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THÈSE DE DOCTORAT

DE L'UNIVERSITÉ PSL

Préparée à l'École Normale Supérieure

**Chaîne de spins intégrable non compacte pour la théorie
fishnet conforme**

**Non-Compact Integrable Spin Chain for the Conformal Fishnet
Theory**

Soutenue par

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Tu sacrifies le temps
À l'éternelle jeunesse de la flamme exacte
Qui voile la nature en la reproduisant

PAUL ÉLUARD, *Facile*

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Contents

Introduction	1
1 Compact Integrable Spin Chains	15
1.1 The $GL(2)$ Heisenberg Spin Chain	15
1.2 R-Matrix and the Yang–Baxter Equation	20
1.2.1 $GL(2)$ -Invariant R-Matrix in Defining Representation	20
1.2.2 $GL(2)$ -Invariant R-Matrix in Symmetric Representations	23
1.2.3 Higher-Rank Symmetry	25
1.3 Nested Bethe Ansatz Equations	27
1.3.1 The $GL(N)$ Heisenberg Spin Chain	27
1.3.2 Transfer Matrix Eigenvalues	29
2 From $\mathcal{N} = 4$ Super Yang–Mills to The Fishnet Theory	31
2.1 $\mathcal{N} = 4$ Super Yang–Mills	31
2.1.1 Fields and Lagrangian	31
2.1.2 Symmetries	32
2.1.3 The Planar Limit	33
2.2 The γ -Deformation of $\mathcal{N} = 4$ SYM	34
2.2.1 Definition	34
2.2.2 The Double-Scaling Limit	35
2.3 The Fishnet Theory	35
2.3.1 Lagrangian and Feynman Rules	35
2.3.2 Mass and Coupling Renormalisation	37
3 Non-compact Conformal Spin Chain	41
3.1 Scalar Principal Series Representations	41
3.1.1 Definitions and Representation Theory Results	41
3.1.2 R-Matrices and Star-Triangle Relation	43
3.2 The Spin Chain	48
3.3 Integrability of the Fishnet Theory	50
3.4 The $J = 2$ Case	53

4	$O(d)$ R-Matrices and Graph-Building Operators	59
4.1	$O(d)$ -Invariant R-Matrices	59
4.1.1	From Fusion	60
4.1.2	Spectral Decomposition	66
4.1.3	Integral Representation	67
4.1.4	Properties of the R-Matrices	70
4.2	Diagonalisation of Graph-Building Operators	75
4.2.1	Construction of the Eigenvectors	75
4.2.2	Symmetry Property	80
4.2.3	Inner Product	81
4.2.4	Completeness	83
4.3	Basso–Dixon Diagrams	84
4.3.1	Ladder Diagrams	85
4.3.2	Two-Layer Diagrams	86
5	Thermodynamic Bethe Ansatz for Fishnet Theories	91
5.1	Graph-building Operators and Scattering Data	91
5.1.1	Magnon Dispersion Relation	93
5.1.2	Magnon S-Matrix	93
5.2	TBA for Ground State	97
5.3	Dual TBA and Y-System	98
5.4	Excited States and Asymptotic Bethe Ansatz	100
5.4.1	One Magnon	101
5.4.2	Two Magnons	101
5.5	Discussion and Prospects	103
6	T-System and Q-System for $SO(2r)$ Spin Chains	105
6.1	A_r Spin Chains	105
6.1.1	T-System	105
6.1.2	Q-System	108
6.2	A D_r Spin Chain	111
6.2.1	Transfer Matrices for Symmetric Representations	111
6.2.2	Diagonalisation of $T_{1,1}$	113
6.2.3	QQ-Relations from Bethe Ansatz Equations	114
6.2.4	Basic (Extremal) Q-Functions	115
6.3	The D_r Q-System	118
6.3.1	Description	118
6.3.2	Wronskian Form of the Bethe Ansatz Equations	122
6.4	Weyl-Type Formulae from Tableaux Representations	123
6.4.1	The Simplest Case	123
6.4.2	Symmetric Representations	125
6.4.3	Antisymmetric Representations	127
6.4.4	Spinorial Representations	129
6.5	QQ'-Type Formulae	130

6.5.1	Symmetric Transfer Matrices	130
6.5.2	Spinorial Transfer Matrices	131
6.5.3	Weyl-Type Formula for $T_{1,1}$ from QQ'-Relations	132
6.5.4	Arbitrary Rectangular Representations	133
6.6	Discussion	134
Conclusion and Outlook		137
A Conventions for Lie Algebras		141
A.1	Special Linear Algebra	141
A.2	Even Orthogonal Algebra	142
A.2.1	Roots, Weights and Characters	142
A.2.2	Clifford Algebra	143
B Feynman Graphs Computations		145
B.1	Basic Integral Relations	145
B.2	Feynman Integrals for Subsection 5.4.2	148
C Details on D_r Q-System		151
C.1	Q-System of $A_3 \simeq D_3$	151
C.2	Details for the Computations of Section 6.5	152
C.2.1	Wronskian Condition from QQ'-Type Constraints	152
C.2.2	Proof of Equation (6.5.28)	153
C.3	More on Weyl-Type Formulae	156
C.3.1	From QQ'-Relations to Weyl-Type Formulae for $D_2 \simeq A_1 \oplus A_1$	156
C.3.2	Additional Formulae in the General Case	157

Introduction

Because they can often be solved exactly, integrable models are of great interest to physicists. Some of these models directly describe realistic physical systems. But even when they do not, they may still provide important insights into physical phenomena. In statistical physics, for instance, a given universality class may contain one integrable model as well as some other, more realistic systems. In that case, all these systems share the same critical exponents, and it suffices to compute these exponents for the integrable model. In high-energy physics, on the other hand, integrability may allow to probe the strongly coupled regime of some quantum field theories, a feat that is in general extremely difficult to achieve. We will focus, in this thesis, on various aspects of integrability related to a particular conformal field theory, the so-called fishnet theory.

The rest of the introduction is organised as follows. We begin with a historical presentation highlighting the main steps in the development of integrability. We then describe the background to the works presented in this thesis, starting with a few specific objects and techniques—T-systems, Q-systems, Q-operators, separated variables—common to many integrable models. After that, we briefly review the solution to the spectral problem of $\mathcal{N} = 4$ super Yang–Mills, where some of these techniques have found applications. We then give an overview of most of the existing literature on the fishnet theory. Finally, we end with an outline of the contents of the thesis.

Classical Integrability and Quantum Integrability

The first models for which integrability was defined were finite-dimensional classical Hamiltonian systems. Given such a system on a $2n$ -dimensional phase space M —a symplectic manifold—we say that it is integrable in the Liouville sense if there exist n independent functions H_1, \dots, H_n such that $H_1 = H$ is the Hamiltonian, and they all Poisson commute with one another: $\{H_i, H_j\} = 0$. Any of these functions could equivalently be taken as the Hamiltonian of the system, and together they constitute a set of (first) integrals of motion, as they are all conserved under Hamiltonian motion. In that case, any level set, i.e. any set of the form $\{x \in M; H_i(x) = h_i\}$ for some fixed h_i 's, that is compact and connected is diffeomorphic to the n -dimensional torus $\{(e^{i\phi_1}, \dots, e^{i\phi_n}) \in U(1) \times \dots \times U(1)\}$, and, on this torus, the angles evolve linearly with time: $\phi_i = \omega_i(h_1, \dots, h_n)$. The equations of motion can moreover be integrated by quadratures. Arnol'd [1] actually showed that, for such systems, one can find symplectic coordinates $(I_1, \dots, I_n, \phi_1, \dots, \phi_n)$, such that I_1, \dots, I_n depend only on the first integrals, and the angles are exactly the coordinates on the torus. These coordinates are called action-angle coordinates.

The next important step in the theory of classical integrable models originated in the study of the Korteweg–de Vries (KdV) equation, a non-linear partial differential equation originally obtained in the description of one-dimensional surface gravity waves on shallow water. Korteweg and de Vries [2] proved that this equation admits progressive wave solutions that are asymptotically constant, i.e. solitary waves. But it was only seventy years later that a numerical study [3] showed the collective behaviour of these waves to be particle-like. Starting from a configuration in which the waves are well separated, it appeared that they will first interact (non-linearly) and mix, but at long times they eventually separate again and regain their respective shapes and velocities, up to a phase shift. Such solitary waves are called solitons. The initial-value problem for the KdV equation was solved [4] soon after, using a method now called the inverse scattering transform. Inspired by this and the existence of an infinite number of conservation laws [5], an interpretation of the KdV equation as an infinite-dimensional classical integrable system was given in [6]. These methods turned out not to be specific to the KdV equation but applicable to a vast array of non-linear systems [7, 8].

Ninety years ago, Bethe [9] solved an integrable quantum system: the XXX Heisenberg spin chain. His method relies on an ansatz which is now known as the coordinate Bethe ansatz: he first made an assumption on the form of the wave function depending on some complex parameters, the momenta or, up to a reparametrisation, the rapidities. Then, he derived an auxiliary set of equations, the Bethe Ansatz Equations (BAE), that these parameters should satisfy for the wave function to be an eigenvector. The energy is the sum of the energies associated to each momentum, thus giving a picture of the eigenstates as states describing a certain number of pseudo-particles, the magnons.

Further progress then came from two-dimensional lattice models of statistical mechanics. In 1944, Onsager [10] gave the exact solution of the two-dimensional Ising model. It was followed, in the sixties, by the solutions to the planar dimer problem [11, 12] and to the six-vertex or ice-type models [13–16]. The latter were explicitly solved using a Bethe ansatz for the eigenvectors of the row-to-row transfer matrices. Around the same time, a quantum mechanical system of particles on a line interacting via a delta function potential, introduced for bosons by Lieb and Liniger in [17], was also solved via a Bethe ansatz [17, 18].

It was thanks to the works of both Baxter [19] and the physicists of the Leningrad school [20–22] that a general framework, called the quantum inverse scattering method (QISM), emerged. QISM can be used to formulate and solve integrable models, including the Heisenberg spin chain studied by Bethe and (most of) the exactly solved two-dimensional lattice models. At the heart of this new algebraic approach lies the Yang–Baxter equation, a cubic relation on an object called the R-matrix or R-operator.

The QISM also found several applications to quantum chromodynamics (QCD). Firstly, it was shown [23, 24] that in the Balitsky–Fadin–Kuraev–Lipatov (BFKL) limit of QCD the effective Hamiltonian exhibits integrability properties of a non-compact $SL(2, \mathbb{C})$ spin chain. Then, it turned out that solving the evolution equations governing the scale dependence of certain correlation functions in QCD is equivalent to solving a non-compact Heisenberg $SL(2, \mathbb{R})$ spin chain [25–29]. Such spin chains are called non-compact because an infinite-dimensional representation of the symmetry group, such as a principal series representation of $SL(2, \mathbb{C})$ with spins $(s, \bar{s}) = (0, 1)$ for the case studied in [23, 24], is associated to each site of the chain. The quantum space in a non-compact spin chain is then some functional

space. Accordingly, the operators—R-matrices, transfer matrices, etc.—are either differential operators or they have an integral kernel.

Another application of integrability to high-energy physics is in one of the realisations of the AdS/CFT correspondence [30–32]. It is indeed believed that type IIB strings in an $AdS_5 \times S^5$ background are equivalent to the four-dimensional maximally supersymmetric Yang–Mills theory ($\mathcal{N} = 4$ SYM), and that both are integrable in the planar limit (this last conjecture is based on numerous results, starting with [33, 34], see also [35–39] for detailed reviews). In that case, integrability is a boon that permits to derive non-perturbative results necessary to test a correspondence relating strongly coupled strings to the weakly coupled gauge theory and vice versa.

Universality of T- and Q-Systems

It was first observed in [19, 40, 41] that the transfer matrices of some two-dimensional lattice models of statistical mechanics provide a solution to a few bilinear difference equations. These results were generalised in [42, 43], where an infinite system of such equations was obtained. It was also realised [43] that such a system can be defined for any simple Lie algebra \mathfrak{g} , this is the T-system associated to \mathfrak{g} . It involves commuting objects, the T-functions $T_{a,s}(x)$ depending on a spectral parameter x , some positive integer s , and an index a labelling the nodes of the Dynkin diagram of \mathfrak{g} . When $\mathfrak{g} = A_r = \mathfrak{sl}(r+1, \mathbb{C})$, for instance, the T-system simply reads [43–45]

$$T_{a,s}^{[1]} T_{a,s}^{[-1]} = T_{a,s+1} T_{a,s-1} + T_{a+1,s} T_{a-1,s}, \quad (0.0.1)$$

where for a function f and an integer k we define the function $f^{[k]}$ by $f^{[k]}(x) = f\left(x + \frac{k}{2}\right)$. In this particular example, the T-system coincides with the Hirota equation of soliton theory [46], thus making a connection with classical integrable systems. There exist also T-systems associated to Lie super algebras [47–50].

As it happens, T-systems turned out to be ubiquitous structures. They arise in the context of cluster algebras, of the ordinary differential equations/integrable models (ODE/IM) correspondence [51], and are also relevant for solving $\mathcal{N} = 4$ super Yang–Mills as we will explain later on. In the case of compact spin chains, this is related to the representation theory of Yangians and, more generally, of quantum affine algebras. In particular, for the latter, the T-functions are q -characters [52], an analogue of usual characters for particular modules introduced by Kirillov and Reshetikhin in [53]. The fact that these q -characters do satisfy the T-system was proven for untwisted simply laced quantum affine algebras in [54, 55], for generic untwisted algebras in [56], and in [57] for the twisted case.

The T-system is not the only structure that is common to many integrable models: there exist others like the Y- or Q-systems. We will not elaborate on the relevance of the Y-system to integrability in this thesis, the interested reader could refer to [58] for a nice review of the interplay between T- and Y-systems. The Q-system is another, equivalent, way to encode the spectrum of integrable models. It is a system of equations on a finite number of functions/commuting operators, whereas there was an infinite number of T-functions. These Q-functions are considered to be more fundamental objects than the T-functions as it is believed that the latter can all be simply expressed in terms of the former. The uncertainty in the previous sentence comes from the fact that the study of Q-systems is far less developed

than that of their T and Y counterparts. Until very recently, the Q-system was actually only known for the Lie algebras $\mathfrak{sl}(r+1, \mathbb{C})$ and their supersymmetric generalisation.

In the case of the A_r symmetry algebra, which is both the simplest and best-understood case, there are 2^{r+1} Q-functions. They are labelled by subsets I of $\{1, \dots, r+1\}$ and satisfy relations of the form

$$Q_{I \cup \{i\}}^{[+1]} Q_{I \cup \{j\}}^{[-1]} - Q_{I \cup \{i\}}^{[-1]} Q_{I \cup \{j\}}^{[+1]} \propto Q_I Q_{I \cup \{i,j\}}. \quad (0.0.2)$$

As for the case of T-functions, these relations imply that the Q-functions are far from being independent. It is actually possible to express all of them in terms of only $r+2$ of them, a convenient choice of the latter being Q_\emptyset and $Q_{\{i\}}$. Moreover, owing to the so-called gauge symmetry exhibited by the Q-system, it is in practice always possible to assume that two of the Q-functions, say Q_\emptyset and $Q_{\{1, \dots, r+1\}}$, are trivial so that one often says that there are only r independent functions. In turn, a solution of the T-system can be obtained for any collection of $r+1$ functions by defining [44]

$$T_{a,s} \propto \det_{1 \leq i, j \leq r+1} \left(Q_{\{i\}}^{[a-s-2j+2s\theta(a-j)]} \right), \quad (0.0.3)$$

where θ is the Heaviside step function (it is 0 for negative arguments and 1 for non-negative ones). The proportionality factor in the previous relation is a function that depends on the model-dependent boundary conditions that one wants to satisfy.

Construction of Q-Operators

The Q-operators appeared for the first time in a work by Baxter on the eight-vertex model [59] as operators satisfying

$$TQ = \phi^{[-1]} Q^{[2]} + \phi^{[1]} Q^{[-2]}, \quad (0.0.4)$$

where T is the transfer matrix and ϕ some fixed function. More generally, the fact that solutions of the T-system can be expressed as determinants of shifted Q-functions, cf. (0.0.3), dates back to [44]. But the Q-functions $Q_{\{i\}}$ were defined at the time as linearly independent solutions of the finite-difference Baxter equation, also called the quantum spectral curve,

$$\sum_{a=0}^{r+1} (-1)^a T_{a,1}^{[1-a]} Q^{[-2a]} = 0. \quad (0.0.5)$$

The direct, i.e. without having to resort to the T-functions, construction of the Q-operators is highly non-trivial and was at first developed on a case-by-case basis for various models [60–67].

For spin chains with infinite-dimensional representations of $\mathfrak{sl}(r+1, \mathbb{C})$ or $GL(r+1, \mathbb{C})$ at each site, either a Verma module or a representation of the principal series, i.e. with or without highest weight, the single-index Q-operators were then constructed in a series of articles by Derkachov and Manashov, first for low ranks in [68–70] and then for arbitrary ranks in [71–74]. They constructed the R-matrices intertwining these representations and showed that they are naturally expressed as a product of $r+1$ simpler operators. This property survives the trace operation involved in the definition of the transfer matrices. As a result, transfer matrices with infinite-dimensional representations in the auxiliary space are written

as a product of $r + 1$ operators which are identified with the single-index Q-operators of the model. Moreover, in the Verma module case, the determinant expression for the transfer matrices with finite-dimensional auxiliary space is a consequence of the Bernstein–Gelfand–Gelfand resolution [75].

Surprisingly, the construction of the Q-operators for the compact spin chains, including the one originally solved by Bethe, only appeared later. Inspired by the construction done in [63,64] for an integrable conformal field theory with $U_q(\widehat{\mathfrak{sl}}(2))$ symmetry, the authors of [76–79] solved this problem for spin chains with $\mathfrak{sl}(n|m)$ symmetry. The 2^{n+m} Q-operators satisfying the Q-system (0.0.2) are constructed as transfer matrices with some infinite-dimensional auxiliary spaces, even though the quantum space is finite-dimensional, that are representation spaces of harmonic oscillator algebras. This involved finding hitherto unknown solutions of the Yang–Baxter equation. These solutions are not representations of the Yangian because they are degenerate; their analogues for quantum affine algebras actually correspond to the so-called prefundamental representations of Borel subalgebras [80,81]. Independently of [76–79], a conceptually different approach, based on the coderivative method introduced in [82], was successfully carried out in [83] for the construction of the Q-operators of the spin chains with $\mathfrak{sl}(n|m)$ symmetry.

Such explicit constructions of the Q-operators make transparent the regularity properties in the spectral parameter of the operators and their eigenvalues. In the compact case, they are all polynomials up to a trivial state-independent prefactor. For principal series representations, however, all the operators are meromorphic. Since, in that case, they are integral operators, it suffices to examine their kernels to determine their asymptotic behaviour at infinity and the position and degree of their poles [67,84]. It appeared in several instances [85,86] that the knowledge of these regularity properties and of the equations satisfied by the operators (Q-system or Baxter’s TQ-equation) is enough to find the eigenvalues. In the compact case, for instance, even if it is straightforward to recover the (nested) Bethe ansatz equations, the Bethe roots being roots of some of the Q-functions, the Q-system formulation of the spectral problem appears to be much more convenient. It does not exhibit the usual drawbacks of the Bethe equations, all its solutions are physical, and it is even possible to prove that it is complete in some cases [87–89].

Another key feature of this approach to finding the spectrum is that it does not require any knowledge about the form of the eigenvectors. In particular, it works in cases for which more conventional methods like algebraic (or coordinate) Bethe ansatz are not applicable. Non-compact spin chains based on principal series representations are typical examples of such a situation.

Separation of Variables

The actual determination of the wave function of the eigenstates is also a very complicated problem. One of the most promising approaches to this question seems to be the technique of separation of variables (SoV) first promoted by Sklyanin, see [90] for a review. For a classical system integrable in the Liouville sense, we say that symplectic coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ are separated if there are relations of the form $\Phi_i(p_i, q_i, H_1, \dots, H_n) = 0$ for all $i \in \{1, \dots, n\}$. The quantum analogue [91] comprises two sets of mutually commut-

ing operators: the separated variables $\{X_1, \dots, X_N\}$ having simple joint spectrum, i.e. an eigenvector common to all the X_i 's is uniquely specified, up to multiplication by a complex number, by its eigenvalues, and a set $\{P_1, \dots, P_N\}$ of operators which act as ladder operators with respect to the separated variables. Moreover, we want that (left) eigenvectors of the X_i 's form a basis $\langle x_1, \dots, x_N |$ of the Hilbert space in which the wave function of any eigenstate $|\Psi\rangle$ (of the original Hamiltonian, transfer matrix, etc.) factorises according to

$$\langle x_1, \dots, x_N | \Psi \rangle = \prod_{i=1}^N \psi_i(x_i), \quad (0.0.6)$$

with each ψ_i a solution to some equation involving the conserved charges as coefficients. The number N of separated variables, which can be seen as the number of degrees of freedom of the quantum system, is *a priori* not easy to determine.

Sklyanin [91, 92] managed to develop such techniques for the compact twisted inhomogeneous spin chain with $GL(2, \mathbb{C})$ symmetry. His construction relies on an identification of the separated variables with the operatorial zeros of a certain family $B(x)$ of commuting operators. He then sets $P_i = A(X_i)$, where $A(x)$ is a second family of commuting operators. If the A and B families satisfy appropriate commutation relation, the shift property of the P_i 's is automatically satisfied. In the compact spin chain we just mentioned, $A(x)$ and $B(x)$ are simply elements of the monodromy matrix, although one has to introduce twists and inhomogeneities in order to ensure that the operators are diagonalisable and possess a non-degenerate spectrum. Separated variables were subsequently found for non-compact models with the same symmetry group [67, 84, 93, 94]. Each ψ_i turns out to satisfy Baxter's equation (0.0.4) evaluated at the eigenvalues x_i of X_i and can be identified with a Q-function. It had actually already been suggested in [95] that the Q-operators could be used, provided one has some explicit expression for them, to construct a unitary transformation (change of basis) going to the SoV representation. This proposal was verified in the non-compact models just mentioned.

The situation is much more complicated for systems with higher-rank symmetry. First of all, the identification of the A- and B-operator is far less clear. A proposal for them was made in [96], based on a study of the classical case [97], for models with $GL(3, \mathbb{C})$ symmetry. However, even in the simplest compact $GL(3, \mathbb{C})$ spin chain, this proposal does not seem to be totally satisfying, as the shift properties of the P_i 's are not verified. Significant progress on this subject has only been achieved recently thanks to the works of several groups [98–100].

On the one hand, Maillet and Niccoli managed to circumvent the need for an explicit construction of the A- and B-operator, and thus of the separated variables themselves, and constructed SoV bases for integrable models associated with the defining representation of $GL(n, \mathbb{C})$ [99, 101], the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}(n))$ [102], the $Y(\mathfrak{gl}(n))$ reflection algebra [103], and the supersymmetric algebras [104] (with Vignoli). Their work is based on the powerful observation that factorisation of the wave function is automatic if the covectors $\langle x_1, \dots, x_N |$ are obtained via the application of the conserved charges themselves on a fixed reference covector. In the simplest spin chain with $GL(2, \mathbb{C})$ symmetry [99], when all sites carry the defining representation, the transfer matrix with defining representation in the auxiliary space is enough to generate all the conserved charges: the number of separated

variables coincides with the length of the chain, $N = L$, and one can define

$$\langle h_1, \dots, h_L | \propto \langle S | \prod_{i=1}^L T_i^h(\xi_i) \quad (0.0.7)$$

for any $(h_1, \dots, h_L) \in \{0, 1\}^L$, with ξ_1, \dots, ξ_L the inhomogeneities (which should be in generic position) and some reference covector $\langle S |$. We stress that the labels of the vectors are not the eigenvalues of some explicit operators anymore, although in this simple case, if $\langle S |$ is conveniently chosen, one can recover Sklyanin's results [92] and the eigenvalues of the separated variables are $x_i = \xi_i + 2h_i - 1$. Maillet and Niccoli proved that, for $\langle S |$ generic enough, the vectors $\langle h_1, \dots, h_L |$ thus generated are linearly independent. For representations other than the defining one, more transfer matrices are needed to generate the basis, but the result still holds [105]. Furthermore, using fusion relations satisfied by the various commuting transfer matrices, like the T-system, as well as the knowledge of their asymptotic behaviour, they were able to give a precise characterisation of the spectrum of the various models they considered. Their analysis is done for a very wide range of twist matrices, but in the particular case of an invertible twist matrix with simple spectrum they show that the spectrum can essentially be characterised by Baxter's equation (0.0.5).

On the other hand, Ryan and Volin followed more closely Sklyanin's approach and focused on compact spin chains with $GL(n, \mathbb{C})$ symmetry, first in the same rectangular representation at each site [100], and then in arbitrary finite-dimensional representations [106]. They used the proposal made in [98] for a polynomial B-operator $B(x)$ that, when evaluated at the momentum-carrying Bethe roots, could be used to create the eigenvectors of the model by repeated application on a reference vector, just as in the $GL(2, \mathbb{C})$ case. This B-operator is explicitly expressed in terms of a sum of products of minors of the monodromy matrix, and it had actually already appeared in [107]. Ryan and Volin first proved that the B-operator is, up to a nilpotent term, given by a product of the generators of the Gelfand–Tsetlin subalgebra of the Yangian [108]. This permitted them to show that the operator is diagonalisable and to determine its spectrum. They further showed that the eigenvectors $\langle x_1, \dots, x_N |$ of $B(x)$ can be constructed, following [99], by application of various transfer matrices of the model, shifted and evaluated at the inhomogeneities, on a suitably chosen reference state. For generic representations at each site, the number of separated variables is $N = \frac{Ln(n-1)}{2}$, and they are labelled X_{kj}^α with $\alpha \in \{1, \dots, L\}$ and $1 \leq j \leq k \leq n - 1$. The wave function of the model factorises into a product of determinants:

$$\langle \{x_{kj}^\alpha\} | \Psi \rangle = \prod_{\alpha=1}^L \prod_{k=1}^{n-1} \det_{1 \leq i, j \leq k} (q_i(x_{kj}^\alpha)) , \quad (0.0.8)$$

where q_i is, up to a trivial normalisation, the eigenvalue of the Q-operators $Q_{\{i\}}$.

The separation of variables is also a convenient approach to compute the norm of the eigenstates and form factors, i.e. matrix elements of operators. The wave functions in the SoV basis being indeed expressed in a particularly compact form, it seems natural to compute such scalar products in this basis. However, the scalar product is defined in the original basis. In order to express it in the SoV basis, one needs an additional piece of information, the SoV measure, also called Sklyanin's measure. It corresponds to the scalar products

$\langle x_1, \dots, x_N | y_1, \dots, y_N \rangle$, where $|y_1, \dots, y_N\rangle$ is a dual SoV basis that need not be equal to $(\langle x_1, \dots, x_N |)^\dagger$. This measure has been known for a long time for models with $SL(2, \mathbb{C})$ symmetry. For models with higher-rank symmetry, it has only started to be understood in the past couple of years. An ingenious method relying on the Baxter equations satisfied by the Q-functions allowed the authors of [109] to bypass the explicit construction of the SoV bases. They obtained a determinant expression for the measure for a spin chain with the Verma module of $\mathfrak{sl}(n, \mathbb{C})$ of highest weight $(-1, 0, \dots, 0)$ attached to each site. The same technique was applied in [110] to the compact spin chain with the defining representation of $GL(3, \mathbb{C})$ at each site, a construction of the dual SoV basis was also presented. These results were further extended in [111] to a particular family of Verma modules and new determinant expressions for several form factors were derived. The SoV measure was also computed in [112] in terms of a solution to some recursion relations for the same model as in [110].

Spectrum of $\mathcal{N} = 4$ SYM

The appearance of integrability in the planar limit of $\mathcal{N} = 4$ SYM dates back to the study by Minahan and Zarembo [33] of the one-loop dilatation operator for operators in the $SO(6)$ sector, i.e. which are traces of products of the scalar fields. They showed that the dilatation operator then coincides with the Hamiltonian of an $SO(6)$ -invariant spin chain with the spins in the defining representation. This notably means that the spectral problem can be reduced to solving Bethe ansatz equations. The complete one-loop dilatation operator with $PSU(2, 2|4)$ symmetry was then obtained in [113, 114].

The one-loop dilatation operator only involves nearest-neighbour interaction, this is however no longer the case when one goes to higher loop orders: the range of the dilatation operator grows with the loop order. The idea that the dilatation operator could be integrable at all loop is due to [115]. It was realised in [116] that in order to determine the all-loop Bethe ansatz equations one should focus on the scattering matrix of the excitations. This S-matrix can be determined up to an overall phase—the dressing factor—from symmetry considerations only; this was done in [117] and allowed to recover the Bethe equations proposed earlier in [118]. The symmetry algebra is a centrally extended $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$.

A physically equivalent S-matrix was shown [119] to describe the scattering of worldsheet excitations in the $AdS_5 \times S^5$ string sigma model. In that case, the S-matrix, up to the dressing factor, follows from the assumption that the excitations are created by the action of a Zamolodchikov–Faddeev algebra [120, 121] and that they have centrally extended $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ symmetry.

An equation that the dressing factor should satisfy, the crossing equation, was proposed in [122] based on the quantum group structure of the symmetry. The solution relevant to $\mathcal{N} = 4$ SYM was conjectured in [123, 124]. It was proven in [125] that this conjecture indeed satisfies the crossing equation. At the same time, the solution to the crossing equation with the minimal number of singularities in the physical strip was shown in [126] to coincide with the conjectured dressing factor.

The Bethe equations thus derived, though valid at all loop orders, are only asymptotic in the sense that they yield the conformal dimensions of infinitely long operators. Their application to operators of finite length is valid only up to terms exponentially small in the

length—the wrapping corrections. These corrections can be added one by one [127], they are then called Lüscher corrections because the expression of the leading correction in relativistic quantum field theories was obtained in [128]. For integrable 1 + 1-dimensional field theories, it is also possible to resum all these terms, the conformal dimension is then encoded in a system of integral equations called the thermodynamic Bethe ansatz (TBA) [129]—TBA equations were actually derived for the first time in [130] in the study of the thermodynamic limit of the Lieb–Liniger model. The formulation of the TBA equations for integrable field theories relies solely on the knowledge of the dispersion relation and the scattering matrix of the excitations in a dual theory dubbed the mirror theory. The mirror S-matrix and the associated bound states for strings in an $AdS_5 \times S^5$ background were determined in a series of articles by Arutyunov and Frolov [125, 131–133], while the complete spectral equations appeared in the form of the Y-system [134] before the TBA equations were formulated [135–137].

The solution to the spectral problem was later greatly simplified [138, 139] by expressing it in terms of a solution to a Q-system. This formulation is the most efficient way to determine the spectrum and is known as the quantum spectral curve (QSC), it is nicely reviewed in [38, 39]. As we explained before, even if the Q-system is entirely fixed by the symmetry algebra, $\mathfrak{psu}(2, 2|4)$ for AdS_5/CFT_4 , the regularity properties of the Q-functions must depend on the specifics of the model. In the case at hand, these properties are quite intricate and were also worked out in [138, 139] starting from the TBA and Y-system formulations. The Q-functions have branch cuts whose positions are determined by the 't Hooft coupling, their asymptotic behaviour is determined by the charges of the operator under consideration, and they satisfy some sewing conditions relating different Q-functions and their monodromies around the branch points. The QSC was then extended in [140] to describe deformations of $\mathcal{N} = 4$ SYM.

Before moving on to the fishnet theory, let us emphasise that these progressive advances in the solution to $\mathcal{N} = 4$ SYM were deeply interwoven with similar progress on the string side of the correspondence, such as the proof that the sigma model on $AdS_5 \times S^5$ is classically integrable [34], or the development of the classical spectral curve [141, 142]. In particular, the TBA, as developed in [129], should *a priori* only be applicable to the string theory but, because of the correspondence, the same equations describe the spectrum of $\mathcal{N} = 4$ SYM. In the much simpler case of the fishnet theory, however, the TBA equations can be derived directly from the conformal field theory as we will show in this thesis.

The Fishnet Theory

The integrability of planar $\mathcal{N} = 4$ SYM gives hope for an exact solution of this strongly coupled field theory. Although this would be a remarkable achievement in itself, it would also, in this case, deepen our understanding of the AdS/ CFT correspondence. It remains nonetheless an arduous task and many results are still conjectural. It thus seems reasonable to begin by studying simpler integrable theories.

The fishnet theory is a non-unitary quantum field theory of two $N_c \times N_c$ matrix complex scalar fields X and Z interacting via a single quartic interaction vertex with coupling ξ^2 . It was originally [143] obtained in four dimensions as a double-scaling limit (strong imaginary twist and weak coupling) of the γ -deformation of $\mathcal{N} = 4$ SYM, and was subsequently generalised to arbitrary dimension [144]. The main interest of this theory comes from the very

simple structure of its Feynman graphs in the planar limit, i.e. $N_c \rightarrow +\infty$ but ξ^2 remains finite. Owing to the chiral nature of the interaction, there are very few of them. What is more, all those graphs are of fishnet type: the bulk of these diagrams is nothing but a piece of a square lattice. This important restriction ensures that the theory is conformal if one includes some specific double-trace counterterms [145] (see also [146] for an investigation of the spontaneous conformal symmetry breaking in the fishnet theory) that were already present in the γ -deformation of $\mathcal{N} = 4$ SYM [147]. This property of the Feynman graphs also ensures integrability of the theory. Zamolodchikov [148] had first observed this several decades ago when he treated fishnet Feynman graphs on the torus as an integrable statistical physics system, and was able to obtain the partition function in the thermodynamic limit. The Yang–Baxter relation guaranteeing integrability being here an integral identity called the star-triangle relation. It turns out that a more explicit connection can be made with some non-compact conformal spin chains studied in [149]. One can indeed exhibit an integral operator which, upon repeated application, yields all the graphs appearing in the perturbative expansion of some correlators. This operator is then explicitly shown to belong to the conserved charges of a non-compact spin chain [143, 150]. Being a non-compact spin chain with a higher-rank symmetry, its solution is not yet known but is expected to be achievable through the methods of Q-operators and separation of variables described above. This thesis will contain results in that direction.

Even though we will only study the fishnet theory just described, let us mention that several theories with a similar regular structure of their Feynman graphs have also been proposed. Firstly, the double-scaling limit can be taken such that three fermions and three scalars remain coupled [143]. In that case, the structure of the graphs is more involved [151] and integrability is already much less apparent (although it was still visible at the level of the graphs for an intermediate theory with three scalars and two fermions [151, 152]). Then, a double-scaling limit of the γ -deformed ABJM model [153, 154] was presented [155] which contains three scalars interacting via a single sextic coupling, the bulk of the graphs being pieces of a triangular lattice. This was further investigated in [156] where theories with graphs based on a hexagonal lattice were also introduced. Finally, it was shown in [157] that a theory of only two fermions and one scalar, which had appeared in [156], can be obtained through the procedure of γ -deformation followed by double-scaling limit applied to an $\mathcal{N} = 2$ superconformal field theory; the authors of [157] also studied integrability and conformality of this theory.

Harking back to the square fishnet theory, the anomalous dimension of a few short operators, as well as some structure constants, can be determined exactly using integrability in a relatively straightforward manner [144, 158, 159]—some four-particle scattering amplitudes can also be computed [160]—but for most operators the result is still very hard to reach. When dealing with the original four-dimensional fishnet theory, one can use the results obtained for $\mathcal{N} = 4$ SYM from which it descends. It is, for instance, possible to take the double-scaling limit in the asymptotic Bethe ansatz equations describing the conformal dimensions of long operators, as was done in [155], or to take the limit directly in the twisted quantum spectral curve as in [150]. This is however not possible for the fishnet theory in other dimensions, or even in four dimensions, if one includes an additional anisotropy parameter. In such cases, it seems that one needs to derive from first principles the relevant quantum spectral curve.

The chirality of the interaction in the fishnet theory induces a dilatation operator which, though integrable, exhibits properties very different from that of $\mathcal{N} = 4$ SYM. In particular, it is non-diagonalisable, and, as a consequence, there exist logarithmic multiplets. This was first realised by Caetano [161] and further investigated in [150]. Moreover, it was noticed [162] that the one-loop dilatation operator (of the fishnet theory and, more generally, of the theory with three fermions and three scalars) coincides with the Hamiltonian of some “eclectic” spin chains. The correspondence with a spin chain was done via a restriction to a certain class of local operators (traces of products of scalars) as in [33]. In the usual Heisenberg spin chain, the interaction is simply a permutation of two adjacent spins, whatever the state they are in. In these eclectic spin chains, however, the interaction is still nearest-neighbour but only some specific configurations can be permuted. The authors of [162, 163] studied these unusual chains in detail. They showed that the quantum inverse scattering method cannot be used to find its eigenvectors, even though the Hamiltonian can still be obtained from a family of commuting transfer matrices. Being non-unitary, it is not a surprise that the Hamiltonian is non-diagonalisable, so the only thing one can hope for is to bring it to Jordan normal form. It was nonetheless pointed out in [163] that the size and the multiplicity of the Jordan blocks seems to be highly structured, thus giving hope for a systematic understanding of this spectrum.

The simplicity of the fishnet theory also allowed a rigorous derivation of its holographic dual. This was achieved by Gromov and Sever: they first showed in [164] that the strong coupling dual of the theory is a classically integrable chain of point-like particles on the light-cone in six-dimensional Minkowski space, which they named the fishchain model. They then proceeded [165] to derive the dual at finite coupling by quantising this classical model for a subsector of the operators in the fishnet theory. They eventually extended their construction to all operators [166]. The fact that the quantum fishchain lives in AdS_5 had already transpired in [167].

Some of the planar integrals relevant to the fishnet theory had already been studied extensively in the past. Such is the case, in four dimensions, of the ladder diagrams computed in [168], or of the (one-)wheel graphs computed in [169], for instance. With the discovery of the fishnet theory came new advances. On the one hand, the (massless) Yangian symmetry of a vast class of single trace correlators of the theory, each given by a single Feynman graph, was derived in [170, 171] thus providing severe constraints on these integrals. A Yangian symmetry for a class of Feynman integrals with massive propagators was then discovered [172, 173], and an interpretation of these integrals as the scattering amplitudes in a massive version of the fishnet theory was also derived [174]. On the other hand, several equivalent expressions for a class of four-point integrals in four dimensions were given in [175], one of which can be obtained rigorously from the integrability of an open non-compact spin chain [176, 177]. These expressions were then used [178] to examine the thermodynamic limit of the four-point integrals. The result was different from that obtained by Zamolodchikov [148] for periodic graphs, thus showing a strong dependence on the boundary conditions. The same integrals in two dimensions were also computed directly from integrability [179]. The results, both in two and four dimensions, admit a remarkable determinant representation.

Outline of the Thesis

The first chapter of this thesis will be devoted to a reminder of some well-known results regarding quantum integrable models, and in particular compact spin chains. After a quick presentation of the Bethe ansatz equations for the simplest XXX spin chain, the concepts of R-matrix and transfer matrix are introduced. The fusion procedure, used to construct new R-matrices starting from known ones, is exemplified in the $GL(2, \mathbb{C})$ case. The simplest R-matrices for symmetry groups of higher ranks are shown, and the appearance of nested Bethe ansatz equations is explained.

We present, in the next two chapters, the fishnet theory and its connection with an integrable non-compact conformal spin chain. We explain how the fishnet theory in four dimensions arose as a double-scaling limit of the γ -deformation of $\mathcal{N} = 4$ SYM. The importance of the chiral interaction for conformality of the theory is reviewed. Scalar principal series of the conformal group, as well as the spin chains involving these infinite-dimensional representations, are described. We show that the graph-building operators producing the perturbative expansion of some correlators are part of the conserved charges of the chain, thus exemplifying integrability of the theory. The solution to the simple case of a closed chain of length two is also reviewed, thus providing the exact results for some conformal dimensions and structure constants.

The diagonalisation of the graph-building operator related to the open spin chain is performed in the fourth chapter. This is done in arbitrary dimension d [180], thus generalising the known results in two and four dimensions. The eigenstates of the operator of length N can be interpreted as N -particle states. There are infinitely many types of particles labelled by integers $l \geq 0$; they all live on a one-dimensional space and have an additional internal $O(d, \mathbb{C})$ symmetry: a particle of type l transforms in the symmetric traceless representation of rank l of the group $O(d, \mathbb{C})$. Though the order of the particles *a priori* matters in the definition of the N -particle states, it is explicitly shown that the exchange of two particles in a given state yields the same state up to mixing of the internal degrees of freedom. This mixing is governed by the $O(d, \mathbb{C})$ -invariant R-matrices acting in the tensor product of two symmetric traceless representations. The N -particle states are also shown to be orthogonal up to the symmetry just mentioned. These computations are based on explicit expressions for the fused R-matrices that were not known before.

The graph-building operators can be seen as the (exponential of minus the) Hamiltonian of the interacting particles. Because of integrability, the energy of an N -particle state is just the sum of the individual energies. What is more, the scattering matrix can be extracted from the asymptotic behaviour of the two-particle states; it contains a matrix part, which is nothing else than the $O(d, \mathbb{C})$ -invariant R-matrix, and a non-trivial scalar phase. From this scattering data, one can write down TBA equations [181] for the dimensions of multi-magnon operators of the type

$$\mathcal{O}_{J,M}(x) = \text{Tr}(X^M Z^J) + \dots \quad (0.0.9)$$

Those TBA equations are presented in the fifth chapter. From them one can immediately extract the asymptotic Bethe ansatz (ABA) equations describing the dimensions of operators in the limit $J \rightarrow +\infty$. The TBA is also explored at finite coupling, following the findings of [167] in four dimensions. A correspondence with a two-dimensional non-linear sigma model

in AdS_{d+1} is established.

After reviewing the A_r Q-system for compact spin chains, we propose, in the sixth chapter, a D_r Q-system. Our proposal [182] is based on the recent construction [183] of some Q-operators for the spin chain with defining representation of the orthogonal group at each site. We also obtain several new expressions for the transfer matrices in terms of the Q-operators. Some of these expressions are the analogues of (0.0.3) for A_r symmetry and constitute a quantisation of the classical Weyl character formula. Other expressions we propose for the transfer matrices seem to be a consequence of relations on infinite-dimensional modules akin to the Bernstein–Gelfand–Gelfand resolution.

We summarise, in a brief conclusion, our main achievements, and we present a few possible future directions of research. Finally, we include three appendices. We present, in the first one, the conventions for Lie algebras used throughout the thesis. The basic integral relations needed for the Feynman diagram computations are contained in the second appendix, as well as the explicit computation of some of the Feynman graphs appearing in the fifth chapter. And we provide, in the last appendix, additional information regarding the results of the sixth chapter.

Our own results are all presented in the last three chapters (and the associated appendices); these are based respectively on the following works:

- *Mirror channel eigenvectors of the d -dimensional fishnets*, arXiv:2108.12620 with Sergey Derkachov and Enrico Olivucci.
- *Thermodynamic Bethe Ansatz for Biscalar Conformal Field Theories in Any Dimension*, Phys. Rev. Lett. **125** (2020) with Benjamin Basso, Vladimir Kazakov, and Deliang Zhong.
- *QQ-systems and Weyl-type transfer matrices in integrable $SO(2r)$ systems*, JHEP **02** (2021) with Vladimir Kazakov and Rouven Frassek.

Chapter 1

Compact Integrable Spin Chains

We review in this chapter some elementary facts about compact spin chains. We first explain the coordinate Bethe ansatz on the simplest spin chain: the XXX Heisenberg spin chain with $GL(2)$ symmetry¹ originally studied by Bethe [9]. After having managed to write the Bethe ansatz equations, we proceed to the construction of a family of operators commuting with the Hamiltonian. This is naturally done in the framework of the quantum inverse scattering method. We thus introduce Yang's R-matrix and the Yang–Baxter relation. We also explain, on this particular example with $GL(2)$ symmetry, the fusion procedure used to construct new solutions to the Yang–Baxter relation. We finally touch upon models with higher-rank symmetry. In particular, we motivate, from the point of view of the coordinate Bethe ansatz for the Heisenberg spin chain with $GL(N)$ symmetry, the appearance of the nested Bethe equations.

Several of the notions introduced here will be useful in the next chapters. Firstly, the definition in Chapter 3 of the non-compact spin chain relevant to the fishnet theory requires particular solutions of the Yang–Baxter equation. Then, the fusion procedure will be used in Chapter 4 to give explicit expressions for some $O(d)$ -invariant R-matrices. The eigenvectors of the non-compact model solved in the same chapter will be shown to satisfy properties analogous, if not identical, to those of the eigenvectors of the compact spin chain with $GL(N)$ symmetry solved here. Finally, the Q-system presented in Chapter 6 will be shown to include the nested Bethe ansatz equations.

1.1 The $GL(2)$ Heisenberg Spin Chain

We consider a discrete quantum system of L spins $\frac{1}{2}$ on a chain with nearest neighbour interactions and periodic boundary conditions. More precisely, the Hamiltonian is

$$H = L - \sum_{i=1}^{L-1} \mathbb{P}_{ii+1} - \mathbb{P}_{L1}, \quad (1.1.1)$$

¹In order to shorten the notations, we denote by $GL(N)$ the group of $N \times N$ invertible complex matrices. The group of invertible real square matrices, which will almost never appear, will be denoted $GL(N, \mathbb{R})$.

and it acts on the quantum space $(\mathbb{C}^2)^{\otimes L}$. In the previous expression, $\mathbb{P}_{ij} = \mathbb{P}_{ji}$ for $i \neq j$ denotes the permutation of spins at sites i and j without modifying the other spins:

$$\mathbb{P}_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_L) = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_L, \quad (1.1.2)$$

where all the v_k 's are in \mathbb{C}^2 . We introduce the usual spin operators at each site

$$\vec{S}_i = \mathbf{I}_2 \otimes \cdots \otimes \frac{\vec{\sigma}}{2} \otimes \cdots \otimes \mathbf{I}_2, \quad (1.1.3)$$

where \mathbf{I}_2 is the 2×2 identity matrix, the operator $\vec{\sigma}$ appears only at the i th site and its components are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.1.4)$$

We will also denote by $|\uparrow\rangle$ and $|\downarrow\rangle$ the two vectors of the basis in which the Pauli matrices are written. In terms of the spin operators, the Hamiltonian can be rewritten

$$H = \frac{L}{2} - 2 \sum_{i=1}^{L-1} \vec{S}_i \cdot \vec{S}_{i+1} - 2 \vec{S}_1 \cdot \vec{S}_L. \quad (1.1.5)$$

The Hamiltonian commutes with the operator \mathcal{P} of translation by one site

$$\mathcal{P}(v_1 \otimes \cdots \otimes v_L) = v_L \otimes v_1 \otimes \cdots \otimes v_{L-1}. \quad (1.1.6)$$

Our goal is to diagonalise simultaneously these two operators, and to determine their spectrum.

We first remark that there is a global $GL(2)$ invariance: for any $G \in GL(2)$, we have

$$[H, G \otimes \cdots \otimes G] = [\mathcal{P}, G \otimes \cdots \otimes G] = 0. \quad (1.1.7)$$

This implies that the Hamiltonian commutes with the associated representation of the Lie algebra $\mathfrak{gl}(2)$, and with the total spin $\vec{S} = \sum_{i=1}^L \vec{S}_i$ of the system in particular. Its spectrum will consequently be degenerate.

There are two obvious eigenvectors, $|\uparrow \dots \uparrow\rangle$ and $|\downarrow \dots \downarrow\rangle$, both having energy 0. We now (arbitrarily) decide to call $|\downarrow \dots \downarrow\rangle$ the vacuum and that M -particle states, for $0 \leq M \leq L$, will refer to linear combinations of states made of M spins up $|\uparrow\rangle$ and $L - M$ spins down $|\downarrow\rangle$. It is clear that M -particle states are exactly the eigenvectors of S_z , the z -component of the total spin, with eigenvalue $M - \frac{L}{2}$, and that they are stable under the action of the Hamiltonian. In order to shorten the notations we shall write $|\emptyset\rangle$ for the vacuum and $|i_1, \dots, i_M\rangle$ with i_1, \dots, i_M distinct integers between 1 and L for the state made of M spins up at sites i_1, \dots, i_M and spins down at the other sites.

One-particle Sector One-particle eigenvectors of \mathcal{P} are trivially found to be

$$|p\rangle = \sum_{j=1}^L e^{ipj} |j\rangle \quad \text{with} \quad e^{iLp} = 1. \quad (1.1.8)$$

These are also eigenvectors of the Hamiltonian:

$$\mathcal{P} |p\rangle = e^{-ip} |p\rangle \quad \text{and} \quad H |p\rangle = 2(1 - \cos p) |p\rangle. \quad (1.1.9)$$

We thus obtained L new eigenvectors, one of which ($p = 0$) has the same energy as the vacuum and is actually in the same $SU(2)$ multiplet:

$$|0\rangle = S^+ |\emptyset\rangle, \quad (1.1.10)$$

where $S^+ = \frac{S_x + iS_y}{2}$ is the usual ladder operator. The other new eigenvectors all have an energy strictly higher than the vacuum.

Two-particle Sector We now look for eigenvectors of the form

$$|p_1, p_2\rangle = \sum_{1 \leq j < k \leq L} \psi_{p_1, p_2}(j, k) |j, k\rangle, \quad (1.1.11)$$

with the wave function given by

$$\psi_{p_1, p_2}(j, k) = e^{ip_1 j + ip_2 k} + S(p_1, p_2) e^{ip_1 k + ip_2 j}. \quad (1.1.12)$$

This wave function in particular satisfies

$$\begin{aligned} \psi_{p_1, p_2}(j+1, k) + \psi_{p_1, p_2}(j-1, k) + \psi_{p_1, p_2}(j, k+1) + \psi_{p_1, p_2}(j, k-1) \\ = 2(\cos p_1 + \cos p_2) \psi_{p_1, p_2}(j, k). \end{aligned} \quad (1.1.13)$$

For $|p_1, p_2\rangle$ to be an eigenvector of \mathcal{P} , the eigenvalue can only be $e^{-i(p_1+p_2)}$, and it is easy to see that

$$\mathcal{P} |p_1, p_2\rangle = e^{-i(p_1+p_2)} |p_1, p_2\rangle \iff e^{-iLp_1} = e^{iLp_2} = S(p_1, p_2). \quad (1.1.14)$$

This is nothing but the appropriate periodicity condition on the wave function

$$\psi_{p_1, p_2}(j-L, k) = \psi_{p_1, p_2}(k, j) = \psi_{p_1, p_2}(j, k+L). \quad (1.1.15)$$

The action of the Hamiltonian on $|j, k\rangle$ for $1 \leq j < k \leq L$ is given by

$$H |j, k\rangle = 4 |j, k\rangle - |j-1, k\rangle - |j+1, k\rangle - |j, k-1\rangle - |j, k+1\rangle, \quad (1.1.16)$$

if $j+1 < k$ and $(j, k) \neq (1, L)$, and else by

$$H |j, j+1\rangle = 2 |j, j+1\rangle - |j-1, j+1\rangle - |j, j+2\rangle, \quad (1.1.17)$$

with the understanding that the chain is periodic (i.e. $|j, L+1\rangle = |j, 1\rangle$ and $|0, k\rangle = |L, k\rangle$). As a consequence, one can write

$$H |p_1, p_2\rangle = 2(2 - \cos p_1 - \cos p_2) |p_1, p_2\rangle + \sum_{j=1}^L (\psi_{p_1, p_2}(j+1, j+1) + \psi_{p_1, p_2}(j, j) - 2\psi_{p_1, p_2}(j, j+1)) |j, j+1\rangle, \quad (1.1.18)$$

where we have used the property (1.1.13) and the condition (1.1.15). We thus get an eigenvector if the coefficients in front of $|j, j+1\rangle$ in the second line vanish, this is equivalent to

$$S(p_1, p_2) = -\frac{1 + e^{ip_1+ip_2} - 2e^{ip_2}}{1 + e^{ip_1+ip_2} - 2e^{ip_1}}. \quad (1.1.19)$$

Together with the condition (1.1.14) this equation restricts the values of e^{ip_1} and e^{ip_2} . It is possible to show that solving these equations would give $\frac{L(L-1)}{2}$ new eigenvectors, one of which would be (proportional to) $S_+^2 |\emptyset\rangle$ (when $p_1 = p_2 = 0$) and $L-1$ others would be (proportional to) $S_+ |p\rangle$ for $p \neq 0$ (when $\{p_1, p_2\} = \{0, p\}$).

Coordinate Bethe Ansatz Now that we have understood the form of the eigenvectors for small number of particles, we are ready to propose an ansatz for an arbitrary number of them: we look for M -particle eigenvectors of the form

$$|p_1, \dots, p_M\rangle = \sum_{j_1 < \dots < j_M} \psi_{\mathbf{p}}(j_1, \dots, j_M) |j_1, \dots, j_M\rangle, \quad (1.1.20)$$

with the wave function given by

$$\psi_{\mathbf{p}}(j_1, \dots, j_M) = \sum_{\sigma \in \mathfrak{S}_M} a(\sigma) e^{i \sum_{k=1}^M p_{\sigma(k)} j_k}, \quad (1.1.21)$$

for some coefficients $a(\sigma)$ to be determined (they will obviously depend on p_k 's, but we do not indicate it in order to lighten the notations). We will refer to the p_k 's as the momenta of the particles. If we first require that $|p_1, \dots, p_M\rangle$ be an eigenvector of \mathcal{P} , then the eigenvalue is necessarily $e^{-i(p_1 + \dots + p_M)}$, and the wave function must satisfy

$$\psi_{\mathbf{p}}(j_1, \dots, j_M + L) = \psi_{\mathbf{p}}(j_M, j_1, \dots, j_{M-1}). \quad (1.1.22)$$

Assuming that the p_k 's are generic, this entails that, for any permutation τ , the coefficients must satisfy

$$\frac{a(\tau(12 \dots M))}{a(\tau)} = e^{-iLp_{\tau(1)}}, \quad (1.1.23)$$

where $(12 \dots M)$ is the cyclic permutation $1 \mapsto 2 \mapsto \dots \mapsto M \mapsto 1$.

If we now act on our ansatz with the Hamiltonian, we get

$$H |p_1, \dots, p_M\rangle = \left(\sum_{k=1}^M \varepsilon(p_k) \right) |p_1, \dots, p_M\rangle + \dots, \quad (1.1.24)$$

where $\varepsilon(p) = 2(1 - \cos p)$ and the last ellipsis stands for terms involving states containing adjacent up spins. The cancellation of these unwanted terms imposes the values of the coefficients $a(\sigma)$. If we consider, for instance, the terms proportional to $|j_1, j_1 + 1, j_3, \dots, j_M\rangle$ with the indices strictly ordered and such that only j_1 and $j_1 + 1$ are neighbours, then the condition reads

$$\psi_{\mathbf{p}}(j_1, j_1, j_3, \dots, j_M) + \psi_{\mathbf{p}}(j_1 + 1, j_1 + 1, j_3, \dots, j_M) = 2\psi_{\mathbf{p}}(j_1, j_1 + 1, j_3, \dots, j_M). \quad (1.1.25)$$

This is in turn equivalent to

$$\frac{a(\sigma(12))}{a(\sigma)} = S(p_{\sigma(1)}, p_{\sigma(2)}), \quad (1.1.26)$$

for any permutation σ , and with S the same function as in (1.1.19) for the 2-particle case. It is easy to see that the following conditions must also hold:

$$\frac{a(\sigma(jj+1))}{a(\sigma)} = S(p_{\sigma(j)}, p_{\sigma(j+1)}), \quad (1.1.27)$$

for any j strictly smaller than M . Since any permutation can be decomposed into a product of elementary transpositions $(jj+1)$, these conditions are already enough to determine all the coefficients. The only question is whether it can be done in a consistent way: the decomposition of a permutation in terms of elementary transpositions is not unique and as such there could *a priori* be several ways of defining $a(\sigma)$ for a generic permutation. In order to show consistency, one simply needs to verify that the two decompositions

$$(13) = (12)(23)(12) = (23)(12)(23) \quad (1.1.28)$$

lead to the same prescription for $a((13))$. It is immediate, both decompositions give

$$\frac{a(13)}{a(\text{Id})} = -\frac{(1 + e^{ip_2+ip_3} - 2e^{ip_3})(1 + e^{ip_1+ip_3} - 2e^{ip_3})(1 + e^{ip_1+ip_2} - 2e^{ip_2})}{(1 + e^{ip_2+ip_3} - 2e^{ip_2})(1 + e^{ip_1+ip_3} - 2e^{ip_1})(1 + e^{ip_1+ip_2} - 2e^{ip_1})}. \quad (1.1.29)$$

As a consequence, we obtain, for an arbitrary permutation σ ,

$$\frac{a(\sigma)}{a(\text{Id})} = \epsilon(\sigma) \prod_{j < k} \frac{1 + e^{ip_{\sigma(j)}+ip_{\sigma(k)} - 2e^{ip_{\sigma(j)}}}}{1 + e^{ip_j+ip_k - 2e^{ip_j}}}, \quad (1.1.30)$$

where $\epsilon(\sigma)$ is the signature of the permutation. It can be shown that such coefficients also guarantee the cancellation of all the other unwanted terms (when more than two up spins are adjacent). One should notice, however, that these conditions are generically not consistent with the periodicity of the wave function expressed in (1.1.22) or (1.1.23). Consistency requires the momenta of the particles to satisfy the so-called Bethe ansatz equations

$$e^{iLp_j} = -\prod_{k=1}^M S(p_k, p_j) = (-1)^{M+1} \prod_{k=1}^M \frac{1 + e^{ip_k+ip_j} - 2e^{ip_j}}{1 + e^{ip_k+ip_j} - 2e^{ip_k}}, \quad (1.1.31)$$

for all j in $\{1, \dots, M\}$. Since everything is algebraic in the exponentials e^{ip_j} , it is common to introduce the rapidity u through

$$u = \frac{1}{2} \cot\left(\frac{p}{2}\right) \iff e^{ip} = \frac{u + \frac{i}{2}}{u - \frac{i}{2}}. \quad (1.1.32)$$

The energy of an excitation and the scattering matrix are then given by

$$\varepsilon(u) = \frac{1}{u^2 + \frac{1}{4}}, \quad S(p_1, p_2) = S(u_1 - u_2) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}, \quad (1.1.33)$$

and the Bethe equations read

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L = - \prod_{k=1}^M \frac{u_j - u_k + i}{u_j - u_k - i}. \quad (1.1.34)$$

1.2 R-Matrix and the Yang–Baxter Equation

We present here a purely algebraic way of constructing models that generalise the Heisenberg spin chain from before. We begin with the simple case of the Heisenberg spin chain and then proceed to present models based on higher-rank symmetry groups.

1.2.1 $GL(2)$ -Invariant R-Matrix in Defining Representation

For x a complex number, let

$$\mathbf{R}(x) = x \mathbf{I}_2 \otimes \mathbf{I}_2 + \mathbb{P} \quad (1.2.1)$$

be an operator acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$, \mathbb{P} being the permutation operator. We shall refer to x as the spectral parameter and to \mathbf{R} as the R-matrix. Introducing, for $1 \leq i, j \leq 2$, the matrices e_{ij} with a single non-zero coefficient, equal to 1, at the i -th row and the j -th column, one can write

$$\mathbb{P} = e_{ij} \otimes e_{ji} \quad \text{and} \quad \mathbf{R}(u) = x \mathbf{I}_2 \otimes \mathbf{I}_2 + e_{ij} \otimes e_{ji}, \quad (1.2.2)$$

where summation is understood with respect to repeated indices. This R-matrix is $GL(2)$ invariant since for any 2×2 matrix G one has

$$[\mathbf{R}(x), G \otimes G] = 0. \quad (1.2.3)$$

We also point out for further use that \mathbf{R} is invertible with inverse given by

$$\mathbf{R}^{-1}(x) = \frac{x \mathbf{I}_2 \otimes \mathbf{I}_2 - \mathbb{P}}{x^2 - 1} = -\frac{\mathbf{R}(-x)}{x^2 - 1}, \quad (1.2.4)$$

and that

$$\mathbf{R}(\pm 1) = \mp 2 \left(\frac{\mathbf{I}_2 \otimes \mathbf{I}_2 \pm \mathbb{P}}{2} \right) = \mp 2 \mathbb{P}_{\pm}, \quad (1.2.5)$$

where \mathbb{P}_{\pm} are the orthogonal projectors onto (anti)symmetric tensors.

We now remark that, if we define three operators on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ in the following fashion:

$$\mathbf{R}_{12}(x) = x \mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \mathbf{I}_2 + \mathbb{P}_{12} = x \mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \mathbf{I}_2 + e_{ij} \otimes e_{ji} \otimes \mathbf{I}_2, \quad (1.2.6)$$

$$\mathbf{R}_{23}(x) = x \mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \mathbf{I}_2 + \mathbb{P}_{23} = x \mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \mathbf{I}_2 + \mathbf{I}_2 \otimes e_{ij} \otimes e_{ji}, \quad (1.2.7)$$

$$\mathbf{R}_{13}(x) = x \mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \mathbf{I}_2 + \mathbb{P}_{13} = x \mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \mathbf{I}_2 + e_{ij} \otimes \mathbf{I}_2 \otimes e_{ji}, \quad (1.2.8)$$

then they satisfy the Yang–Baxter equation

$$R_{12}(x)R_{13}(x+y)R_{23}(y) = R_{23}(y)R_{13}(x+y)R_{12}(x). \quad (1.2.9)$$

The proof is straightforward and goes as follows, one expands both sides and notices that, after some trivial cancellation of the terms cubic and quadratic in the spectral parameters, this is equivalent to

$$\begin{aligned} x(\mathbb{P}_{13}\mathbb{P}_{23} + \mathbb{P}_{12}\mathbb{P}_{23}) + y(\mathbb{P}_{12}\mathbb{P}_{23} + \mathbb{P}_{12}\mathbb{P}_{13}) + \mathbb{P}_{12}\mathbb{P}_{13}\mathbb{P}_{23} \\ = x(\mathbb{P}_{23}\mathbb{P}_{12} + \mathbb{P}_{23}\mathbb{P}_{13}) + y(\mathbb{P}_{13}\mathbb{P}_{12} + \mathbb{P}_{23}\mathbb{P}_{12}) + \mathbb{P}_{23}\mathbb{P}_{13}\mathbb{P}_{12}. \end{aligned} \quad (1.2.10)$$

This last equation holds because the permutations satisfy

$$\mathbb{P}_{13}\mathbb{P}_{23} = \mathbb{P}_{23}\mathbb{P}_{12} = \mathbb{P}_{12}\mathbb{P}_{13} \quad \text{and} \quad \mathbb{P}_{12}\mathbb{P}_{23} = \mathbb{P}_{23}\mathbb{P}_{13} = \mathbb{P}_{13}\mathbb{P}_{12}. \quad (1.2.11)$$

Let us now introduce $L + 1$ copies of \mathbb{C}^2 , we denote them $\mathbb{V}_0, \mathbb{V}_1, \dots, \mathbb{V}_L$. The quantum space of our model will be $\mathcal{H} = \mathbb{V}_1 \otimes \dots \otimes \mathbb{V}_L$, while \mathbb{V}_0 will be called the auxiliary space. We define $R_{0,k}(x) = x \text{Id} + \mathbb{P}_{0k}$ for $k \geq 1$. Let the fundamental monodromy matrix be

$$M(x) = R_{0L}(x) \dots R_{02}(x)R_{01}(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}, \quad (1.2.12)$$

where the matrix structure acts in the auxiliary space, while $A(x), B(x), C(x)$, and $D(x)$ are operators on \mathcal{H} . Let the fundamental transfer matrix be its trace over the auxiliary space:

$$T(x) = \text{Tr}_0(M(x)) = A(x) + D(x). \quad (1.2.13)$$

If we introduce a second auxiliary space $\mathbb{V}'_0 = \mathbb{C}^2$ and define a second monodromy matrix $M'(x) = R_{0'L}(x) \dots R_{0'2}(x)R_{0'1}(x)$, then the following equation holds:

$$R_{0,0'}(x-y)M(x)M'(y) = M'(y)M(x)R_{0,0'}(x-y). \quad (1.2.14)$$

The proof consists in noticing that R_{0j} and $R_{0'k}$ commute when $j \neq k$, and in applying repeatedly the Yang–Baxter equation (1.2.9) for the R-matrix:

$$\begin{aligned} R_{00'}(x-y)M(x)M'(y) &= R_{00'}(x-y)R_{0L}(x)R_{0'L}(y) \dots R_{02}(x)R_{0'2}(y)R_{01}(x)R_{0'1}(y) \\ &= R_{0'L}(y)R_{0L}(x)R_{00'}(x-y) \dots R_{02}(x)R_{0'2}(y)R_{01}(x)R_{0'1}(y) \\ &= R_{0'L}(y)R_{0L}(x) \dots R_{0'2}(y)R_{02}(x)R_{0'1}(y)R_{01}(x)R_{00'}(x-y) \\ &= M'(y)M(x)R_{00'}(x-y). \end{aligned} \quad (1.2.15)$$

Multiplying both sides of (1.2.14) with $R_{00'}^{-1}(x-y)$ from the right and taking the trace with respect to both auxiliary spaces, we immediately get

$$T(x)T(y) = T(y)T(x). \quad (1.2.16)$$

If we write

$$T(x) = 2x^L + \sum_{j=1}^L t_j x^{L-j}, \quad (1.2.17)$$

with t_k an operator on \mathcal{H} , the previous commutation relation is equivalent to

$$[t_j, t_k] = 0 \quad \text{for all } j, k \text{ in } \{1, \dots, L\}. \quad (1.2.18)$$

Let us now take a closer look at the coefficients t_k . It is clear that

$$t_1 = \sum_{k=1}^L \text{Tr}_0 (\mathbb{P}_{0k}) = L \quad (1.2.19)$$

is trivial. The next coefficient is

$$t_2 = \sum_{1 \leq k < j \leq L} \text{Tr}_0 (\mathbb{P}_{0j} \mathbb{P}_{0k}) = \sum_{1 \leq k < j \leq L} \mathbb{P}_{jk}. \quad (1.2.20)$$

Expressing t_2 in terms of the spin operators \vec{S}_i at the different sites we see that it is nothing else than the Casimir operator for the global $\mathfrak{sl}(2)$ symmetry:

$$t_2 = \vec{S} \cdot \vec{S} + \frac{L^2}{4} - L. \quad (1.2.21)$$

On the other hand,

$$t_L = \mathbb{T}(0) = \text{Tr}_0 (\mathbb{P}_{0L} \dots \mathbb{P}_{01}) = \mathbb{P}_{12} \mathbb{P}_{23} \dots \mathbb{P}_{L-1, L} = \mathcal{P}, \quad (1.2.22)$$

with \mathcal{P} the operator of translation by one site (1.1.6). This is a direct consequence of the fact that

$$\mathbb{P}_{0L} \dots \mathbb{P}_{01} = \mathbb{P}_{12} \mathbb{P}_{23} \dots \mathbb{P}_{L-1, L} \mathbb{P}_{L0}. \quad (1.2.23)$$

Similarly, one has

$$\begin{aligned} t_{L-1} &= \mathbb{T}'(0) = \sum_{k=1}^L \text{Tr}_0 \left(\mathbb{P}_{0L} \dots \widehat{\mathbb{P}_{0k}} \dots \mathbb{P}_{01} \right) \\ &= \mathbb{P}_{23} \dots \mathbb{P}_{L-1, L} + \sum_{k=2}^{L-1} \mathbb{P}_{12} \dots \mathbb{P}_{k-1, k+1} \dots \mathbb{P}_{L-1, L} + \mathbb{P}_{12} \dots \mathbb{P}_{L-2, L-1}, \end{aligned} \quad (1.2.24)$$

where the hat over \mathbb{P}_{0k} means that this operator is omitted from the product. It is then easy to check that

$$t_{L-1} \mathcal{P}^{-1} = \sum_{k=1}^L \mathbb{P}_{k, k+1} + \mathbb{P}_{L1} = L - H, \quad (1.2.25)$$

where H is exactly the Hamiltonian (1.1.1) of the $GL(2)$ spin chain. Since $\mathcal{P}^{-1} = \mathbb{T}^{-1}(0)$, this can be rewritten

$$H = L - \mathbb{T}^{-1}(0) \mathbb{T}'(0) = L - (\ln \mathbb{T})'(0). \quad (1.2.26)$$

Since $[\mathbb{T}(x), \mathbb{T}(y)] = 0$ implies $[\mathbb{T}(x), \mathbb{T}'(y)] = 0$, we have in particular $[\mathbb{T}(x), H] = 0$. This means that we have found a whole family of operators commuting with H ; such operators are called (conserved) charges.

1.2.2 $GL(2)$ -Invariant R-Matrix in Symmetric Representations

We now use the so-called fusion procedure [184, 185] to construct new R-matrices that satisfy a cubic relation similar to the Yang–Baxter relation (1.2.9). These R-matrices will be used to obtain new transfer matrices commuting with the fundamental one defined above.

Let $S^l(\mathbb{C}^2)$ be the vector space of rank- l symmetric tensors in 2 dimensions; it is in particular generated by vectors of the form $v \otimes \cdots \otimes v = v^{\otimes l}$ for $v \in \mathbb{C}^2$. For any $x \in \mathbb{C}$, we now define $R^{(1,l)}(x) : \mathbb{C}^2 \otimes S^l(\mathbb{C}^2) \rightarrow \mathbb{C}^2 \otimes S^l(\mathbb{C}^2)$ in the following way:

$$s \otimes t^{\otimes l} \cdot [R^{(1,l)}(x)v \otimes w^{\otimes l}] = \left(x + \frac{1-l}{2}\right) (s \cdot v)(t \cdot w)^l + l(s \cdot w)(t \cdot v)(t \cdot w)^{l-1}, \quad (1.2.27)$$

with v, w, s , and t all in \mathbb{C}^2 , and where $v \cdot w = v_i^* w_i$ is the natural inner product on \mathbb{C}^2 . Equivalently, we could have written

$$[R^{(1,l)}(x)v \otimes C]_{ij_1 \dots j_l} = \left(x + \frac{1-l}{2}\right) v_i C_{j_1 \dots j_l} + \sum_{k=1}^l v_{j_k} C_{ij_1 \dots \hat{j}_k \dots j_l}, \quad (1.2.28)$$

where C is a symmetric tensor of rank l . The first term corresponds to the identity operator while the sum corresponds to the analogue of the permutation operator (we have to sum in order to get a symmetric tensor). Notice that $R^{(1,1)} = R$. For any two spectral parameters x and y , the following relation holds between operators on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S^l(\mathbb{C}^2)$:

$$R_{12}(x)R_{13}^{(1,l)}(x+y)R_{23}^{(1,l)}(y) = R_{23}^{(1,l)}(y)R_{13}^{(1,l)}(x+y)R_{12}(x). \quad (1.2.29)$$

The indices indicate as before on which of the three spaces the operators act non trivially. In order to prove this, let us first show that the new R-matrices satisfy the fusion relations

$$R^{(1,1)}\left(x - \frac{l}{2}\right)R^{(1,l)}\left(x + \frac{1}{2}\right)v \otimes w^{\otimes(l+1)} = \left(x + \frac{2-l}{2}\right)R^{(1,l+1)}(x)v \otimes w^{\otimes(l+1)}. \quad (1.2.30)$$

Before proving this relation, let us emphasise that it does not imply that the operators themselves are proportional, as they are not even defined on the same spaces: $R^{(1,1)}(x)R^{(1,l)}(y)$ acts on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S^l(\mathbb{C}^2)$ whereas $R^{(1,l+1)}(x)$ acts on $\mathbb{C}^2 \otimes S^{l+1}(\mathbb{C}^2)$. What it means is that the restriction of the first one to $\mathbb{C}^2 \otimes S^{l+1}(\mathbb{C}^2) \subset \mathbb{C}^2 \otimes (\mathbb{C}^2 \otimes S^l(\mathbb{C}^2))$ is proportional to the second one. It is fairly easy to check using the explicit form (1.2.28) for the R-matrix:

$$\begin{aligned} & R^{(1,1)}\left(x - \frac{l}{2}\right)R^{(1,l)}\left(x + \frac{1}{2}\right)v \otimes w^{\otimes(l+1)} \\ &= R^{(1,1)}\left(x - \frac{l}{2}\right)\left[\left(x + \frac{2-l}{2}\right)v \otimes w^{\otimes(l+1)} + \sum_{k=1}^l w^{\otimes(1+k)} \otimes v \otimes w^{\otimes(l-k)}\right] \\ &= \left(x + \frac{2-l}{2}\right)\left[\left(x - \frac{l}{2}\right)v \otimes w^{\otimes(l+1)} + \sum_{k=0}^l w^{\otimes(1+k)} \otimes v \otimes w^{\otimes(l-k)}\right] \\ &= \left(x + \frac{2-l}{2}\right)R^{(1,l+1)}(x)v \otimes w^{\otimes(l+1)}. \end{aligned}$$

Now that we have the fusion relations, we can use induction to prove the Yang–Baxter relation (1.2.29). It holds for $l = 1$ because it is then exactly equation (1.2.9) from before. Assuming that it is verified for some $l \geq 1$, we can write, for Z some element of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S^{l+1}(\mathbb{C}^2)$,

$$\begin{aligned}
& \mathbf{R}_{12}(x)\mathbf{R}_{13}^{(1,l+1)}(x+y)\mathbf{R}_{23}^{(1,l+1)}(y)Z \\
&= \mathbf{R}_{12}(x)\frac{\mathbf{R}_{13'}^{(1,1)}\left(x+y-\frac{l}{2}\right)\mathbf{R}_{13''}^{(1,l)}\left(x+y+\frac{1}{2}\right)\mathbf{R}_{23'}^{(1,1)}\left(y-\frac{l}{2}\right)\mathbf{R}_{23''}^{(1,l)}\left(y+\frac{1}{2}\right)}{\left(x+y+\frac{2-l}{2}\right)\left(y+\frac{2-l}{2}\right)}Z \\
&= \mathbf{R}_{12}(x)\frac{\mathbf{R}_{13'}^{(1,1)}\left(x+y-\frac{l}{2}\right)\mathbf{R}_{23'}^{(1,1)}\left(y-\frac{l}{2}\right)\mathbf{R}_{13''}^{(1,l)}\left(x+y+\frac{1}{2}\right)\mathbf{R}_{23''}^{(1,l)}\left(y+\frac{1}{2}\right)}{\left(x+y+\frac{2-l}{2}\right)\left(y+\frac{2-l}{2}\right)}Z \\
&= \frac{\mathbf{R}_{13'}^{(1,1)}\left(x+y-\frac{l}{2}\right)\mathbf{R}_{23'}^{(1,1)}\left(y-\frac{l}{2}\right)\mathbf{R}_{13''}^{(1,l)}\left(x+y+\frac{1}{2}\right)\mathbf{R}_{23''}^{(1,l)}\left(y+\frac{1}{2}\right)}{\left(x+y+\frac{2-l}{2}\right)\left(y+\frac{2-l}{2}\right)}\mathbf{R}_{12}(x)Z \\
&= \frac{\mathbf{R}_{13'}^{(1,1)}\left(x+y-\frac{l}{2}\right)\mathbf{R}_{13''}^{(1,l)}\left(x+y+\frac{1}{2}\right)\mathbf{R}_{23'}^{(1,1)}\left(y-\frac{l}{2}\right)\mathbf{R}_{23''}^{(1,l)}\left(y+\frac{1}{2}\right)}{\left(x+y+\frac{2-l}{2}\right)\left(y+\frac{2-l}{2}\right)}\mathbf{R}_{12}(x)Z \\
&= \mathbf{R}_{23}^{(1,l+1)}(y)\mathbf{R}_{13}^{(1,l+1)}(x+y)\mathbf{R}_{12}(x)Z.
\end{aligned}$$

On the first and last lines we have operators acting on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S^{l+1}(\mathbb{C}^2)$, on all the intermediate lines we have operators acting on a larger space, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes S^l(\mathbb{C}^2)$, which contains the previous one, so that we can still act on Z . The indices of the R-matrices indicate the factors on which they act non-trivially, $3'$ being the third copy of \mathbb{C}^2 while $3''$ stands for $S^l(\mathbb{C}^2)$. We have used the fact that R-matrices with different indices commute among themselves, and, in going from the third to the fourth line, that the Yang–Baxter relation holds for rank-1 and rank- l symmetric tensors.

We use these newly obtained R-matrices to construct new transfer matrices. The procedure is similar to the one for the fundamental transfer matrix save that the auxiliary space \mathbb{V}_0 is now taken to be some $S^l(\mathbb{C}^2)$. More precisely, for any positive integer l , we define

$$\mathbf{T}_l(x) = \text{Tr}_0 \left[\mathbf{R}_{L0}^{(1,l)}(x) \dots \mathbf{R}_{20}^{(1,l)}(x) \mathbf{R}_{10}^{(1,l)}(x) \right]. \quad (1.2.31)$$

Of course, $\mathbf{T}_1 = \mathbf{T}$. The Yang–Baxter relation (1.2.29) ensures that these new transfer matrices commute with the original one:

$$[\mathbf{T}(x), \mathbf{T}_l(y)] = 0. \quad (1.2.32)$$

They thus also commute with the Hamiltonian and give some new conserved charges. One could actually show that they commute among themselves

$$[\mathbf{T}_l(x), \mathbf{T}_{l'}(y)] = 0. \quad (1.2.33)$$

This requires constructing some R-matrix $\mathbf{R}^{(l,l')}$ and showing that it satisfies the Yang–Baxter equation with $\mathbf{R}^{(1,l)}$ and $\mathbf{R}^{(1,l')}$, but we will not do it here.

1.2.3 Higher-Rank Symmetry

Until now, we have only considered a model with $GL(2)$ symmetry. It is however possible to exhibit R-matrices, that is to say solutions of the Yang–Baxter equation, with different symmetry groups. Let us present here the R-matrices in the defining representations of the classical groups.

- For the complex general linear group $GL(N)$, the only invariant operators on $\mathbb{C}^N \otimes \mathbb{C}^N$ being the identity and the permutation operators, the R-matrix is the same as in the $N = 2$ case, namely

$$R^{GL(N)}(x) = x \text{Id} + \mathbb{P}. \quad (1.2.34)$$

It is sometimes referred to as Yang’s R-matrix since it appeared for the first time in Yang’s study [18] of the extension of the Lieb–Liniger model to distinguishable particles.

- For the orthogonal group

$$O(N) = \left\{ M \in GL(N) \mid {}^t M M = I_N \right\}, \quad (1.2.35)$$

there is one more invariant operator, \mathbb{K} , defined by $\mathbb{K}_{i_1 i_2}^{j_1 j_2} = \delta_{i_1 i_2} \delta^{j_1 j_2}$. It satisfies

$$\mathbb{K}^2 = N \mathbb{K}, \quad \mathbb{P} \mathbb{K} = \mathbb{K} \mathbb{P} = \mathbb{K}. \quad (1.2.36)$$

The R-matrix is then given by

$$R^{O(N)}(x) = x(x + \kappa) \text{Id} + (x + \kappa) \mathbb{P} - x \mathbb{K}, \quad (1.2.37)$$

where $\kappa = \frac{N-2}{2}$. It is sometimes referred to as the Zamolodchikovs’ R-matrix since it was found in [120].

- For the symplectic group (in that case N is even)

$$Sp(N) = \left\{ M \in GL(N) \mid {}^t M S M = S \right\} \quad \text{with} \quad S = \begin{pmatrix} 0 & I_{\frac{N}{2}} \\ -I_{\frac{N}{2}} & 0 \end{pmatrix}, \quad (1.2.38)$$

there exists also a third invariant operator $\bar{\mathbb{K}}$ defined by $\bar{\mathbb{K}}_{i_1 i_2}^{j_1 j_2} = S_{i_1 i_2} S^{j_1 j_2}$. It satisfies

$$\bar{\mathbb{K}}^2 = N \bar{\mathbb{K}}, \quad \mathbb{P} \bar{\mathbb{K}} = \bar{\mathbb{K}} \mathbb{P} = -\bar{\mathbb{K}}, \quad (1.2.39)$$

and the R-matrix is given by [186]

$$R^{Sp(N)}(x) = x(x + \bar{\kappa}) \text{Id} + (x + \bar{\kappa}) \mathbb{P} - x \bar{\mathbb{K}}, \quad (1.2.40)$$

where $\bar{\kappa} = \frac{N+2}{2}$.

In each case, the R-matrix that is given is a solution of the Yang–Baxter equation

$$R_{12}^{\mathcal{G}}(x) R_{13}^{\mathcal{G}}(x+y) R_{23}^{\mathcal{G}}(y) = R_{23}^{\mathcal{G}}(y) R_{13}^{\mathcal{G}}(x+y) R_{12}^{\mathcal{G}}(x). \quad (1.2.41)$$

Once we have a solution, we can multiply it by an arbitrary scalar function $R(x) \rightarrow f(x)R(x)$ and get another solution. But if we only look for solutions of the Yang–Baxter equation, such prefactors are inconsequential. They become relevant, however, when the R-matrix is actually the S-matrix of some quantum field theory because then there exist other constraints on the S-matrix, such as unitarity or crossing symmetry, that can only be satisfied thanks to some properly chosen scalar prefactor.

We have also seen that, once we have a solution of the Yang–Baxter equation, it is interesting to look for new R-matrices, acting in different representations of the symmetry group, consistent with the original one. More precisely, we look for a representation ρ of the symmetry group on a vector space \mathbb{V} and an operator $R^{\mathcal{G},\rho}(x)$ on $\mathbb{C}^N \otimes \mathbb{V}$ which is \mathcal{G} -invariant:

$$\forall G \in \mathcal{G}, \quad [R^{\mathcal{G},\rho}(x), G \otimes \rho(G)] = 0, \quad (1.2.42)$$

and satisfies the Yang–Baxter equation

$$R_{12}^{\mathcal{G}}(x)R_{13}^{\mathcal{G},\rho}(x+y)R_{23}^{\mathcal{G},\rho}(y) = R_{23}^{\mathcal{G},\rho}(y)R_{13}^{\mathcal{G},\rho}(x+y)R_{12}^{\mathcal{G}}(x), \quad (1.2.43)$$

where as usual the indices indicate on which of the three factors in $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{V}$ the operators act non trivially. When $\mathcal{G} = GL(N)$, there exists such an R-matrix for any representation ρ , it suffices indeed to imitate the definition of the permutation operator in terms of generators, $\mathbb{P} = e_{ij} \otimes e_{j,i}$, and to set

$$R^{GL(N),\rho}(x) = x \text{Id} + e_{ij} \otimes E_{ji}^{\rho}, \quad (1.2.44)$$

where E_{ji}^{ρ} are the images of the generators in the representation ρ . If we consider in particular the irreducible representation ρ_l on $S^l(\mathbb{C}^N)$, it is clear that this definition coincides with the one given earlier in (1.2.28) (the formula for $N > 2$ being exactly the same as the one for $N = 2$) up to a shift:

$$R^{GL(N),\rho_l}\left(x + \frac{1-l}{2}\right) = R^{(1,l)}(x). \quad (1.2.45)$$

For the other groups, it is not that easy and the R-matrices actually exist only for some specific representations, the Kirillov–Reshetikhin modules [53], of the associated Yangian or extended Yangian. We also point out that such R-matrices, acting in the tensor product of two different representations, are sometimes called L-matrices.

We can then go on and look for \mathcal{G} -invariant R-matrices acting in two representations ρ and ρ' different from the defining one. In that case, all the Yang–Baxter equations that can be written should hold:

$$R_{12}^{\mathcal{G},\rho\rho'}(x)R_{13}^{\mathcal{G},\rho\rho''}(x+y)R_{23}^{\mathcal{G},\rho'\rho''}(y) = R_{23}^{\mathcal{G},\rho'\rho''}(y)R_{13}^{\mathcal{G},\rho\rho''}(x+y)R_{12}^{\mathcal{G},\rho\rho'}(x) \quad (1.2.46)$$

for any three representations for which the R-matrices have been constructed.

In the following, whenever possible without ambiguity, and in order to simplify notations, we will refrain from indicating the symmetry group of the R-matrix. The three R-matrices presented at the beginning of this subsection would thus all be denoted $R(x)$.

1.3 Nested Bethe Ansatz Equations

1.3.1 The $GL(N)$ Heisenberg Spin Chain

We now consider the Heisenberg spin chain with a higher-rank symmetry group. The Hamiltonian is still given by (1.1.1) but it is now defined on $\mathcal{H} = (\mathbb{C}^N)^{\otimes L}$ for some $N \geq 3$, i.e. the symmetry group is now $GL(N)$.

Vectors of the form $v^{\otimes L}$ for arbitrary $v \in \mathbb{C}^N$ are still eigenvectors with zero eigenvalue. If $\{e_1, \dots, e_N\}$ is an orthonormal basis, then we define the vacuum $|\emptyset\rangle = e_N^{\otimes L}$. Turning our attention to excited states, the first thing to notice is that, in contradistinction to the $N = 2$ case, there are now several ways of inserting excitations at a given site: there are $N - 1$ flavours represented by e_1, \dots, e_{N-1} . If i_1, \dots, i_M are distinct integers between 1 and L and a_1, \dots, a_M are all between 1 and $N - 1$ (not necessarily distinct), we will write $|i_1, \dots, i_M\rangle_{a_1 \dots a_M}$ for the state with e_{a_j} at site i_j and e_N at all the other sites. If $C \in (\mathbb{C}^{N-1})^{\otimes M}$ is a rank- M tensor, then we define

$$C \cdot |i_1, \dots, i_M\rangle = C^{a_1 \dots a_M} |i_1, \dots, i_M\rangle_{a_1 \dots a_M}. \quad (1.3.1)$$

One-particle eigenstates are of the form

$$|p; C\rangle = \sum_{j=1}^L e^{ipj} C \cdot |j\rangle \quad \text{with} \quad e^{iLp} = 1 \quad (1.3.2)$$

and C an arbitrary vector in \mathbb{C}^{N-1} . They satisfy $H|p; C\rangle = 2(1 - \cos p)|p; C\rangle = \varepsilon(p)|p; C\rangle$.

For a higher number of particles, we generalise the ansatz (1.1.20)-(1.1.21) and look for eigenvectors of the form

$$|p_1, \dots, p_M; C\rangle = \sum_{j_1 < \dots < j_M} \psi_{\mathbf{p}}(j_1, \dots, j_M; C) \cdot |j_1, \dots, j_M\rangle \quad (1.3.3)$$

with a wave function given by

$$\psi_{\mathbf{p}}(j_1, \dots, j_M; C) = \sum_{\sigma \in \mathfrak{S}_M} e^{i \sum_{k=1}^M p_k j_{\sigma(k)}} \check{\mathfrak{S}}(\sigma) C, \quad (1.3.4)$$

where C is a rank- M tensor, and $\check{\mathfrak{S}}(\sigma) : (\mathbb{C}^{N-1})^{\otimes M} \rightarrow (\mathbb{C}^{N-1})^{\otimes M}$ are operators to be determined. Once again, one can derive two types of conditions on these operators by requiring that $|p_1, \dots, p_M; C\rangle$ be an eigenvector of both \mathcal{P} and H . The first of these are periodicity conditions, they are

$$\mathcal{P}^{(N-1)} \psi_{\mathbf{p}}(j_1, \dots, j_M + L; C) = \psi_{\mathbf{p}}(j_M, j_1, \dots, j_{M-1}; C), \quad (1.3.5)$$

where $\mathcal{P}^{(N-1)}$ is the analogue of \mathcal{P} acting on $(\mathbb{C}^{N-1})^{\otimes M}$. If the momenta are in generic position, this is equivalent to

$$\check{\mathfrak{S}}((1 \dots M)\sigma) C = e^{ip_{\sigma^{-1}(M)}L} \mathcal{P}^{(N-1)} \check{\mathfrak{S}}(\sigma) C \quad (1.3.6)$$

for all permutations $\sigma \in \mathfrak{S}_M$. The second conditions are, as before, much more cumbersome to write explicitly because there are many unwanted terms that arise when acting on

$|p_1, \dots, p_M; C\rangle$ with H . Considering those that involve $|j_1, j_1 + 1, j_3, \dots, j_M\rangle$, with the indices strictly ordered and such that only j_1 and $j_1 + 1$ are neighbours, one obtains the condition

$$\begin{aligned} \psi_{\mathbf{p}}(j_1, j_1, j_3, \dots, j_M; C) + \psi_{\mathbf{p}}(j_1 + 1, j_1 + 1, j_3, \dots, j_M; C) \\ = (\text{Id} + \mathbb{P}_{12}^{(N-1)})\psi_{\mathbf{p}}(j_1, j_1 + 1, j_3, \dots, j_M; C), \end{aligned} \quad (1.3.7)$$

where $\mathbb{P}_{12}^{(N-1)}$ is the permutation operator acting on vectors with $N - 1$ components (not to be confused with $\mathbb{P}_{12}^{(N)} = \mathbb{P}_{12}$ appearing in the Hamiltonian). Inserting the form of the ansatz, one sees that both sides are linear combinations of functions $e^{i(p_{\sigma^{-1}(1)} + p_{\sigma^{-1}(2)})j_1 + i \sum_{k=3}^L p_{\sigma^{-1}(k)} j_k}$. For any permutation σ , there is exactly one other permutation, namely $(12)\sigma$, for which the same function appears. Grouping those terms together, and after some straightforward algebra, one shows that

$$\check{\mathbb{S}}((12)\sigma)C = \check{\mathbb{S}}_{12}(p_{\sigma^{-1}(1)}, p_{\sigma^{-1}(2)})\check{\mathbb{S}}(\sigma)C \quad (1.3.8)$$

must hold. We have introduced the two-body S-matrix

$$\check{\mathbb{S}}_{ij}(p_i, p_j) = \frac{(e^{ip_j} - e^{ip_i})\mathbb{P}_{ij}^{(N-1)} - (e^{ip_i} - 1)(e^{ip_j} - 1)}{1 + e^{i(p_i + p_j)} - 2e^{ip_i}}, \quad (1.3.9)$$

which, in terms of the rapidities $u_i = \frac{1}{2} \cot(\frac{p_i}{2})$, reads more simply

$$\check{\mathbb{S}}_{ij}(p_i, p_j) = \check{\mathbb{S}}_{ij}(u_i - u_j) = \frac{(u_i - u_j)\mathbb{P}_{ij}^{(N-1)} - i}{u_i - u_j + i} = \frac{\mathbb{P}_{ij}^{(N-1)}\mathbf{R}(i(u_i - u_j))}{i(u_i - u_j) - 1}, \quad (1.3.10)$$

where we have recognised the $GL(N - 1)$ -invariant R-matrix (1.2.34). We point out that the S-matrix is unitary

$$\check{\mathbb{S}}_{ij}(p_j, p_i)\check{\mathbb{S}}_{ij}(p_i, p_j) = \text{Id}. \quad (1.3.11)$$

It is clear that, besides (1.3.8), the operators $\check{\mathbb{S}}(\sigma)$ have to satisfy similar conditions for which (12) is replaced with any $(kk + 1)$ for $k \in \{1, \dots, M - 1\}$. All these conditions can be satisfied at once if we define the operators in the following way: if $\sigma = (k_1 k_1 + 1) \cdots (k_l k_l + 1)$, then we set

$$\check{\mathbb{S}}(\sigma) = \check{\mathbb{S}}_{k_1 k_1 + 1} \cdots \check{\mathbb{S}}_{k_l k_l + 1}, \quad (1.3.12)$$

where it is understood that the arguments of $\check{\mathbb{S}}_{kk+1}$ depend on the operators to the right of it. Because the S-matrix is unitary (an equation which now reads $\check{\mathbb{S}}_{ij}\check{\mathbb{S}}_{ij} = \text{Id}$) and satisfies Yang–Baxter relation, this definition is unambiguous. One has for instance

$$\check{\mathbb{S}}((13)) = \check{\mathbb{S}}_{12}(p_2, p_3)\check{\mathbb{S}}_{23}(p_1, p_3)\check{\mathbb{S}}_{12}(p_1, p_2) = \check{\mathbb{S}}_{23}(p_1, p_2)\check{\mathbb{S}}_{12}(p_1, p_3)\check{\mathbb{S}}_{23}(p_2, p_3). \quad (1.3.13)$$

Furthermore, with this definition, one can write

$$\check{\mathbb{S}}(\sigma)\check{\mathbb{S}}(\tau) = \check{\mathbb{S}}(p_{\tau^{-1}(1)}, \dots, p_{\tau^{-1}(M)}; \sigma)\check{\mathbb{S}}(p_1, \dots, p_M; \tau) = \check{\mathbb{S}}(p_1, \dots, p_M; \sigma\tau) = \check{\mathbb{S}}(\sigma\tau), \quad (1.3.14)$$

where, one last time, we have displayed the explicit arguments of the operators. In particular, this means that our putative eigenvectors satisfy

$$|p_1, \dots, p_M; C\rangle = |p_{\sigma^{-1}(1)}, \dots, p_{\sigma^{-1}(M)}; \check{\mathbb{S}}(\sigma)C\rangle \quad (1.3.15)$$

for all permutations $\sigma \in \mathfrak{S}_M$. It turns out that for this choice of operators $\check{S}(\sigma)$, and if (1.3.6) holds, then

$$H |p_1, \dots, p_M; C\rangle = \left(\sum_{k=1}^M \varepsilon(p_k) \right) |p_1, \dots, p_M; C\rangle . \quad (1.3.16)$$

The periodicity conditions (1.3.6) for permutations that are powers of $(1 \cdots M)$ can be rewritten using the explicit form of $\check{S}((1 \cdots M))$, they are then equivalent to

$$S_{kk-1}(u_k - u_{k-1}) \cdots S_{k1}(u_k - u_1) S_{kM}(u_k - u_M) \cdots S_{kk+1}(u_k - u_{k+1}) C = e^{-ip_k L} C \quad (1.3.17)$$

for all $k \in \{1, \dots, M\}$, with $S_{ij} = \mathbb{P}_{ij}^{(N-1)} \check{S}_{ij}$. This is nothing but requiring that C be an eigenvector of the $GL(N-1)$ inhomogeneous spin chain of length M with transfer matrix

$$T(u_1, \dots, u_M; u) = \text{Tr}_0 [S_{0M}(u - u_M) \cdots S_{01}(u - u_1)] . \quad (1.3.18)$$

It indeed trivially holds that

$$T(u_1, \dots, u_M; u_k) = -S_{kk-1}(u_k - u_{k-1}) \cdots S_{k1}(u_k - u_1) S_{kM}(u_k - u_M) \cdots S_{kk+1}(u_k - u_{k+1}) . \quad (1.3.19)$$

In that way we managed to reduce our original problem to a simpler one since the new spin chain is shorter and has a smaller symmetry group. The natural way to proceed is to make a new ansatz for C , similar to (1.3.3) and (1.3.4), and to repeat what has just been done. The only new feature is that one now needs to compute the eigenvalue of the transfer matrix $T(u_1, \dots, u_M; u)$ in order to write the conditions (1.3.17) requiring it to be equal to $e^{-ip_k L} = \left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}} \right)^L$. This can be achieved through the so-called algebraic Bethe ansatz, we shall not explain it here and only present the resulting Bethe equations.

1.3.2 Transfer Matrix Eigenvalues

The authors of [187] found expressions for the transfer matrix eigenvalues valid for more general $GL(r+1)$ -invariant spin chains. Let us give them here for completeness. The spin chain is of length L with quantum space $\mathcal{H} = \mathbb{V}_{\lambda(1)} \otimes \cdots \otimes \mathbb{V}_{\lambda(L)}$, where \mathbb{V}_λ is the space of the finite-dimensional irreducible representation with highest weight $\lambda = \sum_{i=1}^{r+1} \lambda_i \varepsilon_i$ (see Appendix A.1 for an explicit description of the weights, roots, etc.). We also introduce a (constant) diagonal twist matrix $D = \text{Diag}(\tau_1, \dots, \tau_{r+1})$ and some inhomogeneities x_1, \dots, x_L . The transfer matrix of interest is the one with the defining representation in the auxiliary space:

$$T(x_1, \dots, x_L; x) = \text{Tr}_0 \left[D R_{0L}^{\rho_{\lambda(L)}}(x - x_L) \cdots R_{01}^{\rho_{\lambda(1)}}(x - x_1) \right] , \quad (1.3.20)$$

with $R^{\rho_\lambda}(x) = x \text{Id} + e_{ij} \otimes E_{ji}^{\rho_\lambda}$ the $GL(r+1)$ -invariant R-matrix acting on $\mathbb{C}^r \otimes \mathbb{V}_\lambda$ mentioned in Section 1.2.3. The first thing to notice is that, in contradistinction to what we have studied in the previous chapters, the presence of a twist breaks the global $GL(r+1)$ symmetry down to the centraliser of the twist matrix D . For twists in generic position, this is simply the Cartan subgroup of diagonal matrices.

The transfer matrix eigenvalues are characterised by r polynomials $q_a(x) = \prod_{j=1}^{K_a} (x - y_{a,j})$, and given by²

$$T(x_1, \dots, x_L; x) = \tau_1 \prod_{i=1}^L (x - x_i + \lambda_1^{(i)}) \frac{q_1^{[-1]}(x)}{q_1^{[+1]}(x)} + \sum_{a=2}^r \tau_a \prod_{i=1}^L (x - x_i + \lambda_a^{(i)}) \frac{q_{a-1}^{[a+1]}(x) q_a^{[a-2]}(x)}{q_{a-1}^{[a-1]}(x) q_a^{[a]}(x)} + \tau_{r+1} \prod_{i=1}^L (x - x_i + \lambda_{r+1}^{(i)}) \frac{q_r^{[r+2]}(x)}{q_r^{[r]}(x)}, \quad (1.3.21)$$

where $q^{[k]}(x) = q\left(x + \frac{k}{2}\right)$. The roots of the polynomials, called Bethe roots, have to satisfy the Bethe equations

$$-\frac{\tau_1}{\tau_2} \prod_{i=1}^L \frac{y_{1,j} - x_i + \lambda_1^{(i)} - \frac{1}{2}}{y_{1,j} - x_i + \lambda_2^{(i)} - \frac{1}{2}} = \frac{q_1^{[2]}(y_{1,j}) q_2^{[-1]}(y_{1,j})}{q_1^{[-2]}(y_{1,j}) q_2^{[+1]}(y_{1,j})} \quad \text{for } 1 \leq j \leq K_1, \quad (1.3.22)$$

$$-\frac{\tau_a}{\tau_{a+1}} \prod_{i=1}^L \frac{y_{a,j} - x_i + \lambda_a^{(i)} - \frac{a}{2}}{y_{a,j} - x_i + \lambda_{a+1}^{(i)} - \frac{a}{2}} = \frac{q_{a-1}^{[-1]}(y_{a,j}) q_a^{[+2]}(y_{a,j}) q_{a+1}^{[-1]}(y_{a,j})}{q_{a-1}^{[+1]}(y_{a,j}) q_a^{[-2]}(y_{a,j}) q_{a+1}^{[+1]}(y_{a,j})} \quad (1.3.23)$$

for $2 \leq a \leq r-1$ and $1 \leq j \leq K_a$, and

$$-\frac{\tau_r}{\tau_{r+1}} \prod_{i=1}^L \frac{y_{r,j} - x_i + \lambda_r^{(i)} - \frac{r}{2}}{y_{r,j} - x_i + \lambda_{r+1}^{(i)} - \frac{r}{2}} = \frac{q_{r-1}^{[-1]}(y_{r,j}) q_r^{[2]}(y_{r,j})}{q_{r-1}^{[+1]}(y_{r,j}) q_r^{[-2]}(y_{r,j})} \quad \text{for } 1 \leq j \leq K_r. \quad (1.3.24)$$

It is notable that, even though this is not how they were obtained, these equations can be recovered from the final form (1.3.21) of the eigenvalue solely by requiring it to have no pole.

Since the transfer matrix commutes with the (global) action of the Cartan subgroup of $GL(r+1)$, its eigenvectors have definite weights. For the eigenvector associated to the eigenvalue (1.3.21), the weight is given by

$$\lambda = \sum_{i=1}^L \lambda^{(i)} - \sum_{a=1}^{r+1} (K_a - K_{a-1}) \varepsilon_a, \quad \text{where } K_0 = K_{r+1} = 0. \quad (1.3.25)$$

This can be read off from the behaviour of the transfer matrix when $x \rightarrow +\infty$. On the one hand, the definition as the trace of the twisted monodromy matrix indeed implies that

$$T(x_1, \dots, x_L; x) = x^L \text{Tr}(D) + x^{L-1} \left(\sum_{a=1}^{r+1} \tau_a E_{aa} - \text{Tr}(D) \sum_{i=1}^L x_i \right) + \dots, \quad (1.3.26)$$

where $E_{ij} = \sum_{i=1}^L E_{ij}^{\lambda^{(i)}}$ are the generators of the global action of $GL(r+1)$. On the other hand, the asymptotic behaviour of the function (1.3.21) is simply

$$T(x_1, \dots, x_L; x) = x^L \text{Tr}(D) + x^{L-1} \left(\sum_{a=1}^{r+1} \tau_a \left(\sum_{i=1}^L \lambda_a^{(i)} + K_{a-1} - K_a \right) - \text{Tr}(D) \sum_{i=1}^L x_i \right) + \dots \quad (1.3.27)$$

Matching the previous two equations gives exactly the weight (1.3.25) for the eigenvector.

²The roots of [187] were called $u_j^{(a)}$, ours differ from theirs by a constant shift: $y_{a,j} = u_j^{(a)} + \frac{a}{2}$.

Chapter 2

From $\mathcal{N} = 4$ Super Yang–Mills to The Fishnet Theory

We are going to review, in this chapter, the derivation of the fishnet theory in four dimensions as a double-scaling limit of the $\mathcal{N} = 4$ superconformal Yang–Mills (SYM) theory, and its generalisation to arbitrary dimension. The rudiments of $\mathcal{N} = 4$ SYM recalled in the first section, as well as much more information on this theory, can be found in numerous references such as the thesis [35] and the review [36]. The content presented in the last two sections is also reviewed in [188].

2.1 $\mathcal{N} = 4$ Super Yang–Mills

2.1.1 Fields and Lagrangian

$\mathcal{N} = 4$ SYM [189, 190] is a gauge theory with gauge boson A , six real scalar fields ϕ_m and four Weyl spinors ψ_a . All these fields are in the Lie algebra of the gauge group which will be taken to be $SU(N_c)$, they are thus $N_c \times N_c$ traceless Hermitian matrices. They have, however, different transformation properties under a gauge transformation $U(x) \in SU(N_c)$. Namely, the fermions and scalars transform in the adjoint representation, but the gauge boson does not:

$$\phi_m \mapsto U\phi_m U^{-1}, \quad \psi_a \mapsto U\psi_a U^{-1}, \quad A_\mu \mapsto UA_\mu U^{-1} - ig^{-1}(\partial_\mu U)U^{-1}, \quad (2.1.1)$$

where we have introduced a dimensionless coupling constant g . We define the covariant derivative D_μ on any field W by

$$D_\mu W = \partial_\mu W - ig[A_\mu, W]. \quad (2.1.2)$$

This in turn serves to define the field strength F as the commutator of two covariant derivatives:

$$[D_\mu, D_\nu]W = -ig[F_{\mu\nu}, W] \Leftrightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \quad (2.1.3)$$

It transforms in the adjoint representation of the gauge group. We can now write the Lagrangian of the $\mathcal{N} = 4$ super Yang–Mills theory:

$$\begin{aligned} \mathcal{L}_{YM} = -\text{Tr} \left(\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + D^\mu \phi^m D_\mu \phi_m - \frac{g^2}{2} [\phi^m, \phi^n] [\phi_m, \phi_n] \right. \\ \left. + 2\bar{\psi}^a \bar{\sigma}^\mu D_\mu \psi_a - ig \sigma_m^{ab} \psi_{a\alpha} \varepsilon^{\alpha\beta} [\phi^m, \psi_{b\beta}] - ig \bar{\sigma}_{ab}^m \bar{\psi}_\alpha^a \varepsilon^{\dot{\alpha}\dot{\beta}} [\phi_m, \bar{\psi}_\beta^b] \right), \end{aligned} \quad (2.1.4)$$

where ε is the totally antisymmetric tensor of $\mathfrak{su}(2)$ with $\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = 1$, and $\sigma^\mu, \bar{\sigma}^\mu, \sigma^m, \bar{\sigma}^m$ are the chiral projections of the gamma matrices in four and six dimensions (see Appendix A.2.2 for an example of a construction of such matrices). They satisfy the following relations:

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2\eta^{\mu\nu} \mathbf{I}_2, \quad (2.1.5)$$

$$\sigma^m \bar{\sigma}^n + \sigma^n \bar{\sigma}^m = \bar{\sigma}^m \sigma^n + \bar{\sigma}^n \sigma^m = 2\delta^{mn} \mathbf{I}_4, \quad (2.1.6)$$

where we recall that \mathbf{I}_n denotes the $n \times n$ identity matrix.

The equations of motion are

$$D^\mu D_\mu \phi_m = g^2 [\phi_n, [\phi^n, \phi_m]] + \frac{ig}{2} \sigma_m^{ab} \varepsilon^{\alpha\beta} \{\psi_{a\alpha}, \psi_{b\beta}\} + \frac{ig}{2} \bar{\sigma}_{ab}^m \varepsilon^{\dot{\alpha}\dot{\beta}} \{\bar{\psi}_\alpha^a, \bar{\psi}_\beta^b\}, \quad (2.1.7)$$

$$D_\nu F^{\mu\nu} = ig [\phi_m, D^\mu \phi^m] - ig \bar{\sigma}^{\mu,\dot{\alpha}\dot{\beta}} \{\bar{\psi}_\alpha^a, \psi_{a\beta}\}, \quad (2.1.8)$$

$$\bar{\sigma}^{\mu,\dot{\alpha}\dot{\beta}} D_\mu \psi_{a\beta} = ig \bar{\sigma}_{ab}^m \varepsilon^{\dot{\alpha}\dot{\beta}} [\phi_m, \bar{\psi}_\beta^b]. \quad (2.1.9)$$

The action is simply the integral of the Lagrangian density:

$$S_{YM} = \frac{1}{g_{YM}^2} \int \mathcal{L}_{YM}[g=1] d^4x. \quad (2.1.10)$$

It will prove convenient to define $\lambda = g_{YM}^2 N_c$ and to rescale the fields $W \mapsto \lambda W$ such that the action becomes

$$S_{YM} = N_c \int \mathcal{L}_{YM}[g=\lambda] d^4x. \quad (2.1.11)$$

2.1.2 Symmetries

There is a global $SU(4)$ symmetry, called an R-symmetry: the fermions transform in the defining representation, the gauge boson transforms in the trivial one, whereas the scalars transform in the defining representation of $SO(6, \mathbb{R})$ (remember that $SU(4)$ is the universal cover of $SO(6, \mathbb{R})$). More precisely, the Lagrangian is unchanged if we perform the following substitutions:

$$\psi_a \mapsto U_{ab} \psi_b, \quad \bar{\psi}^a \mapsto U_{ab}^* \bar{\psi}^b, \quad \phi_m \mapsto O(U)_{mn} \phi_n, \quad (2.1.12)$$

where $U \in SU(4)$ is arbitrary and $O(U)$ is the (unique) element of $SO(6, \mathbb{R})$ such that $U^* \sigma^m U^\dagger = \sigma^n O(U)_{nm}$.

It can be shown that the theory is also invariant under the action of the $\mathcal{N} = 4$ super Poincaré algebra generated by P_μ for translations, $M_{\mu\nu}$ for Lorentz transformations and

sixteen supercharges Q_α^a and $\bar{Q}_{\dot{\alpha}a}$. In particular, the supercharges generate the following transformation:

$$[Q^a, \phi_m] = \sigma_m^{ab} \psi_b, \quad (2.1.13)$$

$$\{Q_\alpha^a, \psi_{b\beta}\} = -\frac{1}{2} \sigma_{\alpha\dot{\gamma}}^\mu \bar{\sigma}^{\nu\dot{\gamma}\gamma} \varepsilon_{\gamma\beta} \delta_b^a F_{\mu\nu} + \frac{ig}{2} \varepsilon_{\beta\alpha} \bar{\sigma}_{bc}^m \sigma_n^{ca} [\phi_m, \phi^n], \quad (2.1.14)$$

$$\{Q_\alpha^a, \bar{\psi}_{\dot{\beta}}^b\} = \sigma_m^{ba} \sigma_{\alpha\dot{\beta}}^\mu D_\mu \phi^m, \quad (2.1.15)$$

$$[Q^a, A_\mu] = \sigma_\mu \bar{\psi}^a. \quad (2.1.16)$$

Furthermore, the classical scale invariance of the action (all fields are massless and the coupling is dimensionless) survives at the quantum level [191]. In other words, the beta function vanishes, meaning that the theory also has conformal symmetry, which introduces a dilatation generator as well as four special conformal transformation generators. Joined with the previous supercharges, these conformal generators imply the existence of sixteen new supercharges. All these symmetries combine into the full symmetry algebra of $\mathcal{N} = 4$ super Yang–Mills: $\mathfrak{psu}(2, 2|4)$. Apart from the Lorentz and internal symmetries, which are manifestly realised in the quantum theory, the generators of the superconformal algebra receive quantum corrections and as such depend on the coupling constant.

2.1.3 The Planar Limit

It was shown by 't Hooft [192] that if one considers the following limit of the theory

$$g_{YM}^2 \rightarrow 0, \quad N_c \rightarrow +\infty, \quad \lambda = g_{YM}^2 N_c \text{ fixed}, \quad (2.1.17)$$

then only the Feynman graphs with planar topology survive, i.e. those graphs that can be drawn on the plane without crossing lines. This comes from the fact that the planar topology is the one that maximises the number of closed gauge index loops for a given number of vertices. Let us repeat here the short derivation of this fact. First of all, one should notice that because all of the fields transform under the adjoint representation of $SU(N_c)$, they carry two gauge indices. A propagator always identifies the two indices of one field with the two indices of another. A trace involving more than two fields, on the other hand, identifies the indices of one field with those of two different fields. The interaction vertices correspond to such traces.

Let us only consider correlation functions of local gauge-invariant operators, meaning that we consider operators that are products of traces of products of fields and covariant derivatives of fields. Now, each closed index loop of the graph will contribute a factor $\sum_{i=1}^{N_c} \delta_{ii} = N_c$ while, because of the form of the action, each propagator comes with a factor N_c^{-1} and each vertex with a factor N_c . Consequently, if a given graph contains L index loops, P propagators, and V vertices then the corresponding factor will be N_c^{L+V-P} . But the Euler relation states that

$$L + V - P = 2C - 2G - T, \quad (2.1.18)$$

with C the number of connected components of the graph, G the minimal genus of a surface on which the graph can be drawn without lines intersecting, and T the number of traces in

the operators (T is the number of vertices that do not contribute a factor of N_c). This means that the leading contributions come from graphs for which each connected component has planar topology.

In view of the overwhelming evidence (see [35–39], and references therein), there is no doubt that $\mathcal{N} = 4$ SYM is integrable in the planar limit. However, the origin of this integrability is not completely clear yet.

2.2 The γ -Deformation of $\mathcal{N} = 4$ SYM

2.2.1 Definition

The deformation of $\mathcal{N} = 4$ SYM we are about to define dates back to [193–195].

We first group the six real scalars into three complex scalars. If the matrices σ^m and $\bar{\sigma}^m$ are conveniently chosen, one may do it in the following way:

$$\varphi_1 = \phi_1 + i\phi_2, \quad \varphi_2 = \phi_3 + i\phi_4, \quad \varphi_3 = \phi_5 + i\phi_6. \quad (2.2.1)$$

The weight vectors $\mathbf{q}_B = (q_B^1, q_B^2, q_B^3)$ of each of the fundamental fields B with respect to the internal $\mathfrak{su}(4)$ symmetry are given by

B	ψ_1	ψ_2	ψ_3	ψ_4	A	φ_1	φ_2	φ_3
q_B^1	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	1	0	0
q_B^2	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	1	0
q_B^3	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	1

Roughly speaking, the Lagrangian of the γ -deformed theory is obtained by replacing all products BC of two fields with a $*$ -product defined by

$$B * C = e^{\frac{i}{2}\mathbf{q}_B \wedge \mathbf{q}_C} BC, \quad (2.2.2)$$

where

$$\mathbf{q}_B \wedge \mathbf{q}_C = -\det(\mathbf{q}_B, \mathbf{q}_C, \boldsymbol{\gamma}) \quad \text{and} \quad \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3). \quad (2.2.3)$$

It reads (see [147] for instance)

$$\mathcal{L}_\gamma = -N_c \text{Tr} \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} D^\mu \varphi_i^\dagger D_\mu \varphi^i + \bar{\psi}^a \bar{\sigma}^\mu D_\mu \psi_a \right) + \mathcal{L}_{\text{int}}, \quad (2.2.4)$$

with

$$\begin{aligned} \mathcal{L}_{\text{int}} = N_c g \text{Tr} \left(-\frac{g}{4} \{ \varphi_i^\dagger, \varphi^i \} \{ \varphi_j^\dagger, \varphi^j \} + g e^{-i\epsilon^{ijk}\gamma_k} \varphi_i^\dagger \varphi_j^\dagger \varphi^i \varphi^j \right. \\ \left. - e^{-\frac{i}{2}\gamma_j^-} \bar{\psi}^j \varphi_j \bar{\psi}^4 + e^{\frac{i}{2}\gamma_j^-} \bar{\psi}^4 \varphi_j \bar{\psi}^j + i\epsilon_{ijk} e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+} \psi_k \varphi_i \psi_j \right. \\ \left. - e^{\frac{i}{2}\gamma_j^-} \psi_4 \varphi_j^\dagger \psi_j + e^{-\frac{i}{2}\gamma_j^-} \psi_j \varphi_j^\dagger \psi_4 + i\epsilon_{ijk} e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+} \bar{\psi}^k \varphi_i^\dagger \bar{\psi}^j \right), \quad (2.2.5) \end{aligned}$$

where summation over doubly and triply repeated indices is understood and contraction of two undotted/dotted spinors is done through $\epsilon^{\alpha\beta}/\epsilon^{\dot{\alpha}\dot{\beta}}$. The combinations γ_j^\pm of the twists that appear in the Lagrangian are defined in the following way:

$$\gamma_1^\pm = -\frac{\gamma_3 \pm \gamma_2}{2}, \quad \gamma_2^\pm = -\frac{\gamma_1 \pm \gamma_3}{2}, \quad \gamma_3^\pm = -\frac{\gamma_2 \pm \gamma_1}{2}. \quad (2.2.6)$$

Notice that the twists γ_1 , γ_2 , and γ_3 have to be real for the Lagrangian to be Hermitian. When they take arbitrary (real) values, all the supersymmetry is broken but the theory seems to remain integrable [39, 158], provided one includes some specific counterterms [147]. Part of the supersymmetry can be restored if we take particular values for the twists.

2.2.2 The Double-Scaling Limit

The authors of [143] proposed a double-scaling limit of the γ -deformed theory that simplifies the theory while seemingly preserving integrability. The limit is defined by

$$g \rightarrow 0, \quad \gamma_j \rightarrow i\infty, \quad \text{such that} \quad \xi_j = e^{-\frac{1}{2}\gamma_j} g \quad \text{remains fixed} \quad (2.2.7)$$

for all $1 \leq j \leq 3$. Notice that unitarity has been broken when the twists were taken to be imaginary. As mentioned in Introduction, this makes the dilatation generator not diagonalisable anymore [150, 161, 162], and the presence of non-trivial Jordan blocks implies the existence of logarithmic multiplets of operators.

In that limit, the gauge boson A completely decouples from the rest of the fields, and only some of the quartic and Yukawa interactions are preserved:

$$\mathcal{L}_{\varphi\psi} = -N_c \text{Tr} \left(\frac{1}{2} \partial^\mu \varphi_i^\dagger \partial_\mu \varphi^i + \bar{\psi}^a \bar{\sigma}^\mu \partial_\mu \psi_a \right) + \mathcal{L}_{\varphi\psi,\text{int}} \quad (2.2.8)$$

and

$$\begin{aligned} \mathcal{L}_{\varphi\psi,\text{int}} = N_c \text{Tr} \left[\xi_1^2 \varphi_2^\dagger \varphi_3^\dagger \varphi_2 \varphi_3 + \xi_2^2 \varphi_3^\dagger \varphi_1^\dagger \varphi_3 \varphi_1 + \xi_3^2 \varphi_1^\dagger \varphi_2^\dagger \varphi_1 \varphi_2 + i\sqrt{\xi_2 \xi_3} (\psi_3 \varphi_1 \psi_2 + \bar{\psi}^3 \varphi_1^\dagger \bar{\psi}^2) \right. \\ \left. + i\sqrt{\xi_1 \xi_2} (\psi_2 \varphi_3 \psi_1 + \bar{\psi}^2 \varphi_3^\dagger \bar{\psi}^1) + i\sqrt{\xi_3 \xi_1} (\psi_1 \varphi_2 \psi_3 + \bar{\psi}^1 \varphi_2^\dagger \bar{\psi}^3) \right]. \quad (2.2.9) \end{aligned}$$

Now that the gauge boson has decoupled, $U(N_c)$ is only a global (flavour) symmetry.

2.3 The Fishnet Theory

2.3.1 Lagrangian and Feynman Rules

If one sets two of the coupling constants ξ_1 , ξ_2 , ξ_3 to zero in the previous model, the fermions as well as one of the scalars decouple, and only one quartic interaction between two complex scalars remain [143]:

$$\mathcal{L} = -N_c \text{Tr} \left[\partial_\mu X^\dagger \partial^\mu X + \partial_\mu Z^\dagger \partial^\mu Z - (4\pi\xi)^2 X^\dagger Z^\dagger X Z \right], \quad (2.3.1)$$

where we have renamed and rescaled the fields and the coupling. We recall that X and Z are $N_c \times N_c$ matrices.

$$\begin{array}{ccc}
 X \begin{array}{c} \xrightarrow{\text{red}} \\ \xleftarrow{\text{red}} \end{array} X^\dagger & X \begin{array}{c} \xrightarrow{\text{red}} \\ \xleftarrow{\text{red}} \end{array} X^\dagger & \frac{\Gamma(\tilde{\delta})}{\Gamma(\delta)} \frac{4^{-\delta} \pi^{-\frac{d}{2}}}{(x-y)^{2\tilde{\delta}}} \\
 \\
 Z \begin{array}{c} \xrightarrow{\text{black}} \\ \xleftarrow{\text{black}} \end{array} Z^\dagger & Z \begin{array}{c} \xrightarrow{\text{black}} \\ \xleftarrow{\text{black}} \end{array} Z^\dagger & \frac{\Gamma(\delta)}{\Gamma(\delta)} \frac{4^{-\delta} \pi^{-\frac{d}{2}}}{(x-y)^{2\delta}} \\
 \\
 \begin{array}{c} \text{red} \\ \text{black} \end{array} \begin{array}{c} \updownarrow \\ \updownarrow \end{array} \begin{array}{c} \leftarrow \\ \rightarrow \end{array} & \begin{array}{c} \text{red} \\ \text{black} \end{array} \begin{array}{c} \updownarrow \\ \updownarrow \end{array} \begin{array}{c} \leftarrow \\ \rightarrow \end{array} & (4\pi)^{\frac{d}{2}} \xi^2
 \end{array}$$

Figure 2.1: Feynman rules for the fishnet theory (2.3.2). On the left we show the double line notation taking into account the matrix indices of the fields, indices are conserved along a line. The other two columns describe the simplified rules, valid in the planar limit. In that case, the arrows are mainly there to remind us of the chirality of the interaction.

This theory was then generalised in [144] in arbitrary dimension d and with an additional deformation parameter δ such that $0 < \delta < \frac{d}{2}$ in the following way:

$$\mathcal{L}^{(d,\delta)} = -N_c \text{Tr} \left[X^\dagger (-\partial_\mu \partial^\mu)^\delta X + Z^\dagger (-\partial_\mu \partial^\mu)^\delta Z - (4\pi)^{\frac{d}{2}} \xi^2 X^\dagger Z^\dagger X Z \right], \quad (2.3.2)$$

where

$$\tilde{\delta} = \frac{d}{2} - \delta, \quad (2.3.3)$$

and we now work in Euclidean signature so that $(-\partial_\mu \partial^\mu)^\alpha$ can be understood as the operator of multiplication by $(p^2)^\alpha$ in Fourier space. In position space, this becomes

$$[(-\partial_\mu \partial^\mu)^\alpha f](x) = \frac{4^\alpha \Gamma\left(\frac{d}{2} + \alpha\right)}{\Gamma(-\alpha)} \int \frac{f(y)}{(x-y)^{2\left(\frac{d}{2} + \alpha\right)}} \frac{d^d y}{\pi^{\frac{d}{2}}}. \quad (2.3.4)$$

In the deformed theory, the two scalar fields have different bare dimensions: $\tilde{\delta}$ for X and δ for Z .

Feynman Rules The free propagators are given by

$$\langle (X^\dagger)_{ij}(x) X_{kl}(y) \rangle_0 = \frac{\delta_{jk} \delta_{il} \Gamma(\tilde{\delta})}{N_c 4^\delta \pi^{\frac{d}{2}} \Gamma(\delta) (x-y)^{2\tilde{\delta}}}, \quad (2.3.5)$$

$$\langle (Z^\dagger)_{ij}(x) Z_{kl}(y) \rangle_0 = \frac{\delta_{jk} \delta_{il} \Gamma(\delta)}{N_c 4^\delta \pi^{\frac{d}{2}} \Gamma(\tilde{\delta}) (x-y)^{2\delta}}. \quad (2.3.6)$$

The propagators and the single interaction vertex are represented in double-line notation in Figure 2.1.

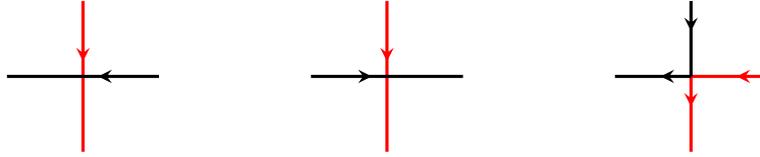


Figure 2.2: Three quartic interactions: we (arbitrarily) decide that the one on the left corresponds to $\text{Tr}(X^\dagger Z^\dagger X Z)$, present in the Lagrangian, whereas the one on the middle corresponds to $\text{Tr}(Z^\dagger X^\dagger Z X)$ absent from the Lagrangian and thus forbidden. The one on the right is then associated to $\text{Tr}(X^\dagger Z^\dagger Z X)$ and is also forbidden, in particular it shows that black and red lines should really cross one another and cannot simply graze.

From now on we will work in the planar limit

$$N_c \rightarrow +\infty, \quad \xi^2 \text{ fixed.} \quad (2.3.7)$$

This means, in particular, that we will only consider planar Feynman graphs. For correlation functions of trace operators, in that limit, the dependence on N_c appears only as an overall trivial factor (cf. Subsection 2.1.3). We can thus drop the double-line notation and use a simplified version of the Feynman rules as depicted in Figure 2.1.

2.3.2 Mass and Coupling Renormalisation

When drawing Feynman graphs for the fishnet theory, the main point to remember is that, because the fields are complex matrix fields and because there is only one allowed interaction vertex, the lines can only cross in a very specific way, see Figure 2.2.

Let us investigate the possible running of the mass and of the coupling constant. If we first consider the possible corrections to the propagator, one can easily convince oneself that the only allowed planar contributions are those depicted on the left of Figure 2.3. There is one graph at each order of the coupling constant. And it is easy to see that they all have the same dependence on the flavour indices and on the number of flavour such that

$$\langle (X^\dagger)_{ij}(x) X_{kl}(y) \rangle = \langle (X^\dagger)_{ij}(x) X_{kl}(y) \rangle_0 + \frac{\delta_{ij} \delta_{kl}}{N_c^2} \left(\sum_{m=1}^{+\infty} \xi^{2m} G_m(x-y) \right), \quad (2.3.8)$$

with G_m that depends neither on ξ^2 nor on N_c . There exists a similar expression for Z fields. Recalling that the free propagator is itself only of order N_c^{-1} , we see that, in the planar limit, there is no mass renormalisation. When N_c is finite, however, the functions G_m need to be taken into account and, since they involve propagators between two coinciding points, this requires regularisation.

Concerning the coupling constant, one realises that it is impossible to draw planar Feynman graphs that would renormalise it. However, even though the coupling ξ^2 does not run, the theory is not automatically conformal. It was indeed proven that some double-trace couplings are perturbatively generated [145]. As a consequence, in order to renormalise the

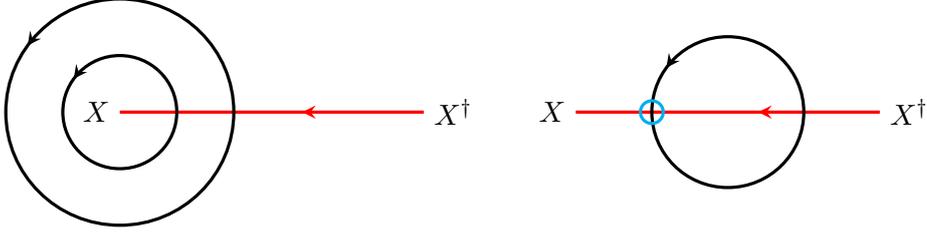


Figure 2.3: Diagrams with an arbitrary number of concentric circles separating the two points would generate the mass if they were not suppressed in the planar limit. The right diagram, on the other hand, would not be suppressed but is not allowed because it involves a forbidden interaction vertex, indicated here with a blue circle (if one had reversed the arrow on the black circle, the blue circle would have been around the other intersection).

theory, one should include the corresponding counterterms from the start, this means adding the following terms to the Lagrangian¹

$$\begin{aligned} \mathcal{L}_{\text{ct}} = \alpha_1^2 & \left[\text{Tr}(XX) \text{Tr}(X^\dagger X^\dagger) + \text{Tr}(ZZ) \text{Tr}(Z^\dagger Z^\dagger) \right] \\ & - \alpha_2^2 \text{Tr}(XZ) \text{Tr}(X^\dagger Z^\dagger) - \alpha_3^2 \text{Tr}(XZ^\dagger) \text{Tr}(ZX^\dagger). \end{aligned} \quad (2.3.9)$$

Let us first notice that, because of the $U(1) \times U(1)$ symmetry associated with the multiplication by a phase of the fields X and Z , the counterterms have to contain the same number of fields X and X^\dagger and, separately, the same number of Z and Z^\dagger . Then, it is also important to remark that, for the new couplings to be dimensionless, one should set $\alpha_1^2 = 0$ when $\delta \neq \frac{d}{4}$.

We now present the results of [144] and [158] concerning the renormalisation of these couplings. One can show that there exist two fixed points for this theory given by

$$(\alpha_1^2, \alpha_2^2, \alpha_3^2) = (\alpha_+^2(\xi^2), \xi^2, \xi^2) \quad \text{or} \quad (\alpha_1^2, \alpha_2^2, \alpha_3^2) = (\alpha_-^2(\xi^2), \xi^2, \xi^2), \quad (2.3.10)$$

where α_\pm^2 are the two roots of the beta function for α_1^2 :

$$\beta_1 = a(\xi^2) + \alpha_1^2 b(\xi^2) + \alpha_1^4 c(\xi^2). \quad (2.3.11)$$

The coefficients a , b , and c were determined perturbatively to order $O(\xi^{14})$ for the original fishnet theory, i.e. $(d, \delta) = (4, 1)$, using dimensional regularisation for the following two-point correlation function

$$\left\langle \text{Tr}(X^2(x)) \text{Tr}((X^\dagger)^2(0)) \right\rangle. \quad (2.3.12)$$

The resulting perturbative expansion for α_\pm^2 is

$$\alpha_\pm^2 = \pm i \frac{\xi^2}{2} - \frac{\xi^4}{2} \mp i \frac{3\xi^6}{4} + \xi^8 \pm i \frac{65\xi^{10}}{48} - \frac{19\xi^{12}}{10} + O(\xi^{14}). \quad (2.3.13)$$

¹These counterterms are enough if the fields are traceless. If this is not true one should include additional multi-trace counterterms such as $\text{Tr}(X) \text{Tr}(X^\dagger Z^\dagger Z)$ [147].

Though these counterterms have to be included, and the new couplings have to take specific values depending on ξ^2 , for the theory to be truly scale invariant, in the following, we will not have to take them into account. They are indeed suppressed in the planar limit with respect to the original interaction vertex. They will thus not contribute to the specific correlation functions we shall be interested in. We stress that these couplings are in general relevant in the planar limit—after all, the results presented above were obtained in that limit—but that there still exist some correlators to which they do not contribute.

Chapter 3

Non-compact Conformal Spin Chain

We will present, in this chapter, the integrable spin chain relevant for the fishnet theory in dimension $d \geq 2$. The symmetry group is the Euclidean conformal group, and the representations under consideration are infinite-dimensional principal series representations. We begin with the construction of the model using the relevant R-matrix. We then explain, in Section 3.3, how it naturally arises when considering two-point correlation functions in the fishnet theory. The last section is devoted to a presentation of the solution in the simple case of a chain of length 2.

3.1 Scalar Principal Series Representations

3.1.1 Definitions and Representation Theory Results

Let us consider the Lorentz group in $d + 2$ dimensions with metric $\eta_{\mu\nu} = \text{Diag}(1, \dots, 1, -1)$, where the indices take values $1, \dots, d + 1, 0$. It is given by

$$O(d + 1, 1) = \left\{ M \in GL(d + 2, \mathbb{R}) \mid {}^t M \eta M = \eta \right\}. \quad (3.1.1)$$

It is well-known that the (global) Euclidean conformal group in d dimensions is isomorphic to the subgroup

$$O^\uparrow(d + 1, 1) = \left\{ M \in O(d + 1, 1) \mid M^0_0 \geq 1 \right\}. \quad (3.1.2)$$

This subgroup has two connected components: the one containing the identity is the subgroup $SO^\uparrow(d + 1, 1)$ of matrices with unit determinant, and the other contains the inversion $I = \text{Diag}(1, \dots, 1, -1, 1)$. At the level of the generators, the isomorphism is realised via the following identification:

$$L_{ij} = X_{ij}, \quad P_i = X_{i0} + X_{id+1}, \quad K_i = X_{i0} - X_{id+1}, \quad D = X_{0d+1}, \quad (3.1.3)$$

where $X_{\mu\nu} = \eta(e_{\mu\nu} - e_{\nu\mu})$ are the natural generators of $\mathfrak{so}(d + 1, 1)$ while L_{ij} , P_i , K_i and D are the generators of rotations, translations, special conformal transformations and dilations.

Here and in the following we assume that Latin indices i, j, k , etc. take values between 1 and d , whereas Greek indices μ, ν, ρ , etc. take values between 0 and $d + 1$. It is clear that special conformal transformations decompose into the successive application of an inversion, a translation and another inversion, $e^{b^i K_i} = I e^{b^i P_i} I$, since this holds at the level of the generators. The quadratic Casimir invariant is

$$X_{\mu\nu} X^{\mu\nu} = L_{ij} L_{ij} - P_i K_i - K_i P_i - 2D^2. \quad (3.1.4)$$

The scalar principal series representations [196, 197] ρ_Δ are characterised by a complex number Δ , called the conformal dimension, and defined on the space

$$\mathcal{V}_\Delta = L^2 \left(\mathbb{R}^d, (1 + x^2)^{2\operatorname{Re}(\Delta) - d} dx \right). \quad (3.1.5)$$

The different elements of the conformal group act as expected:

- rotations $\left[\rho_\Delta \left(e^{\omega_{ij} L^{ij}} \right) f \right] (x) = f(e^{-\omega_{ij} L^{ij}} x)$,
- translations $\left[\rho_\Delta \left(e^{b^i P_i} \right) f \right] (x) = f(x - b)$,
- dilations ($a > 0$) $\left[\rho_\Delta \left(a^D \right) f \right] (x) = \frac{1}{a^\Delta} f\left(\frac{x}{a}\right)$,
- inversion $\left[\rho_\Delta(I) f \right] (x) = \frac{1}{x^{2\Delta}} f\left(\frac{x}{x^2}\right)$.

In general, these representations are not unitary, but there exists an invariant inner product in two important cases:

- unitary principal series: $\Delta = \frac{d}{2} + i\nu$ with ν real, and the representation is unitary for the usual inner product on $L^2(\mathbb{R}^d)$.
- complementary series: $0 < \Delta < d$, and, in this case, the representations are unitary for a new inner product defined by

$$\langle f, g \rangle = \frac{1}{\Gamma\left(\Delta - \frac{d}{2}\right)} \int \frac{f(x)^* g(y)}{(x - y)^{2(d - \Delta)}} d^d x d^d y. \quad (3.1.6)$$

The prefactor is important, it is there to cancel the poles of $(x - y)^{-2(d - \Delta)}$ and to ensure positivity of the inner product. When $\Delta = \frac{d}{2}$, for instance, the latter reduces to that of $L^2(\mathbb{R}^d)$, which is consistent with the fact that we are in the unitary principal series with $\nu = 0$. Representations of the complementary series naturally arise in the study of the fishnet theory, as we shall explain below.

For the reader familiar with the notion of discrete series representations, we point out that the conformal group does not have any when d is even. For odd d , none of these unitary representations appear in the scalar principal series.

The only equivalence between the principal series representations are between representations ρ_Δ and $\rho_{d - \Delta}$ when $-\Delta \notin \mathbb{N}$, $\Delta - d \notin \mathbb{N}$. The intertwining operator $G_\Delta : \mathcal{V}_{d - \Delta} \rightarrow \mathcal{V}_\Delta$ is given by

$$[G_\Delta f](x) = \frac{\Gamma(\Delta)}{\Gamma\left(\frac{d}{2} - \Delta\right)} \int \frac{f(y)}{(x - y)^{2\Delta}} \frac{d^d y}{\pi^{\frac{d}{2}}}. \quad (3.1.7)$$

These equivalent representations are moreover irreducible. When $n \in \mathbb{N}$, however, the space \mathcal{V}_{-n} contains a finite-dimensional subspace of polynomials stable under the action of the conformal group. On the other hand, \mathcal{V}_{d+n} contains an infinite-dimensional invariant subspace on which the conformal group acts unitarily (one of the exceptional series of unitary representations).

As follows from the explicit expression for the representations, the generators of the conformal group act as first-order differential operators:

$$\begin{aligned} \rho_\Delta(P_i) &= -\partial_i, & \rho_\Delta(L_{ij}) &= x_i\partial_j - x_j\partial_i, & \rho_\Delta(D) &= -\Delta - x_i\partial_i, \\ \rho_\Delta(K_i) &= 2x_i(\Delta + x_j\partial_j) - x^2\partial_i, \end{aligned} \quad (3.1.8)$$

where we have also denoted ρ_Δ the representation of the Lie algebra. For unitary representations, these generators are anti-Hermitian. The quadratic Casimir invariant is

$$\rho_\Delta(X_{\mu\nu})\rho_\Delta(X^{\mu\nu}) = 2\Delta(d - \Delta). \quad (3.1.9)$$

This is in accordance with the fact that representations Δ and $d - \Delta$ are equivalent.

Other Principal Series Representations The most general principal series representations of the conformal group are labelled by the highest weight λ of an irreducible finite-dimensional representation of $SO(d, \mathbb{R})$, and a complex number Δ . The representation space is [196, 197]

$$\mathcal{V}_{\lambda, \Delta} = \mathbb{V}_\lambda \otimes L^2\left(\mathbb{R}^d, (1 + x^2)^{2\operatorname{Re}(\Delta) - d} d^d x\right), \quad (3.1.10)$$

where \mathbb{V}_λ is the space of the representation of the orthogonal group.

The scalar representations considered above obviously correspond to the trivial representation, $\lambda = 0$, of $SO(d, \mathbb{R})$. For all λ , the representations are unitary when $\operatorname{Re}(\Delta) = \frac{d}{2}$, those constitute the unitary principal series. In the last section of this chapter we will need representations of the orthogonal group on symmetric traceless tensors, i.e. $\lambda = l\varepsilon_1$, we will then write $\mathcal{V}_{l, \Delta}$ in place of $\mathcal{V}_{l\varepsilon_1, \Delta}$.

3.1.2 R-Matrices and Star-Triangle Relation

We begin with the $\mathfrak{so}(d, \mathbb{C})$ -invariant R-matrix acting in $\mathbb{C}^d \otimes V$, with V the $2^{\lfloor \frac{d}{2} \rfloor}$ -dimensional space of spinor representations (cf. Appendix A.2.2 for a reminder of the relation between gamma matrices and spinor representations) of the orthogonal algebra [198]:

$$\mathbf{R}^{(s)}(x) = x \operatorname{Id} - \frac{1}{8} L_{ij} \otimes [\gamma_i, \gamma_j], \quad (3.1.11)$$

where $L_{ij} = e_{ij} - e_{ji}$ are the generators of the orthogonal algebra, and the superscript (s) stands for spinorial. Let us prove that it satisfies indeed the Yang–Baxter relation

$$\mathbf{R}_{12}(x)\mathbf{R}_{13}^{(s)}(x+y)\mathbf{R}_{23}^{(s)}(y) = \mathbf{R}_{23}^{(s)}(y)\mathbf{R}_{13}^{(s)}(x+y)\mathbf{R}_{12}(x), \quad (3.1.12)$$

with \mathbf{R}_{12} given by (1.2.37).

This equation relates two polynomials in x and y , it suffices to verify that it holds for each coefficient. It is non-trivial only for the coefficients of the monomials x^2 , xy , and x . Let us consider the coefficients of x^2 , one has to show that

$$\begin{aligned} \frac{1}{8}L_{ij} \otimes L_{kl} \otimes [\gamma_i, \gamma_j][\gamma_k, \gamma_l] - (\kappa + \mathbb{P} - \mathbb{K})\mathbf{I}_d \otimes L_{ij} \otimes [\gamma_i, \gamma_j] \\ = \frac{1}{8}L_{ij} \otimes L_{kl} \otimes [\gamma_k, \gamma_l][\gamma_i, \gamma_j] - \mathbf{I}_d \otimes L_{ij} \otimes [\gamma_i, \gamma_j](\kappa + \mathbb{P} - \mathbb{K}), \end{aligned} \quad (3.1.13)$$

where $\kappa = \frac{d}{2} - 1$. It suffices to remark that

$$\mathbb{P} - \mathbb{K} = -\frac{1}{2}L_{ij} \otimes L_{ij}, \quad (3.1.14)$$

and to use the fact that L_{ij} and $\frac{[\gamma_i, \gamma_j]}{4}$ satisfy the same commutation relations to write

$$\frac{1}{4}L_{ij} \otimes L_{kl} \otimes [[\gamma_i, \gamma_j], [\gamma_k, \gamma_l]] = L_{kl} \otimes [L_{ij}, L_{kl}] \otimes [\gamma_i, \gamma_j] = -4L_{ij} \otimes L_{jk} \otimes [\gamma_k, \gamma_i], \quad (3.1.15)$$

which is equivalent to (3.1.13). The computation is similar for the coefficient in front of xy . For the coefficient in front of x , however, one has to prove that

$$\begin{aligned} \frac{1}{8}(\kappa + \mathbb{P} - \mathbb{K})L_{ij} \otimes L_{kl} \otimes [\gamma_i, \gamma_j][\gamma_k, \gamma_l] - \kappa \mathbb{P}\mathbf{I}_d \otimes L_{ij} \otimes [\gamma_i, \gamma_j] \\ - \frac{1}{8}L_{ij} \otimes L_{kl} \otimes [\gamma_k, \gamma_l][\gamma_i, \gamma_j](\kappa + \mathbb{P} - \mathbb{K}) + \kappa \mathbf{I}_d \otimes L_{ij} \otimes [\gamma_i, \gamma_j] \mathbb{P} = 0. \end{aligned} \quad (3.1.16)$$

The terms involving $\frac{\mathbb{P}}{8}$ clearly compensate one another. For the others, we first write

$$\begin{aligned} -\kappa \mathbb{P}\mathbf{I}_d \otimes L_{ij} \otimes [\gamma_i, \gamma_j] + \kappa \mathbf{I}_d \otimes L_{ij} \otimes [\gamma_i, \gamma_j] \mathbb{P} \\ = 2\kappa (-e_{il} \otimes e_{lj} \otimes [\gamma_i, \gamma_j] + e_{lj} \otimes e_{il} \otimes [\gamma_i, \gamma_j]). \end{aligned} \quad (3.1.17)$$

We then look at

$$-\frac{\mathbb{K}}{8}L_{ij} \otimes L_{kl} \otimes [\gamma_i, \gamma_j][\gamma_k, \gamma_l] = -\frac{\mathbb{K}}{2}e_{ij} \otimes e_{kl} \otimes [\gamma_i, \gamma_j][\gamma_k, \gamma_l] \quad (3.1.18)$$

$$= -\frac{1}{2}e_{ij} \otimes e_{il} \otimes [\gamma_k, \gamma_j][\gamma_k, \gamma_l] \quad (3.1.19)$$

$$= 2\kappa e_{ij} \otimes e_{il} \otimes [\gamma_j, \gamma_l] + 2(1 + 2\kappa)\mathbb{K}, \quad (3.1.20)$$

where we used $\gamma_k \gamma_j \gamma_k = -2\kappa \gamma_j$ and $\gamma_k \gamma_j \gamma_l \gamma_k = 2(\kappa - 1)\gamma_j \gamma_l + 4\delta_{jl}$. Similarly, one has

$$\frac{1}{8}L_{ij} \otimes L_{kl} \otimes [\gamma_k, \gamma_l][\gamma_i, \gamma_j] \mathbb{K} = 2\kappa e_{ij} \otimes e_{kj} \otimes [\gamma_i, \gamma_k] - 2(1 + 2\kappa)\mathbb{K}. \quad (3.1.21)$$

Finally, we can add the previous results and write

$$\begin{aligned} \frac{1}{8}L_{ij} \otimes L_{kl} \otimes [\gamma_i, \gamma_j][\gamma_k, \gamma_l] \mathbb{K} - \frac{\mathbb{K}}{8}L_{ij} \otimes L_{kl} \otimes [\gamma_i, \gamma_j][\gamma_k, \gamma_l] \\ - \kappa \mathbb{P}\mathbf{I}_d \otimes L_{ij} \otimes [\gamma_i, \gamma_j] + \kappa \mathbf{I}_d \otimes L_{ij} \otimes [\gamma_i, \gamma_j] \mathbb{P} = 2\kappa L_{ij} \otimes L_{jk} \otimes [\gamma_k, \gamma_i], \end{aligned} \quad (3.1.22)$$

which, according to (3.1.15), cancels out exactly the remaining two terms of (3.1.16).

It is also possible [198, 199] to construct an R-matrix $R^{(s),(s)}$ acting on $V \otimes V$, and satisfying the Yang–Baxter relation both on $\mathbb{C}^d \otimes V \otimes V$ and on $V \otimes V \otimes V$. This R-matrix is not as simple as those we have seen before and, since we do not need it for what follows, we do not give its expression.

Let us remark that, in even dimension d , the space of the spinor representation is reducible, see (A.2.19), and the R-matrix actually decomposes into two smaller matrices:

$$R^{(s)}(x) = \begin{pmatrix} R^{(+)}(x) & 0 \\ 0 & R^{(-)}(x) \end{pmatrix}, \quad (3.1.23)$$

with

$$R^{(+)}(x) = x \text{Id} - \frac{1}{8} L_{ij} \otimes (\sigma_i \bar{\sigma}_j - \sigma_j \bar{\sigma}_i) \quad (3.1.24)$$

and

$$R^{(-)}(x) = x \text{Id} - \frac{1}{8} L_{ij} \otimes (\bar{\sigma}_i \sigma_j - \bar{\sigma}_j \sigma_i). \quad (3.1.25)$$

For the sake of brevity we will, for the moment, restrict ourselves to even dimension $d = 2r$, r is the rank of the orthogonal algebra.

The $\mathfrak{so}(d+1, 1)$ -invariant R-matrices acting in the tensor product of the defining and the spinor representations, or in the tensor product of two spinor representations, are obtained in a similar fashion. We only need a representation of the Clifford algebra $\text{Cl}_{d+1,1}(\mathbb{R})$. Let us construct one starting from the one we used for $\text{Cl}_d(\mathbb{R})$:

$$\Gamma_i = \sigma_x \otimes \gamma_i = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad (3.1.26)$$

for $1 \leq i \leq d$, and

$$\Gamma_{d+1} = \sigma_y \otimes I_{2r} = \begin{pmatrix} 0 & -i I_{2r} \\ i I_{2r} & 0 \end{pmatrix}, \quad \Gamma_0 = i \sigma_x \otimes \sigma_z \otimes I_{2r-1} = \begin{pmatrix} 0 & i \sigma_z \otimes I_{2r-1} \\ i \sigma_z \otimes I_{2r-1} & 0 \end{pmatrix}. \quad (3.1.27)$$

Using the notations of Appendix A.2.2, one has $\Gamma_i = \pi_{d+2}(\gamma_i)$, $\Gamma_{d+1} = \pi_{d+2}(\gamma_{d+2})$, and $\Gamma_0 = i \pi_{d+2}(\gamma_{d+1})$ so that $\{\Gamma_\mu, \Gamma_\nu\} = 2 \eta_{\mu\nu}$ holds indeed. These matrices are of the form

$$\Gamma_\mu = \begin{pmatrix} 0 & \Sigma_\mu \\ \bar{\Sigma}_\mu & 0 \end{pmatrix}, \quad \text{with} \quad \Sigma_\mu \bar{\Sigma}_\nu + \Sigma_\nu \bar{\Sigma}_\mu = \bar{\Sigma}_\mu \Sigma_\nu + \bar{\Sigma}_\nu \Sigma_\mu = 2 \eta_{\mu\nu} I_{2r}. \quad (3.1.28)$$

We are now ready to present the R-matrix acting on $V \otimes \mathcal{V}_\Delta$, it is given by [149]

$$R^{(s,\Delta)}(u) = u - \frac{1}{8} [\Gamma_\mu, \Gamma_\nu] \otimes \rho_\Delta(X_{\mu\nu}) = \begin{pmatrix} R^{(+,\Delta)}(u) & 0 \\ 0 & R^{(-,\Delta)}(u) \end{pmatrix}, \quad (3.1.29)$$

where both blocks are of size 2^r . One of them is

$$R^{(+,\Delta)}(u) = u - \frac{1}{8} (\Sigma_\mu \bar{\Sigma}_\nu - \Sigma_\nu \bar{\Sigma}_\mu) \otimes \rho_\Delta(X^{\mu\nu}) \quad (3.1.30)$$

$$= u - \frac{1}{2} \begin{pmatrix} \rho_\Delta(L_{ij}) \frac{\sigma_i \bar{\sigma}_j - \sigma_j \bar{\sigma}_i}{4} - \rho_\Delta(D) & i \rho_\Delta(P_i) \sigma_i \\ -i \rho_\Delta(K_i) \bar{\sigma}_i & \rho_\Delta(L_{ij}) \frac{\bar{\sigma}_i \sigma_j - \bar{\sigma}_j \sigma_i}{4} + \rho_\Delta(D) \end{pmatrix}, \quad (3.1.31)$$

with now each block of size 2^{r-1} .

The R-matrix $\mathbf{R}^{(\Delta_1, \Delta_2)}$ acting on $\mathcal{V}_{\Delta_1} \otimes \mathcal{V}_{\Delta_2}$ and satisfying the Yang–Baxter relation with $\mathbf{R}^{(+, \Delta_1)}$ and $\mathbf{R}^{(+, \Delta_2)}$ was constructed in [149]. It is an integral operator, and its action on $\Phi \in \mathcal{V}_{\Delta_1} \otimes \mathcal{V}_{\Delta_2}$ reads

$$\left[\mathbf{R}^{(\Delta_1, \Delta_2)}(u) \Phi \right] (x_1, x_2) = \int \mathcal{R}^{(\Delta_1, \Delta_2)}(u; x_1, x_2 | y_1, y_2) \Phi(y_1, y_2) \frac{d^d y_1 d^d y_2}{\pi^d}, \quad (3.1.32)$$

with kernel given by

$$\begin{aligned} \mathcal{R}^{(\Delta_1, \Delta_2)}(u; x_1, x_2 | y_1, y_2) &= 4^{2u} \frac{\Gamma\left(u + \frac{d}{2} + \omega\right) \Gamma\left(u + \frac{d}{2} - \omega\right)}{\Gamma(-u - \omega) \Gamma(-u + \omega)} \\ &\quad \times \frac{1}{x_{12}^{2(-u-\zeta)} (x_2 - y_1)^{2(u+\frac{d}{2}-\omega)} (x_1 - y_2)^{2(u+\frac{d}{2}+\omega)} y_{12}^{2(-u+\zeta)}}, \end{aligned} \quad (3.1.33)$$

where

$$\zeta = \frac{d}{2} - \frac{\Delta_1 + \Delta_2}{2} \quad \text{and} \quad \omega = \frac{\Delta_1 - \Delta_2}{2}. \quad (3.1.34)$$

Let us point out that we conventionally decide to include a factor $\pi^{-\frac{d}{2}}$ in the integration measure over d -dimensional variables. This is done so as to remove all factors of π from the kernels of most of the operators we shall consider in the following. There is one notable exception: the identity operator on functions of L variables now has a kernel given by $\prod_{i=1}^L \pi^{\frac{d}{2}} \delta^{(d)}(x_i - y_i)$.

We first notice that, when $\Delta_1 = \Delta_2 = \Delta$, i.e. $\omega = 0$, the R-matrix reduces to the permutation operator for some specific value of the spectral parameter: $\mathbf{R}^{(\Delta, \Delta)}(0) = \mathbb{P}$. This is because

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{(x - y)^{2(\frac{d}{2} - \epsilon)}} = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \delta^{(d)}(x - y) \quad (3.1.35)$$

as distributions.

Let us now show that the R-matrix satisfies the inversion relation

$$\mathbf{R}^{(\Delta_1, \Delta_2)}(u) \mathbf{R}^{(\Delta_1, \Delta_2)}(-u) = \text{Id} \quad (3.1.36)$$

and the Yang–Baxter relation on $\mathcal{V}_{\Delta_1} \otimes \mathcal{V}_{\Delta_2} \otimes \mathcal{V}_{\Delta_3}$. In terms of kernels, the latter is equivalent to

$$\begin{aligned} &\int \mathcal{R}_{12}^{(\Delta_1, \Delta_2)}(u; x_1, x_2 | z_1, z_2) \mathcal{R}_{13}^{(\Delta_1, \Delta_3)}(u + v; z_1, x_3 | y_1, z_3) \mathcal{R}_{23}^{(\Delta_2, \Delta_3)}(v; z_2, z_3 | y_2, y_3) \frac{\prod_{i=1}^3 d^d z_i}{\pi^{\frac{3d}{2}}} \\ &= \int \mathcal{R}_{23}^{(\Delta_2, \Delta_3)}(v; x_2, x_3 | z_2, z_3) \mathcal{R}_{13}^{(\Delta_1, \Delta_3)}(u + v; x_1, z_3 | z_1, y_3) \mathcal{R}_{12}^{(\Delta_1, \Delta_2)}(v; z_1, z_2 | y_1, y_2) \frac{\prod_{i=1}^3 d^d z_i}{\pi^{\frac{3d}{2}}}. \end{aligned} \quad (3.1.37)$$

Before going further, let us present two well-known integral relations [200–205] that will be used repeatedly in many a computation in this thesis. They are proven, together with several other useful formulae, in Appendix B.1.

$$\begin{aligned}
x \xrightarrow{a} y &= \frac{1}{(x-y)^{2a}} & \bullet &= \int \frac{d^d z}{\pi^{\frac{d}{2}}} \\
x \xrightarrow{a} \bullet \xrightarrow{b} y &= A_0(a)A_0(b)A_0(d-a-b) & x \xrightarrow{a+b-\frac{d}{2}} y & \\
\begin{array}{c} x \\ | \\ a \\ \bullet \\ / \quad \backslash \\ b \quad c \\ y \quad z \end{array} &= A_0(a)A_0(b)A_0(c) & \begin{array}{c} x \\ / \quad \backslash \\ \tilde{c} \quad \tilde{b} \\ \bullet \\ \tilde{a} \\ y \quad z \end{array} &
\end{aligned}$$

Figure 3.1: Conventions for graphical representation of the integrals involving only propagators with arbitrary weights, examples of the chain and star-triangle relations.

- Chain relation (a and b are complex numbers)

$$\int \frac{\pi^{-\frac{d}{2}} d^d z}{(x-z)^{2a}(z-y)^{2b}} = \frac{A_0(a)A_0(b)A_0(d-a-b)}{(x-y)^{2(a+b-\frac{d}{2})}}. \quad (3.1.38)$$

Using the limit (3.1.35), one sees that the particular case $a+b=d$ is actually a representation of the delta distribution:

$$\int \frac{\pi^{-\frac{d}{2}} d^d z}{(x-z)^{2a}(z-y)^{2(d-a)}} = A_0(a)A_0(d-a)\pi^{\frac{d}{2}}\delta^{(d)}(x-y). \quad (3.1.39)$$

- Star-triangle relation, valid for $a+b+c=d$,

$$\int \frac{\pi^{-\frac{d}{2}} d^d w}{(w-x)^{2a}(w-y)^{2b}(w-z)^{2c}} = \frac{A_0(a)A_0(b)A_0(c)}{(x-y)^{2\tilde{c}}(y-z)^{2\tilde{a}}(z-x)^{2\tilde{b}}}. \quad (3.1.40)$$

We use the following notations:

$$\tilde{a} = \frac{d}{2} - a \quad \text{and} \quad A_0(a) = \frac{\Gamma(\tilde{a})}{\Gamma(a)} = \frac{1}{A_0(\tilde{a})}. \quad (3.1.41)$$

The proof of the inversion relation is then a direct consequence of (3.1.39) and reads

$$\begin{aligned}
& \int \mathcal{R}^{(\Delta_1, \Delta_2)}(u; x_1, x_2 | z_1, z_2) \mathcal{R}^{(\Delta_1, \Delta_2)}(-u; z_1, z_2 | y_1, y_2) \frac{d^d z_1 d^d z_2}{\pi^d} \\
&= A_0(-u+\omega)A_0(-u-\omega)A_0(u+\omega)A_0(u-\omega) \frac{1}{x_{12}^{2(-u-\zeta)} y_{12}^{2(u+\zeta)}} \\
&\times \int \frac{\pi^{-d} d^d z_1 d^d z_2}{(x_2 - z_1)^{2(u+\frac{d}{2}-\omega)} (x_1 - z_2)^{2(u+\frac{d}{2}+\omega)} (z_2 - y_1)^{2(-u+\frac{d}{2}-\omega)} (z_1 - y_2)^{2(-u+\frac{d}{2}+\omega)}} \\
&= \left(\frac{x_{12}^2}{y_{12}^2} \right)^{u+\zeta} \pi^d \delta^{(d)}(x_1 - y_1) \delta^{(d)}(x_2 - y_2) = \pi^d \delta^{(d)}(x_1 - y_1) \delta^{(d)}(x_2 - y_2).
\end{aligned}$$

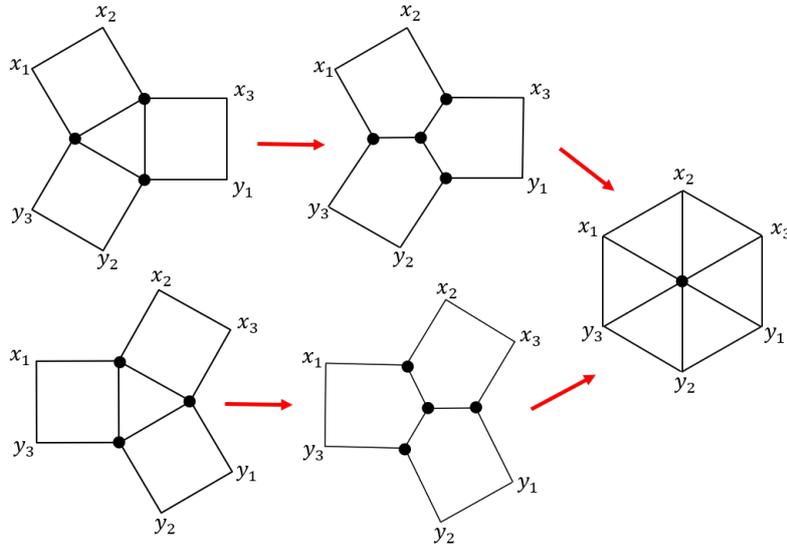


Figure 3.2: Graphical representation of the proof of the Yang–Baxter relation (3.1.37), each drawing is obtained from the previous one applying the star-triangle relation (3.1.40). The line indices have not been displayed for clarity, but they indeed coincide on the rightmost drawing, so do the proportionality constants coming from the star-triangle relation. This graphical proof first appeared in [67], where the $d = 2$ case was studied.

In the following, in order to avoid writing explicitly all the integrals, we will follow a widespread convention and represent them as Feynman graphs as is explained in Figure 3.1. The kernel of the R-matrix, for instance, would then just be a square while the proof of the Yang–Baxter relation (3.1.37) is shown in Figure 3.2.

3.2 The Spin Chain

Thanks to the R-matrices described in the previous section, it is possible to construct an integrable spin chain in which each site carries a scalar principal series representation of the conformal group. Let L be the length of the chain and Δ_i be the conformal dimension at site i , the quantum space is thus $\mathcal{H} = \mathcal{V}_{\Delta_1} \otimes \cdots \otimes \mathcal{V}_{\Delta_L}$. The transfer matrices with spinorial auxiliary space are

$$T_{\pm}(u) = \text{Tr}_{\pm} \left(R_{0L}^{(\pm, \Delta_L)}(u) \cdots R_{01}^{(\pm, \Delta_1)}(u) \right), \quad (3.2.1)$$

with the trace taken over a finite-dimensional space, whereas the transfer matrices

$$T_{\Delta}(u) = \text{Tr}_{\Delta} \left(R_{0L}^{(\Delta, \Delta_L)}(u) \cdots R_{01}^{(\Delta, \Delta_1)}(u) \right) \quad (3.2.2)$$

involve a trace over an infinite-dimensional one. We recall that, because of the various Yang–Baxter relations satisfied by the different R-matrices, all these transfer matrices commute with one another.

Since $R_{0,i}^{(+,\Delta_i)}(u) = u - \frac{1}{4}\Sigma_\mu\bar{\Sigma}_\nu\rho_{\Delta_i}(X^{\mu\nu})$, the transfer matrix T_+ is a polynomial in u of degree L , and its coefficients are differential operators:

$$T_+(u) = 2^r u^L + \sum_{k=1}^L u^{L-k} t_{+,k}, \quad (3.2.3)$$

with

$$t_{+,k} = \sum_{1 \leq i_1 < \dots < i_k \leq L} \frac{\text{Tr}\left(\Sigma_{\mu_k}\bar{\Sigma}_{\nu_k} \dots \Sigma_{\mu_1}\bar{\Sigma}_{\nu_1}\right)}{(-4)^k} \rho_{\Delta_{i_k}}(X^{\mu_k\nu_k}) \dots \rho_{\Delta_{i_1}}(X^{\mu_1\nu_1}). \quad (3.2.4)$$

One can show, using the explicit construction we gave of the matrices Σ_μ and $\bar{\Sigma}_\nu$, that they satisfy the following trace identities:

$$\text{Tr}\left(\Sigma_\mu\bar{\Sigma}_\nu\right) = 2^r \eta_{\mu\nu}, \quad \text{Tr}\left(\Sigma_\mu\bar{\Sigma}_\nu\Sigma_\rho\bar{\Sigma}_\sigma\right) = 2^r [\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\rho\nu}]. \quad (3.2.5)$$

As a consequence, the first two coefficients are

$$t_{+,1} = -2^{r-2} \sum_{i=1}^L \rho_{\Delta_i}(X^\mu_\mu) = 0 \quad (3.2.6)$$

and

$$\begin{aligned} t_{+,2} &= -2^{r-3} \sum_{1 \leq i < j \leq L} \rho_{\Delta_i}(X^{\mu\nu}) \rho_{\Delta_j}(X_{\mu\nu}) \\ &= -2^{r-4} \left(\sum_{i=1}^L \rho_{\Delta_i}(X^{\mu\nu}) \right) \left(\sum_{i=1}^L \rho_{\Delta_i}(X_{\mu\nu}) \right) + 2^{r-3} \sum_{i=1}^L \Delta_i(d - \Delta_i), \end{aligned} \quad (3.2.7)$$

which is, up to a constant, none other than the quadratic Casimir element for the representation of the conformal algebra on \mathcal{H} .

The transfer matrix T_Δ , on the other hand, is an integral operator. Its kernel is

$$\begin{aligned} \mathcal{T}_\Delta(u; x_1, \dots, x_L | y_1, \dots, y_L) &= \int \mathcal{R}_{0L}^{(\Delta, \Delta_L)}(u; w_1, x_L | w_L, y_L) \dots \mathcal{R}_{02}^{(\Delta, \Delta_2)}(u; w_3, x_2 | w_2, y_2) \\ &\quad \times \mathcal{R}_{01}^{(\Delta, \Delta_1)}(u; w_2, x_1 | w_1, y_1) \prod_{i=1}^L \frac{d^d w_i}{\pi^{\frac{d}{2}}}, \end{aligned} \quad (3.2.8)$$

where $w_{L+1} = w_1$. This kernel is depicted in Figure 3.3.

Isotropic Case When all sites carry the same representation, i.e. $\Delta_i = \Delta$ for all i , it is possible to define a Hamiltonian with nearest-neighbour interaction. Making use of the fact that $R^{(\Delta, \Delta)}(0) = \mathbb{P}$, we define

$$H = (\ln T_\Delta)'(0) = \sum_{i=1}^L H_{ii+1}^{(\Delta)}, \quad (3.2.9)$$

with [149]

$$H_{ij}^{(\Delta)} = \mathbb{P}_{ij} R^{(\Delta, \Delta)}(0) = 2 \ln(x_{ij}^2) + x_{ij}^{2(\frac{d}{2} - \Delta)} \ln\left(\partial_i^\mu \partial_{i\mu} \partial_j^\mu \partial_{j\mu}\right) x_{ij}^{2(\Delta - \frac{d}{2})}. \quad (3.2.10)$$

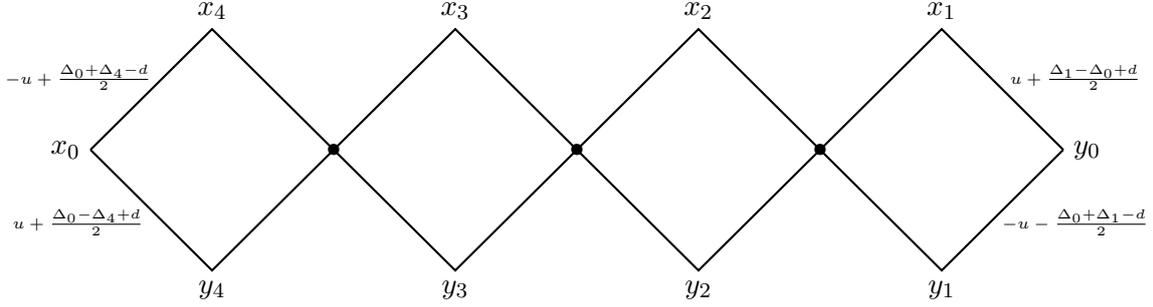


Figure 3.3: Kernel of the monodromy matrix $R_{0,L}^{(\Delta,\Delta_L)}(u) \cdots R_{0,1}^{(\Delta,\Delta_1)}(u)$ for a spin chain of length $L = 4$. For clarity, only some of the line indices have been written, any other index is equal to one already written up to $\Delta_4, \Delta_1 \rightarrow \Delta_i$. The kernel of the transfer matrix T_Δ is obtained by tracing over the auxiliary space, i.e. setting $x_0 = y_0$ and integrating over it (with a factor $\pi^{-\frac{d}{2}}$).

3.3 Integrability of the Fishnet Theory

We now return to the fishnet theory and consider the two-point function

$$G_J(x) = \langle \text{Tr}(Z^J(0)) \text{Tr}((Z^\dagger)^J(x)) \rangle. \quad (3.3.1)$$

If $J \geq 3$, the only graphs contributing to this correlator in the planar limit are the globe graphs shown in Figure 3.4. The important thing to notice is that these graphs can be constructed iteratively if we introduce the graph-building operator $F_{J,\delta}$ with kernel [143, 144]

$$F_{J,\delta}(x_1, \dots, x_J | y_1, \dots, y_J) = \frac{1}{\prod_{i=1}^J (x_i - y_i)^{2\delta} (y_i - y_{i+1})^{2\delta}}, \quad (3.3.2)$$

represented in Figure 3.5. The perturbative expansion of the correlator is indeed given by

$$G_J(x) = J \left(\frac{A_0(\tilde{\delta})}{4^{\tilde{\delta}} \pi^{\frac{d}{2}} x^{2\delta}} \right)^J \left[1 + x^{2J\delta} \sum_{k=1}^{+\infty} \xi^{2k} \int \frac{F_{J,\delta}^k(x, \dots, x | y_1, \dots, y_J)}{y_1^{2\delta} \cdots y_J^{2\delta}} \prod_{i=1}^J \frac{d^d y_i}{\pi^{\frac{d}{2}}} \right], \quad (3.3.3)$$

where, for a given $k \geq 1$, the integral corresponds exactly to the globe graph with k magnons wrapping around the poles. The overall factor of J comes from the different ways of contracting $\text{Tr}(Z^J)$ and $\text{Tr}((Z^\dagger)^J)$ in a planar way. The previous equation is only a formal equality and requires regularisation since the integrals are all divergent. There is indeed an ultraviolet divergence in the globe graphs that comes from the integration regions near the origin and the point x . In other words, $F_{J,\delta}^k(x_1, \dots, x_J | y_1, \dots, y_J)$ diverges when all the x_i 's go to the same point x for $k > 1$, and the previous integral has an extra divergence in the region $y_i \rightarrow 0$. If we work in dimensional regularisation, the dimension becomes $d - 2\epsilon$ and we replace ξ^2 with $\mu^{-2\epsilon} \xi^2$, for some mass scale μ , in order to keep the dimensionless ξ^2 . The divergences are then reduced to a pole in the integrals at $\epsilon = 0$. We also have to renormalise

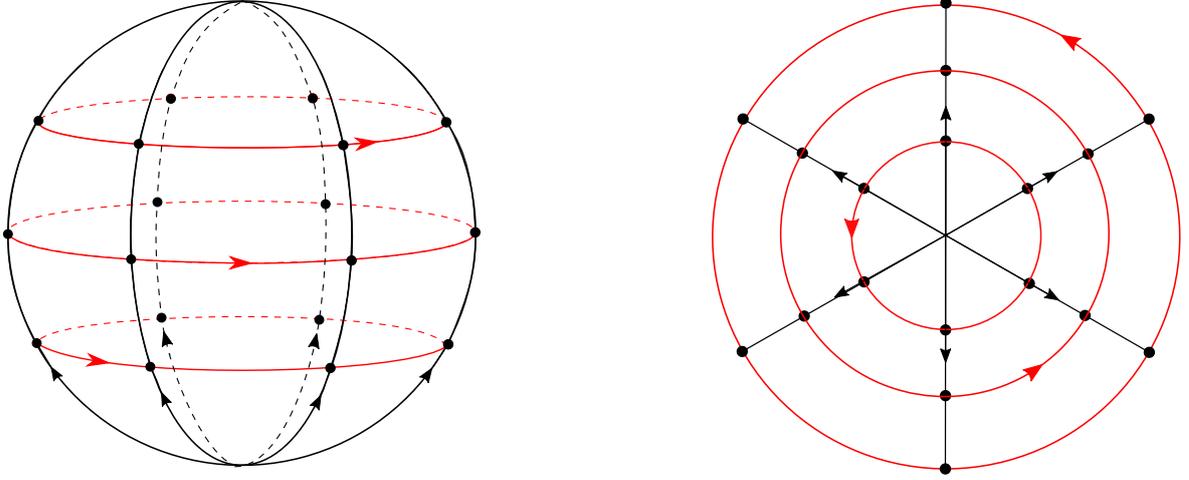


Figure 3.4: Example of a globe graph (left) that appears in the perturbative expansion of $G_6(x)$, and of its amputated version (right), a wheel graph. The North and South poles correspond to positions 0 and x respectively. The divergent part of the globe graph is $2/(x^2)^6$ times that of the wheel graph. A similar drawing with two wrappings can be found in [143].

the operators: $\text{Tr}(Z^J) \rightarrow \sqrt{\mathcal{Z}(\epsilon, \xi)} \text{Tr}(Z^J)$. Since the theory is conformal, the renormalised two-point function $G_{J,R} = \mathcal{Z}G_J$ should have a limit, when $\epsilon \rightarrow 0$, of the form

$$\lim_{\epsilon \rightarrow 0} G_{J,R}(x) = \frac{C_J(\xi)}{x^{2J\delta}(\mu^2 x^2)^{\gamma_J(\xi)}}. \quad (3.3.4)$$

The function γ_J is the anomalous dimension of the operator $\text{Tr}(Z^J)$, whereas C_J is a normalisation constant. Using the fact that G_J depends on μ solely through $\mu^{-2\epsilon}\xi^2$, it is standard to show that, in the minimal subtraction scheme, necessarily $\mathcal{Z}(\epsilon, \xi) = \exp\left(\frac{\mathcal{Z}_1(\xi^2)}{\epsilon}\right)$ for some function \mathcal{Z}_1 , and $\gamma_J(\xi) = -\xi^2 \mathcal{Z}'_1(\xi^2)$.

It was noticed in [144, 150] that the graph-building operator is actually part of the conserved charges of the spin chain constructed in the previous sections. Let us indeed consider a chain of length J with a conformal dimension at each site $\Delta = \delta$. The transfer matrix with auxiliary conformal dimension $\tilde{\delta}$ has a pole at $u = -\frac{d}{4}$, and a direct application of (3.1.35) shows that the behaviour close to this pole is dominated by the operator $F_{J,\delta}$ according to

$$\text{T}_{\tilde{\delta}}\left(-\frac{d}{4} + \epsilon\right) \sim \frac{F_{J,\delta}}{\left(\epsilon \Gamma\left(\frac{d}{2}\right)\right)^J}, \quad \text{when } \epsilon \rightarrow 0. \quad (3.3.5)$$

Since δ and $\tilde{\delta}$ are both real and between 0 and $\frac{d}{2}$, we are considering representations of the complementary series. In order to compute the anomalous dimension γ_J , one would have to diagonalise $F_{J,\delta}$. What we have just observed is that this problem is the same as solving a non-compact conformal spin chain. Apart from $J = 2$, a case that we treat in the next section, this remains highly non-trivial and the solution is not known at the moment.

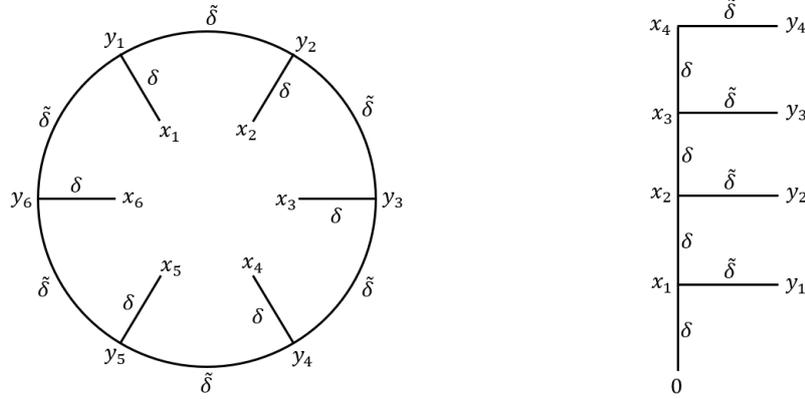


Figure 3.5: Kernels of the graph-building operators $F_{6,\delta}$ (left) and $\Lambda_{4,\tilde{\delta}}$ (right).

In the next two chapters, we will instead consider another family of graph-building operators. We first point out that since all the information we want is contained in the ultraviolet divergence of the globe graphs, it is equivalent to look instead at the wheel graphs, see Figure 3.4. $F_{J,\delta}$ allows to construct all such graphs with a given number J of spokes, but one can also introduce [179] some operator $\Lambda_{N,\tilde{\delta}}$ that builds the graphs with a given number N of circles. The wheel graph with J spokes and N circles is thus formally given by the two (divergent) expressions

$$\mathrm{Tr}\left(\Lambda_{N,\tilde{\delta}}^J\right) = \int F_{J,\delta}^N(0, \dots, 0|y_1, \dots, y_J) \prod_{i=1}^J \frac{d^d y_i}{\pi^{\frac{d}{2}}}. \quad (3.3.6)$$

The operator $\Lambda_{N,\tilde{\delta}}$ acts on $\mathcal{V}_{\frac{d}{2}+\delta}^{\otimes N}$ with a kernel given by

$$\Lambda_{N,\tilde{\delta}}(x_1, \dots, x_N|y_1, \dots, y_N) = \frac{1}{\prod_{i=1}^N (x_i - y_i)^{2\tilde{\delta}} (x_{i-1} - x_i)^{2\delta}}, \quad (3.3.7)$$

for $x_0 = 0$, see Figure 3.5.

While $F_{J,\delta}$ was associated to a closed spin chain, these operators rather correspond to an open one. They turn out to be easier to handle: it is possible to diagonalise them, and, from this, one can extract all the information required to formulate the thermodynamic Bethe ansatz equations for the conformal dimensions γ_J . This will be the subject of the next two chapters. Based on the situation in 2 dimensions, we believe that the diagonalisation of $\Lambda_{N,\tilde{\delta}}$ is an important step towards the solution of this non-compact spin chain via separation of variables. Indeed, when $d = 2$, the conformal group is $SL(2, \mathbb{C})$ and the eigenvectors of $\Lambda_{N,\tilde{\delta}}$ have already been known for some time [206]. They actually correspond to the eigenvectors of the operator A appearing in the 2×2 monodromy matrix, and they are very similar to the basis of separated variables [67]. The situation is however much less clear in higher dimensions for the moment.

3.4 The $J = 2$ Case

If the spin chain contains only two sites, then the only non-trivial coefficient of \mathbb{T}_\pm is the total Casimir operator. In other words, the conformal symmetry alone is enough to determine the eigenvectors of the model: the decomposition of the quantum space into eigenspaces of the chain is the same as the decomposition into irreducible representations of the conformal group.

For any real ν and integer l , there exists an equivariant map $\Pi_{\delta,\delta}^{(l,\nu)}$ from $\mathcal{V}_{d-\delta} \otimes \mathcal{V}_{d-\delta}$ onto $\mathcal{V}_{l, \frac{d}{2} + i\nu}$, cf. (3.1.10), with kernel given by

$$\zeta^{\otimes l} \cdot \Pi_{\delta,\delta}^{(l,\nu)}(x_0|x_1, x_2) = \frac{1}{x_{12}^{2\delta}} \left(\frac{x_{12}^2}{x_{10}^2 x_{20}^2} \right)^{\frac{\Delta-l}{2}} \left[\zeta \cdot \left(\frac{x_{10}}{x_{10}^2} - \frac{x_{20}}{x_{20}^2} \right) \right]^l, \quad (3.4.1)$$

where $\zeta \in \mathbb{C}^d$ is some reference null vector ($\zeta^2 = 0$), $\zeta^{\otimes l} = \zeta \otimes \cdots \otimes \zeta$ is a symmetric traceless tensor of rank l , $\zeta \cdot x = \zeta^\mu x_\mu$, and $\Delta = \frac{d}{2} + i\nu$. The kernel also admits a representation as a conformal three-point function

$$\Pi_{\delta,\delta}^{(l,\nu)}(x_0|x_1, x_2) = \langle \text{Tr}(Z(x_1)Z(x_2))\mathcal{O}_{l,\Delta}(x_0) \rangle, \quad (3.4.2)$$

for some operator $\mathcal{O}_{l,\Delta}$ with scaling dimension Δ and spin l . These projectors satisfy the orthogonality relations [196]

$$\begin{aligned} & \int \overline{\zeta'^{\otimes l'} \cdot \Pi_{\delta,\delta}^{(l',\nu')}(x'_0|x_1, x_2)} \zeta^{\otimes l} \cdot \Pi_{d-\delta, d-\delta}^{(l,\nu)}(x_0|x_1, x_2) \frac{d^d x_1 d^d x_2}{\pi^d} \\ &= \frac{2\pi \delta_{l,\nu}}{\rho_l(\nu)} \left(\delta(\nu - \nu') \pi^{\frac{d}{2}} \delta^{(d)}(x_{00'}) (\bar{\zeta}' \cdot \zeta)^l + c_l(\nu) \delta(\nu + \nu') \left[G_{(l,\Delta)}(x_{00'}) \bar{\zeta}'^{\otimes l} \right] \cdot \zeta^{\otimes l} \right), \end{aligned} \quad (3.4.3)$$

where the measure is

$$\rho_l(\nu) = 2^{l-1} \frac{\Gamma\left(\frac{d}{2} + l\right) \Gamma\left(\frac{d}{2} - 1 + i\nu\right) \Gamma\left(\frac{d}{2} - 1 - i\nu\right)}{l! \Gamma(i\nu) \Gamma(-i\nu)} \left[\left(\frac{d}{2} + l - 1\right)^2 + \nu^2 \right], \quad (3.4.4)$$

$$c_l(\nu) = \left(\frac{\Gamma\left(\frac{d-\Delta+l}{2}\right)}{\Gamma\left(\frac{\Delta+l}{2}\right)} \right)^2 = \left(\frac{\Gamma\left(\frac{d}{4} + \frac{l-i\nu}{2}\right)}{\Gamma\left(\frac{d}{4} + \frac{l+i\nu}{2}\right)} \right)^2, \quad (3.4.5)$$

and one has

$$\left[G_{(l,\Delta)}(x) \zeta^{\otimes l} \right] \cdot \eta^{\otimes l} = k_l(\Delta) \frac{\left(\zeta \cdot \eta - 2 \frac{\zeta \cdot x \eta \cdot x}{x^2} \right)^l}{x^{2\Delta}}, \quad k_l(\Delta) = \frac{\Gamma(\Delta + l) \Gamma(d - \Delta - 1)}{\Gamma(d - \Delta + l - 1) \Gamma\left(\frac{d}{2} - \Delta\right)}. \quad (3.4.6)$$

There are two terms in the orthogonality relation because the representations (l, Δ) and $(l, d - \Delta)$ are equivalent. The intertwining operator between these representations is [196]

$$G_{(l,\Delta)} : \mathcal{V}_{l, d-\Delta} \longrightarrow \mathcal{V}_{l, \Delta}, \quad (3.4.7)$$

with kernel given above. When $l = 0$, this is exactly the operator G_Δ introduced in (3.1.7). The projectors onto these two equivalent representations are related by the application of the intertwining operator according to [207]

$$G_{(l,\Delta)}\Pi_{\delta,\delta}^{(l,-\nu)} = \frac{1}{c_l(\nu)}\Pi_{\delta,\delta}^{(l,\nu)}, \quad (3.4.8)$$

or, in terms of kernels,

$$\int \left[G_{(l,\Delta)}(x_{00'})\Pi_{\delta,\delta}^{(l,-\nu)}(x'_0|x_1, x_2) \right] \cdot \eta^{\otimes l} \frac{d^d x'_0}{\pi^{\frac{d}{2}}} = \frac{1}{c_l(\nu)} \eta^{\otimes l} \cdot \Pi_{\delta,\delta}^{(l,\nu)}(x_0|x_1, x_2), \quad (3.4.9)$$

where $\eta^2 = 0$. In a CFT, the kernel of the operator $G_{(l,d-\Delta)}$ is also used to define the nonlocal “shadow operator” $\tilde{\mathcal{O}}_{l,d-\Delta}$ associated to a rank- l symmetric traceless operator $\mathcal{O}_{l,\Delta}$ [208]. In other words, if the projector is the three-point function (3.4.2) then

$$\left[G_{(l,d-\Delta)}\Pi_{\delta,\delta}^{(l,\nu)} \right] (x_0|x_1, x_2) = \langle \text{Tr}(Z(x_1)Z(x_2))\tilde{\mathcal{O}}_{l,d-\Delta}(x_0) \rangle. \quad (3.4.10)$$

Furthermore, it is known [196] that for $\delta > \frac{d}{4}$ (remember that the fishnet theory was defined for $0 < \delta < \frac{d}{2}$) the tensor product $\mathcal{V}_\delta \otimes \mathcal{V}_\delta$ decomposes only into unitary principal series representations, i.e. one has the following completeness relation:

$$\begin{aligned} \frac{1}{2} \sum_{l=0}^{+\infty} \int_{-\infty}^{+\infty} \rho_l(\nu) \int \overline{\Pi_{\delta,\delta}^{(l,\nu)}(x_0|x_1, x_2)} \cdot \Pi_{d-\delta,d-\delta}^{(l,\nu)}(x_0|y_1, y_2) \frac{d^d x_0}{\pi^{\frac{d}{2}}} \frac{d\nu}{2\pi} \\ = \pi^d \delta^{(d)}(x_1 - y_1) \delta^{(d)}(x_2 - y_2). \end{aligned} \quad (3.4.11)$$

Even if, from conformal invariance, we know that there exists an eigenvalue $T_{\Delta_0}^{(l,\nu)}(u) \in \mathbb{C}$ such that

$$\Pi_{d-\delta,d-\delta}^{(l,\nu)} T_{\Delta_0}(u) = T_{\Delta_0}^{(l,\nu)}(u) \Pi_{d-\delta,d-\delta}^{(l,\nu)}, \quad (3.4.12)$$

there is no simple expression for this eigenvalue in general. We recall that Δ_0 denotes the (arbitrary) conformal dimension of the auxiliary space. However, when $\Delta_0 = \tilde{\delta}$ and $u \rightarrow -\frac{d}{4}$, it is much simpler to compute the eigenvalues $f_l(\nu)$ of $F_{2,\delta}$, they are given by

$$f_l(\nu) = A_0^2(\delta) \frac{\Gamma\left(-\frac{d}{4} + \delta + \frac{l+i\nu}{2}\right) \Gamma\left(-\frac{d}{4} + \delta + \frac{l-i\nu}{2}\right)}{\Gamma\left(\frac{3d}{4} - \delta + \frac{l+i\nu}{2}\right) \Gamma\left(\frac{3d}{4} - \delta + \frac{l-i\nu}{2}\right)}. \quad (3.4.13)$$

Let us now apply these results to the computation of

$$G(x_1, x_2, x_3, x_4) = \langle \text{Tr}(Z(x_1)Z(x_2)) \text{Tr}(Z^\dagger(x_3)Z^\dagger(x_4)) \rangle. \quad (3.4.14)$$

Since the anisotropic case $\delta \neq \frac{d}{4}$ considered here is almost identical to the isotropic case considered in [144, 158, 159], the rest of the section is simply a review of some of the computations of these articles.

If δ is in generic position, then there is no counterterm contributing to this four-point function, and all the graphs are of wheel type. This is similar to what was considered in the

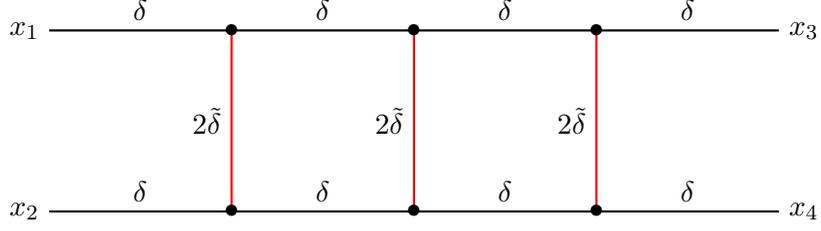


Figure 3.6: One of the graphs appearing in the expansion of $G(x_1, x_2, x_3, x_4)$. The vertical lines have weight $2\tilde{\delta}$ because they contain two magnon propagators. The same graph with $x_1 \leftrightarrow x_2$ should also be considered.

previous section, but now the points are split, see Figure 3.6. As a consequence, if $\delta > \frac{d}{4}$, there is no divergence and we can directly make use of the fact that the sum is of geometric type:

$$G(x_1, x_2, x_3, x_4) = \left(\frac{A_0(\tilde{\delta})}{4\tilde{\delta}\pi^{\frac{d}{2}}} \right)^2 (x_{34}^2)^{2\tilde{\delta}} \left[\frac{F_{2,\delta}}{1 - \xi^4 F_{2,\delta}} \right] (x_1, x_2 | x_3, x_4) + (x_1 \leftrightarrow x_2). \quad (3.4.15)$$

Inserting the completeness relation allows us to write

$$(x_{34}^2)^{2\tilde{\delta}} \left[\frac{F_{2,\delta}}{1 - \xi^4 F_{2,\delta}} \right] (x_1, x_2 | x_3, x_4) = \frac{(x_{34}^2)^{2\tilde{\delta}}}{2} \sum_{l=0}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_l(\nu)}{f_l^{-1}(\nu) - \xi^4} \\ \times \int \overline{\Pi_{\tilde{\delta},\tilde{\delta}}^{(l,\nu)}(x_0 | x_1, x_2)} \cdot \Pi_{d-\tilde{\delta},d-\tilde{\delta}}^{(l,\nu)}(x_0 | x_3, x_4) \frac{d^d x_0}{\pi^{\frac{d}{2}}} \frac{d\nu}{2\pi}. \quad (3.4.16)$$

The projectors clearly satisfy $(x_{34}^2)^{2\tilde{\delta}} \Pi_{d-\tilde{\delta},d-\tilde{\delta}}^{(l,\nu)}(x_0 | x_3, x_4) = \Pi_{\tilde{\delta},\tilde{\delta}}^{(l,\nu)}(x_0 | x_3, x_4)$ and the parity property $\Pi_{\tilde{\delta},\tilde{\delta}}^{(l,\nu)}(x_0 | x_2, x_1) = (-1)^l \Pi_{\tilde{\delta},\tilde{\delta}}^{(l,\nu)}(x_0 | x_1, x_2)$, so that only even spins contribute to the full correlator, which becomes

$$G(x_1, x_2, x_3, x_4) = \left(\frac{A_0(\tilde{\delta})}{4\tilde{\delta}\pi^{\frac{d}{2}}} \right)^2 \sum_{l=0}^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_{2l}(\nu)}{f_{2l}^{-1}(\nu) - \xi^4} \\ \times \int \overline{\Pi_{\tilde{\delta},\tilde{\delta}}^{(2l,\nu)}(x_0 | x_1, x_2)} \cdot \Pi_{\tilde{\delta},\tilde{\delta}}^{(2l,\nu)}(x_0 | x_3, x_4) \frac{d^d x_0}{\pi^{\frac{d}{2}}} \frac{d\nu}{2\pi}. \quad (3.4.17)$$

The integral over x_0 is called a conformal partial wave and can be expressed as a sum of two conformal blocks [196, 208, 209]:

$$\int \overline{\Pi_{\tilde{\delta},\tilde{\delta}}^{(l,\nu)}(x_0 | x_1, x_2)} \cdot \Pi_{\tilde{\delta},\tilde{\delta}}^{(l,\nu)}(x_0 | x_3, x_4) \frac{d^d x_0}{\pi^{\frac{d}{2}}} = \frac{(-2)^{-l}}{(x_{12}^2 x_{34}^2)^{\tilde{\delta}}} \left(\frac{g_{(l,\Delta)}(u, v)}{c_l(\nu) k_l(\nu)} + \frac{g_{(l,d-\Delta)}(u, v)}{c_l(-\nu) k_l(-\nu)} \right), \quad (3.4.18)$$

with c_l, k_l defined in (3.4.5) and (3.4.6) respectively, $\Delta = \frac{d}{2} + i\nu$, and the conformal blocks behave as

$$g_{(l,\Delta)}(u, v) \sim u^{\frac{\Delta-l}{2}} (1-v)^l, \quad \text{when } u \rightarrow 0, v \rightarrow 1. \quad (3.4.19)$$

The two conformal invariants are

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (3.4.20)$$

We thus have

$$G(x_1, x_2, x_3, x_4) = \frac{\mathcal{G}(u, v)}{(x_{12}^2 x_{34}^2)^\delta}, \quad (3.4.21)$$

with

$$\mathcal{G}(u, v) = \left(\frac{A_0(\tilde{\delta})}{4^{\tilde{\delta} \frac{d}{2}}} \right)^2 \sum_{l=0}^{+\infty} \int_{-\infty}^{+\infty} \frac{g_{(2l, \Delta)}(u, v)}{f_{2l}^{-1}(\nu) - \xi^4} \mu_{2l}(\nu) \frac{d\nu}{2\pi}, \quad (3.4.22)$$

and

$$\mu_l(\nu) = \frac{2\rho_l(\nu)}{(-2)^l c_l(\nu) k_l(\nu)} = (-1)^l \frac{\Gamma\left(\frac{d}{2} + l\right) \Gamma\left(\frac{d}{2} + i\nu - 1\right) \Gamma\left(\frac{d}{2} - i\nu + l\right) \Gamma^2\left(\frac{d}{4} + \frac{l+i\nu}{2}\right)}{l! \Gamma(i\nu) \Gamma\left(\frac{d}{2} + i\nu + l - 1\right) \Gamma^2\left(\frac{d}{4} + \frac{l-i\nu}{2}\right)}. \quad (3.4.23)$$

When $u \rightarrow 0$, because of the asymptotic behaviour (3.4.19) of the conformal blocks, one can close the contour in the lower half-plane and compute the integral by residues. The result is of the form

$$\mathcal{G}(u, v) = \sum_{l=0}^{+\infty} \sum_{\Delta} C_{(2l, \Delta)}^2 g_{(2l, \Delta)}(u, v), \quad (3.4.24)$$

expected of the correlation function of four identical scalar fields in a conformal field theory. Each term of the sums represents the total contribution of a multiplet with a primary of spin $2l$ and conformal dimension Δ , and $C_{(2l, \Delta)}$ are the structure constants appearing in the OPE.

There are poles coming from the conformal blocks, from the measure μ_{2l} , and from the denominator. However, the first two series of poles are spurious as they are independent of the model and only come from conformal symmetry. We thus expect that they cancel, and this is what happens, as we shall now explain. The conformal block $g_{(l, \Delta)}$ has a pole at $\Delta = l + d - 1 - 2n$ for integer n such that $1 \leq 2n \leq l$ with residue given by [210, 211]¹

$$g_{(l, l+d-1-2n+\epsilon)} \sim \frac{r_{l, 2n}}{\epsilon} g_{(l-2n, l+d-1)}, \quad (3.4.25)$$

with

$$r_{l, n} = \frac{(-1)^{n+1} n!}{(n!)^2 (l-n)!} \frac{(d+l-n-2)_n}{\left(\frac{d}{2} + l - n\right)_n \left(\frac{d}{2} + l - n - 1\right)_n} \left(\frac{1-n}{2}\right)_n^2. \quad (3.4.26)$$

The contribution of all these residues to the integral in \mathcal{G} is thus

$$\begin{aligned} & - \sum_{l=0}^{+\infty} \sum_{1 \leq 2n \leq l} r_{2l, 2n} \frac{g_{(2l-2n, 2l+d-1)}}{f_{2l}^{-1}\left(i\left(2n+1-2l-\frac{d}{2}\right)\right) - \xi^4} \mu_{2l}\left(i\left(2n+1-2l-\frac{d}{2}\right)\right) \\ & = - \sum_{l=0}^{+\infty} \sum_{n=1}^{+\infty} r_{2l+2n, 2n} \frac{g_{(2l, 2l+2n+d-1)}}{f_{2l+2n}^{-1}\left(i\left(1-2l-\frac{d}{2}\right)\right) - \xi^4} \mu_{2l+2n}\left(i\left(1-2l-\frac{d}{2}\right)\right). \end{aligned} \quad (3.4.27)$$

¹We have used the convention of [212] for the definition of the conformal block; it does not coincide with the definition in [211]: $g_{(l, \Delta)}^{\text{here}} = 2^l g_{\Delta l}^{\text{there}}$. The residues get modified accordingly.

On the other hand, the only poles of the measure μ_l in the lower half-plane $\text{Im}(\nu) < 0 \Leftrightarrow \text{Re}(\Delta) > \frac{d}{2}$ are simple poles located at $\Delta = d + l + 2n - 1$, for integer $n \geq 1$, and it is straightforward to check that the residue is given by

$$\mu_l \left(-i \left(\frac{d}{2} + l + 2n - 1 \right) + \epsilon \right) \sim i \frac{r_{l+2n, 2n}}{\epsilon} \mu_{l+2n} \left(i \left(1 - l - \frac{d}{2} \right) \right). \quad (3.4.28)$$

Consequently, the contributions from the residues of the conformal blocks and the measure combine to give

$$\begin{aligned} & \sum_{l=0}^{+\infty} \sum_{n=1}^{+\infty} r_{2l+2n, 2n} g_{(2l, 2l+2n+d-1)} \mu_{2l+2n} \left(i \left(1 - 2l - \frac{d}{2} \right) \right) \\ & \times \left[\frac{1}{f_{2l}^{-1} \left(-i \left(\frac{d}{2} + 2l + 2n - 1 \right) \right) - \xi^4} - \frac{1}{f_{2l+2n}^{-1} \left(i \left(1 - 2l - \frac{d}{2} \right) \right) - \xi^4} \right] = 0 \end{aligned} \quad (3.4.29)$$

because the eigenvalue (3.4.13) satisfies

$$f_l \left(-i \left(\frac{d}{2} + l + 2n - 1 \right) \right) = f_{l+2n} \left(i \left(1 - l - \frac{d}{2} \right) \right). \quad (3.4.30)$$

Because the conformal partial waves (3.4.18) form a basis of the space of four-point functions of scalar operators with given conformal dimension, the form (3.4.22) of the four-point function is specific neither to the correlation function we are considering nor to the fishnet CFT. In the general case, the same expression (with a sum over all spins, not only even ones) will hold if $(f_{2l}^{-1}(\nu) - \xi^4)^{-1}$ is replaced with some appropriate, model-dependent, function. The cancellation of the spurious poles should thus be universal, and this was proven in [212, 213]. For the fishnet theory, this cancellation was detailed in [159].

Eventually, the only poles left are those found at the solutions of $(f_{2, \delta}^{(2l, \nu)})^{-1} = \xi^4$ which, in terms of the scaling dimension $\Delta = \frac{d}{2} + i\nu$ of the exchanged operator, reads

$$\frac{\Gamma \left(\tilde{\delta} + l + \frac{\Delta}{2} \right) \Gamma \left(\tilde{\delta} + l + \frac{d-\Delta}{2} \right)}{\Gamma \left(-\tilde{\delta} + l + \frac{\Delta}{2} \right) \Gamma \left(-\tilde{\delta} + l + \frac{d-\Delta}{2} \right)} = A_0^2(\delta) \xi^4, \quad \text{Re}(\Delta) > \frac{d}{2}. \quad (3.4.31)$$

When $2\tilde{\delta} \notin \mathbb{N}$, there are infinitely many, generically complex, solutions to the previous equation. They can easily be determined perturbatively because, when $\xi^2 = 0$, they are situated at the zeros of the left-hand side:

$$\Delta_{l, n} = 2\delta + 2l + 2n + 2A_0^2(\delta) \xi^4 \frac{(-1)^{n+1} \Gamma \left(2\delta - \frac{d}{2} + 2l + n \right)}{n! \Gamma \left(\frac{d}{2} + 2l + n \right) \Gamma \left(2\tilde{\delta} - n \right)} + O(\xi^8), \quad (3.4.32)$$

for l and n in \mathbb{N} . When $2\tilde{\delta} = m \in \mathbb{N}$, the left-hand side becomes a polynomial of degree $2m$ (actually, a polynomial of degree m in ν^2), and there are only up to m solutions with $\text{Re}(\Delta) > \frac{d}{2}$.

Finally, let us consider the solution with the smallest possible real value, i.e.

$$\Delta_{0,0} = 2\delta - 2A_0^2(\delta)\xi^4 \frac{\Gamma\left(2\delta - \frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(2\delta\right)} + O(\xi^8). \quad (3.4.33)$$

It is associated to a scalar operator of bare dimension 2δ . It also determines the leading asymptotic behaviour of the four-point function when $u \rightarrow 0$. This solution is thus the conformal dimension of $\text{Tr}(Z^2)$. We now remark that, when $\delta \rightarrow \frac{d}{4}$, the previous expansion of $\Delta_{0,0}$ manifestly fails. Going back to (3.4.31) for $l = 0$ and putting $\delta = \frac{d}{4}$ there, one realises that [144]

$$\Delta_{0,0} = \frac{d}{2} \pm \frac{2i\xi^2}{\Gamma\left(\frac{d}{2}\right)} \pm \frac{i\xi^6 \left[\psi'(1) - \psi'\left(\frac{d}{2}\right)\right]}{\Gamma^3\left(\frac{d}{2}\right)} + O(\xi^{10}), \quad (3.4.34)$$

where $\psi = \frac{\Gamma'}{\Gamma}$. This agrees with the fact that for $\delta = \frac{d}{4}$ the counterterms should be taken into account: the two possible solutions correspond to the two different values of the coupling α_{\pm} , which, as we explained in Subsection 2.3.2, starts at order ξ^2 .

Chapter 4

$O(d)$ R-Matrices and Graph-Building Operators

The content of this chapter and of Appendix B.1 reproduces most of [180], up to some changes in presentation, notably regarding the figures. Appendices B and E of the article have not been included in this thesis, since they correspond to very recent work done by S. Derkachov and E. Olivucci respectively.

The main goal of this chapter will be to find a complete basis of eigenvectors of the graph-building operators $\Lambda_{N,\delta}$ for arbitrary N . An interpretation of these eigenvectors as N -particle states in some $O(d)$ -invariant theory describing infinitely many massless particles will emerge. The theory in question will have one type of particle for each integer $l \in \mathbb{N}$, and such a particle will transform in the rank- l symmetric traceless representation of the orthogonal group. In the spirit of [176, 177, 179], we shall investigate possible applications to the computations of some fishnet four-point correlators such as those studied in [175]. However, because the scattering matrix between the particles will be, up to a scalar phase, the relevant $O(d)$ -invariant R-matrix, we shall need some efficient representation of these matrices. This is what we will begin with, presenting new explicit representations of $O(d)$ -invariant solutions to the Yang–Baxter relation. Of particular interest to us will be integral representations of these matrices.

4.1 $O(d)$ -Invariant R-Matrices

For $l \in \mathbb{N}$, we denote by \mathbb{V}_l the (complex) vector space of symmetric traceless tensors of rank l , in dimension d . We shall denote its dimension by d_l . We will sometimes refer to l as the spin.

As explained above, this section contains explicit expressions for the R-matrices \mathbb{R}_{l_1,l_2} acting in the tensor product $\mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2}$, and satisfying the Yang–Baxter relation in $\mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2} \otimes \mathbb{V}_{l_3}$ for arbitrary l_1, l_2, l_3 . Our starting point will be the Zamolodchikovs' R-matrix [120]

$$\forall C \in \mathbb{V}_1 \otimes \mathbb{V}_1, \quad [\mathbb{R}_{1,1}(u)C]^{\mu\nu} = \frac{1}{u+i} \left[uC^{\mu\nu} + iC^{\nu\mu} - \frac{iu}{u+i\frac{d-2}{2}} C^\rho{}_\rho \delta^{\mu\nu} \right]. \quad (4.1.1)$$

In a first part, we apply the fusion procedure [184, 185] to the construction of the general R-matrices \mathbb{R}_{l_1, l_2} . The fusion procedure was used for the calculation of R-matrices $\mathbb{R}_{1,2}$, $\mathbb{R}_{1,3}$ and $\mathbb{R}_{2,2}$ by N. MacKay [214], and for $\mathbb{R}_{1,l}$ by N. Reshetikhin [215], we therefore generalise their results. We will then argue that there exist equivalent integral representations of these matrices. Those integral representations can be used to give direct proofs of the unitarity of the R-matrices, and of the Yang–Baxter relation. And they will be used extensively in the next section as they also allow to prove symmetry properties of the eigenvectors of the graph-building operators under the exchange of particles.

The equivalence of the two expressions for \mathbb{R}_{l_1, l_2} —the integral representation and the representation obtained directly by fusion procedure—is far from obvious. The proof is very technical and we do not reproduce it here, but it is in Appendix B of [180]. Finally, we should note that the spectral decomposition for the general R-matrices \mathbb{R}_{l_1, l_2} was actually obtained thirty years ago by N. MacKay [214, 216]. His result is in some sense complementary to both our expressions and we check their equivalence in the case of $\mathbb{R}_{1,l}$.

4.1.1 From Fusion

We show in this subsection that the R-matrix acting on $\mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2}$ is given by the following matrix elements

$$\begin{aligned} \zeta_1^{\otimes l_1} \otimes \eta_1^{\otimes l_2} \cdot \left[\mathbb{R}_{l_1, l_2}(u) \zeta_2^{\otimes l_1} \otimes \eta_2^{\otimes l_2} \right] &= \frac{\left(iu + \frac{l_2 - l_1}{2} \right)_{l_1}}{\left(iu - \frac{l_1 + l_2}{2} \right)_{l_1}} \sum_{\substack{k \geq 0, n \geq 0 \\ k+n \leq \min(l_1, l_2)}} \frac{l_1! l_2!}{k! n! (l_1 - k - n)! (l_2 - k - n)!} \\ &\times \frac{(\zeta_1 \cdot \eta_1 \zeta_2 \cdot \eta_2)^k (\zeta_1 \cdot \eta_2 \eta_1 \cdot \zeta_2)^n}{\left(iu + \frac{4 - l_1 - l_2 - d}{2} \right)_k \left(-iu + \frac{2 - l_1 - l_2}{2} \right)_n} (\zeta_1 \cdot \zeta_2)^{l_1 - k - n} (\eta_1 \cdot \eta_2)^{l_2 - k - n}, \quad (4.1.2) \end{aligned}$$

where all contractions of tensor indices, abbreviated with a dot, are done using the Euclidean metric $\delta_{\mu\nu}$, and ζ_1, ζ_2, η_1 , and η_2 are four null vectors in \mathbb{C}^d : one has for instance

$$\zeta_1^2 = \zeta_1 \cdot \zeta_1 = \zeta_{1\mu} \zeta_1^\mu = \zeta_1^\mu \zeta_{1\nu} \delta_{\mu\nu} = 0. \quad (4.1.3)$$

We also use the Pochhammer symbol

$$(a)_l = \frac{\Gamma(a+l)}{\Gamma(a)} = \prod_{k=0}^{l-1} (a+k). \quad (4.1.4)$$

The proof of (4.1.2) is done in two steps. We first apply fusion to increase one of the spins, keeping the other equal to 1. In that case, the previous formula contains only three terms and reads

$$\begin{aligned} \zeta_1 \otimes \eta_1^{\otimes l} \cdot \left[\mathbb{R}_{1,l}(u) \zeta_2 \otimes \eta_2^{\otimes l} \right] &= \frac{1}{u + i \frac{l+1}{2}} \left[\left(u - i \frac{l-1}{2} \right) \zeta_1 \cdot \zeta_2 (\eta_1 \cdot \eta_2)^l \right. \\ &\left. + il \zeta_1 \cdot \eta_2 \eta_1 \cdot \zeta_2 (\eta_1 \cdot \eta_2)^{l-1} - il \frac{u - i \frac{l-1}{2}}{u + i \frac{d+l-3}{2}} \zeta_1 \cdot \eta_1 \zeta_2 \cdot \eta_2 (\eta_1 \cdot \eta_2)^{l-1} \right]. \quad (4.1.5) \end{aligned}$$

Equivalently, we could have written, for $C \in \mathbb{V}_1 \otimes \mathbb{V}_l$,

$$\begin{aligned} [\mathbb{R}_{1,l}(u)C]^{\mu\nu_1 \cdots \nu_l} &= \frac{1}{u + i\frac{l+1}{2}} \left[\left(u - i\frac{l-1}{2} \right) C^{\mu\nu_1 \cdots \nu_l} + i \sum_{j=1}^l C^{\nu_j \mu \nu_1 \cdots \widehat{\nu}_j \cdots \nu_l} \right. \\ &\quad \left. - i \frac{u - i\frac{l-1}{2}}{u + i\frac{d+l-3}{2}} \sum_{j=1}^l \delta^{\mu\nu_j} C^{\rho \nu_1 \cdots \widehat{\nu}_j \cdots \nu_l} + \frac{1}{u + i\frac{d+l-3}{2}} \sum_{1 \leq j < k \leq l} \delta^{\nu_j \nu_k} C^{\rho \mu \nu_1 \cdots \widehat{\nu}_j \cdots \widehat{\nu}_k \cdots \nu_l} \right]. \end{aligned} \quad (4.1.6)$$

Before proving this last formula, we point out that, when $u = i\frac{l+1}{2}$, it reduces to the orthogonal projector $\mathbb{P}_{1,l}^{(l+1)}$ onto $\mathbb{V}_{l+1} \subset \mathbb{V}_1 \otimes \mathbb{V}_l$. This is important because it justifies why the fusion procedure gives new solutions of the Yang–Baxter relation. A proof of this fact goes as follows: first, one notices, from (4.1.6), that $\mathbb{R}_{1,l}(i\frac{l+1}{2})C$ is symmetric traceless in all $l+1$ indices. After that, it is enough to remark that its contraction with any other symmetric traceless tensor $C' \in \mathbb{V}_{l+1}$ is given by $C' \cdot C$.

The proof is made by induction, (4.1.6) clearly holds for $l = 1$, so we assume that it holds for some $l \geq 1$. Let us show it for $l + 1$, where the fusion procedure states that

$$\mathbb{P}_{1',l}^{(l+1)} \mathbb{R}_{1,1'} \left(u - \frac{il}{2} \right) \mathbb{R}_{1,l} \left(u + \frac{i}{2} \right) \mathbb{P}_{1',l}^{(l+1)} = \mathbb{R}_{1,l+1}(u). \quad (4.1.7)$$

We remind the reader that, because of the Yang–Baxter equation, the left projector is useless and could be removed. Consequently, applying the left-hand side to $C \in \mathbb{V}_1 \otimes \mathbb{V}_{l+1} \subset \mathbb{V}_1 \otimes \mathbb{V}'_1 \otimes \mathbb{V}_l$ gives

$$\begin{aligned} \left[\mathbb{R}_{1,1'} \left(u - \frac{il}{2} \right) \mathbb{R}_{1,l} \left(u + \frac{i}{2} \right) C \right]^{\mu\nu_1 \cdots \nu_{l+1}} &= \frac{1}{u + i\frac{2-l}{2}} \left[\left(u - \frac{il}{2} \right) \left[\mathbb{R}_{1,l} \left(u + \frac{i}{2} \right) C \right]^{\mu\nu_1 \cdots \nu_{l+1}} \right. \\ &\quad \left. + i \left[\mathbb{R}_{1,l} \left(u + \frac{i}{2} \right) C \right]^{\nu_1 \mu \nu_2 \cdots \nu_{l+1}} - i \frac{u - \frac{il}{2}}{u + i\frac{d-2-l}{2}} \left[\mathbb{R}_{1,l} \left(u + \frac{i}{2} \right) C \right]^{\rho \nu_2 \cdots \nu_{l+1}} \delta^{\mu\nu_1} \right]. \end{aligned} \quad (4.1.8)$$

We now use equation (4.1.6) to write the second term in the right-hand side as

$$\begin{aligned} \left[\mathbb{R}_{1,l} \left(u + \frac{i}{2} \right) C \right]^{\nu_1 \mu \nu_2 \cdots \nu_{l+1}} &= \frac{1}{u + i\frac{l+2}{2}} \left[\left(u - i\frac{l-2}{2} \right) C^{\nu_1 \mu \nu_2 \cdots \nu_{l+1}} + i \sum_{j=2}^{l+1} C^{\nu_j \mu \nu_1 \cdots \widehat{\nu}_j \cdots \nu_{l+1}} \right. \\ &\quad \left. - i \frac{u - i\frac{l-2}{2}}{u + i\frac{d+l-2}{2}} \sum_{j=2}^{l+1} \delta^{\nu_1 \nu_j} C^{\rho \mu \nu_2 \cdots \widehat{\nu}_j \cdots \nu_{l+1}} + \frac{1}{u + i\frac{d+l-2}{2}} \sum_{2 \leq j < k \leq l+1} \delta^{\nu_j \nu_k} C^{\rho \mu \nu_1 \cdots \widehat{\nu}_j \cdots \widehat{\nu}_k \cdots \nu_{l+1}} \right], \end{aligned} \quad (4.1.9)$$

and, using the fact that C is symmetric traceless in the last $l+1$ indices, the third term is

$$\left[\mathbb{R}_{1,l} \left(u + \frac{i}{2} \right) C \right]^{\rho \mu \nu_2 \cdots \nu_{l+1}} = \frac{\left(u - i\frac{l-2}{2} \right) \left(u + i\frac{d-l-2}{2} \right)}{\left(u + i\frac{l+2}{2} \right) \left(u + i\frac{d+l-2}{2} \right)} C^{\rho \mu \nu_2 \cdots \nu_{l+1}}. \quad (4.1.10)$$

Putting everything together we straightforwardly recover (4.1.6) for $\mathbb{R}_{1,l+1}(u)$.

We now turn our attention to the general case of two arbitrary spins, i.e. Equation (4.1.2). It suffices to prove it for $l_1 \leq l_2$, which we shall do by induction on l_1 for given l_2 . We have just verified it for $l_1 = 1$. If we assume it holds for some $l_1 \leq l_2 - 1$, one just needs to use fusion,

$$\begin{aligned} \zeta_1^{\otimes l_1+1} \otimes \eta_1^{\otimes l_2} \cdot \mathbb{R}_{l_1, l_2} \left(u + \frac{i}{2} \right) \mathbb{R}_{1, l_2} \left(u - \frac{il_1}{2} \right) \zeta_2^{\otimes l_1+1} \otimes \eta_2^{\otimes l_2} \\ = \zeta_1^{\otimes l_1+1} \otimes \eta_1^{\otimes l_2} \cdot \mathbb{R}_{l_1+1, l_2}(u) \zeta_2^{\otimes l_1+1} \otimes \eta_2^{\otimes l_2}, \end{aligned} \quad (4.1.11)$$

to compute $\mathbb{R}_{l_1+1, l_2}(u)$. In the previous equation, the product of the two R-matrices is taken in \mathbb{V}_{l_2} . In order to compute this product, one may insert a resolution of the identity of \mathbb{V}_{l_2} between the two matrices. More explicitly, if $\{C_{j, l_2}\}_{1 \leq j \leq d_l}$ is an orthonormal basis of \mathbb{V}_l (for the inner product $(C, C') = C_{\mu_1 \dots \mu_l}^* C'^{\mu_1 \dots \mu_l} = C^* \cdot C'$), one can write

$$\begin{aligned} \zeta_1^{\otimes l_1+1} \otimes \eta_1^{\otimes l_2} \cdot \mathbb{R}_{l_1, l_2}(a) \mathbb{R}_{1, l_2}(b) \zeta_2^{\otimes l_1+1} \otimes \eta_2^{\otimes l_2} \\ = \sum_{j=1}^{d_{l_2}} \zeta_1^{\otimes l_1} \otimes \eta_1^{\otimes l_2} \cdot \left[\mathbb{R}_{l_1, l_2}(a) \zeta_2^{\otimes l_1} \otimes C_{j, l_2} \right] \zeta_1 \otimes C_{j, l_2}^* \cdot \left[\mathbb{R}_{1, l_2}(b) \zeta_2 \otimes \eta_2^{\otimes l_2} \right]. \end{aligned} \quad (4.1.12)$$

According to formulae (4.1.5) and (4.1.2), for \mathbb{R}_{1, l_2} and \mathbb{R}_{l_1, l_2} respectively, we can thus write

$$\begin{aligned} \zeta_1^{\otimes l_1+1} \otimes \eta_1^{\otimes l_2} \cdot \mathbb{R}_{l_1, l_2} \left(u + \frac{i}{2} \right) \mathbb{R}_{1, l_2} \left(u - \frac{il_1}{2} \right) \zeta_2^{\otimes l_1+1} \otimes \eta_2^{\otimes l_2} \\ = \frac{\left(iu + \frac{l_2 - l_1 - 1}{2} \right)_{l_1+1}}{\left(iu - \frac{l_1 + l_2 + 1}{2} \right)_{l_1+1}} \sum_{k+n \leq l_1} \frac{l_1! l_2!}{k! n! (l_1 - k - n)! (l_2 - k - n)!} \frac{(\zeta_1 \cdot \eta_1)^k (\eta_1 \cdot \zeta_2)^n (\zeta_1 \cdot \zeta_2)^{l_1 - k - n}}{\left(iu + \frac{3 - l_1 - l_2 - d}{2} \right)_k \left(-iu + \frac{3 - l_1 - l_2}{2} \right)_n} \\ \sum_{j=1}^{d_{l_2}} (\zeta_2^{\otimes k} \otimes \zeta_1^{\otimes n} \otimes \eta_1^{\otimes (l_2 - k - n)} \cdot C_{j, l_2}) \left[\zeta_1 \cdot \zeta_2 (C_{j, l_2}^* \cdot \eta_2^{\otimes l_2}) + \frac{l \zeta_1 \cdot \eta_2 (C_{j, l_2}^* \cdot \zeta_2 \otimes \eta_2^{\otimes (l_2 - 1)})}{-iu + \frac{1 - l_1 - l_2}{2}} \right. \\ \left. + \frac{l \zeta_2 \cdot \eta_2 (C_{j, l_2}^* \cdot \zeta_1 \otimes \eta_2^{\otimes (l_2 - 1)})}{iu + \frac{3 + l_1 - l_2 - d}{2}} \right]. \end{aligned} \quad (4.1.13)$$

The only additional formulae needed are

$$\sum_{j=1}^{d_{l_2}} (\zeta_2^{\otimes k} \otimes \zeta_1^{\otimes n} \otimes \eta_1^{\otimes (l_2 - k - n)} \cdot C_{j, l_2}) (C_{j, l_2}^* \cdot \eta_2^{\otimes l_2}) = (\eta_2 \cdot \zeta_2)^k (\eta_2 \cdot \zeta_1)^n (\eta_2 \cdot \eta_1)^{l_2 - k - n} \quad (4.1.14)$$

and

$$\begin{aligned} \sum_{j=1}^{d_2} (\zeta_2^{\otimes k} \otimes \zeta_1^{\otimes n} \otimes \eta_1^{\otimes (l_2-k-n)} \cdot C_{j,l_2}) (C_{j,l_2}^* \cdot \zeta \otimes \eta_2^{\otimes (l_2-1)}) &= \frac{(\eta_2 \cdot \zeta_2)^{k-1} (\eta_2 \cdot \zeta_1)^{n-1} (\eta_2 \cdot \eta_1)^{l_2-k-n-1}}{l_2} \\ &\times \left[(l_2 - k - n) \eta_1 \cdot \zeta \zeta_1 \cdot \eta_2 \zeta_2 \cdot \eta_2 + n \zeta_1 \cdot \zeta \eta_1 \cdot \eta_2 \zeta_2 \cdot \eta_2 + k \zeta_2 \cdot \zeta \eta_1 \cdot \eta_2 \zeta_1 \cdot \eta_2 \right. \\ &\left. - \frac{2\zeta \cdot \eta_2}{d + 2(l_2 - 2)} [(l_2 - k - n)n \eta_1 \cdot \zeta_1 \zeta_2 \cdot \eta_2 + (l_2 - k - n)k \eta_1 \cdot \zeta_2 \zeta_1 \cdot \eta_2 + kn \zeta_1 \cdot \zeta_2 \eta_1 \cdot \eta_2] \right]. \end{aligned} \quad (4.1.15)$$

The first one is trivial since $\eta_2^{\otimes l_2} \in \mathbb{V}_{l_2}$ and $\{C_{j,l_2}\}_{1 \leq j \leq d_2}$ is an orthonormal basis of \mathbb{V}_{l_2} . The second one is a consequence of

$$\begin{aligned} \sum_{j=1}^{d_2} C_{j,l_2}^{\mu_1 \dots \mu_{l_2}} (C_{j,l_2}^* \cdot \zeta \otimes \eta_2^{\otimes (l_2-1)}) &= \frac{1}{l_2} \left[\sum_{i=1}^{l_2} \zeta^{\mu_i} \eta_2^{\mu_1} \dots \widehat{\eta_2^{\mu_i}} \dots \eta_2^{\mu_{l_2}} \right. \\ &\left. - \frac{2}{d + 2(l_2 - 2)} \sum_{1 \leq i < j \leq l_2} \zeta \cdot \eta_2 \delta^{\mu_i \mu_j} \eta_2^{\mu_1} \dots \widehat{\eta_2^{\mu_i}} \dots \widehat{\eta_2^{\mu_j}} \dots \eta_2^{\mu_{l_2}} \right], \end{aligned} \quad (4.1.16)$$

which is the orthogonal projection of $\zeta \otimes \eta_2^{\otimes (l_2-1)} \in \mathbb{V}_1 \otimes \mathbb{V}_{l_2-1}$ onto $\mathbb{V}_{l_2} \subset \mathbb{V}_1 \otimes \mathbb{V}_{l_2-1}$, as we already explained at the beginning of this section (recall that this projector is nothing else than $\mathbb{R}_{1,l_2-1} \left(i \frac{l_2}{2} \right)$).

Using these formulae, we can compute the sum over d_2 appearing in (4.1.13):

$$\begin{aligned} \sum_{j=1}^{d_2} (\zeta_2^{\otimes k} \otimes \zeta_1^{\otimes n} \otimes \eta_1^{\otimes (l_2-k-n)} \cdot C_{j,l_2}) &\left[\zeta_1 \cdot \zeta_2 (C_{j,l_2}^* \cdot \eta_2^{\otimes l_2}) \right. \\ &\left. + \frac{l \zeta_1 \cdot \eta_2 (C_{j,l_2}^* \cdot \zeta_2 \otimes \eta_2^{\otimes (l_2-1)})}{-iu + \frac{1-l_1-l_2}{2}} + \frac{l \zeta_2 \cdot \eta_2 (C_{j,l_2}^* \cdot \zeta_1 \otimes \eta_2^{\otimes (l_2-1)})}{iu + \frac{3+l_1-l_2-d}{2}} \right] \\ &= \frac{(\eta_2 \cdot \zeta_2)^k (\eta_2 \cdot \zeta_1)^n (\eta_2 \cdot \eta_1)^{l_2-k-n-1}}{\left(-iu + \frac{1-l_1-l_2}{2} \right) \left(iu + \frac{3+l_1-l_2-d}{2} \right)} \left[(\zeta_1 \cdot \zeta_2) (\eta_2 \cdot \eta_1) \left(-iu + \frac{1-l_1-l_2}{2} + n \right) \right. \\ &\times \left(iu + \frac{3+l_1-l_2-d}{2} + k \right) + (\eta_1 \cdot \zeta_2) (\eta_2 \cdot \zeta_1) (l_2 - k - n) \left(iu + \frac{3+l_1-l_2-d}{2} + k \right) \\ &\left. + (\eta_1 \cdot \zeta_1) (\eta_2 \cdot \zeta_2) (l_2 - k - n) \left(-iu + \frac{1-l_1-l_2}{2} + n \right) \right]. \end{aligned} \quad (4.1.17)$$

Plugging it back into (4.1.13), the only thing left to do is to rewrite the sum $\sum_{k+n \leq l_1}$ into a sum $\sum_{k'+n' \leq l_1+1}$. The terms contributing to a given pair (k', n') come from $(k, n) \in \{(k', n'), (k'-1, n'), (k', n'-1)\}$. When $(k, n) = (k', n')$, the contribution (without the tensors)

is

$$\frac{l_1!l_2!}{k'!n'!(l_1+1-k'-n')!(l_2-k'-n')!} \frac{l_1+1-k'-n'}{iu + \frac{3+l_1-l_2-d}{2}} \frac{iu + \frac{3+l_1-l_2-d}{2} + k'}{\left(iu + \frac{3-l_1-l_2-d}{2}\right)_{k'} \left(-iu + \frac{1-l_1-l_2}{2}\right)_{n'}}, \quad (4.1.18)$$

when $(k, n) = (k' - 1, n')$, it is

$$\frac{l_1!l_2!}{k'!n'!(l_1+1-k'-n')!(l_2-k'-n')!} \frac{k'}{iu + \frac{3+l_1-l_2-d}{2}} \frac{1}{\left(iu + \frac{3-l_1-l_2-d}{2}\right)_{k'-1} \left(-iu + \frac{1-l_1-l_2}{2}\right)_{n'}}, \quad (4.1.19)$$

and when $(k, n) = (k', n' - 1)$, it is

$$\frac{l_1!l_2!}{k'!n'!(l_1+1-k'-n')!(l_2-k'-n')!} \frac{n'}{iu + \frac{3+l_1-l_2-d}{2}} \frac{iu + \frac{3+l_1-l_2-d}{2} + k'}{\left(iu + \frac{3-l_1-l_2-d}{2}\right)_{k'} \left(-iu + \frac{1-l_1-l_2}{2}\right)_{n'}}. \quad (4.1.20)$$

The sum of the previous three terms is

$$\frac{(l_1+1)!l_2!}{k'!n'!(l_1+1-k'-n')!(l_2-k'-n')!} \frac{1}{\left(iu + \frac{3-l_1-l_2-d}{2}\right)_{k'} \left(-iu + \frac{1-l_1-l_2}{2}\right)_{n'}}, \quad (4.1.21)$$

thus proving formula (4.1.2) for $(l_1 + 1, l_2)$.

Extension of (4.1.2) to Symmetric Tensors We now want to compute $x^{\otimes l_1} \otimes y^{\otimes l_2} \cdot [\mathbb{R}_{l_1, l_2}(u)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}]$ when $\zeta^2 = \eta^2 = 0$ but $x^2 \neq 0$ and $y^2 \neq 0$. Since $\mathbb{R}_{l_1, l_2}(u)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}$ belongs to $\mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2}$, only the symmetric traceless parts of $x^{\otimes l_1}$ and $y^{\otimes l_2}$ are needed. Let us call X_l the symmetric traceless part of $x^{\otimes l}$, it is given by

$$X_l^{\mu_1 \dots \mu_l} = \sum_{p=0}^{\lfloor \frac{l}{2} \rfloor} \frac{(x^2)^p}{\binom{2-l-d}{p} 2^p} \sum_{\{i_1, j_1\}, \dots, \{i_p, j_p\}} \prod_{k=1}^p \delta^{\mu_{i_k} \mu_{j_k}} \prod_{i \notin \{i_1, j_1, \dots, i_p, j_p\}} x^{\mu_i}, \quad (4.1.22)$$

where, for a given p , we sum over the $\frac{l!}{(l-2p)!p!2^p}$ possible ways of forming p pairs among l elements. We can thus write

$$x^{\otimes l_1} \otimes y^{\otimes l_2} \cdot [\mathbb{R}_{l_1, l_2}(u)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}] = X_{l_1} \otimes Y_{l_2} \cdot [\mathbb{R}_{l_1, l_2}(u)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}], \quad (4.1.23)$$

and then apply (4.1.2).

Let us start with only one vector that is not null : $\alpha^2 = 0$ but $y^2 \neq 0$, we have

$$\begin{aligned} \alpha^{\otimes l_1} \otimes y^{\otimes l_2} \cdot [\mathbb{R}_{l_1, l_2}(u)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}] &= \frac{\left(iu + \frac{l_2-l_1}{2}\right)_{l_1}}{\left(iu - \frac{l_1+l_2}{2}\right)_{l_1}} \sum_{k, n \geq 0} \frac{(-l_1)_{k+n} (-l_2)_{k+n}}{k!n!} \\ &\times \frac{(\zeta \cdot \eta)^k (\alpha \cdot \eta)^n (\alpha \cdot \zeta)^{l_2-k-n}}{\left(iu + \frac{4-l_1-l_2-d}{2}\right)_k \left(-iu + \frac{2-l_1-l_2}{2}\right)_n} \left(Y_{l_2} \cdot \alpha^{\otimes k} \otimes \zeta^{\otimes n} \otimes \eta^{\otimes (l_2-k-n)}\right). \end{aligned} \quad (4.1.24)$$

We then use the explicit expression for Y_{l_2} to compute

$$Y_{l_2} \cdot \alpha^{\otimes k} \otimes \zeta^{\otimes n} \otimes \eta^{\otimes (l_2 - k - n)} = \sum_{q=0}^{\lfloor \frac{l_2}{2} \rfloor} \sum_{a+b \leq q} \frac{(-k)_{a+b} (-n)_{q-b} (n+k-l_2)_{q-a}}{a! b! (q-a-b)!} \frac{(y^2)^q}{\left(2-l_2-\frac{d}{2}\right)_q} 2^q \\ \times (\alpha \cdot \zeta)^a (\alpha \cdot \eta)^b (\zeta \cdot \eta)^{q-a-b} (\alpha \cdot y)^{k-a-b} (y \cdot \zeta)^{n+b-q} (y \cdot \eta)^{l_2+a-k-n-q}, \quad (4.1.25)$$

which implies

$$\alpha^{\otimes l_1} \otimes y^{\otimes l_2} \cdot \left[\mathbb{R}_{l_1, l_2}(u) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \\ = \frac{\left(iu + \frac{l_2 - l_1}{2}\right)_{l_1}}{\left(iu - \frac{l_1 + l_2}{2}\right)_{l_1}} \sum_{q=0}^{\lfloor \frac{l_2}{2} \rfloor} \sum_{\substack{K, N \geq q \\ a+b \leq q}} \frac{(-1)^{q+a} (-l_1)_{K+N+a-q} (-l_2)_{K+N}}{(K-q)! (N-q)! a! b! (q-a-b)!} \\ \times \frac{(\alpha \cdot \zeta)^{l_1+q-N-K} (\alpha \cdot \eta)^N (\zeta \cdot \eta)^K (y^2)^q (\alpha \cdot y)^{K-q} (y \cdot \zeta)^{N-q} (y \cdot \eta)^{l_2-N-K}}{2^q \left(2-l_2-\frac{d}{2}\right)_q \left(iu + \frac{4-l_1-l_2-d}{2}\right)_{K+a+b-q} \left(-iu + \frac{2-l_1-l_2}{2}\right)_{N-b}}, \quad (4.1.26)$$

where we have changed summation indices from k, n to $K = k + q - a - b$ and $N = n + b$. Recalling the Gauss identity

$$\sum_{k=0}^n \frac{(-n)_k (u)_k}{k! (v)_k} = \frac{(v-u)_n}{(v)_n}, \quad (4.1.27)$$

one can perform the sums over a and b :

$$\sum_{a+b \leq q} \frac{(-1)^{q+a} (-l_1)_{K+N+a-q}}{a! b! (q-a-b)!} \frac{1}{\left(iu + \frac{4-l_1-l_2-d}{2}\right)_{K+a+b-q} \left(-iu + \frac{2-l_1-l_2}{2}\right)_{N-b}} \\ = \frac{1}{\left(iu + \frac{4-l_1-l_2-d}{2}\right)_K \left(-iu + \frac{2-l_1-l_2}{2}\right)_N} \sum_{a=0}^q \frac{(-l_1)_{K+N+a-q} \left(\frac{d}{2} + l_1 + l_2 - K - N - 1\right)_{q-a}}{a! (q-a)!} \\ = \frac{(-l_1)_{K+N-q}}{q!} \frac{(-1)^q \left(2-l_2-\frac{d}{2}\right)_q}{\left(iu + \frac{4-l_1-l_2-d}{2}\right)_K \left(-iu + \frac{2-l_1-l_2}{2}\right)_N}. \quad (4.1.28)$$

One eventually gets

$$\alpha^{\otimes l_1} \otimes y^{\otimes l_2} \cdot \left[\mathbb{R}_{l_1, l_2}(u) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] = \frac{\left(iu + \frac{l_2 - l_1}{2}\right)_{l_1}}{\left(iu - \frac{l_1 + l_2}{2}\right)_{l_1}} \sum_{q=0}^{\lfloor \frac{l_2}{2} \rfloor} \sum_{K, N \geq q} \frac{(-l_1)_{K+N-q} (-l_2)_{K+N} (-N)_q}{q! (K-q)! N!} \\ \times \frac{(y^2)^q (\alpha \cdot \zeta)^{l_1+q-N-K} (\alpha \cdot \eta)^N (\zeta \cdot \eta)^K (\alpha \cdot y)^{K-q} (y \cdot \zeta)^{N-q} (y \cdot \eta)^{l_2-N-K}}{2^q \left(iu + \frac{4-l_1-l_2-d}{2}\right)_K \left(-iu + \frac{2-l_1-l_2}{2}\right)_N}. \quad (4.1.29)$$

The same procedure allows one to compute $x^{\otimes l_1} \otimes y^{\otimes l_2} \cdot [\mathbb{R}_{l_1, l_2}(u) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}]$: we now have to insert

$$\begin{aligned} & X_{l_1} \cdot y^{\otimes(K-q)} \otimes \zeta^{\otimes(l_1+q-N-K)} \otimes \eta^{\otimes N} \\ &= \sum_{p=0}^{\lfloor \frac{l_1}{2} \rfloor} \sum_{a+b+c \leq p} \frac{(-K+q)_{2p-a-b-2c} (N+K-l_1-q)_{b+c} (-N)_{a+c}}{2^{p-a-b-c} a! b! c! (p-a-b-c)!} \frac{(x^2)^p (y^2)^{p-a-b-c}}{\left(2-l_1-\frac{d}{2}\right)_p 2^p} \\ & \times (y \cdot \eta)^a (y \cdot \zeta)^b (\zeta \cdot \eta)^c (x \cdot y)^{K-q+2c+a+b-2p} (x \cdot \zeta)^{l_1+q-N-K-b-c} (x \cdot \eta)^{N-a-c}. \end{aligned} \quad (4.1.30)$$

After the change of summation indices $q' = q + p - a - b - c$, $k = K + c$, $n = N + p - a - c$ the sums over a , b and c can be performed through repeated application of the Gauss identity (4.1.27). One eventually obtains

$$\begin{aligned} & x^{\otimes l_1} \otimes y^{\otimes l_2} \cdot [\mathbb{R}_{l_1, l_2}(u) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}] \\ &= \frac{\left(iu + \frac{l_2-l_1}{2}\right)_{l_1}}{\left(iu - \frac{l_1+l_2}{2}\right)_{l_1}} \sum_{p=0}^{\lfloor \frac{l_1}{2} \rfloor} \sum_{q=0}^{\lfloor \frac{l_2}{2} \rfloor} \sum_{\substack{k \geq p+q \\ n \geq 0}} \frac{(-l_1)_{k+n-q} (-l_2)_{k+n-p} (-n)_p (-n)_q}{p! q! (k-p-q)! n!} \\ & \times \frac{(x^2)^p (y^2)^q (x \cdot \zeta)^{l_1+q-n-k} (x \cdot \eta)^{n-p} (\zeta \cdot \eta)^k (x \cdot y)^{k-p-q} (y \cdot \zeta)^{n-q} (y \cdot \eta)^{l_2+p-n-k}}{2^{p+q} \left(iu + \frac{4-l_1-l_2-d}{2}\right)_k \left(-iu + \frac{2-l_1-l_2}{2}\right)_n}. \end{aligned} \quad (4.1.31)$$

4.1.2 Spectral Decomposition

The spectral decomposition of the R-matrix was computed by N. MacKay [214, 216]. Since it is clear from our expression (4.1.2) that the completely symmetric traceless tensors are eigenvectors with eigenvalue 1, the normalisation is fixed, and MacKay's result reads

$$\mathbb{R}_{l_1, l_2}(u) = \sum_{0 \leq m \leq n \leq \min(l_1, l_2)} \prod_{p=1}^m \frac{u - i \frac{d+l_1+l_2-2-2q}{2}}{u + i \frac{d+l_1+l_2-2-2q}{2}} \prod_{q=1}^n \frac{u - i \frac{l_1+l_2+2-2q}{2}}{u + i \frac{l_1+l_2+2-2q}{2}} \mathbb{P}_{l_1, l_2}^{(l_1+l_2-2n, n-m)}, \quad (4.1.32)$$

where $\mathbb{P}_{l_1, l_2}^{(n_1, n_2)}$ is the projector onto the subrepresentation of $\mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2}$ with highest weight $n_1 \omega_1 + n_2 \omega_2$, the ω_a 's being fundamental weights (see Appendix A.2.1). When one of the spins is equal to one, the previous decomposition reads

$$\mathbb{R}_{1, l}(u) = \mathbb{P}_{1, l}^{(l+1, 0)} + \frac{u - i \frac{l+1}{2}}{u + i \frac{l+1}{2}} \mathbb{P}_{1, l}^{(l-1, 1)} + \frac{u - i \frac{d+l-3}{2}}{u + i \frac{d+l-3}{2}} \frac{u - i \frac{l+1}{2}}{u + i \frac{l+1}{2}} \mathbb{P}_{1, l}^{(l-1, 0)}. \quad (4.1.33)$$

Let us check that this coincides with the expression (4.1.6) for the R-matrix. We first introduce some operators \mathcal{P} , \mathcal{K}_1 , and \mathcal{K}_2 , in terms of which the R-matrix reads

$$\mathbb{R}_{1, l}(u) = \frac{1}{u + i \frac{l+1}{2}} \left[\left(u - i \frac{l-1}{2}\right) \text{Id} + i \mathcal{P} - i \frac{u - i \frac{l-1}{2}}{u + i \frac{d+l-3}{2}} \mathcal{K}_1 + \frac{1}{u + i \frac{d+l-3}{2}} \mathcal{K}_2 \right]. \quad (4.1.34)$$

We have already explained that $\mathbb{P}_{1,l}^{(l+1,0)} = \mathbb{R}_{1,l} \left(i \frac{l+1}{2} \right)$. In terms of the new operators, this becomes

$$\mathbb{P}_{1,l}^{(l+1,0)} = \frac{1}{l+1} \left[\text{Id} + \mathcal{P} - \frac{2}{d+2l-2} (\mathcal{K}_1 + \mathcal{K}_2) \right]. \quad (4.1.35)$$

We claim that the other two projectors are

$$\mathbb{P}_{1,l}^{(l-1,1)} = \frac{1}{l+1} \left[l \text{Id} - \mathcal{P} + \frac{1}{d+l-3} (2\mathcal{K}_2 - (l-1)\mathcal{K}_1) \right] \quad (4.1.36)$$

and

$$\mathbb{P}_{1,l}^{(l-1,0)} = \frac{1}{(d+2l-2)(d+l-3)} [(d+2l-4)\mathcal{K}_1 - 2\mathcal{K}_2]. \quad (4.1.37)$$

It is clear that $\mathbb{P}_{1,l}^{(l+1,0)} + \mathbb{P}_{1,l}^{(l-1,1)} + \mathbb{P}_{1,l}^{(l-1,0)} = \text{Id}$, and that (4.1.33) is equal to (4.1.34). It remains to check that they are indeed orthogonal projectors; this is a tedious but straightforward computation that we do not show here.

4.1.3 Integral Representation

For $\zeta^2 = \eta^2 = 0$ but x^2 and y^2 arbitrary, one has

$$\begin{aligned} & \left[\mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot (x^{\otimes l_1} \otimes y^{\otimes l_2}) \\ &= F_{l_1, l_2}(\lambda) \int \frac{z^{2(\lambda + \frac{l_1+l_2}{2} - 1)} (\zeta \cdot (y-v))^{l_1} (\eta \cdot (x-v))^{l_2}}{(z-x)^{2(\lambda + \frac{l_2+1}{2})} (z-y)^{2(\lambda + \frac{l_1+2}{2})} (z-v)^{2(d-1 + \frac{l_1+l_2}{2} - \lambda)}} \frac{d^d z d^d v}{\pi^d}, \end{aligned} \quad (4.1.38)$$

where

$$F_{l_1, l_2}(\lambda) = \frac{\Gamma\left(1 - \frac{l_1+l_2}{2} - \lambda\right) \Gamma\left(1 + \frac{l_1+l_2}{2} + \lambda\right) \Gamma\left(d-1 + \frac{l_1+l_2}{2} - \lambda\right)}{\Gamma\left(1 + \frac{l_1+l_2}{2} - \lambda\right) \Gamma\left(\frac{d+l_1+l_2}{2} - 1 - \lambda\right) \Gamma\left(1 - \frac{l_1+l_2+d}{2} + \lambda\right)}. \quad (4.1.39)$$

Though we will not prove¹ here that this integral formula coincides with (4.1.31), we will still perform a few consistency checks. The first one is to restrict ourselves to the case $x^2 = y^2 = 0$, and we show right below how to recover exactly (4.1.2) in this case. Other checks consist in assuming that these integrals define indeed an operator on $\mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2}$, and then verifying that unitarity and the Yang–Baxter relation hold, which we do in the next subsection. Of course, if these integrals are actually homogeneous harmonic polynomials in x and in y , then the restriction to null vectors characterise them uniquely. And we don't have to verify the Yang–Baxter relation because the R-matrices constructed from fusion satisfy it automatically. It is nonetheless interesting to do so because it is not clear that we can perform all the usual manipulations on them. The integral over v is indeed not convergent, and it is actually a representation of a certain sum of derivatives of delta functions. The simplest example of such an integral would be (3.1.39), for $x = 0$ and $y = v$, in which we naïvely perform an inversion $z \mapsto \frac{z}{z^2}$ and $v \mapsto \frac{v}{v^2}$ to obtain

$$\int \frac{\pi^{-\frac{d}{2}} d^d v}{(z-v)^{2(d-a)}} = A_0(a) A_0(d-a) \pi^{\frac{d}{2}} \frac{\delta^{(d)}\left(\frac{z}{z^2}\right)}{z^{2(d-a)}}. \quad (4.1.40)$$

¹The proof is very technical and can be found in Appendix B of [180].

Once we have done this neither the left-hand side nor the right-hand side is well-defined but we understand it as equivalent to the integral before the inversion. In order to apply it to specific wave functions in the next section, we will actually only need representations of the R-matrices before the inversion. Namely, we will use

$$\begin{aligned} & \left[\mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot \left[\left(\frac{x}{x^2} - \frac{w}{w^2} \right)^{\otimes l_1} \otimes \left(\frac{y}{y^2} - \frac{w}{w^2} \right)^{\otimes l_2} \right] = F_{l_1, l_2}(\lambda) \frac{x^{2(\lambda + \frac{l_{21}}{2})} y^{2(\lambda + \frac{l_{12}}{2})}}{w^{2(\lambda + \frac{l_1 + l_2}{2} - 1)}} \\ & \times \int \frac{(z-w)^{2(\lambda + \frac{l_1 + l_2}{2} - 1)} \left[\zeta \cdot \left(\frac{y}{y^2} - \frac{v}{v^2} \right) \right]^{l_1} \left[\eta \cdot \left(\frac{x}{x^2} - \frac{v}{v^2} \right) \right]^{l_2}}{(z-x)^{2(\lambda + \frac{l_{21}}{2})} (z-y)^{2(\lambda + \frac{l_{12}}{2})} (z-v)^{2(d-1 + \frac{l_1 + l_2}{2} - \lambda)} v^{2(1 - \frac{l_1 + l_2}{2} + \lambda)}} \frac{d^d z d^d v}{\pi^d}. \quad (4.1.41) \end{aligned}$$

Integral Formula for Null Vectors When $x^2 = y^2 = 0$, one can perform the integral over z using Symanzik's trick [217]: if the parameters a_1, \dots, a_N satisfy $\sum_{k=1}^N a_k = d$, then it holds that

$$\int \frac{\pi^{-\frac{d}{2}} d^d z}{\prod_{k=1}^N (z-x_k)^{2a_k}} = \frac{1}{\prod_{k=1}^N \Gamma(a_k)} \int_{\mathbb{R}_+^N} \frac{e^{-\frac{\sum_{i,j} \alpha_i \alpha_j (x_i - x_j)^2}{\sum_{k=1}^N \gamma_k a_k}}}{\left(\sum_{k=1}^N \gamma_k a_k \right)^{\frac{d}{2}}} \prod_{k=1}^N \alpha_k^{a_k - 1} d\alpha_k, \quad (4.1.42)$$

where the parameters $\gamma_1, \dots, \gamma_N$ can be chosen arbitrarily, as long as $\gamma_k \geq 0$ and they are not all zero. In our case, $N = 4$ and we choose three of the parameters to be 0 whereas the last one is set to 1, we thus obtain

$$\begin{aligned} & \int \frac{\Gamma\left(\lambda + \frac{l_{21}}{2}\right) \Gamma\left(\lambda + \frac{l_{12}}{2}\right) \Gamma\left(1 - \frac{l_1 + l_2}{2} - \lambda\right) \Gamma\left(d - 1 + \frac{l_1 + l_2}{2} - \lambda\right)}{(z-x)^{2(\lambda + \frac{l_{21}}{2})} (z-y)^{2(\lambda + \frac{l_{12}}{2})} z^{2(1 - \frac{l_1 + l_2}{2} - \lambda)} (z-v)^{2(d-1 + \frac{l_1 + l_2}{2} - \lambda)}} \frac{d^d z}{\pi^{\frac{d}{2}}} \\ & = \int_{\mathbb{R}_+^4} \alpha_1^{\lambda + \frac{l_{21}}{2} - 1} \alpha_2^{\lambda + \frac{l_{12}}{2} - 1} \alpha_3^{-\frac{l_1 + l_2}{2} - \lambda} \alpha_4^{\frac{d}{2} - 2 + \frac{l_1 + l_2}{2} - \lambda} e^{-\frac{\alpha_1 \alpha_2 (x-y)^2 - \alpha_1 (x-v)^2 - \alpha_2 (y-v)^2 - \alpha_3 v^2}{\alpha_4}} \prod_{k=1}^4 d\alpha_k \\ & = \frac{\Gamma\left(\frac{d}{2} + l_1 - 1\right) \Gamma\left(\frac{d}{2} + l_2 - 1\right) \Gamma\left(1 - \frac{l_1 + l_2}{2} - \lambda\right) \Gamma\left(1 - \frac{d + l_1 + l_2}{2} + \lambda\right)}{(y-v)^{2(\frac{d}{2} + l_1 - 1)} (x-v)^{2(\frac{d}{2} + l_2 - 1)} v^{2(1 - \frac{l_1 + l_2}{2} - \lambda)} (x-y)^{2(1 - \frac{d + l_1 + l_2}{2} + \lambda)}}. \quad (4.1.43) \end{aligned}$$

As a consequence, when x and y are null vectors, the formula (4.1.38) reduces to

$$\begin{aligned} & \left[\mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot (x^{\otimes l_1} \otimes y^{\otimes l_2}) = \frac{\Gamma\left(\frac{d}{2} + l_1 - 1\right) \Gamma\left(\frac{d}{2} + l_2 - 1\right) \left(\lambda + \frac{l_1 + l_2}{2}\right) \Gamma\left(\lambda - \frac{l_1 + l_2}{2}\right)}{\Gamma\left(-\lambda + \frac{d - 2 + l_1 + l_2}{2}\right) \Gamma\left(\lambda + \frac{l_1 - l_2}{2}\right) \Gamma\left(\lambda + \frac{l_2 - l_1}{2}\right)} \\ & \times (x-y)^{2(-\lambda + \frac{d + l_1 + l_2 - 2}{2})} \int \frac{(\zeta \cdot (v-y))^{l_1} (\eta \cdot (v-x))^{l_2}}{v^{2(1 - \lambda - \frac{l_1 + l_2}{2})} (y-v)^{2(\frac{d}{2} + l_1 - 1)} (x-v)^{2(\frac{d}{2} + l_2 - 1)}} \frac{d^d v}{\pi^{\frac{d}{2}}}. \quad (4.1.44) \end{aligned}$$

This is now perfectly well-defined. This new identity can be proven as follows. After having stripped the right-hand side (*RHS*) of the (x, y) -independent prefactor, it can be represented

as (it is important to notice that one can only impose $x^2 = y^2 = 0$ after having taken the derivatives):

$$\begin{aligned}
RHS &= (x-y)^{2(-\lambda + \frac{d+l_1+l_2-2}{2})} \frac{(\zeta \cdot \nabla_y)^{l_1} (\eta \cdot \nabla_x)^{l_2}}{2^{l_1+l_2} \binom{\frac{d}{2}-1}_{l_1} \binom{\frac{d}{2}-1}_{l_2}} \int \frac{v^{2(\lambda + \frac{l_1+l_2}{2}-1)}}{(x-v)^{2(\frac{d}{2}-1)} (y-v)^{2(\frac{d}{2}-1)}} \frac{d^d v}{\pi^{\frac{d}{2}}} \\
&= (x-y)^{2(-\lambda + \frac{d+l_1+l_2-2}{2})} \frac{\Gamma(-\lambda + \frac{d-2-l_1-l_2}{2}) (\zeta \cdot \nabla_y)^{l_1} (\eta \cdot \nabla_x)^{l_2}}{2^{l_1+l_2} \Gamma(\frac{d}{2} + l_1 - 1) \Gamma(\frac{d}{2} + l_2 - 1) \Gamma(1 - \lambda - \frac{l_1+l_2}{2})} \\
&\quad \times \int_{[0,1]^3} \alpha_1^{-\lambda - \frac{l_1+l_2}{2}} (\alpha_2 \alpha_3)^{\frac{d}{2}-2} [\alpha_1 \alpha_2 y^2 + \alpha_1 \alpha_3 x^2 + \alpha_2 \alpha_3 (x-y)^2]^{\lambda + \frac{2+l_1+l_2-d}{2}} \\
&\quad \quad \quad \times \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) d\alpha_1 d\alpha_2 d\alpha_3 \\
&= \frac{\Gamma(-\lambda + \frac{d-2-l_1-l_2}{2})}{\Gamma(\frac{d}{2} + l_1 - 1) \Gamma(\frac{d}{2} + l_2 - 1) \Gamma(1 - \lambda - \frac{l_1+l_2}{2})} \sum_{k=0}^{\min(l_1, l_2)} \frac{l_1! l_2!}{k! (l_1 - k)! (l_2 - k)!} \frac{(x-y)^{2k}}{(-2)^k} \\
&\quad \times (\zeta \cdot \eta)^k \frac{\Gamma(\lambda + \frac{4-d+l_1+l_2}{2})}{\Gamma(\lambda + \frac{4-d-l_1-l_2}{2} + k)} \int_{[0,1]^3} (\zeta \cdot (\alpha_1 y + \alpha_3 (y-x)))^{l_1-k} (\eta \cdot (\alpha_1 x + \alpha_2 (x-y)))^{l_2-k} \\
&\quad \quad \quad \times \alpha_1^{-\lambda - \frac{l_1+l_2}{2}} \alpha_2^{\lambda + \frac{l_1-l_2}{2} + k - 1} \alpha_3^{\lambda + \frac{l_2-l_1}{2} + k - 1} \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) d\alpha_1 d\alpha_2 d\alpha_3.
\end{aligned}$$

Since $\alpha_1 + \alpha_2 + \alpha_3 = 1$, one can write

$$(\zeta \cdot (\alpha_1 y + \alpha_3 (y-x)))^{l_1-k} = \sum_{m=0}^{l_1-k} \binom{l_1-k}{m} ((1-\alpha_2)\zeta \cdot y)^m (-\alpha_3 \zeta \cdot x)^{l_1-k-m} \quad (4.1.45)$$

and

$$(\eta \cdot (\alpha_1 x + \alpha_2 (x-y)))^{l_2-k} = \sum_{n=0}^{l_2-k} \binom{l_2-k}{n} ((1-\alpha_3)\eta \cdot x)^n (-\alpha_2 \eta \cdot y)^{l_2-k-n}. \quad (4.1.46)$$

The integral that then appears is of the form

$$\begin{aligned}
&\int_{[0,1]^3} \alpha_1^{a-1} \alpha_2^{b-1} \alpha_3^{c-1} (1-\alpha_2)^m (1-\alpha_3)^n \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) d\alpha_1 d\alpha_2 d\alpha_3 \\
&= \Gamma(a) \sum_{p=0}^m \sum_{q=0}^n \binom{m}{p} \binom{n}{q} (-1)^{p+q} \frac{\Gamma(b+p) \Gamma(c+q)}{\Gamma(a+b+c+p+q)} \\
&= \Gamma(a) \Gamma(c) \sum_{p=0}^m \binom{m}{p} (-1)^p \frac{\Gamma(b+p) (a+b+p)_n}{\Gamma(a+b+c+n+p)} \\
&= \Gamma(a) \Gamma(b) \sum_{q=0}^n \binom{n}{q} (-1)^q \frac{\Gamma(c+q) (a+c+q)_m}{\Gamma(a+b+c+m+q)}, \quad (4.1.47)
\end{aligned}$$

where we used the Gauss identity (4.1.27) in the form $\sum_{l=0}^r \binom{r}{l} (-1)^l \frac{\Gamma(A+l)}{\Gamma(B+l)} = \frac{\Gamma(A)}{\Gamma(B+r)} (B-A)_r$. In our case, the parameters actually are $a = -\lambda + \frac{2-l_1-l_2}{2}$, $b = \lambda + \frac{l_1+l_2}{2} - n$, and $c =$

$\lambda + \frac{l_1+l_2}{2} - m$. In particular, $a + b = 1 - n$ so that $(a + b + p)_n = (1 + p - n)_n = 0$ unless $p \geq n$, and one of the formulae above for the integral shows that it vanishes unless $m \geq n$. Similarly, $a + c = 1 - m$ so that we also need $n \geq m$. In the end, the integral is

$$\int_{[0,1]^3} \alpha_1^{-\lambda - \frac{l_1+l_2}{2}} \alpha_2^{\lambda + \frac{l_1+l_2}{2} - n - 1} \alpha_3^{\lambda + \frac{l_1+l_2}{2} - m - 1} (1 - \alpha_2)^m (1 - \alpha_3)^n \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) d\alpha_1 d\alpha_2 d\alpha_3$$

$$= \delta_{m,n} n! \frac{\Gamma\left(-\lambda + \frac{2-l_1-l_2}{2}\right) \Gamma\left(\lambda + \frac{l_1+l_2}{2}\right)}{\left(\lambda + \frac{l_1+l_2}{2}\right) \left(-\lambda + \frac{2-l_1-l_2}{2}\right)_n}. \quad (4.1.48)$$

Putting everything together yields (we also use $(x - y)^2 = -2x \cdot y$)

$$\begin{aligned} RHS &= \frac{(-1)^{l_1+l_2} \Gamma\left(-\lambda + \frac{d-2-l_1-l_2}{2}\right) \Gamma\left(\lambda + \frac{l_1+l_2}{2}\right)}{\Gamma\left(\frac{d}{2} + l_1 - 1\right) \Gamma\left(\frac{d}{2} + l_2 - 1\right) \left(\lambda + \frac{l_1+l_2}{2}\right)} \sum_{k+n \leq \min(l_1, l_2)} \frac{l_1! l_2! (x \cdot y \zeta \cdot \eta)^k}{k! n! (l_1 - k - n)! (l_2 - k - n)!} \\ &\quad \times \frac{\Gamma\left(\lambda + \frac{4-d+l_1+l_2}{2}\right)}{\Gamma\left(\lambda + \frac{4-d-l_1-l_2}{2} + k\right)} \frac{(\zeta \cdot y \eta \cdot x)^n}{\left(-\lambda + \frac{2-l_1-l_2}{2}\right)_n} (\zeta \cdot x)^{l_1-k-n} (\eta \cdot y)^{l_2-k-n} \\ &= \frac{\Gamma\left(-\lambda + \frac{d-2+l_1+l_2}{2}\right) \Gamma\left(\lambda + \frac{l_1-l_2}{2}\right) \Gamma\left(\lambda + \frac{l_2-l_1}{2}\right)}{\Gamma\left(\frac{d}{2} + l_1 - 1\right) \Gamma\left(\frac{d}{2} + l_2 - 1\right) \left(\lambda + \frac{l_1+l_2}{2}\right) \Gamma\left(\lambda - \frac{l_1+l_2}{2}\right)} \\ &\quad \times \left[\mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot (x^{\otimes l_1} \otimes y^{\otimes l_2}). \quad (4.1.49) \end{aligned}$$

4.1.4 Properties of the R-Matrices

Unitarity The representation (4.1.2) clearly shows that the R-matrices are symmetric and transform simply under complex conjugation:

$${}^t \mathbb{R}_{l_1, l_2} = \mathbb{R}_{l_1, l_2}, \quad \mathbb{R}_{l_1, l_2}(u)^* = \mathbb{R}_{l_1, l_2}(-u^*). \quad (4.1.50)$$

From the integral representation, on the other hand, it is easy to see that the inverse is obtained by changing the sign of the spectral parameter

$$\mathbb{R}_{l_1, l_2}(u) \mathbb{R}_{l_1, l_2}(-u) = \mathbb{I}_{d_{l_1}} \otimes \mathbb{I}_{d_{l_2}}. \quad (4.1.51)$$

With the help of the two previous relations, this amounts to saying that the R-matrix is unitary when u is real.

The proof of unitarity goes as follows. We first use twice the integral representation to

write

$$\begin{aligned}
& \left[\mathbb{R}_{l_1, l_2}(i\lambda) \mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot (x^{\otimes l_1} \otimes y^{\otimes l_2}) \\
&= F_{l_1, l_2}(-\lambda) \int \frac{\left[\mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot ((y-v)^{\otimes l_1} \otimes (x-v)^{\otimes l_2})}{(z-x)^{2(-\lambda+\frac{l_2}{2})} (z-y)^{2(-\lambda+\frac{l_1}{2})} z^{2(1-\frac{l_1+l_2}{2}+\lambda)} (z-v)^{2(d-1+\frac{l_1+l_2}{2}+\lambda)}} \frac{d^d z d^d v}{\pi^d} \\
&= F_{l_1, l_2}(-\lambda) F_{l_1, l_2}(\lambda) \int \frac{1}{(z-x)^{2(-\lambda+\frac{l_2}{2})} (z-y)^{2(-\lambda+\frac{l_1}{2})} z^{2(1-\frac{l_1+l_2}{2}+\lambda)} (z-v)^{2(d-1+\frac{l_1+l_2}{2}+\lambda)}} \\
&\quad \frac{(\zeta \cdot (x-v-v'))^{l_1} (\eta \cdot (y-v-v'))^{l_2}}{(z'+v-y)^{2(\lambda+\frac{l_2}{2})} (z'+v-x)^{2(-\lambda+\frac{l_1}{2})} z'^{2(1-\frac{l_1+l_2}{2}-\lambda)} (z'-v')^{2(d-1+\frac{l_1+l_2}{2}-\lambda)}} \frac{d^d z' d^d v' d^d z d^d v}{\pi^{2d}}.
\end{aligned} \tag{4.1.52}$$

After the change of variables $(v', z') \mapsto (v' - v, z' - v)$, the integral over v reduces to an application of (3.1.39). This produces $\delta^{(d)}(z - z')$ which allows us to perform the integral over z' , we get

$$\begin{aligned}
& \left[\mathbb{R}_{l_1, l_2}(i\lambda) \mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot (x^{\otimes l_1} \otimes y^{\otimes l_2}) \\
&= \frac{\Gamma\left(1 - \frac{l_1+l_2}{2} + \lambda\right) \Gamma\left(d - 1 + \frac{l_1+l_2}{2} - \lambda\right)}{\Gamma\left(\frac{d+l_1+l_2}{2} - 1 - \lambda\right) \Gamma\left(1 - \frac{l_1+l_2+d}{2} + \lambda\right)} \int \frac{(\zeta \cdot (x-v'))^{l_1} (\eta \cdot (y-v'))^{l_2}}{z^{2(1-\frac{l_1+l_2}{2}+\lambda)} (z-v')^{2(d-1+\frac{l_1+l_2}{2}-\lambda)}} \frac{d^d v' d^d z}{\pi^d} \\
&= \int (\zeta \cdot (x-v'))^{l_1} (\eta \cdot (y-v'))^{l_2} \delta^{(d)}(v') d^d v' = (\zeta \cdot x)^{l_1} (\eta \cdot y)^{l_2}.
\end{aligned} \tag{4.1.53}$$

We have once more used (3.1.39).

Crossing Symmetry From the explicit representation (4.1.2) of the R-matrix, one immediately deduces the crossing property

$$t_2 \mathbb{R}_{l_1, l_2} \left(i \frac{2-d}{2} - u \right) = \frac{\left(-iu + \frac{d+l_2-l_1-2}{2} \right)_{l_1} \left(iu - \frac{l_1+l_2}{2} \right)_{l_1}}{\left(-iu + \frac{d-l_1-l_2-2}{2} \right)_{l_1} \left(iu + \frac{l_2-l_1}{2} \right)_{l_1}} \mathbb{R}_{l_1, l_2}(u), \tag{4.1.54}$$

where t_2 denotes transposition in \mathbb{V}_{l_2} only.

Yang–Baxter Relation The fusion procedure being a way to construct new solutions of the Yang–Baxter relation, we know that the expression (4.1.2) satisfies it. It is however also possible to show it directly for the integral representation as we now explain. We want to show that, for arbitrary null vectors ζ , η , and θ , we have

$$\begin{aligned}
& \mathbb{R}_{l_1, l_2}(-i\lambda) \mathbb{R}_{l_1, l_3}(-i(\lambda + \mu)) \mathbb{R}_{l_2, l_3}(-i\mu) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \otimes \theta^{\otimes l_3} \\
&= \mathbb{R}_{l_2, l_3}(-i\mu) \mathbb{R}_{l_1, l_3}(-i(\lambda + \mu)) \mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \otimes \theta^{\otimes l_3}.
\end{aligned} \tag{4.1.55}$$

It suffices to verify that the scalar product with any vector of the form $x^{\otimes l_1} \otimes y^{\otimes l_2} \otimes z^{\otimes l_3}$ for x , y , and z real, is the same for both sides. After taking the scalar product and using

the integral representation (without writing the scalar prefactors F_{l_i, l_j}), the left-hand side becomes

$$\begin{aligned}
& [\mathbb{R}_{l_1, l_2}(-i\lambda)\mathbb{R}_{l_1, l_3}(-i(\lambda + \mu))\mathbb{R}_{l_2, l_3}(-i\mu)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \otimes \theta^{\otimes l_3}] \cdot x^{\otimes l_1} \otimes y^{\otimes l_2} \otimes z^{\otimes l_3} \\
& \propto \int \frac{[\mathbb{R}_{l_1, l_3}(-i(\lambda + \mu))\mathbb{R}_{l_2, l_3}(-i\mu)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \otimes \theta^{\otimes l_3}] \cdot (y - v)^{\otimes l_1} \otimes (x - v)^{\otimes l_2} \otimes z^{\otimes l_3} d^d w d^d v}{(w - x)^{2(\lambda + \frac{l_{21}}{2})} (w - y)^{2(\lambda + \frac{l_{12}}{2})} w^{2(1 - \frac{l_1 + l_2}{2} - \lambda)} (w - v)^{2(d - 1 + \frac{l_1 + l_2}{2} - \lambda)}} \frac{d^d w d^d v}{\pi^d} \\
& \propto \int \frac{\pi^{-d} d^d w d^d v}{(w - x)^{2(\lambda + \frac{l_{21}}{2})} (w - y)^{2(\lambda + \frac{l_{12}}{2})} w^{2(1 - \frac{l_1 + l_2}{2} - \lambda)} (w - v)^{2(d - 1 + \frac{l_1 + l_2}{2} - \lambda)}} \\
& \times \frac{[\mathbb{R}_{l_2, l_3}(-i\mu)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \otimes \theta^{\otimes l_3}] \cdot (z - v')^{\otimes l_1} \otimes (x - v)^{\otimes l_2} \otimes (y - v - v')^{\otimes l_3}}{(w' - y + v)^{2(\lambda + \mu + \frac{l_{31}}{2})} (w' - z)^{2(\lambda + \mu + \frac{l_{13}}{2})} w'^{2(1 - \frac{l_1 + l_3}{2} - \lambda - \mu)} (w' - v')^{2(d - 1 + \frac{l_1 + l_3}{2} - \lambda - \mu)}} \frac{d^d w' d^d v'}{\pi^d} \\
& \propto \int \frac{\pi^{-d} d^d w d^d v}{(w - x)^{2(\lambda + \frac{l_{21}}{2})} (w - y)^{2(\lambda + \frac{l_{12}}{2})} w^{2(1 - \frac{l_1 + l_2}{2} - \lambda)} (w - v)^{2(d - 1 + \frac{l_1 + l_2}{2} - \lambda)}} \\
& \times \frac{\pi^{-d} d^d w' d^d v'}{(w' - y + v)^{2(\lambda + \mu + \frac{l_{31}}{2})} (w' - z)^{2(\lambda + \mu + \frac{l_{13}}{2})} w'^{2(1 - \frac{l_1 + l_3}{2} - \lambda - \mu)} (w' - v')^{2(d - 1 + \frac{l_1 + l_3}{2} - \lambda - \mu)}} \\
& \times \frac{(\zeta \cdot (z - v'))^{l_1} (\eta \cdot (y - v - v' - v''))^{l_2} (\theta \cdot (x - v - v''))^{l_3} \pi^{-d} d^d w'' d^d v''}{(w'' + v - x)^{2(\mu + \frac{l_{32}}{2})} (w'' + v + v' - y)^{2(\mu + \frac{l_{23}}{2})} w''^{2(1 - \frac{l_2 + l_3}{2} - \mu)} (w'' - v'')^{2(d - 1 + \frac{l_2 + l_3}{2} - \mu)}} \\
& \propto \int \frac{\pi^{-d} d^d w d^d v}{(w - x)^{2(\lambda + \frac{l_{21}}{2})} (w - y)^{2(\lambda + \frac{l_{12}}{2})} w^{2(1 - \frac{l_1 + l_2}{2} - \lambda)} (w - v)^{2(d - 1 + \frac{l_1 + l_2}{2} - \lambda)}} \\
& \times \frac{\pi^{-d} d^d w' d^d v'}{(w' - y + v)^{2(\lambda + \mu + \frac{l_{31}}{2})} (w' - z)^{2(\lambda + \mu + \frac{l_{13}}{2})} w'^{2(1 - \frac{l_1 + l_3}{2} - \lambda - \mu)} (w' - v')^{2(d - 1 + \frac{l_1 + l_3}{2} - \lambda - \mu)}} \\
& \times \frac{(\zeta \cdot (z - v'))^{l_1} (\eta \cdot (y - v' - v''))^{l_2} (\theta \cdot (x - v''))^{l_3} \pi^{-d} d^d w'' d^d v''}{(w'' - x)^{2(\mu + \frac{l_{32}}{2})} (w'' + v' - y)^{2(\mu + \frac{l_{23}}{2})} (w'' - v'')^{2(1 - \frac{l_2 + l_3}{2} - \mu)} (w'' - v'')^{2(d - 1 + \frac{l_2 + l_3}{2} - \mu)}}. \tag{4.1.56}
\end{aligned}$$

At the last step we have simply performed the change of variables $(w'', v'') \mapsto (w'' - v, v'' - v)$, so that the integral over v is now computed by a simple application of the star-triangle identity. At the same time, we find it convenient to define $\tilde{z} = z - y$, and to perform the change of variables $(w', v') \mapsto (y - w', y - v')$. We obtain

$$\begin{aligned}
& [\mathbb{R}_{l_1, l_2}(-i\lambda)\mathbb{R}_{l_1, l_3}(-i(\lambda + \mu))\mathbb{R}_{l_2, l_3}(-i\mu)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \otimes \theta^{\otimes l_3}] \cdot x^{\otimes l_1} \otimes y^{\otimes l_2} \otimes z^{\otimes l_3} \\
& = F_{l_1, l_2}(\lambda) F_{l_1, l_3}(\lambda + \mu) F_{l_2, l_3}(\mu) A_0 \left(d - 1 + \frac{l_1 + l_2}{2} - \lambda \right) A_0 \left(\lambda + \mu + \frac{l_{31}}{2} \right) A_0 \left(1 - \frac{l_2 + l_3}{2} - \mu \right) \\
& \times \int \frac{\pi^{-\frac{d}{2}} d^d w}{(w - x)^{2(\lambda + \frac{l_2 - l_1}{2})} (w - y)^{2(\lambda + \frac{l_1 - l_2}{2})} w^{2(1 - \frac{l_1 + l_2}{2} - \lambda)} (w' - w'')^{2(1 - \frac{d + l_1 + l_2}{2} + \lambda)}} \\
& \times \frac{\pi^{-d} d^d w' d^d v'}{(w'' - w)^{2(\frac{d + l_1 - l_3}{2} - \lambda - \mu)} (w' - \tilde{z})^{2(\lambda + \mu + \frac{l_1 - l_3}{2})} (w' - y)^{2(1 - \frac{l_1 + l_3}{2} - \lambda - \mu)} (w' - v')^{2(d - 1 + \frac{l_1 + l_3}{2} - \lambda - \mu)}}
\end{aligned}$$

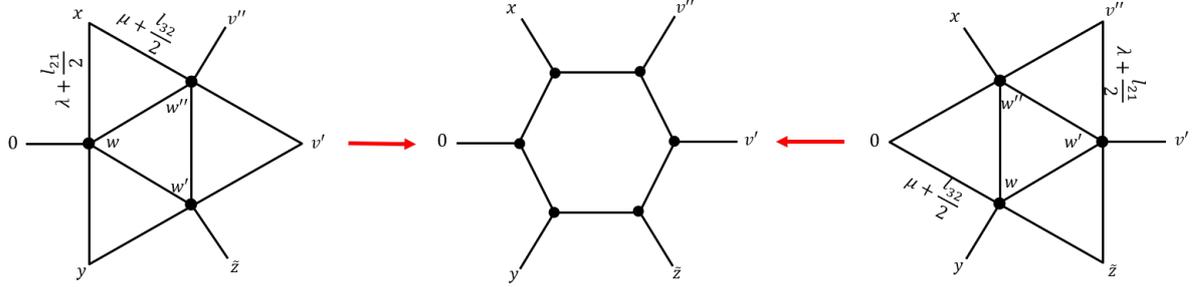


Figure 4.1: The denominators of the integrands of (4.1.57) (on the left) and of (4.1.58) (on the right) are both proportional to the one in the middle. Keeping track of the proportionality constants shows that (4.1.57) = (4.1.58).

$$\times \frac{(\zeta \cdot (\tilde{z} + v'))^{l_1} (\eta \cdot (v' - v''))^{l_2} (\theta \cdot (x - v''))^{l_3} \pi^{-d} d^d w'' d^d v''}{(w'' - x)^{2(\mu + \frac{l_3 - l_2}{2})} (w'' - v')^{2(\mu + \frac{l_2 - l_3}{2})} (w - w')^{2(\frac{d+l_2+l_3}{2} - 1 + \mu)} (w'' - v'')^{2(d - 1 + \frac{l_2 + l_3}{2} - \mu)}} \quad (4.1.57)$$

Similar manipulations for the right-hand side of the Yang–Baxter relation give

$$\begin{aligned} & [\mathbb{R}_{l_2, l_3}(-i\mu) \mathbb{R}_{l_1, l_3}(-i(\lambda + \mu)) \mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \otimes \theta^{\otimes l_3}] \cdot x^{\otimes l_1} \otimes y^{\otimes l_2} \otimes z^{\otimes l_3} \\ &= F_{l_1, l_2}(\lambda) F_{l_1, l_3}(\lambda + \mu) F_{l_2, l_3}(\mu) A_0 \left(d - 1 + \frac{l_2 + l_3}{2} - \mu \right) A_0 \left(\lambda + \mu + \frac{l_{13}}{2} \right) A_0 \left(1 - \frac{l_1 + l_2}{2} - \lambda \right) \\ & \quad \times \int \frac{\pi^{-\frac{d}{2}} d^d w}{w^{2(\mu + \frac{l_3 - l_2}{2})} (w - \tilde{z})^{2(\mu + \frac{l_2 - l_3}{2})} (w - y)^{2(1 - \frac{l_2 + l_3}{2} - \mu)} (w' - w'')^{2(1 - \frac{d+l_2+l_3}{2} + \mu)}} \pi^{-d} d^d w'' d^d v'' \\ & \times \frac{(w'' - x)^{2(\lambda + \mu + \frac{l_3 - l_1}{2})} (w - w')^{2(\frac{d+l_3 - l_1}{2} - \lambda - \mu)} w''^{2(1 - \frac{l_1 + l_3}{2} - \lambda - \mu)} (w'' - v'')^{2(d - 1 + \frac{l_1 + l_3}{2} - \lambda - \mu)}}{(w' - v'')^{2(\lambda + \frac{l_2 - l_1}{2})} (w' - \tilde{z})^{2(\lambda + \frac{l_1 - l_2}{2})} (w - w'')^{2(\frac{d+l_1+l_2}{2} - 1 + \lambda)} (w' - v')^{2(d - 1 + \frac{l_1 + l_2}{2} - \lambda)}} \cdot \\ & \quad \times \frac{(\zeta \cdot (\tilde{z} + v'))^{l_1} (\eta \cdot (v' - v''))^{l_2} (\theta \cdot (x - v''))^{l_3} \pi^{-d} d^d w' d^d v'}{(w' - v'')^{2(\lambda + \frac{l_2 - l_1}{2})} (w' - \tilde{z})^{2(\lambda + \frac{l_1 - l_2}{2})} (w - w'')^{2(\frac{d+l_1+l_2}{2} - 1 + \lambda)} (w' - v')^{2(d - 1 + \frac{l_1 + l_2}{2} - \lambda)}} \quad (4.1.58) \end{aligned}$$

Notice that the numerators of the integrands of the last two formulae are the same, and that these do not involve w , w' , and w'' . Consequently, if we can prove that the integrals over these three variables coincide, then we are done. This is actually a straightforward application of the star-triangle identity as depicted in Figure 4.1.

Additional Property For ζ and η two null vectors, it holds that

$$\begin{aligned} (\zeta \cdot \nabla)^{l_1} (\eta \cdot \nabla)^{l_2} x^{2(\frac{l_1 + l_2 + 2 - d}{2} + \lambda)} &= \frac{\left(\frac{4 - l_1 - l_2 - d}{2} + \lambda \right)_{l_1 + l_2}}{\left(\frac{4 - l_1 - l_2 - d}{2} - \lambda \right)_{l_1 + l_2}} (x^2)^{2\lambda} \\ & \times \left[\mathbb{R}_{l_1, l_2}(-i\lambda) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot \nabla^{\otimes (l_1 + l_2)} x^{2(\frac{l_1 + l_2 + 2 - d}{2} - \lambda)}. \quad (4.1.59) \end{aligned}$$

In order to prove it, one first needs to compute $y^{\otimes(l_1+l_2)} \cdot [\mathbb{R}_{l_1, l_2}(-i\lambda)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}]$ for arbitrary y . We use equation (4.1.38) to write (after having performed the integral over z using the star-triangle relation)

$$\begin{aligned} y^{\otimes(l_1+l_2)} \cdot [\mathbb{R}_{l_1, l_2}(-i\lambda)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}] &= \frac{\Gamma\left(\frac{d}{2} - 2\lambda\right) \Gamma\left(\frac{d+l_1+l_2}{2} - 1 + \lambda\right) \Gamma\left(1 + \frac{l_1+l_2}{2} + \lambda\right)}{\Gamma(2\lambda)\Gamma\left(1 + \frac{l_1+l_2}{2} - \lambda\right) \Gamma\left(\frac{d+l_1+l_2}{2} - 1 - \lambda\right)} \\ &\quad \times y^{2\left(\frac{d+l_1+l_2}{2} - 1 - \lambda\right)} \int \frac{(\zeta \cdot (y-v))^{l_1} (\eta \cdot (y-v))^{l_2}}{v^{2\left(\frac{d}{2} - 2\lambda\right)} (v-y)^{2\left(\frac{d+l_1+l_2}{2} + \lambda - 1\right)}} \frac{d^d v}{\pi^{\frac{d}{2}}} \\ &= \sum_{k=0}^{\min(l_1, l_2)} \frac{l_1! l_2!}{k!(l_1-k)!(l_2-k)!} \frac{(2\lambda)_k (y^2 \zeta \cdot \eta)^k (y \cdot \zeta)^{l_1-k} (y \cdot \eta)^{l_2-k}}{2^k \left(\lambda - \frac{l_1+l_2}{2}\right)_k \left(\lambda + \frac{4-l_1-l_2-d}{2}\right)_k}. \end{aligned} \quad (4.1.60)$$

It is interesting to notice that this last formula is more difficult to obtain starting from the expression (4.1.31) for the R-matrix, as it requires knowing the somewhat mysterious identity

$$\frac{\left(\lambda + \frac{l_2-l_1}{2}\right)_{l_1}}{\left(\lambda - \frac{l_1+l_2}{2}\right)_{l_1}} \sum_{\substack{p, q, n \geq 0 \\ p \leq \lfloor \frac{l_1}{2} \rfloor, q \leq \lfloor \frac{l_2}{2} \rfloor \\ p+q \leq k}} \frac{(-l_1+k)_{n-q} (-l_2+k)_{n-p} (-n)_p (-n)_q}{p! q! (k-p-q)! n! 2^{p+q} \left(-\lambda + \frac{2-l_1-l_2}{2}\right)_n} = \frac{(2\lambda)_k}{2^k k! \left(\lambda - \frac{l_1+l_2}{2}\right)_k}, \quad (4.1.61)$$

which should hold as long as $k \leq \min(l_1, l_2)$.

Returning to the proof of (4.1.59), we can write

$$\begin{aligned} [\mathbb{R}_{l_1, l_2}(-i\lambda)\zeta^{\otimes l_1} \otimes \eta^{\otimes l_2}] \cdot \nabla^{\otimes(l_1+l_2)} x^{2\left(\frac{l_1+l_2+2-d}{2} - \lambda\right)} &= \sum_{k=0}^{\min(l_1, l_2)} \frac{l_1! l_2!}{k!(l_1-k)!(l_2-k)!} (2\zeta \cdot \eta)^k \\ &\quad \times \frac{(2\lambda)_k \left(\lambda + \frac{d-2-l_1-l_2}{2}\right)_k}{\left(\lambda + \frac{4-l_1-l_2-d}{2}\right)_k} (\zeta \cdot \nabla)^{l_1-k} (\eta \cdot \nabla)^{l_2-k} x^{2\left(\frac{l_1+l_2+2-d}{2} - k - \lambda\right)} \\ &= \sum_{k=0}^{\min(l_1, l_2)} \sum_{j=0}^{\min(l_1, l_2) - k} \frac{l_1! l_2! 2^{l_1+l_2-k-j}}{k! j! (l_1-k-j)! (l_2-k-j)!} (\zeta \cdot \eta)^{k+j} (\zeta \cdot x)^{l_1-k-j} (\eta \cdot x)^{l_2-k-j} \\ &\quad \times x^{2\left(\frac{2-l_1-l_2-d}{2} + k+j - \lambda\right)} \frac{(2\lambda)_k (-1)^k}{\left(\lambda + \frac{4-l_1-l_2-d}{2}\right)_k} \left(\frac{4-l_1-l_2-d}{2} + k+j - \lambda\right)_{l_1+l_2-k-j} \\ &= \left(\frac{4-l_1-l_2-d}{2} - \lambda\right)_{l_1+l_2} \sum_{p=0}^{\min(l_1, l_2)} \frac{l_1! l_2! 2^{l_1+l_2-p}}{p! (l_1-p)! (l_2-p)!} \\ &\quad \times \frac{(\zeta \cdot \eta)^p (\zeta \cdot x)^{l_1-p} (\eta \cdot x)^{l_2-p}}{\left(\lambda + \frac{4-l_1-l_2-d}{2}\right)_p} x^{2\left(\frac{2-l_1-l_2-d}{2} + p - \lambda\right)}. \end{aligned} \quad (4.1.62)$$

On the other hand, one has

$$(\zeta \cdot \nabla)^{l_1} (\eta \cdot \nabla)^{l_2} x^{2\left(\frac{l_1+l_2+2-d}{2}+\lambda\right)} = \sum_{p=0}^{\min(l_1, l_2)} \frac{l_1! l_2! 2^{l_1+l_2-p}}{p!(l_1-p)!(l_2-p)!} \left(\frac{4-l_1-l_2-d}{2} + p + \lambda \right)_{l_1+l_2-p} \\ \times (\zeta \cdot \eta)^p (\zeta \cdot x)^{l_1-p} (\eta \cdot x)^{l_2-p} x^{2\left(\frac{2-l_1-l_2-d}{2}+p+\lambda\right)}, \quad (4.1.63)$$

and, since

$$\left(\frac{4-l_1-l_2-d}{2} + p + \lambda \right)_{l_1+l_2-p} = \frac{\left(\frac{4-l_1-l_2-d}{2} + \lambda \right)_{l_1+l_2}}{\left(\frac{4-l_1-l_2-d}{2} + \lambda \right)_p}, \quad (4.1.64)$$

equation (4.1.59) does hold.

Let us define an operator $\mathbb{O}_{l_1, l_2}(u) : \mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2} \rightarrow S^{l_1+l_2}(\mathbb{C}^d)$ that takes values in the space of symmetric tensors of rank $l_1 + l_2$ in the following way:

$$\left[\mathbb{O}_{l_1, l_2}(u) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot x^{\otimes(l_1+l_2)} = \frac{x^{2\left(\frac{l_1+l_2+d-2}{2}-iu\right)} (\zeta \cdot \nabla)^{l_1} (\eta \cdot \nabla)^{l_2} x^{2\left(\frac{l_1+l_2+2-d}{2}+iu\right)}}{2^{l_1+l_2} \left(\frac{4-l_1-l_2-d}{2} + iu \right)_{l_1+l_2}} \quad (4.1.65)$$

or, equivalently, using (4.1.63),

$$\left[\mathbb{O}_{l_1, l_2}(u) \zeta^{\otimes l_1} \otimes \eta^{\otimes l_2} \right] \cdot x^{\otimes(l_1+l_2)} = \sum_{p=0}^{\min(l_1, l_2)} \frac{l_1! l_2!}{p!(l_1-p)!(l_2-p)!} \frac{(x^2 \zeta \cdot \eta)^p (\zeta \cdot x)^{l_1-p} (\eta \cdot x)^{l_2-p}}{2^p \left(\frac{4-l_1-l_2-d}{2} + iu \right)_p}. \quad (4.1.66)$$

The property (4.1.59) we presented above is now written in a concise manner as

$$\mathbb{O}_{l_1, l_2}(u) = \mathbb{O}_{l_1, l_2}(-u) \mathbb{R}_{l_1, l_2}(u). \quad (4.1.67)$$

We also point out that \mathbb{O}_{l_1, l_2} naturally arises in the generalisation of the chain relation:

$$\int \frac{C_1\left(\frac{w}{|w|}\right) C_2\left(\frac{w-x}{|w-x|}\right) d^d w}{w^{2a}(w-x)^{2b}} \frac{d^d w}{\pi^{\frac{d}{2}}} = A_{l_1}(a) A_{l_2}(b) A_{l_1+l_2}(d-a-b) \\ \times \frac{[\mathbb{O}_{l_1, l_2}(i(a+b+1-d)) C_1 \otimes C_2] \cdot x^{\otimes(l_1+l_2)}}{x^{2\left(a+b+\frac{l_1+l_2-d}{2}\right)}}. \quad (4.1.68)$$

4.2 Diagonalisation of Graph-Building Operators

4.2.1 Construction of the Eigenvectors

From now on and until the end of this chapter, forgetting momentarily about the fishnet theory, we consider $\delta \in i\mathbb{R}$. We will only perform analytic continuation in δ at the end. This choice is only motivated by the fact that the inner product in the unitary principal series is simpler than in the complementary series (cf. Subsection 3.1.1), and this slightly simplifies

some of the formulae that follow. We also introduce some reference point $x_0 \in \mathbb{R}^d$ (one could set it to 0 for instance).

For $N \in \mathbb{N}^*$ and $u \in \mathbb{C}$, let $Q_N(u)$ be the operator acting on functions Φ of N variables as

$$\begin{aligned} [Q_N(u)\Phi](x_1, \dots, x_N) &= \int \frac{\Phi(y_1, \dots, y_N)}{\prod_{k=1}^N x_{kk+1}^{2\delta} (y_k - x_k)^{2\alpha} (y_k - x_{k-1})^{2\beta}} \prod_{k=1}^N \frac{d^d y_k}{\pi^{\frac{d}{2}}} \\ &= [A_0(\delta)A_0(\alpha)A_0(\beta)]^N \int \frac{\Phi(y_1, \dots, y_N)}{\prod_{k=1}^N (w_k - y_k)^{2\tilde{\delta}} (w_k - x_k)^{2\tilde{\beta}} (w_k - x_{k-1})^{2\tilde{\alpha}}} \prod_{k=1}^N \frac{d^d w_k d^d y_k}{\pi^d} \end{aligned} \quad (4.2.1)$$

where $x_{kk+1} = x_k - x_{k+1}$, and we have

$$\alpha = \frac{\tilde{\delta}}{2} - iu, \quad \beta = \frac{\tilde{\delta}}{2} + iu \quad \text{so that} \quad \alpha + \beta + \delta = \frac{d}{2}. \quad (4.2.2)$$

Let us recall our convention: because we want to include $\pi^{-\frac{d}{2}}$ in the integration measure over space-time, we define $|x\rangle$ such that $\langle x|y\rangle = \pi^{\frac{d}{2}}\delta(x-y)$. This way, if Φ and Ψ are two functions of N variables, then the inner product is defined by

$$\begin{aligned} \langle \Phi | \Psi \rangle &= \int \langle \Phi | x_1, \dots, x_N \rangle \langle x_1, \dots, x_N | \Psi \rangle \prod_{k=1}^N \frac{d^d x_k}{\pi^{\frac{d}{2}}} \\ &= \int \Phi^*(x_1, \dots, x_N) \Psi(x_1, \dots, x_N) \prod_{k=1}^N \frac{d^d x_k}{\pi^{\frac{d}{2}}}. \end{aligned} \quad (4.2.3)$$

As a consequence, one can write

$$\langle x_1, \dots, x_N | Q_N(u) | y_1, \dots, y_N \rangle = \frac{1}{\prod_{k=1}^N x_{k,k+1}^{2\delta} (y_k - x_k)^{2\alpha} (y_k - x_{k-1})^{2\beta}} \quad (4.2.4)$$

for the kernel of the graph-building operator. It is graphically represented in Figure 4.2.

We point out that, when $x_0 = 0$, the graph-building operator $\Lambda_{N,\tilde{\delta}}$ is part of the family of operators just defined since

$$Q_N\left(i\frac{\tilde{\delta}}{2}\right) = \Lambda_{N,\tilde{\delta}}. \quad (4.2.5)$$

The operators moreover commute for different values of the spectral parameter,

$$[Q_N(u), Q_N(u')] = 0. \quad (4.2.6)$$

The proof is exactly the same as the one in two dimensions that appeared in [206], for instance, though similar computations were already present in [67]. But we show it in Figure 4.3 nevertheless, for completeness. Many computations in this section are way too lengthy to be spelt out entirely, so we will mostly content ourselves with pictures presenting the main steps. In particular, the various proportionality constants will never appear on these pictures, but we have always made sure that they agree with the results stated in the thesis.

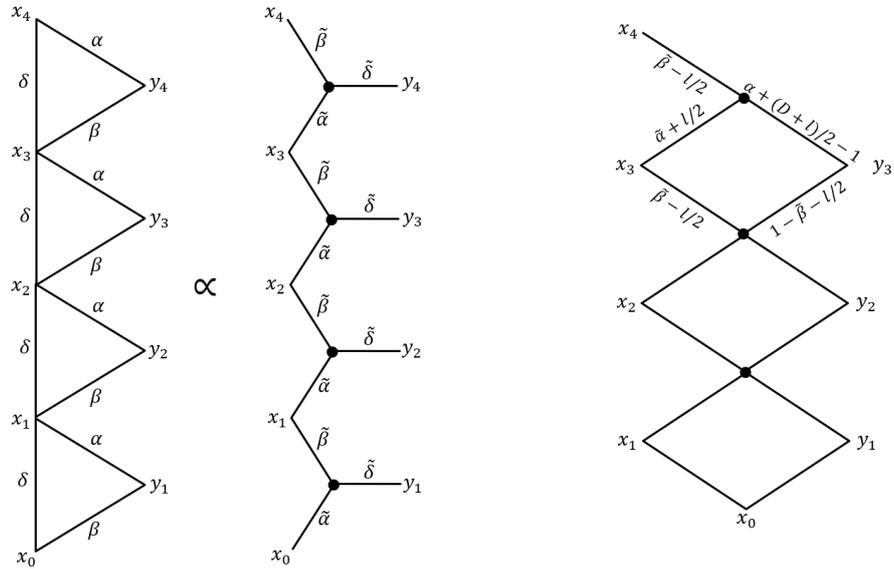


Figure 4.2: Kernels of the graph-building operator $Q_4(u)$ (on the left) and of the operator $\widehat{\Pi}_4^{w_0}(\mathbf{u})$ (on the right)

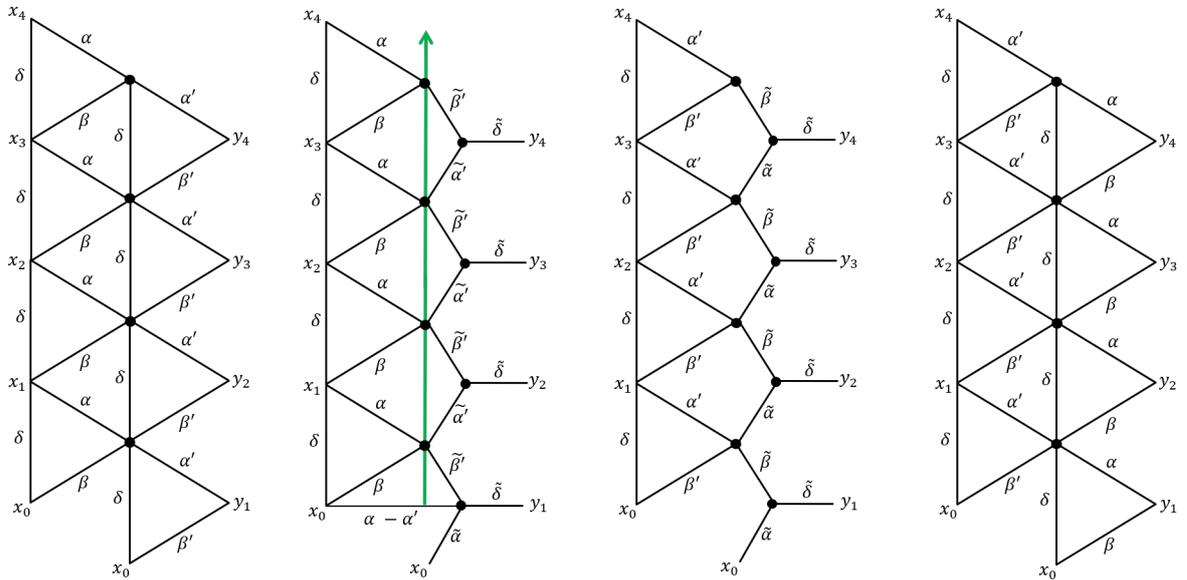


Figure 4.3: Proof of the commutation property of the family $Q_4(u)$ of operators. The first drawing on the left is the kernel of $Q_4(u)Q_4(u')$. We first rewrite the kernel of $Q_4(u')$ in its second form (see Figure 4.2). The green arrow stands for repeated application of the star-triangle identity (3.1.40), starting with the transformation of a triangle into a star. This effectively exchanges u and u' as we wanted.

We shall construct recursively eigenvectors of Q_N . Since these operators commute with global rotations and dilations, the eigenvectors of the first one are immediately found to be

$$\langle x | \mathbf{u}_1; C \rangle = \frac{C(x - x_0)}{(x - x_0)^{2(\tilde{\beta}_1 + \frac{l_1}{2})}}, \quad (4.2.7)$$

where

$$\mathbf{u}_1 = (u_1, l_1) \in \mathbb{C} \times \mathbb{N}, \quad C(y) = C^{\mu_1 \dots \mu_{l_1}} y_{\mu_1} \dots y_{\mu_{l_1}}, \quad (4.2.8)$$

and C belongs to the space \mathbb{V}_{l_1} of symmetric traceless tensors of rank l_1 . One has

$$Q_1(u) | \mathbf{u}_1; C \rangle = Q_{l_1}(u | u_1) | \mathbf{u}_1; C \rangle, \quad (4.2.9)$$

where the eigenvalue, which follows from (B.1.11), is

$$Q_l(u | u') = A_0(\alpha) A_l(\tilde{\alpha}') A_l(\beta + \tilde{\beta}'). \quad (4.2.10)$$

For $N > 1$, $\mathbf{u} \in \mathbb{C} \times \mathbb{N}$, and $C \in \mathbb{V}_l$, we now define some operator $C \cdot \widehat{\Pi}_N(\mathbf{u})$ acting on functions of $N - 1$ variables and returning functions of N variables:

$$\begin{aligned} [C \cdot \widehat{\Pi}_N(\mathbf{u}) \Phi](x_1, \dots, x_N) &= \left[A_0 \left(1 - \alpha - \frac{l}{2} \right) A_0 \left(\beta + \frac{l}{2} \right) \right]^{N-1} \\ &\times \int \frac{C \left(\frac{x_1 - x_0}{(x_1 - x_0)^2} - \frac{y_1 - x_0}{(y_1 - x_0)^2} \right)}{(x_N - w_{N-1})^{2(\tilde{\beta} - \frac{l}{2})} \prod_{k=1}^{N-1} (x_k - w_k)^{2(\tilde{\alpha} + \frac{l}{2})} (x_k - w_{k-1})^{2(\tilde{\beta} - \frac{l}{2})}} \\ &\times \frac{\Phi(y_1, \dots, y_{N-1})}{\prod_{k=1}^{N-1} (y_k - w_k)^{2(\alpha + \frac{d+l}{2} - 1)} (y_k - w_{k-1})^{2(1 - \tilde{\beta} - \frac{l}{2})}} \prod_{k=1}^{N-1} \frac{d^d w_k}{\pi^{\frac{d}{2}}} \frac{d^d y_k}{\pi^{\frac{d}{2}}}, \quad (4.2.11) \end{aligned}$$

with $w_0 = x_0$. The scalar prefactor is merely here for convenience, including it in the definition of $\widehat{\Pi}_N(\mathbf{u})$ simplifies the form of the symmetry property and inner product of the eigenvectors that we will later show. These integrals are ill-defined if $l > 0$, they should be understood as analytic continuations. It would nonetheless seem that we can perform on them all the usual manipulations (like those presented in Appendix B.1). The operator $\widehat{\Pi}_N(\mathbf{u})$ is of course independent of the tensor C , but it is symmetric traceless in its l indices. It is important to notice that, because of (B.1.4), the dependence on the tensor can be rewritten as a derivative if we introduce another operator:

$$C \cdot \widehat{\Pi}_N(\mathbf{u}) = \frac{C(\nabla_{x_0})}{2^l \left(\tilde{\beta} - \frac{l}{2} \right)_l} \widehat{\Pi}_N^{x_0}(\mathbf{u}). \quad (4.2.12)$$

The kernel of this last operator is represented in Figure 4.2.

It follows from the relation (see the different steps of a proof, in the case $N = 3$, in Figure 4.4)

$$Q_N(u) C_N \cdot \widehat{\Pi}_N(\mathbf{u}_N) = Q_{l_N}(u | u_N) C_N \cdot \widehat{\Pi}_N(\mathbf{u}_N) Q_{N-1}(u) \quad (4.2.13)$$

that eigenvectors of $\widehat{\Lambda}_N(u)$ are given by

$$| \mathbf{u}_1, \dots, \mathbf{u}_N; C_1 \otimes \dots \otimes C_N \rangle = C_N \cdot \widehat{\Pi}_N(\mathbf{u}_N) \dots C_2 \cdot \widehat{\Pi}_2(\mathbf{u}_2) | \mathbf{u}_1; C_1 \rangle, \quad (4.2.14)$$

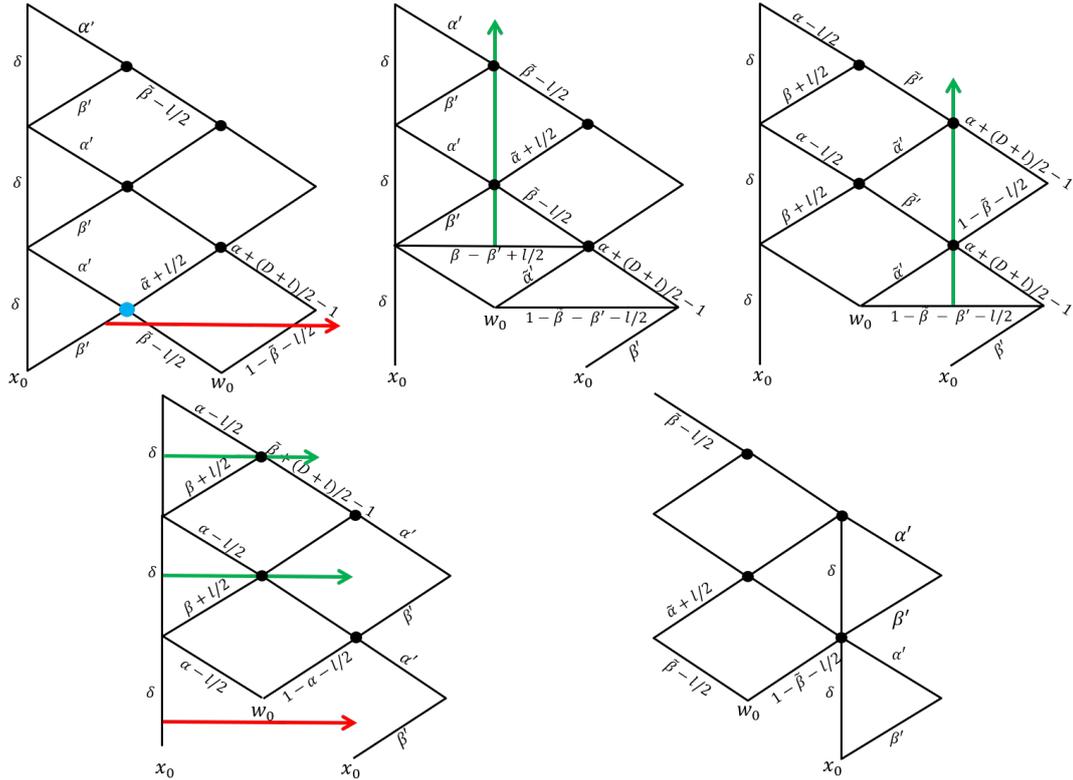


Figure 4.4: Sketch of a proof of relation (4.2.13) when $N = 3$. The first drawing is the kernel of the left hand-side of (4.2.13), whereas the last one is the kernel of the right-hand side (for clarity, we did not display the operator $C(\nabla_{w_0})|_{w_0=x_0}$ acting on each drawing even though it does). We indicate with colours the manipulations needed to pass from one drawing to the next. A red arrow means that we apply identity (B.1.5). The blue dot on the first drawing signifies that we perform the integration. Keeping track of all the proportionality factors, one eventually gets (4.2.13).

with arbitrary $C_i \in \mathbb{V}_{l_i}$. One has indeed

$$\mathbb{Q}_N(u) |\mathbf{u}_1, \dots, \mathbf{u}_N; C\rangle = \prod_{k=1}^N Q_{l_k}(u|u_k) |\mathbf{u}_1, \dots, \mathbf{u}_N; C\rangle \quad (4.2.15)$$

for an arbitrary tensor $C \in \mathbb{V}_{l_1} \otimes \dots \otimes \mathbb{V}_{l_N}$.

4.2.2 Symmetry Property

Using the integral representation of the R-matrix, it is possible to show the following commutation relation for $N > 1$:

$$C \cdot \widehat{\Pi}_{N+1}(\mathbf{u}_1) \otimes \widehat{\Pi}_N(\mathbf{u}_2) = [\mathbb{S}_{l_1, l_2}(u_1 - u_2)C] \cdot \widehat{\Pi}_{N+1}(\mathbf{u}_2) \otimes \widehat{\Pi}_N(\mathbf{u}_1), \quad (4.2.16)$$

where $C \in \mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2}$, the tensor product notation concerns only the finite-dimensional tensor spaces \mathbb{V}_{l_1} and \mathbb{V}_{l_2} , and the scattering matrix is given by the fused R-matrix up to a scalar phase:

$$\mathbb{S}_{l, l'}(u) = (-1)^{l+l'} \frac{\Gamma(1 + \frac{l+l'}{2} - iu) \Gamma(\frac{d}{2} - 1 + \frac{l+l'}{2} - iu)}{\Gamma(1 + \frac{l+l'}{2} + iu) \Gamma(\frac{d}{2} - 1 + \frac{l+l'}{2} + iu)} \mathbb{R}_{l, l'}(u). \quad (4.2.17)$$

Let us define, for any permutation $\sigma \in \mathfrak{S}_N$, the operator $\mathbb{S}(\mathbf{u}_1, \dots, \mathbf{u}_N; \sigma)$ acting on $\mathbb{V}_{l_1} \otimes \dots \otimes \mathbb{V}_{l_N}$. We first set

$$\mathbb{S}(\mathbf{u}_1, \dots, \mathbf{u}_N; \text{id}) = \text{Id}_{l_1} \otimes \dots \otimes \text{Id}_{l_N}, \quad (4.2.18)$$

then, if σ is a transposition of the form $(kk+1)$ for $k \in \{1, \dots, N-1\}$, we define

$$\mathbb{S}(\mathbf{u}_1, \dots, \mathbf{u}_N; (kk+1)) = \text{Id}_{l_1} \otimes \dots \otimes \text{Id}_{l_{k-1}} \otimes \mathbb{S}_{l_k, l_{k+1}}(u_{k+1} - u_k) \otimes \text{Id}_{l_{k+2}} \otimes \dots \otimes \text{Id}_{l_N}. \quad (4.2.19)$$

We then require that

$$\mathbb{S}(\mathbf{u}_1, \dots, \mathbf{u}_N; (kk+1)\sigma) = \mathbb{S}_{l_{\sigma^{-1}(k)}, l_{\sigma^{-1}(k+1)}}(u_{\sigma^{-1}(k+1)} - u_{\sigma^{-1}(k)}) \mathbb{S}(\mathbf{u}_1, \dots, \mathbf{u}_N; \sigma) \quad (4.2.20)$$

for any $k \in \{1, \dots, N-1\}$ and any permutation σ . In the previous formula, we use the (natural) convention that $\mathbb{S}_{l_i, l_j}(u)$ can be seen to act on $\mathbb{V}_{l_1} \otimes \dots \otimes \mathbb{V}_{l_N}$ by being trivial on all the \mathbb{V}_{l_m} for $m \notin \{i, j\}$ (when acting on $\mathbb{V}_{l_1} \otimes \dots \otimes \mathbb{V}_{l_N}$, there is no difference between \mathbb{S}_{l_i, l_j} and \mathbb{S}_{l_j, l_i}). Since any permutation can be decomposed into a product of transpositions of the form $(kk+1)$, this is enough to define $\mathbb{S}(\mathbf{u}_1, \dots, \mathbf{u}_N; \sigma)$ for all $\sigma \in \mathfrak{S}_N$. Furthermore, there is no ambiguity in this definition because $\mathbb{R}_{l, l'}$ satisfies the Yang–Baxter equation.

The eigenvectors satisfy the following symmetry property: for any permutation $\sigma \in \mathfrak{S}_N$, one has

$$|\mathbf{u}_1, \dots, \mathbf{u}_N; C\rangle = \left| \mathbf{u}_{\sigma^{-1}(1)}, \dots, \mathbf{u}_{\sigma^{-1}(N)}; \mathbb{P}_\sigma \mathbb{S}(\mathbf{u}_1, \dots, \mathbf{u}_N; \sigma) C \right\rangle, \quad (4.2.21)$$

where $\mathbb{P}_\sigma : \mathbb{V}_{l_1} \otimes \dots \otimes \mathbb{V}_{l_N} \rightarrow \mathbb{V}_{l_{\sigma^{-1}(1)}} \otimes \dots \otimes \mathbb{V}_{l_{\sigma^{-1}(N)}}$ is the canonical isomorphism. We point out that this symmetry property is exactly the same as Equation (1.3.15) for the compact $GL(N)$ spin chain studied previously. However, since we are now considering a model in infinite volume, the tensor C and the rapidities can be chosen arbitrarily.

4.2.3 Inner Product

The inner product for length one is trivially computed to be

$$\langle \mathbf{u}; C | \mathbf{u}'; C' \rangle = \int \frac{C^*(x-x_0)C'(x-x_0) d^d x}{(x-x_0)^{2(\tilde{\beta}^* + \tilde{\beta}' + \frac{l+l'}{2})}} \frac{d^d x}{\pi^{\frac{d}{2}}} = \delta(\mathbf{u} - \mathbf{u}') \langle C | C' \rangle, \quad (4.2.22)$$

where $\delta(\mathbf{u} - \mathbf{u}') = \delta_{ll'} \delta(u - u')$, and

$$\langle C | C' \rangle = \pi \int_{\mathbb{S}^{d-1}} C^*(n)C'(n) \frac{dn}{\pi^{\frac{d}{2}}} = \frac{l! 2^{1-l} \pi}{\Gamma\left(\frac{d}{2} + l\right)} C_{\mu_1 \dots \mu_l}^* C'^{\mu_1 \dots \mu_l} \quad (4.2.23)$$

is the inner product we choose on \mathbb{V}_l .

Using the integral representation (4.1.38) of the R-matrix and crossing symmetry (4.1.54) (and assuming $\mathbf{u}' \neq \mathbf{u}$), it is possible to show the commutation relations

$$\begin{aligned} \langle x | C' \cdot \hat{\Pi}_2^\dagger(\mathbf{u}') C \cdot \hat{\Pi}_2(\mathbf{u}) | y \rangle &= \frac{1}{\left[(u-u')^2 + \frac{(l-l')^2}{4} \right] \left[(u-u')^2 + \frac{(d-2+l+l')^2}{4} \right]} \\ &\quad \times \frac{[{}^{t_2} \mathcal{S}_{l,l'}(u-u') C \otimes C'] \cdot (x-x_0)^{\otimes l} \otimes (y-x_0)^{\otimes l'}}{(x-x_0)^{2(\tilde{\beta} + \frac{l}{2})} (y-x_0)^{2(\tilde{\beta}' + \frac{l'}{2})}} \end{aligned} \quad (4.2.24)$$

and, for $N > 2$,

$$C' \cdot \hat{\Pi}_N^\dagger(\mathbf{u}') C \cdot \hat{\Pi}_N(\mathbf{u}) = \frac{[{}^{t_2} \mathcal{S}_{l,l'}(u-u') C \otimes C'] \cdot \hat{\Pi}_{N-1}(\mathbf{u}) \otimes \hat{\Pi}_{N-1}^\dagger(\mathbf{u}')}{\left[(u-u')^2 + \frac{(l-l')^2}{4} \right] \left[(u-u')^2 + \frac{(d-2+l+l')^2}{4} \right]}. \quad (4.2.25)$$

This last relation is exemplified in Figure 4.5.

From this and the symmetry property of the eigenvectors, we deduce

$$\langle \mathbf{u}_1, \dots, \mathbf{u}_N; C | \mathbf{u}'_1, \dots, \mathbf{u}'_N; C' \rangle = \frac{\sum_{\sigma \in \mathfrak{S}_N} \prod_{k=1}^N \delta(\mathbf{u}_{\sigma(k)} - \mathbf{u}'_k) \langle C | \mathbb{P}_\sigma \mathcal{S}(\mathbf{u}'_1, \dots, \mathbf{u}'_N; \sigma) C' \rangle}{\mu(\mathbf{u}_1, \dots, \mathbf{u}_N)}, \quad (4.2.26)$$

where

$$\mu(\mathbf{u}_1, \dots, \mathbf{u}_N) = \prod_{1 \leq j < k \leq N} \left[(u_j - u_k)^2 + \frac{(l_j - l_k)^2}{4} \right] \left[(u_j - u_k)^2 + \frac{(d-2+l_j+l_k)^2}{4} \right]. \quad (4.2.27)$$

Let us understand this formula in the case $N = 3$: we are computing

$$\begin{aligned} &\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3; C_1 \otimes C_2 \otimes C_3 | \mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3; C'_1 \otimes C'_2 \otimes C'_3 \rangle \\ &= \langle \mathbf{u}_1; C_1 | C_2 \cdot \hat{\Pi}_2^\dagger(\mathbf{u}_2) C_3 \cdot \hat{\Pi}_3^\dagger(\mathbf{u}_3) C'_3 \cdot \hat{\Pi}_3(\mathbf{u}'_3) C'_2 \cdot \hat{\Pi}_2(\mathbf{u}'_2) | \mathbf{u}'_1; C'_1 \rangle \\ &= \int \langle x | C_2 \cdot \hat{\Pi}_2^\dagger(\mathbf{u}_2) C_3 \cdot \hat{\Pi}_3^\dagger(\mathbf{u}_3) C'_3 \cdot \hat{\Pi}_3(\mathbf{u}'_3) C'_2 \cdot \hat{\Pi}_2(\mathbf{u}'_2) | y \rangle \\ &\quad \times \frac{C_1^*(x-x_0)C'_1(y-x_0)}{(x-x_0)^{2(\tilde{\beta}_1^* + \frac{l_1}{2})} (y-x_0)^{2(\tilde{\beta}'_1 + \frac{l'_1}{2})}} \frac{d^d x d^d y}{\pi^d}. \end{aligned} \quad (4.2.28)$$

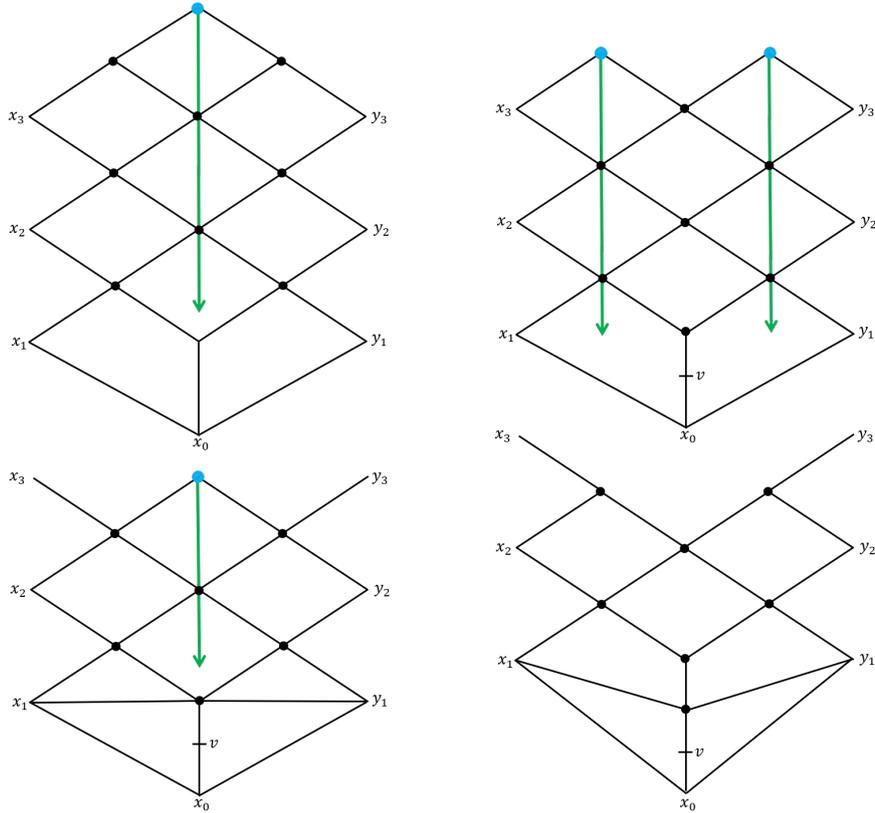


Figure 4.5: Sketch of a proof of relation (4.2.25) when $N = 4$. The first drawing gives the kernel of the left hand-side of (4.2.25) upon multiplication by $C\left(\frac{y_1-x_0}{(y_1-x_0)^2} - \frac{v-x_0}{(v-x_0)^2}\right) C'\left(\frac{x_1-x_0}{(x_1-x_0)^2} - \frac{v-x_0}{(v-x_0)^2}\right)$ and integration over v (divided by $\pi^{\frac{d}{2}}$). If we multiply by the same factor the kernel associated to the last drawing and integrate over v we recognise the integral expression for $[{}^t\mathbb{R}_{l,\nu}(u-u')C \otimes C']\left(\frac{x_1-x_0}{(x_1-x_0)^2} - \frac{w-x_0}{(w-x_0)^2}; \frac{y_1-x_0}{(y_1-x_0)^2} - \frac{w-x_0}{(w-x_0)^2}\right)$. At each step, we first perform the integration (blue dot) and then take the horizontal line down (green arrow). Keeping track of all the proportionality factors one eventually gets (4.2.25).

If we assume that $\mathbf{u}_3 \neq \mathbf{u}'_3$, $\mathbf{u}_3 \neq \mathbf{u}'_2$, and $\mathbf{u}_2 \neq \mathbf{u}'_3$, then, thanks to the above formulae, one can write

$$\begin{aligned} & \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3; C_1 \otimes C_2 \otimes C_3 | \mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3; C'_1 \otimes C'_2 \otimes C'_3 \rangle \\ & \propto \int \left[{}^{t_2} \mathbb{S}_{l'_3, l_2}(u'_3 - u_2) {}^{t_3} \mathbb{S}_{l'_2, l_3}(u'_2 - u_3) {}^{t_3} \mathbb{S}_{l_3, l_3}(u'_3 - u_3) C'_2 \otimes C'_3 \otimes C_2^* \otimes C_3^* \right] (z; x; z; y) \\ & \quad \times \frac{C_1^*(x - x_0) C'_1(y - x_0)}{x^{2\left(\tilde{\beta}_1^* + \tilde{\beta}'_3 + \frac{l_1 + l'_3}{2}\right)} y^{2\left(\tilde{\beta}_3^* + \tilde{\beta}'_1 + \frac{l_3 + l'_1}{2}\right)} z^{2\left(\tilde{\beta}_2^* + \tilde{\beta}'_2 + \frac{l_2 + l'_2}{2}\right)}}{\pi^{\frac{3d}{2}}} d^d x d^d y d^d z. \end{aligned} \quad (4.2.29)$$

Then, since the integrals over x , y and z are of the form of (4.2.22), this simplifies into

$$\begin{aligned} & \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3; C_1 \otimes C_2 \otimes C_3 | \mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3; C'_1 \otimes C'_2 \otimes C'_3 \rangle \propto \delta(\mathbf{u}_1 - \mathbf{u}'_3) \delta(\mathbf{u}_2 - \mathbf{u}'_2) \delta(\mathbf{u}_3 - \mathbf{u}'_1) \\ & \quad \times \left\langle C_3 \otimes C_2 \otimes C_1 \left| \mathbb{S}_{l'_2, l'_3}(u'_3 - u'_2) \mathbb{S}_{l'_1, l'_3}(u'_3 - u'_1) \mathbb{S}_{l'_1, l'_2}(u'_2 - u'_1) C'_1 \otimes C'_2 \otimes C'_3 \right. \right\rangle. \end{aligned} \quad (4.2.30)$$

Thanks to the delta functions, the prefactor is actually exactly $\mu(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^{-1}$. It remains to notice that

$$\mathbb{S}_{l'_2, l'_3}(u'_3 - u'_2) \mathbb{S}_{l'_1, l'_3}(u'_3 - u'_1) \mathbb{S}_{l'_1, l'_2}(u'_2 - u'_1) = \mathbb{S}(\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3; (12)(23)(12)) = \mathbb{S}(\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3; (13)) \quad (4.2.31)$$

because of (4.2.20). On the other hand, when $\mathbf{u}_3 \neq \mathbf{u}'_3$, $\mathbf{u}_3 \neq \mathbf{u}'_2$, and $\mathbf{u}_2 \neq \mathbf{u}'_3$, formula (4.2.26) also reduces to

$$\begin{aligned} & \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3; C_1 \otimes C_2 \otimes C_3 | \mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3; C'_1 \otimes C'_2 \otimes C'_3 \rangle = \mu(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^{-1} \\ & \quad \times \delta(\mathbf{u}_1 - \mathbf{u}'_3) \delta(\mathbf{u}_2 - \mathbf{u}'_2) \delta(\mathbf{u}_3 - \mathbf{u}'_1) \left\langle C_3 \otimes C_2 \otimes C_1 \left| \mathbb{S}(\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3; (13)) C'_1 \otimes C'_2 \otimes C'_3 \right. \right\rangle. \end{aligned} \quad (4.2.32)$$

The other terms of (4.2.26) appear when requiring, following (4.2.21), that the full result for the inner product be invariant under

$$|\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3; C_1 \otimes C_2 \otimes C_3\rangle \mapsto \left| \mathbf{u}_{\sigma^{-1}(1)}, \mathbf{u}_{\sigma^{-1}(2)}, \mathbf{u}_{\sigma^{-1}(3)}; \mathbb{P}_\sigma \mathbb{S}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3; \sigma) C_1 \otimes C_2 \otimes C_3 \right\rangle \quad (4.2.33)$$

for all the permutations $\sigma \in \mathfrak{S}_3$. This whole procedure is easily generalised to arbitrary N .

4.2.4 Completeness

Let $\{C_{m,l}\}_{1 \leq m \leq d_l}$ be an orthonormal basis of \mathbb{V}_l for the inner product defined in (4.2.23) (d_l is the dimension of \mathbb{V}_l). We postulate that, for any N , the following resolution of the identity holds:

$$\begin{aligned} & \sum_{\substack{0 \leq l_1 \leq +\infty \\ 1 \leq m_1 \leq d_{l_1}}} \cdots \sum_{\substack{0 \leq l_N \leq +\infty \\ 1 \leq m_N \leq d_{l_N}}} \int \cdots \int \frac{\mu(\mathbf{u}_1, \dots, \mathbf{u}_N)}{N!} \langle x_1, \dots, x_N | \mathbf{u}_1, \dots, \mathbf{u}_N; C_{m_1, l_1} \otimes \cdots \otimes C_{m_N, l_N} \rangle \\ & \quad \times \langle \mathbf{u}_1, \dots, \mathbf{u}_N; C_{m_1, l_1} \otimes \cdots \otimes C_{m_N, l_N} | y_1, \dots, y_N \rangle \prod_{k=1}^N du_k = \prod_{k=1}^N \pi^{\frac{d}{2}} \delta(x_k - y_k). \end{aligned} \quad (4.2.34)$$

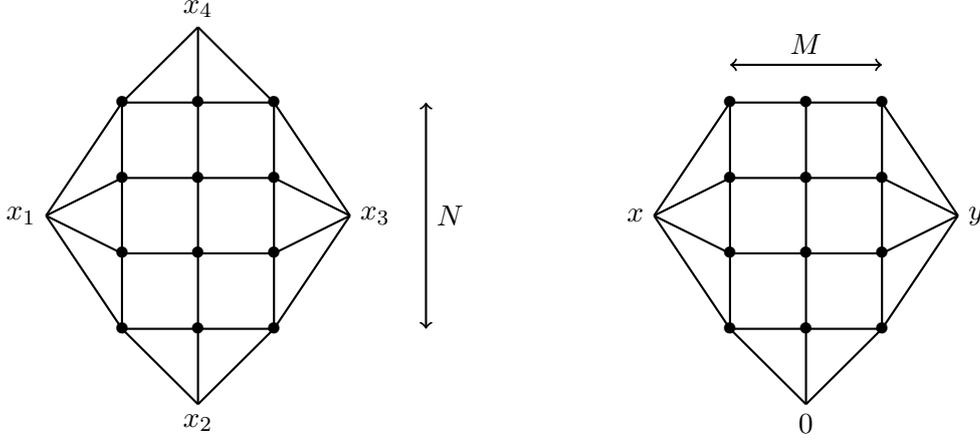


Figure 4.6: Feynman graphs investigated in Section 4.3 in the case $(M, N) = (3, 4)$. We call $G_{M,N}^{(d,\delta)}(x_1, x_2, x_3, x_4)$ and $I_{M,N}^{(d,\delta)}(x, y)$ the integrals represented by the graphs on the left and on the right respectively. All the vertical (or ending on x_2, x_4 , or 0) segments have weight δ , whereas all the horizontal (or ending on x_1, x_3, x , or y) ones have weight $\tilde{\delta}$.

The power of π in the right-hand side comes from the fact that we have defined $|x\rangle$ such that $\langle x|y\rangle = \pi^{\frac{d}{2}}\delta(x-y)$ (see beginning of Section 4.2). This completeness relation is easily verified in the case $N = 1$, but much less clear when $N > 1$.

4.3 Basso–Dixon Diagrams

In this section, we investigate the possible application of these operators to the computations of some fishnet Feynman integrals presented in Figure 4.6. Up to a trivial normalisation factor, the Feynman graph of the left panel has an interpretation as a four-point correlator in the fishnet theory:

$$G_{M,N}^{(d,\delta)}(x_1, x_2, x_3, x_4) \propto \left\langle \text{Tr} \left(X^N(x_1) Z^M(x_2) X^{\dagger N}(x_3) Z^{\dagger M}(x_4) \right) \right\rangle. \quad (4.3.1)$$

Because of the conformal invariance of the integral, it is equivalent to compute the integral associated to the right panel of the figure. A simple change of variables indeed shows that

$$G_{M,N}^{(d,\delta)}(x_1, x_2, x_3, x_4) = \frac{1}{(x_{24}^2)^{M\delta} (x_{14}^2 x_{34}^2)^{N\tilde{\delta}}} I_{M,N}^{(d,\delta)} \left(\frac{x_{14}}{x_{14}^2} - \frac{x_{24}}{x_{24}^2}, \frac{x_{34}}{x_{34}^2} - \frac{x_{24}}{x_{24}^2} \right). \quad (4.3.2)$$

In turn, the integral $I_{M,N}^{(d,\delta)}$ is almost a matrix element of the $M+1$ -th power of the graph-building operator $\Lambda_{N,\tilde{\delta}} = Q_N \left(i \frac{\tilde{\delta}}{2} \right)$, one just has to be careful when sending all the external points to the same value x :

$$I_{M,N}^{(d,\delta)}(x, y) = \pi^{\frac{Nd}{2}} \langle x, \dots, x | \left(\prod_{i=1}^N \hat{x}_{i-1,i}^{2\tilde{\delta}} \right) \Lambda_{N,\tilde{\delta}}^{M+1} | y, \dots, y \rangle, \quad (4.3.3)$$

where we have set $x_0 = 0$. It thus seems natural to use the completeness relation to try to express these integrals in a simpler form. This was successfully achieved in two dimensions in [179] and in four dimensions in [176, 177]. We will now see that for other dimensions the result is actually more complicated, and nice expressions are not as readily obtained.

As seen above, the eigenvalue of $Q_N \left(i \frac{\tilde{\delta}}{2} \right)$ factorises into a product $\prod_{k=1}^N Q_{l_k}(u_k)$ of

$$Q_l(u) = Q_l \left(i \frac{\tilde{\delta}}{2} \middle| u \right) = \frac{\Gamma(\delta) \Gamma \left(\frac{d}{4} - \frac{\delta}{2} + \frac{l}{2} - iu \right) \Gamma \left(\frac{d}{4} - \frac{\delta}{2} + \frac{l}{2} + iu \right)}{\Gamma(\tilde{\delta}) \Gamma \left(\frac{d}{4} + \frac{\delta}{2} + \frac{l}{2} + iu \right) \Gamma \left(\frac{d}{4} + \frac{\delta}{2} + \frac{l}{2} - iu \right)}. \quad (4.3.4)$$

4.3.1 Ladder Diagrams

We first give the expressions for the so-called ladder diagrams [168, 169, 218, 219] in arbitrary dimension:

$$I_{M,1}^{(d,\delta)} = \frac{\Gamma \left(\frac{d}{2} \right)}{(x_3^2 x_4^2)^{\frac{\delta}{2}}} \sum_{l=0}^{+\infty} \frac{2l + d - 2}{d - 2} C_l^{(\frac{d-2}{2})}(\cos \theta) \int \left(\frac{x_3^2}{x_4^2} \right)^{iu} Q_l(u)^{M+1} \frac{du}{2\pi}, \quad (4.3.5)$$

where Q_l is given in equation (4.3.4), $C_l^{(\mu)}$ are Gegenbauer polynomials of degree l , and $\cos \theta = x_3 \cdot x_4 / |x_3| |x_4|$. We assume $d \geq 3$.

The integral is straightforwardly computed by residues, but the eigenvalue Q_l generically has infinitely many poles. However, when δ is a positive integer, Q_l^{-1} is a polynomial of degree 2δ and there is a finite number of poles. In the simplest case, i.e. $\delta = 1$, one has $Q_l(u)^{-1} = \Gamma \left(\frac{d-2}{2} \right) (u^2 + (l + (d-2)/2)^2/4)$ and performing the integral yields

$$I_{M,1}^{(d,1)} = \frac{\Gamma \left(\frac{d-2}{2} \right)^{-M}}{(x_3^2 x_4^2)^{\frac{d-2}{4}}} \sum_{k=M}^{2M} \frac{k! (-2 \ln r)^{2M-k}}{M! (k-M)! (2M-k)!} \sum_{l=0}^{+\infty} C_l^{(\frac{d-2}{2})}(\cos \theta) \frac{r^{l + \frac{d-2}{2}}}{\left(l + \frac{d-2}{2} \right)^k}, \quad (4.3.6)$$

with $r = \sqrt{x_4^2/x_3^2}$.

When d is even we also have the following property of the Gegenbauer polynomials

$$\Gamma \left(\frac{d-2}{2} \right) C_l^{(\frac{d-2}{2})}(x) = 2^{\frac{4-d}{2}} \frac{d^{\frac{d-4}{2}}}{dx^{\frac{d-4}{2}}} \left[C_{l+\frac{d-4}{2}}^{(1)}(x) \right]. \quad (4.3.7)$$

Consequently, for even $d > 2$, we can write ($z = r e^{i\theta}$)

$$I_{M,1}^{(d,1)} = \frac{2^{\frac{4-d}{2}} \Gamma \left(\frac{d-2}{2} \right)^{-M-1}}{(x_3^2 x_4^2)^{\frac{d-2}{4}}} \frac{d^{\frac{d-4}{2}}}{d \cos \theta^{\frac{d-4}{2}}} \left[\frac{L_M(z, \bar{z})}{e^{i\theta} - e^{-i\theta}} \right], \quad (4.3.8)$$

where we have introduced the ladder function L_M defined for $M > 0$ by [168, 220]

$$L_M(z, \bar{z}) = \sum_{k=M}^{2M} \frac{k! [-\ln(z\bar{z})]^{2M-k}}{M! (k-M)! (2M-k)!} [\text{Li}_k(z) - \text{Li}_k(\bar{z})], \quad (4.3.9)$$

with $\text{Li}_k(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^k}$ the polylogarithm of order k .

4.3.2 Two-Layer Diagrams

Inserting the resolution of the identity in the expression of $I_{M,2}^{(d,\delta)}$, it becomes

$$I_{M,2}^{(d,\delta)} = \sum_{\substack{0 \leq l_1 \leq +\infty \\ 1 \leq m_1 \leq d_{l_1}}} \sum_{\substack{0 \leq l_2 \leq +\infty \\ 1 \leq m_2 \leq d_{l_2}}} \int [Q_{l_1}(u_1)Q_{l_2}(u_2)]^{M+1} \frac{\mu(\mathbf{u}_1, \mathbf{u}_2)}{2} \\ \times \langle x_3, x_3 | \hat{x}_{12}^{2\delta} \hat{x}_1^{2\delta} | \mathbf{u}_1, \mathbf{u}_2; C_{m_1, l_1} \otimes C_{m_2, l_2} \rangle \langle \mathbf{u}_1, \mathbf{u}_2; C_{m_1, l_1} \otimes C_{m_2, l_2} | x_4, x_4 \rangle du_1 du_2. \quad (4.3.10)$$

One can show that

$$\langle x_3, x_3 | \hat{x}_{12}^{2\delta} \hat{x}_1^{2\delta} | \mathbf{u}_1, \mathbf{u}_2; C \rangle = A_0(\tilde{\delta}) A_{l_1}(\tilde{\alpha}_1) A_{l_2}(\tilde{\alpha}_2) \frac{4^{1-\frac{d}{2}+iu_{21}} i^{l_1+l_2}}{x_3^{2(2\alpha_1-\frac{d}{2}+1)}} \int \frac{C(p, p) e^{ip \cdot x_3}}{p^{2(1+\frac{l_1+l_2}{2}+iu_{21})}} \frac{d^d p}{\pi^{\frac{d}{2}}}, \quad (4.3.11)$$

$$\langle x_4, x_4 | \mathbf{u}_1, \mathbf{u}_2; C \rangle = A_0\left(\frac{d}{2} + \delta\right) A_{l_1}(\beta_1) A_{l_2}(\beta_2) \frac{4^{1-\frac{d}{2}+iu_{21}} i^{l_1+l_2}}{x_4^{2(2\tilde{\beta}_1-\frac{d}{2}+1)}} \int \frac{C(p, p) e^{ip \cdot x_4}}{p^{2(1+\frac{l_1+l_2}{2}+iu_{21})}} \frac{d^d p}{\pi^{\frac{d}{2}}}. \quad (4.3.12)$$

Both of these equations can be rewritten using the operator \mathbb{O}_{l_1, l_2} introduced in (4.1.66), the first one becomes for instance

$$\langle x_3, x_3 | \hat{x}_{12}^{2\delta} \hat{x}_1^{2\delta} | \mathbf{u}_1, \mathbf{u}_2; C \rangle = A_0(\tilde{\delta}) A_{l_1}(\tilde{\alpha}_1) A_{l_2}(\tilde{\alpha}_2) \\ \times (-1)^{l_1+l_2} A_{l_1+l_2}(1+iu_{21}) \frac{[\mathbb{O}_{l_1, l_2}(u_2 - u_1)C] \cdot x_3^{\otimes(l_1+l_2)}}{x_3^{2(\alpha_1+\alpha_2+\frac{l_1+l_2}{2})}}. \quad (4.3.13)$$

If we want to proceed without using the operator \mathbb{O}_{l_1, l_2} we have to remark that, necessarily, whatever orthonormal basis of symmetric traceless tensors we chose,

$$\sum_{m=1}^{d_l} C_{m, l}(p) C_{m, l}^*(q) = \frac{\Gamma\left(\frac{d}{2}\right) (2l+d-2)}{2\pi(d-2)} |p|^{l_1} |q|^{l_2} C_l^{\left(\frac{d-2}{2}\right)} \left(\frac{p \cdot q}{|p||q|}\right). \quad (4.3.14)$$

We can thus rewrite $I_{M,2}^{(d,\delta)}$ as

$$I_{M,2}^{(d,\delta)} = \frac{\Gamma\left(\frac{d}{2}\right)^2}{2} \sum_{l_1, l_2} \int \frac{4^{2-d} [Q_{l_1}(u_1)Q_{l_2}(u_2)]^{M+2}}{x_3^{2(2\alpha_1-\frac{d}{2}+1)} x_4^{2(2\beta_1-\frac{d}{2}+1)}} \\ \times \left[u_{12}^2 + \frac{l_{12}^2}{4} \right] \left[u_{12}^2 + \frac{(l_1+l_2+d-2)^2}{4} \right] \frac{(2l_1+d-2)(2l_2+d-2)}{(d-2)^2} \\ \times \int \frac{C_{l_1}^{\left(\frac{d-2}{2}\right)} \left(\frac{p \cdot q}{|p||q|}\right) C_{l_2}^{\left(\frac{d-2}{2}\right)} \left(\frac{p \cdot q}{|p||q|}\right)}{p^{2(1+i(u_2-u_1))} q^{2(1+i(u_1-u_2))}} e^{ip \cdot x_3 - iq \cdot x_4} \frac{d^d p d^d q}{\pi^d} \frac{du_1 du_2}{(2\pi)^2}. \quad (4.3.15)$$

If we expect that the integral can be recast as a determinant and hope to generalise this to an arbitrary number of layers, it is possible that this last formula is the most convenient

one. If we however want to perform the integrals over p and q , we may proceed as follows: one first expands the product of two Gegenbauer polynomials according to

$$C_{l_1}^{(\frac{d-2}{2})} \left(\frac{p \cdot q}{|p||q|} \right) C_{l_2}^{(\frac{d-2}{2})} \left(\frac{p \cdot q}{|p||q|} \right) = \sum_{m=0}^{\min(l_1, l_2)} a_{l_1, l_2, m} C_{l_1 + l_2 - 2m}^{(\frac{d-2}{2})} \left(\frac{p \cdot q}{|p||q|} \right), \quad (4.3.16)$$

with

$$a_{l_1, l_2, m} = \frac{(l_1 + l_2 - 2m + \frac{d-2}{2})(l_1 + l_2 - 2m)!}{(l_1 + l_2 - m + \frac{d-2}{2})m!(l_1 - m)!(l_2 - m)!} \frac{\left(\frac{d-2}{2}\right)_m \left(\frac{d-2}{2}\right)_{l_1 - m} \left(\frac{d-2}{2}\right)_{l_2 - m}}{\left(\frac{d-2}{2}\right)_{l_1 + l_2 - m}} (d-2)_{l_1 + l_2 - m}. \quad (4.3.17)$$

Then, one uses the fact that $C_l^{(\frac{d-2}{2})} \left(\frac{p \cdot q}{|p||q|} \right)$ is a spherical harmonic with respect to both p and q (see equation (4.3.14)) to compute the integrals over these variables using (B.1.7):

$$\begin{aligned} \int \frac{C_l^{(\frac{d-2}{2})} \left(\frac{p \cdot q}{|p||q|} \right)}{p^{2(1-iu_{12})} q^{2(1+iu_{12})}} e^{ip \cdot x_3 - iq \cdot x_4} \frac{d^d p d^d q}{\pi^d} \\ = 4^{d-2} \left| \frac{\Gamma\left(\frac{l+d-2}{2} + iu_{12}\right)}{\Gamma\left(\frac{l+2}{2} + iu_{12}\right)} \right|^2 \frac{C_l^{(\frac{d-2}{2})} \left(\frac{x_3 \cdot x_4}{|x_3||x_4|} \right)}{x_3^{2\left(\frac{d}{2}-1+iu_{12}\right)} x_4^{2\left(\frac{d}{2}-1-iu_{12}\right)}}. \end{aligned} \quad (4.3.18)$$

Consequently, one can write

$$\begin{aligned} I_{M,2}^{(d,\delta)} &= \frac{\Gamma\left(\frac{d}{2}\right)^2}{2(x_3^2 x_4^2)^\delta} \sum_{l_1, l_2} \int \left(\frac{x_3^2}{x_4^2}\right)^{i(u_1+u_2)} [Q_{l_1}(u_1) Q_{l_2}(u_2)]^{L+2} \\ &\quad \times \left[u_{12}^2 + \frac{l_{12}^2}{4} \right] \left[u_{12}^2 + \frac{(l_1 + l_2 + d - 2)^2}{4} \right] \frac{(2l_1 + d - 2)(2l_2 + d - 2)}{(d - 2)^2} \\ &\quad \times \sum_{m=0}^{\min(l_1, l_2)} a_{l_1, l_2, m} \left| \frac{\Gamma\left(\frac{l_1 + l_2 - 2m + d - 2}{2} + iu_{12}\right)}{\Gamma\left(\frac{l_1 + l_2 - 2m + 2}{2} + iu_{12}\right)} \right|^2 C_{l_1 + l_2 - 2m}^{(\frac{d-2}{2})}(\cos \theta) \frac{du_1 du_2}{(2\pi)^2}, \end{aligned} \quad (4.3.19)$$

with $x_3 \cdot x_4 = |x_3||x_4| \cos \theta$. Using the operator \mathbb{O}_{l_1, l_2} , this can actually be written in a more concise way:

$$\begin{aligned} I_{M,2}^{(d,\delta)} &= \frac{\Gamma\left(\frac{d}{2}\right)^2}{2(x_3^2 x_4^2)^\delta} \sum_{l_1, l_2} \int \left(\frac{x_3^2}{x_4^2}\right)^{i(u_1+u_2)} [Q_{l_1}(u_1) Q_{l_2}(u_2)]^{M+2} \\ &\quad \times \left[u_{12}^2 + \frac{l_{12}^2}{4} \right] \left[u_{12}^2 + \frac{(l_1 + l_2 + d - 2)^2}{4} \right] \frac{(2l_1 + d - 2)(2l_2 + d - 2)}{(d - 2)^2} \\ &\quad \times \left| \frac{\Gamma\left(\frac{l_1 + l_2 + d - 2}{2} + iu_{12}\right)}{\Gamma\left(\frac{l_1 + l_2 + 2}{2} + iu_{12}\right)} \right|^2 \frac{[\mathbb{O}_{l_1, l_2}(u_{21})]^t \mathbb{O}_{l_1, l_2}(u_{12}) x_3^{\otimes(l_1+l_2)}] \cdot x_4^{\otimes(l_1+l_2)}}{(|x_3||x_4|)^{l_1+l_2}} \frac{du_1 du_2}{(2\pi)^2}. \end{aligned} \quad (4.3.20)$$

Notice that since \mathbb{O}_{l_1, l_2} goes from $\mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2}$ to $S^{l_1+l_2}(\mathbb{C}^d)$ we can only multiply it with its transpose. This is what happens here where we need matrix elements of $\mathbb{O}_{l_1, l_2}(u_{21})^t \mathbb{O}_{l_1, l_2}(u_{12}) : S^{l_1+l_2}(\mathbb{C}^d) \rightarrow S^{l_1+l_2}(\mathbb{C}^d)$. Neither of the previous two expressions for $I_{M,2}^{(d,\delta)}$ seem to be very convenient, but we do not know how to simplify them at the moment.

The limit $d \rightarrow 2$ is seemingly singular but one should remember that the Gegenbauer polynomials for $l > 0$ tend to 0 in this limit so that

$$\forall l \in \mathbb{N}, \quad \frac{2l+d-2}{d-2} C_l^{(\frac{d-2}{2})}(\cos \theta) \xrightarrow{d \rightarrow 2} \frac{2}{1+\delta_{l,0}} \cos l\theta. \quad (4.3.21)$$

Thus, for $\min(l_1, l_2) > 0$, one has

$$\begin{aligned} \frac{(2l_1+d-2)(2l_2+d-2)}{(d-2)^2} a_{l_1, l_2, m} C_{l_1+l_2-2m}^{(\frac{d-2}{2})}(\cos \theta) \\ \xrightarrow{d \rightarrow 2} 2 \left[\delta_{m,0} \cos(l_1+l_2)\theta + \delta_{m, \min(l_1, l_2)} \cos(l_1-l_2)\theta \right]. \end{aligned} \quad (4.3.22)$$

In the end, $I_{M,2}^{(2,\delta)}$ is finite (as it should be) and, using the additional symmetry $Q_l = Q_{-l}$ valid for $l \in \mathbb{Z}$ when $d = 2$, one can extend the sum to $(l_1, l_2) \in \mathbb{Z}^2$ so that

$$I_{M,2}^{(2,\delta)} = \frac{1}{2(x_3^2 x_4^2)^\delta} \sum_{(l_1, l_2) \in \mathbb{Z}^2} e^{i(l_1+l_2)\theta} \int \left(\frac{x_3^2}{x_4^2} \right)^{i(u_1+u_2)} [Q_{l_1}(u_1) Q_{l_2}(u_2)]^{M+2} \left[u_{12}^2 + \frac{l_{12}^2}{4} \right] \frac{du_1 du_2}{(2\pi)^2}. \quad (4.3.23)$$

This coincides with the result of [179].

When $d = 4$, the dependence on u of the sum over m disappears and the sum is then simply

$$C_{l_1}^{(1)}(\cos \theta) C_{l_2}^{(1)}(\cos \theta) = \frac{(e^{i(l_1+1)\theta} - e^{-i(l_1+1)\theta})(e^{i(l_2+1)\theta} - e^{-i(l_2+1)\theta})}{(e^{i\theta} - e^{-i\theta})^2}. \quad (4.3.24)$$

Noticing that $Q_l = Q_{-l-2}$ when $d = 4$, we can keep only one of the four terms from the equation above, if we extend the summation to $(l_1, l_2) \in \mathbb{Z}^2$. We thus obtain (we have also replaced l_j with $a_j = l_j + 1$):

$$\begin{aligned} I_{M,2}^{(4,\delta)} &= \frac{1}{2(x_3^2 x_4^2)^\delta} \frac{1}{(e^{i\theta} - e^{-i\theta})^2} \sum_{(a_1, a_2) \in \mathbb{Z}^2} a_1 a_2 e^{i(a_1+a_2)\theta} \\ &\times \int \left(\frac{x_3^2}{x_4^2} \right)^{i(u_1+u_2)} [Q_{a_1-1}(u_1) Q_{a_2-1}(u_2)]^{M+2} \left[u_{12}^2 + \frac{a_{12}^2}{4} \right] \left[u_{12}^2 + \frac{(a_1+a_2)^2}{4} \right] \frac{du_1 du_2}{(2\pi)^2}. \end{aligned} \quad (4.3.25)$$

When $\delta = 1$, this was found in [175], see also [176, 177].

The next case we could investigate is $(d, \delta) = (6, 1)$, the formula (4.3.19) then reads

$$\begin{aligned}
I_{M,2}^{(6,1)} &= \frac{1}{2(x_3^2 x_4^2)^2} \sum_{l_1, l_2} \int \left(\frac{x_3^2}{x_4^2} \right)^{i(u_1+u_2)} \frac{\left[u_{12}^2 + \frac{l_{12}^2}{4} \right] \left[u_{12}^2 + \frac{(l_1+l_2+4)^2}{4} \right]}{\left[\left(u_1^2 + \frac{(l_1+2)^2}{4} \right) \left(u_2^2 + \frac{(l_2+2)^2}{4} \right) \right]^{M+2}} (l_1+2)(l_2+2) \\
&\quad \times \sum_{m=0}^{\min(l_1, l_2)} \frac{(m+1)(l_1-m+1)(l_2-m+1)(l_1+l_2-m+3)}{(l_1+l_2-2m+1)(l_1+l_2-2m+3)} \\
&\quad \times \left[u_{12}^2 + \frac{(l_1+l_2-2m+2)^2}{4} \right] C_{l_1+l_2-2m}^{(2)}(\cos \theta) \frac{du_1 du_2}{(2\pi)^2}. \quad (4.3.26)
\end{aligned}$$

The integrals are rather easy, at least when M is not too large, but the sums seem to be quite tedious to perform.

We hope that the example of $I_{M,2}^{(d,\delta)}$ shows clearly the difficulties arising when trying to compute Basso–Dixon integrals in arbitrary dimension.

Chapter 5

Thermodynamic Bethe Ansatz for Fishnet Theories

Most of the content of this chapter is the same as that of [181] up to minor changes in presentation. The only two significant additions are the proof of the asymptotic behaviour of the two-particle states in Subsection 5.1.2 and the computations of the two-magnon anomalous dimension in Subsection 5.4.2 and Appendix B.2. This second work was done in collaboration with De-liang Zhong.

We propose here thermodynamic Bethe ansatz (TBA) equations for the dimensions of multi-magnon operators of the type

$$\mathcal{O}_{J,M}(x) = \text{Tr}(X^M Z^J) + \dots \quad (5.0.1)$$

in the fishnet theory (2.3.2), in arbitrary dimension d and for an arbitrary anisotropy parameter δ . The mixing matrix of such operators is entirely defined by multi-wheel or multi-spiral planar Feynman graphs [143, 155], such as those in Figure 5.1. These integrals have attracted a considerable interest in the literature as examples of explicitly calculable multi-loop Feynman graphs [150, 169, 175, 179, 221].

In the case of the original fishnet theory, i.e. when $(d, \delta) = (4, 1)$, these TBA equations can be obtained by taking the double-scaling limit of the full TBA system of twisted $\mathcal{N} = 4$ super Yang–Mills [150, 155, 222]. We no longer have this luxury when we deal with the theory in arbitrary dimension, which does not have its SYM “parent”. In this case, to arrive at the TBA equations we shall rely on the direct fishnet graph computations as well as a certain intuition borrowed from the $d = 4$ case. We will then derive the asymptotic Bethe ansatz (ABA) equations, valid in the limit $J \rightarrow \infty$, and test them against explicit fishnet-type Feynman integrals computations.

5.1 Graph-building Operators and Scattering Data

The TBA construction relies on the knowledge of the asymptotic data, dispersion relation and factorised S-matrix, that characterise the integrable structure of the fishnet graphs. In planar $\mathcal{N} = 4$ SYM, these were determined using supersymmetry and crossing symmetry

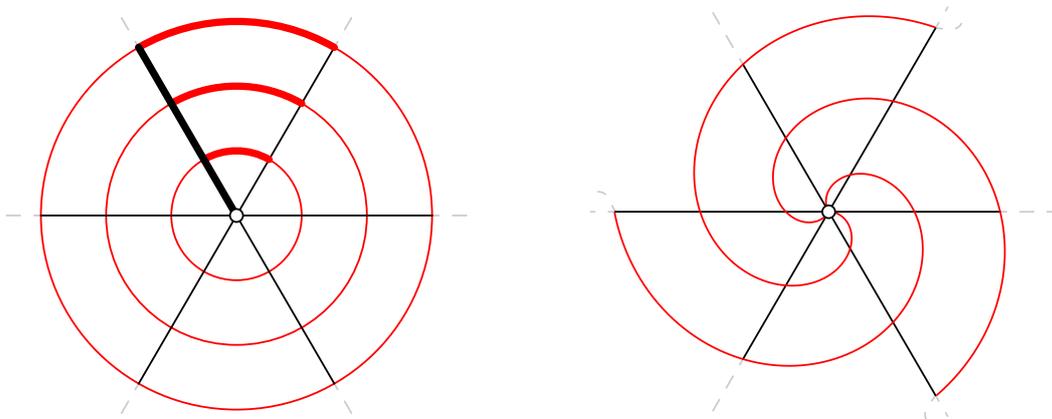


Figure 5.1: Specimens of planar fishnet graphs contributing to the anomalous dimensions of multi-magnon operators (central points) with black and red lines representing propagators of Z and X fields, respectively. Left panel: Multi-wheel graph renormalising the ground-state operator with $M = 0$. The graph can be obtained by iterating the graph-building operator $\Lambda_{N=3,\delta}$ shown here in bold face. Right panel: Multi-spiral graph contributing to the mixing of excited-state operators with here $M = 3$ magnons inserted at the origin.

[117, 122–124]. We cannot follow these steps for the fishnet theory in arbitrary dimension d for lack of symmetries, but we can read off the scattering data directly from the graphs. In fact, the information can all be obtained from the wheel graphs shown in Figure 5.1 (left) and corresponding to the local operator (5.0.1) with $M = 0$, referred to as the vacuum state. General results for the excited states with $M \neq 0$ will be given in a subsequent section.

The S-matrix that is required here is the one controlling the scattering of magnons in the “open string channel” or mirror kinematics. The idea is to treat the X propagators along the angular direction in Figure 5.1 as magnon excitations moving radially along the Z propagators. Geometrically, this mirror 1-dimensional system emerges from the decomposition $\mathbb{R}^d \cong \mathbb{R}_+ \times S^{d-1}$, with $r = e^\sigma \in \mathbb{R}_+$ being the distance to the origin, σ the mirror position, and with the sphere S^{d-1} giving rise to an internal $O(d)$ symmetry.

Mirror magnons evolve in this picture through the action of the graph-building operator

$$[\Lambda_{N,\delta}\Phi](x_1, \dots, x_N) = \int \Phi(y_1, \dots, y_N) \prod_{i=1}^N \frac{\pi^{-d/2} d^d y_i}{(x_{i-1} - x_i)^{2\delta} (x_i - y_i)^{2\delta}}, \quad (5.1.1)$$

with $x_0 = 0$. We recall that it acts on N -magnon wave function $\Phi \in \mathcal{V}_{\frac{d}{2}+\delta}^{\otimes N}$ where $\mathcal{V}_{\frac{d}{2}+\delta}$ is a representation of the conformal group that is part of the scalar complementary series, cf. Section 3.1. Clearly, any wheel graph can be obtained by iteration of a graph-building operator, see Figure 5.1. The significance of these operators in the fishnet theory was unveiled in [179] in the particular case $d = 2$. We show below how their diagonalisation, which was explained in the previous chapter, provides the scattering data for the magnons for general d .

5.1.1 Magnon Dispersion Relation

We saw in the previous chapter that the eigenvectors of $\Lambda_{N,\tilde{\delta}}$ are parametrised by N pairs $\mathbf{u}_k = (u_k, l_k) \in \mathbb{R} \times \mathbb{N}$ and a tensor structure $C \in \mathbb{V}_{l_1} \otimes \cdots \otimes \mathbb{V}_{l_N}$, where \mathbb{V}_l is the vector space of symmetric traceless tensors of rank l . The eigenvalue is then $Q_{l_1}(u_1) \cdots Q_{l_N}(u_N)$ with

$$Q_l(u) = \frac{\Gamma(\tilde{\delta})\Gamma\left(\frac{\delta}{2} + \frac{l}{2} + iu\right)\Gamma\left(\frac{\delta}{2} + \frac{l}{2} - iu\right)}{\Gamma(\delta)\Gamma\left(\frac{d-\delta}{2} + \frac{l}{2} + iu\right)\Gamma\left(\frac{d-\delta}{2} + \frac{l}{2} - iu\right)}. \quad (5.1.2)$$

The quantity $Q_l(u)$ is the weight of propagation of a magnon with rapidity u and spin l , it naturally defines the magnon energy ε_l through

$$\varepsilon_l = -\ln Q_l. \quad (5.1.3)$$

This way, the energy of an N -particle eigenstate is the sum of the individual energies. The momentum conjugate to the (radial) position σ is $p_l(u) = 2u$, as can be read off directly from the expression of the one-particle eigenstates

$$\langle x|\mathbf{u}; C\rangle = \frac{C(x)}{x^{2(\tilde{\beta} + \frac{l}{2})}} = C\left(\frac{x}{|x|}\right) e^{(-\frac{d}{2} - \delta + 2iu)\sigma}, \quad (5.1.4)$$

where β is defined in (4.2.2) and $C(x) = C^{\mu_1 \cdots \mu_l} x_{\mu_1} \cdots x_{\mu_l}$.

5.1.2 Magnon S-Matrix

In order to determine the magnon S-matrix, we need to examine eigenstates with more than one particle. The simplest case is that of two-particle eigenstates whose expression we now recall

$$\begin{aligned} \langle x_1, x_2|\mathbf{u}_1, \mathbf{u}_2; C\rangle &= \frac{A_0\left(1 - \alpha_1 - \frac{l_1}{2}\right) A_0\left(\beta_1 + \frac{l_1}{2}\right)}{x_1^{2(\tilde{\beta}_1 - \frac{l_1}{2})}} \int \frac{1}{x_{2b}^{2(\tilde{\beta}_1 - \frac{l_1}{2})} x_{1b}^{2(\tilde{\alpha}_1 + \frac{l_1}{2})}} \\ &\quad \times \int \frac{C\left(\frac{x_1}{x_1^2} - \frac{x_a}{x_a^2}; x_a\right)}{x_{ab}^{2(\alpha_1 + \frac{d+l_1}{2} - 1)} x_a^{2(1+i(u_1-u_2) + \frac{l_2-l_1}{2})}} \frac{d^d x_a d^d x_b}{\pi^d}, \end{aligned} \quad (5.1.5)$$

where $x_{ij} = x_i - x_j$, the parameters α and β are defined in (4.2.2), $C \in \mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2}$ and

$$C(x; y) = C^{\mu_1 \cdots \mu_{l_1} \nu_1 \cdots \nu_{l_2}} x_{\mu_1} \cdots x_{\mu_{l_1}} y_{\nu_1} \cdots y_{\nu_{l_2}}. \quad (5.1.6)$$

We claim that the asymptotic behaviour of these eigenstates in the limit $x_2^2 \rightarrow +\infty$, with x_1^2 remaining finite, is given by the sum of two plane waves, one incoming and one outgoing:

$$\begin{aligned} \langle x_1, x_2|\mathbf{u}_1, \mathbf{u}_2; C\rangle &\simeq \frac{K_{l_1, l_2}(u_1, u_2)}{(x_1^2 x_2^2)^{\frac{d}{4} + \frac{\delta}{2}}} \left[e^{i(p_1 \sigma_1 + p_2 \sigma_2)} C\left(\frac{x_1}{|x_1|}; \frac{x_2}{|x_2|}\right) \right. \\ &\quad \left. + e^{i(p_2 \sigma_1 + p_1 \sigma_2)} [\mathbf{S}_{l_1, l_2}(u_1, u_2) C]\left(\frac{x_2}{|x_2|}; \frac{x_1}{|x_1|}\right) \right], \end{aligned} \quad (5.1.7)$$

where

$$K_{l_1, l_2}(u_1, u_2) = f_{l_2}(u_2) A_{l_2} \left(\frac{d}{2} + iu_{12} + \frac{l_1}{2} \right) A_{l_2} \left(1 + iu_{12} - \frac{l_1}{2} \right) \quad (5.1.8)$$

is just an overall prefactor, whereas the S-matrix is of the form

$$\mathbf{S}_{l, l'}(u, v) = \frac{f_l(u)}{f_{l'}(v)} \mathcal{S}_{l, l'}(u - v) \mathbb{R}_{l, l'}(u - v). \quad (5.1.9)$$

The dynamical factors are

$$\mathcal{S}_{l, l'}(u) = \frac{\Gamma \left(1 + \frac{l+l'}{2} - iu \right) \Gamma \left(\frac{d}{2} + \frac{l+l'}{2} + iu \right) \Gamma \left(\frac{|l-l'|}{2} + iu \right) \Gamma \left(1 + \frac{|l-l'|}{2} + iu \right)}{\Gamma \left(1 + \frac{l+l'}{2} + iu \right) \Gamma \left(\frac{d}{2} + \frac{l+l'}{2} - iu \right) \Gamma \left(\frac{|l-l'|}{2} - iu \right) \Gamma \left(1 + \frac{|l-l'|}{2} - iu \right)} \quad (5.1.10)$$

and

$$f_l(u) = \frac{A_l(\beta)}{A_l(\alpha)} = \frac{\Gamma \left(\frac{\tilde{\delta}}{2} + \frac{l}{2} - iu \right) \Gamma \left(\frac{d}{4} + \frac{\tilde{\delta}}{2} + \frac{l}{2} - iu \right)}{\Gamma \left(\frac{\tilde{\delta}}{2} + \frac{l}{2} + iu \right) \Gamma \left(\frac{d}{4} + \frac{\tilde{\delta}}{2} + \frac{l}{2} + iu \right)}, \quad (5.1.11)$$

and $\mathbb{R}_{l, l'}$ is the $O(d)$ -invariant rational R-matrix investigated at length in Section 4.1.

The first thing to point out is that the S-matrix $\mathbf{S}_{l, l'}$ appearing here is different from $\mathbb{S}_{l, l'}$ that appeared in the previous chapter, see (4.2.17). This is completely natural, as the latter depends on the normalisation of the eigenvectors whereas the former does not. Moreover, the symmetry of the eigenvector

$$|\mathbf{u}_1, \mathbf{u}_2; C\rangle = |\mathbf{u}_2, \mathbf{u}_1; \mathbb{P}_{(12)} \mathbb{S}_{l_1, l_2}(u_1 - u_2) C\rangle \quad (5.1.12)$$

is consistent with the conjectured asymptotic behaviour, since it holds that

$$\mathbf{S}_{l_1, l_2}(u_1, u_2) = \frac{K_{l_2, l_1}(u_2, u_1)}{K_{l_1, l_2}(u_1, u_2)} \mathbb{S}_{l_1, l_2}(u_1 - u_2). \quad (5.1.13)$$

We are not able to prove the asymptotic behaviour (5.1.7) for arbitrary values of the spins and tensor structure, but we have checked it in two particular cases: when both spins are equal to one and the tensor is arbitrary, and when both spins are arbitrary and the tensor is completely symmetric traceless. Let us explain the computation in the second situation. It is done using the method of expansion by regions [223].

Symmetric Traceless Tensors If $C \in \mathbb{V}_{l_1+l_2} \subset \mathbb{V}_{l_1} \otimes \mathbb{V}_{l_2}$, then the integral over x_a can be performed in the expression for the eigenvector using (B.1.11) so that

$$\begin{aligned} \langle x_1, x_2 | \mathbf{u}_1, \mathbf{u}_2; C \rangle &= \sum_{p=0}^{l_1} \binom{l_1}{p} (-1)^p \frac{A_{l_2+p}(\tilde{\alpha}_2 - \frac{p}{2}) A_0 \left(\beta_1 + \frac{l_1}{2} \right) A_{l_2+p} \left(1 + iu_{12} + \frac{p-l_1}{2} \right)}{x_1^{2(\tilde{\beta}_1 + \frac{l_1}{2} - p)}} \\ &\times \int \frac{C^{(p)} \left(x_1; \frac{x_b}{|x_b|} \right)}{x_{2b}^{2(\tilde{\beta}_1 - \frac{l_1}{2})} x_{1b}^{2(\tilde{\alpha}_1 + \frac{l_1}{2})} x_b^{2(\alpha_2 + \frac{p}{2})}} \frac{d^d x_b}{\pi^{\frac{d}{2}}}, \quad (5.1.14) \end{aligned}$$

where $C^{(p)}(x; y) = C^{\mu_1 \dots \mu_{l_1+l_2}} x_{\mu_1} \dots x_{\mu_{l_1-p}} y_{\mu_{l_1-p+1}} \dots y_{\mu_{l_1+l_2}}$.

We now want to expand this function when $\frac{x_2^2}{x_1^2} \rightarrow +\infty$ up to and including the order $O(x_2^{-2\tilde{\beta}})$. In each integral, we have to consider contributions coming from two regions of integration: x_b close to x_2 , or x_b close to 0 and x_1 . For the p -th term of the sum, the dominant contribution from the first region is obtained by replacing x_{b1}^2 with x_b^2 , it is of order $O(x_2^{-2(\tilde{\beta}+\frac{p}{2})})$, so that in our order of approximation we only need to keep it for the $p = 0$ term:

$$\begin{aligned} \langle x_1, x_2 | \mathbf{u}_1, \mathbf{u}_2; C \rangle_I &\simeq f_{l_2}(u_2) A_{l_2} \left(\frac{d}{2} + iu_{12} + \frac{l_1}{2} \right) A_{l_2} \left(1 + iu_{12} - \frac{l_1}{2} \right) \frac{C \left(\frac{x_1}{|x_1|}; \frac{x_2}{|x_2|} \right)}{x_1^{2\tilde{\beta}_1} x_2^{2\tilde{\beta}_2}} \\ &= K_{l_1, l_2}(u_1, u_2) \frac{C \left(\frac{x_1}{|x_1|}; \frac{x_2}{|x_2|} \right)}{x_1^{2\tilde{\beta}_1} x_2^{2\tilde{\beta}_2}}, \end{aligned} \quad (5.1.15)$$

where we have once again applied (B.1.11).

The contribution from the second region is more difficult to compute. The dominant contribution for each term of the sum is indeed of order $O(x_2^{-2(\tilde{\beta}-\frac{l_2}{2})})$, so that we actually need the first l_2 subleading contributions to each integral. This is done through the following expansion involving Gegenbauer polynomials:

$$\begin{aligned} \frac{1}{x_2^{2(\tilde{\beta}_1-\frac{l_1}{2})}} &= \frac{1}{x_2^{2(\tilde{\beta}_1-\frac{l_1}{2})}} \sum_{n=0}^{+\infty} \left(\frac{|x_b|}{|x_2|} \right)^n C_n^{(\tilde{\beta}_1-\frac{l_1}{2})} \left(\frac{x_2 \cdot x_b}{|x_2||x_b|} \right) \\ &= \frac{1}{x_2^{2(\tilde{\beta}_1-\frac{l_1}{2})}} \sum_{n=0}^{+\infty} \left(\frac{|x_b|}{|x_2|} \right)^n \sum_{q=0}^{\lfloor \frac{n}{2} \rfloor} f_{n,q} \left(\tilde{\beta}_1 - \frac{l_1}{2} \right) C_{n-2q}^{(\frac{d-2}{2})} \left(\frac{x_2 \cdot x_b}{|x_2||x_b|} \right), \end{aligned} \quad (5.1.16)$$

with

$$f_{n,q}(a) = \frac{\left(\frac{d-2}{2} + n - 2q \right) (a)_{n-q} \left(a - \frac{d-2}{2} \right)_q}{\frac{d-2}{2} \left(\frac{d}{2} \right)_{n-q} q!}. \quad (5.1.17)$$

We have expanded in terms of $|x_b|^n C_n^{(\frac{d-2}{2})} \left(\frac{x_2 \cdot x_b}{|x_2||x_b|} \right)$ because those are harmonic in x_b . We can thus compute the integral over x_b , it gives

$$\begin{aligned} \frac{1}{x_1^{2(\tilde{\beta}_1+\frac{l_1}{2}-p)}} \int \frac{C^{(p)} \left(x_1; \frac{x_b}{|x_b|} \right) C_{n-2q}^{(\frac{d-2}{2})} \left(\frac{x_2 \cdot x_b}{|x_2||x_b|} \right) d^d x_b}{x_{1b}^{2(\tilde{\alpha}_1+\frac{l_1}{2})} x_b^{2(\alpha_2+\frac{p-n}{2})}} \frac{1}{\pi^{\frac{d}{2}}} &= \frac{1}{x_1^{2(\tilde{\beta}_1+\frac{l_1}{2}-p)}} \sum_{k=0}^{\min(l_2+p, n-2q)} \binom{l_2+p}{k} \\ &\times 2^{-k} \frac{\Gamma \left(\tilde{\alpha}_2 + \frac{l_2}{2} + n - q - k \right) \Gamma \left(\alpha_1 - \frac{l_1}{2} + k \right) \Gamma \left(iu_{12} + \frac{l_1+l_2}{2} + p - q - k \right)}{\Gamma \left(\alpha_2 + \frac{l_2}{2} + p - q \right) \Gamma \left(\tilde{\alpha}_1 + \frac{l_1}{2} \right) \Gamma \left(\frac{d}{2} + iu_{21} + \frac{l_2-l_1}{2} + n - q \right)} \\ &\times \left(C_{n-2q}^{(\frac{d-2}{2})} \right)^{(k)} \left(\frac{x_2 \cdot x_b}{|x_2||x_b|} \right) C^{(k-l_2)} \left(\frac{x_1}{|x_1|}; \frac{x_2}{|x_2|} \right). \end{aligned} \quad (5.1.18)$$

If we now plug equations (5.1.16) and (5.1.18) into the expression for the eigenvector, we observe that we can perform the sum over p . It is of the form

$$\sum_{p=0}^{l_1} \binom{l_1}{p} (-1)^p P_{l_1, q, k}(p), \quad (5.1.19)$$

with

$$P_{l_1, q, k}(p) = \frac{(l_2 + p) \cdots (l_2 + p + 1 - k)}{k!} \frac{\Gamma\left(\alpha_2 + \frac{l_2}{2} + p\right) \Gamma\left(iu_{12} + \frac{l_1 + l_2}{2} + p - q - k\right)}{\Gamma\left(\alpha_2 + \frac{l_2}{2} + p - q\right) \Gamma\left(iu_{12} + \frac{l_2 - l_1}{2} + p + 1\right)} \quad (5.1.20)$$

a polynomial of degree $l_1 - 1$, unless $q + k \geq l_1$. On the one hand, if $P_{l_1, q, k}$ is of degree $l_1 - 1$, then the sum over p automatically vanishes. On the other hand, since we look for a result valid at order $O(x_2^{-2\tilde{\beta}})$, we can restrict ourselves to $n \leq l_1$. As a consequence, we have $q + k \leq n - q \leq n \leq l_1$, and the only contributions that survive the summation over p are those for $q + k = l_1$, which is equivalent to $q = 0$ and $k = n = l_1$. This miraculous cancellation ensures that the leading contribution coming from the second region of integration is also of order $O(x_2^{-2\tilde{\beta}})$. The sum over p is thus equal to

$$\sum_{p=0}^{l_1} \binom{l_1}{p} (-1)^p P_{l_1, 0, l_1}(p) = \frac{\left(1 - iu_{12} + \frac{l_2 - l_1}{2}\right)_{l_1}}{\left(iu_{12} + \frac{l_2 - l_1}{2}\right)_{l_1 + 1}}. \quad (5.1.21)$$

Finally, we notice that

$$f_{l_1, 0}\left(\tilde{\beta}_1 - \frac{l_1}{2}\right) \left(C_{l_1}^{(\frac{d-2}{2})}\right)^{(l_1)} \left(\frac{x_2 \cdot x_b}{|x_2| |x_b|}\right) = 2^{l_1} \left(\tilde{\beta}_1 - \frac{l_1}{2}\right)_{l_1}. \quad (5.1.22)$$

Putting all the pieces together, we obtain the following contribution from the second region of integration

$$\begin{aligned} \langle x_1, x_2 | \mathbf{u}_1, \mathbf{u}_2; C \rangle_{II} &\simeq f_{l_1}(u_1) \frac{\left(1 - iu_{12} + \frac{l_2 - l_1}{2}\right)_{l_1} \Gamma\left(\frac{d + l_1 + l_2}{2} + iu_{21} - 1\right) C^{(l_1 - l_2)}\left(\frac{x_1}{|x_1|}, \frac{x_2}{|x_2|}\right)}{\left(iu_{12} + \frac{l_2 - l_1}{2}\right)_{l_1 + 1} \Gamma\left(\frac{d + l_1 + l_2}{2} + iu_{21}\right) x_1^{2\tilde{\beta}_2} x_2^{2\tilde{\beta}_1}} \\ &= K_{l_1, l_2}(u_1, u_2) \frac{f_{l_1}(u_1)}{f_{l_2}(u_2)} \mathcal{S}_{l_1, l_2}(u_1 - u_2) \frac{C^{(l_1 - l_2)}\left(\frac{x_1}{|x_1|}, \frac{x_2}{|x_2|}\right)}{x_1^{2\tilde{\beta}_2} x_2^{2\tilde{\beta}_1}}. \end{aligned} \quad (5.1.23)$$

This is in perfect agreement with our claim (5.1.7) since completely symmetric traceless tensors are eigenvectors of the R-matrices with eigenvalue 1, as can be seen from the expression (4.1.2) for instance.

As mentioned before, we have also been able to explicitly reproduce $\mathbb{R}_{1,1}$, the method being the same as for the case we just presented, we do not show these computations. For $d = 2$, the eigenvectors of $\Lambda_{N, \tilde{\delta}}$ were found in [206] (see also [179]) and we checked that their asymptotic behaviour allows one to recover exactly the full S-matrix (5.1.9)-(5.1.11). As a

final remark, for $d = 4$ isotropic fishnets ($\tilde{\delta} = \delta$), we verify agreement with the conjectured S-matrix of the $\mathcal{N} = 4$ SYM theory [124, 125] at weak coupling in the mirror kinematics¹. In fact, our analysis is the first field theory derivation of this mirror S-matrix.

5.2 TBA for Ground State

We turn to the scaling dimension Δ of local operator $\mathcal{O}_{J,0}$. It is encoded in the divergent sum

$$Z = \sum_{N=0}^{+\infty} \xi^{2JN} \text{Tr} \left(\Lambda_{J,\tilde{\delta}}^N \right) \quad (5.2.1)$$

of the (divergent) wheel graphs with a given number J of spokes. The idea of the TBA is to interpret this sum as a partition function for the mirror theory in the thermodynamic limit. In the one-dimensional picture, the scaling dimension corresponds to the free energy of a system of magnons at temperature $1/J$ and chemical potential $\ln \xi^2$, where J is the length of the operator and ξ the coupling constant of the fishnet theory (2.3.2). The factorisation of the S-matrix allows one to compute exactly the free energy at any J and ξ . Following the well-known saddle-point procedure [129, 130, 224, 225], it takes the form

$$\Delta = J\delta - \sum_{l \geq 0} \int_{-\infty}^{+\infty} p'_l(u) \ln(1 + Y_{1,l}(u)) \frac{du}{2\pi}, \quad (5.2.2)$$

where $p'_l(u) = 2$, and the Y-functions $Y_{1,l}$ describe the distribution of energy per magnon, with $l \in \mathbb{N}$ labelling the different types of excitations/symmetric traceless representations of $O(d)$.

The latter Y-functions are part of a larger family of functions $\{Y_{a,l}\}$ needed to account for the matrix degrees of freedom. For the sake of simplicity, we shall restrict ourselves to the simply laced case, corresponding to even dimensions $d > 2$. Hence, $a \in \{1, \dots, r\}$ labels the nodes of the D_r Dynkin diagram, with $r = d/2 + 1$ and incidence matrix I_{ab} .

The Y-functions themselves are determined by an infinite system of non-linear TBA equations. Denoting $L_{a,l} = \ln(1 + Y_{a,l})$, these equations take the form

$$\ln Y_{1,l} = C - J\varepsilon_l + \sum_{l' \geq 0} \mathcal{K}_{l,l'} \star L_{1,l'} + \sum_{l' \geq 1} \mathcal{K}_{l,l'} \star L_{2,l'}, \quad (5.2.3)$$

for the massive nodes ($a = 1, l \geq 0$), where $\varepsilon_l = -\ln Q_l$,

$$C = J \ln \xi^2 - \sum_{l=0}^{\infty} \int_{-\infty}^{+\infty} [i\partial_u \ln f_l(u)] L_{1,l}(u) \frac{du}{2\pi}, \quad (5.2.4)$$

and the \star operation denotes the convolution on the real axis:

$$f \star g(u) = \int_{-\infty}^{+\infty} f(u-v)g(v) \frac{dv}{2\pi}. \quad (5.2.5)$$

¹The comparison with the dressing factor of [124] is not immediate since the latter was written in the physical kinematics. One should first perform an analytic continuation to the mirror kinematics, as was done in [125], before taking the weak coupling limit. The computation of this limit is explained in Appendix B.1 of [222] for instance.

For the remaining, auxiliary, nodes for spin excitations ($a > 1$, $l \geq 1$), the equations are

$$\ln Y_{a,l} = - \sum_{l' \geq 1} \check{\mathbb{K}}_{l,l'} \star L_{a,l'} + \sum_{b, I_{ab} \neq 0} \sum_{l' \geq 1} \mathbb{K}_{l,l'} \star L_{b,l'}, \quad (5.2.6)$$

where we introduced $\check{\mathbb{K}}_{l,l'} = \mathbb{K}_{l,l'+1} + \mathbb{K}_{l,l'-1}$, with symmetric kernels \mathcal{K} and \mathbb{K} defined by

$$\mathcal{K}_{l,l'}(u) = -i \partial_u \ln \mathcal{S}_{l,l'}(u), \quad (5.2.7)$$

and

$$\mathbb{K}_{l,l'}(u) = \sum_{j=-(|l-l'|+1)/2}^{(l+l'-1)/2} \frac{2j}{u^2 + j^2}. \quad (5.2.8)$$

Finally, let us stress that the kernels obey the universal asymptotics

$$\mathcal{K}_{l,l'}(u) = 2 \ln u^2 + O\left(\frac{1}{u}\right) \quad (5.2.9)$$

at large rapidity. Consequently, the scaling dimension (5.2.2) controls the asymptotics of the main Y-functions,

$$\ln Y_{1,l} \sim -\Delta \ln u^2, \quad \text{when } |u| \rightarrow +\infty. \quad (5.2.10)$$

The auxiliary Y-functions are, on the other hand, asymptotically constant at $u \rightarrow \infty$.

5.3 Dual TBA and Y-System

The TBA equations above give us a good handle on the scaling dimension at weak coupling, which is when the massive Y-functions are small, and the equations are solvable iteratively. They are also very useful for the study of fishnet graphs at large order, that is when the coupling constant approaches its critical value [148, 167]. Close to this point, the lightest ($l = 0$) mirror magnons condense, driving the system towards a new phase with gapless excitations. This is analogous to the transition from ferro- to antiferromagnetic order for compact spin chains in a magnetic field. It relates to the continuum limit of the fishnet graphs and to their correspondence with two-dimensional σ -models with AdS target space. This correspondence, which was first discussed in [167] for $d = 4$, also holds in higher dimensions. Namely, there is a dual set of TBA equations looking like that of the familiar $O(d+2)$ σ -model in a finite volume J except that, instead of the standard relativistic dispersion relation, we should use the one dual to (5.1.2).

The duality is established by means of the familiar particle/hole transformation. It involves the operator $\mathbb{1} - \mathbb{K}_{O(d+2)}$, which solves the equation

$$(\mathbb{1} - \mathcal{K}_{0,0}) \star (\mathbb{1} - \mathbb{K}_{O(d+2)}) = \mathbb{1}, \quad (5.3.1)$$

with $\mathbb{1}$ the identity operator and with $\mathcal{K}_{0,0}$ as in (5.2.7). Straightforward algebra gives

$$\mathbb{K}_{O(d+2)}(u) = -i \partial_u \ln S_{O(d+2)}\left(\frac{2\pi u}{d}\right), \quad (5.3.2)$$

with the well-known $O(d+2)$ S matrix [120]

$$S_{O(d+2)}(\theta) = -\frac{\Gamma\left(1 + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{d} - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \frac{1}{d} + \frac{i\theta}{2\pi}\right)}{\Gamma\left(1 - \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{d} + \frac{i\theta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \frac{1}{d} - \frac{i\theta}{2\pi}\right)}, \quad (5.3.3)$$

hinting at the dual σ -model interpretation.

Applying the operator $\mathbb{1} - K_{O(d+2)}$ to (5.2.3) for $l = 0$, we get the dual equation for the scalar node:

$$\ln Y_{1,0} = JE - \mathbb{K}_{O(d+2)} \star L'_{1,0} - \sum_{l \geq 1} \mathbb{K}_{1,l} \star L_{1,l}, \quad (5.3.4)$$

with $L'_{1,0} = \ln(1 + Y_{1,0}^{-1})$, and the new driving term

$$E(u) = \ln \left(\frac{\cosh\left(\frac{2\pi u}{d}\right) + \cos\left(\frac{\pi\delta}{d}\right)}{\cosh\left(\frac{2\pi u}{d}\right) - \cos\left(\frac{\pi\delta}{d}\right)} \right) \quad (5.3.5)$$

is identified as the dual energy. As for the higher harmonics, equations (5.2.3) for $l > 0$, they can be rewritten

$$\ln Y_{1,l} = -\mathbb{K}_{l,1} \star L'_{1,0} - \sum_{l' \geq 1} \check{\mathbb{K}}_{l,l'} \star L_{1,l'} + \sum_{l' \geq 1} \mathbb{K}_{l,l'} \star L_{2,l'}. \quad (5.3.6)$$

The absence of driving terms in these equations indicate that the full symmetry is linearly realised in the dual picture. In fact, if not for the energy, the equations (5.3.4) and (5.3.6), as well as the ones in (5.2.6) which stay untouched, are identical to those for the $O(d+2)$ σ -model.

The non-compactness of the model is seen in the fact that the spectrum is gapless. This is made clearer after introducing a dual momentum P , obtained via a Wick rotation and a reflection² $\delta \rightarrow \tilde{\delta}$. It yields

$$P(u) = -iE \left(u + i\frac{d}{4} \right) \Big|_{\delta \rightarrow \tilde{\delta}}, \quad (5.3.7)$$

resulting in the following dispersion relation

$$\sinh^2 \frac{E}{2} = \tan^2 \left(\frac{\pi\tilde{\delta}}{d} \right) \times \sin^2 \frac{P}{2}, \quad (5.3.8)$$

as in the case of a massless particle on a square lattice. It becomes relativistic at low energy, $E \sim cP$, that is when the continuous σ -model description applies, with the anisotropy being absorbed in the speed of light c .

In the dual TBA equations, the fishnet coupling ξ^2 disappears. It is now captured by the σ -model energy

$$E_{2d}(\Delta, J) = - \int_{-\infty}^{+\infty} L'_{1,0} \partial_u P(u) \frac{du}{2\pi}, \quad (5.3.9)$$

²The reflection is needed because the square lattice is not invariant under a $\frac{\pi}{2}$ rotation when there is anisotropy.

which, as it turns out, can be expressed as

$$E_{2d}(\Delta, J) = J \ln \frac{\xi^2}{\xi_c^2}, \quad (5.3.10)$$

with

$$\ln \xi_c^2 = \int_0^\infty \left[\frac{d}{2} e^{-t} + \frac{e^{-\delta t} - e^{\delta t} + e^{-\tilde{\delta} t} - e^{\tilde{\delta} t}}{(1 - e^{-t})(1 + e^{\frac{dt}{2}})} \right] \frac{dt}{t} \quad (5.3.11)$$

the critical coupling, in agreement with Zamolodchikov's formula [148], up to the normalisation of the coupling constant. This match for any δ and d is yet another check for our expressions.

Lastly, let us note that the TBA equations (5.3.4), (5.3.6) and (5.2.6) can be brought, by inverting the kernels, to the Y-system form:

$$\frac{Y_{a,l}^+ Y_{a,l}^-}{Y_{a,l+1} Y_{a,l-1}} = \frac{\prod_{b=1}^r (1 + Y_{b,l})^{J_{ab}}}{(1 + Y_{a,l+1})(1 + Y_{a,l-1})}, \quad (5.3.12)$$

for all nodes with $1 \leq a \leq r$, $l \geq 1$ (with the convention that $Y_{a,0} = \infty$ for $a > 1$), while $Y_{1,0}$ satisfies

$$\frac{1}{Y_{1,0}^{[r-1]} Y_{1,0}^{[1-r]}} = \left[\prod_{k=1}^{r-2} \left(1 + \frac{1}{Y_{r-k-1,1}^{[k]}} \right) \left(1 + \frac{1}{Y_{r-k-1,1}^{[-k]}} \right) \right] \left(1 + \frac{1}{Y_{r-1,1}} \right) \left(1 + \frac{1}{Y_{r,1}} \right), \quad (5.3.13)$$

with the shorthand notation $f^{[k]}(u) = f\left(u + i\frac{k}{2}\right)$. This agrees with the Y-system equations of the $O(2r)$ sigma model [226].

5.4 Excited States and Asymptotic Bethe Ansatz

The TBA equations can be generalised to the states with an arbitrary number of magnons by the usual trick of the contour deformation [227, 228]. The multi-magnon operators $\mathcal{O}_{J,M}$, associated to spiral graphs shown in the right panel of Figure 5.1, are made out of scalar magnons ($l = 0$) and obtained by exciting the corresponding Y-function. Most of the formulae stay the same, if not for the energy (5.2.2) and the equations (5.2.3) that receive additional driving terms. In particular, the anomalous dimensions

$$\gamma_M = \Delta - (J\delta + M\tilde{\delta}) \quad (5.4.1)$$

of the multi-magnon states read

$$\gamma_M = \sum_{m=1}^M (2iu_m - \tilde{\delta}) - \sum_{l \geq 0} \int_{-\infty}^{+\infty} p'_l(u) L_{1,l}(u) \frac{du}{2\pi}, \quad (5.4.2)$$

with the first term in the right-hand side coming from the logarithmic poles at $Y_{1,0}(u_m) = -1$. The latter conditions are the exact Bethe ansatz equations, which reduce at large J and for

sufficiently weak coupling to the ABA equations

$$1 = \xi^{2J} e^{-\varepsilon_0(u_j)J} \prod_{\substack{k=1 \\ k \neq j}}^M \mathbf{S}_{0,0}(u_j, u_k). \quad (5.4.3)$$

In this case, since all spins $l, l' = 0$, the R-matrix trivialises. It should be supplemented with the trace cyclicity condition

$$\prod_{j=1}^M \xi^2 e^{-\varepsilon_0(u_j)} = 1. \quad (5.4.4)$$

This generalises the 4D ABA equations of [155] to any dimension d and any anisotropy.

5.4.1 One Magnon

As an example, in the simplest $M = 1$ case, the ABA equation predicts that the anomalous dimension is given by $(2iu - \tilde{\delta})$, where u is the solution to $\xi^2 e^{-\varepsilon_0(u)} = 1$. Expanding u perturbatively in ξ^2 around the classical value $-i\tilde{\delta}/2$, we find the one-magnon anomalous dimension

$$\gamma_{M=1} = -\frac{2\xi^2}{\Gamma\left(\frac{d}{2}\right)} + 2\xi^4 \frac{\psi(\tilde{\delta}) + \psi\left(\frac{d}{2}\right) - \psi(1)}{\Gamma\left(\frac{d}{2}\right)^2} + \mathcal{O}(\xi^6), \quad (5.4.5)$$

which agrees with the direct field theory computation.

5.4.2 Two Magnons

Let us now test our ABA prediction against direct Feynman graphs computation for operators with two magnons. Since we will only have results through third-order in ξ^2 , we can restrict ourselves to $J = 3$. In this case, we only need to consider mixing among the following two operators,

$$\mathcal{O}_1 = \text{Tr}\left(X^2 Z^3\right), \quad \mathcal{O}_2 = \text{Tr}\left(X Z X Z^2\right). \quad (5.4.6)$$

Once we take interactions into account, these operators will receive quantum corrections. We will keep the name \mathcal{O}_α for the bare operators, while \mathcal{O}_α^R will refer to their renormalised counterpart. For definiteness, we will use dimensional regularisation in $d - 2\epsilon$ dimensions and the minimal subtraction scheme. We also replace ξ^2 with $\mu^{-2\epsilon}\xi^2$, for some mass scale μ , in order to keep the dimensionless ξ^2 .

The renormalisation procedure will absorb all the divergences. For our purpose, we focus on a particular set of observables, namely, the form factors among those two operators. Recall the definition of a form factor: it is the overlap between an operator \mathcal{O}_α and an asymptotic state β . In our case, we consider only two-point form factors

$$G_{\alpha\beta}(p_1, p_2) = \langle \text{vac} | \mathcal{O}_\alpha(0) | \beta; p_1, p_2 \rangle, \quad (5.4.7)$$

where $|\beta; p_1, p_2\rangle$ is the asymptotic state in momentum space with respect to the operator \mathcal{O}_β , and p_1, p_2 are the momenta associated to the magnon lines. We can obtain the expression

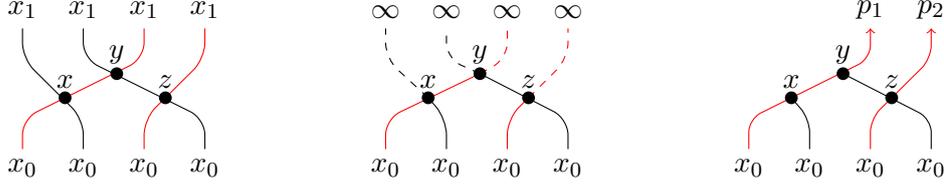


Figure 5.2: Form factor G_{21} at $O(\xi^6)$: we start from the two-point function $\langle \mathcal{O}_2(x_0) \mathcal{O}_1(x_1) \rangle$, then send $x_1 \rightarrow \infty$ and amputate propagators (dashed lines) that are connected to infinity. Finally, we attach $e^{ip_1 y + ip_2 z}$ to the two magnon lines, for simplicity we drop the dashed lines.

from the two-point function by amputating it and associating momenta to the magnon lines, see Figure 5.2.

When $p_1 = -p_2 = p$, the bare form factor takes the following form:

$$G(p, -p) = I_2 + \xi^2 \mu^{-2\epsilon} \begin{pmatrix} 0 & I_1 \\ I_1 & I_1 \end{pmatrix} + \xi^4 \mu^{-4\epsilon} \begin{pmatrix} I_{1,1} & I_2 \\ I_2 & I_1^2 \end{pmatrix} + \xi^6 \mu^{-6\epsilon} \begin{pmatrix} I_3 & I_{1,2} \\ I_{2,1} & I_1 I_2 \end{pmatrix} + O(\xi^8) \quad (5.4.8)$$

where $I_{\#}$'s are Feynman integrals computed in Appendix B.2. The total sum of subscripts describes the order of these integrals, I_3 and $I_{1,2}$ are, for example, three-loop integrals.

The general procedure for renormalising composite operators is the following: we first introduce renormalised operators

$$\mathcal{O}_\alpha^R = \mathcal{Z}_{\alpha\beta} \mathcal{O}_\beta \quad (5.4.9)$$

as linear combinations of the bare operators (there is a summation over β). Then, the renormalised form factor is related to the bare one via

$$G_{\alpha\beta,R}(p_1, p_2) = \langle \text{vac} | \mathcal{O}_\alpha^R(x) | \beta; p_1, p_2 \rangle = \mathcal{Z}_{\alpha\gamma} G_{\gamma\beta}(p_1, p_2). \quad (5.4.10)$$

In the minimal subtraction scheme, $\mathcal{Z}_{\alpha\beta}$ is of the form

$$\mathcal{Z}_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{k=1}^{\infty} \xi^{2k} \mathcal{Z}_{\alpha\beta}^{(k)}, \quad \mathcal{Z}_{\alpha\beta}^{(k)} = \sum_{j=1}^k \frac{\mathcal{Z}_{\alpha\beta}^{(k,j)}}{\epsilon^j}. \quad (5.4.11)$$

Because G depends on the mass scale solely through $\mu^{-2\epsilon} \xi^2$, and \mathcal{Z} does not depend on it, but depends on ξ^2 , taking the derivative of (5.4.10) with respect to μ keeping $\mu^{-2\epsilon} \xi^2$ (as well as p^2 and ϵ) fixed reads

$$\frac{\partial G_R}{\partial \mu} \Big|_{\epsilon, p^2, \xi^2} + 2\epsilon \frac{\xi^2}{\mu} \frac{\partial G_R}{\partial \xi^2} \Big|_{\epsilon, p^2, \mu} = 2\epsilon \frac{\xi^2}{\mu} \frac{\partial \mathcal{Z}}{\partial \xi^2} \Big|_{\epsilon} G = 2\epsilon \frac{\xi^2}{\mu} \frac{\partial \mathcal{Z}}{\partial \xi^2} \Big|_{\epsilon} \mathcal{Z}^{-1} G_R. \quad (5.4.12)$$

Now, since G_R has a finite limit when $\epsilon \rightarrow 0$, the limit of the previous equation is

$$\lim_{\epsilon \rightarrow 0} \frac{\partial G_R}{\partial \ln \mu} \Big|_{\epsilon, p^2, \xi^2} = -2\gamma \lim_{\epsilon \rightarrow 0} G_R, \quad (5.4.13)$$

with the matrix of anomalous dimension given by

$$\gamma = -\xi^2 \lim_{\epsilon \rightarrow 0} \left(\epsilon \frac{\partial \mathcal{Z}}{\partial \xi^2} \mathcal{Z}^{-1} \right) = -\xi^2 \mathcal{Z}'_1(\xi^2), \quad (5.4.14)$$

where $\mathcal{Z}_1 = \sum_{k=1}^{+\infty} \xi^{2k} \mathcal{Z}^{(k,1)}$ is the residue at $\epsilon = 0$. One should also notice that, because γ is finite, all the coefficients for negative powers of ϵ in the Laurent expansion of $\epsilon \frac{\partial \mathcal{Z}}{\partial \xi^2} \mathcal{Z}^{-1}$ vanish. More precisely, this means that taking the limit was not necessary, as \mathcal{Z} should satisfy

$$\frac{\partial \mathcal{Z}}{\partial \xi^2} = \frac{\mathcal{Z}'_1 \mathcal{Z}}{\epsilon}. \quad (5.4.15)$$

Because this is a differential equation on matrices, it cannot be integrated. However, it does imply that the coefficients in front of ϵ^{-k} for $k \geq 2$ are completely determined by \mathcal{Z}_1 .

5.5 Discussion and Prospects

We presented TBA equations for the exact spectrum of arbitrary multimagnon operators in the fishnet CFT in any spacetime dimension d . These operators form an important class of local operators of the theory and contain all of the information about the mirror dynamics. There are other types of operators worth being studied, including spinning operators (i.e. with derivatives) and the conjugate scalars Z^\dagger, X^\dagger . While it should be possible to include the former within the excited state TBA formalism, the latter are more elusive, and relate to the logarithmic property of the fishnet CFTs [150, 162].

The most efficient form of the TBA equations is expected to be given by Baxter's equation. It would be good to derive them for generic d and for a general local operator. This program is already quite advanced in $d = 4$ case [150, 166], but not for other dimensions. Our TBA equations should help filling this gap by providing important information about the analyticity conditions and so-called quantisation conditions specifying the solutions. The Baxter equation formulation would also be instrumental for a thorough study of the correspondence between fishnet graphs and non-compact sigma models, or to reveal relationships with string-bit models in *AdS* [164–166]. Because integrable structures such as Baxter's equation are expected to be universal, we will investigate in the next chapter a simpler model based on the same symmetry.

Another interesting direction concerns the generalisation of our TBA equations to fishnet theories supported on triangular and hexagonal lattices, or dynamical fishnet like the one found in the context of the 3-coupling strongly twisted version of $\mathcal{N} = 4$ SYM theory [143, 151].

Chapter 6

T-System and Q-System for $SO(2r)$ Spin Chains

The results of [182] are presented in this chapter. Apart from Section 6.1, the content of the chapter is the same as that of the article without the introduction and appendices, up to minor changes in presentation. In Appendix C of this thesis, we have collected Appendices D, F, and G of the article.

The goal of the article was to develop the Q-system formalism for integrable systems with D_r symmetry. We thus considered a very simple spin chain with such a symmetry, and we used the explicit construction of some of the Q-operators to formulate a proposal. We also obtained different expressions for the T-functions in terms of the Q-functions. In the first section, we shall first review the results for compact spin chains with A_r symmetry. This will permit a clear comparison with the analogous results we found in the D_r case.

6.1 A_r Spin Chains

6.1.1 T-System

Let us consider, as in Subsection 1.3.2, a $GL(r+1)$ spin chain of length L with quantum space $\mathcal{H} = \mathbb{V}_{\lambda^{(1)}} \otimes \cdots \otimes \mathbb{V}_{\lambda^{(L)}}$, where \mathbb{V}_λ is the space of the finite-dimensional irreducible representation with highest weight $\lambda = \sum_{i=1}^{r+1} \lambda_i \varepsilon_i$ —see Appendix A.1 for an explicit description of the weights, roots, etc. We also introduce a (constant) diagonal twist matrix $D = \text{Diag}(\tau_1, \dots, \tau_{r+1})$ and some inhomogeneities x_1, \dots, x_L . The commuting transfer matrices are defined for arbitrary auxiliary representation ρ_λ with highest weight λ as

$$T_\lambda(x) = \text{Tr}_0 \left[\rho_\lambda(D) R_{0L}^{\lambda\lambda^{(L)}}(x - x_L) \cdots R_{01}^{\lambda\lambda^{(1)}}(x - x_1) \right], \quad (6.1.1)$$

with $R^{\lambda\lambda^{(i)}}$ the $GL(r+1)$ -invariant R-matrix on $\mathbb{V}_\lambda \otimes \mathbb{V}_{\lambda^{(i)}}$. The first thing to notice is that, in contradistinction to what we have studied in the previous chapters, the presence of a twist breaks the global $GL(r+1)$ symmetry down to the centraliser of the twist matrix D . For twists in generic position, this is simply the Cartan subgroup of diagonal matrices and, in this case, the spectrum of the transfer matrices is simple. We also point out that, up to a

trivial rescaling of the transfer matrices, we can (and will) always assume that

$$\det D = \prod_{k=1}^{r+1} \tau_k = 1. \quad (6.1.2)$$

Let us clarify a little bit how to construct the various R-matrices involved in the definition of the transfer matrices. We start from the defining representation in the auxiliary space, i.e. $\lambda = \varepsilon_1$, and, following (1.2.44), we set

$$\mathbf{R}^{\varepsilon_1, \lambda^{(i)}}(x) = x \text{Id} + e_{jk} \otimes E_{kj}^{\lambda^{(i)}}. \quad (6.1.3)$$

The R-matrices for other auxiliary spaces are then constructed via fusion [184, 185], see also [45], without extracting trivial zeros. We have for instance

$$\mathbf{R}^{2\varepsilon_1, \lambda^{(i)}}(x) = \mathbf{R}_{0i}^{\varepsilon_1, \lambda^{(i)}} \left(x - \frac{1}{2} \right) \mathbf{R}_{0'i}^{\varepsilon_1, \lambda^{(i)}} \left(x + \frac{1}{2} \right) \left(\frac{\text{Id} + \mathbb{P}_{0,0'}}{2} \right) \quad (6.1.4)$$

and

$$\mathbf{R}^{\varepsilon_1 + \varepsilon_2, \lambda^{(i)}}(x) = \mathbf{R}_{0i}^{\varepsilon_1, \lambda^{(i)}} \left(x + \frac{1}{2} \right) \mathbf{R}_{0'i}^{\varepsilon_1, \lambda^{(i)}} \left(x - \frac{1}{2} \right) \left(\frac{\text{Id} - \mathbb{P}_{0,0'}}{2} \right). \quad (6.1.5)$$

We saw indeed in Subsection 1.2.2 that when $\lambda^{(i)} = \varepsilon_1$, the R-matrices with symmetric auxiliary spaces contain trivial zeros that can be extracted. More generally, when $\lambda^{(i)} = \varepsilon_1$ all the R-matrices constructed using fusion will be given by Equation (6.1.3) above up to a trivial polynomial prefactor. This is however generically not the case when the representation at site i is different from the defining one. That is why, in order to treat all cases at once, we choose not to discard these potential trivial zeros.

Because of the fusion procedure used to construct the R-matrices, the transfer matrices are far from being independent: they satisfy numerous relations. Of particular interest are the transfer matrices $\mathbf{T}_{a,s} = \mathbf{T}_{\lambda_{a,s}}$ associated to rectangular representations

$$\lambda_{a,s} = \sum_{i=1}^a s \varepsilon_i = s \omega_a \quad \text{for } a \in \{1, \dots, r\}, s \in \mathbb{N}^*. \quad (6.1.6)$$

It is well known that they satisfy the A_r T-system [43–45] (also called Hirota's bilinear equation [46])

$$\mathbf{T}_{a,s}^{[+1]} \mathbf{T}_{a,s}^{[-1]} = \mathbf{T}_{a,s+1} \mathbf{T}_{a,s-1} + \mathbf{T}_{a+1,s} \mathbf{T}_{a-1,s} \quad \text{for } a \in \{1, \dots, r\}, s \in \mathbb{N}^*. \quad (6.1.7)$$

The boundary conditions are

$$\mathbf{T}_{a,0} = \mathbf{T}_{0,s} = 1, \quad \mathbf{T}_{r+1,s} = \frac{Q_{\bar{\emptyset}}^{[r+1+s]}}{Q_{\bar{\emptyset}}^{[r+1-s]}}, \quad (6.1.8)$$

where we have introduced

$$Q_{\bar{\emptyset}} = \prod_{a=1}^{r+1} \Xi_a, \quad \Xi_a(x) = \prod_{i=1}^L \Gamma \left(x - x_i + \lambda_a^{(i)} + 1 - a \right), \quad (6.1.9)$$

and we write $f^{[k]}$ for the function $f^{[k]}(x) = f\left(x + \frac{k}{2}\right)$. This bilinear equation clearly ensures that any $T_{a,s}$ can be expressed in terms of $T_{a,1}$, the resulting expressions date back to [229,230] and read

$$T_{a,s} = \det_{1 \leq i,j \leq s} \left(T_{a+j-i,1}^{[i+j-s-1]} \right) \quad \text{for any } a \geq 0 \text{ and } s \geq 0. \quad (6.1.10)$$

For $s > 0$, we have conventionally set $T_{a,s} = 0$ for $a > r + 1$ or $a < 0$. This formula is proven by induction. It holds trivially when $s = 1$. If we assume that it holds up to some $s \in \mathbb{N}^*$, then it suffices to use the Hirota relation (6.1.7) at (a, s) to express $T_{a,s+1}$ in terms of $T_{a\pm 1,s}$, $T_{a,s}$, and $T_{a,s-1}$. The result then reduces to the Jacobi identity on determinants: if M is a square matrix of size $N \geq 2$, and if I and J are subsets of $\{1, \dots, N\}$, we write $M_{I,J}$ for the matrix obtained by removing the rows i for $i \in I$ and the columns j for $j \in J$, it then holds that

$$\det M \det M_{\{1,N\},\{1,N\}} = \det M_{\{1\},\{1\}} \det M_{\{N\},\{N\}} - \det M_{\{1\},\{N\}} \det M_{\{N\},\{1\}}. \quad (6.1.11)$$

In our case, $N = s + 1$ and the coefficients of M are $M_{i,j} = T_{a+j-i,1}^{[i+j-s-2]}$.

The transfer matrix T_λ can be seen as the quantum analogue of the usual character χ_λ . Since the leading coefficient of the transfer matrix is actually exactly the character, it is customary to call the limit $x \rightarrow +\infty$ classical. In that limit, Equation (6.1.10) reduces to the (second) Jacobi-Trudi formula for characters of $GL(r + 1)$ (Schur polynomials):

$$\chi_{a,s} = \det_{1 \leq i,j \leq s} (\chi_{a+j-i,1}), \quad (6.1.12)$$

where $\chi_{a,s} = \chi_{\lambda_{a,s}}$.

Solution in Terms of Q-Functions Let $Q_{\{1\}}, \dots, Q_{\{r+1\}}$ be arbitrary functions. If, for $a \in \{0, \dots, r + 1\}$ and $s \in \mathbb{Z}$, we define [44]

$$\mathcal{T}_{a,s} = \det_{1 \leq i,j \leq r+1} \left(Q_{\{i\}}^{[a-s-2j+2s\theta(a-j)]} \right), \quad (6.1.13)$$

where θ is the Heaviside step function (it is 0 for negative arguments and 1 for non-negative ones), then we obtain a solution of the Hirota equation (6.1.7). This fact is a direct application of a Plücker identity, also called Sylvester's lemma: if M and N are two matrices of the same size with columns M_1, \dots, M_{r+1} and N_1, \dots, N_{r+1} respectively, then the following identity holds¹ for any $k \in \{1, \dots, r + 1\}$,

$$\det M \det N = \sum_{l=1}^{r+1} |M_1, \dots, M_{k-1}, N_l, M_{k+1}, \dots, M_{r+1}| |N_1, \dots, N_{l-1}, M_k, N_{l+1}, \dots, N_{r+1}|. \quad (6.1.14)$$

In the case at hand, $M = \mathcal{T}_{a,s}^{[+1]} = (C_{a+s-1}, \dots, C_{s-a+1}, C_{-a-s-1}, \dots, C_{a-s-2r-1})$, with C_j the transpose of $(Q_{\{1\}}^{[j]}, \dots, Q_{\{r+1\}}^{[j]})$, and $N = (C_{a+s-3}, \dots, C_{s-a-1}, C_{-a-s-3}, \dots, C_{a-s-2r-3})$

¹We use here another notation for determinants: if M is a $p \times p$ matrix with columns M_1, \dots, M_p , we write $\det M = |M_1, \dots, M_p|$.

have many columns in common, so that if we decide to exchange $M_1 = C_{a+s-1}$, only two terms survive in the sum (when $l = a$ or $l = r + 1$), and they give exactly what we want. This solution of Hirota's equation also satisfies

$$\mathcal{T}_{a,s} = 0, \quad \text{for } s \in \{-r, \dots, -1\}, \quad (6.1.15)$$

and

$$\mathcal{T}_{a,0} = \phi^{[a]}, \quad \mathcal{T}_{0,s} = \phi^{[-s]}, \quad \mathcal{T}_{r+1,s} = \phi^{[r+1+s]}, \quad (6.1.16)$$

where

$$\phi = \det_{1 \leq i, j \leq r+1} \left(Q_{\{i\}}^{[-2j]} \right). \quad (6.1.17)$$

The previous form of a solution of the T-system is actually generic. Let us indeed imagine that we have at our disposal $\mathcal{T}_{a,s}$ satisfying (6.1.7) with boundary conditions (6.1.16) for some unspecified ϕ . Let $\{Q_{\{1\}}, \dots, Q_{\{r+1\}}\}$ be a basis of solutions of the difference equation

$$\sum_{a=0}^{r+1} (-1)^a \mathcal{T}_{a,1}^{[1-a]} Q^{[-2a]} = 0. \quad (6.1.18)$$

Linear independence² of the $Q_{\{i\}}$'s implies that the Casoratian $\det_{1 \leq i, j \leq r+1} \left(Q_{\{i\}}^{[-2j]} \right)$ is non-zero and equal to ϕ up to multiplication by a periodic function. Upon redefinition of one of the functions, say $Q_{\{1\}}$, we can always assume that this periodic function is a constant equal to one, and thus

$$\det_{1 \leq i, j \leq r+1} \left(Q_{\{i\}}^{[-2j]} \right) = \phi. \quad (6.1.19)$$

We then rewrite the difference equations as a linear system of $r + 1$ equations:

$$\phi Q_{\{b\}} = \sum_{a=1}^{r+1} M_{ba} (-1)^{a-1} \mathcal{T}_{a,1}^{[1-a]}, \quad \text{with } M_{ba} = Q_{\{b\}}^{[-2a]}. \quad (6.1.20)$$

Cramer's inversion formula and the condition (6.1.19) immediately results in (6.1.13).

6.1.2 Q-System

We have seen that it is always possible to express all the T-functions in terms of $r + 1$ auxiliary functions $Q_{\{1\}}, \dots, Q_{\{r+1\}}$. In the spin chain model we introduced earlier, it turns out that this even holds at the operatorial level.

It is actually possible [77, 79] to construct 2^{r+1} operators Q_I , labelled by subsets I of $\{1, \dots, r + 1\}$, commuting among themselves and with the transfer matrices and satisfying the following so-called QQ-relations, or Q-system: if $|I| \leq r - 1$, and if i and j are not in I , then

$$Q_{I \cup \{i\}}^{[+1]} Q_{I \cup \{j\}}^{[-1]} - Q_{I \cup \{i\}}^{[-1]} Q_{I \cup \{j\}}^{[+1]} = \frac{\tau_i - \tau_j}{\sqrt{\tau_i \tau_j}} Q_I Q_{I \cup \{i, j\}}. \quad (6.1.21)$$

²For solutions of a difference equation, linear independence means that there is no vanishing linear combination with 1-periodic coefficients.

The construction of these operators involve new singular solutions of the Yang-Baxter relation, the operators themselves being analogues of transfer matrices built from these singular R-matrices. It is conventional to represent the various QQ-relations on a Hasse diagram, see Figure 6.1 for the $r = 2$ case: each Q-operator is attached to a vertex of a hypercube and each face represents one of the relations. We define the level of the Q-operator Q_I as the order $|I|$ of the set I .

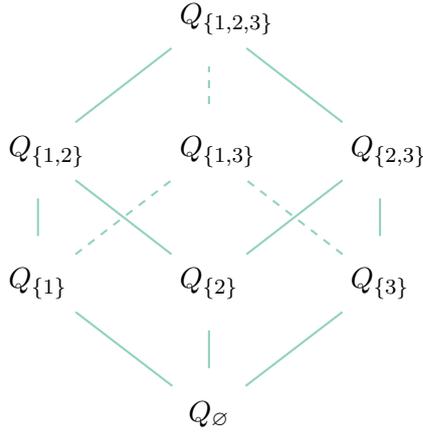


Figure 6.1: Hasse diagram for the A_2 Lie algebra, taken from [182].

The first thing to notice is that, if we have such operators (or functions if we consider their eigenvalues on a particular eigenstate), then they can all be expressed in terms of Q_\emptyset and $Q_{\{i\}}$ through Casoratians:

$$Q_{\{i_1, \dots, i_k\}} = \frac{(\sqrt{\tau_{i_1} \dots \tau_{i_k}})^{k-1} \det_{1 \leq a, b \leq k} \left(Q_{\{i_a\}}^{[k+1-2b]} \right)}{\prod_{1 \leq a < b \leq k} (\tau_{i_a} - \tau_{i_b}) \prod_{l=1}^{k-1} Q_\emptyset^{[k-2l]}}, \quad (6.1.22)$$

where $i_a \neq i_b$ if $a \neq b$. This is once again a trivial application of the Jacobi identity (6.1.11).

Regularity Properties For the spin chain we are considering, the regularity properties of the Q-operators are known [79]. The two operators $Q_\emptyset, Q_{\{1, \dots, r+1\}}$ are trivial and given by

$$Q_\emptyset = \text{Id} \quad \text{and} \quad Q_{\{1, \dots, r+1\}} = Q_{\bar{\emptyset}} \text{Id}, \quad (6.1.23)$$

where $Q_{\bar{\emptyset}}$ is given in Equation (6.1.9). All the other Q-operators are non-trivial, but they are polynomial up to a fixed scalar prefactor. More precisely, for any eigenstate of the chain, there exist $r + 1$ integers m_a such that

$$m_1 + \dots + m_{r+1} = \sum_{i=1}^L \sum_{a=1}^r (\lambda_a^{(i)} - \lambda_{r+1}^{(i)}) \quad (6.1.24)$$

and

$$m_I = \sum_{i \in I} m_i - \sum_{i=1}^L \sum_{a=r+2-|I|}^r (\lambda_a^{(i)} - \lambda_{r+1}^{(i)}) \geq 0, \quad (6.1.25)$$

and the eigenvalue of Q_I is of the form

$$Q_I(x) = \left(\prod_{i \in I} \tau_i^x \right) \left(\prod_{a=r+2-|I|}^{r+1} \Xi_a^{[r+1-|I|]}(x) \right) q_I(x), \quad (6.1.26)$$

where $q_I = \prod_{j=1}^{m_I} (x - z_j^I)$ is of degree m_I .

The Bethe ansatz equations can be recovered from the Q-system and the regularity properties of the Q-functions. Let $\sigma \in \mathfrak{S}_{r+1}$ be a permutation, we define $I_0 = \emptyset$ and $I_a = \{\sigma(1), \dots, \sigma(a)\}$ for $1 \leq a \leq r+1$. We also let, for $1 \leq a \leq r$, $\tilde{I}_a = \{\sigma(1), \dots, \sigma(a-1), \sigma(a+1)\}$ be the only subset of order a such that $I_a \neq \tilde{I}_a$ and $I_{a-1} \subsetneq \tilde{I}_a \subsetneq I_{a+1}$. The QQ-relation for $I_{a-1}, I_a, \tilde{I}_a$, and I_{a+1} , shifted by $\pm \frac{1}{2}$ and evaluated at a root $z_j^{I_a}$ of q_{I_a} (and thus of Q_{I_a}) then yields

$$\pm Q_{I_a}(z_j^{I_a} \pm 1) Q_{\tilde{I}_a}(z_j^{I_a}) = \frac{\tau_{\sigma(a)} - \tau_{\sigma(a+1)}}{\sqrt{\tau_{\sigma(a)} \tau_{\sigma(a+1)}}} Q_{I_{a-1}} \left(z_j^{I_a} \pm \frac{1}{2} \right) Q_{I_{a+1}} \left(z_j^{I_a} \pm \frac{1}{2} \right). \quad (6.1.27)$$

Using the form (6.1.26) of the eigenvalues, the ratio of the two previous equations can be rewritten

$$-\frac{\tau_{\sigma(a+1)}}{\tau_{\sigma(a)}} \prod_{i=1}^L \frac{z_j^{I_a} - x_i + \lambda_{r+1-a}^{(i)} + \frac{a-r-1}{2}}{z_j^{I_a} - x_i + \lambda_{r+2-a}^{(i)} + \frac{a-r-1}{2}} = \frac{q_{I_{a-1}} \left(z_j^{I_a} - \frac{1}{2} \right) q_{I_a} \left(z_j^{I_a} + 1 \right) q_{I_{a+1}} \left(z_j^{I_a} - \frac{1}{2} \right)}{q_{I_{a-1}} \left(z_j^{I_a} + \frac{1}{2} \right) q_{I_a} \left(z_j^{I_a} - 1 \right) q_{I_{a+1}} \left(z_j^{I_a} + \frac{1}{2} \right)}. \quad (6.1.28)$$

For $\sigma(a) = r+2-a$, these are exactly the Bethe equations (1.3.23). It is now clear that there are actually $|\mathfrak{S}_{r+1}| = (r+1)!$ different formulations of them, one for every choice of the permutation σ . There is, however, a better formulation of these equations, which we shall refer to as the Wronskian form of the Bethe equations. It consists simply in writing (6.1.22) for $Q_{\{1, \dots, r+1\}} = Q_{\emptyset}$ using the explicit form of the Q-functions:

$$\det_{1 \leq a, b \leq r+1} \left(\tau_a^{-b} q_{\{a\}}^{[r+2-2b]} \right) = \prod_{1 \leq a < b \leq r+1} (\tau_a - \tau_b) \prod_{i=1}^L \prod_{a=1}^r \frac{\Gamma(x - x_i + \lambda_a^{(i)} + 1 - a)}{\Gamma(x - x_i + \lambda_{r+1}^{(i)} + 1 - a)}, \quad (6.1.29)$$

where the right-hand side is now an explicit polynomial. Alternatively, we could have solved the Q-system in terms of the Q-functions $Q_{\{a\}}$ at level r , the corresponding Wronskian condition is

$$\det_{1 \leq a, b \leq r+1} \left(\tau_a^b q_{\{a\}}^{[r+2-2b]} \right) = \prod_{1 \leq a < b \leq r+1} (\tau_b - \tau_a) \prod_{i=1}^L \prod_{a=2}^{r+1} \frac{\Gamma(x - x_i + \lambda_1^{(i)} + \frac{r+3}{2} - a)}{\Gamma(x - x_i + \lambda_a^{(i)} + \frac{r+3}{2} - a)}. \quad (6.1.30)$$

The degrees of the polynomials on the right-hand sides are not the same so it seems reasonable, when solving them, to choose the equation of lowest degree. In any case, neither of these two

reformulations exhibits any of the drawbacks of the usual (nested) Bethe equations: all the solutions are physical. It was even shown that, when all sites are in the defining representation (in which case (6.1.29) is the most convenient one since it is only of degree L), these equations are actually complete [88, 89]: the number of solutions is the dimension of the quantum space. This alternative method of finding Bethe roots was proposed in [231] and further extended in [39, 86, 140].

The transfer matrices in rectangular representations are given in terms of the Q-operators by

$$T_{a,s} = \frac{\det_{1 \leq i, j \leq r+1} \left(Q_{\{i\}}^{[r+2+a-s-2j+2s\theta(a-j)]} \right)}{\prod_{1 \leq c < b \leq r+1} (\tau_c - \tau_b) Q_{\emptyset}^{[a-s]}} = \frac{\det_{1 \leq i, j \leq r+1} \left(Q_{\{i\}}^{[r+2+a-s-2j+2s\theta(a-j)]} \right)}{\det_{1 \leq i, j \leq r+1} \left(Q_{\{i\}}^{[r+2+a-s-2j]} \right)}. \quad (6.1.31)$$

6.2 A D_r Spin Chain

6.2.1 Transfer Matrices for Symmetric Representations

We consider a spin chain of length L in which each site carries the defining representation of the complex orthogonal group in $2r$ dimensions. The quantum space is thus $(\mathbb{C}^{2r})^{\otimes L}$. For convenience, we take the complex orthogonal group to be defined by

$$O(2r) = \left\{ M \in GL(2r) \mid {}^t M J M = J \right\}, \quad (6.2.1)$$

where J is the matrix with 1 on the antidiagonal and 0 everywhere else: $J_{ij} = \delta_{i+j, 2r+1}$. This choice corresponds to a simple similarity transformation with respect to the usual one. We will also make extensive use of the following notation:

$$i' = 2r + 1 - i. \quad (6.2.2)$$

With this convention for the orthogonal group, the R-matrix (1.2.37) becomes

$$R(x) = x(x + \kappa) \text{Id} + (x + \kappa) \mathbb{P} - x \tilde{\mathbb{K}} \quad (6.2.3)$$

with $\tilde{\mathbb{K}} = e_{ij} \otimes e_{i'j'}$.

The R-matrix (6.2.3) allows to construct the fundamental transfer matrix $T = T_{1,1}$, i.e. with the defining representation in auxiliary space, containing the Hamiltonian of the spin chain. It is also convenient to introduce the symmetric generalisations $T_{1,s}$ at this point. The required R-matrices are the ones given in (4.1.5)-(4.1.6). Up to a constant shift, they can be written

$$\mathcal{L}(x) = x^2 \text{Id} + x f_{ij} \otimes e_{ji} + G_{ij} \otimes e_{ji}, \quad (6.2.4)$$

with

$$G_{ij} = \frac{1}{2} f_{kj} f_{ik} + \frac{\kappa}{2} f_{ij} - \frac{1}{4} \left((\kappa - 1)^2 + 2\kappa s + s^2 \right) \delta_{ij}. \quad (6.2.5)$$

We also remind the reader that $\kappa = r - 1$. We have introduced here the generators f_{ij} of $\mathfrak{so}(2r)$ obeying the commutation relations

$$[f_{ij}, f_{kl}] = \delta_{jk} f_{il} - \delta_{i'l} f_{j'k'} - \delta_{j'l} f_{ik'} + \delta_{il} f_{j'k'}, \quad (6.2.6)$$

with $f_{ij} = -f_{j'i'}$. We stress that the formula for the Lax matrix only holds for symmetric representations ρ_s , with generators acting on the highest weight state $|\text{hws}\rangle$ as follows:

$$\rho_s(f_{ij})|\text{hws}\rangle = 0, \quad \text{for } i < j, \quad \rho_s(f_{ii})|\text{hws}\rangle = s\delta_{1i}|\text{hws}\rangle, \quad (6.2.7)$$

and where $s \in \mathbb{N}$ for finite dimensional representations. In such a representation, the generators satisfy the characteristic identity

$$\sum_{j,k=1}^{2r} (\rho_s(f_{ij}) - \delta_{ij}) (\rho_s(f_{jk}) + s\delta_{jk}) (\rho_s(f_{kl}) - (s + 2\kappa)\delta_{kl}) = 0, \quad (6.2.8)$$

see also [232] for a recent discussion of such constraints. A realisation of the generators f_{ij} for general s in terms of oscillators can be found in [183]. The defining representation $s = 1$ can be realised via

$$f_{ij} = e_{ij} - e_{j'i'}. \quad (6.2.9)$$

We then recover the R-matrix (6.2.3) up to a shift, $\mathcal{L}(x) = R(x - \frac{\kappa}{2})$.

As usual, the first space in (6.2.4), with generators f_{ij} , serves as our auxiliary space, and the quantum space is built from L copies of the second one. The transfer matrix constructed from this monodromy is defined by

$$T_{1,s}(x) = \text{Tr}_s [\rho_s(D)\mathcal{L}_1(x)\mathcal{L}_2(x)\cdots\mathcal{L}_L(x)], \quad (6.2.10)$$

where $\mathcal{L}_i(x)$ denotes the Lax matrix acting non-trivially on the i -th spin chain site, and the trace is taken over the representation with generators f_{ij} . We further introduced a diagonal twist matrix $D = \text{Diag}(\tau_1, \dots, \tau_r, \tau_r^{-1}, \dots, \tau_1^{-1}) \in O(2r)$, i.e. the τ_k 's are arbitrary complex parameters. In an arbitrary representation, the twist matrix becomes

$$\rho(D) = \prod_{k=1}^r \tau_k^{\rho(f_{kk})}. \quad (6.2.11)$$

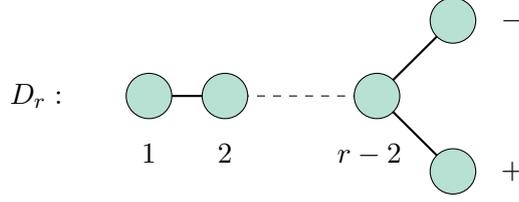
The Hamiltonian of the spin chain is obtained from the fundamental transfer matrix T by taking the logarithmic derivative at the permutation point

$$H = (\ln T)' \left(\frac{\kappa}{2} \right) = \sum_{i=1}^L H_{ii+1}. \quad (6.2.12)$$

The Hamiltonian density is obtained from the logarithmic derivative of the R-matrix at the permutation point. It reads

$$H_{ii+1} = \kappa^{-1} \left(\text{Id} - \tilde{\mathbb{K}} + \kappa \mathbb{P} \right)_{i,i+1}, \quad (6.2.13)$$

and $H_{L,L+1} = D_L H_{L1} D_L^{-1}$, where D_L stands for the twist matrix acting only on the L -th site.

Figure 6.2: Dynkin diagram for D_r Lie algebra.

6.2.2 Diagonalisation of $T_{1,1}$

As discussed above, the fundamental transfer matrix $T = T_{1,1}$ contains the nearest-neighbour Hamiltonian and higher local charges. It has been diagonalised in [233] using the functional Bethe ansatz and in [234] using the algebraic Bethe ansatz, see also [235] for a different nesting procedure and [236] for the trigonometric case. One of the key observations is that the transfer matrix can be written as

$$T(x) = T_+(x) + T_-(x), \quad (6.2.14)$$

where the two terms are related via

$${}^t T_{\pm}(-x)|_{\tau_i \rightarrow \tau_i^{-1}} = T_{\mp}(x). \quad (6.2.15)$$

We note that the twist only slightly modifies the derivations of the spectrum of the transfer matrix in [234]. Following the same logic as in this reference, we find the contributions of T_{\pm} to the eigenvalues of the transfer matrix:

$$T_{\pm} = q_0^{[1-r]} q_0^{[r-1]} \sum_{k=1}^r \tau_k^{\mp 1} \frac{q_{k-1}^{[\pm(k-r+2)]}}{q_{k-1}^{[\pm(k-r)]}} \frac{q_k^{[\pm(k-r-1)]}}{q_k^{[\pm(k-r+1)]}}, \quad (6.2.16)$$

with, as before, the notation $q^{[k]}(x) = q(x + \frac{k}{2})$. The first Q-functions along the tail of the Dynkin diagram are then given by

$$q_0(x) = x^L, \quad q_a(x) = \prod_{j=1}^{m_i} (x - z_j^{(a)}), \quad 1 \leq a \leq r-2. \quad (6.2.17)$$

Here, q_0 does not depend on any Bethe roots and plays a role similar to that of the Q-function for the full set in A -type, cf. (6.1.23). The last two Q-functions factorise:

$$q_{r-1} = s_+ s_-, \quad q_r = s_+^{[+1]} s_+^{[-1]}, \quad (6.2.18)$$

where s_{\pm} are the Q-functions that correspond to the spinorial nodes (see Appendix A.2.2 for the definition of the fundamental spinorial representations). They are polynomials of degree m_{\pm} in the spectral parameter

$$s_{\pm}(x) = \prod_{j=1}^{m_{\pm}} (x - z_j^{(\pm)}). \quad (6.2.19)$$

It immediately follows that the last term in (6.2.16) reduces to the more familiar form

$$\frac{q_{r-1}^{[\pm 2]} q_r^{[\mp 1]}}{q_{r-1}^{[0]} q_r^{[\pm 1]}} = \frac{s_-^{[\pm 2]} s_+^{[\mp 2]}}{s_- s_+}. \quad (6.2.20)$$

Each of the Q-functions q_j for $j \in \{1, 2, \dots, r-2\}$ and s_{\pm} is attached to a different node of the Dynkin diagram, see Figure 6.2. Examining the asymptotic behaviour of the transfer matrix and its eigenvalue, as in Subsection 1.3.2, one sees that the magnon numbers are determined, for a state with given weight λ , through

$$\lambda_a = m_a - m_{a-1} \quad \text{for } 1 \leq a \leq r-2, \quad \lambda_{r-1} = m_+ + m_- - m_{r-2}, \quad \lambda_r = m_+ - m_-, \quad (6.2.21)$$

where $m_0 = L$ is the length of the spin chain.

From the definition of the Hamiltonian (6.2.12) and the eigenvalue equation (6.2.14) of the transfer matrix, we obtain the energy formula. The eigenvalues of the Hamiltonian are parametrised by the Bethe roots and read

$$E = \frac{rL}{r-1} + \frac{q'_1\left(-\frac{1}{2}\right)}{q_1\left(-\frac{1}{2}\right)} - \frac{q'_1\left(\frac{1}{2}\right)}{q_1\left(\frac{1}{2}\right)} = \frac{rL}{r-1} + \sum_{k=1}^{m_1} \left(\frac{1}{z_k^{(1)} - \frac{1}{2}} - \frac{1}{z_k^{(1)} + \frac{1}{2}} \right). \quad (6.2.22)$$

As for the first fundamental representation of A -type, the energy eigenvalues only depend on the Bethe roots at the first nesting level.

6.2.3 QQ-Relations from Bethe Ansatz Equations

The Bethe equations can be read off from the eigenvalue equation of the transfer matrix

$$T = q_0^{[1-r]} q_0^{[r-1]} \sum_{k=1}^r \left[\tau_k^{-1} \frac{q_{k-1}^{[k-r+2]} q_k^{[k-r-1]}}{q_{k-1}^{[k-r]} q_k^{[k-r+1]}} + \tau_k \frac{q_{k-1}^{[r-k-2]} q_k^{[r-k+1]}}{q_{k-1}^{[r-k]} q_k^{[r-k-1]}} \right], \quad (6.2.23)$$

which is obtained by combining (6.2.14) and (6.2.16). Demanding that the transfer matrix be polynomial and the Bethe roots be distinct imply the Bethe equations: they correspond to the vanishing of the residues. They are conveniently written in terms of Q-functions as

$$-\frac{\tau_{k+1}}{\tau_k} = \left(\frac{q_{k-1}^{[-1]} q_k^{[+2]} q_{k+1}^{[-1]}}{q_{k-1}^{[+1]} q_k^{[-2]} q_{k+1}^{[+1]}} \right)_{x=z_j^{(k)}}, \quad 1 \leq k \leq r-3, \quad (6.2.24)$$

$$-\frac{\tau_{r-1}}{\tau_{r-2}} = \left(\frac{q_{r-3}^{[-1]} q_{r-2}^{[+2]} s_+^{[-1]} s_-^{[-1]}}{q_{r-3}^{[+1]} q_{r-2}^{[-2]} s_+^{[+1]} s_-^{[+1]}} \right)_{x=z_j^{(r-2)}}, \quad (6.2.25)$$

$$-\frac{1}{\tau_{r-1} \tau_r} = \left(\frac{q_{r-2}^{[-1]} s_+^{[+2]}}{q_{r-2}^{[+1]} s_+^{[-2]}} \right)_{x=z_j^{(+)}}, \quad (6.2.26)$$

$$-\frac{\tau_r}{\tau_{r-1}} = \left(\frac{q_{r-2}^{[-1]} s_-^{[+2]}}{q_{r-2}^{[+1]} s_-^{[-2]}} \right)_{x=z_j^{(-)}}. \quad (6.2.27)$$

Along the tail of the Dynkin diagram, cf. Figure 6.2, we infer the same QQ-relations as in the A_r case,

$$\frac{\tau_k - \tau_{k+1}}{\sqrt{\tau_k \tau_{k+1}}} q_{k-1} q_{k+1} = \sqrt{\frac{\tau_k}{\tau_{k+1}}} q_k^+ \tilde{q}_k^- - \sqrt{\frac{\tau_{k+1}}{\tau_k}} q_k^- \tilde{q}_k^+, \quad (6.2.28)$$

where q_k and \tilde{q}_k are two different Q-functions at the same level of the Hasse diagram. The form of the eigenvalue equation (6.2.23) is unchanged by such transformation. The apparent difference with (6.1.21) is due to the fact that we have only written here the equation for the polynomial part of the Q-functions (i.e. without including the scalar factor common to all eigenvalues). As before, the Bethe ansatz equations can be restored by shifting the argument of the QQ-relation by $\pm \frac{1}{2}$, evaluating at a root of q_k , and dividing one of the resulting equations by the other. At the fork of the Dynkin diagram, the $(r-2)$ -th node, the QQ-relation takes the form

$$\frac{\tau_{r-2} - \tau_{r-1}}{\sqrt{\tau_{r-2} \tau_{r-1}}} q_{r-3} s_+ s_- = \sqrt{\frac{\tau_{r-2}}{\tau_{r-1}}} q_{r-2}^+ \tilde{q}_{r-2}^- - \sqrt{\frac{\tau_{r-1}}{\tau_{r-2}}} q_{r-2}^- \tilde{q}_{r-2}^+. \quad (6.2.29)$$

At the spinorial nodes \pm , the QQ-relations are

$$\frac{\tau_{r-1} \tau_r - 1}{\sqrt{\tau_{r-1} \tau_r}} q_{r-2} = \sqrt{\tau_{r-1} \tau_r} s_+^+ \tilde{s}_+^- - \frac{1}{\sqrt{\tau_{r-1} \tau_r}} s_+^- \tilde{s}_+^+, \quad (6.2.30)$$

$$\frac{\tau_{r-1} - \tau_r}{\sqrt{\tau_{r-1} \tau_r}} q_{r-2} = \sqrt{\frac{\tau_{r-1}}{\tau_r}} s_-^+ \tilde{s}_-^- - \sqrt{\frac{\tau_r}{\tau_{r-1}}} s_-^- \tilde{s}_-^+. \quad (6.2.31)$$

These QQ-relations for spinorial nodes have appeared in [237] in relation to the ODE/IM correspondence [51] and recently in [238].

6.2.4 Basic (Extremal) Q-Functions

A construction of the Q-operators corresponding to the ends of the Dynkin diagram was recently proposed in [183], we shall briefly review it here. It was inspired by the isomorphism $A_3 \simeq D_3$, admits the asymptotic behaviour expected from (6.2.21) and has been checked by showing some functional relations for $r = 4$ in some examples of finite length. All functional relations in the next section will be consistent with the proposed Q-operators and will have been verified explicitly for several examples of finite length.

First Node We construct $2r$ operators Q_i with $1 \leq i \leq 2r$ corresponding to the first node of the Dynkin diagram. The Lax matrix needed is of the size $2r \times 2r$ with oscillators as entries and of degree two in the spectral parameter. It reads

$$L(z) = \begin{pmatrix} z^2 + z(2 - r - \bar{\mathbf{w}}\mathbf{w}) + \frac{1}{4} \bar{\mathbf{w}}\mathbf{J}^t \bar{\mathbf{w}}^t \mathbf{w}\mathbf{J}\mathbf{w} & z\bar{\mathbf{w}} - \frac{1}{2} \bar{\mathbf{w}}\mathbf{J}^t \bar{\mathbf{w}}^t \mathbf{w}\mathbf{J} & -\frac{1}{2} \bar{\mathbf{w}}\mathbf{J}^t \bar{\mathbf{w}} \\ -z\mathbf{w} + \frac{1}{2} \mathbf{J}^t \bar{\mathbf{w}}^t \mathbf{w}\mathbf{J}\mathbf{w} & z\mathbf{I}_{2r-2} - \mathbf{J}^t \bar{\mathbf{w}}^t \mathbf{w}\mathbf{J} & -\mathbf{J}^t \bar{\mathbf{w}} \\ -\frac{1}{2} \mathbf{w}\mathbf{J}\mathbf{w} & \mathbf{w}\mathbf{J} & 1 \end{pmatrix}. \quad (6.2.32)$$

The Lax matrix above contains $2r - 2$ oscillators arranged into the vectors $\bar{\mathbf{w}}$ and \mathbf{w} as follows

$$\bar{\mathbf{w}} = (\bar{\mathbf{a}}_2, \dots, \bar{\mathbf{a}}_r, \bar{\mathbf{a}}_{r'}, \dots, \bar{\mathbf{a}}_{2r'}), \quad \mathbf{w} = {}^t(\mathbf{a}_2, \dots, \mathbf{a}_r, \mathbf{a}_{r'}, \dots, \mathbf{a}_{2r'}). \quad (6.2.33)$$

They obey the standard commutation relations

$$[\mathbf{a}_i, \bar{\mathbf{a}}_j] = \delta_{ij}. \quad (6.2.34)$$

The matrix J is the same as in (6.2.1) except it is here of size $2r - 2$. The Q-operator Q_1 is defined as the regularised trace over the monodromy of the Lax matrices (6.2.32) which is, as usual, constructed by taking the L -fold tensor product in the matrix space and multiplying in the auxiliary oscillator space:

$$Q_1(x) = \tau_1^x \widehat{\text{Tr}} \left[\mathbf{D} L^{[-1]} \otimes L^{[-1]} \otimes \dots \otimes L^{[-1]} \right]. \quad (6.2.35)$$

The twist matrix \mathbf{D} in the auxiliary space depends on the parameters τ_i , cf. (6.2.11) for the transfer matrix. In the case of the Q-operator Q_1 it reads

$$\mathbf{D} = \prod_{i=2}^r \left(\tau_i \tau_1^{-1} \right)^{\mathbf{N}_i} \left(\tau_i^{-1} \tau_1^{-1} \right)^{\mathbf{N}_{i'}}, \quad (6.2.36)$$

with the number operator $\mathbf{N}_i = \bar{\mathbf{a}}_i \mathbf{a}_i$. The regularised trace is defined by

$$\widehat{\text{Tr}}(\mathbf{D}X) = \frac{\text{Tr}(\mathbf{D}X)}{\text{Tr}(\mathbf{D})}. \quad (6.2.37)$$

Because of the Yang–Baxter relation, Q_1 belongs to the family of commuting operators. For $L = 1$ for instance, all the conserved charges are diagonal and the diagonal elements of $Q_1(x)$ are

$$\begin{aligned} (Q_1(x))_{11} = & \tau_1^x \left[x^2 - x \sum_{k=2}^r \left(1 + \frac{\tau_k}{\tau_1 - \tau_k} + \frac{\tau_k^{-1}}{\tau_1 - \tau_k^{-1}} \right) \right. \\ & \left. + \sum_{k=2}^r \left(\frac{1}{(\tau_1 - \tau_k)(\tau_1 - \tau_k^{-1})} + \frac{\tau_k}{2(\tau_1 - \tau_k)} + \frac{\tau_k^{-1}}{2(\tau_1 - \tau_k^{-1})} \right) + \frac{2r - 3}{4} \right], \end{aligned} \quad (6.2.38)$$

$$(Q_1(x))_{ii} = \tau_1^x \left[x - \frac{1}{2} + \frac{\tau_1^{-1}}{\tau_1^{-1} - \tau_i} \right], \quad 1 < i \leq r, \quad (6.2.39)$$

$$(Q_1(x))_{ii} = \tau_1^x \left[x + \frac{1}{2} - \frac{\tau_{i'}}{\tau_1 - \tau_{i'}} \right], \quad r < i \leq 2r - 1, \quad (6.2.40)$$

$$(Q_1(x))_{2r2r} = \tau_1^x. \quad (6.2.41)$$

In general, one can use Q_1 to define the remaining $2r - 1$ Q-operators at the first fundamental node. For that purpose, we introduce the orthogonal matrix

$$\tilde{B}_{ij} = \sum_{\substack{k=1 \\ k \neq i, j}}^r (e_{k'k'} + e_{kk}) + e_{i'j'} + e_{j'i'} + e_{ij} + e_{ji}, \quad (6.2.42)$$

with $1 \leq i \neq j \leq r$. It is the matrix of the permutation $(ij)(i'j')$. It follows from the $O(2r)$ invariance of the R-matrix that the Q-operators defined via

$$Q_i(x) = (\tilde{B}_{1i} \otimes \dots \otimes \tilde{B}_{1i}) Q_1(x) (\tilde{B}_{1i} \otimes \dots \otimes \tilde{B}_{1i}) \Big|_{\tau_1 \leftrightarrow \tau_i}, \quad i = 2, \dots, r \quad (6.2.43)$$

and

$$Q_i(x) = (J \otimes \dots \otimes J) Q_{i'}(x) (J \otimes \dots \otimes J) \Big|_{\tau_i \rightarrow \tau_i^{-1}}, \quad i = r+1, \dots, 2r \quad (6.2.44)$$

also belong to the family of commuting operators. Up to the exponential prefactor, we shall identify the q-function q_1 with the eigenvalues of the Q-operator Q_1 , but we could have chosen any other single-index Q.

Spinorial Nodes Let us now present the Q-operators corresponding to the spinorial nodes \pm of the Dynkin diagram in Figure 6.2. The Lax matrix of interest is

$$\check{L}(x) = \begin{pmatrix} x I_r + \bar{\mathbf{A}} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{A} & I_r \end{pmatrix}, \quad (6.2.45)$$

where each block is of size $r \times r$. This Lax matrix involves $\frac{r(r-1)}{2}$ pairs of oscillators $[\mathbf{a}_{ij}, \bar{\mathbf{a}}_{kl}] = \delta_{il} \delta_{jk}$: the submatrices $\bar{\mathbf{A}}$ and \mathbf{A} are of the form

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1r'} & \cdots & \bar{\mathbf{a}}_{12'} & 0 \\ \vdots & \ddots & 0 & -\bar{\mathbf{a}}_{12'} \\ \bar{\mathbf{a}}_{r-1r'} & 0 & \ddots & \vdots \\ 0 & -\bar{\mathbf{a}}_{r-1r'} & \cdots & -\bar{\mathbf{a}}_{1r'} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\mathbf{a}_{r'1} & \cdots & -\mathbf{a}_{r'r-1} & 0 \\ \vdots & \ddots & 0 & \mathbf{a}_{r'r-1} \\ -\mathbf{a}_{2'1} & 0 & \ddots & \vdots \\ 0 & \mathbf{a}_{2'1} & \cdots & \mathbf{a}_{r'1} \end{pmatrix}. \quad (6.2.46)$$

As before, we define one Q-operator as the trace of the monodromy built out of the Lax matrix L above as

$$S(x) = (\tau_1 \cdots \tau_r)^{\frac{\alpha}{2}} \widehat{\text{Tr}} \left[\check{\mathbf{D}} \check{L}^{[1-r]} \otimes \check{L}^{[1-r]} \otimes \dots \otimes \check{L}^{[1-r]} \right]. \quad (6.2.47)$$

Here we introduced the twist in the auxiliary space via

$$\check{\mathbf{D}} = \prod_{1 \leq i < j \leq r} (\tau_i \tau_j)^{\bar{\mathbf{a}}_{ij'} \mathbf{a}_{j'i}}. \quad (6.2.48)$$

The remaining Q-operators at the spinorial nodes are obtained through a similarity transformation involving

$$B(\vec{\alpha}) = \frac{1}{2} \sum_{i=1}^r ((1 + \alpha_i)(E_{i'i'} + E_{ii}) + (1 - \alpha_i)(E_{i'i} + E_{ii'})), \quad (6.2.49)$$

with $\alpha_i = \pm 1$, combined with an inversion of some of the twist parameters: we define

$$S_{\vec{\alpha}}(x) = (B(\vec{\alpha}) \otimes \dots \otimes B(\vec{\alpha})) S(x) (B(\vec{\alpha}) \otimes \dots \otimes B(\vec{\alpha})) \Big|_{\tau_i \rightarrow \tau_i^{\alpha_i}}, \quad (6.2.50)$$

labelled by $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ with $\alpha_i = \pm 1$. $B(\vec{\alpha})$ is the matrix of the permutation $\prod_{j; \alpha_j = -1} (jj')$. By construction the 2^r operators $S_{\vec{\alpha}}$ commute with one another and with all the transfer matrices. We choose to identify s_{\pm} with $S_{(+1, \dots, +1, \pm 1)}$ up to the exponential prefactor.

6.3 The D_r Q-System

We introduce in this section the Q-system. It has been verified at small finite length using the construction that we just reviewed. In total we have $3^r - 2^{r-1}r + 2$ Q-functions, see Figure 6.3 and Figure 6.4 for $r = 3, 4$ examples.

6.3.1 Description

The QQ-relations along the tail of the Dynkin diagram have a structure similar to those for A_r , but the labelling of single-index functions is different. We shall say that a subset I of $\{1, \dots, 2r\}$ is acceptable if for all $1 \leq k \leq r$, the integers k and $k' = 2r - k + 1$ do not both belong to I . In particular, an acceptable set cannot have more than r elements: $|I| \leq r$. A Q-function Q_I is associated to each acceptable I , and these functions satisfy the relations

$$Q_{J \cup \{i\}}^{[+1]} Q_{J \cup \{j\}}^{[-1]} - Q_{J \cup \{i\}}^{[-1]} Q_{J \cup \{j\}}^{[+1]} = \frac{\tau_i - \tau_j}{\sqrt{\tau_i \tau_j}} Q_J Q_{J \cup \{i, j\}}, \quad (6.3.1)$$

where $\tau_i = \tau_{i'}^{-1}$, $\{i, i'\} \cap \{j, j'\} = \emptyset$, J is acceptable of order at most $r - 2$ and does not contain i, i', j or j' . We have excluded here the case where k and k' are contained in the same set, as the Q-functions defined this way would not have the expected asymptotic behaviour. For the D_r spin chains under consideration, the Q-operator of the empty set can be conveniently fixed as

$$Q_\emptyset(x) = x^L, \quad (6.3.2)$$

though such a choice for a generic D_r Q-system can be changed via a gauge transformation, see below in this section.

As discussed in Subsection 6.2.2, the Q-functions Q_I with $|I| = r - 1$ or $|I| = r$ factorise into spinorial Q-functions. More precisely,

$$Q_{\{i_1, \dots, i_{r-1}\}} = S_{\{i_1, \dots, i_{r-1}, i_r\}} S_{\{i_1, \dots, i_{r-1}, i'_r\}}, \quad (6.3.3)$$

and

$$Q_{\{i_1, \dots, i_r\}} = S_{\{i_1, \dots, i_r\}}^{[+1]} S_{\{i_1, \dots, i_r\}}^{[-1]}. \quad (6.3.4)$$

The set notation for the Q-operators S_I can be mapped to the notation $S_{\vec{\alpha}}$ of the previous section as follows: to an acceptable set I of order r we associate $\vec{\alpha}$ such that, for $1 \leq i \leq r$,

$$\alpha_i = \begin{cases} +1 & \text{if } i \in I \\ -1 & \text{if } i' \in I \end{cases}. \quad (6.3.5)$$

We thus obtain a one-to-one correspondence between $S_{\{i_1, \dots, i_r\}}$ and $S_{\vec{\alpha}}$ as defined in (6.2.50). We further remark that the polynomial structure of the spinorial Q-functions allows to determine them from the quadratic relations (6.3.3) and (6.3.4).

In total, there are

$$2^k \binom{r}{k} \quad (6.3.6)$$

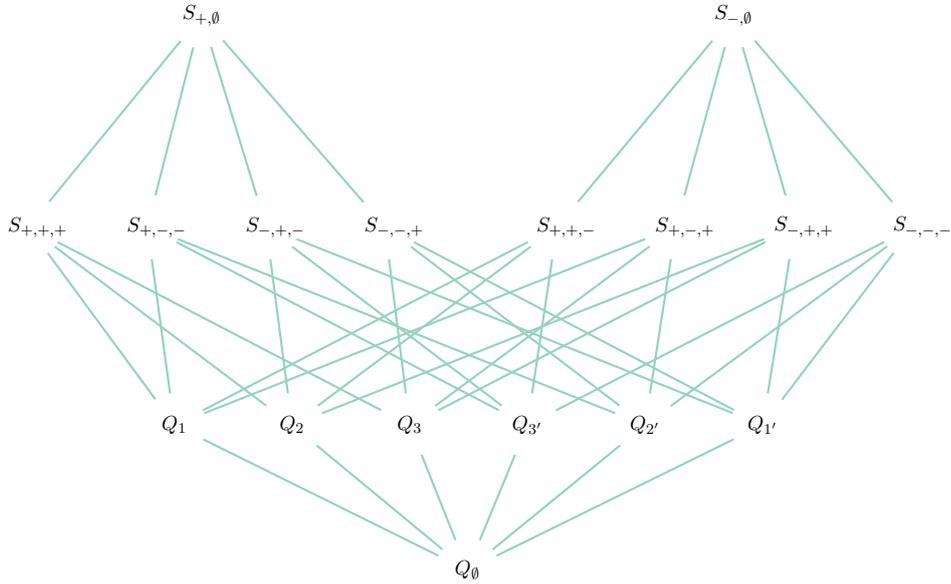


Figure 6.3: Hasse diagram for D_3 . In a particular gauge, the functions at the first and last level nodes can be chosen as in (6.3.17).

Q-functions Q_I at level k . At the last two levels, the Q-functions split according to (6.3.3) and (6.3.4) such that (6.3.6) remains valid for $1 \leq k \leq r - 2$, while $2 \cdot 2^{r-1}$ spinorial Q-functions $S_{\vec{\alpha}}$ distinguished by $\prod_{i=1}^r \alpha_i = \pm 1$ are assigned to the $(r - 1)$ -th and r -th spinor node, respectively.

Let S_I and S_J denote two Q-functions labelled by some acceptable sets I and J verifying $|I \cap J| = r - 2$, i.e.

$$I = \{i_1, \dots, i_{r-2}, i_{r-1}, i_r\} \quad \text{and} \quad J = \{i_1, \dots, i_{r-2}, i'_{r-1}, i'_r\}. \quad (6.3.7)$$

It follows that they must belong to the same node of the Dynkin diagram. Among them, we have the QQ-relations

$$S_I^{[+1]} S_J^{[-1]} - S_I^{[-1]} S_J^{[+1]} = \frac{\tau_{i_{r-1}} \tau_{i_r} - 1}{\sqrt{\tau_{i_{r-1}} \tau_{i_r}}} Q_{I \cap J}, \quad (6.3.8)$$

which relate the spinorial Q-functions to the last Q-functions on the tail of the Dynkin diagram, i.e. at the $(r - 2)$ -th node. Notice that for each Q-function of level $r - 2$ there are two ways to obtain it from spinorial Q-functions, e.g.: when $r = 4$, $I \cap J = \{1, 3\}$ can come from $\{I, J\} = \{\{1, 3, 2, 4\}, \{1, 3, 7, 5\}\}$ or from $\{I, J\} = \{\{1, 3, 2, 5\}, \{1, 3, 7, 4\}\}$. This relation allows us to resolve the last two levels and to represent the Q-functions on a diagram similar to the A_r Hasse diagram, see Figure 6.3 and Figure 6.4 for the cases D_3 and D_4 respectively. Let us note that the D_3 Hasse diagram of Figure 6.3 is (up to a gauge transformation setting Q_0 to 1) the same as the A_3 one, this is not surprising since the two algebras are isomorphic.

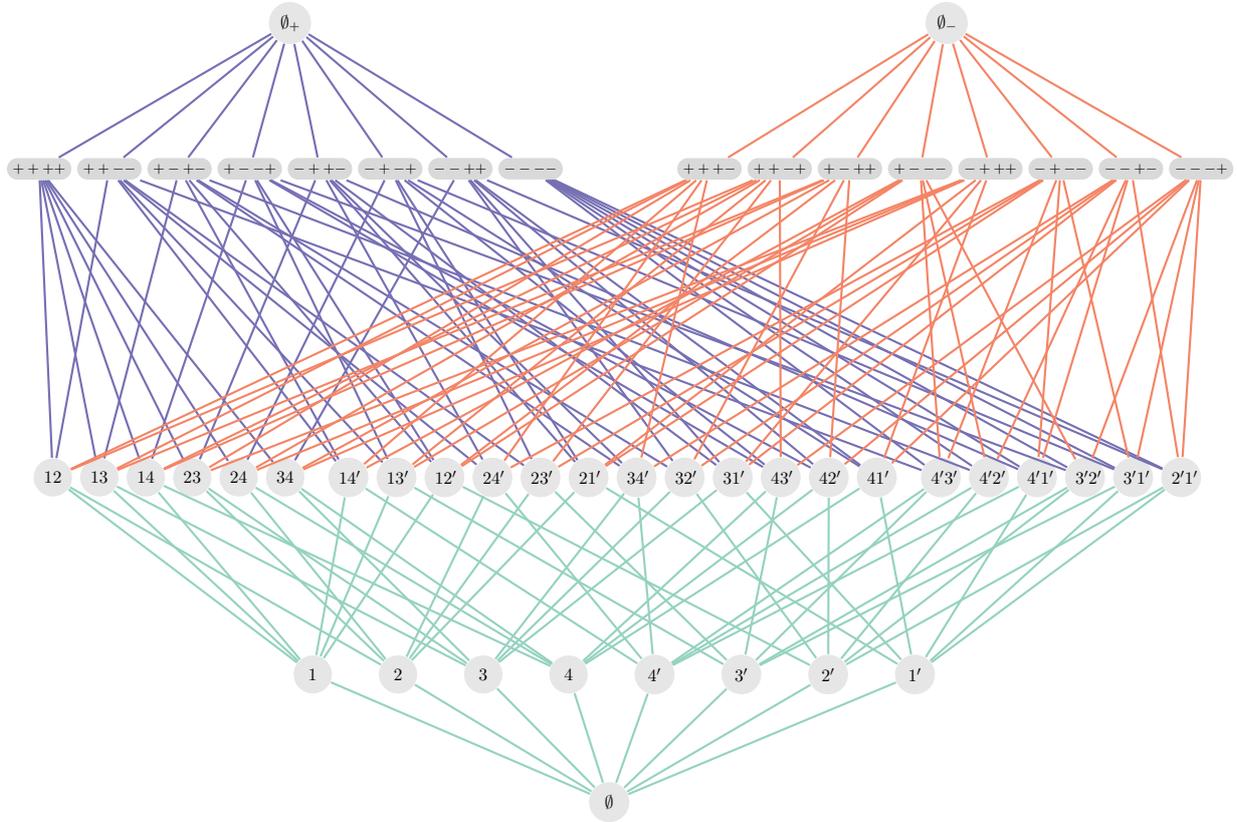


Figure 6.4: Hasse diagram for D_4 . Here, the level 1 and level 2 Q-operators Q_I are abbreviated by their index set I . The third level contains the spinorial Q-operators $S_{\vec{\alpha}}$ which are abbreviated by $\vec{\alpha}$. Finally, we have Q_\emptyset (denoted by \emptyset) at the lowest level and $S_{\pm, \emptyset}$ (denoted by \emptyset_{\pm}) at the highest level. These are proportional to the identity and can be fixed via (6.3.17).

The D_4 Hasse diagram, on the other hand, is new and gives a clear idea of the higher-rank picture.

Using the QQ-relations in (6.3.1) we can express all Q-functions Q_I in terms of Casoratian determinants of single-index ones. We find

$$Q_{\{i_1, \dots, i_k\}} = \frac{(\sqrt{\tau_{i_1} \cdots \tau_{i_k}})^{k-1}}{\prod_{1 \leq a < b \leq k} (\tau_{i_a} - \tau_{i_b})} \frac{|Q_{\{i_a\}}^{[k+1-2b]}|_k}{\prod_{l=1}^{k-1} Q_\emptyset^{[k-2l]}}, \tag{6.3.9}$$

where $i_a \neq i_b$, $i_a \neq i'_b$, and $\tau_i = \tau_i^{-1}$ for $i > r$. Similar formulae exist in terms of spinorial functions: if I is an acceptable set of order $k \leq r - 2$ and the indices i_{k+1}, \dots, i_r are such

that $I_r = I \cup \{i_{k+1}, \dots, i_r\}$ is acceptable of order r , then one has

$$Q_I = \frac{(\sqrt{\tau_{i_{k+1}} \cdots \tau_{i_r}})^{r-k-1}}{\prod_{k+1 \leq a < b \leq r} (\tau_{i_b} - \tau_{i_a})} \frac{\begin{vmatrix} S_{I \cup \{i'_{k+1}, i_{k+2}, \dots, i_r\}}^{[r-k-1]} & S_{I \cup \{i'_{k+1}, i_{k+2}, \dots, i_r\}}^{[r-k-3]} & \cdots & S_{I \cup \{i'_{k+1}, i_{k+2}, \dots, i_r\}}^{[1+k-r]} \\ S_{I \cup \{i_{k+1}, i'_{k+2}, \dots, i_r\}}^{[r-k-1]} & S_{I \cup \{i_{k+1}, i'_{k+2}, \dots, i_r\}}^{[r-k-3]} & \cdots & S_{I \cup \{i_{k+1}, i'_{k+2}, \dots, i_r\}}^{[1+k-r]} \\ \vdots & \vdots & \ddots & \vdots \\ S_{I \cup \{i_{k+1}, \dots, i_{r-1}, i'_r\}}^{[r-k-1]} & S_{I \cup \{i_{k+1}, \dots, i_{r-1}, i'_r\}}^{[r-k-3]} & \cdots & S_{I \cup \{i_{k+1}, \dots, i_{r-1}, i'_r\}}^{[1+k-r]} \end{vmatrix}}{\prod_{l=1}^{r-k-2} S_{I_r}^{[r-k-1-2l]}}. \quad (6.3.10)$$

Gauge Transformation The Q-system as written above corresponds to a particular choice of gauge. In order to describe this gauge freedom, we draw inspiration from the $r = 3$ case, see Appendix C.1. One needs to introduce two new Q-functions $S_{\pm, \emptyset}$. Equations (6.3.1) and (6.3.3) remain unchanged while (6.3.4) and (6.3.8) become

$$Q_I = S_I^{[+1]} S_I^{[-1]} S_{-\epsilon(I), \emptyset} \quad (6.3.11)$$

and

$$S_I^{[+1]} S_J^{[-1]} - S_I^{[-1]} S_J^{[+1]} = \frac{\tau_{i_{r-1}} \tau_{i_r} - 1}{\sqrt{\tau_{i_{r-1}} \tau_{i_r}}} Q_{I \cap J} S_{\epsilon(I), \emptyset}, \quad (6.3.12)$$

where $I = \{i_1, \dots, i_r\}$ and $J = \{i_1, \dots, i_{r-2}, i'_{r-1}, i'_r\}$ are acceptable sets of order r and we define $\epsilon(I) = \prod_{i=1}^r \alpha_i = \epsilon(\vec{\alpha})$, with $\vec{\alpha}$ associated to I according to (6.3.5). These QQ-relations remain unchanged if one applies the gauge transformation, depending on three arbitrary functions g , g_+ and g_- , given by

$$S_{+, \emptyset} \mapsto \frac{g_+^{[+3]} g_-^{[-1]}}{g_+^{[+1]} g_-^{[-3]}} S_{+, \emptyset}, \quad S_{-, \emptyset} \mapsto \frac{g_-^{[+3]} g_+^{[-1]}}{g_-^{[+1]} g_+^{[-3]}} S_{-, \emptyset}, \quad (6.3.13)$$

$$S_{\vec{\alpha}} \mapsto \frac{g_+^{[+2]} g_-}{g_+ g_-^{[-2]}} g S_{\vec{\alpha}} \quad \text{if } \epsilon(\vec{\alpha}) = +, \quad (6.3.14)$$

$$S_{\vec{\alpha}} \mapsto \frac{g_-^{[+2]} g_+}{g_- g_+^{[-2]}} g S_{\vec{\alpha}} \quad \text{if } \epsilon(\vec{\alpha}) = -, \quad (6.3.15)$$

$$Q_I \mapsto \frac{g_+^{[|I|+3-r]} g_-^{[|I|+3-r]}}{g_+^{[r-3-|I|]} g_-^{[r-3-|I|]}} g^{[r-1-|I|]} g^{[|I|+1-r]} Q_I \quad (6.3.16)$$

for I acceptable. In this chapter, we work in the spin-chain gauge

$$Q_{\emptyset}(x) = x^L, \quad S_{\pm, \emptyset}(x) = 1, \quad (6.3.17)$$

and the Q-functions are polynomials in the spectral parameter up to twist-dependent exponential prefactors.

6.3.2 Wronskian Form of the Bethe Ansatz Equations

We investigate here a possible Wronskian relation on $r + 1$ Q -functions which could serve for finding the Bethe roots and, eventually, the energy of the state. We call it the Wronskian BAE, in analogy to the very useful Wronskian BAE (6.1.29) for the A_r spin chains.

We propose to use for this purpose the equation (6.3.10) when $I = \emptyset$:

$$\begin{vmatrix} S_{\{i'_1, i_2, \dots, i_r\}}^{[r-1]} & S_{\{i'_1, i_2, \dots, i_r\}}^{[r-3]} & \cdots & S_{\{i'_1, i_2, \dots, i_r\}}^{[1-r]} \\ S_{\{i_1, i'_2, \dots, i_r\}}^{[r-1]} & S_{\{i_1, i'_2, \dots, i_r\}}^{[r-3]} & \cdots & S_{\{i_1, i'_2, \dots, i_r\}}^{[1-r]} \\ \vdots & \vdots & \ddots & \vdots \\ S_{\{i_1, \dots, i_{r-1}, i'_r\}}^{[r-1]} & S_{\{i_1, \dots, i_{r-1}, i'_r\}}^{[r-3]} & \cdots & S_{\{i_1, \dots, i_{r-1}, i'_r\}}^{[1-r]} \end{vmatrix} = \frac{\prod_{1 \leq a < b \leq r} (\tau_{i_b} - \tau_{i_a})}{(\sqrt{\tau_{i_1} \cdots \tau_{i_r}})^{r-1}} Q_\emptyset \prod_{l=1}^{r-2} S_{I_r}^{[r-1-2l]}, \quad (6.3.18)$$

where we recall that $I_r = \{i_1, \dots, i_r\} \subset \{1, \dots, 2r\}$ is such that $\{i_a, i'_a\} \cap \{i_b, i'_b\} = \emptyset$ for all $a \neq b$. The spinorial Q -functions are polynomials up to a twist-dependent exponential prefactor, their leading asymptotic behaviour is completely determined by the global charges $J_i = -J_{i'} = \sum_{k=1}^L f_{ii}^{(k)}$, it is given by

$$S_{\{i_1, \dots, i_r\}}(x) \underset{x \rightarrow \infty}{\sim} \left(\prod_{a=1}^r \tau_{i_a} \right)^{\frac{x}{2}} x^{\frac{1}{2}(\sum_{k=1}^r J_{i_a} + L)}. \quad (6.3.19)$$

The hope would be that, once the global charges are fixed, it suffices to solve equation (6.3.18) for the unknowns that are the coefficients of the polynomial parts of the spinorial Q -functions. In the A_r case, we thus get exactly all the eigenstates with such a weight. However, this does not seem to be the case here. First of all, one should notice that there are 2^r equations of the type (6.3.18) (as many as there are spinorial Q -functions S_{I_r}). For a given choice of I_r , the number of unknown coefficients can be easily computed to be $L + \frac{r-1}{2}(\sum_{a=1}^r J_{i_a} + L)$, it thus seems natural to chose I_r such that $\sum_{a=1}^r J_{i_a}$ is minimal. Nonetheless, since the degree of the polynomials on each side of the equation is $L + \frac{r-2}{2}(\sum_{a=1}^r J_{i_a} + L)$, as soon as $\sum_{a=1}^r J_{i_a} > -L$, there does not seem to be enough equations to fix all the coefficients. This is understandable, if one looks at the case $r = 3$: the proposed equation does not coincide with (6.1.29), it is instead the expression of Q -functions with three indices in terms of single-index Q -functions. A possible way to resolve this issue would be to solve (6.3.18) for different choices of I_r and to look for common sets of solutions.

Once the Wronskian BAE has been solved, we have enough spinorial Q -functions to recover r single-index Q -functions using (6.3.10). Any of them can be used to compute the energy of the state through (6.2.22).

A Wronskian BAE for single-index Q -functions would be (see (6.4.13))

$$Q_\emptyset^{[r-2]} Q_\emptyset^{[2-r]} \left| Q_{\{i_a\}}^{[r+1-2b]} \right|_{r-1} \left| Q_{\{i_a\}}^{[r-1-2b]} \right|_{r-1} = \prod_{a=1}^{r-1} \frac{\tau_{i_a}}{(\tau_{i_a} - \tau_{i_r})(\tau_{i_a} - \tau_{i'_r})} \left| Q_{\{i_a\}}^{[r+1-2b]} \right|_r \left| Q_{\{j_a\}}^{[r+1-2b]} \right|_r, \quad (6.3.20)$$

where $j_a = i_a$ for $1 \leq a \leq r-1$ and $j_r = i'_r$ and the asymptotic behaviour of the relevant functions is given by

$$Q_{\{i\}}(x) \underset{x \rightarrow \infty}{\sim} \tau_i^x x^{L+J_i}. \quad (6.3.21)$$

Once again there are 2^r equations of this type but they are of higher order than (6.3.18).

6.4 Weyl-Type Formulae from Tableaux Representations

The tableaux sum formulae of [239] give expressions for the transfer matrices of any rectangular representation $T_{a,s}$ through the single terms in the sum of the transfer matrix (6.2.14), as given in (6.2.16). In total, there are $2r$ different terms (boxes), r for T_+ and r for T_- . Instead of using the summands in the form (6.2.16) involving Q 's of different levels, we shall express them either in terms of Q_\emptyset and r single-index Q-functions as in (6.4.2), or in terms of $r+1$ spinorial Q-functions. This will yield new expressions for T-functions associated with totally symmetric ($T_{1,s}$) or antisymmetric ($T_{a,1}$) representations.

6.4.1 The Simplest Case

In Section 6.2.2, we gave the transfer matrix $T = T_{1,1}$ in terms of one single Q-function for each nesting level. We can use the Casoratian formula (6.3.9) to express the transfer matrix only in terms of Q_\emptyset and r fundamental Q-functions $Q_{\{i\}}$. We will show in this subsection that

$$T = Q_\emptyset^{[r-1]} Q_\emptyset^{[3-r]} \frac{|Q_{\{i_a\}}^{[r+2-2b-2\delta_{b,r}]}|_r}{|Q_{\{i_a\}}^{[r+2-2b]}|_r} + Q_\emptyset^{[1-r]} Q_\emptyset^{[r-3]} \frac{|Q_{\{i_a\}}^{[2b-r-2+2\delta_{b,r}]}|_r}{|Q_{\{i_a\}}^{[2b-r-2]}|_r}, \quad (6.4.1)$$

with $i_a \neq i_b$ and $i_a \neq i'_b$ for all $a \neq b$, and the notation $|M_{ab}|_r = \det_{1 \leq a, b \leq r} M_{ab}$.

This formula represents, at least for $T_{1,1}$, the Weyl-type expressions for the transfer matrices of spin chains based on D_r algebra, ‘‘quantising’’ in this way the classical Weyl character formula (A.2.11). The latter can be restored in the classical limit $x \rightarrow \infty$. In that limit $Q_{\{j\}}(x) \underset{x \rightarrow \infty}{\sim} \tau_j^x x^{L+J_j}$, while T behaves as $x^{2N} \sum_{j=1}^r (\tau_j + \frac{1}{\tau_j})$.

Induction We can prove the formula (6.4.1) by expressing the Q-functions q_k in terms of the first r fundamental Q-functions, as in (6.3.9), and inserting it into (6.2.16). We obtain

$$T_\pm = Q_\emptyset^{[\pm(r-1)]} Q_\emptyset^{[\pm(3-r)]} \sum_{k=1}^r \frac{|Q_{\{i\}}^{[\pm(2k-r-2j+2)]}|_{k-1}}{|Q_{\{i\}}^{[\pm(2k-r-2j)]}|_{k-1}} \frac{|Q_{\{i\}}^{[\pm(2k-r-2j)]}|_k}{|Q_{\{i\}}^{[\pm(2k-r-2j+2)]}|_k}. \quad (6.4.2)$$

The desired expression (6.4.1) for the transfer matrix (in the case $i_a = a$) follows from (6.4.2) using the identity

$$\sum_{k=1}^r \frac{|Q_{\{i\}}^{[\pm(2k-r-2j+2)]}|_{k-1}}{|Q_{\{i\}}^{[\pm(2k-r-2j)]}|_{k-1}} \frac{|Q_{\{i\}}^{[\pm(2k-r-2j)]}|_k}{|Q_{\{i\}}^{[\pm(2k-r-2j+2)]}|_k} = \frac{|Q_{\{i\}}^{[\pm(r+2-2j-2\delta_{j,r})]}|_r}{|Q_{\{i\}}^{[\pm(r+2-2j)]}|_r}, \quad (6.4.3)$$

which can be shown by induction on r . It obviously holds true for $r = 1$. It remains to show that

$$\frac{|Q_{\{i\}}^{[\pm(r+3-2j-2\delta_{j,r})]}|_{r+1}}{|Q_{\{i\}}^{[\pm(r+3-2j)]}|_{r+1}} = \frac{|Q_{\{i\}}^{[\pm(r+1-2j-2\delta_{j,r})]}|_r}{|Q_{\{i\}}^{[\pm(r+1-2j)]}|_r} + \frac{|Q_{\{i\}}^{[\pm(r-2j+3)]}|_r}{|Q_{\{i\}}^{[\pm(r-2j+1)]}|_r} \frac{|Q_{\{i\}}^{[\pm(r-2j+1)]}|_{r+1}}{|Q_{\{i\}}^{[\pm(r-2j+3)]}|_{r+1}}, \quad (6.4.4)$$

or equivalently (assuming the determinants are non-vanishing)

$$\begin{aligned} \left| Q_{\{i\}}^{[\mp(2j-1+2\delta_{j,r+1})]} \right|_{r+1} \left| Q_{\{i\}}^{[\mp(2j+1)]} \right|_r &= \left| Q_{\{i\}}^{[\mp(2j-1)]} \right|_{r+1} \left| Q_{\{i\}}^{[\mp(2j+1+2\delta_{j,r})]} \right|_r \\ &+ \left| Q_{\{i\}}^{[\mp(2j-1)]} \right|_r \left| Q_{\{i\}}^{[\mp(2j+1)]} \right|_{r+1}. \end{aligned} \quad (6.4.5)$$

The latter identity can be proven as follows: one first expands each of the $(r+1) \times (r+1)$ determinants with respect to the row involving $Q_{\{r+1\}}$. Both sides become linear combination of $Q_{\{r+1\}}^{[\mp(2j-1)]}$ for $1 \leq j \leq r+2$, and one just has to check that the coefficients on each side are the same. For $j \in \{1, r+1, r+2\}$, this is completely trivial, whereas for $j \in \{2, \dots, r\}$, this becomes

$$\begin{aligned} |C_1, \dots, C_{j-1}, C_{j+1}, \dots, C_r, C_{r+2}| |C_2, \dots, C_{r+1}| \\ = |C_1, \dots, C_{j-1}, C_{j+1}, \dots, C_r, C_{r+1}| |C_2, \dots, C_r, C_{r+2}| \\ - |C_1, \dots, C_r| |C_2, \dots, C_{j-1}, C_{j+1}, \dots, C_{r+2}|, \end{aligned} \quad (6.4.6)$$

where C_j is the transpose of the row vector $(Q_{\{1\}}^{[\mp(2j-1)]}, \dots, Q_{\{r\}}^{[\mp(2j-1)]})$. This last equality is a particular case of the Plücker identity (6.1.14).

Reshuffling Q-Functions We show here that the expression for the transfer matrix (6.4.1) in terms of r fundamental Q-functions is invariant under the replacement $Q_{i_a} \mapsto Q_{i'_a}$ for any a . By obvious symmetry with respect to permutations of the functions Q_i , $i \in \{1, 2, \dots, r\}$, it suffices to show that the transfer matrix is invariant under $Q_{i_r} \mapsto Q_{i'_r}$. This is the case if

$$\frac{Q_{\emptyset}^{[r-1]} Q_{\emptyset}^{[3-r]}}{Q_{\emptyset}^{[r-3]} Q_{\emptyset}^{[1-r]}} = - \frac{\check{T}_{-}^{\{i_1, \dots, i_r\}} - \check{T}_{-}^{\{i_1, \dots, i'_r\}}}{\check{T}_{+}^{\{i_1, \dots, i_r\}} - \check{T}_{+}^{\{i_1, \dots, i'_r\}}}, \quad (6.4.7)$$

where we defined

$$\check{T}_{\pm}^{\{a_1, \dots, a_r\}} = \frac{\left| Q_{\{a_i\}}^{[\mp(2j-r-2+2\delta_{j,r})]} \right|_r}{\left| Q_{\{a_i\}}^{[\mp(2j-r-2)]} \right|_r}. \quad (6.4.8)$$

Using the Jacobi identity on determinants, one can rewrite the numerator and the denominator in the previous condition as

$$\check{T}_{-}^{\{i_1, \dots, i_r\}} - \check{T}_{-}^{\{i_1, \dots, i'_r\}} = (-1)^{1+\lfloor \frac{r}{2} \rfloor} \frac{W_{i_1, \dots, i_{r-1}}^{[-2]} W_{i'_r, i_1, \dots, i_{r-1}, i_r}}{W_{i_1, \dots, i_{r-1}, i_r}^{[-1]} W_{i_1, \dots, i_{r-1}, i'_r}^{[-1]}} \quad (6.4.9)$$

and

$$\check{T}_{+}^{\{i_1, \dots, i_r\}} - \check{T}_{+}^{\{i_1, \dots, i'_r\}} = (-1)^{1+\lfloor \frac{r-1}{2} \rfloor + r} \frac{W_{i_1, \dots, i_{r-1}}^{[-2]} W_{i'_r, i_1, \dots, i_{r-1}, i_r}}{W_{i_1, \dots, i_{r-1}, i_r}^{[+1]} W_{i_1, \dots, i_{r-1}, i'_r}^{[+1]}}, \quad (6.4.10)$$

with

$$W_{i_1, \dots, i_k} := \left| Q_{\{i_a\}}^{[k+1-2b]} \right|_k. \quad (6.4.11)$$

The condition (6.4.7) then reads

$$\frac{Q_{\emptyset}^{[r-1]} Q_{\emptyset}^{[3-r]}}{Q_{\emptyset}^{[r-3]} Q_{\emptyset}^{[1-r]}} = \frac{W_{i_1, \dots, i_{r-1}}^{[-2]} W_{i_1, \dots, i_{r-1}, i_r}^{[+1]} W_{i_1, \dots, i_{r-1}, i'_r}^{[+1]}}{W_{i_1, \dots, i_{r-1}}^{[+2]} W_{i_1, \dots, i_{r-1}, i_r}^{[-1]} W_{i_1, \dots, i_{r-1}, i'_r}^{[-1]}}, \quad (6.4.12)$$

which is indeed satisfied due to the trivial relation

$$\frac{Q_{\{i_1, \dots, i_{r-1}, i_r\}} Q_{\{i_1, \dots, i_{r-1}, i'_r\}}}{Q_{\{i_1, \dots, i_{r-1}\}}^{[+1]} Q_{\{i_1, \dots, i_{r-1}\}}^{[-1]}} = 1, \quad (6.4.13)$$

following immediately from the factorisation properties of the Q-functions (6.3.3) and (6.3.4).

6.4.2 Symmetric Representations

We start from the expressions (6.4.2) for T_{\pm} such that

$$T_{1,1} = T_+ + T_- = \sum_{k=1}^{2r} b_{k,r}, \quad (6.4.14)$$

where $b_{k,r}$ denotes a box as given in [239] for D_r with index k . The expression above, in the character limit $x \rightarrow \infty$, allows to identify $b_{k,r}$ in terms of the single-index Q-functions. We get

$$b_{k,r} = Q_{\emptyset}^{[r-1]} Q_{\emptyset}^{[3-r]} \frac{\left| Q_{\{i\}}^{[2k-r-2j+2]} \right|_{k-1}}{\left| Q_{\{i\}}^{[2k-r-2j]} \right|_{k-1}} \frac{\left| Q_{\{i\}}^{[2k-r-2j]} \right|_k}{\left| Q_{\{i\}}^{[2k-r-2j+2]} \right|_k} \quad (6.4.15)$$

for $1 \leq k \leq r$, and

$$b_{k,r} = Q_{\emptyset}^{[1-r]} Q_{\emptyset}^{[r-3]} \frac{\left| Q_{\{i\}}^{[r-2j-2]} \right|_{k'-1}}{\left| Q_{\{i\}}^{[r-2j]} \right|_{k'-1}} \frac{\left| Q_{\{i\}}^{[r+2-2j]} \right|_{k'}}{\left| Q_{\{i\}}^{[r-2j]} \right|_{k'}} \quad (6.4.16)$$

for $r+1 \leq k \leq 2r$, and we recall that $k' = 2r - k + 1$.

The transfer matrices for generic symmetric representations are given by [239]

$$T_{1,s} = \frac{1}{\prod_{k=1}^{s-1} Q_{\emptyset}^{[r-s-2+2k]} Q_{\emptyset}^{[-(r-s-2+2k)]}} \sum'_{1 \leq i_1 \leq \dots \leq i_s \leq 2r} b_{i_1, r}^{[1-s]} \dots b_{i_s, r}^{[s-1]}, \quad (6.4.17)$$

where the symbol \sum' stands for a sum in which we do not allow for r and $r+1$ to appear at the same time. The denominator appears as a consequence of our boundary conditions for the T-system.

General Symmetric Sum Let us define

$$\tilde{b}_k = \frac{\left| Q_{\{i\}}^{[2k-2j+2]} \right|_{k-1}}{\left| Q_{\{i\}}^{[2k-2j]} \right|_{k-1}} \frac{\left| Q_{\{i\}}^{[2k-2j]} \right|_k}{\left| Q_{\{i\}}^{[2k-2j+2]} \right|_k} \quad (6.4.18)$$

for $1 \leq k \leq r$, and

$$\tilde{b}_k = \frac{\left| Q_{\{i\}}^{[-(2k'-2j+2)]} \right|_{k'-1}}{\left| Q_{\{i\}}^{[-(2k'-2j)]} \right|_{k'-1}} \frac{\left| Q_{\{i\}}^{[-(2k'-2j)]} \right|_{k'}}{\left| Q_{\{i\}}^{[-(2k'-2j+2)]} \right|_{k'}} \quad (6.4.19)$$

for $r+1 \leq k \leq 2r$. This means that

$$b_{k,r} = Q_{\emptyset}^{[r-1]} Q_{\emptyset}^{[3-r]} \tilde{b}_k^{[-r]} \quad \text{if } k \leq r, \quad \text{and} \quad b_{k,r} = Q_{\emptyset}^{[1-r]} Q_{\emptyset}^{[r-3]} \tilde{b}_k^{[r]} \quad \text{if } r+1 \leq k. \quad (6.4.20)$$

For $l \geq 1$, one has

$$\sum_{1 \leq i_1 \leq \dots \leq i_l \leq r} \tilde{b}_{i_1}^{[-2l+1]} \dots \tilde{b}_{i_l}^{[-1]} = \frac{\left| Q_{\{i\}}^{[2r+1-2j-2l\delta_{j,r}]} \right|_r}{\left| Q_{\{i\}}^{[2r+1-2j]} \right|_r} \quad (6.4.21)$$

and

$$\sum_{r+1 \leq i_1 \leq \dots \leq i_l \leq 2r} \tilde{b}_{i_1}^{[1]} \dots \tilde{b}_{i_l}^{[2l-1]} = \frac{\left| Q_{\{i\}}^{[-(2r+1-2j-2l\delta_{j,r})]} \right|_r}{\left| Q_{\{i\}}^{[-(2r+1-2j)]} \right|_r}. \quad (6.4.22)$$

The two identities are equivalent, hence it is enough to prove the first one. We do it by induction on r . It is trivial when $r = 1$. If it is true for some $r_0 \geq 1$, then let us show by induction on l that it is also true for $r_0 + 1$: the case $l = 1$ has been proven earlier in Subsection 6.4.1, so we assume that the identity holds for some $l_0 \geq 1$. We then write

$$\begin{aligned} \sum_{1 \leq i_1 \leq \dots \leq i_{l_0+1} \leq r_0+1} \tilde{b}_{i_1}^{[-2l_0-1]} \dots \tilde{b}_{i_{l_0+1}}^{[-1]} &= \sum_{1 \leq i_1 \leq \dots \leq i_{l_0+1} \leq r_0} \tilde{b}_{i_1}^{[-2l_0-1]} \dots \tilde{b}_{i_{l_0+1}}^{[-1]} \\ &\quad + \tilde{b}_{r_0+1}^{[-1]} \sum_{1 \leq i_1 \leq \dots \leq i_{l_0} \leq r_0+1} \tilde{b}_{i_1}^{[-2l_0-1]} \dots \tilde{b}_{i_{l_0}}^{[-3]}. \end{aligned} \quad (6.4.23)$$

Since we have assumed that the identity holds for r_0 and any l , for $(r_0 + 1, l_0)$ we can write

$$\begin{aligned} \sum_{1 \leq i_1 \leq \dots \leq i_{l_0+1} \leq r_0+1} \tilde{b}_{i_1}^{[-2l_0-1]} \dots \tilde{b}_{i_{l_0+1}}^{[-1]} &= \frac{\left| Q_{\{i\}}^{[2r_0+1-2j-2(l_0+1)\delta_{j,r_0}]} \right|_{r_0}}{\left| Q_{\{i\}}^{[2r_0+1-2j]} \right|_{r_0}} \\ &\quad + \frac{\left| Q_{\{i\}}^{[2r_0-2j+3]} \right|_{r_0} \left| Q_{\{i\}}^{[2r_0+1-2j-2l_0\delta_{j,r_0+1}]} \right|_{r_0+1}}{\left| Q_{\{i\}}^{[2r_0-2j+1]} \right|_{r_0} \left| Q_{\{i\}}^{[2r_0-2j+3]} \right|_{r_0+1}}. \end{aligned} \quad (6.4.24)$$

Consequently, for (6.4.21) to hold for $(r_0 + 1, l_0 + 1)$, one only has to show that

$$\begin{aligned} \left| Q_{\{i\}}^{[-2j]} \right|_{r_0} \left| Q_{\{i\}}^{[2-2j-2(l_0+1)\delta_{j,r_0+1}]} \right|_{r_0+1} &= \left| Q_{\{i\}}^{[2-2j]} \right|_{r_0+1} \left| Q_{\{i\}}^{[-2j-2(l_0+1)\delta_{j,r_0}]} \right|_{r_0} \\ &\quad + \left| Q_{\{i\}}^{[2-2j]} \right|_{r_0} \left| Q_{\{i\}}^{[-2j-2l_0\delta_{j,r_0+1}]} \right|_{r_0+1}. \end{aligned} \quad (6.4.25)$$

This last relation can be proven in much the same way as (6.4.5), which itself corresponds to the case $l_0 = 0$.

Application to the Computation of Transfer Matrices In order to apply the summation formulae (6.4.21) and (6.4.22), we first rewrite equation (6.4.17) as

$$\begin{aligned}
T_{1,s} = & \sum_{l=0}^s Q_{\emptyset}^{[2j+r-s-2]} Q_{\emptyset}^{[2+2j-r-s]} \sum_{\substack{1 \leq i_1 \leq \dots \leq i_l \leq r \\ r+1 \leq i_{l+1} \leq \dots \leq i_s \leq 2r}} \left(\tilde{b}_{i_1}^{[1-s-r]} \dots \tilde{b}_{i_l}^{[2l-s-r-1]} \right) \\
& \times \left(\tilde{b}_{i_{l+1}}^{[2l-s+r+1]} \dots \tilde{b}_{i_s}^{[s+r-1]} \right) - \sum_{l=1}^{s-1} Q_{\emptyset}^{[2l+r-s-2]} Q_{\emptyset}^{[2+2l-r-s]} \tilde{b}_r^{[2l-s-r-1]} \tilde{b}_{r+1}^{[2l-s+r+1]} \\
& \times \sum_{\substack{1 \leq i_1 \leq \dots \leq i_{l-1} \leq r \\ r+1 \leq i_{l+2} \leq \dots \leq i_s \leq 2r}} \left(\tilde{b}_{i_1}^{[1-s-r]} \dots \tilde{b}_{i_{l-1}}^{[2l-s-r-3]} \right) \left(\tilde{b}_{i_{l+2}}^{[2l-s+r+3]} \dots \tilde{b}_{i_s}^{[s+r-1]} \right). \quad (6.4.26)
\end{aligned}$$

By virtue of (6.4.21) and (6.4.22), this becomes

$$\begin{aligned}
T_{1,s} = & \sum_{l=0}^s Q_{\emptyset}^{[2l+r-s-2]} Q_{\emptyset}^{[2+2l-r-s]} \frac{\left| Q_{\{i\}}^{[2l+r-s+1-2j-2l\delta_{j,r}]} \right|_r \left| Q_{\{i\}}^{[-(r+s+1-2l-2j-2(s-l)\delta_{j,r})]} \right|_r}{\left| Q_{\{i\}}^{[2l+r-s+1-2j]} \right|_r \left| Q_{\{i\}}^{[-(r+s+1-2l-2j)]} \right|_r} \\
& - \sum_{l=1}^{s-1} Q_{\emptyset}^{[2l+r-s-2]} Q_{\emptyset}^{[2+2l-r-s]} \frac{\left| Q_{\{i\}}^{[2l+r-s-1-2j-2(l-1)\delta_{j,r}]} \right|_r \left| Q_{\{i\}}^{[-(r+s-1-2l-2j-2(s-1-l)\delta_{j,r})]} \right|_r}{\left| Q_{\{i\}}^{[2l+r-s+1-2j]} \right|_r \left| Q_{\{i\}}^{[-(r+s+1-2l-2j)]} \right|_r}. \quad (6.4.27)
\end{aligned}$$

The terms for $1 \leq l \leq s-1$ of each sum can be combined, thanks to a Plücker identity, to give an explicit and concise Weyl-type representation of symmetric T-functions for D_r algebra

$$T_{1,s} = \sum_{l=0}^s Q_{\emptyset}^{[2l+r-s-2]} Q_{\emptyset}^{[2+2l-r-s]} \frac{\left| Q_{\{i\}}^{[2l+r-s+1-2j+2(s-l)\delta_{j,1}-2l\delta_{j,r}]} \right|_r}{\left| Q_{\{i\}}^{[2l+r-s+1-2j]} \right|_r}. \quad (6.4.28)$$

There are once again 2^r formulae of this type, depending on which set of r single-index Q-functions we use in the right-hand side.

6.4.3 Antisymmetric Representations

The transfer matrices for generic antisymmetric representations are given by [239]

$$\begin{aligned}
T_{a,1} = & \frac{1}{\prod_{k=1}^{a-1} Q_{\emptyset}^{[r-a+2k]} Q_{\emptyset}^{[-(r-a+2k)]}} \sum_{\substack{1 \leq i_1 < \dots < i_k \leq r \\ r+1 \leq j_1 < \dots < j_l \leq 2r \\ a-k-l \in 2\mathbb{N}}} b_{i_1,r}^{[a-1]} \dots b_{i_k,r}^{[a+1-2k]} \\
& \times b_{r+1,r}^{[a-1-2k]} b_{r,r}^{[a-3-2k]} \dots b_{r+1,r}^{[2l+3-a]} b_{r,r}^{[2l+1-a]} b_{j_1,r}^{[2l-a-1]} \dots b_{j_l,r}^{[1-a]}. \quad (6.4.29)
\end{aligned}$$

As it happens, in order to obtain nice expressions for these transfer matrices involving a reduced number of Q-functions, it is more convenient to turn to spinorial Q-functions. If, in order to shorten the notations, we write

$$S_{I_r} = S_{\{1, \dots, r\}}, \quad S_i = S_{\{1, \dots, r-i, r+i, r-i+2, \dots, r\}} \quad \text{for } i \in \{1, \dots, r\}, \quad (6.4.30)$$

then, according to (6.3.10), the boxes are also given by

$$b_{k,r} = Q_\emptyset^{[r-1]} Q_\emptyset^{[1-r]} \frac{S_{I_r}^{[-2]} \left| S_i^{[-2j]} \right|_{r-k} \left| S_i^{[4-2j]} \right|_{r+1-k}}{S_{I_r} \left| S_i^{[2-2j]} \right|_{r-k} \left| S_i^{[2-2j]} \right|_{r+1-k}} \quad (6.4.31)$$

for $1 \leq k \leq r$, and

$$b_{k,r} = Q_\emptyset^{[r-1]} Q_\emptyset^{[1-r]} \frac{S_{I_r}^{[+2]} \left| S_i^{[2j]} \right|_{k-r-1} \left| S_i^{[2j-4]} \right|_{k-r}}{S_{I_r} \left| S_i^{[2j-2]} \right|_{k-r-1} \left| S_i^{[2j-2]} \right|_{k-r}} \quad (6.4.32)$$

for $r+1 \leq k \leq 2r$. The relevant summation formulae read

$$\sum_{1 \leq i_1 < \dots < i_l \leq r} b_{i_1,r}^{[-1]} \dots b_{i_l,r}^{[1-2l]} = \frac{S_{I_r}^{[-1-2l]}}{S_{I_r}^{[-1]}} \left(\prod_{a=1}^l Q_\emptyset^{[r-2l-2+2a]} Q_\emptyset^{[-r-2l+2a]} \right) \frac{\left| S_i^{[1-2j+2\theta(l-j)]} \right|_r}{\left| S_i^{[1-2j]} \right|_r} \quad (6.4.33)$$

and

$$\sum_{r+1 \leq i_1 < \dots < i_l \leq 2r} b_{i_1,r}^{[2l-1]} \dots b_{i_l,r}^{[1]} = \frac{S_{I_r}^{[2l+1]}}{S_{I_r}^{[+1]}} \left(\prod_{a=1}^l Q_\emptyset^{[r-2+2a]} Q_\emptyset^{[-r+2a]} \right) \frac{\left| S_i^{[2r-1-2j+2\theta(r-l-j)]} \right|_r}{\left| S_i^{[2r+1-2j]} \right|_r}, \quad (6.4.34)$$

where we used the Heaviside function θ (it is 0 for negative arguments and 1 for non-negative ones). These formulae can be proven in much the same way as (6.4.21) and (6.4.22).

This permits us to write

$$T_{a,1} = \frac{Q_\emptyset^{[r-a]} Q_\emptyset^{[a-r]}}{S_{I_r}^{[a-1]} S_{I_r}^{[1-a]}} \sum_{\substack{0 \leq k, l \leq a \\ a-k-l \in 2\mathbb{N}}} S_{I_r}^{[a+1-2k]} S_{I_r}^{[2l-1-a]} \frac{\left| S_i^{[a+1-2j+2\theta(k-j)]} \right|_r}{\left| S_i^{[a+1-2j]} \right|_r} \times \frac{\left| S_i^{[2r-a-1-2j+2\theta(r-l-j)]} \right|_r}{\left| S_i^{[2r-a+1-2j]} \right|_r}. \quad (6.4.35)$$

This equation should be compared with the much more complicated expression given in Appendix C.3.2 for the same quantity but in terms of single-index Q-functions. Similarly, the expression for $T_{1,s}$ in terms of spinorial Q-functions is not as simple as (6.4.28).

In the particular case $a = 1$, the previous expression reads

$$T_{1,1} = \frac{Q_\emptyset^{[r-1]} Q_\emptyset^{[1-r]}}{S_{I_r}} \left(S_{I_r}^{[-2]} \frac{\left| S_i^{[2-2j+2\delta_{1,j}]} \right|_r}{\left| S_i^{[2-2j]} \right|_r} + S_{I_r}^{[+2]} \frac{\left| S_i^{[2r-2j-2\delta_{j,r}]} \right|_r}{\left| S_i^{[2r-2j]} \right|_r} \right). \quad (6.4.36)$$

It should be compared with (6.4.1). When $a = r - 1$, there is a factorisation:

$$T_{r-1,1} = \frac{1}{S_{I_r}^{[r-2]} S_{I_r}^{[2-r]}} \frac{Q_\emptyset^{[+1]} Q_\emptyset^{[-1]}}{\left| S_i^{[r+2-2j]} \right|_r \left| S_i^{[r-2j]} \right|_r} \left(\sum_{\substack{k=0 \\ k \text{ even}}}^r S_{I_r}^{[r-2k]} \left| S_i^{[r-2j+2\theta(k-j)]} \right|_r \right) \\ \times \left(\sum_{\substack{k=0 \\ k \text{ odd}}}^r S_{I_r}^{[r-2k]} \left| S_i^{[r-2j+2\theta(k-j)]} \right|_r \right). \quad (6.4.37)$$

6.4.4 Spinorial Representations

Following [58], we express the spinorial T-functions $T_{\pm,1}$ in terms of the Q-functions along a nesting path. One finds

$$T_{\pm,1} = \sum_{|\alpha|=\pm 1} Q_\emptyset^{[-\alpha]} \left(\frac{S_{(+,\dots,+)}^{[\rho_+(\vec{\alpha})+1]}}{S_{(+,\dots,+)}^{[\rho_+(\vec{\alpha})-1]}} \right)^{\frac{\alpha_{r-1}+\alpha_r}{2}} \left(\frac{S_{(+,\dots,+,-)}^{[\rho_-(\vec{\alpha})+1]}}{S_{(+,\dots,+,-)}^{[\rho_-(\vec{\alpha})-1]}} \right)^{\frac{\alpha_{r-1}-\alpha_r}{2}} \prod_{k=1}^{r-2} \left(\frac{Q_{\{1,\dots,k\}}^{[\rho_k(\vec{\alpha})+1]}}{Q_{\{1,\dots,k\}}^{[\rho_k(\vec{\alpha})-1]}} \right)^{\frac{\alpha_k-\alpha_{k+1}}{2}}, \quad (6.4.38)$$

where $Q_\emptyset = x^N$ and the shifts are determined via

$$\rho_k(\vec{\alpha}) = \alpha_1 + \dots + \alpha_{k-1} + \frac{\alpha_k - \alpha_{k+1}}{2} \quad \text{for} \quad 1 \leq k \leq r-2, \\ \rho_\pm(\vec{\alpha}) = \alpha_1 + \dots + \alpha_{r-2} + \frac{\alpha_{r-1} \pm \alpha_r}{2}. \quad (6.4.39)$$

Expressing all Q-functions in terms of spinorial ones using (6.3.10), we obtain determinant formulae for $T_{\pm,1}$. We find

$$T_{+,1} = \frac{(\sqrt{\tau_{i_1} \cdots \tau_{i_r}})^{r-1}}{\prod_{1 \leq a < b \leq r} (\tau_{i_b} - \tau_{i_a})} \frac{1}{\prod_{l=1}^{r-1} S_{I_r}^{[r-2l]}} \sum_{\substack{k=0 \\ k \text{ even}}}^r S_{I_r}^{[r-2k]} \left| S_i^{[2j-r-2\theta(r-k-j)]} \right|_r \quad (6.4.40)$$

and

$$T_{-,1} = \frac{(\sqrt{\tau_{i_1} \cdots \tau_{i_r}})^{r-1}}{\prod_{1 \leq a < b \leq r} (\tau_{i_b} - \tau_{i_a})} \frac{1}{\prod_{l=1}^{r-1} S_{I_r}^{[r-2l]}} \sum_{\substack{k=0 \\ k \text{ odd}}}^r S_{I_r}^{[r-2k]} \left| S_i^{[2j-r-2\theta(r-k-j)]} \right|_r. \quad (6.4.41)$$

These expressions have been verified for $r = 3, 4, 5$ for a particular choice of I_r and we are missing a generic proof. However, the formulae are consistent with the factorisation $T_{r-1,1} = T_{+,1} T_{-,1}$ in (6.4.37) expected from the Hirota relations (see Subsection 6.5.4 below). In principle, one can now generate, from (6.4.35), (6.4.40) and (6.4.41) above, all transfer matrices of rectangular representations using Cherednik–Bazhanov–Reshetikhin type formulae written for D_r symmetry in [58].

6.5 QQ'-Type Formulae

In this section, we present what we call QQ'-type formulae for the symmetric and spinorial T-operators. The reasoning behind our heuristic derivation is based upon [77], where the Bernstein–Gelfand–Gelfand (BGG) resolution [75] was used for a compact spin chain with A_r symmetry, see also [74] for a non-compact spin chain based on the same symmetry algebra. Here we give arguments at the level of characters only, see also [83], which we take as hints to obtain the actual BGG-type relations for the fundamental and spinorial transfer matrices. The final formulae have been checked in several examples for small finite lengths. We also provide a consistency check. Namely, starting from the QQ'-type formulae, we recover the Weyl-type expression for the fundamental transfer matrix (6.4.1). We introduce in the final subsection the T-system [43], and we solve it using the QQ'-type formulae for symmetric T-operators. This yields similar formulae for any rectangular transfer matrix $T_{a,s}$.

6.5.1 Symmetric Transfer Matrices

It was argued in [183] that the product of Lax matrices can be brought to the form

$$L_i^{(1)}(x+x_i)L_{i'}^{(2)}(x-x_i) = S_i \mathfrak{L}_i^{+, (1)}(x) G_i^{(2)} S_i^{-1}, \quad (6.5.1)$$

where the Lax operator L_i is defined by $L_i(x) = \tilde{B}_{1i} L(x) \tilde{B}_{1i}$ for $i \leq r$, and $L_i(x) = \mathbb{J} L_{i'}(x) \mathbb{J}$ for $i \geq r+1$, see Subsection 6.2.4. The superscripts (1, 2) indicate two different families of oscillators. The letter S_i denotes a similarity transformation in the oscillators space. G_i is a dummy matrix which does not depend on the spectral parameter, and commutes with the Lax matrix $\mathfrak{L}_i^+(x)$. Their precise form is given in [183]. We identify the Lax matrix $\mathfrak{L}_i^+(x)$ as a realisation of (6.2.4). The parameter x_i then plays the role of the representation label. We stress that the term linear in the spectral parameter is given by the generators f_{ij} , cf. (6.2.4). In the case (6.5.1), the representation of $\mathfrak{so}(2r)$ is infinite-dimensional in the oscillators space and becomes reducible for certain values of the parameter x_i . The infinite-dimensional representation of $\mathfrak{so}(2r)$ is characterised by its character. For example, for $i = 1$, the Cartan elements are of the form

$$f_{11} = 1 - r + 2x_1 - \sum_{k=2}^{2r-1} \bar{\mathbf{a}}_k \mathbf{a}_k, \quad (6.5.2)$$

$$f_{ii} = \bar{\mathbf{a}}_i \mathbf{a}_i - \bar{\mathbf{a}}_{i'} \mathbf{a}_{i'}, \quad 2 \leq i \leq r. \quad (6.5.3)$$

The character can then be computed:

$$\chi_1^+(x_1) = \text{tr} \prod_{i=1}^r \tau_i^{J_{ii}} = \tau_1^{2x_1} \prod_{k=2}^r \frac{\tau_1}{(\tau_1 - \tau_k)(\tau_1 - \tau_{k'})}. \quad (6.5.4)$$

We find similar formulae for the product of Lax matrices $L_i(x+x_i)L_{i'}(x-x_i)$, by exchanging $\tau_1 \leftrightarrow \tau_i$ and $x_1 \rightarrow x_i$ for $1 \leq i \leq r$ for instance. More precisely, one has

$$\chi_i^+(x_i) = \tau_i^{2x_i} \prod_{\substack{1 \leq k \leq r \\ k \neq i, i'}} \frac{\tau_i}{(\tau_i - \tau_k)(\tau_i - \tau_{k'})}, \quad 1 \leq i \leq 2r. \quad (6.5.5)$$

The product over k is invariant under the simultaneous inversion of all the twists. The finite dimensional characters are related to the ones above by the sum formula

$$\chi_s = \sum_{i=1}^{2r} \chi_i^+ \left(\frac{s+r-1}{2} \right) = \sum_{i=1}^r \left[\prod_{j \neq i} \frac{\tau_i}{(\tau_i - \tau_{j'}) (\tau_i - \tau_j)} \right] \left(\tau_i^{s+r-1} + \tau_{i'}^{s+r-1} \right). \quad (6.5.6)$$

From our results for finite length and the discussion above, we find that the formula can be lifted to transfer matrices and Q-operators. It reads

$$\mathbb{T}_{1,s}(x) = \sum_{i=1}^r \left[\prod_{j \neq i} \frac{\tau_i}{(\tau_i - \tau_{j'}) (\tau_i - \tau_j)} \right] \left(\mathbb{Q}_{\{i\}}^{[s+r-1]} \mathbb{Q}_{\{i'\}}^{[1-r-s]} + \mathbb{Q}_{\{i\}}^{[1-r-s]} \mathbb{Q}_{\{i'\}}^{[s+r-1]} \right). \quad (6.5.7)$$

Notice that, in the limit $x \rightarrow \infty$, (6.5.7) becomes (6.5.6), as it should be.

6.5.2 Spinorial Transfer Matrices

A factorisation formula similar to (6.5.1) exists for the spinorial Lax matrices (6.2.45). It reads

$$\check{L}_{\vec{\alpha}}^{(1)}(x + x_{\vec{\alpha}}) \check{L}_{-\vec{\alpha}}^{(2)}(x - x_{\vec{\alpha}} - \kappa) = \check{S}_{\vec{\alpha}} \check{\mathfrak{L}}_{\vec{\alpha}}^{+, (1)}(x) \check{G}_{\vec{\alpha}}^{(2)} \check{S}_{\vec{\alpha}}^{-1}. \quad (6.5.8)$$

We define here $\check{L}_{\vec{\alpha}}(x) = B(\vec{\alpha}) \check{L}(x) B(\vec{\alpha})$ and use notations similar to those in (6.5.1). The similarity transformation $\check{S}_{\vec{\alpha}}$ only depends on the oscillators, and $\check{G}_{\vec{\alpha}}$ is a matrix that is independent of the spectral parameter and commutes with the Lax matrix $\check{\mathfrak{L}}_{\vec{\alpha}}^+$. The latter denotes an infinite-dimensional realisation of the spinorial Lax matrix

$$\check{\mathfrak{L}}(x) = z \text{Id} + f_{ij} \otimes e_{ji}, \quad (6.5.9)$$

where f_{ij} denote the generators of a spinorial representation. Again the parameter $x_{\vec{\alpha}}$ in (6.5.8) has the role of the representation label. As before, we compute the character of the oscillator representation. In the case $\vec{\alpha} = (+, \dots, +)$, we find

$$\chi_{(+, \dots, +)} = \text{tr} \prod_{i=1}^r \tau_i^{J_{ii}} = \prod_{i=1}^r \tau_i^{x_{(+, \dots, +)}} \prod_{1 \leq j < k \leq r} \frac{\tau_j \tau_k}{\tau_j \tau_k - 1}, \quad (6.5.10)$$

where

$$f_{ii} = x_{(+, \dots, +)} - \sum_{j=i+1}^r \bar{\mathbf{a}}_{ij'} \mathbf{a}_{j'i} - \sum_{j=1}^{i-1} \bar{\mathbf{a}}_{j'i} \mathbf{a}_{i'j}, \quad 1 \leq i \leq r. \quad (6.5.11)$$

The general formula can be obtained using the relations among the spinorial Lax matrices presented in Subsection 6.2.4. We get

$$\chi_{\vec{\alpha}}^{\pm}(x_{\vec{\alpha}}) = \prod_{i=1}^r \tau_i^{\alpha_i x_{\vec{\alpha}}} \prod_{1 \leq j < k \leq r} \frac{\tau_j^{\alpha_j} \tau_k^{\alpha_k}}{\tau_j^{\alpha_j} \tau_k^{\alpha_k} - 1}. \quad (6.5.12)$$

The characters of the finite-dimensional spinor representations \pm with respective highest weights $s\omega_r$ and $s\omega_{r-1}$ can then be written as

$$\chi_{\pm, s} = \sum_{\{\alpha_i\}_{\pm}} \chi_{\vec{\alpha}}^{\pm} \left(\frac{s}{2} \right) = \sum_{\{\alpha_i\}_{\pm}} \prod_{i=1}^r \tau_i^{\frac{s}{2} \alpha_i} \prod_{1 \leq j < k \leq r} \frac{\tau_j^{\alpha_j} \tau_k^{\alpha_k}}{\tau_j^{\alpha_j} \tau_k^{\alpha_k} - 1}. \quad (6.5.13)$$

Here, the sum is taken over all configurations $\{\vec{\alpha}\}_\pm$ such that $\prod_i \alpha_i = \pm 1$.

At the level of transfer matrices, we propose the formula

$$T_{\pm, s} = \sum_{\{\alpha_i\}_\pm} \prod_{1 \leq j < k \leq r} \frac{\tau_j^{\alpha_j} \tau_k^{\alpha_k}}{\tau_j^{\alpha_j} \tau_k^{\alpha_k} - 1} \prod_{i=1}^r \tau_i^{-\frac{\kappa}{2} \alpha_i} S_{\vec{\alpha}}^{[r+s-1]} S_{-\vec{\alpha}}^{[1-s-r]}. \quad (6.5.14)$$

This formula has been verified for small finite lengths by comparing to the transfer matrices directly constructed within the quantum inverse scattering method, using the Lax matrices in (6.5.9) for finite-dimensional spinor representations.

6.5.3 Weyl-Type Formula for $T_{1,1}$ from QQ'-Relations

Let us write (6.5.7) as

$$T_{1,s} = \sum_{i=1}^{2r} h_i Q_{\{i\}}^{[s+r-1]} Q_{\{i'\}}^{[1-r-s]}, \quad (6.5.15)$$

where $h_i = \prod_{j(\neq i, i')}^r (u_i - u_j)^{-1}$ and $u_j = \tau_j + 1/\tau_j$. We further assume that, when s is in $\{1-r, \dots, 0\}$, the identity is still verified if one sets

$$T_{1,0} = Q_{\emptyset}^{[r-2]} Q_{\emptyset}^{[2-r]} \quad \text{and} \quad T_{1,s} = 0 \quad \text{for} \quad 1-r \leq s \leq -1. \quad (6.5.16)$$

We show here that the conditions (6.5.15) and (6.5.16) are enough to recover the expression (6.4.1) giving $T_{1,1}$ in terms of only r of the single-index Q-functions, and so are consistent with it. We also show in Appendix C.2.1 how to retrieve the Wronskian equation (6.3.20) from these conditions.

One simply has to notice that (6.5.15) implies that there exist some Q-dependent coefficients $C_{j,k',k}$ (defined for $0 \leq k' \leq k \leq r$ and $0 \leq j \leq k - k'$) such that

$$\sum_{k'=0}^k \sum_{j=0}^{k-k'} C_{j,k',k} T_{1,k'+1-r}^{[2j+k'-k]} = \sum_{i=1}^r h_i \begin{vmatrix} Q_1^{[-k]} & Q_1^{[-k+2]} & \dots & Q_1^{[k]} \\ \vdots & \vdots & & \vdots \\ Q_k^{[-k]} & Q_k^{[-k+2]} & \dots & Q_k^{[k]} \\ Q_i^{[-k]} & Q_i^{[-k+2]} & \dots & Q_i^{[k]} \end{vmatrix} \left\| \begin{vmatrix} Q_1^{[-k]} & Q_1^{[-k+2]} & \dots & Q_1^{[k]} \\ \vdots & \vdots & & \vdots \\ Q_k^{[-k]} & Q_k^{[-k+2]} & \dots & Q_k^{[k]} \\ Q_{i'}^{[-k]} & Q_{i'}^{[-k+2]} & \dots & Q_{i'}^{[k]} \end{vmatrix} \right. . \quad (6.5.17)$$

It suffices indeed to expand the determinants with respect to their last row and perform the sum over i . One has for instance

$$C_{0,k,k} = (-1)^k \begin{vmatrix} Q_1^{[-k]} & \dots & Q_1^{[k-2]} \\ \vdots & & \vdots \\ Q_k^{[-k]} & \dots & Q_k^{[k-2]} \end{vmatrix} \left\| \begin{vmatrix} Q_1^{[-k+2]} & \dots & Q_1^{[k]} \\ \vdots & & \vdots \\ Q_k^{[-k+2]} & \dots & Q_k^{[k]} \end{vmatrix} \right. . \quad (6.5.18)$$

In particular, plugging the constraints (6.5.16) in the previous relation when $k = r$ gives us

$$C_{0,r-1,r} Q_{\emptyset}^{[r-3]} Q_{\emptyset}^{[1-r]} + C_{1,r-1,r} Q_{\emptyset}^{[r-1]} Q_{\emptyset}^{[3-r]} + C_{0,r,r} T_{1,1} = 0. \quad (6.5.19)$$

Since

$$C_{0,r-1,r} = (-1)^{r+1} \left| Q_i^{[-r+2j]} \right|_r \times \left| Q_i^{[-r+2j-2+2\delta_{j,r}]} \right|_r$$

$$\text{and } C_{1,r-1,r} = (-1)^{r+1} \left| Q_i^{[-r+2j-2]} \right|_r \times \left| Q_i^{[-r+2j-2\delta_{j,1}]} \right|_r, \quad (6.5.20)$$

we recover (6.4.1) in the case $i_a = a$. Notice that, with this derivation, the symmetry under $Q_{\{i\}} \leftrightarrow Q_{\{i'\}}$ is immediate because the equations we started from were already symmetric.

Finally, let us mention that we also checked that one can actually recover formula (6.4.28) for $T_{1,s}$ starting from the conditions (6.5.15) and (6.5.16) using a method similar to the one just shown (or more directly for low ranks, see Appendix C.3.1 for the case $r = 2$).

6.5.4 Arbitrary Rectangular Representations

In this section, we propose relatively simple formulae for T-functions in rectangular representations in terms of bilinear expressions involving Wronskians of both types of single-index Q-functions, Q_i and $Q_{i'}$, where $i = 1, 2, \dots, r$. These formulae follow from (6.5.7) when solving the T-system [43] satisfied by the T-functions, which reads as follows ($s \in \mathbb{N}^*$):

$$T_{a,s}^{[+1]} T_{a,s}^{[-1]} = T_{a,s+1} T_{a,s-1} + T_{a-1,s} T_{a+1,s} \quad (6.5.21)$$

for $1 \leq a \leq r-3$,

$$T_{r-2,s}^{[+1]} T_{r-2,s}^{[-1]} = T_{r-2,s+1} T_{r-2,s-1} + T_{r-3,s} T_{+,s} T_{-,s}, \quad (6.5.22)$$

which can be written in the same form as the previous equation if one sets $T_{r-1,s} = T_{+,s} T_{-,s}$, and

$$T_{\pm,s}^{[+1]} T_{\pm,s}^{[-1]} = T_{\pm,s+1} T_{\pm,s-1} + T_{r-2,s}. \quad (6.5.23)$$

The boundary conditions are ($0 \leq a \leq r-2$, $s \in \mathbb{N}$)

$$T_{a,0} = Q_{\emptyset}^{[r-a-1]} Q_{\emptyset}^{[a+1-r]}, \quad T_{0,s} = Q_{\emptyset}^{[r+s-1]} Q_{\emptyset}^{[1-r-s]}, \quad \text{and } T_{\pm,0}(x) = Q_{\emptyset}(x). \quad (6.5.24)$$

We shall determine here the QQ'-type relations for $T_{a,s}$ for $1 \leq a \leq r-1$, but not for $T_{\pm,s}$. For these spinorial transfer matrices, the spinorial Q-functions seem more suitable, see equation (6.5.14). We start from

$$T_{1,s}^{[+1]} T_{1,s}^{[-1]} - T_{1,s-1} T_{1,s+1} = \sum_{1 \leq i_1 < i_2 \leq 2r} h_{i_1} h_{i_2} \begin{vmatrix} Q_{\{i_1\}}^{[s+r]} & Q_{\{i_1\}}^{[s+r-2]} \\ Q_{\{i_2\}}^{[s+r]} & Q_{\{i_2\}}^{[s+r-2]} \end{vmatrix} \begin{vmatrix} Q_{\{i'_1\}}^{[2-s-r]} & Q_{\{i'_1\}}^{[-s-r]} \\ Q_{\{i'_2\}}^{[2-s-r]} & Q_{\{i'_2\}}^{[-s-r]} \end{vmatrix}, \quad (6.5.25)$$

which can also be written, if the transfer matrices satisfy the Hirota equation (6.5.21) with boundary conditions (6.5.24),

$$T_{1,s}^{[+1]} T_{1,s}^{[-1]} - T_{1,s-1} T_{1,s+1} = T_{0,s} T_{2,s} = Q_{\emptyset}^{[r+s-1]} Q_{\emptyset}^{[1-r-s]} T_{2,s}. \quad (6.5.26)$$

Putting the two expressions together yields the following expression for the second row of transfer matrices:

$$T_{2,s} = \frac{1}{Q_\emptyset^{[r+s-1]} Q_\emptyset^{[1-r-s]}} \sum_{1 \leq i_1 < i_2 \leq 2r} h_{i_1} h_{i_2} \left| \begin{array}{cc} Q_{\{i_1\}}^{[s+r]} & Q_{\{i_1\}}^{[s+r-2]} \\ Q_{\{i_2\}}^{[s+r]} & Q_{\{i_2\}}^{[s+r-2]} \end{array} \right| \left| \begin{array}{cc} Q_{\{i'_1\}}^{[2-s-r]} & Q_{\{i'_1\}}^{[-s-r]} \\ Q_{\{i'_2\}}^{[2-s-r]} & Q_{\{i'_2\}}^{[-s-r]} \end{array} \right|. \quad (6.5.27)$$

This procedure can be continued for $1 \leq a \leq r-1$, it yields

$$T_{a,s} = \frac{1}{\prod_{k=1}^{a-1} Q_\emptyset^{[r+s+2k-a-1]} Q_\emptyset^{[1+a-r-s-2k]}} \sum_{1 \leq i_1 < \dots < i_a \leq 2r} h_{i_1} \cdots h_{i_a} W_{i_1, \dots, i_a}^{[s+r-1]} W_{i'_1, \dots, i'_a}^{[1-s-r]}, \quad (6.5.28)$$

where we recall that $W_{i_1, \dots, i_k} = \left| Q_{\{i_a\}}^{[k+1-2b]} \right|_k$. The proof of this formula, which we present in Appendix C.2.2, boils down to verifying the relation

$$\begin{aligned} & \frac{1}{2} \sum_{\substack{1 \leq i_1 < \dots < i_a \leq 2r \\ 1 \leq j_1 < \dots < j_a \leq 2r}} \left| \begin{array}{cc} W_{i_1, \dots, i_a}^{[s+r]} & W_{i_1, \dots, i_a}^{[s+r-2]} \\ W_{j_1, \dots, j_a}^{[s+r]} & W_{j_1, \dots, j_a}^{[s+r-2]} \end{array} \right| \left| \begin{array}{cc} W_{i'_1, \dots, i'_a}^{[2-s-r]} & W_{i'_1, \dots, i'_a}^{[-s-r]} \\ W_{j'_1, \dots, j'_a}^{[2-s-r]} & W_{j'_1, \dots, j'_a}^{[-s-r]} \end{array} \right| \\ &= \left(\sum_{1 \leq i_1 < \dots < i_{a-1} \leq 2r} W_{i_1, \dots, i_{a-1}}^{[s+r-1]} W_{i'_1, \dots, i'_{a-1}}^{[1-s-r]} \right) \left(\sum_{1 \leq i_1 < \dots < i_{a+1} \leq 2r} W_{i_1, \dots, i_{a+1}}^{[s+r-1]} W_{i'_1, \dots, i'_{a+1}}^{[1-s-r]} \right). \quad (6.5.29) \end{aligned}$$

6.6 Discussion

In this chapter, we proposed the full system of Baxter Q-functions—the Q-system—for the spin chains with $SO(2r)$ symmetry, exemplified for $r=4$ in Figure 6.4. We found Weyl-type formulae for transfer matrices (T-functions) of symmetric and antisymmetric representations in terms of sums of ratios of determinants of a reduced number of basic Q-functions. We proposed QQ'-type formulae expressing the T-functions through $2r$ basic single-index Q-functions. These could be a powerful tool for the study of spin chains and sigma models with D_r symmetry. We also reformulated the Bethe ansatz equations in the form of a single Wronskian relation on $r+1$ basic Q-functions. It is the analogue of a similar Wronskian relation for spin chains with A_r symmetry. However, apart from the Bethe roots, our equation contains extra solutions whose role has yet to be clarified.

Our main assumptions in this chapter are the Plücker QQ-relations (6.3.1) and (6.3.8), as well as the QQ'-relations (6.5.7) and (6.5.14). The QQ-relations (6.3.1) allow to express the fundamental transfer matrix, for which an expression in terms of one Q-function at each level is known from the algebraic Bethe ansatz, in terms of r single-index Q-functions and Q_\emptyset , cf. (6.4.1). This Weyl-type expression has been independently obtained from the QQ'-type relations (6.5.7). We take this as a consistency check. The new formulae for $T_{a,s}$ are obtained either from the T-system or from the tableaux formulae of [239]. Both QQ- and QQ'-relations remain to be proven, but we have tested them explicitly for several examples of small finite length of the chain. A solid proof of these may be possible using the Bernstein–Gelfand–Gelfand resolution (or an analogue of it), or the analogue of the coderivative method proposed in [82], and used for this purpose in [83], see also [240] for a review.

Unlike the well-understood Q-system of A -type, in the case of the D -type Q-system, there are still several questions left and issues to be clarified. Questions exist already at the operator level: the R-matrix—the main building block for Q- and T-functions—is known only for the symmetric and spinorial representations [183, 198, 215] (see also Section 4.1) in the auxiliary space. A full classification of Lax matrices including the ones for the Q-operators was recently given in [241] for A -type. This may shed some light upon the transfer matrices for general rectangular representations and beyond.

Unfortunately, we do not know yet a suitable analogue of Baxter’s TQ-equation (quantum spectral determinant) which appeared to be so useful for the spin chains with A_r symmetry [19, 63], see [140] for the modern description in terms of forms as well as [242] in terms of the quantum determinant. It is possible that the QQ' -type formulae for transfer matrices proposed here can replace the Baxter equations for D_r algebra.

Finally, we hope that our methods can be generalised to B , C and exceptional types of algebras and their deformations, as well as to superalgebras such as $\mathfrak{osp}(m|2n)$ where the Q-system and T-functions are yet to be constructed. This includes the case relevant for AdS_4/CFT_3 for which the QSC has recently been studied in [243–245]. A first step could be the evaluation of the oscillators type Lax matrices for Q-operators using the results of [246, 247] as done in [248] for type A .

Shortly after the results presented in this chapter were put on the arXiv, a preprint with partially overlapping results appeared [249]. The authors of this work approached the Q-system from the ODE/IM correspondence viewpoint. They define an extended Q-system which, in our language, corresponds to imposing no restriction on the sets labelling the Q-functions, $Q_{\{1,1'\}}$ is for instance included in their Q-system. They also treated the exceptional Lie algebras E_6 , E_7 , and E_8 . Two of the authors pursued their investigation of the D -type Q- and T-systems in [250]. They recovered there our Weyl-type formulae for $T_{1,s}$, derived similar formulae for $T_{\pm,s}$, and advocated for an interpretation of them in terms of the representation theory of $\mathfrak{gl}(r) \subset \mathfrak{so}(2r)$.

Conclusion and Outlook

We presented, in this thesis, some advances in the solution of the fishnet theory in arbitrary dimension and of the non-compact spin chain associated to it.

For the isotropic, $\delta = \frac{d}{4}$, fishnet theory in four dimensions one can use the well-established quantum spectral curve (QSC) for $\mathcal{N} = 4$ super Yang–Mills (SYM), and take the double-scaling limit [150] to obtain a solution to the spectral problem. In the same spirit, one can adapt the hexagon factorisation technique for the computation of correlation functions and scattering amplitudes to this simpler theory [251]. As soon as one considers anisotropic fishnet or dimensions different than four, no such tools are available anymore. The relative simplicity of the theory nevertheless incites one to look for similar results in this more general setting. It also gives hope to obtain a first-principle derivation of results that, although they have often been extensively tested, would otherwise remain mainly conjectural.

In order to derive an analogue of the QSC for the fishnet theory in arbitrary dimension d , we started with an investigation of the thermodynamic Bethe ansatz (TBA) equations for the conformal dimensions of multi-magnon operators. The usual prescription of the TBA for integrable quantum field theories [120] requires to solve an auxiliary mirror model. The examination of the Feynman graphs in the case at hand revealed the form of the Hamiltonian in this mirror model as a graph-building operator. The eigenvectors were then constructed explicitly, and they had a clear interpretation as multi-particle states: the mirror system consists of particles living on an infinite one-dimensional space and having internal degrees of freedom with $O(d)$ symmetry. The proof that the eigenvectors indeed provide a realisation of a Zamolodchikov–Faddeev algebra required integral representations for the finite-dimensional $O(d)$ -invariant R-matrices that were not known before.

Under the assumption that the eigenvectors we found form a complete basis of the Hilbert space of the mirror theory, we went on to write the TBA equations. The scattering data was indeed easy to extract once the multi-particle states were found: the dispersion relation is given by the eigenvalue of the Hamiltonian, whereas the scattering matrix, and in particular the dressing phase, could be extracted from the asymptotic behaviour of the states. Generalising the results of [167] in four dimensions, we also showed that our TBA equations admit a dual formulation identical to the TBA for the $O(d+2)$ sigma model, if not for a non-relativistic dispersion relation.

Even though solving the TBA would yield exact results for the conformal dimensions of the multi-magnon operators, this remains a very difficult task. However, from what has been done for $\mathcal{N} = 4$ SYM, we expect that the problem can be further simplified if we reformulate it in terms of Q-functions. Since the symmetry algebra associated to our TBA turned out to

be $\mathfrak{so}(d+2)$, we set out to investigate the associated Q-system. Based on the intuition that the Q-system should be universal, we studied a much simpler model: a compact spin chain with the defining representation of $O(2r)$ at each site. Thanks to the explicit construction of some of the Q-operators [183], and looking at examples for small length and small rank, we were able to propose the full D_r Q-system. We also obtained various expressions for the T-functions—solution to the T-system—in terms of the Q-functions.

We now would like to apply our conjectured Q-system to the fishnet theory. In order to do so, the regularity properties of the Q-functions for the associated non-compact spin chain should be worked out. The Q-system and the regularity properties of the Q-functions should characterise them completely, and allow for a derivation of the quantisation conditions satisfied by the conserved charges of the model. These would in turn relate the coupling constant ξ^2 and the conformal dimension of the operator under consideration. Such an approach has already proven successful for the non-compact $SL(2)$ spin chain [85, 252] (independently of the fishnet theory), and for the fishnet theory in dimension four [150, 166], when the symmetry algebra is $\mathfrak{so}(6) \simeq \mathfrak{sl}(4)$. However, in these two examples, the solution relies heavily on Baxter’s TQ-equation—the quantum spectral curve—for which we know no analogue in the case of $\mathfrak{so}(2r)$ for $r \geq 4$.

Regarding the fishnet theory in arbitrary dimension, another interesting aspect to explore would be the generalisation of the hexagon formalism for the computation of higher-point correlators. The results of Chapter 4 on the diagonalisation of the graph-building operators are an important step in that direction. As we have demonstrated on the example of Basso–Dixon diagrams, the case of general dimension nonetheless appears to be more involved than dimension four and some work remains to be done. It is for instance not clear that nice determinant expressions can be obtained for these diagrams. In any case, our results should prove useful for a better analytical understanding of these multi-loop integrals.

Since the integrability structure underlying the fishnet theory is being better and better understood, one could try to consider a more general theory in the hope of eventually reaching $\mathcal{N} = 4$ SYM. The natural candidate is the theory of three scalars and three fermions introduced in [143]. In this theory, the structure of the Feynman graphs, though richer than in the fishnet theory, remains under control: it exhibits a dynamical fishnet behaviour [151]. We thus hope that the methods presented here could be adapted to this more general situation.

The separation of variables approach to the conformal non-compact spin chains studied in this thesis is still far from being established in dimension $d > 2$. In two dimensions, the global conformal group is $SL(2, \mathbb{C})$, and the non-compact spin chain is well understood. In particular, the graph-building operators of Chapter 4 can be shown to commute with an element of the fundamental monodromy matrix [206]. Their common eigenvectors turn out to be very similar to those of the separated variables. Our results in higher dimensions could thus also prove useful to the development of the separation of variables technique.

Even though we have presented the Q-system for algebras of the D_r series, and the other simply laced algebras were recently treated in [249], the Q-system still remains to be developed for non-simply laced algebras. It would also be useful to investigate the superalgebras $\mathfrak{osp}(m|2n)$ since they include the case relevant to AdS_4/CFT_3 .

More formal questions related to the results presented in this thesis are also worth investigating. One of them concerns the integral representation obtained for the finite-dimensional

$O(d)$ -invariant R-matrices. Since they are reminiscent of the integral kernels of the R-matrices for principal series representations of the conformal group, one could try to derive them from these more general R-matrices. This would require considering the particular principal series representations containing the finite-dimensional representations as subspaces and restricting the R-matrix to this subspace.

We also believe that the QQ'-type relations of Chapter 6 should be a consequence of an analogue of the Bernstein–Gelfand–Gelfand (BGG) resolution. The BGG resolution relates an irreducible finite-dimensional module to several Verma modules. The number of Verma modules involved is the order of the Weyl group of the Lie algebra. For the A_r series of Lie algebras, the BGG resolution allows to express the transfer matrices with finite-dimensional auxiliary space as a determinant of size $r + 1$ of shifted Q-functions. Each of the $(r + 1)!$ terms of the determinants corresponds to a transfer matrix with some Verma module in the auxiliary space, and the number of terms indeed coincide with the order of the Weyl group. In the D_r case, however, the QQ'-type relations only involve $2r$ terms and we do not completely understand how to interpret them. It is possible that this is because, for Lie algebras not belonging to the A_r series, not all representations of the Lie algebra can be lifted to representations of the Yangian.

Appendix A

Conventions for Lie Algebras

Let \mathfrak{g} be a complex semi-simple Lie algebra and \mathfrak{h} be a Cartan subalgebra. Let Δ^+ be the set of the positive roots and W be the Weyl group. We also define $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

If π is an irreducible finite-dimensional representation of \mathfrak{g} with highest weight λ , then we can define the character $\chi_\lambda : \mathfrak{h} \rightarrow \mathbb{C}$ of the representation as the function given by

$$\forall h \in \mathfrak{h}, \quad \chi_\lambda(h) = \text{Tr} \left(e^{\pi(h)} \right). \quad (\text{A.0.1})$$

The Weyl character formula states that

$$\chi_\lambda(h) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)(h)}}{\sum_{w \in W} \epsilon(w) e^{w(\rho)(h)}}, \quad (\text{A.0.2})$$

and it is accompanied by the Weyl denominator formula

$$\sum_{w \in W} \epsilon(w) e^{w(\rho)(h)} = \prod_{\alpha \in \Delta^+} (e^{\frac{\alpha(h)}{2}} - e^{-\frac{\alpha(h)}{2}}). \quad (\text{A.0.3})$$

A.1 Special Linear Algebra

We let the general Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ be that of $n \times n$ matrices and $\mathfrak{sl}(n, \mathbb{C})$ is the subalgebra containing traceless matrices. A basis of the first one is given by the matrices e_{ij} for $1 \leq i, j \leq n$. We choose

$$\mathfrak{h} = \left\{ \text{Diag}(h_{11}, \dots, h_{nn}) \in \mathfrak{gl}(n, \mathbb{C}) \mid \sum_{i=1}^n h_{ii} = 0 \right\} \quad (\text{A.1.1})$$

as a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ and, for any $1 \leq i \leq n$, we define the linear form ε_i on \mathfrak{h} in the following way:

$$\forall h \in \mathfrak{h}, \quad \varepsilon_i(h) = h_{ii}. \quad (\text{A.1.2})$$

These forms clearly generate \mathfrak{h}^* but are not linearly independent since $\sum_{i=1}^n \varepsilon_i = 0$ on \mathfrak{h} . We also introduce a basis $\{h_i = e_{ii} - e_{nn}\}_{1 \leq i \leq n-1}$ of \mathfrak{h} . It is clear that $\{h_i\} \cup \{e_{ij}\}_{i \neq j}$ is a basis of $\mathfrak{sl}(n, \mathbb{C})$ in which the adjoint action of the Cartan subalgebra is diagonal:

$$\forall h \in \mathfrak{h}, \quad [h, e_{ij}] = (\varepsilon_i(h) - \varepsilon_j(h))e_{ij}. \quad (\text{A.1.3})$$

This means that the roots of $\mathfrak{sl}(n, \mathbb{C})$ are the $n(n-1)$ elements of $\{\varepsilon_i - \varepsilon_j\}_{i \neq j}$. We make the following choice of simple roots:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (\text{A.1.4})$$

for $i \leq n-1$, and the fundamental weights are then given by

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i. \quad (\text{A.1.5})$$

If $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i = \sum_{i=1}^{n-1} n_i \omega_i$, then

$$\lambda_i = \sum_{j=i}^{n-1} n_j, \quad \lambda_n = 0, \quad \text{and} \quad n_i = \lambda_i - \lambda_{i+1}. \quad (\text{A.1.6})$$

In particular, half the sum of the positive roots is $\rho = \sum_{i=1}^{n-1} (n-i)\varepsilon_i$. The reflections with respect to the simple roots act as elementary transpositions,

$$s_{\alpha_i}(\lambda) = \lambda_1 \varepsilon_1 + \cdots + \lambda_{i+1} \varepsilon_i + \lambda_i \varepsilon_{i+1} + \cdots + \lambda_n \varepsilon_n, \quad (\text{A.1.7})$$

meaning that the Weyl group is nothing but the group \mathfrak{S}_n of permutations of n elements.

The highest weights of irreducible finite-dimensional representations of $\mathfrak{sl}(n, \mathbb{C})$ (and of $SL(n)$) are thus parametrised by integral λ_i 's such that $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$, and the Weyl character formula, for $h = \sum_{i=1}^n \ln x_i e_{ii}$, with $\prod_{k=1}^n x_k = 1$, reads in that case

$$\chi_\lambda(x_1, \dots, x_n) = \frac{\det_{1 \leq i, j \leq n} (x_i^{\lambda_j + r - j})}{\prod_{i < j} (x_i - x_j)}. \quad (\text{A.1.8})$$

The characters coincide with Schur polynomials: $\chi_\lambda = s_\lambda$.

Finite-dimensional irreducible representations of $GL(n)$ are parametrised by $\lambda_i \in \mathbb{Z}$ such that $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$. They correspond to the representation of $SL(n)$ with labels $\lambda_i - \lambda_n$ tensored with the determinant to the power λ_n . The characters are also given by Equation (A.1.8), but without the restriction $\prod_{k=1}^n x_k = 1$.

A.2 Even Orthogonal Algebra

A.2.1 Roots, Weights and Characters

Let J be the matrix with 1 on the antidiagonal and 0 everywhere else: $J_{ij} = \delta_{i+j, n+1}$. Then the orthogonal Lie algebra is given by

$$\mathfrak{so}(n, \mathbb{C}) = \left\{ M \in \mathfrak{gl}(n, \mathbb{C}) \mid {}^t M J + J M = 0 \right\}. \quad (\text{A.2.1})$$

If $n = 2r$ is even, then the subalgebra

$$\mathfrak{h} = \{ M = \text{Diag}(\tau_1, \dots, \tau_r, -\tau_r, \dots, -\tau_1) \in \mathfrak{gl}(n, \mathbb{C}) \} \quad (\text{A.2.2})$$

of diagonal matrices is a Cartan subalgebra. A natural basis of \mathfrak{h} is given by $\{f_{ii} = e_{ii} - e_{n+1-i, n+1-i}\}_{1 \leq i \leq r}$ and the dual basis is $\{\varepsilon_i\}$ with

$$\forall h \in \mathfrak{h}, \quad \varepsilon_i(h) = \tau_i. \quad (\text{A.2.3})$$

We extend the notations τ_i and ε_i to any $1 \leq i \leq n$ by setting $\tau_i = -\tau_{n+1-i}$ and $\varepsilon_i = -\varepsilon_{n+1-i}$. The matrices $f_{ij} = e_{ij} - e_{n+1-j, n+1-i}$ for $1 \leq i + j \leq n$ form a basis of $\mathfrak{so}(n, \mathbb{C})$. In that basis, the adjoint action of \mathfrak{h} is diagonal: $[h, f_{ij}] = (\varepsilon_i(h) - \varepsilon_j(h))f_{ij}$. As a consequence, the roots of $\mathfrak{so}(n, \mathbb{C})$ are the $2r(r-1)$ elements of $\{\pm\varepsilon_i \pm' \varepsilon_j\}_{1 \leq i \neq j \leq r}$. We make the following choice of simple roots:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for } 1 \leq i \leq r-1, \quad \text{and} \quad \alpha_r = \varepsilon_{r-1} + \varepsilon_r. \quad (\text{A.2.4})$$

Therefore, the fundamental weights are

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \quad \text{for } 1 \leq i \leq r-2, \quad (\text{A.2.5})$$

and

$$\omega_{r-1} = \frac{\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r}{2}, \quad \omega_r = \frac{\varepsilon_1 + \cdots + \varepsilon_{r-1} + \varepsilon_r}{2}. \quad (\text{A.2.6})$$

If $\lambda = \sum_{i=1}^r \lambda_i \varepsilon_i = \sum_{i=1}^r n_i \omega_i$, then

$$\lambda_i = \sum_{j=i}^{r-2} n_j + \frac{n_{r-1} + n_r}{2} \quad \text{for } 1 \leq i \leq r-1, \quad \lambda_r = \frac{n_r - n_{r-1}}{2}, \quad (\text{A.2.7})$$

and

$$n_i = \lambda_i - \lambda_{i+1} \quad \text{for } 1 \leq i \leq r-1, \quad n_r = \lambda_{r-1} + \lambda_r. \quad (\text{A.2.8})$$

In particular, half the sum of the positive roots is $\rho = \sum_{i=1}^r (r-i)\varepsilon_i$. The reflections with respect to the first $r-1$ simple roots act as elementary transpositions,

$$s_{\alpha_i}(\lambda) = \lambda_1 \varepsilon_1 + \cdots + \lambda_{i+1} \varepsilon_i + \lambda_i \varepsilon_{i+1} + \cdots + \lambda_r \varepsilon_r, \quad (\text{A.2.9})$$

while the last one acts as

$$s_{\alpha_r}(\lambda) = \lambda_1 \varepsilon_1 + \cdots + \lambda_{r-2} \varepsilon_{r-2} - \lambda_r \varepsilon_{r-1} - \lambda_{r-1} \varepsilon_r. \quad (\text{A.2.10})$$

The Weyl group is thus of order $2^{r-1}r!$.

The highest weights of irreducible finite-dimensional representations of $\mathfrak{so}(2r, \mathbb{C})$ are parametrised by those $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_r|$ that are either all integral or all in $\mathbb{Z} + \frac{1}{2}$, and the Weyl character formula, for $h = \sum_{i=1}^r \ln x_i f_{ii}$, reads

$$\chi_\lambda(x_1, \dots, x_n) = \frac{\det(x_i^{\lambda_j+r-j} + x_i^{-\lambda_j-r+j}) + \det(x_i^{\lambda_j+r-j} - x_i^{-\lambda_j-r+j})}{2 \prod_{1 \leq i < j \leq r} (x_i + \frac{1}{x_i} - x_j - \frac{1}{x_j})}. \quad (\text{A.2.11})$$

A.2.2 Clifford Algebra

We recall in this appendix some basic facts about the fundamental spinorial representations of the orthogonal Lie algebras. For convenience we now choose this algebra to be that of antisymmetric matrices, its generators are

$$L_{ij} = e_{ij} - e_{ji}, \quad \text{for } i < j. \quad (\text{A.2.12})$$

The Clifford algebra $\text{Cl}_n(\mathbb{C})$ is a complex algebra generated by n elements γ_i satisfying the canonical anticommutation relations

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}. \quad (\text{A.2.13})$$

From that we deduce that, as a vector space, the Clifford algebra is of dimension 2^n and a basis of the algebra is given by

$$\{\gamma_{i_1 \dots i_k} = \gamma_{i_1} \dots \gamma_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}. \quad (\text{A.2.14})$$

It is straightforward to verify that $L_{ij} \mapsto \frac{[\gamma_i, \gamma_j]}{4} = \frac{\gamma_{ij} - \delta_{ij}}{2}$ maps the orthogonal algebra into the Clifford algebra so that any representation of the latter will provide a representation of the former.

When $n = 2r$ is even, the Clifford algebra is a central simple algebra and thus isomorphic to the algebra of $2^r \times 2^r$ matrices over \mathbb{C} . Moreover, all these representations (isomorphisms) are conjugate to one another.

Let us recursively define a particular representation π_n of $\text{Cl}_n(\mathbb{C})$. For $n = 2$ and $n = 3$, we set $\pi_2(\gamma_1) = \pi_3(\gamma_1) = \sigma_x$, $\pi_2(\gamma_2) = \pi_3(\gamma_2) = \sigma_y$, and $\pi_3(\gamma_3) = \sigma_z$. And then, starting from representations π_{2r} and π_{2r+1} , both of dimension 2^r , we define representations π_{2r+2} and π_{2r+3} , of dimension 2^{r+1} , in the following way:

$$\pi_{2r+2}(\gamma_i) = \pi_{2r+3}(\gamma_i) = \sigma_x \otimes \pi_{2r+1}(\gamma_i) \quad (\text{A.2.15})$$

for $1 \leq i \leq 2r + 1$, and

$$\pi_{2r+2}(\gamma_{2r+2}) = \pi_{2r+3}(\gamma_{2r+2}) = \sigma_y \otimes \mathbf{I}_{2^r}, \quad \pi_{2r+3}(\gamma_{2r+3}) = \sigma_z \otimes \mathbf{I}_{2^r}. \quad (\text{A.2.16})$$

The representation π_{2r} is faithful whereas the representation π_{2r+1} is not since it is clear, from our construction, that $\pi_{2r+1}(\gamma_{2r+1}) \propto \pi_{2r+1}(\gamma_1 \dots \gamma_{2r})$.

We remark that, in the representation we have just defined, the gamma matrices (images of the generators of the Clifford algebra) are of the form

$$\pi_{2r}(\gamma_i) = \begin{pmatrix} 0 & \sigma_i \\ \bar{\sigma}_i & 0 \end{pmatrix}, \quad (\text{A.2.17})$$

where the matrices $\sigma_i = \bar{\sigma}_i^\dagger$ are of size 2^{r-1} and satisfy

$$\bar{\sigma}_i \sigma_j + \bar{\sigma}_j \sigma_i = \sigma_i \bar{\sigma}_j + \sigma_j \bar{\sigma}_i = 2\delta_{ij} \mathbf{I}_{2^{r-1}}. \quad (\text{A.2.18})$$

This means that the representation of $\mathfrak{so}(2r, \mathbb{C})$ provided by π_{2r} is given by

$$L_{ij} \mapsto \frac{1}{4} \begin{pmatrix} \sigma_i \bar{\sigma}_j - \sigma_j \bar{\sigma}_i & 0 \\ 0 & \bar{\sigma}_i \sigma_j - \bar{\sigma}_j \sigma_i \end{pmatrix}. \quad (\text{A.2.19})$$

It is the sum of two representations of dimension 2^{r-1} . These two representations are irreducible and inequivalent, they are the two fundamental spinorial representations of $\mathfrak{so}(2r, \mathbb{C})$.

Appendix B

Feynman Graphs Computations

B.1 Basic Integral Relations

We prove, in this appendix, the various integral relations used in this thesis. These relations are obviously not new, they date back to [200–205].

We recall that, for a complex number a and an integer $l \geq 0$, we define

$$\tilde{a} = \frac{d}{2} - a, \quad A_l(a) = \frac{\Gamma(\tilde{a} + \frac{l}{2})}{\Gamma(a + \frac{l}{2})} = \frac{1}{A_l(\tilde{a})}, \quad (a)_l = \frac{\Gamma(a+l)}{\Gamma(a)} = \prod_{k=0}^{l-1} (a+k). \quad (\text{B.1.1})$$

If C is a symmetric traceless tensor of rank l , i.e. $C \in \mathbb{V}_l$, and $x \in \mathbb{R}^d$, we will also write

$$C(x) = C^{\mu_1 \dots \mu_l} x_{\mu_1} \dots x_{\mu_l}. \quad (\text{B.1.2})$$

Because C is traceless, the following two elementary but very useful properties hold for an arbitrary complex number a :

$$\frac{C\left(\frac{x-y}{|x-y|}\right)}{(x-y)^{2a}} = \frac{C(\nabla_x)}{(-2)^l \left(a - \frac{l}{2}\right)_l} \frac{1}{(x-y)^{2(a-\frac{l}{2})}}, \quad (\text{B.1.3})$$

and

$$\frac{C\left(\frac{x-x_0}{|x-x_0|} - \frac{y-x_0}{|y-x_0|}\right)}{(x-x_0)^{2(a-\frac{l}{2})} (y-x_0)^{2(1-a-\frac{l}{2})}} = \frac{C(\nabla_{x_0})}{2^l \left(a - \frac{l}{2}\right)_l} \frac{1}{(x-x_0)^{2(a-\frac{l}{2})} (y-x_0)^{2(1-a-\frac{l}{2})}}. \quad (\text{B.1.4})$$

In particular, the second property implies that, for arbitrary complex numbers a and b , one has

$$\begin{aligned} & C(\nabla_{w_0}) \left[\frac{1}{(x-x_0)^{2b} (x-w_0)^{2(a-\frac{l}{2})} (y-w_0)^{2(1-a-\frac{l}{2})}} \right] \Big|_{w_0=x_0} \\ &= \frac{\left(a - \frac{l}{2}\right)_l}{\left(a + b - \frac{l}{2}\right)_l} C(\nabla_{w_0}) \left[\frac{1}{(x-w_0)^{2(a+b-\frac{l}{2})} (y-w_0)^{2(1-a-b-\frac{l}{2})} (y-x_0)^{2b}} \right] \Big|_{w_0=x_0}. \end{aligned} \quad (\text{B.1.5})$$

We also recall that, if $\text{Re}(a) > 0$, one can write

$$\frac{1}{x^{2a}} = \frac{1}{\Gamma(a)} \int_0^{+\infty} e^{-ux^2} u^{a-1} du. \quad (\text{B.1.6})$$

Fourier Transform of a Propagator For C a rank- l symmetric traceless tensor,

$$\int \frac{C\left(\frac{p}{|p|}\right)}{p^{2a}} e^{ip \cdot x} \frac{d^d p}{\pi^{\frac{d}{2}}} = A_l(a) i^l 4^{\tilde{a}} \frac{C\left(\frac{x}{|x|}\right)}{x^{2\tilde{a}}}. \quad (\text{B.1.7})$$

The proof reads as follows:

$$\begin{aligned} \int \frac{C\left(\frac{p}{|p|}\right)}{p^{2a}} e^{ip \cdot x} \frac{d^d p}{\pi^{\frac{d}{2}}} &= (-i)^l C(\nabla) \int \frac{e^{ip \cdot x}}{p^{2(a+\frac{l}{2})}} \frac{d^d p}{\pi^{\frac{d}{2}}} \\ &= \frac{(-i)^l C(\nabla)}{\Gamma\left(a + \frac{l}{2}\right)} \int \int_0^{+\infty} e^{ip \cdot x - up^2} u^{a+\frac{l}{2}-1} du \frac{d^d p}{\pi^{\frac{d}{2}}} \\ &= \frac{(-i)^l C(\nabla)}{\Gamma\left(a + \frac{l}{2}\right)} \int_0^{+\infty} \int e^{-u(p - \frac{ix}{2u})^2 - \frac{x^2}{4u}} \frac{d^d p}{\pi^{\frac{d}{2}}} u^{a+\frac{l}{2}-1} du \\ &= \frac{(-i)^l C(\nabla)}{\Gamma\left(a + \frac{l}{2}\right)} \int_0^{+\infty} e^{-\frac{x^2}{4u}} u^{a+\frac{l}{2}-1} du \\ &= \frac{(-i)^l \Gamma\left(\tilde{a} - \frac{l}{2}\right)}{\Gamma\left(a + \frac{l}{2}\right)} C(\nabla) \left(\frac{4}{x^2}\right)^{\tilde{a}-\frac{l}{2}} = A_l(a) i^l 4^{\tilde{a}} \frac{C\left(\frac{x}{|x|}\right)}{x^{2\tilde{a}}}. \end{aligned}$$

Chain Relation

$$\int \frac{\pi^{-\frac{d}{2}} d^d z}{(x-z)^{2a} (z-y)^{2b}} = \frac{A_0(a) A_0(b) A_0(d-a-b)}{(x-y)^{2(a+b-\frac{d}{2})}}. \quad (\text{B.1.8})$$

For $0 < \text{Re}(a) < \frac{d}{2}$, $0 < \text{Re}(b) < \frac{d}{2}$, and $\text{Re}(a+b) > \frac{d}{2}$ the integral in the left-hand side is well defined and one can write:

$$\begin{aligned} \int \frac{\pi^{-\frac{d}{2}} d^d z}{(x-z)^{2a} (z-y)^{2b}} &= \frac{1}{\Gamma(a)\Gamma(b)} \int \int_{\mathbb{R}_+^2} e^{-u(x-z)^2 - v(z-y)^2} u^{a-1} v^{b-1} dudv \frac{d^d z}{\pi^{\frac{d}{2}}} \\ &= \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} \int e^{-(u+v)\left(z - \frac{ux+vy}{u+v}\right)^2 - \frac{uv}{u+v}(x-y)^2} \frac{d^d z}{\pi^{\frac{d}{2}}} u^{a-1} v^{b-1} dudv \\ &= \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} e^{-\frac{uv}{u+v}(x-y)^2} \frac{u^{a-1} v^{b-1}}{(u+v)^{\frac{d}{2}}} dudv \\ &= \frac{1}{\Gamma(a)\Gamma(b)} \int_{\mathbb{R}_+^2} e^{-\frac{vw}{w+1}(x-y)^2} \frac{w^{a-1} v^{a+b-\frac{d}{2}-1}}{(w+1)^{\frac{d}{2}}} dw dv \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)} \frac{1}{(x-y)^{2(a+b-\frac{d}{2})}} \int_{\mathbb{R}_+} \frac{w^{\frac{d}{2}-b-1}}{(w+1)^{d-a-b}} dw \\
&= \frac{\Gamma\left(\frac{d}{2}-a\right)\Gamma\left(\frac{d}{2}-b\right)\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)\Gamma(d-a-b)} \frac{1}{(x-y)^{2(a+b-\frac{d}{2})}}.
\end{aligned}$$

Star-Triangle Relation For $a+b+c=d$, one has

$$\int \frac{\pi^{-\frac{d}{2}} d^d w}{(w-x)^{2a}(w-y)^{2b}(w-z)^{2c}} = \frac{A_0(a)A_0(b)A_0(c)}{(x-y)^{2\tilde{c}}(y-z)^{2\tilde{a}}(z-x)^{2\tilde{b}}}. \quad (\text{B.1.9})$$

In order to prove it, one just needs to perform a change of variables in the integral and recover the chain relation. Namely, one first translates w by one of the external points, say x , and then perform an inversion with respect to the origin: $w \mapsto x + \frac{W}{W^2}$. Introducing $Y = \frac{y-x}{(y-x)^2}$ and $Z = \frac{z-x}{(z-x)^2}$, we have

$$(w-x)^2 \mapsto \frac{1}{W^2}, \quad (w-y)^2 \mapsto \frac{(W-Y)^2}{W^2 Y^2}. \quad (\text{B.1.10})$$

Taking into account the conformality condition $a+b+c=d$, one can thus write

$$\begin{aligned}
\int \frac{\pi^{-\frac{d}{2}} d^d w}{(w-x)^{2a}(w-y)^{2b}(w-z)^{2c}} &= Y^{2b} Z^{2c} \int \frac{\pi^{-\frac{d}{2}} d^d W}{(W-Y)^{2b}(W-Z)^{2c}} \\
&= A_0(a)A_0(b)A_0(c) \frac{Y^{2b} Z^{2c}}{(Y-Z)^{2\tilde{a}}} \\
&= \frac{A_0(a)A_0(b)A_0(c)}{(x-y)^{2\tilde{c}}(y-z)^{2\tilde{a}}(z-x)^{2\tilde{b}}}.
\end{aligned}$$

Generalisation of the Chain Relation For $C \in \mathbb{V}_l$, one has

$$\int \frac{C\left(\frac{x-z}{|x-z|}\right)}{(x-z)^{2a}(z-y)^{2b}} \frac{d^d z}{\pi^{\frac{d}{2}}} = A_l(a)A_0(b)A_l(d-a-b) \frac{C\left(\frac{x-y}{|x-y|}\right)}{(x-y)^{2(a+b-\frac{d}{2})}}. \quad (\text{B.1.11})$$

Using (B.1.3) and the basic chain relation itself, the proof is simply

$$\begin{aligned}
\int \frac{C\left(\frac{x-z}{|x-z|}\right)}{(x-z)^{2a}(z-y)^{2b}} \frac{d^d z}{\pi^{\frac{d}{2}}} &= \frac{C(\nabla_x)}{(-2)^l \left(a - \frac{l}{2}\right)_l} \int \frac{\pi^{-\frac{d}{2}} d^d z}{(x-z)^{2(a-\frac{l}{2})}(z-y)^{2b}} \\
&= A_l(a)A_0(b)A_0\left(d-a-b+\frac{l}{2}\right) \frac{C(\nabla_x)}{(-2)^l} \frac{1}{(x-y)^{2(a+b-\frac{l+d}{2})}} \\
&= A_l(a)A_0(b)A_l(d-a-b) \frac{C\left(\frac{x-y}{|x-y|}\right)}{(x-y)^{2(a+b-\frac{d}{2})}}.
\end{aligned}$$

B.2 Feynman Integrals for Subsection 5.4.2

One-Loop Integral At one loop we encounter the following integral, which can be computed by the Fourier transform lemma presented above,

$$I_1(p) = \text{Diagram} = \int \frac{e^{ip \cdot x}}{(x^2)^{\frac{d}{2}}} \frac{d^{d-2\epsilon}x}{\pi^{\frac{d}{2}}} = \left(\frac{p^2}{4\pi}\right)^\epsilon \frac{\Gamma(-\epsilon)}{\Gamma(\frac{d}{2})}. \quad (\text{B.2.1})$$

Two-Loop Integrals At two loops we encounter the following two Feynman integrals:

$$I_2(p) = \text{Diagram} = \int \frac{e^{ip \cdot y}}{(x^2)^{\frac{d}{2}}(x-y)^{2\tilde{\delta}}(y^2)^\delta} \frac{d^{d-2\epsilon}x}{\pi^{\frac{d}{2}}} \frac{d^{d-2\epsilon}y}{\pi^{\frac{d}{2}}} \quad (\text{B.2.2})$$

and

$$I_{1,1}(p_1, p_2) = \text{Diagram} = \int \frac{e^{ip_1 \cdot x + ip_2 \cdot y}}{(x^2)^{\tilde{\delta}}(x-y)^{2\delta}(y^2)^{\frac{d}{2}}} \frac{d^{d-2\epsilon}x}{\pi^{\frac{d}{2}}} \frac{d^{d-2\epsilon}y}{\pi^{\frac{d}{2}}}. \quad (\text{B.2.3})$$

The integral $I_2(p)$ can be computed for any p , however we can only compute $I_{1,1}(p_1, p_2)$ for $p_2 = -p_1$, and the results are given by

$$I_2(p) = I_{1,1}(p, -p) = \left(\frac{p^2}{4\pi}\right)^{2\epsilon} \frac{\Gamma(-\epsilon)\Gamma(-2\epsilon)\Gamma(\tilde{\delta} + \epsilon)\Gamma(\delta - \epsilon)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} + \epsilon)\Gamma(\tilde{\delta})\Gamma(\delta - 2\epsilon)}. \quad (\text{B.2.4})$$

Three-Loop Integrals At three loops we encounter three more Feynman integrals, we present them here only as Feynman diagrams for brevity.

The first one $I_3(p)$ is of single magnon type,

$$I_3(p) = \text{Diagram} \quad (\text{B.2.5})$$

$$= \left(\frac{p^2}{4\pi}\right)^{3\epsilon} \frac{\Gamma(-\epsilon)\Gamma(-2\epsilon)\Gamma(-3\epsilon)\Gamma(\delta - \epsilon)^2\Gamma(\tilde{\delta} + \epsilon)\Gamma(\tilde{\delta} + 2\epsilon)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} + \epsilon)\Gamma(\frac{d}{2} + 2\epsilon)\Gamma(\tilde{\delta})^2\Gamma(\delta - 2\epsilon)\Gamma(\delta - 3\epsilon)}. \quad (\text{B.2.6})$$

The second one $I_{2,1}(p_1, p_2)$ is new, with two magnons,

$$I_{2,1}(p_1, p_2) = \text{Diagram} \quad (\text{B.2.7})$$

$$(\text{B.2.8})$$

We can compute this integral for $p_2 = -p_1$, and the result is given by

$$I_{2,1}(p, -p) = \left(\frac{p^2}{4\pi}\right)^{3\epsilon} \frac{\Gamma(-\epsilon)^2 \Gamma(-3\epsilon) \Gamma(\delta - \epsilon) \Gamma(\tilde{\delta} + 2\epsilon)}{\Gamma\left(\frac{d}{2}\right)^2 \Gamma\left(\frac{d}{2} + 2\epsilon\right) \Gamma(\tilde{\delta}) \Gamma(\delta - 3\epsilon)}. \quad (\text{B.2.9})$$

The last one,

$$I_{1,2}(p_1, p_2) = \text{Diagram} \quad (\text{B.2.10})$$

$$(\text{B.2.11})$$

is the most complicated of the integrals we have to deal with, because even if we take $p_2 = -p_1$ we cannot compute it analytically. However, since we only require its divergent part, we can use its Mellin–Barnes representation [253],

$$\begin{aligned} I_{1,2}(p, -p) &= \frac{\Gamma(-3\epsilon)}{\Gamma\left(\frac{d}{2} + 2\epsilon\right) \Gamma(\delta) \Gamma(\tilde{\delta}) \Gamma\left(\frac{d}{2}\right) \Gamma(-2\epsilon)} \left(\frac{p^2}{4\pi}\right)^{3\epsilon} \\ &\times \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Gamma(-s_1) \Gamma(-s_2) \Gamma\left(\frac{d}{2} + \epsilon + s_1 + s_2\right) \Gamma(-\tilde{\delta} - \epsilon - s_2) \Gamma(-\delta - \epsilon - s_1) \\ &\times \Gamma\left(\frac{d}{2} + s_1 + s_2\right) \frac{\Gamma(\delta - \epsilon + s_1) \Gamma(\tilde{\delta} - \epsilon + s_2) \Gamma(\epsilon - s_1 - s_2) ds_1 ds_2}{\Gamma(\tilde{\delta} - s_1) \Gamma(\delta - s_2) \Gamma\left(\frac{d}{2} - 2\epsilon + s_1 + s_2\right) 2\pi i 2\pi i}. \quad (\text{B.2.12}) \end{aligned}$$

The contours of integration must be such that the poles coming from Gamma functions of the form $\Gamma(-s + \dots)$ lie to the right, while those coming from $\Gamma(s + \dots)$ lie to the left. When $\epsilon \rightarrow 0$, there is no such contour anymore because some left and right poles coalesce. This explains why the integral is divergent. In order to extract this divergent part, one should deform the contours to pick the residues at these poles, the remaining integrals then have a regular limit when $\epsilon \rightarrow 0$. In our case, the divergences in epsilon start at order ϵ^{-3} (as expected) and come from the residues of the poles around $-\delta$ for s_1 and around $-\tilde{\delta}$ for s_2 . Computing their contribution yields

$$\begin{aligned} I_{1,2}(p, -p) &= \left(\frac{p^2}{4\pi}\right)^{3\epsilon} \left[\frac{\Gamma(-\epsilon) \Gamma(-2\epsilon)^2 \Gamma(-3\epsilon) \Gamma\left(\frac{d}{2} + 3\epsilon\right) \Gamma(\delta + \epsilon) \Gamma(\tilde{\delta} + \epsilon)}{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d}{2} + \epsilon\right)^2 \Gamma\left(\frac{d}{2} + 2\epsilon\right) \Gamma(-4\epsilon) \Gamma(\delta) \Gamma(\tilde{\delta})} \right. \\ &\quad \left. + \frac{\psi'(\delta) + \psi'(\tilde{\delta})}{3\epsilon \Gamma\left(\frac{d}{2}\right)^3} + O(1) \right], \quad (\text{B.2.13}) \end{aligned}$$

where $\psi = \frac{\Gamma'}{\Gamma}$.

Appendix C

Details on D_r Q-System

C.1 Q-System of $A_3 \simeq D_3$

We show in this appendix that, as is expected from the isomorphism $A_3 \simeq D_3$, the known Q-system for A_3 can be interpreted as the Q-system for D_3 , albeit in a particular gauge. We start with a reminder of the Q-system for A_3 : in order to avoid confusion, we shall denote \mathcal{Q}_I for $I \subset \{1, 2, 3, 4\}$ the Q-functions for A_3 , and the $SL(4)$ twists will be z_1, z_2, z_3 and z_4 such that $z_1 z_2 z_3 z_4 = 1$. The following relations hold (neither i nor j belongs to I):

$$\mathcal{Q}_{I \cup \{i\}}^{[+1]} \mathcal{Q}_{I \cup \{j\}}^{[-1]} - \mathcal{Q}_{I \cup \{i\}}^{[-1]} \mathcal{Q}_{I \cup \{j\}}^{[+1]} = \frac{z_i - z_j}{\sqrt{z_i z_j}} \mathcal{Q}_I \mathcal{Q}_{I \cup \{i, j\}}. \quad (\text{C.1.1})$$

From these relations, one can easily show that

$$\begin{vmatrix} \mathcal{Q}_{\{i, j\}}^{[-2]} & \mathcal{Q}_{\{i, j\}} & \mathcal{Q}_{\{i, j\}}^{[+2]} \\ \mathcal{Q}_{\{i, k\}}^{[-2]} & \mathcal{Q}_{\{i, k\}} & \mathcal{Q}_{\{i, k\}}^{[+2]} \\ \mathcal{Q}_{\{i, l\}}^{[-2]} & \mathcal{Q}_{\{i, l\}} & \mathcal{Q}_{\{i, l\}}^{[+2]} \end{vmatrix} = \frac{(z_j - z_k)(z_j - z_l)(z_k - z_l)}{z_j z_k z_l} \mathcal{Q}_{\{i\}}^{[-1]} \mathcal{Q}_{\{i\}}^{[+1]} \mathcal{Q}_{\{1, 2, 3, 4\}} \quad (\text{C.1.2})$$

and

$$\begin{vmatrix} \mathcal{Q}_{\{i, j\}}^{[-2]} & \mathcal{Q}_{\{i, j\}} & \mathcal{Q}_{\{i, j\}}^{[+2]} \\ \mathcal{Q}_{\{j, k\}}^{[-2]} & \mathcal{Q}_{\{j, k\}} & \mathcal{Q}_{\{j, k\}}^{[+2]} \\ \mathcal{Q}_{\{i, k\}}^{[-2]} & \mathcal{Q}_{\{i, k\}} & \mathcal{Q}_{\{i, k\}}^{[+2]} \end{vmatrix} = \frac{(z_j - z_i)(z_j - z_k)(z_i - z_k)}{z_i z_j z_k} \mathcal{Q}_{\{i, j, k\}}^{[-1]} \mathcal{Q}_{\{i, j, k\}}^{[+1]} \mathcal{Q}_{\emptyset}. \quad (\text{C.1.3})$$

Both of these equations are identified with equation (6.3.11). More generally, both Q-systems are the same if one makes the following identification between the two sets of Q-functions:

$$Q_{\{1\}} = \mathcal{Q}_{\{1, 2\}}, \quad Q_{\{2\}} = \mathcal{Q}_{\{1, 3\}}, \quad Q_{\{3\}} = \mathcal{Q}_{\{1, 4\}}, \quad (\text{C.1.4})$$

$$Q_{\{1'\}} = \mathcal{Q}_{\overline{\{1, 2\}}} = \mathcal{Q}_{\{3, 4\}}, \quad Q_{\{2'\}} = \mathcal{Q}_{\{2, 4\}}, \quad Q_{\{3'\}} = \mathcal{Q}_{\{2, 3\}}, \quad (\text{C.1.5})$$

$$S_{(+, +, +)} = \mathcal{Q}_{\{1\}}, \quad S_{(+, -, -)} = \mathcal{Q}_{\{2\}}, \quad S_{(-, +, -)} = \mathcal{Q}_{\{3\}}, \quad S_{(-, -, +)} = \mathcal{Q}_{\{4\}}, \quad (\text{C.1.6})$$

$$S_{(-, -, -)} = \mathcal{Q}_{\{2, 3, 4\}}, \quad S_{(+, +, -)} = \mathcal{Q}_{\{1, 2, 3\}}, \quad S_{(+, -, +)} = \mathcal{Q}_{\{1, 2, 4\}}, \quad S_{(-, +, +)} = \mathcal{Q}_{\{1, 3, 4\}}. \quad (\text{C.1.7})$$

The twists are related via

$$\tau_1 = z_1 z_2 = \frac{1}{z_3 z_4}, \quad \tau_2 = z_1 z_3 = \frac{1}{z_2 z_4}, \quad \tau_3 = z_1 z_4 = \frac{1}{z_2 z_3}, \quad (\text{C.1.8})$$

while the remaining Q-functions are

$$Q_\emptyset = 1, \quad S_{+, \emptyset} = Q_\emptyset, \quad \text{and} \quad S_{-, \emptyset} = Q_{\{1,2,3,4\}}. \quad (\text{C.1.9})$$

The previous equation shows that in identifying the two Q-systems we had to partly fix the gauge for D_3 . This explains why there are only two gauge degrees of freedom in the A_3 Q-system [140], whereas there are three of them for the D_3 one.

C.2 Details for the Computations of Section 6.5

C.2.1 Wronskian Condition from QQ'-Type Constraints

Plugging the constraints (6.5.16) into equation (6.5.17) for $k = r - 1$, we get

$$C_{0,r-1,r-1} Q_\emptyset^{[r-2]} Q_\emptyset^{[2-r]} = h_r \begin{vmatrix} Q_1^{[-r+1]} & Q_1^{[-r+3]} & \dots & Q_1^{[r-1]} \\ \vdots & \vdots & & \vdots \\ Q_{r-1}^{[-r+1]} & Q_{r-1}^{[-r+3]} & \dots & Q_{r-1}^{[r-1]} \\ Q_r^{[-r+1]} & Q_r^{[-r+3]} & \dots & Q_r^{[r-1]} \end{vmatrix} \times \begin{vmatrix} Q_1^{[-r+1]} & Q_1^{[-r+3]} & \dots & Q_1^{[r-1]} \\ \vdots & \vdots & & \vdots \\ Q_{r-1}^{[-r+1]} & Q_{r-1}^{[-r+3]} & \dots & Q_{r-1}^{[r-1]} \\ Q_{r'}^{[-r+1]} & Q_{r'}^{[-r+3]} & \dots & Q_{r'}^{[r-1]} \end{vmatrix}. \quad (\text{C.2.1})$$

Using the explicit expression of $C_{0,r-1,r-1}$ gives

$$W_{1,\dots,r-1}^{[-]} W_{1,\dots,r-1}^{[+]} Q_\emptyset^{[r-2]} Q_\emptyset^{[2-r]} = \frac{1}{\prod_{j=1}^{r-1} (u_j - u_r)} W_{1,\dots,r-1,r} W_{1,\dots,r-1,r'}, \quad (\text{C.2.2})$$

where we used the notation $W_{i_1,\dots,i_k} = \left| Q_{\{i_a\}}^{[k+1-2b]} \right|_k = \det_{1 \leq a,b \leq k} \left(Q_{\{i_a\}}^{[k+1-2b]} \right)$. The derivation makes it clear that the previous identity still holds if one exchanges some $Q_{\{i\}}$ with $Q_{\{i'\}}$, so that one may actually write

$$W_{i_1,\dots,i_{r-1}}^{[-]} W_{i_1,\dots,i_{r-1}}^{[+]} Q_\emptyset^{[r-2]} Q_\emptyset^{[2-r]} = \frac{1}{\prod_{j \neq i_r, i'_r}^r (u_j - u_r)} W_{i_1,\dots,i_{r-1},i_r} W_{i_1,\dots,i_{r-1},i'_r}, \quad (\text{C.2.3})$$

where we only assume that for all $1 \leq a \neq b \leq r$, one has $\{i_a, i'_a\} \cap \{i_b, i'_b\} = \emptyset$. This is exactly equation (6.3.20).

C.2.2 Proof of Equation (6.5.28)

We prove here the following claim: if $T_{a,s}$ satisfy the Hirota equations and $T_{1,s}$ is given by equation (6.5.15), then $T_{a,s}$ for $a \leq r-1$ is given by equation (6.5.28).

The proof is made by induction: the claim is true for $a = 1$ by assumption, and we have also shown, in the main text, that it is true for $a = 2$. For higher a , the claim is clearly equivalent to equation (6.5.29), which is itself a particular case of the identity

$$\begin{aligned} & \frac{1}{2} \sum_{\substack{1 \leq i_1 < \dots < i_a \leq 2r \\ 1 \leq j_1 < \dots < j_a \leq 2r}} \left| \begin{array}{cc} W_{i_1, \dots, i_a}^{[+1]} & W_{i_1, \dots, i_a}^{[-1]} \\ W_{j_1, \dots, j_a}^{[+1]} & W_{j_1, \dots, j_a}^{[-1]} \end{array} \right| \left| \begin{array}{cc} \widetilde{W}_{i_1, \dots, i_a}^{[+1]} & \widetilde{W}_{i_1, \dots, i_a}^{[-1]} \\ \widetilde{W}_{j_1, \dots, j_a}^{[+1]} & \widetilde{W}_{j_1, \dots, j_a}^{[-1]} \end{array} \right| \\ &= \left(\sum_{1 \leq i_1 < \dots < i_{a-1} \leq 2r} W_{i_1, \dots, i_{a-1}} \widetilde{W}_{i_1, \dots, i_{a-1}} \right) \left(\sum_{1 \leq i_1 < \dots < i_{a+1} \leq 2r} W_{i_1, \dots, i_{a+1}} \widetilde{W}_{i_1, \dots, i_{a+1}} \right), \quad (\text{C.2.4}) \end{aligned}$$

where $W_{i_1, \dots, i_a} = \left| Q_{i_j}^{[a+1-2k]} \right|_a$ and $\widetilde{W}_{i_1, \dots, i_a} = \left| P_{i_j}^{[a+1-2k]} \right|_a$ for $\{Q_i\}_{1 \leq i \leq 2r}$ and $\{P_i\}_{1 \leq i \leq 2r}$ two sets of arbitrary functions. In this appendix, most of the summation indices run from 1 to $2r$ so we will not write these bounds under each summation symbols in the following. The only indices for which it will be different will be called m , n , \tilde{m} or \tilde{n} , the values they may take will be indicated each time.

We shall now prove (C.2.4). Let us start from the left-hand side, we expand each of the determinants W and \widetilde{W} with respect to the columns with shifts $\pm a$, for instance: $W_{i_1, \dots, i_a}^{[+1]} = \sum_{m=1}^a (-1)^{m+1} Q_{i_m}^{[a]} W_{i_1, \dots, \widehat{i_m}, \dots, i_a}$ and $W_{j_1, \dots, j_a}^{[-1]} = \sum_{n=1}^a (-1)^{n+a} Q_{j_n}^{[-a]} W_{j_1, \dots, \widehat{j_n}, \dots, j_a}$, where the hat over an index means that we omit it. We thus obtain

$$\begin{aligned} & \sum_{\substack{i_1 < \dots < i_a \\ j_1 < \dots < j_a}} \left| \begin{array}{cc} W_{i_1, \dots, i_a}^{[+1]} & W_{i_1, \dots, i_a}^{[-1]} \\ W_{j_1, \dots, j_a}^{[+1]} & W_{j_1, \dots, j_a}^{[-1]} \end{array} \right| \left| \begin{array}{cc} \widetilde{W}_{i_1, \dots, i_a}^{[+1]} & \widetilde{W}_{i_1, \dots, i_a}^{[-1]} \\ \widetilde{W}_{j_1, \dots, j_a}^{[+1]} & \widetilde{W}_{j_1, \dots, j_a}^{[-1]} \end{array} \right| \\ &= \frac{1}{(a!)^2} \sum_{\substack{i_1, \dots, i_a \\ j_1, \dots, j_a}} \left| \begin{array}{cc} W_{i_1, \dots, i_a}^{[+1]} & W_{i_1, \dots, i_a}^{[-1]} \\ W_{j_1, \dots, j_a}^{[+1]} & W_{j_1, \dots, j_a}^{[-1]} \end{array} \right| \left| \begin{array}{cc} \widetilde{W}_{i_1, \dots, i_a}^{[+1]} & \widetilde{W}_{i_1, \dots, i_a}^{[-1]} \\ \widetilde{W}_{j_1, \dots, j_a}^{[+1]} & \widetilde{W}_{j_1, \dots, j_a}^{[-1]} \end{array} \right| \\ &= \frac{1}{(a!)^2} \sum_{\substack{i_1, \dots, i_a \\ j_1, \dots, j_a \\ 1 \leq m, n, \tilde{m}, \tilde{n} \leq a}} (-1)^{m+n+\tilde{m}+\tilde{n}} \left| \begin{array}{cc} Q_{i_m}^{[a]} & Q_{i_m}^{[-a]} \\ Q_{j_n}^{[a]} & Q_{j_n}^{[-a]} \end{array} \right| \left| \begin{array}{cc} P_{j_{\tilde{n}}}^{[a]} & P_{j_{\tilde{n}}}^{[-a]} \\ P_{i_{\tilde{m}}}^{[a]} & P_{i_{\tilde{m}}}^{[-a]} \end{array} \right| \\ & \quad \times W_{i_1, \dots, \widehat{i_m}, \dots, i_a} W_{j_1, \dots, \widehat{j_n}, \dots, j_a} \widetilde{W}_{i_1, \dots, \widehat{i_{\tilde{m}}}, \dots, i_a} \widetilde{W}_{j_1, \dots, \widehat{j_{\tilde{n}}}, \dots, j_a} = L_1 + L_2 + L_3 \quad (\text{C.2.5}) \end{aligned}$$

where we have split the sums over m , n , \tilde{m} and \tilde{n} into three contributions: L_1 , L_2 and L_3 . L_1 contains all the terms with $m = \tilde{m}$ and $n = \tilde{n}$, L_2 all the terms with $m = \tilde{m}$ and $n \neq \tilde{n}$ or $m \neq \tilde{m}$ and $n = \tilde{n}$, while L_3 contains all the terms with $m \neq \tilde{m}$ and $n \neq \tilde{n}$. In each of the three cases, the remaining sums (over i 's and j 's) do not depend on the actual values of m , n , \tilde{m} and \tilde{n} anymore so that we can perform the sums over these latter indices. We thus get

$$L_1 = \left(\frac{1}{(a-1)!} \sum_{i_1, \dots, i_{a-1}} W_{i_1, \dots, i_{a-1}} \widetilde{W}_{i_1, \dots, i_{a-1}} \right)^2 \sum_{i, j} \left| \begin{array}{cc} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{array} \right| \left| \begin{array}{cc} P_i^{[a]} & P_i^{[-a]} \\ P_j^{[a]} & P_j^{[-a]} \end{array} \right|, \quad (\text{C.2.6})$$

$$L_2 = \frac{2}{(a-2)!} \left(\frac{1}{(a-1)!} \sum_{i_1, \dots, i_{a-1}} W_{i_1, \dots, i_{a-1}} \widetilde{W}_{i_1, \dots, i_{a-1}} \right) \\ \times \left(- \sum_{i, j, k, i_1, \dots, i_{a-2}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_i^{[a]} & P_i^{[-a]} \\ P_k^{[a]} & P_k^{[-a]} \end{vmatrix} W_{k, i_1, \dots, i_{a-2}} \widetilde{W}_{j, i_1, \dots, i_{a-2}} \right), \quad (\text{C.2.7})$$

$$L_3 = \frac{1}{((a-2)!)^2} \sum_{\substack{i_1, \dots, i_{a-2} \\ j_1, \dots, j_{a-2} \\ i, j, k, l}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} \\ \times W_{k, i_1, \dots, i_{a-2}} W_{l, j_1, \dots, j_{a-2}} \widetilde{W}_{i, i_1, \dots, i_{a-2}} \widetilde{W}_{j, j_1, \dots, j_{a-2}}. \quad (\text{C.2.8})$$

One can rewrite L_3 using the Plücker identity (6.1.14). We first use it to write

$$W_{k, i_1, \dots, i_{a-2}} W_{l, j_1, \dots, j_{a-1}} = W_{l, i_1, \dots, i_{a-2}} W_{k, j_1, \dots, j_{a-2}} + \sum_{p=1}^{a-2} (-1)^{p-1} W_{k, l, i_1, \dots, \widehat{i}_p, \dots, i_{a-2}} W_{i_p, j_1, \dots, j_{a-2}}, \quad (\text{C.2.9})$$

which we then plug in the expression for L_3 . After some renaming of the indices, this yields

$$L_3 = -L_3 + \frac{1}{(a-3)!(a-2)!} \sum_{\substack{i_1, \dots, i_{a-3} \\ j_1, \dots, j_{a-1} \\ i, j, k, l}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} \\ \times W_{k, l, i_1, \dots, i_{a-3}} W_{j_1, \dots, j_{a-1}} \widetilde{W}_{i, j_1, i_1, \dots, i_{a-3}} \widetilde{W}_{j, j_2, \dots, j_{a-1}}. \quad (\text{C.2.10})$$

This means that

$$L_3 = \frac{1}{2(a-3)!(a-2)!} \sum_{\substack{i_1, \dots, i_{a-3} \\ j_1, \dots, j_{a-1} \\ i, j, k, l}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} \\ \times W_{k, l, i_1, \dots, i_{a-3}} W_{j_1, \dots, j_{a-1}} \widetilde{W}_{i, j_1, i_1, \dots, i_{a-3}} \widetilde{W}_{j, j_2, \dots, j_{a-1}}. \quad (\text{C.2.11})$$

We now apply again the Plücker identity:

$$\widetilde{W}_{i, j_1, i_1, \dots, i_{a-3}} \widetilde{W}_{j, j_2, \dots, j_{a-1}} = \widetilde{W}_{i, j, i_1, \dots, i_{a-3}} \widetilde{W}_{j_1, j_2, \dots, j_{a-1}} \\ + \sum_{p=2}^{a-1} (-1)^p \widetilde{W}_{i, j_p, i_1, \dots, i_{a-3}} \widetilde{W}_{j, j_1, j_2, \dots, \widehat{j}_p, \dots, j_{a-1}} \quad (\text{C.2.12})$$

so that

$$L_3 = \frac{1}{2(a-3)!(a-2)!} \sum_{\substack{i_1, \dots, i_{a-3} \\ j_1, \dots, j_{a-1} \\ i, j, k, l}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} \\ \times W_{k, l, i_1, \dots, i_{a-3}} W_{j_1, \dots, j_{a-1}} \widetilde{W}_{i, j, i_1, \dots, i_{a-3}} \widetilde{W}_{j_1, \dots, j_{a-1}} - (a-2)L_3. \quad (\text{C.2.13})$$

Finally, we arrive at the following expression:

$$L_3 = \frac{1}{2(a-3)!} \left(\frac{1}{(a-1)!} \sum_{i_1, \dots, i_{a-1}} W_{i_1, \dots, i_{a-1}} \widetilde{W}_{i_1, \dots, i_{a-1}} \right) \\ \times \left(\sum_{i, j, k, l, i_1, \dots, i_{a-3}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} W_{k, l, i_1, \dots, i_{a-3}} \widetilde{W}_{i, j, i_1, \dots, i_{a-3}} \right). \quad (\text{C.2.14})$$

In order to prove (C.2.4), we need to show that

$$\frac{L_1 + L_2 + L_3}{2} = \left(\frac{1}{(a-1)!} \sum_{i_1, \dots, i_{a-1}} W_{i_1, \dots, i_{a-1}} \widetilde{W}_{i_1, \dots, i_{a-1}} \right) \\ \times \left(\frac{1}{(a+1)!} \sum_{i_1, \dots, i_{a+1}} W_{i_1, \dots, i_{a+1}} \widetilde{W}_{i_1, \dots, i_{a+1}} \right). \quad (\text{C.2.15})$$

From expressions (C.2.6), (C.2.7), and (C.2.14), this is equivalent to showing that

$$\frac{(a+1)a}{2} \sum_{i, j, i_1, \dots, i_{a-1}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_i^{[a]} & P_i^{[-a]} \\ P_j^{[a]} & P_j^{[-a]} \end{vmatrix} W_{i_1, \dots, i_{a-1}} \widetilde{W}_{i_1, \dots, i_{a-1}} \\ - (a+1)a(a-1) \sum_{i, j, k, i_1, \dots, i_{a-2}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_i^{[a]} & P_i^{[-a]} \\ P_k^{[a]} & P_k^{[-a]} \end{vmatrix} W_{k, i_1, \dots, i_{a-2}} \widetilde{W}_{j, i_1, \dots, i_{a-2}} \\ + \frac{(a+1)a(a-1)(a-2)}{4} \sum_{i, j, k, l, i_1, \dots, i_{a-3}} \begin{vmatrix} Q_i^{[a]} & Q_i^{[-a]} \\ Q_j^{[a]} & Q_j^{[-a]} \end{vmatrix} \begin{vmatrix} P_k^{[a]} & P_k^{[-a]} \\ P_l^{[a]} & P_l^{[-a]} \end{vmatrix} W_{k, l, i_1, \dots, i_{a-3}} \widetilde{W}_{i, j, i_1, \dots, i_{a-3}} \\ = \sum_{i_1, \dots, i_{a+1}} W_{i_1, \dots, i_{a+1}} \widetilde{W}_{i_1, \dots, i_{a+1}}. \quad (\text{C.2.16})$$

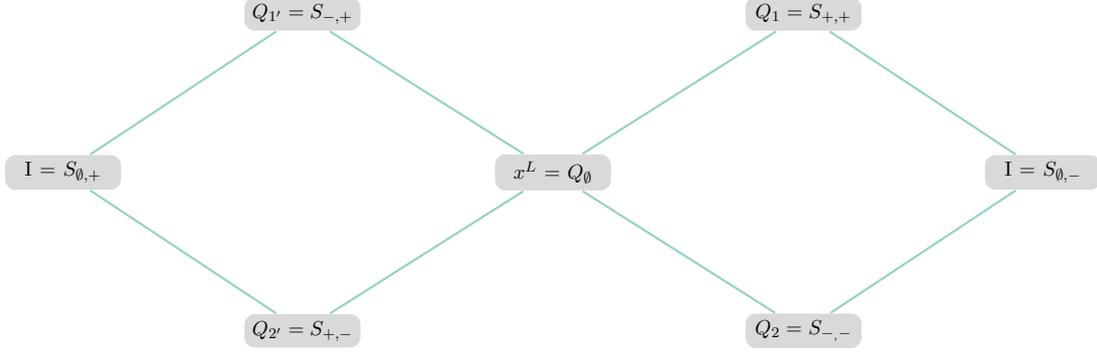
This last identity is proven by expanding the determinants in the right-hand side with respect to their first and last columns:

$$W_{i_1, \dots, i_{a+1}} = \sum_{1 \leq m < n \leq a+1} (-1)^{m+n+a} \begin{vmatrix} Q_{i_m}^{[a]} & Q_{i_m}^{[-a]} \\ Q_{i_n}^{[a]} & Q_{i_n}^{[-a]} \end{vmatrix} W_{i_1, \dots, \widehat{i_m}, \dots, \widehat{i_n}, \dots, i_{a+1}} \quad (\text{C.2.17})$$

and

$$\widetilde{W}_{i_1, \dots, i_{a+1}} = \sum_{1 \leq \tilde{m} < \tilde{n} \leq a+1} (-1)^{\tilde{m} + \tilde{n} + a} \begin{vmatrix} P_{i_{\tilde{m}}}^{[a]} & P_{i_{\tilde{m}}}^{[-a]} \\ P_{i_{\tilde{n}}}^{[a]} & P_{i_{\tilde{n}}}^{[-a]} \end{vmatrix} \widetilde{W}_{i_1, \dots, \widehat{i_{\tilde{m}}}, \dots, \widehat{i_{\tilde{n}}}, \dots, i_{a+1}}. \quad (\text{C.2.18})$$

We then once again group the terms depending on the values of m , n , \tilde{m} and \tilde{n} and recover exactly the identity (C.2.16). There are indeed $\frac{(a+1)a}{2}$ terms with $(m, n) = (\tilde{m}, \tilde{n})$, $(a+1)a(a-1)$ terms with $m = \tilde{m}$ and $n \neq \tilde{n}$ or $m \neq \tilde{m}$ and $n = \tilde{n}$, and $\frac{(a+1)a(a-1)(a-2)}{4}$ terms with $m \neq \tilde{m}$ and $n \neq \tilde{n}$.

Figure C.1: Hasse diagram for $D_2 \simeq A_1 \oplus A_1$

C.3 More on Weyl-Type Formulae

C.3.1 From QQ'-Relations to Weyl-Type Formulae for $D_2 \simeq A_1 \oplus A_1$

We demonstrate here, for the very simple example of D_2 spin chains, how to use the QQ'-relations to recover the Weyl-type formulae for T-functions. The Hasse diagram is depicted in Figure C.1.

From the two constraints

$$Q_1 Q_{1'} + Q_2 Q_{2'} = 0, \quad (\text{C.3.1})$$

and

$$Q_1^{[1]} Q_{1'}^{[-1]} + Q_1^{[-1]} Q_{1'}^{[1]} + Q_2^{[1]} Q_{2'}^{[-1]} + Q_2^{[-1]} Q_{2'}^{[1]} = Q_0^2, \quad (\text{C.3.2})$$

cf. (6.5.16), we obtain

$$\left(\frac{Q_{1'}^{[1]}}{Q_2^{[1]}} - \frac{Q_{1'}^{[-1]}}{Q_2^{[-1]}} \right) = \frac{Q_0^2}{W_1}, \quad (\text{C.3.3})$$

where $W_n = Q_1^{[n]} Q_2^{[-n]} - Q_1^{[-n]} Q_2^{[n]}$.

Moreover, excluding $Q_{2'}$ from

$$T_{1,s} = Q_1^{[s+1]} Q_{1'}^{[-s-1]} + Q_1^{[-s-1]} Q_{1'}^{[s+1]} + Q_2^{[s+1]} Q_{2'}^{[-s-1]} + Q_2^{[-s-1]} Q_{2'}^{[s+1]}, \quad (\text{C.3.4})$$

we get

$$T_{1,s} = \left(\frac{Q_{1'}^{[s+1]}}{Q_2^{[s+1]}} - \frac{Q_{1'}^{[-s-1]}}{Q_2^{[-s-1]}} \right) W_{s+1}. \quad (\text{C.3.5})$$

Excluding the difference in the first bracket in the right-hand side using (C.3.3), we arrive at

$$T_{1,s} = W_{s+1} \sum_{l=0}^s \frac{(Q_0^{[2l-s]})^2}{W_1^{[2l-s]}}. \quad (\text{C.3.6})$$

This coincides with the $r = 2$ case of the determinant formula (6.4.28).

C.3.2 Additional Formulae in the General Case

For the sake of completeness, we give here the Weyl-type formulae complementary to those given in Section 6.4, i.e. for $T_{1,s}$ in terms of spinorial Q-functions:

$$T_{1,s} = Q_{\emptyset}^{[r+s-2]} Q_{\emptyset}^{[2-r-s]} \left(\sum_{l=0}^s \frac{S_{I_r}^{[-s-1]} S_{I_r}^{[s+1]}}{S_{I_r}^{[2l-s-1]} S_{I_r}^{[2l-s+1]}} \frac{|S_i^{[-(2r+s-1-2j-2l)\delta_{j,r}]}|_r |S_i^{[2r+s-1-2j-2(s-l)\delta_{j,r}]}|_r}{|S_i^{[-(2r+s-1-2j)]}|_r |S_i^{[2r+s-1-2j]}|_r} \right. \\ \left. - \sum_{l=1}^{s-1} \frac{S_{I_r}^{[-s-1]} S_{I_r}^{[s+1]}}{S_{I_r}^{[2l-s-1]} S_{I_r}^{[2l-s+1]}} \frac{|S_i^{[-(2r+s-1-2j-2(l-1)\delta_{j,r}]}|_r |S_i^{[2r+s-1-2j-2(s-1-l)\delta_{j,r}]}|_r}{|S_i^{[-(2r+s-1-2j)]}|_r |S_i^{[2r+s-1-2j]}|_r} \right), \quad (\text{C.3.7})$$

and for $T_{a,1}$ in terms of single-index Q-functions:

$$T_{a,1} = \frac{1}{\prod_{k=1}^{a-1} Q_{\emptyset}^{[r-a+2k]} Q_{\emptyset}^{[-(r-a+2k)]}} \sum_{\substack{0 \leq k, l \leq a \\ a-k-l \in 2\mathbb{N}}} \left(\prod_{m=1}^k Q_{\emptyset}^{[r+a-2m]} Q_{\emptyset}^{[4+a-r-2m]} \right) \\ \times \frac{|Q_{\{i\}}^{[r+a+3-2k-2j-2\theta(j+k-r-1)]}|_r^{\frac{a-k-l}{2}}}{|Q_{\{i\}}^{[r+a+3-2k-2j]}|_r} \prod_{m=1}^{\frac{a-k-l}{2}} \left(Q_{\emptyset}^{[r+a-2k-4m]} Q_{\emptyset}^{[4+a-r-2k-4m]} \right)^2 \\ \times \frac{|Q_{\{i\}}^{[r+a+1-2k-2j]}|_r |Q_{\{i\}}^{[r+a-3-2k-2j]}|_r^2 \dots |Q_{\{i\}}^{[2l+r+5-a-2j]}|_r^2 |Q_{\{i\}}^{[2l+r+1-a-2j]}|_r}{|Q_{\{i\}}^{[r+a-1-2k-2j]}|_r^2 \dots |Q_{\{i\}}^{[2l+r+3-a-2j]}|_r^2} \\ \times \left(\prod_{m=1}^l Q_{\emptyset}^{[2+2l-r-a-2m]} Q_{\emptyset}^{[2l+r-a-2m-2]} \right) \frac{|Q_{\{i\}}^{[-(r+a+3-2l-2j-2\theta(j+l-r-1))]}|_r}{|Q_{\{i\}}^{[-(r+a+3-2l-2j)]}|_r}. \quad (\text{C.3.8})$$

Notice that the first formula, expressing $T_{1,s}$ in terms of spinorial Q functions, is much more complicated than the expression (6.4.28) in terms of fundamental Q functions, whereas the second formula, expressing $T_{a,1}$ in terms of fundamental Q functions, is more complicated than (6.4.35) expressing it in terms of spinorial Q functions.

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RÉSUMÉ

Avant d'être étendue à toute dimension, la théorie fishnet a d'abord été obtenue en quatre dimensions comme une limite de fort twist et de faible couplage de la théorie $\mathcal{N} = 4$ super Yang–Mills. C'est une théorie non-unitaire de deux champs scalaires matriciels complexes interagissant d'une manière si restrictive que, dans la limite planaire, très peu de graphes de Feynman survivent. Il est alors possible de montrer que la théorie est conforme. En outre, l'intégrabilité apparaît naturellement à travers une relation avec une chaîne non-compacte de spins dans une représentation de la série principale du groupe conforme. Certaines classes de graphes de Feynman peuvent en effet être obtenues par l'application répétée d'opérateurs coïncidant avec des charges conservées de la chaîne de spins. La théorie fishnet constitue ainsi un exemple simple et rare d'une théorie conforme des champs intégrable sans supersymétrie en dimension arbitraire. Nous présentons, dans cette thèse, la diagonalisation exacte d'opérateurs associés à la chaîne de spins ouverte. Les vecteurs propres ont une interprétation en tant que fonctions d'onde d'états à plusieurs particules dans une théorie unidimensionnelle miroir. La détermination de la relation de dispersion et de la matrice de diffusion de ces particules miroir nous permet de formuler les équations de l'ansatz de Bethe thermodynamique pour les dimensions conformes d'une famille d'opérateurs dans la théorie fishnet. Dans l'intention de simplifier davantage ce problème spectral nous développons le système Q pour des modèles intégrables avec une symétrie $SO(2r)$. Nous obtenons aussi, pour de tels modèles, de nouvelles expressions pour les matrices de transfert en termes des fonctions Q, quantifiant ainsi les formules classiques de Weyl pour les caractères.

MOTS CLÉS

Intégrabilité, Théorie conforme des champs, Chaînes de spins

ABSTRACT

The fishnet theory was first obtained in four dimensions as a strongly twisted, weakly coupled limit of $\mathcal{N} = 4$ super Yang–Mills before being extended to arbitrary dimension. It is a non-unitary theory of two complex matrix scalar fields interacting in such a manner that, in the planar limit, only very few Feynman graphs are allowed and, moreover, the bulk of these graphs must be a piece of a square lattice. As a consequence, the theory can be shown to be conformal and integrability naturally appears through a relation with a non-compact chain of spins in principal series representations of the conformal group. Certain classes of Feynman graphs can indeed be built from the repeated application of operators coinciding with conserved charges of the chain. The fishnet theory thus constitutes a rare and simple example of an integrable non-supersymmetric conformal field theory in arbitrary dimension. We present, in this thesis, the exact diagonalisation of the graph-building operators associated with the open spin chain. The eigenvectors have an interpretation as wave functions of multi-particle states in a mirror one-dimensional theory. Extracting the dispersion relation and the scattering matrix of these mirror particles allow us to formulate the thermodynamic Bethe ansatz equations for the conformal dimensions of a whole class of operators in the fishnet theory. As a first step towards a further simplification of this spectral problem, we develop the Q-system for integrable models with $SO(2r)$ symmetry. We also obtain, for such models, new expressions for the transfer matrices, or T-functions, in terms of the Q-functions, thus quantising the classical Weyl formulae for characters.

KEYWORDS

Integrability, Conformal Field Theory, Spin Chains