic edge distributions encode more general Fredholm determinant solutions of Painlevé II hierarchy equations found in [CCG19]. Section 2.4.3 presents work in progress, where we consider Hermitian Schur measures with “split Fermi seas”. We find new edge statistics for these models using a somewhat different approach.

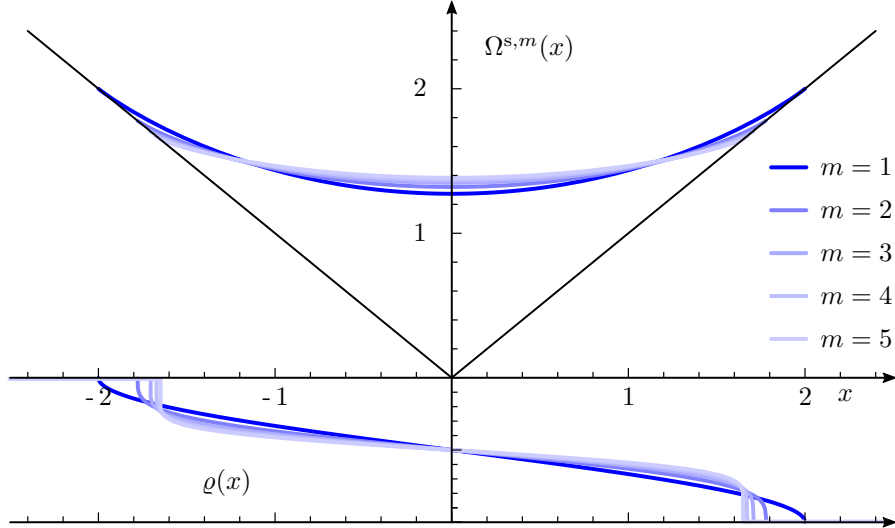


Figure 2.2 Limit curves of the symmetric minimal multicritical measures $\mathbb{P}^{s,m}$ for m from 1 to 5, and the limiting density profiles of the corresponding free fermion models.

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2.2 Fermions on lines and lattices, Hermitian Schur measures

2.2.1 Multicritical fermions on the real line

Flat trap potentials In [LDMS18], the authors considered the following model in the first quantisation formalism: N non-interacting fermions with positions $\mathbf{x}_1, \dots, \mathbf{x}_N$ and momenta $\mathbf{p}_1, \dots, \mathbf{p}_N$ on \mathbb{R} are subject to a flat trap Hamiltonian (written in dimensionless coordinates position space coordinates where $\hbar = 1$ and each particle mass is 1)

$$H = \sum_{i=1}^N -\frac{1}{2} \frac{d^2}{dx_i^2} + x_i^{2m}. \quad (2.22)$$

for integer $m \geq 1$; in the $m = 1$ this is an N particle quantum harmonic oscillator (and not considered flat). Since the potential confines the particles to a region around the origin, there is some Fermi energy E_F so there is an edge position $x_{\text{edge}} = E_F^{1/2m}$, and the arguments of Section 1.3.2 can be applied to show that as $N \rightarrow \infty$, the fluctuations in the position \mathbf{x}_{max} of the rightmost fermion around x_{edge} are driven by the TW-GUE distribution.

Switch to momentum space If instead we consider the consider the model in momentum space, the flat trap Hamiltonian is

$$H = \sum_{i=1}^N (-1)^m \frac{d^{2m}}{dp_i^{2m}} + \frac{1}{2} p_i^2. \quad (2.23)$$

There is an edge in momentum space too, with $p_{\text{edge}} = 2E_F^{1/2}$; considering the single particle Hamiltonian near the edge in coordinates $\tilde{p} = (p - p_{\text{edge}})/\kappa$, in a critical scaling regime $\kappa = p_{\text{edge}}^{-1/(2m+1)}$ we recover the edge Hamiltonian

$$H_{\text{edge}} = (-1)^m \frac{d^{2m}}{d\tilde{p}^{2m}} + \tilde{p} = p_{\text{edge}}^{-\frac{2m}{2m+1}} H + O\left(p_{\text{edge}}^{-\frac{2m}{2m+1}} (p - p_{\text{edge}})^2\right). \quad (2.24)$$

The square integrable eigenfunctions of this operator are given by the higher-order Airy functions defined at (2.6), as we have

$$H_{\text{edge}} \text{Ai}_{2m+1}(x+v) = \left((-1)^m \frac{d^{2m}}{dx^{2m}} + x\right) \text{Ai}_{2m+1}(x+v) = -v \text{Ai}_{2m+1}(x+v). \quad (2.25)$$

As in the position space case discussed in Section 1.3.2, we have an unbounded linear spectrum. In [LDMS18], the authors found the following asymptotic edge fluctuation result (written in our conventions²): *In a system of N fermions under the position space flat trap Hamiltonian (2.23), in the limit as $N \rightarrow \infty$, the law of the maximum momentum \mathbf{p}_{max} is*

$$\mathbb{P}\left[\frac{\mathbf{p}_{\text{max}} - p_{\text{edge}}}{p_{\text{edge}}^{-\frac{1}{2m+1}}} < s\right] = F_{2m+1}(s) := \det(1 - \mathcal{A}_{2m+1})_{L^2([s, \infty))}. \quad (2.26)$$

2.2.2 From lattice fermions to Hermitian Schur measures

Now let us introduce models of non-interacting fermion models on a 1D lattice, indexed (for later convenience) by the half integers $\mathbb{Z} + \frac{1}{2}$, which we will consider in the second quantisation formalism. As we did in Section 1.3.3, we define a Fock space \mathcal{F} with a basis indexed by subsets $S \subset \mathbb{Z} + \frac{1}{2}$, where $|S\rangle$ denotes the state with a particle on each site indexed $k \in S$, and use c_k^\dagger, c_k acting as fermionic creation and annihilation operators by (1.144) and satisfying the anticommutation relations (1.145).

A linear potential We start by placing the fermions a linear potential. Such a system is not generally physical, since a state cannot saturate; however, given an infinite number of particles, we can prepare them in the *domain wall* state $|S(\emptyset)\rangle$, the second quantised Hamiltonian

$$H_0 := \sum_k k :c_k^\dagger c_k: . \quad (2.27)$$

is physical, and has the domain wall state $|S(\emptyset)\rangle$ as a ground state. Each $|S\rangle$ is an eigenstate with energy $Q_1(S)$. The normal ordering $: \cdot :$ is the same one defined at (1.153), but at this point for the sake of physical intuition let us redefine it equivalently as the general operation with respect to the domain wall state

$$:\hat{\mathcal{O}} := \hat{\mathcal{O}} - \langle S(\emptyset) | \hat{\mathcal{O}} | S(\emptyset) \rangle. \quad (2.28)$$

We now modify the model by adding translation invariant kinetic hopping terms, shifting particles by $-r$ sites; the bosonic operators $a_r, a_r^\dagger := a_{-r}$ defined at (1.147) do

²The coordinates used in [LDMS18] have dimensions, with single particle Hamiltonian of the form $H = (-1)^m \hbar^{2m} g \frac{d^{2m}}{dp^{2m}} + \frac{1}{2M} p^2$; the coefficient κ is denoted p_N in their conventions and equal to $\hbar(\frac{Mg}{\hbar p_{\text{edge}}})^{1/(2m+1)}$ (to recover [LDMS18, Equations 2 and 5] exactly, replace m with n , M with m and p_{edge} with p_e).

exactly that. In particular, we choose a sequence of complex parameters $t = (t_1, t_2, \dots)$ with finite support and introduce the unitary operator

$$\mathcal{U} := e^{\sum_{r \geq 1} (t_r a_{-r} - \bar{t}_r a_r)}. \quad (2.29)$$

Then we define the modified Hamiltonian

$$H_t = \mathcal{U} H_0 \mathcal{U}^{-1}. \quad (2.30)$$

Using the commutation relations (1.148) and the additional relation

$$[H_0, a_{-r}] = r a_{-r} \quad (2.31)$$

which follows from (1.145), we find that the new Hamiltonian can be explicitly written

$$H_t = H_0 - \sum_{r \geq 1} r (\bar{t}_r a_r + t_r a_{-r}) + \sum_{r \geq 1} r^2 |t_r|^2. \quad (2.32)$$

In other words, H_t consists of a linear combination of the linear potential H_0 and of finite-range hopping operators (and a scalar term ensuring that the spectra of H_t and H_0 are equal). By (2.30), the ground state of H is given by

$$\mathcal{U}|\emptyset\rangle = e^{-\sum_{r \geq 1} r |t_r|^2 / 2} e^{\sum_{r \geq 1} t_r a_{-r}} |\emptyset\rangle. \quad (2.33)$$

Here, we obtain the right-hand side by performing a normal ordering of the operators a_r and a_{-r} (i.e. moving creation operators to the right by application of the Baker–Campbell–Hausdorff formula (1.162)), noting that $a_r|\emptyset\rangle = 0$ for $r \geq 1$.

Trapped particles again It is instructive to reinterpret this discussion in the language of quantum mechanical harmonic oscillators. In terms of the bosonic operators a_r , with the new addition of the *charge operator*

$$a_0 := \sum_k :c_k^\dagger c_k:, \quad a_0|S\rangle = Q_0(S)|S\rangle, \quad (2.34)$$

the fermionic linear potential becomes

$$H_0 = \sum_{r \geq 1} a_{-r} a_r + a_0^2 / 2. \quad (2.35)$$

We recognise that H_0 corresponds to a collection of single particle harmonic oscillators: considering the $N = 1$, $m = 1$ case of the first quantised Hamiltonian (2.23) in position space, Dirac’s famous “ladder operator” approach defines

$$a_\pm = x \pm \frac{1}{\sqrt{2}} \frac{d}{dx}, \quad (2.36)$$

which satisfy the bosonic commutation relation (1.148) with $a_+ := a_1$, $a_- := a_{-1}$, to write

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + x^2 = a_- a_+ + \frac{1}{\sqrt{2}}. \quad (2.37)$$

Up to the scalars, we identify each $a_{-r} a_r$ with a quantum harmonic oscillator at a level r . Then, from (2.36), the unitary operator \mathcal{U} corresponds to a translation in position space, momentum space, or a combination thereof. This creates the linear terms in the shifted Hamiltonian H_t . Finally, the translated ground state $\mathcal{U}|S(\emptyset)\rangle$ is nothing but a *coherent state*. These eigenstates in the quantum harmonic oscillator are particularly important in different areas of physics, both from a purely theoretical perspective (see e.g. [TAAGM95]) and for their applications in experimental quantum optics (see e.g. [GHFD11]).

Back to random partitions At this point we can relate our discussion to Schur measures, and specifically the setup of Section 1.3.3. In the terms used there, the coherent ground state $\mathcal{U}|S(\emptyset)\rangle$ is just $|\Phi\rangle = Z^{-1/2}\Gamma_-(t)|S(\emptyset)\rangle$ where $t' = \bar{t}$ and $|\Phi'\rangle = |\Phi\rangle$; by (1.174), it also decomposes over partition indexed states as

$$\mathcal{U}|S(\emptyset)\rangle = e^{-\sum_{r \geq 1} r|t_r|^{2/2}} \sum_{\lambda} s_{\lambda}[t]|S(\lambda)\rangle. \quad (2.38)$$

So, we recognise the Schur measure, and that the fermion positions $\mathbf{k}_1, \mathbf{k}_2, \dots$ form precisely the DPP described in Theorem 1.45 for $t' = \bar{t}$. In physical terms, the kernel $K(k, \ell)$ is just the *ground state propagator*

$$\langle c_k^{\dagger} c_{\ell} \rangle_{\text{g.s.}} := \langle \emptyset | \mathcal{U}^{\dagger} c_k^{\dagger} c_{\ell} \mathcal{U} | \emptyset \rangle = K(k, \ell); \quad (2.39)$$

the gap probability $\mathbb{P}(\mathbf{k}_{\max} < k)$ for the rightmost fermion is found from the ‘‘full counting statistics’’ function $\Upsilon(\alpha, k) = \langle \exp(\alpha \sum_k c_k^{\dagger} c_k) \rangle_{\text{g.s.}}$ in the $\alpha \rightarrow -\infty$ limit; we refer to e.g. [St  19, Section 1] for a more physical presentation.

By Theorem 1.45, the ground state propagator can be written in terms of the multivariate Bessel functions defined at (1.151). The same expression can equally be found from the Hamiltonian H_t by noting that (again following discrete versions of arguments in [St  19, Section 1]) if we define the linear combination of fermionic creation operators $\tilde{c}_{\ell}^{\dagger} = \sum_k J_{k+\ell}(t) c_k^{\dagger}$ in terms of $J_n(t) := J_n(t, \bar{t})$ for each half integer ℓ , we use a difference operator eigenvalue relation for the Bessel functions to show

$$\begin{aligned} [H_t, \tilde{c}_{\ell}^{\dagger}] &= \sum_k \left(- \sum_{r \geq 1} (t_r J_{k+\ell+r}(t) + \bar{t}_r J_{k+\ell-r}(t)) + k J_{k+\ell}(t) \right) c_k^{\dagger} \\ &= \sum_k -\ell J_{k+\ell}(t) c_k^{\dagger} = -\ell \tilde{c}_{\ell}^{\dagger}. \end{aligned} \quad (2.40)$$

So, acting on the ground state, $\tilde{c}_{\ell}^{\dagger}$ generates an eigenstate of H_t with energy $-\ell$. From the orthogonality of these states, at $E_F = 0$ (i.e. where the ground state contains only negative energy eigenstates) we have

$$\langle \tilde{c}_k^{\dagger} \tilde{c}_{\ell} \rangle_{\text{g.s.}} = \mathbf{1}_{k=\ell > 0}. \quad (2.41)$$

For instance by considering the inner product $\langle \tilde{c}_k^{\dagger} c_{\ell} \rangle$, we see that $c_k^{\dagger} = \sum_{\ell} J_{k+\ell}(t) \tilde{c}_{\ell}^{\dagger}$, and using this and (2.41) to evaluate the propagator we find precisely the expression

$$\langle c_k^{\dagger} c_{\ell} \rangle_{\text{g.s.}} = \sum_{i=0}^{\infty} J_{k+i+\frac{1}{2}}(t) J_{k+i+\frac{1}{2}}(t) = K(k, \ell). \quad (2.42)$$

2.2.3 Heuristic approach to the continuum limit

Now let us consider the fermion model corresponding to a multicritical Schur measures $\mathbb{P}(\lambda) = e^{-\theta^2 \sum_r r \gamma_r^2} s_{\lambda}[\theta \gamma]^2$, and consider the Hamiltonian

$$\begin{aligned} H_{\theta \gamma} &= \sum_{r \geq 1} \left[a_{-r} a_r - \theta r \gamma_r (a_r + a_{-r}) + \theta^2 \gamma_r^2 r^2 \right] + a_0^2 / 2 \\ &= \sum_k \left[k : c_k^{\dagger} c_k : - \sum_{r \geq 1} \theta r \gamma_r (c_k^{\dagger} c_{k+r} + c_k^{\dagger} c_{k-r}) \right] \end{aligned} \quad (2.43)$$

as θ grows large (here we will only use the second fermionic expression, and note that all of the terms can equivalently be written in normal order).

The bulk First, let us consider this limit at a macroscopic scale of θ , and more precisely a regime where $k := x\theta$ for a finite number x . In this regime we are interested in the *limiting density profile* in the ground state,

$$\varrho(x) = \lim_{\theta \rightarrow \infty} \langle c_{x\theta}^\dagger c_{x\theta} \rangle_{\text{g.s.}}; \quad (2.44)$$

so that in the new continuum coordinates, the probability to find a particle in $[x, x + dx]$ is $\varrho(x)dx$. One approach to this, used in [ADSV16, Section 1] and [BS21], involves a Fourier transform for the fermionic creation operator, with $\hat{c}^\dagger(\xi) = \sum_k e^{ik\xi} c_k^\dagger$; then, under the assumptions of a “local density approximation” (see e.g. [Sté19]) where δ, δ' are at a scale much smaller than the system size but much bigger than the typical gap between particles, the propagator at that scale only “sees” the potential as a fixed Fermi energy, which only limits the Fourier frequencies to $\xi \in [-\chi, \chi]$ for some χ , so we have

$$\langle c_{x\theta+\delta}^\dagger c_{x\theta+\delta'} \rangle_{\text{g.s.}} = \int_{-\chi}^{\chi} \frac{1}{2\pi} e^{i\xi(\delta-\delta')} d\xi = \frac{\sin \chi(\delta - \delta')}{\pi(\delta - \delta')}, \quad (2.45)$$

and in the limit as $\delta \rightarrow \delta'$, we find $\varrho(x) = \chi/\pi$.

Returning to the Hamiltonian, under the above assumptions for positions $x\theta + \delta$ we can remove the sum and look just at a local Hamiltonian H_x at $x\theta$ rescaled by $1/\theta$. In terms of the Fourier transform, the Hamiltonian is diagonal, with $H_x = \int_{-\chi}^{\chi} \epsilon(\xi) c^\dagger(\xi) c(\xi) d\xi$, and for H_x we have

$$x c_{x\theta}^\dagger c_{x\theta} - \sum_{r \geq 1} r \gamma_r (c_{x\theta}^\dagger c_{x\theta+r} + c_{x\theta}^\dagger c_{x\theta-r}) = \int_{-\chi}^{\chi} (x - \sum_{r \geq 1} 2r \gamma_r \cos r\xi) c^\dagger(\xi) c(\xi) d\xi \quad (2.46)$$

Evaluating this on the ground state, the boundary terms in the large θ limit give

$$x - \sum_{r \geq 1} 2r \gamma_r \cos r\chi = 0. \quad (2.47)$$

For a given sequence of coefficients γ , (2.47) gives an explicit formula for that limiting density, which already appeared in the statement of Theorem 2.4. We will revisit this more explicitly and in terms of the corresponding limit for random partitions in Sections 2.3.2 and 2.4.3. One immediate consequence is that (2.47) only has solutions for

$$2 \sum_{r \geq 1} (-1)^r r \gamma_r \leq x \leq 2 \sum_{r \geq 1} r \gamma_r \quad (2.48)$$

or concisely for $x \in [-\tilde{b}, \tilde{b}]$; by (2.1) it has solutions everywhere in that interval, and we have a left edge at $k = -\tilde{b}\theta$ where the density tends to 1 and a right edge at $k = \tilde{b}\theta$ where the density vanishes.

The edge Now let us consider the fluctuations around $k = b\theta$ in a microscopic critical scaling regime, at the level of the Hamiltonian. In particular, we identify $k := b\theta + x(d\theta)^{1/(2m+1)}$ where x is a finite number as a critical regime, as we recover a physical limiting Hamiltonian there (b, d are the coefficients given in Definition 2.1). Then, writing $\tilde{c}_x^\dagger := c_k^\dagger$, we can treat the kinetic hopping terms in this regime just by Taylor expanding, with

$$c_{k+r} = \tilde{c}_{x+r(d\theta)^{-1/(2m+1)}}^\dagger = \sum_{n=0}^{\infty} r^n (d\theta)^{-\frac{n}{2m+1}} \frac{d^n}{dx^n} \tilde{c}_x^\dagger. \quad (2.49)$$

In the expanded Hamiltonian, all of the odd derivatives cancel each other out, and by the multicriticality conditions (2.2) the first $m - 1$ even ones do too; hence, recalling the expressions for b, d , we find

$$H_{\theta\gamma} = (d\theta)^{\frac{1}{2m+1}} \int_{\mathbb{R}} :c_x^\dagger \left[x + (-1)^m \frac{d^{2m}}{dx^{2m}} \right] c(x) : dx + O(\theta^{-\frac{2m+2}{2m+1}}), \quad (2.50)$$

As $\theta \rightarrow \infty$, we have $(d\theta)^{-\frac{1}{2m+1}} H_{\theta\gamma} \rightarrow \hat{H}_{\text{edge}}$ where \hat{H}_{edge} is a second quantised version of the Hamiltonian recovered at the momentum space edge of a flat trap in [LDMS18] (the $Z + \frac{1}{2}$ indexed positions in the lattice model asymptotically coincide with momenta in continuum model). This heuristic is a meaningful one: one would expect as a consequence that the multivariate Bessel function wavefunctions of the lattice model coincide with the higher-order Airy functions of the flat trap edge potential in this asymptotic regime, and that the kernels (or ground state propagators) would be asymptotically equivalent in turn, which we now show to be true directly.

2.3 Proofs: Asymptotic analysis of multicritical Schur measures

2.3.1 The higher-order Bessel kernel in various asymptotic regimes

Theorems 2.2 and 2.4 are proven starting from the immediate consequence of applying Okounkov's DPP formulation of the Schur measure stated in Theorem 1.45 to our multicritical ones.

Proposition 2.10 (Kernel of a multicritical measure). *Let λ be a random partition under an order m multicritical measure $\mathbb{P}_\theta^m(\lambda) = e^{-\theta^2 \sum_r r \gamma_r^2} s_\lambda[\theta\gamma]^2$. Then, its fermion configuration $S(\lambda) \subset \mathbb{Z} + \frac{1}{2}$ is a DPP with kernel*

$$\mathcal{J}_\theta^m(k, \ell) = \frac{1}{(2\pi i)^2} \oint \frac{\exp[\theta \sum_r \gamma_r (z^r - z^{-r})]}{\exp[\theta \sum_r \gamma_r (w^r - w^{-r})]} \frac{1}{z - w} \frac{dz dw}{z^{k+\frac{1}{2}} w^{-\ell+\frac{1}{2}}} \quad (2.51)$$

where the integral in w is taken clockwise along a contour c_- enclosing the origin and the integral in z is taken counter-clockwise over a contour c_+ enclosing c_- .

We call $\mathcal{J}_\theta^m(k, \ell)$ the higher-order Bessel kernel, and note that it can equally be written as an infinite sum of multivariate Bessel functions $\sum_i J_{k+i+\frac{1}{2}}(\theta\gamma) J_{\ell+i+\frac{1}{2}}(\theta\gamma)$. The w and z contours are chosen to encircle poles at ∞ and 0 respectively, and to satisfy the condition $|w| < |z|$ for the generating function of a kernel in Theorem 1.45. The statistics whose large θ limits we are interested in are found from it directly: on the one hand, the distribution of λ is equal to a gap probability on $S(\lambda)$,

$$\mathbb{P}(\lambda_1 < k + \frac{1}{2}) = \det(1 - \mathcal{J}_\theta^m)_{\ell^2([k, \infty))}. \quad (2.52)$$

On the other hand, recalling (1.10), we can see that the one point function

$$\mathbb{P}(k \in S(\lambda)) = \rho_1(k) = \mathcal{J}_\theta^m(k, k) \quad (2.53)$$

gives the expectation of the rescaled profile of λ at integer points, with

$$\mathbb{E}(\psi_{\lambda, \theta}(x)) = x + \frac{2}{\theta} \sum_{k=x\theta+\frac{1}{2}}^{\infty} \mathcal{J}_\theta^m(k, k), \quad x \in \frac{1}{\theta}\mathbb{Z}. \quad (2.54)$$

So, our main task is to find the large θ limit of the kernel in different regimes.

Action notation Let us introduce some useful ways to rewrite the kernel for a given sequence of parameters γ . First, the *potential* (which is always a polynomial in the cases we consider) is

$$V(z) = \sum_{r \geq 1} \gamma_r z^r, \quad (2.55)$$

then the *action* is defined as

$$S(z; x) = \sum_{r \geq 1} \gamma_r z^r - \sum_{r \geq 1} \gamma_r z^{-r} - x \log z = V(z) - V(z^{-1}) - x \log z. \quad (2.56)$$

We will be interested in the large θ behaviour of $\mathcal{J}_\theta^m(k, \ell)$ at points $k = x\theta + k'$, $\ell = x\theta + \ell'$ where x is finite and k', ℓ' are sublinear in θ . Then, we have

$$\mathcal{J}_\theta^m(k, \ell) = \frac{1}{(2\pi i)^2} \iint e^{\theta[S(z;x) - S(w;x)]} \frac{dzdw}{z^{k'+\frac{1}{2}} w^{-\ell'+\frac{1}{2}} (z-w)}. \quad (2.57)$$

The large θ limit is dominated by the saddle points of $S(z; x)$, where its derivative vanishes, as detailed in [FS09, Chapter VIII] (we call $S(z; x)$ an action by analogy: if we interpret \mathcal{J}_θ^m as a Feynman path integral it takes the place of an action and the large θ limit would correspond to a classical limit and the saddle point calculation recovers the least action principle).

Multicritical actions and minimal measures It is useful to write the multicriticality conditions (2.2) in terms of the action, where they are just

$$\frac{d^{2p-1}}{dz^{2p-1}} S(z; b)|_{z=1} = 0, \quad p = 2, \dots, m. \quad (2.58)$$

The edge and fluctuation coefficients given in (2.3) are just solutions to

$$\begin{aligned} \frac{d}{dz} S(z; b)|_{z=1} &= 0, & \frac{d}{dz} S(z; \tilde{b})|_{z=-1} &= 0, \\ \frac{d^{2m+1}}{dz^{2m+1}} S(z; b)|_{z=1} &= (-1)^{m+1} (2m)! d. \end{aligned} \quad (2.59)$$

Making a brief digression, with this expression it is easy to show that

Proposition 2.11 (Multicriticality of the minimal measures). *The measures $\mathbb{P}_\theta^{a,m}$ and $\mathbb{P}_\theta^{s,m}$ are order m multicritical.*

Proof. From (2.15), the action associated with the specialisation of $\mathbb{P}_\theta^{a,m}$ at b satisfies

$$\frac{d}{d \log z} S(z; b) = (-1)^{m+1} \binom{2m}{m-1}^{-1} (z^{1/2} - z^{-1/2})^{2m} \quad (2.60)$$

so we see immediately that it satisfies (2.58). Similarly, from (2.18), the action associated with $\mathbb{P}_\theta^{s,m}$ satisfies

$$\frac{d^2}{d \log z^2} S(z; b) = (-1)^{m+1} \binom{2m-1}{m-1}^{-1} (z - z^{-1})^{2m-1} \quad (2.61)$$

so it satisfies (2.58). ■

A recipe for the saddle point analysis Our approach to the asymptotics of $\mathcal{J}_\theta^m(k, \ell)$ adapts a procedure of Okounkov and coauthors, detailed in the lecture notes [Oko02].

We start by choosing contours $c_+ : |z| = 1 + \delta$ and $c_- : |w| = 1 - \delta$ for a small $\delta > 0$ for the integral (2.51). If $V(z)$ has degree D , $S(z; x)$ has $2D$ saddle points for each x ; from the symmetries of the kernel, if z^* is a saddle point, so are \bar{z}^* and $1/z^*$; each integral is approximated by the contribution near a saddle point which is a maximum of $\operatorname{Re}(S(z; x))$ on the contour.

First let us consider how the analysis of the one point function $\rho_1(k)$ works at a “macroscopic” scale $k = \lfloor x\theta \rfloor$, with reference to the heuristics for the limiting density $\varrho(x) := \lim_{\theta \rightarrow \infty} \rho_1(x\theta)$ we described in Section 2.2.3, and look at the locations of the saddle points, that is the solutions of

$$\frac{d}{d \log z} S(z; x) = 0 = \sum_{r \geq 1} r \gamma_r (z^r - z^{-r}) - x. \quad (2.62)$$

There are three regions of the line:

- (i) *The empty region:* For $x > b$, $S_n(z, x)$ has no saddle points on the unit circle. We find that $\varrho(x) = 0$ by deforming c_+ outwards and c_- inwards, and seeing that the double integral on these contours decays exponentially as $\theta \rightarrow \infty$.
- (ii) *The bulk:* For $-\tilde{b} < x < b$, there are exactly two saddle points on the unit circle, $z_\pm^* = e^{\pm i\chi}$ where $\sum_r r \gamma_r \cos r\chi = x$ and $0 < \chi < \pi$ (it is condition (2.1) that ensures there are no more than two). Then, we can deform both c_+ and c_- through both points z_\pm^* , by pulling c_- outwards over c_+ over the arc of the unit circle from $-\chi$ to χ ; the only contribution in the $\theta \rightarrow \infty$ limit is from integrating the pole at $z = 0$ along that arc, which gives $\varrho = \chi/\pi$.
- (iii) *The frozen region:* For $x < -\tilde{b}$, $S_n(z, x)$ again has no saddle points on the unit circle. In this case, we can deform c_+ inwards and c_- outwards so that the double integral decays exponentially. In doing so, the contours are exchanged completely, and integrating the $z = w$ pole over the unit circle shows that $\varrho(x) = 1$.

This is a very direct generalisation of the asymptotic analysis of the Poissonised Plancherel measure in [BOO00]. On the interfaces between these regions, however, the generalisation is more subtle.

At the right edge $x = b$ where the density ϱ vanishes, $2m$ of the saddle points coalesce at $z^* = 1$, and it is this phenomenon that gives rise to new critical exponents for edge fluctuations. Approximating the kernel on a pair of points near $b\theta$ by the contribution to the double contour integral of this order $2m$ saddle point, we find non-trivial correlations if the points are displaced from $b\theta$ at a new critical scale of $(d\theta)^{1/(2m+1)}$, generalising the $\theta^{1/3}$ critical edge scaling of the Poissonised Plancherel measure (and giving a rather direct meaning to the term “multicritical”). On the left edge $x = -\tilde{b}$, $2\tilde{m}$ saddle points coalesce at $z = -1$ where \tilde{m} is the order of multicriticality of the conjugate measure by Proposition 2.5.

2.3.2 Limit shapes

To prove Theorem 2.4, we start with a very general limit shape result:

Lemma 2.12 (Convergence to a limit shape for a Hermitian Schur measure). *Let λ be a random partition under any Hermitian Schur measure with a single positive parameter θ , such that $\mathbb{P}_\theta^\gamma(\lambda) = e^{-\theta^2 \sum_r r |\gamma|^2} s_\lambda[\theta\gamma] s_\lambda[\theta\bar{\gamma}]$ for some non-zero sequence of complex*

coefficients γ . Then, if there exists a curve Ω such that $\Omega(x) - |x|$ has finite support and for all x $\mathbb{E}(\psi_{\lambda,\theta}(x)) \rightarrow \Omega(x)$ as $\theta \rightarrow \infty$, we also have

$$\sup_x |\psi_{\lambda,\theta}(x) - \Omega(x)| \xrightarrow{P} 0. \quad (2.63)$$

Proof. Let us denote the kernel of the DPP formed by $S(\lambda)$ by K_θ , and put

$$\mathbf{N}(n) = \#\{k \in S(\lambda) | k > n\}. \quad (2.64)$$

The expectation and variance of $\mathbf{N}(n)$ have simple expressions in terms of K_θ , with

$$\mathbb{E}(\mathbf{N}(n)) = \text{tr}_{(n,\infty)} K_\theta, \quad \text{Var}(\mathbf{N}(n)) = \text{tr}_{(n,\infty)} (K_\theta - K_\theta^2). \quad (2.65)$$

Since K_θ is Hermitian, $\text{tr}_{(n,\infty)} K_\theta^2 \geq 0$ and hence

$$\text{Var}(\mathbf{N}(n)) \leq \mathbb{E}(\mathbf{N}(n)). \quad (2.66)$$

Now, considering a regime where $n = x\theta$, we set $\tilde{\mathbf{N}}(x\theta) = \mathbf{N}(x\theta)/\theta$, so

$$\text{Var}(\mathbf{N}(x\theta)) = \theta^{-2} \text{Var}(\tilde{\mathbf{N}}(x\theta)) \leq \theta^{-1} \mathbb{E}(\tilde{\mathbf{N}}(x\theta)). \quad (2.67)$$

Suppose that there exists a fixed function Ω such that we have a limit

$$\Omega(x) = \lim_{\theta \rightarrow \infty} \mathbb{E}(\psi_{\lambda,\theta}(x)) = x + 2 \lim_{\theta \rightarrow \infty} \mathbb{E}(\tilde{\mathbf{N}}(x\theta)) \quad (2.68)$$

(we recall the expression (1.10) for the profile). Then,

$$\lim_{\theta \rightarrow \infty} \text{Var}(\psi_{\lambda,\theta}(x)) \leq \lim_{\theta \rightarrow \infty} \theta^{-1} 2\mathbb{E}(\tilde{\mathbf{N}}(x\theta)) = 0 \quad (2.69)$$

and $\psi_{\lambda,\theta}(x)$ converges in probability to its expectation for each x .

We then have the convergence of the supremum norm as a consequence of the fact that $\psi_{\lambda,\theta}$ is 1-Lipschitz. Let $I \subset \mathbb{R}$ be a bounded interval; then, for each $\varepsilon > 0$, where I' is the set $I \cap \varepsilon\mathbb{Z}$, we have

$$\mathbb{P} \left(\sup_{x \in I} |\psi_{\lambda,\theta}(x) - \Omega(x)| > \varepsilon \right) \leq \mathbb{P} \left(\sup_{x \in I'} |\psi_{\lambda,\theta}(x) - \Omega(x)| > \frac{\varepsilon}{2} \right). \quad (2.70)$$

On right hand side, the supremum is over a finite set, so the convergence to zero at each x implies the convergence of the supremum to zero, in turn implying that the supremum norm over all I converges to zero in probability, and in particular we have convergence over the support of $\Omega(x) - |x|$. To extend to all \mathbb{R} , we reapply the 1-Lipschitz property: let $a > 0$ be finite and let $[-a, a]$ included the support of $\Omega(x) - |x|$, then

$$\sup_{x \in (a,\infty)} |\psi_{\lambda,\theta}(x) - \Omega(x)| = \sup_{x \in (a,\infty)} |\psi_{\lambda,\theta}(x) - |x|| \leq \psi_{\lambda,\theta}(a) - a \quad (2.71)$$

and the final term converges to zero in probability, completing the proof. \blacktriangleleft

With that, we need only find the limiting expectation of the rescaled profile under a multicritical measure. First, we show that

Lemma 2.13 (Limiting bulk kernel and fermion density of a multicritical measure). *For finite integers s, t , as $\theta \rightarrow \infty$ we have*

$$\mathcal{J}_\theta^m(\lfloor x\theta \rfloor + s - \frac{1}{2}, \lfloor x\theta \rfloor + t - \frac{1}{2}) \rightarrow \begin{cases} \mathbf{1}_{s=t}, & x < -\tilde{b} \\ \frac{\sin \chi(x)(s-t)}{\pi(s-t)}, & x \in [-\tilde{b}, b] \\ 0, & x > b \end{cases} \quad (2.72)$$

where $\chi(x)$ is the unique non-negative solution of $\sum_r 2r\gamma_r \cos r\chi(x) = x$, uniformly for x in compact subsets of \mathbb{R} and s, t in compacts of \mathbb{Z} . If $\boldsymbol{\lambda}$ is a random partition under $\mathbb{P}_\theta^m(\boldsymbol{\lambda}) = e^{-\theta^2 \sum_r r\gamma_r^2 s_\lambda [\theta\gamma]^2}$, as $\theta \rightarrow \infty$

$$\mathbb{P}(\lfloor x\theta \rfloor - \frac{1}{2} \in S(\boldsymbol{\lambda})) = \rho_1(\lfloor x\theta \rfloor - \frac{1}{2}) \rightarrow \varrho(x) = \begin{cases} 1, & x < -\tilde{b} \\ \frac{\chi(x)}{\pi}, & x \in [-\tilde{b}, b] \\ 0, & x > b. \end{cases} \quad (2.73)$$

We include the more general statement for the limiting kernel because it is the universal aspect of the asymptotic bulk behaviour: while the limiting density profile ϱ is model dependant (we might consider this a “scaling limit”), the *discrete sine kernel* on the right of (2.72) for points a finite distance apart in the bulk (a “local limit” to some extent) is universal, and even more universal than the asymptotic edge behaviour of Theorem 2.2 since it does not depend on the order of multicriticality m . An analogous continuous sine kernel appears as a local limit in random matrix theory (see Section 3.2), and this can be seen as the trace of universality for free fermions in the bulk of systems.

Proof. The expression for the limiting density follows directly from the $s \rightarrow t = 0$ limit of the limiting kernel, so we only need to find

$$\mathcal{J}_\theta^m(\lfloor x\theta \rfloor + s - \frac{1}{2}, \lfloor x\theta \rfloor + t - \frac{1}{2}) = \frac{1}{(2\pi i)^2} \iint_{c_+, c_-} \frac{e^{\theta[S(z;x) - S(w;x)]} dw dz}{z^s w^t (z - w)} \quad (2.74)$$

as $\theta \rightarrow \infty$ in each of the three “regions” corresponding to ranges for x previously mentioned. Starting from contours c_+ for z passing just outside the unit circle and c_- for w passing just inside it, we deform them to some

$$c'_\pm : R_\pm e^{i\phi}, \quad \phi \in [-\pi, \pi] \quad (2.75)$$

where each $R_\pm := R_\pm(\phi)$ may depend on the angle ϕ but is everywhere close to 1. We will look at

$$\begin{aligned} \operatorname{Re}[S(z;x) - S(w;x)] \Big|_{\substack{z=R_+ e^{i\phi_+} \\ w=R_- e^{i\phi_-}}} &= (R_+ - 1)D(\phi_+; x) - (R_- - 1)D(\phi_-; x) \\ &+ O\left((R_+ - 1)^2 + (R_- - 1)^2\right) \end{aligned} \quad (2.76)$$

where

$$D(\phi; x) := \frac{d}{d \log z} S(z; x) \Big|_{z=e^{i\phi}} = \sum_r r\gamma_r (e^{ir\phi} - e^{-ir\phi}) - x. \quad (2.77)$$

From the conditions of Definition 2.1, for all $\phi \in [-\pi, \pi]$ we have

$$-\tilde{b} - x \leq D(\phi; x) \leq b - x. \quad (2.78)$$

(i) The empty region For $x > b$, we have $D(\phi; x) < 0$ for all $\phi \in [-\pi, \pi]$. Setting $R_+ > 1$ and $R_- < 1$ for all ϕ in c'_\pm as defined in (2.75) we have, for $z \in c'_+$ and $w \in c'_-$,

$$\operatorname{Re}[S(z; x) - S(w; x)] < 0 \quad (2.79)$$

for R_\pm sufficiently small (note that the contours do not need to pass through saddle points to find the required decay). In deforming c_\pm to c'_\pm the contours do not cross one another, so there is no $z = w$ pole to consider; hence for all finite s, t (and indeed for all $s, t = o(\theta)$) we have exponential decay of the kernel which in turn implies dominated convergence, so that

$$\lim_{\theta \rightarrow \infty} \mathcal{J}_\theta^m(x\theta + s, x\theta + t) = \frac{1}{(2\pi i)^2} \lim_{\theta \rightarrow \infty} \iint_{c'_+, c'_-} \frac{e^{\theta[S(z; x) - S(w; x)]} dw dz}{z^{s+\frac{1}{2}} w^{t+\frac{1}{2}} (z - w)} = 0. \quad (2.80)$$

Similarly, $\varrho(x) = 0$ for all $x > b$.

(ii) The bulk For $x \in (-\tilde{b}, b)$, there is an angle $\chi \in (0, \pi)$ such that

$$D(\chi; x) = 2 \sum_r r \gamma_r \cos r\chi - x = 0, \quad (2.81)$$

and $z_\pm^* = e^{\pm i\chi}$ are saddle points $S(z; x)$. By the final condition of (2.1), χ solving (2.81) is unique; from the same condition, we have

$$D(\phi; x) \begin{cases} > 0, & |\phi| < \chi \\ < 0, & |\phi| > \chi. \end{cases} \quad (2.82)$$

Hence, c'_\pm are proper saddle point contours on which $\operatorname{Re}[S(z; x) - S(w; x)] \leq 0$ is maximal and equal to 0 at $z = w = e^{\pm i\chi}$ if we set, respectively,

$$R_+ = R_+(\phi) \begin{cases} < 1, & |\phi| < \chi \\ > 1, & |\phi| > \chi \end{cases}; \quad R_- = R_-(\phi) \begin{cases} > 1, & |\phi| < \chi \\ < 1, & |\phi| > \chi \end{cases} \quad (2.83)$$

sufficiently close to 1. Deforming each of c_\pm to c'_\pm involves pulling them across one another either side of unit circle along the arc $c_{2\chi} : z = e^{i\phi}$, $\phi \in [-\chi, \chi]$. From the exchange, the integral in z picks up a residue of 1 from the $z - w$ pole for all $w = e^{i\phi}$ along $c_{2\chi}$, and we have

$$\mathcal{J}_\theta^m(x\theta + s, x\theta + t) = \frac{1}{2\pi i} \int_{c_{2\chi}} \frac{dw}{w^{s-t+1}} + \frac{1}{(2\pi i)^2} \iint_{c'_+, c'_-} \frac{e^{\theta(S(z; x) - S(w; x))} dw dz}{z^{s+\frac{1}{2}} w^{t+\frac{1}{2}} (z - w)} \quad (2.84)$$

As this is a saddle point approximation we can easily estimate the rate of decay: for all finite s, t , the integral on c'_\pm is $O(\theta^{-1/2})$, since a change of variables to $z = z_\pm^* + i\theta^{-1/2}\zeta$ and $w = z_\pm^* + i\theta^{-1/2}\omega$ shows that, in terms of $f(\zeta, \omega) = (\zeta^2 - \omega^2)S''(z_\pm^*; x)/2$ this integral is equal to

$$\frac{\theta^{-\frac{1}{2}}}{(2\pi)^2} \iint_{-\theta^{\frac{1}{2}}\pi}^{\theta^{\frac{1}{2}}\pi} \frac{e^{\operatorname{Re}f(\zeta, \omega)} \sin(\operatorname{Im}f(\zeta, \omega))}{(z_+^*)^{s+1} (z_-^*)^{t+1}} \frac{d\zeta d\omega}{\zeta - \omega} + O(e^{-\theta}). \quad (2.85)$$

This is sufficient to see that only the integral on $c_{2\chi}$ contributes to the limit, to give

$$\lim_{\theta \rightarrow \infty} \mathcal{J}_\theta^m(x\theta + s, x\theta + t) = \frac{1}{2\pi} \int_{-\chi}^{\chi} e^{-i\phi(s-t)} d\phi = \frac{\sin \chi(s-t)}{\pi(s-t)} \quad (2.86)$$

It follows that $\varrho(x) = \chi/\pi$ where $\chi := \chi(x)$ is the non negative solution of (2.81) for $x \in [\tilde{b}, b]$.

(iii) **The frozen region** For $x < -\tilde{b}$, we have $D(\phi; x) > 0$ for all ϕ . Hence, from (2.76), by setting $R_+ < 1$ and $R_- > 1$ sufficiently close to 1 for all ϕ , we have $\text{Re}[S(z; x) - S(w; x)] < 0$ for z on c'_+ , w on c'_- . Now deforming c_\pm to c'_\pm involves passing them across one another along the whole unit circle $c_1 : |z| = 1$. We have

$$\mathcal{J}_\theta^m(x\theta + s, x\theta + t) = \frac{1}{2\pi i} \oint_{c_1} \frac{dw}{w^{s-t+1}} + \frac{1}{(2\pi i)^2} \iint_{c'_+, c'_-} \frac{e^{\theta(S(z; x) - S(w; x))} dw dz}{z^{s+\frac{1}{2}} w^{t+\frac{1}{2}} (z-w)}. \quad (2.87)$$

The integral on c'_\pm decays to zero exponentially fast as $\theta \rightarrow \infty$, and the residue on c_1 gives

$$\lim_{\theta \rightarrow \infty} \mathcal{J}_\theta^m(x\theta + s, x\theta + t) = \mathbf{1}_{s=t}. \quad (2.88)$$

It follows that $\varrho(x) = 1$ for $x < -\tilde{b}$.

Putting the three regions together, the proof is complete. \blacktriangleleft

With these ingredients we can finally prove the limit shape theorem.

Proof of Theorem 2.4. By Lemma 2.12, it is sufficient to find the limit of the expectation for $\mathbb{E}(\psi_{\lambda, \theta})$ to have convergence in probability; by (2.54) this is

$$\Omega(x) := \lim_{\theta \rightarrow \infty} \mathbb{E}(\psi_{\lambda, \theta}(x)) = x + 2 \int_x^\infty \varrho(x') dx' \quad (2.89)$$

in terms of the limiting fermion density ϱ given in the previous Lemma 2.13. Since $\Omega(b) = b$ and $\Omega(-\tilde{b}) = \tilde{b}$, we can write this as the finite integral

$$\Omega(x) = \begin{cases} x + 2\tilde{b} + \frac{2}{\pi} \int_{-\tilde{b}}^x \chi(v) dv, & x \in [-\tilde{b}, b] \\ |x|, & x > b \text{ and } x < -\tilde{b} \end{cases} \quad (2.90)$$

as required.

Now consider the vanishing of $\varrho(x)$ as $x \rightarrow b$. Noting that $\chi(b) = 0$, we develop $\chi(b - \varepsilon)$ around zero when $\varepsilon > 0$ is small. Expanding (2.81) for χ small and applying the multicriticality condition (2.2) we find

$$b - d\chi^{2m} + O(\chi^{2m+2}) = b - \varepsilon. \quad (2.91)$$

So $\chi(b - \varepsilon) \sim (\varepsilon/d)^{1/2m}$ as $\varepsilon \rightarrow 0$, and as $x \rightarrow b$ we have

$$\chi(x) \sim \left(\frac{b-x}{d} \right)^{\frac{1}{2m}}. \quad (2.92)$$

Then, from (2.90), we recover the edge vanishing behaviour (2.12) as required. \blacksquare

2.3.3 Asymptotic edge fluctuations

We turn our attention proving Theorem 2.2. Starting again with the limiting kernel, now in the critical scaling regime near the right edge, we show that

Lemma 2.14 (Convergence to the higher-order Airy kernel on the edge). *As $\theta \rightarrow \infty$, we have*

$$\begin{aligned} & (d\theta)^{\frac{1}{2m+1}} \mathcal{J}_\theta^m \left(\lfloor b\theta + x(d\theta)^{\frac{1}{2m+1}} \rfloor - \frac{1}{2}, \lfloor b\theta + y(d\theta)^{\frac{1}{2m+1}} \rfloor - \frac{1}{2} \right) \\ & \rightarrow \mathcal{A}_{2m+1}(x, y) = \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}-1} \int_{i\mathbb{R}+1} \frac{\exp \left[(-1)^{m+1} \frac{\zeta^{2m+1}}{2m+1} - x\zeta \right] d\zeta d\omega}{\exp \left[(-1)^{m+1} \frac{\omega^{2n+1}}{2n+1} - y\omega \right] \zeta - \omega} \end{aligned} \quad (2.93)$$

uniformly for x, y in compact subsets of \mathbb{R} .

Here we justify the critical scaling regime directly by the proof, but we refer to Section 2.4.2 for an informal derivation of a similar scaling regime.

Proof. In terms of the action, the kernel in the edge regime is

$$\begin{aligned} \mathcal{J}_\theta^m(\lfloor b\theta + x(d\theta)^{\frac{1}{2m+1}} \rfloor - \frac{1}{2}, \lfloor b\theta + y(d\theta)^{\frac{1}{2m+1}} \rfloor - \frac{1}{2}) \\ = \frac{1}{(2\pi i)^2} \oint_{c_+, c_-} \frac{\exp[\theta S(z; b) - x(d\theta)^{\frac{1}{2m+1}} \log z]}{\exp[\theta S(w; b) - y(d\theta)^{\frac{1}{2m+1}} \log w]} [1 + o(1)] \frac{dw dz}{z - w} \end{aligned} \quad (2.94)$$

where the error term accounts for difference between the continuous coordinates and their integer parts. The action $S(z; b)$ has an order $2m$ saddle point at $z = 1$. Let us take the integral over contours which only approach this point as θ tends to infinity³, with

$$c_+ : |z| = \exp[(d\theta)^{-\frac{1}{2m+1}}], \quad c_- : |w| = \exp[-(d\theta)^{-\frac{1}{2m+1}}]. \quad (2.95)$$

Note that these contours do not cross, so we do not encounter the $z = w$ pole.

As before, let us parametrise c_+ by $\phi \in [-\pi, \pi]$. Then we have

$$\operatorname{Re}(S(e^{(d\theta)^{-\frac{1}{2m+1}}} e^{i\phi}; b)) = 2 \sum_r r \gamma_r (\cos r\phi - 1) (d\theta)^{-\frac{1}{2m+1}} + O(\theta^{-\frac{3}{2m+1}}), \quad (2.96)$$

which is maximal at $\phi = 0$; since the real part of the action is also minimal on c_-^* where it intersects the positive real axis, the $z = 1$ saddle point indeed dominates the integral on these contours. At large θ the integral is dominated by the contribution from a region around this point. Let us show that the contribution from the rest of the contour decays to zero.

On the tails Noting that by the condition (2.1), (2.96) has a unique maximum equal to 0 at $\phi = 0$ and that the leading order term in ϕ is of order $2m$, there is a $C > 1$ such that

$$2 \sum_r r \gamma_r (\cos r\phi - 1) \leq -\frac{\phi^{2m}}{C} \quad \text{for all } \phi \in [-\pi, \pi] \quad (2.97)$$

Then we fix a number $\varepsilon \in (0, \frac{1}{(2m+1)(2m+2)})$, and define the *central region* interval I and complementary *tails* interval I_t as

$$I = \left[-\pi (d\theta)^{-\frac{1}{2m+1} + \varepsilon}, \pi (d\theta)^{-\frac{1}{2m+1} + \varepsilon}\right], \quad I_t = [-\pi, \pi] \setminus I. \quad (2.98)$$

By (2.96), we see that for any points on the tails $z = e^{(d\theta)^{-\frac{1}{2m+1}}} e^{i\phi_t}$, $w = e^{(d\theta)^{-\frac{1}{2m+1}}} e^{i\phi'_t}$ with $\phi_t, \phi'_t \in I_t$, we have a uniform bound on the integrand of the kernel, with

$$e^{\theta(S(z;b) - S(w;b))} = O\left(e^{-(\theta)^{2m\varepsilon}/C}\right). \quad (2.99)$$

So, the limit is dominated by the contribution from the central region I .

³In this choice of contour our approach differs from the one presented in [Oko02] even at $m = 1$; rather, we use the convention of [BB19]. The contours in [Oko02] pass through 1 and leave at an angle of $\pi/3$, which gives an integral expression for the Airy kernel which converges faster than ours. It is possible to pick analogous contours for $m > 1$ at angles of $m\pi/(2m+1)$, and to recover the higher-order Airy function of [CCG19]. Our choice has the same contours in the limit for each m , recovers the integral expression of [LDMS18] and moreover can be directly be applied to cylindrical partition analogues.

In the central region Let us reparametrise the contours c_{\pm} restricted to the central region I_c by the change of variables

$$\begin{aligned} z &= \exp[\zeta(d\theta)^{-\frac{1}{2m+1}}], \quad w = \exp[\omega(d\theta)^{-\frac{1}{2m+1}}], \quad \zeta \in i[-\pi(d\theta)^{\varepsilon}, \pi(d\theta)^{\varepsilon}] + 1, \\ &\omega \in i[-\pi(d\theta)^{\varepsilon}, \pi(d\theta)^{\varepsilon}] - 1. \end{aligned} \quad (2.100)$$

Then, recalling the multicriticality conditions (2.58) and the expressions (2.59) for b, d , the action is uniformly approximated on I by

$$\begin{aligned} S(e^{\zeta(d\theta)^{-\frac{1}{2m+1}}}; b) &= \sum_{n=0}^{\infty} \frac{1}{n!(d\theta)^{\frac{n}{2m+1}}} \frac{d^n}{dz^n} S(z; b)|_{z=1} \zeta^n \\ &= S(1, b) + \frac{(-1)^{m+1}}{\theta} \frac{\zeta^{2m+1}}{2m+1} + O\left(\theta^{(2m+3)\varepsilon - \frac{2m+3}{2m+1}}\right) \end{aligned} \quad (2.101)$$

and we note that $O\left(\theta^{(2m+3)\varepsilon - \frac{2m+3}{2m+1}}\right) = o(1)$. The integrand of \mathcal{J}_{θ}^m has an exponentially decaying upper bound, as uniformly on c_+ and c_- we have

$$\left| \frac{\exp[\theta S(z; b) - x(d\theta)^{\frac{1}{2m+1}} \log z]}{\exp[\theta S(w; b) - y(d\theta)^{\frac{1}{2m+1}} \log w]} \right| \leq C_1 \exp\left[-C_2(d\theta)^{\frac{1}{2m+1}}(x+y)\right] \quad (2.102)$$

for constants C_1, C_2 , so by dominated convergence the limit of its integral converges to the integral of its limit. Since $x \log z = x\zeta(d\theta)^{1/(2m+1)}$, $y \log w = y\omega(d\theta)^{1/(2m+1)}$ and $z - w = (d\theta)^{-1/(2m+1)}(\zeta - \omega) + O(\theta^{-2/(2m+1)})$ and considering the bound on the tails contribution, we have a uniform approximation of the kernel

$$\begin{aligned} &\mathcal{J}_{\theta}^m([b\theta + x(d\theta)^{\frac{1}{2m+1}}] - \frac{1}{2}, [b\theta + y(d\theta)^{\frac{1}{2m+1}}] - \frac{1}{2}) \\ &= (d\theta)^{-\frac{1}{2m+1}} \frac{1}{(2\pi i)^2} \int_{iI_{\theta}-1} \int_{iI_{\theta}+1} \frac{\exp\left[(-1)^{m+1} \frac{\zeta^{2m+1}}{2m+1} - x\zeta\right]}{\exp\left[(-1)^{m+1} \frac{\omega^{2m+1}}{2m+1} - y\omega\right]} [1 + o(1)] \frac{d\zeta d\omega}{\zeta - \omega} \end{aligned} \quad (2.103)$$

where I_{θ} is the interval $[-\pi(d\theta)^{\varepsilon}, \pi(d\theta)^{\varepsilon}]$, and the $o(1)$ collects the error terms from the discretisation, the Taylor expansion of S , and the estimation of $z - w$. As $\theta \rightarrow \infty$, we have $I_{\theta} \rightarrow \mathbb{R}$, and the convergence (2.93) follows immediately, as required. \blacktriangleleft

With this, we can finally prove our main result.

Proof of Theorem 2.2. By Proposition 2.10, if λ is a random partition under \mathbb{P}_{θ}^m the probability of that there is no element of $S(\lambda)$ greater than some half-integer $\ell_s := [b\theta + s(d\theta)^{1/(2m+1)}] - \frac{1}{2}$ is the discrete Fredholm determinant

$$\begin{aligned} \mathcal{F}_{\theta}^m(\ell_s) &= \det(1 - \mathcal{J}_{\theta}^m)_{l^2(\ell_s + \mathbb{Z}_{\geq 0})} = \mathbb{P}_{\theta}^m(\lambda_1 < [b\theta + s(d\theta)^{1/(2m+1)}]) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\ell_s}^{\infty} \cdots \int_{\ell_s}^{\infty} \det_{1 \leq i, j \leq n} \mathcal{J}_{\theta}^m(k_i, k_j) dk_1 \cdots dk_n \end{aligned} \quad (2.104)$$

where the integrals is taken with respect to a reference measure which is zero everywhere but $\mathbb{Z} + \frac{1}{2}$. We want to show that it converges to the continuous Fredholm determinant $\det(1 - \mathcal{A}_{2m+1})_{L^2([s, \infty))}$ of the kernel given in (2.93) as $\theta \rightarrow \infty$.

First, using a change of variables $k_{x_i} := [b\theta + x_i(d\theta)^{1/(2m+1)}] - \frac{1}{2}$, we can write the discrete determinant on $L^2([s, \infty))$ as

$$\mathcal{F}_{\theta}^m(\ell_s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_s^{\infty} \cdots \int_s^{\infty} \det_{1 \leq i, j \leq n} \left[(d\theta)^{\frac{1}{2m+1}} \mathcal{J}_{\theta}^m(k_{x_i}, k_{x_j}) \right] dx_1 \cdots dx_n. \quad (2.105)$$

By Lemma 2.14, we have the convergence of $(d\theta)^{\frac{1}{2m+1}} \mathcal{J}_\theta^m$ to \mathcal{A}_{2m+1} . Then, the convergence of the Fredholm determinant follows from an application of Hadamard's bound of the determinant by a product of column sums; then, we only need to show that the traces of $(d\theta)^{\frac{1}{2m+1}} \mathcal{J}_\theta^m$ converge to the traces of \mathcal{A}_{2m+1} . But we can apply the same exponential decay bound (2.102) once again to bound $(d\theta)^{\frac{1}{2m+1}} \mathcal{J}_\theta^m$ itself on any interval that is bounded below, and by dominated convergence on such an interval we have the convergence of the discrete Fredholm determinant $\mathcal{F}_\theta^m(\ell_s)$ to the continuous one $F_{2m+1}(s) = \det(1 - \mathcal{A}_{2m+1})_{L^2([s, \infty))}$.

It remains to show that the expression (2.93) for \mathcal{A}_{2m+1} in the statement of Lemma 2.14 is indeed equivalent to our original definition (2.6) from Theorem 2.2. First, we insert $1/(\zeta - \omega) = \int_0^\infty e^{v(\zeta - \omega)} dv$ into (2.93) to write

$$\mathcal{A}_{2m+1}(x, y) = \int_0^\infty \text{Ai}_{2m+1}(x+v) \text{Ai}_{2m+1}(y+v) dv. \quad (2.106)$$

From this, to retrieve the second ‘‘Christoffel–Darboux type’’ expression following [LDMS18, Appendix D], we apply the eigenfunction relation (2.25) to write

$$\begin{aligned} (x-y)\mathcal{A}_{2m+1}(x, y) &= \int_0^\infty [(x+v) - (y+v)] \text{Ai}_{2m+1}(x+v) \text{Ai}_{2m+1}(y+v) dv \quad (2.107) \\ &= (-1)^{m+1} \int_0^\infty (\text{Ai}_{2m+1}^{(2m)}(x+v) \text{Ai}_{2m+1}(y+v) - \text{Ai}_{2m+1}(x+v) \text{Ai}_{2m+1}^{(2m)}(y+v)) dv. \end{aligned}$$

Then we note that the integrand can be written

$$\begin{aligned} &\text{Ai}_{2m+1}^{(2m)}(x+v) \text{Ai}_{2m+1}(y+v) - \text{Ai}_{2m+1}(x+v) \text{Ai}_{2m+1}^{(2m)}(y+v) \\ &= \frac{\partial}{\partial v} \sum_{i=0}^{2m-1} (-1)^i \text{Ai}_{2m+1}^{(i)}(x+v) \text{Ai}_{2m+1}^{(2m-1-i)}(y+v). \quad (2.108) \end{aligned}$$

Inserting this back into the integral, only the $v = 0$ boundary term contributes, recovering (2.5) to complete the proof. \blacksquare

2.4 Extensions: Generalised multicritical edge fluctuations

This section presents three extensions of the multicritical measure with asymptotic edge distributions that generalise the higher-order TW-GUE ones $F_{2m+1}(s)$.

2.4.1 Multicritical random cylindric partitions

First let us extend the multicritical measures to analogous Schur processes on cylindric partitions. In [LDMS18], the authors found a direct generalisation of the higher-order TW-GUE distribution asymptotically governing the maximal momentum in a grand canonical ensemble of fermions in a 1D flat trap potential (this follows the arguments for the passage to the grand canonical ensemble in free fermion models outlined in Section 1.3.2). To construct a discrete model with the same asymptotic edge behaviour, we avail of the correspondence between periodic Schur processes and positive temperature fermions found by Betea and Bouttier [BB19]; in particular, the authors proved that in the simplest case of an extension of the Poissonised Plancherel measure to cylindric partition of period one gives rise to fluctuations driven by Johansson's positive temperature extension of the TW-GUE distribution [Joh07] in a suitable asymptotic regime (see Theorem 1.42).

We similarly extend the multicritical measures to rank 1 periodic Schur processes, as follows:

Definition 2.15 (Cylindric multicritical measure). Let $\gamma = (\gamma_1, \gamma_2, \dots)$ be a sequence of real numbers defining an order m multicritical measure by the conditions of Definition 2.1 with right edge and fluctuation coefficients b, d . Then, the measure cylindric partitions $\Lambda = (\mu \subseteq \lambda \supseteq \mu)$ of period one defined by the periodic Schur process

$$\mathbb{P}_{u,\theta}^m(\mu, \lambda) = \frac{1}{Z} u^{|\mu|} s_{\lambda/\mu}[\theta\gamma]^2, \quad Z = \frac{\exp[\frac{\theta^2}{1-u} \sum_r r^2 \gamma_r^2]}{\prod_{i \geq 1} (1 - u^i)} \quad (2.109)$$

is an order m *cylindric multicritical measure*.

It is worth emphasising that for $m > 1$ the specialisation of the Miwa times to $\theta\gamma$ is not Schur positive. In the case of a rank 1 the measure is nonetheless well-defined for a conjugate pair of sets of Miwa times, but since there are negative transition probabilities we cannot readily extend further to a non-periodic process or to a higher rank one. From the partition function Z , we see that

$$\mathbb{E}(|\lambda|) = \frac{\theta^2}{(1-u)^2} \sum_r r^2 \gamma_r^2 - u \frac{d}{du} \log(u; u)_\infty \quad (2.110)$$

so as $\theta/(1-u)$ asymptotically defines a natural length scale for the parts λ_i, λ'_i as $\theta \rightarrow \infty$, $u \rightarrow 1$ (and only in the limit, unlike θ in the case of the usual multicritical measure).

For the cylindric multicritical measures, we have the following positive temperature extension of Theorem 2.2 (or multicritical generalisation of Theorem 1.42):

Theorem 2.16 (Asymptotic edge fluctuations of cylindric multicritical measures). *Let $\Lambda = (\mu \subseteq \lambda \supseteq \mu)$ be a random cylindric partition under a cylindric multicritical measure $P_{u,\theta}^m$ with right edge and fluctuation coefficients b, d . Then, in a critical scaling regime where $\theta(1-u)^{2m} \rightarrow \alpha^{2m+1}d > 0$ as $\theta \rightarrow \infty$ and $u \rightarrow \infty$, we have*

$$\lim_{\theta \rightarrow \infty, u \rightarrow 1} \mathbb{P}_{u,\theta}^m \left(\frac{\lambda_1 - b\Theta}{(d\Theta)^{\frac{1}{2m+1}}} < s \right) = F_{2m+1}^\alpha(s) := \det(1 - \mathcal{A}_{2m+1}^\alpha)_{L^2([s, \infty))} \quad (2.111)$$

$$\text{where } \Theta := \frac{\theta}{1-u}, \quad \Theta \sim \left(\frac{\theta}{\alpha} \right)^{\frac{2m+1}{2m}} d^{-\frac{1}{2m}}$$

where F_{2m+1}^α is the Fredholm determinant of the higher-order α -Airy integral kernel

$$\mathcal{A}_{2m+1}^\alpha(x, y) = \int_{-\infty}^{\infty} \frac{e^{\alpha v}}{1 + e^{\alpha v}} \text{Ai}_{2m+1}(x+v) \text{Ai}_{2m+1}(y+v) dv. \quad (2.112)$$

Here again, α plays the role of a limiting inverse temperature, and in the limit $\alpha \rightarrow \infty$ we have $F_{2m+1}^\alpha \rightarrow F_{2m+1}$. We note that critical exponents are unchanged by the passage to finite temperature in this regime once we replace the large parameter θ with Θ , which also tends to infinity.

The Fredholm determinants F_{2m+1}^α have been related to an integro-differential generalisation of the Painlevé II hierarchy by Krajenbrink [Kra20] generalising an approach of Amir, Corwin and Quastel [ACQ11] from the $m = 1$ case and by Bothner, Cafasso and Tarricone [BCT21] using a rigorous Riemann–Hilbert approach.

A DPP in the grand canonical ensemble As periodic Schur processes, cylindric Schur measures are in general not determinantal, as first observed by Borodin [Bor07]. Specifically, neither of the configurations $S(\boldsymbol{\lambda})$ or $S(\boldsymbol{\mu})$ forms a determinantal point process. While correlation functions for these are in principle explicit, they are complicated multi-contour integrals making their scaling analysis very difficult. This can be remedied, however, via a procedure called *shift-mixing* [Bor07] or *passage to the grand canonical ensemble* [BB19]; the shifted fermion configuration

$$S_c(\boldsymbol{\lambda}) = \{\boldsymbol{\lambda}_i - i + \mathbf{c} + \frac{1}{2}, i \in \mathbb{Z}_{\geq 1}\} \quad (2.113)$$

such that $Q_0(S_c(\boldsymbol{\lambda})) = \mathbf{c}$ and $Q_1(S_c(\boldsymbol{\lambda})) = |\boldsymbol{\lambda}| + \mathbf{c}^2/2$ does form a DPP, where \mathbf{c} is a suitably chosen random integer. Following [BB19], let us now give some intuition for how \mathbf{c} is chosen, starting from a simple case.

For two parameters t, u , we associate to each fermion configuration S a Boltzmann weight of $t^{Q_0(S)}u^{Q_1(S)}$; in physical terms where Q_1 is an energy, u is identified $e^{-\beta}$ where β is the (dimensionless) inverse temperature, and t is identified with $e^{\mu_{\text{ch}}}$ where μ_{ch} is a chemical potential and the charge Q_0 plays the role of a particle number. Summing the weights of configurations $S_c(\boldsymbol{\lambda})$ over all pairs $(\boldsymbol{\lambda}, c)$, we have

$$\sum_{\boldsymbol{\lambda}} u^{|\boldsymbol{\lambda}|} \sum_{c \in \mathbb{Z}} t^c u^{c^2/2} = \frac{1}{(u; u)_{\infty}} \vartheta_3(t; u) = \prod_{i=0}^{\infty} (1 + tu^{i+\frac{1}{2}})(1 + t^{-1}u^{i+\frac{1}{2}}) =: Z_{t,u} \quad (2.114)$$

where $\vartheta_3(t; u) := \sum_{c \in \mathbb{Z}} t^c u^{c^2/2}$ is one of the Jacobi theta functions in multiplicative notation, the q -Pochhammer symbol is $(z; q)_n := \prod_{i=0}^{n-1} (1 - zq^i)$ for $n \in \mathbb{Z} \cup \{\infty\}$, and the final equality is the Jacobi triple product identity. For $t > 0$ and $u \in [0, 1)$, we have a well defined measure on partition-charge pairs $\mathbb{P}(\boldsymbol{\lambda}, c) = t^{|\boldsymbol{\lambda}|} u^{c^2/2} / Z$.

From this construction, some natural statistics are independently distributed. If $(\boldsymbol{\lambda}, \mathbf{c})$ is a random pair under this measure, for each $k \in \mathbb{Z} + \frac{1}{2}$ we have an independent distribution

$$\mathbb{P}(k \in S_c(\boldsymbol{\lambda})) = \frac{tu^k}{1 + tu^k} \quad (2.115)$$

which corresponds precisely to the Fermi density of states factor. The random charge \mathbf{c} has the *discrete Gaussian distribution*

$$\mathbb{P}(c) = \frac{t^c u^{c^2/2}}{\vartheta_3(t; u)}. \quad (2.116)$$

Note that if we condition on \mathbf{c} being zero, the distributions (2.115) are no longer independent; this corresponds to the fact (discussed in Section 1.3.2) that a canonical ensemble of free fermions is not determinantal while a grand canonical ensemble is (meaning that in a canonical ensemble the fermions are not really free).

Coupling a periodic Schur process to these Boltzmann weights corresponds to passing to the grand canonical ensemble, and making it determinantal. Writing a general result only for the case at hand, with \mathbf{c} distributed by (2.116) and $\boldsymbol{\lambda}$ the larger of a pair $(\boldsymbol{\mu} \subseteq \boldsymbol{\lambda})$ distributed by $\mathbb{P}_{u,\theta}^m$, the randomly shifted random configuration $S_c(\boldsymbol{\lambda})$ forms a DPP; to be precise, for any $k_1, \dots, k_n \in \mathbb{Z} + \frac{1}{2}$ we have

$$\rho_n(k_1, \dots, k_n) = \mathbb{P}(\{k_1, \dots, k_n\} \in S_c(\boldsymbol{\lambda})) = \det_{1 \leq i, j \leq n} \mathcal{J}_{u,t,\theta}^m(k_i, k_j) \quad (2.117)$$

where the kernel is

$$\mathcal{J}_{u,t,\theta}^m(k,\ell) = \sum_{i \in \mathbb{Z}} \frac{tu^i}{1+tu^i} J_{k+i+\frac{1}{2}}(\Theta\gamma) J_{\ell+i+\frac{1}{2}}(\Theta\gamma) \quad (2.118)$$

$$= \frac{1}{(2\pi i)^2} \oint_{c_+, c_-} \frac{\exp[\Theta S(z, k/\Theta)]}{\exp[\Theta S(w, \ell/\Theta)]} \cdot \frac{\kappa(z, w) dw dz}{wz}, \quad c_{\pm} : |z| = u^{\mp 1/4},$$

$$\kappa(z, w) = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \frac{tu^i}{1+tu^i} \left(\frac{z}{w}\right)^i = \sqrt{\frac{w}{z}} \cdot \frac{(u; u)_{\infty}^2}{\vartheta_u(w/z)} \cdot \frac{\vartheta_3(tz/w; u)}{\vartheta_3(t; u)}. \quad (2.119)$$

using the notation $\vartheta_u(x) := (x; u)_{\infty}(u/x; u)_{\infty}$ and reusing the action notation for the order m multicritical measure defined at (2.56). The equivalence between the two forms of κ is a special case of Ramanujan's ${}_1\Psi_1$ summation [GR04], and the choice of contours with $|w| < |z|$ ensures the sum converges.

Note the similarity with the integral expression for the zero temperature kernel (1.150). The proof of this in [BB19] adapts Okounkov's fermionic approach (see Theorem 1.45) to a case each expectation with respect to the domain wall vacuum state is replaced with one with respect to a state independent random densities given by (2.115). Then, $\kappa(z, w)$ given in (2.119) is the generating function $\langle c^\dagger(z)c(w) \rangle_{u,t} = \sum_{k,\ell} z^k w^{-\ell} \langle c_k^\dagger c_\ell \rangle_{u,t}$ of propagators on that state.

The crossover regime The asymptotic regime of Theorem 2.16 is the one in which the ‘‘thermal’’ fluctuations coming from the factor of $u^{|\mu|}$ match the order of magnitude of the ‘‘quantum’’ fluctuations coming from the skew Schur functions, so that α parametrises a crossover between regimes where either kind of fluctuation dominate. Heuristically, from the identification $u = e^{-\beta}$, the thermal fluctuations are of order of the temperature β^{-1} , so comparing with scale of the fluctuations in the zero temperature case (i.e. the multicritical Schur measure) we look for a regime in which

$$\beta^{-1} \sim \Theta^{\frac{1}{2m+1}}. \quad (2.120)$$

Fixing a specific regime

$$u := \exp\left[-\alpha(d\Theta)^{-\frac{1}{2m+1}}\right], \quad \theta := \alpha d^{-\frac{1}{2m+1}} \Theta^{\frac{2m}{2m+1}} \quad (2.121)$$

by this reasoning, it is straightforward to see that it is asymptotically equivalent to the crossover regime in the statement.

Proof of Theorem 2.16. Our proof follows that of [BB19], with some adaptations that correspond precisely to the arguments of Section 2.3.3 of this text. It consists of three steps.

(i) Passage to the grand canonical ensemble We choose a real parameter $t > 0$ and we pass to the grand canonical ensemble chemical potential $\mu_{\text{ch}} = -\log t$. Let us consider random partition-charge pairs $(\boldsymbol{\lambda}, \mathbf{c})$ with joint distribution

$$\mathbb{P}_{u,t,\theta}^m(\boldsymbol{\lambda}, \mathbf{c}) = \frac{1}{Z} \sum_{\mu} u^{|\mu| + \frac{c^2}{2}} t^c s_{\lambda/\mu}[\theta\gamma]^2 \quad (2.122)$$

where Z is a normalisation factor. We will look for the asymptotic law of the largest element of the randomly shifted random configuration $S_{\mathbf{c}}(\boldsymbol{\lambda})$; since this forms a DPP, with kernel given in (2.118), we study the distribution

$$\mathbb{P}(\boldsymbol{\lambda}_1 + \mathbf{c} < \ell_s) = \det(1 - \mathcal{J}_{u,t,\theta}^m)_{l^2(\ell_s + \mathbb{Z}_{\geq 0})} \quad (2.123)$$

for $\ell_s := \lfloor b\Theta + (d\Theta)^{\frac{1}{2m+1}} \rfloor - \frac{1}{2}$ in the asymptotic regime of the theorem, where $\theta \rightarrow \infty$, $u \rightarrow 1$ with $\theta(1-u)^{2m} \rightarrow \alpha^{2m+1}$ (and $\Theta := \theta/(1-u)$).

(ii) Asymptotic analysis Let us start from the integrand of $\mathcal{J}_{u,t,\theta}^m(k, \ell)$ in a regime where $k = \lfloor b\Theta + x(d\Theta)^{1/(2m+1)} \rfloor - \frac{1}{2}$ and $\ell = \lfloor b\Theta + y(d\Theta)^{1/(2m+1)} \rfloor - \frac{1}{2}$, which is (suppressing floor functions)

$$\frac{\exp \left[\Theta S(z; b) - x(d\Theta)^{\frac{1}{2m+1}} \log z \right]}{\exp \left[\Theta S(w; b) - y(d\Theta)^{\frac{1}{2m+1}} \log w \right]} \cdot \kappa(z, w). \quad (2.124)$$

Since $\Theta \rightarrow \infty$ in our asymptotic regime, we can directly use Θ as a large parameter, and then for everything except for the function $\kappa(z, w)$, the steepest descent analysis follows precisely the arguments of Section 2.3.3 (with just a change from θ to Θ). At $z = w = 1$ there is an order $2m$ saddle point, and we use the same change of variables

$$z = \exp \left[\zeta (d\Theta)^{-\frac{1}{2m+1}} \right], \quad w = \exp \left[\omega (d\Theta)^{-\frac{1}{2m+1}} \right]. \quad (2.125)$$

The arguments for the tails bound generalise. The contour c_+ of the integral in z is circle on which

$$|z| = u^{-1/4} = \exp[\operatorname{Re}(\zeta)(d\Theta)^{-1/(2m+1)}], \quad (2.126)$$

and as $u \rightarrow 1$ this is satisfied if and $\operatorname{Re}(\zeta) \sim (d\Theta)^{1/(2m+1)}/4(1-u) \sim \alpha/4$, so the central region is asymptotically parametrised by $\zeta \in i\mathbb{R} + \alpha/4$ and $\omega = i\mathbb{R} - \alpha/4$.

At the same time, κ has a reasonable asymptotic behaviour in the above regime and on the contours c_{\pm} . First, when z, w are around around 1, observing that $z = u^{-\zeta/\alpha}$, $w = u^{-\omega/\alpha}$, we have

$$\kappa(z, w) = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \frac{(z/w)^i}{1 + (tu^i)^{-1}} \sim \alpha (d\Theta)^{-\frac{1}{2m+1}} \cdot \frac{\pi}{\sin \frac{\pi(\zeta-\omega)}{\alpha}} \quad \text{as } \Theta \rightarrow \infty. \quad (2.127)$$

This follows by the same argument as that leading to [BB19, Equation (5.32)]: putting $u = e^{-r}$ and $z/w = e^{r/2+i\phi}$ for $\phi \in [-\pi, \pi]$, by the Poisson summation formula we have

$$\kappa(z, w) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{e^{i\phi k}}{2 \cosh \frac{rk}{2}} = \sum_{n \in \mathbb{Z}} (-1)^n \frac{\pi}{r \cosh \frac{\pi(\phi - 2\pi n)}{r}} \quad (2.128)$$

and on the contours c_{\pm} ⁴ as $\Theta \rightarrow \infty$, $u \rightarrow 1$,

$$\kappa(z, w) \sim \frac{\pi}{r \cosh \frac{\pi \operatorname{Im}(\zeta - \omega)}{\alpha}} = \frac{\pi}{r \sin \frac{\pi(\zeta - \omega)}{\alpha}} \quad (2.129)$$

The prefactor $(d\Theta)^{-\frac{1}{2m+1}}$ will be cancelled by part of the Jacobian for the change of variables $(z, w) \mapsto (\zeta, \omega)$. From the same Poisson summation formula, we see that outside of the central region around $z = w = 1$, κ decays exponentially fast to 0, see [BB19, Lemma 5.5].

Putting everything together and noting that the same exponential decay bounds imply dominated convergence, as $\Theta \rightarrow \infty$ and $u \rightarrow 1$ we have

$$\begin{aligned} & (d\Theta)^{\frac{1}{2m+1}} \mathcal{J}_{u,t,\theta}^m \left(\lfloor b\Theta + x(d\Theta)^{1/(2m+1)} \rfloor - \frac{1}{2}, \lfloor b\Theta + y(d\Theta)^{1/(2m+1)} \rfloor - \frac{1}{2} \right) \\ & \rightarrow \frac{1}{(2\pi i)^2} \int_{i\mathbb{R} + \frac{\alpha}{4}} \int_{i\mathbb{R} - \frac{\alpha}{4}} \frac{\exp \left[(-1)^{m+1} \frac{\zeta^{2m+1}}{2m+1} - x\zeta \right]}{\exp \left[(-1)^{m+1} \frac{\omega^{2m+1}}{2m+1} - y\omega \right]} \cdot \frac{\pi}{\alpha \sin \frac{\pi(\zeta-\omega)}{\alpha}} d\omega d\zeta. \end{aligned} \quad (2.130)$$

⁴Let us note that in this instance, we cannot readily switch to contours angled at $m\pi/2m + 1$, due to the poles of κ on the real line.

Using the identity

$$\frac{\pi}{\alpha \sin \frac{\pi(\zeta-\omega)}{\alpha}} = \int_{-\infty}^{\infty} \frac{e^{(\alpha+\omega-\zeta)v} dv}{1 + e^{\alpha v}} \quad (2.131)$$

then noting that the shifts of $\alpha/4$ away from the imaginary axis in the contours for ζ and ω can be changed to 1 (or another positive number) by a change of variables, we see that the limiting kernel is equal to $\mathcal{A}_{2m+1}^{\alpha}(x, y)$.

The same exponential decay arguments for the integrand apply again to the integral, so the traces of $\mathcal{J}_{u,t,\theta}^m$ also converges uniformly to the traces of $\mathcal{A}_{2m+1}^{\alpha}$ on any set that is bounded below. Since the Hadamard bound argument equally applies here, we have convergence of the Fredholm determinants too, with

$$\lim_{\theta \rightarrow \infty, u \rightarrow 1} \mathbb{P} \left[\frac{\lambda_1 + \mathbf{c} - b\Theta}{(d\Theta)^{\frac{1}{2m+1}}} < s \right] = \det(1 - \mathcal{A}_{2m+1}^{\alpha})_{L^2([s, \infty))}. \quad (2.132)$$

(iii) Shift removal The limiting distribution (2.132) above is not quite what we wanted to prove due to the random shift \mathbf{c} . Luckily it can be removed without affecting the result: when the extra “chemical potential” parameter is set to $t = 1$, from its law (2.116), we have that $\mathbf{c}/\Theta^{1/(2m+1)}$ converges to 0 in probability— see Lemma 2.1 in [BB19]. The case for $t \neq 1$ can be recovered through a deterministic shift in our formulas. ■

2.4.2 Generalised higher-order Airy kernel

In this section we extend the multicritical measures to have more general asymptotic edge distributions of a kind shown by Cafasso, Claeys and Girotti [CCG19] to encode Fredholm determinant solutions of the Painlevé II hierarchy; using a Riemann–Hilbert method, the authors showed F_{2m+1} not to be the unique Fredholm determinant solution to the order $2m$ equation, but rather found an element of an $m - 1$ parameter family. To be precise, the previously stated result generalises as follows:⁵

Addendum 2.17 to Theorem 2.3 (Generalised Fredholm determinant solutions of the Painlevé II hierarchy [CCG19]). *Fix a sequence of $m - 1$ real constants $\tau = (\tau_1, \dots, \tau_{m-1})$, then set*

$$p_{\tau;2m+1}(x) = \frac{x^{2m+1}}{2m+1} + \sum_{i=1}^{m-1} \frac{\tau_i}{2i+1} x^{2i+1} \quad (2.133)$$

and define the generalised higher-order Airy kernel

$$\mathcal{A}_{\tau;2m+1}(x, y) = \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}+1} \int_{i\mathbb{R}-1} \frac{\exp[(-1)^{m+1} p_{\tau;2m+1}(\zeta) - x\zeta] d\zeta d\omega}{\exp[(-1)^{m+1} p_{\tau;2m+1}(\omega) - y\omega] \zeta - \omega}. \quad (2.134)$$

Then, the Fredholm determinant

$$F_{\tau;2m+1}(s) = \det(1 - \mathcal{A}_{\tau;2m+1})_{L^2([s, \infty))} \quad (2.135)$$

satisfies the relation (2.8) and boundary behaviour (2.7) stated for $F_{2m+1}(s) = F_{0;2m+1}(s)$ in Theorem 2.3.

⁵This statement is still not as general as the one in [CCG19], which includes distributions $\det(1 - \rho \mathcal{A}_{\tau;2m+1})$ for $\rho \in (0, 1]$ associated with “thinned” DPPs where elements of the configurations are removed independently at random; we will not consider any such processes.

Generalised trapped fermion models As far as we know, the generalised higher-order Airy kernels $\mathcal{A}_{\tau;2m+1}$ have not been studied in the context of edge statistics for trapped fermions. Naively considering the constituent generalised higher order Airy functions

$$\text{Ai}_{\tau;2m+1}(x) = \frac{1}{2\pi i} \int_{i\mathbb{R}+1} \exp[(-1)^{m+1} p_{\tau;2m+1}(\zeta) - x\zeta] d\zeta, \quad (2.136)$$

we see that they satisfy eigenfunction relations generalising (2.25) to

$$(-1)^{m+1} \left[\frac{d^{2m}}{dx^{2m}} + \sum_{i=1}^{m-1} \tau_i \frac{d^{2i}}{dx^{2i}} \right] \text{Ai}_{\tau;2m+1}(x) = -x \text{Ai}_{\tau;2m+1}(x), \quad (2.137)$$

so it would appear that the momentum space approach of [LDMS18] as outlined in Section 2.2.1 would generalise to recover these edge potentials to trapping potentials of the form $V(x) = x^{2m} + \sum_i (-1)^i \tau_i x^{2i}$. We focus on a discrete construction, which coincides with the momentum space edge of such a model in a suitable continuum limit. Our main task is to identify the correct limit.

Generalised discrete models Our construction of multicritical measures on partitions (and the corresponding lattice fermion models) generalises with just some modification of scaling considerations. We again construct each measure with a single real parameter θ , but no longer require each Miwa time in the Schur function specialisation to grow linearly with θ ; once we consider combinations of Miwa times growing at different speeds, we can tune the speeds so that the integrand of the limiting edge kernel has a given odd polynomial in the exponential, from the same saddle point analysis of Section 2.3.3.

To be specific, we combine the coefficients γ_r already used to define multicritical measures, to define generalised ones as follows (where we emphasise that the sequence of constants γ is replaced with a θ -dependant functions $\gamma^\tau(\theta)/\theta$).

Definition 2.18 (Generalised multicritical measure). Fix a sequence of $m - 1$ real constants $\tau = (\tau_1, \dots, \tau_{m-1})$, and choose m sequences of real coefficients $\gamma^{(1)}, \dots, \gamma^{(m)}$ where $\gamma^{(i)}$ satisfies the conditions for an order i multicritical measure and has right edge and fluctuation coefficients b_i, d_i . Then, for a positive parameter θ , define the sequence $\gamma^\tau(\theta)$ of Miwa times, with elements indexed $r \geq 1$

$$\gamma^\tau(\theta)_r = \theta \gamma_r^{(m)} + \sum_{i=1}^{m-1} \theta^{\frac{2i+1}{2m+1}} (-1)^{m-i} \frac{\tau_i}{d_i} \gamma_r^{(i)} \quad (2.138)$$

we define an order m *generalised multicritical measure*

$$\mathbb{P}_\theta^{\tau;m}(\lambda) = \frac{1}{Z} s_\lambda[\gamma^\tau(\theta)]^2, \quad Z = e^{\sum_r r \gamma^\tau(\theta)_r^2} \quad (2.139)$$

along with its edge position function

$$B(\theta) = b_m \theta + \sum_{i=1}^{m-1} b_i (-1)^{m-i} \frac{\tau_i}{d_i} \theta^{\frac{2i+1}{2m+1}}. \quad (2.140)$$

This generalisation is defined so that we have the edge behaviour we would expect in analogy to Theorem 2.2. Namely,

Theorem 2.19 (Edge fluctuations in generalised multicritical measures). *If λ is a random partition under the generalised multicritical measure $\mathbb{P}^{\tau;m}$, then*

$$\lim_{\theta \rightarrow \infty} \mathbb{P}_\theta^{\tau;m} \left[\frac{\lambda_1 - B(\theta)}{(d_m \theta)^{\frac{1}{2m+1}}} \leq s \right] = \det(1 - \mathcal{A}_{\tau;2m+1})_{L^2(s, \infty)} =: F_{\tau;2m+1}(s) \quad (2.141)$$

where $\mathcal{A}_{\tau;2m+1}$ is the kernel (2.134) defined in [CCG19].

It is worth highlighting that the expected edge position $B(\Theta)$ now has quite a non-trivial expansion: it has deterministic terms of orders θ , $\theta^{\frac{2n-1}{2n+1}}$, \dots , $\theta^{\frac{3}{2n+1}}$, and only at order $\theta^{\frac{1}{2n+1}}$ do we encounter the fluctuations. The expected size is also more subtle: since we have $\mathbb{E}(|\lambda|) = \sum_{r \geq 1} r^2 \gamma(\theta)_r^2$, only the leading order term now scales with θ^2 .

Tuning speeds and coefficients The proof of Theorem 2.19 involves no new arguments than the ones of Section 2.3.3, so we find it more instructive to present an informal derivation of Definition 2.18. To do so, let us define additional notation, putting

$$S^{(i)}(z; x) = \sum_{r \geq 1} \gamma_r^{(i)} (z^r - z^{-r}) - x \log z = V^{(i)}(z) - V^{(i)}(z^{-1}) - x \log z \quad (2.142)$$

for the action and potential associated with the coefficients $\gamma^{(i)}$. Since each $\gamma^{(i)}$ defines an order i multicritical measure with right edge and fluctuation coefficients b_i, d_i , we have, by (2.58) and (2.59), the following expansion of $S^{(i)}$ around $z = 1$:

$$S^{(i)}(z; b_i) = \frac{(-1)^{i+1} d_i}{2i+1} (z-1)^{2i+1} + O((z-1)^{2i+3}). \quad (2.143)$$

Let us form a generalized potential, which now scales with θ ,

$$V(z) = \sum_{i=1}^m f_i(\theta) V^{(i)}(z); \quad (2.144)$$

we fix $f_m(\theta) = 1$ for convenience. Our goal is now to find suitable $f_i(\theta)$ so as to obtain the scaling regime of Theorem 2.19 and the limiting edge kernel $\mathcal{A}_{\tau;2m+1}$. We will just look at the integrand in the double contour integral representation in a region near the multicritical saddle point. The discrete kernel we start with is

$$\mathcal{J}_\theta^{\tau;m}(k, \ell) = \frac{1}{(2\pi i)^2} \oint_{c_+, c_-} \frac{\exp[\theta(V(z) - V(z^{-1}))]}{\exp[\theta(V(w) - V(w^{-1}))]} \frac{dz dw}{z^{k+\frac{1}{2}} w^{-\ell+\frac{1}{2}} (z-w)} \quad (2.145)$$

for $k, \ell \in \mathbb{Z} + \frac{1}{2}$, with c_+ for the integration in z passing just outside the unit circle and c_- for w passing just inside. Now we set

$$S(z; x) = V(z) - V(z^{-1}) - x \log z; \quad \mathbf{b}(\theta) := \sum_{i=1}^m f_i(\theta) b_i. \quad (2.146)$$

Then, if we rewrite the coordinates relative to $k = \mathbf{b}(\theta) + k'$, $\ell = \mathbf{b}(\theta) + \ell'$ the kernel may be written

$$\mathcal{J}_\theta^{\tau;m}(k, \ell) = \frac{1}{(2\pi i)^2} \oint_{c_+, c_-} \exp[\theta(S(z; \mathbf{b}(\theta)) - S(w; \mathbf{b}(\theta)))] \frac{dz dw}{z^{k'+1/2} w^{-\ell'+1/2} (z-w)}. \quad (2.147)$$

Since we have

$$S(z; \mathbf{b}(\theta)) = \sum_{i=1}^m f_i(\theta) S^{(i)}(z; b_i), \quad (2.148)$$

near the order $2i$ saddle point for each $S^{(i)}$, we let ε be a small positive number that tends to zero as θ tends to infinity and consider a change of variables

$$z = 1 + \zeta\varepsilon, \quad w = 1 + \omega\varepsilon, \quad k' = \frac{x}{\varepsilon}, \quad \ell' = \frac{y}{\varepsilon} \quad (2.149)$$

(this simple setup is sufficient for our purposes; we will parametrise the contours explicitly once we have suitable ε and $f_i(\theta)$). Expanding in small ε and using (2.143), the leading order approximation of the integrand is

$$\frac{1}{\varepsilon(\zeta - \omega)} \exp \left[\sum_{i=1}^m \theta f_i(\theta) \frac{(-1)^{i+1} d_i}{2i+1} \varepsilon^{2i+1} (\zeta^{2i+1} - \omega^{2i+1}) - x\zeta + y\omega + O(\theta\varepsilon^{2m+3}) \right]. \quad (2.150)$$

It now becomes clear that in the generalised multicritical action, each $f_i(\theta)$ should scale as $\varepsilon^{-2i-1}/\theta$. More precisely, to use our convention that $f_m(\theta) = 1$, we identify $\varepsilon = (d_m\theta)^{-1/(2m+1)}$ (which indeed tends to 0) to be the appropriate scale; taking an action with

$$f_i(\theta) := (-1)^{m-i} \frac{\tau_i}{d_i} \theta^{\frac{2i-2m}{2m+1}}, \quad i = 1, \dots, m-1, \quad (2.151)$$

the leading order term coincides precisely with the integrand of $\mathcal{A}_{\tau, 2m+1}$. At the level of the parametrised specialisations for the corresponding Schur measures, this gives corresponds precisely to Miwa times $\gamma^\tau(\theta)_r$ corresponding the generalised multicritical measure $\mathbb{P}_\theta^{\tau; m}$. The function $\mathbf{b}(\theta)$ determining the edge scaling becomes $B(\theta)$ defined in (2.140).

The edge asymptotics With $f_i(\theta), \varepsilon$ now determined, let us briefly discuss the remaining analysis needed to prove Theorem 2.19. From noting that the Jacobian for the change of variables from z, w to ζ, ω contributes a factor of ε^2 , we see that $(d_m\theta)^{1/(2m+1)} \mathcal{J}_\theta^{\tau; m}$ is the relevant rescaled kernel.

Comparing to the analysis of Section 2.3.3, note that the tails bound and the exponential decay apply immediately to this case. The same contours can be reused along with the same dominated convergence arguments, to show firstly the uniform convergence

$$\begin{aligned} & (d_m\theta)^{\frac{1}{2m+1}} \mathcal{J}_\theta^{\tau; m} (\lfloor B(\theta) + x(d_m\theta)^{\frac{1}{2m+1}} \rfloor - \frac{1}{2}, \lfloor B(\theta) + y(d_m\theta)^{\frac{1}{2m+1}} \rfloor - \frac{1}{2}) \\ & \rightarrow \mathcal{A}_{\tau; 2m+1}(x, y) \end{aligned} \quad (2.152)$$

as $\theta \rightarrow \infty$, and in turn the convergence of traces and finally of Fredholm determinants uniformly on sets bounded below, with

$$\begin{aligned} \lim_{\theta \rightarrow \infty} \mathbb{P} \left[\frac{\lambda_1 - B(\theta)}{(d_m\theta)^{\frac{1}{2m+1}}} < s \right] &= \lim_{\theta \rightarrow \infty} \det(1 - \mathcal{J}_\theta^{\tau; m})_{l^2(\mathbb{Z}_{\geq 0} + \lfloor B(\theta) + s(d_m\theta)^{\frac{1}{2m+1}} \rfloor - \frac{1}{2})} \\ &= \det(1 - \mathcal{A}_{\tau; 2m+1})_{L^2([s, \infty))} \end{aligned} \quad (2.153)$$

as required.

Finally, let us note that the extensions presented in this section and in the previous one are completely compatible; we can directly construct analogous “generalised multicritical cylindric measures” using the Miwa time specialisations of Definition 2.18. The distributions F_{2m+1}^α then generalise to Fredholm determinants of positive temperature kernels composed of the functions $\text{Ai}_{\tau; 2m+1}$.

2.4.3 More general Hermitian Schur measures and Fermi seas

To conclude this chapter, we mention perspectives and work in progress motivated by Definition 2.1 of the multicritical measures itself, and the assumptions it relies on. The multicritical measures are well defined in part because we require the sequences γ to have finite support, and this makes the steepest descent analysis presented of Section 2.3 much simpler since there are only isolated saddle points. A second assumption that was essential to this analysis was the final condition non-negativity of (2.1), which ensures there are at most two (complex conjugate) saddle points on the unit circle. We note, however, that at the level of the measures themselves it is quite natural to lift either of these conditions. In the first instance, if we consider Schur measures defined by specialising the direct parameters to some finite set, the corresponding sequences Miwa times have infinite support. In the second instance, considering the lattice fermion models of Section 2.2, if we tune the parameters for the kinetic hopping terms just beyond the point of multicriticality we violate the non-negativity condition; lifting this condition is important for the universality picture for these models. Let us briefly consider the implications of lifting each of the conditions, with particular focus on the second “single Fermi sea” condition, in which case we discuss non-rigorously how new asymptotic edge statistics arise.

Non-polynomial Hermitian Schur measures from multicriticality The most immediately relevant cases of non-polynomial Hermitian Schur measures are the minimal and symmetric minimal multicritical measures $\mathbb{P}_\theta^{a,m}, \mathbb{P}_\theta^{s,m}$ where we let the order of multicriticality m tend to infinity, or allow m to be non-integer. The explicit coefficients given in Definitions 2.6 and 2.8 yield analytic continuations, and the resulting measures are well defined. Although the Fredholm determinants F_{2m+1} are difficult to study directly as $m \rightarrow \infty$, the limit curves of $\mathbb{P}_\theta^{a,m}, \mathbb{P}_\theta^{s,m}$ can be readily expressed in that limit (it is not yet clear however if the $\theta \rightarrow \infty$ and $m \rightarrow \infty$ limits commute).

Referring back to fermions in flat traps as studied in [LDMS18], the $m \rightarrow \infty$ limit is quite physical, corresponding to a model of free fermions in a square well with infinite walls. Interestingly, such a model escapes the *position space* Airy ensemble universality for free fermions [LACTDMS17]. The non-integer m case, on the other hand, has no trapped fermion analogue as the corresponding potential $V(x) = x^{2m}$ is non-confining. We note, however, that although the lattice fermions corresponding to Schur measures can asymptotically coincide trapped fermion models on the edge, they do not “see” the potential, since they already have an unbounded spectrum and Dirac sea ground state. In this picture, the non-integer m case is equally physical. Moreover, this extension is directly analogous to work of Ambjørn, Budd and Makeenko [ABM16] which extended Kazakov’s multicritical Hermitian matrix models [Kaz89] to non-integer orders of multicriticality. The main challenge in the asymptotic analysis at these new values of m comes from the fact the Miwa times now have infinite support, and the action has a cut on the real axis which prevents us from using the same contours that have served us in the last three sections.

The single Fermi sea condition Turning to the final non-negativity condition of (2.1), we can find a physical interpretation for it by referring back to the local density approximation bulk heuristics for the lattice fermion models, as outlined in Section 2.2.3. From the perspective of the Fourier transformed creation operators, the angle $\chi := \pi \varrho(x)$ solving (2.47) sets an interval for the Fourier frequencies in the ground state— or *momenta*— near x in the bulk (at a scale much smaller than the system size). If

the solutions are $\pm\chi$, then the momentum support is $[-\chi, \chi]$. Consider, however, the case if there are more than two solutions; for example if $\pm\chi_1$ and $\pm\chi_2$ are solutions with $\chi_1 > \chi_2 > 0$, then the momentum support is two disjoint intervals $[-\chi_1, -\chi_2]$ and $[\chi_2, \chi_1]$, and we describe the model as having a *split Fermi sea* or call ground state as being a *Moses state* in the terminology of [FEC14]. Requiring the derivative of (2.47) to be non-positive anywhere to the right of the origin is a necessary and sufficient condition to have connected momentum support at each x ; clearly this condition also ensures that for all x , $S(z; x)$ has only one pair of complex conjugate saddle points on $|z| = 1$.

A simple measure with a split Fermi sea In the simplest (and only simply critical) case with a split Fermi sea, we consider $\mathbb{P}(\lambda) = e^{-\theta^2 \sum_r r \gamma_r^2} s_\lambda[\theta\gamma]$ where only γ_1, γ_2 are nonzero. Explicit limiting densities $\varrho(x)$ were found for any such measure with a split Fermi sea in [BS21, Equation 45]⁶, in the context of lattice fermion models at a points where they coincide exactly with our ones. Let us pick one specific measure, with $\gamma_1 = 1, \gamma_2 = -1/4$, and look at the unit circle saddle points of the action $S(z; x)$ of its kernel K_θ , that is $z_\pm^* = e^{\pm i\chi}$ where

$$2 \cos \chi - \cos 2\chi = x. \tag{2.154}$$

We identify the following regions and points in terms of the position x :

- (i) *The frozen region:* For $x < -3$, there are no solutions to (2.154); we have the same frozen region discussed in Section 2.3.2.
- (ii) *The left edge:* At $x = -3$ we have the usual second order saddle point associated with TW-GUE asymptotic edge statistics at $z = -1$.
- (iii) *The single Fermi sea bulk:* For $-3 < x < 1$, there is one positive solution (this is situation we previously considered in the bulk).
- (iv) *The internal edge:* At $x = 1$, there are first order saddle points at $e^{\pm i\pi/2}$ and a second order saddle point at $\chi = 0$. Approaching this point from the right, the derivative of the density diverges, so it resembles a left edge from that side.
- (v) *The split Fermi sea bulk:* For $1 < x < 3/2$, there are two distinct positive solutions $\chi_1 > \chi_2$ of (2.154), and four first order saddle points on the unit circle.
- (vi) *The right edge:* At $x = 3/2$, there are two double saddle points at $\chi = \pm\pi/3$.
- (vii) *The empty region:* For $x > 3/2$, again there are no solutions to (2.154), and we have an empty region.

As detailed in [BS21, Section 3.3], the limit shape computation proceeds similarly to the one in Section 2.3.2. To compute the large θ limit of $K(x\theta + s, x\theta + t)$ for finite s, t at x in the split Fermi sea bulk, the contours c_+, c_- should now be exchanged for the arcs of the unit circle $c_{+\chi_1, \chi_2} : e^{i\phi}, \phi \in [\chi_2, \chi_1]$ and $c_{-\chi_1, \chi_2} : e^{i\phi}, \phi \in [-\chi_1, \chi_2]$. In the limit, the limiting discrete sine kernel then generalises to

$$K(s, t) = \frac{\sin \chi_1(s - t)}{\pi(s - t)} - \frac{\sin \chi_2(s - t)}{\pi(s - t)} \tag{2.155}$$

⁶In the conventions of [BS21], our γ_1 is equal to $1/2$ and our γ_2 is $\alpha/2$. The authors only give formulas for $\alpha \geq 0$ but note that negative α limit shapes can be obtained via Proposition 2.5; this tells us that if the limiting density at α is ϱ , the limiting density at $-\alpha$ is $\varrho^-(x) = 1 - \varrho(-x)$.

with limiting density $\varrho(x) = (\chi_1 - \chi_2)/\pi$. Beyond the limit shape itself, the fluctuations at the “internal edge” may be interesting to study, since we have first and second order saddle points on the same contour integral.

Looking at the right edge, we can consider the natural gap probability $\mathbb{P}(\lambda_1 < \frac{3}{2}\theta + s\theta^{1/3})$ in the $\theta \rightarrow \infty$ limit starting from the limit of $K_\theta(k, \ell)$ in a regime where $k = \lfloor \frac{3}{2}\theta + x\theta^{1/3} \rfloor - \frac{1}{2}$ and $\ell = \lfloor \frac{3}{2}\theta + y\theta^{1/3} \rfloor - \frac{1}{2}$. Then, using the same approach as before where the contours now pass over two double saddle points at $e^{\pm i\pi/3}$, we find that the leading order term in $\theta^{1/3}K_\theta(k, \ell)$ as $\theta \rightarrow \infty$ is

$$\tilde{K}(k, \ell) = \cos \frac{\pi}{3}(k - \ell) \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}-1} \int_{i\mathbb{R}+1} \frac{\exp[\frac{\zeta^3}{3} - x\zeta] d\zeta d\omega}{\exp[\frac{\omega^3}{3} - y\omega] \zeta - \omega} \quad (2.156)$$

Due to the oscillating prefactor, we do not recover a limiting edge kernel. However, we may approximate the edge gap probability by the discrete Fredholm determinant $\det(1 - \tilde{K})$ on $l^2(\mathbb{Z}_{\geq 0} + \lfloor \frac{3}{2}\theta + x\theta^{1/3} \rfloor - \frac{1}{2})$. Now, within the Fredholm determinant, the sums over elements of $\mathbb{Z}_{\geq 0} + \lfloor \frac{3}{2}\theta + s\theta^{1/3} \rfloor - \frac{1}{2}$ integrate out the products of cosines in the minors of \tilde{K} ; by this reasoning, we expect the limiting distribution to be

$$\begin{aligned} \lim_{\theta \rightarrow \infty} \mathbb{P} \left[\frac{\lambda_1 - \frac{3}{2}\theta}{\theta^{1/3}} < s \right] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_s^{\infty} \cdots \int_s^{\infty} \det_{1 \leq i, j \leq n} \tilde{\mathcal{A}}(x_i, x_j) dx_1 \cdots dx_n, \quad (2.157) \\ &=: \tilde{F}(s), \quad \tilde{\mathcal{A}}(x, y) = \begin{cases} \mathcal{A}(x, y), & x = y \\ \frac{1}{2}\mathcal{A}(x, y), & x \neq y. \end{cases} \end{aligned}$$

It would be interesting (and important) to put our arguments on a rigorous footing. We note that this approach recovers the same distribution for the edge statistics obtained from double saddle points at the edge of any two-interval Fermi sea once those saddle points are at an angle that is a rational multiple of π . The generalisation to a two-interval Fermi sea with a pair of order $2m$ saddle points on the edge is quite direct, the generalisation to a Fermi sea with more cuts is somewhat more complicated. Since the integral operator of $\tilde{\mathcal{A}}$ on $L^2(\mathbb{R})$ is discontinuous, it is not clear if we can write the above formula directly as a Fredholm determinant, but we note that the kernel is Hermitian and ostensibly meets the criteria of the Macchi–Soshnikov theorem 1.44. To make sense of the distribution (2.157), we can look at its logarithm

$$\log \tilde{F}(s) = \sum_{n=1}^{\infty} \frac{1}{n} \int_s^{\infty} \cdots \int_s^{\infty} \tilde{\mathcal{A}}(x_1, x_2) \tilde{\mathcal{A}}(x_2, x_3) \cdots \tilde{\mathcal{A}}(x_{n-1}, x_n) dx_1 \cdots dx_n \quad (2.158)$$

which can be directly compared to a well-defined Fredholm determinant, since the diagonal terms contribute only to the $n = 1$ term of the sum. Hence,

$$\log \tilde{F}(s) = - \int_s^{\infty} \frac{1}{2} \mathcal{A}(x, x) dx + \log \det(1 - \frac{1}{2}\mathcal{A})_{L^2([s, \infty))}, \quad (2.159)$$

and we further expect that

$$\lim_{\theta \rightarrow \infty} \mathbb{P} \left[\frac{\lambda_1 - \frac{3}{2}\theta}{\theta^{1/3}} < s \right] = e^{-\frac{1}{2} \int_s^{\infty} \mathcal{A}(x, x) dx} \det(1 - \frac{1}{2}\mathcal{A})_{L^2([s, \infty))}. \quad (2.160)$$

Random matrices and random maps

In this chapter we discuss new and established connections between random partitions and their continuous analogue, random matrices. The first section, adapted from [1], presents an exact equivalence of edge distributions under the multicritical Schur measures introduced in Chapter 2 and partition functions of certain models of random unitary matrices, explaining the asymptotic appearance of the same Painlevé II hierarchy equations for edge distributions in momentum space trapped fermion models in [LDMS18] and for free energies of matrix models previously studied and called multicritical by Periwal and Shevitz [PS90b]. We give an informal overview of the notion of multicriticality considered in that work, and show that the models corresponding to our Schur measures are also multicritical. The second section discusses models of random Hermitian matrices, in particular the Gaussian unitary matrix ensemble (GUE), whose asymptotic edge distribution coincides with that of the Plancherel measure by the earlier work of Tracy and Widom [TW93]. We discuss the asymptotic behaviour of these models by relating them to free fermions and to the combinatorics of discretised surfaces called maps, which we briefly discuss in their own right. Random matrices and random maps each comprise important models of universality.

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3.1 Unitary matrix models and edge distributions of Schur measures

We consider once again the Schur measures, and in particular the multicritical ones of Definition 2.1. The discussion of this section will be somewhat analogous to that of Section 2.2.2: there we presented fermion models corresponding to Schur measures and discussed multicriticality in the fermion picture, and here we present random matrix

models corresponding to Schur measure and discuss multicriticality for the corresponding matrices. It is worth making two distinctions. Firstly, whereas the asymptotic analysis of statistics on the fermion and the Schur function sides were identical (boiling down to exactly the same DPP kernel), the direct asymptotic analysis of corresponding matrix models is rather different, and in some cases complimentary. Secondly, whereas in Section 2.2.2 we found multicritical analogues both for the nearest-neighbour hopping lattice fermion model corresponding exactly to the Poissonised Plancherel measure and for the harmonic potential continuum model asymptotically coinciding with it on the edge, we only have a partial multicritical generalisation of the matrix model picture associated with the Plancherel measure.

3.1.1 Matrix models correspondences for multicritical measures

Random matrices and Plancherel random partitions In the Plancherel case, there are two main matrix model correspondences. The first, in order of discovery, is exact. In [Joh98], Johansson used an identity of Rains [Rai98] to equate the cumulative edge distribution $\mathbb{P}_\theta(\lambda_1 < \ell)$ of a random partition λ under the Poissonised Plancherel measure \mathbb{P}_θ with a certain integral over $\ell \times \ell$ unitary matrices which had previously appeared as the partition function (or normalisation) of a random matrix model introduced by Gross and Witten [GW80] and, independently, Wadia [Wad80] as a simple lattice field theory with a degree ℓ unitary gauge symmetry group $\mathcal{U}(\ell)$. This correspondence led to an alternative proof of the fact that, $\theta \rightarrow \infty$, λ_1/θ converges in probability to 2 (which, after application of a de-Poissonisation Lemma also proven in [Joh98], is part of Vershik, Kerov, Logan and Shepp's [VK77, LS77] early limit shape theorem 1.35 for the Plancherel measure); in the corresponding $\mathcal{U}(\ell)$ Gross–Witten–Wadia (GWW) model, the 2θ threshold for ℓ corresponds to a *strong-to-weak coupling phase transition*, where there is a discontinuity in the third derivative of the free energy.

The second correspondence is due to Baik, Deift and Johansson [BDJ99], who famously proved that as $\theta \rightarrow \infty$ the same edge distribution $\mathbb{P}_\theta(\lambda_1 < \ell)$ in a regime where $\ell \sim 2\theta + s\theta^{1/3}$ (or, after de-Poissonisation, $\mathbb{P}_n(\lambda_1 < \ell)$ where $\ell \sim 2\sqrt{n} + sn^{1/6}$ as $n \rightarrow \infty$) is equivalent to an analogous asymptotic distribution for the maximal eigenvalue ζ_{\max} of an $N \times N$ GUE random matrix, namely to $\mathbb{P}(\zeta_{\max} < \ell)$ with $\ell \sim \sqrt{2N} + s2^{-1/2}N^{-1/6}$ as $N \rightarrow \infty$. This is the BDJ theorem 1.37 in terms of Tracy and Widom's [TW93] seminal results on level spacing at the edge of the GUE, where the limiting Fredholm determinant distribution F_{GUE} first appeared. For a proof on the Poissonised Plancherel measure side, see Section 2.3.3 and restrict to the case $m = 1$ and to $\gamma = (1, 0, 0, \dots)$. A strong version of this asymptotic correspondence is that the first finite number of parts of a Plancherel random partition of n converge in probability to the largest eigenvalues in the GUE as $n \rightarrow \infty$; this was proven using a connection with fermions in [BOO00] and [Joh01], having previously been proven by Okounkov using an interpretation of GUE expectations as generating functions of discrete surfaces called maps. We will discuss these connections in Section 3.2 and Section 3.2.3 respectively.

The asymptotic edge correspondence with the GUE has a positive temperature analogue. As shown in [BB19], the edge fluctuations of the larger element of a two partition sequence under the *cylindric Plancherel measure* are asymptotically driven by the positive temperature analogue F_{GUE}^α of the TW-GUE distribution (see Theorem 2.16, and Section 2.4.1 for elements of proof in a more general case). The same distribution governs the edge fluctuations in the *Moshe–Neuberger–Shapiro model* [MNS94] random Hermitian matrix model.

For the remainder of this section we discuss the correspondence which we can generalise

to the multicritical measures.

Extension to multicritical measures The exact unitary matrix integral expression for the edge distribution generalises directly to our multicritical Schur measures, defining multicritical analogues for the corresponding unitary matrix models, as follows:

Theorem 3.1 (Multicritical unitary matrix models from Schur measures). *Let λ be a random partition under a multicritical measure $\mathbb{P}_\theta^m(\lambda) = e^{-\theta^2 \sum_r r \gamma_r^2} s_\lambda[\theta \gamma]^2$ and let ℓ be a positive integer. Then*

$$\mathbb{P}_\theta^m(\lambda_1 \leq \ell) = e^{-\sum_r r \theta^2 \gamma_r^2} \int_{\mathcal{U}(\ell)} e^{\theta \operatorname{tr} \sum_r (-1)^{r-1} \gamma_r (U^r + U^{*r})} \mathcal{D}U =: e^{-\sum_r r \theta^2 \gamma_r^2} Z_\ell \quad (3.1)$$

where the integral is taken over the $\ell \times \ell$ unitary matrices with respect to the Haar measure $\mathcal{D}U$. The probability measure

$$\mathbb{P}(U) \mathcal{D}U = \frac{1}{Z_\ell} e^{\theta \operatorname{tr} \sum_r (-1)^{r-1} \gamma_r (U^r + U^{*r})} \mathcal{D}U \quad (3.2)$$

on unitary matrices defines an order m multicritical matrix model, such that in a limit where $\theta, \ell \rightarrow \infty$ with $\ell \sim b\theta$ where b is the right edge coefficient of \mathbb{P}_θ^m , the limiting eigenvalue density has vanishing with an exponent of $2m$.

At $m = 1$ and $\gamma = (1, 0, 0, \dots)$, (3.1) is precisely the equality used in [Joh98], and the linear potential $V(z) = z$ defines the GWW model [GW80, Wad80]. To make sense of the integral in (3.1) and the Haar measure $\mathcal{D}U$, we use the Weyl integration formula (see e.g. [Mec19, Chapter 1] for a thorough discussion, or [AGZ09, Section 4.1.1] for concise examples): noting that the integrand is invariant under conjugation of the matrix U and using the fact that it is unitary, we can perform a change of variables to the eigenvalues u_1, \dots, u_ℓ of U , each of which lies on the unit circle, and find (for a given function f)

$$\int_{\mathcal{U}(\ell)} e^{f(\operatorname{tr} U)} \mathcal{D}U = \frac{1}{(2\pi i)^\ell \ell!} \oint_{c_1} \dots \oint_{c_1} \prod_{i=1}^{\ell} e^{f(u_i)} \prod_{i < j} |u_i - u_j|^2 \frac{du_1}{u_1} \dots \frac{du_\ell}{u_\ell} \quad (3.3)$$

where $c_1 : |u| = 1$ denotes the unit circle. This is equivalently the expectation of $e^{f(\operatorname{tr} U)}$ in the circular unitary ensemble (CUE), and the joint probability density of the eigenvalues can be read from this expression. One factor coming from the Jacobian is the square of the Vandermonde determinant

$$\prod_{i < j} (u_i - u_j) = \det_{1 \leq i, j \leq \ell} (u_j)^{i-1} =: \Delta, \quad (3.4)$$

others account for the volume of $\mathcal{U}(\ell)$. In terms of the ‘‘potentials’’ $V(z) = \sum_r \gamma_r z^r$ introduced in Section 2.3.1, we recover right hand side of (3.1) by inserting

$$f(z) = -\theta[V(-z) + V(-\bar{z})]. \quad (3.5)$$

Here again we have an analogous expression to (3.1) for the distribution of the length $\ell(\lambda)$, by Proposition 2.5.

The original multicritical unitary matrix models If we insert the coefficients γ for the minimal multicritical measure $\mathbb{P}_\theta^{a,m}$ as in Definition 2.6 into the Schur measure, then the

corresponding unitary matrix model by Theorem 3.1 are precisely the ones previously studied by Periwal and Shevitz [PS90b, PS90a]; the same coefficients (differing by an overall factor of $m/(m+1)$) are found in [PS90a, Page 737] (the authors did not consider a “symmetric” potential with only odd terms, such as the ones given in Definition 2.8). In these references, the authors used orthogonal polynomials on the unit circle to compute the partition function Z_ℓ of a $\mathcal{U}(\ell)$ unitary matrix model with probability density of the form of (3.2) in a regime where $\ell := \lfloor x\theta \rfloor$ and as $\theta \rightarrow \infty$. The multicritical potentials were found starting from a potential of the form of (3.5), by tuning a minimum number of nonzero finite coefficients γ_r such that the a particular function whose square is the second derivative of the *free energy* $\mathcal{F}(x) := \lim_{\ell \rightarrow \infty} \ell^{-2} \log Z_\ell$ in that limit would satisfy an equation in x which has $2m$ degenerate solutions at some positive coupling $x = b$ (which turns out to be precisely the right edge coefficient of Definition 2.1).

For each of these minimal multicritical potentials, the authors studied the same second derivative in a critical scaling regime $\ell := \lfloor b\theta + s\theta^{1/(2m+1)} \rfloor$. They found that, if $\mathcal{F}''(b + s\theta^{-2m/(2m+1)}) \rightarrow q^2(s)$ as $\theta \rightarrow \infty$, $q(s)$ would satisfy the m th equation of the Painlevé II hierarchy. The seemingly coincidental appearance of the same integrable hierarchy in these models and in their momentum space flat trap fermion models (described in Section 2.2.1) was noted by Le Doussal, Majumdar and Schehr in [LDMS18]. In light of above correspondence, the origin of that coincidence is explained: by Theorem 2.2, the distribution $\mathbb{P}(\lambda_1 < \ell)$ converges in the critical scaling regime to the higher-order TW-GUE distribution $F_{2m+1}(s)$. Then by Theorem 3.1, the partition function Z_ℓ converges to the same distribution too, up to a prefactor that does not depend on s , and the second derivative of the free energy can be identified with $q^2(s)$ of Theorem 2.3, which is a rigorous result from [CCG19] showing that $q(s)$ satisfies the same Painlevé II equation. In this sense, Theorems 3.1, 2.2 and 2.3 together give a rigorous proof of the orthogonal polynomial calculation of Periwal and Shevitz in [PS90b]. Equally, the unitary matrix model picture gives a new heuristic derivation of the Painlevé II equations for the higher-order TW-GUE distribution.

The eigenvalue density In [PS90b], the authors also found the explicit limiting eigenvalue density $\varrho(\alpha)$ at $e^{-i\alpha}$ on the unit circle, with $\alpha \in [-\pi, \pi]$ (satisfying $\int \varrho(\alpha) d\alpha = 1$), for any potential of degree up to 4. This generalises a calculation detailed in [GW80], and when the potential is multicritical and the coupling tuned to the critical value of $\theta = \ell/b$, the salient feature is that the density goes to zero at π with a vanishing exponent of $2m$; to be precise, in terms of the fluctuation coefficient of Definition 2.1, one can show that

$$\varrho(\alpha) \sim \frac{1}{2\pi} \frac{d}{b} (\pi - \alpha)^{2m}, \quad \alpha \rightarrow \pi. \quad (3.6)$$

In this respect, Periwal and Shevitz’s multicritical matrix models are natural unitary analogues of Kazakov’s earlier multicritical Hermitian matrix models [Kaz89], which exhibit similar behaviour at the edge of the eigenvalue support on the real line. Recalling the vanishing exponents of the limit shapes and limiting fermion densities associated with the multicritical measures as stated in Theorem 2.4, this is a kind of dual behaviour.

The phase transition In terms of the coupling $\theta = \theta(x) := \ell/x$ (we opt in this case to let the matrix size ℓ be a large parameter and let the coupling θ scale accordingly), the point $x = b$ corresponds to a phase transition for the partition function Z_ℓ as ℓ tends to infinity; given Theorem 3.1 and by the multicritical limit shape theorem 2.4, we can see

immediately that as $\ell \rightarrow \infty$

$$e^{-\theta^2 \sum_r r \gamma_r^2} Z_\ell \rightarrow \begin{cases} 0, & x < b \\ 1, & x > b \end{cases}. \quad (3.7)$$

In the $\gamma = (1, 0, 0, \dots)$, $m = 1$ case, by studying the GWW unitary matrix model directly, the free energy was shown to behave in this regime as follows [GW80, Wad80]¹ (in this case, $b = 2$):

$$\mathcal{F}(x) := \lim_{\ell \rightarrow \infty} \frac{1}{\ell^2} \log Z_\ell = \begin{cases} \frac{2}{x} - \frac{3}{4} + \frac{1}{2} \log \frac{x}{2}, & x \in (0, 2] \\ \frac{1}{x^2}, & x > 2, \end{cases} \quad (3.8)$$

and this was the asymptotic result used to find the edge of the limit shape under the Plancherel measure in [Joh98] (where there is also a self-contained proof of the identity above). At $x = 2$, the third derivative $\mathcal{F}'''(x)$ is discontinuous; this associates the generic edge with a third-order phase transition. In [Joh98], this limit behaviour also gave a new proof of a deviation bound for the first part of a Plancherel random partition of Deuschel and Zeitouni [DZ99] (the same bound will be useful to us in the next chapter; see Proposition 4.20).

The association of this phase transition with TW-GUE universal edge behaviour has broad ramifications. In [MS14], Majumdar and Schehr related it to behaviour of the maximal eigenvalue in the GUE and in more general Gaussian ensembles, by studying large deviations and associating each tail with a side of the $x = 2$ coupling threshold of the GWW model. There, the break in the eigenvalue support at the critical coupling was related to the application of a “hard wall” to the GUE, in which the real eigenvalues are constrained to lie below a certain point. The authors make a particularly intuitive link with an early model of a complex system considered by [May72], in which a random matrix determined couplings in the linear evolution of a vector of populations. There, there is a stability threshold for the population vector in terms of the maximum eigenvalue of the random coupling matrix, above which the system is stable and weakly interacting, below which the system is strongly interacting and unstable. For a simply critical Schur measure and the corresponding fermion model, we recognise the same kind of stable, weakly interacting phase in the “empty region” $x > b$ the edge of the limit shape, and identify the “bulk” at $x < b$ with the strongly interacting phase.

Le Doussal, Majumdar and Schehr suggested that the multicritical analogue of this phase transition would have a new scaling exponent of $2 + 1/m$ [LDMS18]. This question was recently considered by Kimura and Zahabi [KZ21a], who found that, starting from the asymptotics of the higher-order TW-GUE distribution found in [LDMS18] (stated here in Theorem 2.3) for the partition function Z_ℓ corresponding to an order m multicritical Schur measure \mathbb{P}_θ^m , one can make a general approximation of $\mathcal{F}(x)$ with

$$\frac{1}{\ell^2} \log Z_\ell = \begin{cases} \mathcal{F}_c + C d^{-\frac{1}{m}} (x - b)^{\frac{2m+1}{m}} + O(e^{-c\ell}), & x \in (0, b) \\ \mathcal{F}_c + O(\ell^{-2}), & x > b, \end{cases} \quad \text{as } \ell \rightarrow \infty \quad (3.9)$$

where \mathcal{F}_c , c and C are constants; see [KZ21a, Section 5.3] for details and explicit formulas. Here again we find a discontinuity only in the third derivative $\mathcal{F}'''(x)$ at $x = b$, but the scaling exponent associated with this third-order phase transition generalises from 3 to $2 + 1/m$ (in the terminology of [PS90a, LDMS18, KZ21a], the order of such a

¹This explicit expression appears in [Joh98, Lemma 2.1], rather than [GW80] or [Wad80] directly; the coefficient x in $\mathcal{F}(x)$ in our conventions corresponds to $2/\gamma$ in Johansson’s $f(\gamma)$. To compare notations with [GW80], our x corresponds to their λ and our θ corresponds to $1/g^2$.

phase transition is $2 + 1/m$). Interestingly, the authors noted that, by the above estimate, as the order of multicriticality m tends to infinity, the phase transition should become a second order one; some related asymptotics of the higher-order TW-GUE distribution are given in [KZ21a, Section 5.3.2]. As mentioned in Section 2.4.3, it would be particularly interesting to study this regime rigorously, starting from analysis of a minimal multicritical Schur measure in this limit.

3.1.2 An equivalence of Toeplitz and Fredholm determinants

Let us turn to proving the main part of Theorem 3.1, the unitary matrix integral expression for the edge distribution under a multicritical Schur measure. In fact, this is a special case of an identity holding for any Schur measure, aspects of which have been used in various works such as [BO00] and [BR01a] – the multicritical case and the particularities of Periwal and Shevitz’s approach will be discussed in the next section. In the form we find to be the clearest, it states that

Theorem 3.2 (Unitary matrix integral edge distribution of any Schur measure). *If λ is a random partition under a Schur measure $\mathbb{P}(\lambda) = e^{-\sum_r r t_r t'_r} s_\lambda[t] s_\lambda[t']$ for some sequences of Miwa times t, t' , then for all positive integers ℓ*

$$e^{\sum_r r t_r t'_r} \mathbb{P}(\lambda_1 \leq \ell) = \det_{1 \leq i, j \leq \ell} f_{j-i} = \int_{\mathcal{U}(\ell)} e^{\text{tr} \sum_r (-1)^{r-1} (t_r U^r + t'_r U^{*r})} \mathcal{D}U \quad (3.10)$$

where the symbol of the central Toeplitz determinant is

$$\sum_{n \in \mathbb{Z}} f_n z^n = \exp \left[\sum_{r \geq 1} (-1)^{r-1} (t_r z^r + t'_r z^{-r}) \right]. \quad (3.11)$$

From the DPP formulation of Schur measures, and specifically Theorem 1.45, the distribution on the left is the discrete Fredholm determinant

$$\mathbb{P}(\lambda_1 \leq \ell) = \det(1 - K)_{l^2(\ell + \mathbb{Z}_{>0})} \quad (3.12)$$

where K is the integral operator of the kernel in (1.150). In [BO00], Borodin and Okounkov used the first equality, which comes from *Gessel’s theorem* from [Ges90], to prove a general Fredholm determinant formula for any determinant of a *Toeplitz matrix*, that is a matrix whose entries $A_{i,j} = A_{i+n,j+n}$ are the same along each diagonal. The reference [BO00] includes further analytic formulations, which equally apply here. The symbol (3.11) of the Toeplitz matrix is a Fourier transform of its elements. The second equality arises in the context of Fourier analysis, and is called *Heine’s identity* (see e.g. [For10]).

Slightly generalising Proposition 2.5, we can see that this implies that

Corollary 3.3 (Equivalent conjugate partition formulation of Theorem 3.2). *If λ is a random partition under a Schur measure $\mathbb{P}(\lambda) = e^{-\sum_r r t_r t'_r} s_\lambda[t] s_\lambda[t']$ for suitable sequences of Miwa times t, t' , then for all positive integers ℓ*

$$e^{\sum_r r t_r t'_r} \mathbb{P}(\ell(\lambda) \leq \ell) = \det_{1 \leq i, j \leq \ell} g_{j-i} = \int_{\mathcal{U}(\ell)} e^{\text{tr} \sum_r (t_r U^r + t'_r U^{*r})} \mathcal{D}U \quad (3.13)$$

where the symbol of the central Toeplitz determinant is

$$\sum_{n \in \mathbb{Z}} g_n z^n = \exp \left[\sum_{r \geq 1} (t_r z^r + t'_r z^{-r}) \right]. \quad (3.14)$$

We note in passing that as $\ell \rightarrow \infty$, the first equality in both versions of the theorem recovers a form of the strong Szegő theorem [Sim05]:

$$\lim_{\ell \rightarrow \infty} \det_{1 \leq i, j \leq \ell} f_{j-i} = \lim_{\ell \rightarrow \infty} \det_{1 \leq i, j \leq \ell} g_{j-i} = \exp \sum_{r \geq 1} r t_r t'_r. \quad (3.15)$$

Proof of Theorem 3.2 . The conjugate formulation in Corollary 3.3 is slightly neater, so we prove it first.

The first equality of Corollary 3.3 may be written

$$\sum_{\lambda: \ell(\lambda) \leq \ell} s_\lambda[t] s_\lambda[t'] = \det_{1 \leq i, j \leq \ell} g_{j-i}, \quad (3.16)$$

which, with g_n as defined in (3.14), was proven by Gessel in [Ges90]. It follows from Definition 1.26 of the Schur measure by the Jacobi–Trudi formula, which gives

$$\sum_{\lambda: \ell(\lambda) \leq \ell} s_\lambda[t] s_\lambda[t'] = \sum_{\lambda} \det_{1 \leq i, j \leq \ell} h_{\lambda_i - i + j}[t] \det_{1 \leq i, j \leq \ell} h_{\lambda_i - i + j}[t'] \quad (3.17)$$

where $h_i[t]$ denotes the complete homogeneous symmetric functions specialised in the Miwa times to t as in Definition 1.23, with generating function

$$\sum_i h_i[t] z^i = e^{\sum_r t_r z^r} =: H[t; z] \quad (3.18)$$

and $h_i = 0$ for all $i < 0$. The expression (3.17) is a sum of products of $\ell \times \ell$ minors of the non-square Toeplitz matrices

$$\begin{aligned} H &= (H_{a,b})_{\substack{1 \leq a \leq \ell, \\ 1 \leq b < \infty}}, & H_{a,b} &= h_{b-a}[t] \\ H' &= (H'_{a,b})_{\substack{1 \leq a \leq \infty, \\ 1 \leq b < \ell}}, & H'_{a,b} &= h_{a-b}[t']. \end{aligned} \quad (3.19)$$

The Cauchy–Binet identity (see e.g. [Ait56, Chapter IV] or [For19]) gives, for any matrices A, B such that AB has dimensions $\ell \times \ell$,

$$\sum_{\substack{L \subset \{1, 2, \dots\} \\ |L| = \ell}} \det A|_L \det B|_L = \det AB \quad (3.20)$$

where $A|_L$ denotes the $\ell \times \ell$ submatrix of A including the rows indexed by L ; noting that $H \cdot H'$ is $\ell \times \ell$, this gives

$$\sum_{\lambda: \ell(\lambda) \leq \ell} s_\lambda[t] s_\lambda[t'] = \sum_{\substack{L \subset \{1, 2, 3, \dots\} \\ |L| = \ell}} \det H|_L \det H'|_L = \det H \cdot H' \quad (3.21)$$

and the entries of the final matrix product are

$$(H \cdot H')_{a,b} = \sum_i h_{i-a}[t] h_{i-b}[t'] = \sum_i h_{i-a+b}[t] h_i[t'], \quad 1 \leq a, b \leq \ell \quad (3.22)$$

(the sum over i can run over all integers thanks to the convention $h_i = 0$ for $i < 0$). Thus $H \cdot H'$ is a Toeplitz matrix, and its symbol is (below we use $z^a = z^{k+a} z^{-k}$)

$$\begin{aligned} \sum_n z^n \sum_i h_{i+n}[t] h_i[t'] &= \sum_n \sum_i z^{i+n} h_{i+n}[t] z^{-i} h_i[t'] \\ &= H[t; z] H[t'; 1/z] = \exp \left[\sum_{r \geq 1} (t_r z^r + t'_r z^{-r}) \right]. \end{aligned} \quad (3.23)$$

This is precisely the Toeplitz determinant symbol generating the entries g_n in the statement, so this proves the first equality.

To prove the second equality, or Heine's identity, we use the Cauchy–Binet identity in its continuous form; this is called the Andreief identity, see e.g. [For19]. For some space R equipped with a measure μ_r and integrable functions Φ_i, Ψ_i on R for $1 \leq i \leq \ell$, this identity gives

$$\int_R \cdots \int_R [\det \Phi_i(z_j)] \cdot [\det \Psi_i(z_j)] d\mu_r(z_1) \cdots d\mu_r(z_\ell) = \ell! \det \left[\int_R \Phi_i(z) \Psi_j(z) d\mu_r(z) \right] \quad (3.24)$$

where each determinant is over indices $1 \leq i, j \leq \ell$. We apply this to the unitary matrix integral on the right hand side, first writing it as an ℓ -fold contour integral on the unit circle

$$\frac{1}{(2\pi i)^{\ell \ell!}} \oint_{c_1} \cdots \oint_{c_1} \prod_{i=1}^{\ell} e^{\sum_{r \geq 1} (t_r u_i^r + t'_r u_i^{-r})} \prod_{i < j} |u_i - u_j|^2 \frac{du_1}{u_1} \cdots \frac{du_\ell}{u_\ell} \quad (3.25)$$

and then as an ℓ -fold integral over determinants: for the squared Vandermonde determinant, we have

$$\prod_{i < j} (u_i - u_j)(\bar{u}_i - \bar{u}_j) = \prod_{i < j} (u_i - u_j)(u_i^{-1} - u_j^{-1}) = \det_{1 \leq i, j \leq \ell} u_j^{i-1} \cdot \det_{1 \leq i, j \leq \ell} u_j^{1-i} \quad (3.26)$$

since $|u_i| = 1$, then split $\prod_i e^{\sum_{r \geq 1} (t_r z^r + t'_r z^{-r})}$ across each determinant to see that this integral is equal to the left hand side of the Andreief identity expression (3.24) where we insert

$$R = c_1, \quad d\mu_r(u) = \frac{du}{2\pi i u}, \quad \Phi_i(u) = u^{1-i} e^{\sum_{r \geq 1} t_r u^r}, \quad \Psi_i(u) = u^{i-1} e^{\sum_{r \geq 1} t'_r u^{-r}}. \quad (3.27)$$

Then, the right hand side of the identity is

$$\det_{1 \leq i, j \leq \ell} \oint_{c_1} e^{\sum_{r \geq 1} (t_r u^r + t'_r u^{-r})} \frac{du}{u^{i-j+1}} \quad (3.28)$$

where the integral extracts precisely the Toeplitz matrix entry g_{i-j} . This gives the second equality and completes the proof.

To prove the original equalities of Theorem 3.2 directly, we can proceed analogously from the dual Jacobi–Trudi formula Theorem 1.31. We have

$$e^{\sum_r r t_r t'_r} \mathbb{P}(\lambda_1 \leq \ell) = \sum_{\lambda: \lambda_1 \leq \ell} \det_{1 \leq i, j \leq \ell} e_{\lambda'_i - i + j}[t] \det_{1 \leq i, j \leq \ell} e_{\lambda'_i - i + j}[t'] \quad (3.29)$$

where $e_i[t]$ are the elementary symmetric functions (see Definition 1.23), generated by

$$\sum_i e_i[t] z^i = e^{\sum_r (-1)^{r+1} t_r z^r} =: E[t; z]. \quad (3.30)$$

Repeating the arguments above, we find

$$e^{\sum_r r t_r t'_r} \mathbb{P}(\lambda_1 \leq \ell) = \det_{1 \leq a, b \leq \ell} \sum_i e_{i-a+b}[t] e_i[t'] \quad (3.31)$$

and the symbol of the Toeplitz determinant is

$$\sum_n z^n \sum_i e_{i+n}[t] e_i[t'] = E[t; z] E[t'; 1/z] = e^{\sum_r (-1)^{r+1} (t_r z^r + t'_r z^{-r})} \quad (3.32)$$

which is the symbol in the statement; so, we have a determinant of f_{i-j} , proving the first equality. This is the dual version of Gessel's theorem.

Now once again we can start from the rightmost unitary matrix integral, and write it in the form of the left hand side of the Andreief identity (3.24) with the same insertions (3.27), except for

$$\Phi_i(u) = u^{1-i} e^{\sum_{r \geq 1} (-1)^{r+1} t_r u^r}, \quad \Psi_i(u) = u^{i-1} e^{\sum_{r \geq 1} (-1)^{r+1} t_r u^{-r}}; \quad (3.33)$$

then, the right hand side of the identity gives us the determinant of a contour integral which extracts the coefficient f_{i-j} from the generating function $E[t; z]E[t'; 1/z]$, proving the second equality.

Of course, the same equalities can be derived from Corollary 3.3 by applying the classical involution $\omega : s_\lambda \leftrightarrow s_{\lambda'}$ of Theorem 1.31 and then, by Proposition 1.32, applying it to the Miwa times, where $\omega(t_r) = (-1)^{r+1} t_r$ and $\omega(t'_r) = (-1)^{r+1} t'_r$. \blacksquare

Extensions and the question of positive temperature With Theorem 3.2 proven, let us note that it can be applied to generalise our original Theorem 3.1, for instance to the generalised multicritical Schur measures of Section 2.4.2, whose asymptotic edge distributions were related to the Painlevé II hierarchy in [CCG19] (we note that the more subtle scaling regimes could be related to the “genus expansions” considered in [PS90a]).

One interesting possible extension would be to periodic Schur measures (see Sections 1.2.3 and 2.4.1). In simplest rank 1 cases presented here, we would need a generalisation of Gessel's theorem to sums of weighted minors of fixed size of bi-infinite matrices. Since we can “pass to the grand canonical ensemble”, and recover a measure with a Fredholm determinant edge distribution, we can hope that the same procedure finds a Toeplitz determinant distribution.

3.1.3 The large matrix limit and multicriticality

Let us now discuss how the unitary matrix models defined by the measure (3.2) are multicritical, first outlining the different approach to tuning a potential to multicriticality used in [PS90b] which happens to coincide with our tuning of Miwa time coefficients in a Schur measure, then showing that our definition is sufficient to have a multicritical vanish exponent in the eigenvalue density to complete the proof of our main result (we present a rather short analysis).

The free energy with orthogonal polynomials The approach used in [PS90b] is to write the partition function Z_ℓ for the $\ell \times \ell$ matrix model in terms of a family polynomials $\{P_i(z) = z^i + \sum_{j=1}^{i-1} a_{i,j} z^j\}$ for i from 0 to ℓ on the unit circle which are orthogonal with respect to a measure μ_r induced by the eigenvalue density, with

$$\oint_{c_1} P_i(z) P_j(z^{-1}) d\mu_r(z) := \frac{1}{2\pi i} \oint_{c_1} P_i(z) P_j(z^{-1}) e^{-\theta[V(-z)+V(-z^{-1})]} \frac{dz}{z} = T_i \mathbf{1}_{i=j} \quad (3.34)$$

(here, we let $V(z) = \sum_r \gamma_r z^r$ denote an arbitrary potential, with γ_r not yet fixed). Then, the Vandermonde determinant Δ may be written $\Delta = \det_{1 \leq i, j \leq \ell} P_i(u_j)$ so that we have

$$Z_\ell = \frac{1}{(2\pi i)^\ell \ell!} \oint_{c_1} \cdots \oint_{c_1} |\Delta|^2 \prod_{i=1}^{\ell} e^{-\theta(V(-u_i)+V(-u_i^{-1}))} \frac{du_i}{u_i} = \prod_{i=1}^{\ell} T_i. \quad (3.35)$$

Writing a recurrence relation for the polynomials

$$P_{i+1}(z) = z P_i(z) + R_i z^i P_i(z^{-1}), \quad (3.36)$$

the normalisation gives (by some standard manipulations of orthogonal polynomials with respect to a measure which is invariant under inversion, see e.g. [Sim05]) that $T_{i+1}/T_i = 1 - R_i^2$, and so

$$Z_\ell = \prod_{i=1}^{\ell} \left(\frac{T_i}{T_{i-1}} \right)^{\ell-i} = \prod_{i=1}^{\ell} (1 - R_{i-1}^2)^{\ell-i}. \quad (3.37)$$

This coefficient is related to the free energy $\mathcal{F}(x) = \lim_{\ell \rightarrow \infty} \ell^{-2} \log Z_\ell$ in the regime $\theta := \ell/x$ as $\ell \rightarrow \infty$ by the heuristic

$$1 - R_\ell^2 = \frac{Z_{\ell+1} Z_{\ell-1}}{Z_\ell^2} \rightarrow 1 - \mathcal{F}''(x). \quad (3.38)$$

Hence, R_ℓ in the limit $\ell \rightarrow \infty$ is a heuristic approximation of the root of the second derivative of the free energy.

The tuning to order m multicriticality involves finding a potential V for which $R(x) := R_{[x\theta]}$ satisfies an equation with $2m$ degenerate solutions as $\theta \rightarrow \infty$ at some x . The main tool used is the following relation, found from $d/dz \oint_{c_1} P_{\ell+1}(z) P_\ell(z^{-1}) d\mu_r = 0$:

$$\begin{aligned} (\ell + 1) \frac{R_\ell^2}{1 - R_\ell^2} &= \frac{\ell}{x} \oint_{c_1} \left[V'(-z) - \frac{1}{z^2} V'(-z^{-1}) \right] P_{\ell+1}(z) P_\ell(z^{-1}) d\mu_r \\ &= \frac{\ell}{x} \oint_{c_1} \frac{1}{z} \sum_{r \geq 1} r (-1)^r \gamma_r (z^r - z^{-r}) P_{\ell+1}(z) P_\ell(z^{-1}) d\mu_r. \end{aligned} \quad (3.39)$$

The right hand side can also be expressed as a combination of coefficients $R_{\ell+i}$ where, if the potential V has degree D , i ranges from $-D$ to D . Making a continuity assumption, these coefficients are all approximated by $R(x)$ in the limit, and we recover a polynomial in $R(x)$ of degree $2D$, with coefficients determined by the potential coefficients γ_r . In [PS90a, Section 2], a general procedure to tune γ_r and x to have an order $2m$ degeneracy is described; with the constraint that $D = m$, the authors find a general expression for the final contour integral above, and by some non-trivial calculations find precisely the coefficients of the minimal multicritical measure and critical coupling $x = b$ given at Definition 2.6. It is natural to expect the general multicriticality conditions of Definition 2.1 to be a simple solution to this problem as well, it would be interesting to show this.

It is from the same equation (3.39) above that, after setting the potential to be multicritical and setting $q(s) := R_\ell$ at $\ell := \lfloor b\theta + s\theta^{\frac{1}{2m+1}} \rfloor$ in a large θ limit, that one can recover an order $2m$ ordinary differential equation (using an assumption of smoothness) for $q(s)$ – as mentioned above, the resulting equation is the m th element of the Painlevé II hierarchy [PS90a].

The limiting eigenvalue density A natural statistic to study in the large matrix and large coupling limit is the limiting density of eigenvalues. For unitary matrices, this is a function on the unit circle c_1 ; letting $\boldsymbol{\xi} = \{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_\ell\}$ with $-\pi \leq \boldsymbol{\xi}_j \leq \pi$ denote the set of arguments of the eigenvalues $e^{i\boldsymbol{\xi}_j}$ of a random unitary matrix \mathbf{U} , the eigenvalue density $\varrho \in [0, 1]$ is such that in the limit as $\ell \rightarrow \infty$,

$$\mathbb{P}(\boldsymbol{\xi} \cap [\beta_0, \beta_1]) = \int_{\beta_0}^{\beta_1} \varrho(\alpha) d\alpha, \quad \beta_0, \beta_1 \in [-\pi, \pi] \quad (3.40)$$

(and hence $\int_{-\pi}^{\pi} \varrho(\alpha) d\alpha = 1$). For a measure of the form (3.2) (where for simplicity we put remove the minus signs from the arguments of the potential), in a regime with

coupling $\theta := \ell/x$, this function is computed by a standard steepest descent analysis, following [BIPZ78] and generalising the approach in [GW80]. First, we write the measure induced on the eigenvalue arguments as

$$\exp \left[\frac{\ell}{x} \sum_{j=1}^{\ell} (V(e^{i\alpha_j}) + V(e^{-i\alpha_j})) + \sum_{j \neq k} \log \left| \sin \frac{\alpha_j - \alpha_k}{2} \right| \right] d\alpha_1 \cdots d\alpha_{\ell} \quad (3.41)$$

(note that, thanks to the inversion invariance, the sum of potentials can be written as a polynomial of cosines; in physical terms, this optimisation problem corresponds to finding the equilibrium distribution of a ‘‘Coulomb gas’’ on c_1). Putting the eigenvalue arguments in non-decreasing order $\alpha_1 \leq \alpha_2 \leq \dots$ and defining rescaled functions $\alpha(u) := \alpha_{[u\ell]}$, the $\ell \rightarrow \infty$ corresponds to an optimisation problem for a functional appearing exponentiated in the integrand:

$$\frac{1}{x} \int_0^1 [V(e^{i\alpha(u)}) + V(e^{-i\alpha(u)})] du + \int_0^1 \int_0^1 \log \left| \sin \frac{\alpha(u) - \alpha(v)}{2} \right| du dv \quad (3.42)$$

where \int denotes the Cauchy principal part. Now if the non-decreasing function α is the equilibrium limiting eigenvalue distribution, it is related to the limiting density by $\varrho(\alpha) = du/d\alpha$, and we can write the optimising condition above as

$$\frac{i}{x} [e^{i\alpha} V'(e^{i\alpha}) - e^{-i\alpha} V'(e^{-i\alpha})] = \int_0^1 \cot \frac{\alpha - \alpha(v)}{2} dv = \int_{-\beta_c}^{\beta_c} \varrho(\beta) \cot \frac{\alpha - \beta}{2} d\beta \quad (3.43)$$

where the support $[-\beta_c, \beta_c]$ of ϱ is also to be determined.

With this toolbox, we can finally claim that the unitary matrix models of Theorem 3.1 are multicritical.

Proof of Theorem 3.1. Applying Theorem 3.2 to \mathbb{P}_{θ}^m by inserting $t = t' = \theta\gamma$, where γ are the coefficients of a multicritical measure with right edge and fluctuation coefficients b, d , we have the first exact identity (3.1) of the theorem. Now let us show that the measure on unitary matrices (3.2) gives rise to multicritical vanishing in the limiting eigenvalue density.

From the formula (3.43) above for the limiting density in a regime $\theta := \ell/x$ as $\ell \rightarrow \infty$ for some x , we make a shift of variables $\alpha \rightarrow \alpha - \pi$ to account for the change in sign in the potentials, and then we have

$$\begin{aligned} \int_{-\beta_c}^{\beta_c} \varrho(\beta) \cot \frac{\alpha - \beta}{2} d\beta &= -\frac{i}{x} [e^{i(\alpha-\pi)} V'(e^{i(\alpha-\pi)}) - e^{-i(\alpha-\pi)} V'(e^{-i(\alpha-\pi)})] \\ &= -\frac{1}{x} \sum_{r \geq 1} 2r\gamma_r \sin r(\alpha - \pi) \end{aligned} \quad (3.44)$$

and the support of ϱ is $[-\beta_c, \beta_c]$. We will approach the critical point from one side only², and let x be sufficiently large that ϱ is supported on $[-\pi, \pi]$. Then, with the additional requirement $\varrho(\pi) = \varrho(\pi)$, we follow the steps of [GW80, Page 449] and simplify the left hand side using the identity

$$\cot \frac{\alpha - \beta}{2} = \sum_{n=1}^{\infty} \sin n\alpha \cos n\beta - \cos n\alpha \sin n\beta \quad (3.45)$$

²The approach from the subcritical $x < b$ side is much more subtle but is still feasible; we refer to the final equations of [PS90b] for an explicit formula for the density and its support below criticality in any degree 4 potential.

and find

$$\varrho(\alpha) = \frac{1}{2\pi} \left[1 - \frac{1}{x} \sum_{r \geq 1} 2r\gamma_r \cos r(\alpha - \pi) \right]. \quad (3.46)$$

Not that this is indeed well normalised on $[-\pi, \pi]$, and that, using the final condition of (2.1) in the definition of the multicritical potential coefficients, there is a unique minimum at $\alpha = \pi$ we have $\varrho \in (0, 1]$ for all $x > b$. At $x \rightarrow b$, we have the appearance of a single cut as $\varrho(\pi) \rightarrow 0$; developing in $\alpha - \pi$ close to zero, we employ the multicriticality conditions (2.2) and the definitions of b, d once again to find that

$$\varrho(\alpha) \sim \frac{1}{2\pi} \frac{d}{b} (\alpha - \pi)^{2m}, \quad \alpha \rightarrow \pi. \quad (3.47)$$

We recognise the multicritical vanishing exponent of $2m$, and classify the unitary matrix model as multicritical to complete the proof. \blacksquare

In light of the discussion in Section 2.4.3 of the asymptotic analysis of Hermitian Schur measures with a “split Fermi sea”, let us remark that lifting the final condition of (2.1) presents an analytic challenge on the unitary matrix model side too, since multiple cuts can now appear in the density support.

3.2 Hermitian matrix models, fermions and combinatorial maps

In this section we consider models of random Hermitian matrices, in particular the GUE, which is the ensemble of $N \times N$ Hermitian matrices distributed by the probability measure

$$\mathbb{P}(M)\mathcal{D}M = \frac{1}{Z_{\text{GUE}}} e^{-\frac{1}{2} \text{tr} M^2} \mathcal{D}M \quad (3.48)$$

where $\mathcal{D}M$ denotes the Lebesgue measure on Hermitian matrices $\mathcal{H}(N)$, explicitly

$$\mathcal{D}M = \prod_{i=1}^N dM_{ii} \prod_{i < j} dM_{ij} d\bar{M}_{ij}, \quad (3.49)$$

and $Z_{\text{GUE}} = (2\pi)^{N^2/2}$ is a normalisation factor from integration over $\mathcal{H}(N)$ (which we will not consider further). This ensemble also had a natural definition in terms of the entries of a random element M (or as a particular type of *Wigner random matrices*): its diagonal entries M_{ii} are i.i.d. real normal random variables of mean zero and variance 1, its off diagonal entries $M_{i,j \neq i}$ are i.i.d. complex normals with mean zero and variance $1/2$. We refer to [AGZ09, Chapter 3] and to [Meh67, Chapters 3 and 4] as general references.

In studying the edge fluctuations of random partitions, this probably the most natural matrix model correspondence to consider, as the universal TW-GUE distribution can be defined, following [TW93] in which it was first observed, in terms of the maximal eigenvalue ζ_{\max} of a GUE random $N \times N$ matrix as

$$F_{\text{GUE}}(s) := \lim_{N \rightarrow \infty} \mathbb{P} \left[\frac{\zeta_{\max} - \sqrt{2N}}{2^{-1/2} N^{-1/6}} < s \right]. \quad (3.50)$$

The higher-order analogues F_{2m+1} of F_{GUE} found in [LDMS18] (here defined in Theorem 2.2) have, for now, no analogous definition. Without a multicritical analogue for this correspondence, in this section we shall only discuss classical results and some perspectives.

At this point let us be somewhat more precise about the correspondence of the edges of the GUE and the Plancherel measure. As conjectured by Baik, Deift and Johansson [BDJ99] and proven by Okounkov [Ok00b], the equivalence of the asymptotic edge distributions may be improved as follows:

Theorem 3.4 (Convergence to GUE eigenvalues under the Plancherel measure [Ok00b]). *Let $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ be a random partition under the Plancherel measure \mathbb{P}_n , and let $(\zeta_1 \geq \zeta_2 \geq \zeta_3 \geq \dots)$ be the eigenvalues of an $n \times n$ random matrix in the GUE (indexed in non-increasing order), and let*

$$\tilde{\lambda}_i := n^{1/3} \left(\frac{\lambda_i}{2n^{1/2}} - 1 \right), \quad \tilde{\zeta}_i = n^{2/3} \left(\frac{\zeta_i}{2^{1/2}n^{1/2}} - 1 \right) \quad (3.51)$$

for each i . Then, in the limit as $n \rightarrow \infty$, the joint distributions of $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_r$ and $\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_r$ are identical for any fixed positive integer r .

In other words, both $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_r$ and $\tilde{\zeta}_1, \tilde{\zeta}_2, \dots, \tilde{\zeta}_r$ converge in distribution to the same limiting ensemble; this limiting ensemble is the Airy ensemble, that is the DPP on the real line whose kernel is the Airy kernel. The convergence to this ensemble under the Poissonised Plancherel measure is discussed in Section 1.3.3, and implicitly proven as a special case in Section 2.3.3. A goal of this section is to justify the fact that the greatest eigenvalues in the GUE converge to the same ensemble. We opt for a heuristic presentation, via the physical models discussed in the last section and in Sections 1.3.2 and 2.2.2.

3.2.1 From the Gaussian unitary ensemble to free fermions

The convergence of the GUE eigenvalues to the Airy ensemble is, as in the Poissonised Plancherel measure case, the convergence of a discrete space DPP to a continuous space one. Just as in the case of Hermitian Schur measures, the DPP formed by the GUE eigenvalues at finite N is naturally related to physical fermions. In particular: *the joint probability density function (PDF) of the eigenvalues in an $N \times N$ GUE matrix is equivalent to the joint PDF of the positions of N fermions in the ground state in a harmonic potential on the real line.*

Orthogonal polynomials again One way to show this is to apply an orthogonal polynomial approach similar to the one discussed in Section 3.1.3 above. Using the measure's $\mathcal{U}(N)$ invariance to diagonalise the integration variable M (which leads to a factor from the Haar measure of $\mathcal{U}(N)$, see [Mec19] or [AGZ09, Section 4.1.1]), we write a joint eigenvalue distribution induced by (3.48)

$$\mathbb{P}(\zeta_1, \dots, \zeta_N) d\zeta_1 \cdots d\zeta_N = \frac{1}{Z_{\text{e.v.}}} \prod_{i < j} (\zeta_i - \zeta_j)^2 e^{-\frac{1}{2} \sum_i \zeta_i^2} d\zeta_1 \cdots d\zeta_N, \quad (3.52)$$

where $Z_{\text{e.v.}}$ is a normalisation factor differing from Z_{GUE} by the ratio of the volumes of $\mathcal{U}(N)$ and $\mathcal{U}(1)^N$. Recognising that we have a square of a Vandermonde determinant $\Delta = \prod_{i < j} (\zeta_i - \zeta_j) = \det \zeta_j^{i-1}$ again, we can immediately write the joint PDF as the L^2 norm of a determinant of a function, having the form of a joint PDF found from a quantum mechanical model in the ground state (see Section 1.3.2), as

$$|\Phi_0(\zeta_1, \dots, \zeta_N)|^2 d\zeta_1 \cdots d\zeta_N, \quad \Phi_0(\zeta_1, \dots, \zeta_N) = \frac{1}{\prod_{1 \leq i, j \leq N} \ell! \sqrt{T_i}} P_i(\zeta_j) e^{-\frac{1}{4} \zeta_i^2} \quad (3.53)$$

for some family of polynomials $\{P_i(x) = x^i + \sum_{j=1}^{i-1} a_{i,j}x^j\}$ and normalisations T_i , for i from 0 to N . Now, in order for the Slater determinant above to correspond to a quantum mechanical state, the “single particle wavefunctions” $\varphi_i(x) := P_i(x)e^{-\frac{1}{4}x^2}/\sqrt{T_i}$ must be orthonormal; hence, we need a set of orthogonal polynomials on the real line such that

$$\int_{-\infty}^{\infty} P_i(x)P_j(x)d\mu_r(x) := \int_{-\infty}^{\infty} P_i(x)P_j(x)e^{-\frac{1}{2}x^2}dx = T_i\mathbf{1}_{i=j}. \quad (3.54)$$

From this orthogonality condition, we can look at $\int xP_i(x)P_j(x)d\mu_r(x)$ to find (again applying some standard procedures; see e.g. [Sim07]) and, noting that it is zero if $|i-j| > 1$, see that we have a recursion of the form

$$xP_j(x) = P_{j+1}(x) + R_jP_j(x) + \frac{T_j}{T_{j-1}}P_{j-1}(x). \quad (3.55)$$

Now, by differentiating $\int P_i(x)P_i(x)d\mu_r(x)$ and then $\int xP_i(x)P_i(x)d\mu_r(x) = 0$ we find $R_j = 0$ and $T_j = j!$, and

$$P_{j+1}(x) = xP_j(x) - jP_{j-1}(x), \quad P'_j(x) = jP_{j-1}(x); \quad (3.56)$$

putting the second equality into the first, this is precisely the recurrence relation of the Hermite polynomials

$$\text{He}_j(x) = (-1)^j e^{\frac{1}{2}x^2} \frac{d^j}{dx^j} e^{-\frac{1}{2}x^2}. \quad (3.57)$$

The single particle eigenfunctions

$$\varphi_j(x) = \frac{1}{\sqrt{j!}} \text{He}_j(x) e^{-\frac{1}{4}x^2} \quad (3.58)$$

arise in quantum mechanics, as the single particle eigenfunctions defining *coherent states* of the harmonic oscillator; recalling the trapped fermions discussion of Section 2.2.1, these are eigenfunctions of $-d^2/dx^2 + x^2$. By the first quantisation approach to DPPs discussed in Section 1.3.2, we recover the classical result that

Proposition 3.5 (DPP from the GUE, see e.g. [Meh67]). *The eigenvalues of an $N \times N$ GUE random matrix form a DPP on the real line with kernel*

$$K_N(x, y) = \sum_{j=1}^{\infty} \frac{1}{j!} \text{He}_j(x) \text{He}_j(y) e^{-\frac{1}{4}(x^2+y^2)} \quad (3.59)$$

where $\text{He}_j(x)$ denotes the Hermite polynomial given at (3.57).

We have an exact correspondence between the GUE and a trapped fermion model (which we have already shown to have the asymptotic edge behaviour characteristic of the GUE).

Local and global limiting behaviour Following Proposition 3.5, the large matrix limit of the GUE can be studied finely. In analogy with the previous discussion of the bulk behaviour of Hermitian Schur measures in Section 2.3.2, the GUE eigenvalue ensemble has both a “local limit”, which is the limit of the kernel K_N of points a finite distance apart as $N \rightarrow \infty$, and a “scaling limit”, which is the limiting density of eigenvalues. In the first case, as $N \rightarrow \infty$ we have

$$K_N(u\sqrt{N} + x, u\sqrt{N} + y) \rightarrow \frac{\sin \pi(x - y)}{\pi(x - y)}, \quad (3.60)$$

and this was used in the work of Dyson, Mehta and Wigner (see [Meh67] for a very clear overview) to study the spacing of eigenvalues in the GUE. This *sine kernel* is an immediate continuum analogue of the limiting discrete sine kernel of the bulk of a Hermitian Schur measure (see Lemma 2.13); in a model of trapped fermions, one can find the same ground state propagator from a local density approximation (see [Sté19], or Section 2.2.3 for a discrete analogue).

In terms of a rescaled coordinate $w := u\sqrt{N}$, in the $N \rightarrow \infty$ limit the density of eigenvalues is given by *Wigner's semi-circle law*, with

$$\varrho(u) = \begin{cases} \frac{1}{\pi}\sqrt{2-u^2}, & |u| \leq \sqrt{2} \\ 0, & |u| \geq \sqrt{2}. \end{cases} \quad (3.61)$$

Comparing with the VKLS limit shape of the Plancherel measure (see Theorem 1.35), the fact their edges lie in the same universality class is indicated by the same vanishing exponent of $1/2$ appearing in either case. At that $u = \sqrt{2}$ edge, we of course have, as $N \rightarrow \infty$,

$$\frac{1}{2^{1/2}N^{1/6}}K_N\left(\sqrt{2N} + \frac{x}{2^{1/2}N^{1/6}}, \sqrt{2N} + \frac{y}{2^{1/2}N^{1/6}}\right) \rightarrow \mathcal{A}(x, y) \quad (3.62)$$

where \mathcal{A} denotes the classical Airy kernel.

3.2.2 Beyond the Gaussian unitary ensemble and beyond free fermions

Universality of Hermitian matrix models Universality in random matrix theory is a particularly rich and well understood subject, with the rare distinction that there exist some rigorous proofs of universality itself for matrices (see for instance Tao and Vu's work on the universality of certain GUE statistics for a wide class of Hermitian and real symmetric random matrices [TV11]). We will not explore this subject in any of the depth it deserves, but just make some brief comparisons with the universality picture for free fermions and for partitions that we have considered until now.

Firstly, let us note that the universality of trapped fermions has a direct analogue for Hermitian matrix models: in the terminology of [TW93], the convergence to the Airy and sine kernels holds when the $-\frac{1}{2}\text{tr} M^2$ term in the exponential is replaced with any “not finely tuned” polynomial potential $\text{tr} V(M)$. This gives a direct analogy to the notion of “generic” confining potentials or generic Hermitian Schur measure specialisations previously discussed. We must make the caveat, however, that there is no general correspondence between confining potentials for fermions on the real line and Hermitian matrix model potentials; the GUE/harmonic potential one is a rare benchmark.

Universality of Gaussian matrix ensembles The GUE serves as an important benchmark for another family of matrix models: different matrix or eigenvalue ensembles with Gaussian measures. There are two natural analogues to the GUE, where the Hermitian matrices are replaced with real symmetric and Hermitian quaternionic matrices; the unitary symmetry group is replaced with an orthogonal one and a symplectic one respectively, and these ensembles are thus called the *GOE* and the *GSE*. In the corresponding measures on eigenvalues, the square of the Vandermonde determinant is replaced by a single Vandermonde determinant in the GOE and a fourth power of the same in the GSE; looking only at the eigenvalues, one can treat the exponent of the Vandermonde determinant, by convention denoted β , as a parameter and let it be real valued. In the general family of “Gaussian β ensembles” ($G\beta E$) of points on the real line, the GUE is

unique in that it defines a DPP, and importantly it has some universal properties. An interesting distinction between the kind of universality seen in this family and in the others we have discussed is that the GUE, GOE and GSE all share a universal limiting eigenvalue density given by the Wigner semi-circle law (3.61), but have asymptotic edge distributions differing from F_{GUE} (we saw the opposite for each order of multicritical Schur measures). In terms of a random partition analogue of this picture, we can refer to the universality connection between a discrete analogue of the $G\beta E$ and certain non-determinantal deformations of the Plancherel measure called *Jack–Plancherel measures* found by Guionnet and Huang [GH19].

The question of a multicritical extension Returning to the universality picture for Hermitian matrix models with generic potentials, it is natural to ask *what asymptotic behaviour can we find for a finely tuned potential?* And, in particular, *can we find the same kind of multicriticality we had for free fermions, unitary matrix models, and Schur measures?* For now, the established finely tuned potentials appear not to exhibit the precise multicritical edge behaviour we considered. From an analytic perspective, different multicritical asymptotics which are also related to the Painlevé II hierarchy have been observed for random Hermitian matrices, for example by Claeys, Its and Krasovsky [CKI10] who tuned even-degree potentials. We do not know if this could be related to our multicritical models. Another candidate is of course Kazakov’s original multicritical Hermitian matrix models [Kaz89]; it would be interesting to study their edge behaviour in depth, but we note that the vanishing exponents in these models generalise in a different way from ours, implying we should not expect them to belong to the same universality class.

The multicritical unitary matrix models may present a path to finding related Hermitian ones— we might note naively that if H is Hermitian then $\exp(iH)$ and $(i - H)(i + H)$ are both unitary, one can pass from one picture to another, but the observables we are comparing on either side (partition functions and edge distributions) are not easily related.

A connection with Laguerre unitary ensembles Let us discuss a connection between another Hermitian matrix model and the Plancherel measure, which is less well understood but which does relate to our previous discussion of unitary matrix models. Consider the *Laguerre unitary ensemble* (LUE) of $N \times N$ matrices for given real $\theta > 0$ and integer $\ell > 0$, with measure

$$\mathbb{P}_N(M) \mathcal{D}M = \frac{1}{Z_{\text{LUE}}} e^{-\text{tr} M} (\det M)^\ell \mathcal{D}M. \quad (3.63)$$

The induced measure on ordered sets of eigenvalues $x_1 < x_2 < \dots < x_N$ is

$$\mathbb{P}_N(x_1, \dots, x_N) dx_1 \cdots dx_N = \frac{1}{Z_{\text{e.v.}}} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{1 \leq i \leq N} e^{-x_i} x_i^\ell dx_1 \cdots dx_N \quad (3.64)$$

(in either case, Z_{LUE} and $Z_{\text{e.v.}}$ are normalisations). The eigenvalues in this model form a DPP with kernel given by Laguerre polynomials—see e.g. [For10]. If we look at the lowest eigenvalue x_1 at the “hard edge” at 0 and rescale the eigenvalues to $\tilde{x} = x_i/N$ and take $N \rightarrow \infty$, we obtain the *continuous* Bessel ensemble DPP of Tracy and Widom [TW94]. Moreover, it can be proven (see e.g. [BF03] and references therein) that the gap probability for the interval $(0, 4\theta^2)$ in the continuous Bessel ensemble equals

a similar gap probability in the *discrete* Bessel ensemble. In terms of a unitary matrix integral, we have [BF03, Equation (2.8)]:

$$\lim_{N \rightarrow \infty} \mathbb{P}_N \left(\frac{x_1}{N} > 4\theta^2 \right) = e^{-\theta^2} \int_{\mathcal{U}(\ell)} e^{\theta \operatorname{tr}(U+U^*)} \mathcal{D}U \quad (3.65)$$

As we showed for the right hand side in Theorem 3.2, both quantities above are Fredholm determinants so we have

$$\det(1 - J^\ell)_{L^2(0,4\theta^2)} = \det(1 - \mathcal{J}_\theta)_{l^2(\ell + \frac{1}{2} + \mathbb{Z}_{\geq 0})} \quad (3.66)$$

where J^ℓ is the continuous Bessel function of [TW94], defined as

$$J^\ell(x, y) = \int_0^1 J_\ell(2\sqrt{ux}) J_\ell(2\sqrt{uy}) du \quad (3.67)$$

and \mathcal{J}_θ is the usual discrete Bessel kernel defined at (1.176). It is possible that the equality (3.65) is not a mere coincidence and so might have a multicritical extension, and hence define a “multicritical Laguerre ensemble” (although we do not know what a natural definition of multicriticality for such a model would be).

3.2.3 From Hermitian matrix models to combinatorics of maps

To conclude our discussion on random Hermitian matrices, we outline a further connection between them and discrete geometric objects called *maps*. Although we will take somewhat of a detour to introduce them, using a perturbative matrix integral calculations, we will ultimately be interested in maps as purely combinatorial objects. In Chapter 4 we present an approach to studying random maps using random partitions.

A diagrammatic approach to Hermitian matrix integrals Let us consider once again random $N \times N$ Hermitian matrices in the GUE, and look at *correlations* of random matrix entries M_{ij} . Denoting an expectation with respect to the Gaussian measure with $\langle \cdot \rangle$, for a two-point correlation we find, by the symmetries of the model,

$$\langle M_{ij} M_{k\ell} \rangle := \frac{1}{Z_{\text{GUE}}} \int_{\mathcal{H}(N)} e^{-\frac{1}{2} \operatorname{tr} M^2} M_{ij} M_{k\ell} \mathcal{D}M = \mathbf{1}_{i=\ell} \mathbf{1}_{j=k}. \quad (3.68)$$

From the symmetry, it is clear that the correlation of an odd number of matrix entries will be exactly zero. Then, the correlation of any even number $2n$ of matrix entries can be reduced to two-point correlations. This is an application of rule we have already seen applied to free fermion ground state propagators, *Wick’s theorem* (see Lemma 1.46). For a $2n$ -point correlation under any Gaussian measure, we have that

$$\begin{aligned} & \langle M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_{2n-1} j_{2n-1}} M_{i_{2n} j_{2n}} \rangle \\ &= \sum_{\substack{\text{disjoint pairs} \\ \sqcup_i (a_i, b_i) = (1, \dots, 2n)}} \langle M_{i_{a_1} j_{a_1}} M_{i_{b_2} j_{b_2}} \rangle \cdots \langle M_{i_{a_{2n-1}} j_{a_{2n-1}}} M_{i_{b_{2n}} j_{b_{2n}}} \rangle \end{aligned} \quad (3.69)$$

Following an approach to random matrix models initiated in [BIPZ78], we apply a diagrammatic method from quantum field theory to this simple situation; see e.g. [Bou15] for a clear overview. We draw a “Feynman diagram” for each GUE expectation, where for each two-point correlation $\langle M_{ij} M_{k\ell} \rangle$, or propagator, we draw an edge connecting two vertices, each labelled with the indices of the entries; so the edge connects a vertex

labelled (i, j) and another labelled (k, ℓ) . If we consider how the propagator changes upon exchanging indices (in particular, that $\langle M_{ij}M_{\ell k} \rangle$ is different but $\langle M_{ji}M_{\ell k} \rangle$ is the same), one can see that an effective way to draw the vertices and edge is to draw two parallel “half-edges”, one connecting i to ℓ and the other connecting j to k , and to give them opposite orientations – namely, we draw one arrow in the i to ℓ direction on that half-edge and another arrow in the k to j direction on the corresponding half-edge (note that this is a particularity about Hermitian matrices; if the matrices were real symmetric, we would have no need for this orientation).

With these ribbon-like edges associated to the propagators, consider the insertion of a trace

$$\text{tr } M^d = \sum_{i_1, \dots, i_d} M_{i_1 i_2} M_{i_2 i_3} \dots M_{i_d i_1} \quad (3.70)$$

into a correlation, for some positive integer d . Diagrammatically, we can associate this with a vertex on which d edges are incident, or where d half-edges with ordered labels meet without crossing one another (a vertex of *degree* d). If we take some correlation

$$C^{d_1, \dots, d_n} = \langle \text{tr } M^{d_1} \cdot \text{tr } M^{d_2} \dots \text{tr } M^{d_n} \rangle \quad (3.71)$$

we will have a series of “ribbon diagrams”, each containing n vertices of degree d_1, \dots, d_n . The diagrams will include all consistent ways of connecting the half-edges consistently so that there are no twists in any edge; meanwhile, the number C^{d_1, \dots, d_n} itself simply counts the number of these ribbon diagrams (with over-counting due to symmetries; we will overlook this for this discussion). So, fixing some set of formal variable w_1, w_2, \dots , we can see that a GUE expectation

$$\langle e^{\sum_n v_n \text{tr } M^n} \rangle = \frac{1}{Z_{\text{GUE}}} \int_{\mathcal{H}(N)} e^{-\frac{1}{2} \text{tr } M^2 + \sum_n v_n \text{tr } M^n} \mathcal{D}M \quad (3.72)$$

generates all numbers of ribbon diagrams, with C^{d_1, \dots, d_n} weighted by $v_{d_1} \dots v_{d_n}$.

Ribbon graphs and maps The ribbon diagrams constructed this way are the main objects that interest us. Considering just an individual ribbon diagram, we can see that the ribbon edges add an extra orientation to a normal graph. If we look at each closed half edge, we can see that they cut off regions of space. The objects in question, more generally called maps, may be defined as follows:

Definition 3.6 (Combinatorial map, its graph and genus). A *graph* is a set of vertices V and a set of edges E , along with a set of incidence relations which associate a vertex to each end of each edge. If it is impossible to separate V into two disjoint sets V_1, V_2 such that no element of V_1 shares an edge with an element of V_2 , the graph is called *connected*. A *map* is a graph equipped with a cyclic ordering of the edges around each vertex, such that there exists a set of faces F where each edge is incident on two faces and two consecutive edges around one vertex are incident on the same face.

A map is called connected if its graph is connected. The *genus* g of a connected map is related to its vertices, edges and faces V, E, F by the *Euler characteristic relation*

$$\chi := \#V - \#E + \#F = 2 - 2g \quad (3.73)$$

and χ is the Euler characteristic.

One could equally define or describe maps in other ways. On one hand, fixing the cyclic ordering of vertices corresponds to deciding how to draw a graph; a map may

be defined as a graph drawn on a surface, with the constraint that the edges cannot overlap. If a connected map is represented as a graph on a closed surface of minimal genus (or number of “handles”), the genus of the surface is the genus g computed by the Euler characteristic relation (3.73). A map could equally be considered as sets of faces and edges with incidence relations and cyclic ordering, or in other words as a gluing of polygons. Starting from a closed surface, a map can be seen as a discretisation. Let us also note that it is particularly natural to encode the data of a map with permutations.

Map ensembles and universality Maps have been a subject of intense interest in combinatorics, since Tutte’s seminal work on the planar ($g = 0$) case [Tut63]. The enumeration of ensembles of maps constructed under different constraints (on vertex and face degrees, for instance, or with rules for alternately colouring vertices or faces) is a rich subject, and related to the main theme of this thesis in an important way: for various ensembles of maps of fixed genus, in the limit as the map size tends to infinity, we have universal exponents in the asymptotic enumeration of $\frac{5}{2}(g - 1)$ (see e.g. [Cha09a]). Uniform random elements of map ensembles also exhibit universality, as geometric objects: as first shown by Le Gall and Miermont [Gal13, Mie13], when considered as metric spaces, an important class of random planar maps converge in a strong sense to a single scaling limit (the Brownian sphere).

Genus expansions from matrix models Returning to the matrix model picture, the planar maps are particularly important there too. From the formal series (3.72) we can write a generating function for *connected* maps as

$$\mathcal{F}_N(v_1, v_2, \dots) = \frac{1}{N^2} \log \frac{1}{Z_{\text{GUE}}} \int_{\mathcal{H}(N)} e^{-\frac{1}{2} \text{tr} M^2 + \sum_n v_n \text{tr} M^n} \mathcal{D}M \quad (3.74)$$

(i.e. a free energy associated with a partition function which is a matrix model generating function). Noting that each vertex corresponds to a trace in the matrix integral and a factor of Nv_n if the vertex has degree n , and that each face contributes another trace (by closing a half edge loop in a ribbon diagram), while each edge contributes a factor of $1/N$, we see that \mathcal{F}_N is an expansion in N weighted by $\chi - 2$; in other words a genus expansion, with

$$\mathcal{F}_N(v_1, v_2, \dots) = \sum_{g=0}^{\infty} N^{-2g} \mathcal{F}^{(g)}(v_1, v_2, \dots) \quad (3.75)$$

where $\mathcal{F}^{(g)}$ is the generating function of maps with genus g . The large matrix limit therefore corresponds to a *planar limit* in these diagrams, as first considered by ’t Hooft [tH74], in which $g = 0$ diagrams dominate.

We should note that this approach involves divergent series, and so is not in itself rigorous. However, it has led to important asymptotic estimates for planar maps that have since been proven bijectively (see for instance [BDFG02]), as well as to the development of powerful tools such as the topological recursion [EO07].

Random maps and asymptotic edge behaviour of random partitions Let us turn our attention back to the asymptotic edge correspondence of the GUE and the Plancherel measure, and to Theorem 3.4. In the seminal paper [Ok00b], Okounkov proved this theorem before the DPP arising from the Poissonised Plancherel measure was known, using a remarkable combinatorial approach. His proof related GUE eigenvalues to map enumeration by the correspondence sketch above, then exploited a connection

between maps and ramified coverings of the sphere and finally related the enumeration of ramified coverings to parts of a Plancherel random partition by way of the Jucys–Murphy elements.

The Plancherel–Hurwitz measure

This chapter presents joint work with Guillaume Chapuy and Baptiste Louf, and is adapted from [2]. We consider random partitions under a deformation of the Plancherel measure which appears naturally in the context of Hurwitz numbers: its normalisation counts transposition factorisations of the identity on symmetric groups, which have topological interpretations, the most combinatorial one being certain maps. We prove that in an asymptotic regime where the number of factors in the corresponding factorisations grows linearly with the order of the group and the maps are of high genus, a random partition exhibits a new twofold limit phenomenon. The first part becomes very large, while the rest of the partition has the VKLS limit shape of the Plancherel measure. As a consequence, we obtain asymptotic estimates for unconnected Hurwitz numbers in this regime, which we use to study corresponding uniform random maps. This is also an estimate for a return probability of a random transposition walk on a symmetric group.

Morally, we follow an inverse approach to that of Chapters 2 and 3. Whereas the measures previously discussed were physically motivated but studied using their intrinsic algebraic structures, here our measure is explicitly motivated by an algebraic identity but we do not (for now) have an algebraic method to study it. Instead, our proofs employ a variety of estimates to optimise entropies of the measure.

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4.1 New twofold limit behaviour for random partitions

In this section we introduce the probability law on partitions that we study, called the Plancherel–Hurwitz measure, and state our main result for its limiting behaviour in a linear regime. We discuss two notable contexts in which the same measure previously appeared (indirectly and without a name): firstly, work by Okounkov [Oko00a] relating a Poissonised version to the Toda integrable hierarchy, and secondly work by Diaconis and Shahshahani [DS81] on random walks on symmetric groups.

Laws on partitions from laws on sequences of permutations The measure we introduce originates from a connection between partitions and elements of symmetric groups. Famously, the classical Plancherel measure

$$\mathbb{P}_n(\lambda) = \frac{f_\lambda^2}{n!} \quad (4.1)$$

on partitions $\lambda \vdash n$ (discussed in Section 1.2.1) arises from the uniform law on the order n symmetric group \mathfrak{S}_n by one such connection: it is the law of the shape of the Young tableaux associated bijectively to a uniform random permutation by the RSK algorithm [Rob38, Sch61] (see Theorem 1.12). This law has been subject to intense interest since, by Greene’s theorem 1.13 [Gre74], the parts of a Plancherel random partition give the lengths of the monotone subsequences of a uniform random permutation. Moreover, the Plancherel measure has elegant asymptotic properties – we highlight once again the VKLS limit shape theorem 1.35 [VK77, LS77] and the BDJ edge fluctuation theorem 1.37 [BDJ99] (thanks to the RSK correspondence, the asymptotic behaviour of the first part has important ramifications since it corresponds to the length of the longest increasing subsequence of a uniform random permutation; we refer again to Romik’s book [Rom15] for a wonderful introduction).

From another perspective, the Plancherel measure arises from the application of purely representation theoretic identity (1.22) to the symmetric group, which equates the size of \mathfrak{S}_n to the sum of the squares of the dimensions of its irreducible representations. Dividing both sides by $n!$, we recognise the normalisation of \mathbb{P}_n . Beyond \mathfrak{S}_n itself, the sizes of other sets of sequences of permutations can be expressed as sums over irreducible representations of \mathfrak{S}_n , i.e. over partitions of n . One powerful classical result *Frobenius’ formula* stated in Theorem 1.16 expresses the number of factorisation of the identity on \mathfrak{S}_n by permutations of given cycle type as a sum of the products of their normalised characters on irreducible representations. This way we can find many measures on partitions of n , each corresponding to a uniform law on sequences of elements of \mathfrak{S}_n with certain constraints. As the correspondence is not bijective, however, the statistics on the partition side cannot be readily interpreted on the factorisation side.

4.1.1 Transposition factorisations and partitions

We consider a measure arising naturally from a simple application of Frobenius’ formula. It is motivated by the study of Hurwitz numbers and *branched coverings of the sphere*, which are realized combinatorially by *transposition factorisations* or equivalently by certain discrete surface called *Hurwitz maps*; we define the measure from the transposition factorisation side, but later focus on the Hurwitz map side, which we discuss in Section 4.2. The Hurwitz maps perspective notably motivates the asymptotic regime in which we study these measures, which is related to “high genus” surfaces. At the level of random partitions, this is a regime in which the limit behaviour generalises the VKLS one in a rather novel way, see Figure 4.1.

For a non-negative integer ℓ , we let $H_{n,\ell}$ be the *unconnected Hurwitz number* of Definition 1.20, counting factorisations of the identity on \mathfrak{S}_n into ℓ transpositions:

$$H_{n,\ell} = \#\{(\tau_1, \tau_2, \dots, \tau_\ell) \in (\mathfrak{S}_n)^\ell, \tau_1 \cdot \tau_2 \cdots \tau_\ell = \mathbb{1}, \text{ each } \tau_i \text{ is a transposition}\}. \quad (4.2)$$

As mentioned in Section 1.1.2, by a classical correspondence between transposition factorisations and branched coverings of the sphere going back to Hurwitz himself ([Hur91], see also e.g. [LZ04]), the number $H_{n,\ell}$ enumerates degree n coverings of the Riemann sphere with ℓ numbered, simple, ramification points, by an oriented surface (connected or not). Recalling Corollary 1.21 of Theorem 1.18, Frobenius' formula together with the combinatorial representation theory of \mathfrak{S}_n provides an explicit expression for this number, with

$$H_{n,\ell} = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 C_\lambda^\ell \quad (4.3)$$

where C_λ denotes the sum of contents of the partition λ (in the notation of (1.16) in Definition 1.8; we recall that this is straightforward data from Young diagram of λ , as is the number of SYT f_λ , see Theorem 1.7). This vastly generalises the identity (1.22) giving the normalisation of the Plancherel measure, which corresponds to $\ell = 0$. If ℓ is odd, this formula gives $H_{n,\ell} = 0$ just as it should, since we have $C_{\lambda'} = -C_\lambda$ and each contribution is cancelled out.

The right hand side of (4.3) naturally gives rise to the following measure on partitions, which is our main object of study:

Definition 4.1 (Plancherel–Hurwitz measure). For $n \in \mathbb{Z}_{>0}$, $\ell \in 2\mathbb{Z}_{\geq 0}$, the Plancherel–Hurwitz measure is the probability measure on partitions of n defined by

$$\mathbb{P}_{n,\ell}(\lambda) := \frac{1}{n! H_{n,\ell}} f_\lambda^2 C_\lambda^\ell. \quad (4.4)$$

For $\ell > 0$, the positive half of the Plancherel–Hurwitz measure is the probability measure on partitions of n with positive content sum

$$\mathbb{P}_{n,\ell}^+(\lambda) := \mathbb{P}_{n,\ell}(\lambda | C_\lambda > 0) = 2 \cdot \mathbf{1}_{C_\lambda > 0} \cdot \mathbb{P}_{n,\ell}(\lambda). \quad (4.5)$$

A partition distributed under $\mathbb{P}_{n,\ell}$ for $\ell > 0$ can be thought of as a partition distributed under $\mathbb{P}_{n,\ell}^+$ whose Young diagram is reflected about a vertical axis with probability $\frac{1}{2}$.

When $\ell = 0$ the measure $\mathbb{P}_{n,\ell}$ is nothing but the Plancherel measure. Our main result deals instead with the case where ℓ grows linearly with n (which we could call the *high-genus regime* after the discussion of Section 4.2 below). We have the following limit behaviour for a large random partition (in terms of the $1/\sqrt{n}$ rescaled profile $\psi_{\lambda, \sqrt{n}}$ of the partition, see Definition 1.5):

Theorem 4.2 (Limit behaviour of the Plancherel–Hurwitz measure at high genus). *Fix $q > 0$ and let $\lambda \vdash n$ be a random partition under the Plancherel–Hurwitz measure $\mathbb{P}_{n,\ell}^+$ in the regime given by $\ell = \ell(n) \sim 2qn$. Then, as $n \rightarrow \infty$:*

(i) *the first part λ_1 is equivalent to $\frac{2\ell}{\log n}$ in probability, with*

$$\frac{\lambda_1 \log n}{2\ell} \xrightarrow{p} 1; \quad (4.6)$$

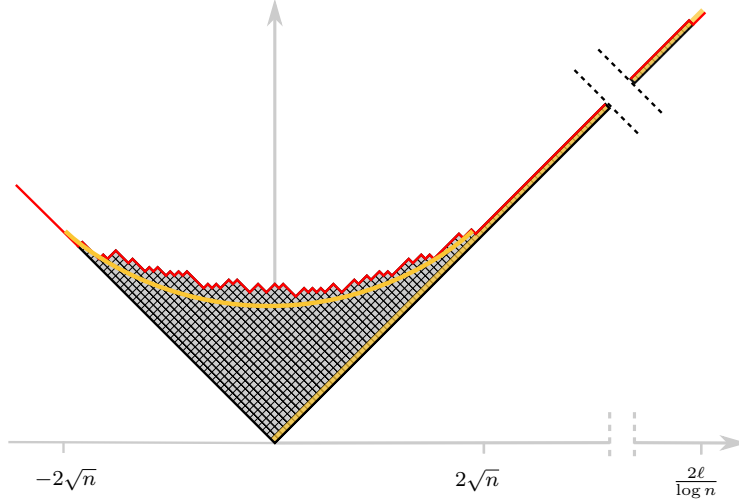


Figure 4.1 The tilted Young diagram of a random partition of $n = 2500$ under the Plancherel–Hurwitz measure $\mathbb{P}_{n,\ell}$ in the high genus regime $\ell = 2\lfloor 1.5n \rfloor$ (sampled via a Metropolis–Hastings algorithm). The twofold asymptotic behaviour is shown in yellow: the first part λ_1 is asymptotic to $\frac{2\ell}{\log(n)}$ and escapes the picture, while the rest of the partition scales in \sqrt{n} with a VKLS limit shape. The profile of the partition is in red, while the VKLS limit shape is the yellow curve, scaled up by a factor $\sqrt{\tilde{n}}$, $\tilde{n} = n - 2\ell/\log n$. See Theorem 4.2.

(ii) the rest of the partition $\tilde{\lambda} = (\lambda_2, \lambda_3, \dots)$ has a VKLS limit shape. Namely,

$$\sup_x |\psi_{\tilde{\lambda}, \sqrt{\tilde{n}}}(x) - \Omega(x)| \xrightarrow{p} 0,$$

$$\text{where } \Omega(x) = \begin{cases} \frac{2}{\pi} \left(\arcsin \frac{x}{2} + \sqrt{4 - x^2} \right), & |x| \leq 2, \\ |x|, & |x| > 2. \end{cases} \quad (4.7)$$

where $\psi_{\tilde{\lambda}, \sqrt{\tilde{n}}}(x)$ is the rescaled profile of $\tilde{\lambda}$;

(iii) the second part and the length are both equivalent to $2\sqrt{\tilde{n}}$ in probability, with

$$\frac{\lambda_2}{\sqrt{\tilde{n}}} \xrightarrow{p} 2 \quad \text{and} \quad \frac{\ell(\tilde{\lambda})}{\sqrt{\tilde{n}}} \xrightarrow{p} 2. \quad (4.8)$$

The two-scale phenomenon We could include λ_1 in the partition $\tilde{\lambda}$ of (4.7), since the supremum norm in this statement is insensitive to a small number of large parts. However, from part (iii) of the theorem, λ_1 is the only part not scaling as $\sqrt{\tilde{n}}$ so we find this formulation more natural. This limiting curve is graphed, along with the Young diagram of a finite sized partition sampled under a Plancherel–Hurwitz measure, in Figure 4.1.

Heuristically, we might think of this theorem as resulting from the two competing “forces” driving a random partition λ under $\mathbb{P}_{n,\ell}^+$. On the one hand, due to the Plancherel factor f_λ^2 , the classical VKLS theorem shows that there is a cost for the partition to deviate from the VKLS shape. On the other hand, the partition may prefer to deviate strongly from the VKLS shape if it gains a sufficiently high content-sum so that the factor C_λ^ℓ compensates the Plancherel loss.

Our result shows that a random partition balances these forces by obtaining a large content-sum exclusively from the first part λ_1 , and then leaving the rest of the partition to maximize only the Plancherel entropy. The different length scales determining the limit behaviour are a rather unique feature of this measure and its “high genus” regime; established generalisations of the Plancherel measure such as the Schur measures and Schur processes discussed previously do not exhibit such behaviour in typical asymptotic regimes [Ok01, OR03].

The corresponding transposition factorisations As a consequence of our analysis of the limit behaviour of the Plancherel–Hurwitz measure, we obtain the following estimate for the unconnected Hurwitz numbers in the same asymptotic regime:

Theorem 4.3 (Asymptotic estimate for unconnected Hurwitz numbers at high genus). *Let $H_{n,\ell}$ be the unconnected Hurwitz number counting (not necessarily transitive) factorisations of the identity on \mathfrak{S}_n by an even number $\ell = \ell(n) \sim 2qn$ of transpositions. Then, as $n \rightarrow \infty$,*

$$H_{n,\ell} = \left(\frac{\ell}{\log n}\right)^{2\ell} \exp [(-2 + \log 2)\ell + o(n)], \quad (4.9)$$

uniformly for ℓ/n in any compact subset of $(0, \infty)$.

This, along with an element of the proof of the limit shape theorem, will allow us to bound the order of the group generated by the largest transitive subsequence of a uniform random transposition factorisation in Section 4.2.2 below. We first interpret the transposition factorisations as maps, making the problem more combinatorial and topological.

4.1.2 Connections: Plancherel–Hurwitz measures in different guises

Let us briefly mention some previous works in which objects related to the Plancherel–Hurwitz measure have appeared.

A generalisation of the Plancherel–Hurwitz measure and integrability In [Ok01], Okounkov presented a fermionic approach to the generating functions of unconnected Hurwitz numbers. The case considered was the somewhat more general one of *double Hurwitz numbers* $\text{Hur}_{n,\ell}(\mu, \nu)$ defined at (1.45), which count transposition factorisations of permutations with a given cycle structure. It is instructive to consider these more general numbers to compare to the Schur measures considered in Sections 1.2.2 and 1.2.2 and Chapter 2.

By the content-sum version of Frobenius’ formula, Theorem 1.18, for the unconnected double Hurwitz numbers¹ we have

$$\text{H}_{n,\ell}(\mu, \nu) = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 C_\lambda^\ell |C_\mu| \chi^\lambda(C_\mu) |C_\nu| \chi^\lambda(C_\nu). \quad (4.10)$$

Now, by Proposition 1.33, the characters χ^λ can be expressed in terms of Schur functions s_λ specialised in the Miwa times; weighting each double Hurwitz numbers $\text{H}_{n,\ell}(\mu, \nu)$ by

¹For simplicity, we have different conventions to [Ok00a], where only the connected numbers are referred to directly; in the notation of that reference, $\text{H}_{n,\ell}(\mu, \nu)$ is $n! \text{Cov}_n(C_\mu, C_\nu, (C_{(2)})^\ell)$.

the powersums $p_\mu \cdot p_\nu$, we can write a generating function for them in terms of the Schur functions as

$$\sum_{\substack{n,\ell \\ \mu,\nu \vdash n}} \frac{u^n v^\ell}{n! \ell!} \mu_1 t_1 \cdots \mu_{\ell(\mu)} t_{\ell(\mu)} \nu_1 t'_1 \cdots \nu_{\ell(\nu)} t'_{\ell(\nu)} H_{n,\ell}(\mu, \nu) = \sum_{\lambda} u^{|\lambda|} e^{v C_\lambda} s_\lambda[t] s_\lambda[t'] \quad (4.11)$$

where the final sum is over all partitions. We recognise a generalisation of Schur measure normalisation (and note that the factor of $u^{|\lambda|}$ is superfluous since it can be removed by rescaling); we equally recognise that if we apply exponential specialisations $t = t' = (\theta, 0, 0 \dots)$ this is just a double Poissonisation of the Plancherel–Hurwitz measure normalisation. Just as for the normalisation of the Schur measure, we can write this as an expectation on a fermionic Fock space over $\mathbb{Z} + \frac{1}{2}$; recalling the notation and arguments of Section 1.3.3 and the eigenvalue relations (1.154), we have

$$\sum_{\lambda} u^{|\lambda|} e^{v C_\lambda} s_\lambda[t] s_\lambda[t'] = \langle S(\emptyset) | \Gamma_+(t) u^{Q_1} e^{v Q_2} | S(\emptyset) \rangle \quad (4.12)$$

$$\text{where } Q_1 := \sum_{k \in \mathbb{Z} + \frac{1}{2}} k : c_k^\dagger c_k :, \quad Q_2 := \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{k^2}{2} : c_k^\dagger c_k : .$$

The most important consequence of this fermionic formulation is that, as proven in [Ok00a], the above generating function is a τ -function of the Toda lattice hierarchy. From the hierarchy of equations satisfied by the logarithm of this generating function, Dubrovin, Yang and Zagier [DYZ17] later found recurrence relations for the *connected* Hurwitz numbers. From the perspective of a corresponding measure on partitions, we would expect the corresponding random fermion configurations to form a DPP. Computing the kernel for this process is challenging, and presents an interesting open question. However, in the linear regime we consider, we cannot expect to find a DPP approach that would allow us to find asymptotics.

Transposition random walks In [DS81], Diaconis and Shahshahani considered a problem directly related to ours. The authors studied a walk on \mathfrak{S}_n by ℓ uniform random transpositions, and considered how fast the number of steps ℓ should grow with n for the walk to be asymptotically “mixed”. They famously showed that

Theorem 4.4 (Mixing threshold for the transposition random walk [DS81]). *If $\sigma^{*\ell} = \tau_1 \cdot \tau_2 \cdots \tau_\ell \in \mathfrak{S}_n$ is a product of ℓ uniform random transpositions on \mathfrak{S}_n and $\mathbb{P}^{*\ell}$ denotes the law of $\sigma^{*\ell}$, then, in a regime where*

$$\ell \geq \frac{1 + \varepsilon}{2} n \log n \quad (4.13)$$

for any $\varepsilon > 0$, the total variation distance from \mathbb{P}^{ℓ} to the uniform measure on \mathfrak{S}_n tends to zero exponentially fast as n tends to infinity.*

The terms of this argument can immediately be related to the unconnected Hurwitz numbers: we have, for all n, ℓ , $\mathbb{P}^{*\ell}(\mathbb{1}) = \binom{n}{2}^{-\ell} H_{n,\ell}$. In fact, the Plancherel–Hurwitz measure itself already appears (with no name) in [DS81], from a Fourier transformation of the uniform measure on transpositions to the irreducible representations of \mathfrak{S}_n ; the proof essentially consists in showing that the Plancherel–Hurwitz measure is dominated by the trivial partition (n) in the superlinear regime (4.13) (indeed, since its content-sum is just $C_{(n)} = \binom{n}{2}$, it is straightforward to see that if $H_{n,\ell}$ is well approximated by $\mathbb{P}_{n,\ell}((n))$

the corresponding $\mathbb{P}^{*\ell}$ is close to uniform). At this point it is worth commenting on a parallel with map enumeration: there, the regime in which the genus is unconstrained, or superlinear, is often much easier to deal with than the linear case.

In the random transposition walk context, our asymptotic estimate (4.9) can also be interpreted as an estimate on the return probability of the random walk after linearly many steps – much before the cut-off time, at a time when the Plancherel–Hurwitz measure still has a more subtle behaviour than the trivial partition.

Corollary 4.5 (Equivalent random walk formulation of Theorem 4.3). *The probability that the transposition random walk on \mathfrak{S}_n returns to the origin after $\ell = \ell(n) \sim 2qn$ steps is*

$$\binom{n}{2}^{-\ell} H_{n,\ell} = \exp[-2\ell(\log \log n - \log q - 2 \log 2 + 1) + o(n)]. \quad (4.14)$$

On the other hand, let us point out a coincidence which may imply a connection between the Diaconis–Shahshahani threshold (4.13) to the limit behaviour of the first part of a random partition in Theorem 2.2: if we write the limiting value of λ_1 as given in Theorem 4.2 put $L := 2\ell/\log \ell$, it is related to $\ell \sim 2qn$ by

$$\ell = \frac{1}{2}L \log L + o(L) \quad (4.15)$$

and L is such that ℓ may be approximated by the mixing threshold for a transposition random walk on \mathfrak{S}_L . This is directly analogous to the limit behaviour of λ_1 when $\ell \sim \frac{1}{2}n \log n$, at the mixing threshold.

A connection to asymptotic factorisations Finally, we note that measures on partitions that would appear to be related to a Poissonised version of the Plancherel–Hurwitz measure in a different asymptotic regime have been studied by Biane in the context of the asymptotic factorisation of characters of \mathfrak{S}_n [Bia01]. The limit-shape phenomena observed in this reference are qualitatively different from ours, and possible connections are left to future work.

4.1.3 Plan of the chapter

In Section 4.2, we present a map model corresponding to the Plancherel–Hurwitz measure and discuss how our results relate to high genus map enumeration. We mention some open questions motivated by this perspective. Using the asymptotic for the unconnected Hurwitz numbers at high genus, we prove a bound on the number of vertices in the largest component of a uniform random element of the corresponding map ensemble. The remaining two sections are dedicated to proofs of the two main theorems stated above; just as the limit behaviour we find has two distinct scaling regimes, the proof of Theorem 4.2 divides naturally into “macroscopic” and “microscopic” parts, where there are respectively large and small costs associated with deviating from the typical behaviour. In Section 4.3, we focus on estimating the order of the first part λ_1 , thus proving part (i) of Theorem 4.2 and by extension part (ii) of the same theorem, along with Theorem 4.3. In Section 4.4, we will study the finer details of the limit shape and control the second part λ_2 and the number of parts $\ell(\lambda)$ to prove Theorem 4.2, part (iii).

Chapter acknowledgements With G. Chapuy and B. Louf, we thank D. Betea, P. Biane, J. Bouttier and A. Sportiello for insightful conversations regarding this project.

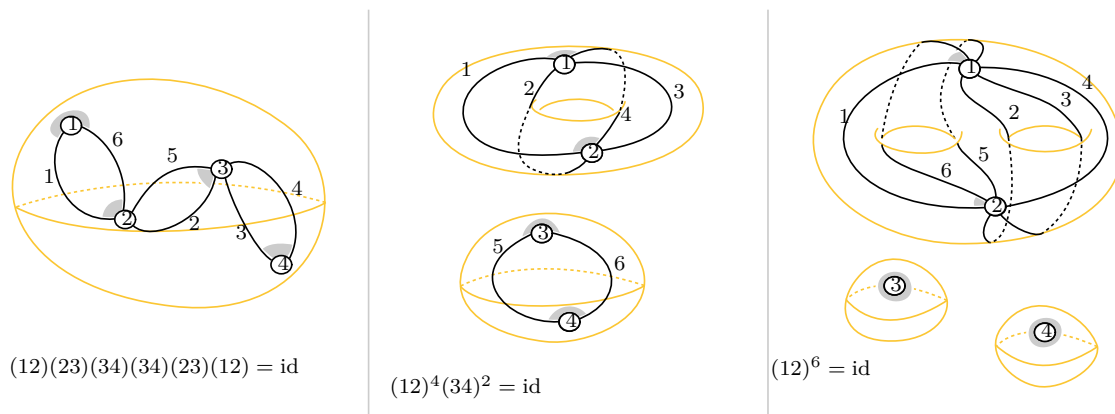


Figure 4.2 Three pure Hurwitz maps, each with 4 vertices, 6 edges and Euler characteristic $\chi = 2$, and the corresponding factorisations of the identity on \mathfrak{S}_4 by 6 transpositions. Left, the map is connected and has genus 0; center, the map has two connected components, of genus 1 and 0; right, has three components, of genus 2, 0 and 0. Edge label descents are shown in grey. See Figure 0.10 for an additional example.

4.2 The associated map model

We now turn to discussing our original motivation for studying the Plancherel–Hurwitz measure: asymptotic map enumeration in a high genus regime.

As mentioned in Section 3.2.3, maps are rich objects, and since the pioneering works of Tutte on planar maps (e.g. [Tut63]) their enumeration has proven to be particularly interesting, borrowing tools from physics, algebra and geometry and revealing their connections within combinatorics. These tools include the matrix integral generating functions outlined in Section 3.2.3 [BIPZ78, LZ04], the topological recursion [EO07], recurrence formulas based on integrable hierarchies [GJ08, CC15], classical generating functions (e.g. [BC86, BMJ06]) and bijective combinatorics (e.g. [Sch99]). Such exact methods have led to the asymptotic enumeration of many types of maps on the plane or on surfaces of fixed genus g , which notably exhibit a universal counting exponent of $\frac{5}{2}(g - 1)$ (e.g. [Cha09a]).

These methods do not, however, extend to maps whose genus grows quickly (in particular, linearly) with the number of polygons. This “high genus” regime, whose study was pioneered in [ACCR13, Ray15] in the case of one-face maps, is one of the most recent and exciting frontiers in the field, due the inefficiency of existing generating-function or bijective methods, requiring the development of new tools.

A major breakthrough in this field was recently made by Budzinski and Louf [BL21], who, as a byproduct of their work on the Benjamini–Curien conjecture [Cur16], found the following estimate by a combination of algebraic, combinatorial, and probabilistic methods:

Theorem 4.6 (Asymptotic estimate for the number of triangulations at high genus [BL21]). *Let $T_{n,g}$ be the number of connected triangulations with n faces on a surface of genus g . Then in a high genus regime where $g = g(n) \sim qn$, as $n \rightarrow \infty$*

$$T_{n,g} = n^{2g} \exp[c(q)n + o(n)] \tag{4.16}$$

uniformly in q , where $c(q) > 0$ is a known continuous function.

Subsequent work by the same authors conjectured the universality of this estimate for a large class of maps [BL22].

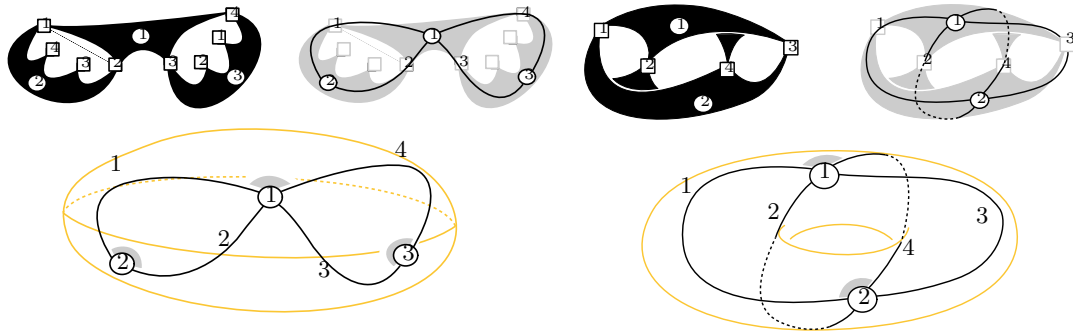


Figure 4.3 Hurwitz maps of genus 0 and 1 respectively, shown with the corresponding constellations. An ℓ -constellation is a generalisation of a map in which each “edge” is incident on ℓ vertices with different labels in cyclic order; Hurwitz maps correspond to ℓ -constellations with n edges constrained to have only one vertex label descent on each face. One identifies vertices with edges and edges with vertices.

4.2.1 Transposition factorisations and Hurwitz maps

We will be interested in a different model of maps:

Definition 4.7 (Hurwitz map). A *Hurwitz map* with n vertices and ℓ edges is a map on a not necessarily connected compact oriented surface, with vertices labelled from 1 to n and edges labelled from 1 to ℓ , where the labels of edges increase cyclically around each vertex counterclockwise. In such a map each vertex is incident to precisely one corner which is an edge-label descent. If moreover each face of the map contains precisely one such corner, the Hurwitz map is called *pure*. We denote the set of pure Hurwitz maps with n vertices and ℓ edges $\mathcal{H}_{n,\ell}$.

It is classical, and easy to see, that Hurwitz maps of parameters n and ℓ are in bijection with sequences of transpositions $(\tau_1, \dots, \tau_\ell)$ in \mathfrak{S}_n , while *pure* Hurwitz maps are in bijection with sequences whose product is equal to the identity. The bijection consists of identifying transpositions with edges of the map, and their index with the edge-label, as illustrated in Figure 4.2. This construction is a special case and an adaptation of the classical construction of “constellations”, see [BMS00, Cha09a, DPS14]; the adaptation is illustrated in Figure 4.3.

Interpreting transposition factorisations in terms of enumerative geometry, pure Hurwitz maps are therefore in bijection with simply ramified, n -sheeted, branched covers of the sphere by an orientable surface, with ℓ numbered simple ramification points, and trivial ramification above ∞ , which is the model considered in [Hur91, Oko00a, DYZ17]. Pure Hurwitz maps have also been studied from the combinatorial and probabilistic viewpoint, and they are known [DPS14], in the planar and fixed-genus cases, to belong to the same universality class as other natural models of maps such as triangulations, quadrangulations, etc. (the convergence to Brownian surfaces is only conjectured, but other properties of the universality class such as counting exponents or the existence of bijections are known).

It is important to make the distinction that the maps we consider are not necessarily connected, which is a significant difference from most models in the literature. A pure Hurwitz map of parameters n and ℓ necessarily has n faces, and its Euler characteristic χ , its number of components κ , and its generalised genus G (the sum of the genera, or number of handles, of each connected component) are related by Euler’s formula:

$$\chi = \#\text{vertices} - \#\text{edges} + \#\text{faces} = 2n - \ell = 2\kappa - 2G. \quad (4.17)$$

For this reason we call regimes where $\ell \gg 2n$ “high genus” regimes. We must insist however that our linear Euler characteristic regime is not the same as a *linear genus* regime of the kind considered by Budzinski and Louf.

By the above correspondence, we have $|\mathcal{H}_{n,\ell}| = H_{n,\ell}$ and we can read (4.9) as an asymptotic enumeration estimate for pure Hurwitz maps in the linear Euler characteristic regime. It is tempting to see this theorem as as strong (for our model) as the Budzinski–Louf estimate (4.16), but unfortunately this is not quite the case. The major difference is that our maps are not necessarily connected. Indeed, we will show that even when $q \geq 1$ and there are sufficiently many edges to connect all the vertices we are predominantly counting unconnected maps, in particular that

Theorem 4.8 (Clustering of edges in a large unconnected Hurwitz map at high genus). *For all $q > 1$, a uniformly random Hurwitz map $\mathbf{h}_{n,\ell} \in \mathcal{H}_{n,\ell}$ with n vertices and an even number $\ell = \ell(n) \sim 2qn$ of edges contains a connected component with at least $\gamma(q)\ell$ edges with high probability, where $\gamma(q) := 2^{2q-1} - 1$. However, in such a map, the number of vertices in the largest connected component is $O_p(n/\log n)$.*

By “with high probability” we mean with probability tending to one as n tends to infinity, and we abbreviate this as w.h.p.; O_p denotes a big O in probability.

Superlinearity of the genus at linear Euler characteristic The fact that the “giant” edge-component has a sublinear number of vertices implies that its genus, viewed as a function of its number of vertices, is superlinear. This seems to rule out the possibility of deducing asymptotics for the *connected* linear genus regime from our results, at least not without new ideas. We can, however, once again note a coincidence, with the limit behaviour of the first part of a corresponding random partition and with the Diaconis–Shahshahani mixing threshold: the best upper bound we can find for the number of vertices in a connected component is proportional to the limiting value of the first part, and so also satisfies (4.15).

Open questions about other regimes The most immediate question following these results is whether we can use the Plancherel–Hurwitz measure approach to study *connected* maps in detail, at superlinear genus or eventually at linear genus. The formulation of the measure opens the possibility of combining our approach with the technology of integrable hierarchies, which have been so fruitful but have so far not directly led to precise asymptotic estimates nor limit theorems for connected random maps or Hurwitz numbers at high genus. In any case, obtaining the analogue the Budzinski–Louf [BL21] estimates for connected pure Hurwitz maps (that is, an exponential estimate up to the linear order, of the kind in (4.16)) is an important question in the direction of completing the universality picture.

Another natural extension of the present work, which we expect to be an important step in understanding the corresponding connected maps, would be to study other regimes for ℓ , especially for sublinear values of ℓ (in the proofs that follow, it will be clear that adapting to superlinear genus simplifies some things while adapting to sublinear genus presents interesting challenges). It is natural to expect that when varying the parameter ℓ from $\ell = 0$ to $\ell = \Theta(n)$, we should be able to make an asymptotic statement interpolating between our main result and the VKLS theorem, with a larger and larger first part λ_1 . Heuristic calculations suggest a possible phase transition around $\ell = \Theta(\sqrt{n})$, which might be the threshold after which the contribution to the content-sum is overwhelmingly made by the first part. We leave these questions to further work.

4.2.2 Uniform random unconnected Hurwitz maps at high genus

In this section we present a proof of Theorem 4.8. We require the estimate on $H_{n,\ell}$ of Theorem 4.3, which is proven in Section 4.3.3, along with the following bound:

Lemma 4.9 (Minimum number of isolated vertices). *There are $n - O_p(n/\log n)$ isolated vertices in a uniform random element $\mathbf{h}_{n,\ell} \in \mathcal{H}_{n,\ell}$.*

This is proven in Section 4.4.1, as a byproduct of proof of the limit behaviour theorem. Throughout the following, $\mathbf{h}_{n,\ell}$ denotes a uniform random element of $\mathcal{H}_{n,\ell}$.

Proof of Theorem 4.8. Lemma 4.9 immediately implies that all connected components of $\mathbf{h}_{n,\ell}$ have $O_p(n/\log n)$ vertices.

Let $\gamma(q) := 2^{2q-1} - 1$, and let $\gamma < \gamma(q)$. For the second part of the proof, we will show that there exists w.h.p. a component of $\mathbf{h}_{n,\ell}$ with at least $(\gamma(q) - o(1))\ell$ edges. To do so, we show that the number of maps with no connected component with more than $\gamma\ell$ edges is $o(H_{n,\ell})$, starting by estimating the number of these maps satisfy some extra constraints.

Let $\mathfrak{h} \in \mathcal{H}_{n,\ell}$ be a map with $n/3$ isolated vertices and no connected component with more than $n/3$ vertices, or more than $\gamma\ell$ edges. Then one can partition the connected components of \mathfrak{h} into two pure Hurwitz maps \mathfrak{h}_1 and \mathfrak{h}_2 that both have less than $(1+\gamma)\ell/2$ edges and more than $n/3$ vertices. Hence, if we let $H_{n,\ell}^{\leq \gamma}$ be the number of such maps, we have the following inequality:

$$H_{n,\ell}^{\leq \gamma} \leq \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq n/3}} \sum_{\substack{\ell_1+\ell_2=n \\ \ell_1, \ell_2 \leq \frac{1+\gamma}{2}\ell}} \binom{n}{n_1} \binom{\ell}{\ell_1} H_{n_1, \ell_1} H_{n_2, \ell_2} \quad (4.18)$$

(the binomials arise because we consider labelled objects).

We rewrite the asymptotic estimation of Theorem 4.3 in a convenient way, as

$$\frac{H_{n,\ell}}{n!\ell!} = \exp(\ell \log \ell - n \log n - (1 - \log 2)\ell + n + 2\ell \log \log n + o(n)), \quad (4.19)$$

where the little o is uniform for ℓ/n in any compact subset of $(0, \infty)$. This holds as long as ℓ and n tend to infinity linearly in one another.

Now, take numbers $(n_1, n_2, \ell_1, \ell_2)$ as above. We can apply the asymptotic estimation (4.19) without restriction because (for $i = 1, 2$)

$$\frac{\ell_i}{n_i} \in \left[\frac{3(1-\gamma)}{4} \frac{\ell}{n}, \frac{3(1+\gamma)}{2} \frac{\ell}{n} \right] \quad (4.20)$$

and hence $\ell_i, n_i \rightarrow \infty$ linearly in each other. Therefore we obtain, since in this case $\log \log n_i = \log \log n + o(1)$ also holds,

$$\frac{\binom{n}{n_1} \binom{\ell}{\ell_1} H_{n_1, \ell_1} H_{n_2, \ell_2}}{H_{n,\ell}} \leq \frac{\exp[\ell_1 \log \ell_1 + \ell_2 \log \ell_2 - \ell \log \ell + o(n)]}{\exp[n_1 \log n_1 + n_2 \log n_2 - n \log n]}. \quad (4.21)$$

Since $\ell_i \leq \frac{1+\gamma}{2}\ell$, we have

$$(\ell_1 \log \ell_1 + \ell_2 \log \ell_2 - \ell \log \ell) \leq \ell \log \left(\frac{1+\gamma}{2} \right). \quad (4.22)$$

On the other hand, we always have

$$n_1 \log n_1 + n_2 \log n_2 - n \log n \geq -n \log 2, \quad (4.23)$$

hence, uniformly,

$$\frac{\binom{n}{n_1} \binom{\ell}{\ell_1} H_{n_1, \ell_1} H_{n_2, \ell_2}}{H_{n, \ell}} \leq \exp \left[n \log 2 + \ell \log \left(\frac{1 + \gamma}{2} \right) + o(n) \right], \quad (4.24)$$

which is exponentially small in n . Plugging this into (4.18) one obtains $H_{n, \ell}^{\leq \gamma} = o(H_{n, \ell})$.

Let $\mathbf{E}^{< \gamma}$ denote the event that all components of $\mathbf{h}_{n, \ell}$ have less than $\gamma \ell$ edges, and let $\mathbf{E}_{n/3}$ denote the event that there are $n/3$ isolated vertices in $\mathbf{h}_{n, \ell}$. Then,

$$\mathbb{P}(\mathbf{E}^{< \gamma}) = \mathbb{P}(\mathbf{E}^{< \gamma} \cap \mathbf{E}_{n/3}) + o(1) = \frac{H_{n, \ell}^{\leq \gamma}}{H_{n, \ell}} + o(1) = o(1), \quad (4.25)$$

where in the first equality we used Lemma 4.9.

This shows that w.h.p., a uniform map of $\mathcal{H}_{n, \ell}$ has a component with more than $(\gamma(q) - o(1))\ell$ edges, which is what we needed to show. \blacksquare

4.3 Proofs: Macroscopic features

In this section we will prove parts (i) and (ii) of Theorem 4.2, which describe the first part and the general shape of a random partition under the measure $\mathbb{P}_{n, \ell}^+$ where $\ell \sim 2qn$. As we will see, to establish these “macroscopic” characteristics we will show that the cost to the measure associated with deviating from the typical behaviour is exponential (this will no longer be true in Section 4.4). We will also prove Theorem 4.3, the approximate asymptotics for $H_{n, \ell}$ at high genus, using the macroscopic limit shape and the intermediate results used to prove it.

Notation and a truncation threshold For any set Λ of partitions of n we write

$$Z_n(\Lambda) := \frac{1}{n!} \sum_{\lambda \in \Lambda} f_\lambda^2(C_\lambda)^\ell. \quad (4.26)$$

Note that the partition function of our model is $H_{n, \ell} = Z_n(\{\lambda \vdash n\}) = \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2(C_\lambda)^\ell$.

We also fix $\varepsilon := \frac{1}{100}$, and we split any partition $\lambda \vdash n$ into

$$\lambda = \lambda^+ \sqcup \lambda^-, \quad (4.27)$$

where λ^+ denotes the parts of λ that are greater than $n^{1-\varepsilon}$ and λ^- the parts that are less than $n^{1-\varepsilon}$, see Figure 4.4. The value of ε is somewhat arbitrary at this stage, and will not affect our final results. With this threshold in mind we establish some convenient sets. For $M \in [n^{1-\varepsilon}, n]$ and given $\mu \vdash n - M$ we let

$$\Lambda(\mu, M) := \{\lambda \mid |\lambda^+| = M, \lambda^- = \mu\}. \quad (4.28)$$

Then for $m \in [0, M]$ we set

$$\Lambda(\mu, M, m) := \{\lambda \in \Lambda(\mu, M) \mid \lambda_1 = M - m\}. \quad (4.29)$$

We also introduce the notation $\lambda^0 = M \sqcup \mu$, such that $\Lambda(\mu, M, 0) = \{\lambda^0\}$.

In addition, we introduce the following set of partitions (which depends implicitly on the integers n and ℓ):

$$\Lambda^* := \left\{ \lambda \vdash n \mid \lambda^+ = (\lambda_1) \text{ and } \lambda_1 \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right] \right\}. \quad (4.30)$$

Finally, we establish the following convention: *throughout Sections 4.3 and 4.4, all little os and big Os are uniform for ℓ/n in any compact subset of $(0, \infty)$ (in addition to uniformity in other quantities, which is stated as appropriate).*

4.3.1 Deterministic estimates and lower bound on $H_{n,\ell}$

We start with some convenient lower bounds. Firstly, three simple ones for the

Lemma 4.10 (Useful bounds). *Let $\lambda \vdash n$ with $\lambda^+ = (\lambda_1, \dots, \lambda_p)$, then*

- (i) $\frac{1}{n!} f_\lambda^2 \leq \frac{n!}{\prod_{i=1}^p (\lambda_i!)^2 (n - |\lambda^+|)!^2} f_{\lambda^-}^2 \leq \frac{n!}{\prod_{i=1}^p (\lambda_i!)^2 (n - |\lambda^+|)!} \leq \frac{n^{|\lambda^+|}}{\prod_{i=1}^p (\lambda_i!)^2},$
- (ii) $C_\lambda \leq \frac{\lambda_1 n}{2},$
- (iii) $C_\lambda = C_{\lambda^+} - p|\lambda^-| + C_{\lambda^-} = \sum_{i=1}^p \frac{\lambda_i(\lambda_i - 2i + 1)}{2} - p|\lambda^-| + C_{\lambda^-}.$

Proof. (i). One can fill a Young diagram of shape $\lambda \vdash n$ with distinct numbers from 1 to n by picking λ_1 numbers then filling the first row with them in increasing order, then doing the same for the second row with λ_2 numbers and so on until the p th row. There are at most $\binom{n}{\lambda_1, \lambda_2, \dots, \lambda_p}$ ways to do so, and once this is done there are at most f_{λ^-} ways to fill the remaining rows. Moreover, from the Plancherel measure normalisation (1.22) we have $f_{\lambda^-}^2 \leq |\lambda^-|! = (n - |\lambda^+|)!$. Therefore we obtain

$$\frac{1}{n!} f_\lambda^2 \leq \frac{n!}{\prod_{i=1}^p (\lambda_i!)^2 (n - |\lambda^+|)!^2} f_{\lambda^-}^2 \leq \frac{n!}{\prod_{i=1}^p (\lambda_i!)^2 (n - |\lambda^+|)!}. \quad (4.31)$$

The last inequality of the claim is straightforward.

(ii). From Definition 1.8 of C_λ , we have

$$C_\lambda = \sum_{i=1}^{\ell(\lambda)} \frac{\lambda_i(\lambda_i - 2i + 1)}{2} \quad (4.32)$$

$$\leq \sum_{i=1}^{\ell(\lambda)} \frac{\lambda_i \cdot \lambda_i}{2} \leq \frac{\lambda_1}{2} \sum_{i=1}^{\ell(\lambda)} \lambda_i = \frac{\lambda_1 n}{2}. \quad (4.33)$$

(iii). Splitting (4.32) into contributions from the first p rows and the subsequent ones, we see that the former just contribute C_{λ^+} to C_λ , while for the latter the content of each box is the content of a box of λ^- shifted by $-p$ (the number of parts in λ^+). This gives the result. \blacktriangleleft

Lemma 4.11 (Bounding the normalisation from below). *We have*

$$H_{n,\ell} \geq \left(\frac{\ell}{\log n} \right)^{2\ell} \exp \left[-\ell(2 - \log 2)\ell + O(\sqrt{n} \log^2 n) \right]. \quad (4.34)$$

Proof. Let $L := \lfloor \frac{2\ell}{\log n} \rfloor$, and let μ be a partition of $n - L$ maximizing f_μ among partitions with $\mu_1 \leq 3\sqrt{n}$ and $\ell(\mu) \leq 3\sqrt{n}$. We let $\lambda^* = L \sqcup \mu$. Using Lemma 4.10 (iii), and noting that $C_\mu \geq -\frac{3}{2}n^{3/2}$ (by Lemma 4.10 (ii)), we have

$$C_{\lambda^*} = \frac{L(L+1)}{2} - |\mu| + C_\mu \geq \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}} \right) \right) \frac{L^2}{2}. \quad (4.35)$$

On the other hand, consider the sum

$$\sum_{\nu \vdash n-L} f_\nu^2 = (n-L)!. \quad (4.36)$$

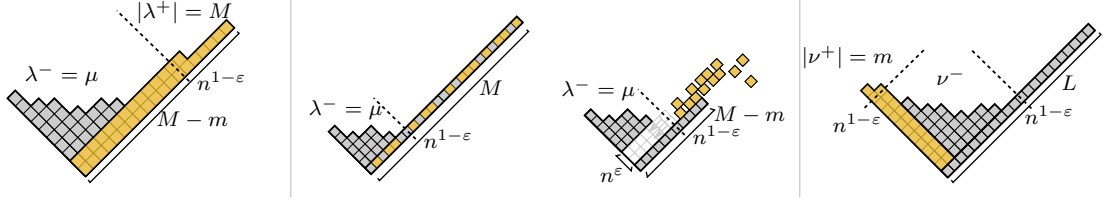


Figure 4.4 Left, a partition $\lambda \vdash n$ in $\Lambda(\mu, M, m)$, with λ^+ and λ^- indicated. Center, a SYT of shape $\lambda^0 \in \Lambda(\mu, M, 0)$ (the filling of the boxes is not shown) is transformed to a SYT of some shape $\lambda \in \Lambda(\mu, M, m)$ or to something else by the surjective operation used to prove Claim 4.15. Right, a partition $\lambda = L \sqcup \nu'$ with $|\nu^+| = m$, used to prove Lemma 4.16.

By the VKLS theorem, we know that this sum is dominated by partitions such that $\nu_1 \leq 3\sqrt{n}$. Since the number of terms is bounded by (1.19), we deduce that $f_\mu^2 \geq (n-L)!e^{O(\sqrt{n})}$, and in fact $f_\mu^2 = (n-L)!e^{O(\sqrt{n})}$.

Then, since $f_{\lambda^*}^2 \geq f_\mu^2$, we have

$$\begin{aligned} Z_n(\{\lambda^*\}) &\geq \frac{(n-L)!e^{O(\sqrt{n})}}{n!} C_{\lambda^*}^\ell \geq n^{-L} e^{O(\sqrt{n})} \left(\frac{L^2}{2}\right)^\ell \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right)^\ell \\ &\geq \exp\left[2\ell(\log \ell - \log \log n) - \ell(2 - \log 2) + O(\sqrt{n} \log^2 n)\right]. \end{aligned} \quad (4.37)$$

This finishes the proof since $H_{n,\ell} = Z_n(\{\lambda \vdash n\}) \geq Z_n(\{\lambda^*\})$. \blacktriangleleft

4.3.2 First bound on the first and second parts

We now proceed with a succession of lemmas that gradually give better control on the partition λ^+ .

Lemma 4.12 (Controlling big parts). *Let λ be a random partition of n under the assumptions of Theorem 4.2. Then $|\lambda^+| \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n}\right]$ w.h.p.*

Proof. Given $\lambda \vdash n$, set $R_\lambda := \frac{|\lambda^+| \log n}{\ell}$. In this proof, all little o s are independent of R_λ . For all $\lambda \vdash n$, by the last inequality in Lemma 4.10(i),

$$\begin{aligned} \frac{1}{n!} f_\lambda^2 &\leq \exp\left[|\lambda^+| \log n - 2 \sum_{i=1}^p \lambda_i \log(\lambda_i) + 2|\lambda^+|\right] \\ &\leq \exp\left[|\lambda^+| \log n - 2 \sum_{i=1}^p (1-\varepsilon)\lambda_i \log(n) + 2|\lambda^+|\right] \\ &\leq \exp\left[-(1-2\varepsilon)R_\lambda \ell + 2R_\lambda \ell / \log n\right]. \end{aligned} \quad (4.38)$$

On the other hand, by Lemma 4.10(ii)-(iii), if $C_\lambda \geq 0$, then

$$\begin{aligned} C_\lambda^\ell &\leq \exp\left[\ell \log\left(\sum_{i=1}^p \frac{\lambda_i^2}{2} + C_{\lambda^-}\right)\right] \\ &\leq \exp\left[\ell \log\left(\frac{|\lambda^+|^2 + n^{2-\varepsilon}}{2}\right)\right] \\ &\leq \exp\left[2\ell(\log \ell - \log \log n) + \ell\left(\log\left(R_\lambda^2 + \frac{n^{2-\varepsilon} \log^2 n}{\ell^2}\right) - \log 2\right)\right]. \end{aligned} \quad (4.39)$$

Combining (4.38) and (4.39), and using (4.34), we obtain

$$\frac{Z_n(\{\lambda\})}{H_{n,\ell}} \leq \exp \left[\ell \left(2 - (1 - 2\varepsilon)R_\lambda + \frac{2R_\lambda}{\log n} + \log \left(R_\lambda^2 + o(1) \right) \right) + o(n) \right]. \quad (4.40)$$

Now, the function $r \mapsto 2 - (1 - 2\varepsilon)r + \log(r^2)$ has a unique maximum on $\mathbb{R}_{>0}$ and goes to $-\infty$ on both ends, so it is less than $-1/100$ outside of a closed interval I , and in fact one can take $I = [0.4, 6]$. Hence for n large enough and $\lambda \vdash n$ with $R_\lambda \notin [0.4, 6]$, $\mathbb{P}_{n,\ell}^+(\lambda) \leq \exp(-\ell/100 + o(n))$, which entails the result since there are $e^{O(\sqrt{n})}$ partitions of n . \blacktriangleleft

Lemma 4.13 (Uniqueness of the big part). *Let λ be a random partition of n under the assumptions of Theorem 4.2. Then $\lambda^+ = (\lambda_1)$, and equivalently $\lambda_2 \leq n^{1-\varepsilon}$, w.h.p.*

The proof of Lemma 4.13 requires a comparison of the contribution of partitions with a single “big part” with the contribution of those with more than one (indeed, because we have neither exact formulas nor precise estimates on the normalisations, we can only rely on “comparison” of probabilities at this stage). We will perform this comparison among partitions having the same “small parts” (called μ), using the sets $\Lambda(\mu, M, m)$ defined at (4.29) with $|\lambda^+| = M$ and $\lambda_1 = M - m$. We will need the following two claims, whose proof is postponed to after that of the lemma.

Claim 4.14. *For all $\lambda \in \Lambda(\mu, M, m)$, we have $C_\lambda \leq C_{\lambda^0} - (m - 1)\frac{M}{2}$.*

Claim 4.15. *If $m > 0$ then, $\sum_{\lambda \in \Lambda(\mu, M, m)} f_\lambda \leq f_{\lambda^0} \exp[m(2\varepsilon \log n + 1)]$.*

Proof of Lemma 4.13. By Lemma 4.12, we know that, w.h.p., $|\lambda^+| \in [0.4\frac{\ell}{\log n}, 6\frac{\ell}{\log n}]$. We can thus assume this event for the rest of this proof.

We now condition on $|\lambda^+| = M$ and $\lambda^- = \mu$, with given $M \in [0.4\frac{\ell}{\log n}, 6\frac{\ell}{\log n}]$ and $\mu \vdash n - M$. In the rest of the proof, the little o s are uniform in M and μ satisfying these conditions. Combining Claims 4.14 and 4.15 for $m > 0$, one obtains

$$\frac{Z_n(\Lambda(\mu, M, m))}{Z_n(\{\lambda^0\})} \leq \exp \left[\ell \log \left(1 - \frac{(m - 1)M}{2C_{\lambda^0}} \right) + 2m(2\varepsilon \log n + 1) \right]. \quad (4.41)$$

But we know by Lemma 4.10 (ii)-(iii) that $C_{\lambda^0} \leq \frac{M^2}{2} + \frac{n^{2-\varepsilon}}{2} = \frac{M^2}{2} + (1 + o(1))\frac{M^2}{2}$ and $M \leq 6\frac{\ell}{\log n}$. Hence

$$\begin{aligned} \frac{Z_n(\Lambda(\mu, M, m))}{Z_n(\{\lambda^0\})} &\leq \exp \left[(1 + o(1))\ell \log \left(1 - \frac{m - 1}{M} \right) + 2m(2\varepsilon \log n + 1) \right] \\ &\leq \exp \left[(1 + o(1))\ell \log \left(1 - \frac{(m - 1) \log n}{6\ell} \right) + 2m(2\varepsilon \log n + 1) \right] \\ &\leq \exp \left[-(1 + o(1))\frac{(m - 1) \log n}{6} + 2m(2\varepsilon \log n + 1) \right] \\ &\leq \exp \left[(1 + o(1))m \log n \left(4\varepsilon - \frac{1}{6} \right) \right] \leq \exp \left(-\frac{m \log n}{100} \right) \end{aligned} \quad (4.42)$$

where the last inequality holds for n large enough.

Summing this over all $m > 0$ (recall that in this case we have $m \geq n^{1-\varepsilon}$), we have

$$\sum_{m>0} Z_n(\Lambda(\mu, M, m)) = o(Z_n(\{\lambda^0\})) \quad (4.43)$$

which is enough to conclude that $\lambda^+ = (\lambda_1)$ w.h.p. \blacktriangleleft

Proof of Claim 4.14 . By Lemma 4.10 (iii), for any $\lambda \in \Lambda(\mu, M)$ we have a simple upper bound

$$C_\lambda \leq C_{\lambda^+} - (n - M) + C_\mu, \quad (4.44)$$

since λ^+ contains at least one part. For λ^0 the same lemma gives the equality

$$C_{\lambda^0} = \frac{M}{2}(M + 1) - (n - M) + C_\mu. \quad (4.45)$$

When $\lambda \in \Lambda(\mu, M, m)$, by Lemma 4.10 (ii) we have $C_{\lambda^+} \leq M(M - m)/2$, and inserting this into (4.44) we have

$$C_\lambda \leq \frac{M}{2}(M - m) - (n - M) + C_\mu = C_{\lambda^0} - \frac{M}{2}(m - 1) \quad (4.46)$$

as required. \blacktriangleleft

Proof of Claim 4.15. To prove this claim we define a “redistribution” operation that enables us to compare the contribution of partitions with one big part to others. Let T be an SYT of shape λ^0 , and consider the following operation.

1. Create n^ε empty rows between the first row of T and the rest,
2. choose m numbers in the first row of T ($\binom{M}{m}$ choices),
3. for each of these numbers, choose one of the newly created rows, and move it there (n^ε choices each time),
4. sort each row and delete the empty rows.

The output is either a SYT of some $\lambda \in \Lambda(\mu, M, m)$, or something else. It is easily checked that this procedure can output any SYT of shape λ , for any $\lambda \in \Lambda(\mu, M, m)$ (indeed, for any λ , λ^+ must have at most $\frac{n}{n^{1-\varepsilon}} = n^\varepsilon$ parts). Hence we have

$$\sum_{\lambda \in \Lambda(\mu, M, m)} f_\lambda \leq \binom{M}{m} n^{\varepsilon m} f_{\lambda^0} \leq \frac{M^m}{m!} n^{\varepsilon m} f_{\lambda^0} \leq f_{\lambda^0} \exp[m(2\varepsilon \log n + 1)] \quad (4.47)$$

where in the last inequality we used the bound $m! \geq (m/e)^m$, along with the facts that $\log M \leq \log n$ and $\log m \geq (1 - \varepsilon) \log n$. \blacktriangleleft

4.3.3 Asymptotic estimate for $H_{n,\ell}$ and macroscopic behaviour of λ

Lemmas 4.12 and 4.13 imply that $\lambda \in \Lambda^*$ w.h.p. (using the notation introduced at (4.30)) and hence

$$H_{n,\ell} = (1 + o(1))Z_n(\Lambda^*), \quad (4.48)$$

and it follows from the proofs that the little o is uniform for ℓ/n in any compact subset of $(0, \infty)$. From this result, we obtain Theorem 4.3 and parts (i) and (ii) of Theorem 4.2.

Proof of Theorem 4.3. Take $\lambda \in \Lambda^*$. We have, by Lemma 4.10 (ii)-(iii) $C_\lambda \leq \frac{\lambda_1^2}{2} + \frac{n^{2-\varepsilon}}{2} = (1 + o(1))\frac{\lambda_1^2}{2}$ where the $o(1)$ is uniform over all $\lambda \in \Lambda^*$, and more generally, from now on, all little o s and big O s will be uniform over all $\lambda \in \Lambda^*$ when applicable.

On the other hand, by Lemma 4.10(i) we have $\frac{1}{n!}f_\lambda^2 \leq \frac{n^{\lambda_1}}{(\lambda_1!)^2}$, hence

$$Z_n(\{\lambda\}) \leq \exp [2\ell \log(\lambda_1) - \ell \log 2 - \lambda_1 \log n + o(n)]. \quad (4.49)$$

Substituting $\lambda_1 = \frac{R_\lambda \ell}{\log n} \sim \frac{R_\lambda 2qn}{\log n}$ into the inequality above, we obtain

$$Z_n(\{\lambda\}) \leq \left(\frac{\ell}{\log n}\right)^{2\ell} \exp[(2 \log R_\lambda - \log 2 - R_\lambda)\ell + o(n)]. \quad (4.50)$$

Now, since the function on the positive reals $x \mapsto 2 \log x - x$ has a unique maximum at $x = 2$, we have

$$Z_n(\{\lambda\}) \leq \left(\frac{\ell}{\log n}\right)^{2\ell} \exp[(-2 + \log 2)\ell + o(n)], \quad (4.51)$$

and since there are $e^{O(\sqrt{n})}$ partitions of n , $Z_n(\Lambda^*) \leq \max_{\lambda \in \Lambda^*} Z_n(\{\lambda\})e^{O(\sqrt{n})}$. Together with the lower bound of Lemma 4.11 and (4.48) this proves that

$$H_{n,\ell} = \left(\frac{\ell}{\log n}\right)^{2\ell} \exp[(-2 + \log 2)\ell + o(n)], \quad (4.52)$$

as required. \blacksquare

Proof of Theorem 4.2, part (i). The upper bound (4.50) in the previous proof along with Lemma 4.11 implies that for $\lambda \in \Lambda^*$ (uniformly),

$$\mathbb{P}_{n,\ell}^+(\lambda) \leq \exp[\ell(2 \log R_\lambda - 2 \log 2 + 2 - R_\lambda) + o(n)]. \quad (4.53)$$

Any non-negligible deviation of R_λ from the unique maximiser of this upper bound thus entails an exponentially decreasing probability, which is enough to conclude that, under the Plancherel–Hurwitz measure at high genus, $R_\lambda \xrightarrow{p} 2$, which is what we wanted. \blacksquare

Proof of Theorem 4.2, part (ii). We refine the previous upper bound on $\mathbb{P}_{n,\ell}^+(\lambda)$ for $\lambda = \lambda_1 \sqcup \tilde{\lambda} \in \Lambda^*$ by writing $\frac{1}{n!} f_\lambda^2 \leq \frac{n!}{\lambda_1!^2 (n-\lambda_1)!^2} f_{\tilde{\lambda}}^2$ by Lemma 4.10 part (i), along with $C_\lambda = (1 + o(1)) \frac{\lambda_1^2}{2}$ as above, uniformly for $\lambda \in \Lambda^*$.

Recall that $R_\lambda = \frac{\lambda_1 \log n}{\ell}$. By Lemma 4.11 and the Plancherel entropy estimate of Proposition 1.36 for $\frac{1}{(n-\lambda_1)!} f_{\tilde{\lambda}}^2$, we have

$$\begin{aligned} \mathbb{P}_{n,\ell}^+(\lambda) &\leq \exp\left[\ell(2 \log R_\lambda - 2 \log 2 + 2 - R_\lambda) - n(1 + 2I_{\text{hook}}(\psi_{\tilde{\lambda}, \sqrt{n}})) + o(n)\right] \\ &\leq \exp\left[-n(1 + I_{\text{hook}}(\psi_{\tilde{\lambda}, \sqrt{n}})) + o(n)\right] \end{aligned} \quad (4.54)$$

for all $\lambda \in \Lambda^*$, uniformly (where the second inequality comes from the unique maximum at $R_\lambda = 2$). We recognise the Plancherel measure estimate from Proposition 1.36. Since $\lambda \in \Lambda^*$ w.h.p., this implies, as in the classical Plancherel case (see [Rom15, Section 1.17]), the almost sure convergence in supremum norm to the VKLS limit shape (4.7). \blacksquare

Note that since the function $x \mapsto \Omega(x) - |x|$ has support $[-2, 2]$, the convergence of the profile directly implies the “lower bound” in Theorem 4.2 (iii):

$$\min(\lambda_2, \ell(\lambda)) \geq (2 - o_p(1))\sqrt{n}. \quad (4.55)$$

Just as in the classical Plancherel case, the upper bound is more delicate and is the subject of the next section.

4.3.4 First bound on the number of parts

To conclude this section, we use Theorem 4.2 (i) to prove a further macroscopic feature of the limit behaviour, which completes the rough bounding box one side of which is determined by Lemma 4.13.

Lemma 4.16 (Bounding the length above). *Let λ be a random partition of n under the assumptions of Theorem 4.2. Then $\ell(\lambda) \leq n^{1-\varepsilon}$ w.h.p.*

Proof. Take $\lambda \in \Lambda^*$, we first use arguments similar to Lemma 4.12 to control the number of boxes in the big parts of the conjugate partition λ' . Let us write $\lambda = L \sqcup \nu'$ (as in the proof of Lemma 4.13, the little o s and big O s are uniform in L and ν) so the conjugate of the small parts λ^- is

$$\nu = (\lambda_2, \lambda_3, \dots, \lambda_{\ell(\lambda)})' = \nu^+ \sqcup \nu^- \quad (4.56)$$

where ν^+ and ν^- denote respectively the parts of ν that are greater and less than $n^{1-\varepsilon}$, and $\ell(\nu) \leq n^{1-\varepsilon}$ (see Figure 4.4). Suppose that the big parts have size $m = m(\lambda) = |\nu^+|$ (note that if $m > 0$ then $m \geq n^{1-\varepsilon}$). By the hook length formula Theorem 1.7 and Lemma 4.10 (i), we have

$$\frac{1}{n!} f_\lambda^2 = \frac{n!}{(n-L)!^2} f_\nu^2 \frac{1}{\prod_{\square \in \lambda_1} \eta_\lambda(\square)^2} \leq \frac{1}{(n-L)!} f_\nu^2 n^{-L} \exp \left[O \left(\frac{n}{\log n} \right) \right], \quad (4.57)$$

then by Lemma 4.10 (i) again (and (4.38))

$$\frac{1}{(n-L)!} f_\nu^2 \leq \exp[-(1-2\varepsilon)m \log(n-L) + 2m]. \quad (4.58)$$

Since $C_\lambda \leq \frac{L^2}{2}(1+n^{2-\varepsilon})$, using (4.34) to bound $H_{n,\ell}$ below we have

$$\frac{Z_n(\{\lambda\})}{H_{n,\ell}} \leq \exp \left[-\frac{98m}{100} \log n + O(m) + O \left(\frac{n}{\log n} \right) \right]. \quad (4.59)$$

Thus,

$$Z_n \left(\left\{ \lambda \in \Lambda^* \mid m(\lambda) \geq \frac{n}{\sqrt{\log n}} \right\} \right) = o(H_{n,\ell}). \quad (4.60)$$

Now, take $\lambda \in \Lambda^*$ with $m(\lambda) < \frac{n}{\sqrt{\log n}}$. Write $\lambda = L \sqcup \nu'$ as above, fix $\nu^- := \mu'$, $|\nu^+| = m$ and fix $M = L + m$. In analogy with the proof of Lemma 4.13, for given M and m , we consider the set $\hat{\Lambda}(\mu, M, m)$ of all such partitions, and compare $Z_n(\hat{\Lambda}(\mu, M, m))$ to $Z_n(\{\lambda^0\})$, where $\lambda^0 = M \sqcup \mu = L \sqcup \mu$ as before, such that $\hat{\Lambda}(\mu, M, 0) = \{\lambda^0\}$. It follows immediately from this definition that Claim 4.15 also applies to $\hat{\Lambda}(\mu, M, m)$, that is to say

$$\sum_{\lambda \in \hat{\Lambda}(\mu, M, m)} f_\lambda \leq f_{\lambda^0} \exp[m(2\varepsilon \log n + 1)]. \quad (4.61)$$

Then, by Lemma 4.10 (iii) and (4.45), we get

$$C_\lambda = C_{\lambda^0} - \frac{m(2M - m - 1)}{2} + \ell(\nu^+)(n - M) - C_{\nu^+}. \quad (4.62)$$

It is clear that $\ell(\nu^+) \leq m/n^{1-\varepsilon}$, and if ν^+ is non-empty, C_{ν^+} has a straightforward lower bound of

$$C_{\nu^+} \geq \frac{(\nu_{\ell(\nu)} - 2\ell(\nu) + 1)n}{2} \geq \left(\frac{n^{1-\varepsilon}}{2} - \frac{m}{n^{1-\varepsilon}} + 1 \right) n \geq -mn^\varepsilon, \quad (4.63)$$

and, since $m = o(M)$ and $M \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right]$,

$$C_\lambda \leq C_{\lambda^0} - mM(1 + o(1)). \quad (4.64)$$

As $C_{\lambda^0} \leq (1 + o(1))\frac{M^2}{2}$, repeating precisely the arguments of Lemma 4.13 we have

$$\begin{aligned} \frac{Z_n(\hat{\Lambda}(\mu, M, m))}{Z_n(\{\lambda^0\})} &\leq \exp \left[\ell \log \left(1 - \frac{m \log n}{2\ell} \right) (1 + o(1)) + 2m(2\varepsilon \log n + 1) \right] \\ &\leq \exp \left(-\frac{m \log n}{4} (1 + o(1)) \right) \end{aligned} \quad (4.65)$$

and after considering enumeration of partitions again we find that the probability that $m(\boldsymbol{\lambda}) > 0$ is $o(1)$, so $\ell(\boldsymbol{\lambda}) \leq n^{1-\varepsilon}$ w.h.p. as required. \blacktriangleleft

4.4 Proofs: Microscopic features

In this section we consider the smaller scale of the limit shape and prove Theorem 4.2 part (iii) by bounding the size of $\boldsymbol{\lambda}_2$ and $\ell(\boldsymbol{\lambda})$ above. As previously mentioned, the VKLS limit shape result in supremum norm does not imply such a bound, and even in the Plancherel case extra arguments are needed to obtain the sharp bound $(2 + o_p(1))\sqrt{n}$ on $\boldsymbol{\lambda}_1$. In the case of the classical Plancherel measure and the VKLS limit shape theorem 1.35, a good way to do this is to use Kerov's Plancherel growth process [Ker98] described in Section 1.2.1, which provides one with a coupling between the measures at sizes n and $n - 1$, through which the evolution of $\boldsymbol{\lambda}_1$ is tractable inductively.

In our context, we do not have such a coupling, however we will be able to compare the behaviour of random partitions in sizes n and $n - 1$ by calculations which in some sense provide an approximation of a corresponding growth process. We will directly use one result from the Plancherel growth process approach, which just follows from the fact that it is a well defined probability law:

Proposition 4.17 (Plancherel growth process normalisation [Ker98]). *Let $\mu \vdash n - 1$ and let $\mu \nearrow \nu$ denote that $\nu \vdash n$ is obtained from μ by adding one box. We have*

$$\sum_{\nu: \mu \nearrow \nu} f_\nu = n f_\mu. \quad (4.66)$$

From now on we will work with a given value of $\lambda_1 = L$ with $L \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right]$, and in the notation of Theorem 4.2, we let $\tilde{\lambda} = \lambda \setminus \lambda_1$. We introduce notation for the normalisation with a fixed first part, putting

$$Z_n[L] := Z_n(\{\lambda \vdash n \mid \lambda_1 = L\}). \quad (4.67)$$

4.4.1 An intermediate bounding box for $\tilde{\lambda}$

Proposition 4.18. *Under the hypotheses of Theorem 4.2, the second part satisfies $\boldsymbol{\lambda}_2 \leq (e + o_p(1))\sqrt{n}$, and similarly the length satisfies $\ell(\boldsymbol{\lambda}) \leq (e + o_p(1))\sqrt{n}$.*

Our proof relies on comparisons between partitions of n and partitions of $n - 1$.

Lemma 4.19. *Uniformly for $L \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right]$, we have*

$$1 - O(1/\log n) \leq \frac{Z_{n-1}[L]}{Z_n[L]} \leq 1 + O(\log^2 n/n^\varepsilon). \quad (4.68)$$

Proof. We first bound $Z_{n-1}[L]/Z_n[L]$ above. For partitions of $n - 1$ whose first part is equal to L , the normalisation of the conditioned measure is

$$Z_{n-1}[L] = \frac{1}{(n-1)!} \sum_{\substack{\mu \vdash n-1 \\ \mu_1=L}} f_\mu^2 C_\mu^\ell = \frac{1}{n!} \sum_{\substack{\mu \vdash n-1 \\ \mu_1=L}} f_\mu C_\mu^\ell \sum_{\lambda: \mu \nearrow \lambda} f_\lambda, \quad (4.69)$$

where the final equality comes from the normalisation of the Plancherel growth process, as given in Proposition 4.17. This can equally be expressed as a sum over partitions of n with first part equal to L , as

$$Z_{n-1}[L] = \frac{1}{n!} \sum_{\substack{\lambda \vdash n \\ \lambda_1=L}} f_\lambda \sum_{\substack{\mu: \mu \nearrow \lambda \\ \mu_1=L}} f_\mu C_\mu^\ell + \frac{1}{n!} \sum_{\substack{\mu \vdash n-1 \\ \mu_1=L}} f_{\mu^{1+}} f_\mu C_\mu^\ell \quad (4.70)$$

where μ^{1+} denotes the partition $\mu^{1+} = (\mu_1 + 1, \mu_2, \mu_3, \dots)$.

Considering the first term, we note that by Lemma 4.13 and Lemma 4.16, the contribution to $Z_{n-1}[L]$ from any partition with more than one big part will be exponentially suppressed. Then, for $\mu \nearrow \lambda$, $\mu_1 = \lambda_1 = L$ and $\mu_2 \leq n^{1-\varepsilon}$, the modulus of the content of the additional box will be at most $n^{1-\varepsilon}$, and

$$C_\mu^\ell = (C_\lambda + O(n^{1-\varepsilon}))^\ell = C_\lambda^\ell (1 + O(\log^2 n/n^\varepsilon)). \quad (4.71)$$

Then we have

$$Z_{n-1}[L] \leq \frac{1}{n!} \sum_{\substack{\lambda \vdash n \\ \lambda_1=L}} f_\lambda C_\lambda^\ell (1 + O(\log^2 n/n^\varepsilon)) \sum_{\mu: \mu \nearrow \lambda} f_\mu + \frac{1}{n!} \sum_{\substack{\mu \vdash n-1 \\ \mu_1=L}} f_{\mu^{1+}} f_\mu C_\mu^\ell \quad (4.72)$$

where we over-count only by adding the partition $(L - 1, \lambda_2, \lambda_3, \dots)$ in the sum over $\{\mu : \mu \nearrow \lambda\}$. Then, using the identity $\sum_{\mu: \mu \nearrow \lambda} f_\mu = f_\lambda$ from recursively counting SYT, we have

$$Z_{n-1}[L] \leq Z_n[L] (1 + O(\log^2 n/n^\varepsilon)) + \frac{1}{n!} \sum_{\substack{\mu \vdash n-1 \\ \mu_1=L}} f_{\mu^{1+}} f_\mu C_\mu^\ell. \quad (4.73)$$

The second term on the right is finally estimated using the hook-length formula Theorem 1.7. For $\mu \vdash n - 1$, $\mu_1 = L$ and $\mu_2 \leq n^{1-\varepsilon}$ we have

$$f_{\mu^{1+}} = n f_\mu \exp \left[\sum_{j=1}^{\mu_2} \log \frac{L + \mu'_j - j}{L + \mu'_j - j + 1} + \sum_{j=\mu_2+1}^L \log \frac{L - j + 1}{L - j + 2} \right] = n f_\mu O\left(\frac{\log n}{n}\right) \quad (4.74)$$

so the second term is just $Z_{n-1}[L] O(\log n/n)$, which is absorbed into the left hand side of (4.73), proving the upper bound in (4.68).

The ratio is similarly bounded below by writing $Z_n[L]$ as a sum over partitions of $n - 1$, to find

$$Z_n[L] \leq \frac{1}{(n-1)!} \sum_{\substack{\mu \vdash n-1 \\ \mu_1=L}} f_\mu^2 C_\mu^\ell (1 - O(\log^2 n/n^\varepsilon)) + \frac{1}{n!} \sum_{\substack{\lambda \vdash n \\ \lambda_1=L}} f_{\lambda^{1-}} f_\lambda C_\lambda^\ell \quad (4.75)$$

where λ^{1-} denotes the partition $(\lambda_1 - 1, \lambda_2, \lambda_3, \dots)$. Then we have

$$f_{\lambda^{1-}} = \frac{1}{n} f_\lambda \exp \left[\sum_{j=1}^{\lambda_2} \log \frac{L + \lambda'_j - j}{L + \lambda'_j - j - 1} + \sum_{j=\lambda_2+1}^{L-1} \log \frac{L - j + 1}{L - j} \right] = \frac{1}{n} f_\lambda O\left(\frac{n}{\log n}\right) \quad (4.76)$$

and finally $Z_{n-1}[L]/Z_n[L] \geq 1 - O(1/\log n)$, which concludes the proof. \blacktriangleleft

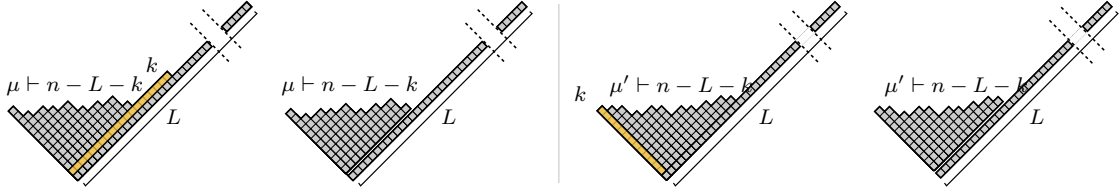


Figure 4.5 Left, partitions $L \sqcup k \sqcup \mu \vdash n$ and $L \sqcup \mu \vdash n - k$, as used in the proof of Proposition 4.18 to bound the second part λ_2 . Right, partitions $L \sqcup (k \sqcup \mu') \vdash n$ and $L \sqcup \mu' \vdash n - k$ with a box removed from each as used to bound the length $\ell(\lambda)$.

This result immediately implies Lemma 4.9 used in Section 4.2.2:

Proof of Lemma 4.9. The expected number of isolated vertices in $\mathbf{H}_{n,\ell}$ is easily shown to be

$$\frac{nZ_{n-1}}{Z_n}. \quad (4.77)$$

Since, uniformly for $L \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right]$,

$$\frac{Z_{n-1}[L]}{Z_n[L]} \geq 1 - O(1/\log n) \quad (4.78)$$

and since the contribution from any partitions without $\lambda_1 \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right]$ will be exponentially suppressed, we deduce from (4.78) that this expectation is equal to $n - O(n/\log n)$, which implies the result from the fact that the number of isolated vertices is at most n and from the Markov inequality. \blacktriangleleft

We are now ready to prove Proposition 4.18.

Proof of Proposition 4.18. Under the Plancherel–Hurwitz measure conditioned on the first part being $\lambda_1 = L$ (with $L \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right]$), the distribution of the second part is

$$\mathbb{P}(\lambda_2 = k | \lambda_1 = L) = \frac{1}{n!Z_n[L]} \sum_{\substack{\mu \vdash n-L-k \\ \mu_1 \leq k}} f_{L \sqcup k \sqcup \mu}^2 C_{L \sqcup k \sqcup \mu}^\ell. \quad (4.79)$$

Comparing SYT of shape $L \sqcup k \sqcup \mu \vdash n$ with ones of shape $L \sqcup \mu \vdash n - k$, obtained by removing the second part, we have a rough upper bound of

$$f_{L \sqcup k \sqcup \mu} \leq \binom{n}{k} f_{L \sqcup \mu} \quad (4.80)$$

by over-counting the ways the boxes of the second part could be labelled. The content-sum of each of these partitions are related by

$$C_{L \sqcup k \sqcup \mu} = C_{L \sqcup \mu} - |\mu| + \frac{k(k-3)}{2} = C_{L \sqcup \mu}(1 + o(1)) \quad (4.81)$$

as long as $k \leq n^{1-\varepsilon}$. By Lemma 4.13 we may consider only partitions with $\lambda_2 \leq n^{1-\varepsilon}$, we thus get

$$\begin{aligned} \mathbb{P}(\lambda_2 = k | \lambda_1 = L) &\leq \frac{1}{n!Z_n[L]} \sum_{\mu \vdash n-L-k} \binom{n}{k}^2 f_{L \sqcup \mu}^2 C_{L \sqcup \mu}^\ell (1 + o(1)) \\ &= \binom{n}{k}^2 \frac{(n-k)!}{n!} \frac{Z_{n-k}[L]}{Z_n[L]} (1 + o(1)). \end{aligned} \quad (4.82)$$

Now, by Lemma 4.19, it follows that

$$\mathbb{P}(\lambda_2 = k | \lambda_1 = L) \leq \frac{n!}{k!^2(n-k)!} e^{o(k)} \leq \frac{n^k}{(k/e)^{2k}} e^{o(k)}, \quad (4.83)$$

where the little o is uniform for $L \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right]$. Hence

$$\forall \varepsilon > 0, \quad \mathbb{P}(\lambda_2 = (1 + \varepsilon)e\sqrt{n} | \lambda_1 = L) = (1 + \varepsilon)^{-2k+o(k)}, \quad (4.84)$$

with the same uniformity. From Theorem 4.2, part (i), this implies that $\lambda_2 \leq e(1 + o_p(1))\sqrt{n}$ as $n \rightarrow \infty$.

The proof that $\ell(\lambda) \leq e(1 + o_p(1))\sqrt{n}$ works exactly the same way, except that, instead of removing the second part, one removes one box from each part except the first one (see Figure 4.5), which explicitly means comparing partitions $L \sqcup (k \sqcup \mu)'$ and $L \sqcup \mu'$ to find, by Lemma 4.16,

$$\begin{aligned} \mathbb{P}(\ell(\lambda) = k + 1 | \lambda_1 = L) &= \frac{1}{n!Z_n[L]} \sum_{\substack{\mu \vdash n-L-k \\ \mu_1 \leq k}} f_{L \sqcup (k \sqcup \mu)'}^2 C_{L \sqcup (k \sqcup \mu)'}^\ell \\ &\leq \frac{1}{n!Z_n[L]} \sum_{\mu \vdash n-L-k} \binom{n}{k}^2 f_{L \sqcup \mu}^2 C_{L \sqcup \mu}^\ell (1 + o(1)) \end{aligned} \quad (4.85)$$

leading to the upper bound by precisely the same steps as for the second part. \blacksquare

4.4.2 Deviation bounds for the Plancherel measure

In order to refine our last estimate, we need some deviation bounds for quantities under the classical Plancherel measure. For the deviation of λ_1 from $2\sqrt{n}$, precise bounds can luckily be found in the literature; we state one of them here²:

Proposition 4.20 (Deviation bound for the first part of a Plancherel random partition [DZ99], Equation (1.5)). *Let $\lambda \vdash n$ be a random partition under the Plancherel measure \mathbb{P}_n . Then, for all $\varepsilon > 0$, there exists $K > 0$ such that*

$$\mathbb{P}_n(\lambda_1 \geq (2 + \varepsilon)\sqrt{n}) \leq \exp[-(1 + o(1))K\sqrt{n}]. \quad (4.86)$$

With this, we can prove a deviation bound for the content-sum of under the Plancherel measure. To the best of our knowledge, such a bound is new, and we think it might be of independent interest. However, we do not think this bound is optimal, nor that conditions on λ_1 and $\ell(\lambda)$ are necessary – finding the best upper bound on these deviation probabilities seems an interesting problem that we leave open.

Theorem 4.21 (Deviation bound for the sum of contents under the Plancherel measure). *Let $\lambda \vdash n$ be a random partition under the Plancherel measure \mathbb{P}_n . Then, for all $t \gg (\log n / \sqrt{n})^{1/6}$ and constants $c \geq 2$, there exists a constant B such that*

$$\mathbb{P}_n \left(C_\lambda \geq n^{3/2}t \text{ and } \max(\lambda_1, \ell(\lambda_1)) \leq c\sqrt{n} \right) \leq \exp[-(B + o(1))nt^6]. \quad (4.87)$$

Proof. Let us recall the discussion of the Plancherel entropy estimate of Section 1.2.1, and once again define the height function of the rescaled profile of a given partition λ

²The result in [DZ99] is actually more precise, but this is sufficient for our purposes.

as $h(x) := \psi_{\lambda, \sqrt{n}}(x) - |x|$, its distance above the lines $y = |x|$. We associate the usual Fourier transform \hat{h} to h :

$$\hat{h}(u) = \int_{-\infty}^{\infty} e^{-ixu} h(x) dx. \quad (4.88)$$

We can calculate the content sum of $\lambda \vdash n$ easily from h :

$$C_\lambda = \frac{n^{3/2}}{\sqrt{2}} \int_{-\infty}^{\infty} xh(x) dx = \frac{n^{3/2}}{\sqrt{2}} \hat{h}'(0). \quad (4.89)$$

The Plancherel entropy estimate I_{hook} is written as a functional of h at (1.95). Letting h_0 denote the height function of the VKLS limit curve Ω , we cite an additional expression given at [Rom15, equation 1.19]:

$$\frac{f_\lambda^2}{n!} = \exp[-2n(I_{\text{hook}}(h) - I_{\text{hook}}(h_0)) + O(\sqrt{n} \log n)]. \quad (4.90)$$

For our proof, we will only need the following inequality ([Rom15, Equations (1.35) and (1.38)]):

$$I_{\text{hook}}(h) - I_{\text{hook}}(h_0) \geq \frac{1}{2} Q(h - h_0) := \frac{1}{8} \int_{-\infty}^{\infty} |x| |\hat{h}(x) - \hat{h}_0(x)|^2 dx. \quad (4.91)$$

Let $h_* := h - h_0$, where h is the height function for λ satisfying $\max(\lambda_1, \ell(\lambda)) \leq c\sqrt{n}$. Then h_λ is supported on $[-c, c]$, and thus so is h_* (because h_0 is supported on $[-2, 2]$). We have

$$\begin{aligned} |\hat{h}'_*(u) - \hat{h}'_*(0)| &= \left| \int_{-\infty}^{\infty} (e^{ixu} - 1) x h_*(x) dx \right| \\ &\leq \int_{-c}^c x^2 |u| |h_*(x)| dx \\ &\leq |u| \int_{-c}^c c^2 (h_\lambda(x) + h_0(x)) dx = 2c^2 |u|, \end{aligned}$$

where in the second line we use the inequality $|e^{it} - 1| \leq |t|$, in the third line we use the fact that $h_0 \geq 0$ and $h \geq 0$, and in the last line we used the fact that $\int_{-c}^c h(x) dx = \int_{-c}^c h_0(x) dx = 1$. Now, for all $v > 0$, we have

$$|\hat{h}_*(v) - v \hat{h}'_*(0)| \leq \int_0^v |\hat{h}'_*(u) - \hat{h}'_*(0)| du \leq \int_0^v 2c^2 u du = Cv^2 \quad (4.92)$$

for some constant $C > 0$. Therefore by the triangular inequality, we have

$$|\hat{h}_*(v)| \geq v |\hat{h}'_*(0)| - Cv^2. \quad (4.93)$$

Taking $\varepsilon = \frac{|\hat{h}'_*(0)|}{2C}$ and noticing that for all $v \in [0, \varepsilon]$ one has $v |\hat{h}'_*(0)| - Cv^2 \geq \frac{v |\hat{h}'_*(0)|}{2} \geq 0$, we obtain

$$\begin{aligned} Q(h - h_0) &\geq \int_0^\varepsilon v |\hat{h}_*(v)|^2 dv \\ &\geq \int_0^\varepsilon v \left(v |\hat{h}'_*(0)| - Cv^2 \right)^2 dv \\ &\geq \frac{|\hat{h}'_*(0)|^2}{4} \int_0^\varepsilon v^3 dv \geq C' |\hat{h}'_*(0)|^6 \end{aligned} \quad (4.94)$$

for some constant $C' > 0$.

Now, if $C_\lambda \geq n^{3/2}t$ it means, by (4.89), that $\hat{h}'_*(0) = \hat{h}'(0) \geq \sqrt{2}t$, which by (4.94) implies

$$Q(h - h_0) \geq 8C't^6, \quad (4.95)$$

which by (4.90) and (4.91) implies the result. \blacksquare

4.4.3 Final bounding box for $\tilde{\lambda}$

Thanks to the previous results we can now focus (for n large enough) on partitions $\lambda = L \sqcup \tilde{\lambda}$ such that $\lambda_2, \ell(\lambda) \leq 2e\sqrt{n}$, which by Lemma 4.10 (i) implies $C_{\tilde{\lambda}} \leq en^{3/2}$.

Proposition 4.22. *We have*

$$\mathbb{P}_{n,\ell}^+(\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \dots) = \tilde{\lambda} | \boldsymbol{\lambda}_1 = L \leq \exp \left[O \left(\frac{C_{\tilde{\lambda}} \log^2 n}{n} + \log n \right) \right] \frac{(f_{\tilde{\lambda}})^2}{(n-L)!}. \quad (4.96)$$

uniformly for $L \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right]$ and all $\tilde{\lambda} \vdash n - L$ satisfying $C_{\tilde{\lambda}} \leq en^{3/2}$.

Proposition 4.23. *We have*

$$\mathbb{P}_{n,\ell}^+(\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \dots) \leq \tilde{\lambda} | \boldsymbol{\lambda}_1 = L \leq \exp \left[O \left(\frac{C_{\tilde{\lambda}} \log^2 n}{n} + \log^2 n \right) \right] \frac{(f_{\tilde{\lambda}})^2}{(n-L)!}. \quad (4.97)$$

uniformly for $L \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n} \right]$ and all $\tilde{\lambda} \vdash n - L$ satisfying $C_{\tilde{\lambda}} \leq en^{3/2}$.

Proof. Take any λ satisfying the conditions above. It is easily shown from the hook-length formula that

$$f_\lambda \leq \binom{n}{L} f_{\tilde{\lambda}} \quad (4.98)$$

and also

$$C_\lambda \leq \frac{L(L-1)}{2} + C_{\tilde{\lambda}}. \quad (4.99)$$

Hence

$$\begin{aligned} Z_n(\{\lambda\}) &\leq \frac{1}{n!} \binom{n}{L}^2 f_{\tilde{\lambda}}^2 \left(\frac{L(L-1)}{2} + C_{\tilde{\lambda}} \right)^\ell \\ &\leq \frac{(n-L)!}{n!} \binom{n}{L}^2 \left(\frac{L(L-1)}{2} \right)^\ell \exp \left[\Theta \left(\frac{C_{\tilde{\lambda}} \log^2 n}{n} \right) \right] \frac{(f_{\tilde{\lambda}})^2}{(n-L)!}. \end{aligned} \quad (4.100)$$

Let us now bound the normalisation factor $Z_n[L]$ below. Let $\Lambda_L = \{\mu \vdash n - L | C_\mu \geq 0 \text{ and } \mu_1, \ell(\mu) \leq 3\sqrt{n}\}$. Take $\lambda = L \sqcup \mu$ with $\mu \in \Lambda_L$. In the rest of the proof, uniformity will also be over $\mu \in \Lambda_L$. Then, by Lemma 4.10-(iii),

$$C_\lambda \geq \frac{L(L-1)}{2} - (n-L) = \frac{L(L-1)}{2} \left(1 + O \left(\frac{\log^2 n}{n} \right) \right) \quad (4.101)$$

and, by the hook-length formula,

$$f_\lambda = \frac{n!}{(n-L)!} \frac{1}{(L-\mu_1)!} \frac{1}{\prod_{i=1}^{\mu_1} (\mu'_i + L - i)} f_\mu. \quad (4.102)$$

and we have

$$\prod_{i=1}^{\mu_1} (\mu'_i + L - i) \leq \prod_{i=1}^{\mu_1} (L - i + 3\sqrt{n}) \leq \exp(O(\log n)) \prod_{i=1}^{\mu_1} (L - i) \quad (4.103)$$

as $n \rightarrow \infty$, hence

$$f_\lambda \geq \binom{n}{L} f_\mu \exp(O(\log n)). \quad (4.104)$$

By Theorem 1.35 (together with the fact that $\mathbb{P}_n(C_\lambda \geq 0) \geq 1/2$), we have

$$\sum_{\mu \in \Lambda_L} f_\mu^2 \geq \left(\frac{1}{2} - o(1)\right) (n - L)! \quad (4.105)$$

hence

$$\sum_{\substack{\lambda=L \sqcup \mu \\ \mu \in \Lambda_L}} f_\lambda^2 \geq (n - L)! \binom{n}{L}^2 \exp(O(\log n)), \quad (4.106)$$

and therefore, using (4.101),

$$\mathbb{Z}_n[L] \geq \mathbb{Z}_n(\{\lambda = L \sqcup \mu \mid \mu \in \Lambda_L\}) \geq \frac{(n - L)!}{n!} \binom{n}{L}^2 \left(\frac{L(L - 1)}{2}\right)^\ell \exp(O(\log^2 n)). \quad (4.107)$$

Combining (4.100) and (4.107) proves (4.97). The uniformity condition is easily checked. \blacksquare

We can now prove the last part of Theorem 4.2.

Proof of Theorem 4.2, part (iii). We set $L \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n}\right]$, $m = n - L$ and $\varepsilon > 0$, and introduce the sets

$$\mathcal{Z}_m = \{\mu \vdash m \mid \mu_1, \ell(\mu) \leq 3\sqrt{m}\} \quad (4.108)$$

and, respectively,

$$\mathcal{X}_m^{\leq t} = \{\mu \in \mathcal{Z}_m \mid C_\mu \leq m^{3/2}t\}, \quad \mathcal{X}_m^{\geq t} = \{\mu \in \mathcal{Z}_m \mid C_\mu \geq m^{3/2}t\} \quad (4.109)$$

and finally

$$\mathcal{Y}_m^\varepsilon = \{\mu \in \mathcal{Z}_m \mid \max(\mu_1, \ell(\mu)) \geq (2 + \varepsilon)\sqrt{m}\}. \quad (4.110)$$

In accordance with (4.97), we will show that for any constant C , uniformly in C and L ,

$$\sum_{\mu \in \mathcal{Y}_m^\varepsilon} \exp\left(C \frac{C_\mu \log^2 n}{n}\right) \frac{f_\mu^2}{m!} \leq \exp[-(1 + o(1))K\sqrt{n}] \quad (4.111)$$

where K is the constant of Proposition 4.20.

Let us fix $t = 1/\log^3 n$. To establish this inequality we will split the sum into two sums, over $\mathcal{Y}_m^\varepsilon \cap \mathcal{X}_m^{\leq t}$ and $\mathcal{Y}_m^\varepsilon \cap \mathcal{X}_m^{\geq t}$. For all $\mu \in \mathcal{Y}_m^\varepsilon \cap \mathcal{X}_m^{\leq t}$, we have $C_\mu \log^2 n/n = o(\sqrt{n})$ uniformly in μ hence by Proposition 4.20, we have

$$\sum_{\mu \in \mathcal{Y}_m^\varepsilon \cap \mathcal{X}_m^{\leq t}} \exp\left(C \frac{C_\mu \log^2 n}{n}\right) \frac{f_\mu^2}{m!} \leq \exp[-(1 + o(1))K\sqrt{n}]. \quad (4.112)$$

On the other hand, for all $\mu \in \mathcal{Y}_m^\varepsilon \cap \mathcal{X}_m^{\geq t}$, we have both $C_\mu = O(n^{3/2})$ and, by Theorem 4.21 (very roughly), $f_\mu^2/m! \leq \exp(-n^{0.9})$. Since $|\mathcal{Y}_m^\varepsilon \cap \mathcal{X}_m^{\geq t}| = e^{O(\sqrt{n})}$ as well, we have

$$\sum_{\mu \in \mathcal{Y}_m^\varepsilon \cap \mathcal{X}_m^{\geq t}} \exp\left(C \frac{C_\mu \log^2 n}{n}\right) \frac{f_\mu^2}{m!} \leq \exp[-(1 + o(1))n^{0.9}]. \quad (4.113)$$

Putting (4.112) and (4.113) together establishes (4.111).

Now, by Proposition 4.18, we have $\tilde{\lambda} \in \mathcal{Z}_m$ w.h.p., hence we can combine (4.111) with (4.97) to establish that for all $\varepsilon > 0$ there is $K > 0$ such that

$$\mathbb{P}_{n,\ell}^+(\max(\lambda_2, \ell(\lambda)) \geq (2 + \varepsilon)\sqrt{n} | \lambda_1 = L) \leq \exp[-(1 + o(1))K\sqrt{n}] \quad (4.114)$$

uniformly for $L \in \left[\frac{0.4\ell}{\log n}, \frac{6\ell}{\log n}\right]$; since λ_1 is in this interval w.h.p., one we can remove the conditioning and get

$$\mathbb{P}_{n,\ell}^+(\max(\lambda_2, \ell(\lambda)) \geq (2 + \varepsilon)\sqrt{n}) \leq \exp[-(1 + o(1))K\sqrt{n}]. \quad (4.115)$$

Together with (4.55), this directly implies that

$$\frac{\lambda_2}{\sqrt{n}} \xrightarrow{p} 2 \quad \text{and} \quad \frac{\ell(\lambda)}{\sqrt{n}} \xrightarrow{p} 2 \quad (4.116)$$

which is what we wanted to show. ■

Perspectives

The main contributions presented in this thesis were new probability laws on integer partitions which exhibit interesting limiting behaviour in certain asymptotic regimes. In each case, the measures we introduced are generalisations of the Plancherel measure, which is very well understood. Its rich asymptotic behaviour –most notably, its TW-GUE edge fluctuations and its deterministic limit shape– have broad ramifications, relating different physical, probabilistic and combinatorial models. Thanks in particular to the integrability of its Poissonisation, the Plancherel measure is an excellent benchmark model for universal phenomena, and profound connections have been established between this measure and models of random matrices, maps, permutations, and fermions, to name but a few.

For the multicritical Schur measures and Plancherel–Hurwitz measures we introduced, we have demonstrated novel asymptotic behaviour, making them interesting candidate models to study new universality classes. Each of these measures is directly related to certain other models. For the multicritical Schur measures, we have exact and asymptotic correspondences with fermions on a 1D lattice and line respectively, and an exact correspondence with certain unitary matrix models. For the Plancherel–Hurwitz measure, we have a corresponding model of random maps, or random sequences of elements of symmetric groups. We hope to extend this picture.

In the broadest terms, the main question raised by this work is: *can we find analogues of everything we know about the Plancherel measure for multicritical Schur measures and the Plancherel–Hurwitz measure?* Here we present some axes along which we hope to explore that question.

The first parts of a random partition In both of the models we considered, the atypical asymptotic behaviour was characterised by the behaviour of the first part of a random partition (or its length, depending on how we choose our conventions). In the case of the multicritical measures, the first part exhibits new fluctuations, distinguished by smaller critical exponents than the universal $1/3$, and asymptotic distributions which are higher-order analogues of the TW-GUE distribution. Under the Plancherel–Hurwitz measure in a linear asymptotic regime, the first part scales much faster than all of the other parts.

Under the Plancherel measure the first part is an important statistic: by the RSK bijection, it is equivalent in law to the length of the longest increasing subsequence of a uniform random permutation. One important question is to interpret this statistic for each of the cases we considered. Since the Plancherel–Hurwitz measure corresponds to a model of transposition factorisations on symmetric groups, one would expect it could be interpreted in terms of those factorisations. This fits into another natural question,

which is to

Question 5.1. *Find a bijection relating pairs of Young tableaux with n boxes to some sequences of permutations of n elements which proves Frobenius' formula for the unconnected Hurwitz numbers.*

There may be some hints for this problem in the limit behaviour we found in the linear regime, where the limiting value of the first part is proportional to an upper bound for the order of the group generated by the longest transitive subsequence of a corresponding random transposition factorisation. This limiting value also coincides the asymptotic mixing threshold for the transposition random walk.

The multicritical Schur measures, on the other hand, have no immediate combinatorial interpretation, as they are constructed in such a way that the RSK bijection does not define a corresponding model (this is due to the non-positivity of the Miwa time specialisations we use). A more natural way to interpret the first part is then to generalise the connection between parts of a Plancherel random partition and eigenvalues in the GUE, and namely to

Question 5.2. *Find a Hermitian matrix model whose largest eigenvalues converge to the higher order Airy ensemble asymptotically found on the edge under a multicritical Schur measure.*

The exact correspondence we have between multicritical Schur measures and unitary matrix models presents one approach to this question.

Connections with random maps In terms of the connection between the Plancherel measure and the GUE, let us also remark that Okounkov's first celebrated proof of the asymptotic equivalence of elements from either model exploited a correspondences between the Plancherel measure and coverings of the Riemann sphere, those coverings and maps, and map generating functions and the GUE. In that sense, another interesting question is to

Question 5.3. *Relate Schur measures specialised in the Miwa times (other than the Poissonized Plancherel measure) to generating functions of maps.*

Let us remark that it's particularly natural to ask this question for Schur measures specialised in the Miwa times, since it is in terms of the Miwa times that we have more general formulas relating the enumeration of branched coverings to Schur functions; this would appear to be the most natural combinatorial interpretation for Schur measures specialised this way.

For the Plancherel–Hurwitz measure, there is an explicit connection to a family of maps. However, for now we have only studied unconnected maps in this context; using the connection between the Toda lattice hierarchy and Hurwitz numbers, an interesting application of this correspondence would be to

Question 5.4. *Study connected Hurwitz maps using integrable generating functions related to the Plancherel–Hurwitz measure.*

Questions from integrability Another interesting direction for the measures we introduced would be to further explore connections with integrability. In the case of the

Plancherel–Hurwitz measure, the apparent connection with a determinantal point process (as well as with integrable hierarchies) presents the possibility to find an interesting new physical (or quasi-physical) model. In particular, it would be interesting to

Question 5.5. *Interpret a Poissonised Plancherel–Hurwitz measures as models of fermions.*

The particular asymptotic behaviour of the Plancherel–Hurwitz measure is quite different from limiting phenomena we have seen for free fermion models, so this represents an interesting new direction.

In the case of the multicritical Schur measures, where we used the fermionic integrability of the measures themselves to find asymptotic distributions which are related to solutions of integrable hierarchies, one interesting direction would be to

Question 5.6. *Understand the classical integrable hierarchies solved by asymptotic edge distributions of multicritical Schur measures at the level of specializations.*

In other words, we could interpret our tuning of Miwa times to multicriticality as specialising solutions of the KP hierarchy to Painlevé II solutions. While this is an ambitious project, from the multicritical Schur measures we have explicit specialisations which may make this problem tractable.

A natural extension to both measures To conclude, let us propose a new family of measures which are natural extensions of both the cases considered here, arising naturally from the generating function of double Hurwitz numbers:

Definition 5.7 (Schur–Hurwitz measure). The *Schur–Hurwitz* measures on all integer partitions λ are, suitable sequences t, t' of complex numbers and real parameter β ,

$$\mathbb{P}(\lambda) = \frac{1}{Z} s_\lambda[t] s_\lambda[t'] e^{\beta C_\lambda}.$$

where C_λ is the sum of contents of the partition λ and Z is a normalisation factor.

Due to the correspondence with double Hurwitz numbers, it would be most natural to study these measures in terms of Miwa time specialisations; in that sense, we can look for generalisations of multicritical behaviour in the presence of a content-sum term. At the level of integrability, since expectations of these measures are τ -functions of a more general integrable hierarchy, we can hope to find more diverse asymptotic statistics. It would be particularly interesting to study these measures using fermionic methods, which are unlikely to apply to a linear asymptotic regime such as the one we studied in the case of the Plancherel–Hurwitz measure. However, even in simpler asymptotic regimes, we believe this kind of deformation of a Schur measure could lead to interesting behaviour.

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