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Lohan Sartore

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Pour obtenir le grade de

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**Théories multi-échelles au-delà de l'ordre dominant et groupe de renormalisation : Théorie et applications**

**Multi-scale theories beyond tree-level and the renormalisation group: Theory and applications**

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# Abstract

Since the discovery of the Higgs boson, the field of high energy physics has entered a precision era. The absence of clear deviations from the Standard Model predictions calls for a reassessment and overall improvement of the frameworks and practical methods used to put models of new physics to the test. We report in this thesis the development of novel methods and tools dedicated to the study of models with multiple mass scales. In part I, we demonstrate that, using renormalisation group techniques, the logarithmic contributions stemming from the regularisation of ultraviolet divergences can be systematically resummed if one supplements the  $\overline{\text{MS}}$  scheme with multiple renormalisation scales. This method is in principle applicable to any general gauge theory and appears as a powerful tool to enhance the predictivity and overall insight into the structure of a large class of models of new physics. The only prerequisite is the knowledge of the beta-functions of the model, which can be obtained using the computer tool PyR@TE 3, presented in this thesis. Part II is dedicated to the study of two specific multi-scale extensions of the Standard Model. First, we develop a framework allowing to describe all interactions in the general Two-Higgs Doublet model in terms of gauge-invariant quantities. Second, we show that a novel kind of constraints on the scalar potential of Grand Unified theories can be derived from the requirement of proper symmetry breaking towards the Standard Model gauge group. These findings are exemplified in the context of a classically scale-invariant  $SO(10)$  model.

# Résumé

Depuis la découverte du boson de Higgs, la physique des hautes énergies est entrée dans l'ère de la précision. L'absence de nettes déviations par rapport aux prédictions du Modèle Standard nous amène à reconsidérer et perfectionner le cadre théorique et les méthodes utilisés pour mettre à l'épreuve les modèles de nouvelle physique. Dans cette thèse, nous présentons le développement de nouveaux outils et méthodes dédiés à l'étude de modèles impliquant plusieurs échelles de masse. En partie I, nous démontrons la possibilité, au travers des propriétés du groupe de renormalisation, de resommer systématiquement les corrections logarithmiques liées à la régularisation des divergences ultraviolettes en incluant dans le schéma  $\overline{\text{MS}}$  plusieurs échelles de renormalisation. Cette méthode est en principe applicable à toute théorie de jauge et ouvre la voie à une prédictivité accrue et une compréhension plus fine de la structure d'une large classe de modèles de nouvelle physique. Il suffit pour cela de disposer des fonctions bêta pour les couplages du modèle, calculables en pratique grâce au logiciel PyR@TE 3, présenté dans cette thèse. La seconde partie est dédiée à l'étude de deux extensions spécifiques du Modèle Standard. Nous présentons, dans un premier temps, un nouveau formalisme pour décrire l'ensemble des interactions dans le modèle à deux doublets de Higgs en termes de quantités invariants de jauge. Enfin, nous introduisons un nouveau type de contraintes sur le potentiel scalaire des théories de grande unification, résultant d'un schéma de brisure de symétrie compatible avec le groupe de jauge du Modèle Standard. Ces résultats sont illustrés au travers de l'exemple d'un modèle  $SO(10)$  classiquement invariant d'échelle.



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# Introduction

The second half of the past century was a golden age for particle physics. It has been marked by outstanding progress in the development of quantum field theory, together with a largely successful experimental programme designed to probe the subatomic world at increasingly high precision and energy. The understanding of fundamental interactions as stemming from local symmetries alongside the study of their spontaneous breaking has led to the development of a unified theory of electromagnetic and weak interactions. Together with quantum chromodynamics, a consistent description of all three fundamental interactions based on the gauge group  $SU(3)_C \times SU(2)_L \times U(1)_Y$  constitutes the Standard Model of particle physics. Among many other achievements, the discovery in the early 80's of the bosons W and Z [1, 2, 3, 4], followed after decades by the measurement of the Higgs boson at the LHC in 2012 [5, 6] have proven the robustness and the predictive power of this theoretical framework.

There are, on the other hand, reasons to believe that the Standard Model only provides an effective description of particle physics phenomena, and emerges as the low-energy limit of a more fundamental theory: for instance, the absence of neutrino mass terms or the inadequate amount of CP violation needed to explain the observed asymmetry between matter and antimatter are, in this regard, often put forward. While the conviction that new physics would lie around the TeV scale has prevailed in the past decades, the absence of clear deviations from the Standard Model predictions has led the field of particle physics to a precision era. Beyond the paradigm of TeV-scale supersymmetry, numerous scenarios of new physics have emerged as many possible solutions to the shortcomings of the Standard Model. The proliferation of such models and their possibly complicated structure calls for an improvement of the presently available methods, in a quest for precision and predictivity.

The content of this thesis is overall related to the study of a general class of extensions of the Standard Model, in which the scalar sector is extended with new degrees of freedom – which we shall refer to as multi-scalar extensions of the Standard Model. Such models are in many aspects phenomenologically relevant, yet largely unconstrained by experiment. In this context, it is crucial to predict to the highest level of precision the phenomenological implications stemming from these additional scalar degrees of freedom. A promising direction in this regard is the analysis, beyond leading-order, of the perturbative structure of multi-scalar theories. Besides a valuable increase in precision for quantitative predictions, some scalar theories may exhibit qualitatively new features beyond tree-level. Classically scale-invariant theories are such an example, in which spontaneous symmetry breaking is triggered by the quantum corrections to the scalar potential. In addition, the complexity of the calculations in multi-scalar models, oftentimes already at tree-level, calls for an overall reassessment and improvement of the methods from a theoretical, computational and algorithmic point of view. These various aspects are covered in this thesis, where we present a number of original results.

After providing in chapter 1 an overview of the renormalisation procedure in general scalar theories and introducing the perturbative renormalisation group, we discuss in chapter 2 the practical computation of beta-functions in general gauge theories, their extension to dimensionful couplings [10], and their implementation in a public computer code, PyR@TE 3 [9]. In the subsequent chapters, we discuss one of the most powerful applications of the renormalisation group, allowing to resum – to all orders in perturbation theory – the logarithmic contributions stemming from the regularisation of ultraviolet divergences. First, we focus in chapter 3 on theories containing a single scalar field. A number of crucial results are derived, and a novel formalism is established to perform the resummation of all logarithmic contributions. These results are extended in chapter 4 to general multi-scalar theories, where we propose a generalisation of the traditional  $\overline{\text{MS}}$  scheme allowing for multiple renormalisation scales. In this framework, we demonstrate that the results previously obtained in single-scale theories can be extended, in a straightforward manner, to any general theory with interacting scalar fields. This will constitute the main result of part I of this thesis.

Part II is based on the publications [7] and [8], where we approach from a novel perspective two well-motivated extensions of the Standard Model.

- In chapter 5, based on [7], we explore the structure of one of the most studied extensions of the Standard Model, where a second Higgs doublet is added in the scalar sector – the Two-Higgs Doublet model (THDM). It has been previously shown that the scalar potential in the THDM can be expressed in terms of gauge invariant operators (the bilinear fields) that transform covariantly under unitary mixing of the doublets. Based on these bilinear fields, a compact and elegant formalism has been developed, allowing to carry out in a very efficient manner the study of CP symmetry, electroweak symmetry breaking and stability of the potential. We show in chapter 5 that such a formalism can be extended beyond the scalar sector, allowing for a fully gauge invariant description of the interactions among scalars, gauge bosons and fermions, and we derive the full set of corresponding vertices at leading order.
- In chapter 6, based on [8], we study the mechanism of radiative symmetry breaking in a Grand Unified Theory (GUT) based on the  $SO(10)$  gauge group. We show that the consideration of the breaking patterns of the  $SO(10)$  symmetry group which do not lead to the Standard Model provides a novel kind of constraint on the parameters of the scalar sector. We propose a general method, applicable in practice to any GUT, allowing to identify regions of the scalar parameter space which cannot lead to proper breakdown towards the Standard Model, hence restricting the phenomenologically allowed parameter space. In this study, scale invariance is imposed at the level of the tree-level scalar potential, so that the mechanism of spontaneous symmetry breaking is triggered by the renormalisation group evolution of the quartic couplings. Possible extensions of our results as well as implications in the context of theories of quantum gravity are discussed.

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## List of publications

- [7] L. Sartore, M. Maniatis, I. Schienbein and B. Herrmann, *The general Two-Higgs Doublet Model in a gauge-invariant form*, *JHEP* **12** (2022) 051 [2208.13719]
- [8] A. Held, J. Kwapisz and L. Sartore, *Grand unification and the Planck scale: an  $SO(10)$  example of radiative symmetry breaking*, *JHEP* **08** (2022) 122 [2204.03001]
- [9] L. Sartore and I. Schienbein, *PyR@TE 3*, *Comput. Phys. Commun.* **261** (2021) 107819 [2007.12700]
- [10] L. Sartore, *General renormalization group equations for dimensionful couplings in the  $\overline{\text{MS}}$  scheme*, *Phys. Rev. D* **102** (2020) 076002 [2006.12307]
- [11] M. Maniatis, L. Sartore and I. Schienbein, *Multiple point principle in the general Two-Higgs-Doublet model*, *JHEP* **08** (2020) 158 [2001.10541]

## Part I

# The renormalisation group



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# Chapter 1

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## Perturbative renormalisation in multi-scalar theories

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This chapter is an introduction to the elementary concepts of renormalisation in quantum field theory (QFT), through the application of the dimensional regularisation (DR) procedure [12, 13]. In particular, sections 1 to 3 consist in a generalisation of the usual textbook approach to dimensional regularisation (see e.g. [14, 15]), in a general theory with  $N$  interacting scalars. In this context, we shall establish a number of definitions and central results which will allow us to (i) extend the discussion to general gauge theories in chapter 2, and (ii) explore in a novel approach the perturbative structure of general scalar theories in chapters 3 and 4.

We establish in section 1 the formalism and the main definitions on which we shall rely throughout this thesis. We explicitly carry out in section 2 the regularisation of the theory up to the two-loop level. The results hence obtained will be useful to illustrate the discussion on the beta-functions and the renormalisation group (RG) in sections 3 and 4. We discuss in section 5 the physically relevant case where the ground state of the theory lies away from the origin, giving rise to spontaneous evolution of scalar fields. We introduce for such theories the notion of effective potential, a central object whose properties determine the location of the ground state in the field space. Finally, we provide in section 6 two alternative renormalisation prescriptions, laying the basis for the generalisation of  $\overline{\text{MS}}$  renormalisation to multi-scale theories in chapter 4.

### 1 Definitions

Throughout this chapter, we will consider a 4-dimensional theory with  $N$  real scalars, gathered in the scalar multiplet  $\phi^a$  ( $a = 1, \dots, N$ ). Using a Minkowskian metric with signature  $(+, -, -, -)$ , the Lagrangian density is given by:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi^a - \frac{1}{2} M_{ab} \phi^a \phi^b - \frac{1}{4!} \lambda_{abcd} \phi^a \phi^b \phi^c \phi^d. \quad (1.1)$$

Physical observables such as cross-sections and decay rates can be computed from the one-particle irreducible (1PI) correlation functions (or  $n$ -point functions), most easily obtained in a perturbative, diagrammatic approach using the Feynman rules

$$\text{-----} = i [p^2 \mathbb{1} - M + i\epsilon \mathbb{1}]^{-1}, \quad (1.2)$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = -i \lambda_{abcd}, \quad (1.3)$$

where  $M$  stands for the  $N \times N$  matrix with components  $M_{ab}$ . Depending on the basis chosen to express the scalar fields and couplings, the propagator matrix is not necessarily diagonal. It is, however, manifestly diagonal in the basis where the mass matrix  $M$  is, referred to as the mass-basis in the following. Assuming

at this stage that every scalar quantity has been rotated to such a basis, we may write

$$----- = \frac{i\delta^{ab}}{p^2 - M_{ab} + i\epsilon}. \quad (1.4)$$

Beyond leading order, the  $n$ -point functions are decomposed as a perturbative series, most easily written in terms of loop-diagrams. For instance, the amputated<sup>1</sup> 4-point function  $\Gamma_{abcd}$  reads

$$\Gamma_{abcd}(p_i; M, \lambda) = \Gamma_{abcd}^{(0)}(p_i; M, \lambda) + \Gamma_{abcd}^{(1)}(p_i; M, \lambda) + \dots = i \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} + \left[ i \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \text{ with a loop } + \dots \right] + \dots, \quad (1.5)$$

and is formally a function of the couplings and external momenta. Loop-contributions such as the one depicted above may be formally infinite due to the integration over internal four-momenta. For instance, one encounters a logarithmic divergence when computing

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \sim \lambda_{abij} \lambda_{klcd} \int \frac{d^4 k}{(2\pi)^4} \frac{\delta^{ik}}{k^2 - M_{ik} + i\epsilon} \frac{\delta^{jl}}{k^2 - M_{jl} + i\epsilon} \sim \log \infty. \quad (1.6)$$

Therefore, with ill-defined correlation functions, the theory is unpredictable beyond leading-order and must be regularised. The standard approach consists in viewing the couplings and fields in the original Lagrangian (1.1) as unphysical quantities. As such, those can be formally infinite as long as the physical observables computed from them are well-behaved. To insist on the unphysical nature of the original couplings and fields, we re-parameterise the Lagrangian density as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_0^a \partial^\mu \phi_0^a - \frac{1}{2} M_{ab}^0 \phi_0^a \phi_0^b - \frac{1}{4!} \lambda_{abcd}^0 \phi_0^a \phi_0^b \phi_0^c \phi_0^d, \quad (1.7)$$

where  $\phi_0$ ,  $M^0$  and  $\lambda^0$  denote the *bare* (*i.e.* unphysical) fields and couplings. The next step to regularise the theory is to express the bare quantities as functions of renormalised (*i.e.* finite) couplings and fields, noted  $\phi$ ,  $M$  and  $\lambda$ . One possible way is to resort to dimensional regularisation (DR) [12, 13], where the dimension of the spacetime is shifted by a small but finite amount, noted  $\varepsilon$ . In  $d = 4 - \varepsilon$  dimensions, loop-integrals such as (1.6) are finite and scale as  $\mathcal{O}(\varepsilon^{-k})$  with  $k$  some positive integer. With a Lagrangian density of mass-dimension  $d = 4 - \varepsilon$ , the bare coupling and fields must satisfy

$$[\phi_0] = 1 - \frac{\varepsilon}{2}, \quad [M^0] = 2, \quad [\lambda^0] = \varepsilon, \quad (1.8)$$

so, in particular,  $\lambda^0$  is no longer a dimensionless quantity. We first define the field-strength renormalisation matrix  $Z$  relating the bare field multiplet to its renormalised counterpart, such that<sup>2</sup>

$$\phi_0 = Z\phi. \quad (1.9)$$

The formally infinite contributions to the couplings in the 4D theory are absorbed by counterterms, which are conventionally introduced as

$$M_{ab}^0 = M_{ab} + \delta M_{ab}, \quad \lambda_{abcd}^0 = \lambda_{abcd} + \delta \lambda_{abcd}, \quad (1.10)$$

where  $M$  and  $\lambda$  stand for the renormalised, finite couplings. Here, we propose a different (yet equivalent) prescription for the relation between the bare and renormalised quantities that conveniently allows to remove any occurrence of the field-strength renormalisation matrix in the non-kinetic Lagrangian. Namely, defining instead

$$M_{ab}^0 = (M_{ef} + \delta M_{ef}) Z_{ea}^{-1} Z_{fb}^{-1} \quad (1.11)$$

$$\lambda_{abcd}^0 = \mu^\varepsilon (\lambda_{efgh} + \delta \lambda_{efgh}) Z_{ea}^{-1} Z_{fb}^{-1} Z_{gc}^{-1} Z_{hd}^{-1}, \quad (1.12)$$

<sup>1</sup>If not otherwise stated, we will for simplicity use the term  $n$ -point function to refer to the amputated  $n$ -point functions, of which the external legs do not receive self-energy loop-corrections.

<sup>2</sup>For a single scalar field, the renormalisation strength constant is conventionally defined as  $Z^{1/2}$ . Here, with  $\phi$  an  $N$ -components multiplet, we rather choose to avoid the introduction of the square root of  $Z$ , which is only well defined if  $Z$  is a positive-definite matrix.

the Lagrangian density reads

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a (Z^\text{T} Z)^{ab} \partial^\mu \phi^b - \frac{1}{2} (M_{ab} + \delta M_{ab}) \phi^a \phi^b - \frac{\mu^\varepsilon}{4!} (\lambda_{abcd} + \delta \lambda_{abcd}) \phi^a \phi^b \phi^c \phi^d, \quad (1.13)$$

and the positive-definite matrix product  $Z^\text{T} Z$  is expanded in terms of the counterterm  $\delta_\phi$  according to

$$Z^\text{T} Z = \mathbb{1} + \delta_\phi. \quad (1.14)$$

An arbitrary energy scale  $\mu$  has been introduced in Eqs. (1.12) and (1.13) in order for the renormalised quartic couplings to remain dimensionless in  $d$  dimensions. Having expressed the Lagrangian density in terms of renormalised couplings and fields, the amputated  $n$ -point functions take the same form as previously (see e.g. (1.5)) but are now given as functions of the renormalised couplings and counterterms

$$\Gamma_{[n]}^R(p_i; M, \lambda) = \Gamma_{[n]}(p_i; M + \delta M, \lambda + \delta \lambda). \quad (1.15)$$

The theory can now be renormalised, requiring every  $\Gamma_{[n]}^R$  to be finite order by order in perturbation theory. For instance, the renormalised 2-point function (the inverse propagator) is given, at leading order, by

$$i\Gamma_{ab}^R(p^2; M) = i [p^2 \mathbb{1} - M - \delta M + p^2 \delta_\phi - i\epsilon \mathbb{1}]^{ab} + (\text{loops}). \quad (1.16)$$

Gathering the loop contributions in the matrix of scalar self-energies, noted  $\Pi(p^2)$ , the inverse propagator reads

$$i\Gamma_{ab}^R(p^2; M) = i [p^2 \mathbb{1} - M - i\epsilon \mathbb{1} - \Pi(p^2) - \delta M + p^2 \delta_\phi]^{ab}, \quad (1.17)$$

and must be finite order-by-order. Hence, the renormalised scalar self-energy,

$$\Pi_R(p^2) = \Pi(p^2) + \delta M - p^2 \delta_\phi, \quad (1.18)$$

must be finite, allowing to compute the expression of the field-strength and mass counterterms order-by-order in perturbation theory. In turn, the renormalised propagator reads

$$iG_R(p^2) = \frac{i}{p^2 \mathbb{1} - M + i\epsilon \mathbb{1} - \Pi_R(p^2)}, \quad (1.19)$$

and is expanded perturbatively as

$$iG_R(p^2) = \frac{i}{p^2 \mathbb{1} - M + i\epsilon \mathbb{1}} + \frac{i}{p^2 \mathbb{1} - M + i\epsilon \mathbb{1}} [-i\Pi_R(p^2)] \frac{i}{p^2 \mathbb{1} - M + i\epsilon \mathbb{1}} + \dots \quad (1.20)$$

Finally, we note that the field-strength renormalisation matrix can always be made symmetric by first expressing the polar decomposition of  $Z$ ,

$$Z = Z' O, \quad Z' = (Z Z^\text{T})^{1/2} = Z'^\text{T}, \quad O O^\text{T} = \mathbb{1}, \quad (1.21)$$

and by performing a re-definition of the renormalised couplings and fields. Namely, if the relation between the bare and renormalised scalar multiplet is rewritten as

$$\phi_0 = Z \phi = Z' O \phi = Z' \phi', \quad \phi' = O \phi, \quad (1.22)$$

the Lagrangian density reads

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi'^a Z'^2_{ab} \phi'^b - \frac{1}{2} (M'_{ab} + \delta M'_{ab}) \phi'^a \phi'^b - \frac{1}{4!} (\lambda'_{abcd} + \delta \lambda'_{abcd}) \phi'^a \phi'^b \phi'^c \phi'^d, \quad (1.23)$$

where the primed couplings and counterterms have been rotated to the basis where the field-strength renormalisation matrix is  $Z' = Z'^\text{T}$ . Re-defining the renormalised fields and couplings according to

$$\phi' \rightarrow \phi, \quad Z' \rightarrow Z, \quad M' \rightarrow M, \quad \lambda' \rightarrow \lambda \quad (1.24)$$

proves the above statement that the field-strength renormalisation matrix can always be made symmetric. In addition, we will make the assumption that the mass matrix  $M$  is diagonal in the basis where  $Z = Z^\text{T}$  without impact on the main results derived in the following. Proper treatment of the case where  $M$  is not diagonal will later be given in section 6.2.



## 2 Renormalisation up to two-loop

We may now proceed with the practical regularisation of the theory. After introducing some useful results, we successively perform the renormalisation of the 2- and 4-point functions to first and second order in perturbation theory. For the formulae given in section 2.1, we refer to [15].

### 2.1 Technicalities

We first provide some useful results which will allow to compute the relevant one-loop 2- and 4-point corrections in the following. What is perhaps the most central tool to handle loop-integrals in DR is the Feynman parameterisation formula:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}. \quad (1.25)$$

Applying this formula will lead in the following to  $d$ -dimensional integrals of the form

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 - \Delta + i\epsilon]^n} = \int \widetilde{d^d k} \frac{1}{[k^2 - \Delta + i\epsilon]^n}, \quad (1.26)$$

where we have defined the shortcut notation  $\widetilde{d^d k} = \frac{d^d k}{(2\pi)^d}$ . Such integrals can be expressed as

$$I_n^{(1)}(\Delta) = \int \widetilde{d^d k} \frac{1}{[k^2 - \Delta + i\epsilon]^n} = i(-1)^n \frac{(4\pi)^{\varepsilon/2} \Gamma(n-2+\frac{\varepsilon}{2})}{16\pi^2 \Gamma(n)} \frac{1}{\Delta^{n-2+\varepsilon/2}}, \quad (1.27)$$

with  $\Gamma$  the Euler gamma-function. We see that, as expected, the integral is only divergent for  $n \leq 2$  (quadratically for  $n = 1$ , logarithmically for  $n = 2$ ). It is convenient to define the modified scale  $\tilde{\mu}$  such that

$$\tilde{\mu}^2 = 4\pi e^{-\gamma_E} \mu^2, \quad (1.28)$$

with  $\gamma_E$  the Euler–Mascheroni constant, appearing in the series expansions of  $\Gamma$ . We may then rewrite (1.27) as

$$\mu^\varepsilon I_n^{(1)}(\Delta) = i \frac{1}{16\pi^2} (-1)^n \frac{1}{\Delta^{n-2}} \frac{\tilde{\Gamma}(n-2+\frac{\varepsilon}{2})}{\Gamma(n)} \left(\frac{\Delta}{\tilde{\mu}^2}\right)^{-\varepsilon/2}, \quad (1.29)$$

with

$$\tilde{\Gamma}(x) = e^{\gamma_E \varepsilon/2} \Gamma\left(x + \frac{\varepsilon}{2}\right) \quad (1.30)$$

a modified gamma-function, of which the series expansion in  $\varepsilon$  around integer values no longer involves the constant  $\gamma_E$ . In particular, one obtains for  $n = 1, 2$  the following  $\varepsilon$ -expansions:

$$\mu^\varepsilon I_1^{(1)}(\Delta) = \frac{i}{16\pi^2} \Delta \left\{ \frac{2}{\varepsilon} - \left( \log \frac{\Delta}{\tilde{\mu}^2} - 1 \right) + \frac{1}{4} \left[ 1 + \zeta_2 + \left( \log \frac{\Delta}{\tilde{\mu}^2} - 1 \right)^2 \right] \varepsilon \right\} + \mathcal{O}(\varepsilon^2), \quad (1.31)$$

$$\mu^\varepsilon I_2^{(1)}(\Delta) = \frac{i}{16\pi^2} \left\{ \frac{2}{\varepsilon} - \log \frac{\Delta}{\tilde{\mu}^2} + \frac{1}{4} \left[ \zeta_2 + \left( \log \frac{\Delta}{\tilde{\mu}^2} \right)^2 \right] \varepsilon \right\} + \mathcal{O}(\varepsilon^2), \quad (1.32)$$

where  $\zeta$  stands for the Riemann zeta-function, with  $\zeta_2 = \frac{\pi^2}{6}$ .

### 2.2 One-loop 2-point function

At one-loop, a single diagram contributes to the scalar self-energy:

$$-i\Pi^{ab}(p^2) = \text{---} \text{---} \text{---} \text{---} = \frac{-i}{2} \mu^\varepsilon \lambda_{abij} \int \widetilde{d^d k} \frac{i\delta^{ij}}{k^2 - M_{ij} + i\epsilon} = \frac{\mu^\varepsilon}{2} \lambda_{abii} I_1^{(1)}(M_{ii}). \quad (1.33)$$

From Eq. (1.31), we directly obtain

$$\begin{aligned}\Pi^{ab}(p^2) &= -\frac{1}{16\pi^2}\lambda_{abii}M_{ii}\left[\frac{1}{\varepsilon}-\frac{1}{2}\left(\log\frac{M_{ii}}{\tilde{\mu}^2}-1\right)\right]+\mathcal{O}(\varepsilon) \\ &= -\frac{1}{16\pi^2}\lambda_{abij}\left[\frac{1}{\varepsilon}M_{ij}-\frac{1}{2}\left(M\text{Log}\frac{M}{\tilde{\mu}^2}-M\right)\right]_{ij}+\mathcal{O}(\varepsilon),\end{aligned}\quad (1.34)$$

where the second equality follows from  $M$  (and its logarithm,  $\text{Log } M$ ) being diagonal. We stress that, here and in the following, repeated indices are systematically summed over. We may now fix the counterterms from the definition of the renormalised self-energy (1.18), by minimally subtracting the divergent contribution,

$$\delta_\phi = 0, \quad \delta M_{ab} = \frac{1}{\varepsilon(16\pi^2)}\lambda_{abij}M_{ij}, \quad (1.35)$$

We hence obtain the following expression of  $\Pi_R(p^2)$  at one-loop in the minimal subtraction (MS) scheme

$$\Pi_R^{ab}(p^2) = \frac{1}{16\pi^2}\frac{1}{2}\lambda_{abij}\left[M\left(\log\frac{M}{\tilde{\mu}^2}-\mathbb{1}\right)\right]^{ij}. \quad (1.36)$$

It should be emphasised that the absence of contributions to the field-strength renormalisation constant comes from the fact that  $\Pi(p^2)$  is independent of  $p^2$  at one-loop.

It is usual to adopt a modified prescription (defining the  $\overline{\text{MS}}$  scheme) where the numerical factor  $\log(4\pi e^{-\gamma_E})$  is absorbed in the counterterm, effectively replacing  $\tilde{\mu}$  with  $\mu$  in the argument of the logarithms involved in the finite contributions, yielding

$$-i\Pi^{ab}(p^2) = -\frac{1}{16\pi^2}\frac{1}{2}\lambda_{abii}M_{ii}\left[\left(\frac{2}{\varepsilon}+\log 4\pi-\gamma_E\right)-\left(\log\frac{M_{ii}}{\mu^2}-1\right)\right]+\mathcal{O}(\varepsilon), \quad (1.37)$$

so that the  $\overline{\text{MS}}$  counterterm reads

$$\delta M_{ab} = \lambda_{abij}M_{ij}\frac{1}{16\pi^2}\left[\frac{1}{\varepsilon}+\frac{1}{2}\log 4\pi-\frac{\gamma_E}{2}\right]. \quad (1.38)$$

In order to remove the explicit occurrence of such numerical factors, it is always possible to either re-scale the arbitrary parameter  $\mu$  as

$$\mu^2 \rightarrow \frac{\mu^2}{4\pi e^{-\gamma_E}}, \quad (1.39)$$

or to introduce the modified quantity  $\varepsilon'$  such that

$$\frac{1}{\varepsilon'} = \frac{1}{\varepsilon} + \frac{1}{2}\log 4\pi - \frac{\gamma_E}{2}, \quad (1.40)$$

followed by the re-definition  $\varepsilon' \rightarrow \varepsilon$ . Either way, the  $\log(4\pi e^{-\gamma_E})$  factors no longer explicitly appear in the expression of the  $\overline{\text{MS}}$  renormalised  $n$ -point functions and counterterms, and we obtain in particular

$$\delta M_{ab} = \frac{1}{16\pi^2}\frac{1}{\varepsilon}\lambda_{abij}M_{ij}, \quad \Pi_R^{ab}(p^2) = \frac{1}{16\pi^2}\frac{1}{2}\lambda_{abij}\left[M\left(\log\frac{M}{\mu^2}-\mathbb{1}\right)\right]^{ij} \equiv \frac{1}{16\pi^2}\frac{1}{2}\lambda_{abij}A(M_{ij}), \quad (1.41)$$

where we have defined, following [16],

$$A(x) = x\left[\log\frac{x}{\mu^2}-1\right]. \quad (1.42)$$

### 2.3 One-loop 4-point function

We now proceed with the computation of the one-loop contributions to the 4-point function. In that case, three diagrams should be computed that are related by a crossing symmetry. Hence, we only need in practice to derive the  $s$ -channel diagram<sup>3</sup>

$$-i\Gamma_{abcd}^{(1)} = \begin{array}{c} a \qquad \qquad c \\ \diagdown \quad \diagup \\ \text{---} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ b \qquad \qquad d \end{array}, \quad (1.43)$$

which is expressed, noting  $p = p_a + p_b$  such that  $p^2 = s = (p_a + p_b)^2$ , as

$$\begin{aligned} -i\Gamma_{abcd}^{(1)} &= \frac{1}{2}\mu^\varepsilon (-i\lambda_{abij})(-i\lambda_{klcd}) \int \widetilde{d^d k} \frac{i\delta_{ik}}{(p-k)^2 - M_{ik} + i\epsilon} \frac{i\delta_{jl}}{k^2 - M_{jl} + i\epsilon} \\ &= \frac{1}{2}\mu^\varepsilon \lambda_{abij} \lambda_{klcd} \delta_{ik} \delta_{jl} \int_0^1 dx \int \widetilde{d^d k} \frac{1}{[x(p-k)^2 + (1-x)k^2 - xM_{ik} - (1-x)M_{jl} + i\epsilon]^2} \\ &= \frac{1}{2}\mu^\varepsilon \lambda_{abij} \lambda_{klcd} \delta_{ik} \delta_{jl} \int_0^1 dx \int \widetilde{d^d k} \frac{1}{[k^2 + x(1-x)p^2 - xM_{ik} - (1-x)M_{jl} + i\epsilon]^2} \\ &\equiv i\lambda_{abij} \lambda_{klcd} V_{ijkl}(p^2), \end{aligned} \quad (1.44)$$

where in the two last steps we have successively applied a shift on the integration variable,  $k \rightarrow k + xp$ , and defined the auxiliary function

$$V_{ijkl}(p^2) = -\frac{i}{2}\mu^\varepsilon \delta_{ik} \delta_{jl} \int_0^1 dx I_2^{(1)}(xM_{ik} + (1-x)M_{jl} - x(1-x)p^2). \quad (1.45)$$

Using Eq. (1.32), we find

$$\begin{aligned} V_{ijkl}(p^2) &= \frac{\delta_{ik} \delta_{jl}}{(16\pi^2)} \left[ \frac{1}{\varepsilon} - \frac{1}{2} \int_0^1 dx \log \left( \frac{|xM_{ik} + (1-x)M_{jl} - x(1-x)p^2|}{\tilde{\mu}^2} \right) \right] \\ &= \frac{\delta^{ik} \delta^{jl}}{(16\pi^2)} \left[ \frac{1}{\varepsilon'} + \frac{1}{2} B(M_{ik}, M_{jl}, p^2) \right] + \mathcal{O}(\varepsilon'), \end{aligned} \quad (1.46)$$

where, following [16], we have defined the function

$$B(M_1, M_2, p^2) = -\int_0^1 dx \log \left( \frac{|xM_1 + (1-x)M_2 - x(1-x)p^2|}{\mu^2} \right). \quad (1.47)$$

In turn,  $\Gamma_{abcd}^{(1)}$  reads, after re-defining  $\varepsilon' \rightarrow \varepsilon$ ,

$$\Gamma_{abcd}^{(1)} = -\frac{1}{(16\pi^2)} \lambda_{abij} \lambda_{ijcd} \left[ \frac{1}{\varepsilon} + \frac{1}{2} B(M_{ii}, M_{jj}, (p_a + p_b)^2) \right] + \mathcal{O}(\varepsilon). \quad (1.48)$$

The two remaining contributions – namely the  $t$ - and  $u$ -channel counterparts of (1.43) – are obtained from a permutation of the external indices:

$$\Gamma_{abcd}^{(1,\text{all})} = \Gamma_{abcd}^{(1)} + \Gamma_{acbd}^{(1)} + \Gamma_{adbc}^{(1)}. \quad (1.49)$$

In particular, from the overall one-loop contribution  $\Gamma_{abcd}^{(1,\text{all})}$ , we directly find the expression of the 4-leg counterterm, requiring  $\lambda_{abcd} + \delta\lambda_{abcd} + \Gamma_{abcd}$  to be finite at one-loop:

$$\delta\lambda_{abcd} = \frac{1}{\varepsilon(16\pi^2)} [\lambda_{abij} \lambda_{ijcd} + \lambda_{acij} \lambda_{ijbd} + \lambda_{adij} \lambda_{ijbc}]. \quad (1.50)$$

<sup>3</sup>We note that, since we are considering the amputated  $n$ -point functions, the self-energy corrections to the external lines need not be considered.

## 2.4 Two-loop 2-point function

To the next order in perturbation theory, the self-energy is given by the sum of four contributions

$$\begin{aligned}
 -i\Pi^{(2)}(p^2) &= \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\
 &= -i\Pi^{(2,A)}(p^2) - i\Pi^{(2,B)}(p^2) - i\Pi^{(2,C)}(p^2) - i\Pi^{(2,D)}(p^2),
 \end{aligned} \tag{1.51}$$

where a cross on a vertex or internal line denotes a counterterm insertion. To begin with, the 4-leg-counterterm diagram (A) can directly be read from the one-loop self-energy (1.34):

$$-i\Pi^{(2,A)}(p^2) = \frac{i}{(16\pi^2)} \delta\lambda_{abij} \left[ \frac{1}{\varepsilon} M_{ij} - \frac{1}{2} A(M_{ij}) + \mathcal{O}(\varepsilon) \right].$$

Since  $\delta\lambda_{abcd}$  has a  $\varepsilon^{-1}$ -pole, the above expression is only sufficient if one is interested in studying the 2-loop pole structure. On the other hand, to capture the  $\mathcal{O}(1)$  contributions, one would need to expand the Taylor series up to order  $\mathcal{O}(\varepsilon)$ , yielding from Eq. (1.31)

$$-i\Pi^{(2,A)}(p^2) = \frac{i}{(16\pi^2)} \delta\lambda_{abij} \left\{ \frac{1}{\varepsilon} M_{ij} - \frac{1}{2} A(M_{ij}) + \frac{\varepsilon}{4} M_{ij} \left[ 1 + \zeta_2 + \left( \log \frac{M_{ij}}{\mu^2} - 1 \right)^2 \right] \right\} + \mathcal{O}(\varepsilon^2). \tag{1.52}$$

However, we restrict the discussion to the singular parts, allowing us to compute beta-functions up to the two-loop level in section 3. From the expression of the quartic counterterm in Eq. (1.50), we obtain

$$-i\Pi^{(2,A)}(p^2) = \frac{i}{(16\pi^2)^2} (\lambda_{abkl}\lambda_{kl ij} + 2\lambda_{aikl}\lambda_{bjkl}) \left[ \frac{1}{\varepsilon^2} M_{ij} - \frac{1}{2\varepsilon} A(M_{ij}) \right] + \mathcal{O}(1). \tag{1.53}$$

For the contribution  $\Pi^{(2,B)}(p^2)$ , we find

$$-i\Pi^{(2,B)}(p^2) = \frac{i}{(16\pi^2)^2} \lambda_{abij}\lambda_{ijkl}M_{kl} \left[ \frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon} B(M_{ii}, M_{jj}, 0) \right] + \mathcal{O}(1). \tag{1.54}$$

The third diagram, corresponding to  $\Pi^{(2,C)}$ , contains two separate one-loop subdivergences and is readily expressed as

$$-i\Pi^{(2,C)}(p^2) = -\frac{i}{(16\pi^2)^2} \lambda_{abij}\lambda_{ijkl} \left\{ \left[ \frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon} B(M_{ii}, M_{jj}, 0) \right] M_{kl} - \frac{1}{2\varepsilon} A(M_{ij}) \right\} + \mathcal{O}(1). \tag{1.55}$$

The fourth diagram (often referred to in the literature as the sunset diagram) involves overlapping divergences and is less straightforward to compute. Deferring this computation to appendix B, we obtain

$$\begin{aligned}
 -i\Pi^{(2,D)}(p^2) &= -\frac{i}{6} \frac{1}{(16\pi^2)^2} \lambda_{aijk}\lambda_{ijkb} \left\{ \frac{2}{\varepsilon^2} (M_{ii} + M_{jj} + M_{kk}) - \frac{1}{\varepsilon} \left( \frac{p^2}{2} - M_{ii} - M_{jj} - M_{kk} \right) \right. \\
 &\quad \left. - \frac{2}{\varepsilon} \left[ A(M_{ii}) + A(M_{jj}) + A(M_{kk}) \right] \right\} + \mathcal{O}(1) \\
 &= -\frac{i}{(16\pi^2)^2} \lambda_{aijk}\lambda_{ijkb} \left[ \frac{1}{\varepsilon^2} M_{ii} + \frac{1}{2\varepsilon} M_{ii} - \frac{1}{\varepsilon} A(M_{ii}) - \frac{1}{12\varepsilon} p^2 \right] + \mathcal{O}(1),
 \end{aligned} \tag{1.56}$$

where, in the second equality, we have used the permutation symmetry under internal indices to simplify the expression. Summing all the above contributions, we get the final result for the two relevant topologies:

$$-i\Pi^{(2)} \left( \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \right) = \frac{i}{(16\pi^2)^2} \lambda_{abij}\lambda_{ijkl}M_{kl} \left[ \frac{1}{\varepsilon^2} + \mathcal{O}(1) \right], \tag{1.57}$$

$$-i\Pi^{(2)} \left( \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \right) = \frac{i}{(16\pi^2)^2} \lambda_{aijk}\lambda_{bijk} \left[ M_{ii} \left( \frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} \right) + \frac{p^2}{12\varepsilon} + \mathcal{O}(1) \right]. \tag{1.58}$$

We may finally infer the two-loop contributions to the field-strength and mass-counterterms:

$$\delta_\phi^{ab} = -\frac{1}{(16\pi^2)^2} \frac{1}{12\varepsilon} \lambda_{aijk} \lambda_{bijk}, \quad (1.59)$$

$$\delta M_{ab} = \frac{1}{(16\pi^2)^2} \left[ \frac{1}{\varepsilon^2} \lambda_{abij} \lambda_{ijkl} M_{kl} + \left( \frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} \right) \lambda_{aijk} \lambda_{bijk} M_{ii} \right]. \quad (1.60)$$

## 2.5 Two-loop 4-point function

We finally compute the two-loop contributions to the 4-leg vertex in order to derive the corresponding beta-functions in the next section. As previously, we will restrict the discussion to the singular parts of the diagrams. In addition, all external momenta will be taken to vanish in order to simplify the computations. In the  $s$ -channel, five diagrams contribute

$$\begin{aligned} -i\Gamma_{abcd}^{(2)} &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} \\ &= -i\Gamma_{abcd}^{(2,A)} - i\Gamma_{abcd}^{(2,B)} - i\Gamma_{abcd}^{(2,C)} - i\Gamma_{abcd}^{(2,D)} - i\Gamma_{abcd}^{(2,E)} \\ &\quad - i\Gamma_{abcd}^{(2,A')} - i\Gamma_{abcd}^{(2,E')}, \end{aligned} \quad (1.61)$$

where  $\Gamma_{abcd}^{(2,A')}$  and  $\Gamma_{abcd}^{(2,E')}$  stand for the diagrams obtained by mirroring horizontally  $\Gamma_{abcd}^{(2,A)}$  and  $\Gamma_{abcd}^{(2,E)}$ . The expression of the first diagram can directly be read from Eq. (1.48)–(1.49):

$$\begin{aligned} -i\Gamma_{abcd}^{(2,A)} &= \frac{i}{(16\pi^2)^2} \lambda_{abij} \delta \lambda_{ijcd} \left[ \frac{1}{\varepsilon} + \frac{1}{2} B(M_{ii}, M_{jj}, 0) \right] + \mathcal{O}(1) \\ &= \frac{i}{(16\pi^2)^2} \lambda_{abij} (\lambda_{ijkl} \lambda_{klcd} + 2\lambda_{ickl} \lambda_{kljd}) \left[ \frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon} B(M_{ii}, M_{jj}, 0) \right] + \mathcal{O}(1), \end{aligned} \quad (1.62)$$

and

$$-i\Gamma_{abcd}^{(2,A')} = \frac{i}{(16\pi^2)^2} (\lambda_{abij} \lambda_{ijkl} + 2\lambda_{aijk} \lambda_{bijl}) \lambda_{klcd} \left[ \frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon} B(M_{kk}, M_{ll}, 0) \right] + \mathcal{O}(1). \quad (1.63)$$

Diagram  $B$  is expressed without difficulty as

$$\begin{aligned} -i\Gamma_{abcd}^{(2,B)} &= -\frac{i}{(16\pi^2)^2} \lambda_{abij} \lambda_{ijkl} \lambda_{klcd} \left[ \frac{1}{\varepsilon} + \frac{1}{2} B(M_{ii}, M_{jj}, 0) \right] \left[ \frac{1}{\varepsilon} + \frac{1}{2} B(M_{kk}, M_{ll}, 0) \right] + \mathcal{O}(1) \\ &= -\frac{i}{(16\pi^2)^2} \lambda_{abij} \lambda_{ijkl} \lambda_{klcd} \left( \frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon} [B(M_{ii}, M_{jj}, 0) + B(M_{kk}, M_{ll}, 0)] \right) + \mathcal{O}(1). \end{aligned}$$

Without direct computation, it is easy to see that the poles of  $\Gamma_{abcd}^{(2,C)}$  and  $\Gamma_{abcd}^{(2,D)}$  cancel at order  $\mathcal{O}(\varepsilon^{-1})$ . In turn, as explained in the next section, these diagrams do not contribute to the beta-function of  $\lambda$ . Therefore, despite a non-zero contribution at order  $\mathcal{O}(\varepsilon^{-2})$ , we will skip for conciseness the computation of  $\Gamma_{abcd}^{(2,D)}$ . Finally, the fifth diagram can, in fact, be inferred from the sunset diagram introduced in the previous section: the former can be constructed from the latter by replacing a mass insertion with a 4-leg vertex, thereby doubling the corresponding propagator. We show in appendix A that any diagram obtained by doubling a propagator can be straightforwardly expressed without the need for direct computation. For instance, writing Eq. (1.56) in the form

$$\begin{aligned} -i\Pi^{(2,D)} &= -\frac{i}{(16\pi^2)^2} \lambda_{aijk} \lambda_{ijkb} \left[ \frac{1}{\varepsilon^2} M_{ii} + \frac{1}{2\varepsilon} M_{ii} - \frac{1}{\varepsilon} A(M_{ii}) - \frac{1}{12\varepsilon} p^2 \right] \\ &\equiv \frac{i}{(16\pi^2)^2} \lambda_{aijk} \lambda_{ijkb} I_3(M_{ii}), \end{aligned} \quad (1.64)$$

the four-leg diagram  $\Gamma_{abcd}^{(2,E)}$  is in fact equal to

$$\begin{aligned} -i\Gamma_{abcd}^{(2,E)} &= \frac{i}{(16\pi^2)^2} \lambda_{abij} \lambda_{ickl} \lambda_{kljd} \frac{I_3(M_{ii}) - I_3(M_{jj})}{M_{ii} - M_{jj}} \\ &= \frac{i}{(16\pi^2)^2} \lambda_{abij} \lambda_{ickl} \lambda_{kljd} \left[ -\frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} + \frac{1}{\varepsilon} \frac{A(M_{ii}) - A(M_{jj})}{M_{ii} - M_{jj}} \right]. \end{aligned} \quad (1.65)$$

It can be shown that

$$\frac{A(M_{ii}) - A(M_{jj})}{M_{ii} - M_{jj}} = -B(M_{ii}, M_{jj}, 0), \quad (1.66)$$

so we may write

$$-i\Gamma_{abcd}^{(2,E)} = \frac{i}{(16\pi^2)^2} \lambda_{abij} \lambda_{ickl} \lambda_{kljd} \left[ -\frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} - \frac{1}{\varepsilon} B(M_{ii}, M_{jj}, 0) \right], \quad (1.67)$$

and

$$-i\Gamma_{abcd}^{(2,E')} = \frac{i}{(16\pi^2)^2} \lambda_{aijk} \lambda_{bijl} \lambda_{klcd} \left[ -\frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} - \frac{1}{\varepsilon} B(M_{kk}, M_{ll}, 0) \right]. \quad (1.68)$$

Gathering all previous results, we finally obtain

$$-i\Gamma_{abcd}^{(2)} \left( \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) = \frac{i}{(16\pi^2)^2} \lambda_{abij} \lambda_{ijkl} \lambda_{klcd} \left[ \frac{1}{\varepsilon^2} + \mathcal{O}(1) \right], \quad (1.69)$$

$$-i\Gamma_{abcd}^{(2)} \left( \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) = \frac{i}{(16\pi^2)^2} (\lambda_{abij} \lambda_{ickl} \lambda_{kljd} + \lambda_{aijk} \lambda_{bijl} \lambda_{klcd}) \left[ \frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} + \mathcal{O}(1) \right], \quad (1.70)$$

and the two-loop counterterm reads

$$\begin{aligned} \delta^{(2)} \lambda_{abcd} &= \frac{1}{(16\pi^2)^2} \left\{ \frac{1}{\varepsilon^2} [\lambda_{abij} \lambda_{ijkl} \lambda_{klcd} + \lambda_{abij} \lambda_{ickl} \lambda_{kljd} + \lambda_{aijk} \lambda_{bijl} \lambda_{klcd}] \right. \\ &\quad \left. - \frac{1}{2\varepsilon} [\lambda_{abij} \lambda_{ickl} \lambda_{kljd} + \lambda_{aijk} \lambda_{bijl} \lambda_{klcd}] \right\} + (b \leftrightarrow c) + (b \leftrightarrow d). \end{aligned} \quad (1.71)$$

### 3 Beta-functions

If all couplings of the theory have been made finite through the process of renormalisation, they cannot, however, be viewed as physical quantities. In particular, all couplings and fields have acquired a dependence on the arbitrary scale  $\mu$  introduced in the process of dimensional regularisation. Such a dependence is governed by a set of differential equations: the *renormalisation group equation* (RGEs). We call *beta-function* of a coupling its derivative with respect to the logarithm of the arbitrary scale  $\mu$ , in the 4-dimensional regularised theory:

$$\beta(X) \equiv \mu \frac{dX}{d\mu} \Big|_{\varepsilon \rightarrow 0}. \quad (1.72)$$

In a single-scale theory, one similarly defines the *anomalous dimension* of a scalar field with field-strength renormalisation constant  $Z_\phi$  as

$$\gamma_\phi = \frac{d \log Z_\phi}{d \log \mu} \Big|_{\varepsilon \rightarrow 0} = \frac{\mu}{Z_\phi} \frac{dZ_\phi}{d\mu} \Big|_{\varepsilon \rightarrow 0}. \quad (1.73)$$

As illustrated in the following, both the beta-functions and the anomalous dimensions are computed from the requirement that the bare couplings and fields are independent of the scale  $\mu$ . In particular, to see how the definition of the anomalous dimension generalises to the case of a multi-scalar theory, we write

$$\mu \frac{d}{d\mu} \phi_0 = 0 = \mu \frac{d}{d\mu} (Z\phi) = \mu \frac{dZ}{d\mu} \phi + Z \frac{d\phi}{d\mu}. \quad (1.74)$$

Multiplying the right-hand side from the left by  $Z^{-1}$ , one gets

$$\mu \frac{d\phi}{d\mu} = -Z^{-1} \mu \frac{dZ}{d\mu} \phi, \quad (1.75)$$

so the natural generalisation of the single-scale anomalous dimension  $\gamma_\phi$  is the matrix  $\gamma$  such that

$$\mu \left. \frac{d\phi}{d\mu} \right|_{\varepsilon \rightarrow 0} = -\gamma \phi, \quad (1.76)$$

meaning in turn that

$$\gamma = Z^{-1} \mu \left. \frac{dZ}{d\mu} \right|_{\varepsilon \rightarrow 0}. \quad (1.77)$$

We note that, in addition, an alternative expression for  $\gamma$  can be found<sup>4</sup> starting instead from the relation  $\phi = Z^{-1} \phi_0$ :

$$\phi = Z^{-1} \phi_0 \quad \rightarrow \quad \gamma = -\mu \left. \frac{dZ^{-1}}{d\mu} Z \right|_{\varepsilon \rightarrow 0}. \quad (1.78)$$

Finally, it will be useful in the following to define the shortcut notations

$$\gamma_\varepsilon = Z^{-1} \mu \frac{dZ}{d\mu} = -\mu \frac{dZ^{-1}}{d\mu} Z, \quad \beta_\varepsilon(X) = \mu \frac{dX}{d\mu}, \quad (1.79)$$

such that

$$\gamma = \gamma_\varepsilon|_{\varepsilon \rightarrow 0}, \quad \beta(X) = \beta_\varepsilon(X)|_{\varepsilon \rightarrow 0}. \quad (1.80)$$

Now turning to the computation of the quartic beta-function, we have

$$\lambda_{\varepsilon fgh}^0 = \mu^\varepsilon (\lambda_{abcd} + \delta\lambda_{abcd}) Z_{ae}^{-1} Z_{bf}^{-1} Z_{cg}^{-1} Z_{dh}^{-1}, \quad (1.81)$$

so that taking the logarithmic derivative with respect to the RG-scale on both sides yields, using Eq. (1.78),

$$\begin{aligned} 0 = \mu^\varepsilon \left[ \varepsilon (\lambda_{abcd} + \delta\lambda_{abcd}) + \beta_\varepsilon(\lambda_{abcd}) + D\delta\lambda_{abcd} \right. \\ \left. - (\lambda_{ibcd}\gamma_\varepsilon^{ia} + \lambda_{aicd}\gamma_\varepsilon^{ib} + \lambda_{abid}\gamma_\varepsilon^{ic} + \lambda_{abci}\gamma_\varepsilon^{id}) \right. \\ \left. - (\delta\lambda_{ibcd}\gamma_\varepsilon^{ia} + \delta\lambda_{aicd}\gamma_\varepsilon^{ib} + \delta\lambda_{abid}\gamma_\varepsilon^{ic} + \delta\lambda_{abci}\gamma_\varepsilon^{id}) \right] Z_{ae}^{-1} Z_{bf}^{-1} Z_{cg}^{-1} Z_{dh}^{-1}, \end{aligned} \quad (1.82)$$

where we have defined the differential operator

$$D = \mu \frac{d}{d\mu}. \quad (1.83)$$

In order to simplify the expressions, it is convenient to define a generalised anti-commutation operation between a matrix  $X$  and a rank- $n$  tensor  $T$ :

$$\{X \rightarrow T\}^{a_1, \dots, a_n} \equiv X^{a_1 b} T_{b, a_2, \dots, a_n} + \dots + X^{a_n b} T_{a_1, \dots, a_{n-1}, b}, \quad (1.84a)$$

$$\{T \leftarrow X\}^{a_1, \dots, a_n} \equiv T_{b, a_2, \dots, a_n} X^{ba_1} + \dots + T_{a_1, \dots, a_{n-1}, b} X^{ba_n} = \{X^T \rightarrow T\}^{a_1, \dots, a_n}, \quad (1.84b)$$

so that Eq. (1.82) can be put into the more compact form

$$0 = \mu^\varepsilon \left[ \varepsilon (\lambda_{abcd} + \delta\lambda_{abcd}) + \beta_\varepsilon(\lambda_{abcd}) + D\delta\lambda_{abcd} - \{\lambda \leftarrow \gamma_\varepsilon\}_{abcd} - \{\delta\lambda \leftarrow \gamma_\varepsilon\}_{abcd} \right] Z_{ae}^{-1} Z_{bf}^{-1} Z_{cg}^{-1} Z_{dh}^{-1}. \quad (1.85)$$

Since the  $\varepsilon$ -dependent beta-functions and anomalous dimensions are finite in the limit  $\varepsilon \rightarrow 0$ , those can generally be expanded in a power series in  $\varepsilon$ :

$$\beta_\varepsilon = \beta_{[0]} + \varepsilon \beta_{[1]} + \varepsilon^2 \beta_{[2]} + \dots \quad (1.86)$$

$$\gamma_\varepsilon = \gamma_{[0]} + \varepsilon \gamma_{[1]} + \varepsilon^2 \gamma_{[2]} + \dots, \quad (1.87)$$

<sup>4</sup>Equivalently, we may use the fact that  $\mu \frac{d}{d\mu} (Z^{-1} Z) = Z^{-1} \mu \frac{dZ}{d\mu} + \mu \frac{dZ^{-1}}{d\mu} Z = 0$ .

and we have in particular that

$$\beta = \beta_{[0]}, \quad \gamma = \gamma_{[0]}. \quad (1.88)$$

In fact, in addition to  $\beta_{[0]}$ , only the  $\mathcal{O}(\varepsilon)$  contribution  $\beta_{[1]}$  can be non-zero. It is of leading order in perturbation theory and characterises the anomalous scaling of the renormalised couplings compared to their bare counterpart in the  $(4 - \varepsilon)$ -dimensional theory. The quartic coupling beta-function is then equal to

$$\beta_\varepsilon(\lambda_{abcd}) = \beta(\lambda_{abcd}) + \beta_{[1]}(\lambda_{abcd})\varepsilon, \quad (1.89)$$

and one can show that  $\gamma_{[1]} = 0$  by expanding the relation

$$Z\gamma_\varepsilon = DZ = \beta_\varepsilon(\lambda_{ijkl}) \frac{\partial Z}{\partial \lambda_{ijkl}} \quad (1.90)$$

in terms of powers of  $\varepsilon$ . That  $\gamma_{[1]}$  vanishes is thus consistent with  $\phi_0$  and  $\phi$  having the same mass-dimension. To further express  $\beta_\varepsilon(\lambda_{abcd})$ , we define the  $\varepsilon$ -expansion of the counterterm as

$$\delta\lambda_{abcd} = \sum_{k=1}^{\infty} \frac{\delta_{[k]}\lambda_{abcd}}{\varepsilon^k}, \quad (1.91)$$

and write from Eq. (1.85):

$$0 = \varepsilon(\lambda_{abcd} + \delta\lambda_{abcd}) + \beta_\varepsilon(\lambda_{abcd}) + \beta_\varepsilon(\lambda_{ijkl}) \frac{\partial \delta\lambda_{abcd}}{\partial \lambda_{ijkl}} - \{\lambda \leftarrow \gamma\}_{abcd} - \{\delta\lambda \leftarrow \gamma\}_{abcd}. \quad (1.92)$$

At order  $\mathcal{O}(\varepsilon)$ , we have

$$\varepsilon\lambda_{abcd} + \varepsilon\beta_{[1]}(\lambda_{abcd}) = 0, \quad (1.93)$$

and therefore

$$\beta_{[1]}(\lambda_{abcd}) = -\lambda_{abcd}. \quad (1.94)$$

At order  $\mathcal{O}(\varepsilon^0)$ , one obtains in turn the generic form of the quartic beta-function:

$$\beta(\lambda_{abcd}) = -\delta_{[1]}\lambda_{abcd} + \lambda_{efgh} \frac{\partial \delta_{[1]}\lambda_{abcd}}{\partial \lambda_{efgh}} + \{\lambda \leftarrow \gamma\}_{abcd}. \quad (1.95)$$

We recover the standard result that the beta-functions are completely determined by the first-order pole of the counterterms. For negative powers of  $\varepsilon$ , we obtain consistency conditions on the pole structure of the counterterms. For instance, noting  $\delta_{[k]}^{(n)}$  the  $n$ -loop contribution to the counterterm  $\delta_{[k]}$ , we have at two-loop that

$$\delta_{[2]}^{(2)}\lambda_{abcd} = \frac{1}{2} \frac{\partial \delta_{[1]}^{(1)}\lambda_{abcd}}{\partial \lambda_{efgh}} \delta_{[1]}^{(1)}\lambda_{efgh}. \quad (1.96)$$

Inserting the expression of the one-loop counterterm (1.71) in (1.96) and only considering for simplicity the  $s$ -channel contributions, we find

$$\delta_{[2]}^{(2)}\lambda_{abcd} \supset \lambda_{abij}\lambda_{ijkl}\lambda_{klcd}, \quad (1.97)$$

consistent/in agreement with Eq. (1.71).

From Eq. (1.95), we see that the quartic coupling beta-function contains contributions involving the anomalous dimension matrix, which we now compute. From Eq. (1.77) and the fact that  $Z$  can be taken to be a symmetric matrix without loss of generality (see the discussion at the end of section 1), we have

$$Z\gamma = DZ \quad \rightarrow \quad Z\gamma Z = (DZ)Z = \frac{1}{2}D(Z^2). \quad (1.98)$$

From the decomposition

$$Z^2 = 1 + \delta_\phi = 1 + \sum_{k=1}^{\infty} \frac{\delta_\phi^{[k]}}{\varepsilon^k}, \quad (1.99)$$



we obtain, evaluating Eq. (1.98) at order  $\mathcal{O}(\varepsilon)$ ,

$$\gamma = -\frac{1}{2}\lambda_{abcd}\frac{\partial\delta_\phi^{[1]}}{\partial\lambda_{abcd}}. \quad (1.100)$$

We finally turn to the RG-dependence of the mass parameter  $M_{ab}$ . In this regard, it is usual to define the anomalous mass-dimension,

$$\gamma_M = \mu\frac{d\log M}{d\mu}. \quad (1.101)$$

However, such a definition is only appropriate to work with if the masses are renormalised multiplicatively, which is indeed true for the scalar theory considered in this chapter. Had we included in the Lagrangian density cubic couplings of the form

$$\mathcal{L} \supset -\tau_{abc}\phi^a\phi^b\phi^c, \quad (1.102)$$

the mass parameter would have received, already at the one-loop level, contributions of the type

$$\text{---} \circ \text{---} \propto \tau_{aij}\tau_{ijb}. \quad (1.103)$$

Hence, in a general scalar theory, the mass parameter is not renormalised multiplicatively and we will prefer in this discussion the use of the beta-function

$$\beta(M) = \mu\frac{dM}{d\mu}\Big|_{\varepsilon\rightarrow 0} = \beta_\varepsilon(M)\Big|_{\varepsilon\rightarrow 0}. \quad (1.104)$$

Starting from the relation between the bare and renormalised mass matrices,

$$M_{ef}^0 = (M_{ab} + \delta M_{ab})Z_{ae}^{-1}Z_{bf}^{-1}, \quad (1.105)$$

one obtains

$$0 = \beta_\varepsilon(M_{ab}) + \beta_\varepsilon(M_{ij})\frac{\partial\delta M_{ab}}{\partial M_{ij}} + \beta_\varepsilon(\lambda_{efgh})\frac{\partial\delta M_{ab}}{\partial\lambda_{efgh}} - \{M \leftarrow \gamma\}_{ab}, \quad (1.106)$$

Evaluating the above expression at order  $\mathcal{O}(\varepsilon^0)$ , we find

$$\beta(M_{ab}) = \lambda_{efgh}\frac{\partial\delta_{[1]}M_{ab}}{\partial\lambda_{efgh}} + \{M \leftarrow \gamma\}_{ab}. \quad (1.107)$$

Using  $\gamma = \gamma^T$ , we summarise the main results obtained so far,

$$\gamma = -\frac{1}{2}\lambda_{abcd}\frac{\partial\delta_\phi^{[1]}}{\partial\lambda_{abcd}}, \quad (1.108)$$

$$\beta(M_{ab}) = \lambda_{efgh}\frac{\partial\delta_{[1]}M_{ab}}{\partial\lambda_{efgh}} + \{\gamma \rightarrow M\}_{ab}, \quad (1.109)$$

$$\beta(\lambda_{abcd}) = -\delta_{[1]}\lambda_{abcd} + \lambda_{efgh}\frac{\partial\delta_{[1]}\lambda_{abcd}}{\partial\lambda_{efgh}} + \{\gamma \rightarrow \lambda\}_{abcd}, \quad (1.110)$$

recalling that the anti-commutation operation between a matrix and a tensor was defined in Eq. (1.84). The above expressions can be further simplified by observing that the operator

$$\lambda_{efgh}\frac{\partial}{\partial\lambda_{efgh}}$$

counts the number of 4-leg vertices in a diagram. As mentioned above, for the scalar theory considered here, such a number is in fact directly related to the loop-order  $n$ , and

$$\left(\lambda_{efgh}\frac{\partial}{\partial\lambda_{efgh}}\right)\delta\lambda = (n+1)\delta\lambda, \quad (1.111)$$

$$\left(\lambda_{efgh}\frac{\partial}{\partial\lambda_{efgh}}\right)\delta M = n\delta M, \quad (1.112)$$

$$\left(\lambda_{efgh}\frac{\partial}{\partial\lambda_{efgh}}\right)\delta\phi = n\delta\phi, \quad (1.113)$$

Hence, perturbatively expanding the beta- and gamma-functions as

$$\beta = \beta^{(1)} + \beta^{(2)} + \dots, \quad \gamma = \gamma^{(1)} + \gamma^{(2)} + \dots, \quad (1.114)$$

we can rewrite Eqs. (1.108)–(1.110) in the simpler form

$$\gamma^{(n)} = -\frac{n}{2} \delta_\phi^{[1],(n)}, \quad (1.115)$$

$$\beta^{(n)}(M_{ab}) = n \delta_{[1]}^{(n)} M_{ab} + \left\{ \gamma^{(n)} \rightarrow M \right\}_{ab}, \quad (1.116)$$

$$\beta^{(n)}(\lambda_{abcd}) = n \delta_{[1]}^{(n)} \lambda_{abcd} + \left\{ \gamma^{(n)} \rightarrow \lambda \right\}_{abcd}. \quad (1.117)$$

At one-loop, applying Eqs. (1.115)–(1.117) gives

$$\gamma_{ab}^{(1)} = 0, \quad \beta^{(1)}(M_{ab}) = \frac{1}{16\pi^2} \lambda_{abij} M_{ij}, \quad \beta^{(1)}(\lambda_{abcd}) = \frac{1}{16\pi^2} (\lambda_{abij} \lambda_{ijcd} + \lambda_{acij} \lambda_{ijbd} + \lambda_{adij} \lambda_{ijbc}). \quad (1.118)$$

At two-loop, we obtain from the expression of the counterterms given in Eqs. (1.59), (1.60) and (1.71):

$$\gamma_{ab}^{(2)} = \frac{1}{(16\pi^2)^2} \frac{1}{12} \lambda_{aijk} \lambda_{ijkb}, \quad (1.119)$$

$$\beta^{(2)}(M_{ab}) = \frac{1}{(16\pi^2)^2} \left[ -\lambda_{aikl} \lambda_{bjkl} M_{ij} + \frac{1}{12} (\lambda_{aijk} \lambda_{ijke} M_{eb} + M_{ae} \lambda_{eijk} \lambda_{ijkb}) \right], \quad (1.120)$$

$$\beta^{(2)}(\lambda_{abcd}) = \frac{1}{(16\pi^2)^2} \left\{ -[\lambda_{abij} \lambda_{ickl} \lambda_{kljd} + \lambda_{aijk} \lambda_{bijl} \lambda_{klcd} + (b \leftrightarrow c) + (b \leftrightarrow d)] \right. \\ \left. + \frac{1}{12} [\lambda_{aijk} \lambda_{ijke} \lambda_{ebcd} + (a \leftrightarrow b) + (a \leftrightarrow c) + (a \leftrightarrow d)] \right\}. \quad (1.121)$$

## 4 The Callan-Symanzik equation

It is remarkable that, despite the unphysical and arbitrary nature of the scale  $\mu$ , all couplings and fields maintain a residual, non-trivial dependence on this parameter in the 4-dimensional limit of a dimensionally regularised theory. The transformation properties of the couplings and fields under variations of the scale  $\mu$  define the renormalisation group (RG), and  $\mu$  is then referred to as the renormalisation scale (or RG scale). On the other hand, we have seen that  $\mu$  explicitly appears in the logarithmic contributions to the renormalised  $n$ -point functions. As we shall now demonstrate, the implicit dependence (through the beta-functions) and the explicit dependence (through the logarithmic contributions) exactly cancel, so the renormalised (non-amputated)  $n$ -point functions are exactly RG-invariant. For instance, the renormalised, amputated 2-point function, noted  $\Gamma_{ab}^R$ , relates to its bare counterpart according to

$$\Gamma_{ab}^R = [Z^T \Gamma^0 Z]^{ab}. \quad (1.122)$$

Taking the RG-scale derivative on both sides, and using the independence of  $\Gamma_{ab}^0$  with respect to  $\mu$  gives

$$\mu \frac{d\Gamma_{ab}^R}{d\mu} = \left[ \left( \mu \frac{dZ}{d\mu} \right)^T \Gamma^0 Z + Z^T \Gamma^0 \left( \mu \frac{dZ}{d\mu} \right) \right]^{ab} = [\gamma^T \Gamma^R + \Gamma^R \gamma]^{ab} = \Gamma_{ib}^R \gamma^{ia} + \Gamma_{ai}^R \gamma^{ib}. \quad (1.123)$$

This relation is easily generalised to any  $n$ -point function  $\Gamma_{(n)}^R$  using the matrix-tensor anti-commutator (see Eq. (1.84)),

$$\mu \frac{d}{d\mu} \Gamma_{(n)}^R = \left\{ \Gamma_{(n)}^R \leftarrow \gamma \right\}. \quad (1.124)$$

Decomposing the total RG-derivative in terms of its explicit and implicit components, we may write in turn

$$\mu \frac{\partial}{\partial \mu} \Gamma_{(n)}^R + \sum_{\lambda_i} \beta(\lambda_i) \frac{\partial}{\partial \lambda_i} \Gamma_{(n)}^R - \left\{ \Gamma_{(n)}^R \leftarrow \gamma \right\} = 0, \quad (1.125)$$

where  $\lambda_i$  generically denotes any coupling of the theory appearing in the expression of  $\Gamma_{(n)}^R$ . The relation (1.124) – together with its expanded form, Eq. (1.125) – is usually referred to as the Callan-Symanzik equation [17, 18], and is to be interpreted in the following way: the dependence of the amputated, renormalised  $n$ -point functions  $\Gamma_{(n)}^R$  on the renormalisation scale only stems from its open indices (legs). This means in turn that the non-amputated  $n$ -point functions – of which the external lines receive self-energy loop corrections – are exactly RG-invariant. This is a necessary condition to maintain overall physical consistency in the renormalised theory. This brings us to the conclusion that a consistent theoretical description of any scalar QFT not only relies on the expression of the renormalised  $n$ -point functions, but also on that of the beta-functions.

Furthermore, the Callan-Symanzik equation reveals in fact an intimate connection between the logarithmic structure of the loop contributions and the beta-functions. As such, it is possible, as we shall illustrate, to recover the expression of the beta-functions in the 4-dimensional regularised theory directly from that of the renormalised  $n$ -point functions. Let us for instance consider the 4-point function, which is expanded perturbatively as:

$$\Gamma_{abcd}^R = \lambda_{abcd} + \left( \delta_{abcd}^{(1)} + \Gamma_{abcd}^{(1)} \right) + \left( \delta_{abcd}^{(2)} + \Gamma_{abcd}^{(2)} \right) + \dots \quad (1.126)$$

By construction, the terms in brackets stand for the finite (i.e. minimally subtracted)  $n$ -loop contributions. Hence, we may alternatively rewrite the above perturbative expansion in terms of the  $n$ -loop finite contributions to the renormalised 4-point function:

$$\Gamma_{abcd}^R = \lambda_{abcd} + \Gamma_{abcd}^{R(1)} + \Gamma_{abcd}^{R(2)} + \dots \quad (1.127)$$

We reiterate that the loop contributions to  $\Gamma_{abcd}^R$  thereby defined depend on the RG-scale  $\mu$  in two ways: (i) explicitly, from the occurrence of  $\mu$  in the logarithms and (ii) implicitly, from their dependence on the running couplings. A crucial consequence of this property is that the total RG-derivative of  $\Gamma_{abcd}^{R(n)}$  always contains two distinct contributions, with perturbative order  $n$  – from the explicit dependence – and  $(n+1) + (n+2) + \dots$  – from the implicit dependence and the beta-functions – respectively. Since the Callan Symanzik holds order-by order, we may write<sup>5</sup>

$$\mu \frac{d}{d\mu} \Gamma_{abcd}^R = \left[ \beta^{(1)}(\lambda_{abcd}) + \mu \frac{\partial}{\partial \mu} \Gamma_{abcd}^{R(1)} \right] + \left[ \beta^{(1)}(\Gamma_{abcd}^{R(1)}) + \beta^{(2)}(\lambda_{abcd}) + \mu \frac{\partial}{\partial \mu} \Gamma_{abcd}^{R(2)} \right] + \dots, \quad (1.128)$$

so the  $n$ -loop contributions to the beta-functions can be computed iteratively from the finite part of the  $n$ -loops diagrams. For instance, at one-loop, we have from Eqs. (1.48)–(1.50):

$$\Gamma_{abcd}^{R(1)} = -\frac{1}{2} \frac{1}{16\pi^2} [\lambda_{abij} \lambda_{ijcd} B(M_{ii}, M_{jj}, s) + \lambda_{acij} \lambda_{ijbd} B(M_{ii}, M_{jj}, t) + \lambda_{adij} \lambda_{ijbc} B(M_{ii}, M_{jj}, u)] \quad (1.129)$$

where  $s$ ,  $t$  and  $u$  are the Mandelstam variables, describing the kinematics in the  $s$ ,  $t$  and  $u$  channels. The next step is to differentiate Eq. (1.129) with respect to the renormalisation scale. We remind the expression of the  $B$  function,

$$B(M_1, M_2, p^2) = -\int_0^1 dx \log \left( \frac{|xM_1 + (1-x)M_2 - x(1-x)p^2|}{\mu^2} \right). \quad (1.130)$$

simply giving

$$\mu \frac{\partial}{\partial \mu} B(M_1, M_2, p^2) = 2. \quad (1.131)$$

<sup>5</sup>By a slight abuse of notation, we use  $\beta^{(n)}$  as a shortcut notation for the differential operator

$$\beta^{(n)} \equiv \sum_{\lambda_i} \beta^{(n)}(\lambda_i) \frac{\partial}{\partial \lambda_i}.$$

Therefore, from Eq. (1.128), we obtain at one-loop

$$\mu \frac{d}{d\mu} \Gamma_{abcd}^R = \left[ \beta^{(1)}(\lambda_{abcd}) - \frac{1}{16\pi^2} (\lambda_{abij}\lambda_{ijcd} + \lambda_{acij}\lambda_{ijbd} + \lambda_{adij}\lambda_{ijbc}) \right] + \dots, \quad (1.132)$$

from which we directly read the pure-vertex contribution to the one-loop beta-function: With  $\gamma^{(1)} = 0$ , Eq.(1.128) simply gives, to first order,

$$\beta^{(1)}(\lambda_{abcd}) = \frac{1}{16\pi^2} (\lambda_{abij}\lambda_{ijcd} + \lambda_{acij}\lambda_{ijbd} + \lambda_{adij}\lambda_{ijbc}). \quad (1.133)$$

Therefore, we have been able to recover the expression of the one-loop quartic beta-function from that of the renormalised 4-point function. The same conclusion holds at higher orders in perturbation theory. However, such a property is not very useful in practice: Deriving the beta-functions is in general a simpler exercise than obtaining the renormalised  $n$ -point functions. On the other hand, one truly remarkable consequence of Eq. (1.128) is uncovered adopting the opposite perspective: We will show in chapters 3 and 4 that the expression of the beta-functions provides information – to *all orders* in perturbation theory – on the structure of the logarithmic contributions to the  $n$ -point functions.

## 5 The effective scalar potential

### 5.1 Ground states away from the origin

In the previous sections, we have considered a scalar theory with a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi^a - \frac{1}{2} M_{ab} \phi^a \phi^b - \frac{1}{4!} \lambda_{abcd} \phi^a \phi^b \phi^c \phi^d = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi^a - V(\phi), \quad (1.134)$$

with  $V$  the scalar potential, *i.e.* the part of the Lagrangian density describing the scalar interactions:

$$V(\phi) = \frac{1}{2} M_{ab} \phi^a \phi^b + \frac{1}{4!} \lambda_{abcd} \phi^a \phi^b \phi^c \phi^d. \quad (1.135)$$

In such a description,  $M$  is viewed as the leading-order contribution to the mass of the propagating (physical) fields, and  $\phi$  as the dynamical component of these propagating fields around their expectation value. In other words, we have implicitly assumed that the potential was minimal (at least locally) and convex, that is with a positive (semi)definite Hessian matrix – the scalar mass matrix.

If, on the other hand, the mass matrix possesses at least one negative eigenvalue, the scalar potential is no longer minimal at the stationary point  $\langle \phi \rangle = 0$  which therefore constitutes either a saddle point or a local maximum. Any variation of the dynamical fields around their expectation value  $\langle \phi \rangle = 0$  will in turn trigger spontaneous evolution of the system towards a configuration  $\langle \phi' \rangle$  which minimises the scalar potential, *i.e.* where

$$\partial_a V(\langle \phi' \rangle) \equiv \frac{\partial V}{\partial \phi^a}(\langle \phi' \rangle) = 0, \quad a = 1, \dots, N, \quad (1.136)$$

and where the matrix of the second derivatives of  $V$  is positive (semi)definite. In this new configuration, the classical action is extremal and the ground state is now stable under fluctuations of the scalar fields around their expectation value. It may still be possible, however, that the scalar potential is only locally minimal at the point  $\langle \phi' \rangle$ , *i.e.* that some  $\langle \phi'' \rangle$  satisfies

$$\partial_a V(\langle \phi'' \rangle) \equiv \frac{\partial V}{\partial \phi^a}(\langle \phi'' \rangle) = 0, \quad V(\langle \phi'' \rangle) < V(\langle \phi' \rangle). \quad (1.137)$$

In that case, the state  $\langle \phi' \rangle$  is only meta-stable and can decay, with non-zero probability, to the deeper, stable state  $\langle \phi'' \rangle$ . For instance, the question of knowing whether the Standard Model vacuum is either stable or meta-stable constitutes an active field of research [19, 20, 21, 22, 23, 24, 25, 26, 27].

Most importantly, this mechanism is at the origin of the phenomenon of spontaneous symmetry breaking in theories satisfying global or gauge symmetries, upon which is based our current understanding

of fundamental interactions. For these reasons, the in-depth study of the overall structure of the scalar potential is crucial for many applications. In such cases, one wishes in particular to evaluate with the highest accuracy the position of the minima of the scalar potential, defining the set of physically accessible meta-(stable) ground states. The quantity obtained after inclusion of the quantum corrections is referred to as the effective potential, and coincides with the zero-momentum limit of the 1PI effective action evaluated at constant field values.

## 5.2 The renormalised vacuum energy

We have discussed in the previous sections the renormalisation procedure for the 2- and 4-point functions, describing respectively the propagation and self-interactions of the scalar fields. In fact, one must in general include a vacuum energy contribution to the Lagrangian at leading order, namely

$$V(\phi) = \Lambda + \frac{1}{2}M_{ab}\phi^a\phi^b + \frac{1}{4!}\lambda_{abcd}\phi^a\phi^b\phi^c\phi^d. \quad (1.138)$$

As any other  $n$ -point function, the vacuum energy (that is, the 0-point function) receives loop-contributions beyond leading order and must be renormalised. Diagrammatically, one has

$$\Gamma_0 = \Lambda + \Gamma_0^{(1)} + \Gamma_0^{(2)} + \dots = \bullet + \text{loop} + \text{2-loop} + \dots \quad (1.139)$$

To compute the one-loop contribution,  $\Gamma_0^{(1)}$ , can be computed using the formula presented in appendix A for the doubling of propagators. Schematically, we may write

$$\text{loop} = \frac{d}{dM_{ii}} \text{loop}, \quad (1.140)$$

where the dot on the propagator represents an open index  $i$ . This means in turn that

$$\frac{d}{dM_{ii}} \left( -i\Gamma_0^{(1)} \right) = \frac{1}{2} \int \widetilde{d^d k} \frac{i}{k^2 - M_{ii}}, \quad (1.141)$$

and hence

$$\Gamma_0^{(1)} = \frac{1}{2} \sum_i \int \widetilde{d^d k} \log(k^2 - M_{ii}) + C, \quad (1.142)$$

where the sum over  $i$  is taken to include the contributions of every scalar in the theory. The integration constant can only vanish in order to obtain a finite result. Alternatively, we may perform integration with respect to the mass parameter after evaluation of the loop-integral, yielding

$$\text{loop} = \frac{1}{16\pi^2} M_{ij} \delta^{ij} \left[ -\frac{1}{\varepsilon} + \frac{1}{2} \left( \log \frac{M_{ii}}{\mu^2} - 1 \right) \right] + \mathcal{O}(\varepsilon), \quad (1.143)$$

and, in turn,

$$\Gamma_0^{(1)} = \frac{1}{4} \frac{1}{16\pi^2} \left[ -2 \frac{M_{ii}^2}{\varepsilon} + M_{ii}^2 \left( \log \frac{M_{ii}}{\mu^2} - \frac{3}{2} \right) \right] + \mathcal{O}(\varepsilon), \quad (1.144)$$

where the integration constant was taken to vanish consistently with our previous observation. Minimally subtracting the pole finally gives the expression of the renormalised one-loop vacuum energy:

$$\Gamma_0^{(1)} + \delta^{(1)}\Lambda = \frac{1}{4} \frac{1}{16\pi^2} M_{ii}^2 \left[ \log \frac{M_{ii}}{\mu^2} - \frac{3}{2} \right]. \quad (1.145)$$

At two-loop, a direct computation gives

$$\Gamma_0^{(2)} + \delta^{(2)}\Lambda = \frac{1}{8} \frac{1}{(16\pi^2)^2} \lambda_{ijj} M_{ii} M_{jj} \left[ \log \frac{M_{ii}}{\mu^2} - 1 \right] \left[ \log \frac{M_{jj}}{\mu^2} - 1 \right] = \frac{1}{8} \lambda_{ijj} A(M_{ii}) A(M_{jj}). \quad (1.146)$$

As previously done for the renormalised 4-point function in Eq. (1.126), we define at each loop-order an effective 0-point function,  $\Lambda^{(n)}$ , so the renormalised vacuum energy reads

$$\Gamma_0^R = \Lambda + \Lambda^{(1)} + \Lambda^{(2)} + \dots, \quad (1.147)$$

with, as shown above,

$$\Lambda^{(1)} = \frac{1}{4} M_{ii}^2 \left[ \log \frac{M_{ii}}{\mu^2} - \frac{3}{2} \right], \quad \Lambda^{(2)} = \frac{1}{8} \frac{1}{(16\pi^2)^2} \lambda_{ijj} A(M_{ii}) A(M_{jj}). \quad (1.148)$$

### 5.3 From the renormalised vacuum energy to the effective potential

Having expressed the renormalised vacuum energy in a theory where the fields have zero expectation value (where the true vacuum lies at the origin), we may now generalise our results to a scalar theory with a minimum away from the origin. In this case, the scalar fields can be decomposed in terms of a background component (the classical field, corresponding to the expectation value) and a dynamical component  $\varphi$ :

$$\phi = \phi_{\mathbf{c}} + \varphi. \quad (1.149)$$

Expressing the Lagrangian density, and in particular the scalar potential in terms of the newly defined dynamical field  $\varphi$ , we obtain

$$\begin{aligned} V(\phi) = & V(\phi_{\mathbf{c}}) + \frac{\partial V}{\partial \phi^a}(\phi_{\mathbf{c}}) \varphi^a + \frac{1}{2} \frac{\partial^2 V}{\partial \phi^a \partial \phi^b}(\phi_{\mathbf{c}}) \varphi^a \varphi^b + \frac{1}{3!} \frac{\partial^3 V}{\partial \phi^a \partial \phi^b \partial \phi^c}(\phi_{\mathbf{c}}) \varphi^a \varphi^b \varphi^c \\ & + \frac{1}{4!} \frac{\partial^4 V}{\partial \phi^a \partial \phi^b \partial \phi^c \partial \phi^d}(\phi_{\mathbf{c}}) \varphi^a \varphi^b \varphi^c \varphi^d, \end{aligned} \quad (1.150)$$

Defining background-dependent couplings, the field  $\varphi$  is described by a theory with scalar potential

$$\tilde{V}(\varphi) = \tilde{\Lambda} + \tilde{\sigma}_a \varphi^a + \frac{1}{2} \tilde{M}_{ab} \varphi^a \varphi^b + \frac{1}{3!} \tilde{\tau}_{abc} \varphi^a \varphi^b \varphi^c + \frac{1}{4!} \tilde{\lambda}_{abcd} \varphi^a \varphi^b \varphi^c \varphi^d. \quad (1.151)$$

As for any scalar theory, every result derived above applies in the shifted-theory (we insist that  $\phi_{\mathbf{c}}$  is not a quantum field but only a number). In particular, the renormalised vacuum energy reads

$$\tilde{\Gamma}_0^R = \tilde{\Lambda} + \tilde{\Gamma}_0^{R(1)} + \tilde{\Gamma}_0^{R(2)} + \dots, \quad (1.152)$$

and must be minimal (extremal) to cancel the tadpole term  $\tilde{\sigma}$  to all orders and minimise (extremise) the quantum action. The renormalised vacuum energy in the shifted theory is referred to as the effective potential,

$$V_{\text{eff}} \equiv \tilde{\Gamma}_0^R, \quad (1.153)$$

of which the perturbative expansion is usually rewritten

$$V_{\text{eff}} = V^{(0)} + V^{(1)} + V^{(2)} + \dots, \quad (1.154)$$

and where, using the results of the previous section,

$$V^{(0)} = V(\phi_{\mathbf{c}}) = \frac{1}{2} M_{ab} \phi_{\mathbf{c}}^a \phi_{\mathbf{c}}^b + \frac{1}{4!} \lambda_{abcd} \phi_{\mathbf{c}}^a \phi_{\mathbf{c}}^b \phi_{\mathbf{c}}^c \phi_{\mathbf{c}}^d, \quad (1.155)$$

$$V^{(1)} = \frac{1}{16\pi^2} \tilde{M}_{ii}^2 \left( \log \frac{\tilde{M}_{ii}}{\mu^2} - \frac{3}{2} \right), \quad (1.156)$$

$$V^{(2)} \supset \frac{1}{8} \frac{1}{(16\pi^2)^2} \tilde{\lambda}_{ijjj} A(\tilde{M}_{ii}) A(\tilde{M}_{jj}). \quad (1.157)$$

The background-field-dependent mass matrix reads

$$\tilde{M}_{ab} = \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} = M_{ab} + \frac{1}{2} \lambda_{abcd} \phi_{\mathbf{c}}^c \phi_{\mathbf{c}}^d, \quad (1.158)$$

and the two-loop contribution  $V^{(2)}$  is in fact incomplete due to the absence in the above expression of the contributions involving the background-field-dependent cubic coupling  $\tilde{\tau}$ . Finally, as any renormalised  $n$ -point function, the effective potential satisfies the Callan-Symanzik equation

$$\frac{d}{d\mu} V_{\text{eff}} = \left[ \mu \frac{\partial}{\partial \mu} + \sum_I \beta(g^I) \frac{\partial}{\partial g^I} - \phi_{\mathbf{c}}^a \gamma^{ab} \frac{\partial}{\partial \phi^b} \right] V_{\text{eff}} = 0, \quad (1.159)$$

and is therefore exactly RG-invariant.

## 6 Alternative renormalisation prescriptions

In this last section, we introduce two variations of the renormalisation procedure presented above that will be extensively used in chapter 4 to extend the  $\overline{\text{MS}}$  scheme to multiple RG-scales. The first one consists in re-defining the renormalised couplings and fields in the  $(4 - \varepsilon)$ -dimensional theory, by altering the mass-dimension of the dimensionful quantities, so that the shift in the dimensionality compared to a 4D theory is conveniently carried by the legs of every operator and tensor coupling. We will then introduce a renormalisation scheme in which the diagonal structure of the mass matrix is preserved along the RG-flow, through the introduction of a skew-symmetric contribution to the scalar-field anomalous dimension matrix.

### 6.1 Consistent occurrences of the renormalisation scale

In the previous sections, we have used a renormalisation prescription where, to maintain the dimensionless property of the quartic couplings in the  $(4 - \varepsilon)$ -dimensional theory, an arbitrary scale  $\mu$  was introduced such that

$$\mathcal{L} \supset -\frac{\mu^\varepsilon}{4!} \lambda_{abcd} \phi^a \phi^b \phi^c \phi^d, \quad (1.160)$$

with renormalised fields of mass-dimension  $[\phi] = 1 - \varepsilon/2$ . The starting point of the present discussion is to observe that it is possible to re-define the scalar fields such that  $\mu$  no longer appears in the above expression. This is achieved by defining the auxiliary field

$$\phi' = \mu^{\varepsilon/4} \phi, \quad (1.161)$$

with mass-dimension  $[\phi'] = 1 - \frac{\varepsilon}{4}$ , such that

$$\mathcal{L} \supset -\frac{1}{4!} \lambda_{abcd} \phi'_a \phi'_b \phi'_c \phi'_d. \quad (1.162)$$

Such a prescription is fact rather natural, since it enforces the constraint that dimensionless couplings in the 4-dimensional theory must remain dimensionless in  $(4 - \varepsilon)$  dimensions, the required shift being now carried by the renormalised scalar fields. Furthermore, it introduces an interesting regularity in the dimensionality of every operator and coupling in the theory. To illustrate this point, we consider an arbitrary operator built from  $n$  scalar fields ( $n \geq 0$ ) and the corresponding coupling  $C_{a_1 \dots a_n}$ :

$$\mathcal{L} \supset -C_{a_1 \dots a_n} \phi^{a_1} \dots \phi^{a_n}. \quad (1.163)$$

In the  $(4 - \varepsilon)$ -dimensional theory, with  $[\phi] = 1 - \frac{\varepsilon}{2}$ , the tensor  $C$  must have mass-dimension

$$[C] = (4 - n) - \frac{\varepsilon}{2}(2 - n). \quad (1.164)$$

Expressing instead the operator (1.163) in terms of the auxiliary field  $\phi'$ ,

$$\mathcal{L} \supset -C'_{a_1 \dots a_n} \phi'_{a_1} \dots \phi'_{a_n}, \quad (1.165)$$

the newly defined coupling  $C'$  has now mass-dimension

$$[C'] = (4 - n) - \frac{\varepsilon}{4}(4 - n) = \left(1 - \frac{\varepsilon}{4}\right) (4 - n) = \left(1 - \frac{\varepsilon}{4}\right) [C']_{\varepsilon \rightarrow 0}, \quad (1.166)$$

so that, manifestly, the quartic coupling remains dimensionless independently of the value of  $\varepsilon$ . The dimension of every other coupling is simply obtained by multiplying its canonical mass-dimension (in the 4-dimensional theory) with the factor  $(1 - \varepsilon/4)$ . Furthermore, every term appearing in the interaction Lagrangian is guaranteed to be of dimension  $d = 4 - \varepsilon$  without the need to explicitly introduce powers of the renormalisation scale. On the other hand, the auxiliary renormalised field  $\phi'$  being related to its bare counterpart through

$$\phi_0 = \mu^{-\varepsilon/4} Z \phi', \quad (1.167)$$

one introduces instead powers of  $\mu$  in the kinetic Lagrangian:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} \mu^{-\varepsilon/2} \partial_\mu \phi'^T Z^2 \partial^\mu \phi'. \quad (1.168)$$

As a direct consequence, the powers of the scale  $\mu$  needed to maintain dimensional consistency during the process of renormalisation now originate from the propagators instead of the vertices. This interesting property, (already mentioned in [28]), will be central in chapter 4 to generalise dimensional regularisation to multiple renormalisation scales  $\mu_i$ .

For the remaining of this discussion, we shall explore the consequences on the regularisation process of the theory as compared to the standard prescription. Starting from the most general (perturbatively renormalisable) scalar theory with a bare Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_0^a \partial^\mu \phi_0^b - \Lambda^0 - \sigma_a^0 \phi^a - \frac{1}{2} M_{ab}^0 \phi_0^a \phi_0^b - \frac{1}{3!} \tau_{abc}^0 \phi_0^a \phi_0^b \phi_0^c - \frac{1}{4!} \lambda_{abcd}^0 \phi_0^a \phi_0^b \phi_0^c \phi_0^d, \quad (1.169)$$

we define the renormalised field multiplet  $\phi$  with mass-dimension  $1 - \varepsilon/4$ , satisfying

$$\phi_0 = \mu^{-\varepsilon/4} Z \phi. \quad (1.170)$$

In turn, the bare Lagrangian is rewritten in terms of the renormalised couplings as<sup>6</sup>

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \mu^{-\varepsilon/2} \partial_\mu \phi^a Z_{ab}^2 \partial^\mu \phi^b - (\Lambda + \delta\Lambda) - (\sigma_a + \delta\sigma_a) \phi^a - \frac{1}{2} (M_{ab} + \delta M_{ab}) \phi^a \phi^b \\ & - \frac{1}{3!} (\tau_{abc} + \delta\tau_{abc}) \phi^a \phi^b \phi^c - \frac{1}{4!} (\lambda_{abcd} + \delta\lambda_{abcd}) \phi^a \phi^b \phi^c \phi^d. \end{aligned} \quad (1.171)$$

where

$$\begin{aligned} \Lambda^0 &= \Lambda + \delta\Lambda, \\ \sigma_a^0 &= \mu^{\varepsilon/4} (\sigma_e + \delta\sigma_e) Z_{ea}^{-1}, \\ M_{ab}^0 &= \mu^{\varepsilon/2} (M_{ef} + \delta M_{ef}) Z_{ea}^{-1} Z_{fb}^{-1}, \\ \tau_{abc}^0 &= \mu^{3\varepsilon/4} (\tau_{efg} + \delta\tau_{efg}) Z_{ea}^{-1} Z_{fb}^{-1} Z_{gc}^{-1}, \\ \lambda_{abcd}^0 &= \mu^\varepsilon (\lambda_{efgh} + \delta\lambda_{efgh}) Z_{ea}^{-1} Z_{fb}^{-1} Z_{gc}^{-1} Z_{hd}^{-1}. \end{aligned} \quad (1.172)$$

Based on these relations, both the scale  $\mu$  and the field-strength normalisation constant now exclusively appear in the kinetic Lagrangian. A subtlety arises however regarding the mass term, which in this configuration is no longer homogeneous to the momentum-squared  $p^2$  since  $[M] = 2 - \varepsilon/2$ . Hence, in order to maintain dimensional consistency in the expression of 2-point functions, it will be useful to define the auxiliary renormalised mass couplings  $m$  with mass-dimension 2,

$$m \equiv \mu^{\varepsilon/2} M, \quad (1.173)$$

and similarly for the corresponding counterterm  $\delta m$ , such that the bare mass matrix satisfies

$$M_{ab}^0 = \mu^{\varepsilon/2} (M_{ef} + \delta M_{ef}) Z_{ea}^{-1} Z_{fb}^{-1} = (m_{ef} + \delta m_{ef}) Z_{ea}^{-1} Z_{fb}^{-1}. \quad (1.174)$$

The quadratic part of the Lagrangian may then be rewritten in the form

$$\mathcal{L} \supset \frac{1}{2} \mu^{-\varepsilon/2} \left( \partial_\mu \phi^a Z_{ab}^2 \partial^\mu \phi^b - \frac{1}{2} (m_{ab} + \delta m_{ab}) \right) \phi^a \phi^b. \quad (1.175)$$

We reiterate that, as compared to the conventional prescription, the powers of  $\mu$  maintaining the dimensional consistency in loop computations here originate from the propagators. Restricting for simplicity the discussion to a theory with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \mu^{-\varepsilon/2} \left( \partial_\mu \phi^a Z_{ab}^2 \partial^\mu \phi^b - \frac{1}{2} (m_{ab} + \delta m_{ab}) \phi^a \phi^b \right) - \frac{1}{4!} (\lambda_{abcd} + \delta\lambda_{abcd}) \phi^a \phi^b \phi^c \phi^d, \quad (1.176)$$

<sup>6</sup>We reiterate that one can always define the renormalised fields in a basis where  $Z = Z^T$ .



the leading-order propagator is modified according to

$$iG_{\text{tree}}(p^2) = \frac{i}{\mu^{-\varepsilon/2}p^2 - \mu^{-\varepsilon/2}m + i\epsilon} = \frac{i\mu^{\varepsilon/2}}{p^2 - m + i\epsilon}, \quad (1.177)$$

and, for instance, the one-loop contribution to the 4-point function reads

$$-i\Gamma_{abcd}^{(1)}(s^2) = \frac{1}{2}\lambda_{abij}\lambda_{ijcd} \int \widetilde{d^d k} \frac{\mu^{\varepsilon/2}}{(s-k)^2 - m_{ii} + i\epsilon} \frac{\mu^{\varepsilon/2}}{k^2 - m_{jj} + i\epsilon}, \quad (1.178)$$

which is equivalent to Eq. (1.44) if one performs the replacement

$$M \leftrightarrow m. \quad (1.179)$$

In fact, the only notable difference compared to the conventional prescription occurs in the expression of the  $\varepsilon$ -dependent beta-functions and anomalous dimension. Reiterating the relation between the bare and renormalised scalar fields, one has

$$\phi_0 = \mu^{-\varepsilon/4} Z \phi, \quad (1.180)$$

and it is natural to define a modified field-strength renormalisation matrix

$$\widehat{Z} = \mu^{-\varepsilon/4} Z, \quad (1.181)$$

so that all occurrences of the renormalisation scale are contained in  $\widehat{Z}$ . The corresponding anomalous dimension matrix, noted  $\widehat{\gamma}_\varepsilon$ , hence reads

$$\widehat{\gamma}_\varepsilon = \widehat{Z}^{-1} \mu \frac{d\widehat{Z}}{d\mu} = Z^{-1} \mu \frac{dZ}{d\mu} - \frac{\varepsilon}{4} \mathbb{1} = \gamma - \frac{\varepsilon}{4} \mathbb{1}. \quad (1.182)$$

The bare quartic coupling is then rewritten as

$$\lambda_{efgh}^0 = (\lambda_{abcd} + \delta\lambda_{abcd}) \widehat{Z}_{ae}^{-1} \widehat{Z}_{bf}^{-1} \widehat{Z}_{cg}^{-1} \widehat{Z}_{dh}^{-1}, \quad (1.183)$$

so that (the anti-commutation operation between a tensor and a matrix was defined in Eq. (1.84))

$$0 = \beta_\varepsilon(\lambda_{abcd}) + \mu \frac{d\delta\lambda_{abcd}}{d\mu} - \{\lambda \leftarrow \widehat{\gamma}_\varepsilon\}_{abcd} - \{\delta\lambda \leftarrow \widehat{\gamma}_\varepsilon\}_{abcd}. \quad (1.184)$$

Defining the auxiliary quartic coupling and counterterm

$$\widehat{\lambda}_{efgh} = \lambda_{abcd} \widehat{Z}_{ae}^{-1} \widehat{Z}_{bf}^{-1} \widehat{Z}_{cg}^{-1} \widehat{Z}_{dh}^{-1} \quad (1.185)$$

$$\delta\widehat{\lambda}_{efgh} = \delta\lambda_{abcd} \widehat{Z}_{ae}^{-1} \widehat{Z}_{bf}^{-1} \widehat{Z}_{cg}^{-1} \widehat{Z}_{dh}^{-1}, \quad (1.186)$$

with mass-dimension  $[\widehat{\lambda}] = \varepsilon$ , Eq.(1.183) simply becomes

$$\lambda_{abcd}^0 = \widehat{\lambda}_{abcd} + \delta\widehat{\lambda}_{abcd}. \quad (1.187)$$

We then observe that the counterterm can be rewritten in terms of the hatted couplings. For instance, from the one-loop quartic counterterm

$$\delta\lambda_{abcd} = \frac{1}{\varepsilon(16\pi^2)} \lambda_{abij} \lambda_{ijcd}, \quad (1.188)$$

we obtain

$$\delta\widehat{\lambda}_{abcd} = \frac{1}{\varepsilon(16\pi^2)} \widehat{\lambda}_{abij} \widehat{Z}_{ik}^2 \widehat{Z}_{jl}^2 \widehat{\lambda}_{klcd}. \quad (1.189)$$

More generally,  $\delta\widehat{\lambda}_{abcd}$  is obtained by replacing every vertex with its hatted counterpart, and every internal index contraction by  $\widehat{Z}^2$ , namely

$$\lambda_{ijkl} \rightarrow \widehat{\lambda}_{ijkl}, \quad \delta^{ij} \rightarrow \widehat{Z}_{ij}^2. \quad (1.190)$$

Since  $\widehat{\lambda}_{abcd}$  has canonical mass-dimension in the  $(4 - \varepsilon)$ -dimensional theory, its beta-function is  $\varepsilon$ -independent. Taking the RG-derivative of Eq. (1.187), one gets

$$0 = \beta(\widehat{\lambda}_{abcd}) + \mu \frac{d\widehat{\lambda}_{abcd}}{d\mu} = \beta(\widehat{\lambda}_{abcd}) + \beta(\widehat{\lambda}_{efgh}) \frac{\partial \widehat{\lambda}_{abcd}}{\partial \widehat{\lambda}_{efgh}} + \mu \frac{d\widehat{Z}_{ef}^2}{d\mu} \frac{\partial \widehat{\lambda}_{abcd}}{\partial \widehat{Z}_{ef}^2}. \quad (1.191)$$

Then, using

$$\mu \frac{d\widehat{Z}^2}{d\mu} = 2\widehat{Z}\widehat{\gamma}_\varepsilon\widehat{Z} = 2\widehat{Z}\gamma\widehat{Z} - \frac{\varepsilon}{2}\widehat{Z}^2, \quad (1.192)$$

we obtain, at order  $\mathcal{O}(\varepsilon^0)$ ,

$$0 = \beta(\widehat{\lambda}_{abcd}) - \frac{1}{2}\mathcal{N}_I \left[ \delta_{[1]} \widehat{\lambda}_{abcd} \right], \quad (1.193)$$

where  $\mathcal{N}_I$  is an operator that ‘‘counts’’ the number of internal lines. From the Euler formula for planar graphs, stating that

$$n = I - V + 1, \quad (1.194)$$

with  $n$  the number of loops,  $I$  the number of internal lines and  $V$  the number of vertices; and from the fact that in the considered theory, the  $n$ -loop contributions to the 4-point function contain  $n + 1$  quartic vertices, we have that

$$I = 2n. \quad (1.195)$$

Therefore, expanding the beta-function perturbatively as

$$\beta(\widehat{\lambda}_{abcd}) = \beta^{(1)}(\widehat{\lambda}_{abcd}) + \beta^{(2)}(\widehat{\lambda}_{abcd}) + \dots, \quad (1.196)$$

one directly obtains

$$\beta^{(n)}(\widehat{\lambda}_{abcd}) = n \delta_{[1]}^{(n)} \widehat{\lambda}_{abcd}, \quad (1.197)$$

with  $\delta_{[1]}^{(n)} \widehat{\lambda}_{abcd}$  standing for the  $n$ -loop contribution to the first-order pole of the quartic counterterm. Finally, to obtain the beta-function of the original quartic couplings, one simply has to write

$$\beta_\varepsilon(\widehat{\lambda}_{abcd}) = \left[ \beta(\lambda_{efgh}) - \{\lambda \leftarrow \widehat{\gamma}_\varepsilon\}_{efgh} \right] Z_{ea}^{-1} Z_{fb}^{-1} Z_{gc}^{-1} Z_{hd}^{-1}, \quad (1.198)$$

yielding

$$\beta_\varepsilon(\lambda_{abcd}) = -\varepsilon \lambda_{abcd} + \beta(\widehat{\lambda}_{efgh}) Z_{ea} Z_{fb} Z_{gc} Z_{hd} + \{\lambda \leftarrow \gamma\}_{abcd}. \quad (1.199)$$

Hence, we finally obtain for the  $n$ -loop contribution to  $\beta(\lambda_{abcd})$ :

$$\beta^{(n)}(\lambda_{abcd}) = n \delta_{[1]}^{(n)} \lambda_{abcd} + \left\{ \lambda \leftarrow \gamma^{(n)} \right\}_{abcd}, \quad (1.200)$$

consistently with the result obtain from the standard  $\mu$ -prescription (see Eq. (1.117)).

The computation of the mass matrix beta-function is can either be carried out in the same way as in section 3, starting from the relation

$$M_{ab}^0 = (m_{ef} + \delta m_{ef}) Z_{ea}^{-1} Z_{fb}^{-1}, \quad (1.201)$$

or in analogy with the above computations, if one differentiates instead the relation

$$M_{ab}^0 = (M_{ef} + \delta M_{ef}) \widehat{Z}_{ea}^{-1} \widehat{Z}_{fb}^{-1}. \quad (1.202)$$

As compared to the standard prescription, the mass matrix  $M$  here exhibits an anomalous scaling, consistently with the fact that its mass-dimension equals  $2 - \frac{\varepsilon}{2}$  in the  $(4 - \varepsilon)$ -dimensional theory. Defining

$$\widehat{M}_{ab} = M_{ef} \widehat{Z}_{ea}^{-1} \widehat{Z}_{fb}^{-1} \quad (1.203)$$

and reiterating the above procedure, it is easy to obtain, starting from Eq. (1.202),

$$0 = \beta(\widehat{M}_{ab}) - \frac{1}{2}\mathcal{N}_I \left[ \delta_{[1]} \widehat{M}_{ab} \right], \quad (1.204)$$

and in turn

$$\beta^{(n)}(M_{ab}) = n \delta_{[1]}^{(n)} M_{ab} + \{M \leftarrow \gamma\}_{ab}, \quad (1.205)$$

as expected from Eq.(1.116).

## 6.2 Diagonal mass-matrix across scales

In the previous sections, we have made the assumption that the mass matrix  $M$  (and in turn the leading-order propagator) is diagonal at the renormalisation scale  $\mu$ . On the other hand, from the expression of the beta-functions derived in section 3, we see that, starting from the one-loop level the mass matrix will in general develop off-diagonal components when varying the renormalisation scale. For instance, the one-loop contributions to the mass-matrix beta-function was shown in Eq. (1.118) to be equal to

$$\beta^{(1)}(M_{ab}) = \frac{1}{16\pi^2} \lambda_{abij} M_{ij}. \quad (1.206)$$

We show in this section that a modified renormalisation scheme allows to maintain the diagonal structure of the mass matrix across scales, by introducing an RG-scale-dependent orthogonal matrix  $U$  such that

$$\widetilde{M} = U M U^T = \text{diag}(M_1, \dots, M_N). \quad (1.207)$$

In turn, the Lagrangian density in the  $(4 - \varepsilon)$ -dimensional theory can be rewritten, using the modified prescription introduced in the previous section, as

$$\mathcal{L} = \frac{1}{2} \mu^{-\varepsilon/2} \left[ \partial_\mu \widetilde{\phi}^a \widetilde{Z}_{ab}^2 \partial^\mu \widetilde{\phi}^b - \frac{1}{2} (\widetilde{m}_{ab} + \delta \widetilde{m}_{ab}) \widetilde{\phi}^a \widetilde{\phi}^b \right] - \frac{1}{4!} \left( \widetilde{\lambda}_{abcd} + \delta \widetilde{\lambda}_{abcd} \right) \widetilde{\phi}^a \widetilde{\phi}^b \widetilde{\phi}^c \widetilde{\phi}^d, \quad (1.208)$$

where, in particular,

$$\widetilde{\phi} = U \phi, \quad \widetilde{Z} = Z U^T. \quad (1.209)$$

The tilded renormalised couplings and fields are expressed in the basis that diagonalises the mass matrix and are related to their bare counterpart via

$$\phi_0^a = \mu^{-\varepsilon/4} \widetilde{Z}^{ab} \widetilde{\phi}^b, \quad (1.210)$$

$$M_{ab}^0 = (\widetilde{m}_{ef} + \delta \widetilde{m}_{ef}) \widetilde{Z}_{-1}^{ea} \widetilde{Z}_{-1}^{fb}, \quad (1.211)$$

$$\lambda_{abcd}^0 = \mu^\varepsilon \left( \widetilde{\lambda}_{efgh} + \delta \widetilde{\lambda}_{efgh} \right) \widetilde{Z}_{-1}^{ea} \widetilde{Z}_{-1}^{fb} \widetilde{Z}_{-1}^{gc} \widetilde{Z}_{-1}^{hd}. \quad (1.212)$$

The renormalised propagator reads, in the mass basis:

$$i \widetilde{G}_R(p^2) = \frac{i}{\mu^{-\varepsilon/2} p^2 \mathbb{1} - \widetilde{M} - \widetilde{\Pi}_R(p^2)} = \mu^{\varepsilon/2} \frac{i}{p^2 \mathbb{1} - \widetilde{m} - \widetilde{\pi}_R(p^2)}, \quad (1.213)$$

where  $\widetilde{\Pi}_R(p^2)$  stands for the renormalised matrix of self-energies expressed in the mass basis, and  $\widetilde{\pi}_R(p^2)$  for its counterpart with mass-dimension 2. The latter reads

$$\widetilde{\pi}_R(p^2) = \widetilde{\pi}(p^2) + \delta \widetilde{m} - p^2 \widetilde{\delta}_\phi, \quad (1.214)$$

where the field-strength counterterm  $\widetilde{\delta}_\phi$  satisfies

$$\widetilde{Z}^T \widetilde{Z} = 1 + \widetilde{\delta}_\phi = U Z^2 U^T = 1 + U \delta_\phi U^T, \quad (1.215)$$

and hence  $\widetilde{\delta}_\phi = U \delta_\phi U^T$ . A perturbative expansion of the renormalised propagator yields, as previously,

$$\begin{aligned} i \widetilde{G}_R(p^2) &= \frac{i}{\mu^{-\varepsilon/2} p^2 \mathbb{1} - \widetilde{M}} + \frac{i}{\mu^{-\varepsilon/2} p^2 \mathbb{1} - \widetilde{M}} \left( -i \widetilde{\Pi}_R(p^2) \right) \frac{i}{\mu^{-\varepsilon/2} p^2 \mathbb{1} - \widetilde{M}} + \dots \\ &= \frac{i \mu^{\varepsilon/2}}{p^2 \mathbb{1} - \widetilde{m}} + \frac{i \mu^{\varepsilon/2}}{p^2 \mathbb{1} - \widetilde{m}} \left( -i \widetilde{\pi}_R(p^2) \right) \frac{i \mu^{\varepsilon/2}}{p^2 \mathbb{1} - \widetilde{m}} + \dots, \end{aligned} \quad (1.216)$$

so the leading-order propagator is diagonal. In turn, the renormalisation of the theory from counterterms is performed in the same way as in section 2 where we had assumed that the mass matrix (and therefore

the leading-order propagator) was diagonal. For instance, we find at one-loop the following expressions for the counterterms

$$\delta^{(1)} \widetilde{M}_{ab} = \frac{1}{\varepsilon (16\pi^2)} \widetilde{\lambda}_{abef} \widetilde{M}^{ef}, \quad (1.217)$$

$$\delta^{(1)} \widetilde{\lambda}_{abcd} = \frac{1}{\varepsilon (16\pi^2)} \left( \widetilde{\lambda}_{abef} \widetilde{\lambda}_{efcd} + \widetilde{\lambda}_{acef} \widetilde{\lambda}_{efbd} + \widetilde{\lambda}_{adef} \widetilde{\lambda}_{efbc} \right). \quad (1.218)$$

It is useful to note that, in the mass-basis, the counterterms take the same form as previously. This means that, rotating the fields and couplings back to the original basis we exactly recover the expressions obtained in section 2 where the mass-matrix was assumed to be diagonal. In fact, this non-trivial feature is due to the absence of non-local divergences, *i.e.* of contributions involving logarithms of the mass matrix in the counterterms. As previously, the  $n$ -loop beta-functions for the mass matrix and the quartic coupling are given (see section 3 and 6.1) by

$$\beta^{(n)} \left( \widetilde{M}_{ab} \right) = n \delta_{[1]}^{(n)} \widetilde{M}_{ab} + \left\{ \widetilde{M} \leftarrow \widetilde{\gamma}^{(n)} \right\}_{ab} \quad (1.219)$$

$$\beta^{(n)} \left( \widetilde{\lambda}_{abcd} \right) = n \delta_{[1]}^{(n)} \widetilde{\lambda}_{abcd} + \left\{ \widetilde{\lambda} \leftarrow \widetilde{\gamma}^{(n)} \right\}_{abcd}. \quad (1.220)$$

On the other hand, the anomalous dimension matrix receives here an additional contribution stemming from the rotation to the mass-basis:

$$\widetilde{\gamma} = \widetilde{Z}^{-1} D \widetilde{Z} = U Z^{-1} [(DZ) U^T + Z (DU^T)] = U \gamma U^T + U (DU^T). \quad (1.221)$$

The additional contribution involves the  $\mu$ -derivative of the rotation matrix  $U$ , and can be shown to be skew-symmetric by observing that

$$D(UU^T) = 0 = (DU)U^T + U(DU^T) = V + V^T, \quad (1.222)$$

where

$$V \equiv (DU)U^T = -V^T = -U(DU^T). \quad (1.223)$$

It is in fact a general feature that the anomalous dimension matrix acquires a skew-symmetric component as soon as an RG-dependent rotation of the scalar fields and couplings is introduced in the theory. In our case, the anomalous dimension matrix takes the form

$$\widetilde{\gamma} = U \gamma U^T - V, \quad (1.224)$$

and the mass matrix beta function reads (in matrix form)

$$\beta^{(n)} \left( \widetilde{M} \right) = U \beta^{(n)}(M) U^T + \left[ V^{(n)}, \widetilde{M} \right] = U \beta^{(n)}(M) U^T + \left\{ V^{(n)} \rightarrow \widetilde{M} \right\}, \quad (1.225)$$

where  $V^{(n)}$  stands for the  $n$ -loop contribution in the perturbative expansion of  $V$ . In other words, the beta-function of the diagonalised mass matrix  $\widetilde{M}$  is not only obtained by rotating  $\beta(M)$  to the mass basis, but receives an additional contribution involving the skew-symmetric part of the anomalous dimension matrix. The same observation holds for the quartic coupling  $\widetilde{\lambda}$ , and one obtains

$$\beta^{(n)} \left( \widetilde{\lambda}_{abcd} \right) = U^{ae} U^{bf} U^{cg} U^{dh} \beta^{(n)}(\lambda_{abcd}) + \left\{ V^{(n)} \rightarrow \widetilde{\lambda} \right\}_{abcd}. \quad (1.226)$$

The explicit form of  $V$  (and of every  $V^{(n)}$ ) is computed from the requirement that the rotated mass matrix  $\widetilde{M}$  must preserve its diagonal form along the RG-flow, namely that the off-diagonal components of  $\beta(\widetilde{M})$  must vanish. Thus, we directly obtain from Eq. (1.225), for  $a \neq b$ :

$$V_{ab}^{(n)} = \frac{[U \beta^{(n)}(M) U^T]^{ab}}{M_a - M_b}, \quad (1.227)$$

where  $M_i = \widetilde{M}_{ii}$  denotes the  $i^{\text{th}}$  eigenvalue of the mass matrix. Hence, for instance, the one-loop beta-function of the diagonal mass-matrix becomes

$$\beta^{(1)}(\widetilde{M}_{ab}) = \frac{1}{16\pi^2} \left[ \widetilde{\lambda}_{abij} \widetilde{M}_{ij} + \sum_{e \neq a} \frac{\widetilde{\lambda}_{aeij} \widetilde{M}_{ij}}{M_a - M_e} \widetilde{M}_{eb} + \sum_{e \neq b} \frac{\widetilde{\lambda}_{beij} \widetilde{M}_{ij}}{M_b - M_e} \widetilde{M}_{ae} \right], \quad (1.228)$$

from which one can explicitly check that for  $a \neq b$ ,  $\beta^{(1)}(\widetilde{M}_{ab})$  indeed vanishes.

# Appendix

## A Double propagators

We consider a generic loop diagram with  $n+1$  internal propagators, with masses squared  $m_i^2, m_{a_1}^2, \dots, m_{a_n}^2$ . Such a diagram involves a loop integral of the generic form

$$I(m_i^2, m_{a_1}^2, \dots, m_{a_n}^2) = \int \widetilde{d^d k} \frac{1}{k^2 - m_i^2} X(k^2, m_{a_1}^2, \dots, m_{a_n}^2), \quad (1.A1)$$

where  $X$  is an arbitrary function of the masses of the  $n$  remaining propagating scalars. Doubling the propagator of the scalar with mass  $m_i^2$  yields

$$J(m_i^2, m_j^2, m_{a_1}^2, \dots, m_{a_n}^2) = \int \widetilde{d^d k} \frac{1}{k^2 - m_i^2} \frac{1}{k^2 - m_j^2} X(k^2, m_{a_1}^2, \dots, m_{a_n}^2). \quad (1.A2)$$

Then, since

$$\frac{1}{k^2 - m_i^2} - \frac{1}{k^2 - m_j^2} = \frac{m_i^2 - m_j^2}{(k^2 - m_i^2)(k^2 - m_j^2)}, \quad (1.A3)$$

we directly obtain, for  $m_i^2 \neq m_j^2$ ,

$$J(m_i^2, m_j^2, m_{a_1}^2, \dots, m_{a_n}^2) = \frac{I(m_i^2, m_{a_1}^2, \dots, m_{a_n}^2) - I(m_j^2, m_{a_1}^2, \dots, m_{a_n}^2)}{m_i^2 - m_j^2}. \quad (1.A4)$$

The case where  $m_i^2 = m_j^2$  is simply handled by expressing the derivative of  $I$  with respect to its first argument, noted  $I'$ ,

$$I'(m_i^2, m_{a_1}^2, \dots, m_{a_n}^2) = \int \widetilde{d^d k} \frac{1}{(k^2 - m_i^2)^2} X(k^2, m_{a_1}^2, \dots, m_{a_n}^2), \quad (1.A5)$$

directly leading to the result

$$J(m_i^2, m_i^2, m_{a_1}^2, \dots, m_{a_n}^2) = I'(m_i^2, m_{a_1}^2, \dots, m_{a_n}^2) = \lim_{m_j^2 \rightarrow m_i^2} \frac{I(m_i^2, m_{a_1}^2, \dots, m_{a_n}^2) - I(m_j^2, m_{a_1}^2, \dots, m_{a_n}^2)}{m_i^2 - m_j^2}. \quad (1.A6)$$

## B Two-loop sunset integral

We provide in this appendix the complete derivation of the pole structure of the two-loop sunset integral:

$$S(x, y, z, p^2) = \begin{array}{c} \begin{array}{c} \xleftarrow{x, k} \\ \text{---} \bigcirc \text{---} \\ \xrightarrow{y, p+k+q} \\ \text{---} \bigcirc \text{---} \\ \xleftarrow{z, q} \end{array} \end{array} \quad (1.B1)$$

$$= \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{1}{k^2 - x} \frac{1}{(p+k+q)^2 - y} \frac{1}{q^2 - z} \equiv \int \widetilde{d^d k} \int \widetilde{d^d q} \sigma(x, y, z, p, k, q)$$

It is convenient to first define and compute the auxiliary quantity  $\Sigma$  defined as

$$\Sigma(x, y, z, p^2) = xS(x', y, z, p^2) + yS(x, y', z, p^2) + zS(x, y, z', p^2) + p^2 \frac{dS}{dp^2}(x, y, z, p^2). \quad (1.B2)$$

The first term can be expressed in the form

$$\begin{aligned}
xS(x', y, z, p^2) &= \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{x}{[k^2 - x]^2} \frac{1}{(p + k + q)^2 - y} \frac{1}{q^2 - z} \\
&= \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{[x - k^2] + l^2}{[k^2 - x]^2} \frac{1}{(p + k + q)^2 - y} \frac{1}{q^2 - z} \\
&= -S(x, y, z, p^2) + \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{k^2}{[k^2 - x]^2} \frac{1}{(p + k + q)^2 - y} \frac{1}{q^2 - z}
\end{aligned} \tag{1.B3}$$

and similarly

$$yS(x, y', z, p^2) = -S(x, y, z, p^2) + \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{1}{k^2 - x} \frac{(p + k + q)^2}{[(p + k + q)^2 - y]^2} \frac{1}{q^2 - z} \tag{1.B4}$$

$$zS(x, y, z, p^2) = -S(x, y, z, p^2) + \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{1}{k^2 - x} \frac{1}{(p + k + q)^2 - y} \frac{q^2}{[q^2 - z]^2}. \tag{1.B5}$$

The  $p^2$ -derivative is computed as

$$\begin{aligned}
p^2 \frac{dS}{dp^2}(x, y, z, p^2) &= \frac{1}{2} p^\mu \frac{dS}{dp^\mu}(x, y, z, p^2) \\
&= - \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{1}{k^2 - x} \frac{p^\mu (p + k + q)_\mu}{[(p + k + q)^2 - y]^2} \frac{1}{q^2 - z} \\
&= \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{1}{k^2 - x} \frac{-(p + k + q)^2 + k^\mu (p + k + q)_\mu + q^\mu (p + k + q)_\mu}{[(p + k + q)^2 - y]^2} \frac{1}{q^2 - z}.
\end{aligned} \tag{1.B6}$$

Gathering all above expressions, we observe that  $\Sigma$  can be written as

$$\Sigma(x, y, z, p^2) = -3S(x, y, z, p^2) - \frac{1}{2} \int \widetilde{d^d k} \int \widetilde{d^d q} k^\mu \frac{\partial \sigma}{\partial k^\mu} + q^\mu \frac{\partial \sigma}{\partial q^\mu}. \tag{1.B7}$$

Integrating by parts allows to further rewrite

$$\Sigma(x, y, z, p^2) = -3S(x, y, z, p^2) + \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{\delta^\mu_\nu}{2} \left( \frac{\partial k^\nu}{\partial k^\mu} + \frac{\partial q^\nu}{\partial q^\mu} \right) \sigma(x, y, z, p, k, q), \tag{1.B8}$$

and since, in  $d = 4 - \varepsilon$  dimensions, one has

$$\delta^\mu_\nu \frac{\partial k^\nu}{\partial k^\mu} = \delta^\mu_\nu \frac{\partial q^\nu}{\partial q^\mu} = d = 4 - \varepsilon, \tag{1.B9}$$

we finally obtain

$$\Sigma(x, y, z, p^2) = (1 - \varepsilon) S(x, y, z, p^2). \tag{1.B10}$$

In fact, one would have obtained the same result directly from dimensional analysis, given that the mass-dimension of  $S$  equals  $2 - 2\varepsilon$ .

It is useful at this stage to define the master integral (for  $n = 1, 2$ )

$$I_n(x, y, z, p^2) = \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{1}{[k^2 - x]^n} \frac{1}{(p + k + q)^2 - y} \frac{1}{q^2 - z}, \tag{1.B11}$$

so that Eq. (1.B10) takes the form

$$\Sigma(x, y, z, p^2) = (1 - \varepsilon) S(x, y, z, p^2) = \left[ x I_2(x, y, z, p^2) + (x \leftrightarrow y) + (x \leftrightarrow z) \right] + p^2 \frac{\partial I_1}{\partial p^2}(x, y, z, p^2). \tag{1.B12}$$

First performing the integral over  $q$ , we obtain (omitting the arguments of  $I_n$  for clarity):

$$\begin{aligned}
I_n &= \int \widetilde{d^d k} \int \widetilde{d^d q} \int_0^1 d\eta \frac{1}{[k^2 - x]^n} \frac{1}{[q^2 + \eta(1-\eta)(k+p)^2 - \eta y - (1-\eta)z]^2} \\
&= i \frac{(4\pi)^{\varepsilon/2}}{16\pi^2} \Gamma\left(\frac{\varepsilon}{2}\right) \int \widetilde{d^d k} \int_0^1 d\eta \frac{1}{[k^2 - x]^n} \frac{1}{[-\eta(1-\eta)(k+p)^2 + \eta y + (1-\eta)z]^{\varepsilon/2}} \\
&= i(-1)^{-\varepsilon/2} \frac{(4\pi)^{\varepsilon/2}}{16\pi^2} \Gamma\left(\frac{\varepsilon}{2}\right) \int \widetilde{d^d k} \int_0^1 d\eta \frac{1}{[k^2 - x]^n} \frac{[\eta(1-\eta)]^{-\varepsilon/2}}{\left[(k+p)^2 - \frac{\eta y + (1-\eta)z}{\eta(1-\eta)}\right]^{\varepsilon/2}}. \tag{1.B13}
\end{aligned}$$

Applying the generalised Feynman parameterisation formula,

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 d\rho \frac{\rho^{\alpha-1} (1-\rho)^{\beta-1}}{[\rho A + (1-\rho)B]^{\alpha+\beta}}, \tag{1.B14}$$

yields, after a change of variables,

$$I_n = i \frac{(4\pi)^{\varepsilon/2}}{16\pi^2} (-1)^{-\varepsilon/2} \frac{\Gamma(n + \frac{\varepsilon}{2})}{\Gamma(n)} \int \widetilde{d^d k} \int_0^1 d\eta \int_0^1 d\rho \frac{(1-\rho)^{n-1} \rho^{\varepsilon/2-1} [\eta(1-\eta)]^{-\varepsilon/2}}{\left[k^2 + \rho(1-\rho)p^2 - (1-\rho)x - \rho \frac{\eta y + (1-\eta)z}{\eta(1-\eta)}\right]^{n+\varepsilon/2}} \tag{1.B15}$$

Integration over  $k$  gives

$$I_n = (-1)^{n-1} \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2} e^{\varepsilon\gamma_E} \frac{\Gamma(n-2+\varepsilon)}{\Gamma(n)} \int_0^1 d\eta [\eta(1-\eta)]^{-\varepsilon/2} \int_0^1 d\rho \Delta^{2-n} (1-\rho)^{n-1} \rho^{\varepsilon/2-1} \tilde{\Delta}^{-\varepsilon}, \tag{1.B16}$$

where we have introduced the quantity

$$\Delta = \rho \frac{\eta y + (1-\eta)z}{\eta(1-\eta)} + (1-\rho)x - \rho(1-\rho)p^2 \tag{1.B17}$$

and its dimensionless counterpart

$$\tilde{\Delta} = \frac{\Delta}{\mu^2}. \tag{1.B18}$$

In order to apply Eq. (1.B12), we first proceed with the computation of  $I_2$ . From Eq. (1.B16), we get

$$I_2 = -\frac{\mu^{-2\varepsilon}}{(16\pi^2)^2} e^{\varepsilon\gamma_E} \Gamma(\varepsilon) \int_0^1 d\eta [\eta(1-\eta)]^{-\varepsilon/2} \int_0^1 d\rho (1-\rho) \rho^{\varepsilon/2-1} \tilde{\Delta}^{-\varepsilon}. \tag{1.B19}$$

The second order pole structure arises from the  $\varepsilon$ -expansion of  $\Gamma(\varepsilon)$ , combined with the fact that the integral over  $\rho$  diverges as  $\varepsilon^{-1}$ . We have in particular that

$$e^{\varepsilon\gamma_E} \Gamma(\varepsilon) = \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon), \tag{1.B20}$$

and

$$\int_0^1 d\eta [\eta(1-\eta)]^{-\varepsilon/2} = B\left(1 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right) = \frac{\Gamma(1 - \frac{\varepsilon}{2}) \Gamma(1 - \frac{\varepsilon}{2})}{\Gamma(2 - \varepsilon)} = 1 + \varepsilon + \mathcal{O}(\varepsilon^2), \tag{1.B21}$$

$$\int_0^1 d\rho (1-\rho) \rho^{\varepsilon/2-1} = B\left(2, -\frac{\varepsilon}{2}\right) = \frac{\Gamma(2) \Gamma(-\frac{\varepsilon}{2})}{\Gamma(2 - \frac{\varepsilon}{2})} = \frac{2}{\varepsilon} - 1 + \mathcal{O}(\varepsilon), \tag{1.B22}$$

where  $B$  denotes here the Euler beta-function, defined as

$$B(a, b) = \int_0^1 dx x^{a-1} (1-x)^{b-1}. \tag{1.B23}$$



In fact, the above expansions are all we need to express the diverging contributions to  $I_2$ . The reason is that we may rewrite  $\Delta$  as

$$\Delta = x \left( 1 + \frac{\rho}{x} \left[ \frac{\eta y + (1-\eta)z}{\eta(1-\eta)} - (x+p^2) \right] + \rho^2 \frac{p^2}{x} \right), \quad (1.B24)$$

so that

$$\tilde{\Delta}^{-\varepsilon} = \left( \frac{x}{\tilde{\mu}^2} \right)^{-\varepsilon} \left[ 1 - \varepsilon \log \left( 1 + \frac{\rho}{x} \left[ \frac{\eta y + (1-\eta)z}{\eta(1-\eta)} - (x+p^2) \right] + \rho^2 \frac{p^2}{x} \right) + \mathcal{O}(\varepsilon^2) \right]. \quad (1.B25)$$

In this form, we see that the logarithm scales as  $\mathcal{O}(\rho)$  when  $\rho \rightarrow 0$ , hence taming the divergence of the integral over  $\rho$ . In turn, the  $\mathcal{O}(\varepsilon)$  contributions in the expansion of  $\tilde{\Delta}^{-\varepsilon}$  do not contribute to the pole structure of  $I_2$ , and we finally obtain

$$I_2(x, y, z, p^2) = -\frac{\mu^{-2\varepsilon}}{(16\pi^2)^2} \left[ \frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} \left( 1 - 2 \log \frac{x}{\tilde{\mu}^2} \right) + \mathcal{O}(1) \right]. \quad (1.B26)$$

Turning to the computation of the last term in Eq. (1.B12), we have from Eq. (1.B16)

$$I_1 = \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2} e^{\varepsilon\gamma_E} \Gamma(-1+\varepsilon) \int_0^1 d\eta [\eta(1-\eta)]^{-\varepsilon/2} \int_0^1 d\rho \Delta \rho^{\varepsilon/2-1} \tilde{\Delta}^{-\varepsilon}. \quad (1.B27)$$

Taking the  $p^2$ -derivative tames the divergence in the integral over  $\rho$  since

$$\frac{\partial \Delta}{\partial p^2} = -\rho(1-\rho). \quad (1.B28)$$

It is then straightforward to arrive at the result

$$p^2 \frac{\partial I_1}{\partial p^2} = \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2} \left[ \frac{p^2}{2\varepsilon} + \mathcal{O}(1) \right]. \quad (1.B29)$$

Inserting the above results in Eq. (1.B12), we obtain the final form of the diverging contributions to  $S$ :

$$S(x, y, z, p^2) = \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2} \left\{ \frac{-2}{\varepsilon^2} (x+y+z) + \frac{1}{\varepsilon} \left( \frac{p^2}{2} - x - y - z \right) + \frac{2}{\varepsilon} [A(x) + A(y) + A(z)] \right\} + \mathcal{O}(1). \quad (1.B30)$$

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## Chapter 2

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# Renormalisation group equations for general gauge theories

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We have arrived in the previous chapter at the conclusion that a consistent perturbative description of a scalar field theory relies on (i) the expression of the renormalised  $n$ -point functions and (ii) the knowledge of the full set of beta-functions and anomalous dimensions. This observation equally holds when it comes to more general theories, and particular to gauge-Yukawa theories which constitute the standard framework for the study of the Standard Model and its postulated extensions. We call renormalisation group equations (RGEs) the set of differential equations governing the evolution of the couplings and fields under variations of the renormalisation scale – among which the Callan-Symanzik equation, introduced in section 4. The knowledge of the RGEs is therefore an essential component of the theoretical description of a given perturbative QFT, and is required for many phenomenological applications.

We have exemplified in chapter 1 the procedure allowing to obtain the RGEs in a simple scalar theory up to the two-loop level. If the resulting expressions, Eqs. (1.118), (1.119)–(1.121), have been obtained at the cost of a reasonable effort, this task becomes increasingly difficult as additional degrees of freedom are introduced in the theory. On the other hand, performing the computations without explicitly specifying the field content of the theory has the major advantage that the resulting expressions are, in turn, applicable to any particular model sharing the same structure. For this reason, a long-standing programme was initiated forty years ago [29, 30, 31, 32, 33, 34, 35] with the aim of deriving *once and for all* the RGEs of a general gauge theory up to the two-loop level. Over the years, the formalism was extended to take into account Abelian kinetic mixing [36, 37] and the mixing between scalar degrees of freedom [38] induced by off-diagonal components in the anomalous dimension matrix. The application of the dummy field method to derive the beta-functions of dimensionful couplings (as discussed in section 2) was reviewed in [38]. Additionally, various mistakes and misprints in the original results have since been uncovered.

More recently, an elegant formalism was developed in [39] to express and derive the RGEs of a general gauge theory. In addition to systematically taking into account kinetic and scalar mixing, the full set of 3-loop beta-functions for the gauge couplings was obtained, for the first time, in a theory based on a semi-simple gauge group (generalising the results of [40] in theories based on a simple gauge group). The approach of [39] relies on the local renormalisation group framework [41, 42, 43, 44, 45], which reveals non-trivial relations between the beta-functions of gauge, Yukawa and scalar quartic couplings at different loop-orders: the so-called Weyl consistency conditions. Not only these relations provide a powerful way of cross-checking existing results; they also constitute a powerful tool in the determination of still unknown higher-loop order beta-functions, as demonstrated in [46] where the four-loop gauge-coupling beta-functions in the Standard Model were obtained for the first time.

With these general expressions at hand, one has in turn to specialise the formulae to any particular gauge-theory of interest, whose phenomenological properties depend on the form of the beta-functions. If performed by hand, this task proves in most cases remarkably difficult, and a more straightforward

(and reliable) approach consists in implementing the general expressions in a computer code. For this reason, a module of the Mathematica package SARAH [47] implements since its version 4.0 the general results of [29, 30, 31, 35], and has been developed jointly with the dedicated Python tool PyR@TE [48, 49].

In this chapter, we present a new version of the tool PyR@TE – PyR@TE 3 [9] –, implementing the new formalism developed in [39]. In addition to the possibility of computing the gauge-coupling beta-functions up to three-loop, many improvements were made to the code which has, on this occasion, been essentially rewritten. As compared to its previous versions, PyR@TE 3 is more flexible in the implementation of the model, offering the possibility to compute the RGEs of a larger class of theories. In addition, its performance was increased by a factor  $\mathcal{O}(10^2)$  to  $\mathcal{O}(10^5)$  for a class of relatively simple extensions of the Standard Model.

We review section 1 the general formalism [39] allowing to parameterise the Lagrangian density of the most general gauge-Yukawa theory. In section 2, we present the general form of the beta-functions in this formalism and explicitly show how to obtain the RGEs for the dimensionful couplings using the dummy-field method [10] – deferring to appendix (A) the presentation of the resulting expression. Finally, we thoroughly discuss in section 3 their implementation in PyR@TE 3, including in particular a description of some of the newly implemented algorithms and the validation of the results against other available tools.

## 1 Definitions

### 1.1 Semi-simple gauge groups: a unified notation

We consider in this chapter a general theory, invariant under gauge transformation belonging to the semi-simple gauge group

$$\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_M, \quad (2.1)$$

written as the product of  $M \geq 1$  simple Lie groups. We will at this stage make the assumption that  $\mathcal{G}$  contains at most one  $U(1)$  factor, ensuring the absence of kinetic mixing in the abelian sector. It will be shown in a second step how to generalise the present formalism to the most general case, where an arbitrary number of  $U(1)$  gauge factors can be considered.

To each gauge factor  $\mathcal{G}_p$  ( $p = 1, \dots, M$ ) corresponds a set of  $\mathcal{N}_p = \dim \mathcal{G}_p$  vector fields  $(V_p)_\mu^a$  ( $a = 1, \dots, \mathcal{N}_p$ ) linearly transforming in the adjoint representation of  $\mathcal{G}_p$ ,

$$(V_p)_\mu^a \rightarrow (V_p)_\mu^a + \frac{1}{g_p} \partial_\mu \theta^a(x) - (f_p)^{abc} \theta^b(x) (V_p)_\mu^c, \quad (2.2)$$

where  $g_p$  is the gauge coupling associated to  $\mathcal{G}_p$ , and  $f_p$  denotes the structure constants of the corresponding Lie algebra. Hence, noting  $(T_p)^a$  the gauge generators in some representation of the algebra of  $\mathcal{G}_p$ , one has

$$\left[ (T_p)^a, (T_p)^b \right] = i (f_p)^{abc} (T_p)^c. \quad (2.3)$$

The notation is greatly simplified by introducing a set of indices  $A, B, C, \dots$  ranging from 1 to  $\dim \mathcal{G}$ , where

$$\dim \mathcal{G} = \sum_{p=1}^M \dim \mathcal{G}_p = \sum_{p=1}^M \mathcal{N}_p \equiv N_G. \quad (2.4)$$

In turn, all vector fields may be gathered in the  $N_G$ -dimensional multiplet  $V_\mu^A$  ( $A = 1, \dots, N_G$ ), such that

$$V_\mu^A = \left( (V_1)_\mu^1, \dots, (V_1)_\mu^{\mathcal{N}_1}, \dots, (V_M)_\mu^1, \dots, (V_M)_\mu^{\mathcal{N}_M} \right). \quad (2.5)$$

This newly defined vector multiplet linearly transforms in the adjoint representation of  $\mathcal{G}$ :

$$V_\mu^A \rightarrow V_\mu^A + \frac{1}{g_A} \partial^\mu \theta^A(x) - f^{ABC} \theta_B(x) V_\mu^C, \quad (2.6)$$

where

$$g_A = (g_1, \dots, g_1, \dots, g_M, \dots, g_M) , \quad (2.7)$$

and where  $f$  now stands for the structure constants of the whole semi-simple Lie algebra. In any of its representation, the gauge generators  $T^A$  ( $A = 1, \dots, N_G$ ) hence satisfy

$$[T^A, T^B] = if^{ABC}T^C . \quad (2.8)$$

In a next step, the field-strength tensor  $F_{\mu\nu}^A$  is defined in the usual way

$$F_{\mu\nu}^A = \partial_\mu V_\nu^A - \partial_\nu V_\mu^A + g^A f^{ABC} V_\mu^B V_\nu^C , \quad (2.9)$$

so that the kinetic Lagrangian for the gauge fields reads

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} F_{\mu\nu}^A F_A^{\mu\nu} . \quad (2.10)$$

Following [39], we may absorb the gauge coupling constants in the vector fields by defining the auxiliary multiplet

$$\bar{V}_\mu^A = g_A V_\mu^A , \quad (2.11)$$

and the corresponding field-strength tensor  $\bar{F}_{\mu\nu}^A$ , expressed as<sup>1</sup>

$$\bar{F}_{\mu\nu}^A = g_A F_{\mu\nu}^A = \partial_\mu \bar{V}_\nu^A - \partial_\nu \bar{V}_\mu^A + f^{ABC} \bar{V}_\mu^B \bar{V}_\nu^C , \quad (2.12)$$

and the kinetic Lagrangian may be rewritten in terms of  $\bar{F}_{\mu\nu}^A$

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} g_A^{-2} \bar{F}_{\mu\nu}^A \bar{F}_A^{\mu\nu} = -\frac{1}{4} G_{AB}^{-2} \bar{F}_{\mu\nu}^A \bar{F}_B^{\mu\nu} , \quad (2.13)$$

where we have conveniently defined the diagonal matrix  $G$ , such that

$$G_{AB} = \delta_{AB} g_A . \quad (2.14)$$

As discussed in detail in [39, 9] the above notations can be straightforwardly generalised to the case where multiple  $U(1)$  gauge factors are present, by letting  $G_{AB}^2$  develop non-vanishing off-diagonal components in the abelian sector. Considering a general gauge group with  $p$  abelian gauge factors,

$$\mathcal{G} = U(1)^p \times \mathcal{G}_{\text{N.A.}} , \quad (2.15)$$

the symmetric matrix  $G^2$  takes the generic block-diagonal form:

$$G_{AB}^2 = \begin{pmatrix} h_{11} & \cdots & h_{1p} & & & \\ \vdots & \ddots & \vdots & & & \\ h_{p1} & \cdots & h_{pp} & & & \\ & & & g_{p+1}^2 & & \\ & & & & \ddots & \\ & & & & & g_n^2 \end{pmatrix}_{AB} \equiv \begin{pmatrix} H^2 & 0 \\ 0 & G_{\text{N.A.}}^2 \end{pmatrix}_{AB} . \quad (2.16)$$

The  $p \times p$  symmetric block  $H^2$  can always be decomposed as

$$H^2 = G_{\text{mix}} G_{\text{mix}}^T , \quad (2.17)$$

where  $G_{\text{mix}}$  is only defined up orthogonal transformations of the form

$$G_{\text{mix}} \rightarrow G_{\text{mix}} O , \quad O O^T = \mathbb{1} , \quad (2.18)$$

and can therefore always be chosen to be triangular (either upper or lower).

Finally re-defining  $\bar{V}_\mu \rightarrow V_\mu$  and  $\bar{F}_{\mu\nu} \rightarrow F_{\mu\nu}$ , we arrive at the following general form for the gauge kinetic Lagrangian

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} G_{AB}^{-2} F_{\mu\nu}^A F_B^{\mu\nu} . \quad (2.19)$$

---

<sup>1</sup>We have used the fact that  $g_A^2 f^{ABC} = g_B g_C f^{ABC}$  due to the semi-simple structure of the algebra:  $f^{ABC}$  can only be non-zero if all three indices correspond to the same simple sub-algebra.

## 1.2 General gauge-Yukawa theories

We introduce in the gauge theory considered above a set of  $N_F$  left-handed Weyl fermions gathered in the multiplet  $\psi^i$ , and a set  $N_S$  of real scalars noted  $\phi^a$ . Under infinitesimal gauge transformations, these multiplets undergo

$$\psi^i \rightarrow \psi^i + i\theta^A(x) (T_\psi^A)^{ij} \psi^j, \quad (2.20)$$

$$\phi^a \rightarrow \phi^a + i\theta^A(x) (T_\phi^A)^{ab} \phi^b, \quad (2.21)$$

with  $T_\psi$  and  $T_\phi$  the corresponding gauge generators. In turn, the fermionic and scalar covariant derivatives read

$$D_\mu \psi^i = \partial_\mu \psi^i - iV_\mu^A (T_\psi^A)^{ij} \psi^j, \quad (2.22)$$

$$D_\mu \phi^a = \partial_\mu \phi^a - iV_\mu^A (T_\phi^A)^{ab} \phi^b, \quad (2.23)$$

respectively. We note that the gauge couplings do not appear explicitly in the expansion of the covariant derivatives since they have been absorbed in the re-definition of the vector fields. The kinetic Lagrangian involving the vector, fermion and scalar fields is then given in the form

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} G_{AB}^{-2} F_{\mu\nu}^A F_B^{\mu\nu} + i\psi_i^\dagger \bar{\sigma}^\mu D_\mu \psi^i + \frac{1}{2} D_\mu \phi_a D^\mu \phi^a, \quad (2.24)$$

where  $\bar{\sigma}^\mu = (\mathbb{1}, -\boldsymbol{\sigma})^\mu$ , with  $\sigma_i$  ( $i = 1, 2, 3$ ) the usual Pauli matrices. The most general interaction Lagrangian is parameterised as

$$\begin{aligned} \mathcal{L}_{\text{int}} = & -\frac{1}{2} (Y_{aij} \psi^i \psi^j \phi^a + \text{h.c.}) - \frac{1}{2} (M_{ij} \psi^i \psi^j + \text{h.c.}) \\ & - \frac{1}{2} \mu_{ab} \phi^a \phi^b - \frac{1}{3!} t_{abc} \phi^a \phi^b \phi^c - \frac{1}{4!} \lambda_{abcd} \phi^a \phi^b \phi^c \phi^d, \end{aligned} \quad (2.25)$$

where h.c. denotes the hermitian conjugate. The Yukawa and fermion-mass couplings both are symmetric under exchange of their fermionic indices, while  $\mu$ ,  $t$  and  $\lambda$  are fully symmetric tensors populated with real entries.

The authors of [39, 45] have introduced a convenient notation where the left-handed Weyl spinors and their right-handed hermitian conjugate are gathered in the Majorana-like spinor  $\Psi$ , defined as

$$\Psi^i \equiv \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix}^i, \quad i = 1, \dots, 2N_F. \quad (2.26)$$

The corresponding covariant derivative reads

$$D_\mu \Psi^i = \partial_\mu \Psi^i - iV_\mu^A (T_\Psi^A)^{ij} \Psi^j, \quad (2.27)$$

where the newly defined fermionic gauge generators take the following block-diagonal form

$$T_\Psi^A = \begin{pmatrix} T_\psi^A & 0 \\ 0 & -(T_\psi^A)^* \end{pmatrix}. \quad (2.28)$$

In turn,  $\mathcal{L}_{\text{kin}}$  can be rewritten in terms of  $\Psi$  as [39]

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4} G_{AB}^{-2} F_{\mu\nu}^A F_B^{\mu\nu} + \frac{i}{2} \Psi^\text{T} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} D_\mu \Psi + \frac{1}{2} D_\mu \phi_a D^\mu \phi^a, \quad (2.29)$$

where  $\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma})^\mu$ . Similarly, defining the Yukawa and fermion-mass couplings in the  $2N_F$ -dimensional space as

$$y_{aij} \equiv \begin{pmatrix} Y_a & 0 \\ 0 & Y_a^* \end{pmatrix}_{ij}, \quad m_{ij} \equiv \begin{pmatrix} M & 0 \\ 0 & M^* \end{pmatrix}_{ij}, \quad (2.30)$$

allows rewrite the interaction Lagrangian (2.25) in the form

$$\mathcal{L}_{\text{int}} = -\frac{1}{2} y_{aij} \Psi^i \Psi^j \phi^a - \frac{1}{2} m_{ij} \Psi^i \Psi^j - \frac{1}{2} \mu_{ab} \phi^a \phi^b - \frac{1}{3!} t_{abc} \phi^a \phi^b \phi^c - \frac{1}{4!} \lambda_{abcd} \phi^a \phi^b \phi^c \phi^d. \quad (2.31)$$

## 2 Beta-functions

Following [39], we parameterise the all-order beta-functions of the dimensionless (*i.e.* gauge, Yukawa and quartic) couplings as

$$\beta_{AB} \equiv \frac{dG_{AB}^2}{dt} = \frac{1}{2} \sum_{n=1}^{\infty} \kappa^n \sum_{\text{perm}} G_{AC}^2 \beta_{CD}^{(n)} G_{DB}^2, \quad (2.32)$$

$$\beta_{aij} \equiv \frac{dy_{aij}}{dt} = \frac{1}{2} \sum_{n=1}^{\infty} \kappa^n \sum_{\text{perm}} \beta_{aij}^{(n)}, \quad (2.33)$$

$$\beta_{abcd} \equiv \frac{d\lambda_{abcd}}{dt} = \frac{1}{4!} \sum_{n=1}^{\infty} \kappa^n \sum_{\text{perm}} \beta_{abcd}^{(n)}, \quad (2.34)$$

where  $\kappa = 1/(16\pi^2)$ . The ‘‘RG-time’’  $t$  is defined such that

$$\frac{d}{dt} = \mu \frac{d}{d\mu}, \quad (2.35)$$

meaning that  $t = \log \frac{\mu}{\mu_0}$  with  $\mu_0$  some arbitrary energy scale. The permutations are understood to be taken among the two gauge indices  $A, B$ , the two fermionic indices  $i, j$  and the four scalar indices  $a, b, c, d$  in Eqs. (2.32), (2.33) and (2.34), respectively. The beta-functions are given as a perturbative expansion in terms of the  $\beta^{(n)}$ , with  $n$  denoting the loop-order. As mentioned in the introduction of this chapter, while the expressions of  $\beta_{AB}^{(1,2)}$ ,  $\beta_{aij}^{(1,2)}$  and  $\beta_{abcd}^{(1,2)}$  have been known for decades [29, 30, 31, 32, 33, 34, 35], the authors of [39] have derived in this formalism the gauge coupling beta-functions up to the three-loop level,  $\beta_{AB}^{(3)}$ , and reformulated the known expressions in the Yukawa and quartic sectors. They have however not provided the corresponding beta-functions for the dimensionful couplings  $m_{ij}$ ,  $\mu_{ab}$  and  $t_{abcd}$ , which we have computed in [10]. Following Eqs. (2.32) - (2.34), we define the beta-functions for fermion mass, trilinear and scalar mass couplings as

$$\beta_{ij} \equiv \frac{dm_{ij}}{dt} = \frac{1}{2} \sum_{n=1}^{\infty} \kappa^n \sum_{\text{perm}} \beta_{ij}^{(n)}, \quad (2.36)$$

$$\beta_{abc} \equiv \frac{dt_{abc}}{dt} = \frac{1}{3!} \sum_{n=1}^{\infty} \kappa^n \sum_{\text{perm}} \beta_{abc}^{(n)}, \quad (2.37)$$

$$\beta_{ab} \equiv \frac{d\mu_{ab}}{dt} = \frac{1}{2} \sum_{n=1}^{\infty} \kappa^n \sum_{\text{perm}} \beta_{ab}^{(n)}, \quad (2.38)$$

respectively. Assuming that the beta-functions for the Yukawa and quartic couplings are known, it is in fact always possible to determine the full set of corresponding expressions for the dimensionful couplings. Such a procedure has been referred to in the literature as the dummy-field method [35, 50, 38] and used in the past to extend the formulae derived in the seminal works [29, 30, 31, 32, 33, 34]. In the following, we briefly summarise this procedure as usually presented in the literature, and will show in a second step that a conceptually different but strictly equivalent approach involving background fields can also be followed.

We consider a general theory containing only dimensionless couplings, for which the expressions of the beta-functions are known. In our notation, the interacting part of the associated Lagrangian density is given by

$$\mathcal{L}_{\text{int}} = -\frac{1}{2} y_{aij} \Psi_i \Psi_j \phi^a - \frac{1}{4!} \lambda_{abcd} \phi^a \phi^b \phi^c \phi^d. \quad (2.39)$$

The starting point of the dummy-field method is to extend the theory with a non-propagating scalar field with no gauge interactions. Reusing the notation from [38], this dummy field, noted  $\phi_{\hat{d}}$ , hence satisfies  $D_\mu \phi_{\hat{d}} = 0$ . Making explicit the terms involving  $\phi_{\hat{d}}$  and discarding a constant as well as a term linear in

$\phi_{\hat{j}}$ , the Lagrangian (2.39) may be rewritten

$$\begin{aligned} \mathcal{L}_{\text{int}} = & -\frac{1}{2}y_{aij}\Psi_i\Psi_j\phi^a - \frac{1}{4!}\lambda_{abcd}\phi^a\phi^b\phi^c\phi^d \\ & -\frac{1}{2}y_{\hat{a}ij}\Psi_i\Psi_j\phi_{\hat{a}} - \frac{1}{4}\lambda_{ab\hat{a}\hat{a}}\phi^a\phi^b\phi_{\hat{a}}\phi_{\hat{a}} - \frac{1}{3!}\lambda_{abcd}\phi^a\phi^b\phi^c\phi_{\hat{a}}. \end{aligned} \quad (2.40)$$

The above form makes it clear that the Lagrangian (2.31) of a theory containing dimensionful coupling can be recovered if the following identifications are made:

$$y_{\hat{a}ij}\phi_{\hat{a}} = m_{ij}, \quad \lambda_{ab\hat{a}\hat{a}}\phi_{\hat{a}}\phi_{\hat{a}} = 2\mu_{ab}, \quad \lambda_{abcd}\phi_{\hat{a}} = t_{abc}. \quad (2.41)$$

The mapping (2.41) allows for a straightforward derivation of the individual contributions to the beta-functions of the dimensionful couplings, starting from the known expressions in the dimensionless case. For instance, the diagrams contributing to the fermion mass beta-function,  $\beta_{ij}^{(n)}$ , are obtained from the individual contributions to the Yukawa couplings beta-function,  $\beta_{aij}^{(n)}$ , when the external scalar field  $\phi_a$  is taken to be the dummy field  $\phi_{\hat{j}}$ . Consequently, one should in practice perform the following replacements:

$$a \rightarrow \hat{a}, \quad y_{aij} \rightarrow m_{ij}, \quad \lambda_{abcd} \rightarrow t_{abc}.$$

Similarly, starting from the quartic couplings RGEs, one may derive the beta-functions for the trilinear and scalar mass couplings when one or two of the external scalar legs are replaced by a dummy field. This procedure is applied on a diagrammatic basis, allowing in particular the identification of unphysical tadpole contributions which need to be discarded [38].

In fact, the same results can be obtained without the need to introduce a dummy field, but rather by introducing for the scalar fields in the original theory with Lagrangian (2.39) a classical background component (see also [51]). Namely, we may define the dynamical fields  $\varphi$  around the classical background<sup>2</sup>  $\phi_{\mathbf{c}}$ ,

$$\phi = \phi_{\mathbf{c}} + \varphi, \quad (2.42)$$

so that the Lagrangian (2.39) takes the form

$$\begin{aligned} \mathcal{L}_{\text{int}}(\varphi) = & -\frac{1}{2}m_{ij}(\phi_{\mathbf{c}})\Psi^i\Psi^j - \frac{1}{2}y_{aij}\Psi^i\Psi^j\varphi^a \\ & -\Lambda(\phi_{\mathbf{c}}) - \sigma_a(\phi_{\mathbf{c}})\varphi^a - \frac{1}{2}\mu_{ab}(\phi_{\mathbf{c}})\varphi^a\varphi^b - \frac{1}{3!}t_{abc}(\phi_{\mathbf{c}})\varphi^a\varphi^b\varphi^c - \frac{1}{4!}\lambda_{abcd}\varphi^a\varphi^b\varphi^c\varphi^d, \end{aligned} \quad (2.43)$$

with background field-dependent dimensionful couplings:

$$m_{ij}(\phi_{\mathbf{c}}) = y_{aij}\phi_{\mathbf{c}}^a, \quad (2.44a)$$

$$\Lambda(\phi_{\mathbf{c}}) = \frac{1}{4!}\lambda_{abcd}\phi_{\mathbf{c}}^a\phi_{\mathbf{c}}^b\phi_{\mathbf{c}}^c\phi_{\mathbf{c}}^d \quad (2.44b)$$

$$\sigma_a(\phi_{\mathbf{c}}) = \frac{1}{3!}\lambda_{abcd}\phi_{\mathbf{c}}^b\phi_{\mathbf{c}}^c\phi_{\mathbf{c}}^d, \quad (2.44c)$$

$$\mu_{ab}(\phi_{\mathbf{c}}) = \frac{1}{2}\lambda_{abcd}\phi_{\mathbf{c}}^c\phi_{\mathbf{c}}^d, \quad (2.44d)$$

$$t_{abc}(\phi_{\mathbf{c}}) = \lambda_{abcd}\phi_{\mathbf{c}}^d. \quad (2.44e)$$

The Lagrangian density thereby obtained is very similar to Eq. (2.40), with the background-field multiplet playing the role of the dummy field. In particular, since  $\phi_{\mathbf{c}}$  is a background field, it does not propagate, has no gauge interactions, and satisfies  $\partial_{\mu}\phi_{\mathbf{c}} = 0$ .

The beta-functions of the dimensionful couplings are then simply expressed starting from their expression in terms of the dimensionless ones and the background-field. For instance, the fermion mass parameter satisfies (dropping  $\phi_{\mathbf{c}}$  in the argument of  $m_{ij}$  for better clarity)

$$\beta_{ij} \equiv \beta(m_{ij}) = \frac{d}{dt}m_{ij} = \frac{d}{dt}(y_{aij}\phi_{\mathbf{c}}^a) = \beta_{aij}\phi_{\mathbf{c}}^a - y_{aij}\gamma^{ab}\phi_{\mathbf{c}}^b. \quad (2.45)$$

<sup>2</sup>A similar procedure was applied in chapter 1, section 5.3 to define the effective potential.

The first term is obtained by contracting the scalar leg of the Yukawa-coupling beta-function with the background field (similar to the dummy-field prescription). The second term systematically removes any contribution stemming from the leg corrections to  $y_{aij}$ , in correspondence with the dummy-field method prescription stating that the tadpole contributions must be individually discarded [38]. We obtain similar relations for the dimensionful couplings of the scalar potential, involving the beta-function of the quartic couplings. For instance,

$$\beta_{abc} \equiv \beta(t_{abc}) = \beta_{abcd}\phi_{\mathbf{c}}^d - \lambda_{abcd}\gamma^{de}\phi_{\mathbf{c}}^e, \quad (2.46)$$

$$\beta_{ab} \equiv \beta(\mu_{ab}) = \frac{1}{2} (\beta_{abcd}\phi_{\mathbf{c}}^c\phi_{\mathbf{c}}^d - \lambda_{abcd}\gamma^{ce}\phi_{\mathbf{c}}^e\phi_{\mathbf{c}}^d - \lambda_{abcd}\phi_{\mathbf{c}}^c\gamma^{de}\phi_{\mathbf{c}}^e). \quad (2.47)$$

In a second step, with the expression of the dimensionless-coupling beta-functions at hand, any contraction involving the background field is substituted with its expression in terms of the dimensionful couplings. For instance, the one-loop pure-scalar contribution to the quartic beta-function (computed in chapter 1) reads

$$\frac{1}{4!} \sum_{\text{perm}} \beta_{abcd}^{(1)} \supset \frac{1}{4!} \sum_{\text{perm}} 3\lambda_{abij}\lambda_{ijcd} = \lambda_{abij}\lambda_{ijcd} + \lambda_{acij}\lambda_{ijbd} + \lambda_{adij}\lambda_{ijbc}, \quad (2.48)$$

and, with view on Eqs. (2.46) and (2.47), contributes to the trilinear and scalar mass couplings beta-functions according to

$$\begin{aligned} \frac{1}{3!} \sum_{\text{perm}} \beta_{abc}^{(1)} &\supset (\lambda_{abij}\lambda_{ijcd} + \lambda_{acij}\lambda_{ijbd} + \lambda_{adij}\lambda_{ijbc}) \phi_{\mathbf{c}}^d \\ &= \lambda_{abij}t_{ijc} + \lambda_{acij}t_{ijb} + t_{aij}\lambda_{ijbc} \\ &= \frac{1}{3!} \sum_{\text{perm}} 3\lambda_{abij}t_{ijc}, \end{aligned} \quad (2.49)$$

and

$$\begin{aligned} \frac{1}{2!} \sum_{\text{perm}} \beta_{ab}^{(1)} &\supset \frac{1}{2} (\lambda_{abij}\lambda_{ijcd} + \lambda_{acij}\lambda_{ijbd} + \lambda_{adij}\lambda_{ijbc}) \phi_{\mathbf{c}}^c\phi_{\mathbf{c}}^d \\ &= \frac{1}{2} (2\lambda_{abij}m_{ij} + t_{aij}t_{ijb} + t_{aij}t_{ijb}) \\ &= \frac{1}{2!} \sum_{\text{perm}} \lambda_{abij}m_{ij} + t_{aij}t_{ijb}, \end{aligned} \quad (2.50)$$

respectively. We therefore conclude from Eqs. (2.49) and (2.50) that the pure-scalar contributions to the one-loop beta-functions of  $t_{abc}$  and  $\mu_{ab}$  read

$$\beta_{acd}^{(1)} \supset 3\lambda_{abij}t_{ijc}, \quad \beta_{ab}^{(1)} \supset \lambda_{abij}m_{ij} + t_{aij}t_{ijb}, \quad (2.51)$$

in agreement with our previous findings [10] based on the dummy-field method. Either way, one is able to derive all contributions to the beta-functions of the dimensionful parameters  $m_{ij}$ ,  $t_{abc}$  and  $\mu_{ab}$ . The resulting expressions, which we have obtained in [10], are listed in appendix A. We finally note that it could be useful in the future to also extend these results to the tadpole term  $\sigma_a$  and the vacuum energy  $\Lambda$ : while the former can be absorbed by a shift of the scalar fields and the latter is irrelevant in the determination of the vacuum of the theory at a fixed renormalisation scale, their RG-evolution is in general non-trivial and needed to properly understand the properties of the scalar potential across RG-scales.

### 3 Implementation in PyR@TE 3

Having presented the general formalism allowing to express the beta-functions for every (dimensionless and dimensionful) coupling of a general gauge theory, we now proceed with their implementation in PyR@TE 3. First, we discuss in section 3.1 the general structure of the programme, as well as some



important technical aspects related to the improved performance compared to PyR@TE 2 [49]. The subsequent sections will provide a pedagogical introduction to PyR@TE 3, covering the implementation of the model and the various options and functionalities made available to the user. We shall finally discuss in section 3.6 the validation of the output against previous versions of the software and other available tools.

### 3.1 Technical overview

We provide in figure 2.1 a schematic representation of the main steps implemented in the program that allows, starting from the model file, to eventually obtain the set of beta-functions for all couplings of the model. Each of these main steps is detailed in the following, and will be illustrated for concreteness with the example of the Standard Model (SM), based on the gauge group  $\mathcal{G}_{\text{SM}} = U(1) \times SU(2) \times SU(3)$ . To keep this presentation concise, we will more precisely focus on the quartic self-interaction of the Higgs doublet, denoted  $\Phi$ , and decomposed in terms of its real components according to<sup>3</sup>

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_1 + i\sigma_1 \\ \pi_2 + i\sigma_2 \end{pmatrix}. \quad (2.52)$$

In turn, the quartic part of the Higgs potential is given by

$$V(\Phi) \supset \frac{\lambda}{2} (\Phi^\dagger \Phi)^2 = \frac{\lambda}{8} (\pi_1^2 + \pi_2^2 + \sigma_1^2 + \sigma_2^2)^2, \quad (2.53)$$

where the factor 1/2 is purely conventional and will slightly simplify the forthcoming expressions.

#### Mapping onto the general formalism

In input, all relevant information on the model is gathered in a model file with a syntax which will be extensively described in the next sections; the minimal amount of information required to compute the beta-functions being (i) the gauge group, (ii) the field content of the model, and (iii) the interaction Lagrangian. The next step, corresponding to the block **Mapping (1)** in figure 2.1, consists in mapping the model of interest onto the general formalism introduced in section 1. In this regard, one needs to

1. Express the general scalar multiplet  $\phi^a$  in terms of the real components of every scalar field present in the model,
2. Express the general Weyl multiplet  $\Psi^i$  in terms of every fermion present in the model,
3. Express the gauge coupling matrix  $G_{AB}$  in terms of the gauge couplings of each individual gauge factor composing semi-simple gauge group,
4. Obtain the explicit form of the gauge generators,  $T_\Psi^A$  and  $T_\phi^A$ ,
5. Express the various tensors involved in the expression of the general interaction Lagrangian density ( $y_{aij}, \lambda_{abcd}, \mu_{ab}, \dots$ ).

For the Standard Model, the real scalar multiplet  $\phi$  – step (1.) – has  $N_S = 4$  components and takes the form

$$\phi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \sigma_1 \\ \sigma_2 \end{pmatrix}, \quad (2.54)$$

while, for the Weyl multiplet  $\Psi$  – step (2.) – we arrive at  $N_F = 15$  components when taking into account the chirality, weak-isospin and colour of every fermion (but not the generations). In step (3.), we obtain for the SM gauge-coupling matrix the following structure:

$$G_{AB} = \text{diag} \left( g_1, g_2, g_2, g_2, \underbrace{g_3, \dots, g_3}_{8 \text{ times}} \right), \quad (2.55)$$

---

<sup>3</sup>The factor  $1/\sqrt{2}$  enforces canonical normalisation in the kinetic Lagrangian written in terms of the real components.

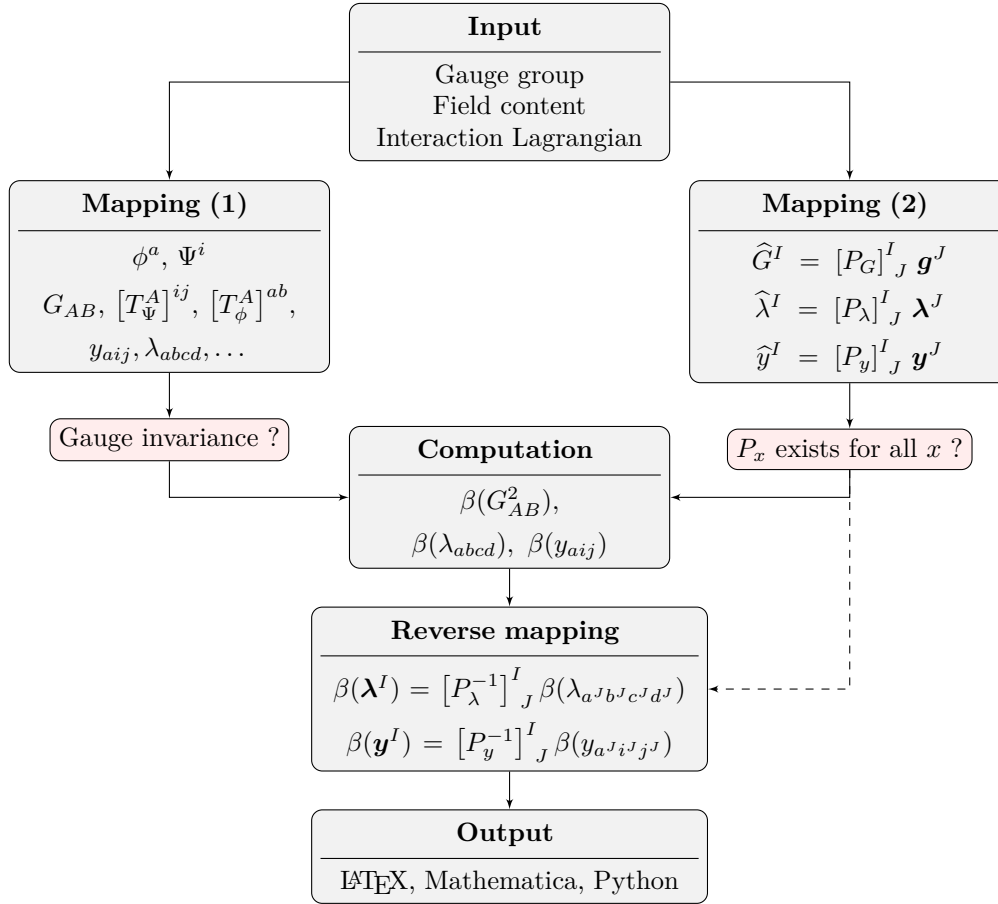


Figure 2.1: General structure of the program. The various steps are described in detail and exemplified in the main text.

with, in total,  $N_G = 12$  gauge degrees of freedom. The gauge couplings  $g_1$ ,  $g_2$  and  $g_3$  correspond to the  $U(1)$ ,  $SU(2)$  and  $SU(3)$  gauge factors, respectively.

To obtain the expression of the gauge generators in step (4.), PyR@TE relies since version 2 on an internal module called PyLie [49], allowing to obtain an explicit form for the representation matrices in any given representation of a simple Lie group<sup>4</sup>. On subtlety arises however for complex scalars, for which a translation must be given between the generation matrices in the complex representation of the Lie algebra and the representation matrices of the real multiplet  $\phi$ . Let us consider a general  $n$ -component complex field of the form

$$\Phi = \frac{1}{\sqrt{2}} (\pi + i\sigma), \quad (2.56)$$

and with a set of gauge generators  $T_\Phi^A$  in the complex representation of the Lie algebra. We define the real multiplet

$$\phi = \begin{pmatrix} \pi \\ \sigma \end{pmatrix}, \quad (2.57)$$

with  $2n$  components, and the corresponding set of gauge generators  $T_\phi^A$ . It can be shown that  $T_\phi^A$  is given, in terms of  $T_\Phi^A$  as

$$T_\phi^A = i \operatorname{Im} \left\{ \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \otimes T_\Phi^A \right\}, \quad (2.58)$$

<sup>4</sup>In version 2, only the  $SU(N)$  and  $SO(N)$  Lie groups were included, while since version 3 it is possible, in addition, to implement the symplectic  $Sp(2N)$  family as well as the exceptional Lie groups  $G_2$ ,  $F_4$  and  $E_{6,7,8}$ .

where  $\otimes$  denotes the Kronecker product between two matrices. As required by the  $\phi$  being a real scalar field,  $T_\phi^A$  is always a purely imaginary, skew-symmetric matrix. While the former property is manifest from Eq. (2.58), the latter can be explicitly checked by writing<sup>5</sup>

$$(T_\phi^A)^T = i \operatorname{Im} \left\{ \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \otimes (T_\Phi^A)^* \right\} = i \operatorname{Im} \left\{ \left[ \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \otimes (T_\Phi^A) \right]^* \right\} = -iT_\phi^A. \quad (2.59)$$

Back to the example of the SM Higgs doublet (with hypercharge 1/2), the generators  $T_\Phi^A$  ( $A = 1, 2, 3, 4$ ) under the electroweak gauge group  $U(1) \times SU(2)$  are given by

$$T_\Phi^{1,2,3,4} = \frac{1}{2} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \quad (2.60)$$

and applying Eq. (2.58) yields

$$T_\phi^{1,2,3,4} = \frac{i}{2} \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}. \quad (2.61)$$

We finally note that, for  $A = 5, \dots, 12$ , the matrices  $T_\phi^A$  identically vanish since  $\Phi$  transforms trivially under  $SU(3)$ .

Finally, step (5.) requires for instance to express the rank 4,  $N_S$ -dimensional symmetric tensor  $\lambda_{abcd}$  in terms of the couplings of the scalar potential. One way to proceed is to compute the fourth derivatives of  $V$  with respect to the real field multiplet  $\phi$ . Equivalently, PyR@TE 3 expands the Lagrangian in terms of the real components  $\phi^i$  ( $i = 1, \dots, N_S$ ), and subsequently identifies each coefficient with a particular component of the tensor  $\lambda_{abcd}$ . For instance, in the SM, we have from Eq.(2.53)

$$V(\Phi) \supset \frac{\lambda}{8} (\pi_1^4 + 2\pi_1^2\pi_2^2 + \dots + \sigma_2^4) \equiv \frac{1}{4!} \lambda_{abcd} \phi^a \phi^b \phi^c \phi^d, \quad (2.62)$$

yielding for instance (properly taking into account the symmetry of  $\lambda_{abcd}$  under permutation of its indices)

$$\lambda_{1111} = 3\lambda, \quad \lambda_{1122} = \lambda, \quad \dots \quad \lambda_{4444} = 3\lambda. \quad (2.63)$$

Representing the 4-dimensional, rank 4 tensor  $\lambda$  in the form of a  $4 \times 4$  array of  $4 \times 4$  matrices yields the overall structure<sup>6</sup>

$$[\lambda]_{abcd} = \begin{bmatrix} \begin{pmatrix} 3\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} & \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 3\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 3\lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} & \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 3\lambda \end{pmatrix} \end{bmatrix}, \quad (2.64)$$

<sup>5</sup>Note that the Kronecker product satisfies  $(A \otimes B)^T = A^T \otimes B^T$  and  $(A \otimes B)^* = A^* \otimes B^*$ . In addition, we have used the hermiticity of the representation matrices  $T_\Phi^A$ .

<sup>6</sup>To the human eye, this structure looks already rather complicated in case as simple as the Standard Model, with only one quartic coupling and 4 scalar components.

which, due to the symmetries of the model, is essentially composed of zero entries. This predominance of null entries in the tensors (we will call such tensors *sparse*) is in fact characteristic of models with a high degree of symmetry (and hence of any gauge theory). This observation is rather intuitive: the constraint of a symmetric Lagrangian only allows a very specific subset of contractions between the components of the scalar and fermion multiplets. Quite importantly, the large increase in performance compared to the previous versions of the program stems from this property: The algorithms have been specifically designed in PyR@TE 3 to take advantage of the sparsity of the tensors. In this regard, the highly efficient `dict()` data structure in Python conveniently allows to store the content of the sparse tensors in the form of a dictionary of keys.

If not otherwise specified by the user, PyR@TE 3 will check before continuing further that gauge invariance is satisfied at the level of the full interaction Lagrangian. For instance, in the quartic sector, gauge invariance translates into the following constraint on the tensor of quartic couplings  $\lambda_{abcd}$ :

$$(T_\phi^A)_{ae}\lambda_{ebcd} + (T_\phi^A)_{be}\lambda_{aecd} + (T_\phi^A)_{ce}\lambda_{abed} + (T_\phi^A)_{de}\lambda_{abce} = 0. \quad \forall a, b, c, d, A. \quad (2.65)$$

### Mapping onto a minimal subset of couplings

Once every tensor in the theory has been computed in terms of the user input (the representation matrices and every tensor of couplings), an additional step before the actual computation consists in identifying a minimal subset of beta-functions to be computed from the general formulae. For instance, it is clear from Eq. (2.64) that computing  $\beta(\lambda_{1111})$  is sufficient to recover the expression of  $\beta(\lambda)$ . In the general case, PyR@TE first determines a minimal set of linearly independent components in each tensor. For the quartic couplings, we shall denote this list of linearly independent components by  $\hat{\lambda}$ . In this process, it is explicitly checked that (i)  $\hat{\lambda}$  contains as many elements as the number of quartic couplings in the model, and (ii) that the matrix  $P_\lambda$  relating the elements of  $\hat{\lambda}$  and  $\lambda$ , the  $N_\lambda$ -component vector populated with the quartic couplings is invertible. In our example,  $N_\lambda = 1$  and we (trivially) obtain

$$\hat{\lambda} = (\lambda_{1111}) = (3\lambda) = (3) \cdot (\lambda), \quad \lambda = (\lambda), \quad (2.66)$$

so the  $1 \times 1$  matrix  $P_\lambda$  and its inverse read

$$P_\lambda = (3), \quad P_\lambda^{-1} = \left(\frac{1}{3}\right). \quad (2.67)$$

The computation will be aborted if there are less than  $N_\lambda$  linearly independent elements among the components of the tensor  $\lambda$  (and similarly for every other tensor of couplings). Such a situation occurs if the user has implemented a Lagrangian density containing a redundant basis of operators. In that case, too many free parameters (couplings) are introduced in the model, and it is only possible to compute the beta-functions for some linear combinations of these couplings.

### Computation of the beta-functions

The final step is then to compute the beta-functions for each element of the vector  $\hat{\lambda}$  (and similarly for  $\hat{G}$ ,  $\hat{y}$ , ...) using the general formulae of [39] and [10]. In our example, only one beta-function needs to be computed:  $\beta(\lambda_{1111})$ . For instance, the pure scalar contributions to this beta function at one-loop are computed from Eq. (2.48):

$$\beta^{(1)}(\lambda_{1111}) \supset \frac{1}{4!} \sum_{\text{perm}} 3\lambda_{11ij}\lambda_{ij11} = 3\lambda_{11ij}\lambda_{ij11} = 36\lambda^2. \quad (2.68)$$

Finally, inverting the relation between  $\hat{\lambda}$  and  $\lambda$  allows to obtain the beta-functions of each quartic coupling of the model, through

$$\lambda = P_\lambda^{-1}\hat{\lambda} \quad \rightarrow \quad \beta(\lambda^I) = [P_\lambda^{-1}]^I_J \beta(\hat{\lambda}^J). \quad (2.69)$$

In our simple example, this yields

$$\beta(\lambda) = \frac{1}{3}\beta(\lambda_{1111}) \quad \rightarrow \quad \beta^{(1)}(\lambda) \supset 12\lambda^2. \quad (2.70)$$

### 3.2 Download and installation

The new version of PyR@TE may be downloaded from the following GitHub web page:

<https://github.com/LSartore/pyrate>

No further installation procedure is needed and the user may start working in PyR@TE's main folder. However, PyR@TE relies on a number of Python modules which need to be installed by the user if not already present on his/her machine. We show here the full list of dependencies:

- `Python`  $\geq$  3.6
- `PyYAML`  $\geq^*$  5.3
- `Sympy`  $\geq$  1.5
- `h5py`  $\geq^*$  2.10
- `Numpy`  $\geq^*$  1.18
- `Scipy`  $\geq^*$  1.4
- `Matplotlib`  $\geq^*$  3.1

Version requirements marked with an asterisk are not *critical* requirements, in the sense that PyR@TE is likely to work properly with older versions of these modules. However, we note that during its development, PyR@TE was only tested with the above listed versions. We also note that `Scipy` and `Matplotlib` are solely required to run the Python output of PyR@TE (see section 3.5).

### 3.3 Definition of the model

The information on the particle physics model is contained in the so-called model file, which is the starting point of any computation performed by PyR@TE. The overall structure of the model file is essentially based on the previous versions [48, 49] of the software. However, a number of improvements were made that we will review in the following. Let us begin this discussion with, as usual, the example of the Standard Model. The full SM model file is provided in B. In the following we will go through this model file step by step, making comments wherever useful.

**General information** – Three fields can be provided in this section, namely the `Author`, `Date` of creation and `Name` of the model file. The first two are essentially informative while the last one will be used by PyR@TE to generate its output (see section 3.5).

```
Author: Lohan Sartore
Date: 08.06.2020
Name: SM
```

**Gauge groups** – This is the first essential information to provide to PyR@TE. Here the full gauge group of the SM is  $U(1)_Y \times SU(2)_L \times SU(3)_c$ :

```
Groups: {U1Y: U1, SU2L: SU2, SU3c: SU3}
```

Each gauge factor must be given a custom label and is described by the usual name of the associated Lie group. Possible choices cover the entirety of the gauge groups associated with a simple Lie algebra, namely  $U(1)$ ,  $SU(N)$ ,  $SO(N)$ ,  $Sp(N)$  and the five exceptional groups  $G_2$ ,  $F_4$  and  $E_{6,7,8}$ . We emphasise that this is an improvement compared to PyR@TE 2, where symplectic and exceptional groups were not implemented.

**Field content** – The next step consists in defining the particle content of the model. Three kinds of fields may be implemented in the model file, namely `Fermions`, `RealScalars` and `ComplexScalars`:

```

Fermions: {
  Q : {Gen: 3, Qnb: {U1Y: 1/6, SU2L: 2, SU3c: 3}},
  L : {Gen: 3, Qnb: {U1Y: -1/2, SU2L: 2, SU3c: 1}},
  uR : {Gen: 3, Qnb: {U1Y: 2/3, SU2L: 1, SU3c: 3}},
  dR : {Gen: 3, Qnb: {U1Y: -1/3, SU2L: 1, SU3c: 3}},
  eR : {Gen: 3, Qnb: {U1Y: -1, SU2L: 1, SU3c: 1}},
}

RealScalars: {
}

ComplexScalars: {
  H : {RealFields: [Pi, Sigma], Norm: 1/sqrt(2), Qnb: {U1Y: 1/2, SU2L: 2, SU3c: 1}},
}

```

In every case, the user must specify the quantum numbers of each particle under the gauge group of the model. For Abelian gauge factors, we call *quantum number* the charge of the fields (*e.g.* the hypercharge under  $U(1)_Y$ ). For non-Abelian gauge factors, it corresponds instead to the irreducible representation under which the field transforms. Most of the time, the usual notation based on the dimension of the representation (*e.g.* **2** of  $SU(2)$ , **3** of  $SU(3)$ , ...) is sufficient to identify unambiguously a given representation. In this notation, the conjugate representations are indicated by a negative quantum number in the model file. For instance, it is understood that `Qnb: {..., SU3c: -3}` refers to the anti-fundamental representation of  $SU(3)_c$ . In addition, the quantum number `1` indicates that the field transforms in the trivial representation, *i.e.*, is unaffected by the gauge transformations of the associated gauge factor. Another way to indicate that a field is not charged under one or several gauge factors is to simply omit them in the definition of the quantum numbers. For instance, it is understood that `eR` is a singlet under both `SU2L` and `SU3c` if the user writes:

```
eR : {Gen: 3, Qnb: {U1Y: -1}}
```

This remark also applies to Abelian gauge factors, in which case the charge of the corresponding field is automatically set to `0`.

In a situation where a given gauge group possesses several representations with the same dimension (*e.g.* the **15** and **15'** of  $SU(3)$ ), the usual notation breaks down and the user must use the Dynkin labels notation instead. To keep this presentation straightforward we defer the discussion on Dynkin labels and their implementation in the model file to C.

In addition to the quantum numbers, the `Fermions` of the model may be assigned a generation number. This number can either be a positive integer or a symbolic number, for instance: `Q : {Gen: nG, ...}`. In the latter case, the beta-functions will be expressed in terms of the symbolic generation numbers they may explicitly depend on.

For `ComplexScalars`, the user must indicate the decomposition of the fields in terms of a real and imaginary part. To this end, he/she must use the `RealFields` and `Norm` keywords. In our case, the complex Higgs doublet is expressed as

$$H = \frac{1}{\sqrt{2}} (\Pi + i \Sigma) . \quad (2.71)$$

We note that there is no need to define the real fields beforehand in the `RealScalars` section of the model file. We also want to draw the reader's attention to the influence of the norm of complex scalars on the actual expression of the resulting beta-functions: varying the norm of a complex scalar field can be seen as a rescaling of its real components, or, equivalently, a rescaling of the couplings of the model.

Finally, since the SM contains no real scalars (other than the real components of the Higgs doublet), we show here an example of syntax:

```
RealScalars: {
  S : {Qnb: {U1Y: 1/2, SU2L: 1, SU3c: 1}}
}
```

A short-hand syntax may also be employed, removing the `Qnb` keyword:

```
RealScalars: {
  S : {U1Y: 1/2, SU2L: 1, SU3c: 1}
}
```

A last important point concerns the conjugate fields of the model. In contrast to the previous version of PyR@TE, the anti-particles should never be defined in the model file. Instead, every fermion and complex scalar is automatically assigned a conjugate counterpart which can be accessed by appending `bar` to its name. For instance, in our case, `Qbar` and `Hbar` would respectively correspond to the fields  $\bar{Q}$  and  $H^\dagger$  of the Standard Model.

**Potential** – This section contains the expression of the Lagrangian density of the theory. Since kinetic terms are not needed in PyR@TE, the only types of couplings which must be defined are:

- The Yukawa couplings ( `Yukawas` )
- The quartic scalar couplings ( `QuarticTerms` )
- The trilinear scalar couplings ( `TrilinearTerms` )
- The scalar mass couplings ( `ScalarMasses` )
- The fermion mass couplings ( `FermionMasses` )

The syntax used in these five sections to define the Lagrangian was revisited in PyR@TE 3. In order for the user to have full control over the expressions of the various terms, the new syntax consists in writing the terms as an explicit mathematical expression with contracted indices. As an illustration, the down-type Yukawa coupling of the Standard Model,

$$Y_{df_1, f_2} \bar{Q}_{f_1, i, a} H^i d_{R, f_2}^a + \text{h.c.}, \quad (2.72)$$

is implemented in the model file as:

```
Yukawas: {
  Yd : Qbar[i,a] H[i] dR[a]
}
```

A few important remarks are in order.

First, in the definition of the Yukawa couplings (and fermion masses), the flavour indices are implicit. Consequently, the order in which the two fermions appear in the expression has an influence on the structure of the Yukawa matrices. In the example above, it is understood that `Qbar` is contracted with the first flavour index of `Yd` and `dR` with the second one.

Then, a field which transforms non-trivially under more than one non-Abelian gauge factor will carry more than one index. In this case, the order in which the indices should be written is based on the order in which the gauge factors were defined. Therefore, in the above example, the `SU2` index of `Qbar` comes before the `SU3` index.

Finally, we note that the hermitian conjugate is automatically inferred by PyR@TE and therefore should not need to be defined explicitly. This behaviour always applies to Yukawa and fermion mass couplings, but also to couplings involving scalar fields. In the latter case, however, the user has the choice

to include or not the conjugate couplings in the model file. Let us consider as an illustration the  $\lambda_5$  quartic coupling of the Two-Higgs-doublet model:

$$\frac{1}{2} \left[ \lambda_5 \left( \phi_1^\dagger \phi_2 \right)^2 + \lambda_5^* \left( \phi_2^\dagger \phi_1 \right)^2 \right]. \quad (2.73)$$

There are essentially two ways of implementing this term:

```

QuarticCouplings: {
  # First possibility - the conjugate counterpart is explicitly defined
  lambda5 : 1/2 (phi1bar[i] phi2[i])**2,
  lambda5star : 1/2 (phi2bar[i] phi1[i])**2

  # Second possibility - let PyR@TE infer the conjugate automatically
  lambda5 : 1/2 (phi1bar[i] phi2[i])**2
}

```

In the first implementation, the  $\lambda_5^*$  term is explicitly defined. In this situation, the suffix `star` must be appended to the name of the coupling (here, `lambda5star`) to enforce the relation of conjugation among the two couplings. In the second implementation, where the conjugate counterpart is omitted, PyR@TE will automatically detect that the term is not invariant under complex conjugation, and consequently generate the  $\lambda_5^*$  term when constructing the Lagrangian of the model. Finally, there may be situations where  $\lambda_5$  is assumed real. In this case, the natural way to proceed is to set  $\lambda_5^* = \lambda_5$  in Eq. (2.73), leading to the first implementation shown below. Equivalently, the assumption<sup>7</sup> `real` may be used to achieve the same outcome:

```

QuarticCouplings: {
  # lambda5 is assumed real
  lambda5 : 1/2 (phi1bar[i] phi2[i])**2 + 1/2 (phi2bar[i] phi1[i])**2

  # lambda5 is assumed real, shorthand
  lambda5 : {1/2 (phi1bar[i] phi2[i])**2 , real}
}

```

**Definitions** – In order to help the user deal with this new syntax and produce a clear and readable model file, a new section was introduced in PyR@TE 3. This is the **Definitions** section, which must be implemented inside the **Potential** part of the model file (see the full SM model file in B). In this section, the user may define quantities that will be used in the expression of the Lagrangian density. For instance, in the SM, one has to introduce the conjugated Higgs field:

$$\tilde{H} \equiv \varepsilon H^\dagger, \quad (2.74)$$

where  $\varepsilon = i\sigma_2$  is the Levi-Civita tensor of rank 2. Such an auxiliary quantity may be defined in the model file in the following way:

```

Definitions: {
  Htilde[i] : Eps[i,j]*Hbar[j]
}

```

The tensor `Eps` is a pre-defined object in PyR@TE 3. Levi-Civita tensors of rank higher than 2 may be defined in the same fashion, increasing the number of indices accordingly. We emphasise that the gauge indices must always appear explicitly, both on the right- and left-hand side. According to the usual convention, repeated indices will be summed over internally. Any object defined this way can now be used in the Lagrangian expression. In our case, the up-type Yukawa couplings may be written as:

<sup>7</sup>More detail about assumptions is given below.



```

Yukawas: {
  Yu : Qbar[i,a] Htilde[i] uR[a],
  # ... is equivalent to ...
  Yu : Qbar[i,a] Eps[i,j] Hbar[j] uR[a],
}

```

Not only the `Definitions` sections may help produce a clear and structured model file, but it also introduces two additional features, namely:

1. The use of gauge group generators in the expression of the Lagrangian,
2. The possibility of using Clebsch-Gordan coefficients (CGCs) to produce terms that cannot be expressed simply in terms of products of fields with contracted indices.

Both rely on the group-theoretical module `PyLie`, first introduced in `PyR@TE 2` [49]. This module underwent a number of modifications and improvements presented in C. We also defer to this appendix the treatment of the CGCs and their implementation in the model file. Here we focus on the first feature, namely the definition of gauge generators in the model file.

As an illustration, let us consider a toy model in which the scalar sector of the SM is extended by a complex  $SU(2)$  triplet  $\delta$ . It is convenient to rewrite this triplet in the form of a  $2 \times 2$  matrix by contraction with the generators of the fundamental representation:

$$\Delta \equiv t_a \delta^a. \quad (2.75)$$

In the following, we demonstrate how to use the `Definitions` section to implement the quartic part of the scalar potential:

$$V = \lambda_1 (H^\dagger H)^2 + \lambda_2 \text{Tr}(\Delta^\dagger \Delta) H^\dagger H + \lambda_3 H^\dagger \Delta \Delta^\dagger H + \lambda_4 \text{Tr}(\Delta^\dagger \Delta)^2 + \lambda_5 \text{Tr}(\Delta^\dagger \Delta \Delta^\dagger \Delta). \quad (2.76)$$

First, we have to introduce the generators  $t_a$  in order to define the matrix  $\Delta$ . The general syntax to define the generators of a given representation is `t(group, representation)`, where the representation may be labelled either by its dimension or its Dynkin labels. We note that the generators thus defined carry three indices with respective ranges  $D$ ,  $N_r$  and  $N_r$ , where  $D$  is the dimension of the Lie group (*i.e.* of its adjoint representation) and  $N_r$  that of the considered representation. The matrix  $\Delta$  and its hermitian conjugate  $\Delta^\dagger$  may therefore be implemented as<sup>8</sup>:

```

Definitions: {
  # Define the generators of the fundamental rep of SU2 (indices are implicit)
  tFund : t(SU2, 2),

  # Define the matrix Delta and its adjoint
  Delta[i,j] : tFund[a,i,j] delta[a],
  DeltaDag[i,j] : tFund[a,i,j] deltabar[a]

  # We may also define the traces
  Tr2 : DeltaDag[i,j] Delta[j,i],
  Tr4 : DeltaDag[i,j] Delta[j,k] DeltaDag[k,l] Delta[l,i]
}

```

Having also pre-defined the traces  $\text{Tr}(\Delta^\dagger \Delta)$  and  $\text{Tr}(\Delta^\dagger \Delta \Delta^\dagger \Delta)$ , the scalar potential (2.76) can now be implemented in a simple, concise form:

```

QuarticTerms: {
  lambda1 : (Hbar[i] H[i])**2,
  lambda2 : Hbar[i] H[i] Tr2,
  lambda3 : Hbar[i] Delta[i,j] DeltaDag[j,k] H[k],
}

```

<sup>8</sup>In this example, we used the fact that  $SU(N)$  generators are hermitian. In general, we may conjugate the representation matrices appending `bar` to the name of the generators. For instance, `tFundbar[a,j,i]` would correspond to  $(t_a^*)^j_i = (t_a^\dagger)^i_j$ .

```

lambda4 : Tr2**2,
lambda5 : Tr4
}

```

**Assumptions** – In some cases, the user may want to assume some particular properties for the Yukawa (or fermion mass) matrices. Four different assumptions can be implemented for such couplings in the model file, namely: `real`, `symmetric`, `hermitian` and `unitary`. To impose one or more of these properties to a Yukawa matrix, the general syntax is:

```

coupling: {expression, assumption1, assumption2, ...}

```

For instance, let us assume a real and symmetric Yukawa matrix in the leptonic sector of the SM:

```

Yukawas: {
  # Without assumptions
  Ye : Lbar[i] H[i] eR

  # With assumptions
  Ye : {Lbar[i] H[i] eR, real, symmetric}
},

```

Based on these assumptions, PyR@TE will automatically perform the appropriate simplifications in the resulting RGEs. In the case illustrated above,  $Y_e^\dagger$  would be systematically simplified as  $Y_e$ . We note that in some cases, such assumptions should be necessarily included in order to guarantee explicit gauge invariance of the Lagrangian, checked by PyR@TE 3 using the appropriate options (see section 3.4). As stated previously, the `real` assumption may also be used for quartic, trilinear and scalar mass couplings. At this point, it is important to note that depending on the model considered, such assumptions might not be stable along the RG-flow. This may happen for instance if a coupling which is assumed real has a beta-function with a non-vanishing imaginary part. In such cases, the simplifications made in the expressions of the beta-functions would not be valid at all scales and would generate an inconsistent RG-flow.

Finally, another possible assumption, `squared`, concerns only the scalar mass terms, and allows to make the distinction between the two following notations:

$$\mathcal{L} \supset \mu \phi^\dagger \phi \quad \text{and} \quad \mathcal{L} \supset \mu^2 \phi^\dagger \phi. \quad (2.77)$$

The former case, where the mass dimension of  $\mu$  equals 2, is the one assumed in PyR@TE by default. In the latter case, the `squared` keyword may be added in order to indicate that the scalar mass coupling has a mass dimension of 1:

```

ScalarMasses: {
  mu : {Phibar[i] Phi[i], squared}
}

```

In this case, PyR@TE will internally use the relation

$$\beta(\mu) = \frac{1}{2\mu} \beta(\mu^2) \quad (2.78)$$

to infer the value of  $\beta(\mu)$ .

**Vacuum-expectation values** – The 2-loop RGEs of vacuum-expectation values (VeVs) were implemented in PyR@TE 3, based on the  $\overline{\text{MS}}$  expressions of Refs. [52, 53]. To define a VeV, the only thing to do is to identify the real scalar component which develops a non-zero expectation value. For instance, considering that in the SM the VeV is developed by the real part of the second component of the Higgs doublet, we simply have to write:

```

Vevs: {
  v : Pi[2]
}

```

Since VEVs break gauge invariance, the resulting RGEs are given in a general  $R_\xi$  gauge and therefore explicitly depend on the  $\xi$  parameter. If the user wishes to fix the gauge, he/she may use the `GaugeParameter` keyword:

```

# Let's work in the Landau gauge
GaugeParameter: 0

```

**Anomalous dimensions** – The 2-point anomalous dimensions of scalars and fermions may be computed by PyR@TE. This is achieved by adding the optional `ScalarAnomalous` and `FermionAnomalous` sections in the model file. In order to compute the anomalous dimensions of one or several pairs of fields, the following syntax may be used:

```

ScalarAnomalous: {
  (Pi[1], Pi[1]),
  (Sigma[2], Sigma[2])
}

FermionAnomalous: {
  (Q[1,1], Q[1,1]),
  (eR, eR),
  (uR[1], dR[1])
}

```

We emphasise that the fields used as an input for computing the anomalous dimensions are the individual components of the gauge eigenstates, and not the fields themselves (except of course if a given field is a gauge singlet).

To compute all the possible anomalous dimensions of the model, the user may instead use the keyword `all`:

```

ScalarAnomalous : all
FermionAnomalous : all

```

In this case, PyR@TE will display in its output all the non-vanishing anomalous dimensions of the fields of the model.

We note finally that the anomalous dimensions depend in general on the gauge fixing parameter  $\xi$ . If the gauge was fixed using the `GaugeParameter` keyword introduced above, its value will be substituted accordingly in the expression of the anomalous dimensions.

**Substitutions** – Another new feature in PyR@TE 3 is the possibility of performing various kinds of substitutions after the computation of the RGEs is carried out. To this end, the `Substitutions` section can be implemented in the model file. The possible types of substitutions which may be performed are listed and exemplified below.

(1) The first type of substitution consists in renaming some of the couplings. This is mainly useful to rename the gauge couplings whose names are set by default to `g[GroupName]`. For instance, in the SM, the three gauge couplings would respectively be labelled as `gU1Y`, `gSU2L`, `gSU3c`. If one wishes to rename them to `g1`, `g2` and `g3`, the following syntax must be used:

```

Substitutions: {
  gU1Y : g1,

```

```

gSU2L : g2,
gSU3c : g3
}

```

(2) Another possibility is to apply GUT normalisation factors to some of the gauge couplings. The following code indicates that the normalisation  $g_1 \rightarrow \sqrt{5/3} g_1$  must be adopted and substituted in the beta-functions:

```

Substitutions: {
  g1 : sqrt(5/3)*g1
}

```

(3) In some cases, the user may want to constrain the form of the Yukawa (or fermion mass) matrices, for instance neglecting the first two generations or the off-diagonal couplings. The examples below show how to proceed in these two cases:

```

Substitutions: {
  # First example - Neglect the first and second generations
  Yu : [[0, 0, 0 ],
        [0, 0, 0 ],
        [0, 0, 'yt']],
  # A short-hand notation may be employed for diagonal matrices
  Yd : [0, 0, 'yb'],
  Ye : [0, 0, 'ytau'],

  # Second example - Neglect the off-diagonal couplings
  Yu : ['yu', 'yc', 'yt'],
  Yd : ['yd', 'ys', 'yb'],
  Ye : ['ye', 'ymu', 'ytau'],
}

```

It is important to note that the newly defined couplings (e.g. `'yt'`, `'yb'`, ...) are put inside quotation marks `' '`. Furthermore, the user must be aware that in some cases, the form of the Yukawa matrices as defined in the `Substitutions` section might not be preserved along the RG-flow. In other words, some components of the Yukawa matrices which were set to `0` in the model file may have a non-vanishing beta-function starting from the one- or two-loop level. In such cases, PyR@TE will generate a warning message in the Latex output, informing the user that the RG-flow is inconsistent.

(4) The last possibility consists in defining new quantities, expressed in terms of the couplings of the model. For instance, if the user wishes to obtain the beta-functions in terms of the  $\alpha_i = g_i^2/(4\pi)$  instead of the coupling constants  $g_i$ , he/she would have to define the following substitutions:

```

Substitutions: {
  alpha1 : g1**2 / (4 pi),
  alpha2 : g2**2 / (4 pi),
  alpha3 : g3**2 / (4 pi)
}

```

In this situation, PyR@TE will internally take care of computing the beta-functions of the new couplings, using the relation

$$\beta(\alpha_i) = \frac{1}{4\pi} 2g_i \beta(g_i) , \quad (2.79)$$

and then substituting  $g_i^2$  with  $4\pi \alpha_i$  everywhere in the expression of the beta-functions.

**Latex** – In this section, the user is invited to define any substitution that should be performed in the Latex output. Although it is possible to choose as the name of a coupling a simple Latex expression (e.g. `g_1`), we recommend to avoid it in practice (and more generally to avoid the use of any special

character in the name of the couplings) since it might lead to some unexpected behaviour. Instead, the mapping to Latex expressions should be defined in the dedicated `Latex` section, so the user has full control over the output.

### 3.4 Running PyR@TE

As for previous versions of the software, PyR@TE 3 can be run either from the console or from an interactive IPython notebook. In the former case, the general syntax is:

```
$ python pyR@TE.py -m [path to model file] [-args]
```

We emphasise that the `python` alias must be associated with a Python 3 executable correctly linked to the dependencies listed in section 3.2. In an interactive IPython notebook, PyR@TE would be run using:

```
%run pyR@TE.py -m [path to model file] [-args]
```

The only mandatory argument is the `-m` argument, used to specify the path (relative or absolute) to the model file. The list of optional arguments is presented in Table 2.1. Compared to the previous versions, the number of available command-line arguments was reduced in order to simplify to a maximum the use of the software. Instead, most of the settings are gathered in a configuration file, `'default.settings'`, which can be found in PyR@TE's main directory. The list of options that are available in this file is given in Table 2.2. For some of the options, there is an overlap between the default and command-line settings. In this case, the latter always has priority over the former. For example, even if the gauge invariance of the model is to be checked by default, the user may use the `-no-gi` command-line option to prevent this behaviour.

We note in passing that the gauge invariance check of the model is a new feature in PyR@TE 3. When enabled, the invariance of all types of couplings under infinitesimal gauge transformations is tested. For instance, in the scalar quartic sector (as already mentioned in Eq.(2.65)), gauge invariance requires that

$$(T_\phi^A)_{ae}\lambda_{ebcd} + (T_\phi^A)_{be}\lambda_{aecd} + (T_\phi^A)_{ce}\lambda_{abcd} + (T_\phi^A)_{de}\lambda_{abce} = 0 \quad (2.80)$$

for all  $a, b, c, d$  spanning the space of the scalar components of the model, and for all  $A = 1, \dots, N_G$ . It is strongly advised to check the gauge invariance of a model at least once, after which the option `-no-gi` may be used to speed up the computation<sup>9</sup>.

The `-l` and `DefaultLoopLevel` settings should be followed either by a positive integer (1, 2 or 3) or the keyword `max`, meaning that the computation should be carried out at the maximum loop-order, *i.e.* 3-loop for the gauge couplings and 2-loop for the other couplings. Another possibility is to use a list of three elements<sup>10</sup>, *e.g.* `[3,2,1]` to set the respective orders of the gauge, Yukawa, and quartic couplings RGEs to different values.

The `-rb` and `RealBasis` settings enable the use of a new feature available in PyR@TE 3, namely the rotation of the generators of real representations. These representations have the property that there exists a set of bases where their generators  $t^A$  are imaginary, skew-symmetric matrices. In the following, such bases will be referred to as a *real bases*. In addition, for adjoint representations (which are always real), there exists a basis where the usual relation

$$(t^A)^B{}_C = -if^{AB}{}_C, \quad (2.81)$$

with  $f^{AB}{}_C$  the structure constants of the Lie algebra, is satisfied. However, the generators of real representations originally computed by PyLie never satisfy the two above properties, preventing the user from properly implementing models where one or several fields transform under real representations. In order to circumvent this issue, a new algorithm was implemented in PyLie, allowing to systematically

<sup>9</sup>For large gauge groups and/or representations, the gauge invariance check may take up to a few minutes due to the large number of tensor contractions to be computed. For simpler models, the computation takes a few seconds at most.

<sup>10</sup>On some operating systems, the use of square brackets in the command line may not work properly. In this case, putting the list inside quotation marks should solve the problem.

Option	Short-hand	Description
<code>--Loops</code>	<code>-l</code>	Set the loop-level of the computation
<code>--Results</code>	<code>-res</code>	Set the directory in which the output is stored
<code>--RealBasis</code>	<code>-rb</code>	Set the behaviour regarding the generators of real representations
<code>--Quiet</code>	<code>-q</code>	Switch off informative output
<code>--no-KinMix</code>	<code>-no-kin</code>	Neglect the Abelian kinetic mixing
<code>--CheckGaugeInvariance</code>	<code>-gi</code>	Switch on the Lagrangian gauge invariance check
<code>--no-CheckGaugeInvariance</code>	<code>-no-gi</code>	Switch off the Lagrangian gauge invariance check
<code>--LatexOutput</code>	<code>-tex</code>	Switch on the Latex output
<code>--no-LatexOutput</code>	<code>-no-tex</code>	Switch off the Latex output
<code>--MathematicaOutput</code>	<code>-math</code>	Switch on the Mathematica output
<code>--no-MathematicaOutput</code>	<code>-no-math</code>	Switch off the Mathematica output
<code>--PythonOutput</code>	<code>-py</code>	Switch on the Python output
<code>--no-PythonOutput</code>	<code>-no-py</code>	Switch off the Python output

Table 2.1: Command-line options in PyR@TE 3. The second part of the table shows the various switches which may be used by the user to override some of the default settings.

Option	Description
<code>VerboseLevel</code>	Set the verbose level : <b>Info</b> , <b>Critical</b> (only errors and warnings).
<code>ResultsFolder</code>	Set the default folder in which the results are stored.
<code>LogFolder</code>	Set the default folder in which the log files are stored.
<code>DisableLogFiles</code>	<b>True/False</b> .
<code>DefaultLoopLevel</code>	Default loop level of the computation.
<code>CheckGaugeInvariance</code>	<b>True/False</b> .
<code>PrintComputationTimes</code>	Display the computation times : <b>True/False</b> .
<code>RealBasis</code>	Default behaviour regarding the generators of real representations.
<code>CreateFolder</code>	Store the output in a folder named after the model: <b>True/False</b> .
<code>CopyModelFile</code>	Copy the original model file in the results folder : <b>True/False</b> .
<code>LatexOutput</code>	<b>True/False</b> .
<code>AbsReplacements</code>	In the Latex output, replace $x*x$ with $ x ^2$ : <b>True/False</b> .
<code>GroupTheoryInfo</code>	In the Latex output, display some information about the gauge groups and their representations: <b>True/False</b> .
<code>MoreGroupTheoryInfo</code>	Display information about the first N representations of the gauge groups: <b>N/False</b> .
<code>MathematicaOutput</code>	<b>True/False</b> .
<code>MathematicaSolver</code>	In the Mathematica output, generate a ready-to-use RGE solver: <b>True/False</b> .
<code>PythonOutput</code>	<b>True/False</b> .
<code>EndCommands</code>	Commands to be automatically run from the console after the output is generated.

Table 2.2: Default settings available in PyR@TE's configuration file.

rotate the generators of real representations to a real basis; and in the case of adjoint representations, to the basis where Eq. (2.81) is satisfied. To trigger this behaviour, the `-rb` (or `RealBasis`) option may be assigned three possible values:

- `-rb adjoint` rotates only adjoint representations,
- `-rb all` rotates all real representations,
- `-rb False` or `-rb None` disables the rotation to real bases.

We emphasise that complex and pseudo-real representations are unaffected by this setting.

The `EndCommands` default settings allows to run one or several shell commands after the run. It was designed to compile the Latex output automatically after its generation by PyR@TE. The general syntax is:

```
EndCommands "command 1, command 2, ..."
```

For instance, we may call `pdflatex` twice using

```
EndCommands "pdflatex [name].tex, pdflatex [name].tex"
```

where `[name]` will be automatically replaced by the name of the model (and therefore of the output files) before running the commands.

### 3.5 Output

After the computation, PyR@TE may generate three types of output, namely Latex, Mathematica, and Python files. With the `CreateFolder` option enabled, all the output files will be gathered in a folder named after the model (*e.g.* `SM/` in our case).

The Latex output contains a detailed summary of the content of the model (gauge groups, particles, Lagrangian, substitutions), and the results of the computations, *i.e.* the beta-functions of the various couplings of the model. If the `GroupTheoryInfo` setting is enabled, some information about the gauge groups and their representations will be added in an appendix, along with the expressions of the CGCs possibly used in the definition of the Lagrangian. Using the `MoreGroupTheoryInfo` setting followed by a positive integer will display information about the first few representations of the gauge groups, in addition to the representations actually populated by the fields of the model.

The Mathematica output consists of a single file, containing the expressions of the beta-functions. In addition, if the `MathematicaSolver` default setting is set to `True`, the file is enhanced with a ready-to-use RGE solver. To solve the RGEs, the first thing that the user needs to do is to define the boundary conditions, that is the value of the couplings of the model at some initial energy scale. Then, after having defined the running scheme (the loop level for each type of couplings), the user may solve the system of RGEs using the `RGsolve[]` function and finally plot the results using the `RGplot[]` function.

Finally, the Python output consists of three files, gathered in a `PythonOutput` folder. For instance, in the case of the SM:

- `RGEs.py` contains the expressions of the beta-functions in the form of Python functions.
- `SM.py` contains various classes used to define the couplings of the model, the RGEs, and several functions used to solve the RGEs and plot the running couplings. In principle, this file should not be modified by the user, but may still serve as a basis for more sophisticated analyses.
- `run.py` is the file that should be modified and run by the user to perform the RG analysis. For more details about the use of the classes and functions called in this file, we invite the reader to refer to the documentation provided with PyR@TE in the `doc/` folder.

### 3.6 Validation

The output of PyR@TE 3 was validated against some of the results found in the literature, in particular in the 3-loop gauge sector which is a new feature compared to the previous version. Full agreement was found between PyR@TE 3 and the following references :

- SM gauge couplings RGEs at three-loop [54, 55, 56].
- Gauge, Yukawa and quartic sector of the general Two-Higgs-doublet model at respective orders 3-2-1 [57].
- 2-loop dimensionless and dimensionful RGEs of the toy model described in [38] in the SUSY limit<sup>11</sup>.
- 1-loop quartic RGEs of simple  $SU(5)$  and  $SO(10)$  models [58].
- 1-loop Yukawa sector of the minimal  $SU(5)$  model [59].
- 1-loop quartic, trilinear and scalar mass RGEs of the most general Georgi-Machacek model [60]. We find agreement for the 1-loop gauge coupling RGEs after correcting a trivial mistake in Eq. (C1) of [60].

In addition, we compared the results of PyR@TE 3 against those of PyR@TE 2 for some of the models presented in [48] and [49]. In doing so, we were able to compare the performance of the two versions of the software in terms of execution time, as illustrated in Table 2.3. We first present the comparison of the results for some of the models listed in [48]:

- Standard Model with a real scalar singlet: full agreement at 2-loop.
- Standard Model with a complex scalar doublet: full agreement at 2-loop.
- Standard Model with Majorana triplet fermion and Dirac doublet: full agreement in the gauge, Yukawa and quartic sectors at 2-loop. Agreement is found for the scalar mass RGEs after correcting a mistake in the latest version of the PyR@TE 2 code. For the fermion mass RGEs, we find agreement once the corrected formulae presented in [38] are taken into account.
- SM extended by a complex triplet and vectorlike doublets: at 1-loop, we find many disagreements in the quartic sector due to an incomplete implementation of the model. Indeed, in PyR@TE 1 it was not possible to contract the same set of fields in different ways. As a consequence, couplings (operators) allowed by gauge invariance were missing. In such a situation, the resulting RGEs are arbitrary and vary depending on the choice of the  $n$  legs chosen to compute the various  $n$ -point contributions to the beta-functions (see also footnote <sup>11</sup>).

The limitation of PyR@TE 1 mentioned in the last point was overcome in PyR@TE 2, as illustrated in [49] with the example of a toy model where the SM is extended with a scalar triplet. For this model, the scalar potential is given by Eq. (2.76), and we were able to compare the results given by PyR@TE 2 and 3. Full agreement is found at 2-loop, after an obvious error in the model file from PyR@TE 2 is corrected (an additional lepton doublet was unintentionally included).

Finally, we validated the output of PyR@TE 3 for the  $U(1)_{B-L}$  extension of the SM described in [49]. In the gauge, Yukawa and quartic sector, full agreement is found between the two versions of the code in presence of kinetic mixing in the Abelian sector. However, there is a disagreement for the scalar mass beta-functions starting at 2-loop which we describe in the following. Letting

$$V \supset \mu_H H^\dagger H + \mu_\chi \chi^\dagger \chi \quad (2.82)$$

<sup>11</sup>As explained in [38], the toy model in its non-SUSY form is not well defined, leading to arbitrary results depending on the details of the implementation. It is possible to get full agreement between the results from SARAH and PyR@TE before taking the SUSY limit by a simple mapping. However, we refrain from providing these details here since they concern unphysical results and the mapping is only simple because the implementation conventions used in SARAH and PyR@TE are quite similar.



Model	Loop order	PyR@TE 2	PyR@TE 3
SM B-L	1	114	1.5
	2	8823	11
	2 + 3 (gauge)	/	23
SM + complex triplet	1	385	1.0
	2	59936	3.2
	2 + 3 (gauge)	/	5.7
SM + scalar singlet	1	79	0.9
	2	5765	4.3
	2 + 3 (gauge)	/	5.6
SM + complex doublet	1	153	1.2
	2	39666	6.2
	2 + 3 (gauge)	/	9.4
SM + Majorana triplet + Vectorlike doublet	1	262	1.3
	2	15653	10.7
	2 + 3 (gauge)	/	13.2

Table 2.3: Comparison of the running times of PyR@TE 2 and PyR@TE 3 for several models, using a machine with a 1.8 GHz Intel Core i7 processor. The running times are expressed in seconds.

be the quadratic part of the scalar potential  $V$ , where  $H$  is the usual SM Higgs doublet and  $\chi$  is a complex scalar field which is a SM singlet with (B-L)-charge 2, the disagreement occurs in the two-loop contributions of  $\beta(\mu_H)$  and  $\beta(\mu_\chi)$ . More precisely, the problematic terms are of the generic form  $g^4\mu_H$  and  $g^4\mu_\chi$ , respectively, so we only include the terms of these forms in the expressions below. We note that the gauge coupling matrix is taken to be

$$\begin{pmatrix} g_1 & g_{12} \\ g_{21} & g_{BL} \end{pmatrix}, \quad (2.83)$$

while  $g_2$  stands for the  $SU(2)_L$  coupling constant.

### Expressions of $\beta^{(2)}(\mu_H)$

Using PyR@TE 2:

$$\begin{aligned} \frac{1}{\mu_H}\beta^{(2)}(\mu_H) \supset & \frac{1105}{96}g_1^4 - \frac{145}{16}g_2^4 + \frac{1105}{96}g_{12}^4 + \frac{7}{2}g_1^2g_2^2 + \frac{1105}{48}g_1^2g_{12}^2 + 17g_1^2g_{21}^2 + 17g_{12}^2g_{BL}^2 \\ & + \frac{7}{2}g_{12}^2g_2^2 + \frac{40}{3}g_1^3g_{21} + \frac{40}{3}g_{12}^3g_{BL} + \frac{40}{3}g_1^2g_{12}g_{BL} + \frac{40}{3}g_1g_{12}^2g_{21} + 34g_1g_{12}g_{21}g_{BL} \end{aligned} \quad (2.84)$$

Using PyR@TE 3:

$$\begin{aligned} \frac{1}{\mu_H}\beta^{(2)}(\mu_H) \supset & \frac{557}{48}g_1^4 - \frac{145}{16}g_2^4 + \frac{557}{48}g_{12}^4 + \frac{15}{8}g_1^2g_2^2 + \frac{557}{24}g_1^2g_{12}^2 + 17g_1^2g_{21}^2 + 17g_{12}^2g_{BL}^2 \\ & + \frac{15}{8}g_{12}^2g_2^2 + \frac{40}{3}g_1^3g_{21} + \frac{40}{3}g_{12}^3g_{BL} + \frac{40}{3}g_1^2g_{12}g_{BL} + \frac{40}{3}g_1g_{12}^2g_{21} + 34g_1g_{12}g_{21}g_{BL} \end{aligned} \quad (2.85)$$

### Expressions of $\beta^{(2)}(\mu_\chi)$

Using PyR@TE 2:

$$\begin{aligned} \frac{1}{\mu_\chi}\beta^{(2)}(\mu_\chi) \supset & 648g_{BL}^4 + 648g_{21}^4 + \frac{422}{3}g_1^2g_{21}^2 + \frac{422}{3}g_{12}^2g_{BL}^2 + 1296g_{21}^2g_{BL}^2 + \frac{640}{3}g_1g_{21}^3 \\ & + \frac{640}{3}g_{12}g_{BL}^3 + \frac{640}{3}g_1g_{21}g_{BL}^2 + \frac{640}{3}g_{12}g_{21}^2g_{BL} + \frac{844}{3}g_1g_{12}g_{21}g_{BL} \end{aligned} \quad (2.86)$$

Using PyR@TE 3:

$$\begin{aligned} \frac{1}{\mu_\chi} \beta^{(2)}(\mu_\chi) \supset & 672g_{BL}^4 + 672g_{21}^4 + \frac{422}{3}g_1^2g_{21}^2 + \frac{422}{3}g_{12}^2g_{BL}^2 + 1344g_{21}^2g_{BL}^2 + \frac{640}{3}g_1g_{21}^3 \\ & + \frac{640}{3}g_{12}g_{BL}^3 + \frac{640}{3}g_1g_{21}g_{BL}^2 + \frac{640}{3}g_{12}g_{21}g_{BL} + \frac{844}{3}g_1g_{12}g_{21}g_{BL} \end{aligned} \quad (2.87)$$

Although the above expressions alone do not allow one to discard any of the two sets of results, there is a fact indicating that PyR@TE 2 is internally inconsistent when it comes to kinetic mixing: running PyR@TE 2 with the option `--KinMix` to disable the effects of kinetic mixing, or taking the limit  $g_{12}, g_{21} \rightarrow 0$  in the two expressions above do not yield the same result, as would be expected. Interestingly, the results obtained with the `--KinMix` option (*i.e.* disabling kinetic mixing in PyR@TE 2) completely agree with those of PyR@TE 3 in the limit where  $g_{12}$  and  $g_{21}$  vanish.

As a concluding remark, while we cannot make the claim that the results obtained with PyR@TE 3 are correct<sup>12</sup>, the treatment of kinetic mixing for the scalar mass RGEs in PyR@TE 2 appears to be inconsistent. Furthermore, the implementation of kinetic mixing in the formalism described in Sec. 1.2 is quite natural whereas it has been incorporated *a posteriori* in PyR@TE 2 [49] by enhancing [37] the expressions derived by Machacek & Vaughn<sup>13</sup>. As a result, the implementation of the kinetic mixing in PyR@TE 2 is quite involved and we were not able to directly identify the source of the inconsistency. In any case, the process of validation performed above allowed us to highlight at least two flaws present in the PyR@TE 2 code, and we recommend to systematically use PyR@TE 3 in the future instead of its previous versions which are now deprecated.

<sup>12</sup>It is worth noting that complete agreement was found with Thomsen's code RGBeta [61], based on the same formalism.

<sup>13</sup>A similar comment holds for the Mathematica package SARAH [47].

# Appendix

## A Two-loop beta-functions for dimensionful couplings

We present in this appendix the general formulae for the beta-functions of fermion mass, scalar mass and trilinear couplings obtained from the application of the dummy field method in the formalism of [39]. We give in a first step the definition of various auxiliary quantities appearing in the resulting expressions, allowing to provide in turn the comprehensive list of tensor structures and corresponding coefficients as computed in [10].

We use a notation where the fermion indices are made implicit. In this context,  $m$ ,  $y_a$ , and any other tensor carrying two fermion indices may be seen as matrices in the space of the fermions of the theory. Following the notations of [39], to each such matrix  $Y$  corresponds a tilded counterpart  $\tilde{Y}$  with opposite chirality,

$$\tilde{Y} = \Sigma_1 Y \Sigma_1, \quad (2.A1)$$

where

$$\Sigma_1 = \sigma_1 \otimes \mathbb{1}_{N_F} = \begin{pmatrix} 0 & \mathbb{1}_{N_F} \\ \mathbb{1}_{N_F} & 0 \end{pmatrix}, \quad (2.A2)$$

so, for instance:

$$\tilde{y}_a = \Sigma_1 y_a \Sigma_1 = \begin{pmatrix} Y_a^* & 0 \\ 0 & Y_a \end{pmatrix}. \quad (2.A3)$$

In a next step, closely following [39], we define a set of two-index quantities, which regularly appear as substructures in the expressions of the beta-functions. We provide below the definition of each of these auxiliary tensors structures, gathered by the nature of their indices and the number of loops they involve.

1-loop gauge:

$$[S_2(F)]_{AB} = \text{Tr} [T_\Psi^A T_\Psi^B], \quad [S_2(S)]_{AB} = (T_\phi^A T_\phi^B)_{aa}, \quad [C_2(G)]_{AB} = f^{ACD} f^{CDB}. \quad (2.A4)$$

1-loop fermion:

$$C_2(F) = T_\Psi^A T_\Psi^B G_{AB}^2, \quad Y_2(F) = y_a \tilde{y}_a. \quad (2.A5)$$

1-loop scalar:

$$[C_2(S)]_{ab} = (T_\phi^A T_\phi^B)_{ab} G_{AB}^2, \quad [Y_2(S)]_{ab} = \text{Tr} [y_a \tilde{y}_b]. \quad (2.A6)$$

2-loop gauge:

$$\begin{aligned} [S_2(F, C_F)]_{AB} &= \text{Tr} [T_\Psi^A T_\Psi^B C_2(F)], & [S_2(F, Y_F)]_{AB} &= \text{Tr} [T_\Psi^A T_\Psi^B \tilde{Y}_2(F)], \\ [S_2(S, C_S)]_{AB} &= (T_\Psi^A T_\Psi^B)_{ab} [C_2(S)]_{ba}, & [S_2(S, Y_S)]_{AB} &= (T_\Psi^A T_\Psi^B)_{ab} [Y_2(S)]_{ba}. \end{aligned} \quad (2.A7)$$

2-loop fermion:

$$\begin{aligned} C_2(F, G) &= T_\Psi^A T_\Psi^B (G^2 [C_2(G)] G^2)_{AB}, & C_2(F, S) &= T_\Psi^A T_\Psi^B (G^2 [S_2(S)] G^2)_{AB}, \\ C_2(F, F) &= T_\Psi^A T_\Psi^B (G^2 [S_2(F)] G^2)_{AB}, & Y_2(F, C_F) &= y_a C_2(F) \tilde{y}_a, \\ Y_2(F, C_S) &= y_a \tilde{y}_b [C_2(S)]_{ab}, & Y_2(F, Y_F) &= y_a \tilde{Y}_2(F) \tilde{y}_a, \\ Y_2(F, Y_S) &= y_a \tilde{y}_b [Y_2(S)]_{ab}, & Y_4(F) &= y_a \tilde{y}_b y_a \tilde{y}_b. \end{aligned} \quad (2.A8)$$

2-loop scalar:

$$\begin{aligned} [C_2(S, G)]_{ab} &= (T_\phi^A T_\phi^B)_{ab} (G^2 [C_2(G)] G^2)_{AB}, & [C_2(S, S)]_{ab} &= (T_\phi^A T_\phi^B)_{ab} (G^2 [S_2(S)] G^2)_{AB}, \\ [C_2(S, F)]_{ab} &= (T_\phi^A T_\phi^B)_{ab} (G^2 [S_2(F)] G^2)_{AB}, & [Y_2(S, C_F)]_{ab} &= \text{Tr} [y_a C_2(F) \tilde{y}_b], \\ [Y_2(S, Y_F)]_{ab} &= \text{Tr} [y_a \tilde{Y}_2(F) \tilde{y}_b], & [Y_4(S)]_{ab} &= \text{Tr} [y_a \tilde{y}_c y_b \tilde{y}_c]. \end{aligned} \quad (2.A9)$$

With these definitions at hand, we may now provide the expressions of the one- and two-loop beta-functions for the fermion mass, trilinear and scalar mass couplings respectively.

### Fermion mass beta-functions

At 1-loop:

$$\beta^{(1)} = \xi_1^{(1)} m C_2(F) + \xi_2^{(1)} y_b m y_b + \xi_3^{(1)} m \tilde{Y}_2(F). \quad (2.A10)$$

At 2-loop:

$$\begin{aligned} \beta^{(2)} = & \xi_1^{(2)} \tilde{C}_2(F) m C_2(F) & + \xi_2^{(2)} m C_2(F) C_2(F) & + \xi_3^{(2)} m C_2(F, G) \\ & + \xi_4^{(2)} m C_2(F, S) & + \xi_5^{(2)} m C_2(F, F) & + \xi_6^{(2)} y_b T_\Psi^A \tilde{y}_b m T_\Psi^B G_{AB}^2 \\ & + \xi_7^{(2)} Y_2(F) \tilde{T}^A m T_\Psi^B G_{AB}^2 & + \xi_8^{(2)} y_b \tilde{m} y_c [C_2(S)]_{bc} & + \xi_9^{(2)} y_b C_2(F) \tilde{m} y_b \\ & + \xi_{10}^{(2)} y_b \tilde{m} y_b C_2(F) & + \xi_{11}^{(2)} Y_2(F, C_S) m & + \xi_{12}^{(2)} Y_2(F, C_F) m \\ & + \xi_{13}^{(2)} m \tilde{Y}_2(F) C_2(F) & + \xi_{14}^{(2)} y_b \tilde{y}_c y_d t_{bcd} & + \xi_{15}^{(2)} y_b \tilde{y}_c m \tilde{y}_b y_c \\ & + \xi_{16}^{(2)} y_b \tilde{m} y_c \tilde{y}_b y_c & + \xi_{17}^{(2)} y_b \tilde{y}_c m \tilde{y}_c y_b & + \xi_{18}^{(2)} Y_4(F) m \\ & + \xi_{19}^{(2)} y_b \tilde{m} Y_2(F) y_b & + \xi_{20}^{(2)} m \tilde{Y}_2(F, Y_F) & + \xi_{21}^{(2)} y_b \tilde{m} y_c [Y_2(S)]_{bc} \\ & + \xi_{22}^{(2)} Y_2(F, Y_S) m. \end{aligned} \quad (2.A11)$$

### Trilinear couplings beta-functions

At 1-loop:

$$\beta_{abc}^{(1)} = \tau_1^{(1)} [C_2(S)]_{ae} t_{ebc} + \tau_2^{(1)} \lambda_{abef} t_{efc} + \tau_3^{(1)} [Y_2(S)]_{ae} t_{ebc} + \tau_4^{(1)} \text{Tr} [m \tilde{y}_a y_b \tilde{y}_c]. \quad (2.A12)$$

At 2-loop:

$$\begin{aligned} \beta_{abc}^{(2)} = & \tau_1^{(2)} (T_\phi^A T_\phi^C)_{ae} G_{AB}^2 G_{CD}^2 (T_\phi^B T_\phi^D)_{bf} t_{efc} & + \tau_2^{(2)} (T_\phi^A T_\phi^C)_{ab} G_{AB}^2 G_{CD}^2 (T_\phi^B T_\phi^D)_{ef} t_{efc} \\ & + \tau_3^{(2)} [C_2(S)]_{ae} [C_2(S)]_{bf} t_{efc} & + \tau_4^{(2)} [C_2(S)]_{ae} [C_2(S)]_{ef} t_{fbc} \\ & + \tau_5^{(2)} [C_2(S, G)]_{ae} t_{ebc} & + \tau_6^{(2)} [C_2(S, S)]_{ae} t_{ebc} \\ & + \tau_7^{(2)} [C_2(S, F)]_{ae} t_{ebc} & + \tau_8^{(2)} (T_\phi^A)_{ae} (T_\phi^B)_{bf} G_{AB}^2 \lambda_{efgh} t_{ghc} \\ & + \tau_9^{(2)} \lambda_{abef} [C_2(S)]_{fg} t_{egc} & + \tau_{10}^{(2)} [C_2(S)]_{ae} t_{efg} \lambda_{fgbc} \\ & + \tau_{11}^{(2)} [C_2(S)]_{ae} \lambda_{ebfg} t_{fgc} & + \tau_{12}^{(2)} \lambda_{aefg} \lambda_{efgh} t_{hbc} \\ & + \tau_{13}^{(2)} t_{aef} \lambda_{eghb} \lambda_{fghc} & + \tau_{14}^{(2)} \lambda_{abef} \lambda_{eghc} t_{fgh} \\ & + \tau_{15}^{(2)} \lambda_{abef} \lambda_{efgh} t_{ghc} & + \tau_{16}^{(2)} (T_\phi^A T_\phi^C)_{ab} G_{AB}^2 G_{CD}^2 \text{Tr} [T_\Psi^D T_\Psi^B \tilde{m} y_c] \\ & + \tau_{17}^{(2)} (T_\phi^A T_\phi^C)_{ab} G_{AB}^2 G_{CD}^2 \text{Tr} [T_\Psi^D T_\Psi^B \tilde{y}_c m] & + \tau_{18}^{(2)} [Y_2(S, C_F)]_{ae} t_{ebc} \\ & + \tau_{19}^{(2)} [C_2(S)]_{ae} [Y_2(S)]_{ef} t_{fbc} & + \tau_{20}^{(2)} \lambda_{abef} [Y_2(S)]_{fg} t_{egc} \\ & + \tau_{21}^{(2)} G_{AB}^2 \text{Tr} [m T_\Psi^A \tilde{y}_a y_b T_\Psi^B \tilde{y}_c] & + \tau_{22}^{(2)} G_{AB}^2 \text{Tr} [y_a T_\Psi^A \tilde{m} y_b T_\Psi^B \tilde{y}_c] \\ & + \tau_{23}^{(2)} [C_2(S)]_{ae} \text{Tr} [y_e \tilde{m} y_b \tilde{y}_c] & + \tau_{24}^{(2)} [C_2(S)]_{ae} \text{Tr} [y_e \tilde{y}_b m \tilde{y}_c] \\ & + \tau_{25}^{(2)} \text{Tr} [m \tilde{y}_a y_b \tilde{y}_c \tilde{C}_2(F)] & + \tau_{26}^{(2)} \text{Tr} [y_a m y_b \tilde{y}_c \tilde{C}_2(F)] \\ & + \tau_{27}^{(2)} \text{Tr} [m \tilde{y}_e y_a \tilde{y}_f] \lambda_{efbc} & + \tau_{28}^{(2)} \text{Tr} [y_a \tilde{y}_e y_b \tilde{y}_f] t_{efc} \\ & + \tau_{29}^{(2)} \text{Tr} [m \tilde{y}_a y_e \tilde{y}_f] \lambda_{efbc} & + \tau_{30}^{(2)} \text{Tr} [y_a \tilde{y}_b y_e \tilde{y}_f] t_{efc} \\ & + \tau_{31}^{(2)} [Y_4(S)]_{ae} t_{ebc} & + \tau_{32}^{(2)} [Y_2(S, Y_F)]_{ae} t_{ebc} \\ & + \tau_{33}^{(2)} \text{Tr} [m \tilde{y}_a y_b \tilde{y}_c y_e \tilde{y}_e] & + \tau_{34}^{(2)} \text{Tr} [y_a \tilde{m} y_b \tilde{y}_c y_e \tilde{y}_e] \\ & + \tau_{35}^{(2)} \text{Tr} [y_a \tilde{y}_b m \tilde{y}_c y_e \tilde{y}_e] & + \tau_{36}^{(2)} \text{Tr} [m \tilde{y}_a y_e \tilde{y}_b y_c \tilde{y}_e] \\ & + \tau_{37}^{(2)} \text{Tr} [m \tilde{y}_a y_b \tilde{y}_c Y_2(F)] & + \tau_{38}^{(2)} \text{Tr} [y_a \tilde{m} y_b \tilde{y}_c Y_2(F)]. \end{aligned} \quad (2.A13)$$

### Scalar mass beta-functions

At 1-loop:

$$\begin{aligned} \beta_{ab}^{(1)} = & \sigma_1^{(1)} [C_2(S)]_{ae} \mu_{eb} & + \sigma_2^{(1)} \lambda_{abef} \mu_{ef} & + \sigma_3^{(1)} t_{aef} t_{efb} \\ & + \sigma_4^{(1)} [Y_2(S)]_{ae} \mu_{eb} & + \sigma_5^{(1)} \text{Tr} [m \tilde{m} y_a \tilde{y}_b] & + \sigma_6^{(1)} \text{Tr} [m \tilde{y}_a m \tilde{y}_b] . \end{aligned} \quad (2.A14)$$

At 2-loop:

$$\begin{aligned} \beta_{ab}^{(2)} = & \sigma_1^{(2)} (T_\phi^A T_\phi^C)_{ae} G_{AB}^2 G_{CD}^2 (T_\phi^B T_\phi^D)_{bf} \mu_{ef} & + \sigma_2^{(2)} (T_\phi^A T_\phi^C)_{ab} G_{AB}^2 G_{CD}^2 (T_\phi^B T_\phi^D)_{ef} \mu_{ef} \\ & + \sigma_3^{(2)} [C_2(S)]_{ae} [C_2(S)]_{bf} \mu_{ef} & + \sigma_4^{(2)} [C_2(S)]_{ae} [C_2(S)]_{ef} \mu_{fb} \\ & + \sigma_5^{(2)} [C_2(S, G)]_{ae} \mu_{eb} & + \sigma_6^{(2)} [C_2(S, S)]_{ae} \mu_{eb} \\ & + \sigma_7^{(2)} [C_2(S, F)]_{ae} \mu_{eb} & + \sigma_8^{(2)} (T_\phi^A)_{ae} (T_\phi^B)_{bf} G_{AB}^2 \lambda_{efgh} \mu_{gh} \\ & + \sigma_9^{(2)} \lambda_{abef} [C_2(S)]_{fg} \mu_{eg} & + \sigma_{10}^{(2)} t_{aef} [C_2(S)]_{fg} t_{egb} \\ & + \sigma_{11}^{(2)} [C_2(S)]_{ae} \lambda_{ebfg} \mu_{fg} & + \sigma_{12}^{(2)} [C_2(S)]_{ae} t_{efg} t_{fgb} \\ & + \sigma_{13}^{(2)} \lambda_{aefg} \lambda_{efgh} \mu_{hb} & + \sigma_{14}^{(2)} \lambda_{aegh} \lambda_{bfg} \mu_{ef} \\ & + \sigma_{15}^{(2)} \lambda_{abef} t_{egh} t_{fgh} & + \sigma_{16}^{(2)} t_{aef} t_{egh} \lambda_{bfg} \\ & + \sigma_{17}^{(2)} \lambda_{abef} \lambda_{efgh} \mu_{gh} & + \sigma_{18}^{(2)} t_{aef} \lambda_{efgh} t_{ghb} \\ & + \sigma_{19}^{(2)} (T_\phi^A T_\phi^C)_{ab} G_{AB}^2 G_{CD}^2 \text{Tr} [T_\Psi^D T_\Psi^B \tilde{m} m] & + \sigma_{20}^{(2)} [Y_2(S, C_F)]_{ae} \mu_{eb} \\ & + \sigma_{21}^{(2)} [C_2(S)]_{ae} [Y_2(S)]_{ef} \mu_{fb} & + \sigma_{22}^{(2)} \lambda_{abef} [Y_2(S)]_{fg} \mu_{eg} \\ & + \sigma_{23}^{(2)} t_{aef} [Y_2(S)]_{fg} t_{egb} & + \sigma_{24}^{(2)} G_{AB}^2 \text{Tr} [y_a T_\Psi^A \tilde{m} m T_\Psi^B \tilde{y}_b] \\ & + \sigma_{25}^{(2)} G_{AB}^2 \text{Tr} [y_a T_\Psi^A \tilde{y}_b m T_\Psi^B \tilde{m}] & + \sigma_{26}^{(2)} G_{AB}^2 \text{Tr} [y_a T_\Psi^A \tilde{m} y_b T_\Psi^B \tilde{m}] \\ & + \sigma_{27}^{(2)} [C_2(S)]_{ae} \text{Tr} [y_e \tilde{m} y_b m] & + \sigma_{28}^{(2)} [C_2(S)]_{ae} \text{Tr} [y_e \tilde{m} m y_b] \\ & + \sigma_{29}^{(2)} \text{Tr} [y_a \tilde{y}_b m \tilde{m} \tilde{C}_2(F)] & + \sigma_{30}^{(2)} \text{Tr} [y_a \tilde{m} y_b \tilde{m} \tilde{C}_2(F)] \\ & + \sigma_{31}^{(2)} \text{Tr} [y_a \tilde{m} m \tilde{y}_b \tilde{C}_2(F)] & + \sigma_{32}^{(2)} \text{Tr} [m \tilde{y}_a y_b \tilde{m} \tilde{C}_2(F)] \\ & + \sigma_{33}^{(2)} \text{Tr} [y_a \tilde{y}_e y_b \tilde{y}_f] \mu_{ef} & + \sigma_{34}^{(2)} \text{Tr} [y_a \tilde{y}_e m \tilde{y}_f] t_{efb} \\ & + \sigma_{35}^{(2)} \text{Tr} [m \tilde{y}_e m \tilde{y}_f] \lambda_{efab} & + \sigma_{36}^{(2)} \text{Tr} [y_a \tilde{y}_b y_e \tilde{y}_f] \mu_{ef} \\ & + \sigma_{37}^{(2)} \text{Tr} [y_a \tilde{m} y_e \tilde{y}_f] t_{efb} & + \sigma_{38}^{(2)} \text{Tr} [m \tilde{m} y_e \tilde{y}_f] \lambda_{efab} \\ & + \sigma_{39}^{(2)} [Y_4(S)]_{ae} \mu_{eb} & + \sigma_{40}^{(2)} [Y_2(S, Y_F)]_{ae} \mu_{eb} \\ & + \sigma_{41}^{(2)} \text{Tr} [m \tilde{y}_a m \tilde{y}_e y_b \tilde{y}_e] & + \sigma_{42}^{(2)} \text{Tr} [y_a \tilde{y}_b m \tilde{y}_e m \tilde{y}_e] \\ & + \sigma_{43}^{(2)} \text{Tr} [y_a \tilde{m} m \tilde{y}_e y_b \tilde{y}_e] & + \sigma_{44}^{(2)} \text{Tr} [y_a \tilde{m} y_b \tilde{y}_e m \tilde{y}_e] \\ & + \sigma_{45}^{(2)} \text{Tr} [y_a \tilde{y}_b y_e \tilde{m} m \tilde{y}_e] & + \sigma_{46}^{(2)} \text{Tr} [y_a \tilde{m} y_e \tilde{m} y_b \tilde{y}_e] \\ & + \sigma_{47}^{(2)} \text{Tr} [y_a \tilde{m} y_e \tilde{y}_b m \tilde{y}_e] & + \sigma_{48}^{(2)} \text{Tr} [y_a \tilde{y}_b m \tilde{m} Y_2(F)] \\ & + \sigma_{49}^{(2)} \text{Tr} [y_a \tilde{m} y_b \tilde{m} Y_2(F)] & + \sigma_{50}^{(2)} \text{Tr} [y_a \tilde{m} m \tilde{y}_b Y_2(F)] \\ & + \sigma_{51}^{(2)} \text{Tr} [m \tilde{y}_a y_b \tilde{m} Y_2(F)] . \end{aligned} \quad (2.A15)$$

### Fermion mass coefficients

At 1-loop:

$$\xi_1^{(1)} = -6, \quad \xi_2^{(1)} = 2, \quad \xi_3^{(1)} = 1. \quad (2.A16)$$

At 2-loop:

$$\begin{aligned}
\xi_1^{(2)} &= 0, & \xi_2^{(2)} &= -3, & \xi_3^{(2)} &= -\frac{97}{3}, & \xi_4^{(2)} &= \frac{11}{6}, & \xi_5^{(2)} &= \frac{5}{3}, & \xi_6^{(2)} &= 12, \\
\xi_7^{(2)} &= 0, & \xi_8^{(2)} &= 6, & \xi_9^{(2)} &= 10, & \xi_{10}^{(2)} &= 6, & \xi_{11}^{(2)} &= 9, & \xi_{12}^{(2)} &= -\frac{1}{2}, \\
\xi_{13}^{(2)} &= -\frac{7}{2}, & \xi_{14}^{(2)} &= -2, & \xi_{15}^{(2)} &= 2, & \xi_{16}^{(2)} &= 0, & \xi_{17}^{(2)} &= -2, & \xi_{18}^{(2)} &= 0, \\
\xi_{19}^{(2)} &= -2, & \xi_{20}^{(2)} &= -\frac{1}{4}, & \xi_{21}^{(2)} &= -1, & \xi_{22}^{(2)} &= -\frac{3}{4}.
\end{aligned} \tag{2.A17}$$

### Trilinear coefficients

At 1-loop:

$$\tau_1^{(1)} = -9, \quad \tau_2^{(1)} = 3, \quad \tau_3^{(1)} = \frac{3}{2}, \quad \tau_4^{(1)} = -12. \tag{2.A18}$$

At 2-loop:

$$\begin{aligned}
\tau_1^{(2)} &= 6, & \tau_2^{(2)} &= 30, & \tau_3^{(2)} &= 0, & \tau_4^{(2)} &= \frac{9}{2}, & \tau_5^{(2)} &= -\frac{143}{4}, & \tau_6^{(2)} &= \frac{11}{4}, \\
\tau_7^{(2)} &= \frac{10}{4}, & \tau_8^{(2)} &= -9, & \tau_9^{(2)} &= 24, & \tau_{10}^{(2)} &= -\frac{9}{2}, & \tau_{11}^{(2)} &= -9, & \tau_{12}^{(2)} &= \frac{1}{4}, \\
\tau_{13}^{(2)} &= -3, & \tau_{14}^{(2)} &= -3, & \tau_{15}^{(2)} &= 0, & \tau_{16}^{(2)} &= -36, & \tau_{17}^{(2)} &= -36, & \tau_{18}^{(2)} &= \frac{15}{2}, \\
\tau_{19}^{(2)} &= 0, & \tau_{20}^{(2)} &= -3, & \tau_{21}^{(2)} &= 0, & \tau_{22}^{(2)} &= 0, & \tau_{23}^{(2)} &= 12, & \tau_{24}^{(2)} &= 6, \\
\tau_{25}^{(2)} &= -24, & \tau_{26}^{(2)} &= -24, & \tau_{27}^{(2)} &= 6, & \tau_{28}^{(2)} &= 6, & \tau_{29}^{(2)} &= 0, & \tau_{30}^{(2)} &= 0, \\
\tau_{31}^{(2)} &= -\frac{3}{2}, & \tau_{32}^{(2)} &= -\frac{9}{4}, & \tau_{33}^{(2)} &= 24, & \tau_{34}^{(2)} &= 12, & \tau_{35}^{(2)} &= 12, & \tau_{36}^{(2)} &= 24, \\
\tau_{37}^{(2)} &= 12, & \tau_{38}^{(2)} &= 12.
\end{aligned} \tag{2.A19}$$

### Scalar mass coefficients

At 1-loop:

$$\sigma_1^{(1)} = -6, \quad \sigma_2^{(1)} = 1, \quad \sigma_3^{(1)} = 1, \quad \sigma_4^{(1)} = 1, \quad \sigma_5^{(1)} = -4, \quad \sigma_6^{(1)} = -2. \tag{2.A20}$$

At 2-loop:

$$\begin{aligned}
\sigma_1^{(2)} &= 2, & \sigma_2^{(2)} &= 10, & \sigma_3^{(2)} &= 0, & \sigma_4^{(2)} &= 3, & \sigma_5^{(2)} &= -\frac{143}{6}, & \sigma_6^{(2)} &= \frac{11}{6}, \\
\sigma_7^{(2)} &= \frac{10}{6}, & \sigma_8^{(2)} &= -3, & \sigma_9^{(2)} &= 8, & \sigma_{10}^{(2)} &= 8, & \sigma_{11}^{(2)} &= -3, & \sigma_{12}^{(2)} &= -3, \\
\sigma_{13}^{(2)} &= \frac{1}{6}, & \sigma_{14}^{(2)} &= -1, & \sigma_{15}^{(2)} &= -\frac{1}{2}, & \sigma_{16}^{(2)} &= -2, & \sigma_{17}^{(2)} &= 0, & \sigma_{18}^{(2)} &= 0, \\
\sigma_{19}^{(2)} &= -12, & \sigma_{20}^{(2)} &= 5, & \sigma_{21}^{(2)} &= 0, & \sigma_{22}^{(2)} &= -1, & \sigma_{23}^{(2)} &= -1, & \sigma_{24}^{(2)} &= 0, \\
\sigma_{25}^{(2)} &= 0, & \sigma_{26}^{(2)} &= 0, & \sigma_{27}^{(2)} &= 2, & \sigma_{28}^{(2)} &= 4, & \sigma_{29}^{(2)} &= -8, & \sigma_{30}^{(2)} &= -8, \\
\sigma_{31}^{(2)} &= -4, & \sigma_{32}^{(2)} &= -4, & \sigma_{33}^{(2)} &= 2, & \sigma_{34}^{(2)} &= 4, & \sigma_{35}^{(2)} &= 1, & \sigma_{36}^{(2)} &= 0, \\
\sigma_{37}^{(2)} &= 0, & \sigma_{38}^{(2)} &= 0, & \sigma_{39}^{(2)} &= -1, & \sigma_{40}^{(2)} &= -\frac{3}{2}, & \sigma_{41}^{(2)} &= 4, & \sigma_{42}^{(2)} &= 8, \\
\sigma_{43}^{(2)} &= 8, & \sigma_{44}^{(2)} &= 4, & \sigma_{45}^{(2)} &= 4, & \sigma_{46}^{(2)} &= 4, & \sigma_{47}^{(2)} &= 4, & \sigma_{48}^{(2)} &= 4, \\
\sigma_{49}^{(2)} &= 4, & \sigma_{50}^{(2)} &= 2, & \sigma_{51}^{(2)} &= 2.
\end{aligned} \tag{2.A21}$$

## B The SM model file

As an illustration of the overall structure of a model file in PyR@TE 3, we show below the full SM model file. We note that this file can be found in PyR@TE's `models/` subdirectory, along with several BSM model files.

```
# YAML 1.1
---
Author: Lohan Sartore
Date: 08.06.2020
Name: SM
Groups: {U1Y: U1, SU2L: SU2, SU3c: SU3}

Fermions: {
  Q : {Gen: 3, Qnb: {U1Y: 1/6, SU2L: 2, SU3c: 3}},
  L : {Gen: 3, Qnb: {U1Y: -1/2, SU2L: 2}},
  uR : {Gen: 3, Qnb: {U1Y: 2/3, SU3c: 3}},
  dR : {Gen: 3, Qnb: {U1Y: -1/3, SU3c: 3}},
  eR : {Gen: 3, Qnb: {U1Y: -1}},
}

RealScalars: {
}

ComplexScalars: {
  H : {RealFields: [Pi, Sigma], Norm: 1/sqrt(2), Qnb: {U1Y: 1/2, SU2L: 2}},
}

Potential: {
  Definitions: {
    Htilde[i] : Eps[i,j]*Hbar[j]
  },
  Yukawas: {
    Yu : Qbar[i,a] Htilde[i] uR[a],
    Yd : Qbar[i,a] H[i] dR[a],
    Ye : Lbar[i] H[i] eR
  },
  QuarticTerms: {
    lambda : (Hbar[i] H[i])**2
  },
  ScalarMasses: {
    mu : -Hbar[i] H[i]
  }
}

Vevs: {
  v: Pi[2]
}

Substitutions: {
  # Rename the gauge coupling constants
  g_U1Y : g1,
  g_SU2L : g2,
  g_SU3c : g3,

  # Possibly define GUT normalisations
  g1 : sqrt(5/3) * g1
}

Latex: {
```

```

# Particles

uR : u_R,
dR : d_R,
eR : e_R,

Pi : \Pi,
Sigma : \Sigma,

Htilde : \tilde{H},

# Couplings

g1 : g_1,
g2 : g_2,
g3 : g_3,

Yu : Y_u,
Yd : Y_d,
Ye : Y_e,

lambda : \lambda,
mu : \mu
}

```

## C Group theory and PyLie

This appendix is dedicated to the description of the group theoretical functionalities available in PyR@TE 3. All the group theory related computations are handled, since the previous version of PyR@TE, by the PyLie module. PyLie is essentially a Python rewrite of the group theory module of SUSYNO [62] (also part of the more recent tool GroupMath [63]). Such computations comprise for instance the possibility of computing the generators of a given representation, the structure constants of the Lie algebra, or the Clebsch-Gordan coefficients (CGCs) of a given set of representations. A lot of effort was put in the development of PyR@TE 3 to improve the performances of PyLie’s main functions. This concerns in particular the three kinds of calculations listed above. However, despite these efforts, some calculations may be quite time-consuming when it comes to high-dimensional representations. Therefore, based on the previous version of PyLie, we developed a database in which the results of the group-theoretical computations are systematically stored for any later use. At various steps of the computation of the RGEs, PyR@TE 3 interacts with this database through the PyLieDB module. The user may interact as well with this database through an interactive IPython session or a Jupyter notebook in order to access the results of PyLie’s computations.

After a discussion about the Dynkin labels as a way to uniquely identify the representations of a given gauge group, we introduce the use of CGCs to build a Lagrangian in PyR@TE 3. A short tutorial (in the form of an interactive Python notebook) available in the `doc/` repository of PyR@TE 3 is dedicated to the interaction with PyLie’s database. In order to keep this appendix as concise as possible, we invite the interested user to refer to this tutorial if needed.

### C.1 Dynkin labels

The Dynkin labels of an irreducible representation are a set of  $N$  positive integers, where  $N$  is the rank of the algebra. They characterise the decomposition of the highest weight of the representation in terms of the  $N$  fundamental weights of the algebra. In PyLie, they are used to identify uniquely the representations of a given Lie algebra. In practice, the Dynkin labels take the form of Python lists, and can be used in the model file in place of the usual notation for representations based on their dimensions. For instance, one could define the quantum numbers of the quark doublet  $\mathbf{Q}$  as:

```
Q: {Gen: 3, Qnb: {U1Y: 1/6, SU2L: [1], SU3c: [1,0]}}
```



We invite the user who is not familiar with the Dynkin labels notation to refer to the document provided with PyR@TE 3 in the `doc/` directory, where the first few representations of some usual gauge groups are listed along with their Dynkin labels. Several functions were also implemented in the `PyLieDB` module that can be called by the user to get information on the representations and their Dynkin labels.

## C.2 Clebsch-Gordan coefficients

In some cases, it may not be possible to express a given gauge invariant combination of the fields using a notation with contracted indices involving only the gauge generators, the Levi-Civita tensor and the fields themselves. In such cases, the user may use the Clebsch-Gordan coefficients (CGCs) generated by PyLie.

Given a set of  $N$  fields  $F_k, k \leq N$  transforming under the irreducible representations  $r_k, k \leq N$  of a given Lie algebra, we call CGCs all the tensors  $I_{i_1, \dots, i_N}^A$  such that the quantity

$$I_{i_1, \dots, i_N}^A F_1^{i_1} \dots F_N^{i_N} \quad (2.C1)$$

is gauge invariant. Since in PyR@TE we consider only renormalisable theories,  $N$  may only equal 2, 3 or 4. We note that there are  $M$  linearly independent CGCs, where  $M$  is the multiplicity of the trivial representation in the decomposition of the tensor product  $r_1 \otimes \dots \otimes r_N$ .

One of the main functionalities of PyLie consists in finding a basis of CGCs given a set of  $N$  fields. These CGCs may be used by the user in the model file to build the Lagrangian. To this end, the CGCs must be defined in the `Definitions` section of the model file using the two following possible syntaxes :

```
Definitions: {
  C : cgc(groupName, field_1, ... field_N, P),
  # or, equivalently,
  C : cgc(group, representation 1, ... representation N, P)
}
```

The last argument, `P`, is an integer indicating that we are asking for the  $P$ th CGCs returned by PyLie. If omitted, the first invariant tensor will be returned. Defined this way, `C` is a tensor with `N` indices that may be used in the expression of the Lagrangian as any other tensor quantity defined in the model file.

In order to illustrate the use of CGCs in a simple case, let us first consider the example of the up-type Yukawa coupling in the SM. In a notation with contracted indices, one would simply write:

```
Yukawas: {
  Yu : Qbar[i,a] Htilde[i] uR[a]
}
```

where `Htilde` is the conjugated Higgs field, defined in the `Definitions` section of the SM model file. The  $SU(2)$  and  $SU(3)$  contractions rely on the two simple decompositions

$$SU(2) : \mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}, \quad (2.C2)$$

$$SU(3) : \bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}, \quad (2.C3)$$

from which it can be seen that only one gauge singlet may be constructed in each case. Using the CGCs instead, the `Yu` term would be defined as:

```
Definitions: {
# SU2 contraction
  c1 : cgc(SU2L, Qbar, Hbar),
  # or, equivalently,
```

```

c1 : cgc(SU2, -2, -2),
# SU3 contraction
c2 : cgc(SU3c, Qbar, uR),
# or, equivalently,
c2 : cgc(SU3, -3, 3)
},
Yukawas: {
  Yu : c1[i,j] c2[a,b] Qbar[i,a] Hbar[j] uR[b]
}

```

Of course, the above example looks like an unnecessary complication since `c1` and `c2` actually correspond to the rank 2 Levi-Civita tensor and the 3-dimensional Kronecker delta.

As a more sophisticated example, let us now consider a  $SU(5)$  model with the following field content:

$$\begin{aligned}
 \text{Fermions} & : \psi_5 \sim \bar{\mathbf{5}} \quad \text{and} \quad \psi_{10} \sim \mathbf{10} \\
 \text{Scalars} & : \phi \sim \mathbf{5}
 \end{aligned}
 \tag{2.C4}$$

In order to construct the Yukawa sector, we make use of the decompositions

$$\bar{\mathbf{5}} \otimes \bar{\mathbf{5}} \otimes \mathbf{10} = \mathbf{126} \oplus \mathbf{75} \oplus \mathbf{24} \oplus \mathbf{24} \oplus \mathbf{1},
 \tag{2.C5}$$

$$\mathbf{10} \otimes \mathbf{5} \otimes \mathbf{10} = \overline{\mathbf{175}'} \oplus \overline{\mathbf{126}} \oplus \mathbf{75} \oplus \mathbf{75} \oplus \mathbf{24} \oplus \mathbf{24} \oplus \mathbf{1},
 \tag{2.C6}$$

enabling one to define the two following Yukawa couplings

$$Y_5 \psi_5 \phi^\dagger \psi_{10} \quad \text{and} \quad Y_{10} \psi_{10} \phi \psi_{10}.
 \tag{2.C7}$$

The two above terms are usually expressed in a simple form, noticing that the 10-plet can be reorganised in a  $5 \times 5$  antisymmetric matrix. However, in PyR@TE, the field  $\psi_{10}$  only carries a single  $SU(5)$  index and we must use the CGC notation in order to build the Yukawa Lagrangian. We show below a minimal implementation of this  $SU(5)$  toy model.

```

Groups: {SU5: SU5}

Fermions: {
  psi5 : {Gen: 3, Qnb: {SU5: -5}},
  psi10 : {Gen: 3, Qnb: {SU5: 10}}
}

ComplexScalars: {
  phi : {RealFields: [phiR, phiI], Norm: 1/sqrt(2), Qnb: {SU5: 5}}
}

Potential: {
  Definitions: {
    c5 : cgc(SU5, psi5, phibar, psi10),
    c10 : cgc(SU5, psi10, phi, psi10)
  },
  Yukawas: {
    Y5 : c5[i,j,k] psi5[i] phibar[j] psi10[k],
    Y10 : {c10[i,j,k] psi10[i] phi[j] psi10[k], symmetric}
  }
}

```



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## Chapter 3

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# Renormalisation group improvement in single-scale theories

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It has been shown in chapter 1 that the expression of the renormalised  $n$ -point functions together with the set of beta-functions describing the evolution of the couplings with the renormalisation scale define the perturbatively renormalised theory in 4 dimensions. The physical consistency is in turn guaranteed by the Callan-Symanzik equation, stating that the effective action is invariant under simultaneous evolution of the explicitly  $\mu$ -dependent contributions and of the couplings according to their beta-functions. For practical applications, one is however limited by the difficulty of obtaining both the expressions of the renormalised  $N$ -point functions and those of the beta-functions at high orders in perturbation theory. The consequences of working at fixed order in perturbation theory are in turn not only quantitative –  $N$ -point functions and observables cannot be computed at an arbitrary level of precision – but also qualitative: By construction, the validity of the Callan-Symanzik equation relies on an interplay between contributions at orders  $n$  and  $n + 1$  in perturbation theory; any fixed-order truncation hence spoils the invariance under the arbitrary renormalisation scale.

As we shall demonstrate in the following, the renormalisation group evolution of the couplings is intimately connected to the structure of the logarithmic contributions to the renormalised  $N$ -point functions. It is customary, as a way to alleviate the impact of the unphysical scale  $\mu$  in practical calculations, to evolve the couplings of the theory towards a renormalisation scale such that the logarithms are *not too large*. Another simpler possibility, when  $\mu$  is a free parameter of the theory, is to set its value so that this (rather vague) requirement is satisfied. While for some applications such prescriptions may lead to a sufficient level of precision, it is in general rather unsatisfactory or even unachievable. This is true in particular for models with multiple, largely separated mass scales, where multiple logarithms have to be kept under control with a single parameter  $\mu$ . Another problematic situation concerns the study of the effective potential, whose logarithmic contributions depend on the classical background field. Therefore, tuning the renormalisation scale so as to suppress the logarithms for a particular value of the background field cannot prevent the occurrence of large such logarithms in other regions of the field space.

It is important to stress that the occurrence of large logarithms in the UV does not invalidate the perturbative expansion, contrary for instance to the occurrence of Landau poles in the RG-flow. On the other hand, their presence signals a strong dependence of the various quantities under variations of the scale  $\mu$ . In turn, if large logarithms do not spoil perturbativity, they do, on the other hand, essentially spoil predictivity. Consequently, proper treatment of the logarithmic contributions is in many cases a necessary condition to establish reliable quantitative predictions, as well as to understand the qualitative features of a number of theories with interacting fields and/or with a ground state away from the origin: In brief, necessary for any phenomenologically relevant application.

Historically, the connection between UV-logarithms and the renormalisation group has led to call renormalisation group improvement (RG-improvement) the procedure allowing to resum the logarithmic

contributions stemming from regularisation of the UV divergences<sup>1</sup>. The main reason, as we will see, is that the infinite series of logarithmic contributions takes the form of a Taylor expansion around  $\log \mu$  of the  $N$ -point functions. Resumming the logarithmic contributions thus means reformulating this series in a closed form, amounting, in turn, to evaluate an RG-dependent quantity at some RG-scale different from the initial scale  $\mu$ . Many works in the literature (see *e.g.* [64, 65, 66, 67, 68, 69]) have focused on single-scale theories, *i.e.* theories where a single mass scale can occur in the UV-logarithms. In this chapter, we shall likewise restrict the discussion to such theories, since there are many interesting lessons one can learn about the interplay between the renormalisation group and the perturbative structure of the  $N$ -point functions. Most of the results derived below will be crucial when generalising the discussion to any general scalar theory in chapter 4.

In the first section, we consider a massive  $\phi^4$  theory with a positive mass term – the  $\phi_+^4$  theory. As we shall see, interesting conclusions can be drawn from this simple case: in particular, we derive the all-order resummed  $N$ -point functions in a closed form and provide a method to systematically perform their practical computations. In subsequent sections, we will consider a more general situation where the mass term is allowed to take negative values – the  $\phi_-^4$  theory. In this context, the true ground state of the theory is in general non-trivial, and one must resort to the study of the effective potential to determine its precise location. In particular, we derive the stationary point equation satisfied by the all-order resummed potential, and outline a general method allowing to find its solutions in practice. Overall, the results obtained in this chapter will lay the bases for an extension of the resummation procedure to general scalar theories in chapter 4.

## 1 Resummed $N$ -point functions in the massive $\phi_+^4$ theory

We consider in this section a simple scalar theory, with a single scalar field  $\phi$  and a positive mass parameter  $M$ :

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \Lambda - \frac{1}{2} M \phi^2 - \frac{1}{4!} \phi^4. \quad (3.1)$$

In what follows, we establish some important results regarding the structure of the renormalised  $N$ -point functions to all orders in perturbation theory, based on the application of the Callan-Symanzik equation.

### 1.1 All-order parameterisation of the $N$ -point functions

In a slightly different notation compared to chapter 1, we note  $\Gamma_{[N]}$  the  $\overline{\text{MS}}$  renormalised  $N$ -point functions in the 4-dimensional theory. For any  $N$ ,  $\Gamma_{[N]}$  can be perturbatively expanded as

$$\Gamma_{[N]} = \Gamma_{[N]}^{(0)} + \kappa \Gamma_{[N]}^{(1)} + \kappa^2 \Gamma_{[N]}^{(2)} + \dots = \sum_{n=0}^{\infty} \kappa^n \Gamma_{[N]}^{(n)}, \quad (3.2)$$

Where the loop-counting parameter  $\kappa = 1/(16\pi^2)$  has been explicitly factored out from every fixed-order contribution  $\Gamma_{[N]}^{(n)}$ , and where the leading-order renormalised  $N$ -point function  $\Gamma_{[N]}^{(0)}$  simply stands for the tree-level coupling. In particular,

$$\Gamma_{[0]}^{(0)} = \Lambda, \quad \Gamma_{[2]}^{(0)} = M, \quad \Gamma_{[4]}^{(0)} = \lambda. \quad (3.3)$$

To any order in perturbation theory, the contribution  $\Gamma_{[N]}^{(n)}$  can be conveniently decomposed in terms of the powers of the mass-logarithm,  $\log \frac{M}{\mu^2}$ . Furthermore, it is a general feature that the  $n$ -loop contributions depend at most on the  $n$ -th power of the mass logarithm. In turn, one can always write

$$\Gamma_{[N]}^{(n)} = \sum_{p=0}^n c_{[N],p}^{(n)} \left( \log \frac{M}{\mu^2} \right)^p. \quad (3.4)$$

<sup>1</sup>In the remaining of this chapter, we will use the terms *RG-improvement*, *logarithmic resummation* or simply *resummation* interchangeably.

To prevent the proliferation of factors of 2 in the forthcoming expressions, we define the mass-logarithm  $\bar{L}$  such that

$$\bar{L} \equiv \frac{1}{2} \log \frac{M}{\mu^2} = \log \frac{\sqrt{M}}{\mu}, \quad (3.5)$$

and, by a re-definition of the coefficients  $c_{[N],p}^{(n)}$  appearing in Eq.(3.4), we rewrite the logarithmic expansion of  $\Gamma_{[N]}^{(n)}$  in the form

$$\Gamma_{[N]}^{(n)} = \sum_{p=0}^n c_{[N],p}^{(n)} \bar{L}^p. \quad (3.6)$$

Inserting this expression in the perturbative expansion of  $\Gamma_{[N]}$ , Eq. (3.2), we arrive at the most general parameterisation

$$\Gamma_{[N]} = \sum_{n=0}^{\infty} \kappa^n \sum_{p=0}^n c_{[N],p}^{(n)} \bar{L}^p. \quad (3.7)$$

A second parameterisation will be extremely useful in the following to study the overall structure of the logarithmic series, which is obtained by exchanging the two sums in Eq. (3.7):

$$\Gamma_{[N]} = \sum_{p=0}^{\infty} \left( \sum_{n=p}^{\infty} \kappa^n c_{[N],p}^{(n)} \right) \bar{L}^p \equiv \sum_{p=0}^{\infty} c_{[N],p} \bar{L}^p. \quad (3.8)$$

In this parameterisation, the renormalised  $n$ -point functions are simply given as a power series in the mass-logarithm, with coefficients  $c_{[N],p}$  containing an infinite number of loop-contributions with perturbative order  $n \geq p$ . We insist that, in both parameterisations (3.7) and (3.8), the explicit dependence on the renormalisation scale is exclusively contained in the powers of the mass-logarithm. On the other hand, the coefficients  $c_{[N],p}^{(n)}$  and  $c_{[N],p}$  are functions of the couplings of the theory and, as such, implicitly depend on  $\mu$  through their beta-functions.

## 1.2 Logarithmic contributions: resummation to all orders

We have shown in chapter 1 that the explicit and implicit dependence on the renormalisation scale  $\mu$  should properly cancel in order to maintain physical consistency of the theory – up to the RG-dependence stemming from the self-energy corrections to the external legs. This invariance translates into the Callan-Symanzik equation (1.124):

$$\mu \frac{d}{d\mu} \Gamma_{[N]} = \{ \Gamma_{[N]} \leftarrow \gamma \} = N\gamma, \quad (3.9)$$

where the last equality follows from the specialisation to a theory with a single scalar field. It will be useful in the following to define several differential operators, depending on the nature of the RG-dependence they encompass. First of all, we define

$$D = \mu \frac{d}{d\mu}, \quad (3.10)$$

the operator giving the total derivative with respect to the renormalisation scale. It can be decomposed as

$$D = \partial_\mu + \mathcal{D}, \quad (3.11)$$

where

$$\partial_\mu \equiv \mu \frac{\partial}{\partial \mu} \quad (3.12)$$

stands for the explicit derivative, while

$$\mathcal{D} \equiv \beta(\Lambda) \frac{d}{d\Lambda} + \beta(M) \frac{d}{dM} + \beta(\lambda) \frac{d}{d\lambda}, \quad (3.13)$$

accounts for the implicit dependence through the beta-function of the couplings.

In a next preliminary step, we shall define an auxiliary scalar field  $\widehat{\phi}$  which is exactly invariant under RG-transformations. Since the renormalised scalar field  $\phi$  implicitly depends on  $\mu$  according to

$$D\phi = \mathcal{D}\phi = -\gamma\phi, \quad (3.14)$$

it is always possible to explicitly integrate the above relation in order to obtain

$$\phi(\log \mu) = \exp \left\{ - \int_{\log \mu_0}^{\log \mu} \gamma(t) dt \right\} \phi(\log \mu_0) \equiv K(\log \mu) \phi(\log \mu_0), \quad (3.15)$$

for some initial scale  $\mu_0$ . Taking  $\widehat{\phi} = \phi(\log \mu_0)$  and going back to a simplified notation where the arguments are made implicit, we get

$$\phi = K\widehat{\phi}, \quad D\widehat{\phi} = 0, \quad DK = -\gamma K. \quad (3.16)$$

In turn, we define the modified  $N$ -point functions  $\widehat{\Gamma}_{[N]}$ ,

$$\widehat{\Gamma}_{[N]} \equiv K^N \Gamma_{[N]}, \quad (3.17)$$

which are exactly invariant under RG-transformations, since

$$D\widehat{\Gamma}_{[N]} = NDKK^{N-1}\Gamma_{[N]} + K^N D\Gamma_{[N]} = -N\gamma\widehat{\Gamma}_{[N]} + N\gamma\widehat{\Gamma}_{[N]} = 0. \quad (3.18)$$

Finally, defining the new set of coefficients

$$\widehat{c}_{[N],p}^{(n)} \equiv K^N c_{[N],p}^{(n)}, \quad \widehat{c}_{[N],p} \equiv K^N c_{[N],p}, \quad (3.19)$$

we arrive at the following set of equations which will constitute the starting point of the forthcoming analysis

$$\widehat{\Gamma}_{[N]} = \sum_{n=0}^{\infty} \kappa^n \sum_{p=0}^n \widehat{c}_{[N],p}^{(n)} \overline{L}^p, \quad (3.20)$$

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \widehat{c}_{[N],p} \overline{L}^p, \quad (3.21)$$

$$D\widehat{\Gamma}_{[N]} = 0. \quad (3.22)$$

With these definitions at hand, we will now explore the consequences of the Callan-Symanzik equation (3.22) on the all-order structure of the (modified)  $N$ -point functions  $\widehat{\Gamma}_{[N]}$ . Using in a first step the logarithmic series expansion, Eq. (3.21), we obtain:

$$\begin{aligned} 0 = D\widehat{\Gamma}_{[N]} &= \sum_{p=0}^{\infty} (D\widehat{c}_{[N],p}) \overline{L}^p + p\widehat{c}_{[N],p} (D\overline{L}) \overline{L}^{p-1} \\ &= \sum_{p=0}^{\infty} (D\widehat{c}_{[N],p}) \overline{L}^p + \sum_{p=0}^{\infty} (p+1)\widehat{c}_{[N],p+1} (D\overline{L}) \overline{L}^p \\ &= \sum_{p=0}^{\infty} [(D\widehat{c}_{[N],p}) + (p+1)\widehat{c}_{[N],p+1} (D\overline{L})] \overline{L}^p. \end{aligned} \quad (3.23)$$

Defining  $L = \frac{1}{2} \log M$ , the derivative of the mass logarithm simply reads

$$D\overline{L} = D \left( \frac{1}{2} \log \frac{M}{\mu^2} \right) = -1 + DL, \quad (3.24)$$

so that Eq.(3.23) becomes

$$0 = \sum_{p=0}^{\infty} [(D\widehat{c}_{[N],p}) - (p+1)\widehat{c}_{[N],p+1}(1 - DL)] \overline{L}^p. \quad (3.25)$$

Since the above equation must hold for any value of the RG-scale (and hence of the mass-logarithm  $\bar{L}$ ), we arrive, for any  $p \geq 0$ , at the recurrence relation

$$(p+1)\widehat{c}_{[N],p+1} = \frac{1}{1-DL}D\widehat{c}_{[N],p}, \quad (3.26)$$

allowing to obtain every coefficient  $\widehat{c}_{[N],p}$  in terms of  $\widehat{c}_{[N],0}$ , which we shall refer to as the all-order leading coefficient. We insist that the operator  $D$  accounts for the RG-derivatives involving the all-order beta-functions, which are in practice inaccessible. Similarly, only the first few perturbative orders in the decomposition of the coefficient  $\widehat{c}_{[N],0}$  are known. Nonetheless, relation (3.26) gives valuable information on the structure of the all-order renormalised  $N$ -point functions, with important consequences in the limit of a truncated perturbative expansion.

### An insightful approximation

Since the derivative of the mass-logarithm  $L = \frac{1}{2} \log M$  is only implicit, it is a most of order  $\mathcal{O}(\kappa^1)$  in perturbation theory, and one may express the denominator in Eq. (3.26) in the form of a power series,

$$\frac{1}{1-DL} = \sum_{k=0}^{\infty} (DL)^k. \quad (3.27)$$

Retaining only the first term in the above series yields the approximate relation

$$(p+1)\widehat{c}_{[N],p+1} \approx D\widehat{c}_{[N],p}, \quad (3.28)$$

which can be analytically solved to give, for all  $p \geq 0$ ,

$$\widehat{c}_{[N],p} = \frac{1}{p!} D^p \widehat{c}_{[N],0}. \quad (3.29)$$

Inserting this expression in the logarithmic-series expansion of  $\widehat{\Gamma}_{[N]}$  yields in turn

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \frac{1}{p!} D^p \widehat{c}_{[N],0} \bar{L}^p = \sum_{p=0}^{\infty} \frac{1}{p!} \frac{d^p}{d(\log \mu)^p} \widehat{c}_{[N],0} \left( \frac{1}{2} \log M - \log \mu \right)^p. \quad (3.30)$$

In the last equality, we have rewritten the expansion in a form which, manifestly, is nothing else than the Taylor expansion of  $\widehat{c}_{[N],0}$  around  $\log \mu$ , evaluated at  $\frac{1}{2} \log M$ . In other words, in the limit where  $DL$  is negligible, the resummation of the logarithmic power-series yields

$$\widehat{\Gamma}_{[N]} \approx \widehat{c}_{[N],0} \left( \frac{1}{2} \log M(\log \mu) \right). \quad (3.31)$$

This means in turn that the renormalised  $N$ -point functions obtained after resumming the infinite tower of logarithmic contributions is equal, in a first approximation, to its leading coefficient evaluated at the scale  $\mu_1 = \sqrt{M(\log \mu)}$ . Of course, the quantity hence obtained is not exactly RG-invariant, but can be expected in practice to provide a good approximation of the all-order result.

From the above approximation, it then appears that the explicit resummation of the logarithmic contributions is equivalent to evaluating the leading coefficient at a scale which cancels the mass-logarithm. Starting with the  $N$ -point function expressed at some scale  $\mu_0$ , we may evolve all couplings according to their beta-functions up to the scale<sup>2</sup>  $\log \mu_1 = \frac{1}{2} \log M(\mu_0)$ . Since  $\widehat{\Gamma}_{[N]}$  is exactly RG-invariant, we then obtain

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \widehat{c}_{[N],p}(\log \mu_1) \left( \frac{1}{2} \log M(\log \mu_1) - \log \mu_1 \right)^p. \quad (3.32)$$

If the running of  $M$  is indeed negligible – *i.e.* if  $DL \approx 0$  – we recover the above result, since

$$\frac{1}{2} M(\log \mu_1) \approx \frac{1}{2} M(\log \mu_0) = \log \mu_1, \quad (3.33)$$

and therefore

$$\widehat{\Gamma}_{[N]} \approx \widehat{c}_{[N],0}(\log \mu_1) = \widehat{c}_{[N],0} \left( \frac{1}{2} \log M(\mu_0) \right). \quad (3.34)$$

<sup>2</sup>Although, given some RG-scale  $\mu$ , its logarithm  $\log \mu$  is a dimensionless quantity, we will for simplicity sometimes refer to  $\log \mu$  as a “scale”.



### Towards exact RG-invariance

From the above discussion, we observe in fact that a very natural and remarkably simple way of removing *exactly* every logarithm in the perturbative expansion consists in evaluating every quantity at a scale  $\mu_*$  such that

$$\frac{1}{2} \log M(\log \mu_*) = \log \mu_*. \quad (3.35)$$

If such a scale exists (which is indeed the case as long as  $M > 0$ , and more generally, as long as the couplings remain perturbative), and since  $\widehat{\Gamma}_{[N]}$  is exactly invariant under variations of the RG-scale, we simply have to perform the running of every coupling from the initial scale  $\mu$  to  $\mu_*$  in order to obtain the exact all-order result

$$\widehat{\Gamma}_{[N]} = \widehat{c}_{[N],0}(\log \mu_*) = \widehat{c}_{[N],0}\left(\frac{1}{2} \log M(\mu_*)\right), \quad (3.36)$$

in which every mass-logarithm has been cancelled by the constraint (3.35). It is crucial to note that, by construction, the scale  $\mu_*$  possesses no dependence whatsoever on the arbitrary initial scale. In fact, Eq. (3.35) defines a fixed point for the function  $f$  such that

$$f : \log \mu \mapsto \frac{1}{2} \log M(\log \mu) = \log \sqrt{M(\log \mu)}. \quad (3.37)$$

The same conclusions can in fact be drawn by iterating the procedure described around Eq. (3.32). To see how, let us define an initial scale  $\mu_0$ , and, as previously, the scale  $\mu_1$  such that

$$\log \mu_1 = \frac{1}{2} \log M(\log \mu_0). \quad (3.38)$$

As we have seen, the  $N$ -point function evaluated at  $\log \mu_1$  takes the form

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \widehat{c}_{[N],p}(\log \mu_1) \left( \frac{1}{2} \log M(\log \mu_1) - \log \mu_1 \right)^p. \quad (3.39)$$

We may now proceed and iterate the procedure by defining the scale  $\mu_2$  such that

$$\log \mu_2 = \frac{1}{2} \log M(\log \mu_1) = \log \sqrt{M(\log \mu_1)}, \quad (3.40)$$

and in turn, the all-order  $N$ -point function can be expressed as

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \widehat{c}_{[N],p}(\log \mu_2) \left( \frac{1}{2} \log M(\log \mu_2) - \log \mu_2 \right)^p. \quad (3.41)$$

For all  $n \geq 0$ , we may then define the scale  $\mu_n$  such that

$$\log \mu_{n+1} = \frac{1}{2} \log M(\log \mu_n) = \log \sqrt{M(\log \mu_n)}, \quad (3.42)$$

bringing the  $N$ -point function into the form

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \widehat{c}_{[N],p}(\log \mu_n) \left( \frac{1}{2} \log M(\log \mu_n) - \log \mu_n \right)^p, \quad (3.43)$$

valid for any  $n \geq 0$ , since  $\widehat{\Gamma}_{[N]}$  is overall independent of the RG-scale. We finally observe the scale  $\mu_n$  can be expressed in terms of  $\mu_0$ , the initial scale, in the form of repeated compositions of the RG-dependent function  $f$  defined in Eq.(3.37):

$$\log \mu_n = \log \sqrt{M(\log \mu_{n-1})} = \log \sqrt{M(\log \sqrt{M(\log \mu_{n-2})})} = \dots = \underbrace{(f \circ \dots \circ f)}_{n \text{ times}}(\log \mu_0), \quad (3.44)$$

for all  $n \geq 0$ . Therefore, the scale  $\mu_*$  defined such that

$$\log \mu_* = \lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \underbrace{(f \circ \dots \circ f)}_{n \text{ times}}(\log \mu_0) \quad (3.45)$$

is nothing else than the fixed point of the function  $f$ , satisfying

$$\log \mu_* = \frac{1}{2} \log M(\log \mu_*), \quad (3.46)$$

and therefore cancelling every power of the mass-logarithm.

To summarise: we have arrived at the conclusion that a scale  $\mu_*$  exists such that, when evaluated at  $\mu_*$ , the expression of the all-order  $N$ -point function,  $\widehat{\Gamma}_{[N]}$ , does no longer involve the mass-logarithm. Instead,  $\widehat{\Gamma}_{[N]}$  is simply given by its leading coefficient  $\widehat{c}_{[N],0}$  evaluated at the RG-scale  $\mu_*$ . Additionally, we have shown that at any arbitrary scale,

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \widehat{c}_{[N],p} \overline{L}^p \quad (3.47)$$

where, for all  $p \geq 1$

$$\widehat{c}_{[N],p} = \frac{1}{p} \frac{1}{1 - DL} D \widehat{c}_{[N],p-1}. \quad (3.48)$$

Interestingly, we may also conclude from the above discussion that the denominator in Eq. (3.48) is at the origin of the infinite number of compositions of the function  $f : \log \mu \mapsto \log \sqrt{M(\log \mu)}$ , generating the fixed point structure and the convergence towards  $\mu_*$ . Its presence is crucial to explain the exact RG-independence of any renormalised quantity. As we shall explain in the next section, the presence of this denominator can be taken into account in a compact and rather elegant way, by introducing a set of modified beta-functions which automatically drive the couplings towards their value at the fixed point  $\mu_*$ .

### 1.3 The resummation operator

Let us now examine more closely the impact of the denominator  $1/(1 - DL)$  in the recurrence relations between the various coefficients in the logarithmic series. From Eq.(3.48), we have

$$\begin{aligned} \widehat{c}_{[N],1} &= \frac{1}{1 - DL} D \widehat{c}_{[N],0}, \\ \widehat{c}_{[N],2} &= \frac{1}{2} \frac{1}{1 - DL} D \widehat{c}_{[N],1} = \frac{1}{2} \frac{1}{1 - DL} D \left[ \frac{1}{1 - DL} D \widehat{c}_{[N],0} \right], \\ \widehat{c}_{[N],3} &= \dots \end{aligned} \quad (3.49)$$

Explicitly performing the derivatives is not very insightful, since this leads to a proliferation of terms involving higher derivatives of the mass-logarithm  $L$  (it is this complicated structure that, when resummed explicitly, will lead to convergence towards the fixed point). On the other hand, it is possible to define the differential operator  $\mathbf{D}$  such that

$$\mathbf{D} \equiv \frac{1}{1 - DL} D, \quad (3.50)$$

and in turn,

$$\begin{aligned} \widehat{c}_{[N],1} &= \mathbf{D} \widehat{c}_{[N],0}, \\ \widehat{c}_{[N],2} &= \frac{1}{2} \mathbf{D} \widehat{c}_{[N],1} = \frac{1}{2} \mathbf{D}^2 \widehat{c}_{[N],0}, \\ &\vdots \\ \widehat{c}_{[N],p} &= \frac{1}{p!} \mathbf{D}^p \widehat{c}_{[N],0}. \end{aligned} \quad (3.51)$$

Consequently, the all-order  $N$ -point function can be put in the rather compact form

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \frac{1}{p!} \mathbf{D}^p \widehat{c}_{[N],0} \bar{L}^p, \quad (3.52)$$

or, more explicitly,

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \frac{1}{p!} \mathbf{D}^p \widehat{c}_{[N],0}(\log \mu_0) \left( \frac{1}{2} \log \frac{M(\log \mu_0)}{\mu_0} \right)^p, \quad (3.53)$$

for any arbitrary fixed scale  $\mu_0$ . To further rewrite this expression, it is useful to define the functional operator  $\mathcal{E}_{\mu_0}$  with parameter  $\mu_0$  acting in the space of RG-dependent function, such that

$$\mathcal{E}_{\mu_0} [f] = \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{1}{2} \log \frac{M(\log \mu_0)}{\mu_0} \right)^p [\mathbf{D}^p f] (\log \mu_0), \quad (3.54)$$

and which can be rewritten in the very compact form

$$\mathcal{E}_{\mu_0} [f] = \left[ e^{\bar{L}(\mu_0) \mathbf{D}} (f) \right] (\log \mu_0), \quad (3.55)$$

where

$$\bar{L}(\mu_0) = \frac{1}{2} \log \frac{M(\log \mu_0)}{\mu_0} = \log \frac{\sqrt{M(\log \mu_0)}}{\mu_0}. \quad (3.56)$$

By construction, this operator is defined such that the function

$$F : \mu_0 \mapsto \mathcal{E}_{\mu_0} [f] \quad (3.57)$$

is constant on  $\mathbb{R}$ . To explicitly prove this statement, we write

$$\begin{aligned} \mu_0 \frac{d}{d\mu_0} F &= \mu_0 \frac{d}{d\mu_0} \left\{ \sum_{p=0}^{\infty} \frac{1}{p!} [\mathbf{D}^p f] \bar{L} \right\} \\ &= \sum_{p=0}^{\infty} \frac{1}{p!} D (\mathbf{D}^p f) \bar{L} + p \mathbf{D}^p f (DL - 1) \\ &= (1 - DL) \left\{ \sum_{p=0}^{\infty} \frac{1}{p!} (\mathbf{D}^{p+1} f) \bar{L} - \sum_{p=1}^{\infty} \frac{1}{(p-1)!} (\mathbf{D}^p f) \right\} \\ &= (1 - DL) \left\{ \sum_{p=0}^{\infty} \frac{1}{p!} (\mathbf{D}^{p+1} f) \bar{L} - \sum_{p=0}^{\infty} \frac{1}{p!} (\mathbf{D}^{p+1} f) \right\} \\ &= 0. \end{aligned} \quad (3.58)$$

Consequently, it is always possible to write, for any arbitrary scale  $\mu_0$

$$\mathcal{E}_{\mu_0} [f] = \mathcal{E}_{\mu_*} [f], \quad (3.59)$$

that is, the parameter of the resummation functional operator can be equally taken to be  $\mu_*$ . Hence, one convenient way to interpret the effect of the resummation operator is the following<sup>3</sup>: When applied

<sup>3</sup>This operator can in fact be generalised as the following abstract object. Given some function  $g : x \mapsto g(x)$ , and given the differential operator

$$\mathbf{D}_g : f \mapsto \frac{1}{1-g'} f',$$

the functional operator  $\mathcal{E}_{g,x_0}$  defined such that

$$\mathcal{E}_{g,x_0} = \exp \left[ (g(x_0) - x_0) \mathbf{D}_g \right]$$

acts on any function  $f$  according to

$$\mathcal{E}_{g,x_0} [f] = f(x_*) \quad \forall x_0, \quad (3.60)$$

where  $x_*$  is a fixed point of  $g$ . If  $g$  has many fixed point, we expect the function  $x_0 \mapsto \mathcal{E}_{g,x_0} [f]$  to be locally constant, but equal to  $f(x_*^i)$  where  $x_*^i$  is a fixed point of  $g$ . However,  $g$  must certainly satisfy a number of constraints in order for the operator to be well defined.

on an RG-dependent function  $f$ ,  $\mathcal{E}_{\mu_0}$  gives the value of  $f$  at the fixed point of the function  $g : \log \mu_0 \mapsto \frac{1}{2} \log M(\log \mu_0)$ . With these definitions, the all-order resummed  $N$ -point functions are simply given by

$$\widehat{\Gamma}_{[N]} = \mathcal{E}_{\mu_0} [\widehat{c}_{[N],0}] = \widehat{c}_{[N],0}(\log \mu_*) , \quad (3.61)$$

with  $\mu_0$  any arbitrary energy scale.

#### 1.4 RG-running towards $\mu_*$

If the resummation operator defined above allows for a compact and elegant description of the resummed  $N$ -point functions, it is not clear at this point how the above results can be used to perform practical computations. This is the goal of the following discussion, where we show that it is possible to trade the RG-scale for a dimensionless parameter  $z \in [0, 1]$ , such that, for any RG-dependent function  $f$ ,

$$f(z=0) = f(\mu_0), \quad f(z=1) = f(\mu_*). \quad (3.62)$$

In turn, based on the initial value of the couplings at some scale  $\mu_0$  and a set of differential equations, we will implement a method allowing to systematically evolve the couplings towards the fixed-point scale  $\mu_*$ , where the  $N$ -point functions are free of any logarithmic contributions.

To begin with, we define for each RG-dependent quantity  $f(\log \mu)$  its counterpart  $\mathbf{f}(z)$ , where the parameter  $z$  satisfies  $0 \leq z \leq 1$ . In particular, we define

$$\begin{aligned} \boldsymbol{\lambda}(z) & \text{ such that } \boldsymbol{\lambda}(0) = \lambda(\log \mu_0), \\ \mathbf{M}(z) & \text{ such that } \mathbf{M}(0) = M(\log \mu_0), \end{aligned}$$

and similarly for coefficients of the  $N$ -point functions, for which we define

$$\widehat{\mathbf{c}}_{[N],p}(z) \text{ such that } \widehat{\mathbf{c}}_{[N],p}(0) = \widehat{c}_{[N],p}(\log \mu_0).$$

It will also be useful to define the  $z$ -dependent mass-logarithm

$$\mathbf{L}(z) = \frac{1}{2} \log \mathbf{M}(z). \quad (3.63)$$

Next, we define the beta-functions of the  $z$ -dependent couplings as

$$\beta(\boldsymbol{\lambda}) \equiv \beta(\lambda) \Big|_{\lambda \rightarrow \boldsymbol{\lambda}}, \quad \beta(\mathbf{M}) \equiv \beta(M) \Big|_{M \rightarrow \mathbf{M}}, \quad (3.64)$$

that is, we only replace every coupling with its  $z$ -dependent counterpart in the expression of the beta-functions. In addition, the operator  $D$ , when applied to some combination of  $z$ -dependent couplings, will simply be assumed to act as on its  $\mu$ -dependent counterpart. Namely,

$$Df(\boldsymbol{\lambda}, \mathbf{M}) \equiv Df(\lambda, M) \Big|_{\substack{\lambda \rightarrow \boldsymbol{\lambda} \\ M \rightarrow \mathbf{M}}}. \quad (3.65)$$

Finally, the operator  $\mathbf{D}$  defined in Eq. (3.50) as

$$\mathbf{D} = \frac{1}{1 - DL} D, \quad (3.66)$$

will simply act on a  $z$ -dependent quantity as

$$\mathbf{D}f(\boldsymbol{\lambda}, \mathbf{M}) = \frac{1}{1 - D\mathbf{L}} Df(\boldsymbol{\lambda}, \mathbf{M}). \quad (3.67)$$

Based on these definitions, the renormalised  $N$ -point functions (which are in the following formally considered as numbers, and not functions of the RG-scale) can now be expressed as

$$\widehat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \widehat{\mathbf{c}}_{[N],p}(0) \overline{L}_0^p, \quad (3.68)$$

where, as previously,  $\bar{L}_0 = \frac{1}{2} \log \frac{M(\log \mu_0)}{\mu_0^2}$  (this quantity will remain a constant for the remaining of the discussion). This expansion can be rewritten in terms of the operator  $\mathbf{D}$  as previously (see Eq. (3.52)), but now in terms of the  $z$ -dependent coefficient  $\hat{\mathbf{c}}_{[N],0}$ :

$$\hat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \frac{1}{p!} \mathbf{D}^p \hat{\mathbf{c}}_{[N],0}(0) \bar{L}_0^p. \quad (3.69)$$

The next step is to observe that, if we define the derivative of  $\hat{\mathbf{c}}_{[N],0}$  with respect to the parameter  $z$  as

$$\frac{d\hat{\mathbf{c}}_{[N],0}}{dz}(z) = \frac{\bar{L}_0}{1 - D\mathbf{L}(z)} D \hat{\mathbf{c}}_{[N],0}(z), \quad (3.70)$$

then Eq. (3.69) simply becomes

$$\hat{\Gamma}_{[N]} = \sum_{p=0}^{\infty} \frac{1}{p!} \frac{d^p}{dz^p} \hat{\mathbf{c}}_{[N],0}(0), \quad (3.71)$$

which is nothing less than the Taylor expansion of  $\hat{\mathbf{c}}_{[N],0}$  around  $z = 0$ , evaluated at  $z = 1$ . In turn, we obtain the very following simple expression for the all-order, resummed  $N$ -point function:

$$\hat{\Gamma}_{[N]} = \hat{\mathbf{c}}_{[N],0}(1). \quad (3.72)$$

In other words, the all-order resummed  $N$ -point functions are obtained by evolving the coefficient  $\hat{\mathbf{c}}_{[N],0}(z)$  from its initial value,

$$\hat{\mathbf{c}}_{[N],0}(0) = \hat{\mathbf{c}}_{[N],0}(\log \mu_0), \quad (3.73)$$

to its value at  $z = 1$ . Since the derivative of  $\hat{\mathbf{c}}_{[N],0}(z)$  is known for all  $0 \leq z \leq 1$  (see Eq. (3.70)), what one needs to do in practice is simply to solve (numerically) the differential equation

$$\frac{d\hat{\mathbf{c}}_{[N],0}}{dz}(z) = \frac{\bar{L}_0}{1 - D\mathbf{L}(z)} D \hat{\mathbf{c}}_{[N],0}(z), \quad \hat{\mathbf{c}}_{[N],0}(0) = \hat{\mathbf{c}}_{[N],0}(\log \mu_0), \quad (3.74)$$

from  $z = 0$  to  $z = 1$ . Such a procedure will *systematically* drive the coefficient  $\hat{\mathbf{c}}_{[N],0}(\log \mu_0)$  towards its value at the scale  $\mu_*$  at which all logarithms cancel. In other words,

$$\hat{\mathbf{c}}_{[N],0}(1) = \hat{\mathbf{c}}_{[N],0}(\log \mu_*). \quad (3.75)$$

In fact, since the  $z$ -derivative of  $\hat{\mathbf{c}}_{[N],0}$  explicitly depend on the couplings  $\boldsymbol{\lambda}$  and  $\mathbf{M}$ , one needs in practice to solve the system of differential equations

$$\frac{d\boldsymbol{\lambda}}{dz}(z) = \frac{\bar{L}_0}{1 - D\mathbf{L}(z)} D \boldsymbol{\lambda}(z), \quad \boldsymbol{\lambda}(0) = \boldsymbol{\lambda}(\log \mu_0), \quad (3.76)$$

$$\frac{d\mathbf{M}}{dz}(z) = \frac{\bar{L}_0}{1 - D\mathbf{L}(z)} D \mathbf{M}(z), \quad \mathbf{M}(0) = \mathbf{M}(\log \mu_0), \quad (3.77)$$

in order to obtain

$$\boldsymbol{\lambda}(1) = \boldsymbol{\lambda}(\log \mu_*), \quad \mathbf{M}(1) = \mathbf{M}(\log \mu_*), \quad (3.78)$$

and infer the value of  $\hat{\mathbf{c}}_{[N],0}(1)$  from its expression in terms of the couplings.

To summarise: The all-order resummed renormalised  $N$ -point functions are simply given by their leading coefficient  $\hat{\mathbf{c}}_{[N],0}$  evaluated at the scale  $\mu_*$  which satisfies  $\log \mu_* = \frac{1}{2} \log M(\log \mu_*)$ . The dependence on  $\mu$  of every coupling in the model can be traded for a dependence on a dimensionless parameter  $z$  ranging from 0 to 1. The  $z$ -dependent couplings  $\boldsymbol{\lambda}$  and  $\mathbf{M}$  satisfy

$$\boldsymbol{\lambda}(0) = \boldsymbol{\lambda}(\log \mu_0), \quad \boldsymbol{\lambda}(1) = \boldsymbol{\lambda}(\log \mu_*), \quad (3.79)$$

$$\mathbf{M}(0) = \mathbf{M}(\log \mu_0), \quad \mathbf{M}(1) = \mathbf{M}(\log \mu_*), \quad (3.80)$$

and their derivatives with respect to  $z$  are known for all  $0 \leq z \leq 1$ :

$$\frac{d\boldsymbol{\lambda}}{dz}(z) = \frac{\bar{L}_0}{1 - D\mathbf{L}(z)} D\boldsymbol{\lambda}(z) \quad (3.81)$$

$$\frac{dM}{dz}(z) = \frac{\bar{L}_0}{1 - D\mathbf{L}(z)} D M(z). \quad (3.82)$$

Therefore, to obtain the value of the  $z$ -dependent couplings at  $z = 1$ , one simply needs in practice to solve (numerically) the above system of differential equations. Evolving the  $z$ -dependent couplings from their initial values at  $z = 0$  towards  $z = 1$  will automatically drive them to the values  $\lambda(\log \mu_*)$  and  $M(\log \mu_*)$ . Inserting these values in the expression of the leading coefficient  $\hat{c}_{[N],0}$  (or, equivalently,  $\hat{c}_{[N],0}$ ) will in turn directly give the value of the all-order resummed  $N$ -point function, since

$$\hat{\Gamma}_{[N]} = \hat{c}_{[N],0}(1). \quad (3.83)$$

## 1.5 Fixed-order truncations in perturbation theory

Having examined the structure of the resummed  $N$ -point functions to all orders in perturbation theory, we shall now discuss the practical implications for fixed-order calculations. From the above results, there are in fact two components in the definition of a fixed-order truncation: (i) The perturbative order at which the leading coefficient  $\hat{c}_{[N],0}$  is expressed and (ii) The loop-level at which the beta-function are computed. As we shall explain, these two components can in fact be considered independently from each other.

First of all, let us study the impact of the perturbative order of the beta-functions. As we have seen, by construction, the resummation operator (3.55) can be applied to *any* RG-dependent function, and will systematically drive this function to the fixed-point scale  $\mu_*$  such that

$$\log \frac{M(\log \mu_*)}{\mu_*^2} = 0. \quad (3.84)$$

Therefore, the resummation operator  $\mathcal{E}_{\mu_0}$  is blind to the precise form of the leading coefficient  $\hat{c}_{[N],0}$  on which it is applied to yield the resummed  $N$ -point function. It is however dependent on the precise form of the beta-functions, which determine the RG-trajectory of the mass parameter  $M$ . Therefore, with the initial value of the couplings being fixed at some scale  $\mu_0$ , we arrive in fact at a set of scales  $\mu_*^{(n)}$ , depending on the perturbative truncation used to generate the RG-trajectories. No matter the precise value of  $n$  – the perturbative order of the beta-functions – the scale  $\mu_*^{(n)}$  is guaranteed, by construction, to be independent of the initial scale. In turn, the  $N$ -point function obtained after resummation from the  $n$ -loop RG-trajectories is guaranteed to be exactly independent under the  $n$ -loop RG-flow. For most practical applications,  $\mu_*^{(n)}$  can be expected to converge rather quickly towards its all-order value, with  $\mu_*^{(1)}$  already giving a reasonable approximation.

Then, since the  $\mathcal{E}_{\mu_0}$  is a linear operator, any perturbative truncation at order  $m$  of the leading coefficient will give the resummed  $N$ -point function

$$\hat{\Gamma}_{[N]} = \mathcal{E}_{\mu_0} \left[ \sum_{k=0}^m \hat{c}_{[N],0}^{(k)} \right] = \sum_{k=0}^m \mathcal{E}_{\mu_0} \left[ \hat{c}_{[N],0}^{(k)} \right], \quad (3.85)$$

with each individual contribution in the rightmost sum being exactly invariant under the  $n$ -loop RG-flow.

### Example: resummed vacuum energy at two-loop

To exemplify the above results, we shall now provide the explicit form of the two-loop resummed vacuum energy (*i.e.* the 0-point function) in the  $\phi_+^4$  theory, of which we recall the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \Lambda - \frac{1}{2} M \phi^2 - \frac{1}{4!} \phi^4. \quad (3.86)$$

We will therefore need the expression of the renormalised vacuum energy (noted  $\Gamma_0$  consistently with the previous notation) to the two-loop level,

$$\Gamma_0 = \Gamma_0^{(0)} + \kappa\Gamma_0^{(1)} + \kappa^2\Gamma_0^{(2)} + \mathcal{O}(\kappa^3), \quad (3.87)$$

where

$$\begin{aligned} \Gamma_0^{(0)} &= \Lambda, \\ \Gamma_0^{(1)} &= -\frac{3}{8}M^2 + \frac{M^2}{4} \log \frac{M}{\mu^2}, \\ \Gamma_0^{(2)} &= \frac{1}{8}\lambda M^2 - \frac{1}{4}M^2\lambda \log \frac{M}{\mu^2} + \frac{1}{8}\lambda M^2 \left( \log \frac{M}{\mu^2} \right)^2. \end{aligned} \quad (3.88)$$

From the above expressions, we may also directly write down the leading coefficient, noted  $c_0$ ,

$$\begin{aligned} c_0 &= c_0^{(0)} + \kappa c_0^{(1)} + \kappa^2 c_0^{(2)} + \mathcal{O}(\kappa^3) \\ &= \Lambda - \frac{3}{8}\kappa M^2 + \frac{1}{8}\kappa^2 \lambda M^2 + \mathcal{O}(\kappa^3). \end{aligned} \quad (3.89)$$

We will also need the two-loop beta functions of  $\lambda$  and  $M$ :

$$\begin{aligned} \beta(\lambda) &= \kappa\beta^{(1)}(\lambda) + \kappa^2\beta^{(2)}(\lambda) + \mathcal{O}(\kappa^3) \\ &= 3\kappa\lambda^2 - \frac{17}{3}\kappa^2\lambda^3 + \mathcal{O}(\kappa^3), \end{aligned} \quad (3.90)$$

$$\begin{aligned} \beta(M) &= \kappa\beta^{(1)}(M) + \kappa^2\beta^{(2)}(M) + \mathcal{O}(\kappa^3) \\ &= \kappa\lambda M - \frac{5}{6}\kappa^2\lambda^2 M + \mathcal{O}(\kappa^3). \end{aligned} \quad (3.91)$$

$$\begin{aligned} \beta(\Lambda) &= \kappa\beta^{(1)}(\Lambda) + \kappa^2\beta^{(2)}(\Lambda) + \mathcal{O}(\kappa^3) \\ &= \frac{1}{2}\kappa M^2 - \mathcal{O}(\kappa^3). \end{aligned} \quad (3.92)$$

One of the conclusions of the previous sections was that, in order to obtain the resummed  $N$ -point function (here, the vacuum energy), one only needs to express the derivatives of the  $z$ -dependent parameters, of which we recall the expression:

$$\frac{d\boldsymbol{\lambda}}{dz}(z) = \frac{\bar{L}_0}{1 - D\mathbf{L}(z)} D\boldsymbol{\lambda}(z) \quad (3.93)$$

$$\frac{dM}{dz}(z) = \frac{\bar{L}_0}{1 - D\mathbf{L}(z)} D M(z), \quad (3.94)$$

$$\frac{d\Lambda}{dz}(z) = \frac{\bar{L}_0}{1 - D\mathbf{L}(z)} D\Lambda(z). \quad (3.95)$$

We reiterate that  $\bar{L}_0 = \frac{1}{2} \log \frac{M(\log \mu_0)}{\mu_0^2}$  is only a constant, and that  $D\boldsymbol{\lambda}(z)$  and  $DM(z)$  are simply obtained from the expression of the beta-functions, where the  $\mu$ -dependent couplings are replaced with their  $z$ -dependent counterpart. Therefore, it only remains to express the derivative of the mass-logarithm:

$$D\mathbf{L} = D \left( \frac{1}{2} \log M \right) = \frac{DM}{2M}, \quad (3.96)$$

which, from the expression of the beta-function of the mass parameter, Eq. (3.90), is explicitly computed as

$$D\mathbf{L} = \frac{1}{2}\kappa\lambda - \frac{5}{12}\kappa^2\lambda^2 + \mathcal{O}(\kappa^3). \quad (3.97)$$

In turn, we obtain the explicit form of the  $z$ -derivatives:

$$\frac{d\lambda}{dz} = \frac{\bar{L}_0}{1 - \frac{1}{2}\kappa\lambda + \frac{5}{12}\kappa^2\lambda^2} \left( 3 - \frac{17}{3}\kappa\lambda \right) \kappa\lambda^2 + \mathcal{O}(\kappa^3), \quad (3.98)$$

$$\frac{dM}{dz} = \frac{\bar{L}_0}{1 - \frac{1}{2}\kappa\lambda + \frac{5}{12}\kappa^2\lambda^2} \left( 1 - \frac{5}{6}\kappa\lambda \right) \kappa\lambda M + \mathcal{O}(\kappa^3), \quad (3.99)$$

$$\frac{d\Lambda}{dz} = \frac{\bar{L}_0}{1 - \frac{1}{2}\kappa\lambda + \frac{5}{12}\kappa^2\lambda^2} \frac{\kappa}{2} M^2 + \mathcal{O}(\kappa^3). \quad (3.100)$$

Numerically solving this differential system is an easy task, and for any initial values of  $\lambda$ ,  $M$  and  $\mu_0$ , and it can be explicitly checked that the running of  $M$  always ends its course at the value  $M(1)$  such that  $\frac{1}{2} \log M(1) = \frac{1}{2} \log M(\log \mu_*) = \log \mu_*$ . From the values of  $\lambda(1)$ ,  $M(1)$  and  $\Lambda(1)$ , the 2-loop resummed vacuum energy is simply obtained from Eq. (3.89):

$$\Gamma_0 = c_0(\log \mu_*) = c_0(1) = \Lambda(1) - \frac{3}{8}\kappa M(1)^2 + \frac{1}{8}\kappa^2\lambda M(1)^2 + \mathcal{O}(\kappa^3) \quad (3.101)$$

## 2 RG-improved potential in the massive $\phi_-^4$ theory

Having shown how to properly perform the resummation of logarithmic contributions in the  $\phi_+^4$  theory, we now release the constraint of a positive mass parameter. In particular, as discussed in chapter 1, the ground state of the theory is in general no longer located at the origin of the field space, and we must introduce a background field component to account for such a shift. In fact, every result previously derived remains valid if the couplings are replaced with their background-field-dependent counterparts. In the  $\phi^4$  theory considered here, the mass  $M$  is now traded for its field-dependent counterpart, noted  $m(\phi)$ , such that

$$m(\phi) = \frac{\partial^2 V(\phi)}{\partial \phi^2} = M + \frac{1}{2}\lambda\phi^2. \quad (3.102)$$

In addition, every renormalised  $N$ -point function is now a function of the field  $\phi$ , and in particular the renormalised vacuum energy – that is, the effective potential.

The first step prior to being able to compute physical observables consists in finding the global minimum of  $V_{\text{eff}}(\phi)$ , defining the true vacuum of the theory. Once such a minimum is identified, the background field can be set to the constant value

$$\phi = \phi_{\text{min}}, \quad (3.103)$$

and we recover the case of a  $\phi_+^4$  theory, in which every  $N$ -point function can be computed and resummed according to the procedure described above. In the remaining of this section, we shall therefore concentrate on the determination of the true vacuum of the theory, that is, the minimisation of the effective potential.

### 2.1 Field derivatives of the resummed potential

As the renormalised 0-point of the shifted theory, the effective potential (simply noted  $V$  for simplicity) can be given in the form of a power series in the mass-logarithm

$$V(\phi) = \sum c_p(\log \mu, \phi) \bar{L}(\phi)^p, \quad (3.104)$$

where, as before,

$$\bar{L}(\phi) \equiv \frac{1}{2} \log \frac{m(\log \mu, \phi)}{\mu^2}. \quad (3.105)$$

As we have seen, resumming the logarithmic contributions simply brings  $V(\phi)$  into the form

$$V(\phi) = c_0(\log \mu_*(\phi), \phi), \quad (3.106)$$



where  $\mu_*(\phi)$  is the RG-scale at which the mass-logarithm cancels exactly. Namely,

$$\log \mu_*(\phi) = \frac{1}{2} \log m(\log \mu_*(\phi), \phi). \quad (3.107)$$

The minimisation condition for the effective potential (the stationary point equation) reads

$$\frac{dV}{d\phi}(\phi) \equiv \nabla_\phi V(\phi) = 0. \quad (3.108)$$

The field-derivative  $\nabla_\phi V(\phi)$  can be explicitly computed from the resummed expression, Eq.(3.106),

$$\nabla_\phi V(\phi) = \nabla_\phi c_0(\log \mu_*(\phi), \phi) = \partial_\phi c_0(\log \mu_*(\phi), \phi) + (\nabla_\phi \log \mu_*(\phi)) Dc_0(\log \mu_*(\phi), \phi), \quad (3.109)$$

where two contributions arise, depending on the explicit or implicit nature of the  $\phi$ -dependence. Dropping the various arguments for clarity (keeping in mind in particular that  $\mu_*$  is field-dependent), we rewrite

$$\nabla_\phi V = \partial_\phi c_0 + (\nabla_\phi \log \mu_*) Dc_0. \quad (3.110)$$

To further express Eq.(3.110), one needs to compute the field derivative of the fixed-point scale  $\mu_*(\phi)$ . This is achieved by taking the derivative of its defining relation, Eq.(3.107):

$$\nabla_\phi \log \mu_* = \frac{1}{2} \nabla_\phi \log m = \frac{1}{2m} [\partial_\phi m + (\nabla_\phi \log \mu_*) Dm]. \quad (3.111)$$

Gathering the occurrences of  $\nabla_\phi \log \mu_*$  on the same side of the equation allows to obtain the expression

$$\nabla_\phi \log \mu_* = \frac{\partial_\phi m}{2m - Dm} = \frac{\partial_\phi L}{1 - DL}, \quad (3.112)$$

where  $L = \frac{1}{2} \log m$ . Consequently, the total field-derivative of the resummed potential takes the simple form

$$\nabla_\phi V = \partial_\phi c_0 + \frac{\partial_\phi L}{1 - DL} Dc_0 = \partial_\phi c_0 + \partial_\phi L c_1, \quad (3.113)$$

where  $c_1$  is the next-to-leading coefficient in the logarithmic series expansion of  $V(\phi)$ . The same result would have been obtained by first taking the derivative of Eq.(3.104) and subsequently resum the logarithms.

It will be very useful in the following to obtain the derivatives of  $V$  in the formalism where the couplings are traded for their  $z$ -dependent counterpart. In particular, with at hand the expression of  $\partial_\phi c_0$  for any  $0 \leq z \leq 1$ , we will show in the next section that the minima of  $V$  can be obtained as solutions, at  $z = 1$ , of a system of differential equations. Based on the results obtained in the previous section, we define  $\mathbf{c}_0(z, \phi)$  such that

$$\mathbf{c}_0(0, \phi) = c_0(\log \mu_0, \phi), \quad \frac{d\mathbf{c}_0}{dz} = \frac{\bar{L}_0(\phi)}{1 - D\mathbf{L}(z, \phi)} D\mathbf{c}_0, \quad (3.114)$$

where  $\bar{L}_0(\phi) = \frac{1}{2} \log m(\log \mu_0, \phi) - \log \mu_0$ . In addition, we have shown that taking  $z = 1$  is equivalent to evaluating the RG-dependent quantities at the fixed-point scale  $\mu_*$ , so that

$$\mathbf{c}_0(1, \phi) = c_0(\log \mu_*(\phi), \phi). \quad (3.115)$$

The procedure discussed in the next section to minimise the resummed potential will necessitate the expression of the field-derivative of  $\mathbf{c}_0(z, \phi)$  for any  $0 \leq z \leq 1$ . At this time, we only have computed this derivative at  $z = 1$ , leading to Eq. (3.113). We therefore need to examine more closely the behaviour of  $\mathbf{c}_0$  for intermediate values of  $z$ . There should exist a mapping of the form

$$\mathbf{c}_0(z, \phi) = c_0(\log \boldsymbol{\mu}(z, \phi), \phi), \quad (3.116)$$

where we have introduced the  $z$ -dependent renormalisation scale  $\boldsymbol{\mu}$ , such that

$$\log \boldsymbol{\mu}(0, \phi) = \log \mu_0, \quad \log \boldsymbol{\mu}(1, \phi) = \log \mu_*(\phi). \quad (3.117)$$

In order to determine the explicit form of  $\boldsymbol{\mu}$ , we may first compute the  $z$ -derivative of the  $z$ -dependent mass-logarithm  $\bar{\mathbf{L}}(z, \phi)$ , recalling that

$$\frac{d}{dz} = \frac{\bar{L}_0(\phi)}{1 - D\mathbf{L}(z, \phi)} D, \quad (3.118)$$

yielding

$$\begin{aligned} \frac{d\bar{\mathbf{L}}}{dz}(z, \phi) &= \frac{\bar{L}_0(\phi)}{1 - D\mathbf{L}(z, \phi)} D\bar{\mathbf{L}}(z, \phi) \\ &= \frac{\bar{L}_0(\phi)}{1 - D\mathbf{L}(z, \phi)} (D\mathbf{L}(z, \phi) - 1) \\ &= -\bar{L}_0(\phi). \end{aligned}$$

Solving the above equation for  $z$  with the initial condition  $\bar{\mathbf{L}}(0, \phi) = \bar{L}_0(\phi)$  gives

$$\bar{\mathbf{L}}(z, \phi) = (1 - z)\bar{L}_0(\phi). \quad (3.119)$$

Finally, since  $\bar{\mathbf{L}}(z, \phi) = \mathbf{L}(z, \phi) - \log \boldsymbol{\mu}(z, \phi)$ , we obtain

$$\log \boldsymbol{\mu}(z, \phi) = \mathbf{L}(z, \phi) - (1 - z)\bar{L}_0. \quad (3.120)$$

In a next step, we wish to express the field-derivative of the newly introduced  $z$ -dependent RG-scale  $\boldsymbol{\mu}$ . It is useful in this regard to explicitly replace  $\mathbf{L}(z, \phi)$  with its expression in terms of  $\log \boldsymbol{\mu}(z, \phi)$  in Eq. (3.120) prior to taking the field derivative:

$$\begin{aligned} \nabla_\phi \log \boldsymbol{\mu}(z, \phi) &= \nabla_\phi [\mathbf{L}(z, \phi) - (1 - z)\bar{L}_0] \\ &= \nabla_\phi [L(\log \boldsymbol{\mu}(z, \phi), \phi) - (1 - z)\bar{L}_0] \\ &= \nabla_\phi \log \boldsymbol{\mu}(z, \phi) DL(\boldsymbol{\mu}(z, \phi), \phi) + \partial_\phi L(\boldsymbol{\mu}(z, \phi), \phi) - (1 - z)\partial_\phi \bar{L}_0(\phi). \end{aligned}$$

Gathering the occurrences of  $\nabla_\phi \log \boldsymbol{\mu}(z, \phi)$  on the same side of the equation leads to

$$\nabla_\phi \log \boldsymbol{\mu}(z, \phi) = \frac{\partial_\phi \mathbf{L}(z, \phi) - (1 - z)\partial_\phi \bar{L}_0(\phi)}{1 - D\mathbf{L}(z, \phi)}. \quad (3.121)$$

In the limit where  $z \rightarrow 1$ , we recover the expression of the field-derivative of  $\log \mu_*$ :

$$\nabla_\phi \boldsymbol{\mu}(1, \phi) = \frac{\partial_\phi \mathbf{L}(1, \phi)}{1 - D\mathbf{L}(1, \phi)} = \frac{\partial_\phi \mathbf{L}(1, \phi)}{1 - D\mathbf{L}(1, \phi)}, \quad (3.122)$$

previously derived in Eq. (3.112) in terms of the RG-dependent couplings. With this expression at hand, we can finally express the field-derivative of  $\mathbf{c}_0$  for any  $z$  between 0 and 1

$$\begin{aligned} \nabla_\phi \mathbf{c}_0(z, \phi) &= \nabla_\phi \mathbf{c}_0(\log \boldsymbol{\mu}(z, \phi), \phi) \\ &= [\nabla_\phi \log \boldsymbol{\mu}(z, \phi)] D\mathbf{c}_0(\log \boldsymbol{\mu}(z, \phi), \phi) + \partial_\phi \mathbf{c}_0(\log \boldsymbol{\mu}(z, \phi), \phi) \\ \nabla_\phi \mathbf{c}_0 &= \partial_\phi \mathbf{c}_0 + \frac{\partial_\phi \mathbf{L} - (1 - z)\partial_\phi \bar{L}_0}{1 - D\mathbf{L}} D\mathbf{c}_0, \end{aligned} \quad (3.123)$$

where, in the last equality, we have dropped the arguments for better clarity.

## 2.2 Minimisation of the resummed potential

Having established a set of expressions for the first field-derivatives of the resummed potential, we present in this section a procedure allowing to determine the position of its minima. At least in cases where the coefficient  $c_0$  is polynomial in the field, this procedure will allow for the determination of all solutions to the stationary point equation.

The proposed minimisation procedure is based on the following observation: each solution  $\phi_{\min}(1)$  of the stationary point equation at  $z = 1$  can be continuously evolved, using the differential evolution of the  $z$ -dependent couplings, back to  $z = 0$  where the input parameters of the theory are initially expressed. We have derived in Eq. (3.123) the explicit form of the stationary point equation at every  $z$ :

$$\nabla_{\phi} \mathbf{c}_0(z) = \partial_{\phi} \mathbf{c}_0 + \frac{\partial_{\phi} \mathbf{L} - (1-z) \partial_{\phi} \bar{L}_0}{1 - D\mathbf{L}} D\mathbf{c}_0 = 0. \quad (3.124)$$

In order to simplify the expressions, we note that the full set of solutions at  $z = 1$  is independent of the  $\partial_{\phi} \bar{L}_0$  contribution, which can therefore be safely discarded. By doing so, we obtain the modified  $z$ -dependent stationary point equation

$$\nabla_{\phi} \mathbf{c}_0(z) = \partial_{\phi} \mathbf{c}_0 + \frac{\partial_{\phi} \mathbf{L}}{1 - D\mathbf{L}} D\mathbf{c}_0 = 0, \quad (3.125)$$

which, once again, captures every solution at  $z = 1$ . It may be further simplified by multiplying both sides with the denominator of the second contribution:

$$\partial_{\phi} \mathbf{c}_0 (1 - D\mathbf{L}) + (\partial_{\phi} \mathbf{L}) D\mathbf{c}_0 = 0. \quad (3.126)$$

We may also insert the explicit form of the derivatives of the logarithms,

$$D\mathbf{L} = \frac{1}{2} \frac{D\mathbf{m}}{\mathbf{m}}, \quad \partial_{\phi} \mathbf{L} = \frac{1}{2} \frac{\partial_{\phi} \mathbf{m}}{\mathbf{m}}, \quad (3.127)$$

leading to

$$\mathbf{P} \equiv 2\mathbf{m} \partial_{\phi} \mathbf{c}_0 + D\mathbf{c}_0 \partial_{\phi} \mathbf{m} - D\mathbf{m} \partial_{\phi} \mathbf{c}_0 = 0. \quad (3.128)$$

In this form, the stationary point equation shares its overall structure with the coefficient  $\mathbf{c}_0$ . Of particular interest is the case where  $\mathbf{c}_0$  is polynomial in the fields, implying that Eq.(3.128) is also polynomial (we recall that  $\mathbf{m}$  is obtained from the second field-derivatives of  $\mathbf{c}_0$ ). Based on this expression, the minimisation procedure can be summarised in the following steps

1. We determine (in general, numerically) the full set of solutions of Eq. (3.128) at  $z = 0$ .
2. For each solution  $\phi_{\min}^0$ , we define the  $z$ -dependent quantity  $\phi_{\min}(z)$  such that  $\phi_{\min}(0) = \phi_{\min}^0$  and

$$\mathbf{P}(z, \phi_{\min}(z)) = 0, \quad 0 \leq z \leq 1. \quad (3.129)$$

In other words, we require  $\phi_{\min}(z)$  to remain a solution of the (modified)  $z$ -dependent stationary point equation (3.128) all along the way from  $z = 0$  to  $z = 1$ .

3. The above constraint translates into a differential equation on  $\phi_{\min}(z)$  of the form

$$\frac{d\phi_{\min}}{dz}(z) = f(z). \quad (3.130)$$

This equation is solved numerically for every solution  $\phi_{\min}$  initially found at  $z = 0$ , yielding in turn the value of  $\phi_{\min}(1)$ . The quantity hence obtained is guaranteed to be a solution of the stationary point equation (3.128), *i.e.* to correspond to a stationary point of the resummed effective potential.

In step 2, the differential equation governing the evolution of  $\phi_{\min}(z)$  can be obtained by first writing the  $z$ -dependent stationary point equation in the form

$$\mathbf{P}(z, \phi_{\min}(z)) = 0, \quad (3.131)$$

and requiring that  $P(z, \phi_{\min}(z)) = 0$  for all  $z$ . Since  $\phi_{\min}(0)$  is a solution of the initial stationary point equation, *i.e.*

$$\mathbf{P}(0, \phi_{\min}(0)) = 0, \quad (3.132)$$

the above requirement translates to a vanishing  $z$ -derivative:

$$\frac{d}{dz} [\mathbf{P}(z, \phi_{\min}(z))] = 0. \quad (3.133)$$

Explicitly, the above constraint yields

$$\frac{\partial \mathbf{P}}{\partial z}(z, \phi_{\min}(z)) + \frac{d\phi_{\min}}{dz}(z) \frac{\partial \mathbf{P}}{\partial \phi} = 0, \quad (3.134)$$

which can be solved for  $\frac{d\phi_{\min}}{dz}$ :

$$\frac{d\phi_{\min}}{dz}(z) = -\frac{\partial_z \mathbf{P}}{\partial_\phi \mathbf{P}}(z, \phi_{\min}(z)). \quad (3.135)$$

Numerically solving this equation jointly with the set of differential equations for the  $z$ -dependent couplings allows to evolve every solution  $\phi_{\min}(z)$  from  $\phi_{\min}(0)$  to  $\phi_{\min}(1)$ . Namely, we have found a method allowing to obtain the solutions of the stationary point equation at  $z = 1$ , that is, the solutions to the stationary point equation satisfied by the resummed potential. It is useful to note that, some solutions to the (*e.g.* polynomial) stationary point equation at  $z = 0$  may be complex, and turn into real solutions at some  $0 < z_{\text{crit}} < 1$ . Therefore, one way to proceed to capture such solutions as well is to solve the differential equation in the complex plane. In the end, only the solutions whose imaginary part vanishes will be retained at  $z = 1$ .



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## Chapter 4

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# Renormalisation group improvement in multi-scale theories

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Having established a number of important results in the framework of theories with a single scalar field, we now extend the discussion to general scalar theories containing  $N$  interacting scalar fields. The resummation of logarithmic contributions (in particular in the effective potential) in such theories leads to several conceptual difficulties, and for this reason, has been an active field of research in the last decades [70, 71, 72, 73, 74, 75, 28, 76, 77, 78, 79, 80, 51]. The task is tedious, but the stakes are high: understanding the perturbative structure of theories with interacting scalar fields (and more generally interacting, non-degenerate massive states) is crucial in countless theoretical studies and phenomenological applications. Proper understanding of spontaneous symmetry breaking (in particular in classically scale-invariant theories [81]) in the Standard Model and its extensions, the study of decoupling in theories with vastly separated mass scales (and the related hierarchy problem), the study of vacuum stability in the Standard Model and beyond, and the overall search for precision and predictivity in any interacting theory – to only cite a few – are such examples.

The main conceptual difficulty one encounters when dealing with theories with multiple mass scales (or multi-scale theories) is usually formulated in the following way. While it is possible, in single-scale theories, to tune the renormalisation scale so as to cancel the logarithms, when at least two masses occur in the theory, it can only be tuned to cancel one such logarithm. For instance, in a theory with two mass scales  $m_1^2$  and  $m_2^2$  where renormalised quantities contain UV-logarithmic contributions of the form

$$\left(\log \frac{m_1^2}{\mu^2}\right)^p \left(\log \frac{m_2^2}{\mu^2}\right)^q, \quad (4.1)$$

setting  $\mu = m_1$  (or, equivalently  $\mu = m_2$ ) only resums part of the logarithmic series. In addition, logarithms of the form

$$\log \frac{m_1}{m_2} \quad (4.2)$$

are generated, which generally violate the decoupling theorem in the case of widely separated mass scales. In such cases, one solution consists in implementing “by hand” the decoupling theorem [65, 71, 77, 80], or addressing the problem from the effective field theory (EFT) perspective [51]. Another path one can follow is to solve this conceptual limitation by introducing multiple renormalisation scales in the theory [28, 70, 73, 74, 75, 76], which can therefore be adjusted independently to cancel the logarithms. However, at this time, none of these solutions is fully satisfactory. In the former case, the heavy physics is forced to decouple, and the question of whether such a decoupling could *naturally* occur remains unanswered. In the latter case, it has been shown that the introduction of multiple renormalisation scales inevitably (unless the scalars are decoupled) induces an explicit dependence of the renormalised quantities on logarithms of the ratios of such scales, *e.g.* of the form  $\log \frac{\mu_1}{\mu_2}$ . In turn, reusing the above example, setting  $\mu_1^2 = m_1$  and  $\mu_2^2 = m_2$  generates, once again, logarithms of the form  $\log \frac{m_1}{m_2}$ .

Throughout this chapter, we will adopt the second point of view and introduce multiple renormalisation scales in the process of dimensional regularisation of the theory. While this has been discussed on several occasions in the literature, there has been, to the best of our knowledge, no attempt to properly generalise and perform an in-depth study of such a procedure beyond a few examples of two-mass systems. The forthcoming presentation constitutes an attempt to fill this gap, by rigorously generalising, in section 1, the standard  $\overline{\text{MS}}$  renormalisation procedure in the presence of several renormalisation scales. This approach defines a new renormalisation scheme, referred to as the multi-scale  $\overline{\text{MS}}$  ( $\text{m}\overline{\text{MS}}$ ) scheme in the following. We shall in turn derive and examine the general structure of the multi-scale beta-functions in section 2. Section 3 will finally be dedicated to the resummation of the logarithmic contributions in the expression of the renormalised  $N$ -point functions. We will show that, from the structure of the  $\text{m}\overline{\text{MS}}$ -renormalised perturbation expansion, it is in fact possible to generalise the single-scale results in a rather straightforward way, providing a theoretical framework as well as practical methods allowing to systematically resum UV logarithms in a general scalar theory. Finally, we will summarise in section 4 our main findings and discuss their conceptual and practical implications.

## 1 Multi-scale renormalisation and the $\text{m}\overline{\text{MS}}$ scheme

As advertised in introduction, we now proceed with the renormalisation of a theory with  $N$  interacting scalars in which several renormalisation scales are introduced. As mentioned above, the renormalisation scheme hence obtained is referred to as the multi-scale  $\overline{\text{MS}}$  ( $\text{m}\overline{\text{MS}}$ ) scheme. The starting point, as we shall see, is to promote the renormalisation scale  $\mu$  to an  $N \times N$  matrix, whose structure will be discussed in section 1.2. The modified prescription for the occurrences of the renormalisation scale (matrix) introduced at the end of chapter 1 will be extremely helpful to understand the precise role of the newly introduced RG-scales in the process of computing the loop diagrams (section 1.3) and the beta-functions (section 2).

### 1.1 Definitions

We start from the alternative  $\mu$ -prescription described in chapter 1, section 6.1:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \mu^{-\varepsilon/2} \partial_\mu \phi^a Z_{ab}^2 \partial^\mu \phi^b - \frac{1}{2} (M_{ab} + \delta M_{ab}) \phi^a \phi^b - \frac{1}{4!} (\lambda_{abcd} + \delta \lambda_{abcd}) \phi^a \phi^b \phi^c \phi^d \\ &= \frac{1}{2} \mu^{-\varepsilon/2} \left[ \partial_\mu \phi^a Z_{ab}^2 \partial^\mu \phi^b - \frac{1}{2} (m_{ab} + \delta m_{ab}) \phi^a \phi^b \right] - \frac{1}{4!} (\lambda_{abcd} + \delta \lambda_{abcd}) \phi^a \phi^b \phi^c \phi^d, \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} \phi_0^a &= \mu^{-\varepsilon/4} Z^{ae} \phi^e, \\ M_{ab}^0 &= \mu^{\varepsilon/2} (M_{ef} + \delta M_{ef}) Z_{ea}^{-1} Z_{fb}^{-1} = (m_{ef} + \delta m_{ef}) Z_{ea}^{-1} Z_{fb}^{-1}, \\ \lambda_{abcd}^0 &= \mu^\varepsilon (\lambda_{efgh} + \delta \lambda_{efgh}) Z_{ea}^{-1} Z_{fb}^{-1} Z_{gc}^{-1} Z_{hd}^{-1}. \end{aligned} \quad (4.4)$$

In order to introduce multiple renormalisation scales in the theory, we promote the single RG-scale  $\mu$  to a matrix  $\mathcal{M}$ , and first rewrite the relation between the bare and renormalised scalar fields as (in matrix form)

$$\phi_0 = Z \mathcal{M}^{-\varepsilon/4} \phi. \quad (4.5)$$

While the matrix  $\mathcal{M}$  is *a priori* general<sup>1</sup>, it will be shown in the next section that it can always be made symmetric by a harmless field re-definition. Therefore, we will assume in the following that  $\mathcal{M} = \mathcal{M}^T$ , simplifying the forthcoming results and formulae accordingly.

<sup>1</sup>Apart from the fact that its eigenvalues must be positive in order to avoid the introduction of an imaginary part to the Lagrangian density.

From Eq. (4.5), the couplings are renormalised according to<sup>2</sup>:

$$\begin{aligned} M_{ab}^0 &= (M_{ef} + \delta M_{ef}) \mathcal{M}_{ei}^{\varepsilon/4} \mathcal{M}_{fj}^{\varepsilon/4} Z_{ia}^{-1} Z_{jb}^{-1} = (m_{ef} + \delta m_{ef}) Z_{ia}^{-1} Z_{jb}^{-1}, \\ \lambda_{abcd}^0 &= (\lambda_{efgh} + \delta \lambda_{efgh}) \mathcal{M}_{ei}^{\varepsilon/4} \mathcal{M}_{fj}^{\varepsilon/4} \mathcal{M}_{gk}^{\varepsilon/4} \mathcal{M}_{hl}^{\varepsilon/4} Z_{ia}^{-1} Z_{jb}^{-1} Z_{kc}^{-1} Z_{ld}^{-1}, \end{aligned} \quad (4.6)$$

where the auxiliary mass matrix  $m$  with mass-dimension 2 and the corresponding counterterm satisfy

$$m = \mathcal{M}^{\varepsilon/4} M \mathcal{M}^{\varepsilon/4}, \quad \delta m = \mathcal{M}^{\varepsilon/4} \delta M \mathcal{M}^{\varepsilon/4}. \quad (4.7)$$

It will be useful in the following to similarly define the auxiliary quartic couplings  $\ell$  with mass-dimension  $\varepsilon$  as

$$\ell_{abcd} = \lambda_{efgh} \mathcal{M}_{ea}^{\varepsilon/4} \mathcal{M}_{fb}^{\varepsilon/4} \mathcal{M}_{gc}^{\varepsilon/4} \mathcal{M}_{hd}^{\varepsilon/4}, \quad (4.8)$$

and similarly for the corresponding counterterm,  $\delta \ell$ . In turn, the relation between the renormalised couplings read, in terms of  $m$  and  $\ell$ ,

$$\begin{aligned} M_{ab}^0 &= (m_{ef} + \delta m_{ef}) Z_{ea}^{-1} Z_{fb}^{-1}, \\ \lambda_{abcd}^0 &= (\ell_{efgh} + \delta \ell_{efgh}) Z_{ea}^{-1} Z_{fb}^{-1} Z_{gc}^{-1} Z_{hd}^{-1}. \end{aligned} \quad (4.9)$$

When written in terms of  $M$  and  $\lambda$ , promoting the renormalisation scale to a matrix only affects the kinetic part of the Lagrangian which takes the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \left[ \mathcal{M}^{-\varepsilon/4} Z^2 \mathcal{M}^{-\varepsilon/4} \right]_{ab} \partial^\mu \phi^a - \frac{1}{2} (M_{ab} + \delta M_{ab}) \phi^a \phi^b - \frac{1}{4!} (\lambda_{abcd} + \delta \lambda_{abcd}) \phi^a \phi^b \phi^c \phi^d. \quad (4.10)$$

On the other hand, a dimensionally consistent description of the quadratic part of the Lagrangian involves the mass-matrix  $m$ , such that

$$\mathcal{L} \supset \frac{1}{2} \partial_\mu \phi^a \left[ \mathcal{M}^{-\varepsilon/4} Z^2 \mathcal{M}^{-\varepsilon/4} \right]_{ab} \partial^\mu \phi^a - \frac{1}{2} \phi^a \left[ \mathcal{M}^{-\varepsilon/4} (m + \delta m) \mathcal{M}^{-\varepsilon/4} \right]_{ab} \phi^b. \quad (4.11)$$

In a next step, we apply a rotation of the scalar fields such that the mass matrix, and in turn the leading-order propagator, is diagonal (see chapter 1, section 6.2). This is achieved by defining the orthogonal matrix  $U$  such that

$$\tilde{m} \equiv U m U^T = \text{diag}(m_1, \dots, m_N). \quad (4.12)$$

Applying this rotation to every tensor quantity in the theory, we define in particular

$$\tilde{Z} = Z U^T, \quad \tilde{\mathcal{M}} = U \mathcal{M} U^T, \quad \tilde{\phi} = U \phi, \quad (4.13)$$

and arrive at a new set relations between the bare and renormalised quantities<sup>3</sup>

$$\begin{aligned} \phi_0 &= \tilde{Z} \tilde{\mathcal{M}}^{-\varepsilon/4} \tilde{\phi} \\ M_{ab}^0 &= (\tilde{m}_{ef} + \delta \tilde{m}_{ef}) \tilde{Z}_{ea}^{-1} \tilde{Z}_{fb}^{-1}, \\ \lambda_{abcd}^0 &= \left( \tilde{\ell}_{efgh} + \delta \tilde{\ell}_{efgh} \right) \tilde{Z}_{ea}^{-1} \tilde{Z}_{fb}^{-1} \tilde{Z}_{gc}^{-1} \tilde{Z}_{hd}^{-1}. \end{aligned} \quad (4.14)$$

The propagator of the scalar mass eigenstates  $\tilde{\phi}$  reads

$$\begin{aligned} i\tilde{G} &= \frac{i}{\tilde{\mathcal{M}}^{-\varepsilon/4} \tilde{Z}^T \tilde{Z} \tilde{\mathcal{M}}^{-\varepsilon/4} p^2 - \tilde{M} - \delta \tilde{M}} \\ &= \tilde{\mathcal{M}}^{\varepsilon/4} \frac{i}{\tilde{Z}^T \tilde{Z} p^2 - \tilde{m} - \delta \tilde{m}} \tilde{\mathcal{M}}^{\varepsilon/4} \\ &= \tilde{\mathcal{M}}^{\varepsilon/4} \frac{i}{p^2 \mathbb{1} - \tilde{m} - \delta \tilde{m} + p^2 \tilde{\delta}_\phi} \tilde{\mathcal{M}}^{\varepsilon/4}, \end{aligned} \quad (4.15)$$

<sup>2</sup>If not otherwise stated,  $\mathcal{M}_{ab}^{\varepsilon/2}$  is used as a shorthand notation for  $(\mathcal{M}^{\varepsilon/2})_{ab}$ . That is, we first perform matrix exponentiation, then take the element at position  $(a, b)$ . This is in general not equivalent to exponentiating the element  $(a, b)$  of  $\mathcal{M}$ , in which case we shall use the explicit notation  $(\mathcal{M}_{ab})^{\varepsilon/2}$ . A similar remark holds for the field-strength matrix  $Z$ , with expressions of the type  $Z_{ab}^{-1} \equiv (Z^{-1})_{ab}$ .

<sup>3</sup>Note that  $U \mathcal{M}^x U^T = (U \mathcal{M} U^T)^x$  for any real exponent  $x$ .



and is renormalised according to

$$\begin{aligned} i\widetilde{G}_R &= \widetilde{\mathcal{M}}^{\varepsilon/4} \frac{i}{p^2 \mathbb{1} - \widetilde{m} - \widetilde{\pi}_R(p^2)} \widetilde{\mathcal{M}}^{\varepsilon/4} \\ &= \widetilde{\mathcal{M}}^{\varepsilon/4} \left[ \frac{i}{p^2 \mathbb{1} - \widetilde{m}} + \frac{i}{p^2 \mathbb{1} - \widetilde{m}} (-i\widetilde{\pi}_R(p^2)) \frac{i}{p^2 \mathbb{1} - \widetilde{m}} + \dots \right] \widetilde{\mathcal{M}}^{\varepsilon/4}. \end{aligned} \quad (4.16)$$

In the above expression, we have defined the renormalised matrix of self-energies with mass-dimension 2,  $\widetilde{\pi}_R(p^2)$ , such that

$$\widetilde{\pi}_R(p^2) = \widetilde{\mathcal{M}}^{\varepsilon/4} \widetilde{\Pi}_R(p^2) \widetilde{\mathcal{M}}^{\varepsilon/4}, \quad (4.17)$$

with  $\widetilde{\Pi}_R$  the renormalised matrix of self-energies with mass-dimension  $2 - \varepsilon/2$  expressed in the mass-basis. From Eq. (4.15), one has

$$\widetilde{\pi}_R(p^2) = U \pi_R(p^2) U^T = U [\pi(p^2) + \delta m - p^2 \delta_\phi] U^T = \widetilde{\pi}(p^2) + \delta \widetilde{m} - p^2 \widetilde{\delta}_\phi, \quad (4.18)$$

or, equivalently,

$$\widetilde{\Pi}_R(p^2) = \widetilde{\mathcal{M}}^{-\varepsilon/4} \widetilde{\pi}_R(p^2) \widetilde{\mathcal{M}}^{-\varepsilon/4} = \widetilde{\Pi}(p^2) + \delta \widetilde{M} - p^2 \left( \widetilde{\mathcal{M}}^{-\varepsilon/4} \widetilde{\delta}_\phi \widetilde{\mathcal{M}}^{-\varepsilon/4} \right), \quad (4.19)$$

which will be used in the next sections to fix the mass- and field-strength counterterms.

## 1.2 The structure of the renormalisation-scale matrix

As mentioned in the previous section, while the matrix  $\mathcal{M}$  is *a priori* general, one is able by a re-definition of the scalar fields to effectively promote it to a symmetric matrix. To see how, we first rewrite  $\mathcal{M}^{-\varepsilon/4}$  in terms of its polar decomposition:

$$\mathcal{M}^{-\varepsilon/4} = \left[ \mathcal{M}^{-\varepsilon/4} \left( \mathcal{M}^{-\varepsilon/4} \right)^T \right]^{1/2} R_\varepsilon, \quad (4.20)$$

with  $R$  an orthogonal matrix of the form  $R = \mathbb{1} + \mathcal{O}(\varepsilon)$ . The positive-definite matrix under the square root can be approximated up to order  $\mathcal{O}(\varepsilon^2)$  using the Baker–Campbell–Hausdorff formula,

$$\mathcal{M}^{-\varepsilon/4} \left( \mathcal{M}^{-\varepsilon/4} \right)^T = \exp \left( -\frac{\varepsilon}{4} \mathcal{L} \right) \exp \left( -\frac{\varepsilon}{4} \mathcal{L}^T \right) = \exp \left( -\frac{\varepsilon}{4} (\mathcal{L} + \mathcal{L}^T) \right) + \mathcal{O}(\varepsilon^2), \quad (4.21)$$

where we have defined the matrix-logarithm<sup>4</sup> of  $\mathcal{M}$ :

$$\mathcal{L} = \text{Log } \mathcal{M}. \quad (4.22)$$

We may write in turn

$$\mathcal{M}^{-\varepsilon/4} = \left[ \exp \left( -\frac{\varepsilon}{4} (\mathcal{L} + \mathcal{L}^T) \right) + \mathcal{O}(\varepsilon^2) \right]^{1/2} R_\varepsilon = \exp \left( -\frac{\varepsilon}{4} \frac{\mathcal{L} + \mathcal{L}^T}{2} \right) K_\varepsilon, \quad (4.23)$$

where the matrix  $K$  defined in the right-hand side satisfies  $K = \mathbb{1} + \mathcal{O}(\varepsilon)$ . Hence, Eq. (4.5) can be put in the form

$$\phi_0 = Z \mathcal{M}_{\text{sym}}^{-\varepsilon/4} K_\varepsilon \phi, \quad (4.24)$$

where  $\mathcal{M}_{\text{sym}}$  is a symmetric matrix with logarithm  $\text{Log } \mathcal{M}_{\text{sym}} = (\mathcal{L} + \mathcal{L}^T)/2$ . Since the dimensionless matrix  $K_\varepsilon$  satisfies  $K_\varepsilon \rightarrow \mathbb{1}$  in the 4-dimensional theory, we may absorb it in the scalar fields through the re-definition

$$K_\varepsilon \phi \rightarrow \phi, \quad (4.25)$$

without effect on the physical properties of the regularised theory. Re-defining in addition

$$\mathcal{M}_{\text{sym}} \rightarrow \mathcal{M} \quad (4.26)$$

<sup>4</sup>Here and in the following, we use for better clarity a different notation for the matrix logarithm, e.g.  $\text{Log } X$ , and the logarithm of a scalar quantity, e.g.  $\log x$ .

proves our previous statement that  $\mathcal{M}$ , although *a priori* general, can always be made symmetric. Consequently, the matrix  $\mathcal{M}$  can now be diagonalised by an orthogonal transformation, according to

$$\boldsymbol{\mu} = O\mathcal{M}O^T = \text{diag}(\mu_1, \dots, \mu_N). \quad (4.27)$$

Hence, the new degrees of freedom introduced by promoting the single renormalisation scale  $\mu$  to a matrix can be factored in terms of  $N$  renormalisation scales  $\mu_i$  ( $i = 1, \dots, N$ ) and a set of angles parameterising the orthogonal transformation  $O$ . In section 2, we will define, in a natural generalisation of the single-scale case, the beta-functions of the couplings as their derivative with respect to the eigenvalues of the RG-scale matrix, *i.e.* with respect to  $\mu_i$  ( $i = 1, \dots, N$ ). It will be shown in this context that the explicit computation of the beta-functions involves the derivatives of  $\mathcal{M}$  with respect to its own eigenvalues, namely

$$\mu_i \frac{d}{d\mu_i} \mathcal{M}^{ab} = O^{ia} O^{ib}. \quad (4.28)$$

It is then important to note that a limiting case arises when  $\mathcal{M}$  is proportional to the identity matrix, *i.e.* when  $\mathcal{M} = \mu \mathbb{1}$  for some scale  $\mu$ . In such a case, the matrix  $O$  is not uniquely defined, meaning in turn that neither are the beta-functions. Based on this observation, it therefore appears that an additional renormalisation condition must be introduced in order for the theory to be properly renormalised. Such a condition must unambiguously fix the orthogonal basis that defines the RG-scales  $\mu_i$  ( $i = 1, \dots, N$ ) at each point in the RG-space (*i.e.* the  $N$ -dimensional space of renormalisation scales). While an infinite set of such conditions may be given, defining a family of  $\overline{\text{mMS}}$  renormalisation schemes, one choice of particular interest consists in defining the renormalisation scales in the mass-basis, namely:

$$\mathcal{M}^{ab} = U^T \boldsymbol{\mu} U. \quad (4.29)$$

As expected, this condition extends the validity of (4.28) even in the case where  $\mathcal{M} \propto \mathbb{1}$ , and comes with an interesting interpretation: to each propagating mass-eigenstate  $\tilde{\phi}^i$  unambiguously corresponds a renormalisation scale  $\mu_i$ . In the remaining of this chapter, we shall assume that such a renormalisation condition always holds, so that  $\mathcal{M}$  and its matrix-logarithm  $\mathcal{L} = \text{Log } \mathcal{M}$  are diagonal in the mass basis, namely:

$$\tilde{\mathcal{M}} \equiv U\mathcal{M}U^T = \boldsymbol{\mu} = \text{diag}(\mu_1, \dots, \mu_N) \quad (4.30)$$

$$\tilde{\mathcal{L}} \equiv U\mathcal{L}U^T = \text{Log } \boldsymbol{\mu} = \text{diag}(\log \mu_1, \dots, \log \mu_N). \quad (4.31)$$

Nonetheless, when possible, we will carry out the forthcoming computations by keeping  $\tilde{\mathcal{M}}$  and  $\tilde{\mathcal{L}}$  in a general matrix form in order to establish a set of general results holding independently of the above renormalisation condition.

### 1.3 Two-loop renormalisation

One of the main goals of this chapter is to obtain the form of the beta-functions and anomalous dimensions in the newly defined  $\overline{\text{mMS}}$  scheme. In analogy with the single-scale case, one needs to express the  $n$ -point functions beyond leading order, and to subsequently regularise them by minimally subtracting the divergent contributions. From the expression of the counterterms, we will in turn be able in the next section to compute the beta-functions from the requirement that the bare couplings are independent of *all* renormalisation scales.

#### One-loop two-point function

We begin this discussion with the renormalisation of the two-point function at the one-loop level. In order to consistently keep track of the non-integer powers of dimensionful quantities, it is a necessary first step to compute the corrections to the matrix of self-energies with mass-dimension 2,  $\tilde{\pi}(p^2)$  from which  $\tilde{\Pi}(p^2)$  can be later recovered using

$$\left[ \tilde{\Pi}(p^2) \right]_{ab} = \tilde{\mathcal{M}}_{ae}^{-\varepsilon/4} \tilde{\mathcal{M}}_{bf}^{-\varepsilon/4} \left[ \tilde{\pi}(p^2) \right]_{ef}. \quad (4.32)$$

As previously, the only diagram contributing to  $\tilde{\pi}(p^2)$  at one-loop reads<sup>5</sup>

$$-i\tilde{\pi}_{ab}^{(1)} = \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} = \frac{1}{2} \widetilde{\mathcal{M}}_{ae}^{\varepsilon/4} \widetilde{\mathcal{M}}_{bf}^{\varepsilon/4} \tilde{\lambda}_{efij} \widetilde{\mathcal{M}}_{ik}^{\varepsilon/4} \widetilde{\mathcal{M}}_{jl}^{\varepsilon/4} \int d^d k \frac{\delta^{kl}}{k^2 - \tilde{m}_{kl} + i\varepsilon}. \quad (4.33)$$

It is then useful to define the dimensionally-incomplete basis integral  $\left[\mathbf{I}_2^{(1)}\right]_{ij}$ ,

$$\left[\mathbf{I}_2^{(1)}\right]_{ij} = \int d^d k \frac{\delta^{ij}}{k^2 - \tilde{m}_{ij} + i\varepsilon}, \quad (4.34)$$

with mass-dimension  $2 - \varepsilon$ . We may construct in turn the basis integral integral noted  $\left[\mathbf{J}_2^{(1)}\right]_{ij}^{abef}$  with canonical mass-dimension:

$$\left[\mathbf{J}_2^{(1)}\right]_{ij}^{abef} = \left(\widetilde{\mathcal{M}}_{ae}^{\varepsilon/4} \widetilde{\mathcal{M}}_{bf}^{\varepsilon/4}\right) \left(\widetilde{\mathcal{M}}_{ik}^{\varepsilon/4} \widetilde{\mathcal{M}}_{jl}^{\varepsilon/4} \left[\mathbf{I}_2^{(1)}\right]_{kl}\right), \quad (4.35)$$

bringing  $\tilde{\pi}_{ab}^{(1)}$  into the form

$$-i\tilde{\pi}_{ab}^{(1)} = \text{---} \text{---} \text{---} \text{---} \text{---} = \frac{1}{2} \tilde{\lambda}_{efij} \left[\mathbf{J}_2^{(1)}\right]_{ij}^{abef}. \quad (4.36)$$

Reusing the results of chapter 1, the integral  $\mathbf{I}_2^{(1)}$  is explicitly given by

$$\left[\mathbf{I}_2^{(1)}\right]_{ij} = i \frac{\tilde{m}_{ij}}{16\pi^2} (4\pi^2 e^{-\gamma_E})^{\varepsilon/2} \left[ \frac{2}{\varepsilon} - (\log \tilde{m}_{ij} - 1) \right] + \mathcal{O}(\varepsilon), \quad (4.37)$$

and inserting the  $\varepsilon$ -expansion of  $\mathcal{M}^{\varepsilon/4}$  in Eq. (4.35) yields, up to order  $\mathcal{O}(\varepsilon^0)$ ,

$$\left[\mathbf{J}_2^{(1)}\right]_{ij}^{abef} = \frac{i}{16\pi^2} \left\{ \frac{2}{\varepsilon} \tilde{m}_{ij} \delta_{ae} \delta_{bf} - \left[ \tilde{m} \text{Log} \tilde{m} - \frac{1}{2} \left\{ \tilde{m}, \tilde{\mathcal{L}}' \right\} - \mathbb{1} \right]_{ij} \delta_{ae} \delta_{bf} - \frac{1}{2} \tilde{m}_{ij} \left( \tilde{\mathcal{L}}'_{ae} \delta_{bf} + \delta_{ae} \tilde{\mathcal{L}}'_{bf} \right) \right\} + \mathcal{O}(\varepsilon), \quad (4.38)$$

where we have defined the logarithm of the  $\overline{\text{MS}}$ -corrected renormalisation scale matrix,

$$\tilde{\mathcal{L}}' = \text{Log} \left[ (4\pi^2 e^{-\gamma_E})^{1/2} \widetilde{\mathcal{M}} \right] = \text{Log} \widetilde{\mathcal{M}}'. \quad (4.39)$$

As compared to the single-scale computations, the logarithms of the RG-scales are now localised in the diagram, giving rise to a rather intricate tensor structure already at the one-loop level. In particular, the adimensionality of the logarithmic contributions is obtained from a non-trivial combination of the logarithm of the mass matrix,  $\tilde{m}$ , with occurrences of  $\tilde{\mathcal{L}}'$  stemming from both internal and external legs.

It will help simplify the forthcoming expressions to define the tensor structure  $\left[\mathbf{A}(\tilde{m})\right]_{ij}^{abef}$  as

$$\left[\mathbf{A}(\tilde{m})\right]_{ij}^{abef} = \left[ \tilde{m} \text{Log} \tilde{m} - \frac{1}{2} \left\{ \tilde{m}, \tilde{\mathcal{L}}' \right\} \right]_{ij} \delta_{ae} \delta_{bf} - \frac{1}{2} \tilde{m}_{ij} \left( \tilde{\mathcal{L}}'_{ae} \delta_{bf} + \delta_{ae} \tilde{\mathcal{L}}'_{bf} \right), \quad (4.40)$$

so that Eq. (4.38) takes the form

$$\left[\mathbf{J}_2^{(1)}\right]_{ij}^{abef} = \frac{2i}{16\pi^2} \left( \frac{1}{\varepsilon} \tilde{m}_{ij} \delta_{ae} \delta_{bf} - \frac{1}{2} \left[\mathbf{A}(\tilde{m})\right]_{ij}^{abef} \right) + \mathcal{O}(\varepsilon), \quad (4.41)$$

yielding in turn,

$$-i\tilde{\pi}_{ab}^{(1)} = \frac{1}{2} \tilde{\lambda}_{efij} \left[\mathbf{J}_2^{(1)}\right]_{ij}^{abef} = \frac{i}{16\pi^2} \left( \frac{1}{\varepsilon} \tilde{\lambda}_{abij} \tilde{m}_{ij} - \frac{1}{2} \tilde{\lambda}_{efij} \left[\mathbf{A}(\tilde{m})\right]_{ij}^{abef} \right) + \mathcal{O}(\varepsilon). \quad (4.42)$$

<sup>5</sup>We omit for clarity the argument  $p^2$  of which the self-energy is independent at one-loop.

Finally, we may recover from Eq. (4.32) the expression of  $\widetilde{\Pi}^{(1)}$  in terms of the renormalised mass matrix  $M$ , properly performing in addition the  $\varepsilon$ -expansion of each occurrence of  $\widetilde{m}$  outside the argument of the logarithms. Doing so removes the logarithms of the RG-scales attached to the external legs, so that  $\widetilde{\Pi}^{(1)}$  can be put into the form

$$-i\widetilde{\Pi}_{ab}^{(1)} = \frac{i}{16\pi^2} \widetilde{\lambda}_{abij} \left( \frac{1}{\varepsilon} \widetilde{M}_{ij} - \frac{1}{2} \left[ \mathbf{A}(\widetilde{M}) \right]_{ij} \right) + \mathcal{O}(\varepsilon), \quad (4.43)$$

where

$$\left[ \mathbf{A}(\widetilde{M}) \right]_{ij} = \left[ \widetilde{M} \text{Log } \widetilde{m} - \left\{ \widetilde{M}, \widetilde{\mathcal{L}}' \right\} - \mathbb{1} \right]_{ij} \quad (4.44)$$

is the analogue of the single-scale function  $A(x) = x(\log(x/\mu^2) - 1)$  defined in Eq. (1.42). We then observe from Eq. (4.43) that the one-loop mass-matrix counterterm takes the same form as in the case of single-scale renormalisation, that is

$$\delta \widetilde{M}_{ab} = \frac{1}{\varepsilon(16\pi^2)} \widetilde{\lambda}_{abij} \widetilde{M}_{ij}. \quad (4.45)$$

### One-loop four-point function

Turning to the case of the 4-point function (which we shall only compute at zero-momentum), we write

$$-i\widetilde{\Gamma}_{abcd}^{(1)} = \frac{1}{2} \lambda_{abef} \lambda_{ghcd} \widetilde{\mathcal{M}}_{ei}^{\varepsilon/4} \widetilde{\mathcal{M}}_{fj}^{\varepsilon/4} \widetilde{\mathcal{M}}_{gk}^{\varepsilon/4} \widetilde{\mathcal{M}}_{hl}^{\varepsilon/4} \left[ \mathbf{I}_4^{(1)} \right]_{ik,jl} = \frac{1}{2} \lambda_{abef} \left[ \mathbf{J}_4^{(1)} \right]_{eg,fh} \lambda_{ghcd}, \quad (4.46)$$

where the 4-point basis integral  $\mathbf{I}_4^{(1)}$ , with mass-dimension  $-\varepsilon$ , is defined as

$$\left[ \mathbf{I}_4^{(1)} \right]_{ab,cd} = \int \widetilde{d^d k} \frac{\delta^{ab}}{k^2 - \widetilde{m}_{ab}} \frac{\delta^{cd}}{k^2 - \widetilde{m}_{cd}}, \quad (4.47)$$

and can be directly expressed by reusing the results of chapter 1:

$$\left[ \mathbf{I}_4^{(1)} \right]_{ab,cd} = i \frac{\delta^{ab} \delta^{cd}}{16\pi^2} (4\pi^2 e^{-\gamma_E})^{\varepsilon/2} \left[ \frac{2}{\varepsilon} - \int_0^1 dx \log [x\widetilde{m}_{ab} + (1-x)\widetilde{m}_{cd}] \right] + \mathcal{O}(\varepsilon). \quad (4.48)$$

Its dimensionless counterpart, noted  $\mathbf{J}_4^{(1)}$ , is then computed as

$$\begin{aligned} \left[ \mathbf{J}_4^{(1)} \right]_{ab,cd} &= \widetilde{\mathcal{M}}_{ai}^{\varepsilon/4} \widetilde{\mathcal{M}}_{bj}^{\varepsilon/4} \widetilde{\mathcal{M}}_{ck}^{\varepsilon/4} \widetilde{\mathcal{M}}_{dl}^{\varepsilon/4} \left[ \mathbf{I}_4^{(1)} \right]_{ij,kl} \\ &= \frac{i}{16\pi^2} \left[ 2 \frac{\delta^{ab} \delta^{cd}}{\varepsilon} - \left( \int_0^1 dx \delta^{ab} \delta^{cd} \log [x\widetilde{m}_{ab} + (1-x)\widetilde{m}_{cd}] - \widetilde{\mathcal{L}}'_{ab} \delta_{cd} - \delta_{ab} \widetilde{\mathcal{L}}'_{cd} \right) \right] + \mathcal{O}(\varepsilon) \\ &= \frac{2i}{16\pi^2} \left( \frac{\delta^{ab} \delta^{cd}}{\varepsilon} + \frac{1}{2} \left[ \mathbf{B}(\widetilde{M}) \right]_{ab,cd} \right) + \mathcal{O}(\varepsilon), \end{aligned} \quad (4.49)$$

where we have defined the multi-scale analogue of the function  $B(x)$  (see Eq. (1.47)) as

$$\left[ \mathbf{B}(\widetilde{M}) \right]_{ab,cd} = - \int_0^1 dx \delta^{ab} \delta^{cd} \log [x\widetilde{m}_{ab} + (1-x)\widetilde{m}_{cd}] - \widetilde{\mathcal{L}}'_{ab} \delta_{cd} - \delta_{ab} \widetilde{\mathcal{L}}'_{cd}. \quad (4.50)$$

From Eq. (4.49) we infer the expression of the one-loop quartic counterterm

$$\delta \widetilde{\lambda}_{abcd} = \frac{1}{\varepsilon(16\pi^2)} \left( \widetilde{\lambda}_{abij} \widetilde{\lambda}_{ijcd} + \widetilde{\lambda}_{abij} \widetilde{\lambda}_{ijcd} + \widetilde{\lambda}_{abij} \widetilde{\lambda}_{ijcd} \right), \quad (4.51)$$

taking the same form as in single-scale renormalisation.

### Two-loop two-point function

Turning to the two-loop contributions to  $\tilde{\pi}$ , the two relevant topologies can be schematically decomposed as

$$-i\tilde{\pi}^{(2)} \left[ \text{Sunset} \right] \sim \text{Sunset}^{\times} + \text{Sunset}^{\times} + \text{Sunset}, \quad (4.52)$$

and

$$-i\tilde{\pi}^{(2)}(p^2) \left[ \text{Sunset} \right] \sim \text{Sunset}^{\times} + \text{Sunset}, \quad (4.53)$$

where it is understood that, in both cases, only the relevant quartic counterterm contributions are retained. Conveniently expressing the first topology in terms of the one-loop basis integrals, we obtain

$$\begin{aligned} -i\tilde{\pi}_{ab}^{(2)} \left[ \text{Sunset} \right] &= \frac{1}{2} \delta \tilde{\lambda}_{efij} \left[ \mathbf{J}_2^{(1)} \right]_{ij}^{abef} + \frac{1}{2} \tilde{\lambda}_{abij} \left[ \mathbf{J}_4^{(1)} \right]_{ij,kl} \delta \tilde{m}_{kl} + \frac{i}{4} \tilde{\lambda}_{efij} \left[ \mathbf{J}_4^{(1)} \right]_{ij,kl} \tilde{\lambda}_{klmn} \left[ \mathbf{J}_2^{(1)} \right]_{mn}^{abef} \\ &= \frac{i}{16\pi^2} \left[ \frac{1}{\varepsilon^2} \tilde{\lambda}_{abij} \tilde{\lambda}_{ijkl} \tilde{m}_{kl} + \mathcal{O}(1) \right], \end{aligned} \quad (4.54)$$

and observe that the corresponding counterterm is, once again, equivalent to that obtained in the usual  $\overline{\text{MS}}$  scheme. Turning to the sunset topology, we define the momentum-dependent two-loop basis integral

$$\left[ \mathbf{I}_2^{(2)}(p^2) \right]_{ij,kl,mn} = \int \widetilde{d^d k} \int \widetilde{d^d q} \frac{\delta^{ij}}{k^2 - \tilde{m}_{ij}} \frac{\delta^{kl}}{(p+k+q)^2 - \tilde{m}_{kl}} \frac{\delta^{mn}}{q^2 - \tilde{m}_{mn}}, \quad (4.55)$$

and its counterpart with canonical mass-dimension,

$$\left[ \mathbf{J}_2^{(2)}(p^2) \right]_{ij,kl,mn}^{abef} = \left( \widetilde{\mathcal{M}}_{ae}^{\varepsilon/4} \widetilde{\mathcal{M}}_{bf}^{\varepsilon/4} \right) \left( \widetilde{\mathcal{M}}_{il}^{\varepsilon/4} \widetilde{\mathcal{M}}_{jk}^{\varepsilon/4} \right) \left( \widetilde{\mathcal{M}}_{kk}^{\varepsilon/4} \widetilde{\mathcal{M}}_{kk}^{\varepsilon/4} \right) \left( \widetilde{\mathcal{M}}_{mM}^{\varepsilon/4} \widetilde{\mathcal{M}}_{nN}^{\varepsilon/4} \right) \left[ \mathbf{I}_2^{(2)}(p^2) \right]_{IJ,KL,MN}, \quad (4.56)$$

so that the contributions of the sunset topology to  $\tilde{\pi}$  are given by

$$-i\tilde{\pi}_{ab}^{(2)}(p^2) \left[ \text{Sunset} \right] = \frac{1}{2} \delta \tilde{\lambda}_{efij} \left[ \mathbf{J}_2^{(1)} \right]_{ij}^{abef} + \frac{i}{6} \tilde{\lambda}_{eikm} \tilde{\lambda}_{fjln} \left[ \mathbf{J}_2^{(2)}(p^2) \right]_{ij,kl,mn}^{abef}. \quad (4.57)$$

The sunset loop integral was computed in chapter 1, appendix B, from which the expression of  $\left[ \mathbf{J}_2^{(2)}(p^2) \right]_{ij,kl,mn}^{abef}$  is obtained by properly expanding of the  $\varepsilon$ -dependent contributions. It is convenient to isolate the momentum dependence from the basis integral as follows:

$$\left[ \mathbf{J}_2^{(2)}(p^2) \right]_{ij,kl,mn}^{abef} = \frac{p^2}{2\varepsilon} \delta_{ij} \delta_{kl} \delta_{mn} + \left( \widetilde{\mathcal{M}}_{ae}^{\varepsilon/4} \widetilde{\mathcal{M}}_{bf}^{\varepsilon/4} \right) \left[ \mathbf{J}_2^{(2)} \right]_{ij,kl,mn} + \mathcal{O}(1), \quad (4.58)$$

where we have gathered in  $\left[ \mathbf{J}_2^{(2)} \right]_{ij,kl,mn}$  the remaining pole terms with no dependence on  $p^2$ , and conveniently factored out powers of the RG-scale matrix attached to the external legs: those will eventually cancel when recovering  $\tilde{\Pi}^{(2)}(p^2)$  from the expression of  $\tilde{\pi}^{(2)}(p^2)$ . Explicitly, we obtain:

$$\begin{aligned} \left[ \mathbf{J}_2^{(2)} \right]_{ij,kl,mn} &= \left\{ \left( -\frac{2}{\varepsilon^2} \widetilde{M}_{ij} + \frac{1}{\varepsilon} \widetilde{M}_{ij} + \frac{2}{\varepsilon} \left[ \mathbf{A}(\widetilde{M}) \right]_{ij} \right) \delta_{kl} \delta_{mn} \right. \\ &\quad \left. + \frac{1}{\varepsilon} \left( \left\{ \widetilde{M}, \widetilde{\mathcal{L}}' \right\}_{ij} \delta_{kl} \delta_{mn} - \widetilde{M}_{ij} \widetilde{\mathcal{L}}'_{kl} \delta_{mn} - \widetilde{M}_{ij} \delta_{kl} \widetilde{\mathcal{L}}'_{mn} \right) \right. \\ &\quad \left. + (ij \leftrightarrow kl) + (ij \leftrightarrow mn) \right\} + \mathcal{O}(1). \end{aligned} \quad (4.59)$$

We recognise in the first line of the above expression the same contributions which we had found from the single-scale  $\overline{\text{MS}}$  scheme computations (see Eq. (1.B30)). In turn, taking into account the counterterm contributions is expected to properly cancel the mass-logarithms contained in  $\mathbf{A}(\widetilde{M})$  and guarantee as usual the absence of non-local divergences. On the other hand, additional contributions arise in the second line of Eq. (4.59) involving logarithms of the RG-scale matrix. Such contributions are characteristic of the presence of multiple RG-scales in the theory and naturally cancel in the limit  $\mathcal{M} \rightarrow \mu\mathbb{1}$ . This is clear from Eq. (4.59), since

$$\left\{ \widetilde{M}, \widetilde{\mathcal{L}}' \right\}_{ab} \delta_{cd} \delta_{ef} - \widetilde{M}_{ab} \widetilde{\mathcal{L}}'_{cd} \delta_{ef} - \widetilde{M}_{ab} \delta_{cd} \widetilde{\mathcal{L}}'_{ef} \xrightarrow{\mathcal{M} \rightarrow \mu\mathbb{1}} \left( 2\widetilde{M}_{ab} - \widetilde{M}_{ab} - \widetilde{M}_{ab} \right) \delta^{cd} \delta^{ef} \log \mu' = 0. \quad (4.60)$$

Interestingly, these new contributions carry information on the structure of overlapping UV-divergences (this justifies in particular their absence from the diagram with topology (4.54)): The non-trivial structure of the RG-scale matrix indeed implies that each occurrence of  $\mathcal{L} = \text{Log } \mathcal{M}$  is attached to an internal propagator, *i.e.* is localised in the loop-diagram. We have in particular that the terms in the second line of Eq. (4.59) signal the presence of a purely two-loop overlapping divergence, which cannot be reproduced (and cancelled) by the composition of two one-loop integrals.

From Eqs (4.57)–(4.58), we obtain for the sunset contributions to  $\widetilde{\Pi}^{(2)}(p^2)$  the following expression:

$$\begin{aligned} -i\widetilde{\Pi}_{ab}^{(2)}(p^2) \left[ \text{---} \overline{\text{---}} \text{---} \right] &= \frac{i}{(16\pi^2)^2} \left\{ \left( \widetilde{M}_{ae}^{-\varepsilon/4} \widetilde{M}_{bf}^{-\varepsilon/4} \right) \frac{p^2}{12\varepsilon} \widetilde{\lambda}_{eijk} \widetilde{\lambda}_{ijkf} + \widetilde{\lambda}_{aief} \widetilde{\lambda}_{bjef} \left( \frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} \right) \widetilde{M}_{ij} \right. \\ &\quad \left. + \frac{1}{2\varepsilon} \widetilde{\lambda}_{aike} \widetilde{\lambda}_{bjle} \left( \left\{ \widetilde{M}, \widetilde{\mathcal{L}}' \right\}_{ij} \delta_{kl} - 2\widetilde{M}_{ij} \widetilde{\mathcal{L}}'_{kl} \right) \right\} + \mathcal{O}(1), \end{aligned} \quad (4.61)$$

and, as expected from the above discussion, the two-loop counterterm acquires an explicit RG-scale dependence:

$$\delta^{(2)} \widetilde{M}_{ab} = \frac{1}{(16\pi^2)^2} \left\{ \widetilde{\lambda}_{aief} \widetilde{\lambda}_{bjef} \left( \frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} \right) \widetilde{M}_{ij} + \frac{1}{2\varepsilon} \widetilde{\lambda}_{aike} \widetilde{\lambda}_{bjle} \left( \left\{ \widetilde{M}, \widetilde{\mathcal{L}}' \right\}_{ij} \delta_{kl} - 2\widetilde{M}_{ij} \widetilde{\mathcal{L}}'_{kl} \right) \right\}. \quad (4.62)$$

We also obtain that the field-strength counterterm takes the same form as previously, namely:

$$\left[ \widetilde{\delta}_\phi^{(2)} \right]_{ab} = -\frac{1}{(16\pi^2)^2} \frac{1}{12\varepsilon} \widetilde{\lambda}_{aijk} \widetilde{\lambda}_{bjik}. \quad (4.63)$$

### Two-loop four-point function

The case of the two-loop contributions to the four-point function is similar to that of the two-point function, and we shall only report here the final result obtained using  $\text{m}\overline{\text{MS}}$  renormalisation in the  $s$ -channel configuration:

$$-i\widetilde{\Gamma}_{abcd}^{(2)} \left[ \text{---} \overline{\text{---}} \text{---} \right] = \frac{i}{(16\pi^2)^2} \widetilde{\lambda}_{abij} \widetilde{\lambda}_{ijkl} \widetilde{\lambda}_{klcd} \left[ \frac{1}{\varepsilon^2} + \mathcal{O}(1) \right], \quad (4.64)$$

$$\begin{aligned} -i\widetilde{\Gamma}_{abcd}^{(2)} \left[ \text{---} \overline{\text{---}} \text{---} \right] &= \frac{i}{(16\pi^2)^2} \left\{ \lambda_{abij} \lambda_{iklc} \lambda_{jkld} \left( \frac{1}{\varepsilon^2} - \frac{1}{2\varepsilon} \right) \right. \\ &\quad \left. + \frac{1}{2\varepsilon} \lambda_{abij} \lambda_{i'k'lc} \lambda_{j'k'l'd} \left[ \left( \widetilde{\mathcal{L}}'_{ii'} \delta_{jj'} + \delta_{ii'} \widetilde{\mathcal{L}}'_{jj'} \right) \delta_{kk'} \delta_{ll'} - \delta_{ii'} \delta_{jj'} \left( \widetilde{\mathcal{L}}'_{kk'} \delta_{ll'} + \delta_{kk'} \widetilde{\mathcal{L}}'_{ll'} \right) \right] \right. \\ &\quad \left. + (ab \leftrightarrow cd) \right\} + \mathcal{O}(1), \end{aligned} \quad (4.65)$$

## 2 Multi-scale beta-functions

With the expression of the two-loop counterterms at hand, one is able to derive the beta-functions and anomalous dimensions in the  $\text{m}\overline{\text{MS}}$  scheme. With multiple RG-scales at play, one has to define, for each

coupling, a set of  $N$  beta-functions characterising their evolution along the multi-dimensional RG-flow. In addition, a new feature compared to the single-scale case is the appearance of explicit logarithms of the RG-scales in the counterterms, which will propagate to the expression of the beta-functions starting from the two-loop level.

## 2.1 General considerations

As mentioned in section 1.2, a natural generalisation of single-scale renormalisation consists in defining the beta-functions in terms of the eigenvalues of the RG-scale matrix  $\mathcal{M}$ . We note, as previously,

$$\boldsymbol{\mu} = \text{diag}(\mu_1, \dots, \mu_N), \quad (4.66)$$

the diagonal counterpart of  $\mathcal{M}$ . We shall use the renormalisation condition introduced in section 1.2 by imposing that the RG-scale matrix is diagonal in the mass-basis, namely:

$$\boldsymbol{\mu} = \widetilde{M} = U\mathcal{M}U^T. \quad (4.67)$$

Hence, in the mass-basis, to each coupling corresponds a set of  $N$  beta-functions, defined as the  $\varepsilon$ -independent contribution in the RG-derivative expansion, namely:

$$\beta_A(\widetilde{\lambda}_{abcd}) \equiv D_A \widetilde{\lambda}_{abcd} \Big|_{\varepsilon \rightarrow 0}, \quad \beta_A(\widetilde{M}_{ab}) \equiv D_A \widetilde{M}_{ab} \Big|_{\varepsilon \rightarrow 0}, \quad (4.68)$$

where we have defined the differential operators

$$D_A = \mu_A \frac{d}{d\mu_A}, \quad A = 1, \dots, N. \quad (4.69)$$

Similarly,  $N$  anomalous dimension matrices are defined such that

$$D_A \widetilde{\phi} \Big|_{\varepsilon \rightarrow 0} = -\widetilde{\gamma}_A \widetilde{\phi}, \quad (4.70)$$

in analogy with standard  $\overline{\text{MS}}$  renormalisation. The derivation of the beta-functions and anomalous dimension matrices is similar to the single-scale case although rather technical, and shall be deferred to appendix A to keep this presentation concise. We will therefore concentrate on the main results, commenting on the major differences that arise in the  $\text{m}\overline{\text{MS}}$  scheme compared to its single-scale counterpart.

We find in appendix A that the beta-functions can be written in a rather compact form, by first defining the set of matrices  $\gamma_A^{[1]}$  such that

$$[\gamma_A^{[1]}]^{ij} = -\frac{1}{4} \delta^{iA} \delta^{Aj}, \quad (4.71)$$

and which occur in the  $\varepsilon$ -expansion of the anomalous dimension matrix in  $(4 - \varepsilon)$  dimensions. Their presence is associated to the anomalous scaling of the renormalised fields with mass-dimension  $1 - \varepsilon/4$ . We similarly obtain that the RG-derivatives of the couplings contain at order  $\mathcal{O}(\varepsilon)$  contributions of the form

$$\beta_A^{[1]}(\widetilde{\lambda}_{abcd}) = \left\{ \widetilde{\lambda} \leftarrow \gamma_A^{[1]} \right\}_{abcd}, \quad \beta_A^{[1]}(\widetilde{M}_{ab}) = \left\{ \widetilde{M} \leftarrow \gamma_A^{[1]} \right\}_{ab}. \quad (4.72)$$

For instance, for the mass term, we explicitly have

$$D_A \widetilde{M}_{ab} \supset \varepsilon \beta_A^{[1]}(\widetilde{M}_{ab}) = -\frac{\varepsilon}{4} \left( \delta^{aA} \widetilde{M}_{Ab} + \widetilde{M}_{aA} \delta^{Ab} \right). \quad (4.73)$$

With these definitions at hand, we show in appendix A that the multi-scale beta-functions and anomalous dimension matrices take the form

$$\widetilde{\gamma}_A = \frac{1}{2} \beta_A^{[1]}(\widetilde{\lambda}_{efgh}) \frac{\partial \widetilde{\delta}_\phi^{[1]}}{\partial \widetilde{\lambda}_{efgh}} - V_A \quad (4.74)$$

$$\beta_A(\widetilde{\lambda}_{abcd}) = -\beta_A^{[1]}(\widetilde{\lambda}_{efgh}) \frac{\partial \delta_{[1]} \widetilde{\lambda}_{abcd}}{\partial \widetilde{\lambda}_{efgh}} + \left\{ \delta_{[1]} \widetilde{\lambda} \leftarrow \gamma_A^{[1]} \right\}_{abcd} + \left\{ \widetilde{\lambda} \leftarrow \widetilde{\gamma}_A \right\}_{abcd}, \quad (4.75)$$

$$\beta_A(\widetilde{M}_{ab}) = -\beta_A^{[1]}(\widetilde{\lambda}_{efgh}) \frac{\partial \delta_{[1]} \widetilde{M}_{ab}}{\partial \widetilde{\lambda}_{efgh}} - \beta_A^{[1]}(\widetilde{M}_{ef}) \frac{\partial \delta_{[1]} \widetilde{M}_{ab}}{\partial \widetilde{M}_{ef}} + \left\{ \delta_{[1]} \widetilde{M} \leftarrow \gamma_A^{[1]} \right\}_{ab} + \left\{ \widetilde{M} \leftarrow \widetilde{\gamma}_A \right\}_{ab}, \quad (4.76)$$

where  $\tilde{\delta}_\phi^{[1]}$ ,  $\delta_{[1]}\tilde{\lambda}$  and  $\delta_{[1]}\tilde{M}$  respectively denote the first-order pole contributions to the counterterms  $\tilde{\delta}_\phi$ ,  $\tilde{\delta}\tilde{\lambda}$  and  $\tilde{\delta}\tilde{M}$ . The skew-symmetric matrices  $V_A$  appearing in Eq. (4.74) are defined as

$$V_A = (D_A U) U^T, \quad (4.77)$$

where  $U$  is the orthogonal matrix which diagonalises the mass matrix according to

$$\tilde{M} = U M U^T. \quad (4.78)$$

The explicit form of  $V_A$  is obtained from the requirement that the mass matrix preserves its diagonal structure along the RG-flow, and reads

$$[V_A]^{ab} = \frac{[U \beta_A(M) U^T]^{ab}}{M_a - M_b}, \quad (4.79)$$

with  $M_p = \tilde{M}_{pp}$  the  $p$ -th eigenvalue of the mass matrix.

Intuitively, the interpretation of the first two terms in Eq. (4.75) and of the first three terms in Eq. (4.76) is the following: each internal index contraction between two vertices, generically noted  $\delta^{ij}$ , is successively replaced by

$$\delta^{ij} \rightarrow \frac{1}{2} \delta^{iA} \delta^{Aj}. \quad (4.80)$$

It is in fact useful to define an operator  $\mathcal{D}_A$  which, when applied to a particular tensor structure, performs such a replacement. As an illustration, let us consider the tensor structure

$$\tilde{\lambda}_{abij} \tilde{\lambda}_{ijcd} \quad (4.81)$$

appearing in the expression of the one-loop quartic counterterm. When then have

$$\mathcal{D}_A [\tilde{\lambda}_{abij} \tilde{\lambda}_{ijcd}] = \frac{1}{2} (\tilde{\lambda}_{abAj} \tilde{\lambda}_{Ajcd} + \tilde{\lambda}_{abiA} \tilde{\lambda}_{iAcd}), \quad (4.82)$$

or, in a diagrammatic notation,

$$\mathcal{D}_A \left[ \text{diagram} \right] = \frac{1}{2} \text{diagram}_1 + \frac{1}{2} \text{diagram}_2, \quad (4.83)$$

where the small circles signal the presence of an open index  $A$ . In turn, the beta-functions take a more compact form when expressed in terms of  $\mathcal{D}_A$ :

$$\beta_A(\tilde{\lambda}_{abcd}) = \mathcal{D}_A [\delta^{[1]}\tilde{\lambda}_{abcd}] + \left\{ \tilde{\lambda} \leftarrow \tilde{\gamma}_A \right\}_{abcd}, \quad (4.84)$$

$$\beta_A(\tilde{M}_{ab}) = \mathcal{D}_A [\delta^{[1]}\tilde{M}_{ab}] + \left\{ \tilde{M} \leftarrow \tilde{\gamma}_A \right\}_{ab}. \quad (4.85)$$

A subtlety arises starting from the two-loop level, where the logarithms of the RG-scales may explicitly appear in the expression of the counterterms. In such a case, applying the operator  $\mathcal{D}_A$  on a tensor structure involving a contraction of the type

$$\lambda_{abci} \tilde{\mathcal{L}}_{ij} \lambda_{jdef} \quad (4.86)$$

yields

$$\mathcal{D}_A [\lambda_{abci} \tilde{\mathcal{L}}_{ij} \lambda_{jdef}] = \frac{1}{4} (\tilde{\lambda}_{abcA} \tilde{\mathcal{L}}_{Aj} \lambda_{jdef} + \tilde{\lambda}_{abci} \tilde{\mathcal{L}}_{iA} \tilde{\lambda}_{Adef}), \quad (4.87)$$

which, since the RG-scale matrix is diagonal in the mass-basis, reduces to

$$\mathcal{D}_A [\lambda_{abci} \tilde{\mathcal{L}}_{ij} \lambda_{jdef}] = \frac{1}{2} \log \mu_A \tilde{\lambda}_{abcA} \tilde{\lambda}_{Adef}. \quad (4.88)$$



## 2.2 Towards a basis-independent formulation

In the above discussion, the multi-scale beta-functions have been defined in the basis where the RG-scale matrix  $\mathcal{M}$  is diagonal, which also coincides with the mass-basis from the additional renormalisation condition introduced in section 1.2. In their current form, the resulting expressions can hardly be extended to the more general case where the couplings and fields are not initially expressed in such a basis, hence breaking the manifest covariance which characterises the tensor formalism we have made use of throughout this chapter. Let us consider as an illustration the one-loop quartic beta-function,  $\beta(\tilde{\lambda}_{abcd})$ , obtained from Eqs. (4.75) and (4.82):

$$\beta_A^{(1)}(\tilde{\lambda}_{abcd}) = \frac{1}{16\pi^2} \left[ \tilde{\lambda}_{abAi} \tilde{\lambda}_{Aicd} + (b \leftrightarrow c) + (b \leftrightarrow d) \right] + \left\{ \tilde{\lambda} \leftarrow \tilde{\gamma}_A \right\}_{abcd}. \quad (4.89)$$

From  $\beta_A^{(1)}(\tilde{\lambda}_{abcd})$ , we may recover the beta-function of the quartic coupling  $\lambda_{abcd}$  expressed in the arbitrary, original basis:

$$\begin{aligned} \beta_A^{(1)}(\lambda_{abcd}) &= \beta_A^{(1)} \left( \tilde{\lambda}_{efgh} U_{ea} U_{fb} U_{gc} U_{hd} \right) \\ &= \left[ \beta_A^{(1)}(\tilde{\lambda}_{efgh}) + \left\{ \tilde{\lambda} \leftarrow V_A \right\}_{efgh} \right] U_{ea} U_{fb} U_{gc} U_{hd} \\ &= \frac{1}{16\pi^2} \left[ \lambda_{abCi} \lambda_{Dicd} + (b \leftrightarrow c) + (b \leftrightarrow d) \right] U^{AC} U^{AD}. \end{aligned} \quad (4.90)$$

Two comments are in order. First, having rotated the quartic couplings back to the original basis has removed as expected the contributions coming from the skew-symmetric part of the anomalous dimension. Second, expressing  $\beta_A^{(1)}(\lambda_{abcd})$  exclusively in terms of  $\lambda_{abcd}$  has generated occurrences of the orthogonal matrix  $U$ , since the index  $A$  has not been rotated back to the original basis. In fact, in the above form, it is not possible to covariantly transform the index  $A$  which merely labels the position of the renormalisation scale  $\mu_A$  of which we are taking the derivative.

Nonetheless, we observe that the non-covariant contribution  $U^{AC}U^{AD}$  appearing in Eq.(4.90) can in fact be obtained by differentiating the logarithm of the RG-scale matrix,  $\mathcal{L}$  with respect to one of its own eigenvalues:

$$\frac{d\mathcal{L}^{CD}}{d \log \mu_A} = \mu_A \frac{d}{d\mu_A} \mathcal{L}^{CD} = D_A \left[ U^T \tilde{\mathcal{L}} U \right]^{CD} = U^{AC} U^{AD}. \quad (4.91)$$

This means in turn that  $\beta^{(1)}(\lambda_{abcd})$  can be expressed as

$$\beta_A^{(1)}(\lambda_{abcd}) = \frac{1}{16\pi^2} \left[ \lambda_{abCi} \lambda_{Dicd} + (b \leftrightarrow c) + (b \leftrightarrow d) \right] (D_A \mathcal{L}^{CD}) \equiv \beta_{CD}(\lambda_{abcd}) (D_A \mathcal{L}^{CD}), \quad (4.92)$$

where we have defined the two-index auxiliary beta-function  $\beta_{CD}^{(1)}(\lambda_{abcd})$  as

$$\beta_{CD}^{(1)}(\lambda_{abcd}) = \frac{1}{16\pi^2} \left[ \lambda_{abCi} \lambda_{Dicd} + (b \leftrightarrow c) + (b \leftrightarrow d) \right]. \quad (4.93)$$

It will be useful to introduce a compact notation allowing to rewrite Eq. (4.92) in the form

$$\beta_A^{(1)}(\lambda_{abcd}) = (D_A \mathcal{L}^{CD}) \beta_{CD}(\lambda_{abcd}) \equiv [D_A \mathcal{L} \cdot \beta](\lambda_{abcd}), \quad (4.94)$$

where we have introduced the shortcut notation  $\mathcal{L} \cdot \beta = \mathcal{L}^{CD} \cdot \beta_{CD}$ .

Generalising the above example, we may in fact always re-define the actual beta-functions in terms of a set of 2-index auxiliary beta-functions. This holds for the anomalous dimension matrix as well, and one may take as the new definition of the beta-functions the following relations

$$\gamma_A^{ab} = D_A \mathcal{L} \cdot \gamma^{ab}, \quad (4.95)$$

$$\beta_A(\lambda_{abcd}) = [D_A \mathcal{L} \cdot \beta](\lambda_{abcd}), \quad (4.96)$$

$$\beta_A(M_{ab}) = [D_A \mathcal{L} \cdot \beta](M_{ab}), \quad (4.97)$$

holding in any basis, and with the two-index beta functions  $\beta$  and anomalous dimensions  $\gamma$  properly transforming (*i.e.* covariantly) under an orthogonal change of basis. These newly defined two-index beta-functions and anomalous dimensions are defined in complete analogy with their one-index counterparts: With view on Eqs. (4.71) and (4.72), we define

$$\left[\gamma_{AB}^{[1]}\right]^{ij} = -\frac{1}{4}\delta^i_A\delta^j_B, \quad \beta_{AB}^{[1]}(\lambda_{abcd}) = \left\{\lambda \leftarrow \gamma_{AB}^{[1]}\right\}_{abcd}, \quad \beta_{AB}^{[1]}(M_{ab}) = \left\{M \leftarrow \gamma_{AB}^{[1]}\right\}_{ab}, \quad (4.98)$$

and establish in turn the two-index analogues of Eqs. (4.74)–(4.76) in the arbitrary, original basis:

$$\gamma_{AB} = \frac{1}{2}\beta_{AB}^{[1]}(\lambda_{efgh})\frac{\partial\delta_\phi^{[1]}}{\partial\lambda_{efgh}} \quad (4.99)$$

$$\beta_{AB}(\lambda_{abcd}) = -\beta_{AB}^{[1]}(\lambda_{efgh})\frac{\partial\delta_{[1]}\lambda_{abcd}}{\partial\lambda_{efgh}} + \left\{\delta_{[1]}\lambda \leftarrow \gamma_{AB}^{[1]}\right\}_{abcd} + \left\{\lambda \leftarrow \gamma_{AB}\right\}_{abcd}, \quad (4.100)$$

$$\beta_{AB}(M_{ab}) = -\beta_{AB}^{[1]}(\lambda_{efgh})\frac{\partial\delta_{[1]}M_{ab}}{\partial\lambda_{efgh}} - \beta_{AB}^{[1]}(M_{ef})\frac{\partial\delta_{[1]}M_{ab}}{\partial M_{ef}} + \left\{\delta_{[1]}M \leftarrow \gamma_{AB}^{[1]}\right\}_{ab} + \left\{M \leftarrow \gamma_{AB}\right\}_{ab}. \quad (4.101)$$

We note that it is always possible to enforce the symmetry of the two-index beta-functions and anomalous dimensions under exchange of their indices (*e.g.*  $A$  and  $B$  in the above expression) without impact on the expression of the actual beta-functions. To do so, one simply needs to define  $\gamma_{AB}^{[1]}$  as

$$\left[\gamma_{AB}^{[1]}\right]^{ij} = -\frac{1}{4}\frac{\delta^i_A\delta^j_B + \delta^i_B\delta^j_A}{2}, \quad (4.102)$$

instead of

$$\left[\gamma_{AB}^{[1]}\right]^{ij} = -\frac{1}{4}\delta^i_A\delta^j_B. \quad (4.103)$$

Finally, before turning to the next section, it is convenient to introduce an additional notation, allowing to disentangle the pure-vertex contributions to the beta-functions of  $\lambda$  and  $M$  and those stemming from the anomalous dimensions. In particular, we decompose the above general beta-functions as

$$\beta_{AB}(\lambda_{abcd}) = \widehat{\beta}_{AB}(\lambda_{abcd}) + \left\{\lambda \leftarrow \gamma_{AB}\right\}_{abcd}, \quad (4.104)$$

$$\beta_{AB}(M_{ab}) = \widehat{\beta}_{AB}(M_{ab}) + \left\{M \leftarrow \gamma_{AB}\right\}_{ab}, \quad (4.105)$$

where the definition of  $\widehat{\beta}_{AB}(\lambda_{abcd})$  and  $\widehat{\beta}_{AB}(M_{ab})$  – the pure-vertex contributions to  $\beta_{AB}(\lambda_{abcd})$  and  $\beta_{AB}(M_{ab})$ , respectively – directly follow from Eqs (4.100) and (4.101).

### 2.3 Two-loop beta-functions

Having established the general formalism allowing to compute the multi-scale beta-functions, we may now specialise the above formulae to express the one- and two-loop beta functions of the general scalar theory considered throughout this section. To keep the presentation general, we shall provide the relevant expressions in the original basis using the two-index notation introduced above, keeping in mind that their counterpart in the mass basis are straightforwardly obtained by (i) rotating all fields and couplings to the mass-basis and (ii) taking into account the additional skew-symmetric contribution to the anomalous dimension matrices, stemming from the relation

$$D_A\widetilde{\phi} = D_A(U\phi) = -[U\gamma_A U^T - V_A]\widetilde{\phi}. \quad (4.106)$$

At one-loop, anomalous dimension matrix vanishes and we obtain for the quartic coupling

$$\begin{aligned} \beta_{AB}^{(1)}(\lambda_{abcd}) &= \left[ \frac{1}{2} \text{diagram}_1 + \frac{1}{2} \text{diagram}_2 \right] + (t\text{-channel}) + (u\text{-channel}) \\ &= \frac{1}{16\pi^2} \frac{1}{2} [\lambda_{abAi}\lambda_{Bicd} + (b \leftrightarrow c) + (b \leftrightarrow d)] + (A \leftrightarrow B), \end{aligned} \quad (4.107)$$

where we have used the symmetric definition of  $\beta_{AB}$ . We similarly obtain for the mass-matrix:

$$\beta_{AB}^{(1)}(M_{ab}) = \frac{1}{2} \text{---} \overset{\circ}{\bullet} \text{---} \overset{\circ}{\bullet} \text{---} + \frac{1}{2} \text{---} \overset{\circ}{\bullet} \text{---} \overset{\circ}{\bullet} \text{---} = \frac{1}{16\pi^2} \frac{1}{2} [\lambda_{abAi} M_{Bi} + \lambda_{abBi} M_{Ai}], \quad (4.108)$$

where a black dot on an internal line indicates a mass insertion. It is interesting to note that the single-scale  $\overline{\text{MS}}$  beta-functions may be recovered by contracting the two open indices, namely:

$$\beta^{(1)}(\lambda_{abcd}) \Big|_{\overline{\text{MS}}} = \delta^{AB} \beta_{AB}^{(1)}(\lambda_{abcd}), \quad \beta^{(1)}(M_{ab}) \Big|_{\overline{\text{MS}}} = \delta^{AB} \beta_{AB}^{(1)}(M_{ab}). \quad (4.109)$$

At two-loop, the anomalous dimension matrix acquires its first non-zero contributions, obtained from the expression of the field-strength counterterm in Eq. (4.63). From Eq. (4.99), we find

$$\begin{aligned} [\gamma_{AB}^{(2)}]_{ab} &= \frac{1}{24} \left[ \frac{1}{4} \text{---} \circ \text{---} \circ \text{---} \text{---} + \frac{1}{4} \text{---} \text{---} \text{---} \circ \text{---} \circ \text{---} + \frac{3}{2} \text{---} \text{---} \text{---} \right] \\ &= \frac{1}{(16\pi^2)^2} \frac{1}{24} \left[ \frac{1}{4} \delta_{aA} \lambda_{Bijk} \lambda_{ijkb} + \frac{1}{4} \lambda_{aijk} \lambda_{ijkA} \delta_{Bb} + \frac{3}{2} \lambda_{aijA} \lambda_{ijBb} \right], \end{aligned} \quad (4.110)$$

and contracting the indices  $A$  and  $B$  allows to recover the single-scale  $\overline{\text{MS}}$ -result, namely

$$\delta^{AB} [\gamma_{AB}^{(2)}]_{ab} = \frac{1}{(16\pi^2)^2} \frac{1}{12} \lambda_{aijk} \lambda_{ijkb} = \gamma_{ab}^{(2)} \Big|_{\overline{\text{MS}}}. \quad (4.111)$$

Turning to the two-loop mass-matrix beta-function, it is convenient to express the first order pole terms in Eq. (4.62) in a diagrammatic form:

$$\delta_{[1]}^{(2)} M = -\frac{1}{2\varepsilon} \text{---} \text{---} \text{---} + \frac{1}{2\varepsilon} \left[ \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + (-2) \text{---} \text{---} \text{---} \right], \quad (4.112)$$

where empty squares signal an insertion of the RG-scale matrix logarithm,  $\mathcal{L} = \text{Log } \mathcal{M}$ . For the contribution independent of the RG-scale logarithms, we find

$$\beta_{AB}^{(2)}(M_{ab}) \supset -\frac{1}{(16\pi^2)^2} \frac{1}{8} [\lambda_{aAjk} M_{Bf} \lambda_{fjkb} + \lambda_{aejk} M_{eA} \lambda_{Bjkb} + 2\lambda_{aeAk} M_{ef} \lambda_{fBkb}] + (A \leftrightarrow B), \quad (4.113)$$

and recover once again the  $\overline{\text{MS}}$  result from the contraction of  $A$  and  $B$ . For the contributions which explicitly depend on the RG-scale logarithm, the greater number of vertices produces a rather large amount of terms, which for conciseness we will not report here. We reiterate however that, in the limit  $\mathcal{M} \propto \mathbb{1}$ , the sum of the RG-scale-dependent contributions vanishes, so that their occurrence is a new feature compared to the single-scale  $\overline{\text{MS}}$  scheme. In fact, for some generic coupling  $\lambda$ , the two-index beta-function can be parameterised, to all orders, as

$$\beta_{AB}(\lambda) = \beta_{AB}^{[0]}(\lambda) + \beta_{AB,CD}^{[1]}(\lambda) \mathcal{L}^{CD} + \beta_{AB,CD,EF}^{[2]}(\lambda) \mathcal{L}^{CD} \mathcal{L}^{EF} + \dots, \quad (4.114)$$

where the notation  $\beta_{AB}^{[k]}(\lambda)$  is introduced to label the contributions according to the power  $k$  of the logarithms of the RG-scale they involve. Hence, the constraint of recovering the  $\overline{\text{MS}}$  beta-functions in the limit  $\mathcal{L} \rightarrow \log(\mu) \mathbb{1}$  translates into:

$$\begin{aligned} \delta^{AB} \beta_{AB}^{[0]}(\lambda) &= \beta(\lambda) \Big|_{\overline{\text{MS}}}, \\ \delta^{AB} \delta^{CD} \beta_{AB,CD}^{[1]}(\lambda) &= 0, \\ \delta^{AB} \delta^{CD} \delta^{EF} \beta_{AB,CD,EF}^{[2]}(\lambda) &= 0, \end{aligned} \quad (4.115)$$

and similarly for the contributions with  $k > 2$ . In other words, for  $k \geq 1$ , the quantity

$$\beta_{AB,A_1 B_1, \dots, A_k B_k}^{[k]}(\lambda) \quad (4.116)$$

is a traceless tensor, which furthermore can always be taken to be symmetric under the exchange of any couple of indices with  $k \geq 1$ , *i.e.* under any permutation of the type

$$(A_i B_i) \leftrightarrow (A_j B_j), \quad i, j \geq 1. \quad (4.117)$$

Finally, as noted in [28], a set of consistency conditions exists on the structure of the beta-functions, stemming from the requirement that differentiation with respect to  $\mu_A$  and with respect to  $\mu_B$  commute for all  $A, B = 1, \dots, N$ . In other words, recalling that

$$D_A \equiv \mu_A \frac{d}{d\mu_A}, \quad (4.118)$$

this consistency condition (referred to as *integrability condition* in [28]) translates into

$$[D_A, D_B] = 0, \quad A, B = 1, \dots, N. \quad (4.119)$$

It is a non-trivial observation that the validity of Eq.(4.119) is in fact guaranteed by the presence of the logarithmic contributions in the beta-functions. For instance, for some generic coupling  $\lambda$ , we obtain up to first-order in the RG-scale logarithm

$$\begin{aligned} [D_A, D_B](\lambda) &= D_A \beta_B(\lambda) - D_B \beta_A(\lambda) \\ &= D_A \left( \beta_B^{[0]}(\lambda) + \beta_{B,C}^{[1]}(\lambda) \log \mu_C + \dots \right) - (B \leftrightarrow A) \\ &= \left( \beta_A^{[0]} \circ \beta_B^{[0]}(\lambda) - \beta_B^{[0]} \circ \beta_A^{[0]}(\lambda) \right) + \left[ \beta_{B,A}^{[1]}(\lambda) - \beta_{A,B}^{[1]}(\lambda) \right], \end{aligned} \quad (4.120)$$

so that the requirement  $[D_A, D_B](\lambda) = 0$  implies, at leading order in the logarithm of the RG-scale, a cancellation between contributions of the type  $\beta^{[0]} \circ \beta^{[0]}(\lambda)$  and  $\beta^{[1]}(\lambda)$ . In other words, a set of non-trivial relations exist among the contributions to the beta-functions at different orders in the logarithm of the RG-scales, and furthermore, across various loop-orders.

## 2.4 Application to a two-scalar model

To provide better insight into the form of the multi-scale beta-functions, we specialise in this section the above formulae to the case of a theory with  $N = 2$  scalar fields  $\phi_1$  and  $\phi_2$ . The corresponding Lagrangian density reads

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1, \phi_2), \quad (4.121)$$

and we shall for simplicity restrict the scalar potential to the form

$$V(\phi_1, \phi_2) = \frac{1}{2} M_1 \phi_1^2 + \frac{1}{2} M_2 \phi_2^2 + \frac{1}{24} \lambda_1 \phi_1^4 + \frac{1}{24} \lambda_2 \phi_2^4 + \frac{1}{4} \lambda_3 \phi_1^2 \phi_2^2. \quad (4.122)$$

### One-loop beta-functions

At one-loop, defining for conciseness the loop-counting parameter  $\kappa = 1/(16\pi^2)$ , we find for the masses the following two-index beta-functions,

$$\beta_{AB}^{(1)}(M_1) = \kappa \begin{pmatrix} \lambda_1 M_1 & 0 \\ 0 & \lambda_3 M_2 \end{pmatrix}, \quad \beta_{AB}^{(1)}(M_2) = \kappa \begin{pmatrix} \lambda_3 M_1 & 0 \\ 0 & \lambda_2 M_2 \end{pmatrix}. \quad (4.123)$$

In the quartic sector we find

$$\begin{aligned} \beta_{AB}^{(1)}(\lambda_1) &= \kappa \begin{pmatrix} 3\lambda_1^2 & 0 \\ 0 & 3\lambda_3^2 \end{pmatrix}, & \beta_{AB}^{(1)}(\lambda_2) &= \kappa \begin{pmatrix} 3\lambda_3^2 & 0 \\ 0 & 3\lambda_2^2 \end{pmatrix}, \\ \beta_{AB}^{(1)}(\lambda_3) &= \kappa \begin{pmatrix} \lambda_3(\lambda_1 + 2\lambda_3) & 0 \\ 0 & \lambda_3(\lambda_2 + 2\lambda_3) \end{pmatrix}. \end{aligned} \quad (4.124)$$

It is easily checked that taking the trace of each two-index beta-function (*i.e.* contracting  $\beta_{AB}$  with  $\delta^{AB}$ ) gives the  $\overline{\text{MS}}$  result.

### Two-loop beta-functions

At two-loop, we obtain the following anomalous dimensions:

$$\gamma_{AB}^{(2)}(\phi_1) = \frac{\kappa^2}{24} \begin{pmatrix} 2\lambda_1^2 + 3\lambda_3^2 & 0 \\ 0 & 3\lambda_3^2 \end{pmatrix}, \quad \gamma_{AB}^{(2)}(\phi_2) = \frac{\kappa^2}{24} \begin{pmatrix} 3\lambda_3^2 & 0 \\ 0 & 2\lambda_2^2 + 3\lambda_3^2 \end{pmatrix}, \quad (4.125)$$

and recover the  $\overline{\text{MS}}$  result by taking the trace of  $\gamma_{AB}^{(2)}(\phi_{1,2})$ . Noting as previously the pure-vertex contributions to the beta-functions  $\widehat{\beta}$ , we get at two-loop for the mass couplings:

$$\widehat{\beta}_{AB}^{(2)}(M_1) = -\frac{\kappa^2}{2} \left[ \begin{pmatrix} (2\lambda_1^2 + \lambda_3^2) M_1 + \lambda_3^2 M_2 & 0 \\ 0 & \lambda_3^2 (M_1 + 3M_2) \end{pmatrix} - \log \frac{\mu_1}{\mu_2} \begin{pmatrix} \lambda_3^2 (2M_1 - M_2) & 0 \\ 0 & \lambda_3^2 (2M_1 - 3M_2) \end{pmatrix} \right], \quad (4.126)$$

$$\widehat{\beta}_{AB}^{(2)}(M_2) = -\frac{\kappa^2}{2} \left[ \begin{pmatrix} \lambda_3^2 (M_2 + 3M_1) & 0 \\ 0 & (2\lambda_2^2 + \lambda_3^2) M_2 + \lambda_3^2 M_1 \end{pmatrix} + \log \frac{\mu_1}{\mu_2} \begin{pmatrix} \lambda_3^2 (2M_2 - 3M_1) & 0 \\ 0 & \lambda_3^2 (2M_2 - M_1) \end{pmatrix} \right], \quad (4.127)$$

As expected, the RG-scale dependent contributions only depend on ratios of the multiple RG-scales, properly cancelling in the limit where  $\mu_1 = \mu_2$ . In addition, taking the trace of the two-index beta-functions yields

$$\begin{aligned} \delta^{AB} \widehat{\beta}_{AB}^{(2)}(M_1) &= -\kappa^2 \left[ (\lambda_1^2 + \lambda_3^2) M_1 + 2\lambda_3^2 M_2 - 2\lambda_3^2 (M_1 - M_2) \log \frac{\mu_1}{\mu_2} \right] \\ \delta^{AB} \widehat{\beta}_{AB}^{(2)}(M_2) &= -\kappa^2 \left[ (\lambda_2^2 + \lambda_3^2) M_2 + 2\lambda_3^2 M_1 + 2\lambda_3^2 (M_1 - M_2) \log \frac{\mu_1}{\mu_2} \right]. \end{aligned} \quad (4.128)$$

While the RG-scale-independent contributions indeed correspond in both cases to the  $\overline{\text{MS}}$  beta-functions, we observe that the terms proportional to  $\log \frac{\mu_1}{\mu_2}$  persist, and only cancel in the limit  $\mu_1 = \mu_2$ . Furthermore, it can be checked explicitly that the consistency condition on the commutator  $[D_1, D_2]$  exactly holds up to the two-loop level. For instance, with view on Eq. (4.120), we may compute on the one hand

$$\beta_1^{[0]} \circ \beta_2^{[0]}(M_1) - \beta_2^{[0]} \circ \beta_1^{[0]}(M_1) = 2\kappa^2 (M_1 - M_2) \lambda_3^2 + \mathcal{O}(\kappa^3), \quad (4.129)$$

and on the other hand

$$\beta_1^{[1]}(M_1) - \beta_2^{[1]}(M_1) = \left[ \frac{\kappa^2}{2} (M_2 - 2M_1) \lambda_3^2 \right] - \left[ \frac{\kappa^2}{2} (2M_1 - 3M_2) \right] = -2\kappa^2 (M_1 - M_2) \lambda_3^2, \quad (4.130)$$

so that

$$[D_1, D_2](M_1) = \mathcal{O}(\kappa^3), \quad (4.131)$$

providing an explicit check that the consistency condition holds up to the two-loop level.

Finally, the expression of the quartic beta-functions does not qualitatively differ from those of the mass parameters, so there is not much interest in reporting them here.

## 3 Resummation of logarithms

### 3.1 Parameterisation of the all-order $N$ -point functions

We consider any renormalised  $N$ -point function, noted  $\Gamma$  for simplicity. In the traditional  $\overline{\text{MS}}$  scheme, it can be parameterised as

$$\Gamma = c + c_{i_1 j_1} \left[ \text{Log} \frac{M}{\mu^2} \right]^{i_1 j_1} + c_{i_1 j_1, i_2 j_2} \left[ \text{Log} \frac{M}{\mu^2} \right]^{i_1 j_1} \left[ \text{Log} \frac{M}{\mu^2} \right]^{i_2 j_2} + \dots, \quad (4.132)$$

where the coefficients  $c$  themselves contain an infinite number of contributions in perturbation theory<sup>6</sup>. Regardless, we are mainly interested at this stage in the logarithmic structure of the full perturbative expansion. The various coefficients in the logarithmic expansion are now tensors, transforming covariantly under basis transformations. However, it should be noted that, in the basis where  $M$  is diagonal, for each couple of indices  $i_k j_k$ , we have that

$$c_{i_1 j_1, \dots, i_p j_p} = 0 \quad \text{if} \quad \bigvee_{k=1}^p i_k \neq j_k. \quad (4.133)$$

In other words, in the mass-basis, one necessarily has  $i_k = j_k$  for every  $k$  and for every coefficient of the logarithmic series. Any non-zero component such that  $i_k \neq j_k$  would be irrelevant since the diagonal structure of the mass logarithm only selects the diagonal elements of the tensors it is contracted with. Hence, in the mass basis, a more compact expansion can be given in terms of the eigenvalues of the mass logarithm:

$$\Gamma = c + c_{i_1} \log \frac{M_{i_1}}{\mu^2} + c_{i_1, i_2} \log \frac{M_{i_1}}{\mu^2} \frac{M_{i_2}}{\mu^2} + \dots \quad (4.134)$$

Now turning to the  $\overline{\text{MS}}$  parameterisation, we have seen in the previous sections that the renormalised  $N$ -point functions exactly reduce to their  $\overline{\text{MS}}$  counterpart in the limit where  $\mathcal{M} \rightarrow \mu \mathbb{1}$ . We have also observed the appearance of explicit logarithms of the RG-scales in the expression of the beta-functions at two-loop. Such logarithms also generally occur in the expression of the renormalised  $N$ -point functions beyond one-loop. These explicit occurrences of logarithms of the RG scales have the property to exactly cancel in the limit  $\mathcal{M} \rightarrow \mu \mathbb{1}$ , meaning that they only appear in the form of ratios, *i.e.*

$$\log \frac{\mu_i}{\mu_j}, \quad (4.135)$$

for some  $i, j \leq N$ . In a matrix formalism, this means that, in the logarithmic expansion, such terms arise from the contraction of the RG-scale matrix logarithm  $\mathcal{L}$  with some traceless two-index tensor. That is, schematically, contractions of the form

$$c_{\dots ij \dots} \mathcal{L}^{ij}, \quad \delta^{ij} c_{\dots ij \dots} = 0. \quad (4.136)$$

In turn, when  $\mathcal{L}^{ij}$  becomes proportional to the identity matrix,  $\delta^{ij}$ , the contraction overall cancels as expected. Another way of understanding this feature is to decompose the RG-scale matrix logarithm as

$$\mathcal{L} = \frac{1}{N} \text{Tr}(\mathcal{L}) \mathbb{1} + \left( \mathcal{L} - \frac{1}{N} \text{Tr}(\mathcal{L}) \mathbb{1} \right) \equiv \mathcal{L}_1 + \Delta. \quad (4.137)$$

The first contribution is proportional to the identity, and with coefficient

$$\frac{1}{N} \text{Tr}(\mathcal{L}) = \frac{1}{N} \sum_{i=1}^N \log \mu_i = \frac{1}{N} \log \prod_{i=1}^N \mu_i = \log(\mu_1 \cdots \mu_N)^{1/N} \equiv \log \tilde{\mu}. \quad (4.138)$$

The scale  $\tilde{\mu}$  appearing in the rightmost logarithm is obtained from the geometric mean of all eigenvalues of the RG-scale matrix<sup>7</sup>. For this reason, we shall refer to  $\tilde{\mu}$  as the average RG-scale. The matrix  $\Delta$  defined in Eq. (4.137) is, by construction, traceless. In the mass-basis, where  $\mathcal{L}$  is diagonal,  $\Delta$  takes the explicit form

$$\Delta = \frac{1}{N} \begin{pmatrix} (\log \mu_1 - \log \mu_2) + \cdots + (\log \mu_1 - \log \mu_N) & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & (\log \mu_N - \log \mu_1) + \cdots + (\log \mu_N - \log \mu_{N-1}) \end{pmatrix}. \quad (4.139)$$

<sup>6</sup>We note also that, contrary to theories with a single scalar field, non-polynomial functions of the scalar masses can also be involved in the coefficients  $c$ , and in particular logarithms of the form  $\log \frac{M_i}{M_j}$  for some  $i, j \leq N$ . For  $M_i < M_j$ , a series expansion can be given for such non-polynomial functions in terms of  $\alpha_{ij} \equiv \frac{M_i}{M_j} < 1$ , involving contributions of the type

$\mathcal{O}(\alpha_{ij}^p \log(\alpha_{ij})^q)$ , with  $p, q \geq 1$ . The absence of  $\mathcal{O}(\log \alpha_{ij})$  contributions prevents the occurrence of IR divergences.

<sup>7</sup>We can also write  $\frac{1}{N} \text{Tr}(\mathcal{L}) = \frac{1}{N} \text{Tr}(\text{Log } \mathcal{M}) = \frac{1}{N} \log(\det \mathcal{M})$ , so that  $\tilde{\mu} = (\det \mathcal{M})^{1/N}$ .

More concisely, the diagonal entry  $\Delta^{ii}$  ( $i = 1, \dots, N$ ) reads in the mass basis

$$\frac{1}{N} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \log \mu_i - \log \mu_j = \frac{1}{N} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \log \frac{\mu_i}{\mu_j}, \quad (4.140)$$

and therefore only depends on logarithms of ratios of the RG scales. In turn, any such ratio occurring in the beta-functions or the renormalised  $N$ -point functions stems from the contraction of some tensor with the traceless matrix  $\Delta$ .

From the above discussion, we arrive at the conclusion that any renormalised  $N$ -point function  $\Gamma$  can be parameterised, in the  $\overline{\text{mMS}}$  scheme, in the form

$$\begin{aligned} \Gamma = & c + [c_{a_1 b_1} (\text{Log } M - \mathcal{L}) + d_{a_1 b_1} \Delta^{a_1 b_1}] \\ & + [c_{a_1 b_1, a_2 b_2} (\text{Log } M - \mathcal{L})^{a_1 b_1} (\text{Log } M - \mathcal{L})^{a_2 b_2} + d_{a_1 b_1, a_2 b_2} \Delta^{a_1 b_1} (\text{Log } M - \mathcal{L})^{a_2 b_2} \\ & + e_{a_1 b_1, a_2 b_2} (\text{Log } M - \mathcal{L})^{a_1 b_1} \Delta^{a_2 b_2} + f_{a_1 b_1, a_2 b_2} \Delta^{a_1 b_1} \Delta^{a_2 b_2}] \\ & + \dots, \end{aligned} \quad (4.141)$$

where the tensors  $c_{i_1 j_1, \dots, i_p j_p}$  are equal to those defined above, in the conventional  $\overline{\text{MS}}$  scheme. This rather intricate structure leads in fact to two different parameterisations, both of which will be helpful in the resummation of logarithmic contributions. As a last preliminary step, we define the shorthand notations

$$L \equiv \frac{1}{2} \text{Log } M, \quad \bar{L} \equiv \frac{1}{2} \text{Log } M - \text{Log } \mathcal{M} = L - \mathcal{L} = \frac{1}{2} \text{Log } \frac{M}{\mathcal{M}^2}, \quad (4.142)$$

where the matrix  $\text{Log } \frac{M}{\mathcal{M}^2}$  is defined, by a slight abuse of notation, as the matrix with diagonal entries  $\log \frac{M_i}{\mu_i^2}$  in the mass-basis<sup>8</sup> (we recall that  $\mathcal{M}$  is diagonal in the mass basis).

A first parameterisation can be given in terms of the logarithms  $\bar{L}$  exclusively. Namely, allowing for  $\Delta$ -dependent coefficients, noted  $C_{i_1 j_1, \dots, i_p j_p}$ , we can write

$$\Gamma = C + C_{i_1 j_1} \bar{L}^{i_1 j_1} + C_{i_1 j_1, i_2 j_2} \bar{L}^{i_1 j_1} \bar{L}^{i_2 j_2} + C_{i_1 j_1, i_2 j_2, i_3 j_3} \bar{L}^{i_1 j_1} \bar{L}^{i_2 j_2} \bar{L}^{i_3 j_3} + \dots, \quad (4.143)$$

where the individual tensors  $C$  can themselves be decomposed in a series expansion in  $\Delta$ . The leading contributions to each of the  $C$  (that is, its  $\Delta$ -independent term) corresponds to the  $\overline{\text{MS}}$  coefficient  $c$ .

A second parameterisation is obtained from the opposite perspective, expressing the perturbative series in terms of  $\Delta$ . The  $\bar{L}$ -dependent coefficients hence obtained are noted  $B$ , and we write

$$\Gamma = B + B_{i_1 j_1} \Delta^{i_1 j_1} + B_{i_1 j_1, i_2 j_2} \Delta^{i_1 j_1} \Delta^{i_2 j_2} + B_{i_1 j_1, i_2 j_2, i_3 j_3} \Delta^{i_1 j_1} \Delta^{i_2 j_2} \Delta^{i_3 j_3} + \dots. \quad (4.144)$$

Rather importantly, the leading coefficient  $B$  corresponds in fact to the  $\overline{\text{MS}}$  renormalised  $N$ -point function (where  $\log \frac{M_i}{\mu_i^2} \rightarrow \log \frac{M_i}{\mu_i}$ ). From the contraction with  $\Delta$ , every high order coefficient is traceless:

$$B_{\dots, i_k j_k, \dots} \delta^{i_k j_k} = 0 \quad \forall k. \quad (4.145)$$

In the following, both parameterisations (4.143) and (4.144) will be successively inserted into the Callan-Symanzik equation to infer valuable information on the overall structure of the perturbative expansion. From this point onward, we shall consider for clarity that every tensor and coupling is expressed in the mass-basis, without using the tilde-notation used throughout the previous sections.

<sup>8</sup>A more rigorous definition could be  $\text{Log } \frac{M}{\mathcal{M}^2} \equiv \text{Log } \mathcal{M}^{-1} M \mathcal{M}^{-1}$ .

### 3.2 Callan-Symanzik equation in the $\Delta$ -parameterisation

In this parameterisation, we first need to express the derivatives of  $\Delta$  with respect to the RG-scale. This is most easily achieved by using the trace-decomposition of the RG-scale logarithm,

$$\Delta = \mathcal{L} - \mathcal{L}_\perp. \quad (4.146)$$

The RG-derivatives of  $\mathcal{L}$  and  $\mathcal{L}_\perp$  being given by

$$D_A \mathcal{L}^{ij} = \delta_A^i \delta_A^j, \quad D_A \mathcal{L}_\perp^{ij} = \frac{\delta^{ij}}{N} \sum_{k=1}^N D_A \log \mu_k = \frac{\delta^{ij}}{N}, \quad (4.147)$$

one has in turn

$$D_A \Delta^{ij} = D_A (\mathcal{L} - \mathcal{L}_\perp)^{ij} = \delta_A^i \delta_A^j - \frac{\delta^{ij}}{N}. \quad (4.148)$$

We shall in addition use the shorthand notation

$$B_{i_1 j_1, \dots, i_p j_p} \Delta^{i_1 j_1} \dots \Delta^{i_p j_p} \equiv B_{\{p\}} \cdot \Delta^{\{p\}}, \quad B_{\{0\}} \cdot \Delta^{\{0\}} \equiv B, \quad (4.149)$$

so the  $\Delta$ -parameterisation, Eq. (4.144), more simply reads

$$\Gamma = \sum_{p=0}^{\infty} B_{\{p\}} \cdot \Delta^{\{p\}}. \quad (4.150)$$

Finally, when taking the RG-derivatives to apply the Callan-Symanzik equation, we shall systematically contract the open index  $A$  of  $D_A$  derivative with  $\Delta_{AA}$ , defining in turn an auxiliary differential operator

$$(\Delta \cdot D) \equiv \sum_{A=1}^N \Delta^{AA} D_A. \quad (4.151)$$

This newly defined operator is introduced for mere convenience and will allow to maintain the forthcoming expressions in a compact form. For instance, the  $(\Delta \cdot D)$ -derivative of  $\Delta$  itself yields, from Eq. (4.148),

$$(\Delta \cdot D) \Delta^{ij} = \sum_{A=1}^N \Delta^{AA} D_A \Delta^{ij} = \sum_{A=1}^N \Delta^{AA} \delta^{iA} \delta^{jA} = \Delta^{ij}, \quad (4.152)$$

where we have used in the last step the fact that  $\Delta$  is diagonal in the mass-basis.

As a last preliminary step, we define the modified  $N$ -point function  $\widehat{\Gamma}$  in exact analogy with the single-scale case (see section 1.2), by contracting its legs with the matrix  $K$  which factors the RG-dependence out from the scalar field multiplet:

$$\phi = K \widehat{\phi}, \quad D_A K = -\gamma_A K, \quad D_A \widehat{\phi} = 0. \quad (4.153)$$

Doing so allows to recover an exactly RG-invariant  $N$ -point function, bringing the Callan-Symanzik in the simpler form

$$D_A \widehat{\Gamma} = 0. \quad (4.154)$$

Contracting the above open index  $A$  with  $\Delta^{AA}$  finally yields from Eq. (4.151)

$$(\Delta \cdot D) \widehat{\Gamma} = 0, \quad (4.155)$$

which constitutes our starting point to study the impact of the Callan-Symanzik equation on the structure of the  $\Delta$ -parameterisation. Let us then proceed and insert Eq. (4.150) in the above relation<sup>9</sup>:

$$\begin{aligned} 0 &= (\Delta \cdot D) \widehat{\Gamma} \\ &= (\Delta \cdot D) \sum_{p=0}^{\infty} \widehat{B}_{\{p\}} \cdot \Delta^{\{p\}} \\ &= \sum_{p=0}^{\infty} \left[ (\Delta \cdot D) \widehat{B} \right]_{\{p\}} \cdot \Delta^{\{p\}} + \sum_{p=0}^{\infty} p \widehat{B}_{\{p\}} \cdot \Delta^{\{p\}}. \end{aligned} \quad (4.156)$$

<sup>9</sup>The hatted coefficients  $\widehat{B}$  are simply obtained after contraction of the open legs of  $B$  with the matrix  $K$ .



Our notations allow indeed to maintain the expressions in a compact form, without the need to explicitly write every index contraction. Gathering every contribution according to the powers of the matrix  $\Delta$  it involves gives

$$\begin{aligned}
0 &= \sum_{p=0}^{\infty} \left[ (\Delta \cdot D) \widehat{B} \right]_{\{p\}} \cdot \Delta^{\{p\}} + \sum_{p=0}^{\infty} p \widehat{B}_{\{p\}} \cdot \Delta^{\{p\}} \\
&= \sum_{p=0}^{\infty} \left[ (\Delta \cdot D) \widehat{B} \right]_{\{p\}} \cdot \Delta^{\{p\}} + \sum_{p=0}^{\infty} (p+1) \widehat{B}_{\{p+1\}} \cdot \Delta^{\{p+1\}} \\
&= \sum_{p=0}^{\infty} \left[ (\Delta \cdot D) \widehat{B} \right]_{\{p\}} \cdot \Delta^{\{p\}} + (p+1) \widehat{B}_{\{p+1\}} \cdot \Delta^{\{p+1\}} .
\end{aligned} \tag{4.157}$$

This equation must hold independently of the precise values of the RG-scales, and hence independently of the precise form  $\Delta$ . Consequently, it must hold order-by-order in  $\Delta$  and we obtain the relation

$$\left[ (\Delta \cdot D) \widehat{B} \right]_{\{p\}} \cdot \Delta^{\{p\}} + (p+1) \widehat{B}_{\{p+1\}} \cdot \Delta^{\{p+1\}} = 0 , \tag{4.158}$$

allowing to obtain, for all  $p$ , the expression of  $\widehat{B}_{\{p+1\}} \cdot \Delta^{\{p+1\}}$  in terms of the previous coefficients:

$$\widehat{B}_{\{p+1\}} \cdot \Delta^{\{p+1\}} = -\frac{1}{p+1} \left[ (\Delta \cdot D) \widehat{B} \right]_{\{p\}} \cdot \Delta^{\{p\}} . \tag{4.159}$$

For  $p = 0$ , we infer from this recurrence relation that

$$\widehat{B}_{\{1\}} \cdot \Delta^{\{1\}} = -(\Delta \cdot D) \widehat{B} = -\sum_{A=1}^N \Delta^{AA} D_A \widehat{B} \equiv -\Delta^A D_A \widehat{B} , \tag{4.160}$$

where we have defined  $\Delta^A = \Delta^{AA}$  ( $A = 1, \dots, N$ ) the vector populated with the diagonal entries of  $\Delta$ . For higher values of  $p$ , we obtain

$$\begin{aligned}
\widehat{B}_{\{2\}} \cdot \Delta^{\{2\}} &= -\frac{1}{2} \left[ (\Delta \cdot D) \widehat{B}_{\{1\}} \cdot \Delta \right] = \frac{1}{2} \Delta^A \Delta^B D_A D_B \widehat{B} \\
\widehat{B}_{\{3\}} \cdot \Delta^{\{3\}} &= -\frac{1}{3!} \Delta^A \Delta^B \Delta^C D_A D_B D_C \widehat{B} \\
&\dots \\
\widehat{B}_{\{p\}} \cdot \Delta^{\{p\}} &= (-1)^p \frac{1}{p!} \Delta^{A_1} \dots \Delta^{A_p} D_{A_1} \dots D_{A_p} \widehat{B} .
\end{aligned} \tag{4.161}$$

The last relation, valid for any  $p \geq 0$ , allows to rewrite the entire  $\Delta$ -expansion into the form

$$\widehat{\Gamma} = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \Delta^{A_1} \dots \Delta^{A_p} D_{A_1} \dots D_{A_p} \widehat{B} , \tag{4.162}$$

where each term in the expansion is now given in terms of the derivatives of the leading coefficient,  $B$ . This expansion is very similar to a multivariate Taylor-expansion, and we shall now demonstrate that we can indeed rewrite it in such a form. To see how, we recall that  $\Delta$  can be given in terms of  $\mathcal{L}$  and  $\mathcal{L}_{\mathbb{1}}$  according to

$$\Delta = \mathcal{L} - \mathcal{L}_{\mathbb{1}} = -(\mathcal{L}_{\mathbb{1}} - \mathcal{L}) . \tag{4.163}$$

Since all coefficients are evaluated at the arbitrary RG-scales ( $\log \mu_1, \dots, \log \mu_N$ ) populating the diagonal entries of  $\mathcal{L}$ , having expressed  $\Delta$  in the form (4.163) is indeed what we need to put Eq. (4.162) in the form of a genuine multivariate Taylor expansion:

$$\widehat{\Gamma} = \sum_{p=0}^{\infty} \frac{1}{p!} \left( D_{A_1} \dots D_{A_p} \widehat{B} \right) [\mathcal{L}_{\mathbb{1}} - \mathcal{L}]^{A_1} \dots [\mathcal{L}_{\mathbb{1}} - \mathcal{L}]^{A_p} . \tag{4.164}$$

This crucial result has a remarkable consequence: With every coefficient  $\widehat{B}\dots$  initially evaluated at the scales  $(\log \mu_1, \dots, \log \mu_N)$ , resumming the infinite tower of explicit logarithmic contributions amounts to evolve every scale  $\mu_i$  towards the average scale  $\widetilde{\mu} = (\mu_1 \cdots \mu_N)^{1/N}$ :

$$\widehat{\Gamma} = \sum_{p=0}^{\infty} \frac{1}{p!} \left( D_{A_1} \cdots D_{A_p} \widehat{B} \right) [\mathcal{L}_1 - \mathcal{L}]^{A_1} \cdots [\mathcal{L}_1 - \mathcal{L}]^{A_p} = \widehat{B}_{\mathcal{L} \rightarrow \mathcal{L}_1} = \widehat{B}_{\log \mu_i \rightarrow \log \widetilde{\mu}}. \quad (4.165)$$

In other words, resummation to all orders of the ratios  $\log \frac{\mu_i}{\mu_j}$  appearing in the expression of the  $N$ -point functions exactly cancels each of them by taking  $\mu_i \rightarrow \widetilde{\mu}$  for all  $i = 1, \dots, N$ . Once again, this is a remarkable result: the multi-scale description of the theory is systematically brought back, upon resummation of explicit RG-scale logarithms, to a  $\overline{\text{MS}}$ -like description with a single renormalisation scale  $\widetilde{\mu}$ . Recalling that the leading coefficient  $B$  corresponds to the  $\overline{\text{MS}}$  resummed  $N$ -point function, this means

$$\Gamma_{\text{m}\overline{\text{MS}}} = \Gamma_{\overline{\text{MS}}}|_{\mu=\widetilde{\mu}}. \quad (4.166)$$

We insist that this result is completely general since it holds to all orders in perturbation theory, and for any  $N$ -point function.

### 3.3 Callan-Symanzik equation in the $\overline{L}$ -parameterisation

It remains at this stage to explore the other possible parameterisation for the renormalised  $N$ -point functions, expanded in terms of the mass-logarithms:

$$\Gamma = C + C_{i_1 j_1} \overline{L}^{i_1 j_1} + C_{i_1 j_1, i_2 j_2} \overline{L}^{i_1 j_1} \overline{L}^{i_2 j_2} + C_{i_1 j_1, i_2 j_2, i_3 j_3} \overline{L}^{i_1 j_1} \overline{L}^{i_2 j_2} \overline{L}^{i_3 j_3} + \dots \quad (4.167)$$

Adapting our previous notations, we define

$$C_{\{p\}} \overline{L}^{\{p\}} \equiv C_{i_1 j_1, \dots, i_p j_p} \overline{L}^{i_1 j_1} \cdots \overline{L}^{i_p j_p}, \quad C_{\{0\}} \overline{L}^{\{0\}} \equiv C, \quad (4.168)$$

so as to rewrite

$$\widehat{\Gamma} = \sum_{p=0}^{\infty} \widehat{C}_{\{p\}} \overline{L}^{\{p\}}. \quad (4.169)$$

Inserting this expression into the Callan-Symanzik equation gives

$$0 = D_A \widehat{\Gamma} = \sum_{p=0}^{\infty} \left( D_A \widehat{C}_{\{p\}} \right) \overline{L}^{\{p\}} + \widehat{C}_{\{p\}} D_A \left( \overline{L}^{\{p\}} \right). \quad (4.170)$$

The situation is more involved here than in the case of the  $\Delta$ -parameterisation, since the mass-logarithm  $\overline{L}$  depends implicitly on the scales  $\mu_A$  ( $A = 1, \dots, N$ ) through the beta-functions of  $M$ . Namely,

$$D_A \overline{L}^{ij} = D_A \left( \frac{1}{2} \text{Log } M - \mathcal{L} \right) = D_A L^{ij} - \delta_A^i \delta_A^j. \quad (4.171)$$

From this point onward, we shall switch to a notation where the components of the (diagonal) matrices are traded for vectors of eigenvalues. For instance,

$$\overline{L}^A \equiv \overline{L}^{AA} = \frac{1}{2} \log \frac{M_A}{\mu_A^2}, \quad L^A \equiv L^{AA} = \frac{1}{2} \log M_A, \quad \mathcal{L}^A \equiv \mathcal{L}^{AA} = \log \mu_A. \quad (4.172)$$

In turn, we have

$$\widehat{C}_{\{p\}} \overline{L}^{\{p\}} \equiv \widehat{C}_{A_1, \dots, A_p} \overline{L}^{A_1} \cdots \overline{L}^{A_p}, \quad (4.173)$$

and, from Eq.(4.171),

$$D_A \overline{L}^B = D_A L^B - \delta_A^B \equiv -H^{AB}, \quad (4.174)$$

where we have defined the matrix  $H$  such that

$$H^{AB} = \delta^{AB} - \frac{1}{2} D_A L^B \quad (4.175)$$

In turn, the Callan-Symanzik equation (4.170) takes the explicit form

$$\begin{aligned}
0 = D_A \hat{\Gamma} &= \sum_{p=0}^{\infty} D_A \hat{C}_{A_1, \dots, A_p} \bar{L}^{A_1} \dots \bar{L}^{A_p} \\
&- \sum_{p=0}^{\infty} H^{AB} \left[ \hat{C}_{B, A_1, \dots, A_p} + \hat{C}_{A_1, B, \dots, A_p} + \dots + \hat{C}_{A_1, \dots, A_p, B} \right] \bar{L}^{A_1} \dots \bar{L}^{A_p} \\
&= \sum_{p=0}^{\infty} D_A \hat{C}_{A_1, \dots, A_p} \bar{L}^{A_1} \dots \bar{L}^{A_p} - \sum_{p=0}^{\infty} (p+1) H^{AB} \hat{C}_{B, A_1, \dots, A_p} \bar{L}^{A_1} \dots \bar{L}^{A_p}, \quad (4.176)
\end{aligned}$$

where the last equality follows from the symmetry of the coefficients under permutation of their indices. Since this equality must hold independently of the precise value of the mass-logarithm eigenvalues, we obtain the following recurrence relation:

$$H^{AB} \hat{C}_{B, A_1, \dots, A_p} = \frac{1}{p+1} D_A \hat{C}_{A_1, \dots, A_p}. \quad (4.177)$$

The matrix  $H$  departs from the identity starting at first-order in perturbation theory and can hence be inverted. We define the differential operator

$$\mathbf{D}_A \equiv [H^{-1}]^{AB} D_B, \quad (4.178)$$

as the analogue of the single-scale operator  $\mathbf{D}$  defined in chapter 3 as

$$\mathbf{D} = \frac{1}{1 - DL} D. \quad (4.179)$$

Iteratively applying Eq. (4.177) then gives

$$H^{AB} \hat{C}_B = D_A \hat{C} \quad \rightarrow \quad \hat{C}_B = \mathbf{D}_A \hat{C} \quad (4.180)$$

$$H^{AB} \hat{C}_{BA_1} = \frac{1}{2} D_A \hat{C}_{A_1} \quad \rightarrow \quad \hat{C}_{AB} = \frac{1}{2} \mathbf{D}_A \hat{C}_B = \frac{1}{2} \mathbf{D}_A \mathbf{D}_B \hat{C}, \quad (4.181)$$

from which we conclude that, since  $\hat{C}_{AB}$  is symmetric under  $A \leftrightarrow B$ , the differential operators  $\mathbf{D}_A$  ( $A = 1, \dots, N$ ) necessarily commute:

$$[\mathbf{D}_A, \mathbf{D}_B] = 0 \quad \forall (A, B) = 1, \dots, N. \quad (4.182)$$

For arbitrary  $p$ , the tensor contracted  $p$  times with the mass-logarithm then reads

$$\hat{C}_{A_1, \dots, A_p} = \frac{1}{p!} \mathbf{D}_{A_1} \dots \mathbf{D}_{A_p} \hat{C}, \quad (4.183)$$

and the  $\bar{L}$ -expansion of the renormalised  $n$ -point function can be rewritten in the compact form

$$\hat{\Gamma} = \sum_{p=0}^{\infty} \frac{1}{p!} \left( \mathbf{D}_{A_1} \dots \mathbf{D}_{A_p} \hat{C} \right) \bar{L}^{A_1} \dots \bar{L}^{A_p}. \quad (4.184)$$

It is important to reiterate that the derivatives involved in the above expression contain explicit occurrences of the RG-scale logarithms, which would only cancel after resummation of the  $\Delta$ -expansion, as explained in the previous section. In fact, the occurrence of these logarithms is a necessary condition in order to for the differential operators  $\mathbf{D}_{A_k}$  to commute with one another.

### 3.4 All-order resummation

We are now ready to proceed with the complete resummation of the logarithmic contributions to the  $\overline{\text{mMS}}$  resummation  $N$ -point functions. Starting with the  $\overline{L}$ -expansion computed above,

$$\widehat{\Gamma} = \sum_{p=0}^{\infty} \frac{1}{p!} \left( \mathbf{D}_{A_1} \dots \mathbf{D}_{A_p} \widehat{\mathbf{C}} \right) \overline{L}^{A_1} \dots \overline{L}^{A_p}, \quad (4.185)$$

we first resum every ratio of RG-scale logarithms using the results obtained in the  $\Delta$ -expansion. This amounts to run every coupling from  $(\log \mu_1, \dots, \log \mu_N)$  to  $(\log \tilde{\mu}, \dots, \log \tilde{\mu})$ . At this stage, the beta-functions involved in the above expression are now completely independent of the RG-scale since  $\mu_i = \tilde{\mu}$  for all  $i = 1, \dots, N$ . Reusing the results of section 2, the operators  $D_A$  applied to any function of the couplings are now given by

$$D_A = \sum_i \beta_A^{[0]}(\lambda_i) \frac{\partial}{\partial \lambda_i}, \quad (4.186)$$

where  $\beta_A^{[0]}$  is the contribution to the beta-functions which was independent of the RG-scale logarithms before their resummation, and where  $\lambda_i$  denotes any generic coupling of the theory.

The very final step allowing to resum the logarithmic contributions is, from a generalisation of the procedure established in the single-scale case, to define a new set of  $z$ -dependent couplings, with  $z$  a parameter ranging from 0 to 1. In particular, we define

$$\boldsymbol{\lambda}_{abcd}(z), \quad \boldsymbol{\lambda}_{abcd}(0) = \boldsymbol{\lambda}_{abcd}(\log \tilde{\mu} \mathbb{1}), \quad (4.187)$$

$$\mathbf{M}_{ab}(z), \quad \mathbf{M}_{ab}(0) = \mathbf{M}_{ab}(\log \tilde{\mu} \mathbb{1}), \quad (4.188)$$

and similarly for any other coupling that would have been included in the Lagrangian density. We define the  $z$ -derivatives of the couplings as

$$\frac{d\boldsymbol{\lambda}_{abcd}}{dz}(z) = \overline{L}_0^A \mathbf{D}_A \boldsymbol{\lambda}_{abcd} = \overline{L}_0^A [H^{-1}]^{AB} D_B \boldsymbol{\lambda}_{abcd} = \overline{L}_0^A [H^{-1}]^{AB} \beta_B^{[0]}(\boldsymbol{\lambda}_{abcd}), \quad (4.189)$$

$$\frac{d\mathbf{M}_{ab}}{dz}(z) = \overline{L}_0^A \mathbf{D}_A \mathbf{M}_{abcd} = \overline{L}_0^A [H^{-1}]^{AB} D_B \mathbf{M}_{ab} = \overline{L}_0^A [H^{-1}]^{AB} \beta_B^{[0]}(\mathbf{M}_{ab}), \quad (4.190)$$

where

$$\overline{L}_0^A = \left[ \frac{1}{2} \log M(\log \tilde{\mu} \mathbb{1}) - \log \tilde{\mu} \mathbb{1} \right]^A = \frac{1}{2} \log \frac{M^A(\log \tilde{\mu} \mathbb{1})}{\tilde{\mu}^2}, \quad (4.191)$$

is kept constant under evolution of the parameter  $z$  (the subscript 0 emphasises this feature). Hence,  $\overline{L}_0^A$  simply stands for the eigenvalues of mass-logarithm evaluated at the initial average scale  $\tilde{\mu}$ .

Exactly as in the single-scale case, having defined these  $z$ -dependent parameters and their derivatives, the  $\overline{L}$ -expansion of the renormalised  $N$ -point functions now takes the very simple form

$$\widehat{\Gamma} = \sum_{p=0}^{\infty} \frac{1}{p!} \frac{d^p}{dz^p} \widehat{\mathbf{C}}, \quad (4.192)$$

and, as previously, is nothing else than the Taylor expansion of the coefficient  $\widehat{\mathbf{C}}$  around the initial value  $z = 0$ , evaluated at  $z = 1$ . This brings us to the most important result of this chapter,

$$\widehat{\Gamma} = \widehat{\mathbf{C}}(1). \quad (4.193)$$

Namely, the renormalised  $N$ -point functions in which logarithmic contributions have been resummed to all orders are obtained from their leading coefficient evaluated at  $z = 1$ . The  $z$ -dependence of the couplings is governed by the set of differential equations, Eq. (4.189) and Eq. (4.190) (trivially generalised to models with additional interactions), and their value at  $z = 0$  simply coincides with their initial value in the  $\overline{\text{MS}}$  scheme at the scale  $\tilde{\mu}$ .

## 4 Summary and discussion

### 4.1 Summary of the results

Before discussing the possible implications of our findings, let us summarise the main steps that have led us to the above results. First, we have carried out the renormalisation of a general scalar theory using dimensional regularisation in the presence of multiple renormalisation scales, and obtained general expression for the beta-functions in such a renormalisation scheme, called  $\overline{\text{mMS}}$ . These multi-scale beta-functions explicitly depend on (ratios of) the logarithms of the RG-scales. Following the discussion in section 3.1, this means that, to all orders in perturbation theory, the  $\overline{\text{mMS}}$  beta-functions of a generic coupling  $\lambda_i$  take the form (for  $A = 1, \dots, N$ )

$$\mu_A \frac{d}{d\mu_A}(\lambda_i) \equiv \beta_A(\lambda_i) = \beta_A^{[0]}(\lambda_i) + \Delta^A \beta_A^{[1]}(\lambda_i) + \Delta^A \Delta^B \beta_{A,B}^{[2]}(\lambda_i) + \dots, \quad (4.194)$$

where  $\Delta^A$  stands for the eigenvalues of the traceless component of the RG-scale matrix logarithm. In this logarithmic series, the leading contributions  $\beta_A^{[0]}$  are directly related to the conventional  $\overline{\text{MS}}$  beta-functions, since

$$\sum_A \beta_A^{[0]}(\lambda_i) = \beta_{\overline{\text{MS}}}(\lambda_i). \quad (4.195)$$

In addition,  $\beta_A^{[0]}(\lambda_i)$  can be directly obtained from the general expression of the  $\overline{\text{mMS}}$  beta function, successively “opening” the internal propagators in the loop diagrams, replacing them with

$$\delta^{ij} \rightarrow \frac{1}{2} \delta_A^i \delta_A^j, \quad (4.196)$$

and taking the sum of the diagrams with open index  $A$  thereby obtained. For instance, for the one-loop quartic beta-functions, this procedure can be schematically summarised as

$$\text{[diagram]} \rightarrow \frac{1}{2} \text{[diagram with top circle]} + \frac{1}{2} \text{[diagram with bottom circle]}. \quad (4.197)$$

In the  $\overline{\text{mMS}}$  scheme, the renormalised  $N$ -point functions have been shown to take the general form of a logarithmic series involving both the mass-logarithm  $\bar{L}$ ,

$$\bar{L} = \frac{1}{2} \text{Log } M - \mathcal{L} \xrightarrow{\text{mass basis}} \bar{L}^{ii} = \frac{1}{2} \log \frac{M_{ii}}{\mu_i^2}, \quad (4.198)$$

and the traceless matrix  $\Delta$ , populated with the logarithms of ratios of the RG-scales. Starting with a description of the theory at an arbitrary set of scales  $(\mu_1, \dots, \mu_N)$ , we have demonstrated in section 3.2 that the resummation of  $\Delta$  contributions was equivalent to performing the running of all couplings towards the average scale  $\tilde{\mu} = (\mu_1 \cdots \mu_N)^{1/N}$ , hence cancelling every ratio of the form  $\log \frac{\mu_i}{\mu_j}$  (that is, resumming them). Such a resummation therefore brings the expression of every renormalised quantity back into its  $\overline{\text{MS}}$  form, evaluated at the single scale  $\tilde{\mu}$ . On the other hand, instead of performing the  $\Delta$ -resummation, inserting in the Callan-Symanzik equation the mass-logarithm parameterisation gives a set of recurrence relations between the coefficients, which one could not have obtained from  $\overline{\text{MS}}$  renormalisation. First removing the leg-dependence of the renormalised  $N$ -point function  $\Gamma$  by defining

$$\hat{\Gamma} = K^N \Gamma, \quad \text{where} \quad D_A K = -\gamma_A, \quad (4.199)$$

these recurrence relations bring the all-order RG-invariant quantity  $\hat{\Gamma}$  into the form

$$\hat{\Gamma} = \sum_{p=0}^{\infty} \frac{1}{p!} \left( \mathbf{D}_{A_1} \dots \mathbf{D}_{A_p} \hat{C} \right) \bar{L}^{A_1} \dots \bar{L}^{A_p}, \quad (4.200)$$

where the differential operator  $\mathbf{D}_{A_1}$  is the direct multi-variate generalisation of the operator  $\mathbf{D}$  obtained in chapter 3 in the process of resummation in single-scale theories (see Eq. 3.50):

$$\mathbf{D}_A = [H^{-1}]^{AB} D_B, \quad H_{IJ} = \delta_{IJ} - D_I L^J, \quad (4.201)$$

where  $L^J$  denotes the  $J$ -th eigenvalue of  $L = \frac{1}{2} \text{Log } M$ . Similar to the denominator  $1/(1 - DL)$  in single-scale theories,  $H^{-1}$  is crucial in guaranteeing the exact RG-scale independence of the perturbative expansion. Finally, directly generalising the methods established in chapter 3, we may trade the RG-dependence of every coupling for a dependence on a dimensionless parameter  $0 \leq z \leq 1$ . With couplings originally evaluated at the  $\overline{\text{MS}}$ -scale  $\mu_0$ , we define, for any generic coupling  $\lambda$ , its counterpart  $\boldsymbol{\lambda}(z)$  such that

$$\boldsymbol{\lambda}(0) = \lambda(\log \mu_0 \mathbb{1}), \quad (4.202)$$

and

$$\frac{d\boldsymbol{\lambda}}{dz}(z) = \bar{L}_0^A \mathbf{D}_A(\boldsymbol{\lambda}), \quad (4.203)$$

where  $\bar{L}_0^A = \frac{1}{2} \log \frac{M_A(\log \mu_0)}{\mu_0^2}$ . Noting  $\widehat{\mathbf{C}}(z)$  the  $z$ -dependent leading coefficient of the logarithmic expansion of  $\widehat{\Gamma}$ , we have finally shown that, to all orders in perturbation theory,

$$\widehat{\Gamma} = \widehat{\mathbf{C}}(1). \quad (4.204)$$

In other words, running the  $z$ -dependent couplings towards  $z = 1$  resums every mass-logarithm in every  $N$ -point function of the theory (that is, in the full effective action), bringing them into a form where only their leading coefficient contributes, and has to be evaluated at  $z = 1$ . We reiterate that, by construction, this procedure fully eliminates the dependence on the initial renormalisation scale. Finally, we note that the non-modified  $N$  point function  $\Gamma$  is simply recovered from

$$\Gamma = [\mathbf{K}(1)]^{-2} \widehat{\Gamma}, \quad (4.205)$$

where the  $z$ -dependent matrix  $\mathbf{K}(z)$  is defined such that

$$\mathbf{K}(0) = \mathbb{1}, \quad \frac{d}{dz} \mathbf{K} = -\boldsymbol{\gamma}_A, \quad (4.206)$$

where  $\widehat{\boldsymbol{\gamma}}$  results from the application of the operator  $\mathbf{D}_A$  on the scalar fields, namely

$$\boldsymbol{\gamma}_A = [H^{-1}]_{AB} \boldsymbol{\gamma}_B. \quad (4.207)$$

## 4.2 Interpretation in terms of multi-scale RG-running

Similar to the single-scale case, the differential evolution of the couplings from  $z = 0$  to  $z = 1$  can be interpreted in terms of a multi-scale RG-flow, where each of the  $\mu_i$ , initially set at  $\mu_i = \mu_i$ , will be driven towards their fixed-point value  $\mu_{i*}$  such that

$$\log \mu_{i*} = \frac{1}{2} \log M_i (\log \mu_{1*}, \dots, \log \mu_{N*}). \quad (4.208)$$

This means in turn that the cancellation of all logarithms can still be interpreted, as in the single-scale case, in terms of the simultaneous evolution of the  $\mu_i$  (at  $z = 0$ ) towards  $\mu_{i*}$  (at  $z = 1$ ). To understand this property, we first compute the derivative of the  $z$ -dependent mass logarithms  $\bar{L}^A$  ( $A = 1, \dots, N$ ):

$$\begin{aligned} \frac{d\bar{L}^A}{dz} &= \bar{L}_0^B [H^{-1}]_{BC} D_C \bar{L}^A \\ &= \bar{L}_0^B [H^{-1}]_{BC} (D_C \bar{L}^A - \delta_{CA}) \\ &= -\bar{L}_0^B [H^{-1}]_{BC} H_{CA} \\ &= -\bar{L}_0^A. \end{aligned} \quad (4.209)$$

In turn, since  $\bar{L}^A(0) = \bar{L}_0^A$ , this yields the following explicit dependence on  $z$

$$\bar{L}^A(z) = (1 - z) \bar{L}_0^A, \quad \forall A = 1, \dots, N. \quad (4.210)$$

Solving the above relation for  $\log \mu_{\mathbf{A}}$ , the  $z$ -dependent counterpart of the scale  $\log \mu_{\mathbf{A}}$ , yields

$$\log \mu_{\mathbf{A}}(z) = \mathbf{L}^A(z) - (1-z)\bar{L}_0^A, \quad (4.211)$$

and we recover that

$$\begin{aligned} \log \mu_{\mathbf{A}}(0) &= \mathbf{L}^A(0) - \bar{L}_0^A = \log \mu, \\ \log \mu_{\mathbf{A}}(1) &= \mathbf{L}^A(1) = \frac{1}{2} \log \mathbf{M}(1) \equiv \log \mu_{\mathbf{A}*}. \end{aligned} \quad (4.212)$$

Therefore, although made implicit in the formulation of the theory in terms of the  $z$ -evolution, there are indeed multiple RG-scales  $\mu_{\mathbf{A}}$  involved in the process of resummation. All of them are set to the same initial value  $\mu$ , but individually run towards their fixed-point value  $\log \mu_{\mathbf{A}*}$  when  $z$  departs from  $z = 0$  and approaches  $z = 1$ . In conclusion, this completes the proof that the multi-scale approach to renormalisation group improvement is indeed able to resum the UV logarithms in the effective action in a straightforward way. On the other hand, starting from the two-loop level, residual logarithms of the form  $\log \frac{M_i}{M_j}$  are generated, for which it might be possible to perform a further resummation. The investigation of such a possibility is left for future work.

### 4.3 Generalisations

There are a number of straightforward generalisations of the above results.

#### The effective potential

First, we have assumed in the previous discussion that the theory was formulated at its true vacuum (or at least at a stable minimum of the effective potential). Exactly as in the single-scale case, one may introduce a non-zero background-field component, so as to determine the location of the ground state from minimisation of the resummed effective potential. Every result presented above still holds when the couplings are replaced with their background-field-dependent counterparts. For instance,  $M$  has to be replaced with the field-dependent mass matrix  $m(\phi)$  such that

$$m^{ab}(\phi) = \frac{\partial^2 V^{(0)}}{\partial \phi^a \partial \phi^b}(\phi). \quad (4.213)$$

Most of the discussion of chapter 3 in the single-scale case can be transposed and generalised. In particular, skipping the details of the computation, it is not difficult to compute the field derivatives of RG-improved potential. In particular, the field derivatives of the  $z$ -dependent leading coefficient  $\widehat{\mathbf{C}}(z, \phi)$  can be shown to take the form (omitting the arguments for clarity)

$$\frac{d}{d\phi^i} \widehat{\mathbf{C}} \equiv \nabla_i \widehat{\mathbf{C}} = \partial_i \widehat{\mathbf{C}} + \left[ \partial_i \mathbf{L}^A - (1-z)\partial_i \bar{L}_0^A \right] \mathbf{D}_A \widehat{\mathbf{C}}, \quad (4.214)$$

in exact analogy with the single-scale result. Following the discussion in chapter 3, section 2.2, the stationary point equation can be solved by continuously evolving the solutions  $\phi_{\min}(z)$  from  $z = 0$  towards  $z = 1$  based on a system of differential equations imposing  $\nabla_i \widehat{\mathbf{C}}(z, \phi_{\min}(z)) = 0$  for all  $z \in [0, 1]$ .

#### Global symmetries and degenerate masses

When global symmetries are imposed on the theory, degeneracies may in general appear in the scalar spectrum. As long as the symmetry remains unbroken, the mass matrix takes, in the mass basis, a block-diagonal form

$$M = \begin{pmatrix} M_{[1]} \mathbb{1}_{n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{[P]} \mathbb{1}_{n_P} \end{pmatrix}, \quad (4.215)$$

where  $P$  is the number of sub-multiplets transforming under particular representations of the Lie algebra, and where  $n_k$  denotes the dimension of the  $k$ -th such representation. In such a case, a consistent approach to  $\overline{\text{mMS}}$  renormalisation only requires to introduce  $P$  distinct RG-scales in the regularisation process. That is, the general prescription consists in introducing one renormalisation scale per massive propagating multiplet transforming under some representation of the Lie algebra.

### Extension to general gauge-Yukawa theories

Up to this point, we have only considered in our discussion the case of a theory containing  $N$  interacting scalars. In practice, any phenomenologically relevant application would require to extend our formalism to general gauge-Yukawa theories. We believe that such a generalisation is in fact very straightforward. One simply needs in practice to assign to every propagating state a renormalisation scale  $\mu_A$ ,  $\mu_i$  or  $\mu_a$ , depending on the species of the field – vector boson, fermion or scalar, respectively. In the  $(4 - \varepsilon)$ -dimensional theory, we use a generalised prescription where the renormalisation scales are attached to the kinetic terms. One should however carefully inspect the mass-dimension of each species in such a prescription. The treatment of vector bosons, with canonical mass-dimension 1, is analogue to that of the scalar field. However, for the fermion fields, one should inspect the dimensionless Yukawa couplings

$$y_{aij} \phi^a \psi^i \psi^j, \quad (4.216)$$

in order to infer the mass-dimension of the fermion fields. Recalling that  $[\phi] = 1 - \varepsilon/4$ , we obtain

$$1 - \varepsilon/4 + 2[\psi] = 4 - \varepsilon \quad \rightarrow \quad [\psi] = \frac{3}{2} \left(1 - \frac{\varepsilon}{4}\right), \quad (4.217)$$

consistently with the observation that, in this prescription, the mass-dimension of the renormalised fields is obtained by multiplying their canonical mass-dimension in 4-dimension with the factor  $(1 - \varepsilon/4)$ . Since the bare fermion fields have mass-dimension

$$[\psi_0] = \frac{3}{2} - \frac{\varepsilon}{4}, \quad (4.218)$$

an anomalous scaling for the renormalised fields will stem from the difference

$$[\psi] - [\psi_0] = -\frac{\varepsilon}{8}, \quad (4.219)$$

to be compared with the scalar- (or vector-)field anomalous scaling

$$[\phi] - [\phi_0] = \frac{\varepsilon}{4}. \quad (4.220)$$

As a direct consequence, the general prescription to compute the  $\overline{\text{mMS}}$  beta-functions based on the  $\overline{\text{MS}}$  result is generalised to (introducing a new type of index,  $u, v, w, \dots$ , to label the multiple RG-scales)

$$\begin{aligned} \text{Scalar propagators: } \delta^{ab} & \xrightarrow{\beta^{\overline{\text{MS}}} \rightarrow \beta_u^{\overline{\text{mMS}}}} \frac{1}{2} \delta_u^a \delta_u^b, \\ \text{Vector propagators: } \delta^{AB} & \xrightarrow{\beta^{\overline{\text{MS}}} \rightarrow \beta_u^{\overline{\text{mMS}}}} \frac{1}{2} \delta_u^A \delta_u^B, \\ \text{Fermion propagators: } \delta^{ij} & \xrightarrow{\beta^{\overline{\text{MS}}} \rightarrow \beta_u^{\overline{\text{mMS}}}} \frac{1}{4} \delta_u^i \delta_u^j. \end{aligned} \quad (4.221)$$

Every subsequent result obtained in this chapter then follows from the expression of the multi-scale beta-functions. In particular, every mass-logarithm involving scalars, fermions and massive bosons will be resummed in the process of evolving every  $z$ -dependent coupling from  $z = 0$  to  $z = 1$ . In this regard, the general formalism presented in chapter 2 as well as its implementation in PyR@TE 3 [9] can prove very useful.



# Appendix

## A Multi-scale beta-functions: general derivation

While the explicit derivation of the beta-functions in the  $\overline{\text{mMS}}$  scheme is analogue to the single-case, the occurrence of a renormalisation-scale matrix,  $\mathcal{M}$ , instead of a single renormalisation scale makes the computation rather technical. We provide below the details of this calculation, the main results being summarised in the main text in section 2.

### A.1 Quartic beta-functions

In the quartic sector, it is convenient to use the relation between the bare couplings and  $\tilde{\ell}_{abcd}$  (the auxiliary quartic couplings with mass-dimension  $\varepsilon$ ) as a starting point to derive the beta-functions. Namely, we have

$$\lambda_{abcd}^0 = \left( \tilde{\ell}_{efgh} + \delta \tilde{\ell}_{efgh} \right) \tilde{Z}_{ea}^{-1} \tilde{Z}_{fb}^{-1} \tilde{Z}_{gc}^{-1} \tilde{Z}_{hd}^{-1}, \quad (4.A1)$$

leading after differentiation to

$$0 = D_A \tilde{\ell}_{abcd} + D_A \delta \tilde{\ell}_{abcd} - \left\{ \tilde{\ell} \leftarrow \tilde{\gamma}'_A \right\}, \quad (4.A2)$$

where the differential operator  $D_A$  has been defined in Eq. (4.69) as

$$D_A = \mu_A \frac{d}{d\mu_A}, \quad (4.A3)$$

and where the auxiliary anomalous dimension matrix  $\tilde{\gamma}'_A$  is defined as

$$\tilde{\gamma}'_A = \tilde{Z}^{-1} D_A \tilde{Z}, \quad (4.A4)$$

and is to be distinguished from  $\tilde{\gamma}_A$ , the anomalous dimension of the  $(1 - \varepsilon/4)$ -dimensional renormalised scalar fields. The latter is defined from the relation

$$\phi^0 = \tilde{Z} \mathcal{M}^{-\varepsilon/4} \tilde{\phi}, \quad (4.A5)$$

yielding after differentiation

$$D_A \tilde{\phi} = - \left( \tilde{Z} \mathcal{M}^{-\varepsilon/4} \right)^{-1} D_A \left( \tilde{Z} \mathcal{M}^{-\varepsilon/4} \right) \equiv -\tilde{\gamma}_A \tilde{\phi}. \quad (4.A6)$$

In turn,  $\tilde{\gamma}_A$  can be expressed in terms of  $\tilde{\gamma}'_A$  as

$$\tilde{\gamma}_A = \mathcal{M}^{\varepsilon/4} \tilde{\gamma}'_A \mathcal{M}^{-\varepsilon/4} + \mathcal{M}^{\varepsilon/4} D_A \mathcal{M}^{-\varepsilon/4} \equiv \mathcal{M}^{\varepsilon/4} \tilde{\gamma}'_A \mathcal{M}^{-\varepsilon/4} + \gamma_A^{[1]} \varepsilon, \quad (4.A7)$$

where, since  $\tilde{\mathcal{M}}$  is diagonal,

$$\left[ \gamma_A^{[1]} \right]^{ab} = \left[ \mathcal{M}^{\varepsilon/4} D_A \mathcal{M}^{-\varepsilon/4} \right] = -\frac{1}{4} \delta_A^a \delta^b_1. \quad (4.A8)$$

The auxiliary anomalous dimension matrix  $\tilde{\gamma}'_A$  has to be regarded as the anomalous dimension of the scalar fields  $\tilde{\phi}'$  with canonical mass-dimension in the  $(4 - \varepsilon)$ -dimensional theory,

$$\tilde{\phi}' = \mathcal{M}^{-\varepsilon/4} \tilde{\phi}. \quad (4.A9)$$

Since  $\tilde{\ell}$  has canonical mass-dimension, we expect the  $\varepsilon$ -dependent contributions to its beta-function to vanish. For the same reason we shall assume that  $\tilde{\gamma}'_A$  is independent of  $\varepsilon$ . To obtain the beta-function of  $\tilde{\ell}$ , we then need to express the contributions at order  $\mathcal{O}(\varepsilon^0)$  in the derivative of the counterterm. This is

most easily achieved by observing that, to all orders,  $\delta\tilde{\ell}$  takes the same form as  $\delta\tilde{\lambda}$  if one replaces  $\tilde{\lambda}$  with  $\tilde{\ell}$  and every internal index contraction by powers of the RG-scale matrix  $\tilde{\mathcal{M}}$ . For instance, at one-loop, with view on Eq. (4.A1), one has

$$\begin{aligned}\delta^{(1)}\tilde{\ell}_{abcd} &= \delta\tilde{\lambda}_{efgh}\tilde{\mathcal{M}}_{ea}^{\varepsilon/4}\tilde{\mathcal{M}}_{fb}^{\varepsilon/4}\tilde{\mathcal{M}}_{gc}^{\varepsilon/4}\tilde{\mathcal{M}}_{hd}^{\varepsilon/4} \supset \frac{1}{\varepsilon(16\pi^2)}\tilde{\lambda}_{efij}\tilde{\lambda}_{ijgh}\tilde{\mathcal{M}}_{ea}^{\varepsilon/4}\tilde{\mathcal{M}}_{fb}^{\varepsilon/4}\tilde{\mathcal{M}}_{gc}^{\varepsilon/4}\tilde{\mathcal{M}}_{hd}^{\varepsilon/4} \\ &= \frac{1}{\varepsilon(16\pi^2)}\tilde{\ell}_{abij}\tilde{\mathcal{M}}_{ik}^{-\varepsilon/2}\tilde{\mathcal{M}}_{jl}^{-\varepsilon/2}\tilde{\ell}_{klcd} \\ &= \frac{1}{\varepsilon(16\pi^2)}\tilde{\ell}_{abij}\mu_i^{-\varepsilon/2}\mu_j^{-\varepsilon/2}\tilde{\ell}_{ijcd}.\end{aligned}\quad (4.A10)$$

At higher orders, we have seen that powers of the logarithm of the RG-scale matrix,  $\tilde{\mathcal{L}}$ , may occur in the expression of the counterterms. In that case, the expression of  $\delta\tilde{\ell}$  is simply obtained by performing the additional replacement

$$\left[\tilde{\mathcal{L}}^p\right]_{ab} = (\log\mu_a)^p\delta_{ab} \longrightarrow \left[\tilde{\mathcal{M}}^{-\varepsilon/4}\tilde{\mathcal{L}}^p\tilde{\mathcal{M}}^{-\varepsilon/4}\right]_{ab} = \mu_a^{-\varepsilon/2}(\log\mu_a)^p\delta_{ab}.\quad (4.A11)$$

We further note that the explicit dependence of the counterterms on the logarithms of the renormalisation scales does not introduce additional sources of order- $\mathcal{O}(\varepsilon^0)$  contributions in Eq. (4.A2). It is then clear that applying  $D_A$  on the counterterms will select the first-order pole contributions through the derivatives of the powers of the renormalisation scales, namely

$$D_A\mu_i^{-\varepsilon/2} = \frac{-\varepsilon}{2}\delta_{Ai}\mu_i^{-\varepsilon/2}.\quad (4.A12)$$

Contrary to the single-scale case, the selection of the first-order pole term  $\delta_{[1]}\tilde{\ell}$  keeps track of the position of the renormalisation scales in the diagram, and we obtain for instance from Eq. (4.A10) that

$$D_A\delta^{(1)}\tilde{\ell}_{abcd} \supset -\frac{1}{2(16\pi)^2}\left[\tilde{\ell}_{abAj}\mu_A^{-\varepsilon/2}\mu_j^{-\varepsilon/2}\tilde{\ell}_{Ajcd} + \tilde{\ell}_{abiA}\mu_i^{-\varepsilon/2}\mu_A^{-\varepsilon/2}\tilde{\ell}_{iAcd}\right].\quad (4.A13)$$

Commuting the renormalisation scales through  $\tilde{\ell}$  back to their original position, we obtain

$$D_A\delta^{(1)}\tilde{\ell}_{abcd} \supset -\frac{1}{2(16\pi)^2}\left[\tilde{\lambda}_{efAj}\tilde{\lambda}_{Ajgh} + \tilde{\lambda}_{efiA}\tilde{\lambda}_{iAgh}\right]\tilde{\mathcal{M}}_{ea}^{\varepsilon/4}\tilde{\mathcal{M}}_{fb}^{\varepsilon/4}\tilde{\mathcal{M}}_{gc}^{\varepsilon/4}\tilde{\mathcal{M}}_{hd}^{\varepsilon/4}.\quad (4.A14)$$

More generally, we conclude that the contributions of order  $\mathcal{O}(\varepsilon^0)$  in Eq. (4.A2) are obtained by ‘‘opening’’ each internal line in the diagrams contributing to the first-order pole contributions, noted  $\delta_{[1]}\tilde{\ell}$ , and inserting at this position an open index  $A$  associated with the scale  $\mu_A$  of which we are taking the derivative. It is convenient to define an operator, noted  $\mathcal{D}_A$ , which performs this operation when applied on a particular tensor structure. This allows in turn to derive a simple expression for the beta-function of  $\tilde{\ell}$ ,

$$\beta_A(\tilde{\ell}) = D_A(\tilde{\ell}) = \frac{1}{2}\mathcal{D}_A\left(\delta_{[1]}\tilde{\ell}\right) + \left\{\tilde{\ell} \leftarrow \tilde{\gamma}'_A\right\},\quad (4.A15)$$

which in the limit  $\varepsilon \rightarrow 0$  coincides with the beta-function of the dimensionless, renormalised coupling  $\tilde{\lambda}$ . Otherwise, one needs to differentiate the relation

$$\ell_{abcd} = \lambda_{efgh}\tilde{\mathcal{M}}_{ea}^{\varepsilon/4}\tilde{\mathcal{M}}_{fb}^{\varepsilon/4}\tilde{\mathcal{M}}_{gc}^{\varepsilon/4}\tilde{\mathcal{M}}_{hd}^{\varepsilon/4},\quad (4.A16)$$

yielding

$$D_A\tilde{\lambda}_{abcd} = \beta_A\left(\tilde{\ell}_{efgh}\right)\tilde{\mathcal{M}}_{ea}^{-\varepsilon/4}\tilde{\mathcal{M}}_{fb}^{-\varepsilon/4}\tilde{\mathcal{M}}_{gc}^{-\varepsilon/4}\tilde{\mathcal{M}}_{hd}^{-\varepsilon/4} + \varepsilon\left\{\tilde{\lambda} \leftarrow \gamma_A^{[1]}\right\}_{abcd},\quad (4.A17)$$

where  $\gamma_A^{[1]}$  was defined in Eq. (4.A8). The inverse powers of the RG-scale matrix in the first term systematically cancel those appearing in the expression of  $\ell$  in Eq. (4.A15), yielding

$$\beta_A\left(\tilde{\ell}_{efgh}\right)\tilde{\mathcal{M}}_{ea}^{-\varepsilon/4}\tilde{\mathcal{M}}_{fb}^{-\varepsilon/4}\tilde{\mathcal{M}}_{gc}^{-\varepsilon/4}\tilde{\mathcal{M}}_{hd}^{-\varepsilon/4} = \frac{1}{2}\mathcal{D}_A\left(\delta_{[1]}\tilde{\lambda}\right) + \left\{\tilde{\lambda} \leftarrow \tilde{\mathcal{M}}^{\varepsilon/4}\tilde{\gamma}_A\tilde{\mathcal{M}}^{-\varepsilon/4}\right\},\quad (4.A18)$$

while the second term in Eq.(4.A17) is easily evaluated using Eq. (4.A8):

$$\left\{ \tilde{\lambda} \leftarrow \gamma_A^{[1]} \right\}_{abcd} = -\frac{1}{4} \left( \delta_{aA} \tilde{\lambda}_{Abcd} + \delta_{bA} \tilde{\lambda}_{aAcd} + \delta_{cA} \tilde{\lambda}_{AcAd} + \delta_{ad} \tilde{\lambda}_{abcA} \right). \quad (4.A19)$$

Conveniently defining

$$\beta_A^{[1]}(\tilde{\lambda}_{abcd}) = \left\{ \tilde{\lambda} \leftarrow \gamma_A^{[1]} \right\}_{abcd}, \quad (4.A20)$$

we arrive at the following expression for  $D_A \tilde{\lambda}$ :

$$D_A \tilde{\lambda}_{abcd} = \varepsilon \beta_A^{[1]}(\tilde{\lambda}_{abcd}) + \frac{1}{2} \mathcal{D}_A \left( \delta_{[1]} \tilde{\lambda} \right) + \left\{ \tilde{\lambda} \leftarrow \tilde{\mathcal{M}}^{\varepsilon/4} \tilde{\gamma}'_A \tilde{\mathcal{M}}^{-\varepsilon/4} \right\}. \quad (4.A21)$$

The first contribution, of order  $\mathcal{O}(\varepsilon)$  stems from the anomalous scaling of  $\tilde{\lambda}$  in the  $(4 - \varepsilon)$ -dimensional theory, and is the analogue of the single-scale  $\overline{\text{MS}}$  contribution  $\beta(\lambda_{abcd}) \supset -\varepsilon \lambda_{abcd}$ . It is in fact interesting to note that taking the sum of Eq. (4.A21) over  $A$ , one effectively recovers the standard single-scale result:

$$\begin{aligned} \sum_{A=1}^N D_A \tilde{\lambda}_{abcd} &\supset \sum_{A=1}^N \varepsilon \beta_A^{[1]}(\tilde{\lambda}_{abcd}) \\ &= -\frac{\varepsilon}{4} \sum_{A=1}^N \left( \delta_{aA} \tilde{\lambda}_{Abcd} + \delta_{bA} \tilde{\lambda}_{aAcd} + \delta_{cA} \tilde{\lambda}_{AcAd} + \delta_{ad} \tilde{\lambda}_{abcA} \right) \\ &= -\varepsilon \lambda_{abcd}. \end{aligned} \quad (4.A22)$$

Finally, while the second term in Eq. (4.A21) only contributes at order  $\mathcal{O}(\varepsilon^0)$  (i.e. to the beta-function), the third term involving the anomalous dimension matrix produces an infinite series of positive powers of  $\varepsilon$ . This is a new feature compared to the single-scale case, where the beta-functions of the renormalised couplings were at most of order  $\mathcal{O}(\varepsilon)$ .

## A.2 Anomalous dimension matrix

It remains at this stage to determine the expression of the anomalous dimension matrix  $\tilde{\gamma}'_A$ . For this purpose, we shall consider as our starting point the following relation:

$$D_A \left( \tilde{Z}^T \tilde{Z} \right) = \left( D_A \tilde{Z}^T \right) \tilde{Z} + \tilde{Z}^T \left( D_A \tilde{Z} \right) = \tilde{\gamma}'_A \tilde{Z}^T \tilde{Z} + \tilde{Z}^T \tilde{\gamma}'_A, \quad (4.A23)$$

and, inserting the decomposition of the field-strength renormalisation matrix in terms of the corresponding counterterm,

$$\tilde{Z}^T \tilde{Z} = U Z^2 U^T = 1 + U \delta_\phi U^T = 1 + \tilde{\delta}_\phi, \quad (4.A24)$$

we obtain

$$D_A \tilde{\delta}_\phi = \tilde{\gamma}'_A + \tilde{\gamma}'_A{}^T + \tilde{\delta}_\phi \tilde{\gamma}'_A + \tilde{\gamma}'_A{}^T \tilde{\delta}_\phi. \quad (4.A25)$$

In analogy with single-scale renormalisation – and with the expression of  $\tilde{\delta}_\phi$  and hand – the next step to compute  $\tilde{\gamma}'_A$  is to decompose the RG-derivative of the counterterm according to

$$D_A \tilde{\delta}_\phi = D_A \tilde{\lambda}_{efgh} \frac{\partial \tilde{\delta}_\phi}{\tilde{\lambda}_{efgh}}, \quad (4.A26)$$

in order to isolate the order  $\mathcal{O}(\varepsilon^0)$  contributions. However, we have seen from Eq. (4.A21) that the RG-derivatives of  $\tilde{\lambda}$  involve an infinite series of  $\varepsilon$ -dependent terms, which by cancellation with the poles of  $\tilde{\delta}_\phi$  will a priori contribute to the finite part of the anomalous dimension. Fortunately, it is possible to avoid such a difficult computation by providing another expression for  $D_A \tilde{\delta}_\phi$ , first observing that

$$D_A \left( \tilde{\mathcal{M}}^{\varepsilon/4} \tilde{\delta}_\phi \tilde{\mathcal{M}}^{\varepsilon/4} \right) = \tilde{\mathcal{M}}^{\varepsilon/4} \left[ D_A \tilde{\delta}_\phi + \tilde{\mathcal{M}}^{-\varepsilon/4} \left( D_A \tilde{\mathcal{M}}^{\varepsilon/4} \right) \tilde{\delta}_\phi + \tilde{\delta}_\phi \left( D_A \tilde{\mathcal{M}}^{\varepsilon/4} \right) \tilde{\mathcal{M}}^{-\varepsilon/4} \right] \tilde{\mathcal{M}}^{\varepsilon/4}, \quad (4.A27)$$

yielding in turn

$$D_A \tilde{\delta}_\phi = \tilde{\mathcal{M}}^{-\varepsilon/4} \left[ D_A \left( \tilde{\mathcal{M}}^{\varepsilon/4} \tilde{\delta}_\phi \tilde{\mathcal{M}}^{\varepsilon/4} \right) \right] \tilde{\mathcal{M}}^{-\varepsilon/4} - \tilde{\mathcal{M}}^{-\varepsilon/4} \left( D_A \tilde{\mathcal{M}}^{\varepsilon/4} \right) \tilde{\delta}_\phi - \tilde{\delta}_\phi \left( D_A \tilde{\mathcal{M}}^{\varepsilon/4} \right) \tilde{\mathcal{M}}^{-\varepsilon/4}. \quad (4.A28)$$

As compared to Eq. (4.A26), the  $\varepsilon$ -expansion of every term in the above expression is finite, rendering the  $\mathcal{O}(\varepsilon^0)$  contributions tractable. The computation of the first term is in fact completely analogue to the case of  $D_A \tilde{\ell}$  presented above, since the powers of  $\mathcal{M}$  can be commuted through the quartic couplings, effectively trading  $\tilde{\lambda}$  for  $\tilde{\ell}$ , whose beta-function does not contain powers of  $\varepsilon$ . For instance, at two-loop, we have

$$\begin{aligned} \left[ \tilde{\mathcal{M}}^{\varepsilon/4} \tilde{\delta}_\phi \tilde{\mathcal{M}}^{\varepsilon/4} \right]_{ab} &\supset -\frac{1}{12\varepsilon(16\pi^2)^2} \left[ \tilde{\mathcal{M}}^{\varepsilon/4} \right]_{ai} \tilde{\lambda}_{iefg} \tilde{\lambda}_{efgj} \left[ \tilde{\mathcal{M}}^{\varepsilon/4} \right]_{jb} \\ &= -\frac{1}{12\varepsilon(16\pi^2)^2} \tilde{\ell}_{iefg} \mu_e^{-\varepsilon/4} \mu_f^{-\varepsilon/4} \mu_g^{-\varepsilon/4} \tilde{\ell}_{efgj}. \end{aligned} \quad (4.A29)$$

As previously, differentiation with respect to the RG-scales will only generate positive powers of  $\varepsilon$  through the explicit derivatives of  $\mu_x^{-\varepsilon/2}$  ( $x = e, f, g$ ). Hence, we obtain after commuting powers of the RG-scale matrix back to their original position:

$$D_A \left[ \tilde{\mathcal{M}}^{\varepsilon/4} \tilde{\delta}_\phi \tilde{\mathcal{M}}^{\varepsilon/4} \right]_{ab} \supset \frac{1}{24(16\pi^2)^2} \left[ \tilde{\mathcal{M}}^{\varepsilon/4} \right]_{ai} \left( \tilde{\lambda}_{iAfg} \tilde{\lambda}_{Afgj} + \tilde{\lambda}_{ieAg} \tilde{\lambda}_{eAgj} + \tilde{\lambda}_{iefA} \tilde{\lambda}_{efAj} \right) \left[ \tilde{\mathcal{M}}^{\varepsilon/4} \right]_{jb}. \quad (4.A30)$$

Making use of the operator  $\mathcal{D}_A$  defined above, we then arrive at the simple result

$$\tilde{\mathcal{M}}^{-\varepsilon/4} \left[ D_A \left( \tilde{\mathcal{M}}^{\varepsilon/4} \tilde{\delta}_\phi \tilde{\mathcal{M}}^{\varepsilon/4} \right) \right] \tilde{\mathcal{M}}^{-\varepsilon/4} = -\frac{1}{2} \mathcal{D}_A \left( \tilde{\delta}_\phi^{[1]} \right) + \mathcal{O}(\varepsilon^{-1}), \quad (4.A31)$$

where  $\tilde{\delta}_\phi^{[1]}$  stands for the first-order pole contributions to  $\tilde{\delta}_\phi$ . With  $\tilde{\mathcal{M}}$  a diagonal matrix, the last two terms of Eq. (4.A28) are easily evaluated and the components of  $D_A \tilde{\delta}_\phi$  can eventually be brought into the form

$$\left[ D_A \tilde{\delta}_\phi \right]_{ab} = -\frac{1}{2} \left[ \mathcal{D}_A \left( \tilde{\delta}_\phi^{[1]} \right) \right]_{ab} - \frac{1}{4} \delta_{aA} \left[ \tilde{\delta}_\phi^{[1]} \right]_{Ab} - \frac{1}{4} \left[ \tilde{\delta}_\phi^{[1]} \right]_{aA} \delta_{Ab} + \mathcal{O}(\varepsilon^{-1}). \quad (4.A32)$$

We are finally able to determine the expression of the symmetric part of  $\tilde{\gamma}'_A$  by collecting the  $\varepsilon$ -independent contributions in Eq. (4.A25), yielding

$$\left[ \tilde{\gamma}'_A + \tilde{\gamma}'_A{}^T \right]_{ab} = -\frac{1}{2} \left[ \mathcal{D}_A \left( \tilde{\delta}_\phi^{[1]} \right) \right]_{ab} - \frac{1}{4} \delta_{aA} \left[ \tilde{\delta}_\phi^{[1]} \right]_{Ab} - \frac{1}{4} \left[ \tilde{\delta}_\phi^{[1]} \right]_{aA} \delta_{Ab}, \quad (4.A33)$$

which corresponds exactly to the expression one would have obtained from (see Eq. (4.A26))

$$\tilde{\gamma}'_A + \tilde{\gamma}'_A{}^T = \beta_A^{[1]}(\tilde{\lambda}_{abcd}) \frac{\partial \tilde{\delta}_\phi}{\partial \tilde{\lambda}_{abcd}}, \quad (4.A34)$$

with

$$\beta_A^{[1]}(\tilde{\lambda}_{abcd}) = -\frac{1}{4} \left( \delta_{aA} \tilde{\lambda}_{Abcd} + \delta_{bA} \tilde{\lambda}_{aAcd} + \delta_{cA} \tilde{\lambda}_{AcAd} + \delta_{dA} \tilde{\lambda}_{abcA} \right) \quad (4.A35)$$

the order- $\mathcal{O}(\varepsilon)$  contributions to  $D_A \tilde{\lambda}$  stemming only from the first term in Eq. (4.A21).

In a final step, we may recover the expression of  $\gamma'_A$ , the anomalous dimension matrix in the original basis,

$$\gamma'_A = Z^{-1} D_A Z, \quad (4.A36)$$

recalling that

$$\tilde{Z} = Z U^T, \quad (4.A37)$$

with  $U$  is the orthogonal matrix which diagonalises the mass-matrix,  $m$ . From Eq. (4.A37), we have

$$\tilde{\gamma}'_A = \tilde{Z}^{-1} D_A \tilde{Z} = U \gamma'_A U^T + U D_A U^T. \quad (4.A38)$$

As discussed in chapter 1, section 6.2, the introduction of the RG-dependent orthogonal matrix  $U$  effectively induces a skew-symmetric component to the anomalous dimension matrix. Defining

$$V_A = (D_A U) U^T = -V_A^T, \quad (4.A39)$$

we obtain from Eq. (4.A38), rewritten in the form

$$\tilde{\gamma}'_A = U \gamma'_A U^T - V_A, \quad (4.A40)$$

that

$$\tilde{\gamma}'_A + \tilde{\gamma}'_A{}^T = 2U \gamma_A U^T. \quad (4.A41)$$

In turn, the expression of  $\gamma'_A$  rotated to the mass basis is simply obtained from Eq. (4.A34), yielding

$$U \gamma_A U^T = \frac{1}{2} \beta_A^{[1]} (\tilde{\lambda}_{efgh}) \frac{\partial \tilde{\delta}_\phi}{\partial \tilde{\lambda}_{efgh}}, \quad (4.A42)$$

in analogy with single-scale renormalisation. Finally, the skew-symmetric component of  $\tilde{\gamma}'_A$  is computed from the requirement of a diagonal mass matrix at all scales, namely that the off-diagonal components of  $D_A \tilde{m}$  vanish. Explicitly, we write<sup>10</sup>

$$D_A \tilde{m} = D_A (U m U^T) = [(D_A U) U^T, \tilde{m}] + U (D_A m) U^T = [V_A, \tilde{m}] + U (D_A m) U^T, \quad (4.A43)$$

and obtain from the evaluation of the vanishing off-diagonal components,

$$[V_A]^{ab} = \frac{[U (D_A m) U^T]^{ab}}{m_a - m_b}, \quad (4.A44)$$

where  $m_p = \tilde{m}_{pp}$  stands for the  $p$ -th eigenvalue of  $m$ . In the limit of a 4-dimensional theory where  $\varepsilon \rightarrow 0$ , we arrive at

$$[V_A]^{ab} = \frac{[U \beta_A(M) U^T]^{ab}}{M_a - M_b}, \quad (4.A45)$$

in exact analogy with the single-scale case (see Eq.(1.227)).

### A.3 Summary of the results

At this stage, it is useful to reiterate the main results that we have obtained from the above discussion. First, the anomalous dimension matrix for the renormalised scalar fields  $\tilde{\phi}$  decomposes as

$$\tilde{\gamma}_A = \mathcal{M}^{\varepsilon/4} \tilde{\gamma}'_A \mathcal{M}^{-\varepsilon/4} + \mathcal{M}^{\varepsilon/4} D_A \mathcal{M}^{-\varepsilon/4} \equiv \mathcal{M}^{\varepsilon/4} \tilde{\gamma}'_A \mathcal{M}^{-\varepsilon/4} + \gamma_A^{[1]} \varepsilon, \quad (4.A46)$$

where

$$[\gamma_A^{[1]}]^{ab} = -\frac{1}{4} \delta^a_A \delta^b_A. \quad (4.A47)$$

The auxiliary anomalous dimension matrix of the scalar fields with canonical mass-dimension,  $\tilde{\gamma}'_A$ , decomposes according to

$$\tilde{\gamma}'_A = U \gamma'_A U^T - V_A. \quad (4.A48)$$

The first contribution is a symmetric matrix, and is obtained from the first-order pole contributions to the field-strength counterterm,

$$U \gamma'_A U^T = \frac{1}{2} \beta_A^{[1]} (\tilde{\lambda}_{efgh}) \frac{\partial \tilde{\delta}_\phi}{\partial \tilde{\lambda}_{efgh}}, \quad (4.A49)$$

while  $V_A$  is a skew-symmetric matrix accounting for the RG-dependent orthogonal rotation maintaining the fields and couplings in the mass-basis.

<sup>10</sup>We reiterate that the orthogonality of  $U$  translates into the relation  $(D_A U) U^T + U (D_A U^T) = V_A + V_A^T = 0$ .

In the quartic sector, we have shown that the RG-derivatives of the couplings are expressed in terms of the first-order pole contributions to the counterterm,

$$D_A \tilde{\lambda}_{abcd} = \varepsilon \beta_A^{[1]}(\tilde{\lambda}_{abcd}) + \mathcal{D}_A \left( \delta_{[1]} \tilde{\lambda} \right) + \left\{ \tilde{\lambda} \leftarrow \tilde{\mathcal{M}}^{\varepsilon/4} \tilde{\gamma}'_A \tilde{\mathcal{M}}^{-\varepsilon/4} \right\} \quad (4.A50)$$

where

$$\beta_A^{[1]}(\tilde{\lambda}_{abcd}) = \left\{ \tilde{\lambda} \leftarrow \gamma_A^{[1]} \right\}_{abcd} = -\frac{1}{4} \left( \delta_{aA} \tilde{\lambda}_{Abcd} + \delta_{bA} \tilde{\lambda}_{aAc d} + \delta_{cA} \tilde{\lambda}_{AcAd} + \delta_{dA} \tilde{\lambda}_{abcA} \right). \quad (4.A51)$$

Skipping the explicit computation for the mass matrix – carried out in exact analogy with the quartic couplings – we only report here the final result,

$$D_A \tilde{M}_{ab} = \varepsilon \beta_A^{[1]}(\tilde{M}_{ab}) + \mathcal{D}_A \left( \delta_{[1]} \tilde{M} \right) + \left\{ \tilde{M} \leftarrow \tilde{\mathcal{M}}^{\varepsilon/4} \tilde{\gamma}'_A \tilde{\mathcal{M}}^{-\varepsilon/4} \right\}, \quad (4.A52)$$

with

$$\beta_A^{[1]}(\tilde{M}_{ab}) = \left\{ \tilde{M} \leftarrow \gamma_A^{[1]} \right\}_{ab} = -\frac{1}{4} \left( \delta_{aA} \tilde{M}_{Ab} + \delta_{bA} \tilde{M}_{aA} \right). \quad (4.A53)$$

The operator  $\mathcal{D}_A$  is defined in such a way that, when applied on a particular tensor structure, each internal contraction  $\delta_{ij}$  is successively substituted with

$$\delta_{ij} \rightarrow \frac{1}{2} \delta_{iA} \delta_{Aj}. \quad (4.A54)$$

It can be more rigorously defined in terms of the auxiliary differential operator  $\mathcal{D}_A^{[1]}$  acting on the couplings of the theory, such that

$$\mathcal{D}_A^{[1]}(\tilde{\lambda}_{abcd}) = -\beta_A^{[1]}(\tilde{\lambda}_{abcd}), \quad \mathcal{D}_A^{[1]}(\tilde{M}_{ab}) = -\beta_A^{[1]}(\tilde{M}_{ab}). \quad (4.A55)$$

In turn, for any tensor  $T$  with rank  $n$ ,  $\mathcal{D}_A(T)$  can be defined as

$$\mathcal{D}_A(T) = \mathcal{D}_A^{[1]}(T) + \left\{ T \leftarrow \gamma_A^{[1]} \right\}. \quad (4.A56)$$

Explicitly, the corresponding contributions appearing in the RG-derivatives of the quartic couplings and of the mass matrix then read

$$D_A(\tilde{\lambda}_{abcd}) \supset \mathcal{D}_A(\delta_{[1]} \tilde{\lambda}_{abcd}) = -\beta_A^{[1]}(\tilde{\lambda}_{efgh}) \frac{\partial \delta_{[1]} \tilde{\lambda}_{abcd}}{\partial \tilde{\lambda}_{efgh}} + \left\{ \delta_{[1]} \tilde{\lambda} \leftarrow \gamma_A^{[1]} \right\}_{abcd}, \quad (4.A57)$$

$$D_A(\tilde{M}_{ab}) \supset \mathcal{D}_A(\delta_{[1]} \tilde{M}_{ab}) = -\beta_A^{[1]}(\tilde{\lambda}_{efgh}) \frac{\partial \delta_{[1]} \tilde{M}_{ab}}{\partial \tilde{\lambda}_{efgh}} - \beta_A^{[1]}(\tilde{M}_{ef}) \frac{\partial \delta_{[1]} \tilde{M}_{ab}}{\partial \tilde{M}_{ef}} + \left\{ \delta_{[1]} \tilde{M} \leftarrow \gamma_A^{[1]} \right\}_{ab}. \quad (4.A58)$$

Finally defining the beta-functions as

$$\beta_A(\tilde{\lambda}_{abcd}) = D_A \tilde{\lambda}_{abcd} \Big|_{\varepsilon \rightarrow 0}, \quad \beta_A(\tilde{M}_{ab}) = D_A \tilde{M}_{ab} \Big|_{\varepsilon \rightarrow 0}, \quad (4.A59)$$

we arrive at the final result of this appendix, namely the expression of the multi-scale beta-functions and anomalous dimensions in the 4-dimensional regularised theory:

$$\tilde{\gamma}_A = \frac{1}{2} \beta_A^{[1]}(\tilde{\lambda}_{efgh}) \frac{\partial \tilde{\delta}_\phi^{[1]}}{\partial \tilde{\lambda}_{efgh}} - V_A, \quad (4.A60)$$

$$\beta_A(\tilde{\lambda}_{abcd}) = -\beta_A^{[1]}(\tilde{\lambda}_{efgh}) \frac{\partial \delta_{[1]} \tilde{\lambda}_{abcd}}{\partial \tilde{\lambda}_{efgh}} + \left\{ \delta_{[1]} \tilde{\lambda} \leftarrow \gamma_A^{[1]} \right\}_{abcd} + \left\{ \tilde{\lambda} \leftarrow \tilde{\gamma}_A \right\}_{abcd}, \quad (4.A61)$$

$$\beta_A(\tilde{M}_{ab}) = -\beta_A^{[1]}(\tilde{\lambda}_{efgh}) \frac{\partial \delta_{[1]} \tilde{M}_{ab}}{\partial \tilde{\lambda}_{efgh}} - \beta_A^{[1]}(\tilde{M}_{ef}) \frac{\partial \delta_{[1]} \tilde{M}_{ab}}{\partial \tilde{M}_{ef}} + \left\{ \delta_{[1]} \tilde{M} \leftarrow \gamma_A^{[1]} \right\}_{ab} + \left\{ \tilde{M} \leftarrow \tilde{\gamma}_A \right\}_{ab}. \quad (4.A62)$$



## Part II

# Multi-scalar extensions of the Standard Model





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## Chapter 5

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# The Two-Higgs Doublet model in a gauge-invariant form

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### 1 Introduction

The Two-Higgs Doublet Model (THDM) was introduced by T. D. Lee decades ago [82] in order to have another source for CP violation – necessary in particular to address the imbalance problem of baryonic matter over baryonic antimatter in the observable Universe [83]. Still today this argument remains one of the main motivations to look for physics beyond the Standard Model of elementary particles [84].

The electroweak precision measurements give strong restrictions for extensions of the Standard Model with respect to further Higgs multiplets. In particular the  $\rho$  parameter [84], relating the electroweak gauge boson masses with the weak mixing angle, is in very good agreement with the prediction of the Standard Model. In general, the  $\rho$  parameter is very sensitive to any new Higgs multiplets [85] which couple to the electroweak gauge bosons. However, at least at tree level, the  $\rho$  parameter remains unchanged for an arbitrary number of Higgs-boson doublets in the model. This opens the possibility to study models with an extended number of Higgs-boson doublets. Here we want to consider the simplest extension with two Higgs-boson doublets, the Two-Higgs Doublet model (THDM). In the past decades lots of effort has been spent since the seminal work of T. D. Lee in the THDM; we refer in this respect to the review [86] and some newer works like [87, 88, 89, 90, 91, 92, 93, 94, 95, 96].

Extending the Standard Model to two Higgs-boson doublets, it is clear that the model has a much richer structure: First of all, the most general Higgs potential [97] which can be formed out of two doublets contains already 14 real parameters compared to two in the Standard Model. Of course, not all parameters can be chosen arbitrarily. For instance, in order to achieve stability, that is, a potential bounded from below, the parameters are restricted. Similar, in order to achieve the observed electroweak-symmetry breaking  $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{em}}$ , further restrictions appear. In addition, following the philosophy of the Standard Model to write down all Lagrangian terms not violating any symmetry of the model and not violating renormalisability, the most general Yukawa couplings of the two-Higgs boson doublets with the fermions of the model yield large flavour-changing neutral currents – not observed in Nature. The usual way to avoid these unobserved interactions is to impose symmetries on the model. A simple example is a  $\mathbb{Z}_2$  symmetry, such that only one Higgs-boson doublet transforms non-trivially with a sign flip. This restricts the Yukawa couplings to only one of the two Higgs-boson doublets. Symmetries of the general THDM have been studied in some detail; see for instance [98, 99, 91, 100].

It has been shown that stability and electroweak symmetry breaking of any THDM can be studied concisely with the help of *bilinears* [101, 102, 103]. In particular, all unphysical gauge degrees of freedom of the Higgs doublets are systematically avoided. The principal idea is to consider the gauge-invariant scalar products of the doublets instead of the gauge-dependent doublet components themselves. This formalism has turned out to be rather powerful not only for the study of stability and the electroweak symmetry breaking but also to study symmetries of the potential or the renormalisation-group equations [11, 9]. Since the gauge dependency obscures the physical content, new insights could be gained based on the gauge-invariant formalism.

Let us also mention some of the interesting works based on bilinears. For an extension of bilinears to

models beyond Higgs-boson doublets we refer to [104]. Custodial symmetry of the general NHDM has been studied based on bilinears in [105]. We also refer to [106], where the bilinear formalism has been extended to the Yukawa couplings.

In this work we want to show how to extend this formalism to the complete model, avoiding systematically gauge redundancies in the scalar trilinear and quartic couplings, the gauge-boson-Higgs interactions, and the most general Yukawa couplings. We shall illustrate in examples how this can be achieved together with the advantages of the method.

In the scalar sector, the starting point is to derive the squared mass matrix in a gauge-invariant way. We shall see that this can be done without resorting to the actual form of the potential. Therefore, the resulting expressions are valid for any THDM potential, at tree-level or effective, to any perturbation order. In turn, the trilinear and quartic couplings are obtained by taking suitable derivatives of the mass matrix, leading to concise and gauge-invariant expressions which reveal new insights. For instance, we find an all-order expression for the mass of the charged Higgs boson, valid for any THDM.

Let us briefly outline this chapter: In order to make this work self-contained, we briefly recall in section 2 the bilinear formalism and show how the electroweak-symmetry breaking behaviour can be studied concisely. In section 3, we derive the scalar mass matrix to all orders in perturbation theory in the THDM and apply these results to a general tree-level THDM potential. Then, we derive gauge invariant expressions for the trilinear and quartic scalar interactions, the gauge-boson-Higgs interactions, and the Yukawa interactions in the respective sections 4–6. Finally, in section 7, we summarise our main results and draw some conclusions. A more detailed discussion of the structure of the scalar mass matrix, extending section 3, is presented in appendix A. A list of analytic expressions for all relevant tree-level couplings of the general THDM in gauge-invariant form can be found in appendix B.

## 2 Review of the bilinear formalism

Here, we briefly review the bilinears in the THDM [101, 103, 102] in order to make this article self contained. We will also discuss basis transformations. In the convention with both Higgs-boson doublets carrying hypercharge  $y = +1/2$  corresponding to upper charged components, we write

$$\begin{aligned}\varphi_1(x) &= \begin{pmatrix} \varphi_1^+(x) \\ \varphi_1^0(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_1^1(x) + i\sigma_1^1(x) \\ \pi_1^2(x) + i\sigma_1^2(x) \end{pmatrix}, \\ \varphi_2(x) &= \begin{pmatrix} \varphi_2^+(x) \\ \varphi_2^0(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_2^1(x) + i\sigma_2^1(x) \\ \pi_2^2(x) + i\sigma_2^2(x) \end{pmatrix}.\end{aligned}\tag{5.1}$$

Explicitly we have decomposed the complex fields into their real and imaginary components. From now on we suppress the space-time argument of the fields. We will also write the real and imaginary upper and lower components of the two doublets in the form of one eight-component vector:

$$\phi = (\pi_1^1, \pi_1^2, \sigma_1^1, \sigma_1^2, \pi_2^1, \pi_2^2, \sigma_2^1, \sigma_2^2)^T, \quad \text{with } \phi \in \mathbb{R}^8.\tag{5.2}$$

The most general, tree-level, gauge-invariant potential with two Higgs-boson doublets reads [97]

$$\begin{aligned}V_{\text{THDM}}^0(\varphi_1, \varphi_2) &= m_{11}^2(\varphi_1^\dagger\varphi_1) + m_{22}^2(\varphi_2^\dagger\varphi_2) - m_{12}^2(\varphi_1^\dagger\varphi_2) - (m_{12}^2)^*(\varphi_2^\dagger\varphi_1) \\ &+ \frac{1}{2}\lambda_1(\varphi_1^\dagger\varphi_1)^2 + \frac{1}{2}\lambda_2(\varphi_2^\dagger\varphi_2)^2 + \lambda_3(\varphi_1^\dagger\varphi_1)(\varphi_2^\dagger\varphi_2) \\ &+ \lambda_4(\varphi_1^\dagger\varphi_2)(\varphi_2^\dagger\varphi_1) + \frac{1}{2}[\lambda_5(\varphi_1^\dagger\varphi_2)^2 + \lambda_5^*(\varphi_2^\dagger\varphi_1)^2] \\ &+ [\lambda_6(\varphi_1^\dagger\varphi_2) + \lambda_6^*(\varphi_2^\dagger\varphi_1)](\varphi_1^\dagger\varphi_1) + [\lambda_7(\varphi_1^\dagger\varphi_2) + \lambda_7^*(\varphi_2^\dagger\varphi_1)](\varphi_2^\dagger\varphi_2).\end{aligned}\tag{5.3}$$

In this potential we have two real quadratic parameters  $m_{11}^2$ ,  $m_{22}^2$ , and one complex quadratic parameter  $m_{12}^2$  as well as seven quartic parameters  $\lambda_1, \dots, \lambda_7$ , where the first four are real and  $\lambda_5, \lambda_6, \lambda_7$  complex. Altogether this corresponds to 14 real parameters in the potential.

Bilinears systematically avoid unphysical gauge degrees of freedom and are defined in the following way: First, the  $2 \times 2$  matrix of the two doublets,

$$\psi = \begin{pmatrix} \varphi_1^T \\ \varphi_2^T \end{pmatrix} = \begin{pmatrix} \varphi_1^+ & \varphi_1^0 \\ \varphi_2^+ & \varphi_2^0 \end{pmatrix},\tag{5.4}$$

is introduced. All gauge-invariant scalar products of the two doublets  $\varphi_1$  and  $\varphi_2$  can be arranged in one matrix

$$\underline{K} = \psi\psi^\dagger = \begin{pmatrix} \varphi_1^\dagger\varphi_1 & \varphi_2^\dagger\varphi_1 \\ \varphi_1^\dagger\varphi_2 & \varphi_2^\dagger\varphi_2 \end{pmatrix}. \quad (5.5)$$

This by construction hermitian  $2 \times 2$  matrix  $\underline{K}$  can be decomposed into a basis of the unit matrix and the Pauli matrices,

$$\underline{K} = \frac{1}{2} \left( K_0 \mathbb{1}_2 + K_a \sigma_a \right), \quad a = 1, 2, 3, \quad (5.6)$$

with the usual convention to sum over repeated indices. The four real coefficients  $K_0, K_a$  are called bilinears. Building traces on both sides of this equation (also with products of Pauli matrices) we get the four real bilinears explicitly,

$$\begin{aligned} K_0 &= \varphi_1^\dagger\varphi_1 + \varphi_2^\dagger\varphi_2, & K_1 &= \varphi_1^\dagger\varphi_2 + \varphi_2^\dagger\varphi_1, \\ K_2 &= i(\varphi_2^\dagger\varphi_1 - \varphi_1^\dagger\varphi_2), & K_3 &= \varphi_1^\dagger\varphi_1 - \varphi_2^\dagger\varphi_2. \end{aligned} \quad (5.7)$$

Inverting these relations we can express any THDM potential in terms of bilinears:

$$\begin{aligned} \varphi_1^\dagger\varphi_1 &= \frac{1}{2}(K_0 + K_3), & \varphi_1^\dagger\varphi_2 &= \frac{1}{2}(K_1 + iK_2), \\ \varphi_2^\dagger\varphi_1 &= \frac{1}{2}(K_1 - iK_2), & \varphi_2^\dagger\varphi_2 &= \frac{1}{2}(K_0 - K_3). \end{aligned} \quad (5.8)$$

The matrix  $\underline{K}$  is positive semi-definite. From  $K_0 = \text{Tr}(\underline{K})$  and  $\det(\underline{K}) = \frac{1}{4}(K_0^2 - K_a K_a)$  we obtain

$$K_0 \geq 0, \quad K_0^2 - K_a K_a \geq 0. \quad (5.9)$$

As has been shown in [102] there is a one-to-one correspondence between the original doublet fields and the bilinears apart from unphysical gauge-degrees of freedom. In terms of bilinears we can write any tree-level THDM potential (a constant term can always be dropped) as

$$V_{\text{THDM}}^0(K_0, K_a) = \xi_0 K_0 + \xi_a K_a + \eta_{00} K_0^2 + 2K_0 \eta_a K_a + K_a E_{ab} K_b, \quad (5.10)$$

with real parameters  $\xi_0, \xi_a, \eta_{00}, \eta_a, E_{ab} = E_{ba}, a, b \in \{1, 2, 3\}$ . Expressed in terms of the conventional parameters appearing in (5.3), these new parameters read

$$\xi_0 = \frac{1}{2}(m_{11}^2 + m_{22}^2), \quad \boldsymbol{\xi} = (\xi_a) = \frac{1}{2}(-2\text{Re}(m_{12}^2), 2\text{Im}(m_{12}^2), m_{11}^2 - m_{22}^2)^\text{T}, \quad (5.11)$$

$$\eta_{00} = \frac{1}{8}(\lambda_1 + \lambda_2) + \frac{1}{4}\lambda_3, \quad \boldsymbol{\eta} = (\eta_a) = \frac{1}{4}(\text{Re}(\lambda_6 + \lambda_7), -\text{Im}(\lambda_6 + \lambda_7), \frac{1}{2}(\lambda_1 - \lambda_2))^\text{T}, \quad (5.12)$$

$$E = (E_{ab}) = \frac{1}{4} \begin{pmatrix} \lambda_4 + \text{Re}(\lambda_5) & -\text{Im}(\lambda_5) & \text{Re}(\lambda_6 - \lambda_7) \\ -\text{Im}(\lambda_5) & \lambda_4 - \text{Re}(\lambda_5) & -\text{Im}(\lambda_6 - \lambda_7) \\ \text{Re}(\lambda_6 - \lambda_7) & -\text{Im}(\lambda_6 - \lambda_7) & \frac{1}{2}(\lambda_1 + \lambda_2) - \lambda_3 \end{pmatrix}. \quad (5.13)$$

As has been shown in [102], we can also form Minkowski-type four-vectors from the bilinears,

$$\widetilde{\mathbf{K}} = \begin{pmatrix} K_0 \\ \mathbf{K} \end{pmatrix}, \quad \text{with } \mathbf{K} = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}. \quad (5.14)$$

Writing the parameters of the potential in the form

$$\tilde{\boldsymbol{\xi}} = \begin{pmatrix} \xi_0 \\ \boldsymbol{\xi} \end{pmatrix}, \quad \tilde{E} = \begin{pmatrix} \eta_{00} & \boldsymbol{\eta}^\text{T} \\ \boldsymbol{\eta} & E \end{pmatrix}, \quad (5.15)$$

the tree-level potential (5.10) can be written as

$$V_{\text{THDM}}^0(\widetilde{\mathbf{K}}) = \widetilde{\mathbf{K}}^\text{T} \tilde{\boldsymbol{\xi}} + \widetilde{\mathbf{K}}^\text{T} \tilde{E} \widetilde{\mathbf{K}}. \quad (5.16)$$

The conditions (5.9) for the bilinears can, with the help of

$$\tilde{g} = \text{diag}(1, -\mathbb{1}_3), \quad (5.17)$$

be expressed as

$$K_0 \geq 0, \quad \widetilde{\mathbf{K}}^T \tilde{g} \widetilde{\mathbf{K}} \geq 0. \quad (5.18)$$

We will consider unitary mixing of the two doublets,

$$\varphi'_i = U_{ij} \varphi_j, \quad \text{with } U = (U_{ij}), \quad U^\dagger U = \mathbb{1}_2. \quad (5.19)$$

These basis transformations correspond in terms of bilinears to

$$K'_0 = K_0, \quad K'_a = R_{ab}(U) K_b, \quad (5.20)$$

with  $R_{ab}(U)$  defined by

$$U^\dagger \sigma^a U = R_{ab}(U) \sigma^b. \quad (5.21)$$

It follows that  $R(U) \in SO(3)$ , that is,  $R(U)$  is a proper rotation in three dimensions. We see that the potential (5.10) stays invariant under a change of basis of the bilinears (5.20) if we simultaneously transform the parameters [102]

$$\xi'_0 = \xi_0, \quad \xi'_a = R_{ab} \xi_b, \quad \eta'_{00} = \eta_{00}, \quad \eta'_a = R_{ab} \eta_b, \quad E'_{cd} = R_{ca} E_{ab} R_{bd}^T. \quad (5.22)$$

Note that by a change of basis we can diagonalise the real symmetric matrix  $E$ .

Let us also briefly recall from [102] how electroweak symmetry breaking translates into conditions on the gauge-invariant bilinears. First, we suppose that the potential of the THDM is stable, that is, it is bounded from below. Then at a minimum of the potential (5.4) becomes,

$$\langle \psi \rangle = \begin{pmatrix} v_1^+ & v_1^0 \\ v_2^+ & v_2^0 \end{pmatrix}. \quad (5.23)$$

Here, we have introduced the vacuum expectation values of the components of the doublets, that is,  $v_{1/2}^+ = \langle \varphi_{1/2}^+ \rangle$  and  $v_{1/2}^0 = \langle \varphi_{1/2}^0 \rangle$ .

In case that we have a charge-breaking (CB) minimum, the matrix  $\langle \psi \rangle$ , and therefore also  $\langle \underline{K} \rangle = \langle \psi \rangle \langle \psi \rangle^\dagger$ , has full rank and this translates to the condition (suppressing the angular brackets in the notation indicating the vacuum)

$$\text{CB:} \quad K_0 > 0, \quad K_0^2 - \mathbf{K}^T \mathbf{K} = \widetilde{\mathbf{K}}^T \tilde{g} \widetilde{\mathbf{K}} > 0. \quad (5.24)$$

In contrast, in case we have a neutral, charge-conserving (CC) vacuum, the matrix  $\langle \psi \rangle$  together with  $\langle \underline{K} \rangle$  has to have rank one resulting in

$$\text{CC:} \quad K_0 > 0, \quad K_0^2 - \mathbf{K}^T \mathbf{K} = \widetilde{\mathbf{K}}^T \tilde{g} \widetilde{\mathbf{K}} = 0. \quad (5.25)$$

A vacuum which does not break the electroweak symmetry at all corresponds to

$$K_0 = 0, \quad (5.26)$$

where we also have  $\widetilde{\mathbf{K}} = 0$ .

In case of a charge-conserving minimum we can, by a basis change, achieve that only the component  $\varphi_1^0$  gets a non-vanishing vacuum-expectation value and we set in this basis

$$\langle \varphi_1^{0'} \rangle = v_1^{0'} \equiv \frac{v}{\sqrt{2}} \quad (5.27)$$

with the conventional factor  $1/\sqrt{2}$ . This basis, in which only the neutral component of  $\varphi_1$  gets a non-vanishing vacuum-expectation value, is sometimes called Higgs basis. The bilinears in this basis read  $\widetilde{\mathbf{K}} = (v^2/2, 0, 0, v^2/2)^T$ .

For a stable potential, the minimum can be found from the gradient of the potential. In the case of a charge-breaking minimum, the conditions read

$$\text{CB:} \quad \partial_\mu V \equiv \frac{\partial V}{\partial K^\mu} = 0 \quad (5.28)$$

on the domain (5.24). An unbroken minimum is simply given for vanishing doublets, corresponding to  $K_0 = 0$  and therefore for a vanishing potential. The correct electroweak-symmetry breaking minimum can be found by introducing a Lagrange multiplier  $u$  in order to satisfy the second equation in (5.25), that is, for a minimum with the correct electroweak symmetry breaking,

$$\text{CC:} \quad \partial_\mu V = 2u(\tilde{g}\widetilde{\mathbf{K}})_\mu \quad (5.29)$$

on the domain (5.25).

Let us also recall the dimensionless expressions of the bilinears. First we note that for  $K_0 = 0$  the potential is trivially vanishing. We define for  $K_0 > 0$  [102]

$$k_a = \frac{K_a}{K_0}, \quad a \in \{1, 2, 3\} \quad \text{and} \quad \mathbf{k} = (k_1, k_2, k_3)^T. \quad (5.30)$$

With (5.30) we can write the tree-level potential (5.10) in the form

$$V_{\text{THDM}}^0 = K_0 (\xi_0 + \boldsymbol{\xi}^T \mathbf{k}) + K_0^2 (\eta_{00} + 2\boldsymbol{\eta}^T \mathbf{k} + \mathbf{k}^T E \mathbf{k}) \quad (5.31)$$

defined on the compact domain, as follows from (5.9),

$$|\mathbf{k}| \leq 1. \quad (5.32)$$

We will follow the convention to use Greek indices  $\mu, \nu, \dots \in \{0, \dots, 3\}$  for the Minkowski-type four-vectors, for instance we write  $K_\mu$ . The two doublets themselves are distinguished by Latin indices  $i, j, \dots \in \{1, 2\}$ . For the component fields of the two doublets (5.2) we also use Latin indices  $i, j, \dots \in \{1, \dots, 8\}$ . Let us note that we do not distinguish between upper and lower indices, that is, for instance we have  $K^\mu = K_\mu$  with  $(K^\mu) = (K_\mu) = (K_0, K_1, K_2, K_3)^T$  as well as for the 8-component vector  $\phi^i = \phi_i$ .

### 3 Gauge-invariant scalar mass matrices

The conventional procedure to compute the scalar mass matrices in the THDM is to firstly check that the potential is stable, that is, bounded from below. Secondly, a minimum (typically the global minimum) has to be found. Thirdly, the electroweak symmetry-breaking behaviour of the minimum has to be checked. For a physically acceptable minimum it has to be verified that the electroweak symmetry is broken  $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{em}}$ . At a fixed gauge, the second derivatives of the potential with respect to the excitation fields about the vacuum give then the squared mass matrices.

In contrast, here we want to express the mass matrices in terms of gauge-invariant quantities. The mass matrices are defined as the second derivative of the Lagrangian with respect to the fields. Therefore we have first to establish the connection of the component fields of the doublets (5.2),  $\phi^i$ ,  $i \in \{1, \dots, 8\}$  to the bilinear fields  $K^\mu$ ,  $\mu \in \{0, \dots, 3\}$ .

The gauge invariants, that is, the bilinear fields can be written in terms of the components of the doublets (5.2)

$$K^\mu \equiv \frac{1}{2} \Delta_{ij}^\mu \phi^i \phi^j, \quad i, j \in \{1, \dots, 8\}. \quad (5.33)$$

For instance, in the basis given in (5.2), we find explicitly the four real and symmetric  $8 \times 8$  matrices  $\Delta_{ij}^\mu$ :

$$\begin{aligned} \Delta^0 &= \begin{pmatrix} \mathbb{1}_2 & & & \\ & \mathbb{1}_2 & & \\ & & \mathbb{1}_2 & \\ & & & \mathbb{1}_2 \end{pmatrix}, & \Delta^1 &= \begin{pmatrix} & & \mathbb{1}_2 & \\ & & & \mathbb{1}_2 \\ \mathbb{1}_2 & & & \\ & \mathbb{1}_2 & & \end{pmatrix}, \\ \Delta^2 &= \begin{pmatrix} & & & \mathbb{1}_2 \\ & & -\mathbb{1}_2 & \\ & -\mathbb{1}_2 & -\mathbb{1}_2 & \\ \mathbb{1}_2 & & & \end{pmatrix}, & \Delta^3 &= \begin{pmatrix} \mathbb{1}_2 & & & \\ & \mathbb{1}_2 & & \\ & & -\mathbb{1}_2 & \\ & & & -\mathbb{1}_2 \end{pmatrix}, \end{aligned} \quad (5.34)$$

where  $\mathbb{1}_2$  is the  $2 \times 2$  unit matrix and the blank spaces have to be filled with zeros. It should be noted that the matrices  $\Delta^{1,2,3}$  resemble the Pauli matrices  $\sigma^{1,2,3}$ , respectively, in that they satisfy a Clifford algebra  $\{\Delta^a, \Delta^b\} = 2\delta^{ab}\mathbb{1}_8$  (but not a Lie algebra).

The connection between the bilinears and the component fields are therefore given by the four  $8 \times 4$  matrices  $\Gamma$ , defined as

$$\Gamma_i^\mu \equiv \frac{\partial K^\mu}{\partial \phi^i} = \partial_i K^\mu = \Delta_{ij}^\mu \phi^j. \quad (5.35)$$

Note that the Greek indices correspond to the gauge invariants, whereas the Latin indices correspond to the component fields  $\phi^i$ .

We would now like to derive the algebra of the four  $8 \times 8$  matrices  $\Delta_{ij}^\mu$ . To this end we compute

$$(\Gamma^2)^{\mu\nu} = (\Gamma^T \Gamma)^{\mu\nu} = \Gamma_i^\mu \Gamma_i^\nu = \Delta_{ij}^\mu \Delta_{ik}^\nu \phi^j \phi^k = \frac{1}{2} \{\Delta^\mu, \Delta^\nu\}_{jk} \phi^j \phi^k, \quad (5.36)$$

where  $\{\Delta^\mu, \Delta^\nu\}$  is the anti-commutator of the  $\Delta$ -matrices. From (5.36), we observe that  $\Gamma^2$  is a gauge-independent object (since it only carries Greek bilinear-field indices) and that it depends quadratically on the fields. Therefore, a rank-3 tensor  $T_\lambda^{\mu\nu}$  can be defined such that

$$(\Gamma^2)^{\mu\nu} = T_\lambda^{\mu\nu} K^\lambda. \quad (5.37)$$

It then follows from Eqs. (5.33), (5.36) and (5.37) that

$$\{\Delta^\mu, \Delta^\nu\}_{ij} = T_\lambda^{\mu\nu} \Delta_{ij}^\lambda. \quad (5.38)$$

Observing that the symmetric  $\Delta$ -matrices obey a closure relation

$$\Delta_{ij}^\mu \Delta_{ij}^\nu = \Delta_{ij}^\mu \Delta_{ji}^\nu = \text{Tr}(\Delta^\mu \Delta^\nu) = 8\delta^{\mu\nu}, \quad (5.39)$$

we can provide an explicit expression<sup>1</sup> for  $T_\lambda^{\mu\nu}$ :

$$T_\lambda^{\mu\nu} = \frac{1}{8} \text{Tr}(\{\Delta^\mu, \Delta^\nu\} \Delta_\lambda). \quad (5.40)$$

In particular, we obtain

$$\Gamma^2 = 2 \begin{pmatrix} K_0 & K_1 & K_2 & K_3 \\ K_1 & K_0 & 0 & 0 \\ K_2 & 0 & K_0 & 0 \\ K_3 & 0 & 0 & K_0 \end{pmatrix} = 2 \begin{pmatrix} K_0 & \mathbf{K}^T \\ \mathbf{K} & K_0 \mathbb{1}_3 \end{pmatrix}. \quad (5.41)$$

Since  $\det(\Gamma^2) = 16K_0^2 (K_0^2 - \mathbf{K}^2) = 16K_0^2 \widetilde{\mathbf{K}}^T \widetilde{g} \widetilde{\mathbf{K}}$ , the electroweak symmetry-breaking behaviour depends on the nullspace structure of  $\Gamma^2$ . As discussed in the last section, a charge-breaking minimum satisfying (5.24) leads to a complete breakdown of the  $SU(2)_L \times U(1)_Y$  group, whereas (5.25) leads to the viable, charge-conserving  $SU(2)_L \times U(1)_Y \rightarrow U(1)_{em}$  breaking. We now consider the scalar mass matrix with respect to the charge-breaking (CB) and charge-conserving (CC) case separately.

The field-dependent scalar mass matrix is expressed in our formalism as

$$(M_s^2)_{ij} = \partial_i \partial_j V = \partial_i (\Gamma_j^\mu \partial_\mu V) = \Delta_{ij}^\mu \partial_\mu V + \Gamma_i^\mu \Gamma_j^\nu \partial_\mu \partial_\nu V. \quad (5.42)$$

With the definition

$$\mathcal{M}_{\mu\nu} = \partial_\mu \partial_\nu V \quad (5.43)$$

and writing  $\mathcal{M} = (\mathcal{M}_{\mu\nu})$  as well as  $\Delta^\mu = (\Delta_{ij}^\mu)$  we express (5.42) in matrix form,

$$M_s^2 = \Delta^\mu \partial_\mu V + \Gamma \mathcal{M} \Gamma^T. \quad (5.44)$$

<sup>1</sup>Note that an alternative expression can be given in terms of Pauli matrices:

$$T_\lambda^{\mu\nu} = \frac{1}{2} \text{Tr}(\{\sigma^\mu, \sigma^\nu\} \sigma_\lambda).$$

With this form we have achieved an intuitive understanding of the scalar spectrum at a minimum of the potential. In particular, for a charge-breaking minimum, where  $\partial_\mu V = 0$  according to (5.28), the  $8 \times 8$  scalar mass matrix reduces to

$$M_s^2 \stackrel{CB}{=} \Gamma \mathcal{M} \Gamma^T \quad (5.45)$$

and can only possess four non-zero eigenvalues (*i.e.* as many as the number of independent gauge-invariant scalar operators). The four remaining massless states correspond to the would-be Goldstone bosons associated with the four broken generators of the  $SU(2)_L \times U(1)_Y$  gauge group. On the other hand, as we will show below, at a charge-conserving minimum, where  $\partial_\mu V = 2u(\tilde{g}\tilde{\mathbf{K}})_\mu$  or  $\det(\Gamma^2) = 0$  according to (5.29), the  $\Gamma \mathcal{M} \Gamma^T$  matrix possesses at most three non-zero eigenvalues and the full scalar mass matrix

$$M_s^2 \stackrel{CC}{=} 2u(\tilde{g}\tilde{\mathbf{K}})_\mu \Delta^\mu + \Gamma \mathcal{M} \Gamma^T \quad (5.46)$$

contains five massive states and three Goldstone modes corresponding to the broken generators of  $SU(2)_L \times U(1)_Y \rightarrow U(1)_{em}$ . Now we want to consider an orthogonal rotation of the component fields,

$$\hat{\phi}^i = U^{ij} \phi^j \quad \text{with} \quad U^{ij} U^{kj} = \delta^{ik}, \quad i, j, k \in \{1, \dots, 8\}. \quad (5.47)$$

In matrix notation we write simply  $\hat{\phi} = U \phi$  with  $U^T U = \mathbb{1}_8$ . By an orthogonal rotation - only applied to the component fields, leaving the gauge invariants  $\tilde{\mathbf{K}}$  unchanged - we can always achieve a form for the  $8 \times 4$  matrix  $\Gamma$  with the first four rows vanishing<sup>2</sup>,

$$\hat{\Gamma} = U_c \Gamma = \begin{pmatrix} 0_{4 \times 4} \\ \gamma \end{pmatrix}, \quad (5.48)$$

where we will call  $U_c$  a canonical rotation and the corresponding bases, where  $\hat{\Gamma}$  has this form, canonical bases. The matrix  $\gamma$  is obviously a  $4 \times 4$  matrix. We get

$$\Gamma^2 = \Gamma^T \Gamma = \hat{\Gamma}^T \hat{\Gamma} = \gamma^T \gamma, \quad (5.49)$$

meaning that the  $\gamma$  matrix can be obtained from a Cholesky-like<sup>3</sup> decomposition of  $\Gamma^2$ , and that the resulting expression only depends on gauge invariants, *i.e.* the bilinear fields. Requiring  $\gamma$  to be lower-triangular fixes its components uniquely, and we find

$$\gamma = \sqrt{2K_0} \begin{pmatrix} \sqrt{1 - \mathbf{k}^T \mathbf{k}} & \mathbf{0}^T \\ \mathbf{k} & \mathbb{1}_3 \end{pmatrix}. \quad (5.50)$$

In a canonical basis we can always express  $\hat{\Gamma}$  in terms of the field components  $\hat{\phi} = U_c \phi$ :

$$\hat{\Gamma} = U_c \Gamma = U_c \Delta \phi = U_c \Delta U_c^T \hat{\phi} = \hat{\Delta} \hat{\phi}, \quad (5.51)$$

where  $\hat{\Delta} = U_c \Delta U_c^T$  is simply obtained from the canonical orthogonal rotation  $U_c$ .

Moreover, two canonical bases can always be related by an orthogonal transformation since

$$\begin{pmatrix} U_{4 \times 4} & 0_{4 \times 4} \\ 0_{4 \times 4} & \mathbb{1}_4 \end{pmatrix} \hat{\Gamma} = \hat{\Gamma}. \quad (5.52)$$

We want to show now that in a canonical basis the electroweak-symmetry breaking behaviour of the scalar mass matrix at the vacuum becomes manifest. First, at a charge-breaking minimum, Eq. (5.45) directly yields<sup>4</sup>

$$\hat{M}_s^2 \stackrel{CB}{=} \hat{\Gamma} \mathcal{M} \hat{\Gamma}^T = \begin{pmatrix} 0_{4 \times 4} & 0_{4 \times 4} \\ 0_{4 \times 4} & \gamma \mathcal{M} \gamma^T \end{pmatrix}. \quad (5.53)$$

<sup>2</sup>The matrix  $\Gamma$  consists of four 8-component vectors spanning a 4-dimensional hypersurface. It is always possible to choose a basis such that the 8-4 dimensional space orthogonal to the hypersurface is represented by the first four basis vectors such that the corresponding components of the four 8-component vectors are zero.

<sup>3</sup>The usual Cholesky decomposition would have made  $\gamma$  an upper-triangular matrix, while it is in our case lower-triangular.

<sup>4</sup>Here and in the remainder of the section, hatted quantities are understood to be expressed in a canonical basis.



Clearly, the mass matrix expressed in this basis takes a block-diagonal form, where the Goldstone and physical sectors have been manifestly disentangled. In addition, the number of massive states, the number of gauge singlets and the number of independent vacuum-expectation values all equal the number of independent gauge invariant operators, namely four. This equality is characteristic of a maximal breakdown of the gauge group<sup>5</sup>.

Turning to the case of a charge-conserving breaking, we show in App. A that the mass matrix (5.46) in a canonical basis takes a block-diagonal form,

$$\begin{aligned}\widehat{M}_s^2 &\stackrel{CC}{=} 2u (\widetilde{g}\widetilde{\mathbf{K}})_\mu \widehat{\Delta}^\mu + \widehat{\Gamma}\mathcal{M}\widehat{\Gamma}^\text{T} \\ &= 2u \begin{pmatrix} A_{55} & 0_{5\times 3} \\ 0_{3\times 5} & B_{33} \end{pmatrix} + \begin{pmatrix} 0_{5\times 5} & 0_{5\times 3} \\ 0_{3\times 5} & \gamma_3\mathcal{M}\gamma_3^\text{T} \end{pmatrix} = \begin{pmatrix} \widehat{\mathcal{M}}_{\text{CC}}^2 & 0_{5\times 3} \\ 0_{3\times 5} & \widehat{\mathcal{M}}_{\text{neutral}}^2 \end{pmatrix},\end{aligned}\quad (5.54)$$

where  $\widehat{\mathcal{M}}_{\text{CC}}^2$  accounts for mixed contributions between massive and Goldstone states, both possibly charged under the new residual subgroup (here,  $U(1)_{\text{em}}$ ). The matrix  $\gamma_3$ , which appears in (5.54) is defined to be the non-vanishing, lower  $3 \times 4$  block of the matrix  $\gamma$  evaluated at a charge-conserving minimum where  $1 - \mathbf{k}^2 = 0$ :

$$\gamma \stackrel{CC}{=} \sqrt{2K_0} \begin{pmatrix} 0 & \mathbf{0}^\text{T} \\ \mathbf{k} & \mathbb{1}_3 \end{pmatrix} \equiv \begin{pmatrix} 0_{1\times 4} \\ \gamma_3 \end{pmatrix}.\quad (5.55)$$

The matrices  $A_{55}$  and  $B_{33}$  read explicitly

$$A_{55} = 2K_0 \begin{pmatrix} 0 & \mathbf{0}^\text{T} & 0 \\ \mathbf{0} & \mathbf{k}\mathbf{k}^\text{T} & \mathbf{0} \\ 0 & \mathbf{0}^\text{T} & 1 \end{pmatrix}, \quad B_{33} = 2K_0 (\mathbb{1}_3 - \mathbf{k}\mathbf{k}^\text{T}) = -\gamma_3\widetilde{g}\gamma_3^\text{T}.\quad (5.56)$$

By a further transformation from one canonical basis to another we can completely disentangle the electrically neutral from the electrically charged contributions,

$$\widehat{M}_s^2 \stackrel{CC}{=} \begin{pmatrix} 0_{3\times 3} & & \\ & \widehat{\mathcal{M}}_{\text{charged}}^2 & \\ & & \widehat{\mathcal{M}}_{\text{neutral}}^2 \end{pmatrix}.\quad (5.57)$$

In App. A we show in detail how we arrive at the form of the scalar mass matrix (5.57). Similarly, we show that the charged mass matrix can be diagonalised,

$$\bar{\mathcal{M}}_{\text{charged}}^2 = \text{diag}(m_{H^\pm}^2, m_{H^\pm}^2) = \text{diag}(4uK_0, 4uK_0),\quad (5.58)$$

and the neutral scalar mass matrix reads

$$\widehat{\mathcal{M}}_{\text{neutral}}^2 = \gamma_3 (\mathcal{M} - 2u\widetilde{g}) \gamma_3^\text{T}.\quad (5.59)$$

Having found the scalar mass matrices for the charge-conserving minimum, we go ahead and diagonalise the neutral part (5.59). The similarity transformation to diagonalise the real symmetric matrix (5.59) corresponds to a change of basis of the bilinears. Let us denote with  $R$  the corresponding rotation in the 3-dimensional bilinear field space, defined as

$$\bar{\mathcal{M}}_{\text{neutral}}^2 = R\widehat{\mathcal{M}}_{\text{neutral}}^2 R^\text{T} = \text{diag}(m_1^2, m_2^2, m_3^2).\quad (5.60)$$

As shown in Sec. 2, by this basis transformation the bilinears transform as

$$\bar{K}_0 = K_0, \quad \bar{\mathbf{K}} = R\mathbf{K}.\quad (5.61)$$

With respect to the four vectors we write also

$$\widetilde{\bar{\mathbf{K}}} = \widetilde{R}\widetilde{\mathbf{K}} \quad \text{with} \quad \widetilde{R} = \begin{pmatrix} 1 & \mathbf{0}^\text{T} \\ \mathbf{0} & R \end{pmatrix},\quad (5.62)$$

<sup>5</sup>Generally speaking, *maximally broken* does not mean that the residual gauge group is trivial, but instead that the number of gauge singlets is maximal given a specific field content. In the present case however,  $SU(2)_L \times U(1)_Y$  is indeed broken down to the trivial group.

that is,  $R$  is a  $3 \times 3$  matrix, whereas  $\tilde{R}$  a  $4 \times 4$  matrix. The dimensionless bilinears transform as  $\bar{\mathbf{k}} = R\mathbf{k}$ . Similar, we find for  $\Gamma_i^\mu$  and  $\Delta_{ij}^\mu$  the following transformation behaviour under a change of basis:

$$\bar{\Gamma}_i^\mu = \tilde{R}_\nu^\mu \hat{\Gamma}_i^\nu, \quad \bar{\Delta}_{ij}^\mu = \tilde{R}_\nu^\mu \hat{\Delta}_{ij}^\nu \quad \text{with } (\tilde{R}_\nu^\mu) = \tilde{R}. \quad (5.63)$$

After applying this rotation in the bilinear field space, we obtain the expression of  $M_s^2$  in a basis which will be referred to as the *mass basis* in the following. In this basis, the scalar mass matrix finally reads

$$\bar{M}_s^2 = \text{diag} (0, 0, 0, m_{H^\pm}^2, m_{H^\pm}^2, m_1^2, m_2^2, m_3^2). \quad (5.64)$$

We finally note that the vacuum-expectation value of the scalar multiplet  $\phi$ , noted  $\langle \phi \rangle$ , can readily be computed in the mass basis from the relation

$$\bar{\Gamma}_i^\mu = \bar{\Delta}_{ij}^\mu \langle \bar{\phi} \rangle^j, \quad (5.65)$$

and using the expressions of  $\bar{\Gamma}$  and  $\bar{\Delta}$  given in Appendix A. Explicitly, one has

$$\langle \bar{\phi} \rangle = \sqrt{2K_0} \begin{pmatrix} 0_{4 \times 1} \\ \alpha \\ \bar{\mathbf{k}} \end{pmatrix} \quad (5.66)$$

with  $\alpha = \sqrt{1 - \bar{\mathbf{k}}^T \bar{\mathbf{k}}}$  vanishing at a charge-conserving minimum.

Let us briefly recap how we get the mass spectrum in any THDM for the charge-conserving vacuum, that is, a vacuum respecting the observed electroweak symmetry breaking. First, we express the potential  $V$  in terms of gauge-invariant bilinears employing (5.8). From the potential we compute  $\mathcal{M} = \partial_\mu \partial_\nu V$ . In turn we can compute  $\widehat{\mathcal{M}}_{\text{neutral}}^2$  in (5.59) with  $\gamma_3$  given in (5.55). Diagonalising the matrix  $\widehat{\mathcal{M}}_{\text{neutral}}^2$  according to (5.60) we get the scalar mass squared eigenvalues. The degenerate charged Higgs-boson masses squared follow directly from the minimum of the potential from (5.58).

We emphasise that we have not specified the explicit form of the potential  $V$  in this section. Indeed, it can be any THDM potential, for instance the tree-level potential (5.16) or a higher loop effective THDM potential. All the expressions given above in this section remain valid. This holds in particular for the scalar mass spectrum given in (5.58) and (5.60). The charged Higgs-boson mass squared is  $m_{H^\pm}^2 = 4uK_0$  for any THDM to any loop order – a quite remarkable result: From the minimum of the potential, that is, from the solutions of (5.28)-(5.29), we get  $K_0$  and  $u$  at the minimum directly giving the charged Higgs-boson mass squared. These results illustrate the benefits of working with gauge invariants in general, and in particular in studying THDMs using the bilinear formalism. We emphasise that we have derived the mass matrices in a completely gauge-invariant way.

### 3.1 Tree-level scalar mass matrix

Eventually, we would like to explicitly show our result in the case of a general THDM tree-level potential. The charged scalar masses squared are

$$m_{H^\pm}^2 = 4uK_0 \quad (5.67)$$

as discussed before. Furthermore, the neutral  $3 \times 3$  matrix (5.59) becomes in this case

$$\widehat{\mathcal{M}}_{\text{neutral}}^2 = 4K_0 [\eta_{00} \mathbf{k} \mathbf{k}^T + \boldsymbol{\eta} \mathbf{k}^T + \mathbf{k} \boldsymbol{\eta}^T + E + u(\mathbb{1}_3 - \mathbf{k} \mathbf{k}^T)]. \quad (5.68)$$

which is explicitly gauge invariant. This real symmetric matrix can be diagonalised with the rotation matrix  $R$ , (5.60). Under this change of basis, the bilinears transform as shown in (5.61) and the parameters of the tree-level potential transform as, (5.22),

$$\bar{\xi}_0 = \xi_0, \quad \bar{\boldsymbol{\xi}} = R\boldsymbol{\xi}, \quad \bar{\eta}_{00} = \eta_{00}, \quad \bar{\boldsymbol{\eta}} = R\boldsymbol{\eta}, \quad \bar{E} = RER^T. \quad (5.69)$$

Therefore, the neutral mass matrix becomes

$$\bar{\mathcal{M}}_{\text{neutral}}^2 = 4K_0 [\eta_{00} \bar{\mathbf{k}} \bar{\mathbf{k}}^T + \bar{\boldsymbol{\eta}} \bar{\mathbf{k}}^T + \bar{\mathbf{k}} \bar{\boldsymbol{\eta}}^T + \bar{E} + u(\mathbb{1}_3 - \bar{\mathbf{k}} \bar{\mathbf{k}}^T)]. \quad (5.70)$$

Let us note that in practical calculations we can use (5.70) and (5.67) together with the parameters  $\eta_{00}$  and  $\bar{\eta}$  to fix the parameter matrix  $\bar{E}$  of the THDM in terms of the scalar masses. The scalar mass squared matrix for the tree level case, given in (5.70), agrees with the known result in the Higgs basis (where  $\bar{k} = (0, 0, 1)^T$ ); see for instance [102]. In particular we get the squared masses of the scalar sector as known from the conventional formalism.

## 4 Scalar trilinear and quartic interactions

Having found the scalar squared mass matrix in terms of gauge invariants in the last section, we can now proceed and express the scalar trilinear and quartic couplings in a gauge-invariant way. The connection between the bilinears and the component fields (5.1) is given in terms of the  $\Gamma$  matrices (5.35). From these  $\Gamma$  matrices we can express the cubic and quartic couplings in terms of the squared mass matrix (5.42). We emphasise that this matrix can be any THDM mass matrix, not restricted to the tree level THDM.

We find for the cubic and quartic interactions in any THDM

$$\lambda_{ijk} = (\partial_i M_s^2)^{jk} = \left( \Delta_{ij}^\mu \Gamma_k^\nu + \Delta_{ik}^\mu \Gamma_j^\nu + \Delta_{jk}^\mu \Gamma_i^\nu \right) \mathcal{M}_{\mu\nu}, \quad (5.71)$$

$$\lambda_{ijkl} = (\partial_i \partial_j M_s^2)^{kl} = \left( \Delta_{ij}^\mu \Delta_{kl}^\nu + \Delta_{ik}^\mu \Delta_{jl}^\nu + \Delta_{il}^\mu \Delta_{jk}^\nu \right) \mathcal{M}_{\mu\nu}. \quad (5.72)$$

In the mass basis (5.64), this yields

$$\bar{\lambda}_{ijk} = \left( \bar{\Delta}_{ij}^\mu \bar{\Gamma}_k^\nu + \bar{\Delta}_{ik}^\mu \bar{\Gamma}_j^\nu + \bar{\Delta}_{jk}^\mu \bar{\Gamma}_i^\nu \right) \bar{\mathcal{M}}_{\mu\nu}, \quad (5.73)$$

$$\bar{\lambda}_{ijkl} = \left( \bar{\Delta}_{ij}^\mu \bar{\Delta}_{kl}^\nu + \bar{\Delta}_{ik}^\mu \bar{\Delta}_{jl}^\nu + \bar{\Delta}_{il}^\mu \bar{\Delta}_{jk}^\nu \right) \bar{\mathcal{M}}_{\mu\nu}. \quad (5.74)$$

The expressions of  $\bar{\Delta}$  and  $\bar{\Gamma}$  can be inferred from (5.63) in the last section. The matrix  $\mathcal{M}$  is defined in (5.43) and  $\bar{\mathcal{M}}$  follows from  $\mathcal{M}$  by (5.59) and then (5.60).

In order to get simple expressions, we first want to introduce new indices for the different scalars appearing in (5.64). For the  $8 \times 8$  matrix (5.64), we have  $(\bar{M}_s^2)_{ij}$  with  $i, j \in \{1, \dots, 8\}$ . The entries  $i, j = 1$  correspond to the neutral Goldstone mode  $G^0$ , the two following entries, that is,  $i, j \in \{2, 3\}$  to the charged Goldstone modes  $G^\pm$ , whereas  $i, j \in \{4, 5\}$  correspond to the charged Higgs bosons  $H^\pm$ , and the entries  $i, j \in \{6, 7, 8\}$  to the three neutral scalars – in the CP conserving case usually denoted by  $h, H, A$  or  $h^0, H^0, A^0$ .

We use in the following the index  $G^0 = 1$  corresponding to  $i = 1$ , the indices  $G_p^\pm, G_q^\pm \in \{1, 2\}$  referring to the second and third index  $i, j, \dots \in \{2, 3\}$ . This means that the indices  $G_p^\pm, G_q^\pm$  are shifted by one unit down with respect to the indices  $i, j$ . Similar, for the charged Higgs bosons we use the indices  $H_p^\pm, H_q^\pm \in \{1, 2\}$  referring to the original indices  $\{4, 5\}$ , that is, shifted by three units. Eventually, we use the indices  $s, t, \dots \in \{1, 2, 3\}$  corresponding to  $i, j, \dots \in \{6, 7, 8\}$ , hence, shifted five units. With this notation, we can write down all the non-vanishing components of the matrices  $\bar{\Gamma}$  and  $\bar{\Delta}$ :

$$\bar{\Gamma}_s^0 = \sqrt{2K_0} \bar{k}^s, \quad \bar{\Gamma}_s^a = \sqrt{2K_0} \delta^{as}, \quad (5.75)$$

$$\bar{\Delta}_{st}^0 = \delta^{st}, \quad \bar{\Delta}_{st}^a = \delta^{sa} \bar{k}^t + \delta^{ta} \bar{k}^s - \delta^{st} \bar{k}^a, \quad (5.76)$$

$$\bar{\Delta}_{G_p^\pm G_q^\pm}^\mu = \frac{\bar{K}^\mu}{K_0} \delta^{G_p^\pm G_q^\pm}, \quad \bar{\Delta}_{H_p^\pm H_q^\pm}^\mu = \frac{\bar{g} \bar{K}^\mu}{K_0} \delta^{H_p^\pm H_q^\pm}, \quad (5.77)$$

$$\bar{\Delta}_{G^0 G^0}^\mu = \frac{\bar{K}^\mu}{K_0}, \quad \bar{\Delta}_{sG^0}^a = -\varepsilon_{ast} \bar{k}^t. \quad (5.78)$$

We recall that the index  $a \in \{1, 2, 3\}$  denotes the three spatial components of Minkowski-type vectors with indices  $\mu, \nu, \dots \in \{0, 1, 2, 3\}$ . Similar, we have non-vanishing  $\bar{\Delta}^\mu$  matrices mixing the charged Higgs and Goldstone components. The corresponding entries are not uniquely fixed due to the possibility of rotating the corresponding real components (*i.e.* performing phase redefinitions of the charged fields). However, one can always write

$$\bar{\Delta}_{G_p^\pm H_q^\pm}^a = \begin{pmatrix} x_a & -y_a \\ y_a & x_a \end{pmatrix}^{G_p^\pm H_q^\pm} \equiv (\chi_a)_{G_p^\pm H_q^\pm}, \quad (5.79)$$

where the 3-vectors  $\mathbf{x}$  and  $\mathbf{y}$  are defined such that  $(\mathbf{x}, \mathbf{y}, \bar{\mathbf{k}})$  constitutes an orthonormal basis of the 3-dimensional bilinear field space, *i.e.*

$$\mathbf{x}^T \bar{\mathbf{k}} = \mathbf{y}^T \bar{\mathbf{k}} = \mathbf{x}^T \mathbf{y} = 0 \quad \text{and} \quad \mathbf{x}^T \mathbf{x} = \mathbf{y}^T \mathbf{y} = \bar{\mathbf{k}}^T \bar{\mathbf{k}} = 1, \quad (5.80)$$

oriented such that  $\mathbf{x} \times \mathbf{y} = \bar{\mathbf{k}}$ . In fact,  $\mathbf{x}$  and  $\mathbf{y}$  characterise the orthogonal transformation  $R_H$  allowing to rotate  $\bar{\mathbf{k}}$  to the Higgs basis which was used in App. A to diagonalise the mass matrix. Namely, one has

$$R_H = \begin{pmatrix} -\mathbf{y}^T \\ \mathbf{x}^T \\ \bar{\mathbf{k}}^T \end{pmatrix} = \begin{pmatrix} -y_1 & -y_2 & -y_3 \\ x_1 & x_2 & x_3 \\ \bar{k}_1 & \bar{k}_2 & \bar{k}_3 \end{pmatrix}, \quad R_H \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (5.81)$$

which, from (5.80) is manifestly orthogonal. In turn, the reason why  $\mathbf{x}$  and  $\mathbf{y}$  are not uniquely defined is clear: there exist infinitely many orthogonal transformations  $R'_H$  rotating  $\mathbf{k}$  to the Higgs basis, related by arbitrary rotations around the  $z$ -axis.

While the matrices  $\chi_a$  are themselves not uniquely defined, it is useful to observe that since

$$\mathbf{x} \mathbf{x}^T + \mathbf{y} \mathbf{y}^T + \bar{\mathbf{k}} \bar{\mathbf{k}}^T = \mathbb{1}_3 \quad \text{and} \quad \mathbf{x} \times \mathbf{y} = \bar{\mathbf{k}}, \quad (5.82)$$

we obtain

$$x_a x_b + y_a y_b = \delta_{ab} - \bar{k}_a \bar{k}_b \quad \text{and} \quad x_a y_b - x_b y_a = \varepsilon_{abc} \bar{k}^c. \quad (5.83)$$

Therefore, for any two matrices  $\chi_a$  and  $\chi_b$ , one has

$$\chi_a \chi_b^T = \begin{pmatrix} \delta_{ab} - \bar{k}_a \bar{k}_b & \varepsilon_{abc} \bar{k}^c \\ -\varepsilon_{abc} \bar{k}^c & \delta_{ab} - \bar{k}_a \bar{k}_b \end{pmatrix}, \quad (5.84)$$

independently of the basis chosen to express  $\mathbf{x}$  and  $\mathbf{y}$ .

#### 4.1 Tree level scalar cubic and quartic interactions

Eventually, we would like to give the explicit expressions for the tree-level cubic and quartic interactions in gauge-invariant form. These interactions are derived from the tree-level potential.

The neutral scalar mass matrix in the mass basis has been computed already in (5.70). Together with the expressions (5.75)–(5.79) we get the cubic and quartic interactions (5.73), and (5.74), respectively. It turns out to be convenient to define the following 3-vectors:

$$\bar{f}^a = 8K_0 (\eta_{00} \bar{k}^a + \bar{\eta}^a) - \bar{k}^a m_a^2, \quad (5.85)$$

$$\bar{f}_{\pm}^a = 8K_0 [(\eta_{00} - u) \bar{k}^a + \bar{\eta}^a] = 8K_0 (\eta_{00} \bar{k}^a + \bar{\eta}^a) - 2\bar{k}^a m_{H\pm}^2. \quad (5.86)$$

Let us note that an index appearing with a mass does not imply summation, for instance in the expression  $\bar{k}^a m_a^2$ .

The part of the Lagrangian containing the scalar cubic interactions can be written, after spontaneous symmetry breaking and rotation to the mass basis, as

$$\begin{aligned} -\mathcal{L} \supset & \frac{1}{3!} \bar{\lambda}_{stu} h^s h^t h^u + \frac{1}{2} \bar{\lambda}_{stG^0} h^s h^t G^0 + \frac{1}{2} \bar{\lambda}_{sG^0 G^0} h^s G^0 G^0 \\ & + \bar{\lambda}_{sH^\pm H^\pm} h^s H^- H^+ + \bar{\lambda}_{sG^\pm G^\pm} h^s G^- G^+ + [\bar{\lambda}_{sG^\pm H^\pm} h^s G^- H^+ + \text{h.c.}] . \end{aligned} \quad (5.87)$$

The complex charged fields  $H^\pm$ ,  $G^\pm$  are given by

$$H^\pm = \frac{e^{\pm i\omega_H}}{\sqrt{2}} (H_1^\pm \pm iH_2^\pm), \quad (5.88)$$

$$G^\pm = \frac{e^{\pm i\omega_G}}{\sqrt{2}} (G_1^\pm \pm iG_2^\pm), \quad (5.89)$$

where the inclusion of  $\omega_H$  and  $\omega_G$  reflects the possibility of performing arbitrary phase redefinitions of the complex fields. The analytic expressions for the various cubic couplings are readily derived through (5.73),

(5.75)–(5.79) and put into the form

$$\begin{aligned}
\bar{\lambda}_{stu} &= \frac{1}{\sqrt{2K_0}} \left\{ (\delta^{st} - \bar{k}^s \bar{k}^t) \bar{f}_\pm^u + \delta^{st} \bar{k}^u (m_s^2 + m_t^2 - m_u^2) \right\} + (s \leftrightarrow u) + (t \leftrightarrow u), \\
\bar{\lambda}_{stG^0} &= \varepsilon_{stu} \frac{m_t^2 - m_s^2}{\sqrt{2K_0}} \bar{k}^u, \\
\bar{\lambda}_{sG^0G^0} &= \frac{m_s^2}{\sqrt{2K_0}} \bar{k}^s, \\
\bar{\lambda}_{sH^\pm H^\pm} &= \frac{\bar{f}^s}{\sqrt{2K_0}}, \\
\bar{\lambda}_{sG^\pm G^\pm} &= \frac{m_s^2}{\sqrt{2K_0}} \bar{k}^s, \\
\bar{\lambda}_{sG^\pm H^\pm} &= \frac{m_s^2 - m_{H^\pm}^2}{\sqrt{2K_0}} e^{i(\omega_G - \omega_H)} (x^s + iy^s).
\end{aligned} \tag{5.90}$$

From the last expression, we see that a phase redefinition involving  $\omega_G - \omega_H$  can always be compensated by a redefinition of the vectors  $\mathbf{x}$  and  $\mathbf{y}$  already mentioned above. More precisely, by defining a new pair of vectors  $\mathbf{x}'$  and  $\mathbf{y}'$  such that

$$\begin{pmatrix} x'_i \\ y'_i \end{pmatrix} = \begin{pmatrix} \cos(\omega_G - \omega_H) & -\sin(\omega_G - \omega_H) \\ \sin(\omega_G - \omega_H) & \cos(\omega_G - \omega_H) \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \tag{5.91}$$

the  $h_s G^\pm H^\pm$  vertex is simply rewritten as

$$\bar{\lambda}_{sG^\pm H^\pm} = \frac{m_s^2 - m_{H^\pm}^2}{\sqrt{2K_0}} (x'_s + iy'_s). \tag{5.92}$$

Therefore, here and in the following, we shall for clarity drop the arbitrary phases  $\omega_{G,H}$  from the expressions of the tree-level vertices without any loss of generality. Instead, one should keep in mind that vertices involving two distinct charged fields can only be defined up to unphysical phase redefinitions.

The part of the Lagrangian corresponding to the scalar quartic interactions reads

$$\begin{aligned}
-\mathcal{L} \supset & \frac{1}{4!} \bar{\lambda}_{stuv} h^s h^t h^u h^v + \frac{1}{3!} \bar{\lambda}_{stG^0} h^s h^t h^u G^0 + \frac{1}{4} \bar{\lambda}_{stG^0G^0} h^s h^t G^0 G^0 \\
& + \frac{1}{3!} \bar{\lambda}_{sG^0G^0G^0} h^s G^0 G^0 G^0 + \frac{1}{4!} \bar{\lambda}_{G^0G^0G^0G^0} G^0 G^0 G^0 G^0 \\
& + \frac{1}{2} \bar{\lambda}_{stH^\pm H^\pm} h^s h^t H^- H^+ + \bar{\lambda}_{sG^0H^\pm H^\pm} h^s G^0 H^- H^+ + \frac{1}{2} \bar{\lambda}_{G^0G^0H^\pm H^\pm} G^0 G^0 H^- H^+ \\
& + \frac{1}{2} \bar{\lambda}_{stG^\pm G^\pm} h^s h^t G^- G^+ + \bar{\lambda}_{sG^0G^\pm G^\pm} h^s G^0 G^- G^+ + \frac{1}{2} \bar{\lambda}_{G^0G^0G^\pm G^\pm} G^0 G^0 G^- G^+ \\
& + \left[ \frac{1}{2} \bar{\lambda}_{stG^\pm H^\pm} h^s h^t G^- H^+ + \bar{\lambda}_{sG^0G^\pm H^\pm} h^s G^0 G^- H^+ + \frac{1}{2} \bar{\lambda}_{G^0G^0G^\pm H^\pm} G^0 G^0 G^- H^+ + \text{h.c.} \right] \\
& + \frac{1}{4} \bar{\lambda}_{H^\pm H^\pm H^\pm H^\pm} H^- H^+ H^- H^+ + \bar{\lambda}_{G^\pm G^\pm H^\pm H^\pm} G^- G^+ H^- H^+ \\
& + \frac{1}{4} \bar{\lambda}_{G^\pm G^\pm G^\pm G^\pm} G^- G^+ G^- G^+ \\
& + \left[ \frac{1}{2} \bar{\lambda}_{G^\pm H^\pm H^\pm H^\pm} G^- H^+ H^- H^+ + \frac{1}{2} \bar{\lambda}_{G^\pm G^\pm G^\pm H^\pm} G^- G^+ G^- H^+ + \text{h.c.} \right].
\end{aligned} \tag{5.93}$$

The analytic expressions for the various quartic couplings in gauge-invariant form can be derived using Eqs. (5.74), (5.75)–(5.79) and are given in App. B.1.

## 5 Gauge sector

### 5.1 Definitions

Turning to the gauge sector, we are interested in expressing all relevant quantities (gauge boson masses and vertices) in terms of gauge-invariant quantities. The gauge generators of the 8-component scalar multiplet  $\phi$  under  $SU(2)_L \times U(1)_Y$  are denoted by  $T_A^\phi$ ,  $A = 1, \dots, 4$ , where  $T_1^\phi$  stands for the  $U(1)_Y$  generator while  $T_{2,3,4}^\phi$  correspond to  $SU(2)_L$  transformations. The kinetic part of the Lagrangian which couples the scalar fields to the gauge bosons is given, before symmetry breaking, by<sup>6</sup>

$$\mathcal{L} \supset \frac{1}{2} D^\mu \phi_i D_\mu \phi^i, \quad (5.94)$$

where the scalar field covariant derivative can be generically written as

$$D_\mu \phi^i = \partial_\mu \phi^i + i V_\mu^A G^{AB} \left( T_B^\phi \right)^{ij} \phi^j. \quad (5.95)$$

In the above expression  $V_\mu^A$  stands for the vector fields whereas  $G$  is a  $4 \times 4$  diagonal matrix populated with the gauge couplings  $g_1$  and  $g_2$  according to

$$G = \text{diag} (g_1, g_2, g_2, g_2). \quad (5.96)$$

Expanding (5.94) yields

$$\mathcal{L} \supset \frac{1}{2} \partial^\mu \phi_i \partial_\mu \phi^i - i V_\mu^A G^{AB} \left( T_B^\phi \right)^{ij} \phi^i \partial_\mu \phi^j + \frac{1}{4} V_\mu^A V_B^\mu G^{AC} G^{BD} \left\{ T_C^\phi, T_D^\phi \right\}^{ij} \phi^i \phi^j. \quad (5.97)$$

After spontaneous symmetry breaking, the scalar multiplet is shifted according to

$$\phi \rightarrow \langle \phi \rangle + \omega \quad (5.98)$$

such that the Lagrangian contains in particular the following interactions

$$\mathcal{L}_{\text{broken}} \supset \frac{1}{2} (M_G^2)^{AB} V_\mu^A V_B^\mu + g_{Aij} V_A^\mu \omega^i \partial_\mu \omega^j + \frac{1}{2} g_{ABi} V_\mu^A V_B^\mu \omega^i + \frac{1}{4} g_{ABij} V_\mu^A V_B^\mu \omega^i \omega^j, \quad (5.99)$$

where the gauge boson mass matrix and the vector-scalar-scalar (VSS), vector-vector-scalar (VVS) and vector-vector-scalar-scalar (VVSS) couplings are respectively given by<sup>7</sup>

$$(M_G^2)^{AB} = \frac{1}{2} G^{AC} G^{BD} \left\{ T_C^\phi, T_D^\phi \right\}^{ij} \langle \phi \rangle^i \langle \phi \rangle^j, \quad (5.100)$$

$$g_{Aij} = -i G^{AB} \left( T_B^\phi \right)^{ij}, \quad (5.101)$$

$$g_{ABi} = G^{AC} G^{BD} \left\{ T_C^\phi, T_D^\phi \right\}^{ij} \langle \phi \rangle^j, \quad (5.102)$$

$$g_{ABij} = G^{AC} G^{BD} \left\{ T_C^\phi, T_D^\phi \right\}^{ij}. \quad (5.103)$$

### 5.2 Gauge-boson mass matrix

With the definitions of the previous subsection let us first examine the gauge-boson mass matrix  $M_G^2$ . With standard conventions for the gauge symmetry generators, the entries of  $M_G^2$  cannot generally be expressed in terms of bilinear fields in a basis-independent way. On the other hand, its matrix invariants – and in particular its eigenvalues – can be expressed in such a way. From a direct calculation, it is straightforward to show that  $M_G^2$  can be rotated into the diagonal form

$$\bar{M}_G^2 = \text{diag} (M_\gamma^2, M_W^2, M_W^2, M_Z^2), \quad (5.104)$$

<sup>6</sup>Here and in the following, we use a metric of signature  $(+, -, -, -)$ .

<sup>7</sup>We have defined the couplings so that they match the conventions in [107], despite a different convention for the Minkowski metric.

where the eigenvalues are given in terms of the bilinear fields by

$$M_W^2 = \frac{1}{2} (g_+^2 - g_-^2) K_0 = \frac{1}{2} g_2^2 K_0, \quad M_{Z,\gamma}^2 = \frac{1}{2} \left[ g_+^2 K_0 \pm \sqrt{g_+^4 \mathbf{K}^2 + g_-^4 (K_0^2 - \mathbf{K}^2)} \right], \quad (5.105)$$

with

$$g_+^2 \equiv \frac{g_1^2 + g_2^2}{2} \quad \text{and} \quad g_-^2 \equiv \frac{g_1^2 - g_2^2}{2}. \quad (5.106)$$

At a charge-conserving minimum where  $K_0^2 = \mathbf{K}^2$ , we obtain from (5.105) the familiar result

$$M_Z^2 = g_+^2 K_0 = \frac{1}{2} (g_1^2 + g_2^2) K_0, \quad M_\gamma = 0. \quad (5.107)$$

In what follows, we are only interested in the physically relevant case of a charge-conserving minimum where  $K_0^2 = \mathbf{K}^2$ , that is a spontaneous symmetry breaking of the type  $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{em}}$ . In that case, the gauge-boson mass matrix in the original basis (*i.e.* before diagonalisation) conveniently reduces to

$$M_G^2 = \frac{K_0}{2} \begin{pmatrix} g_1^2 & 0 & 0 & -g_1 g_2 \\ 0 & g_2^2 & 0 & 0 \\ 0 & 0 & g_2^2 & 0 \\ -g_1 g_2 & 0 & 0 & g_2^2 \end{pmatrix} \quad (5.108)$$

and is diagonalised according to

$$\bar{M}_G^2 = R_W M_G^2 R_W^T = \text{diag} (0, M_W^2, M_W^2, M_Z^2), \quad (5.109)$$

where

$$R_W = \begin{pmatrix} \cos \theta_W & 0 & 0 & \sin \theta_W \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin \theta_W & 0 & 0 & \cos \theta_W \end{pmatrix}, \quad (5.110)$$

with  $\theta_W$  the weak-mixing angle, satisfying

$$\cos \theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} = \frac{M_W}{M_Z}. \quad (5.111)$$

### 5.3 Scalar generators in the mass basis

While expressing the gauge-boson mass matrix in terms of bilinear fields at a charge-conserving minimum was a trivial task, the derivation of the scalar-vector vertices/couplings in the physical basis is considerably more involved. With view on (5.101)–(5.103), we see that such a computation reduces in fact to the determination of the expression of the gauge generators in the mass basis. Recalling that the bilinear fields are given by

$$K^\mu = \frac{1}{2} \Delta_{ij}^\mu \phi^i \phi^j, \quad (5.112)$$

the invariance of  $K^\mu$  under infinitesimal gauge transformations of the form

$$\phi^i \rightarrow \left( \mathbb{1}_8 + i\theta^A T_A^\phi \right)^{ij} \phi^j \quad (5.113)$$

translates, for all  $\mu \in \{0, \dots, 3\}$  and  $A \in \{1, \dots, 4\}$ , into the condition

$$\left[ \Delta^\mu, T_A^\phi \right]^{ij} = 0. \quad (5.114)$$

Clearly, the above relation remains valid after rotation of the scalar multiplet. In particular, defining the scalar generators  $\bar{T}^\phi$  rotated to the mass basis, (5.114) translates to

$$\left[ \bar{\Delta}^\mu, \bar{T}_A^\phi \right]^{ij} = 0. \quad (5.115)$$

With the analytic expressions of  $\bar{\Delta}^\mu$  in terms of bilinear fields at hand, the derivation of  $\bar{T}_A^\phi$  simply amounts to solving a system of linear equations. Doing so yields a four-dimensional basis of purely imaginary skew-symmetric matrices, noted  $\{\Theta_B\}$ , such that

$$\left(\bar{T}_A^\phi\right)^{ij} = X^{AB}\Theta_B^{ij}. \quad (5.116)$$

The matrix of coefficients  $X^{AB}$  is most easily fixed by considering a set of quantities independent of the basis chosen for the scalar multiplet. For instance, requiring that the quantity

$$\Gamma_i^\mu \Gamma_j^\nu \left(T_A^\phi\right)^{ij} = \Gamma^\mu T_A^\phi \Gamma^{\nu T} \quad (5.117)$$

remains unchanged (for all indices  $\mu, \nu, A$ ) after rotation to the mass basis provides a set of linear constraints on the coefficients  $X^{AB}$ . Requiring in addition the gauge boson mass matrix  $M_G^2$  to be invariant under a change of basis allows to uniquely determine the expressions of  $\bar{T}_A^\phi$ . Before presenting the final expression, it is useful to define two generators  $\bar{T}_\pm^\phi$  such that

$$\begin{pmatrix} \bar{T}_2^\phi \\ \bar{T}_3^\phi \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{T}_+^\phi \\ \bar{T}_-^\phi \end{pmatrix} = \begin{pmatrix} \cos\theta \bar{T}_+^\phi - \sin\theta \bar{T}_-^\phi \\ \sin\theta \bar{T}_+^\phi + \cos\theta \bar{T}_-^\phi \end{pmatrix}, \quad (5.118)$$

where the precise value of  $\theta$  depends on the basis chosen to express the vectors  $\mathbf{x}$  and  $\mathbf{y}$  (we will come back to this point further below). We may now provide the main result of this section, namely the expression of the gauge generators  $\bar{T}_{1,4}^\phi$  and  $\bar{T}_\pm^\phi$  in the mass basis. It is useful to reiterate that, in this basis, the dynamical field multiplet reads

$$\bar{\varphi} = (G^0, G_1^\pm, G_2^\pm, H_1^\pm, H_2^\pm, h_1^0, h_2^0, h_3^0). \quad (5.119)$$

We have:

$$\bar{T}_1^\phi = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i\bar{\mathbf{k}}^T \\ 0 & & \sigma_2 & 0 & 0 & 0_{2 \times 3} \\ 0 & 0 & 0 & 0 & 0 & 0_{2 \times 3} \\ 0 & 0 & 0 & \sigma_2 & 0 & 0_{2 \times 3} \\ 0 & 0 & 0 & 0 & \sigma_2 & 0_{2 \times 3} \\ -i\bar{\mathbf{k}} & 0_{3 \times 2} & 0_{3 \times 2} & 0_{3 \times 2} & i\Lambda_{\bar{\mathbf{k}}} & 0_{3 \times 3} \end{pmatrix}, \quad \bar{T}_+^\phi = \frac{1}{2} \begin{pmatrix} 0 & 0 & i & 0 & 0 & 0_{1 \times 3} \\ 0 & 0 & 0 & 0 & 0 & -i\bar{\mathbf{k}}^T \\ -i & 0 & 0 & 0 & 0 & 0_{1 \times 3} \\ 0 & 0 & 0 & 0 & 0 & i\mathbf{y}^T \\ 0 & 0 & 0 & 0 & 0 & i\mathbf{x}^T \\ 0_{3 \times 1} & i\bar{\mathbf{k}} & 0_{3 \times 1} & -i\mathbf{y} & -i\mathbf{x} & 0_{3 \times 3} \end{pmatrix}, \quad (5.120)$$

$$\bar{T}_4^\phi = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i\bar{\mathbf{k}}^T \\ 0 & & \sigma_2 & 0 & 0 & 0_{2 \times 3} \\ 0 & 0 & 0 & 0 & 0 & 0_{2 \times 3} \\ 0 & 0 & 0 & \sigma_2 & 0 & 0_{2 \times 3} \\ 0 & 0 & 0 & 0 & \sigma_2 & 0_{2 \times 3} \\ i\bar{\mathbf{k}} & 0_{3 \times 2} & 0_{3 \times 2} & 0_{3 \times 2} & -i\Lambda_{\bar{\mathbf{k}}} & 0_{3 \times 3} \end{pmatrix}, \quad \bar{T}_-^\phi = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0_{1 \times 3} \\ i & 0 & 0 & 0 & 0 & 0_{1 \times 3} \\ 0 & 0 & 0 & 0 & 0 & -i\bar{\mathbf{k}}^T \\ 0 & 0 & 0 & 0 & 0 & -i\mathbf{x}^T \\ 0 & 0 & 0 & 0 & 0 & i\mathbf{y}^T \\ 0_{3 \times 1} & 0_{3 \times 1} & i\bar{\mathbf{k}} & i\mathbf{x} & -i\mathbf{y} & 0_{3 \times 3} \end{pmatrix}, \quad (5.121)$$

where  $0_{m \times n}$  is the  $m \times n$  zero matrix and where the auxiliary matrix  $\Lambda_{\bar{\mathbf{k}}}$  is given by

$$(\Lambda_{\bar{\mathbf{k}}})^{ij} = \varepsilon_{ijk} \bar{k}^k. \quad (5.122)$$

A few remarks are in order. First, we may check that the generator of the residual gauge group  $U(1)_{\text{em}}$  is given by

$$\bar{T}_{\text{em}}^\phi = \bar{T}_1^\phi + \bar{T}_4^\phi = \begin{pmatrix} 0 & & & & & \\ & \sigma_2 & & & & \\ & & \sigma_2 & & & \\ & & & & & \\ & & & & & \\ & & & & & 0_{3 \times 3} \end{pmatrix}, \quad (5.123)$$

as expected from the fact that only  $G^\pm$  and  $H^\pm$  are electrically charged. Then, the presence of  $\mathbf{x}$  and  $\mathbf{y}$  in the expressions of  $\bar{T}_\pm^\phi$  (and, hence, of  $\bar{T}_{2,3}^\phi$ ) indicates an unphysical dependence of the gauge generators on the basis chosen to express these vectors (see also the discussion in Sec. 4). A similar statement holds



regarding the angle  $\theta$  introduced in (5.118). More precisely, if we apply a phase shift to every charged field in the scalar-gauge sector,

$$\begin{aligned} W^\pm &\rightarrow e^{i\omega_W} W^\pm, \\ H^\pm &\rightarrow e^{i\omega_H} H^\pm, \\ G^\pm &\rightarrow e^{i\omega_G} G^\pm, \end{aligned} \quad (5.124)$$

the analytic expression of  $\bar{T}_{2,3}^\phi$  remains unchanged if  $\mathbf{x}$ ,  $\mathbf{y}$  are redefined according to (5.91) and  $\theta$  according to

$$\theta \rightarrow \theta' = \theta - \omega_G - \omega_W. \quad (5.125)$$

In other words,  $\theta$  is arbitrary and may always be absorbed by a phase redefinition of  $G^\pm$  and/or  $W^\pm$ . Hence, we shall take  $\theta = 0$  in the forthcoming analytic expressions without any loss of generality.

As a final remark, we note that the generators  $(\bar{T}_A^\phi)^{ij}$  presented above have only been rotated to the basis of the scalar mass eigenstates. To obtain in addition their expression in the basis of the gauge mass eigenstates, one only needs to apply the rotation  $R_W$ , defined in (5.110), to the gauge index:

$$(\bar{T}_A^\phi)^{ij} \rightarrow R_W^{AB} (\bar{T}_B^\phi)^{ij}. \quad (5.126)$$

#### 5.4 Vector-scalar interactions

Having determined the expression of the scalar generators in the mass basis, it is straightforward to compute the expression of all vector-scalar interaction vertices at tree-level in terms of the bilinear fields, using (5.101)–(5.103). The part of the Lagrangian density describing the vector-scalar interactions reads

$$\mathcal{L}_{V-S} = \mathcal{L}_{VSS} + \mathcal{L}_{VVS} + \mathcal{L}_{VVSS}, \quad (5.127)$$

where, from (5.99),

$$\mathcal{L}_{VSS} = g_{Aij} V_A^\mu \varphi^i \partial_\mu \varphi^j, \quad (5.128)$$

$$\mathcal{L}_{VVS} = \frac{1}{2} g_{ABi} V_\mu^A V_B^\mu \varphi^i, \quad (5.129)$$

$$\mathcal{L}_{VVSS} = \frac{1}{4} g_{ABij} V_\mu^A V_B^\mu \varphi^i \varphi^j. \quad (5.130)$$

Applying (5.101)–(5.103) (the expectation value of the scalar multiplet in the mass basis can be read from (5.66)) and retaining only non-zero couplings gives

$$\begin{aligned} \mathcal{L}_{VSS} &= g_{sG^0} Z h^s \partial_\mu G^0 Z^\mu + g_{H^\pm H^\pm Z} H^- \partial_\mu H^+ Z^\mu + g_{G^\pm G^\pm Z} G^- \partial_\mu G^+ Z^\mu \\ &\quad + g_{H^\pm H^\pm \gamma} H^- \partial_\mu H^+ A^\mu + g_{G^\pm G^\pm \gamma} G^- \partial_\mu G^+ A^\mu \\ &\quad + g_{sH^\pm W^\pm} h^s \partial_\mu H^\pm W^\mp + g_{sG^\pm W^\pm} h^s \partial_\mu G^\pm W^\mp + g_{G^0 G^\pm W^\pm} G^0 \partial_\mu G^\pm W^\mp, \end{aligned} \quad (5.131)$$

$$\mathcal{L}_{VVS} = \frac{1}{2} g_{sZZ} h^s Z^\mu Z_\mu + g_{sW^\pm W^\pm} h^s W^{-\mu} W_\mu^+ + g_{G^\pm ZW^\pm} G^- Z^\mu W_\mu^+ + g_{G^\pm \gamma W^\pm} G^- A^\mu W_\mu^+, \quad (5.132)$$

$$\begin{aligned} \mathcal{L}_{VVSS} &= \frac{1}{4} g_{stZZ} h^s h^t Z^\mu Z_\mu + \frac{1}{2} g_{stW^\pm W^\pm} h^s h^t W^{-\mu} W_\mu^+ + g_{sH^\pm ZW^\pm} h^s H^- Z^\mu W_\mu^+ \\ &\quad + g_{sG^\pm ZW^\pm} h^s G^- Z^\mu W_\mu^+ + g_{sH^\pm \gamma W^\pm} h^s H^- A^\mu W_\mu^+ + g_{sG^\pm \gamma W^\pm} h^s G^- A^\mu W_\mu^+ \\ &\quad + \frac{1}{4} g_{G^0 G^0 ZZ} G^0 G^0 Z^\mu Z_\mu + \frac{1}{2} g_{G^0 G^0 W^\pm W^\pm} G^0 G^0 W^{-\mu} W_\mu^+ + g_{G^0 G^\pm ZW^\pm} G^0 G^\pm Z^\mu W_\mu^+ \\ &\quad + g_{G^0 G^\pm \gamma W^\pm} G^0 G^\pm A^\mu W_\mu^+ + \frac{1}{2} g_{H^\pm H^\pm ZZ} H^- H^+ Z^\mu Z_\mu + \frac{1}{2} g_{G^\pm G^\pm ZZ} G^- G^+ Z^\mu Z_\mu \\ &\quad + g_{H^\pm H^\pm Z\gamma} H^- H^+ Z^\mu A_\mu + g_{G^\pm G^\pm Z\gamma} G^- G^+ Z^\mu A_\mu + g_{H^\pm H^\pm \gamma\gamma} H^- H^+ A^\mu A_\mu \\ &\quad + g_{G^\pm G^\pm \gamma\gamma} G^- G^+ A^\mu A_\mu + g_{H^\pm H^\pm W^\pm W^\pm} H^- H^+ W^{-\mu} W_\mu^+ \\ &\quad + g_{G^\pm G^\pm W^\pm W^\pm} G^- G^+ W^{-\mu} W_\mu^+. \end{aligned} \quad (5.133)$$

All couplings can be found in App. B.3.

## 6 Yukawa sector

### 6.1 A basis-invariant description

While the bilinear formalism offers, for any type of THDM, a compact and elegant basis-independent formulation of the scalar interactions, it is a priori unable to describe the scalar-fermion interactions which are by construction linear in the scalar fields. The purpose of this section is to show how such a basis-independent formalism can in fact be extended to the Yukawa sector. Combined with the gauge-invariant approach developed in this work, we will be able to derive every Yukawa coupling in terms of gauge-invariant quantities, thus providing a complete description of the interactions among all physical states (scalars, vectors and fermions) in the most general THDM.

The part of the Lagrangian describing the Yukawa interactions involving the two Higgs doublets  $\varphi_1$  and  $\varphi_2$  can be first parameterised in the standard way:

$$-\mathcal{L}_Y = \left[ \overline{Q}_L (y_u \tilde{\varphi}_1 + \epsilon_u \tilde{\varphi}_2) u_R + \overline{Q}_L (y_d \varphi_1 + \epsilon_d \varphi_2) d_R + \overline{L} (y_e \varphi_1 + \epsilon_e \varphi_2) e_R \right] + \text{h.c.} . \quad (5.134)$$

As usual,  $Q_L$  denotes the left-handed quark doublets and  $L$  the left handed lepton doublets,  $u_R$ ,  $d_R$  are the right-handed up- and down-type quark singlets, and  $e_R$  the right-handed leptons. The corresponding Yukawa coupling matrices are denoted by  $y_u$ ,  $y_d$ ,  $y_e$ , as well as  $\epsilon_u$ ,  $\epsilon_d$ ,  $\epsilon_e$ . The conjugate doublets  $\tilde{\varphi}_a$  are given, as usual, by

$$(\tilde{\varphi}_a)^i = \varepsilon^{ij} (\varphi_a^*)_j, \quad \text{with} \quad \varepsilon = i\sigma_2, \quad (5.135)$$

where the indices  $i$  and  $j$  refer to the weak-isospin components of the doublets  $\varphi_a$  ( $a = 1, 2$ ). We note that the position of the family index  $a$  is relevant, since  $\varphi_a$  transforms under the fundamental representation of  $U(2)_H$  (the group describing unitary mixing of the two doublets), while  $\varphi_a^*$  and  $\tilde{\varphi}_a$  transform under the anti-fundamental representation of the family symmetry group.

For each fermion species  $f = u, d, e$ , the Yukawa matrices  $y_f$  and  $\epsilon_f$  can in fact be unified into a single object  $\mathcal{F} = \mathcal{U}, \mathcal{D}, \mathcal{E}$ , transforming under the (anti-)fundamental representation of the family group. Namely, we define

$$\mathcal{U}^a = \begin{pmatrix} y_u \\ \epsilon_u \end{pmatrix}, \quad \mathcal{U}_a^\dagger = \begin{pmatrix} y_u^\dagger & \epsilon_u^\dagger \end{pmatrix}, \quad (5.136a)$$

$$\mathcal{D}_a = \begin{pmatrix} y_d & \epsilon_d \end{pmatrix}, \quad \mathcal{D}^{\dagger a} = \begin{pmatrix} y_d^\dagger \\ \epsilon_d^\dagger \end{pmatrix}, \quad (5.136b)$$

$$\mathcal{E}_a = \begin{pmatrix} y_e & \epsilon_e \end{pmatrix}, \quad \mathcal{E}^{\dagger a} = \begin{pmatrix} y_e^\dagger \\ \epsilon_e^\dagger \end{pmatrix}, \quad (5.136c)$$

in order to rewrite the Yukawa Lagrangian (5.134) in the more compact form

$$-\mathcal{L}_Y = \left[ \overline{Q}_L \mathcal{U}^a \tilde{\varphi}_a u_R + \overline{Q}_L \mathcal{D}_a \varphi^a d_R + \overline{L} \mathcal{E}_a \varphi^a e_R \right] + \text{h.c.}, \quad (5.137)$$

manifestly invariant under a change of basis in the Higgs family space, that is, a unitary transformation  $U$  of the two doublets, provided

$$\varphi^a \rightarrow U^a_b \varphi^b \quad \Rightarrow \quad \mathcal{D}_a \rightarrow \mathcal{D}_b (U^\dagger)^b_a, \quad \mathcal{E}_a \rightarrow \mathcal{E}_b (U^\dagger)^b_a, \quad \mathcal{U}^a \rightarrow U^a_b \mathcal{U}^b. \quad (5.138)$$

While the Yukawa interactions themselves are only linear in the scalar fields (and the Yukawa matrices), some physically relevant quantities such as the fermion mass matrix squared,  $M_F^2$ , depend on them quadratically. For such quantities, the bilinear formalism can be appropriately used and can in fact be extended to the basis-dependent objects  $\mathcal{D}$ ,  $\mathcal{E}$  and  $\mathcal{U}$ . In analogy with the definition of the bilinear fields in (5.6) which we repeat here for clarity,

$$\underline{K}^a_b = \varphi^a \varphi_b^\dagger = \frac{1}{2} K^\mu (\sigma_\mu)^a_b, \quad (5.139)$$

we may write

$$\mathcal{U}^a \mathcal{U}_b^\dagger \equiv \frac{1}{2} Y_u^\mu (\sigma_\mu)^a{}_b, \quad (5.140)$$

as the definition of the bilinear up-type Yukawa coupling  $Y_u$ . Similarly, the four-component bilinear Yukawa couplings  $Y_d$  and  $Y_e$  can be defined through

$$\mathcal{D}_a \mathcal{D}^{\dagger b} = \frac{1}{2} Y_d^\mu (\sigma_\mu)^b{}_a, \quad \mathcal{E}_a \mathcal{E}^{\dagger b} = \frac{1}{2} Y_e^\mu (\sigma_\mu)^b{}_a \quad (5.141)$$

with

$$(D_a (D^\dagger)^b)^{ik} = D_a^{ij} (D^*)^b{}_{kj} = (D^*)^b{}_{kj} D_a^{ij}. \quad (5.142)$$

Contracting both sides of (5.140) with  $(\sigma^\nu)^b{}_a$ , and both sides of (5.141) with  $(\sigma^\nu)^a{}_b$  allows to obtain the expressions of  $Y_{u,d,e}$ :

$$Y_u^\nu = (\sigma^\nu)^b{}_a \mathcal{U}^a \mathcal{U}_b^\dagger, \quad Y_d^\nu = (\sigma^\nu)^a{}_b \mathcal{D}_a \mathcal{D}^{\dagger b}, \quad Y_e^\nu = (\sigma^\nu)^a{}_b \mathcal{E}_a \mathcal{E}^{\dagger b}. \quad (5.143)$$

Explicitly, the components of  $Y_f$  ( $f = u, d, e$ ) are given by the hermitian matrices

$$Y_u = \begin{pmatrix} y_u y_u^\dagger + \epsilon_u \epsilon_u^\dagger \\ y_u \epsilon_u^\dagger + \epsilon_u y_u^\dagger \\ i(y_u \epsilon_u^\dagger - \epsilon_u y_u^\dagger) \\ y_u y_u^\dagger - \epsilon_u \epsilon_u^\dagger \end{pmatrix}, \quad Y_d = \begin{pmatrix} y_d y_d^\dagger + \epsilon_d \epsilon_d^\dagger \\ y_d \epsilon_d^\dagger + \epsilon_d y_d^\dagger \\ -i(y_d \epsilon_d^\dagger - \epsilon_d y_d^\dagger) \\ y_d y_d^\dagger - \epsilon_d \epsilon_d^\dagger \end{pmatrix}, \quad Y_e = \begin{pmatrix} y_e y_e^\dagger + \epsilon_e \epsilon_e^\dagger \\ y_e \epsilon_e^\dagger + \epsilon_e y_e^\dagger \\ -i(y_e \epsilon_e^\dagger - \epsilon_e y_e^\dagger) \\ y_e y_e^\dagger - \epsilon_e \epsilon_e^\dagger \end{pmatrix}. \quad (5.144)$$

Under changes of basis, the four-component bilinear Yukawa couplings transform in the same way as  $\widetilde{\mathbf{K}}$ , namely (for  $f = u, d, e$ )

$$Y_f^0 \rightarrow Y_f^0, \quad Y_f^a \rightarrow R(U)^{ab} Y_f^b, \quad a = 1, 2, 3, \quad (5.145)$$

where  $R(U)$  is an orthogonal  $3 \times 3$  matrix defined in Eq. (5.21).

From these transformation properties, we may define in particular the Yukawa couplings  $\bar{U}$ ,  $\bar{D}$  and  $\bar{E}$  obtained after performing the change of basis which diagonalises the neutral mass matrix, described in Eqs. (5.60) and (5.61). Similarly, the four-component bilinear Yukawa couplings in such a basis are denoted by

$$\bar{Y}_f = \begin{pmatrix} Y_f^0 \\ \bar{Y}_f \end{pmatrix} \quad (5.146)$$

for  $f = u, d, e$ .

## 6.2 Yukawa couplings in the mass basis

Having established a basis-independent formalism to describe the Yukawa sector of any THDM, we are now interested in computing the tree-level Yukawa vertices involving the physical states, *i.e.* in expressing such vertices in the mass basis. Having determined the form of the scalar generators in such a basis in Sec. 5.3, one way to proceed to compute the Yukawa vertices is to rely on the gauge-invariance of the Yukawa interaction Lagrangian. First, let us note that the Yukawa interactions can generically be written as

$$-\mathcal{L}_Y = \frac{1}{2} y_i^{IJ} \phi^i \psi^I \psi^J + \text{h.c.} \quad (5.147)$$

where all left-handed two-component spinors of the model were gathered in the fermion multiplet  $\psi$ . Under  $SU(3)_C \times SU(2)_L \times U(1)_Y$ , the scalar and fermion multiplets simultaneously transform as

$$\phi^i \rightarrow \left( \mathbb{1} + i\theta^A T_A^\phi \right)^{ij} \phi^j, \quad \psi^I \rightarrow \left( \mathbb{1} + i\theta^A T_A^\psi \right)^{IJ} \psi^J, \quad A = 1, \dots, 12, \quad (5.148)$$

with  $T^\psi$  the gauge generators of the left-handed fermion multiplet. Extending the notations of the previous section, the generators  $T_1^{\phi,\psi}$ ,  $T_{2,3,4}^{\phi,\psi}$  and  $T_{5,\dots,12}^{\phi,\psi}$  correspond to  $U(1)_Y$ ,  $SU(2)_L$ , and  $SU(3)_C$

transformations respectively. The invariance of (5.147) under infinitesimal gauge transformations implies<sup>8</sup>

$$\left(T_A^{\psi*}\right)^{IK} y_i^{KJ} + y_i^{IK} \left(T_A^\psi\right)^{KJ} - \left(T_A^\phi\right)^{ij} y_j^{IJ} = 0 \quad (5.149)$$

for all values of the indices  $A, I, J$  and  $i$ . Since this relation must hold independently of the basis chosen to express the various types of fields, we may in particular express it in the scalar mass basis, thus obtaining

$$\left(T_A^{\psi*}\right)^{IK} \bar{y}_i^{KJ} + \bar{y}_i^{IK} \left(T_A^\psi\right)^{KJ} - \left(\bar{T}_A^\phi\right)^{ij} \bar{y}_j^{IJ} = 0, \quad (5.150)$$

with  $\bar{y}$  denoting the tensor of Yukawa couplings obtained after rotation of its scalar index towards the mass basis. This relation is linear in  $\bar{y}$  and can be solved without too much difficulty, based on the known expressions of  $T^\psi$  and  $\bar{T}^\phi$ . While the linear system of equations thus obtained is underdetermined, we may, as previously done in the gauge sector, supplement it with a set of relations which are independent on the basis chosen to express the scalar fields. For instance, imposing

$$\Gamma_i^\mu \Gamma_j^\nu y_i^{JJ} y_j^{JI} = \bar{\Gamma}_i^\mu \bar{\Gamma}_j^\nu \bar{y}_i^{JJ} \bar{y}_j^{JI} \quad (5.151)$$

suffices to compute the entries of  $\bar{y}_i^{IJ}$  unambiguously, completing the determination of the Yukawa interactions in the mass basis.

After symmetry breaking towards  $U(1)_{\text{em}}$ , the Yukawa interaction Lagrangian can be expanded, in terms of two-component fermions, as

$$\begin{aligned} -\mathcal{L}_Y = & \left\{ h^a \left[ \bar{\xi}_a^u u_L^\dagger u_R + \bar{\xi}_a^d d_L^\dagger d_R + \bar{\xi}_a^e e_L^\dagger e_R \right] + G^0 \left[ \bar{\xi}_{G^0}^u u_L^\dagger u_R + \bar{\xi}_{G^0}^d d_L^\dagger d_R + \bar{\xi}_{G^0}^e e_L^\dagger e_R \right] \right. \\ & + H^- \left[ \bar{\xi}_{H^-}^{ud} u_L^\dagger d_R + \bar{\xi}_{H^-}^{\nu e} \nu_L^\dagger e_R \right] + G^- \left[ \bar{\xi}_{G^-}^{ud} u_L^\dagger d_R + \bar{\xi}_{G^-}^{\nu e} \nu_L^\dagger e_R \right] \\ & \left. + H^+ \left[ \bar{\xi}_{H^+}^{ud} d_L^\dagger u_R + G^+ \bar{\xi}_{G^+}^{ud} d_L^\dagger u_R \right] \right\} + \text{h.c.}, \end{aligned} \quad (5.152)$$

where a sum on the index  $a = 1, 2, 3$  (*i.e.* on the three neutral scalars) is implied in the first term. In addition, if more than one generation of fermions are considered, we assume that the fermion mass matrices have been properly (bi-)diagonalised so that  $u_{L,R}$ ,  $d_{L,R}$  and  $e_{L,R}$  correspond to fermion mass eigenstates. The expressions of all Yukawa matrices involved in (5.152) in terms of bilinear fields are given in App. B.4. These expressions involve the complex two-vector  $\bar{\kappa}^a$ , defined such that

$$\bar{K}^a{}_b = \frac{1}{2} \bar{K}^\mu (\sigma_\mu)^a{}_b \equiv \bar{\kappa}^a \bar{\kappa}_b^*. \quad (5.153)$$

It should be noted that such a decomposition is made possible by  $\bar{K}$  being of rank 1 at a charge-conserving minimum (and, more generally, within the charge-conserving hypersurface). Explicitly,  $\bar{\kappa}$  can be written

$$\bar{\kappa} = \sqrt{\frac{K_0}{2}} \frac{1}{\sqrt{1+\bar{k}_3}} \begin{pmatrix} 1 + \bar{k}_3 \\ \bar{k}_1 + i\bar{k}_2 \end{pmatrix} = \sqrt{\frac{K_0}{2}} \begin{pmatrix} \sqrt{1+\bar{k}_3} \\ \sqrt{1-\bar{k}_3} e^{i\zeta} \end{pmatrix}, \quad (5.154)$$

where the phase  $\zeta$  was defined such that

$$\bar{k}_1 + i\bar{k}_2 = \sqrt{\bar{k}_1^2 + \bar{k}_2^2} e^{i\zeta} = \sqrt{1 - \bar{k}_3^2} e^{i\zeta}. \quad (5.155)$$

From (5.154), it is straightforward to check that

$$\bar{\kappa}^a \bar{\kappa}_b^* = \frac{K_0}{2} \begin{pmatrix} 1 + \bar{k}_3 & \bar{k}_1 - i\bar{k}_2 \\ \bar{k}_1 + i\bar{k}_2 & 1 - \bar{k}_3 \end{pmatrix} \quad (5.156)$$

as expected from (5.153). The complex 2-vector  $\bar{\kappa}$  thus defined allows to write the various Yukawa vertices in a very compact form. For instance, the  $h^a u_L^\dagger u_R$  and  $h^a d_L^\dagger d_R$  interactions given in (5.B58a) and (5.B58b) read

$$\bar{\xi}_a^u = \frac{1}{\sqrt{2K_0}} (\sigma_a)^\alpha{}_\beta \bar{\kappa}_\alpha^* \bar{U}^\beta, \quad \bar{\xi}_a^d = \frac{1}{\sqrt{2K_0}} (\sigma_a)^\alpha{}_\beta \bar{D}_\alpha \bar{\kappa}^\beta. \quad (5.157)$$

<sup>8</sup>Note that the generators  $T_A^\phi$  are purely imaginary such that  $T_A^{\phi*} = -T_A^\phi$ .

### 6.3 Fermion masses

After symmetry breaking, the scalar multiplet acquires a vacuum expectation value according to (5.66), and the part of the Lagrangian describing the fermion mass terms reads

$$-\mathcal{L} \supset \left\{ u_L^\dagger M_u u_R + d_L^\dagger M_d d_R + e_L^\dagger M_e e_R \right\} + \text{h.c.}, \quad (5.158)$$

where, for  $f = u, d, e$ ,

$$M_f = \sqrt{2K_0} \bar{k}^a \bar{\xi}_a^f = \sqrt{2K_0} k^a \xi_a^f. \quad (5.159)$$

For instance, we find for the up-type quarks

$$M_u = \sqrt{2K_0} k^a \xi_a^u = (k^a \sigma_a)^\alpha{}_\beta \kappa_\alpha^* \mathcal{U}^\beta. \quad (5.160)$$

To further simplify this expression, we note that since

$$\underline{K}^\alpha{}_\beta = \frac{1}{2} K^\mu (\sigma_\mu)^\alpha{}_\beta = \frac{K_0}{2} \left[ \delta^\alpha{}_\beta + (k^a \sigma_a)^\alpha{}_\beta \right] = \kappa^\alpha \kappa_\beta^*, \quad (5.161)$$

we find

$$(k^a \sigma_a)^\alpha{}_\beta = \frac{2}{K_0} \kappa^\alpha \kappa_\beta^* - \delta^\alpha{}_\beta, \quad (5.162)$$

so that

$$M_u = \left[ \frac{2}{K_0} \kappa^\alpha \kappa_\beta^* - \delta^\alpha{}_\beta \right] \kappa_\alpha^* \mathcal{U}^\beta = \left( \frac{2}{K_0} |\kappa|^2 - 1 \right) \kappa_\alpha^* \mathcal{U}^\alpha. \quad (5.163)$$

Finally,

$$|\kappa|^2 = \kappa_\alpha^* \kappa^\alpha = \text{Tr} \underline{K} = K_0 \quad (5.164)$$

so  $M_u$  is simply given by

$$M_u = \kappa_\alpha^* \mathcal{U}^\alpha. \quad (5.165)$$

In turn, the up-type quark mass matrix squared,  $M_u^2$  is computed as

$$M_u^2 = M_u M_u^\dagger = \kappa_\alpha^* \kappa^\beta \mathcal{U}^\alpha \mathcal{U}_\beta^\dagger = \underline{K}^\beta{}_\alpha \mathcal{U}^\alpha \mathcal{U}_\beta^\dagger = \frac{1}{2} K^\mu (\sigma_\mu)^\beta{}_\alpha \mathcal{U}^\alpha \mathcal{U}_\beta^\dagger = \frac{1}{2} K_\mu Y_u^\mu. \quad (5.166)$$

Similarly, we obtain for the down-type quark and lepton mass matrices:

$$M_d = \bar{\mathcal{D}}_\alpha \bar{k}^\alpha, \quad M_d^2 = \frac{1}{2} K_\mu Y_d^\mu, \quad (5.167)$$

$$M_e = \bar{\mathcal{E}}_\alpha \bar{k}^\alpha, \quad M_e^2 = \frac{1}{2} K_\mu Y_e^\mu. \quad (5.168)$$

## 7 Conclusions

In the THDM potential it has been shown that gauge-invariant expressions give new insights [108]. With the introduction of bilinears [101, 103, 102] gauge-invariants have been introduced which have a one-to-one correspondence to the Higgs-boson doublets - except for the non-physical gauge redundancies. In particular, it has been shown, that bilinears form Minkowski-type four vectors. Stability, electroweak symmetry breaking and the symmetries of the potential can be studied in a concise form based on bilinears [102, 99, 98].

Here we have extended this gauge-invariant bilinear formalism to the squared mass matrices and the interaction terms of the THDM, that is, the trilinear and quartic scalar couplings, the gauge and Higgs boson interactions, and the Yukawa couplings.

With the help of the connection  $\Gamma$ , (5.35), between the component fields of the two doublets and the bilinears we were able to get a completely gauge-invariant form of the scalar squared mass matrix. We revealed the general expressions for the cases of full electroweak symmetry breaking as well as for the charge-conserving breaking case, without specifying the actual form of the potential. We derived the scalar mass spectrum for the general case in terms of gauge-invariant expressions. In particular,

the respective Goldstone modes appeared in a very transparent way. From the gauge-invariant form of the mass matrix we then derived the cubic and quartic scalar interactions. Then, from the expressions of the scalar gauge generators in the mass-basis, we calculated the full set of vector-scalar couplings in a gauge-invariant form. Finally, we have extended the bilinear formalism to the Yukawa sector and obtained expressions for the fermion mass matrices squared that are invariant under mixing of the two Higgs doublets. Starting again from the expression of the scalar generators and employing the gauge-invariance of the Yukawa Lagrangian, we obtained gauge-invariant expressions for the Yukawa couplings after electroweak symmetry breaking.

With these results, we have extended the bilinear formalism to all interactions in a general THDM, making it a truly powerful tool to study any specific THDM.

# Appendix

## A Structure of the scalar mass matrix

This appendix extends the discussion of section 3 on the structure of the scalar mass matrix. In particular, we provide explicit forms for the various quantities involved in the expression of the mass matrix, both, at a charge-breaking minimum and at a charge-conserving minimum. The various quantities expressed in this appendix will allow in turn to compute analytically the derivatives of scalar eigenvalues, as required to express the derivatives of the effective potential. First, let us recall the general form of the scalar mass matrix in our formalism:

$$\widehat{M}_s^2 = \widehat{\Delta}^\mu \partial_\mu V + \widehat{\Gamma} \mathcal{M} \widehat{\Gamma}^\top. \quad (5.A1)$$

In particular, the hatted quantities correspond to the canonical bases; see section 3. In this basis, the connection matrices  $\Gamma$  are of the form (5.48). To simplify the forthcoming expressions, we define

$$\alpha = \left( \frac{\widetilde{\mathbf{K}}^\top \widetilde{g} \widetilde{\mathbf{K}}}{K_0^2} \right)^{1/2} = \sqrt{1 - \mathbf{k}^\top \mathbf{k}} \quad (5.A2)$$

so that  $\gamma$  (5.50) can be written

$$\gamma = \sqrt{2K_0} \begin{pmatrix} \alpha & \mathbf{0}^\top \\ \mathbf{k} & \mathbb{1}_3 \end{pmatrix}. \quad (5.A3)$$

We now want to derive the explicit form of  $\widehat{M}_s^2$  in a canonical basis. We first need to examine the form taken by the  $\widehat{\Delta}^\mu$  matrices. We may decompose the four symmetric  $8 \times 8$  matrices  $\widehat{\Delta}^\mu$  into  $4 \times 4$  blocks

$$\widehat{\Delta}^\mu = \begin{pmatrix} A_{44}^\mu & C_{44}^\mu \\ (C_{44}^\mu)^\top & B_{44}^\mu \end{pmatrix}. \quad (5.A4)$$

As we will show,  $A_{44}^\mu$  and  $C_{44}^\mu$  depend on the chosen basis, while  $B_{44}^\mu$  is fixed by the requirement of working in a canonical basis. In fact, it is possible to determine  $B_{44}^\mu$  by first computing the quantity

$$\Sigma^{\mu\rho\sigma} = \Gamma_i^\rho \Delta_{ij}^\mu \Gamma_j^\sigma = \Gamma_i^\rho \Delta_{ji}^\mu \Gamma_j^\sigma = \Sigma^{\mu\sigma\rho}. \quad (5.A5)$$

or, in matrix notation  $\Sigma^\mu = \Gamma^\top \Delta^\mu \Gamma$ . This quantity is constructed in a way that all field component indices are contracted and therefore it is gauge invariant. In order to get an explicit expression for  $\Sigma^\mu$  we first compute

$$\begin{aligned} \Gamma_i^\mu \Delta_{ia}^\nu &= \Delta_{ij}^\mu \Delta_{ia}^\nu \phi^j \\ &= \partial_a (\Delta_{ij}^\mu \Delta_{ik}^\nu \phi^j \phi^k) - \Delta_{ia}^\mu \Delta_{ik}^\nu \phi^k \\ &= \partial_a (\Gamma^2)^{\mu\nu} - \Delta_{ia}^\mu \Delta_{ij}^\nu \phi^j \\ &= \Gamma_a^\lambda T_\lambda^{\mu\nu} - \Gamma_i^\nu \Delta_{ia}^\mu, \end{aligned} \quad (5.A6)$$

where the rank-3 symmetric tensor  $T$  has been defined in (5.37) and we have used (5.33). By repeatedly applying (5.A6) in the definition of  $\Sigma^\mu$ , recalling that  $\Gamma^2 \equiv \Gamma^\top \Gamma$ , that is,  $(\Gamma^2)^{\rho\sigma} = \Gamma_i^\rho \Gamma_i^\sigma$ , we arrive at

$$\Sigma^\mu = \frac{1}{2} \left[ \Gamma^2 T^\mu + T^\mu \Gamma^2 - (\Gamma^2)^{\mu\nu} T^\nu \right]. \quad (5.A7)$$

Let us show (5.A7) in detail, starting from the definition of  $\Sigma^\mu$ , (5.A5), writing all indices explicitly and using (5.A6) twice,

$$\Sigma^{\mu\rho\sigma} = \Gamma_i^\rho \Delta_{ij}^\mu \Gamma_j^\sigma = \Gamma_j^\lambda T_\lambda^{\rho\mu} \Gamma_j^\sigma - \Gamma_i^\mu \Delta_{ij}^\rho \Gamma_j^\sigma = \Gamma_j^\lambda T_\lambda^{\rho\mu} \Gamma_j^\sigma - \Gamma_i^\mu (\Gamma_i^\lambda T_\lambda^{\rho\sigma} - \Gamma_j^\rho \Delta_{ij}^\sigma). \quad (5.A8)$$

For the first term on the right-hand side we get  $\Gamma_j^\lambda T_\lambda^{\rho\mu} \Gamma_j^\sigma = T^\mu (\Gamma^2)^{\rho\sigma}$  and for the second term  $\Gamma_i^\mu \Gamma_i^\lambda T_\lambda^{\rho\sigma} = (\Gamma^2)^{\mu\lambda} T_\lambda^{\rho\sigma}$ , for the third term  $\Gamma_i^\mu \Gamma_j^\rho \Delta_{ij}^\sigma = \Gamma_i^\mu \Delta_{ij}^\sigma \Gamma_j^\rho = \Gamma_j^\lambda T_\lambda^{\mu\sigma} \Gamma_j^\rho - \Gamma_i^\sigma \Delta_{ij}^\mu \Gamma_j^\rho = (\Gamma^2)^{\rho\sigma} T^\mu - \Sigma^{\mu\rho\sigma}$  altogether,

$$\Sigma^{\mu\rho\sigma} = T^\mu (\Gamma^2)^{\rho\sigma} - (\Gamma^2)^{\mu\lambda} T_\lambda^{\rho\sigma} + (\Gamma^2)^{\rho\sigma} T^\mu - \Sigma^{\mu\rho\sigma}. \quad (5.A9)$$

This is equivalent to (5.A7).

With the explicit form for the  $T_\mu$  matrices given in (5.40),

$$T_0 = 2\mathbb{1}_4, \quad T_i = 2 \begin{pmatrix} 0 & \mathbf{e}_i^T \\ \mathbf{e}_i & 0_{3 \times 3} \end{pmatrix}, \quad (5.A10)$$

we find

$$\Sigma^0 = \Gamma^2 = 2 \begin{pmatrix} K_0 & \mathbf{K}^T \\ \mathbf{K} & K_0 \mathbb{1}_3 \end{pmatrix}, \quad \Sigma^i = 2K_i \tilde{g} + 2 \begin{pmatrix} 0 & K_0 \mathbf{e}_i^T \\ K_0 \mathbf{e}_i & \mathbf{e}_i \mathbf{K}^T + \mathbf{K} \mathbf{e}_i^T \end{pmatrix}. \quad (5.A11)$$

On the other hand, we can express  $\Sigma^\mu$ , (5.A5) in any basis, since it is gauge-invariantly defined. In the canonical basis we have

$$\Sigma^\mu = \Gamma^T \Delta^\mu \Gamma = \widehat{\Gamma}^T \widehat{\Delta}^\mu \widehat{\Gamma} = \begin{pmatrix} 0_{4 \times 4} & \gamma^T \\ (C_{44}^T)^\mu & B_{44}^\mu \end{pmatrix} \begin{pmatrix} 0_{4 \times 4} \\ \gamma \end{pmatrix} = \gamma^T B_{44}^\mu \gamma. \quad (5.A12)$$

At a point of the bilinear field space where  $\gamma$  is non-singular (i.e., where  $\det(\gamma) = 4K_0^2 \alpha \neq 0$ ), the relation can be inverted using

$$\gamma^{-1} = \frac{1}{\alpha \sqrt{2K_0}} \begin{pmatrix} 1 & \mathbf{0}^T \\ -\mathbf{k} & \alpha \mathbb{1}_3 \end{pmatrix} \quad (5.A13)$$

and we finally get with (5.A11):

$$B_{44}^\mu = (\gamma^T)^{-1} \Sigma^\mu \gamma^{-1} \quad \text{that is} \quad B_{44}^0 = \mathbb{1}_4, \quad B_{44}^i = -k_i \mathbb{1}_4 + \begin{pmatrix} 0 & \alpha \mathbf{e}_i^T \\ \alpha \mathbf{e}_i & \mathbf{e}_i \mathbf{k}^T + \mathbf{k} \mathbf{e}_i^T \end{pmatrix}. \quad (5.A14)$$

Having found the result for  $B_{44}^\mu$  valid in any canonical basis, we want to determine a possible form for  $A_{44}^\mu$  and  $C_{44}^\mu$ . As stated above, these two matrices are not uniquely determined since a continuous set of canonical bases exist. However, requiring that  $\widehat{\Delta}^\mu$  satisfies the same anti-commutation properties as  $\Delta^\mu$  in (5.38), namely

$$\{\widehat{\Delta}^\mu, \widehat{\Delta}^\nu\} = T_\lambda^{\mu\nu} \widehat{\Delta}^\lambda, \quad (5.A15)$$

and that the components  $\widehat{\Delta}^a$ ,  $a = 1, 2, 3$  properly transform under rotations in the bilinear field space, allows us to first determine the set of possible expressions for  $C_{44}^\mu$ , and then for  $A_{44}^\mu$ . These expressions are valid anywhere in the field space, hence, in particular at the charge-conserving and the charge-breaking hypersurfaces. We may now provide a complete set of  $A_{44}^\mu$ ,  $B_{44}^\mu$  and  $C_{44}^\mu$  matrices valid for any values of the bilinear fields but assuming a canonical basis:

$$A_{44}^0 = \mathbb{1}_4, \quad A_{44}^a = k_a \mathbb{1}_4 - \frac{1}{|\mathbf{k}|^2} \begin{pmatrix} 0 & \alpha k_a \mathbf{k}^T \\ \alpha k_a \mathbf{k} & 2k_a \mathbf{k} \mathbf{k}^T - ([\mathbf{e}_a \times \mathbf{k}] \mathbf{k}^T + \mathbf{k} [\mathbf{e}_a \times \mathbf{k}]^T) \end{pmatrix} \quad (5.A16)$$

$$B_{44}^0 = \mathbb{1}_4, \quad B_{44}^a = -k_a \mathbb{1}_4 + \begin{pmatrix} 0 & \alpha \mathbf{e}_a^T \\ \alpha \mathbf{e}_a & \mathbf{e}_a \mathbf{k}^T + \mathbf{k} \mathbf{e}_a^T \end{pmatrix}, \quad (5.A17)$$

$$C_{44}^0 = 0_{4 \times 4}, \quad C_{44}^a = \frac{1}{|\mathbf{k}|} \begin{pmatrix} 0 & [\mathbf{e}_a \times \mathbf{k}]^T \\ |\mathbf{k}|^2 \mathbf{e}_a - k_a \mathbf{k} & \alpha (k_a \mathbb{1}_3 - \mathbf{e}_a \mathbf{k}^T) \end{pmatrix}. \quad (5.A18)$$

For the case of a charge-conserving minimum we have  $|\mathbf{k}| = 1$ , corresponding to  $\alpha = 0$ . In this case (see (5.48))

$$\widehat{\Gamma} = \begin{pmatrix} 0_{4 \times 4} \\ \gamma \end{pmatrix} = \begin{pmatrix} 0_{4 \times 4} \\ 0_{1 \times 4} \\ \gamma_3 \end{pmatrix} \quad (5.A19)$$

since

$$\gamma = \sqrt{2K_0} \begin{pmatrix} \alpha & \mathbf{0}^T \\ \mathbf{k} & \mathbb{1}_3 \end{pmatrix} \stackrel{CC}{\equiv} \sqrt{2K_0} \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{k} & \mathbb{1}_3 \end{pmatrix} \equiv \begin{pmatrix} 0_{1 \times 4} \\ \gamma_3 \end{pmatrix}. \quad (5.A20)$$



Here we have defined the  $3 \times 4$  matrix  $\gamma_3$  with index 3 in order to distinguish it from the  $4 \times 4$  matrix  $\gamma$ . As a consequence, the term  $\widehat{\Gamma}\mathcal{M}\widehat{\Gamma}^T$  in the mass matrix (5.A1) takes the form

$$\widehat{\Gamma}\mathcal{M}\widehat{\Gamma}^T = \begin{pmatrix} 0_{5 \times 5} & 0_{5 \times 3} \\ 0_{3 \times 5} & \gamma_3\mathcal{M}\gamma_3^T \end{pmatrix}. \quad (5.A21)$$

Hence, at a charge-conserving minimum, the block-diagonal structure of  $\widehat{\Delta}^\mu$  is more appropriately reorganised as

$$\widehat{\Delta}^\mu = \begin{pmatrix} A_{55}^\mu & C_{53}^\mu \\ C_{35}^\mu & B_{33}^\mu \end{pmatrix}, \quad (5.A22)$$

where the expression of the newly defined blocks  $A_{55}^\mu$ ,  $B_{33}^\mu$  and  $C_{53}^\mu$  can be readily inferred from Eqs. (5.A16)–(5.A18). Next, we express the first term of (5.A1) in the form

$$\widehat{\Delta}^\mu \partial_\mu V \stackrel{CC}{=} 2u (\widetilde{g}\widetilde{\mathbf{K}})_\mu \widehat{\Delta}^\mu = 2u \begin{pmatrix} A_{55} & C_{53} \\ C_{35} & B_{33} \end{pmatrix}, \quad (5.A23)$$

with

$$A_{55} = 2K_0 \begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbf{k}\mathbf{k}^T & \mathbf{0} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \quad B_{33} = 2K_0 (\mathbb{1}_3 - \mathbf{k}\mathbf{k}^T) = -\gamma_3 \widetilde{g} \gamma_3^T, \quad (5.A24)$$

$$C_{53} = \alpha K_0 \begin{pmatrix} \mathbf{0}^T \\ \mathbf{k}\mathbf{k}^T - \mathbb{1}_3 \\ -\mathbf{k}^T \end{pmatrix} = 0.$$

Note that although  $\alpha$  and therefore  $C_{53}$  vanish at a charge-conserving minimum, for later convenience we want to have the expression for  $\widehat{\Delta}^\mu$  not only at the minimum of the potential. Eventually, we have found that the off-diagonal blocks of  $\widehat{M}_s^2$  of (5.A1) vanish, simultaneously for the term  $\widehat{\Delta}^\mu \partial_\mu V$  and for the term  $\widehat{\Gamma}\mathcal{M}\widehat{\Gamma}^T$ . This shows the convenience to choose a canonical basis, where this simple structure of the scalar mass matrix appears. We may thus finally write with (5.A21),

$$\widehat{M}_s^2 \stackrel{CC}{=} 2u (\widetilde{g}\widetilde{\mathbf{K}})_\mu \widehat{\Delta}^\mu + \widehat{\Gamma}\mathcal{M}\widehat{\Gamma}^T = \begin{pmatrix} \widehat{M}_{CC}^2 & 0_{5 \times 3} \\ 0_{3 \times 5} & \widehat{M}_{\text{neutral}}^2 \end{pmatrix}. \quad (5.A25)$$

The block

$$\widehat{M}_{CC}^2 = 2u A_{55} \quad (5.A26)$$

fully contains the Goldstone sector as well as the two massive charged Higgs fields  $H^\pm$ . We can diagonalise this matrix

$$\bar{M}_{CC}^2 = U_{CC} \widehat{M}_{CC}^2 U_{CC}^T = \text{diag}(0, 0, 0, 4uK_0, 4uK_0) = \text{diag}(0, 0, 0, m_{H^\pm}^2, m_{H^\pm}^2). \quad (5.A27)$$

The rotation matrix  $U_{CC}$  can explicitly be decomposed as

$$U_{CC} = \begin{pmatrix} 1 & \mathbf{0}^T & 0 \\ \mathbf{0} & R_H & \mathbf{0} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix} \quad (5.A28)$$

with  $R_H$  given by the  $3 \times 3$  matrix in the bilinear field space which rotates  $\mathbf{k}$  to the Higgs basis with  $\mathbf{k} = (0, 0, 1)^T$ . Turning to the neutral sector, the matrix

$$\widehat{M}_{\text{neutral}}^2 = \gamma_3 (\mathcal{M} - 2u\widetilde{g}) \gamma_3^T \quad (5.A29)$$

contains the masses of the three neutral states  $h$ ,  $H$  and  $A$ . It is remarkable that  $\widehat{M}_{\text{neutral}}^2$  can in fact always (that is, not only for the tree-level case) be diagonalised by a rotation in the bilinear field space, *i.e.* a change of basis:

$$\bar{M}_{\text{neutral}} = R \widehat{M}_{\text{neutral}}^2 R^T = \text{diag}(m_1^2, m_2^2, m_3^2). \quad (5.A30)$$

The orthogonal rotation which diagonalises the  $8 \times 8$  scalar mass matrix in the canonical basis introduced above reads

$$\bar{U} = \begin{pmatrix} U_{CC} & 0_{5 \times 3} \\ 0_{3 \times 5} & R \end{pmatrix}. \quad (5.A31)$$

With this rotation matrix  $\bar{U}$  we get the transformations of  $\widehat{\Delta}$ ,  $\widehat{\Gamma}$  from the canonical basis to the Higgs basis:

$$\bar{\Delta}^\mu = \bar{U} \widehat{\Delta}^\mu \bar{U}^T, \quad \bar{\Gamma} = \bar{U} \widehat{\Gamma}. \quad (5.A32)$$

Gathering up the above results, the scalar mass matrix in the diagonal basis reads

$$\bar{M}_s^2 = 2u(\widetilde{g\mathbf{K}})_\mu \bar{\Delta}^\mu + \bar{\Gamma} \mathcal{M} \bar{\Gamma}^T = \text{diag}(0, 0, 0, m_{H^\pm}^2, m_{H^\pm}^2, m_1^2, m_2^2, m_3^2) \quad (5.A33)$$

This concludes our discussion on the structure of the scalar mass matrix at a charge-conserving minimum. We stress that all of the above is valid at any order in perturbation theory. We think that this illustrates the convenience of working with the gauge-invariant formalism.

As an example the scalar mass matrix squared (5.A29) for the tree-level THDM potential is computed in (5.68).

## B THDM couplings

In this section we present the analytic couplings of the general THDM in gauge invariant form. We recall that the indices  $a, b, c, d, e \in \{1, 2, 3\}$  denote the three neutral Higgs-boson scalars. The squared masses of the three neutral scalars are denoted by  $m_a^2$ . For repeatedly appearing indices  $a, b, c, d, e$  we employ as usual the sum convention. Note, that we do not distinguish between upper and lower indices. We note that the index of the masses of the scalars does not imply any summation, for instance the expression  $k^a m_a$  in (5.B1) is simply a vector with component  $a$ , whereas in an expression like  $\varepsilon_{abc} \bar{k}^a m_a^2$  we have to sum over the index  $a$  because it appears twice in tensor components, that is, in  $\varepsilon_{abc}$  and in  $\bar{k}^a$ .

### B.1 Scalar couplings

The cubic and quartic neutral scalar couplings can be found in section 4. We repeat here for convenience the 3-vector  $\bar{f}^a$  in terms of which some of the couplings below are expressed:

$$\bar{f}^a = 8K_0 (\eta_{00} \bar{k}^a + \bar{\eta}^a) - \bar{k}^a m_a^2. \quad (5.B1)$$

The scalar mass squared matrix  $\bar{\mathcal{M}}_{\text{neutral}}^2$  is given in (5.70). The 3-vectors  $\mathbf{x}$  and  $\mathbf{y}$  are defined in (5.80). The couplings then read:

$$h_a - h_b - G^0 \quad \bar{\lambda}_{abG^0} = \frac{1}{\sqrt{2K_0}} \varepsilon_{abc} \bar{k}^c (m_b^2 - m_a^2) \quad (5.B2)$$

$$h - G^0 - G^0 \quad \bar{\lambda}_{aG^0G^0} = \frac{1}{\sqrt{2K_0}} \bar{k}^a m_a^2 \quad (5.B3)$$

$$h_a - H^\pm - H^\pm \quad \bar{\lambda}_{aH^\pm H^\pm} = \frac{\bar{f}^a}{\sqrt{2K_0}} \quad (5.B4)$$

$$h_a - G^\pm - G^\pm \quad \bar{\lambda}_{aG^\pm G^\pm} = \frac{1}{\sqrt{2K_0}} \bar{k}^a m_a^2 \quad (5.B5)$$

$$h_a - G^\pm - H^\pm \quad \bar{\lambda}_{aG^\pm H^\pm} = \frac{1}{\sqrt{2K_0}} (m_a^2 - m_{H^\pm}^2) (x^a + iy^a) \quad (5.B6)$$

$$h_a - h_b - h_c - h_d \quad \bar{\lambda}_{abcd} = g_{abcd} + g_{cdab} + g_{acbd} + g_{bdac} + g_{adbc} + g_{bcad}, \quad (5.B7)$$

with the auxiliary function

$$g_{abcd} = \frac{\delta^{ab}}{2K_0} \left( \frac{\delta^{cd}}{2} \bar{\mathbf{k}}^T \bar{\mathcal{M}}_{\text{neutral}}^2 \bar{\mathbf{k}} + \bar{k}^c \bar{k}^d (m_a^2 + m_b^2 - m_c^2 - m_d^2) \right) + 4(\delta^{ab} - \bar{k}^a \bar{k}^b) (\bar{k}^c \bar{k}^d (\eta_{00} - \frac{m_{H^\pm}^2}{4K_0}) + \bar{\eta}^c \bar{k}^d + \bar{\eta}^d \bar{k}^c - \bar{\eta}^T \bar{\mathbf{k}} \delta^{cd}). \quad (5.B8)$$

$h_a - h_b - h_c - G^0$

$$\bar{\lambda}_{abcG^0} = f'_{abc} + f'_{bca} + f'_{cab}, \quad (5.B9)$$

with the auxiliary function

$$f'_{abc} = \frac{1}{2K_0} \left( 8K_0 (\delta^{ab} - \bar{k}^a \bar{k}^b) (\bar{\eta} \times \bar{\mathbf{k}})^c - \frac{1}{2} \delta^{ab} \varepsilon^{cde} (m_d^2 - m_e^2) \bar{k}^d \bar{k}^e - \varepsilon^{abd} \bar{k}^d (m_a^2 - m_b^2) \bar{k}^c \right). \quad (5.B10)$$

$h_a - h_b - G^0 - G^0$

$$\bar{\lambda}_{abG^0G^0} = \frac{1}{2K_0} \left( 8K_0 (\delta^{ab} - \bar{k}^a \bar{k}^b) (\eta_{00} - \frac{m_{H^\pm}^2}{4K_0} + \bar{\eta}^T \bar{\mathbf{k}}) + \bar{k}^a \bar{k}^b (m_a^2 + m_b^2) - \delta^{ab} \bar{\mathbf{k}}^T \bar{\mathcal{M}}_{\text{neutral}}^2 \bar{\mathbf{k}} + 2\varepsilon^{cad} \varepsilon^{cbe} \bar{k}^d \bar{k}^e m_c^2 \right) \quad (5.B11)$$

$h_a - G^0 - G^0 - G^0$

$$\bar{\lambda}_{aG^0G^0G^0} = \frac{3}{4K_0} \varepsilon^{abc} \bar{k}^b \bar{k}^c (m_b^2 - m_c^2) \quad (5.B12)$$

$G^0 - G^0 - G^0 - G^0$

$$\bar{\lambda}_{G^0G^0G^0G^0} = \frac{3}{2K_0} \bar{\mathbf{k}}^T \bar{\mathcal{M}}_{\text{neutral}}^2 \bar{\mathbf{k}} \quad (5.B13)$$

$h_a - h_b - H^\pm - H^\pm$

$$\bar{\lambda}_{abH^\pm H^\pm} = \frac{1}{2K_0} \left( \delta^{ab} (\bar{\mathbf{k}}^T \bar{\mathcal{M}}_{\text{neutral}}^2 \bar{\mathbf{k}} - 16K_0 \bar{\eta}^T \bar{\mathbf{k}}) + \bar{k}^a \bar{k}^b (8K_0 (\eta_{00} + \bar{\eta}^T \bar{\mathbf{k}}) - (m_a^2 + m_b^2)) + 8K_0 (\bar{k}^a \bar{\eta}^b + \bar{k}^b \bar{\eta}^a) \right) \quad (5.B14)$$

$h_a - G^0 - H^\pm - H^\pm$

$$\bar{\lambda}_{aG^0H^\pm H^\pm} = \frac{1}{2K_0} \varepsilon^{abc} \bar{f}^b \bar{k}^b \bar{k}^c \quad (5.B15)$$

$G^0 - G^0 - H^\pm - H^\pm$

$$\bar{\lambda}_{G^0G^0H^\pm H^\pm} = \frac{1}{2K_0} \bar{k}^a \bar{f}^a \quad (5.B16)$$

$h_a - h_b - G^\pm - G^\pm$

$$\bar{\lambda}_{abG^\pm G^\pm} = \frac{1}{2K_0} \left( 8K_0 (\delta^{ab} - \bar{k}^a \bar{k}^b) (\eta_{00} + \bar{\eta}^T \bar{\mathbf{k}}) - \delta^{ab} \bar{\mathbf{k}}^T \bar{\mathcal{M}}_{\text{neutral}}^2 \bar{\mathbf{k}} + \bar{k}^a \bar{k}^b (m_a^2 + m_b^2) \right) \quad (5.B17)$$

$h_a - G^0 - G^\pm - G^\pm$

$$\bar{\lambda}_{aG^0G^\pm G^\pm} = \frac{1}{4K_0} \varepsilon^{abc} \bar{k}^b \bar{k}^c (m_b^2 - m_c^2) \quad (5.B18)$$

$G^0 - G^0 - G^\pm - G^\pm$

$$\bar{\lambda}_{G^0G^0G^\pm G^\pm} = \frac{1}{2K_0} \bar{\mathbf{k}}^T \bar{\mathcal{M}}_{\text{neutral}}^2 \bar{\mathbf{k}} \quad (5.B19)$$

$h_a - h_b - G^\pm - H^\pm$

$$\bar{\lambda}_{abG^\pm H^\pm} = 4\bar{\eta}^c (x^c + iy^c) (\delta^{ab} - \bar{k}^a \bar{k}^b) + \frac{1}{2K_0} \left( \bar{k}^a (x^b + iy^b) (m_b^2 - m_{H^\pm}^2) + \bar{k}^b (x^a + iy^a) (m_a^2 - m_{H^\pm}^2) - \bar{k}^c (x^c + iy^c) m_c^2 \delta^{ab} \right) \quad (5.B20)$$

$h_a - G^0 - G^\pm - H^\pm$ 

$$\bar{\lambda}_{aG^0G^\pm H^\pm} = \frac{1}{2K_0} \varepsilon^{abc} \bar{k}^b (x^c + iy^c) (m_{H^\pm}^2 - m_c^2) \quad (5.B21)$$

 $G^0 - G^0 - G^\pm - H^\pm$ 

$$\bar{\lambda}_{G^0G^0G^\pm H^\pm} = \frac{1}{2K_0} \bar{k}^a (x^a + iy^a) m_a^2 \quad (5.B22)$$

## B.2 Charged quartic couplings

 $H^\pm - H^\pm - H^\pm - H^\pm$ 

$$\bar{\lambda}_{H^\pm H^\pm H^\pm H^\pm} = \frac{1}{K_0} \left( \bar{\mathbf{k}}^T \bar{\mathcal{M}}_{neutral}^2 \bar{\mathbf{k}} - 16K_0 \bar{\boldsymbol{\eta}}^T \bar{\mathbf{k}} \right) \quad (5.B23)$$

 $G^\pm - G^\pm - H^\pm - H^\pm$ 

$$\bar{\lambda}_{G^\pm G^\pm H^\pm H^\pm} = \frac{1}{2K_0} \left( 8K_0 (\eta_{00} - \frac{m_{H^\pm}^2}{4K_0} + \bar{\boldsymbol{\eta}}^T \bar{\mathbf{k}}) - 2\bar{\mathbf{k}}^T \bar{\mathcal{M}}_{neutral}^2 \bar{\mathbf{k}} + \text{Tr}(\bar{\mathcal{M}}_{neutral}^2) \right) \quad (5.B24)$$

 $G^\pm - G^\pm - G^\pm - G^\pm$ 

$$\bar{\lambda}_{G^\pm G^\pm G^\pm G^\pm} = \frac{1}{K_0} \bar{\mathbf{k}}^T \bar{\mathcal{M}}_{neutral}^2 \bar{\mathbf{k}} \quad (5.B25)$$

 $G^\pm - H^\pm - H^\pm - H^\pm$ 

$$\bar{\lambda}_{G^\pm H^\pm H^\pm H^\pm} = (\bar{\boldsymbol{\eta}}^a - \bar{k}^a \frac{m_a^2}{K_0}) (x^a + iy^a) \quad (5.B26)$$

 $G^\pm - G^\pm - G^\pm - H^\pm$ 

$$\bar{\lambda}_{G^\pm G^\pm G^\pm H^\pm} = \frac{1}{K_0} \bar{k}^a (x^a + iy^a) m_a^2 \quad (5.B27)$$

## B.3 Vector - scalar couplings

 $h_a - G^0 - Z$ 

$$\bar{g}_{aG^0Z} = 2 \frac{\sqrt{m_Z}}{\sqrt{K_0}} \bar{k}^a \quad (5.B28)$$

 $H^+ - H^+ - Z$ 

$$\bar{g}_{H^+H^+Z} = -i\sqrt{2} \frac{m_Z - 2m_W}{\sqrt{K_0} m_Z} \quad (5.B29)$$

 $G^+ - G^+ - Z$ 

$$\bar{g}_{G^+G^+Z} = -i\sqrt{2} \frac{m_Z - 2m_W}{\sqrt{K_0} m_Z} \quad (5.B30)$$

 $H^+ - H^+ - \gamma$ 

$$\bar{g}_{H^+H^+\gamma} = 2i \frac{\sqrt{2m_W(m_Z - m_W)}}{\sqrt{K_0} m_Z} \quad (5.B31)$$

 $G^+ - G^+ - \gamma$ 

$$\bar{g}_{G^+G^+\gamma} = 2i \frac{\sqrt{2m_W(m_Z - m_W)}}{\sqrt{K_0} m_Z} \quad (5.B32)$$

 $h_a - H^+ - W^+$ 

$$\bar{g}_{aH^+W^+} = i \frac{\sqrt{2m_W}}{\sqrt{K_0}} (x^a + iy^a) \quad (5.B33)$$

 $h_a - G^+ - W^+$ 

$$\bar{g}_{aG^+W^+} = 2 \frac{\sqrt{m_W}}{\sqrt{2K_0}} \bar{k}^a \quad (5.B34)$$

$$G^0 - G^+ - W^+ \quad \bar{g}_{G^0 G^+ W^+} = -i \frac{\sqrt{2m_W}}{\sqrt{K_0}} \quad (5.B35)$$

$$h_a - Z - Z \quad \bar{g}_{aZZ} = 4 \frac{m_Z}{\sqrt{2K_0}} \bar{k}^a \quad (5.B36)$$

$$h_a - W^+ - W^+ \quad \bar{g}_{aW^+W^+} = 4 \frac{m_W}{\sqrt{2K_0}} \bar{k}^a \quad (5.B37)$$

$$G^+ - Z - W^+ \quad \bar{g}_{G^+ Z W^+} = 2i \frac{\sqrt{2m_W}}{\sqrt{K_0 m_Z}} (m_W - m_Z) \quad (5.B38)$$

$$G^+ - \gamma - W^+ \quad \bar{g}_{G^+ \gamma W^+} = 2i \frac{\sqrt{2(m_Z - m_W)}}{\sqrt{K_0 m_Z}} m_W \quad (5.B39)$$

$$h_a - h_b - Z - Z \quad \bar{g}_{abZZ} = \frac{4m_Z^2}{K_0} \delta^{ab} \quad (5.B40)$$

$$h_a - h_b - W^+ - W^+ \quad \bar{g}_{abW^+W^+} = \frac{4m_W^2}{K_0} \delta^{ab} \quad (5.B41)$$

$$h_a - H^+ - Z - W^+ \quad \bar{g}_{aH^+ Z W^+} = \frac{2 \cos(\theta_W)}{K_0} \left( (2m_W^2 - m_Z^2)(x^a - iy^a) + im_Z^2 \varepsilon^{abc} (x^b - iy^b) \bar{k}^b \right) \quad (5.B42)$$

$$h_a - G^+ - Z - W^+ \quad g_{aG^+ Z W^+} = 4i \frac{m_W}{m_Z} \frac{m_W^2 - m_Z^2}{K_0} \bar{k}^a \quad (5.B43)$$

$$h_a - H^+ - \gamma - W^+ \quad \bar{g}_{aG^+ \gamma W^+} = 4 \frac{m_W^2}{K_0} \frac{\sqrt{m_Z^2 - m_W^2}}{m_Z} (x^a - iy^a) \quad (5.B44)$$

$$h_a - G^+ - \gamma - W^+ \quad \bar{g}_{aG^+ \gamma W^+} = 4i \frac{m_W^2}{K_0} \frac{\sqrt{m_Z^2 - m_W^2}}{m_Z} \bar{k}^a \quad (5.B45)$$

$$G^0 - G^0 - Z - Z \quad \bar{g}_{G^0 G^0 ZZ} = 4 \frac{m_Z^2}{K_0} \quad (5.B46)$$

$$G^0 - G^0 - W^+ - W^+ \quad \bar{g}_{G^0 G^0 W^+ W^+} = 4 \frac{m_W^2}{K_0} \quad (5.B47)$$

$$G^0 - G^+ - Z - W^+ \quad \bar{g}_{G^0 G^+ Z W^+} = 4 \frac{m_W}{m_Z} \frac{m_Z^2 - m_W^2}{K_0} \quad (5.B48)$$

$$G^0 - G^+ - \gamma - W^+ \quad \bar{g}_{G^0 G^+ \gamma W^+} = -4 \frac{m_W^2}{K_0} \frac{\sqrt{m_Z^2 - m_W^2}}{m_Z} \quad (5.B49)$$

$$H^+ - H^+ - Z - Z \quad \bar{g}_{H^+ H^+ ZZ} = 4 \frac{(m_Z^2 - 2m_W^2)^2}{K_0 m_Z^2} \quad (5.B50)$$

$G^+ - G^+ - Z - Z$ 

$$\bar{g}_{G^+G^+ZZ} = 4 \frac{(m_Z^2 - 2m_W^2)^2}{K_0 m_Z^2} \quad (5.B51)$$

 $H^+ - H^+ - Z - \gamma$ 

$$\bar{g}_{H^+H^+ZZ} = 8m_W(2m_W^2 - m_Z^2) \frac{\sqrt{m_Z^2 - m_W^2}}{K_0 m_Z^2} \quad (5.B52)$$

 $G^+ - G^+ - Z - \gamma$ 

$$\bar{g}_{G^+G^+Z\gamma} = 8m_W(2m_W^2 - m_Z^2) \frac{\sqrt{m_Z^2 - m_W^2}}{K_0 m_Z^2} \quad (5.B53)$$

 $H^+ - H^+ - \gamma - \gamma$ 

$$\bar{g}_{H^+H^+\gamma\gamma} = 8m_W^2 \frac{m_Z^2 - m_W^2}{K_0 m_Z^2} \quad (5.B54)$$

 $G^+ - G^+ - \gamma - \gamma$ 

$$\bar{g}_{G^+G^+\gamma\gamma} = 8m_W^2 \frac{m_Z^2 - m_W^2}{K_0 m_Z^2} \quad (5.B55)$$

 $H^+ - H^+ - W^+ - W^+$ 

$$\bar{g}_{H^+H^+W^+W^+} = \frac{4m_W^2}{K_0} \quad (5.B56)$$

 $G^+ - G^+ - W^+ - W^+$ 

$$\bar{g}_{G^+G^+W^+W^+} = \frac{4m_W^2}{K_0} \quad (5.B57)$$

#### B.4 Yukawa couplings

We note that we only give here the Yukawa couplings involving right-handed Weyl spinors. Their analogous parts for left-handed spinors are simply obtained through complex conjugation.

 $h - f_L^\dagger - f_R$ 

$$\bar{\xi}_a^u = \frac{1}{\sqrt{2K_0}} (\sigma_a)^\alpha{}_\beta \bar{\kappa}_\alpha^* \bar{U}^\beta \quad (5.B58a)$$

$$\bar{\xi}_a^d = \frac{1}{\sqrt{2K_0}} (\sigma_a)^\alpha{}_\beta \bar{D}_\alpha \bar{\kappa}^\beta \quad (5.B58b)$$

$$\bar{\xi}_a^e = \frac{1}{\sqrt{2K_0}} (\sigma_a)^\alpha{}_\beta \bar{\mathcal{E}}_\alpha \bar{\kappa}^\beta \quad (5.B58c)$$

 $G^0 - f_L^\dagger - f_R$ 

$$\bar{\xi}_{G^0}^u = \frac{i}{\sqrt{2K_0}} \bar{\kappa}_\alpha^* \bar{U}^\alpha \quad (5.B59a)$$

$$\bar{\xi}_{G^0}^d = -\frac{i}{\sqrt{2K_0}} \bar{D}_\alpha \bar{\kappa}^\alpha \quad (5.B59b)$$

$$\bar{\xi}_{G^0}^e = -\frac{i}{\sqrt{2K_0}} \bar{\mathcal{E}}_\alpha \bar{\kappa}^\alpha \quad (5.B59c)$$

 $H^\pm - f_L^\dagger - f'_R$ 

$$\bar{\xi}_{H^+}^{ud} = -\frac{i}{\sqrt{K_0}} \varepsilon_{\alpha\beta} \bar{\kappa}^\alpha \bar{U}^\beta \quad (5.B60a)$$

$$\bar{\xi}_{H^-}^{ud} = -\frac{i}{\sqrt{K_0}} \varepsilon^{\alpha\beta} \bar{D}_\alpha \bar{\kappa}_\beta^* \quad (5.B60b)$$

$$\bar{\xi}_{H^-}^{\nu e} = -\frac{i}{\sqrt{K_0}} \varepsilon^{\alpha\beta} \bar{\mathcal{E}}_\alpha \bar{\kappa}_\beta^* \quad (5.B60c)$$

$$G^\pm - f_L^\dagger - f'_R$$

$$\bar{\xi}_{G^+}^{ud} = \frac{i}{\sqrt{K_0}} \bar{\kappa}_\alpha^* \bar{U}^\alpha \quad (5.B61a)$$

$$\bar{\xi}_{G^-}^{ud} = \frac{i}{\sqrt{K_0}} \bar{D}_\alpha \bar{\kappa}^\alpha \quad (5.B61b)$$

$$\bar{\xi}_{G^-}^{\nu e} = \frac{i}{\sqrt{K_0}} \bar{E}_\alpha \bar{\kappa}^\alpha \quad (5.B61c)$$

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## Chapter 6

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# Grand unification and the Planck scale: An $SO(10)$ example of radiative symmetry breaking

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## 1 Introduction

Unification of the Standard Model (SM) gauge groups into a grand unified theory (GUT) [109, 110, 111] remains an attractive new-physics scenario: GUTs have the potential to (i) provide an explanation for the seemingly coincidental near-equality of SM gauge couplings at the high-energy scale  $M_{\text{GUT}} \approx 10^{15}$  GeV, see *e.g.* [112, 113]; (ii) (partially) explain the observed mass spectrum by unifying the fermionic representations [114, 115, 116]; (iii) account for neutrino masses [117, 118, 119, 120, 121] with a suitable see-saw mechanism [122, 123, 124]; and (iv) offer a scenario for leptogenesis, see *e.g.* [125, 126, 127, 128].

Yet, the explanatory power of a GUT – manifest in relations among SM couplings and charges – comes with the caveat of having to construct a viable mechanism to break the large gauge group in just the right way such as to obtain the SM. The unified gauge group can be reduced to the SM via spontaneous symmetry breaking in a suitable scalar potential. Most GUT analyses to date simply assume that all breaking chains, which are group-theoretically possible, can be realised by some – potentially contrived and complicated – scalar potential. Oftentimes, the latter is not explicitly specified. Indeed, such potentials remain largely arbitrary without specific knowledge about microscopic boundary conditions in the theory space of couplings, for instance, at the Planck scale. As a result, the plethora of SM parameters is effectively traded for a plethora of admissible breaking potentials. In particular, currently viable GUTs require more free parameters than the SM itself<sup>1</sup>. In contrast to the Yukawa and gauge couplings, the (quartic) couplings entering the GUT potential are not directly constrained by the experimental data.

On a seemingly unrelated note, in quantum gravity, any phenomenology is hard to come by. However, several quantum-gravity scenarios hold the promise to predict Planck-scale boundary conditions, both for the gauge-Yukawa sector and the scalar potential; in the context of GUTs, see [129, 130] for asymptotic safety and *e.g.* [131, 132, 133, 134] for string theory.

Quantum gravity (QG) and grand unification are thus two friends in need. Quantitative progress requires a link between Planck-scale initial conditions and GUT phenomenology. Naturally, such a link would benefit GUT model builders and QG phenomenologists alike:

- GUTs would aid QG phenomenology: The requirement of viable initial conditions promises to indirectly constrain any predictive QG scenario.
- QG would aid GUT model-building: Any predictive QG scenario will, in turn, predict/constrain the Planckian parameter space and thereby may exclude (*i.e.*, be incompatible with) specific GUTs.

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<sup>1</sup>For instance, the  $SO(10)$  model with  $\mathbf{45}_H \oplus \mathbf{126}_H$  possesses 16 parameters in the scalar potential [124]. Considering its realistic  $\mathbf{10}_H \oplus \mathbf{45}_H \oplus \mathbf{126}_H$  extension would further increase this number.



To build this link, progress on both ends is required: On the one hand, Planck-scale predictions of QG scenarios have to be obtained and solidified. On the other hand, viable Planck-scale initial conditions have to be identified in specific GUTs.

In the present work, we focus on the GUT side of progress. In particular, we point out that the requirement of viable radiative symmetry breaking – or rather the absence of non-viable radiative symmetry breaking – places strong constraints on the underlying Planck-scale initial conditions. To do so, we treat the GUT as an effective field theory (EFT)<sup>2</sup>. The respective grand-unified effective field theory (GUEFT) is fully specified by its symmetry group  $\mathcal{G}_{\text{GUT}}$  – including a local gauge group  $\mathcal{G}_{\text{GUT}}^{(\text{local})}$  as well as potential additional global symmetries  $\mathcal{G}_{\text{GUT}}^{(\text{global})}$  – and the set of fermionic as well as scalar representations  $\mathcal{F}_{\text{GUT}}$  and  $\mathcal{S}_{\text{GUT}}$ , respectively. The resulting EFT action includes all symmetry invariants that can be constructed from the gauge and matter fields. The initial conditions for the corresponding couplings specify an explicit realisation of the GUEFT.

We then assume that some UV dynamics provides said initial conditions of the GUEFT at some ultraviolet (UV) scale. In the following, we identify this scale with the Planck scale and hence the UV dynamics with QG. Still, the general framework presented here applies more widely.

Once the initial conditions are specified at the Planck scale  $M_{\text{Pl}}$ , the renormalisation group (RG) equations evolve each such realisation towards lower energies, in particular down to the electroweak scale where (some of) the couplings need to be matched to experiment. The evolution with RG scale  $\mu$  is given by the beta-functions, *i.e.*,

$$\beta_{c_i} = \mu \frac{\partial}{\partial \mu} c_i. \quad (6.1)$$

Here, we focus solely on the perturbative regime. This allows us to make use of (i) the computational toolkit PyR@TE 3 [9] to determine the full set of perturbative beta-functions and of (ii) perturbative techniques for multidimensional effective potentials [136] (see also [137]). Non-perturbative RG schemes such as the functional RG [138] (see [139, 140, 141, 142, 143] for multidimensional effective potentials in the context of condensed-matter theory), in principle, allow to extend our framework to the non-perturbative regime. We leave such an extension and, in particular, the inclusion of gravitational fluctuations and thus any trans-Planckian dynamics at  $\mu > M_{\text{Pl}}$  (cf. Sec. 6.3 for an outlook), for future work.

In this perturbative GUEFT setup, we will analyse the question of how radiative symmetry breaking constrains the viable parameter space: We propose a blueprint that can be applied to any GUEFT and demonstrate its application in a specific  $SO(10)$  example.

This chapter is organised as follows. In Sec. 2, we present the abstract blueprint for how to place theoretical and phenomenological constraints on the parameter space at  $M_{\text{Pl}}$ . In particular, our blueprint encompasses a novel set of systematic constraints on a viable (perturbative) scalar potential. In the Sec 3, we review the required and previously mentioned (see (i) and (ii) in the paragraph above) perturbative techniques. In Sec. 4 we focus on a particular non-supersymmetric  $SO(10)$  model. We discuss the possible breaking chains, including those that lead to the SM (admissible) but also many that do not (non-admissible). In Sec. 5, we present the explicit results for said model. In particular, we demonstrate how the Planckian parameter space is constrained with each individual constraint in the blueprint. In Sec. 6, we close with a wider discussion of our results and an outlook on future work. In particular, we briefly comment on how to (i) extend our results to a GUEFT with a realistic Yukawa sector and (ii) eventually connect these to QG scenarios that may set the Planck-scale initial conditions. Technical details on the one-loop RG-improved potential (App. A), the tree-level stability conditions (App. B), a quantitative measure of perturbativity (App. C), cf. [144], the explicit scalar potentials (App. D), and the perturbative beta-functions (App. E) are delegated into appendices.

Readers who are not interested in the methodology of RG-improvement or the details of the specific GUT model are encouraged to read Sec. 2, 5, and 6, which are kept accessible to a broad audience.

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<sup>2</sup>To prevent potential confusion, we note that by the term EFT, we refer to a quantum field theory with unknown initial conditions and a finite cutoff scale. In principle, such a theory includes all dimensional couplings but in the present chapter we will focus on marginal couplings only. In particular, we will not address the treatment of higher-order operators such as, for instance, in Standard Model Effective Field Theory (SMEFT) [135].

## 2 The blueprint: How to constrain grand-unified effective field theories

The following section can be read in two ways: either as a physical description of the methodology applied to the specific  $SO(10)$  models in this work; or as a more general blueprint applicable to any grand-unified effective field theory (GUEFT).

We define a GUEFT by its symmetry group  $\mathcal{G}_{\text{GUT}}$  – including a local gauge group  $\mathcal{G}_{\text{GUT}}^{(\text{local})}$  as well as potential additional global symmetries  $\mathcal{G}_{\text{GUT}}^{(\text{global})}$  – and the set of fermionic as well as scalar representations  $\mathcal{F}_{\text{GUT}}$  and  $\mathcal{S}_{\text{GUT}}$ , respectively. For instance, the two models that we will investigate in Sec. 5 as an explicit example, are denoted by

$$\left\{ \mathcal{G}_{\text{GUT}}^{(\text{local})}, \mathcal{F}_{\text{GUT}}, \mathcal{S}_{\text{GUT}} \right\} = \left\{ SO(10), \mathbf{16}_F^{(i)}, \mathbf{45}_H \right\}, \quad (\text{see Sec. 5.1}) \quad (6.2)$$

$$\left\{ \mathcal{G}_{\text{GUT}}^{(\text{local})}, \mathcal{F}_{\text{GUT}}, \mathcal{S}_{\text{GUT}} \right\} = \left\{ SO(10), \mathbf{16}_F^{(i)}, \mathbf{16}_H \oplus \mathbf{45}_H \right\}. \quad (\text{see Sec. 5.2}) \quad (6.3)$$

Herein,  $i = 1, 2, 3$  denotes a family index. The model in Eq. (6.3) also exhibits an additional global symmetry,  $\mathcal{G}_{\text{GUT}}^{(\text{global})} = U(1)$ , cf. App. D.2.

The purpose of the following blueprint is to constrain the possibility that such a GUEFT is a UV extension of the SM. We distinguish the notions of UV extension and UV completion. By UV extension of the SM, we refer to some high-energy EFT which contains the SM at lower scales. In particular, we do not demand that the UV extension itself is UV-complete, *i.e.*, extends to arbitrarily high energies without developing pathologies. By UV completion of the SM, we refer to a UV extension which moreover is UV-complete.

In principle, the respective EFT action includes all possible symmetry invariants that can be constructed from the gauge and matter fields. For this work, however, we will focus on the marginal couplings only. This amounts to restricting the EFT-analysis to the perturbative regime around the free fixed point. Close to the free fixed point, canonically irrelevant couplings will be power-law suppressed.

Moreover, we omit potentially sizeable mass terms. In the presence of mass terms, the following constraints have to be re-interpreted but are still of relevance for phenomenology. We discuss this further in Sec. 6.2.

In consequence, the GUEFT is parameterised by the initial conditions of all its marginal couplings at an *a priori* unknown high-energy cutoff scale. In the following, we will tentatively identify the cutoff with the Planck scale  $M_{\text{Pl}}$ .

In this setup, we first focus on a set of constraints in the scalar sector. These arise from radiative symmetry breaking and are necessary but not sufficient for the GUEFT to be a UV extension of the SM.

(I.a) We demand tree-level stability at  $M_{\text{Pl}}$ .

(I.b) We demand the absence of Landau poles between the Planck scale  $M_{\text{Pl}}$  and the first symmetry-breaking scale  $M_{\text{GUT}}$ . (In addition, we define a perturbativity criterion, cf. [144] as well as App. C, and demand that the GUEFT remains perturbative between  $M_{\text{GUT}}$  and  $M_{\text{Pl}}$ .)

(I.c) We demand that the deepest minimum induced by radiative symmetry breaking<sup>3</sup>, is admissible, *i.e.*, the respective vacuum expectation value (vev) remains invariant under the Standard Model gauge group  $\mathcal{G}_{\text{SM}} \subset \mathcal{G}_{\text{GUT}}^{(\text{local})}$ .

Each of these necessary conditions may be applied on their own to constrain the set of initial conditions at  $M_{\text{Pl}}$ . Applying the constraints in the above order turns out to be most efficient as we will explicitly demonstrate in Sec. 5.

On top of these constraints on the scalar potential, one may apply more commonly addressed phenomenological constraints on the gauge-Yukawa sector, namely:

<sup>3</sup>Here, we do not account for the possibility of a meta-stable but sufficiently long-lived minimum, cf. Sec. 6.

- (II.a) The requirement of gauge coupling unification and of a sufficiently long lifetime of the proton to avoid experimental proton-decay bounds, cf. [145, 146, 147, 148] for previous work;
- (II.b) The requirement of a viable Yukawa sector. (Realising a viable Yukawa sector is in itself a very non-trivial question [149, 115, 150, 122, 151, 152, 153, 154, 155, 156, 157, 158, 159, 148, 153, 134].)

The necessary conditions (I) in the scalar sector and (II) in the gauge-Yukawa sector do, in principle, depend on each other<sup>4</sup>. Ideally, one would thus want to include the gauge and Yukawa couplings in the set of random initial conditions and apply (I) and (II) simultaneously. Alternatively, one may fix the gauge and Yukawa couplings to approximate phenomenological values, see [160, 148, 161, 113]. Subsequently, one has to then crosscheck that the constraints which we obtain from (I) are not significantly altered when varying initial conditions in the gauge-Yukawa sector (see Sec. 5).

The above two sets of constraints can be viewed as necessary consistency constraints for a specific realisation of a GUEFT to be a viable UV extension of the SM. In that sense, they realise a set of exclusion principles in a top-down approach to grand unification. We refer to a specific realisation of a GUEFT as admissible if it obeys the first set of constraints (I.a), (I.b), and, in particular, (I.c). We refer to a specific realisation of a GUEFT as viable if it obeys both sets of constraints (I) and (II) (see Sec. 4.2).

In addition, one may specify an underlying UV completion. This extends the GUEFT to arbitrarily high scales above  $M_{\text{Pl}}$  and – for each individual underlying quantum-gravity scenario – typically results in additional constraints.

- (III) Strong additional constraints may arise from demanding that the initial conditions can arise from a specific assumption about the transplanckian theory, *i.e.*, from a specific model or assumption about quantum gravity.

We review the significance of such constraints alongside existing literature as part of the discussion in Sec. 6. An explicit implementation is left to future work.

### 3 Methodology: RG-flow, effective potential, and breaking patterns

#### 3.1 Renormalisation group-improved one-loop potential

In this work we are interested in the radiative minima of the potential generated due to the renormalisation group (RG) flow of the quartic couplings. Hence the renormalisation group equations (RGEs) constitute the principal tool in our analysis. The schematic form of the one-loop RGEs are given in the seminal papers [29, 30, 31], see also the recent discussion [38, 39, 10].

In the absence of mass terms in the tree-level potential, any non-trivial minimum must be generated by higher-order corrections to the scalar potential. The dependence of loop corrections on the arbitrary RG scale can be alleviated using perturbative techniques of RG-improvement of the scalar potential [81, 70, 71, 162, 72, 75, 28, 77, 78]. We caution that the computation of the full quantum potential may be impacted by higher-order operators, scheme dependence, and non-perturbative effects, cf. [163, 164, 165, 166] for analysis in the context of the SM Higgs potential. In any case, perturbative RG-improvement techniques provide an important step towards the full quantum potential and are expected to provide a better approximation than fixed-order perturbation theory. For these reasons, we employ the RG-improved one-loop potential to study the breaking patterns of a GUT model, in a formalism that we now briefly review.

Considering a gauge theory with a scalar multiplet denoted by  $\phi$ , and using the conventions of [136], the one-loop contributions to the effective potential can be put in the form

$$V^{(1)} = \mathbb{A} + \mathbb{B} \log \frac{\varphi^2}{\mu_0^2} \tag{6.4}$$

---

<sup>4</sup>Interdependence of (I) and (II) occurs not only via higher-loop corrections. For instance, the gauge coupling will impact the radiative symmetry-breaking scale. At the same time, the symmetry-breaking scale will impact the RG flow of the gauge couplings, even at one-loop order.

where  $\mu_0$  is the arbitrary renormalisation scale and where  $\varphi = \sqrt{\phi_i \phi^i}$ . The quantities  $\mathbb{A}$  and  $\mathbb{B}$  receive contributions from the scalar, gauge and Yukawa sectors of the theory. In the  $\overline{\text{MS}}$  scheme and working in the Landau gauge, they can be expressed (see *e.g.* [136]) as

$$\mathbb{A} = \frac{1}{64\pi^2} \sum_{i=s,g,f} n_i \text{Tr} \left[ M_i^4 \left( \log \frac{M_i^2}{\varphi^2} - C_i \right) \right], \quad (6.5)$$

$$\mathbb{B} = \frac{1}{64\pi^2} \sum_{i=s,g,f} n_i \text{Tr} (M_i^4). \quad (6.6)$$

where the numerical constants  $n_i$  and  $C_i$  take the values

$$\begin{aligned} n_s &= 1, & n_g &= 3, & n_f &= -2, \\ C_s &= \frac{3}{2}, & C_g &= \frac{5}{6}, & C_f &= \frac{3}{2}, \end{aligned} \quad (6.7)$$

and where  $M_{s,g,f}$  respectively stand for the field-dependent mass matrices of the scalars, gauge bosons and fermions of the model. The first two matrices can be straightforwardly computed once the scalar potential and the gauge generators of the scalar representations have been fixed:

$$(M_s^2)_{ij} = \frac{\partial^2 V^{(0)}}{\partial \phi^i \partial \phi^j} \quad (6.8)$$

$$(M_g^2)_{AB} = \frac{1}{2} g^2 \{T_A, T_B\}_{ij} \phi^i \phi^j \quad (6.9)$$

The  $\mathbf{45}_H$  and  $\mathbf{16}_H \oplus \mathbf{45}_H$  models considered in this work contain no Yukawa interactions, hence we will take the  $M_f$  mass matrix to vanish.

The dependence of  $V^{(1)}$  on the renormalisation scale  $\mu_0$  is an artefact of working at fixed order in perturbation theory, and introduces arbitrariness in the computations. In some circumstances, simple prescriptions on the value of  $\mu_0$  may be given that are appropriate for computations involving the quantum potential. Such prescriptions are in particular suitable for single-scale models, thus giving a reasonable approximation of the effective potential around this one scale. For computations involving a wider range of energy scales, or in theories with multiple characteristic scales (*e.g.* several vevs and/or masses, possibly spanning over orders of magnitude), one inevitably encounters large logarithms. Various RG-improvement techniques were developed to resum such large logarithms (see *e.g.* [70, 71, 162, 72, 75, 28, 77, 78]), with the aim of yielding a well-behaved quantum potential for multi-scale theories and/or over large energy ranges.

The  $SO(10)$  model considered in this work (and generally any GUT model) is a multi-scale theory, requiring an appropriate procedure of RG-improvement. Here we briefly review the method developed in [136] and further extended in [137] to the case of classically scale-invariant potentials. The starting point is to consider the Callan-Symanzik equation satisfied by the all-order quantum potential, stating that the total derivative of the effective potential with respect to the renormalisation scale vanishes:

$$\frac{dV^{\text{eff}}}{d \log \mu_0} = \left( \frac{\partial}{\partial \log \mu_0} + \sum_i \beta(g_i) \frac{\partial}{\partial g_i} - \phi^i \gamma^{ij} \frac{\partial}{\partial \phi^j} \right) V^{\text{eff}} = 0. \quad (6.10)$$

The above relation describes the independence of the quantum potential on the renormalisation scale, given that the couplings of the theory are evolved according to their beta-functions, and the field strength renormalisation values according to their anomalous dimension matrix  $\gamma$ . Following [136, 137] and using Eq. (6.10), we may simultaneously promote the RG-scale  $\mu_0$  to a field-dependent quantity  $\mu(\phi^i)$ , and the couplings and fields to  $\mu$ -dependent quantities. Formally, we have

$$\begin{aligned} \mu_0 &\longrightarrow \mu(\phi^i), \\ \lambda &\longrightarrow \lambda(\mu(\phi^i)), \\ \phi &\longrightarrow \phi(\mu(\phi^i)). \end{aligned} \quad (6.11)$$

The cornerstone of the RG-improvement procedure presented in [136] is to note that for each point in the field space, and as long as perturbation theory holds, there exists a renormalisation scale  $\mu_*$  such that the one-loop corrections  $V^{(1)}$  vanish<sup>5</sup>:

$$V^{(1)}(\phi^i, \lambda^i; \mu_*) = \mathbb{A}(\phi^i(\mu_*), \lambda^i(\mu_*)) + \mathbb{B}(\phi^i(\mu_*), \lambda^i(\mu_*)) \log \frac{\varphi^2}{\mu_*^2} = 0. \quad (6.12)$$

The above relation gives the implicit definition of the field-dependent scale  $\mu_*(\phi^i)$ , and allows for resummation of a certain class of logarithmic contributions [136]. The full one-loop effective potential is then given by its tree-level contribution, with the couplings and fields are evaluated at the scale  $\mu_*$ :

$$V^{\text{eff}}(\phi^i) = V^{(0)}(\phi^i; \mu_*(\phi^i)). \quad (6.13)$$

The RG-improved effective potential takes the same form as the tree-level potential with field-dependent couplings. This provides valuable insight on the conditions of radiative symmetry breaking in classically scale-invariant models [136, 167, 137]. In particular, a necessary condition for symmetry breaking to occur is that the tree-level stability conditions of the scalar potential must be violated at some scale along the RG-flow. Crucially, this observation allows to determine whether the breaking of the  $SO(10)$  symmetry towards a specific subgroup will happen at all, given some initial conditions for the quartic couplings at the high energy scale.

### 3.2 Minimisation of the RG-improved potential

In order to identify the breaking patterns of the model, one needs to evaluate the depth of the RG-improved potential at the minimum for each relevant vacuum configuration. The set of stationary-point equations of the RG-improved potential are derived in App. A, and would in principle need to be solved numerically in order to determine the position of its global minimum. Such a numerical minimisation procedure, however, can be computationally very costly and therefore rather inappropriate in the context of this work, where a scan over a large number of points is to be performed. Instead, we propose a simple procedure allowing to estimate (rather accurately) the position and depth of the minimum of the RG-improved potential.

In App. A, we derive the radial stationary-point equation (6.A9) which restricts the position of the minimum to an  $(n-1)$ -dimensional hypersurface in the  $n$ -dimensional field space. In the  $\mathcal{O}(\hbar)$  approximation, where only contributions that are formally of first order in perturbation theory are retained, it reads:

$$4V^{\text{eff}} + 2\mathbb{B} = 0, \quad \frac{d\mathbb{A}}{dt} \approx 0 \quad \text{and} \quad \frac{d\mathbb{B}}{dt} \approx 0. \quad (6.14)$$

As discussed in App. A.1, the quantity  $\mathbb{B}$  must be strictly positive at a minimum, thus implying

$$V^{\text{eff}} < 0. \quad (6.15)$$

For all field values,  $V^{\text{eff}}$  takes its classically scale-invariant tree-level form. In turn, the tree-level stability conditions have to be violated at the RG-scale  $\mu_*^{\text{min}}$ , defined such that

$$\frac{\partial V^{\text{eff}}}{\partial \langle \phi \rangle^i}(\langle \phi \rangle^i; \mu_*^{\text{min}}(\langle \phi \rangle^i)) = 0 \quad \text{and} \quad V^{(1)}(\langle \phi \rangle^i; \mu_*^{\text{min}}(\langle \phi \rangle^i)) = 0. \quad (6.16)$$

More concretely,  $\mu_*^{\text{min}}$  is the value of the RG-improved scale  $\mu_*$  evaluated at the vacuum  $\langle \phi \rangle$ . For some arbitrary high-energy scale  $\mu_0$  at which the tree-level potential is assumed to be stable, there must exist a scale  $\mu_{\text{GW}}$  characterising the breaking of tree-level stability, such that

$$\mu_*^{\text{min}} < \mu_{\text{GW}} < \mu_0. \quad (6.17)$$

Hence, at the RG-scale  $\mu_{\text{GW}}$  the tree-level potential (without RG-improvement) develops flat directions, along which a minimum will be radiatively generated through the Gildener-Weinberg mechanism [168], see also App. A.2.

<sup>5</sup>In presence of negative eigenvalues in the mass matrices, one may instead require the real part of the one-loop corrections to vanish.

A first important observation is that  $\mu_{\text{GW}}$  gives an upper bound on the value of  $\mu_*$  at the minimum. This bound can be further refined by observing that an additional scale  $\tilde{\mu}$  can be identified, at which the quantity  $\tilde{V}^{(0)}$  defined as

$$\tilde{V}^{(0)} \equiv V^{\text{eff}} + \frac{1}{2}\mathbb{B} \quad (6.18)$$

develops flat directions, see App. A. Since  $\mathbb{B} > 0$  near the minimum, one has

$$\mu_*^{\text{min}} < \tilde{\mu} < \mu_{\text{GW}}, \quad (6.19)$$

such that  $\tilde{\mu}$  provides an improved upper bound for  $\mu_*^{\text{min}}$ . In practice, the scale  $\tilde{\mu}$  provides, in most cases, a remarkably accurate estimate for  $\mu_*^{\text{min}}$ , cf. Appendix A for further explanation and explicit numerical comparison.

Based on this observation, we employ an efficient simplified procedure to identify and characterise the minima of RG-improved potentials — the alternative being the minimisation via numerical methods, the numerical cost of which increases for vacuum structures with increasing number of vevs. For a given vacuum configuration, this minimisation procedure is summarised as follows:

1. Starting with random values for the quartic couplings at some high scale  $\mu_0$ , the stability of the tree-level potential is asserted and unstable configurations are discarded.
2. Evolution of the quartic couplings according to their one-loop beta-functions is performed down to some sufficiently low scale  $\mu_1$ . In the present context of Planck-scale initial conditions for an  $SO(10)$  GUTs and for any phenomenologically meaningful choice of gauge coupling  $g$ , a natural choice for this scale is  $\mu_1 \approx 10^{11}$  GeV, where the gauge coupling usually runs into a Landau pole<sup>6</sup>.
3. The scale  $\tilde{\mu}$  at which  $\tilde{V}^{(0)}$  develops flat directions is identified. To determine  $\tilde{\mu}$  in practice, we assert the tree-level stability conditions at each integration step over the considered energy range.
4. At the scale  $\tilde{\mu}$ , depending on the considered vacuum structure, the flat direction  $\vec{n}$  is identified (see App. B). Along this flat direction, the field values take the form

$$\phi = \varphi \vec{n} \quad (6.20)$$

5. The unique value of  $\langle \varphi \rangle$  such that

$$V^{(1)}(\langle \varphi \rangle \vec{n}; \tilde{\mu}) = 0 \quad (6.21)$$

is identified. The field vector  $\langle \phi \rangle = \langle \varphi \rangle \vec{n}$  constitutes an estimate of the exact position of the minimum.

6. Finally, the depth of the RG-improved potential at the minimum, *i.e.*,

$$V^{\text{eff}}(\langle \phi \rangle) = V^{(0)}(\langle \phi \rangle; \tilde{\mu}) \quad (6.22)$$

is evaluated.

In this form, the above procedure is essentially equivalent to a Gildener-Weinberg minimisation (see App. A.2). However, as explained in App. A.3, it can be straightforwardly extended to include  $\mathcal{O}(\hbar^2)$  corrections characteristic of the one-loop RG-improvement procedure. The accuracy of this procedure compared to a full-fledged numerical minimisation of the RG-improved potential is studied in Appendix A.4. From an algorithmic point of view, our method proves remarkably more efficient, in particular for multidimensional vacuum manifolds. The reason is rather simple: Here, one avoids the numerical minimisation of a multivariate function, whose evaluation at a point  $\phi \in \mathbb{R}^N$  is itself rather costly. (Evaluating the potential at some given field value involves a root-finding algorithm to determine the RG-improved scale  $\mu_*$ .) Instead, two one-dimensional numerical scans are performed, to find the value of  $\tilde{\mu}$  at step 3, then and the value of  $\langle \varphi \rangle$  at step 5, respectively.

<sup>6</sup>The precise value of  $\mu_1$  is mostly arbitrary, since, in practice, one observes either the breakdown of  $SO(10)$  or the occurrence of Landau poles along the way from  $\mu_0$  down to  $\mu_1$ .

### 3.3 Breaking patterns triggered by the RG-flow

As discussed above, the spontaneous breakdown of  $SO(10)$  — *i.e.*, the occurrence of a non-trivial minimum of the RG-improved potential — is triggered close to the RG scale at which the tree-level potential turns unstable. While the knowledge of necessary stability conditions allows to discard points from the parameter space for which the scalar potential is clearly unstable (see step 1. in the minimisation procedure described above), the determination of the breaking patterns of the model requires additional information.

Given some vacuum manifold, there are in general several qualitatively different ways of violating the stability conditions (see App. B). More precisely, the set of stability conditions for a given vacuum structure can in general be expressed as the conjunction of  $n$  individual constraints:

$$S = S_1 \wedge \cdots \wedge S_n . \quad (6.23)$$

Defining  $\bar{S}$  as the condition for an unstable potential, one clearly has

$$\bar{S} = \bar{S}_1 \vee \cdots \vee \bar{S}_n . \quad (6.24)$$

The violation of any one of the  $S_i$  will trigger spontaneous symmetry breaking, in general towards different subgroups of original symmetry group. To illustrate this rather general statement, let us consider a concrete example. For instance, for the  $\mathbf{16}_H \oplus \mathbf{45}_H$   $SO(10)$  model considered in the next section, a possible vacuum configuration leading to a  $SU(5)$  breaking is obtained from Eq. (6.39) in the limit  $\omega_R = \omega_B = \omega/\sqrt{5}$ ,  $\chi_R = 0$ , where  $\omega$  and  $\chi$  are the  $\mathbf{45}_H$  and  $\mathbf{16}_H$  vevs, see Sec. 4,

$$\langle V \rangle_{SU(5)} = \left( \lambda_1 + \frac{13}{20} \lambda_2 \right) \omega^4 + \left( 2\lambda_8 + \frac{5}{2} \lambda_9 \right) \omega^2 \chi^2 + \lambda_6 \chi^4 , \quad (6.25)$$

and matches the definition of a general 2-vev vacuum manifold given in Appendix B. Using the results in Appendix B, we derive the following tree-level stability conditions<sup>7</sup>:

$$S_1 : \lambda_1 + \frac{13}{20} \lambda_2 > 0 , \quad (6.26)$$

$$S_2 : \lambda_6 > 0 , \quad (6.27)$$

$$S_3 : 2\lambda_8 + \frac{5}{2} \lambda_9 + 2\sqrt{\lambda_6 \left( \lambda_1 + \frac{13}{20} \lambda_2 \right)} > 0 . \quad (6.28)$$

With these definitions at hand, the sufficient and necessary stability condition for this vacuum manifold is given by

$$S = S_1 \wedge S_2 \wedge S_3 . \quad (6.29)$$

Note that Eq. (6.29) only constitutes a set of necessary but not sufficient conditions for the stability of the full  $SO(10)$  potential. Starting at an RG-scale  $\mu_0$  where  $S$  holds, spontaneous symmetry breaking will occur around the scale  $\mu_{\text{GW}} < \mu_0$  at which any one of the  $S_i$  is violated. This can occur in three distinct manners, generating in each case different vacuum configurations along the flat directions appearing at  $\mu_{\text{GW}}$ :

$$\bar{S}_1 : \lambda_1(\mu_{\text{GW}}) + \frac{13}{20} \lambda_2(\mu_{\text{GW}}) = 0 \quad \rightarrow (\omega, \chi) = (\langle \omega \rangle, 0) , \quad (6.30)$$

$$\bar{S}_2 : \lambda_6(\mu_{\text{GW}}) = 0 \quad \rightarrow (\omega, \chi) = (0, \langle \chi \rangle) , \quad (6.31)$$

$$\bar{S}_3 : \left[ 2\lambda_8 + \frac{5}{2} \lambda_9 + 2\sqrt{\lambda_6 \left( \lambda_1 + \frac{13}{20} \lambda_2 \right)} \right] (\mu_{\text{GW}}) = 0 \quad \rightarrow (\omega, \chi) = (\langle \omega \rangle, \lambda \langle \omega \rangle) . \quad (6.32)$$

Finally, based on group theoretical arguments, the residual symmetry group can be determined for each vacuum configuration. Here,  $\bar{S}_2$  and  $\bar{S}_3$  generate an  $SU(5)$  minimum, although in the former case  $\omega$  vanishes. In contrast, the minimum associated with  $\bar{S}_1$  preserves an additional  $U(1)$  gauge factor, such

<sup>7</sup>It is implicitly understood that in the definition of  $S_3$ ,  $S_1$  and  $S_2$  must be satisfied.

that the residual symmetry group is  $SU(5) \times U(1)$ .

The above example shows how to determine the residual gauge symmetry associated with a flat direction of the tree-level potential in a specific vacuum configuration. In addition, one must be able to determine the location and depth of the minimum of the effective potential. For this purpose, the procedure described in the previous section can be used in practice, allowing to estimate the position and depth of the minimum based on the study of the flat directions of  $\tilde{V}_0$ . Such a procedure is reiterated for every relevant vacuum configuration, so that a deepest minimum can be identified. In turn, all other minima can at most be local and their respective symmetry-breaking patterns do not occur in practice.

## 4 The model: minimal $SO(10)$

In this section, we present the specific  $SO(10)$ -GUT model to be investigated. In the persistent absence of any observational hints for supersymmetry, we focus on non-supersymmetric GUTs. After constructing the corresponding tree-level scalar potential, we establish a (non-exhaustive) classification of the possible breaking patterns of the model, clarifying in passing the distinction between the standard and flipped embeddings of the Standard Model into  $SU(5) \times U(1) \subset SO(10)$ . Our classification includes breaking patterns towards subgroups of  $SO(10)$  that do not contain the Standard Model gauge group, allowing us in Sec. 5 to establish a novel kind of theoretical constraint on the parameters of the scalar sector. Finally, we explain why some of the breaking patterns never occur (at least at tree-level), despite being allowed by the group-theoretical structure of the model.

### 4.1 $SO(10)$ with fermionic $\mathbf{16}_F$ , scalar $\mathbf{16}_H$ and $\mathbf{45}_H$

For the  $SO(10)$ -GUT the fermionic content of the Standard Model (together with right-handed neutrinos) fits into a unifying  $\mathbf{16}_F$  spinor representation [111]. The minimal scalar content to reproduce the Standard Model electroweak theory is  $\mathbf{16}_H \oplus \mathbf{45}_H$ . In terms of its group-theoretical specification, the model reads

$$(\mathcal{G}_{\text{GUT}}, \mathcal{F}_{\text{GUT}}, \mathcal{S}_{\text{GUT}}) = (SO(10), \mathbf{16}_F, \mathbf{16}_H \oplus \mathbf{45}_H), \quad (6.33)$$

The Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_K - V, \quad (6.34)$$

where the  $\mathcal{L}_K$  is the fermionic, scalar and gauge kinetic part and  $V$  is the  $\mathbf{16}_H \oplus \mathbf{45}_H$  potential. In order to break the electroweak symmetry, an additional real  $\mathbf{10}_H$  representation is introduced. In this case, a Yukawa interaction of the form  $\mathbf{16}_F \mathbf{10}_H \mathbf{16}_F$  must be included and the Lagrangian reads

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_Y - V. \quad (6.35)$$

We give in App. D a possible parameterisation of the most general scalar potential (and of  $\mathcal{L}_Y$ ) including scalar representations  $\mathbf{10}_H \oplus \mathbf{16}_H \oplus \mathbf{45}_H$ .

We stress that the studied model fails to produce a viable (*i.e.* SM-like) fermion sector at low energies for the simple reason that the single Yukawa matrix characterising the  $\mathbf{16}_F \mathbf{10}_H \mathbf{16}_F$  interaction can always be diagonalised by a redefinition of the fermion fields. The question of constructing a minimal viable  $SO(10)$  Yukawa sector cf. [149, 115, 150, 122, 152, 153, 154, 155, 156, 157, 158, 159, 148, 153, 134], will not be discussed here. Nevertheless, one model with a potentially viable gauge-Yukawa sector consists of a  $\mathbf{10}_H \oplus \mathbf{45}_H \oplus \mathbf{126}_H$  scalar sector and features the same admissible breaking patterns as the  $\mathbf{10}_H \oplus \mathbf{16}_H \oplus \mathbf{45}_H$  model considered here, hence justifying our motivation to investigate its main features despite its non-viable low-energy phenomenology.

We further simplify the setup by omitting the  $\mathbf{10}_H$  representation and investigate models based on the scalar representations  $\mathbf{16}_H \oplus \mathbf{45}_H$  and  $\mathbf{45}_H$ , respectively. In the former case, the tree-level scalar potential reduces to

$$\begin{aligned} V(\chi, \phi) = & \frac{\lambda_1}{4} \text{Tr}(\Phi_{16}^2)^2 + \lambda_2 \text{Tr}(\Phi_{16}^4) + 4\lambda_6 (\chi^\dagger \chi)^2 + \lambda_7 (\chi_+^\dagger \Gamma_i \chi_-) (\chi_-^\dagger \Gamma^i \chi_+) \\ & + 2\lambda_8 (\chi^\dagger \chi) \text{Tr}(\Phi_{16}^2) + 8\lambda_9 \chi^\dagger \Phi_{16}^2 \chi. \end{aligned} \quad (6.36)$$



where the  $\lambda_i$  denote the six quartic couplings of the potential and  $\chi$  denotes the  $\mathbf{16}_H$  multiplet. The auxiliary fields  $\Phi_{16}$  and  $\chi_{\pm}$  (associated with the  $\mathbf{45}_H$  and  $\mathbf{16}_F$  multiplets respectively) as well as the  $\Gamma_i$  matrices are defined in App. D.

A series of articles in the early 1980's studying the  $\mathbf{16}_H \oplus \mathbf{45}_H$  model [169, 170, 171, 172] pointed out that the only potentially viable minima of the (tree-level) potential induce a breaking towards either  $SU(5) \times U(1)$  (which require large threshold corrections to be consistent with gauge-coupling unification [148]) or the non-viable (*i.e.*, phenomenologically excluded) standard  $SU(5)$ . The other possible vacuum configurations in the Pati-Salam directions (see below for more detail) are not minima but saddle points of the potential. For this reason, the model has been discarded 30 years ago. More recently, models featuring a  $\mathbf{45}_H$  have been revived in [173, 160, 161] where it was shown that one-loop quantum corrections to the potential could turn admissible (*cf.* Sec. 2) saddle points into actual minima. This observation further motivates the inclusion of quantum corrections to the scalar potential based on the formalism introduced in the previous section.

## 4.2 Potential breaking chains of the minimal $SO(10)$ model

A comprehensive discussion of symmetry breaking in the minimal  $SO(10)$  model introduced above would require one to classify *all* potential breaking directions allowed by group-theoretical considerations. Here we study a subset of possible breaking patterns, yet considerably extending existing results [169, 170, 171, 172, 113, 173, 174, 160, 161]. This includes all admissible breaking chains towards the Standard Model (in fact, all of the potentially viable ones) as well as several non-admissible vacuum configurations that break the  $SO(10)$  towards non-SM directions. As will become clear later, the inclusion of additional non-admissible breaking patterns can impose significant additional constraints on the parameter space of Planck-scale initial conditions.

### Admissible breaking patterns

Following [161, 173], we observe that, in order to break  $SO(10)$  towards the Standard Model, the adjoint field  $\mathbf{45}_H$  must have, up to arbitrary gauge transformations, the following anti-diagonal<sup>8</sup> vev texture:

$$\phi_{ij} = \text{antidiag}(\omega_R, \omega_R, \omega_B, \omega_B, \omega_B, -\omega_B, -\omega_B, -\omega_B, -\omega_R, -\omega_R), \quad (6.37)$$

where  $\sqrt{3}\omega_B$  and  $\sqrt{2}\omega_R$  respectively denote the vevs of the  $(\mathbf{1}, \mathbf{1}, \mathbf{1}, 0)$  singlet and of the  $(\mathbf{1}, \mathbf{1}, \mathbf{3}, 0)$  triplet contained in the  $\mathbf{45}_H$ , following a  $3_C 2_L 2_R 1_{B-L}$  labelling convention<sup>9</sup>. The above vev structure generally corresponds to a breaking towards  $3_C 2_L 1_R 1_{B-L}$ , and different breaking chains can be conveniently recovered as particular cases:

$$\begin{aligned} \omega_R = 0 : & \quad SO(10) \longrightarrow 3_C 2_L 2_R 1_{B-L}, \\ \omega_B = 0 : & \quad SO(10) \longrightarrow 4_C 2_L 1_R, \\ \omega_R = \omega_B = \omega_5 : & \quad SO(10) \longrightarrow SU(5) \times U(1)_X. \end{aligned}$$

In comparison to [161, 173], the *standard* and *flipped*  $SU(5) \times U(1)$  configurations are not distinguished at the level of the first breaking stage. The reason is that these two breaking chains are characterised by a different embedding of the SM gauge group within  $SU(5) \times U(1)$ , independently of the embedding of  $SU(5) \times U(1)$  within  $SO(10)$  (which is essentially<sup>10</sup> unique). In practice, in the case where  $\omega_R = -\omega_B$  (identified in [161, 173] as the flipped  $SU(5) \times U(1)$  vacuum structure), one can always perform a gauge transformation effectively leading to  $\omega_R \rightarrow -\omega_R$ , and hence to  $\omega_B = \omega_R$ . Of course, such a transformation also affects the other scalar multiplets, and in particular  $\mathbf{16}_H$ . This will be discussed in more detail in what follows.

<sup>8</sup>We use the anti-diagonal matrix notation from bottom left to upper-right entry.

<sup>9</sup>We follow *e.g.* [161] and denote a multiplet of a semi-simple gauge group with  $n$  special unitary subgroups and  $m$  Abelian subgroups by  $SU(N_1) \times \dots \times SU(N_n) \times U(1)_1 \times \dots \times U(1)_m$ . The respective multiplets are denoted by the dimensions  $\mathbf{D}_i$  with which they transform under each  $SU(N_i)$  and their hypercharge  $Y_j$  with respect to each  $U(1)_j$  factor, *i.e.*, by  $(\mathbf{D}_1, \dots, \mathbf{D}_n, Y_1, \dots, Y_m)$ . Further, we use the standard shorthands  $N_n$  for  $SU(N_n)$  and  $1_m$  for  $U(1)_m$ .

<sup>10</sup>For instance, one always has the freedom to embed  $SU(5) \times U(1)$  into  $SO(10)$  such that the branching rule of  $\mathbf{16}$  is either given by  $\mathbf{16} \rightarrow (\mathbf{10}, -1) \oplus (\mathbf{5}, 3) \oplus (\mathbf{1}, -5)$ , or  $\mathbf{16} \rightarrow (\mathbf{10}, 1) \oplus (\mathbf{5}, -3) \oplus (\mathbf{1}, 5)$ , or any other physically equivalent decomposition (see *e.g.* the discussion on symmetry breaking in [63]).

We now turn to the vev structure of  $\mathbf{16}_H$ . In addition to the vev of the  $(\mathbf{1}, \mathbf{1}, \mathbf{2}, +\frac{1}{2})$  doublet, denoted by  $\chi_R$ , we also consider a possibly non vanishing vev for the  $(\mathbf{1}, -5)$  singlet under  $SU(5) \times U(1)$ , denoted by  $\chi_5$ . For the labelling convention of  $3_C 2_L 2_R 1_{B-L}$  multiplets, introducing this additional vev,  $\chi_5$ , simply results in two possible SM vevs in the  $(\mathbf{1}, \mathbf{1}, \mathbf{2}, +\frac{1}{2})$  doublet. With this notation (and in a basis in which the adjoint field has the vev structure Eq. (6.37)) the scalar 16-plet can be put in the form<sup>11</sup>:

$$\chi = \frac{1}{\sqrt{2}} \text{diag} (0, -i\chi_5, 0, -\chi_R, 0, \chi_R, 0, i\chi_5, 0, \chi_5, 0, -i\chi_R, 0, -i\chi_R, 0, \chi_5)^T. \quad (6.38)$$

Inserting Eq. (6.37) and Eq. (6.38) into the expression of the tree-level scalar potential Eq. (6.36), we find:

$$\begin{aligned} \langle V \rangle = & \lambda_1 (3\omega_B^2 + 2\omega_R^2)^2 + \frac{\lambda_2}{4} (21\omega_B^4 + 36\omega_B^2\omega_R^2 + 8\omega_R^4) + 4\lambda_6 (|\chi_R|^2 + |\chi_5|^2)^2 \\ & + 4\lambda_8 (|\chi_R|^2 + |\chi_5|^2) (3\omega_B^2 + 2\omega_R^2) + \lambda_9 (|\chi_R|^2 (3\omega_B - 2\omega_R)^2 + |\chi_5|^2 (3\omega_B + 2\omega_R)^2). \end{aligned} \quad (6.39)$$

When  $\omega_R, \omega_B \neq 0$ , a direct breakdown of  $SO(10)$  towards  $\mathcal{G}_{\text{SM}} = 3_C 2_L 1_Y$  is triggered as soon as one of the components of  $\mathbf{16}_H$  acquires a non-zero expectation value:

$$\begin{aligned} \chi_R \neq 0, \chi_5 = 0 : & \quad SO(10) \xrightarrow{(1)} \mathcal{G}_{\text{SM}}, \\ \chi_R = 0, \chi_5 \neq 0 : & \quad SO(10) \xrightarrow{(2)} \mathcal{G}_{\text{SM}}, \end{aligned}$$

where the embedding of  $\mathcal{G}_{\text{SM}}$  into  $SO(10)$  may *a priori* differ between the breakings (1) and (2). In the case (1), taking  $\omega_R = \omega_B$ , one exactly recovers the vacuum structure described in [161, 173] in a situation where  $\omega_R = -\omega_B$  (therein identified as a flipped embedding of the SM into  $SU(5) \times U(1)$ ), while the latter breaking would correspond to  $\omega_R = \omega_B$  (identified as the standard embedding). However, as stated previously, the latter relation can be recovered from the former making use of a class of gauge transformations effectively leading to

$$\omega_R \longleftrightarrow -\omega_R, \quad |\chi_R| \longleftrightarrow |\chi_5|. \quad (6.40)$$

The gauge generators leading to Eq. (6.40) are clearly broken in any one of the SM vacua. Hence at a minimum they are associated with Goldstone modes, so both minima belong to a larger, continuous set of degenerate minima. Therefore, considering that the breaking towards the SM occurs at once – *i.e.* corresponds to a one-step breaking – the standard and flipped embeddings are formally equivalent. The degeneracy can however be removed if a large hierarchy exists among the vevs, allowing to adopt an effective description of the theory based on either  $SU(5)$  or  $SU(5) \times U(1)$  over a given energy range. In particular, if  $\chi_5 \gg \omega_R, \omega_B$  (or equivalently  $\chi_R \gg \omega_R, \omega_B$ ), a first breaking towards  $SU(5)$  is triggered at the GUT scale  $M_{\text{GUT}}$ , and the breaking towards the SM will be assumed to occur at an intermediate scale  $M_I \ll M_{\text{GUT}}$ . This case obviously corresponds to a standard embedding, since no  $U(1)$  factor can enter in the definition of the hypercharge generator. Conversely, a first breaking can occur at  $M_{\text{GUT}}$  towards  $SU(5) \times U(1)$ , with a subsequent breaking towards the SM occurring at the lower scale  $M_I$ . In this case, one expects that the precise form of the scalar potential in the  $SU(5) \times U(1)$  phase will determine whether the SM embedding is standard or flipped, since RG-running effects in the  $SU(5) \times U(1)$  phase would have spoiled the  $SO(10)$  invariance of the vacuum manifold and therefore removed the degeneracy of the minima.

### Non-admissible breaking patterns

In addition to the admissible breaking patterns above, *i.e.* those involving intermediate gauge groups which contain the SM, it is vital to also consider possible symmetry breakings towards other gauge groups. Thereby, one can identify and exclude regions of the parameter space that specifically trigger such breaking patterns. In particular, for the  $\mathbf{16}_H \oplus \mathbf{45}_H$  model, we have identified a family of non-admissible breaking patterns towards subgroups of  $SO(8) \times U(1)$  (one of the maximal subgroups of

<sup>11</sup>This form is only unique up to gauge transformations preserving the vev structure of  $\mathbf{45}_H$ .

$SO(10)$ ). Similar to the SM case, this family of non-admissible breakings can be parameterised by a general vacuum structure for the scalar fields  $\mathbf{45}_H$  and  $\mathbf{16}_H$ , namely:

$$\phi_{ij} = \text{antidiag}(-\omega_8, \omega_4, \omega_4, \omega_4, \omega_4, -\omega_4, -\omega_4, -\omega_4, -\omega_4, \omega_8), \quad (6.41)$$

and

$$\chi = \frac{1}{\sqrt{2}} (i\chi_4, -i\chi_5, 0, 0, 0, 0, i\chi_4, i\chi_5, -\chi_4, \chi_5, 0, 0, 0, 0, \chi_4, \chi_5)^T. \quad (6.42)$$

We stress that the vev  $\chi_5$  that appears in the above expression has the same origin as the vev texture, cf. Eq. (6.38), for the SM breakings. This common origin is manifest in the present choice of gauge. With these vev textures, the vacuum manifold takes the general form

$$\begin{aligned} \langle V \rangle = & \lambda_1 (\omega_8^2 + 4\omega_4^2)^2 + \frac{\lambda_2}{4} (\omega_8^4 + 24\omega_8^2\omega_4^2 + 40\omega_4^4) + 4\lambda_6 (|\chi_4|^2 + |\chi_5|^2)^2 + 8\lambda_7 |\chi_4|^2 |\chi_5|^2 \\ & + 4\lambda_8 (|\chi_4|^2 + |\chi_5|^2) (\omega_8^2 + 4\omega_4^2) + \lambda_9 (|\chi_4|^2 (\omega_8 - 4\omega_4)^2 + |\chi_5|^2 (\omega_8 + 4\omega_4)^2). \end{aligned} \quad (6.43)$$

Its similarity with the SM vacuum manifold is worth noticing. When  $\omega_4, \omega_8 \neq 0$  and either  $\chi_4 \neq 0$  or  $\chi_5 \neq 0$ , this vacuum manifold corresponds to a breaking towards  $SU(4) \times U(1)$ . Imposing particular relations on the vevs yields larger residual gauge groups such as  $SO(8) \times U(1)$ ,  $SO(7)$  and  $SU(4) \times U(1)^2$ , as reported in Table 6.1.

Finally, we note that an additional vev texture for  $\mathbf{45}_H$  is considered in this work, leading to an alternative embedding of  $SU(4) \times U(1)^2$  within  $SO(8) \times U(1)$  (stemming from the so-called triality property of  $SO(8)$ ). This additional embedding only involves a non-trivial vev texture for  $\mathbf{45}_H$ , given by

$$\phi_{ij} = \text{antidiag}(\omega_8, \omega'_4, 0, 0, 0, 0, 0, 0, -\omega'_4, -\omega_8). \quad (6.44)$$

It is interesting to note that the constraint  $\omega'_4 = \omega_8$  induces a breaking towards  $4_C 2_L 1_R$ , of which  $SU(4) \times U(1)^2$  is indeed a subgroup. As discussed in the next section, this alternative breaking can only be triggered by a local minimum of the scalar potential and does not occur in practice.

### Non-observable broken phases

In a gauge theory with a specified particle content, a limited number of gauge invariants can be formed. Allowing these invariants to get non-zero expectation values defines orbits of gauge-equivalent field configurations. Each orbit is associated with a residual symmetry group (the orbit's little group), and the set of orbits associated with the same residual symmetry forms a stratum [175]. Specifying the scalar potential of the theory fixes the stratum structure, and therefore the number of subgroups that can be obtained after spontaneous breakdown of the original symmetry. Those strata (and associated phases) will be called observable if there exists a field configuration minimising the scalar potential and leading to the spontaneous breakdown of the gauge group towards the associated subgroup [176, 177].

For the  $\mathbf{45}_H \oplus \mathbf{16}_H$  model at hand, one can show that the strata corresponding to symmetry breaking towards  $3_C 2_L 1_R 1_{B-L}$ ,  $SU(4) \times U(1)^2$  and  $[SU(4) \times U(1)^2]'$  are in fact non-observable. For concreteness, we now provide a proof of this statement for the  $3_C 2_L 1_R 1_{B-L}$  breaking. While the model considered in this work is classically scale-invariant, it is convenient for the purposes of the present discussion to include the scalar mass term:

$$V^{(0)} \supset -\frac{1}{2} \mu_\phi \text{Tr}(\Phi_{16}^2). \quad (6.45)$$

In this case, the  $3_C 2_L 1_R 1_{B-L}$  vacuum manifold reads:

$$V^{(0)} = -\mu_\phi (3\omega_B^2 + 2\omega_R^2) + \lambda_1 (3\omega_B^2 + 2\omega_R^2)^2 + \frac{\lambda_2}{4} (21\omega_B^4 + 36\omega_B^2\omega_R^2 + 8\omega_R^4). \quad (6.46)$$

Solving the stationary point equations with respect to  $\omega_{B,R}$  yields the set of solutions (excluding the trivial solution  $\omega_B = \omega_R = 0$ )

$$(\omega_B^2, \omega_R^2) \in \left\{ \left( 0, \frac{\mu_\phi}{4\lambda_1 + 2\lambda_2} \right), \left( \frac{2\mu_\phi}{12\lambda_1 + 7\lambda_2}, 0 \right), \left( \frac{2\mu_\phi}{20\lambda_1 + 13\lambda_2}, \frac{2\mu_\phi}{20\lambda_1 + 13\lambda_2} \right) \right\}. \quad (6.47)$$

These three solutions respectively belong to orbits associated with the residual subgroups  $4_C2_L1_R$ ,  $3_C2_L2_R1_{B-L}$  and  $SU(5) \times U(1)$ , and we conclude that the  $3_C2_L1_R1_{B-L}$  broken phase is non-observable. In fact, this statement persists in the classically scale-invariant model considered in this work (*i.e.* in the absence of scalar mass terms), when one examines the residual gauge group along the flat directions of the vacuum manifold. A similar reasoning applies to the  $SO(10) \rightarrow [SU(4) \times U(1)^2]'$  vacuum manifold which effectively corresponds to a  $SO(10) \rightarrow SU(5) \times U(1)$  breaking after minimisation.

At this point, we would like to make an important comment regarding the  $3_C2_L2_R1_{B-L}$  and  $4_C2_L1_R1_{B-L}$  breakings studied in *e.g.* [161, 113, 173, 160]. In particular, still in presence of a scalar mass term in the tree-level potential, it is straightforward to compute the depth of the minimum in the following vacuum configurations:

$$V_{3_C2_L2_R1_{B-L}}^{\min} = \frac{-3\mu_\phi^2}{12\lambda_1 + 7\lambda_2}, \quad (6.48)$$

$$V_{4_C2_L1_R1_{B-L}}^{\min} = \frac{-\mu_\phi^2}{4\lambda_1 + 2\lambda_2}, \quad (6.49)$$

$$V_{SU(5) \times U(1)}^{\min} = \frac{-5\mu_\phi^2}{20\lambda_1 + 13\lambda_2}. \quad (6.50)$$

We can add to this list the depth of the minimum in a  $SO(8) \times U(1)$  vacuum configuration:

$$V_{SO(8) \times U(1)}^{\min} = \frac{-\mu_\phi^2}{4\lambda_1 + \lambda_2}. \quad (6.51)$$

With these expressions at hand, we may establish a hierarchy for the depth of the minima in these four vacuum configurations<sup>12</sup>, see also [178]:

$$V_{SO(8) \times U(1)}^{\min} < V_{4_C2_L1_R1_{B-L}}^{\min} < V_{3_C2_L2_R1_{B-L}}^{\min} < V_{SU(5) \times U(1)}^{\min} \quad \text{if } \frac{\lambda_2}{\lambda_1} > 0, \quad (6.52)$$

$$V_{SO(8) \times U(1)}^{\min} > V_{4_C2_L1_R1_{B-L}}^{\min} > V_{3_C2_L2_R1_{B-L}}^{\min} > V_{SU(5) \times U(1)}^{\min} \quad \text{if } \frac{\lambda_2}{\lambda_1} < 0, \quad (6.53)$$

$$V_{SO(8) \times U(1)}^{\min} = V_{4_C2_L1_R1_{B-L}}^{\min} = V_{3_C2_L2_R1_{B-L}}^{\min} = V_{SU(5) \times U(1)}^{\min} \quad \text{if } \lambda_2 = 0. \quad (6.54)$$

Crucially, we observe that the  $3_C2_L2_R1_{B-L}$  and  $4_C2_L1_R1_{B-L}$  vacua cannot correspond to global minima, except perhaps in the limiting case where  $\lambda_2 = 0$ , in which loop corrections to the scalar potential would have to be included to lift the degeneracy. Such phases are referred to as *locally* observable in Table 6.1, reflecting their property to only corresponding to local minima at tree-level. They belong, at best, to a degenerate set of global minima in a rather fine-tuned set of Planck-scale initial conditions.

### Viability of admissible breaking patterns

We conclude this section with a discussion of the viability of the breaking chains eventually leading to the Standard Model. These admissible breakings are summarised in Table 6.1. Independently of the observable property of the  $SO(10)$  vacua (which is *a priori* dependent of the perturbative order of the quantum scalar potential), a viable breaking is understood to feature desirable (and non-excluded) phenomenological properties in the low-energy regime (*e.g.* down to the electroweak scale). In its strongest version, such a definition encompasses a large number of criteria such as proper gauge coupling unification, a proton decay constant large enough to evade current experimental bounds, a fermion and scalar spectrum containing the Standard Model and compatible with negative new physics searches, among many others. In this work, we will solely retain the first two criteria since any further considerations are beyond the scope of the present analysis<sup>13</sup>.

First focusing on the Georgi-Glashow route, the one-step unification  $SU(5) \rightarrow 3_C2_L1_Y$  is not supported by the current measurements of the Standard Model gauge couplings, thus bringing us to regard

<sup>12</sup>Note that when  $\lambda_1 = 0$ , the conditions  $\frac{\lambda_2}{\lambda_1} > 0$  and  $\frac{\lambda_2}{\lambda_1} < 0$  must be respectively replaced by  $\lambda_2 > 0$  and  $\lambda_2 < 0$ .

<sup>13</sup>Furthermore, as mentioned in Sec. 4.1, the  $SO(10)$  model investigated here is anyways unable to reproduce some phenomenological features of the Standard Model.

Table 6.1: Summary of the considered breaking patterns. In each case, we indicate which vevs should be non-zero in order to trigger spontaneous breakdown towards the relevant subgroups. A starred vev (*e.g.*  $\omega_8^*$ ) can however vanish without altering the nature of the vacuum. As explained in the main text, non-observable phases correspond to minima which cannot be global, while non-admissible breakings occur towards subgroups of  $SO(10)$  which do not contain the Standard Model. An admissible breaking is called viable if it can obey the proton stability and gauge coupling unification constraints (more detail is given in Sec. 4.2).

Breaking chain	Vevs	Observable?	Admissible?	Viable?
$SU(5) \times U(1)$	$\omega_R = \omega_B$	Yes	Yes	Yes
$SU(5)$	$\omega_R^* = \omega_B^*, \chi_5$	Yes	Yes	No
$3C2_L2_R1_{B-L}$	$\omega_B, \chi_5$	Yes, locally	Yes	Yes
$4C2_L1_R$	$\omega_R, \chi_5$	Yes, locally	Yes	Yes
$3C2_L1_R1_{B-L}$	$\omega_R, \omega_B$	No	Yes	Yes
$3C2_L1_Y$	$\omega_R, \omega_B, \chi_5$ or $\chi_R$	Yes	Yes	Yes
$SO(8) \times U(1)$	$\omega_8$	Yes	No	No
$SO(7)$	$\omega_8^*, \chi_4 = \chi_5$	Yes	No	No
$SU(4) \times U(1)^2$	$\omega_8^*, \omega_4$	Yes	No	No
$[SU(4) \times U(1)^2]'$	$\omega_8^*, \omega_4'$	No	No	No
$SU(4) \times U(1)$	$\omega_8, \omega_4, \chi_4$ or $\chi_5$	Yes	No	No

the  $SU(5)$  breaking as non-viable. On the other hand, the  $SU(5) \times U(1) \rightarrow 3C2_L1_Y$  embedding (flipped or standard, see Sec. 4.2) can be realised if large thresholds<sup>14</sup> are present [148], thus implying large hierarchies in the scalar and gauge boson spectrum at the  $SO(10)$ -breaking scale. Combined with constraints stemming from the proton decay, this scenario is rather tightly constrained yet not ruled out.

For the Pati-Salam route<sup>15</sup> including the breakdown of  $SO(10)$  towards  $4C2_L1_R$ ,  $3C2_L2_R1_{B-L}$  and  $3C2_L1_R1_{B-L}$ , we refer to [113] and conclude that gauge coupling unification and proton-decay constraints can be satisfied for the first two breakings. On the other hand,  $SO(10) \rightarrow 3C2_L1_R1_{B-L}$  is shown in [148] to require sizeable threshold corrections in order to allow for a proper unification of the gauge-couplings. This being said, we have mentioned in the introduction of this section that the  $4C2_L1_R$ ,  $3C2_L2_R1_{B-L}$  and  $3C2_L1_R1_{B-L}$  breakings have long been disregarded due to the presence of tachyonic scalar modes in their tree-level spectrum (put differently, the corresponding extrema can only be saddle points [181, 170, 171, 172]). More recently, it has been shown that the inclusion of one-loop corrections could stabilise the scalar potential [173, 160, 161], rendering such breaking patterns potentially viable. What the authors did not consider however is the eventuality that a deeper minimum triggering a breakdown towards  $SO(8) \times U(1)$  would prevent the Pati-Salam vacua to correspond to global minima. While this statement was proven at tree-level in the previous section, one cannot infer *a priori* that the non-observability of the Pati-Salam vacua would persist after including loop-corrections. In Sec. 5, we investigate this matter and show that in fact, the inclusion of one-loop corrections does not change this overall picture (at least in the particular model considered here).

Finally, we comment on the viability of the one-step  $SO(10) \rightarrow 3C2_L1_Y$  breaking. As compared to the  $SU(5) \rightarrow 3C2_L1_Y$  embedding mentioned above, gauge coupling unification does not necessarily have to occur at once (*i.e.* at a single unification scale). In fact, an effective description of the model from the UV to the IR regime can include multiple intermediate scales at which massive gauge bosons are integrated out, and between which different sets of gauge couplings are assumed to run. This happens in particular if a clear hierarchy appears between the various vevs involved in the description of the vacuum manifold after minimisation of the scalar potential (see also the discussion on standard and flipped  $SU(5) \times U(1)$  embeddings in Sec. 4.2). Such a situation most likely involves rather fine-tuned relations among the parameters of the scalar potential, which we however do not consider as a criterion for the non-viability

<sup>14</sup>Let us note that these corrections can be straightforwardly calculated within our approach and are indeed subject to the investigation in the following paper [179]. This calculation will prove crucial for the low-energy observables such as proton lifetime [180]. Here we simply assume that such a scenario can take place.

<sup>15</sup>This encompasses  $3C2_L2_R1_{B-L}$  and  $4C2_L1_R$  as maximal subgroups of the Pati-Salam gauge group,  $SU(4)_c \times SU(2)_L \times SU(2)_R$  [110].

of the model.

## 5 Results

With the formalism for the one-loop RG-improved potential, cf. Sec. 3, and the group-theoretical structure of the specified  $SO(10)$  GUTs, cf. Sec. 4.2, at hand, we now demonstrate how the EFT parameter space of said GUTs is restricted by the various constraints introduced in Sec. 2. In order to determine these constraints, we sample random initial conditions at  $M_{\text{Pl}}$  to map out how each constraint reduces the available parameter space. We do so first for a model with  $\mathbf{45}_H$  as the only scalar representation, cf. Sec. 5.1. We then extend the analysis to a model with  $\mathbf{45}_H$  and  $\mathbf{16}_H$  scalar representations, cf. Sec. 5.2. An extension of our analysis to a model with realistic Yukawa sector will be discussed in future work [179].

The RG evolution of the scalar couplings is not just driven by self-interactions but, more importantly, also by contributions of the gauge coupling  $g$ . The gauge coupling  $g$  tends to destabilise the quartic scalar potential, *i.e.*, tends to induce radiative symmetry breaking [130]. Hence, the RG scale-dependent value of  $g$  is a crucial input to explicitly determine constraints on the scalar potential. At the same time,  $g$  itself also needs to be matched to the observed low-energy gauge-coupling values. To maintain this matching, its viable initial value at  $M_{\text{Pl}}$  thus needs to be varied with any variations of the RG dependence of  $g$  and of the intermediate gauge groups' gauge couplings between  $M_{\text{Pl}}$  and  $M_{\text{EW}}$ . The respective uncertainties include higher-loop corrections but are dominated by the dependence on different breaking schemes and breaking scales, cf. *e.g.* [113]. In principle, one thus also needs to sample over different values of  $g|_{M_{\text{Pl}}}$ .

A phenomenologically meaningful value for  $g|_{M_{\text{Pl}}}$  can be obtained by matching to the relevant breaking schemes in [113]<sup>16</sup>. Evolving the gauge coupling from the respective GUT scale value  $g|_{M_{\text{GUT}}}$  up to the Planck scale using Eq. (6.E19) results in  $g|_{M_{\text{Pl}}} \in [0.41, 0.44]$ . In the following, we work with the central value

$$g|_{M_{\text{Pl}}} = 0.425 . \quad (6.55)$$

We caution that [113] only includes Pati-Salam type breaking chains. As we will see below, other breaking chains turn out to be the most relevant ones. When cross checking the dependence of our most important results on varying gauge coupling, we thus vary over a significantly larger range, *i.e.*,  $g|_{M_{\text{Pl}}} \in [0.35, 0.5]$ .

In principle, all of the above also holds for Yukawa couplings. Based on [130], we expect Yukawa couplings to stabilise<sup>17</sup> the quartic scalar potential, *i.e.*, to delay the onset of radiative symmetry breaking. Of course, this applies only to quartic couplings of representations involved in the Yukawa interaction. Most excitingly, this provides a potential mechanism for a hierarchy of several breaking scales since some of the scalar representations in a GUEFT couple to fermions via Yukawa couplings while others do not. However, neither of the presently investigated scalar potentials admits Yukawa couplings to the fermionic  $\mathbf{16}_F$ , *i.e.*, to SM fermions.

In order to demonstrate the restrictive power of each constraint, cf. Sec. 2, we apply the constraints individually: first, we demand tree-level stability (I.a); second, we demand perturbativity between  $M_{\text{Pl}}$  and  $M_{\text{GUT}}$  (I.b); third, we demand that the deepest vacuum be an admissible one (I.c), *i.e.*, one which still remains invariant under the SM gauge group  $\mathcal{G}_{\text{SM}}$ . Since the last constraint is conceptually new, we emphasise two important remarks.

The first remark concerns the inclusion of non-admissible breaking chains. On the one hand, a successful application requires the inclusion of *all* admissible vacua in order to make sure that the ruled-out EFT parameter space is in fact not admissible. On the other hand, it does *not* require the inclusion of all non-admissible vacua. Yet, the more non-admissible vacua are included, the more the EFT parameter space will be restricted, cf. Sec. 4.2 for the respective group-theoretical discussion for the examples at hand.

<sup>16</sup>In the notation of [113], the relevant breaking chains are VIIIb and XIIb. The respective values of  $M_{\text{GUT}}$  can be found in Tab. III in [113]. The respective values for  $g|_{M_{\text{GUT}}}$  can be read off from Fig. II in [113].

<sup>17</sup>As we evolve the potential from the UV to the IR, Yukawa couplings have a stabilising effect. This is in line with the notion that Yukawa couplings tend to destabilise the Higgs potential when the RG flow is reversed and evolved from the IR to the UV.

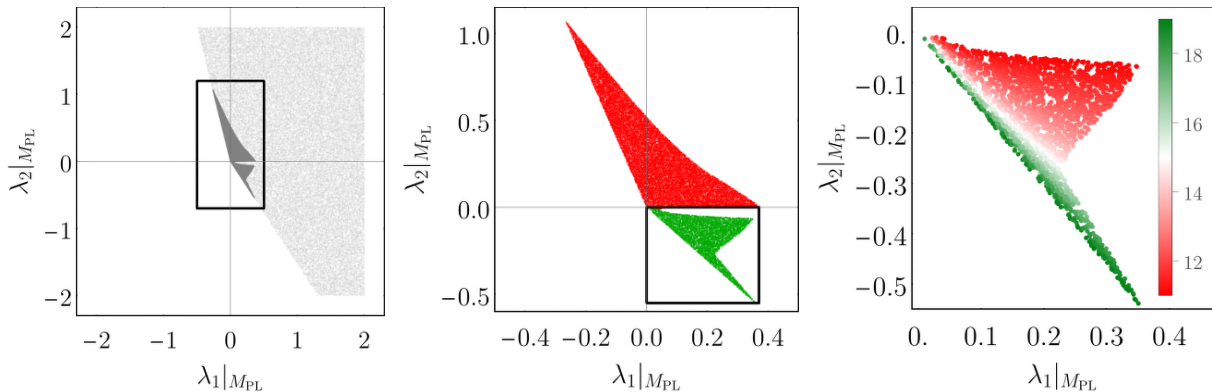


Figure 6.1: Successive constraints arising from an admissible scalar potential, cf. Sec. 2, on the Planck-scale theory space of quartic couplings in an  $SO(10)$  GUT with  $\mathbf{45}_H$  scalar matter content and with  $g|_{M_{\text{Pl}}} = 0.425$ . The left-hand panel shows allowed regions arising from (I.a) stability (light-gray region) and from (I.b) perturbativity (dark-gray region). The middle panel zooms in on the resulting stable and perturbative region and shows the additional constraint (I.c) arising from the deepest radiative minimum occurring in a non-admissible direction (red upper region). The right-hand panel zooms in on the remaining admissible region (lower green region in the middle panel) and shows the additional constraint (II.a) from proton decay. The colour scale indicates  $\log_{10}(M_{\text{GUT}})$ , with  $M_{\text{GUT}}$  the breaking scale. The breaking scale is constrained by proton decay, with only the lower green stripe remaining potentially viable.

The second remark concerns the possible spontaneous symmetry breaking by additional mass terms, which we neglect in the present study. We expect the admissibility-constraints to remain applicable as long as additional mass terms are significantly smaller than the relevant radiative symmetry breaking scale, cf. Sec. 6.2 for further discussion. Nevertheless, we consider an inclusion of mass terms as an important future extension of our work.

In addition to these sets of constraints (I.a-I.c) arising from an admissible scalar potential, one may also apply more commonly discussed constraints arising from (II.a) viable gauge unification and (II.b) a viable Yukawa sector. As mentioned, (II.b) does not apply to the investigated models. The application of the gauge-unification constraint (II.a) to the remaining admissible parameter space after application of (I.a-I.c), is briefly discussed in case of the  $\mathbf{45}_H$ , cf. Sec. 5.1.

## 5.1 Constraints on an $SO(10)$ model with $\mathbf{45}_H$ scalar potential

An  $SO(10)$  GUT with the  $\mathbf{45}_H$  as the only scalar representation cannot fully break to the SM. Nevertheless, the  $\mathbf{45}_H$  is responsible for the first breaking step in many realistic  $SO(10)$ -breaking chains, cf. Tab. 6.1. It thus serves as a simplified toy-model for the first breaking step. The simplification is justified whenever portal couplings to other scalar representations remain negligibly small. Note that it is not consistent to simply set the portal couplings to zero since they are not protected by any global symmetry and thus induced by gauge-boson loop corrections. The  $\mathbf{45}_H$ -model is thus a good approximation for a realistic first breaking step only in a regime in which portal couplings remain negligibly small. The subsequent extension to a scalar potential with  $\mathbf{45}_H$  and  $\mathbf{16}_H$  representation, cf. Sec. 5.2, can be interpreted as a test of this approximation. Indeed, we will see that the main constraint on the EFT parameter space – while smeared out – will still remain important if the scalar potential is extended.

Group-theoretically, the  $\mathbf{45}_H$  can break  $SO(10)$  to three different classes of observable vacua, cf. Tab. 6.1 and Sec. 4 for details:

- Admissible Georgi-Glashow direction: The  $\mathbf{45}_H$  can break towards  $SU(5) \times U(1)$  which still contains the SM.
- Admissible Pati-Salam directions: The  $\mathbf{45}_H$  can break towards two different Pati-Salam-type direc-

tions, *i.e.*, to  $3_C 2_L 2_R 1_{B-L}$  or  $4_C 2_L 1_R$ , which also still contain the SM.

- Non-admissible directions: The  $45_H$  can break  $SO(10)$  towards two non-admissible directions, *i.e.*, to  $SO(8) \times U(1)$  or  $SU(4) \times U(1)^2$ , which can no longer contain the SM and are therefore excluded.

One of the important results of this work is that we find that the Pati-Salam directions can never occur as global minima: Either the Georgi-Glashow minima or the non-admissible minima is always deeper. This statement is proven at tree-level in Sec. 4.2. We find that it persists when radiative effects are included. In fact, we find that all initial conditions either break towards  $SU(5) \times U(1)$  or towards  $SO(8) \times U(1)$ .

The  $45_H$  toy-model also demonstrates clearly how the three scalar-potential constraints, *i.e.*, tree-level stability (I.a), perturbativity (I.b), and admissibility (I.c), successively constrain the EFT parameter space at  $M_{\text{Pl}}$ . This is presented in Fig. 6.1, where we summarise the results of a successive analysis of uniformly distributed random initial conditions for  $\lambda_1$  and  $\lambda_2$ .

First, we determine tree-level stability: Stable initial conditions are marked with light-gray points in the left-hand panel of Fig. 6.1. As is apparent in this simple toy-model, the boundary of the tree-level stable region simply corresponds to the analytical tree-level stability conditions such as Eq. (6.26).

Second, we apply the perturbativity constraint: Initial conditions which remain sufficiently perturbative along the relevant RG-flow towards lower scales are marked with dark-gray points in the left-hand panel of Fig. 6.1. Unfortunately, there is no strict perturbativity criterion since the perturbative series is assumed to have zero radius of convergence [182]. The practical perturbativity criterion, extending a proposed criterion in [144], is discussed in App. C. It amounts to the demand that the theory-space norm of neglected 2-loop contributions does not outgrow a specified fraction  $\alpha$  of the theory-space norm of one-loop contributions. For all the results in this work, we pick  $\alpha = 0.1$  which might be overly conservative but allows us to avoid convergence issues in the subsequent numerical determination of the one-loop effective potential and its deepest minimum. In practice, perturbativity is determined as follows: We pick a random point in the interval  $\lambda_{1/2} \in [-2, 2]$ . (If the point violates tree-level stability, we pick again.) We evolve the respective initial conditions towards lower scales until the analytical conditions in App. B suggest that radiative symmetry breaking occurs. If the RG-flow remains perturbative until radiative symmetry breaking occurs, the respective initial conditions pass the perturbativity criterion (dark-gray region in the left-hand panel of Fig. 6.1). We iterate this procedure for  $> 10^7$  points.

Third, we apply the admissibility criterion: For initial conditions which pass tree-level stability and perturbativity, we determine the one-loop effective potential, the deepest minimum, and the respective invariant subgroup (breaking direction). Construction of the one-loop effective potential following [136, 137] and the numerical method are detailed in Sec. 3. In agreement with tree-level expectations in Sec. 4.2, we find that the global minimum occurs either in the admissible  $SU(5) \times U(1)$  direction (green region in the right-hand panel in Fig. 6.1) or in the non-admissible  $SO(8) \times U(1)$  direction (red region in the right-hand panel in Fig. 6.1).

We find that minima along Pati-Salam-type directions never occur as the deepest minimum of the RG-improved potential. For example, Fig. 6.2 shows the ratio of logarithms of the depths of the respective minima, depending on  $\lambda_2|_{M_{\text{Pl}}}$  and at fixed  $\lambda_1|_{M_{\text{Pl}}}$ . For any value of  $\lambda_2|_{M_{\text{Pl}}}$  in Fig. 6.2 (and any combination of  $\lambda_i|_{M_{\text{Pl}}}$  in general), there is always (at least) one such ratio which is larger than one, *i.e.*, there is always a non-Pati-Salam minimum which is deeper than the deepest Pati-Salam minimum. Hence, radiative symmetry breaking towards a Pati-Salam minimum does not occur.

Fig. 6.2 also exemplifies that violations of the perturbativity criterion, cf. App. C, close to  $\lambda_2|_{M_{\text{Pl}}} \approx 0$ , cf. middle panel in Fig. 6.1, are accompanied by a near-degeneracy of different minima. The underlying reason is that for initial conditions with  $\lambda_2|_{M_{\text{Pl}}} \approx 0$ , the scale of radiative symmetry breaking is delayed and approaches the onset of the gauge-coupling Landau pole near  $\mu = 10^{11} \text{GeV}$ , cf. right-hand panel in Fig. 6.1. As this happens, perturbativity is violated and the vacua become near-degenerate.

Moreover, Fig. 2 provides the opportunity to demonstrate that our results and, in particular, the absence of deepest minima breaking towards Pati-Salam directions, are robust under varying the gauge coupling in the range  $g|_{M_{\text{Pl}}} \in [0.35, 0.5]$ . While the ratios of the depths of the minima quantitatively change when varying  $g|_{M_{\text{Pl}}}$ , the result – *i.e.*, the absence of deepest minima breaking towards Pati-Salam directions – persists.



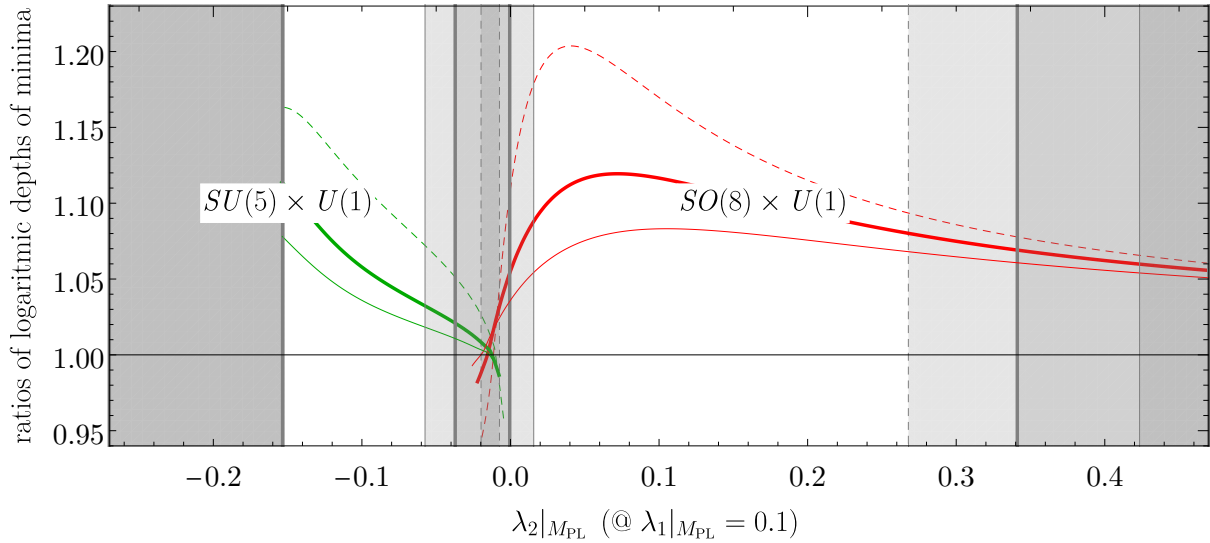


Figure 6.2: Ratio of the logarithm of depths, *i.e.*,  $\mathcal{V}_i/\mathcal{V}_{\text{Pati-Salam}}$  with  $\mathcal{V}_i = \log |V_i^{\min}|$ , of the minima corresponding to non-Pati-Salam breakings ( $SU(5) \times U(1)$  in green and  $SO(8) \times U(1)$  in red dashed) and the depth of the deepest Pati-Salam minimum  $\mathcal{V}_{\text{Pati-Salam}}$  (breaking towards either  $3_C 2_L 2_R 1_{B-L}$  or  $4_C 2_L 1_R$ ). For any Planck-scale initial condition, there is always a non-Pati-Salam minimum which is deeper than the Pati-Salam ones (*i.e.*, there is always one of the ratios which remains larger than one). As an example, we show the  $\lambda_2$ -dependence at  $\lambda_1 = 0.1$  at  $g_{M_{\text{Pl}}} = 0.425$ , cf. Eq. (6.55). As a result, symmetry breaking towards the Pati-Salam subgroups cannot occur. Gray regions indicate regions in which the perturbativity (according to App. C) is violated, cf. Fig. 6.1. All curves and regions are also given for  $g_{M_{\text{Pl}}} = 0.35$  (thin dashed) and  $g_{M_{\text{Pl}}} = 0.5$  (thin).

Finally, we also demonstrate that viability in the gauge-Yukawa sector (see (II) in Sec. 2) further constrains the Planck-scale parameter space. In particular, the right-hand panel in Fig. 6.1 shows the logarithm of the breaking scale for each admissible (*i.e.* towards  $SU(5) \times U(1)$ ) point in the Planck-scale parameter space. Increasingly green-coloured points (below the falling diagonal) indicate a higher and higher breaking scale. Increasingly red-coloured points (above the falling diagonal) indicate a lower and lower breaking scale.

Gauge unification and experimental proton-decay bounds demand that  $M_{\text{GUT}}$  remains sufficiently high. A precise constraint depends on threshold effects [148]. Given that the present model lacks a realistic Yukawa sector, we refrain from a more detailed analysis. Nevertheless, the right-hand panel in Fig. 6.1 clearly demonstrates that such additional constraints from a viable gauge-Yukawa sector can be addressed within our formalism.

In summary: First, the Planck-scale theory space of quartic couplings is significantly constrained by demanding an admissible scalar potential. Second, the one-loop effective potential will never develop a deepest minimum along a Pati-Salam-type breaking direction.

We proceed to test how robust these conclusions are, when including a  $\mathbf{16}_H$  along with the  $\mathbf{45}_H$  scalar representation.

## 5.2 Constraints on an $SO(10)$ model with $\mathbf{16}_H \oplus \mathbf{45}_H$ scalar potential

Including the  $\mathbf{16}_H$  alongside the  $\mathbf{45}_H$  scalar representation, the quartic parameter space, cf. Eq. (6.36), is 6-dimensional, cf. App. (D.3). This entails additional breaking chains which are discussed in detail in Sec. 4.2 and summarised in Tab. 6.1.

We obtain a large sample of uniformly distributed random points in the region  $\lambda_i \in [-1, 1]$  and proceed as in Sec. 5.1 by successively applying the three constraints on the scalar potential, cf. Sec. 2. For each successive constraint, we only take into account points which have passed the previous constraints.

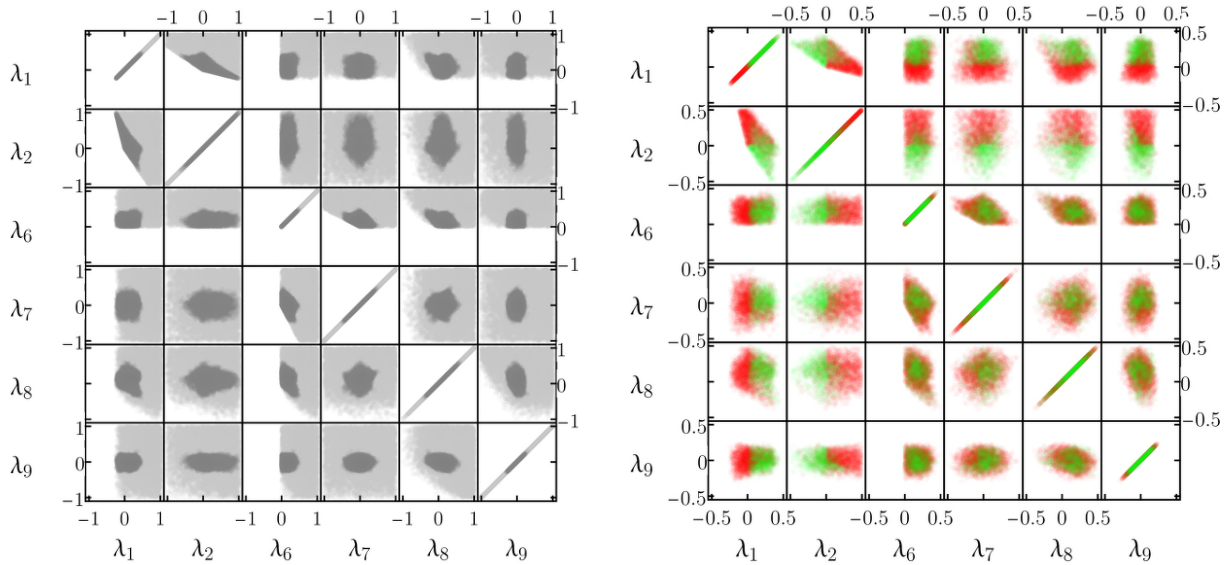


Figure 6.3: Successive constraints (I.a-c), cf. Sec. 2, on the 6-parameter Planck-scale theory space of quartic couplings (we suppress the subscript  $M_{\text{Pl}}$  in the  $\lambda_i|_{M_{\text{Pl}}}$ ) in an  $SO(10)$  GUT with  $\mathbf{16}_H + \mathbf{45}_H$  scalar matter content. The left-hand panel shows a statistical scatter-plot matrix of constraints arising from (I.a) stability (light-gray regions) and from (I.b) perturbativity (dark-gray regions). The right-hand panel shows a zoomed-in scatter-plot matrix of additional constraints (I.c) arising from the deepest radiative minimum occurring in a non-admissible direction (red/darker points). The green/lighter points in the right-hand panel remain potentially viable, see also Fig. 6.4.

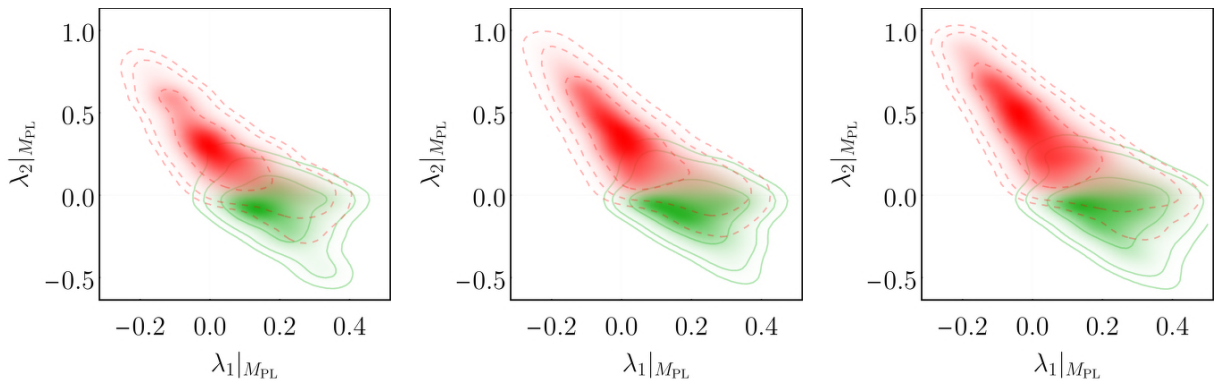


Figure 6.4: Probability-density functions (PDFs) of the admissible (green) and non-admissible regions projected into the two-dimensional  $(\lambda_1|_{M_{\text{Pl}}}, \lambda_2|_{M_{\text{Pl}}})$ -slice of theory space. Contour lines indicate the  $1\sigma$ -,  $2\sigma$ -, and  $3\sigma$ -regions obtained with a Gaussian kernel of width 0.05. The different panels quantify the mild gauge-coupling dependence ( $g|_{M_{\text{Pl}}} = 0.35$ ,  $g|_{M_{\text{Pl}}} = 0.425$ , and  $g|_{M_{\text{Pl}}} = 0.5$  from left to right, respectively) of the regions in theory space.

The results are shown in Fig. 6.3. We present them in the form of statistical scatter-plot matrices which project the 6D parameter space onto a full set of 2D slices. While such a projection reveals important correlations, it also leads to a perceived blurring of presumably sharp boundaries in the full higher-dimensional parameter space.

First, we determine tree-level stability: Stable initial conditions are marked with light-gray points in the left-hand panel of Fig. 6.3. The stability-constraints on the pure- $\mathbf{45}_H$ -couplings  $\lambda_1$  and  $\lambda_2$  remain the same as for the case without  $\mathbf{16}_H$ . There are similar constraints on the pure- $\mathbf{16}_H$ -couplings  $\lambda_6$  and  $\lambda_7$ . Finally, also the portal couplings  $\lambda_8$  and  $\lambda_9$  are constrained by demanding tree-level stability of the

initial conditions.

Second, we apply the perturbativity constraint: Initial conditions which remain perturbative between  $M_{\text{Pl}}$  and  $M_{\text{GUT}}$ , are marked in the left-hand panel of Fig. 6.3 as dark-gray points. We remind the reader that we determine perturbativity by demanding that the theory-space norm of neglected 2-loop contributions does not outgrow a fraction  $\alpha = 1/10$  of the theory-space norm of one-loop contributions, cf. App. C for details. In keeping with an intuitive notion of perturbativity, the remaining points cluster around  $\lambda_i = 0$ .

Third, we apply the admissibility criterion: We obtain the one-loop effective potential, the deepest minimum, and the respective invariant subgroup (breaking direction) to determine whether the latter is admissible, *i.e.*, remains invariant under the SM subgroup. The results are presented in the right-hand panel of Fig. 6.3 where due to significant constraints from perturbativity, we focus only on the remaining subregion  $\lambda_i \in [-0.5, 0.5]$ . (Non-) Admissible points are shown in green (red).

We find that the pure- $\mathbf{45}_H$  couplings  $\lambda_1$  and  $\lambda_2$  are still dominant in determining whether stable and perturbative initial conditions are also admissible or not. Fig. 6.4 presents this dominant correlation in the form of probability-density functions (PDFs).

At the same time, non-vanishing portal couplings can apparently alter admissibility-constraints on the pure- $\mathbf{45}_H$  couplings  $\lambda_1$  and  $\lambda_2$ . This can, for instance, be seen by comparing the middle panel in Fig. 6.1 with Fig. 6.4: Without the  $\mathbf{16}_H$ , the boundary is  $\lambda_2|_{M_{\text{Pl}}} \approx 0$ , with  $\lambda_2|_{M_{\text{Pl}}} < 0$  ( $\lambda_2|_{M_{\text{Pl}}} > 0$ ) admissible (non-admissible). With the  $\mathbf{16}_H$ , the projected boundary is smeared out. In particular, the admissible region in a  $(\lambda_1|_{M_{\text{Pl}}}, \lambda_2|_{M_{\text{Pl}}})$  projection of theory space grows when the  $\mathbf{16}_H$  is included: This occurs because there are additional admissible breaking patterns in comparison to the pure- $\mathbf{45}_H$  case, cf. Tab. 6.1. The additional breaking patterns include an admissible  $SU(5)$  breaking and, in principle, an admissible direct breaking to the SM, *i.e.*, to  $SU(3) \times SU(2) \times U(1)$ . In particular, the  $SU(5)$ -breaking can occur as the deepest vacuum, even for  $\lambda_2 > 0$ . This adds admissible initial conditions with  $\lambda_1 - \lambda_2$  which were previously excluded in the pure- $\mathbf{45}_H$  model.

Whether and if so which of these admissible vacua is the deepest one depends on the initial conditions of all of the six quartic couplings. Fig. 6.5 shows the most prominent correlations that we were able to identify. We emphasise that there are still no initial conditions with observable Pati-Salam type breaking.

Multiple local minima in the scalar potential of a GUT, cf. Sec 5, may be subject to additional constraints in the context of cosmology. In cosmology, multiple (near-)degenerate minima may result in long-lived cosmological domain walls [183, 184] which could obstruct a viable cosmological evolution [184, 185, 186, 187]. Indeed, we find regions in the parameter space of quartic couplings which result in multiple (near-) degenerate minima, cf. *e.g.* Fig. 6.4. We caution that such near-degenerate minima seem to be connected to regions of parameter space in which perturbativity (cf. App. C for our criterion) breaks down.

In summary: The main constraints from stability and perturbativity are robust under the extension of the  $\mathbf{45}_H$  to the  $\mathbf{16}_H \oplus \mathbf{45}_H$  scalar potential. While Pati-Salam type breakings remain non-observable, multiple other admissible breakings can occur.

## 6 Discussion

We close with (i) a brief summary of our results, some comments on (ii) low-energy predictivity in grand unification and on (iii) trans-Planckian extensions in existing quantum-gravity scenarios, and with (iv) an outlook on what we consider the most important open questions.

### 6.1 Summary of the main results

We have initiated a systematic study of how radiative symmetry breaking to non-admissible vacua places significant constraints on the initial conditions of any potentially viable grand-unified effective field theory (GUEFT). We embed this novel constraint in a systematic set of constraints, some of which have been previously discussed in the literature. The resulting blueprint is given in Sec. 2. It encompasses several constraints on the scalar sector: (I.a) a tree-level stability constraint, (I.b) a perturbativity constraint on quartic couplings, and (I.c) the above-mentioned novel requirement of admissible vacua. These scalar-potential constraints supplement well-known requirements on a viable gauge-Yukawa sector, cf. Sec. 2 as

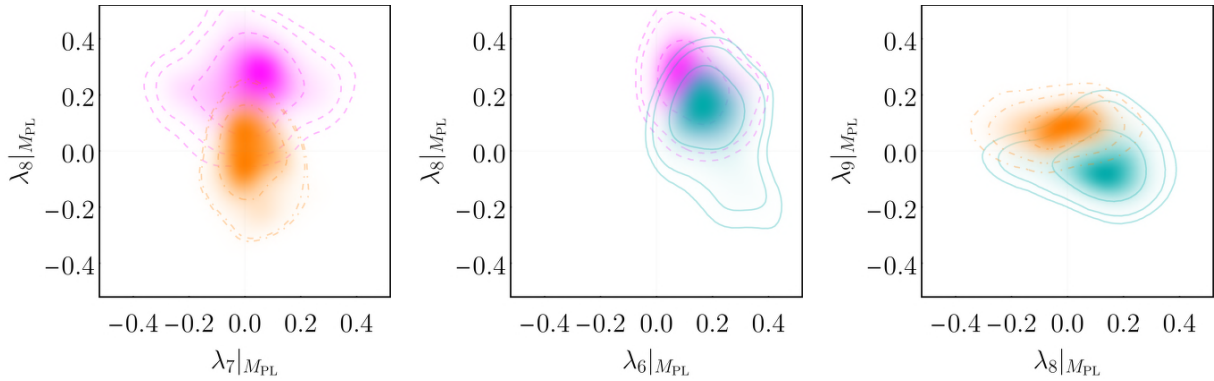


Figure 6.5: Probability-density functions (PDFs) of pairs of different admissible breakings ( $SU(5)$  in cyan;  $SU(5) \times U(1)$  in dashed and magenta;  $SU(3) \times SU(2) \times U(1)$  in dot-dashed and orange) for selected projections into 2-dimensional slices of theory space. Contour lines indicate the  $1\sigma$ -,  $2\sigma$ -, and  $3\sigma$ -regions obtained with a Gaussian kernel of width 0.05. We only show those projections which best discriminate the two respective breaking patterns.

well as [145, 146, 147, 148, 149, 115, 150, 122, 152, 153, 154, 155, 156, 157, 158, 159, 148, 153, 134].

As a first application, we exemplify these constraints in an  $SO(10)$  GUT with three families of  $\mathbf{16}_F$  fermionic representations and a scalar potential build from a  $\mathbf{16}_H$  and a  $\mathbf{45}_H$  representation. Therein, we have demonstrated how each successive application of the constraints (I.a), (I.b), and (I.c) reduces the admissible parameter space of initial conditions.

In the absence of Yukawa couplings, the above concrete model cannot reproduce the SM fermion sector. Still, we were able to draw some important specific conclusions. In particular, we find that previously neglected non-admissible breaking directions prohibit any possibility of radiative symmetry breaking to the Standard Model via Pati-Salam-type intermediate vacua. This conclusion exemplifies that (I.c) poses a novel and highly restrictive constraint on GUEFTs.

## 6.2 Towards low-energy-predictive grand unification

Our first main motivation is to phenomenologically constrain GUT models. Here, we put our results into context and emphasise why a quantitative understanding of scalar potentials is key to progress in GUT phenomenology. Phenomenological viability and predictive power of grand unification have indeed been extensively studied in the gauge-Yukawa sector (see *e.g.* [145, 146, 147, 148, 149, 115, 150, 122, 152, 153, 154, 155, 156, 157, 158, 159, 148, 153, 134]). Less focus has been given to complement such studies with a quantitative analysis of the required symmetry breaking via scalar potentials. The latter is however, for reasons detailed below, no less crucial.

On the one hand, the gauge-coupling unification paradigm provides strong constraints in the gauge sector, enhanced by the consideration of a proton decay constant compatible with the current experimental bounds. In the Yukawa sector, the common origin of SM fermionic representation tends to reduce the number of free parameters in the unified description of the theory. Matching fermion masses and mixing angles in the low-energy regime provides stringent constraints, conferring in some cases a certain predictive power to the GUT framework (in particular in the neutrino sector). Both in the gauge and Yukawa sectors, the lack of precise information on the structure of the scalar potential is handily compensated by the introduction of a limited number of additional free parameters (threshold corrections in the gauge sector, linear combinations of scalar expectation values in the Yukawa sector). Crucially, at one-loop (and up to two-loop for the gauge couplings), the scalars do not contribute to the running in the gauge-Yukawa sector.

On the other hand, the scalar potential determines both the scalar expectation values and the threshold corrections, see *e.g.* the recent work [188]. Quantitative considerations in the scalar sector are thus a crucial input for the gauge-Yukawa sector. However, the scalar sector comes with conceptual as well as practical limitations. On the conceptual side, while the unification of vector and fermionic representation

tends to reduce the size of the corresponding parameter space, the group-theoretical properties of large scalar representations generally result in a vastly extended scalar potential<sup>18</sup>, hence negatively (and, often, drastically) impacting the overall predictive power of the model<sup>19</sup>. On the practical side, the required coexistence of vastly separated symmetry-breaking scales within the theory poses difficulties in the construction of viable scalar sectors:

- One option is to introduce a sophisticated mechanism generating large hierarchies in the scalar spectrum, possibly between states belonging to the same unified representation. This includes the emblematic doublet-triplet splitting problem and its solutions in Georgi-Glashow SUSY models (see *e.g.* [189, 190, 191]) or the "missing vev" mechanism and its extensions in  $SO(10)$  GUTs [192, 193, 194, 195, 196]. Oftentimes, additional scalar representations must be introduced, hence reducing predictive power.
- Another option is to fine-tune relations among the free mass parameters to achieve cancellations in the physical spectrum. Putting aside the question of whether and why such fine-tuned relations would be expected to occur in nature, a fine-tuned parameter space poses practical difficulties in attempting to make physical predictions. For example, a scalar mass coupling<sup>20</sup> could be taken of order  $M_{EW}$  while others are typically of order  $M_{GUT}$ . Such a setting is actually extremely unstable along the RG flow (see for instance the expression of the beta-functions in Eqs. (6.E14)-(6.E18)). In turn, physical observables become highly dependent on the renormalisation scale prescription, rendering current perturbative methods unreliable.
- As a third option, radiative symmetry breaking presents an appealing mechanism to dynamically generate mass scales within the theory. However, to our knowledge, the question of knowing whether significantly separated mass scales can thereby be generated remains to be addressed.

In any case, understanding whether, and if so how, the unified theory can reproduce a viable phenomenology from the GUT scale down to the EW scale requires a quantitative exploration of the scalar sector. We note that, once the physical vev has been identified, identifying the correct gauge-invariant bound-state spectrum may deviate from perturbative expectations [197, 198], cf. [199, 200] for studies in the context of GUT gauge groups.

In this work, we have chosen to focus on radiative symmetry breaking. We have introduced a set of perturbative methods to exclude regions of the scalar parameter space by examining the structure of the scalar potential. In addition to applying stability and perturbativity constraints, we have discussed how the consideration of non-admissible breaking patterns provides valuable insights on the local or global property of minima otherwise leading to potentially viable breaking scenarios. Indeed, we have demonstrated that such considerations can entirely rule out specific breaking patterns. In the  $\mathbf{16}_H \oplus \mathbf{45}_H$  model, the non-observability of Pati-Salam breaking chains essentially obviates the need for a detailed analysis of tachyons in the scalar spectrum<sup>21</sup> [169, 170, 171, 172, 173, 160, 161, 188]. While our methods were applied to a specific  $SO(10)$  model, we stress that these can be straightforwardly generalised to any other GUEFT possessing non-admissible breaking chains in addition to those leading to  $3_C 2_L 1_Y$ . A natural extension of the present work would be to transpose the analysis to the realistic  $\mathbf{10}_H \oplus \mathbf{45}_H \oplus \mathbf{126}_H$  model and to address, among other things, whether Pati-Salam breaking chains are observable or not.

A natural remaining question is how to generalise to scalar potentials including mass terms. We expect the answer to depend on the ratio of the radiative symmetry-breaking scale and the bare mass terms. First, mass terms can safely be neglected whenever they are significantly smaller than the radiative symmetry-breaking scale. In this case, we expect all of our results to persist. Second, mass terms will dominate if they are significantly larger than the radiative symmetry-breaking scale (obtained in the scalar potential without mass terms). In this case, the mass terms dominantly drive symmetry breaking and the

<sup>18</sup>In particular, the lack of knowledge of the threshold corrections leads to very large uncertainties in the proton lifetime and the Weinberg angle [180].

<sup>19</sup>The situation worsens if one goes beyond a perturbatively renormalisable description of the GUEFT, including higher-dimensional operators in the scalar potential.

<sup>20</sup>More generally, a combination of scalar mass and/or trilinear couplings.

<sup>21</sup>We caution however that we have not proven that such breaking patterns would not become observable with other parameterisations of the quantum potential (higher-order truncations or a different method of RG-improvement)

presented constraints do not apply. Finally, the case of radiative symmetry breaking with comparably sized mass terms requires a renewed analysis. Similar statements hold for the effect of higher-order couplings.

Further, we do not account for the possibility of a meta-stable admissible vacuum in the presence of a deeper non-admissible vacuum. In particular, for near-degenerate vacua, meta-stability could present a way to evade the excluded regions in parameter space inferred with our blueprint.

After decades of active development, phenomenological studies of the gauge-Yukawa sector have mostly saturated: In particular for  $SO(10)$ , the complete perturbative Yukawa couplings to the fermionic  $\mathbf{16}_F$  (containing the SM fermions) have been studied [157] and gauge-coupling unification has been studied at the 2-loop level [113]. Moreover, we cannot expect to rule out this minimal potentially viable model from improvement of direct experimental constraints, which are essentially limited to proton-decay bounds.

Progress to rule out (or vice versa identify the most promising) GUT models needs to thus focus on the scalar potential. Our results present an explicit first example as well as a more generally applicable blueprint and technical toolkit to systematically rule out GUT models based on the scalar sector.

However, even if the investigated constraints can fully fix the scalar potential, the gauge-Yukawa sector of the  $\mathbf{10}_H \oplus \mathbf{45}_H \oplus \mathbf{126}_H$  still contains a similar amount of free parameters as the SM and some degree of predictivity may be possible [159]. In order to significantly reduce the plethora of free parameters of the SM, further theoretical input is necessary. The vicinity of the GUT and the Planck scale hints at quantum gravity as a promising candidate.

### 6.3 Towards constraints on quantum-gravity scenarios

Our second main motivation is to systematically constrain the viable Planck-scale parameter space of GUEFTs and thereby connect to quantum-gravity (QG) scenarios. If such a link can be drawn, the predictive power of QG scenarios may provide a further set of constraints, cf. (III) in Sec. 2, on the Planck-scale initial conditions. Vice versa, GUEFTs provide a possible arena for indirect experimental tests of QG. This is because, as our work clearly demonstrates, viable IR phenomenology is impacted by Planck-scale initial conditions.

Before providing an outlook on future work, we thus briefly comment on three QG scenarios<sup>22</sup> in which we see a promising route to make this link explicit. It is useful to distinguish between two possibilities.

On the one hand, there are QG scenarios which remain within the framework of quantum field theory. In this case, gravitational fluctuations will provide additional contributions to the Renormalisation Group (RG) flow of beta functions of the GUEFT, *i.e.*,

$$\beta_{c_i} = \beta_{c_i}^{(\text{GUEFT})} + \beta_{c_i}^{(\text{gravity})} . \quad (6.56)$$

Herein,  $c_i$  denotes the collection of all GUEFT couplings. Typically, one then demands  $\beta_{c_i}$  to lead to a UV-complete theory. With sufficient insight into the gravitational contributions  $\beta_{c_i}^{(\text{gravity})}$ , such a UV-completion implies additional constraints on the parameter space at the Planck scale. We will comment on two such scenarios – Complete Asymptotic Freedom as well as Asymptotic Safety – below.

On the other hand, there are QG scenarios which cannot be phrased in the framework of quantum field theory. Nevertheless, in order to be consistent with observations, they have to provide a limit – typically associated with the Planck scale – in which an EFT description emerges as a low-energy limit. Hence, there again must be some way of extracting predictions about the GUEFT couplings at the Planck scale. We will briefly comment on the case of string theory below.

#### Complete Asymptotic Freedom

The QG scenario in [201, 202, 203] suggests that, even at trans-Planckian scales, gravity decouples from the matter sector. In practice, such a scenario thus amount to simply extrapolating the SM or, in the

<sup>22</sup>We are unaware of other QG scenarios in which predictions or constraints on the Planck-scale parameter space of gauge-Yukawa theories have been obtained.

present context, the respective GUEFT beyond the Planck scale, *i.e.*,

$$\beta_{c_i}^{(\text{gravity})} = 0, \quad (6.57)$$

in Eq. 6.56. We note that, in any such QG scenario, the Standard Model remains UV-incomplete due to the U(1) Landau-pole. This obstruction, however, can be avoided in a GUT where the U(1) Abelian gauge group at high energies is part of a non-Abelian gauge group with self-interactions. Said self-interactions can – depending on the respective gauge group and matter content – be sufficiently antiscreening to provide for asymptotic freedom of the gauge coupling. Even asymptotically free gauge sectors can be sufficient to also render Yukawa couplings and quartic couplings asymptotically free: a proposal known as Complete Asymptotic Freedom (CAF) of gauge-Yukawa theories<sup>23</sup>. The conditions to achieve CAF in gauge-Yukawa theories have been analysed, for instance, in [206, 207, 208, 209]. The requirement of CAF without gravity, places additional non-trivial constraints on the viable parameter space at the Planck scale. Such a QG scenario is thus probably the most straightforward example of additional constraints from demanding a UV-completion.

### Asymptotic Safety

The asymptotic-safety scenario for QG [210, 211] (see [212, 213] for textbooks and [214] for a recent review) predicts quantum scale symmetry of gravity and matter at scales  $k$  beyond the Planck scale  $M_{\text{Pl}}$ . (We use  $k$  instead of  $\mu$  here to distinguish non-perturbative and perturbative RG schemes.) If asymptotic safety is realised, the dimensionless Newton coupling  $g = G k^2$  (with  $G$  the usual dimensionful Newton coupling) transitions between classical power-law scaling  $g \sim k^2$  below  $M_{\text{Pl}}$  and scale symmetry, *i.e.*, scale-independent behaviour  $g = g_* = \text{const}$ , above  $M_{\text{Pl}}$ . The leading-order gravitational contribution to matter couplings acts like an anomalous dimension and the transition can be described by, cf. [215, 216, 217] and [218] for a recent review,

$$\beta_{c_i}^{(\text{gravity})} = \begin{cases} f_{c_i} c_i + \mathcal{O}(c_i^2) + \dots & k > M_{\text{Pl}} \\ 0 & k < M_{\text{Pl}} \end{cases}. \quad (6.58)$$

with  $f_{c_i}$  constant, dependent on the gravitational fixed-point values, *e.g.*  $g_*$ , and, in principle, calculable from first principles. As long as all GUEFT couplings  $c_i$  remain in the perturbative regime, neglecting  $\mathcal{O}(c_i^2)$  is a good approximation. Dots denote further terms arising from non-minimal couplings [219, 220, 221, 222, 223, 224, 225] and induced higher-order matter [226, 227, 228, 229, 230, 222, 223, 231, 232] couplings.

Due to the universal nature of gravity,  $f_{c_i} \equiv f_g$  is universal for all gauge couplings;  $f_{c_i} \equiv f_y$  is universal for all Yukawa couplings (invariant under the same global symmetry); and  $f_{c_i} \equiv f_\lambda$  is universal for all quartic couplings. Functional RG calculations provide the following picture<sup>24</sup>: The gravitational contribution to gauge couplings is found to be antiscreening, *i.e.*,  $f_g \geq 0$  [245, 246, 247, 230, 216, 248]; the screening or antiscreening nature of the gravitational contribution to Yukawa couplings depends on the matter content of the universe [249, 250, 228, 229]; The gravitational contribution to quartic couplings is found to be screening, *i.e.*,  $f_\lambda < 0$  [219, 220, 251, 252, 253, 225, 254].

Clearly, the additional antiscreening contribution  $f_g$  to the RG flow of gauge couplings will (in comparison to CAF without gravity) enlarge the Planck-scale parameter space with underlying UV-complete dynamics, cf. [129] for an application in the context of GUTs.

To the contrary, the screening contribution to scalar quartic couplings (and scalar potentials in general) is expected to provide sharp predictions for the shape of scalar potentials, cf. [255] for an application to the SM Higgs potential, [251, 256, 257, 258, 225, 259] for applications to dark-matter, and [130] for a previous discussion of GUT potentials. This is most exciting in the present context of GUEFTs since it suggests that Asymptotic Safety may fully predict the scalar potentials and thus the breaking scales and breaking directions of GUEFT models [130]. The methods developed in this work provide the basis

<sup>23</sup>More recently, it has also been found that gauge-Yukawa theories can develop interacting fixed points with UV-attractive directions and may thus, in principle, be asymptotically safe without the presence of gravitational fluctuations [204]. We caution that it is unclear whether commonly discussed GUTs can realise such a scenario [205].

<sup>24</sup>In perturbative dimensional regularisation schemes, gravitational contributions to matter couplings have also been calculated [233, 234, 235, 236, 237, 238, 239, 240, 241], see also [242, 243] for recent progress on the relation between functional RG and dimensional regularisation schemes and [244] for an application to gravity-matter systems.

for a systematic study of these promising ideas. The biggest outstanding caveat is the question how gravitational contributions and contributions from dimensionful terms will impact the presently applied method to determine the multidimensional RG-improved potential, cf. Sec. 3.

### String Theory

Let us first note that supersymmetric GUTs are quite natural in the context of string theory, see for example [131, 260, 261, 262, 134] and references therein. Naturally, the blueprint in Sec. 2 can also be applied to supersymmetric theories but requires renewed analysis. Since low-energy supersymmetry has not been found, a viable breaking of supersymmetry adds to the constraints on scalar potentials.

Concerning non-supersymmetric GUTs, non-supersymmetric vacua are notoriously hard to construct in string compactifications. Yet, there exists a non-tachyonic  $SO(16) \times SO(16)'$  string theory without supersymmetry [263, 264], in which – when compactified to four dimensions – an  $SO(10)$  GUT with  $\mathbf{16}_F$  spinor representation and  $\mathbf{45}_H$  potential can be found [133].

In string theory all of the low energy couplings stem from expectation values of the radion fields, that is the diagonal part of the metric in the compactified dimensions, see [265] for review. Hence, in a given compactification of  $SO(16) \times SO(16)'$  to 4-dimensions, the quartic couplings  $\lambda_i$  are not free parameters, but can, in principle, be calculated and compared with the Planck-scale parameter-space constraints arising from an analysis as presented here.

## 6.4 Outlook

We see the following important extensions and applications.

First and foremost, a viable Yukawa sector [149, 115, 150, 122, 152, 153, 154, 155, 156, 157, 158, 159, 148, 153, 134] requires an extension of the specified scalar representations from  $\mathbf{16}_H \oplus \mathbf{45}_H$  to  $\mathbf{10}_H \oplus \mathbf{45}_H \oplus \mathbf{126}_H$ . Without such an extension, any application to specific quantum-gravity approaches may still give tentative insights but does not address the full picture.

Within such a minimal but potentially viable grand-unified effective field theory (GUEFT), the presented blueprint will determine which regions in parameter space correspond to a potentially viable IR phenomenology. In such a setup, it may prove important to reconsider the respective constraints arising from non-admissible breaking directions in view of mass terms, cf. Sec. 2. In any case, the methods presented here are general and can serve as useful starting point for further analyses such as the characterisation of the physical spectrum, providing new input to traditional explorations of gauge-Yukawa sectors.

With regards to applications to concrete quantum-gravity (QG) scenarios, it seems promising to study all three approaches in Sec. 6.3. QG scenarios which reduce a study of Complete Asymptotic Freedom (CAF) are directly applicable. The QG scenario of asymptotic safety requires an extension of the RG-improved potential to include gravitational contributions.

Overall, we are convinced that this work is only the first step and that there is a promising route to quantitatively connect QG approaches and EW-scale physics to restore predictive power in GUEFTs.



# Appendix

## A One-loop RG-improved potential

In this appendix, we review a certain number of useful properties and relations satisfied by the one-loop RG-improved potential introduced in section 3. In particular, we analytically justify the numerical approach used in this work to efficiently estimate the depth of the RG-improved potential in every relevant vacuum configuration in sections A.2 and A.3, and numerically evaluate its accuracy in section A.4.

### A.1 Stationary point equation

In order to derive the station point equation for the RG-improved potential, it is first useful to note that the field derivatives of any RG-improved quantity  $V(\phi^i, \mu_*(\phi^i))$  can be decomposed in the following way:

$$\nabla_i V(\phi, \mu_*(\phi)) = \frac{dV}{d\phi^i}(\phi, \mu_*(\phi)) = \frac{\partial V}{\partial \phi^i}(\phi, \mu_*(\phi)) + \nabla_i t_* \frac{dV}{dt}(\phi, \mu_*(\phi)). \quad (6.A1)$$

Namely, the total derivative with respect to the field component  $\phi^i$  splits into a partial derivative, and a contribution stemming from the implicit dependence of the RG-scale  $t_* = \log \mu_*^2$  on the field values. Next, we can readily express the stationary point equations satisfied by the RG-improved potential at an extremum:

$$\begin{aligned} \nabla_i V^{\text{eff}}(\phi, \mu_*(\phi^i)) &= \frac{\partial V^{\text{eff}}}{\partial \phi^i}(\phi, \mu_*(\phi)) + \nabla_i t_* \frac{dV^{\text{eff}}}{dt}(\phi, \mu_*(\phi)) \\ &= \partial_i V^{\text{eff}}(\phi, \mu_*(\phi)) + 2\mathbb{B}(\phi, \mu_*(\phi)) \nabla_i t_*(\phi) = 0, \end{aligned} \quad (6.A2)$$

where we used in the last step the first order truncated Callan-Symanzik equation (6.10) (see *e.g.* [137]):

$$\frac{dV^{(0)}}{dt} = 2\mathbb{B}. \quad (6.A3)$$

An expression for  $\nabla_i t_*$  may be derived, using the fact that  $V^{(1)}(\phi; \mu_*(\phi))$  identically vanishes:

$$V^{(1)}(\phi; \mu_*(\phi)) = \mathbb{A}(\phi; \mu_*(\phi)) + \mathbb{B}(\phi; \mu_*(\phi)) \log \frac{\varphi^2}{\mu_*^2} = 0 \quad \forall \phi. \quad (6.A4)$$

Taking the field derivatives of the previous equation yields (we temporarily omit the functions' arguments for clarity):

$$\begin{aligned} \nabla_i V^{(1)} &= \nabla_i \mathbb{A} + \nabla_i \mathbb{B} \log \frac{\varphi^2}{\mu_*^2} + 2\mathbb{B} \left( \frac{\phi^i}{\varphi^2} - 2\nabla_i t_* \right) = 0 \\ \Rightarrow \quad \nabla_i t_* &= \frac{\phi^i}{\varphi^2} + \frac{1}{2\mathbb{B}} \left( \nabla_i \mathbb{A} + \nabla_i \mathbb{B} \log \frac{\varphi^2}{\mu_*^2} \right). \end{aligned} \quad (6.A5)$$

Neglecting terms of order 2 in perturbation theory, the field derivatives of  $\mathbb{A}$  and  $\mathbb{B}$  can be simplified as<sup>25</sup>

$$\nabla_i \mathbb{A} = \frac{\partial \mathbb{A}}{\partial \phi^i} + \nabla_i t_* \frac{d\mathbb{A}}{dt} \approx \frac{\partial \mathbb{A}}{\partial \phi^i}, \quad (6.A6)$$

$$\nabla_i \mathbb{B} = \frac{\partial \mathbb{B}}{\partial \phi^i} + \nabla_i t_* \frac{d\mathbb{B}}{dt} \approx \frac{\partial \mathbb{B}}{\partial \phi^i}. \quad (6.A7)$$

In this approximation, it is straightforward to derive the radial stationary-point equation (6.14), by first noting that

$$\begin{aligned} \phi^i \nabla_i t_* &= 1 + \frac{1}{2\mathbb{B}} \left( \phi^i \frac{\partial \mathbb{A}}{\partial \phi^i} + \phi^i \frac{\partial \mathbb{B}}{\partial \phi^i} \log \frac{\varphi^2}{\mu_*^2} \right) \\ &= 1 + \frac{1}{2\mathbb{B}} 4V^{(1)} = 1, \end{aligned} \quad (6.A8)$$

<sup>25</sup>This is the  $t_*^{(0)}$  approximation mentioned in [136, 137].

where the last line stems from the homogeneity of  $\mathbb{A}$  and  $\mathbb{B}$  with respect to  $\phi$ . Hence, we may write

$$\phi^i \nabla_i V^{\text{eff}} = \phi_i \frac{\partial V^{\text{eff}}}{\partial \phi^i} + 2\mathbb{B} \phi^i \nabla_i t_* = 4V^{\text{eff}} + 2\mathbb{B} = 0, \quad (6.A9)$$

which is the radial stationary point equation derived in [137]. While this equation alone does not allow to locate the minimum, it restricts its position to a  $(n-1)$ -dimensional hypersurface in the  $n$ -dimensional field space. Going one step further, it is possible to reiterate the derivation beyond the one-loop approximation. We first derive the exact form of  $\nabla_i t_*$ , starting from Eq. (6.A5) and using once again the decomposition in Eq. (6.A1):

$$\begin{aligned} \nabla_i t_* &= \frac{\phi^i}{\varphi^2} + \frac{1}{2\mathbb{B}} \left( \nabla_i \mathbb{A} + \nabla_i \mathbb{B} \log \frac{\varphi^2}{\mu_*^2} \right) \\ &= \frac{\phi^i}{\varphi^2} + \frac{1}{2\mathbb{B}} \left( \frac{\partial \mathbb{A}}{\partial \phi^i} + \frac{\partial \mathbb{B}}{\partial \phi^i} \log \frac{\varphi^2}{\mu_*^2} \right) + \frac{1}{2\mathbb{B}} \nabla_i t_* \left( \frac{d\mathbb{A}}{dt} + \frac{d\mathbb{B}}{dt} \log \frac{\varphi^2}{\mu_*^2} \right). \end{aligned} \quad (6.A10)$$

Collecting all the  $\nabla_i t_*$  terms in the left-hand side, one finally gets

$$\nabla_i t_* = \eta \left[ \frac{\phi^i}{\varphi^2} + \frac{1}{2\mathbb{B}} \left( \frac{\partial \mathbb{A}}{\partial \phi^i} + \frac{\partial \mathbb{B}}{\partial \phi^i} \log \frac{\varphi^2}{\mu_*^2} \right) \right] \quad (6.A11)$$

where

$$\eta \equiv \left[ 1 - \frac{1}{2\mathbb{B}} \left( \frac{d\mathbb{A}}{dt} + \frac{d\mathbb{B}}{dt} \log \frac{\varphi^2}{\mu_*^2} \right) \right]^{-1}. \quad (6.A12)$$

In other words, including the order 2 contributions introduces a multiplicative factor  $\eta$  in the expression of  $\nabla_i t_*$ , satisfying

$$\eta = 1 + \mathcal{O}(\hbar^2). \quad (6.A13)$$

In particular, the set of stationary point equations now reads

$$\nabla_i V^{\text{eff}} = \partial_i V^{\text{eff}} + 2\eta \mathbb{B} \nabla_i t_* = 0, \quad (6.A14)$$

and the radial stationary point equation becomes

$$4V^{\text{eff}} + 2\eta \mathbb{B} = 0. \quad (6.A15)$$

Clearly, in the  $\mathcal{O}(\hbar)$  approximation, where the running of  $\mathbb{A}$  and  $\mathbb{B}$  is neglected, Eq. (6.A15) reduces to Eq. (6.A9):

$$4V^{\text{eff}} + 2\mathbb{B} = 0. \quad (6.A16)$$

The main advantage of the above approximation is that Eq. (6.A16) always takes a polynomial form in the fields. More precisely, the quantity

$$\tilde{V}^{(0)} \equiv V^{\text{eff}} + \frac{1}{2}\mathbb{B} \quad (6.A17)$$

takes the same polynomial form as  $V^{(0)}$  with one-loop corrected numerical coefficients and vanishes at a minimum by (6.A9). It can be shown [137] that the second derivative of the potential along the radial direction is proportional to  $\mathbb{B}$ . Therefore, according to Eq. (6.A9), at a minimum, one has  $\mathbb{B} > 0$  and  $V^{\text{eff}} < 0$ .

## A.2 RG-improvement and the Gildener-Weinberg approximation

In the Gildener-Weinberg approach [168], the renormalisation scale prescription consists in identifying the RG-scale  $\mu_{\text{GW}}$  at which the tree-level potential develops a flat direction. Along this flat direction, the field values are expressed as

$$\phi = \varphi \vec{n}. \quad (6.A18)$$

Based on the general expression of the one loop contributions to the scalar potential in Eq. (6.4) and on the homogeneity of  $\mathbb{A}$  and  $\mathbb{B}$  with respect to the radial coordinate, one may write

$$V^{(1)}(\phi; \mu) = \mathbb{A}(\phi; \mu) + \mathbb{B}(\phi; \mu) \log \frac{\varphi^2}{\mu^2} = \mathbb{A}(\vec{n}; \mu) \varphi^4 + \mathbb{B}(\vec{n}; \mu) \varphi^4 \log \frac{\varphi^2}{\mu^2} \quad (6.A19)$$

so the tree-level and one-loop contributions to the scalar potential take the following form along the flat direction:

$$V^{(0)}(\phi; \mu_{\text{GW}}) = \lambda(\vec{n}; \mu_{\text{GW}}) \varphi^4 = 0, \quad (6.A20)$$

$$V^{(1)}(\phi; \mu_{\text{GW}}) = \mathbb{A}(\vec{n}; \mu_{\text{GW}}) \varphi^4 + \mathbb{B}(\vec{n}; \mu_{\text{GW}}) \varphi^4 \log \frac{\varphi^2}{\mu_{\text{GW}}^2}. \quad (6.A21)$$

Taking the derivative with respect to  $\varphi$  yields

$$\frac{\partial V^{(0)}}{\partial \varphi}(\phi; \mu_{\text{GW}}) = 4\lambda(\vec{n}; \mu_{\text{GW}}^2) \varphi^3 = 0, \quad (6.A22)$$

$$\frac{\partial V^{(1)}}{\partial \varphi}(\phi; \mu_{\text{GW}}) = 4\varphi^3 \left[ \mathbb{A}(\vec{n}; \mu_{\text{GW}}) + \mathbb{B}(\vec{n}; \mu_{\text{GW}}) \left( \log \frac{\varphi^2}{\mu_{\text{GW}}^2} + \frac{1}{2} \right) \right]. \quad (6.A23)$$

Hence at the minimum the radial coordinate satisfies the relation

$$\log \frac{\langle \varphi \rangle^2}{\mu_{\text{GW}}^2} = -\frac{1}{2} - \frac{\mathbb{A}}{\mathbb{B}}. \quad (6.A24)$$

Getting back to the RG improvement procedure described in section 3.1, one may define a RG-scale  $\tilde{\mu}$  such that the one loop corrections vanish at the field value  $\langle \phi \rangle = \vec{n} \langle \varphi \rangle$ , namely:

$$V^{(1)}(\langle \phi \rangle; \tilde{\mu}) = 0. \quad (6.A25)$$

Let  $\delta t = \tilde{t} - t_{\text{GW}} = \log \frac{\tilde{\mu}^2}{\mu_{\text{GW}}^2}$  be the associated shift in the logarithm of the RG-scales. To first order in  $\delta t$ , one has

$$V^{(0)}(\langle \phi \rangle, \tilde{\mu}) = V^{(0)}(\langle \phi \rangle, \mu_{\text{GW}}) + \delta t \frac{dV^{(0)}}{dt}(\langle \phi \rangle, \mu_{\text{GW}}) + \mathcal{O}(\delta t^2), \quad (6.A26)$$

$$\mathbb{A}(\vec{n}, \tilde{\mu}) = \mathbb{A}(\vec{n}, \mu_{\text{GW}}) + \delta t \frac{d\mathbb{A}}{dt}(\vec{n}, \mu_{\text{GW}}) + \mathcal{O}(\delta t^2), \quad (6.A27)$$

$$\mathbb{B}(\vec{n}, \tilde{\mu}) = \mathbb{B}(\vec{n}, \mu_{\text{GW}}) + \delta t \frac{d\mathbb{B}}{dt}(\vec{n}, \mu_{\text{GW}}) + \mathcal{O}(\delta t^2). \quad (6.A28)$$

Discarding terms that are formally of order 2 in perturbation theory allows to simplify the last two relations:

$$\mathbb{A}(\vec{n}, \tilde{\mu}) = \mathbb{A}(\vec{n}, \mu_{\text{GW}}) + \mathcal{O}(\delta t^2), \quad (6.A29)$$

$$\mathbb{B}(\vec{n}, \tilde{\mu}) = \mathbb{B}(\vec{n}, \mu_{\text{GW}}) + \mathcal{O}(\delta t^2). \quad (6.A30)$$

Had we retained terms of order  $(\delta t)^2$  in the above expansions, working in the one-loop approximation would have yielded Eqs. (6.A26), (6.A29), and (6.A30) anyways, since the  $\mathcal{O}(\delta t^2)$  terms formally encompass  $\mathcal{O}(\hbar^2)$  quantities. Combining Eqs. (6.A23) and (6.A29), Eq. (6.A30) allows to rewrite the Gildener-Weinberg radial stationary point equation at the shifted scale  $\tilde{\mu}$ :

$$\begin{aligned} 0 &= \frac{\partial}{\partial \varphi} \left( V^{(0)}(\phi; \mu_{\text{GW}}) + V^{(1)}(\phi; \mu_{\text{GW}}) \right) \\ &= 4\varphi^3 \left[ \mathbb{A}(\vec{n}, \tilde{\mu}) + \mathbb{B}(\vec{n}; \tilde{\mu}) \left( \log \frac{\varphi^2}{\tilde{\mu}^2} + 2\delta t + \frac{1}{2} \right) \right] \\ &= 4\varphi^3 \mathbb{B}(\vec{n}; \tilde{\mu}) \left( \frac{1}{2} + 2\delta t \right), \end{aligned} \quad (6.A31)$$

where Eq. (6.A25) was used in the last step. We conclude that, in the one loop approximation,

$$\delta t = -\frac{1}{4}. \quad (6.A32)$$

Considering the first order truncation of Callan-Symanzik equation (6.10),

$$\frac{dV^{(0)}}{dt} = 2\mathbb{B}. \quad (6.A33)$$

we finally obtain the relation

$$V^{(0)}(\phi; \mu_{\text{GW}}) = V^{(0)}(\phi; \tilde{\mu}) - 2\mathbb{B}(\phi; \mu_{\text{GW}})\delta t = V^{(0)}(\phi; \tilde{\mu}) + \frac{1}{2}\mathbb{B}(\phi; \tilde{\mu}) \equiv \tilde{V}^{(0)}(\phi; \tilde{\mu}), \quad (6.A34)$$

where the quantity  $\tilde{V}^{(0)}$  is defined similarly as in Eq. (6.A17). Quite importantly, the above relation implies that in the one loop approximation, at the scale  $\tilde{\mu}$ , the corrected tree-level potential  $\tilde{V}^{(0)}$  has the same structure than the tree-level potential evaluated at the scale  $\mu_{\text{GW}}$ . In particular,  $\tilde{V}^{(0)}(\phi; \tilde{\mu})$  inherits the flat direction of  $V^{(0)}(\phi; \mu_{\text{GW}})$ , and therefore

$$\tilde{V}^{(0)}(\langle\phi\rangle; \tilde{\mu}) = 0. \quad (6.A35)$$

From the above equation, we may finally conclude that the Gildener-Weinberg vev  $\langle\phi\rangle$  satisfies the RG-improved radial stationary-point equation (6.A9). It is worth emphasising that,  $\langle\phi\rangle$  is not, in general, a solution of the full set of stationary-point equations (6.A2), *i.e.* it does not minimise the RG-improved potential  $V^{\text{eff}}$ . In what follows, we will show however that it constitutes a first order approximation of the actual vev  $\langle\phi\rangle^{\text{min}}$ .

Denoting  $\delta\phi = \langle\phi\rangle^{\text{min}} - \langle\phi\rangle$  the shift between the actual vev and the Gildener-Weinberg solution, one may write, to first order in  $\delta\phi$ :

$$\tilde{V}^{(0)}(\langle\phi\rangle^{\text{min}}) = \tilde{V}^{(0)}(\langle\phi\rangle) + \delta\phi^i \nabla_i \tilde{V}^{(0)}(\langle\phi\rangle) + \mathcal{O}(\delta\phi^2). \quad (6.A36)$$

Since both  $\langle\phi\rangle^{\text{min}}$  and  $\langle\phi\rangle$  belong to the hypersurface where the radial stationary point equation is satisfied (*i.e.* where  $\tilde{V}^{(0)} = 0$ ) and using the decomposition in Eq. (6.A1), the above relation reduces to

$$0 = \delta\phi^i \nabla_i \tilde{V}^{(0)}(\langle\phi\rangle) = \delta\phi^i \partial_i \tilde{V}^{(0)}(\langle\phi\rangle) + \delta\phi^i \nabla_i t_*(\langle\phi\rangle) \frac{d\tilde{V}^{(0)}}{dt}(\langle\phi\rangle). \quad (6.A37)$$

The  $\partial_i$  derivative in the right-hand side of the above equation vanishes since  $\langle\phi\rangle$  lies along the flat direction of  $\tilde{V}^{(0)}(\phi; \tilde{\mu})$ . In addition, to first order in perturbation theory, one can approximate

$$\frac{d\tilde{V}^{(0)}}{dt}(\langle\phi\rangle) = \frac{dV^{(0)}}{dt}(\langle\phi\rangle) + \mathcal{O}(\hbar^2), \quad (6.A38)$$

and Eq. (6.A37) implies

$$\delta\phi^i \nabla_i t_*(\langle\phi\rangle) = 0. \quad (6.A39)$$

Furthermore, to first order in  $\delta\phi^i$

$$t_*(\langle\phi\rangle^{\text{min}}) = t_*(\langle\phi\rangle) + \delta\phi^i \nabla t_*(\langle\phi\rangle), \quad (6.A40)$$

so we can finally establish that, to first order in perturbation theory,

$$t_*(\langle\phi\rangle^{\text{min}}) = t_*(\langle\phi\rangle) + \mathcal{O}(\delta\phi^2) \quad \Rightarrow \quad \mu_*^{\text{min}} \approx \tilde{\mu}. \quad (6.A41)$$

The above approximation constitutes the main result of this appendix, which can be summarised as follows: The RG-scale  $\tilde{\mu}$  at which the corrected tree-level potential  $\tilde{V}^{(0)} = V^{(0)} + \mathbb{B}/2$  develops a flat direction is a first order approximation of the value taken by field-dependent RG-scale  $\mu_*^{\text{min}}$  at the minimum of the RG-improved potential. This observation justifies the procedure described in section 3.2 to estimate, in an algorithmically efficient way, the position and depth of the minimum of the RG-improved potential.

### A.3 Minimisation beyond the one-loop approximation

The numerical procedure described in section 3.2 in the one-loop approximation of the radial stationary point equation can be slightly improved by taking into account corrections that are formally of order 2 in perturbation theory. For convenience, we rewrite below the exact radial stationary-point equation (6.A15) obtained beyond the one-loop approximation:

$$4V^{\text{eff}} + 2\eta\mathbb{B} = 0. \quad (6.A42)$$

where

$$\eta = \left[ 1 - \frac{1}{2\mathbb{B}} \left( \frac{d\mathbb{A}}{dt} + \frac{d\mathbb{B}}{dt} \log \frac{\varphi^2}{\mu_*^2} \right) \right]^{-1}. \quad (6.A43)$$

The expression of  $\eta$  can be further simplified by using

$$V^{(1)} = \mathbb{A} + \mathbb{B} \log \frac{\varphi^2}{\mu_*^2} = 0 \quad \Rightarrow \quad \log \frac{\varphi^2}{\mu_*^2} = -\frac{\mathbb{A}}{\mathbb{B}}, \quad (6.A44)$$

namely:

$$\eta = \left[ 1 - \frac{1}{2} \frac{\frac{d\mathbb{A}}{dt}\mathbb{B} - \mathbb{A}\frac{d\mathbb{B}}{dt}}{\mathbb{B}^2} \right]^{-1} = \left[ 1 - \frac{1}{2} \frac{d}{dt} \frac{\mathbb{A}}{\mathbb{B}} \right]^{-1}. \quad (6.A45)$$

In this latter form, it is clear that  $\eta$  does not depend on the radial field coordinate, but only on the direction of the field vector in the field space. We conveniently make use of this property in an iterative method allowing to estimate the position of the minimum beyond the one-loop approximation. We restate below the minimisation procedure described in Section 3.2, where steps 3 and 4 have been modified to include the effect of 2-loop corrections stemming from  $\eta$ , namely:

1. Starting with random values for the quartic couplings at some high scale  $\mu_0$ , the stability of the tree-level potential is asserted, and unstable configurations are discarded.
2. Evolution of the quartic couplings according to their RG running is performed down to some lower scale  $\mu_1$ .
3. At this point, we initialise the iterative procedure taking into account the effects of  $\eta \neq 1$ . For the first iteration, we set

$$k = 0, \quad \eta_k = \eta_0 = 1.$$

Defining

$$\tilde{V}^{(0)} \Big|_k = \tilde{V}^{(0)} \Big|_{\eta=\eta_k} = V^{(0)} + 2\eta_k\mathbb{B},$$

the scale  $\tilde{\mu}_k$  at which  $\tilde{V}^{(0)} \Big|_k$  develops flat directions is identified.

4. At the scale  $\tilde{\mu}_k$ , the flat direction  $\vec{n}_k$  is identified. The value of

$$\eta_{k+1} = \eta(\vec{n}_k; \tilde{\mu}_k)$$

does not depend on the radial field coordinate, and is evaluated using Eq. (6.A45). If  $|\eta_{k+1} - \eta_k| > \varepsilon$ , we repeat step 3 with

$$k \rightarrow k + 1, \quad \eta_k \rightarrow \eta_{k+1}.$$

Otherwise, we consider that the iteration has converged (in practice we set  $\varepsilon = 10^{-5}$ ), and will use  $\vec{n} = \vec{n}_k$  as the corrected generating vector of the flat direction, along which the field values take the form

$$\phi = \varphi \vec{n}.$$

5. The unique value of  $\langle \varphi \rangle$  such that

$$V^{(1)}(\langle \varphi \rangle \vec{n}; \tilde{\mu}) = 0$$

is identified. The field vector  $\langle \phi \rangle = \langle \varphi \rangle \vec{n}$  constitutes an estimation of the exact position of the minimum.

6. Finally, the depth of the RG-improved potential at the minimum, *i.e.* the quantity

$$V^{\text{eff}}(\langle\phi\rangle) = V^{(0)}(\langle\phi\rangle; \tilde{\mu})$$

is evaluated.

This modified minimisation procedure allows to achieve better accuracy (see the next section) on the estimation of the position and depth of the minimum. This is the procedure that was systematically used in our numerical study of the breaking patterns of the model.

#### A.4 Numerical performance and accuracy of the minimisation procedure

In order to confirm that the simplified minimisation procedure described in Section 3.2 and improved above does provide a reasonable estimation of the depth and position of the minimum, we have compared its outcome with that of a full-fledged numerical minimisation of the RG-improved potential. This comparison has been performed on a random sample of points, both in the case of 2- and 3-vev manifolds<sup>26</sup>, for which the number of minima hence characterised amounts to  $N^{(2)} = 2000$  and  $N^{(3)} = 500$ , respectively.

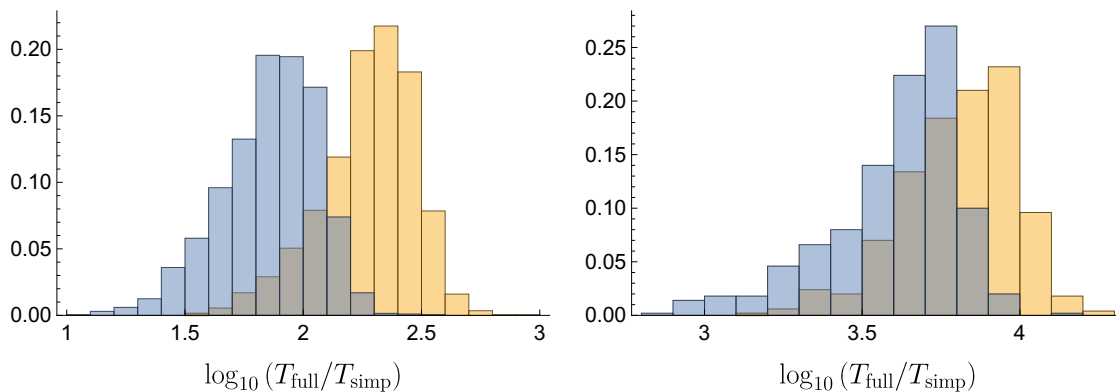


Figure A.1: Gain in performance for 2-vev (left panel) and 3-vev (right panel) manifolds, using the minimisation procedure described in Section 3.2 (yellow bars) and its improved version (blue bars), compared to a full-fledged numerical minimisation.

As stated before, the main motivation for using a simplified minimisation procedure is to speed-up the computations, therefore enabling one to perform a random scan over a large sample of points. Fig. A.1 illustrates the performance improvement in terms of execution time for both 2- and 3-vev manifolds. Given a minimum, we define  $T_{\text{full}}$  as the execution time of a full numerical minimisation and  $T_{\text{simp}}$  as the execution time of the simplified algorithms. The gain in performance  $T_{\text{full}}/T_{\text{simp}}$  is of order  $\mathcal{O}(10^2)$  and  $\mathcal{O}(10^3) - \mathcal{O}(10^4)$  for 2- and 3-vev manifolds respectively. Note that, on the computer used to perform this analysis, the average execution time of the full numerical minimisation is 4s for 2-vev manifolds and 460s for 3-vev manifolds.

Of course, the major gain in performance comes at a price: our minimisation procedure only provides an approximation of the position and depth of a minimum. However, as shown in Figures A.2–A.4, the relative error on the quantities  $V_{\text{min}}$ ,  $\varphi^{\text{min}} = \sqrt{\langle\phi\rangle_i^{\text{min}} \langle\phi\rangle_i^{\text{min}}}$  and  $t_*^{\text{min}} = t_*(\langle\phi\rangle^{\text{min}})$  are kept at a reasonable level. Concretely speaking, defining the logarithmic relative error on the parameter  $X$  (with  $X = V_{\text{min}}, \varphi^{\text{min}}, t_*^{\text{min}}$ ) as

$$\log_{\text{rel}}(X) = \log_{10} |\delta(X)| = \log_{10} \left| 100 \times \frac{X_{\text{simp}} - X_{\text{full}}}{X_{\text{full}}} \right|, \quad (6.A46)$$

<sup>26</sup>By construction, the improved minimisation procedure described above is ensured to converge towards the true minimum in the case of 1-vev manifolds.

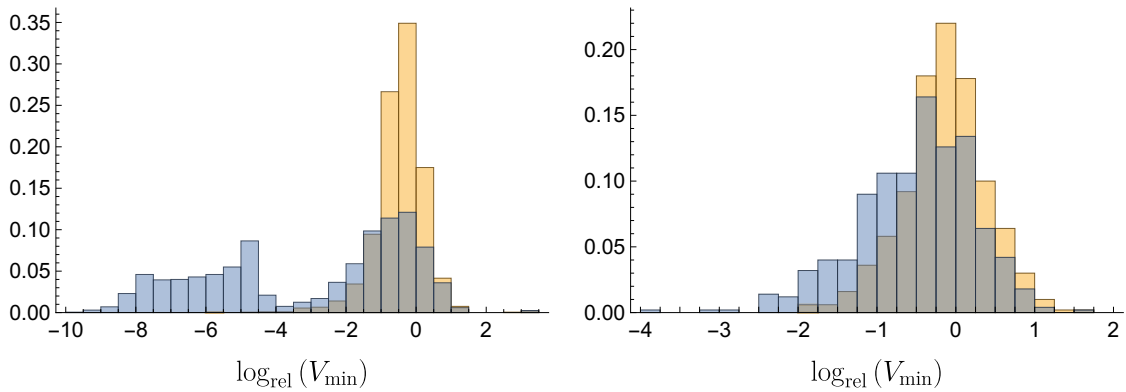


Figure A.2: Logarithmic relative error on  $V_{\min}$  for 2-vev (left panel) and 3-vev (right panel) manifolds, using the minimisation procedure described in Section 3.2 (yellow bars) and its improved version (blue bars), compared to a full-fledged numerical minimisation.

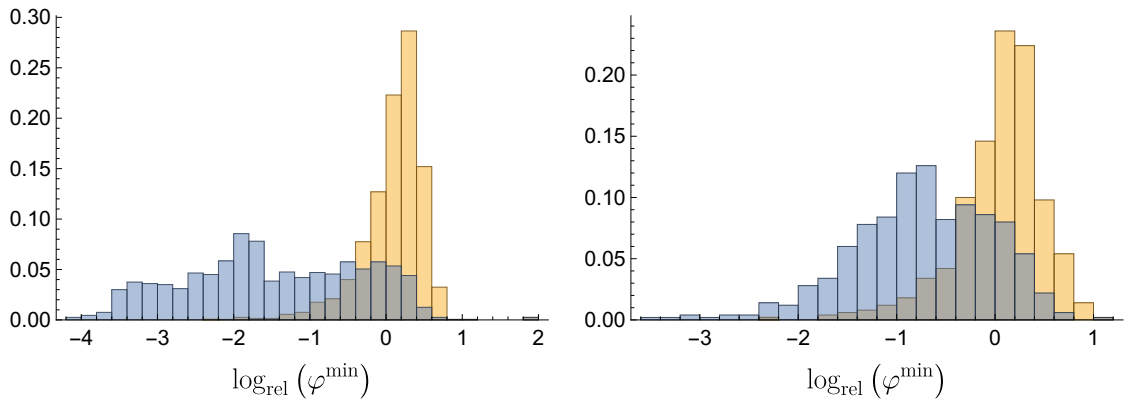


Figure A.3: Logarithmic relative error on  $\varphi^{\min}$  for 2-vev (left panel) and 3-vev (right panel) manifolds, using the minimisation procedure described in Section 3.2 (yellow bars) and its improved version (blue bars), compared to a full-fledged numerical minimisation.

rare are the points for which  $\log_{\text{rel}}(X) > 1$ . In other words, the relative error (in particular on the depth of the potential at a minimum, which is the most important quantity in this analysis) is almost always kept under the 10% level. In fact, focusing on the quantity of interest,  $V_{\min}$ , we observe that:

- For 2-vev manifolds,  $\delta(V_{\min}) > 10\%$  for 0.8% of the points and  $\delta(V_{\min}) > 5\%$  for 2.4% of the points,
- For 3-vev manifolds,  $\delta(V_{\min}) > 10\%$  for 0.6% of the points and  $\delta(V_{\min}) > 5\%$  for 3% of the points.

In addition, two comments are worth making regarding the left panel of Fig. A.2, showing the relative error on  $V_{\min}$  in the case of 2-vev manifolds:

- The excess of points with a relative error of order  $\mathcal{O}(10^{-10}\%)$  to  $\mathcal{O}(10^{-4}\%)$  corresponds to situations where one of the two vevs actually vanishes along the flat direction. In such cases, one effectively ends up minimising a 1-vev manifold, for which the improved minimisation procedure in ensured to converge towards the true minimum (up to numerical errors).
- A small number of points (5 out of 2000) give  $\log_{\text{rel}}(V_{\min}) \approx 3$  or, equivalently,  $\delta(V_{\min}) \approx 1000\%$ . We have explicitly checked that those points are in fact characterised by the occurrence of two flat directions of different nature at RG-scales very close to each other, leading to a situation where either (i) two extrema coexist, one of them possibly corresponding to a local maximum [137], or

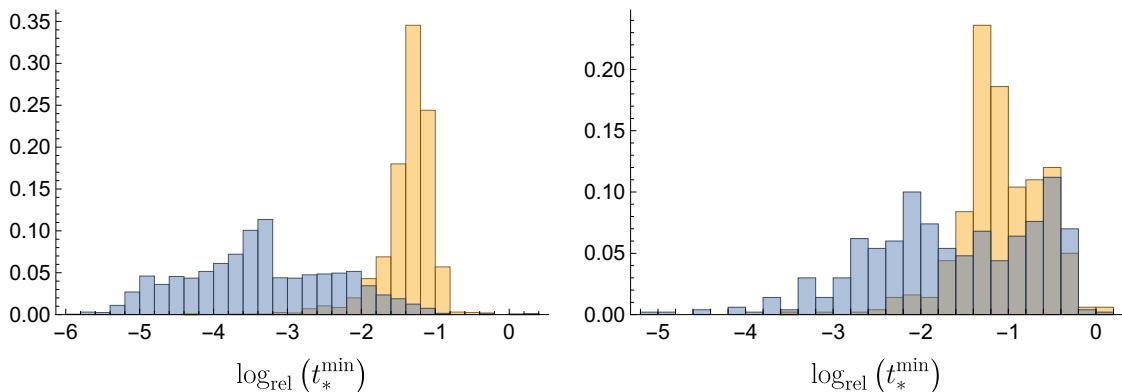


Figure A.4: Logarithmic relative error on  $t_*^{\min}$  for 2-vev (left panel) and 3-vev (right panel) manifolds, using the minimisation procedure described in Section 3.2 (yellow bars) and its improved version (blue bars), compared to a full-fledged numerical minimisation.

(ii) a single minimum emerges as a non-trivial combination of quantum corrections to the potential around the two flat directions [266]. In the former case, our algorithm may wrongly characterise a local maximum instead of the true minimum. In the latter case, it may fail to identify the true minimum which could be rather distant from one of the flat directions. In any case, the rarity of such events make them harmless in the overall interpretation of the results.

Finally, Figures A.2–A.4 show that, as expected, the improved minimisation algorithm described in App. A.3 overall yields a better characterisation of the minima, both in the case of 2- and 3-vev manifolds (at the reasonable cost of a slightly increased execution time).

## B General tree-level stability conditions

The various breaking patterns studied in this work are characterised by vacuum manifolds consisting of at most 3 vevs. In this appendix, we establish the conditions of tree-level stability for general potentials of 1, 2 and 3 variables, as well as the circumstances of their violation along the RG-flow. We give in particular a characterisation of the flat directions that appear at the precise energy scale at which the violation of tree-level stability occurs.

For completeness, let us start the discussion with 1-vev vacuum manifolds (occurring for instance in the  $SO(8) \times U(1)$  breaking), for which the study of stability and its violation is trivial. Such a vacuum structure is parameterised by

$$V(x) = ax^4, \quad (6.B1)$$

so the condition for a stable (*i.e.* bounded from below) potential is simply

$$a > 0. \quad (6.B2)$$

Therefore, symmetry breaking will be uniquely triggered along the RG-flow as soon as the quartic coupling  $a$  turns negative.

### B.1 Stability of 2-vev vacuum manifolds

Now turning to vacuum manifolds consisting of 2 variables, we have in general:

$$V(x, y) = a_0x^4 + a_1x^2y^2 + a_2y^4, \quad (6.B3)$$

and it is straightforward to derive the conditions of a stable potential:

$$a_0 > 0 \wedge a_2 > 0 \wedge a_1 + 2\sqrt{a_0a_2} > 0. \quad (6.B4)$$



Following the discussion in Sec. 3.3, the stability constraints can be violated in three distinct manners, corresponding to the violation of any one of the three conditions in Eq. (6.B4). Below we examine the generated flat direction in each of these three cases. We make the assumption that only one of the three conditions in Eq. (6.B4) gets violated along the RG-flow, *i.e.* that the other two remain satisfied.

**Case 1:**  $a_0 = 0$ . The potential simplifies as

$$V(x, y) = (a_1x^2 + a_2y^2)y^2, \quad (6.B5)$$

and the other stability constraints are satisfied, namely

$$a_2 > 0 \wedge a_1 > 0. \quad (6.B6)$$

Clearly, the flat direction is parameterised by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}. \quad (6.B7)$$

**Case 2:**  $a_2 = 0$ . Similarly, the potential simplifies as

$$V(x, y) = (a_0x^2 + a_1y^2)x^2, \quad (6.B8)$$

and since  $a_0, a_1 > 0$ , the flat direction occurs in the direction

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (6.B9)$$

**Case 3:**  $a_1 + 2\sqrt{a_0a_2} = 0$ . In this case, the potential can be factored in the form

$$V(x, y) = (\sqrt{a_0}x^2 - \sqrt{a_2}y^2)^2. \quad (6.B10)$$

The other constraints are satisfied, namely

$$a_0 > 0 \wedge a_2 > 0, \quad (6.B11)$$

and two flat directions occur in the directions

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} a_2^{1/4} \\ \pm a_0^{1/4} \end{pmatrix}. \quad (6.B12)$$

## B.2 Stability of 3-vev vacuum manifolds

We finally turn to the study of 3-vev manifolds, which will need a much more elaborate discussion. However it will be helpful to note that the 3-vev structures considered in this work can be put in the form

$$V(\chi, \omega_1, \omega_2) = \alpha\chi^4 + \beta(\omega_1, \omega_2)\chi^2 + \gamma(\omega_1, \omega_2), \quad (6.B13)$$

where  $\beta$  and  $\gamma$  take the general forms

$$\beta(\omega_1, \omega_2) = b_0\omega_1^2 + b_1\omega_1\omega_2 + b_2\omega_2^2 = (b_0 + b_1X + b_2X^2)\omega_1^2 = \tilde{\beta}(X)\omega_1^2, \quad (6.B14)$$

$$\gamma(\omega_1, \omega_2) = c_0\omega_1^4 + c_1\omega_1^2\omega_2^2 + c_2\omega_2^4 = (c_0 + c_1X^2 + c_2X^4)\omega_1^4 = \tilde{\gamma}(X)\omega_1^4, \quad (6.B15)$$

with  $X = \omega_2/\omega_1$ . For the above potential to be bounded from below, it must be non-negative for all values of the vevs. First of all, positivity at  $\omega_1 = \omega_2 = 0$  and at  $\chi = 0$  imposes the constraints

$$\alpha > 0 \wedge \gamma(\omega_1, \omega_2) > 0, \quad \forall(\omega_1, \omega_2). \quad (6.B16)$$

Reusing the results established above for 2-vev functions, the latter inequality requires

$$c_0 > 0 \wedge c_2 > 0 \wedge c_1 + 2\sqrt{c_0c_2} > 0. \quad (6.B17)$$

Then, taking  $V$  as a quadratic polynomial in  $\chi^2$ , positivity requires its roots to be either complex or negative. Defining<sup>27</sup>  $\Delta = 4\alpha\gamma - \beta^2$ , the condition to have either complex roots or non-positive roots reads

$$\Delta(\omega_1, \omega_2) > 0 \vee \left( \Delta(\omega_1, \omega_2) \leq 0 \wedge \beta(\omega_1, \omega_2) > 0 \right), \quad \forall(\omega_1, \omega_2), \quad (6.B18)$$

or, more concisely

$$\Delta(\omega_1, \omega_2) > 0 \vee \beta(\omega_1, \omega_2) > 0, \quad \forall(\omega_1, \omega_2). \quad (6.B19)$$

The quantity  $\Delta(\omega_1, \omega_2)$  can be generically expressed as

$$\begin{aligned} \Delta(\omega_1, \omega_2) &= 4\alpha\gamma(\omega_1, \omega_2) - \beta(\omega_1, \omega_2)^2 \\ &= a_0\omega_1^4 + a_1\omega_1^3\omega_2 + a_2\omega_1^2\omega_2^2 + a_3\omega_1\omega_2^3 + a_4\omega_2^4 \\ &= (a_0 + a_1X + a_2X^2 + a_3X^3 + a_4X^4)\omega_1^4 \\ &= \tilde{\Delta}(X)\omega_1^4, \end{aligned} \quad (6.B20)$$

and in practice, when  $\omega_2 \neq 0$ , one only needs to consider the simplified stability constraint

$$\tilde{\Delta}(X) > 0 \vee \tilde{\beta}(X) > 0, \quad \forall X. \quad (6.B21)$$

Since  $\tilde{\beta}$  and  $\tilde{\Delta}(X)$  are polynomials of respective degree 2 and 4 in  $X$ , one could determine analytic conditions for them to be positive for all  $X$ . However, we insist that the constraint in Eq. (6.B21) is *not* equivalent to the following:

$$\left( \tilde{\Delta}(X) > 0, \quad \forall X \right) \vee \left( \tilde{\beta}(X) > 0, \quad \forall X \right), \quad (6.B22)$$

since the latter is only a sufficient condition for the former to be satisfied. Instead, one should simultaneously inspect the shape of both polynomials in terms of their number of real roots and the sign of their leading coefficient, in order to identify the regions where either one or the other is positive. Such a case-by-case study is readily performed, as reported in Table B.1. Here, since we aim at determining the conditions of a stable potential in Eq. (6.B13), we will consider that all other necessary conditions determined previously must hold. In particular, Eq. (6.B16) holds. Hence, a useful observation to make is that if  $\tilde{\beta} = 0$ , then

$$\tilde{\Delta}(X) = 4\alpha\tilde{\gamma}(X) > 0. \quad (6.B23)$$

In other words,  $\tilde{\Delta}(X)$  is always strictly positive at the locations of the roots of  $\tilde{\beta}$ . This greatly reduces the number of possibilities when inspecting the shapes of the two polynomials. Overall,  $\tilde{\Delta}$  can have either 0, 2 or 4 real roots, with a positive or negative leading coefficient  $a_4$ , while  $\tilde{\beta}$  can have either 0 or 2 real roots with a positive or negative leading coefficient  $b_2$ . At this point, a comment is worth making: we do not consider the cases of multiple roots, nor those of a vanishing leading coefficient. The reason is that, at the initial scale where potential stability must be asserted, the couplings (and therefore the value of the coefficients appearing in the polynomials) are generated randomly. Hence, exact relations such that a vanishing discriminant or coefficient will never occur. On the other hand, such quantities can very well vanish at a given scale along the RG-flow and possibly trigger spontaneous breaking of the model. Such situations are described in subsection B.3 below.

Finally, although this procedure can readily be performed numerically, we review here the analytical conditions allowing to determine the number of real roots of the quartic polynomial  $\tilde{\Delta}$  [267, 268, 269]. Those conditions will also help understand how the stability condition in Eq. (6.B21) can be violated along the RG-flow. The main quantity of interest here is the discriminant  $D$  of the polynomial  $\tilde{\Delta}$ :

$$\begin{aligned} D &= 256a_0^3a_4^3 - 4a_1^3a_3^3 - 27a_0^2a_3^4 + 16a_0a_2^4a_4 - 6a_0a_1^2a_3^2a_4 - 27a_1^4a_4^2 \\ &\quad - 192a_0^2a_1a_3a_4^2 - 4a_2^3(a_0a_3^2 + a_1^2a_4) + 18a_2(a_1a_3 + 8a_0a_4)(a_0a_3^2 + a_1^2a_4) \\ &\quad + a_2^2(a_1^2a_3^2 - 80a_0a_1a_3a_4 - 128a_0^2a_4^2). \end{aligned} \quad (6.B24)$$

<sup>27</sup>Note the negative sign compared to the usual definition of the quadratic discriminant.

Table B.1: Realisation of the stability constraint in Eq. (6.B21), depending on the number of roots of the polynomials  $\tilde{\Delta}$  and  $\tilde{\beta}$  and the sign of their leading coefficient. We write  $\Delta(n)^s$  and  $\beta(n)^s$  to respectively denote the number  $n$  of roots and the sign  $s$  of the leading coefficient of the polynomials  $\tilde{\Delta}$  and  $\tilde{\beta}$ . The roots of  $\tilde{\Delta}$  are noted  $\delta_i$  with  $\delta_1 < \dots < \delta_n$ , those of  $\tilde{\beta}$  are noted  $\beta_i$  with  $\beta_1 < \beta_2$ . Cases where the stability condition in Eq. (6.B21) is satisfied for any values of the roots are referred to as Stable, and cases where the condition cannot be satisfied are referred to as Unstable. For cases where the realisation of Eq. (6.B21) depends on the value of the roots, the additional constraints to be satisfied by them are reported. Finally, two cases never occur because of the constraint of a positive  $\tilde{\Delta}$  at the location of the roots of  $\tilde{\beta}$ .

	$\beta(0)^+$	$\beta(0)^-$	$\beta(2)^+$	$\beta(2)^-$
$\Delta(0)^+$	Stable	Stable	Stable	Stable
$\Delta(0)^-$	Stable	Unstable	/	/
$\Delta(2)^+$	Stable	Unstable	$\beta_2 < \delta_1 \vee \beta_1 > \delta_2$	$\beta_2 < \delta_1 \wedge \beta_1 > \delta_2$
$\Delta(2)^-$	Stable	Unstable	$\beta_1 > \delta_1 \wedge \beta_2 < \delta_2$	Unstable
$\Delta(4)^+$	Stable	Unstable	$\beta_2 < \delta_1 \vee \beta_1 > \delta_4 \vee (\beta_1 > \delta_2 \wedge \beta_2 < \delta_3)$	$\beta_1 < \delta_1 \vee \beta_2 > \delta_4$
$\Delta(4)^-$	Stable	Unstable	$(\beta_1 > \delta_1 \wedge \beta_2 < \delta_2) \vee (\beta_1 > \delta_3 \wedge \beta_2 < \delta_4)$	Unstable

We will not show here the expression of  $D$  as a function of  $\alpha, b_i, c_i$  here since it is rather lengthy. However, we make the important remark that

$$D \propto \alpha^2, \quad (6.B25)$$

which will help understand the symmetry breaking patterns in the next subsection. The nature of the roots depend on the sign of  $D$ :

$$D > 0 : \text{The four roots are either all complex or all real} \quad (6.B26)$$

$$D = 0 : \text{There exists multiple roots} \quad (6.B27)$$

$$D < 0 : \text{Two roots are complex, the other two are real} \quad (6.B28)$$

In the first case,  $D > 0$ , the nature of the roots can be determined by defining the following additional quantities [269]

$$Q = 8a_2a_4 - 3a_3^2, \quad R = 64a_0a_4^3 + 16a_2a_3^2a_4 - 16a_4^2(a_2^2 + a_1a_3) - 3a_3^4, \quad (6.B29)$$

such that the four roots are complex if either  $Q > 0$  or  $R > 0$ . In summary, we have:

$$D > 0 \wedge (Q > 0 \vee R > 0) : \text{No real roots} \quad (6.B30)$$

$$D < 0 : \text{Two real roots} \quad (6.B31)$$

$$D > 0 \wedge (Q \leq 0 \wedge R \leq 0) : \text{Four real roots} \quad (6.B32)$$

### B.3 Stability violation for 3-vev manifolds

As previously done in the case of 2-vev manifolds, we now inspect the different ways in which the stability conditions of 3-vev manifolds can be violated. Obviously, more cases will have to be considered here, due to the richer structure of the potential and its stability conditions.

**Case 1:**  $\alpha = 0$ . The potential simplifies as

$$V(\chi, \omega_1, \omega_2) = \beta(\omega_1, \omega_2)\chi^2 + \gamma(\omega_1, \omega_2) \quad (6.B33)$$

We consider that all other stability condition are satisfied. In particular,  $\gamma$  is always positive, and since  $\alpha = 0$ , one has  $\Delta(\omega_1, \omega_2) = -\beta(\omega_1, \omega_2)^2$ . Therefore, according to Eq. (6.B19),  $\beta$  is always positive, and  $V$  can only vanish in the region where  $\omega_1 = \omega_2 = 0$ . In this case, the flat direction lies along

$$\begin{pmatrix} \chi \\ \omega_1 \\ \omega_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (6.B34)$$

This corresponds to a symmetry breaking exclusively driven by the vev  $\chi$ . Specialising this result to the SM vacuum manifold in Eq. (6.39) with  $\chi = \chi_5$  (or equivalently  $\chi = \chi_R$ ) yields a breaking towards the  $SU(5)$  subgroup.

**Case 2:**  $c_0 = 0$  or  $c_2 = 0$ . Let us first consider the case where  $c_0 = 0$ . In this case, the quantity  $\gamma(\omega_1, \omega_2)$  vanishes along the flat direction

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (6.B35)$$

and according to Eq. (6.B19),  $\beta(\omega_1, \omega_2) > 0$ . The potential simplifies as

$$V(\chi, \omega_1, \omega_2) = (\alpha\chi^2 + \beta(\omega_1, \omega_2)) \chi^2, \quad (6.B36)$$

and can only vanish if  $\chi = 0$ . Hence, the flat direction is

$$\begin{pmatrix} \chi \\ \omega_1 \\ \omega_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (6.B37)$$

This is a breaking triggered by the vev  $\omega_1$  exclusively. Similarly, in the case where  $c_2 = 0$ , the flat direction is given by

$$\begin{pmatrix} \chi \\ \omega_1 \\ \omega_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (6.B38)$$

and corresponds to a breaking driven by  $\omega_2$ . Considering the SM vacuum manifold in Eq. (6.39) with  $\omega_1 = \omega_B$ ,  $\omega_2 = \omega_R$  and either  $\chi = \chi_5$  or  $\chi = \chi_5$ , the above cases respectively correspond to the  $3_C 2_L 2_R 1_{B-L}$  and  $4_C 2_L 1_R$  breakings.

**Case 2:**  $c_1 + 2\sqrt{c_0 c_2} = 0$ . In this case, the quantity  $\gamma$  has a flat direction along

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \lambda \begin{pmatrix} c_2^{1/4} \\ \pm c_0^{1/4} \end{pmatrix}. \quad (6.B39)$$

Here again, the potential takes the form in Eq. (6.B36). Therefore the flat directions are given by

$$\begin{pmatrix} \chi \\ \omega_1 \\ \omega_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ c_2^{1/4} \\ \pm c_0^{1/4} \end{pmatrix} \quad (6.B40)$$

and the breaking is driven by the vevs  $\omega_1$  and  $\omega_2$ . In the SM vacuum, this corresponds to the  $3_C 2_L 1_R 1_{B-L}$  breaking<sup>28</sup>.

**Case 3:**  $a_0 = 0$  or  $a_4 = 0$ . Here we consider the possibility that Eq. (6.B19) gets violated, in the particular situation where the leading coefficient of either  $\omega_1^4$  or  $\omega_2^4$  vanishes along the RG-flow. Starting with the case where  $a_0 = 0$ , we have

$$\Delta(\omega_1, \omega_2) = (a_1\omega_1^3 + a_2\omega_1^2\omega_2 + a_3\omega_1\omega_2^2 + a_4\omega_2^3) \omega_2. \quad (6.B41)$$

Hence,  $\Delta$  clearly vanishes when  $\omega_2 = 0$ . We note that other roots with  $\omega_1, \omega_2 \neq 0$  may exist, but this situation is taken into account in the more general **Case 4** below. Here we restrict the discussion to the case where  $\omega_2 = 0$ . In this case, we have

$$a_0 = 4\alpha c_0 - b_0^2 = 0, \quad (6.B42)$$

$$\beta(\omega_1, 0) = b_0\omega_1^2, \quad (6.B43)$$

<sup>28</sup>However, for reasons explained in Sec. 4.2, in this case the actual breaking direction is  $SU(5) \times U(1)$  since in practice one always has  $\eta = \sqrt{2/3}$ .

so the full potential simplifies as

$$V(\chi, \omega_1, 0) = (\sqrt{\alpha}\chi^2 \pm \sqrt{c_0}\omega_1^2)^2. \quad (6.B44)$$

Since  $\alpha > 0$  and  $c_0 > 0$ , the only case yielding a flat direction corresponds to  $b_0 < 0$ , and hence

$$V(\chi, \omega_1, 0) = (\sqrt{\alpha}\chi^2 - \sqrt{c_0}\omega_1^2)^2. \quad (6.B45)$$

In this case, the flat directions are given by

$$\begin{pmatrix} \chi \\ \omega_1 \\ \omega_2 \end{pmatrix} = \lambda \begin{pmatrix} c_0^{1/4} \\ \pm \alpha^{1/4} \\ 0 \end{pmatrix}. \quad (6.B46)$$

Reiterating the above calculations in the case where  $a_4 = 0$  yields the following flat directions

$$\begin{pmatrix} \chi \\ \omega_1 \\ \omega_2 \end{pmatrix} = \lambda \begin{pmatrix} c_0^{1/4} \\ 0 \\ \pm \alpha^{1/4} \end{pmatrix}. \quad (6.B47)$$

When considering the SM vacuum manifold, such flat directions correspond to a complete breaking of  $SO(10)$  towards the SM, despite the fact that one of the  $\omega_i$  vanishes.

**Case 4.** Whereas all previous cases involved only one or two of the vevs, we now turn to the possibility of violating the stability conditions in a non-trivial way, where none of the vevs vanishes. Concretely, it means that the condition in Eq. (6.B19) or, equivalently, Eq. (6.B21) needs to be violated along the RG-flow, in a case where  $\chi, \omega_1, \omega_2 \neq 0$ . A closer look at Eq. (6.B21) shows that the transition from a stable to an unstable potential can only occur at a given value of  $X$  in the two following pictures

$$\tilde{\Delta}(X) > 0 \wedge \tilde{\beta}(X) < 0 \longrightarrow \tilde{\Delta}(X) = 0 \wedge \tilde{\beta}(X) < 0 \longrightarrow \tilde{\Delta}(X) < 0 \wedge \tilde{\beta}(X) < 0, \quad (6.B48)$$

$$\tilde{\Delta}(X) < 0 \wedge \tilde{\beta}(X) > 0 \longrightarrow \tilde{\Delta}(X) < 0 \wedge \tilde{\beta}(X) = 0 \longrightarrow \tilde{\Delta}(X) < 0 \wedge \tilde{\beta}(X) < 0. \quad (6.B49)$$

However, as mentioned before,  $\tilde{\Delta}(X)$  can only be positive when evaluated a root of  $\tilde{\beta}$ . This observation allows to rule out the scenario in Eq. (6.B49), making Eq. (6.B48) the only way of generating a flat direction. Furthermore, the change of sign of  $\tilde{\Delta}(X)$  due to a sign flip of its leading coefficient was already covered in **Case 3** above, so we can discard this possibility. The only remaining way to achieve the transition in Eq. (6.B48) is for  $\tilde{\Delta}$  to acquire a multiple root at some value of  $X$ . This happens when the discriminant  $D$  of  $\tilde{\Delta}$  vanishes at some RG-scale. The multiple real root that appears will be denoted  $\delta$ , and we have

$$X = \delta \Rightarrow \omega_2 = \delta\omega_1. \quad (6.B50)$$

Since  $\tilde{\Delta}(\delta) = \Delta(\omega_1, \delta\omega_1) = 4\alpha\gamma(\omega_1, \delta\omega_1) - \beta(\omega_1, \delta\omega_1)^2 = 0$  and  $\beta(\omega_1, \delta\omega_1) < 0$ , the potential takes the form

$$V(\chi, \omega_1, \delta\omega_1) = \left( \sqrt{\alpha}\chi^2 - \sqrt{\gamma(\omega_1, \delta\omega_1)} \right)^2, \quad (6.B51)$$

where

$$\sqrt{\gamma(\omega_1, \delta\omega_1)} = \sqrt{c_0 + c_1\delta^2 + c_2\delta^4}\omega_1^2. \quad (6.B52)$$

This means in turn that the potential vanishes if

$$\chi = \pm \left( \frac{c_0 + c_1\delta^2 + c_2\delta^4}{\alpha} \right)^{1/4} \omega_1, \quad (6.B53)$$

so one concludes that the flat directions are given by

$$\begin{pmatrix} \chi \\ \omega_1 \\ \omega_2 \end{pmatrix} = \lambda \begin{pmatrix} \pm \left( \frac{c_0 + c_1\delta^2 + c_2\delta^4}{\alpha} \right)^{1/4} \\ 1 \\ \delta \end{pmatrix}. \quad (6.B54)$$

This completes our discussion on the stability of 1-, 2- and 3-vev manifolds and on the classification of the possible flat directions generated by the RG evolution of the quartic couplings. The symmetry breaking patterns occurring in each case identified above are summarised in Table 6.1.

## C A quantitative measure of perturbativity

In this appendix, we develop a method allowing to obtain a quantitative measure of perturbativity based on the comparison of the size of the one- and two-loop contributions to the beta-functions. In [144], one of us has proposed a simple perturbativity criterion translating, using definition in Eq. (6.E1), into

$$\left| \beta^{(2)}(g_i) \right| < \frac{1}{2} \left| \beta^{(1)}(g_i) \right|, \quad \forall g_i \quad (6.C1)$$

where  $g_i$ ,  $i = 1, \dots, N$  generically denotes all the couplings of the theory. Note that the inclusion of the factor  $\frac{1}{2}$  is rather arbitrary, since the boundary (in the space of the couplings of the model) between the perturbative and non-perturbative regimes is anyways equivocal. This being said, the above criterion allows in particular to systematically detect the occurrence of Landau poles in the RG-flow, indicating a breakdown of perturbation theory.

While this criterion was successfully applied in [144] as a way to phenomenologically constrain (extensions of) the SM, it comes with a caveat: A change of sign in the one-loop beta-function of any of the couplings systematically violates Eq. (6.C1), even in a region of the coupling space where the regime is clearly perturbative. To circumvent this issue, we generalise the above criterion involving simultaneously all the couplings of the theory. For  $p$  a positive integer and  $\alpha > 0$ , this generalised perturbativity criterion reads

$$\left( \sum_{i=1}^N \left| \beta^{(2)}(g_i) \right|^p \right)^{1/p} < \alpha \left( \sum_{i=1}^N \left| \beta^{(1)}(g_i) \right|^p \right)^{1/p}, \quad (6.C2)$$

or, in a more compact form,

$$\left\| \boldsymbol{\beta}^{(2)}(\mathbf{g}) \right\|_p < \alpha \left\| \boldsymbol{\beta}^{(1)}(\mathbf{g}) \right\|_p, \quad (6.C3)$$

where  $\|\cdot\|_p$  denotes the usual  $\ell_p$ -norm and where

$$\boldsymbol{\beta}^{(n)}(\mathbf{g}) = \begin{pmatrix} \beta^{(n)}(g_1) \\ \vdots \\ \beta^{(n)}(g_N) \end{pmatrix}. \quad (6.C4)$$

This generalised criterion will no longer fail if *some* of the one-loop beta-functions vanish. It will, however, fail if *all* one-loop beta-functions vanish, *i.e.*, if the one-loop system approaches a fixed point.

The free parameters  $p$  and  $\alpha$  conveniently allow to adapt the (non-)conservative property of the criterion. As a particular case of Eq. (6.C3), note that taking  $p \rightarrow \infty$  yields

$$\max_i \left| \beta^{(2)}(g_i) \right| < \alpha \max_i \left| \beta^{(1)}(g_i) \right| \quad (6.C5)$$

whereas  $p = 1$  gives

$$\sum_{i=1}^N \left| \beta^{(2)}(g_i) \right| < \alpha \sum_{i=1}^N \left| \beta^{(1)}(g_i) \right|. \quad (6.C6)$$

In a theory with a single coupling  $g$ , taking in addition  $\alpha = \frac{1}{2}$  in the above expression allows to recover the formula

$$\left| \beta^{(2)}(g) \right| < \frac{1}{2} \left| \beta^{(1)}(g) \right|, \quad (6.C7)$$

which coincides with the original criterion in Eq. (6.C1). As a final remark, we have observed that, in practice, the impact of a change in the value of  $p$  can be roughly compensated by a change in the value of  $\alpha$ . For all results in this chapter, cf. Sec. 5, we fix  $p = 1$ , therefore using Eq. (6.C6) as a quantitative measure of the perturbativity of the studied models. Further, we specify to  $\alpha = 0.1$ . This may be overly conservative. However, such a conservative choice avoids convergence issues in the subsequent numerical analysis of the one-loop effective potential, cf. Sec. 3 and App. A.

## D Scalar potential for the considered models

In this appendix, we provide the expression of the most general perturbatively renormalisable scalar potential for the  $\mathbf{10}_H \oplus \mathbf{16}_H \oplus \mathbf{45}_H$   $SO(10)$  model. This will allow us in turn to specialise this expression to the two simplified models considered in this work, where the scalar sector is reduced to  $\mathbf{16}_H \oplus \mathbf{45}_H$  and  $\mathbf{45}_H$  respectively.

### D.1 Definitions and conventions

The fundamental  $\mathbf{10}_H$  multiplet is noted  $H_i$  in the following. We use a standard convention for the 45 gauge generators of the fundamental representation,

$$(T_{AB})^{ij} = \frac{i}{\sqrt{2}} \left( \delta_A^i \delta_B^j - \delta_A^j \delta_B^i \right), \quad A, B = 1, \dots, 10, \quad (6.D1)$$

where the factor  $\sqrt{2}$  in the denominator fixes the value of the Dynkin index to  $T_{\mathbf{10}} = 1$ . Next, based on the decomposition

$$\mathbf{10} \otimes \mathbf{10} = \mathbf{54}_S \oplus \mathbf{45}_A \oplus \mathbf{1}, \quad (6.D2)$$

the adjoint  $\mathbf{45}_H$  (as a special case of  $\mathbf{45}_A$ ) field is conveniently expressed as an antisymmetric  $10 \times 10$  matrix, noted  $\phi_{ij}$ . Finally, reusing the notations from [161], the reducible 32-dimensional spinor field is noted  $\Xi$  and can be decomposed under  $\mathbf{32} = \mathbf{16}_R \oplus \mathbf{16}_L$  as

$$\Xi = \begin{pmatrix} \chi \\ \chi^c \end{pmatrix}. \quad (6.D3)$$

The generators of the reducible 32-dimensional representation are given by

$$S_{ij} = \frac{1}{4\sqrt{2}i} [\Gamma_i, \Gamma_j] \equiv \frac{1}{2} \begin{pmatrix} \sigma_{ij} & 0 \\ 0 & \tilde{\sigma}_{ij} \end{pmatrix} \quad (6.D4)$$

with  $i, j = 1, \dots, 10$ . The  $\Gamma_i$ 's are  $32 \times 32$  matrices satisfying the anticommutation relations

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \mathbb{1}_{32}, \quad (6.D5)$$

characteristic of a Clifford algebra. An explicit form for  $\Gamma_i$  will not be provided here but can be found in [172, 161]. Note that, as compared to [161], an additional factor of  $\sqrt{2}$  was included in the denominator of Eq. (6.D4) (and in the definition of  $\sigma_{ij}$ ) in order to match the convention where the Dynkin index of  $\mathbf{16}$  equals 2 (instead of 4). Right- and left-handed projectors  $P_+$  and  $P_-$  can be constructed such that

$$P_+ \Xi = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \equiv \chi_+, \quad P_- \Xi = \begin{pmatrix} 0 \\ \chi^c \end{pmatrix} \equiv \chi_-. \quad (6.D6)$$

We note in passing that the spinor field  $\chi^c$  is obtained from a conjugation operation

$$\chi^c = C\chi, \quad C \in \mathbf{16}_{SO(10)}, \quad (6.D7)$$

characteristic of the discrete left-right symmetry  $D \subset SO(10)$ , usually referred to as D-parity. Finally, it will be useful to construct the auxiliary adjoint fields

$$\Phi_{16} = \frac{1}{4} \sigma_{ij} \phi^{ij} \quad \text{and} \quad \Phi_{32} = \frac{1}{2} S_{ij} \phi^{ij}. \quad (6.D8)$$

in order to construct the various gauge invariant operators in a notation adapted to the presence of a scalar spinorial representation.

## D.2 Scalar potential for the $10_H \oplus 16_H \oplus 45_H$ model

With the above definitions at hand, we may now write down the most general renormalisable scalar potential built from the scalar representations  $10_H$ ,  $16_H$  and  $45_H$ :

$$\begin{aligned}
V(H, \chi, \phi) = & \mu_1 H_i H^i + \mu_2 \chi^\dagger \chi + \mu_3 \text{Tr}(\Phi_{16}^2) \\
& + \tau_1 (\chi_-^\dagger \Gamma_i \chi_+) H^i + \tau_1^* (\chi_+^\dagger \Gamma_i \chi_-) H^i + \tau_2 \chi^\dagger \Phi_{16} \chi \\
& + \Lambda_1 \text{Tr}(\Phi_{16}^2)^2 + \Lambda_2 \text{Tr}(\Phi_{16}^4) \\
& + \Lambda_3 (H_i H^i)^2 + \Lambda_4 (H_i H^i) \text{Tr}(\Phi_{16}^2) + \Lambda_5 H_i H_j \text{Tr}(\Gamma^i \Phi_{32} \Gamma^j \Phi_{32}) \\
& + \Lambda_6 (\chi^\dagger \chi)^2 + \Lambda_7 (\chi_+^\dagger \Gamma_i \chi_-) (\chi_-^\dagger \Gamma^i \chi_+) + \Lambda_8 (\chi^\dagger \chi) \text{Tr}(\Phi_{16}^2) + \Lambda_9 \chi^\dagger \Phi_{16}^2 \chi \\
& + \Lambda_{10} (H_i H^i) (\chi^\dagger \chi).
\end{aligned} \tag{6.D9}$$

It is worth noticing that in the limit  $\tau_1 \rightarrow 0$ , the above scalar potential is invariant under a global  $U(1)$  transformation under which only  $16_H$  is charged<sup>29</sup>, *i.e.* under

$$\chi \rightarrow e^{i\alpha} \chi. \tag{6.D10}$$

Finally, the  $10_H$  multiplet couples to fermions  $\psi \sim 16_F$  through the Yukawa term

$$-\mathcal{L}_Y = Y_{10} (\psi_-^\dagger \Gamma_i \psi_+) H^i + \text{h.c.}, \tag{6.D11}$$

where  $\psi_\pm$  was defined similarly to  $\chi_\pm$  (see Eq. (6.D6)).

## D.3 Scalar potential for the $16_H \oplus 45_H$ and $45_H$ models

We now specialise expression in Eq. (6.D9) to the case of the simplified model considered in this work, where the scalar sector only consists of  $16_H \oplus 45_H$ . Discarding in addition the relevant operators in order to achieve scale invariance at the classical level, we write

$$\begin{aligned}
V(\chi, \phi) = & \frac{\lambda_1}{4} \text{Tr}(\Phi_{16}^2)^2 + \lambda_2 \text{Tr}(\Phi_{16}^4) + 4\lambda_6 (\chi^\dagger \chi)^2 + \lambda_7 (\chi_+^\dagger \Gamma_i \chi_-) (\chi_-^\dagger \Gamma^i \chi_+) \\
& + 2\lambda_8 (\chi^\dagger \chi) \text{Tr}(\Phi_{16}^2) + 8\lambda_9 \chi^\dagger \Phi_{16}^2 \chi.
\end{aligned} \tag{6.D12}$$

We note that the normalisation of the various operators is arbitrary and that the six quartic couplings  $\lambda_i$  were defined such that perturbativity is lost around  $\lambda_i \gtrsim 1$ . Our notation and conventions translate to those of [161, 173, 160] according to:

$$\lambda_1 \leftrightarrow 4a_1, \quad \lambda_2 \leftrightarrow a_2, \quad \lambda_6 \leftrightarrow \frac{\lambda_1}{16}, \quad \lambda_7 \leftrightarrow \frac{\lambda_2}{4}, \quad \lambda_8 \leftrightarrow \alpha, \quad \lambda_9 \leftrightarrow \frac{\beta}{4}. \tag{6.D13}$$

Following the comment made above, the absence of relevant operators in Eq. (6.D12) implies invariance under the  $U(1)$  global symmetry in Eq. (6.D10). Finally, the scalar potential for the  $45_H$  model simply reads

$$V(\phi) = \frac{\lambda_1}{4} \text{Tr}(\Phi_{16}^2)^2 + \lambda_2 \text{Tr}(\Phi_{16}^4). \tag{6.D14}$$

## E Beta-functions

The beta-functions for the couplings of the three models presented in the previous section were computed up to the two-loop level using the tool PyR@TE 3<sup>30</sup> [9]. We report here the obtained expressions at one-loop, first in the case of the  $10_H \oplus 16_H \oplus 45_H$  model<sup>31</sup>, then in the case of the simplified  $16_H \oplus 45_H$

<sup>29</sup>Note that this global symmetry could be restored for  $\tau_1 \neq 0$  by complexifying and assigning a  $U(1)$  charge to the  $10_H$  multiplet. Invariance of the Yukawa term in Eq. (6.D11) would in turn require to give a charge to the fermionic  $16_F$  multiplet.

<sup>30</sup>Note that the one-loop beta-functions can also be obtained from the one-loop scalar potential [188].

<sup>31</sup>Although this model was not studied in the present work, we provide the corresponding set of beta-functions since these might be useful to the reader.



model. The two-loop contributions to the beta-functions being rather lengthy, we will not report them here and we invite the interested reader to refer to the ancillary file containing the full expressions in a computer-readable form. In the following, we use the convention

$$\beta(X) \equiv \mu \frac{dX}{d\mu} \equiv \frac{1}{(4\pi)^2} \beta^{(1)}(X) + \frac{1}{(4\pi)^4} \beta^{(2)}(X). \quad (6.E1)$$

### E.1 $10_H \oplus 16_H \oplus 45_H$ model

We provide below the one-loop beta-functions for the full  $10_H \oplus 16_H \oplus 45_H$  model.

**Gauge coupling.**

$$\beta^{(1)}(g) = -\frac{139}{6} g^3 \quad (6.E2)$$

**Yukawa coupling.**

$$\beta^{(1)}(Y_{10}) = -24Y_{10}Y_{10}^*Y_{10} + 64 \text{Tr}(Y_{10}Y_{10}^*)Y_{10} - \frac{135}{4}g^2Y_{10} \quad (6.E3)$$

**Quartic couplings.**

$$\begin{aligned} \beta^{(1)}(\Lambda_1) &= 1696\Lambda_1^2 + 412\Lambda_1\Lambda_2 + \frac{279}{8}\Lambda_2^2 + 20\Lambda_4^2 + 48\Lambda_4\Lambda_5 + 112\Lambda_5^2 + 64\Lambda_8^2 \\ &\quad + 4\Lambda_8\Lambda_9 - 96\Lambda_1g^2 + \frac{27}{16}g^4 \end{aligned} \quad (6.E4)$$

$$\beta^{(1)}(\Lambda_2) = 384\Lambda_1\Lambda_2 - 4\Lambda_2^2 - 512\Lambda_5^2 + \Lambda_9^2 - 96\Lambda_2g^2 - 3g^4 \quad (6.E5)$$

$$\begin{aligned} \beta^{(1)}(\Lambda_3) &= 144\Lambda_3^2 + 360\Lambda_4^2 + 864\Lambda_4\Lambda_5 + 1440\Lambda_5^2 + 16\Lambda_{10}^2 - 54\Lambda_3g^2 + \frac{27}{8}g^4 \\ &\quad + 256\Lambda_3 \text{Tr}(Y_{10}Y_{10}^*) - 256 \text{Tr}(Y_{10}Y_{10}^*Y_{10}Y_{10}^*) \end{aligned} \quad (6.E6)$$

$$\begin{aligned} \beta^{(1)}(\Lambda_4) &= 1504\Lambda_1\Lambda_4 + 1728\Lambda_1\Lambda_5 + 206\Lambda_2\Lambda_4 + 276\Lambda_2\Lambda_5 + 96\Lambda_3\Lambda_4 + 96\Lambda_3\Lambda_5 \\ &\quad + 32\Lambda_4^2 + 768\Lambda_5^2 + 64\Lambda_{10}\Lambda_8 + 2\Lambda_{10}\Lambda_9 - 75\Lambda_4g^2 + \frac{15}{8}g^4 \\ &\quad + 128\Lambda_4 \text{Tr}(Y_{10}Y_{10}^*) \end{aligned} \quad (6.E7)$$

$$\begin{aligned} \beta^{(1)}(\Lambda_5) &= 64\Lambda_1\Lambda_5 - 24\Lambda_2\Lambda_5 + 16\Lambda_3\Lambda_5 + 64\Lambda_4\Lambda_5 - 192\Lambda_5^2 - 75\Lambda_5g^2 - \frac{9}{16}g^4 \\ &\quad + 128\Lambda_5 \text{Tr}(Y_{10}Y_{10}^*) \end{aligned} \quad (6.E8)$$

$$\begin{aligned} \beta^{(1)}(\Lambda_6) &= 80\Lambda_6^2 + 160\Lambda_6\Lambda_7 + 320\Lambda_7^2 + 1440\Lambda_8^2 + 90\Lambda_8\Lambda_9 + \frac{105}{32}\Lambda_9^2 + 20\Lambda_{10}^2 \\ &\quad - \frac{135}{2}\Lambda_6g^2 + \frac{315}{32}g^4 \end{aligned} \quad (6.E9)$$

$$\beta^{(1)}(\Lambda_7) = 24\Lambda_6\Lambda_7 + \frac{3}{8}\Lambda_9^2 - \frac{135}{2}\Lambda_7g^2 + \frac{9}{8}g^4 \quad (6.E10)$$

$$\begin{aligned} \beta^{(1)}(\Lambda_8) &= 1504\Lambda_1\Lambda_8 + 45\Lambda_1\Lambda_9 + 206\Lambda_2\Lambda_8 + \frac{93}{16}\Lambda_2\Lambda_9 + 68\Lambda_6\Lambda_8 + 2\Lambda_6\Lambda_9 \\ &\quad + 80\Lambda_7\Lambda_8 + 2\Lambda_7\Lambda_9 + 32\Lambda_8^2 + \frac{3}{8}\Lambda_9^2 + 20\Lambda_{10}\Lambda_4 + 24\Lambda_{10}\Lambda_5 \\ &\quad - \frac{327}{4}\Lambda_8g^2 + \frac{9}{8}g^4 \end{aligned} \quad (6.E11)$$

$$\begin{aligned} \beta^{(1)}(\Lambda_9) &= 64\Lambda_1\Lambda_9 + 20\Lambda_2\Lambda_9 + 4\Lambda_6\Lambda_9 + 16\Lambda_7\Lambda_9 + 64\Lambda_8\Lambda_9 + 17\Lambda_9^2 \\ &\quad - \frac{327}{4}\Lambda_9g^2 + 12g^4 \end{aligned} \quad (6.E12)$$

$$\begin{aligned} \beta^{(1)}(\Lambda_{10}) &= 1440\Lambda_4\Lambda_8 + 45\Lambda_4\Lambda_9 + 1728\Lambda_5\Lambda_8 + 54\Lambda_5\Lambda_9 + 96\Lambda_{10}\Lambda_3 + 68\Lambda_{10}\Lambda_6 \\ &\quad + 80\Lambda_{10}\Lambda_7 + 8\Lambda_{10}^2 - \frac{243}{4}\Lambda_{10}g^2 + \frac{27}{8}g^4 + 128\Lambda_{10} \text{Tr}(Y_{10}Y_{10}^*) \end{aligned} \quad (6.E13)$$

**Scalar mass and cubic couplings.**

$$\begin{aligned} \beta^{(1)}(\mu_1) &= 96\Lambda_3\mu_1 + 32\Lambda_{10}\mu_2 + 720\Lambda_4\mu_3 + 864\Lambda_5\mu_3 - 27g^2\mu_1 \\ &\quad + 128\mu_1 \operatorname{Tr}(Y_{10}Y_{10}^*) + 64|\tau_1|^2 \end{aligned} \quad (6.E14)$$

$$\begin{aligned} \beta^{(1)}(\mu_2) &= \frac{45}{4}\tau_2^2 + 40\Lambda_{10}\mu_1 + 68\Lambda_6\mu_2 + 80\Lambda_7\mu_2 + 1440\Lambda_8\mu_3 + 45\Lambda_9\mu_3 \\ &\quad - \frac{135}{4}g^2\mu_2 + 80|\tau_1|^2 \end{aligned} \quad (6.E15)$$

$$\begin{aligned} \beta^{(1)}(\mu_3) &= 40\Lambda_4\mu_1 + 48\Lambda_5\mu_1 + 64\Lambda_8\mu_2 + 2\Lambda_9\mu_2 + 1504\Lambda_1\mu_3 + 206\Lambda_2\mu_3 \\ &\quad - 48g^2\mu_3 + \tau_2^2 \end{aligned} \quad (6.E16)$$

$$\beta^{(1)}(\tau_1) = 4\Lambda_6\tau_1 + 64\Lambda_7\tau_1 + 8\Lambda_{10}\tau_1 - \frac{189}{4}g^2\tau_1 + 64\tau_1 \operatorname{Tr}(Y_{10}Y_{10}^*) \quad (6.E17)$$

$$\beta^{(1)}(\tau_2) = 4\Lambda_6\tau_2 - 48\Lambda_7\tau_2 + 32\Lambda_8\tau_2 + 29\Lambda_9\tau_2 - \frac{231}{4}g^2\tau_2 \quad (6.E18)$$

**E.2  $16_H \oplus 45_H$  and  $45_H$  models**

We provide below the beta-functions for the simplified  $16_H \oplus 45_H$  model up to the one-loop level. The two-loop contributions being used in the present analysis to establish a quantitative measure of perturbativity (see Appendix C) can be found in the ancillary file.

**Gauge coupling.**

$$\beta^{(1)}(g) = -\frac{70}{3}g^3 \quad (6.E19)$$

**Quartic couplings.**

$$\beta^{(1)}(\lambda_1) = 424\lambda_1^2 + 412\lambda_1\lambda_2 + \frac{279}{2}\lambda_2^2 + 256\lambda_8^2 + 128\lambda_8\lambda_9 - 96g^2\lambda_1 + \frac{27}{4}g^4 \quad (6.E20)$$

$$\beta^{(1)}(\lambda_2) = 96\lambda_1\lambda_2 - 4\lambda_2^2 + 64\lambda_9^2 - 96g^2\lambda_2 - 3g^4 \quad (6.E21)$$

$$\begin{aligned} \beta^{(1)}(\lambda_6) &= 320\lambda_6^2 + 160\lambda_6\lambda_7 + 80\lambda_7^2 + 360\lambda_8^2 + 180\lambda_8\lambda_9 + \frac{105}{2}\lambda_9^2 \\ &\quad - \frac{135}{2}g^2\lambda_6 + \frac{315}{128}g^4 \end{aligned} \quad (6.E22)$$

$$\beta^{(1)}(\lambda_7) = 96\lambda_6\lambda_7 + 24\lambda_9^2 - \frac{135}{2}g^2\lambda_7 + \frac{9}{8}g^4 \quad (6.E23)$$

$$\begin{aligned} \beta^{(1)}(\lambda_8) &= 376\lambda_1\lambda_8 + 90\lambda_1\lambda_9 + 206\lambda_2\lambda_8 + \frac{93}{2}\lambda_2\lambda_9 + 272\lambda_6\lambda_8 + 64\lambda_6\lambda_9 + 80\lambda_7\lambda_8 \\ &\quad + 16\lambda_7\lambda_9 + 32\lambda_8^2 + 24\lambda_9^2 - \frac{327}{4}g^2\lambda_8 + \frac{9}{8}g^4 \end{aligned} \quad (6.E24)$$

$$\begin{aligned} \beta^{(1)}(\lambda_9) &= 16\lambda_1\lambda_9 + 20\lambda_2\lambda_9 + 16\lambda_6\lambda_9 + 16\lambda_7\lambda_9 + 64\lambda_8\lambda_9 + 136\lambda_9^2 \\ &\quad - \frac{327}{4}g^2\lambda_9 + \frac{3}{2}g^4 \end{aligned} \quad (6.E25)$$

The beta-functions of  $\lambda_{1,2}$  in the case of the  $45_H$ -only model are simply found by taking the limit  $\lambda_{6,7,8,9} \rightarrow 0$  in the above expressions. Note however that the gauge coupling beta-function reduces to

$$\beta^{(1)}(g) = -24g^3. \quad (6.E26)$$



# Bibliography

- [1] UA1 collaboration, *Experimental Observation of Isolated Large Transverse Energy Electrons with Associated Missing Energy at  $\sqrt{s} = 540$  GeV*, *Phys. Lett. B* **122** (1983) 103.
- [2] UA1 collaboration, *Experimental Observation of Lepton Pairs of Invariant Mass Around 95-GeV/c\*\*2 at the CERN SPS Collider*, *Phys. Lett. B* **126** (1983) 398.
- [3] UA2 collaboration, *Evidence for  $Z^0 \rightarrow e^+e^-$  at the CERN  $\bar{p}p$  Collider*, *Phys. Lett. B* **129** (1983) 130.
- [4] UA2 collaboration, *Observation of Single Isolated Electrons of High Transverse Momentum in Events with Missing Transverse Energy at the CERN anti-p p Collider*, *Phys. Lett. B* **122** (1983) 476.
- [5] ATLAS collaboration, *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett. B* **716** (2012) 1 [1207.7214].
- [6] CMS collaboration, *Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC*, *Phys. Lett. B* **716** (2012) 30 [1207.7235].
- [7] L. Sartore, M. Maniatis, I. Schienbein and B. Herrmann, *The general Two-Higgs Doublet Model in a gauge-invariant form*, *JHEP* **12** (2022) 051 [2208.13719].
- [8] A. Held, J. Kwapisz and L. Sartore, *Grand unification and the Planck scale: an  $SO(10)$  example of radiative symmetry breaking*, *JHEP* **08** (2022) 122 [2204.03001].
- [9] L. Sartore and I. Schienbein, *PyR@TE 3*, *Comput. Phys. Commun.* **261** (2021) 107819 [2007.12700].
- [10] L. Sartore, *General renormalization group equations for dimensionful couplings in the  $\overline{\text{MS}}$  scheme*, *Phys. Rev. D* **102** (2020) 076002 [2006.12307].
- [11] M. Maniatis, L. Sartore and I. Schienbein, *Multiple point principle in the general Two-Higgs-Doublet model*, *JHEP* **08** (2020) 158 [2001.10541].
- [12] C.G. Bollini and J.J. Giambiagi, *Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter*, *Nuovo Cim. B* **12** (1972) 20.
- [13] G. 't Hooft and M.J.G. Veltman, *Regularization and Renormalization of Gauge Fields*, *Nucl. Phys. B* **44** (1972) 189.
- [14] M.E. Peskin and D.V. Schroeder, *An Introduction to quantum field theory*, Addison-Wesley, Reading, USA (1995).
- [15] M.D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press (3, 2014).
- [16] S.P. Martin, *Evaluation of two loop selfenergy basis integrals using differential equations*, *Phys. Rev. D* **68** (2003) 075002 [hep-ph/0307101].
- [17] C.G. Callan, Jr., *Broken scale invariance in scalar field theory*, *Phys. Rev. D* **2** (1970) 1541.
- [18] K. Symanzik, *Small distance behavior in field theory and power counting*, *Commun. Math. Phys.* **18** (1970) 227.
- [19] G. Degrassi, S. Di Vita, J. Elias-Miro, J.R. Espinosa, G.F. Giudice, G. Isidori et al., *Higgs mass and vacuum stability in the Standard Model at NNLO*, *JHEP* **08** (2012) 098 [1205.6497].

- [20] I. Masina, *Higgs boson and top quark masses as tests of electroweak vacuum stability*, *Phys. Rev. D* **87** (2013) 053001 [1209.0393].
- [21] F. Bezrukov, M.Y. Kalmykov, B.A. Kniehl and M. Shaposhnikov, *Higgs Boson Mass and New Physics*, *JHEP* **10** (2012) 140 [1205.2893].
- [22] D. Buttazzo, G. Degrassi, P.P. Giardino, G.F. Giudice, F. Sala, A. Salvio et al., *Investigating the near-criticality of the Higgs boson*, *JHEP* **12** (2013) 089 [1307.3536].
- [23] A.V. Bednyakov, B.A. Kniehl, A.F. Pikelner and O.L. Veretin, *Stability of the Electroweak Vacuum: Gauge Independence and Advanced Precision*, *Phys. Rev. Lett.* **115** (2015) 201802 [1507.08833].
- [24] G. Iacobellis and I. Masina, *Stationary configurations of the Standard Model Higgs potential: electroweak stability and rising inflection point*, *Phys. Rev. D* **94** (2016) 073005 [1604.06046].
- [25] A. Andreassen, W. Frost and M.D. Schwartz, *Scale Invariant Instantons and the Complete Lifetime of the Standard Model*, *Phys. Rev. D* **97** (2018) 056006 [1707.08124].
- [26] S. Chigusa, T. Moroi and Y. Shoji, *State-of-the-Art Calculation of the Decay Rate of Electroweak Vacuum in the Standard Model*, *Phys. Rev. Lett.* **119** (2017) 211801 [1707.09301].
- [27] J.R. Espinosa, *Vacuum Decay in the Standard Model: Analytical Results with Running and Gravity*, *JCAP* **06** (2020) 052 [2003.06219].
- [28] C. Ford and C. Wiesendanger, *Multiscale renormalization*, *Phys. Lett. B* **398** (1997) 342 [hep-th/9612193].
- [29] M.E. Machacek and M.T. Vaughn, *Two Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization*, *Nucl. Phys. B* **222** (1983) 83.
- [30] M.E. Machacek and M.T. Vaughn, *Two Loop Renormalization Group Equations in a General Quantum Field Theory. 2. Yukawa Couplings*, *Nucl. Phys. B* **236** (1984) 221.
- [31] M.E. Machacek and M.T. Vaughn, *Two Loop Renormalization Group Equations in a General Quantum Field Theory. 3. Scalar Quartic Couplings*, *Nucl. Phys. B* **249** (1985) 70.
- [32] I. Jack and H. Osborn, *Two Loop Background Field Calculations for Arbitrary Background Fields*, *Nucl. Phys. B* **207** (1982) 474.
- [33] I. Jack and H. Osborn, *General Two Loop Beta Functions for Gauge Theories With Arbitrary Scalar Fields*, *J. Phys. A* **16** (1983) 1101.
- [34] I. Jack and H. Osborn, *General Background Field Calculations With Fermion Fields*, *Nucl. Phys. B* **249** (1985) 472.
- [35] M.-x. Luo, H.-w. Wang and Y. Xiao, *Two loop renormalization group equations in general gauge field theories*, *Phys. Rev. D* **67** (2003) 065019 [hep-ph/0211440].
- [36] M.-x. Luo and Y. Xiao, *Renormalization group equations in gauge theories with multiple  $U(1)$  groups*, *Phys. Lett. B* **555** (2003) 279 [hep-ph/0212152].
- [37] R.M. Fonseca, M. Malinský and F. Staub, *Renormalization group equations and matching in a general quantum field theory with kinetic mixing*, *Phys. Lett. B* **726** (2013) 882 [1308.1674].
- [38] I. Schienbein, F. Staub, T. Steudtner and K. Svirina, *Revisiting RGEs for general gauge theories*, *Nucl. Phys. B* **939** (2019) 1 [1809.06797].
- [39] C. Poole and A.E. Thomsen, *Constraints on 3- and 4-loop  $\beta$ -functions in a general four-dimensional Quantum Field Theory*, *JHEP* **09** (2019) 055 [1906.04625].

- [40] A.G.M. Pickering, J.A. Gracey and D.R.T. Jones, *Three loop gauge beta function for the most general single gauge coupling theory*, *Phys. Lett. B* **510** (2001) 347 [[hep-ph/0104247](#)].
- [41] H. Osborn, *Derivation of a Four-dimensional c Theorem*, *Phys. Lett. B* **222** (1989) 97.
- [42] I. Jack and H. Osborn, *Analogues for the c Theorem for Four-dimensional Renormalizable Field Theories*, *Nucl. Phys. B* **343** (1990) 647.
- [43] H. Osborn, *Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories*, *Nucl. Phys. B* **363** (1991) 486.
- [44] O. Antipin, M. Gillioz, E. Mølgaard and F. Sannino, *The a theorem for gauge-Yukawa theories beyond Banks-Zaks fixed point*, *Phys. Rev. D* **87** (2013) 125017 [[1303.1525](#)].
- [45] I. Jack and C. Poole, *The a-function for gauge theories*, *JHEP* **01** (2015) 138 [[1411.1301](#)].
- [46] J. Davies, F. Herren, C. Poole, M. Steinhauser and A.E. Thomsen, *Gauge Coupling  $\beta$  Functions to Four-Loop Order in the Standard Model*, *Phys. Rev. Lett.* **124** (2020) 071803 [[1912.07624](#)].
- [47] F. Staub, *SARAH 4 : A tool for (not only SUSY) model builders*, *Comput. Phys. Commun.* **185** (2014) 1773 [[1309.7223](#)].
- [48] F. Lyonnet, I. Schienbein, F. Staub and A. Wingerter, *PyR@TE: Renormalization Group Equations for General Gauge Theories*, *Comput. Phys. Commun.* **185** (2014) 1130 [[1309.7030](#)].
- [49] F. Lyonnet and I. Schienbein, *PyR@TE 2: A Python tool for computing RGEs at two-loop*, *Comput. Phys. Commun.* **213** (2017) 181 [[1608.07274](#)].
- [50] S.P. Martin and M.T. Vaughn, *Two loop renormalization group equations for soft supersymmetry breaking couplings*, *Phys. Rev. D* **50** (1994) 2282 [[hep-ph/9311340](#)].
- [51] A.V. Manohar and E. Nardoni, *Renormalization Group Improvement of the Effective Potential: an EFT Approach*, *JHEP* **04** (2021) 093 [[2010.15806](#)].
- [52] M. Sperling, D. Stöckinger and A. Voigt, *Renormalization of vacuum expectation values in spontaneously broken gauge theories*, *JHEP* **07** (2013) 132 [[1305.1548](#)].
- [53] M. Sperling, D. Stöckinger and A. Voigt, *Renormalization of vacuum expectation values in spontaneously broken gauge theories: Two-loop results*, *JHEP* **01** (2014) 068 [[1310.7629](#)].
- [54] L.N. Mihaila, J. Salomon and M. Steinhauser, *Gauge Coupling Beta Functions in the Standard Model to Three Loops*, *Phys. Rev. Lett.* **108** (2012) 151602 [[1201.5868](#)].
- [55] L.N. Mihaila, J. Salomon and M. Steinhauser, *Renormalization constants and beta functions for the gauge couplings of the Standard Model to three-loop order*, *Phys. Rev. D* **86** (2012) 096008 [[1208.3357](#)].
- [56] A.V. Bednyakov, A.F. Pikelner and V.N. Velizhanin, *Anomalous dimensions of gauge fields and gauge coupling beta-functions in the Standard Model at three loops*, *JHEP* **01** (2013) 017 [[1210.6873](#)].
- [57] F. Herren, L. Mihaila and M. Steinhauser, *Gauge and Yukawa coupling beta functions of two-Higgs-doublet models to three-loop order*, *Phys. Rev. D* **97** (2018) 015016 [[1712.06614](#)].
- [58] M.B. Einhorn and D.R.T. Jones, *Asymptotic freedom in certain  $SO(N)$  and  $SU(N)$  models*, *Phys. Rev. D* **96** (2017) 055035 [[1705.00751](#)].
- [59] M.T. Vaughn, *Renormalization Group Constraints on Unified Gauge Theories. 2. Yukawa and Scalar Quartic Couplings*, *Z. Phys. C* **13** (1982) 139.
- [60] S. Blasi, S. De Curtis and K. Yagyu, *Effects of custodial symmetry breaking in the Georgi-Machacek model at high energies*, *Phys. Rev. D* **96** (2017) 015001 [[1704.08512](#)].

- [61] A.E. Thomsen, *Introducing RGBeta: a Mathematica package for the evaluation of renormalization group  $\beta$ -functions*, *Eur. Phys. J. C* **81** (2021) 408 [2101.08265].
- [62] R.M. Fonseca, *Calculating the renormalisation group equations of a SUSY model with Susyno*, *Comput. Phys. Commun.* **183** (2012) 2298 [1106.5016].
- [63] R.M. Fonseca, *GroupMath: A Mathematica package for group theory calculations*, *Comput. Phys. Commun.* **267** (2021) 108085 [2011.01764].
- [64] B.M. Kastening, *Renormalization group improvement of the effective potential in massive  $\phi^4$  theory*, *Phys. Lett. B* **283** (1992) 287.
- [65] M. Bando, T. Kugo, N. Maekawa and H. Nakano, *Improving the effective potential*, *Phys. Lett. B* **301** (1993) 83 [hep-ph/9210228].
- [66] J.M. Chung and B.K. Chung, *Renormalization group improvement of the effective potential in massive  $\phi^4$  theory*, *Phys. Rev. D* **60** (1999) 105001 [hep-th/9905086].
- [67] J.M. Chung and B.K. Chung, *Renormalization group improvement of the effective potential in massive  $\phi^4$  theory: Next-next-next-to-leading logarithm resummation*, *J. Korean Phys. Soc.* **39** (2001) 971 [hep-th/9911196].
- [68] F.T. Brandt, F.A. Chishtie and D.G.C. McKeon, *The Renormalization group improved effective potential in massless models*, *Mod. Phys. Lett. A* **20** (2005) 2215 [hep-th/0407171].
- [69] F.T. Brandt, F.A. Chishtie and D.G.C. McKeon, *The effective potential in the massive  $\phi^4(4)$  model*, *Int. J. Mod. Phys. A* **22** (2007) 1 [hep-th/0511161].
- [70] M.B. Einhorn and D.R.T. Jones, *A NEW RENORMALIZATION GROUP APPROACH TO MULTISCALE PROBLEMS*, *Nucl. Phys. B* **230** (1984) 261.
- [71] M. Bando, T. Kugo, N. Maekawa and H. Nakano, *Improving the effective potential: Multimass scale case*, *Prog. Theor. Phys.* **90** (1993) 405 [hep-ph/9210229].
- [72] C. Ford, *Multiscale renormalization group improvement of the effective potential*, *Phys. Rev. D* **50** (1994) 7531 [hep-th/9404085].
- [73] C. Wiesendanger, *Resummation of the two distinct large logarithms in the broken  $O(N)$  symmetric  $\phi^4$  model*, in *3rd International Conference on Renormalization Group (RG 96)*, 9, 1996 [hep-th/9609129].
- [74] C. Ford, *Multiscale renormalization and decoupling*, in *3rd International Conference on Renormalization Group (RG 96)*, 8, 1996 [hep-th/9609127].
- [75] C. Ford and C. Wiesendanger, *A Multiscale subtraction scheme and partial renormalization group equations in the  $O(N)$  symmetric  $\phi^4$  theory*, *Phys. Rev. D* **55** (1997) 2202 [hep-ph/9604392].
- [76] C.R. Stephens, *Why two renormalization groups are better than one*, *Int. J. Mod. Phys. B* **12** (1998) 1379 [hep-th/9611062].
- [77] J.A. Casas, V. Di Clemente and M. Quiros, *The Effective potential in the presence of several mass scales*, *Nucl. Phys. B* **553** (1999) 511 [hep-ph/9809275].
- [78] T.G. Steele, Z.-W. Wang and D.G.C. McKeon, *Multiscale renormalization group methods for effective potentials with multiple scalar fields*, *Phys. Rev. D* **90** (2014) 105012 [1409.3489].
- [79] A. Pilaftsis and D. Teresi, *Exact RG Invariance and Symmetry Improved 2PI Effective Potential*, *Nucl. Phys. B* **920** (2017) 298 [1703.02079].
- [80] S. Iso and K. Kawana, *RG-improvement of the effective action with multiple mass scales*, *JHEP* **03** (2018) 165 [1801.01731].

- [81] S.R. Coleman and E.J. Weinberg, *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking*, *Phys. Rev. D* **7** (1973) 1888.
- [82] T.D. Lee, *A Theory of Spontaneous T Violation*, *Phys. Rev. D* **8** (1973) 1226.
- [83] A.D. Sakharov, *Violation of CP Invariance, C asymmetry, and baryon asymmetry of the universe*, *Pisma Zh. Eksp. Teor. Fiz.* **5** (1967) 32.
- [84] PARTICLE DATA GROUP collaboration, *Review of Particle Physics*, *PTEP* **2020** (2020) 083C01.
- [85] W. Bernreuther and O. Nachtmann, *Flavor dynamics with general scalar fields*, *Eur. Phys. J. C* **9** (1999) 319 [hep-ph/9812259].
- [86] G.C. Branco, P.M. Ferreira, L. Lavoura, M.N. Rebelo, M. Sher and J.P. Silva, *Theory and phenomenology of two-Higgs-doublet models*, *Phys. Rept.* **516** (2012) 1 [1106.0034].
- [87] Anisha, L. Biermann, C. Englert and M. Mühlleitner, *Two Higgs doublets, effective interactions and a strong first-order electroweak phase transition*, *JHEP* **08** (2022) 091 [2204.06966].
- [88] P. Basler, M. Mühlleitner and J. Wittbrodt, *The CP-Violating 2HDM in Light of a Strong First Order Electroweak Phase Transition and Implications for Higgs Pair Production*, *JHEP* **03** (2018) 061 [1711.04097].
- [89] P. Athron, C. Balazs, A. Cherchiglia, D.H.J. Jacob, D. Stoeckinger, H. Stoeckinger-Kim et al., *GM2Calc - 2 for the 2HDM*, *PoS CompTools2021* (2022) 009 [2207.09039].
- [90] F. Arco, S. Heinemeyer and M.J. Herrero, *Triple Higgs couplings in the 2HDM: the complete picture*, *Eur. Phys. J. C* **82** (2022) 536 [2203.12684].
- [91] P.M. Ferreira, B. Grzadkowski, O.M. Ogreid and P. Osland, *Symmetries of the 2HDM: an invariant formulation and consequences*, *JHEP* **02** (2021) 196 [2010.13698].
- [92] S. Jana, V.P. K. and S. Saad, *Resolving electron and muon  $g - 2$  within the 2HDM*, *Phys. Rev. D* **101** (2020) 115037 [2003.03386].
- [93] A.V. Bednyakov, *On three-loop RGE for the Higgs sector of 2HDM*, *JHEP* **11** (2018) 154 [1809.04527].
- [94] C. Cai, G. Cacciapaglia and H.-H. Zhang, *Vacuum alignment in a composite 2HDM*, *JHEP* **01** (2019) 130 [1805.07619].
- [95] A. Goudelis, B. Herrmann and O. Stål, *Dark matter in the Inert Doublet Model after the discovery of a Higgs-like boson at the LHC*, *JHEP* **09** (2013) 106 [1303.3010].
- [96] G. Belanger, B. Dumont, A. Goudelis, B. Herrmann, S. Kraml and D. Sengupta, *Dilepton constraints in the Inert Doublet Model from Run 1 of the LHC*, *Phys. Rev. D* **91** (2015) 115011 [1503.07367].
- [97] J.F. Gunion, H.E. Haber, G.L. Kane and S. Dawson, *The Higgs Hunter's Guide*, *Front. Phys.* **80** (2000) 1.
- [98] I.P. Ivanov, *Minkowski space structure of the Higgs potential in 2HDM. II. Minima, symmetries, and topology*, *Phys. Rev. D* **77** (2008) 015017 [0710.3490].
- [99] M. Maniatis and O. Nachtmann, *Symmetries and renormalisation in two-Higgs-doublet models*, *JHEP* **11** (2011) 151 [1106.1436].
- [100] M.P. Bento, R. Boto, J.a.P. Silva and A. Trautner, *A fully basis invariant Symmetry Map of the 2HDM*, *JHEP* **21** (2020) 229 [2009.01264].
- [101] F. Nagel, *New aspects of gauge-boson couplings and the Higgs sector*, Ph.D. thesis, Heidelberg U., 2004.



- [102] M. Maniatis, A. von Manteuffel, O. Nachtmann and F. Nagel, *Stability and symmetry breaking in the general two-Higgs-doublet model*, *Eur. Phys. J.* **C48** (2006) 805 [hep-ph/0605184].
- [103] C.C. Nishi, *CP violation conditions in N-Higgs-doublet potentials*, *Phys. Rev.* **D74** (2006) 036003 [hep-ph/0605153].
- [104] K. Kannike, *Vacuum stability conditions and potential minima for a matrix representation in lightcone orbit space*, *Eur. Phys. J. C* **81** (2021) 940 [2109.01671].
- [105] M.A. Solberg, *Conditions for the custodial symmetry in multi-Higgs-doublet models*, *JHEP* **05** (2018) 163 [1801.00519].
- [106] Q.-H. Cao, K. Cheng and C. Xu, *CP Phases in 2HDM and Effective Potential: A Geometrical View*, 2201.02989.
- [107] S.P. Martin, *Effective potential at three loops*, *Phys. Rev. D* **96** (2017) 096005 [1709.02397].
- [108] J. Velhinho, R. Santos and A. Barroso, *Tree level vacuum stability in two Higgs doublet models*, *Phys. Lett. B* **322** (1994) 213.
- [109] H. Georgi and S.L. Glashow, *Unity of All Elementary Particle Forces*, *Phys. Rev. Lett.* **32** (1974) 438.
- [110] J.C. Pati and A. Salam, *Lepton Number as the Fourth Color*, *Phys. Rev. D* **10** (1974) 275.
- [111] H. Fritzsch and P. Minkowski, *Unified Interactions of Leptons and Hadrons*, *Annals Phys.* **93** (1975) 193.
- [112] D. Chang, R.N. Mohapatra, J. Gipson, R.E. Marshak and M.K. Parida, *Experimental Tests of New SO(10) Grand Unification*, *Phys. Rev. D* **31** (1985) 1718.
- [113] S. Bertolini, L. Di Luzio and M. Malinsky, *Intermediate mass scales in the non-supersymmetric SO(10) grand unification: A Reappraisal*, *Phys. Rev.* **D80** (2009) 015013 [0903.4049].
- [114] A.J. Buras, J.R. Ellis, M.K. Gaillard and D.V. Nanopoulos, *Aspects of the Grand Unification of Strong, Weak and Electromagnetic Interactions*, *Nucl. Phys. B* **135** (1978) 66.
- [115] H. Georgi and C. Jarlskog, *A New Lepton - Quark Mass Relation in a Unified Theory*, *Phys. Lett. B* **86** (1979) 297.
- [116] G. Lazarides, Q. Shafi and C. Wetterich, *Proton Lifetime and Fermion Masses in an SO(10) Model*, *Nucl. Phys. B* **181** (1981) 287.
- [117] R. Barbieri, D.V. Nanopoulos, G. Morchio and F. Strocchi, *Neutrino Masses in Grand Unified Theories*, *Phys. Lett. B* **90** (1980) 91.
- [118] E. Witten, *Neutrino Masses in the Minimal O(10) Theory*, *Phys. Lett. B* **91** (1980) 81.
- [119] R.N. Mohapatra and G. Senjanovic, *Neutrino Masses and Mixings in Gauge Models with Spontaneous Parity Violation*, *Phys. Rev. D* **23** (1981) 165.
- [120] K.S. Babu and S.M. Barr, *Large neutrino mixing angles in unified theories*, *Phys. Lett. B* **381** (1996) 202 [hep-ph/9511446].
- [121] E. Nezri and J. Orloff, *Neutrino oscillations versus leptogenesis in SO(10) models*, *JHEP* **04** (2003) 020 [hep-ph/0004227].
- [122] K.S. Babu and R.N. Mohapatra, *Predictive neutrino spectrum in minimal SO(10) grand unification*, *Phys. Rev. Lett.* **70** (1993) 2845 [hep-ph/9209215].
- [123] B. Bajc, G. Senjanovic and F. Vissani, *b - tau unification and large atmospheric mixing: A Case for noncanonical seesaw*, *Phys. Rev. Lett.* **90** (2003) 051802 [hep-ph/0210207].

- [124] S. Bertolini, L. Di Luzio and M. Malinsky, *Seesaw Scale in the Minimal Renormalizable  $SO(10)$  Grand Unification*, *Phys. Rev. D* **85** (2012) 095014 [1202.0807].
- [125] M. Yoshimura, *Unified Gauge Theories and the Baryon Number of the Universe*, *Phys. Rev. Lett.* **41** (1978) 281.
- [126] A.Y. Ignatiev, V.A. Kuzmin and M.E. Shaposhnikov, *Baryon Asymmetry of the Universe in Grand Unified Theories*, *Phys. Lett. B* **87** (1979) 114.
- [127] V.A. Kuzmin and M.E. Shaposhnikov, *Baryon Asymmetry of the Universe Versus Left-right Symmetry*, *Phys. Lett. B* **92** (1980) 115.
- [128] E.K. Akhmedov, M. Frigerio and A.Y. Smirnov, *Probing the seesaw mechanism with neutrino data and leptogenesis*, *JHEP* **09** (2003) 021 [hep-ph/0305322].
- [129] A. Eichhorn, A. Held and C. Wetterich, *Quantum-gravity predictions for the fine-structure constant*, *Phys. Lett. B* **782** (2018) 198 [1711.02949].
- [130] A. Eichhorn, A. Held and C. Wetterich, *Predictive power of grand unification from quantum gravity*, *JHEP* **08** (2020) 111 [1909.07318].
- [131] V. Braun, Y.-H. He, B.A. Ovrut and T. Pantev, *A Heterotic standard model*, *Phys. Lett. B* **618** (2005) 252 [hep-th/0501070].
- [132] L.B. Anderson, J. Gray, D. Grayson, Y.-H. He and A. Lukas, *Yukawa Couplings in Heterotic Compactification*, *Commun. Math. Phys.* **297** (2010) 95 [0904.2186].
- [133] M. McGuigan, *Dark Horse, Dark Matter: Revisiting the  $SO(16) \times SO(16)$ ' Nonsupersymmetric Model in the LHC and Dark Energy Era*, 1907.01944.
- [134] L.B. Anderson, J. Gray, M. Larfors, M. Magill and R. Schneider, *Generalized Vanishing Theorems for Yukawa Couplings in Heterotic Compactifications*, 2103.10454.
- [135] I. Brivio and M. Trott, *The Standard Model as an Effective Field Theory*, *Phys. Rept.* **793** (2019) 1 [1706.08945].
- [136] L. Chataignier, T. Prokopec, M.G. Schmidt and B. Swiezewska, *Single-scale Renormalisation Group Improvement of Multi-scale Effective Potentials*, *JHEP* **03** (2018) 014 [1801.05258].
- [137] K. Kannike, K. Loos and L. Marzola, *Minima of classically scale-invariant potentials*, *JHEP* **06** (2021) 128 [2011.12304].
- [138] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J.M. Pawłowski, M. Tissier et al., *The nonperturbative functional renormalization group and its applications*, *Phys. Rept.* **910** (2021) 1 [2006.04853].
- [139] S. Bornholdt, N. Tetradis and C. Wetterich, *Coleman-Weinberg phase transition in two scalar models*, *Phys. Lett. B* **348** (1995) 89 [hep-th/9408132].
- [140] S. Bornholdt, P. Buttner, N. Tetradis and C. Wetterich, *Flow of the coarse grained free energy for crossover phenomena*, *Int. J. Mod. Phys. A* **14** (1999) 899 [cond-mat/9603129].
- [141] A. Eichhorn, D. Mesterházy and M.M. Scherer, *Multicritical behavior in models with two competing order parameters*, *Phys. Rev. E* **88** (2013) 042141 [1306.2952].
- [142] I. Boettcher, *Scaling relations and multicritical phenomena from Functional Renormalization*, *Phys. Rev. E* **91** (2015) 062112 [1503.07817].
- [143] A. Chlebicki and P. Jakubczyk, *Criticality of the  $O(2)$  model with cubic anisotropies from nonperturbative renormalization*, *Phys. Rev. E* **100** (2019) 052106 [1909.10600].

- [144] A. Held, *Effective asymptotic safety and its predictive power: Gauge-Yukawa theories*, *Front. in Phys.* **8** (2020) 341 [2003.13642].
- [145] N.G. Deshpande, E. Keith and P.B. Pal, *Implications of LEP results for  $SO(10)$  grand unification*, *Phys. Rev. D* **46** (1993) 2261.
- [146] K.S. Babu and S. Khan, *Minimal nonsupersymmetric  $SO(10)$  model: Gauge coupling unification, proton decay, and fermion masses*, *Phys. Rev. D* **92** (2015) 075018 [1507.06712].
- [147] R. Lal Awasthi and M.K. Parida, *Inverse Seesaw Mechanism in Nonsupersymmetric  $SO(10)$ , Proton Lifetime, Nonunitarity Effects, and a Low-mass  $Z'$  Boson*, *Phys. Rev. D* **86** (2012) 093004 [1112.1826].
- [148] T. Ohlsson, M. Pernow and E. Sönnerlind, *Realizing unification in two different  $SO(10)$  models with one intermediate breaking scale*, *Eur. Phys. J. C* **80** (2020) 1089 [2006.13936].
- [149] R.N. Mohapatra and B. Sakita,  *$SO(2n)$  Grand Unification in an  $SU(N)$  Basis*, *Phys. Rev. D* **21** (1980) 1062.
- [150] F. Wilczek and A. Zee, *Families from Spinors*, *Phys. Rev. D* **25** (1982) 553.
- [151] P.H. Chankowski and Z. Pluciennik, *Renormalization group equations for seesaw neutrino masses*, *Phys. Lett. B* **316** (1993) 312 [hep-ph/9306333].
- [152] K. Matsuda, Y. Koide and T. Fukuyama, *Can the  $SO(10)$  model with two Higgs doublets reproduce the observed fermion masses?*, *Phys. Rev. D* **64** (2001) 053015 [hep-ph/0010026].
- [153] B. Bajc, A. Melfo, G. Senjanovic and F. Vissani, *Yukawa sector in non-supersymmetric renormalizable  $SO(10)$* , *Phys. Rev. D* **73** (2006) 055001 [hep-ph/0510139].
- [154] A.S. Joshipura and K.M. Patel, *Fermion Masses in  $SO(10)$  Models*, *Phys. Rev. D* **83** (2011) 095002 [1102.5148].
- [155] G. Altarelli and D. Meloni, *A non supersymmetric  $SO(10)$  grand unified model for all the physics below  $M_{GUT}$* , *JHEP* **08** (2013) 021 [1305.1001].
- [156] A. Dueck and W. Rodejohann, *Fits to  $SO(10)$  Grand Unified Models*, *JHEP* **09** (2013) 024 [1306.4468].
- [157] K.S. Babu, B. Bajc and S. Saad, *Resurrecting Minimal Yukawa Sector of SUSY  $SO(10)$* , *JHEP* **10** (2018) 135 [1805.10631].
- [158] T. Ohlsson and M. Pernow, *Running of Fermion Observables in Non-Supersymmetric  $SO(10)$  Models*, *JHEP* **11** (2018) 028 [1804.04560].
- [159] T. Ohlsson and M. Pernow, *Fits to Non-Supersymmetric  $SO(10)$  Models with Type I and II Seesaw Mechanisms Using Renormalization Group Evolution*, *JHEP* **06** (2019) 085 [1903.08241].
- [160] L. Di Luzio, *Aspects of symmetry breaking in Grand Unified Theories*, Ph.D. thesis, SISSA, Trieste, 2011. 1110.3210.
- [161] S. Bertolini, L. Di Luzio and M. Malinsky, *On the vacuum of the minimal nonsupersymmetric  $SO(10)$  unification*, *Phys. Rev. D* **81** (2010) 035015 [0912.1796].
- [162] C. Ford, D.R.T. Jones, P.W. Stephenson and M.B. Einhorn, *The Effective potential and the renormalization group*, *Nucl. Phys. B* **395** (1993) 17 [hep-lat/9210033].
- [163] H. Gies, C. Gneiting and R. Sondenheimer, *Higgs Mass Bounds from Renormalization Flow for a simple Yukawa model*, *Phys. Rev. D* **89** (2014) 045012 [1308.5075].
- [164] H. Gies and R. Sondenheimer, *Higgs Mass Bounds from Renormalization Flow for a Higgs-top-bottom model*, *Eur. Phys. J. C* **75** (2015) 68 [1407.8124].

- [165] J. Borchardt, H. Gies and R. Sondenheimer, *Global flow of the Higgs potential in a Yukawa model*, *Eur. Phys. J. C* **76** (2016) 472 [1603.05861].
- [166] H. Gies, R. Sondenheimer and M. Warschinke, *Impact of generalized Yukawa interactions on the lower Higgs mass bound*, *Eur. Phys. J. C* **77** (2017) 743 [1707.04394].
- [167] L. Chataignier, T. Prokopec, M.G. Schmidt and B. Świeżewska, *Systematic analysis of radiative symmetry breaking in models with extended scalar sector*, *JHEP* **08** (2018) 083 [1805.09292].
- [168] E. Gildener and S. Weinberg, *Symmetry Breaking and Scalar Bosons*, *Phys. Rev. D* **13** (1976) 3333.
- [169] M. Yasuè, *Symmetry breaking of  $so(10)$  and constraints on the higgs potential: Adjoint 45 and spinorial 16 representations*, *Phys. Rev. D* **24** (1981) 1005.
- [170] M. Yasue, *HOW TO BREAK  $SO(10)$  VIA  $SO(4) \times SO(6)$  DOWN TO  $SU(2)(L) \times SU(3)(C) \times U(1)$* , *Phys. Lett. B* **103** (1981) 33.
- [171] G. Anastaze, J.P. Derendinger and F. Buccella, *INTERMEDIATE SYMMETRIES IN THE  $SO(10)$  MODEL WITH  $(16+16) + 45$  HIGGSES*, *Z. Phys. C* **20** (1983) 269.
- [172] K.S. Babu and E. Ma, *Symmetry Breaking in  $SO(10)$ : Higgs Boson Structure*, *Phys. Rev. D* **31** (1985) 2316.
- [173] S. Bertolini, L. Di Luzio and M. Malinsky, *The quantum vacuum of the minimal  $SO(10)$  GUT*, *J. Phys. Conf. Ser.* **259** (2010) 012098 [1010.0338].
- [174] S. Bertolini, L. Di Luzio and M. Malinsky, *Towards a New Minimal  $SO(10)$  Unification*, *AIP Conf. Proc.* **1467** (2012) 37 [1205.5637].
- [175] J. Kim, *General Method for Analyzing Higgs Potentials*, *Nucl. Phys. B* **196** (1982) 285.
- [176] G. Sartori and G. Valente, *Allowed and observable phases in two Higgs doublet standard models*, [hep-ph/0304026](#).
- [177] G. Sartori and G. Valente, *Symmetry allowed, but unobservable, phases in renormalizable gauge field theory models*, [hep-th/0405021](#).
- [178] L.-F. Li, *Group Theory of the Spontaneously Broken Gauge Symmetries*, *Phys. Rev. D* **9** (1974) 1723.
- [179] A. Held, J.H. Kwapisz and L. Sartore. to appear, 2022.
- [180] V.V. Dixit and M. Sher, *The Futility of High Precision  $SO(10)$  Calculations*, *Phys. Rev. D* **40** (1989) 3765.
- [181] M. Yasue, *Symmetry Breaking of  $SO(10)$  and Constraints on Higgs Potential. 1. Adjoint (45) and Spinorial (16)*, *Phys. Rev. D* **24** (1981) 1005.
- [182] C.M. Bender and T.T. Wu, *Analytic structure of energy levels in a field theory model*, *Phys. Rev. Lett.* **21** (1968) 406.
- [183] A. Vilenkin, *Cosmic Strings and Domain Walls*, *Phys. Rept.* **121** (1985) 263.
- [184] W.H. Press, B.S. Ryden and D.N. Spergel, *Dynamical Evolution of Domain Walls in an Expanding Universe*, *Astrophys. J.* **347** (1989) 590.
- [185] D. Coulson, Z. Lalak and B.A. Ovrut, *Biased domain walls*, *Phys. Rev. D* **53** (1996) 4237.
- [186] Z. Lalak, S. Lola, B.A. Ovrut and G.G. Ross, *Large scale structure from biased nonequilibrium phase transitions: Percolation theory picture*, *Nucl. Phys. B* **434** (1995) 675 [[hep-ph/9404218](#)].

- [187] T. Krajewski, J.H. Kwapisz, Z. Lalak and M. Lewicki, *Stability of domain walls in models with asymmetric potentials*, 2103.03225.
- [188] K. Jarkovská, M. Malinský, T. Mede and V. Susič, *Quantum guts of the minimal potentially realistic SO(10) Higgs model*, 2109.06784.
- [189] H. Georgi, *AN ALMOST REALISTIC GAUGE HIERARCHY*, *Phys. Lett. B* **108** (1982) 283.
- [190] A. Masiero, D.V. Nanopoulos, K. Tamvakis and T. Yanagida, *Naturally Massless Higgs Doublets in Supersymmetric SU(5)*, *Phys. Lett. B* **115** (1982) 380.
- [191] B. Grinstein, *A Supersymmetric SU(5) Gauge Theory with No Gauge Hierarchy Problem*, *Nucl. Phys. B* **206** (1982) 387.
- [192] M. Srednicki, *Supersymmetric Grand Unified Theories and the Early Universe*, *Nucl. Phys. B* **202** (1982) 327.
- [193] K.S. Babu and S.M. Barr, *Natural suppression of Higgsino mediated proton decay in supersymmetric SO(10)*, *Phys. Rev. D* **48** (1993) 5354 [hep-ph/9306242].
- [194] K.S. Babu and S.M. Barr, *Supersymmetric SO(10) simplified*, *Phys. Rev. D* **51** (1995) 2463 [hep-ph/9409285].
- [195] K.S. Babu and S.M. Barr, *Natural gauge hierarchy in SO(10)*, *Phys. Rev. D* **50** (1994) 3529 [hep-ph/9402291].
- [196] K.S. Babu and R.N. Mohapatra, *Mass matrix textures from superstring inspired SO(10) models*, *Phys. Rev. Lett.* **74** (1995) 2418 [hep-ph/9410326].
- [197] J. Frohlich, G. Morchio and F. Strocchi, *HIGGS PHENOMENON WITHOUT SYMMETRY BREAKING ORDER PARAMETER*, *Nucl. Phys. B* **190** (1981) 553.
- [198] A. Maas, *Describing gauge bosons at zero and finite temperature*, *Phys. Rept.* **524** (2013) 203 [1106.3942].
- [199] P. Törek, A. Maas and R. Sondenheimer, *A study of how the particle spectra of SU(N) gauge theories with a fundamental Higgs emerge*, *EPJ Web Conf.* **175** (2018) 08002 [1710.01941].
- [200] R. Sondenheimer, *Analytical relations for the bound state spectrum of gauge theories with a Brout-Englert-Higgs mechanism*, *Phys. Rev. D* **101** (2020) 056006 [1912.08680].
- [201] E.S. Fradkin and A.A. Tseytlin, *Renormalizable asymptotically free quantum theory of gravity*, *Nucl. Phys. B* **201** (1982) 469.
- [202] A. Salvio and A. Strumia, *Agravity*, *JHEP* **06** (2014) 080 [1403.4226].
- [203] J.F. Donoghue and G. Menezes, *Gauge Assisted Quadratic Gravity: A Framework for UV Complete Quantum Gravity*, *Phys. Rev. D* **97** (2018) 126005 [1804.04980].
- [204] D.F. Litim and F. Sannino, *Asymptotic safety guaranteed*, *JHEP* **12** (2014) 178 [1406.2337].
- [205] A.D. Bond, D.F. Litim, G. Medina Vazquez and T. Steudtner, *UV conformal window for asymptotic safety*, *Phys. Rev. D* **97** (2018) 036019 [1710.07615].
- [206] T.P. Cheng, E. Eichten and L.-F. Li, *Higgs Phenomena in Asymptotically Free Gauge Theories*, *Phys. Rev. D* **9** (1974) 2259.
- [207] G.F. Giudice, G. Isidori, A. Salvio and A. Strumia, *Softened Gravity and the Extension of the Standard Model up to Infinite Energy*, *JHEP* **02** (2015) 137 [1412.2769].
- [208] B. Holdom, J. Ren and C. Zhang, *Stable Asymptotically Free Extensions (SAFEs) of the Standard Model*, *JHEP* **03** (2015) 028 [1412.5540].

- [209] H. Gies and L. Zambelli, *Asymptotically free scaling solutions in non-Abelian Higgs models*, *Phys. Rev. D* **92** (2015) 025016 [1502.05907].
- [210] S. Weinberg, *ULTRAVIOLET DIVERGENCES IN QUANTUM THEORIES OF GRAVITATION*, in *General Relativity: An Einstein Centenary Survey*, pp. 790–831 (1980).
- [211] M. Reuter, *Nonperturbative evolution equation for quantum gravity*, *Phys. Rev. D* **57** (1998) 971 [hep-th/9605030].
- [212] R. Percacci, *An Introduction to Covariant Quantum Gravity and Asymptotic Safety*, vol. 3 of *100 Years of General Relativity*, World Scientific (2017), 10.1142/10369.
- [213] M. Reuter and F. Saueressig, *Quantum Gravity and the Functional Renormalization Group: The Road towards Asymptotic Safety*, Cambridge University Press (1, 2019).
- [214] A. Bonanno, A. Eichhorn, H. Gies, J.M. Pawłowski, R. Percacci, M. Reuter et al., *Critical reflections on asymptotically safe gravity*, *Front. in Phys.* **8** (2020) 269 [2004.06810].
- [215] A. Eichhorn and A. Held, *Top mass from asymptotic safety*, *Phys. Lett. B* **777** (2018) 217 [1707.01107].
- [216] A. Eichhorn and F. Versteegen, *Upper bound on the Abelian gauge coupling from asymptotic safety*, *JHEP* **01** (2018) 030 [1709.07252].
- [217] A. Eichhorn and A. Held, *Mass difference for charged quarks from asymptotically safe quantum gravity*, *Phys. Rev. Lett.* **121** (2018) 151302 [1803.04027].
- [218] A. Eichhorn, *Status update: asymptotically safe gravity-matter systems*, 1, 2022 [2201.11543].
- [219] G. Narain and R. Percacci, *Renormalization Group Flow in Scalar-Tensor Theories. I*, *Class. Quant. Grav.* **27** (2010) 075001 [0911.0386].
- [220] R. Percacci and G.P. Vacca, *Search of scaling solutions in scalar-tensor gravity*, *Eur. Phys. J. C* **75** (2015) 188 [1501.00888].
- [221] A. Eichhorn and S. Lippoldt, *Quantum gravity and Standard-Model-like fermions*, *Phys. Lett. B* **767** (2017) 142 [1611.05878].
- [222] A. Eichhorn, S. Lippoldt and V. Skrinjar, *Nonminimal hints for asymptotic safety*, *Phys. Rev. D* **97** (2018) 026002 [1710.03005].
- [223] A. Eichhorn, S. Lippoldt and M. Schiffer, *Zooming in on fermions and quantum gravity*, *Phys. Rev. D* **99** (2019) 086002 [1812.08782].
- [224] J. Daas, W. Oosters, F. Saueressig and J. Wang, *Asymptotically Safe Gravity-Fermion Systems on Curved Backgrounds*, *Universe* **7** (2021) 306 [2107.01071].
- [225] A. Eichhorn and M. Pauly, *Constraining power of asymptotic safety for scalar fields*, *Phys. Rev. D* **103** (2021) 026006 [2009.13543].
- [226] A. Eichhorn and H. Gies, *Light fermions in quantum gravity*, *New J. Phys.* **13** (2011) 125012 [1104.5366].
- [227] A. Eichhorn, *Quantum-gravity-induced matter self-interactions in the asymptotic-safety scenario*, *Phys. Rev. D* **86** (2012) 105021 [1204.0965].
- [228] A. Eichhorn, A. Held and J.M. Pawłowski, *Quantum-gravity effects on a Higgs-Yukawa model*, *Phys. Rev. D* **94** (2016) 104027 [1604.02041].
- [229] A. Eichhorn and A. Held, *Viability of quantum-gravity induced ultraviolet completions for matter*, *Phys. Rev. D* **96** (2017) 086025 [1705.02342].

- [230] N. Christiansen and A. Eichhorn, *An asymptotically safe solution to the  $U(1)$  triviality problem*, *Phys. Lett. B* **770** (2017) 154 [1702.07724].
- [231] G.P. de Brito, A. Eichhorn and R.R.L.d. Santos, *The weak-gravity bound and the need for spin in asymptotically safe matter-gravity models*, *JHEP* **11** (2021) 110 [2107.03839].
- [232] A. Eichhorn, J.H. Kwapisz and M. Schiffer, *The weak-gravity bound in asymptotically safe gravity-gauge systems*, 2112.09772.
- [233] S.P. Robinson and F. Wilczek, *Gravitational correction to running of gauge couplings*, *Phys. Rev. Lett.* **96** (2006) 231601 [hep-th/0509050].
- [234] A.R. Pietrykowski, *Gauge dependence of gravitational correction to running of gauge couplings*, *Phys. Rev. Lett.* **98** (2007) 061801 [hep-th/0606208].
- [235] D.J. Toms, *Quantum gravity and charge renormalization*, *Phys. Rev. D* **76** (2007) 045015 [0708.2990].
- [236] D. Ebert, J. Plefka and A. Rodigast, *Absence of gravitational contributions to the running Yang-Mills coupling*, *Phys. Lett. B* **660** (2008) 579 [0710.1002].
- [237] Y. Tang and Y.-L. Wu, *Gravitational Contributions to the Running of Gauge Couplings*, *Commun. Theor. Phys.* **54** (2010) 1040 [0807.0331].
- [238] D.J. Toms, *Quantum gravity, gauge coupling constants, and the cosmological constant*, *Phys. Rev. D* **80** (2009) 064040 [0908.3100].
- [239] M.M. Anber, J.F. Donoghue and M. El-Houssieny, *Running couplings and operator mixing in the gravitational corrections to coupling constants*, *Phys. Rev. D* **83** (2011) 124003 [1011.3229].
- [240] J. Ellis and N.E. Mavromatos, *On the Interpretation of Gravitational Corrections to Gauge Couplings*, *Phys. Lett. B* **711** (2012) 139 [1012.4353].
- [241] L.I. Bevilaqua, M. Dias, A.C. Lehum, C.R. Senise, Jr., A.J. da Silva and H. Souza, *Gravitational corrections to two-loop beta function in quantum electrodynamics*, *Phys. Rev. D* **104** (2021) 125001 [2105.12577].
- [242] A. Baldazzi, R. Percacci and L. Zambelli, *Functional renormalization and the  $\overline{MS}$  scheme*, *Phys. Rev. D* **103** (2021) 076012 [2009.03255].
- [243] A. Baldazzi, R. Percacci and L. Zambelli, *Vanishing Regulators*, 2105.05778.
- [244] G.P. de Brito and A. Eichhorn, *Nonvanishing gravitational contribution to matter beta functions for vanishing dimensional regulators*, 2201.11402.
- [245] J.-E. Daum, U. Harst and M. Reuter, *Running Gauge Coupling in Asymptotically Safe Quantum Gravity*, *JHEP* **01** (2010) 084 [0910.4938].
- [246] S. Folkerts, D.F. Litim and J.M. Pawłowski, *Asymptotic freedom of Yang-Mills theory with gravity*, *Phys. Lett. B* **709** (2012) 234 [1101.5552].
- [247] U. Harst and M. Reuter, *QED coupled to QEG*, *JHEP* **05** (2011) 119 [1101.6007].
- [248] N. Christiansen, D.F. Litim, J.M. Pawłowski and M. Reichert, *Asymptotic safety of gravity with matter*, *Phys. Rev. D* **97** (2018) 106012 [1710.04669].
- [249] O. Zanusso, L. Zambelli, G.P. Vacca and R. Percacci, *Gravitational corrections to Yukawa systems*, *Phys. Lett. B* **689** (2010) 90 [0904.0938].
- [250] K.-y. Oda and M. Yamada, *Non-minimal coupling in Higgs–Yukawa model with asymptotically safe gravity*, *Class. Quant. Grav.* **33** (2016) 125011 [1510.03734].

- [251] A. Eichhorn, Y. Hamada, J. Lumma and M. Yamada, *Quantum gravity fluctuations flatten the Planck-scale Higgs potential*, *Phys. Rev. D* **97** (2018) 086004 [1712.00319].
- [252] J.M. Pawłowski, M. Reichert, C. Wetterich and M. Yamada, *Higgs scalar potential in asymptotically safe quantum gravity*, *Phys. Rev. D* **99** (2019) 086010 [1811.11706].
- [253] C. Wetterich and M. Yamada, *Variable Planck mass from the gauge invariant flow equation*, *Phys. Rev. D* **100** (2019) 066017 [1906.01721].
- [254] G.P. De Brito, A. Eichhorn and A.D. Pereira, *A link that matters: Towards phenomenological tests of unimodular asymptotic safety*, *JHEP* **09** (2019) 100 [1907.11173].
- [255] M. Shaposhnikov and C. Wetterich, *Asymptotic safety of gravity and the Higgs boson mass*, *Phys. Lett. B* **683** (2010) 196 [0912.0208].
- [256] M. Reichert and J. Smirnov, *Dark Matter meets Quantum Gravity*, *Phys. Rev. D* **101** (2020) 063015 [1911.00012].
- [257] Y. Hamada, K. Tsumura and M. Yamada, *Scalegenesis and fermionic dark matters in the flatland scenario*, *Eur. Phys. J. C* **80** (2020) 368 [2002.03666].
- [258] A. Eichhorn and M. Pauly, *Safety in darkness: Higgs portal to simple Yukawa systems*, *Phys. Lett. B* **819** (2021) 136455 [2005.03661].
- [259] A. Eichhorn, M. Pauly and S. Ray, *Towards a Higgs mass determination in asymptotically safe gravity with a dark portal*, *JHEP* **10** (2021) 100 [2107.07949].
- [260] V. Braun, Y.-H. He, B.A. Ovrut and T. Pantev, *A Standard model from the  $E(8) \times E(8)$  heterotic superstring*, *JHEP* **06** (2005) 039 [hep-th/0502155].
- [261] V. Braun, Y.-H. He and B.A. Ovrut, *Yukawa couplings in heterotic standard models*, *JHEP* **04** (2006) 019 [hep-th/0601204].
- [262] L.B. Anderson, J. Gray, D. Grayson, Y.-H. He and A. Lukas, *Yukawa Couplings in Heterotic Compactification*, *Commun. Math. Phys.* **297** (2010) 95 [0904.2186].
- [263] L.J. Dixon and J.A. Harvey, *String Theories in Ten-Dimensions Without Space-Time Supersymmetry*, *Nucl. Phys. B* **274** (1986) 93.
- [264] L. Alvarez-Gaume, P.H. Ginsparg, G.W. Moore and C. Vafa, *An  $O(16) \times O(16)$  Heterotic String*, *Phys. Lett. B* **171** (1986) 155.
- [265] D. Tong, *String Theory*, 0908.0333.
- [266] K. Kannike, L. Marzola, M. Raidal and A. Strumia, *Light Higgs boson from multi-phase criticality in dynamical symmetry breaking*, *Phys. Lett. B* **816** (2021) 136241 [2102.01084].
- [267] E.L. Rees, *Graphical discussion of the roots of a quartic equation*, *The American Mathematical Monthly* **29** (1922) 51.
- [268] D. Lazard, *Quantifier elimination: Optimal solution for two classical examples*, *Journal of Symbolic Computation* **5** (1988) 261.
- [269] K. Kannike, *Vacuum Stability of a General Scalar Potential of a Few Fields*, *Eur. Phys. J. C* **76** (2016) 324 [1603.02680].