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CENTRALE LILLE

THESE

Présentée en vue
d'obtenir le grade de

DOCTEUR

En

Spécialité : **Automatique et Productique**

Par

Xinyong WANG

DOCTORAT DELIVRE PAR CENTRALE LILLE

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Soutenue le 14 février 2022 devant le jury d'examen :

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Thèse préparée au Centre de Recherche en Informatique Signal et Automatique de Lille
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Ecole Doctorale SPI 072

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In

Speciality : **Automation control**

By

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Contents

Acknowledgements	i
Contents	iii
Acronyms	v
Notation	vii
Résumé	1
Summary	3
1 Introduction	5
State of the art and objectives	5
1.1 Context and motivation	6
1.2 Continuous-time control for hyperbolic systems	7
1.3 Sampled-data control for parabolic systems	20
1.4 Sampled-data control for hyperbolic systems	27
1.5 Goal of the thesis	41
1.6 Contributions and structure of the thesis	46
2 Hyperbolic system of balance laws with sampled-data control	49
2.1 Introduction	50
2.2 System description and problem formulation	50
2.3 Main result	53

2.4	Numerical simulation	63
2.5	Conclusion	66
2.6	Appendix of Chapter 2	67
3	Hyperbolic system with spatially sampled state measurements	69
3.1	Introduction	70
3.2	System description and problem formulation	70
3.3	Main result	72
3.4	Numerical examples	85
3.5	Conclusion	90
3.6	Appendix of Chapter 3	92
4	Sampled-data hyperbolic systems via an impulsive approach	99
4.1	Introduction	100
4.2	Problem statement	100
4.3	System remodeling	101
4.4	Stability analysis	102
4.5	Numerical example	111
4.6	Conclusion	114
	Conclusion and perspectives	117
	Appendix A Some useful theorems and lemmas	119
	Bibliography	134

Acronyms

- LSS Large-Scale Systems
- LMI Linear Matrix Inequality
- PDE Partial Differential Equation
- ZOH Zero-Order Hold
- KSE Kuramoto-Sivashinsky Equation
- IQC Integral Quadratic Constraint
- ODE Ordinary Differential Equation
- GES Globally Exponentially Stable
- ISS Input-to-State-Stable/Stability
- MASl Maximum Allowable Sampling Interval
- RFDE Retarded Functional Differential Equation
- PI Proportional-Integral
- ACC Adaptive Cruise Control

Notations

Some notations and preliminary definitions used throughout the thesis are as follows:

- \mathbb{N} - the set of nonnegative integers from 0 to infinity.
- \mathbb{R} - the set of reals.
- \mathbb{R}_+^0 - the set of non-negative reals.
- \mathbb{R}_+ - the set of positive reals.
- \mathcal{C}^0 - the space of continuous functions.
- \mathcal{C}^1 - the space of continuously differentiable functions.
- \mathcal{C}^2 - the space of twice continuously differentiable functions.
- \mathcal{C}_{lpw} - the set of all piecewise left continuous functions.
- \mathbb{X} - the set of left continuous and piecewise \mathcal{C}^1 functions.
- \mathbb{R}^n - the set of n -dimensional Euclidean space with the norm $|\cdot|$.
- $|\cdot|$ - the absolute value in \mathbb{R} .
- $M \succeq 0$ - the matrix M is symmetric and positive semidefinite.
- $M \succ 0$ - the matrix M is symmetric and positive definite.

- $M \preceq 0$ - the matrix M is symmetric and negative semidefinite.
- $M \prec 0$ - the matrix M is symmetric and negative definite.
- M^\top - the transpose of the matrix M .
- $L^2(0, L)$ - the Hilbert space of square integrable scalar functions on $(0, L)$ with the corresponding norm $\|\cdot\|_{L^2(0,L)}$, defined by $\|y\|_{L^2(0,L)} = \sqrt{\int_0^L |y(x)|^2 dx}$.
- $\|q(t)\|_A^2$ - for every $q(t) := q(\cdot, t) \in L^2((0, L); \mathbb{R}^n)$, $\|q(t)\|_A^2 = \int_0^L q^\top(t, x) A q(t, x) dx$ for a positive definite matrix A .
- $H^n(0, L)$ - the Sobolev space of absolutely continuous functions $y : (0, L) \rightarrow \mathbb{R}^n$ whose spatial derivatives up to n^{th} order are square integrable, defined by

$$\|y\|_{H^n(0,L)} = \sqrt{\int_0^L (|y(x)|^2 + |y'(x)|^2 + \cdots + |y^{(n)}(x)|^2) dx}.$$

- \mathcal{L}_p - the temporal norm $\|f\|_{\mathcal{L}_p} = \left(\int_0^{+\infty} |f(t)|^p dt\right)^{\frac{1}{p}}$, for $p < +\infty$, and $\|f\|_{\mathcal{L}_p} = \sup_{t \geq 0} |f(t)|$, for $p = +\infty$.
- \mathbf{I} - the identity matrix (of appropriate dimension).
- $\mathbf{M}_{n \times m}$ - for a matrix \mathbf{M} with n rows and m columns, we use the notation $\mathbf{M}_{n \times m}$ to specify the dimension of \mathbf{M} (when relevant).
- $\text{Sym}(A)$ - the symmetric part of matrix A , defined by $\text{Sym}(A) = \frac{1}{2}(A + A^\top)$.
- $\lambda_{\min}(\Theta)$ - the minimum eigenvalue of the matrix Θ .
- $\lambda_{\max}(\Theta)$ - the maximum eigenvalue of the matrix Θ .

-
- $\lceil \cdot \rceil$ - the ceiling function: for $x \in \mathbb{R}$, $\lceil x \rceil = \min\{n \in \mathbb{N} \mid n \geq x\}$.
 - $*$ - the elements that can be inferred by symmetry in a matrix.
 - $\partial_x f \equiv f_x$ - the partial derivative of a function f with respect to the variable x .
 - $\partial_t f \equiv f_t$ - the partial derivative of a function f with respect to the variable t .
 - $\partial_{xt} f \equiv f_{xt} \equiv \frac{\partial^2 f}{\partial_t \partial_x} \equiv \frac{\partial}{\partial_t} \left(\frac{\partial f}{\partial_x} \right)$ - the partial derivative of a function $\frac{\partial f}{\partial_x}$ with respect to the variable t .
 - $\partial_{tx} f \equiv f_{tx} \equiv \frac{\partial^2 f}{\partial_x \partial_t} \equiv \frac{\partial}{\partial_x} \left(\frac{\partial f}{\partial_t} \right)$ - the partial derivative of a function $\frac{\partial f}{\partial_t}$ with respect to the variable x .

Résumé

Dans les systèmes d'ingénierie, la commande mise est généralement en œuvre sous la forme d'un algorithme informatique et une attention particulière doit être portée à la gestion rigoureuse de la mise en œuvre numérique. Pour les systèmes décrits par des Équations aux Dérivées Partielles (en anglais Partial Differential Equations-**PDEs**), ce problème est très difficile car la dynamique se produit dans un espace de dimension infinie et l'échantillonnage induit des discontinuités difficiles à gérer d'un point de vue théorique.

Le problème de l'analyse de stabilité pour une classe de systèmes hyperboliques linéaires avec des lois de contrôle distribuées à données échantillonnées est étudié dans cette thèse. Les **PDEs** hyperboliques considérées sont dotées de lois d'équilibre qui sont utilisées pour représenter la dynamique fondamentale de nombreux systèmes physiques. Les questions suivantes se posent naturellement lorsque l'on pense à la stabilité de tels systèmes dynamiques :

1. Est-il possible de garantir la stabilité d'un système hyperbolique linéaire des lois d'équilibre avec des correcteurs échantillonnés spatio-temporels ?
2. Quelles sont les conditions à vérifier pour garantir la stabilité du système ?

Cette thèse vise à répondre aux questions ci-dessus. Nous proposons des méthodes pour l'étude du système hyperbolique 1-D des lois d'équilibre basées sur les techniques de Lyapunov. En utilisant la technique de Lyapunov-Razumikhin, un cas générique dans lequel le contrôleur est discrétisé à la fois dans le temps et dans l'espace peut être abordé. De plus, nous utilisons une méthode de modélisation impulsive pour obtenir des conditions de stabilité exponentielle globale pour une classe de systèmes hyperboliques linéaires échantillonnés.

Présentons brièvement les principales contributions de cette thèse.

Contributions et structure de la thèse

Ce manuscrit comprend quatre chapitres, ainsi qu'une conclusion avec des perspectives.

1. Une introduction au domaine des systèmes hyperboliques à données échantillonnées est présentée dans le Chapitre 1. Dans ce chapitre, nous présentons le contexte de notre travail, la motivation ainsi que plusieurs résultats liés au contrôle continu aux bords et dans le domaine pour les systèmes hyperboliques. Une revue de la littérature

pour le contrôle à données échantillonnées des systèmes paraboliques et hyperboliques est également fournie. Enfin, nous formalisons les principaux objectifs de la thèse.

2. Dans le Chapitre 2, nous proposons des méthodes pour analyser la stabilité d'une classe de systèmes hyperboliques linéaires avec des correcteurs à données échantillonnées distribuées. On obtient des conditions suffisantes basées sur les techniques de Lyapunov pour la stabilité pratique locale ($R\epsilon$ -stabilité) du système, exprimée sous forme d'inégalités matricielles linéaires (en anglais Linear Matrix Inequalities-**LMIs**).
3. Dans le Chapitre 3, nous étendons les résultats du Chapitre 2 pour les correcteurs qui présentent à la fois la discrétisation temporelle et spatiale, en utilisant une approche basée sur la méthode de Lyapunov-Razumikhin. Nous proposons des conditions suffisantes pour la stabilité pratique locale ($R\epsilon$ -stabilité) pour les systèmes hyperboliques linéaires de lois d'équilibre proposés en reformulant le système original sous une forme entrée-sortie équivalente avec une boucle de contrôle en temps continu et des opérateurs représentant les erreurs de discrétisation (erreurs d'échantillonnage spatio-temporelles). De plus, le cas particulier où l'espace n'est pas discrétisé est également considéré.
4. Dans le Chapitre 4, nous traitons le problème global de stabilité exponentielle pour une classe de systèmes hyperboliques linéaires échantillonnés. En utilisant l'approche impulsive, la relation entre l'intervalle d'échantillonnage, l'état du système et son vecteur échantillonné est caractérisée par une Contrainte Quadratique Intégrale (en anglais Integral Quadratic Constraint-**IQC**). L'**IQC** obtenue est utilisée pour déduire des critères de stabilité numériquement tractable.

Tout au long de la thèse, nous utilisons la méthode des représentations entrées-sorties et la méthode impulsive pour analyser les conditions de stabilité des systèmes hyperboliques avec commande à données échantillonnées. De plus, la méthode de Lyapunov-Razumikhin peut réduire la conservation de l'analyse. La stabilité exponentielle globale peut également être considérée en utilisant un système impulsive augmenté.

Summary

In engineering systems, the control is usually implemented as a computer algorithm and attention has to be paid to rigorously handle the digital implementation. For systems described by *Partial Differential Equations* (**PDEs**), this problem is very challenging since the dynamics occur in an infinite dimensional space and sampling induces discontinuities which are difficult to handle from a theoretical point of view.

The problem of stability analysis for a class of linear hyperbolic systems with distributed sampled-data control laws is investigated in this thesis. The considered **PDEs** are hyperbolic systems of balance laws which are used to represent the fundamental dynamics of many systems in physics. The following questions arise naturally when thinking about the stability of such dynamical systems:

1. Is it possible to guarantee the stability of linear hyperbolic systems of balance laws with spatio-temporal sampled controllers?
2. What are the conditions that have to be checked to guarantee the stability of such systems?

This thesis aims at answering the above questions. We propose methods for the study of 1-D hyperbolic system of balance laws based on Lyapunov techniques. By using the Lyapunov-Razumikhin technique, a generic case in which the controller is discretized both in time and space can be addressed. Moreover, we employ an impulsive modeling method to derive global exponential stability conditions for a class of sampled linear hyperbolic systems.

This manuscript includes four chapters, as well as a conclusion and perspectives.

An introduction to the domain of sampled-data hyperbolic system is presented in Chapter 1. In this chapter, we introduce the context of our work, the motivations as well as several results related to continuous time boundary and in-domain control for hyperbolic systems. A literature review for sampled-data control of parabolic and hyperbolic systems is also given. At last, we formalize the main objectives of the thesis.

In Chapter 2, we provide methods for analysing the stability of a class of linear hyperbolic systems of balance laws with distributed sampled-data controllers. Sufficient conditions based on Lyapunov techniques for the local practical stability ($R\epsilon$ -stability) of the system expressed as *Linear Matrix Inequalities* (**LMIs**) are obtained.

In Chapter 3, we extend the results from Chapter 2 for controllers with both time and space discretization based on the Lyapunov-Razumikhin method. We introduce sufficient conditions for the local practical stability ($R\epsilon$ -stability) of the proposed linear hyperbolic systems of balance laws by recasting the original system into an equivalent input-output form with a continuous time control loop and operators representing the discretization errors (spatio-temporal sampling errors). In addition, the special case where the space is not discretized is also considered.

In Chapter 4, we deal with the global exponential stability problem for a class of sampled linear hyperbolic systems. By employing the impulsive approach, the relation between the sampling interval, the system state and its sampled vector is characterized by an *Integral Quadratic Constraint (IQC)*. The acquired **IQC** is utilized to derive numerically tractable stability criteria.

Throughout the thesis, we use the input-output and the impulsive modeling methods to provide stability conditions for the hyperbolic systems with sampled-data control. In addition, the Lyapunov-Razumikhin method can reduce the conservatism of the analysis methods. The global exponential stability can also be considered by using an augmented impulsive system.

Introduction

Contents

1.1	Context and motivation	6
1.2	Continuous-time control for hyperbolic systems	7
1.2.1	Hyperbolic systems	8
1.2.2	Basic stability conditons	9
1.2.3	Boundary control	10
1.2.4	Examples of in-domain control problems	17
1.3	Sampled-data control for parabolic systems	20
1.3.1	Semilinear diffusion equations	21
1.3.2	Reaction-advection-diffusion equation	23
1.3.3	1-D Kuramoto-Sivashinsky equation	25
1.4	Sampled-data control for hyperbolic systems	27
1.4.1	Periodic and aperiodic boundary control	27
1.4.2	Event-based boundary control	30
1.4.3	Distributed sampled-data control	36
1.5	Goal of the thesis	41
1.6	Contributions and structure of the thesis	46

General introduction

This thesis is devoted to the stability analysis for a class of linear hyperbolic system with distributed sampled-data control. We propose methods for the study of 1-D hyperbolic system with balance laws based on Lyapunov techniques. A generic case in which the controller is discretized both in time and space can be addressed by using the Lyapunov-Razumikhin technique. Furthermore, we employ an impulsive modelling method to derive global exponential stability conditions for a class of sampled linear hyperbolic systems. This chapter is dedicated to an introduction to the domain of sampled-data hyperbolic system. In the following sections, we introduce the context of our work, the motivation as well as a literature review for sampled-data control of parabolic and hyperbolic systems, respectively. At last, we formalize the main objectives of the thesis.

1.1 Context and motivation

In real life, due to the increasing complexity of controlled engineering systems, *Large-Scale Systems (LSS)* are frequently encountered in many fields, such as cyber-physical systems, power engineering, biology systems, micro-electro mechanics and so on. Modelling of these phenomena often leads to very high-order systems consisting of sets of interconnected dynamics. Such systems are widely distributed in space and the subsystems are interdependent, which poses huge challenges to analysis, design, and control problems. The study of such systems is very important because even small improvements in **LSS** operation can bring substantial savings and significant economic effects.

PDEs provide a stylish way to handle **LSS** in various domains [82, 131, 70, 4, 89]: the flow of rivers [82], the vibrations of beams [131], the road traffic [4], etc. When controlling such systems, the use of communications and computational resources is ubiquitous. The control is usually implemented as a computer algorithm and attention has to be paid to rigorously handle the digital implementation. Sensor and actuator data is communicated over digital channels with finite communication capacity. It is important to take into account such limitations. Therefore, extending the *Sampled-Data Control Theory* to **PDEs** is nowadays a very active topic [79, 65]. An important problem in this context is to provide methods for estimating the *Maximum Allowable Sampling Interval (MASI)* for which the sampled-data implementation preserves the characteristics of the continuous-time controller. For **PDEs**, this problem is very challenging since the dynamics occur in an infinite dimensional space and sampling induces discontinuities which are difficult to handle from a theoretical point of view.

Roughly speaking, the study of sampled-data systems can be carried out using four

basic types of models (see [51] for a survey): discrete-time [61], time-delay [109], input-output [42] and hybrid [99] models. A large number of results are available in the case of sampled-data *Ordinary Differential Equations* (ODEs) [57]. Compared to the ODE case, fewer results are suitable for the case of PDEs. For PDEs with distributed sampled-data control, we can mention the results in [105, 107, 37] and [60] where the case of parabolic systems was studied using the time-delay method.

For hyperbolic systems, results for sampled-data control are rare. Event-triggered control was studied in [31] for linear conservation laws and in [32] for systems with quantization. A backstepping approach for event-triggered control was developed in [30]. In [22] the stability analysis problem was investigated for hyperbolic PDEs via a looped-functional method. See also [63] for linear transport PDEs with non-local terms. Nevertheless, most of the existing papers address cases of boundary sampled-data control. Despite the frequent occurrence of distributed sampled-data controllers in practical applications, the results in this context are sparse and only particular cases of hyperbolic system have been addressed in the literature [121, 119]. In [121] a sampled-data observer was studied for a one-dimensional semi-linear wave equation. Therein, sufficient conditions for the stability of the estimation error are derived by using the time-delay approach. A network-based distributed controller was designed for the damped semi-linear beam equation in [119].

As follows, before presenting more details in several existing recent works on sampled-data control of PDEs, we first introduce some results concerning the continuous time (boundary and in-domain) control of hyperbolic systems (Section 1.2). The remaining presentation consists of 4 sections. Section 1.3 presents some results concerning the distributed sampled-data control of parabolic systems, which have impact for the study of hyperbolic systems. Section 1.4 is concerned with the description of sampled-data control analysis methods for hyperbolic systems, including boundary sampled-data control, event-triggered boundary control and distributed sampled-data control. We mainly give the problem formulation and stability conditions in the existing papers. Then in Section 1.5 we present our main goal as a result of the literature review.

1.2 Continuous-time control for hyperbolic systems

In this section, we will present several results related to continuous time boundary and in-domain control for hyperbolic systems. We first introduce briefly the hyperbolic systems under study.

1.2.1 Hyperbolic systems

Among various classes of systems described by infinite dimensional systems and **PDEs**, hyperbolic **PDEs** stand out as relevant in various domains. They have been widely used to model the dynamics of physical systems: hydrodynamical models for semiconductors [1], shallow waters [93], traffic flows [17], age-dependent epidemic models [66], etc. In all of these applications described above, although natural dynamics are three-dimensional, dynamics can be effectively expressed by the one-dimensional hyperbolic balance laws, since the main phenomena evolve in a privileged coordinate dimension and the phenomena in other directions are negligible. This motivates the kernel of this thesis to be the 1-D hyperbolic systems with balance laws. Referring to [4] and the references therein, we give a brief mathematical background to hyperbolic systems in the sequel. In this thesis, we focus on a class of linear hyperbolic systems of balance laws described by the following **PDE**:

$$y_t(x, t) + \Lambda y_x(x, t) + \Upsilon y(x, t) = 0, x \in [0, L], t \in [0, +\infty), \quad (1.1)$$

where $y : [0, L] \times [0, +\infty) \rightarrow \mathbb{R}^n$, Υ is a constant matrix in $\mathbb{R}^{n \times n}$, Λ is a diagonal matrix in $\mathbb{R}^{n \times n}$ and defined as

$$\Lambda = \begin{pmatrix} \Lambda^+ & 0 \\ 0 & -\Lambda^- \end{pmatrix}$$

with

$$\begin{aligned} \Lambda^+ &= \text{diag} \{ \lambda_1, \dots, \lambda_m \}, & \lambda_i &> 0 \ \forall i = 1, \dots, m. \\ \Lambda^- &= \text{diag} \{ \lambda_{m+1}, \dots, \lambda_n \}, \end{aligned}$$

The boundary condition is defined as

$$\begin{pmatrix} y^+(0, t) \\ y^-(L, t) \end{pmatrix} = G \begin{pmatrix} y^+(L, t) \\ y^-(0, t) \end{pmatrix}, t \in [0, +\infty) \quad (1.2)$$

where G is a constant matrix in $\mathbb{R}^{n \times n}$ and the initial condition is defined as

$$y(x, 0) = y_0(x), x \in [0, L]. \quad (1.3)$$

System (1.1)-(1.2) can be related to system

$$y_t(x, t) + \Lambda y_x(x, t) + M y(x, t) + u(x, t) = 0, x \in [0, L], t \in [0, +\infty) \quad (1.4)$$

with boundary condition (1.2) and in-domain control

$$u(x, t) = K y(x, t), \quad (1.5)$$

or to system (1.1) subject to boundary condition

$$\begin{pmatrix} y^+(0, t) \\ y^-(L, t) \end{pmatrix} = B \begin{pmatrix} y^+(L, t) \\ y^-(0, t) \end{pmatrix} + u(x, t), t \in [0, +\infty) \quad (1.6)$$

with boundary control

$$u(x, t) = K \begin{pmatrix} y^+(L, t) \\ y^-(0, t) \end{pmatrix}, \quad (1.7)$$

where M, B, K are constant matrices in $\mathbb{R}^{n \times n}$.

Controlling and stabilizing these systems is an important and fundamental issue in the field of control theory.

1.2.2 Basic stability conditons

In the following, we present the definition of exponential stability in L^2 -norm, which is useful in the subsequent theorems.

Definition 1.1 (Bastin 2016 [4])

System (1.1)-(1.2) is exponentially stable in L^2 -norm if there exist $\nu > 0$ and $C > 0$ such that, for every $y_0 \in L^2((0, L); \mathbb{R}^n)$, the L^2 -solution of the system (1.1)-(1.3) satisfies

$$\|y(\cdot, t)\|_{L^2((0, L); \mathbb{R}^n)} \leq C e^{-\nu t} \|y_0\|_{L^2((0, L); \mathbb{R}^n)}, \quad \forall t \in [0, +\infty).$$

Based on the early stability results in [101, 103], the authors in [4] adopted a L^2 Lyapunov function candidate:

$$V(y) = \int_0^L y^\top Q(x) y dx$$

with

$$\begin{aligned} Q(x) &\triangleq \text{diag} \{Q^+(x), Q^-(x)\}, \\ Q^+ &\in \mathcal{C}^1([0, L]; \mathbb{R}^{m \times m}), \\ Q^- &\in \mathcal{C}^1([0, L]; \mathbb{R}^{(n-m) \times (n-m)}). \end{aligned} \quad (1.8)$$

From the time derivative of function V along the solutions of the system (1.1)-(1.3), the following stability result is obtained.

Proposition 1.1 (Bastin 2016 [4])

The solution $y(x, t)$ of the system (1.1)-(1.3) exponentially converges to 0 in L^2 -norm if there exists a map Q satisfying (1.8) such that the following (LMIs) hold:

$$1. \quad \begin{pmatrix} Q^+(L)\Lambda^+ & 0 \\ 0 & Q^-(0)\Lambda^- \end{pmatrix} - G^\top \begin{pmatrix} Q^+(0)\Lambda^+ & 0 \\ 0 & Q^-(L)\Lambda^- \end{pmatrix} G \succeq 0$$

$$2. \quad -(Q(x)\Lambda)_x + Q(x)\Upsilon + \Upsilon^\top Q(x) \succ 0, \forall x \in [0, L].$$

In the special case where

$$m = n, \Lambda = \Lambda^+ \in \mathbb{R}^{n \times n}$$

and

$$Q(x) \triangleq P(\mu x) = \text{diag} \{p_i e^{-\mu x}\}, \quad p_i > 0 \text{ for } i = 1, \dots, n, \quad (1.9)$$

the following theorem is a particular case of Proposition 1.1 with an appropriate choice of the weighting matrix $Q(x)$ characterizing the Lyapunov function.

Theorem 1.1 (Bastin 2016 [4])

If there exists $P \in \mathbb{R}^{n \times n}$ satisfying (1.9) such that

$$\Upsilon^\top P + P\Upsilon \succeq 0,$$

and

$$\|\Delta G \Delta^{-1}\| < 1$$

with $\Delta \triangleq \sqrt{P|\Lambda|}$, then the system (1.1)-(1.3) is exponentially stable in L^2 -norm.

This result in [4] is an important stepping stone for various control design methods.

1.2.3 Boundary control

In this subsection, we present some results related to boundary control for 1-D hyperbolic systems. The asymptotic stability of a class of time-varying hyperbolic **PDEs** was studied in [100] by using *Input-to-State-Stable (ISS)*-Lyapunov functions. In [130], the exponential stability for a class of linear hyperbolic systems with relaxation was considered. *Proportional-Integral (PI)* boundary feedback control was introduced for hyperbolic systems in [137].

1.2.3.1 Time-varying hyperbolic PDEs

For time-varying hyperbolic **PDEs**, the authors in [100] have designed **ISS**-Lyapunov functions through a strictification approach. They have ensured the asymptotic stability for the general case, as well as robustness properties with respect to the additive perturbations. The following linear **PDEs** was considered

$$\frac{\partial y}{\partial t}(z, t) + \Lambda(z, t) \frac{\partial y}{\partial z}(z, t) = F(z, t)y(z, t) + \delta(z, t) \quad (1.10)$$

with boundary conditions

$$\begin{pmatrix} y_+(0, t) \\ y_-(L, t) \end{pmatrix} = K \begin{pmatrix} y_+(L, t) \\ y_-(0, t) \end{pmatrix} \quad (1.11)$$

and the initial condition

$$y(z, 0) = y_0(z), \forall z \in (0, L), \quad (1.12)$$

where $z \in [0, L], t \in [0, +\infty)$ and $\Lambda(z, t) = \text{diag}\{\lambda_1(z, t), \dots, \lambda_n(z, t)\}$ is a diagonal matrix in $\mathbb{R}^{n \times n}$ whose m first diagonal terms are nonnegative and the $n - m$ last diagonal terms are nonpositive. $y = [y_+, y_-]^\top, y_+ \in \mathbb{R}^m, y_- \in \mathbb{R}^{n-m}$ and $K \in \mathbb{R}^{n \times n}$ is a constant matrix representing a boundary control law. The function δ is a disturbance of class \mathcal{C}^1, F is a periodic function with respect to t of period T and of class \mathcal{C}^1, Λ is a function of class \mathcal{C}^1 , periodic with respect to t of period T . X_0 is a function $[0, L] \rightarrow \mathbb{R}^n$ of class \mathcal{C}^1 .

Since $m < n$, $y(z, t)$ can be replaced by $[y_+(z, t), y_-(L - z, t)]^\top$, then the boundary conditions (1.11) becomes

$$y(0, t) = Ky(L, t). \quad (1.13)$$

The initial condition (1.12) satisfies the following zero-order compatibility condition:

$$\begin{pmatrix} y_+^0(0) \\ y_-^0(L) \end{pmatrix} = K \begin{pmatrix} y_+^0(L) \\ y_-^0(0) \end{pmatrix}$$

and the following first-order compatibility condition:

$$= K \begin{pmatrix} -\lambda_1(0, 0) \frac{dX_1^0}{dz}(0) + (F(0, 0)X^0(0) + \delta(0, 0))_1 \\ \vdots \\ -\lambda_m(0, 0) \frac{dX_m^0}{dz}(0) + (F(0, 0)X^0(0) + \delta(0, 0))_m \\ -\lambda_{m+1}(L, 0) \frac{dX_{m+1}^0}{dz}(L) + (F(L, 0)X^0(L) + \delta(L, 0))_{m+1} \\ \vdots \\ -\lambda_n(L, 0) \frac{dX_n^0}{dz}(L) + (F(L, 0)X^0(L) + \delta(L, 0))_n \\ -\lambda_1(L, 0) \frac{dX_1^0}{dz}(L) + (F(L, 0)X^0(L) + \delta(L, 0))_1 \\ \vdots \\ -\lambda_m(L, 0) \frac{dX_m^0}{dz}(L) + (F(L, 0)X^0(L) + \delta(L, 0))_m \\ -\lambda_{m+1}(0, 0) \frac{dX_{m+1}^0}{dz}(0) + (F(0, 0)X^0(0) + \delta(0, 0))_{m+1} \\ \vdots \\ -\lambda_n(0, 0) \frac{dX_n^0}{dz}(0) + (F(0, 0)X^0(0) + \delta(0, 0))_n \end{pmatrix}.$$

Based on [84], the authors introduced the following assumption.

Assumption 1.1 (Priour 2012 [100])

For all $t \geq 0$ and for all $z \in [0, L]$, all the entries of the diagonal matrix $\Lambda(z, t)$ are nonnegative. There exist a symmetric positive definite matrix Q , a real number $\alpha \in (0, 1)$, a continuous real-valued function r , periodic of period $T > 0$ such that the constant

$$B = \int_0^T \left[\frac{\max\{r(m), 0\}}{\|Q\|} + \frac{\min\{r(m), 0\}}{\lambda_Q} \right] dm$$

where λ_Q is the smallest positive eigenvalue of Q . Moreover, for all $t \geq 0$ and for all $z \in [0, L]$, the following inequalities:

$$\begin{aligned} \text{Sym}(\alpha Q \Lambda(L, t) - K^\top Q \Lambda(0, t) K) &\geq 0, \\ \text{Sym}(Q \Lambda(z, t)) &\geq r(t) \mathbf{I}, \\ \text{Sym} \left(Q \frac{\partial \Lambda}{\partial z}(z, t) + 2Q F(z, t) \right) &\leq 0 \end{aligned}$$

are satisfied.

In the light of the above assumption, the author constructed an **ISS**-Lyapunov function to analyze the robustness of a linear time-varying hyperbolic **PDE** with uncertainty. The results are summarized in the following theorem.

Theorem 1.2 (Priour 2012 [100])

Assume the system (1.10) with the boundary conditions (1.13) satisfies Assumption 1.1. Let μ be any real number such that

$$0 < \mu \leq -\frac{1}{L} \ln(\alpha).$$

Then the function $U : L^2(0, L) \times \mathbb{R} \rightarrow \mathbb{R}$ defined, for all $\phi \in L^2(0, L)$ and $t \in \mathbb{R}$, by

$$U(\phi, t) = \exp \left(\frac{1}{T} \int_{t-T}^t \int_\ell^t q(m) dm d\ell \right) \int_0^L \phi(z)^\top Q \phi(z) e^{-\mu z} dz$$

where q is defined as

$$q(t) = \mu \left[\frac{\max\{r(t), 0\}}{\|Q\|} + \frac{\min\{r(t), 0\}}{\lambda_Q} \right] - \frac{\mu B}{2T},$$

is an **ISS**-Lyapunov function for the system (1.10) with the boundary conditions (1.13).

This is a very generic result with potential to several class of 1-D hyperbolic systems of balance laws.

In [100], the authors applied their result to two problems. The first one belongs to a benchmark hyperbolic equation, the second one pertains Saint-Venant-Exner equations. It will be interesting to experimentally verify the prediction of the offset inferred from the proposed **ISS**-Lyapunov function. Recently, the authors in [130] studied a linear hyperbolic system of balance laws with boundary disturbances in one dimension by using an explicit **ISS**-Lyapunov function in L^2 -norm. In addition, the numerical discretization and stabilization of a balance law with boundary disturbance were also discussed in [130].

1.2.3.2 Hyperbolic systems with relaxation

Boundary stabilization for 1-D linear hyperbolic systems with relaxation was considered in [50]. This seems to be the first place where the structure is explicitly used to demonstrate exponential stability. Inspired by [24] and [133], the authors in [50] studied a 1-D linear system

$$y_t(t, x) + Ay_x(t, x) + By(t, x) = 0 \quad (1.14)$$

where $x \in [0, 1]$ and $t \geq 0$, $y = [u, q]^\top$, $u : [0, \infty) \times [0, 1] \rightarrow \mathbb{R}^{n-r}$, $q : [0, \infty) \times [0, 1] \rightarrow \mathbb{R}^r$, $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{n \times n}$, $B := \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathbb{R}^{n \times n}$, $e \in \mathbb{R}^{r \times r}$.

With respect to system (1.14), the authors used the following two assumptions.

Assumption 1.2 (Herty 2016 [50])

There exists a symmetric positive definite matrix $A_0 \in \mathbb{R}^{n \times n}$ such that

$$A_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is symmetric and } A_0 = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$$

with $X_1 \in \mathbb{R}^{(n-r) \times (n-r)}$ and $X_2 \in \mathbb{R}^{r \times r}$.

Assumption 1.3 (Herty 2016 [50])

The matrix

$$X_2 e + e^\top X_2$$

is positive definite.

Assumptions 1.2 and 1.3 are exactly the structural stability conditions proposed in [134] for the general system

$$Y_t + AY_x = QY.$$

The coefficient matrix A can be diagonalized with a transformation matrix $T \in \mathbb{R}^{n \times n}$ such that

$$T^{-1}AT = \Lambda, \Lambda := \begin{pmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{pmatrix}, \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = T^{-1}Y, \quad (1.15)$$

where Λ_{\pm} are diagonal and contain the positive and negative eigenvalues of A , the boundary conditions are specified as

$$\begin{aligned} \xi_+(t, 0) &= K_{00}\xi_+(t, 1), \xi_+ \in \mathbb{R}^m, \\ \xi_-(t, 1) &= K_{11}\xi_-(t, 0), \xi_- \in \mathbb{R}^{n-m}. \end{aligned} \quad (1.16)$$

Further, equation (1.14) is accompanied by appropriate initial conditions

$$y(0, x) = y_0(x). \quad (1.17)$$

The main result of [50] on exponential stability was stated by the following theorem.

Theorem 1.3 (Herty 2016 [50])

Suppose the system (1.14) fulfills the assumptions 1.2 and 1.3. Then there exist K_{00} and K_{11} such that the system (1.14) together with (1.16) and (1.17) is exponentially stable.

In order to prove Theorem 1.3, a suitable Lyapunov function was given below

$$\begin{aligned} V(t) &= \int_0^1 Y^\top (\alpha A_0 + \mu(x)) Y dx \\ &= \alpha \|(u, q)(t)\|_{A_0}^2 + \|(u, q)(t)\|_{\mu}^2 \end{aligned}$$

for some $\alpha > 0$ and a family of matrices $\mu(x) \in \mathbb{R}^{n \times n}$ given by

$$\mu(x) := (T^{-1})^\top e^{(-\Lambda x)} T^{-1}$$

for $x \in [0, 1]$ and T and Λ given by equation (1.15).

Instead of requiring the source term to be marginally diagonally stable as in [24], the authors applied a different method to explore the relaxation construction. This result was applied to the boundary stabilization problem for water flows in open canals.

1.2.3.3 Hyperbolic systems under PI boundary control

The **PI** boundary feedback control for the class of linear hyperbolic systems of balance laws has been investigated in [137]. The authors developed exponential stability conditions

of closed-loop systems by means of a new weighted Lyapunov function. The work focused on the 1-D $n \times n$ linear hyperbolic systems as shown below

$$y_t + \Lambda y_x = My, t \in [0, \infty), x \in (0, L) \quad (1.18)$$

where $y : (0, L) \times [0, \infty) \rightarrow \mathbb{R}^n$, $\Lambda \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{R}^{n \times n}$. $\Lambda = \text{diag}\{\Lambda^+, -\Lambda^-\}$, $\Lambda^+ = \text{diag}\{\lambda_1, \dots, \lambda_m\}$, $\Lambda^- = \text{diag}\{\lambda_{m+1}, \dots, \lambda_n\}$, $1 \leq m \leq n$ and $\lambda_i > 0$, $\forall i \in \{1, \dots, n\}$.

The input and the output of the system (1.18) on the left and the right boundaries were defined by

$$y_{\text{in}}(t) \triangleq \begin{bmatrix} y^+(0, t) \\ y^-(L, t) \end{bmatrix}, \quad y_{\text{out}}(t) \triangleq \begin{bmatrix} y^+(L, t) \\ y^-(0, t) \end{bmatrix}$$

where $y^+ = [y_1, \dots, y_m]^\top \in \mathbb{R}^m$ and $y^- = [y_{m+1}, \dots, y_n]^\top \in \mathbb{R}^{n-m}$.

In [137], the authors considered the following type **PI** feedback control

$$y_{\text{in}}(t) = K_p y_{\text{out}}(t) + K_I \int_0^t y_{\text{out}}(\tau) d\tau \quad (1.19)$$

where $K_p \in \mathbb{R}^{n \times n}$ and $K_I \in \mathbb{R}^{n \times n}$ are matrix gains.

The initial condition was considered of the following form

$$y(x, 0) = y^0(x), \quad x \in (0, L).$$

Theorem 1.4 (Zhang 2019 [137])

The linear hyperbolic system (1.18)-(1.19) is exponentially stable in L^2 -norm if there exist a diagonal matrix $P_1 \in \mathbb{R}^{n \times n}$, a symmetric matrix $P_2 \in \mathbb{R}^{n \times n}$ and a matrix $P_3 \in \mathbb{R}^{n \times n}$, a real constant μ , such that the following matrix inequalities hold, for all $x \in [0, L]$,

$$P = \begin{bmatrix} P_1 & P_3 \\ * & P_2 \end{bmatrix} > 0,$$

$$\Omega^e(x) = \begin{bmatrix} \Omega_{11}^e(x) & \Omega_{12}^e(x) & P_3(x) \\ * & \Omega_{22}^e(x) & \Omega_{23}^e(x) \\ * & * & \Omega_{33}^e(x) \end{bmatrix} < 0$$

where

$$\Omega_{11}^e(x) = M^\top P_1(x) + P_1(x)M - 2\mu|\Lambda|P_1(x),$$

$$\Omega_{12}^e(x) = M^\top P_3(x) - \mu|\Lambda|P_3(x),$$

$$\Omega_{22}^e = \frac{1}{L} (K_I^\top |\Lambda| P_1 E_1 K_I + K_I^\top |\Lambda| P_3 E_3 + E_3 P_3^\top |\Lambda| K_I),$$

$$\Omega_{23}^e = \frac{1}{L} (K_I^\top |\Lambda| P_1 E_1 K_p - |\Lambda| P_3 E_4 + E_3 P_3^\top |\Lambda| K_p) + P_2,$$

$$\Omega_{33}^e = \frac{1}{L} (K_p^\top |\Lambda| P_1 E_1 K_p - |\Lambda| P_1 E_2)$$

with

$$P_1(x) = P_1 \operatorname{diag} \{e^{2\mu x} I_m, e^{2\mu x} I_{n-m}\},$$

$$P_3(x) = P_3 \operatorname{diag} \{e^{-\mu x} I_m, e^{\mu x} I_{n-m}\},$$

$$|\Lambda| = \operatorname{diag} \{\Lambda^+, \Lambda^-\},$$

$$E_1 = \operatorname{diag} \{I_m, e^{2\mu L} I_{n-m}\},$$

$$E_2 = \operatorname{diag} \{e^{-2\mu L} I_m, I_{n-m}\},$$

$$E_3 = \sqrt{E_1}, \quad E_4 = \sqrt{E_2}.$$

In order to prove the exponential stabilization of system (1.18)-(1.19) with the **PI** controller, the authors constructed the following candidate Lyapunov function $V : L^2((0; L); \mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}$

$$\begin{aligned} V(y, \zeta) &= \int_0^L [y^\top P_1(x)y + y^\top P_3(x)\zeta + \zeta^\top P_3^\top(x)y] dx + L\zeta^\top P_2\zeta \\ &= \int_0^L \begin{bmatrix} y \\ \zeta \end{bmatrix}^\top \begin{bmatrix} P_1(x) & P_3(x) \\ * & P_2 \end{bmatrix} \begin{bmatrix} y \\ \zeta \end{bmatrix} dx \end{aligned}$$

where $\zeta : [0, \infty) \rightarrow \mathbb{R}$ is the integral of the output of the system (1.18), i.e.,

$$\zeta(t) = \int_0^t y_{\text{out}}(\tau) d\tau.$$

Their results were illustrated by the linearized Aw-Rascle-Zhang traffic flow model. The issue of feedback stabilization and disturbance rejection under **PI** boundary control for a class of density-flow systems has been addressed in [5], which provided explicit necessary and sufficient stability conditions in the frequency domain.

1.2.4 Examples of in-domain control problems

Most of the contributions of hyperbolic equation control design involve boundary control rather than in-domain control. For quasi-linear first-order hyperbolic systems (such as convective reaction processes) and parabolic systems (such as diffusion reaction processes), control methods that directly consider the process distribution characteristics have been developed [15]. Next, we will present two particular applications considering the stabilization of hyperbolic systems with in-domain control.

1.2.4.1 Freeway traffic flow

For traffic consisting of both *Adaptive Cruise Control-equipped* (**ACC**-equipped) and manual vehicles, the problem of feedback control design for stabilization of traffic flow in congested regime by time-gap manipulation has been addressed in [6]. The following 2×2 nonlinear first-order hyperbolic **PDEs** with in-domain actuation was considered

$$\begin{aligned}
\rho_t(x, t) &= -\rho_x(x, t)v(x, t) - \rho(x, t)v_x(x, t), \\
v_t(x, t) &= -\left(v(x, t) + \rho(x, t)\frac{\partial V_{\text{mix}}(\rho(x, t), h_{\text{acc}}(x, t))}{\partial \rho}\right), \\
&\quad \times v_x(x, t) + \frac{V_{\text{mix}}(\rho(x, t), h_{\text{acc}}(x, t)) - v(x, t)}{\tau_{\text{mix}}}, \\
q_{\text{in}} &= \rho(0, t)v(0, t), \\
v_t(D, t) &= \frac{V_{\text{mix}}(\rho(D, t), h_{\text{acc}}(D, t)) - v(D, t)}{\tau_{\text{mix}}}
\end{aligned} \tag{1.20}$$

where

$$\begin{aligned}
V_{\text{mix}}(\rho, h_{\text{acc}}) &= \tau_{\text{mix}} \left(\frac{\alpha}{\tau_{\text{acc}}} V_{\text{acc}}(\rho, h_{\text{acc}}) + \frac{1-\alpha}{\tau_{\text{m}}} V_{\text{m}}(\rho) \right) \\
V_{\text{acc}}(\rho, h_{\text{acc}}) &= \frac{1}{h_{\text{acc}}} \left(\frac{1}{\rho} - L \right), \quad \rho_{\text{min}} < \rho < \frac{1}{L} \\
V_{\text{m}}(\rho) &= \frac{1}{h_{\text{m}}} \left(\frac{1}{\rho} - L \right), \quad \rho_{\text{min}} < \rho < \frac{1}{L} \\
\tau_{\text{mix}} &= \frac{1}{\frac{\alpha}{\tau_{\text{acc}}} + \frac{1-\alpha}{\tau_{\text{m}}}}.
\end{aligned}$$

Here ρ is the traffic density, $x \in [0, D]$ is the spatial variable, $D > 0$ is the length of a freeway, $t \geq 0$ is the time, $0 < v \leq v_f$ is the traffic speed, where v_f is the free-flow speed, $L > 0$ is the average effective length of each vehicle, $\alpha \in [0, 1]$ is the percentage of **ACC**-equipped vehicles with respect to total vehicles, $\rho_{\min} > 0$ is the lowest value for density for which the model is accurate. $q_{\text{in}} > 0$ is a constant external inflow, $\tau_{\text{acc}}, \tau_{\text{m}} > 0$ are the time constants of the **ACC**-equipped and manual vehicles, respectively, $h_{\text{m}} > 0$ is the time-gap of manual vehicles and $h_{\text{acc}} > 0$ is the time-gap of **ACC**-equipped vehicles (which is the control input).

By linearizing and diagonalizing the system, the authors designed the following in-domain controller

$$\begin{aligned} h_{\text{acc}}(x, t) &= \bar{h}_{\text{acc}} + \frac{1}{c_3} \left(-c_1 e^{-\frac{c_2 x}{\bar{v}}} \tilde{z}(x, t) + k \tilde{v}(x, t) \right) \\ &= \bar{h}_{\text{acc}} + \frac{1}{c_3} \left(-c_1 \bar{\rho}(x, t) + (k - c_2) \tilde{v}(x, t) \right) \end{aligned} \quad (1.21)$$

to get

$$\tilde{z}_t(x, t) + \bar{v} \tilde{z}_x(x, t) = c_2 \tilde{z}(x, t) - k e^{\frac{c_2 x}{\bar{v}}} \bar{h}_{\text{mix}} \bar{\rho}^2 \tilde{v}(x, t), \quad (1.22)$$

$$\tilde{v}_t(x, t) - c_4 \tilde{v}_x(x, t) = -k \tilde{v}(x, t), \quad (1.23)$$

$$\tilde{z}(0, t) = -L \frac{\bar{\rho}^2}{\bar{v}} \tilde{v}(0, t), \quad (1.24)$$

$$\tilde{v}_t(D, t) = -k \tilde{v}(D, t) \quad (1.25)$$

where \bar{h}_{acc} and $\bar{\rho}$ are steady-state time-gap and steady density for **ACC**-equipped vehicles, $\bar{v} = q_{\text{in}}/\bar{\rho}$, $k > 0$ being arbitrary, $c_1 = \frac{1}{\bar{\rho}^2 \tau_{\text{mix}} h_{\text{mix}}}$, $c_2 = \frac{1}{\tau_{\text{mix}}}$, $c_3 = \frac{\alpha}{\tau_{\text{acc}} h_{\text{acc}}^2} \left(\frac{1}{\bar{\rho}} - L \right)$, $c_4 = \frac{L}{h_{\text{mix}}}$, $\tilde{z}(x) = e^{\frac{c_2 x}{\bar{v}}} (\bar{\rho}(x) + \bar{h}_{\text{mix}} \bar{\rho}^2 \tilde{v}(x))$, $\tilde{v} = v - \bar{v}$, $\bar{\rho} = \rho - \bar{\rho}$.

System (1.22)-(1.23) has a form which is similar to (1.1). Applying an input-output approach, the following result was established.

Theorem 1.5 (Bekiaris 2020 [6])

System (1.23), (1.25) is \mathcal{L}_p , $p \in [1, \infty]$, convectively stable in the sense that for any $0 \leq x_2 < x_1 \leq D$ such that $\tilde{v}(x_1) \in \mathcal{L}_p$ and $\tilde{v}_x(x_1) \in \mathcal{L}_p$, the following hold

$$\|\tilde{v}(x_2)\|_{\mathcal{L}_p} < \|\tilde{v}(x_1)\|_{\mathcal{L}_p},$$

$$\|\tilde{v}_x(x_2)\|_{\mathcal{L}_p} < \|\tilde{v}_x(x_1)\|_{\mathcal{L}_p}.$$

The control design method proposed by the authors was used to stabilize traffic flow in congested conditions. The closed-loop system has been proven to be convectively stable under the developed control law. In the future, it will be interesting to consider the problem

of nonlinear feedback control design. In addition, since the vehicles have decentralized actuators and sampling phenomenon cannot be ignored. It would be interested to study the sampling implementation of the controller (1.21).

1.2.4.2 Nonisothermal plug-flow reactor

In [16], the authors presented an output feedback control methodology which was implemented on a nonisothermal plug-flow reactor through simulations. The nonisothermal plug-flow reactor is modeled by three quasi-linear hyperbolic **PDEs**:

$$\begin{aligned}
\frac{\partial C_A}{\partial t} &= -v_l \frac{\partial C_A}{\partial z} - k_{10} e^{\frac{-E_1}{RT_r}} C_A, \\
\frac{\partial C_B}{\partial t} &= -v_l \frac{\partial C_B}{\partial z} + k_{10} e^{\frac{-E_1}{RT_r}} C_A - k_{20} e^{\frac{-E_2}{RT_r}} C_B, \\
\frac{\partial T_r}{\partial t} &= -v_l \frac{\partial T_r}{\partial z} + \frac{(-\Delta H_{r_1})}{\rho_m c_{pm}} k_{10} e^{\frac{-E_1}{RT_r}} C_A + \frac{(-\Delta H_{r_2})}{\rho_m c_{pm}} k_{20} e^{\frac{-E_2}{RT_r}} C_B \\
&\quad + \frac{U_w}{\rho_m c_{pm} V_r} (T_j - T_r)
\end{aligned} \tag{1.26}$$

subject to the following boundary conditions:

$$C_A(0, t) = C_{A0}, \quad C_B(0, t) = 0, \quad T_r(0, t) = T_{A0}$$

where k_{10}, k_{20}, E_1, E_2 denote the kinetic constants and the activation energies of the reactions, v_l is the superficial fluid velocity, R is the ideal gas constant, C_A, C_B denote the concentrations of the species A and B in the reactor and T_r denotes the temperature of the reactor. $\Delta H_{r_1}, \Delta H_{r_2}$ denote the enthalpies of the two reactions, ρ_m, c_{pm} denote the density and heat capacity of the fluid in the reactor, V_r denotes the volume of the reactor, U_w denotes the heat transfer coefficient and T_j denotes the spatially uniform temperature in the jacket.

The control objective is the regulation of the concentration of the product species B throughout the reactor by manipulating the jacket temperature T_j . In practice, T_j is usually operated indirectly through control of the jacket inlet flow rate. Setting:

$$u = T_j - T_{j_s}, \quad x_1 = C_A, \quad x_2 = C_B, \quad x_3 = T_r, \quad y = C_B$$

where T_{j_s} is the steady-state profile for T_j . The process model of (1.26) can be expressed in the following form:

$$\begin{aligned}
\frac{\partial x}{\partial t} &= A(x) \frac{\partial x}{\partial z} + f(x) + g(x)u, \\
y &= h(x), \quad q = p(x)
\end{aligned} \tag{1.27}$$

with the boundary condition:

$$C_1x(a, t) + C_2x(b, t) = R(t) \quad (1.28)$$

and the initial condition:

$$x(z, 0) = x_0(z) \quad (1.29)$$

where $x(z, t) \in H^n([a, b]; \mathbb{R}^n)$ is the vector of state variables, $z \in [a, b] \subset \mathbb{R}$ is the position and $t \in [0, \infty)$ is the time, $u(z, t)$ denotes the manipulated variable, $y(z, t)$ denotes the controlled variable and $q(z, t)$ denotes the measured variable. $A(x)$ is a sufficiently smooth matrix, $f(x)$ and $g(x)$ are sufficiently smooth vector functions, $h(x), p(x)$ are sufficiently smooth scalar functions, $R(t)$ is a column vector that is assumed to be a sufficiently smooth function of time, $x_0(z) \in H([a, b]; \mathbb{R}^n)$ and C_1, C_2 are constant matrices of dimension $n \times n$.

It is important to remark that, linearizing system (1.27) around the equilibrium, we obtain a system of the form (1.1). By introducing a concept of characteristic index for the synthesis of distributed state feedback laws, the output tracking in distributed parameter system (1.27)-(1.29) was guaranteed by combining the theory of **PDEs** with methods of geometric control. A distributed control law was obtained. In [16], the conditions to ensure the exponential stability of the closed-loop system were derived.

It is significant to highlight the fact that for chemical reactors, networked control [110, 138, 58] implementations are usual used. The impact of data communication can not be ignored, especially when wireless networks are being used [7]. In this context, it is of interest to study the impact of the sampled-data control implementation on the overall system behaviour.

1.3 Sampled-data control for parabolic systems

In this section, we present several results considering the stability of sampled-time parabolic **PDEs**. This results are of interest since the proposed techniques could also be relevant in the case of sampled-time hyperbolic **PDEs**.

Parabolic **PDEs** are used to describe a wide variety of time-dependent phenomena [71, 40, 33] including heat conduction [34], particle diffusion [115] and pricing of derivative investment instruments [52]. The theory of the heat equation was first developed in [34] for the purpose of modeling how a quantity such as heat diffuses through a given region. In modern science, the first systematic experimental study of diffusion was performed by Thomas Graham in [115]. He studied diffusion in gases and the main mathematical description of the phenomenon was described by him in 1829. Motivations can also be

found in other domains. In [52], the authors briefly discussed the logic behind the method of pricing securities that leads to the use of parabolic **PDEs**. Few results are allowable for the study of sampled-data parabolic systems. The study of parabolic systems with distributed sampled-data control has been developed in [37, 107, 60]. In the following, the main contributions in three mentioned papers will be presented in detail.

1.3.1 Semilinear diffusion equations

For parabolic systems governed by uncertain semilinear diffusion equations with distributed control on a finite interval, the sampled-data controllers have been investigated in [37]. The authors considered the following semilinear scalar diffusion equation

$$z_t(x, t) = \frac{\partial}{\partial x} [a(x)z_x(x, t)] + \phi(z(x, t), x, t)z(x, t) + u(x, t), \quad t \geq t_0, x \in [0, l], l > 0 \quad (1.30)$$

with Dirichlet boundary conditions

$$z(0, t) = z(l, t) = 0, \quad (1.31)$$

or with mixed boundary conditions

$$z_x(0, t) = \gamma z(0, t), \quad z(l, t) = 0, \quad \gamma \geq 0, \quad (1.32)$$

where γ may be unknown. In (1.30), $u(x, t)$ is the control input. The functions a and ϕ are of class \mathcal{C}^1 and may be unknown. These functions satisfy the inequalities $a \geq a_0 > 0$, $\phi_m \leq \phi \leq \phi_M$, where a_0 , ϕ_m and ϕ_M are known bounds.

The open-loop system under the above boundary conditions may become unstable if ϕ_M is too big (see [21] for $\phi \equiv \phi_M$). It is assumed that there exists a linear infinite-dimensional state feedback $u(x, t) = -Kz(x, t)$ with big enough $K > 0$ which exponentially stabilizes the system.

Then a sampled-data controller implementation is carried out for the case when the sampled (in time) measurements of the state are taken in a finite number of fixed sampling points in the spatial domain. It was assumed that both the sampling intervals in time and in space are bounded. The sampled-data static output feedback enters the equation through a finite number of shape functions (which are localized in the space) multiplied by the corresponding state measurements. It is piecewise-constant in time and it may possess an additional time-delay.

The space variable $x = [0, L]$ has been divided into N on space sampling intervals as

$$0 = x_0 < x_1 < \cdots < x_N = L, \quad (1.33)$$

and the time sampling intervals have been considered as follows

$$0 = t_0 < t_1 < \cdots < t_k < \cdots, \quad \lim_{k \rightarrow \infty} t_k = \infty. \quad (1.34)$$

The sampling intervals in time and in space are assumed to be variable but bounded,

$$0 \leq t_{k+1} - t_k \leq h, 0 < x_j - x_{j-1} = \Delta_j \leq \Delta \quad (1.35)$$

where h and Δ are the corresponding upper bounds. Sensors provide discrete measurements of the state:

$$y_{jk} = z(\bar{x}_j, t_k), \quad \bar{x}_j = \frac{x_{j+1} + x_j}{2}, j \in \{0, \dots, N-1\}, t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad (1.36)$$

The following sampled-data controller is obtained:

$$u(x, t) = -Kz(\bar{x}_j, t_k), x \in [x_j, x_{j+1}), j \in \{0, \dots, N-1\}, t \in [t_k, t_{k+1}), k \in \mathbb{N} \quad (1.37)$$

with the gain $K > 0$. This controller presents both time and space discretization.

To check the stability of the closed-loop system (1.30) and (1.37), a direct Lyapunov method for the stability analysis of the resulting closed-loop system was developed, which was based on the application of Wirtinger's [46] (see Theorem A.4 in Appendix A) and Halanay's inequalities [45] (see Lemma A.3 in Appendix A). The following Lyapunov function was used

$$V(z(x, t)) = \int_0^l z^2(x, t) dx. \quad (1.38)$$

An extension to the case of sampled-data control with delay is also presented, in which a more general controller was obtained

$$u(x, t) = -Kz(\bar{x}_j, t_k - \eta_k)$$

where $\eta_k \in [0, \eta_M]$ is an additional delay. The Lyapunov function was modified as follows:

$$\begin{aligned} V(z(x, t)) = & p_1 \int_0^l z^2(x, t) dx + p_3 \int_0^l a(x) z_x^2(x, t) dx \\ & + \int_0^l \left[\tau_M r \int_{-\tau_M}^0 \int_{t+\theta}^t e^{2\delta(s-t)} z_s^2(x, s) ds d\theta \right. \\ & \left. + g \int_{t-\tau_M}^t e^{2\delta(s-t)} z^2(x, s) ds \right] dx + qz^2(0, t) \end{aligned} \quad (1.39)$$

where $p_3 > 0, p_1 > 0, r \geq 0, g \geq 0, \tau_M = h + \eta_M$.

The result was also applicable to the convection-diffusion equation

$$\begin{aligned} z_t(x, t) &= a_0 z_{xx}(x, t) - \beta z_x(x, t) + \phi(z(x, t), x, t)z(x, t) + u(x, t), \\ t &\geq t_0, x \in [0, l], l > 0 \end{aligned} \quad (1.40)$$

with constant and known $\beta \in \mathbb{R}$, $a_0 > 0$ and unknown $\phi_m \leq \phi \leq \phi_M$ of class \mathcal{C}^1 under the Dirichlet boundary conditions (1.31) or under the mixed boundary conditions

$$z_x(0, t) = \gamma_0 z(0, t), \quad z(l, t) = 0, \quad \gamma_0 \geq \frac{\beta}{2a_0}, \quad (1.41)$$

where the measurements are given by (1.36).

This method in [37] gives a general framework for robust control of parabolic systems: being formulated in terms of **LMIs**. The conditions can be further adapted to systems with saturated actuators and to the **ISS** stabilization problem. It gives tools for network-based control, where data packet dropouts (resulting in variable in time sampling) and network-induced delays are taken into account. Extension of the method to various classes of parabolic systems, as well as its improvement may be topics for the future research.

1.3.2 Reaction-advection-diffusion equation

Reaction-advection-diffusion equations are **PDE** models that are used to represent the evolution of a substance (e.g., a drug) in a medium described by spatial coordinates involving privileged transport (or advection) according to a physical or chemical force represented by a velocity vector, diffusion, that is, random motion of the substance molecules in the medium, and reaction (e.g., chemical) with other constituents present in the medium represented by source or loss terms in the equations [27, 87]. In biology, the reaction-advection-diffusion equation is used to model chemotaxis observed in bacteria, population migration, evolutionary adaptation to changing environments and the spatiotemporal dynamics of molecular species including morphogenesis [11]. For the sampled-data control case, a vector reaction-advection-diffusion equation on a hypercube was considered in [107]. The measurements were weighted averages of the state over different subdomains. These measurements were asynchronously sampled in time. The discrete control signals were applied through shape functions and *Zero-Order Hold (ZOH)* and were subject to matched disturbances. In this work, the authors have focused on a generalized relay control, that is, the control signals take their values in a finite set. The following semilinear parabolic system was considered:

$$\begin{aligned} z_t(x, t) &= \Delta_D z(x, t) + \beta \nabla z(x, t) + Az(x, t) + f(x, t, z) \\ &+ B \sum_{j=1}^{N_s} b_j(x) [u_j(t) + w_j(t)], \quad x \in \Omega \end{aligned} \quad (1.42)$$

with the space domain $\Omega = [0, 1]^N$, state $z : \Omega \times [t_0, \infty) \rightarrow \mathbb{R}^M$, matched disturbances $w_j(t)$ and matrices $\beta \in \mathbb{R}^{M \times MN}$, $A \in \mathbb{R}^{M \times M}$, $B \in \mathbb{R}^{M \times L}$. The diffusion term is defined as

$$\Delta_D z = (\Delta_D^1 z^1, \dots, \Delta_D^M z^M)^\top,$$

where

$$\Delta_D^m z^m(x, t) = \operatorname{div} (D^m(x) \nabla z^m(x, t))$$

with

$$D^m(x) = (D^m(x))^\top \in \mathcal{C}^1(\Omega, \mathbb{R}^{N \times N}) \text{ for } m \in \{1, \dots, M\}.$$

The space domain Ω was divided into N_s rectangular subdomains Ω_j , where the control signals were applied through shape functions $b_j(x) \in H^1(0, 1)$ such that

$$\begin{cases} b_j(x) = 0, & x \notin \Omega_j, \\ b_j(x) = 1, & x \in \Omega_j^\varepsilon \\ b_j(x) \in [0, 1], & x \in \Omega_j \setminus \Omega_j^\varepsilon \end{cases} \quad (1.43)$$

with Ω_j^ε being subsets of Ω_j .

Each control signal u_j is applied through **ZOH** changing its value at asynchronous sampling instants. It was assumed that the measurements of the system (1.42), (1.43) were given by

$$y_{j,p} = \int_{\Omega_j} b_j(x) z(x, s_{j,p}) dx, \quad j \in \{1, \dots, N_s\}, p \in \mathbb{N},$$

where $s_{j,p}$ presents the sampling time of the j^{th} sensor. In addition, it was assumed that $s_{j,0} = t_0, \forall j \in \{1, \dots, N_s\}$ and $s_{j,p+1} - s_{j,p} \in (0, h]$. Let $\mathcal{V} = \{v_1, \dots, v_q\} \subset \mathbb{R}^L$ be a set of control values. The following generalized sampled-data relay control was considered:

$$u_j(t) = \operatorname{argmin}_{v \in \mathcal{V}} y_{j,p}^\top P_1 B v, \quad t \in [s_{j,p}, s_{j,p+1}), j \in \{1, \dots, N_s\}, p \in \mathbb{N} \quad (1.44)$$

with

$$P_1 = \operatorname{diag} \{p_1^1, \dots, p_1^m\} \geq 0.$$

For instance, if $\mathcal{V} = \{-v, v\}$ with $0 < v \in \mathbb{R}$, the minimum in (1.44) is delivered by

$$u_j(t) = -v \operatorname{sign} \left\{ (P_1 B)^\top y_{j,p} \right\}, \quad j \in \{1, \dots, N_s\}$$

which is consistent with the classical relay control.

The goal was to provide a switching control law (1.44) that ensures semi-global practical stability, that is, for any arbitrarily large domain of initial conditions the trajectories converge to a set whose size does not depend on the domain size. The control design procedure was based on Lyapunov-Krasovskii functional

$$V = V_1 + V_2 + V_W$$

with

$$\begin{aligned}
V_1 &= \int_{\Omega} z^\top(x, t) P_1 z(x, t) dx, \\
V_2 &= h \sum_{m=1}^M \int_{\Omega} p_3^m (\nabla z^m(x, t))^\top D^m(x) \nabla z^m(x, t) dx, \\
V_W &= h e^{2\alpha h} \sum_{j=1}^{N_s} \int_{\Omega_j} \int_{t_{j,k}}^t e^{-2\alpha(t-s)} z_s^\top(x, s) W z_s(x, s) ds dx \\
&\quad - \frac{\pi^2 h}{4} \sum_{j=1}^{N_s} \int_{\Omega_j} \int_{t_{j,k}}^t e^{-2\alpha(t-s)} \eta^\top(x, s) W \eta(x, s) ds dx, \quad t \in [t_k, t_{k+1}), k \in \mathbb{N}, \quad (1.45)
\end{aligned}$$

where

$$\eta(x, t) = \frac{1}{h} [z(x, t) - z(x, t_{j,k})]$$

and the notation

$$t_{j,k} = \max_{p \in \mathbb{N}} \{s_{j,p} \mid s_{j,p} \leq t_k\}, [t_k, t_{k+1}) = \bigcap_{j=1}^{N_s} [t_{j,k}, t_{j,k+1})$$

were adopted.

This work is an extension of the preliminary results given in [108], that is, generalizing the preliminary results to a vector system with multi-dimensional domain, convection term, reaction term and asynchronous sampling. For disturbance-free systems the constructive switching procedure in [107] guarantees exponential convergence to the origin.

1.3.3 1-D Kuramoto-Sivashinsky equation

Kuramoto-Sivashinsky Equation (KSE) is a fourth-order nonlinear **PDE**. It was derived in the late 1970s to model the diffusive instabilities in a laminar flame front [112, 72, 113]. The **KSE** is known for its chaotic behavior [95]. Applications of the **KSE** extend beyond its original context of flame propagation and reaction-diffusion systems. These additional applications include flows in pipes and at interfaces, plasmas, chemical reaction dynamics and models of ion-sputtered surfaces [20, 59]. In the context of sampled-data systems, the case of distributed sampled-data control of nonlinear **PDE** system governed by 1-D **KSE** has been addressed in [60]. A standard 1-D **KSE** is shown as follows

$$u_t(x, t) + u_{xx}(x, t) + v u_{xxxx}(x, t) + u(x, t) u_x(x, t) = \sum_{j=1}^N b_j(x) U_j(t), \quad 0 < x < L, t \geq 0. \quad (1.46)$$

In [60], the authors considered both the case of systems subject to Dirichlet boundary conditions:

$$u(0, t) = u(L, t) = 0, u_x(0, t) = u_x(L, t) = 0$$

or to periodic boundary conditions:

$$\frac{\partial^m u}{\partial x^m}(0, t) = \frac{\partial^m u}{\partial x^m}(L, t), m \in \{0, 1, 2, 3\}.$$

Here v is a positive constant, $u(x, t)$ is the state of **KSE** and $U_j(t) \in \mathbb{R}, j \in \{1, \dots, N\}$ are the control inputs.

It was assumed that N sensors provide sampled in time spatially distributed (either point or averaged) measurements of the state. In terms of (1.33)-(1.35), the control inputs $U_j(t)$ enter (1.46) through the shape functions

$$\begin{cases} b_j(x) = 1, x \in \Omega_j, \\ b_j(x) = 0, x \notin \Omega_j, \end{cases} \Omega_j = [x_{j-1}, x_j], j \in \{1, \dots, N\}.$$

Sensors provide either point as shown in (1.36) or averaged measurements of the state

$$y_{jk} = \frac{\int_{x_{j-1}}^{x_j} u(x, t_k) dx}{\Delta_j}, j \in \{1, \dots, N\}, k \in \mathbb{N}. \quad (1.47)$$

An exponentially stabilizing sampled-data controller for (1.46) was designed and implemented by **ZOH**:

$$U_j(t) = -\mu y_{jk}, \quad t \in [t_k, t_{k+1}), \quad (1.48)$$

where μ is a positive controller gain and y_{jk} is given by (1.36) or (1.47).

Given upper bounds on the sampling intervals in time and in space, sufficient conditions for regional exponential stability were established by applying the time-delay approach [78, 36]. The results were obtained by employing the following Lyapunov-Krasovskii functional

$$\begin{aligned} V_1(t) &= p_1 \int_0^L u^2(x, t) dx + p_3 v \int_0^L u_{xx}^2(x, t) dx + r (t_{k+1} - t) \int_0^L \int_{t_k}^t e^{2\delta(s-t)} u_s^2(x, s) ds dx, \\ t &\in [t_k, t_{k+1}), p_1 > 0, p_3 > 0, r > 0. \end{aligned} \quad (1.49)$$

Here p_1 and p_3 terms are extensions of the corresponding terms of [37] to **KSE**, whereas r -term treats sampled-data control as introduced in [35]. The parameter $\delta > 0$ stands for the decay rate.

Distributed control of **KSE** under the point or averaged state measurements was initiated in [80]. However, for practical use of such controllers, their sampled-data implementation was missing. The method in [60] gives efficient tools for sampled-data observer design. An interesting, yet technically challenging, open question regarding this approach is how to extend it to the case of observer-based sampled-data controllers.

1.4 Sampled-data control for hyperbolic systems

Some physical systems are described by hyperbolic **PDEs**, such as those used to model the flow of a river and the vibrations of beams, the road traffic [82, 131, 4], etc. Therefore, the study of hyperbolic equations is of substantial contemporary interest [47, 75]. For hyperbolic **PDEs**, we focus on results concerning both sampled-data control and event-triggered control.

1.4.1 Periodic and aperiodic boundary control

1.4.1.1 1-D linear transport PDEs with non-local terms

The application of boundary feedback control with **ZOH** to 1-D inhomogeneous, linear, transport partial differential equations on bounded domains with constant velocity and non-local terms was provided in [63]. The authors considered the following control system

$$\frac{\partial y}{\partial t}(t, z) + \frac{\partial y}{\partial z}(t, z) = g(z)y(t, 1) + \int_z^1 f(z, s)y(t, s)ds, \quad (1.50)$$

$$y(t, 0) = u(t) - \int_0^1 p(s)y(t, s)ds, \text{ for } t \geq 0 \quad (1.51)$$

where $(t, z) \in \mathbb{R}_+^0 \times [0, 1]$, $g \in \mathcal{C}^0([0, 1]; \mathbb{R})$, $p \in \mathcal{C}^1([0, 1]; \mathbb{R})$, $f \in \mathcal{C}^0([0, 1]^2; \mathbb{R})$ are given functions, $y[t]$ denotes the profile of the state y at certain $t \geq 0$, i.e., $(y[t])(z) = y(t, z)$ for all $z \in [0, 1]$ and $u(t)$ is the control input. More specifically, the solution of the initial-boundary value problem (1.50), (1.51) was considered under boundary sampled-data control with **ZOH**:

$$u(t) = u_k, \text{ for } t \in [t_k, t_{k+1}) \text{ and for all } k \in \mathbb{N} \quad (1.52)$$

where $\{t_k \geq 0, k \in \mathbb{N}\}$ is an increasing sequence (the sequence of sampling times) with $t_0 = 0, \lim_{k \rightarrow +\infty} (t_k) = +\infty$ and $\{u_k \in \mathbb{R}, k \in \mathbb{N}\}$ is the sequence of applied inputs and initial condition

$$y(0, z) = y_0(z), \text{ for all } z \in [0, 1], \quad (1.53)$$

where $y_0 : [0, 1] \rightarrow \mathbb{R}$ is a given function. The authors adapt the backstepping design approach from [69] to the sampled-data case. The following sampled-data controller is considered

$$u_i = \int_0^1 (p(s) + k(0, s))y(t_k, s) ds, \text{ for all } i \in \mathbb{N}. \quad (1.54)$$

The function k in (1.54) corresponds to the kernel function in backstepping approach (see Theorem 1 in [69]).

Theorem 1.6 (Karafyllis 2017 [63])

For every $\sigma > 0$ there exist constants $T, G > 0$ with the following property: for every $y_0 \in \mathbb{X}$ and for every increasing sequence $\{t_k \geq 0, k = 0, 1, 2, \dots\}$ with $t_0 = 0, \sup_{k \geq 0} (t_{k+1} - t_k) \leq T$ and $\lim_{k \rightarrow +\infty} (t_k) = +\infty$ the initial-boundary value problem (1.50)-(1.54) satisfies the following estimate:

$$\|y[t]\|_\infty \leq G \exp(-\sigma t) \|y_0\|_\infty, \quad \text{for all } t \geq 0. \quad (1.55)$$

It was shown in [63] that the closed-loop system is exponentially stable with the proposed sampled-data controller for sufficiently small sampling period. Moreover, it was also shown that, contrary to the parabolic case [64], a smaller sampling period implies a faster convergence rate with no upper bound for the achieved convergence rate.

1.4.1.2 2×2 linear hyperbolic systems

The global exponential stability of a 2×2 linear hyperbolic system with a sampled-data boundary feedback control designed by means of the backstepping method was studied in [22]. The work focused on the following linear hyperbolic system:

$$\begin{aligned} u_t(t, x) + \lambda_1 u_x(t, x) &= \sigma_1(x)v(t, x) \\ v_t(t, x) - \lambda_2 v_x(t, x) &= \sigma_2(x)u(t, x) \end{aligned} \quad (1.56)$$

with boundary conditions

$$u(t, 0) = qv(t, 0), v(t, 1) = \rho u(t, 1) + U(t) \quad (1.57)$$

where $u, v : \mathbb{R}_+^0 \times [0, 1] \rightarrow \mathbb{R}$ are the system states, $\lambda_1, \lambda_2 \in \mathbb{R}_+, \sigma_1, \sigma_2 \in \mathcal{C}^0([0, 1], \mathbb{R}), \rho, q \in \mathbb{R}$ are such that $|\rho q| < 1$ and $U : \mathbb{R}_+^0 \rightarrow \mathbb{R}$ is the control signal.

In this paper, the authors investigated the effect of sampling the control signal on the exponential convergence of the solution to (1.56)-(1.57) with a backstepping controller. It is assumed that there exists a strictly increasing sequence of instants $\{t_k\}_{k \in \mathbb{N}}$ satisfying

$$t_0 = 0, \quad T_k = t_{k+1} - t_k \in (0, h], \quad \lim_{k \rightarrow \infty} t_k = \infty$$

with $h > 0$. The following control signal was considered

$$U(t) = \kappa \left((u(t_k, \cdot), v(t_k, \cdot))^T \right), \quad t \in [t_k, t_{k+1}), \quad (1.58)$$

where

$$\kappa \left((u(t, \cdot), v(t, \cdot))^\top \right) = -k_0 \rho u(t, 1) + \int_0^1 \begin{bmatrix} K^u(x) & K^v(x) \end{bmatrix} \begin{bmatrix} u(t, x) \\ v(t, x) \end{bmatrix} dx \quad (1.59)$$

where $K^u(x)$ and $K^v(x)$ are kernel functions derived as in [125].

Two theorems were given to show that there exists a sufficiently small inter sampling time (that encompasses both periodic and aperiodic sampling) for which the global exponential stability of the closed-loop system is guaranteed.

The first result considers the case of systems with boundary reflection (i.e., $\rho \neq 0$).

Theorem 1.7 (Davó 2018 [22])

Consider the set $\mathcal{T} = \{T \in \mathbb{R}_+ : \exists m \in \mathbb{N}, T_p = mT\}$ with $T_p = \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2}$. There exist constants $\eta, C > 0$ and

$$T^* \in \begin{cases} \mathbb{R}_+, & \text{if } k_0 = 0 \\ \mathcal{T}, & \text{if } k_0 \in (0, 1] \end{cases} \quad (1.60)$$

such that for all constant $T \leq T^*$ and for all initial conditions $(u^0, v^0)^\top \in \mathcal{C}^1([0, 1], \mathbb{R}^2)$, the solution to system (1.56)-(1.57) satisfies

$$\|s(t, \cdot)\|_{L^2([0, 1], \mathbb{R}^4)}^2 \leq C e^{-\eta t} \left\| (u^0, v^0, u_x^0, v_x^0)^\top \right\|_{L^2([0, 1], \mathbb{R}^4)}^2$$

with

$$s(t, \cdot) = (u(t, \cdot), v(t, \cdot), \partial_x^- u(t, \cdot), \partial_x^+ v(t, \cdot))^\top,$$

when the control law (1.58) is applied with the following:

- 1) $T_k \in (0, T], k \in \mathbb{N}$ for $k_0 = 0$ (aperiodic sampling);
- 2) $T_k = T \in \mathcal{T}, k \in \mathbb{N}$ for $k_0 \in (0, 1]$ (periodic sampling).

The following theorem considers the case of system without boundary reflection (i.e., $\rho = 0$).

Theorem 1.8 (Davó 2018 [22])

There exist constants $T^*, \eta, C > 0$ such that for all $(u^0, v^0) \in \mathcal{C}^1([0, 1], \mathbb{R}^2)$ the solution to system (1.56)-(1.57) with $\rho = 0$ and control law (1.58) with $T_k \in (0, T^*]$ for all $k \in \mathbb{N}$, satisfies

$$\left\| (u(t, \cdot), v(t, \cdot))^\top \right\|_{L^2([0, 1], \mathbb{R}^2)} \leq C e^{-\eta t} \left\| (u^0, v^0)^\top \right\|_{L^2([0, 1], \mathbb{R}^2)}. \quad (1.61)$$

The controller under study is the modified version, proposed in [2], of the full state boundary feedback controller propounded in [125] by means of the backstepping method.

Two different cases depending on the boundary conditions were analyzed. It could be interesting to generalize the obtained results to a more general class of systems such as $n \times n$ linear hyperbolic systems and also to relax equation (1.60) for $k_0 \neq 0$. In addition, the proposed method might be combined with an event-triggering mechanism, in order to design an event-triggering strategy with a greater dwell-time, or a periodic event-triggered controller following the ideas in [48].

1.4.2 Event-based boundary control

During the last decade, event-triggered control has received increasing attention in real-time control systems [83, 48, 26, 111, 116]. One distinguished characteristic is that event-triggered control provides a strategy under which the control task is executed only when necessary. Compared with traditional time-triggered control, event-triggered control can efficiently reduce the number of execution of control tasks while preserving the desired closed-loop performance [81]. Under an event-triggering scheme, extra hardware is commonly used to monitor the instantaneous system state so that the next event time can be calculated.

1.4.2.1 An event-based boundary control strategy

Event-based boundary controls for 1-D linear hyperbolic systems of conservation laws was introduced in [31]. The system was given in Riemann coordinates:

$$\partial_t y(t, x) + \Lambda \partial_x y(t, x) = 0 \quad x \in [0, 1], t \in \mathbb{R}_+^0 \quad (1.62)$$

where $y : \mathbb{R}_+^0 \times [0, 1] \rightarrow \mathbb{R}^n$, Λ is a diagonal matrix in $\mathbb{R}^{n \times n}$ such that $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ with $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$. The following boundary condition was considered:

$$y(t, 0) = Hy(t, 1) + Bu(t), \quad t \in \mathbb{R}_+^0 \quad (1.63)$$

where $H \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $u : \mathbb{R}_+^0 \rightarrow \mathbb{R}^m$. The initial condition was given by

$$y(0, x) = y^0(x), \quad x \in [0, 1] \quad (1.64)$$

where $y^0 \in \mathcal{C}_{\text{lpw}}([0, 1], \mathbb{R}^n)$. The output function was defined as follows:

$$z(t) = y(t, 1) \quad (1.65)$$

The following assumption was considered to guarantee that the system (1.62), (1.64) and (1.65) with boundary condition

$$y(t, 0) = Gz(t), \quad t \in \mathbb{R}_+^0 \quad (1.66)$$

where $G = H + BK$, $K \in \mathbb{R}^{m \times n}$ is globally exponentially stable.

Assumption 1.4 (Coron 2008 [18])

The following inequality holds:

$$\rho_1(G) = \text{Inf} \{ \|\Delta G \Delta^{-1}\|; \Delta \in \mathcal{D}_{n,+} \} < 1 \quad (1.67)$$

where $\|\cdot\|$ denotes the usual 2-norm of matrices in $\mathbb{R}^{n \times n}$ and $\mathcal{D}_{n,+}$ denotes the set of diagonal matrices whose elements on the diagonal are strictly positive.

Proposition 1.2 (Diagne 2012 [24])

Under Assumption 1.4, there exist $\mu > 0$ and a diagonal positive definite matrix $Q \in \mathbb{R}^{n \times n}$ (with $Q = \Lambda^{-1} \Delta^2$) such that the following matrix inequality holds

$$G^\top Q \Lambda G < e^{-2\mu} Q \Lambda. \quad (1.68)$$

Then, the linear hyperbolic system (1.62), (1.64)-(1.66) is Globally Exponentially Stable (GES).

In regard of (1.68), the Lyapunov function was defined, for all $y(\cdot) \in L^2([0, 1], \mathbb{R}^n)$, by

$$V(y) = \int_0^1 y(x)^\top Q y(x) e^{-2\mu x} dx. \quad (1.69)$$

Then an event-based control scheme relying on both the ISS property with respect to deviations to sampling and Lyapunov techniques was introduced for hyperbolic systems of conservation laws.

Definition 1.2 (Espitia 2016 [31])

(Definition of φ_1): φ_1 is defined as an operator which maps z to u as follows:

The control function, $z \mapsto \varphi_1(z)(t) = u(t)$, is defined as

$$\varphi_1 : \begin{cases} u(t) = 0, & \forall t \in [t_0^u, t_1^u) \\ u(t) = Kz(t_k^u), & \forall t \in [t_k^u, t_{k+1}^u), k \geq 1, \end{cases}$$

where (t_k^u) denotes the sampling instants. More precisely, the sampling instants are defined by the following event-triggered law

$$t_{k+1}^u = \inf \{ t \in \mathbb{R}^+ \mid t > t_k^u \wedge \|BK(-z(t) + z(t_k^u))\|^2 \geq \kappa W_1(t) + \varepsilon_1(t) \} \quad (1.70)$$

where

$$W_1 \left(\frac{1}{\lambda} \right) = \sum_{i=1}^n Q_{ii} \int_0^1 \left(H_i z \left(t - \frac{x}{\lambda_i} \right) \right)^2 e^{-2\mu x} dx \quad (1.71)$$

and, for all $t > \frac{1}{\underline{\lambda}}$, by

$$W_1(t) = \sum_{i=1}^n Q_{ii} \int_0^1 \left(H_i z \left(t - \frac{x}{\lambda_i} \right) + B_i u \left(t - \frac{x}{\lambda_i} \right) \right)^2 e^{-2\mu x} dx, \quad (1.72)$$

$$\varepsilon_1(t) = \varsigma_1 W_1 \left(\frac{1}{\underline{\lambda}} \right) e^{-\eta t} \forall t \geq \frac{1}{\underline{\lambda}}.$$

where $\underline{\lambda} = \min_{1 \leq i \leq n} \{\lambda_i\}$, K in $\mathbb{R}^{m \times n}$, Q is a diagonal positive matrix in $\mathbb{R}^{n \times n}$ and $\varsigma_1, \kappa, \eta, \mu > 0$ are design parameters.

The following theorem provides stability condition for the event-triggered control system (1.62), (1.65), (1.70), (1.71), (1.72).

Theorem 1.9 (Espitia 2016 [31])

Let K be in $\mathbb{R}^{n \times n}$ such that Assumption 1.4 holds for $G = H + BK$. Let $\mu > 0$, Q a diagonal positive matrix in $\mathbb{R}^{n \times n}$ and $v = \mu \underline{\lambda}$ be as in Proposition 1.2. Let σ be in $(0, 1)$ and $\alpha > 0$ such that

$$(1 + \alpha)G^\top Q \Lambda G \leq e^{-2\mu} Q \Lambda.$$

Let ρ be the largest eigenvalue of $(1 + \frac{1}{\alpha}) Q \Lambda$,

$$\kappa = \frac{2v\sigma}{\rho}, \eta > 2v(1 - \sigma)$$

and ε_1 and φ_1 be given in Definition 1.2. Let V be given by (1.69). Then the system (1.62)-(1.65) with the controller (1.70) has a unique solution and is globally exponentially stable.

Highly inspired by [118] and [31], an extension to a more general case was given in [32]. Therein the approach is extended to the case of space-quantized measurements. A sampling algorithm was designed so that L^2 -norm of $y(t, \cdot)$ converges to a bound parameterized by the quantization error. Then the boundary controller was given by $u = \varphi_s(z)$ where the operator $\varphi_s(z)$ enclosed the triggering condition, the quantizer and the control function characterized as follows:

Definition 1.3 (Espitia 2017 [32])

(Definition of φ_s): Let $\sigma \in (0, 1)$, $\gamma_s, \xi, \delta, \mu, \nu > 0$ and $K \in \mathbb{R}^{m \times n}$. Let

$$\varepsilon_1(t) = \varepsilon_1(0)e^{-\delta t}, \quad \forall t \geq \frac{1}{\underline{\lambda}},$$

with

$$\varepsilon_1(0) \leq \xi W_1 \left(\frac{1}{\underline{\lambda}} \right).$$

The static control function, $z \mapsto \varphi_s(z)(t) = u(t)$, is described by:

$$\varphi_s : \begin{cases} u(t) = 0 & \forall t \in [t_0, t_1), \\ u(t) = Kq(z(t_k)) & \forall t \in [t_k, t_{k+1}), \quad k \geq 1, \end{cases} \quad (1.73)$$

where $q : \mathbb{R}^n \rightarrow \mathcal{Q}$ is the quantizer having the property that

$$|q(x) - x| \leq \Delta_q,$$

for some countable set \mathcal{Q} and a scalar $\Delta_q > 0$. The sampling instants are defined by the following event-triggered law

$$t_{k+1} = \inf \{t \in \mathbb{R}^+ \mid t > t_k \wedge \gamma_s \|BK(-z(t) + z(t_k))\|^2 \geq 2\nu\sigma W_1(t) + \varepsilon_1(t)\}. \quad (1.74)$$

If $\tilde{V}\left(\frac{1}{\lambda}\right) = 0$, the time instants are $t_0 = 0, t_1 = \frac{1}{\lambda}$ and $t_2 = \infty$.

For each $t \geq \frac{1}{\lambda}$, the boundary condition (1.63), with (1.65), under static boundary control, (1.73) as

$$u(t) = Kq(z(t_k)), t \in [t_k, t_{k+1}),$$

was rewritten as a nominal continuous time control loop

$$y(t, 0) = (H + BK)z(t) + d_q(t) + d_s(t) \quad (1.75)$$

with perturbations

$$\begin{cases} d_q(t) := BK(q(z(t_k)) - z(t_k)) \\ d_s(t) := BK(z(t_k) - z(t)), \quad \forall t \in [t_k, t_{k+1}) \end{cases} \quad (1.76)$$

that can be seen as errors related to the quantization (d_q) and to the sampling (d_s) respectively.

A condition for the stability in the presence of event-based sampling and quantization errors is provided in the following theorem.

Theorem 1.10 (Espitia 2017 [32])

(L^2 -stability): Let $K \in \mathbb{R}^{m \times n}$ be such that Assumption 1.4 holds. Let $\mu > 0, Q \in \mathcal{D}_{n,+}, \nu = \mu\lambda, \sigma \in (0, 1)$ and $\delta > 2\nu(1 - \sigma)$. Let $\varepsilon_s(t)$ be the decreasing function as in Definition 1.3 and assume that there exist γ_q and $\gamma_s > 0$ such that

$$M_c = \begin{pmatrix} G^\top Q \Lambda G - Q \Lambda e^{-2\mu} & G^\top Q \Lambda & G^\top Q \Lambda \\ \star & Q \Lambda - \gamma_q I & Q \Lambda \\ \star & Q \Lambda & Q \Lambda - \gamma_s I \end{pmatrix} \leq 0 \quad (1.77)$$

Then the closed-loop system (1.62)-(1.65), (1.75) with controller u in (1.73) is **ISS** in L^2 -norm with respect to d_q .

The paper [31] might be considered as the first contribution to event-based control of hyperbolic **PDEs** and complements the work of [37] and [106] on sampled data control of parabolic **PDEs** and on event-based control of parabolic **PDEs**, respectively. However, the event-based stabilization approaches may be applied to a linear hyperbolic system of balance laws. In [32], the authors studied **ISS** in L^2 - and H^1 - norms. It could be fruitful to consider also sampling algorithms for the control input in order to keep it constant until an update is necessary.

1.4.2.2 2×2 coupled linear hyperbolic system

A well-established backstepping controller was used to stabilize a 2×2 coupled linear hyperbolic system along with a dynamic triggering condition in [30]. The linear hyperbolic system under consideration was

$$u_t(t, x) + \lambda_1 u_x(t, x) = c_1 v(t, x) \quad (1.78)$$

$$v_t(t, x) - \lambda_2 v_x(t, x) = c_2 u(t, x) \quad (1.79)$$

along with the following boundary conditions:

$$u(t, 0) = qv(t, 0) \quad (1.80)$$

$$v(t, 1) = U(t) \quad (1.81)$$

where $u, v : \mathbb{R}_+^0 \times [0, 1] \rightarrow \mathbb{R}$ are the system states with $x \in [0, 1]$, $t \geq 0$, $U(t)$ is the control input and $\lambda_1 > 0, \lambda_2 > 0$. In addition, for technical issues related to the existence of solutions, it was assumed that $c_1, c_2 \neq 0, q \neq 0$,

$$\cos(w) - q \frac{\lambda_1}{c_1} w \sin(w) \neq 0, \quad \text{if } c_1 c_2 > 0$$

and

$$\cos h(w) + q \frac{\lambda_1}{c_1} w \sin h(w) \neq 0, \quad \text{if } c_1 c_2 < 0,$$

where

$$w = \sqrt{\frac{|c_1 c_2|}{\lambda_1 \lambda_2}}.$$

As it can be seen in [19], $U(t)$ is a continuous full-state feedback control which is designed to ensure that the closed-loop system is **GES** in L^2 norm. The backstepping Volterra transformation

$$\begin{aligned} \alpha(t, x) &= u(t, x) - \int_0^x K^{uu}(x, \xi) u(t, \xi) d\xi - \int_0^x K^{uv}(x, \xi) v(t, \xi) d\xi \\ \beta(t, x) &= v(t, x) - \int_0^x K^{vu}(x, \xi) u(t, \xi) d\xi - \int_0^x K^{vv}(x, \xi) v(t, \xi) d\xi \end{aligned} \quad (1.82)$$

with the kernel $K = \begin{pmatrix} K^{uu}(x, \xi) & K^{uv}(x, \xi) \\ K^{vu}(x, \xi) & K^{vv}(x, \xi) \end{pmatrix}$ is used to get $U(t)$ under the form

$$U(t) = \int_0^1 K^{vu}(1, \xi)u(t, \xi)d\xi + \int_0^1 K^{vv}(1, \xi)v(t, \xi)d\xi. \quad (1.83)$$

Since the system transformation (1.82) is invertible, when applying the continuous control (1.83), the original is **GES**. In addition, the definition of an event-based controller under which global exponential stability of the system is achieved and, furthermore, the existence of a minimal dwell-time between two triggering times was guaranteed in [30].

The event-triggered approach was extended to output feedback where the design makes use of observers in [29]. The system under consideration was the same as (1.78)-(1.80), but (1.81) was changed to

$$v(t, 1) = \rho u(t, 1) + U(t), \quad (1.84)$$

where ρ is the proximal reflection terms. It was assumed that $|\rho q| < \frac{1}{2}$ and the initial conditions

$$(u(0, x), v(0, x))^{\top} = (u^0, v^0)^{\top} \in L^2((0, 1), \mathbb{R}^2).$$

In [125] and [74] an observer was proposed by using a collocated output, i.e. having $u(t, 1)$ available as a measurement output. The observer represents a copy of the system (1.78)-(1.80) and (1.84) with output injections terms. It was stated as follows:

$$\begin{aligned} \hat{u}_t(t, x) + \lambda_1 \hat{u}_x(t, x) &= c_1(x) \hat{v}(t, x) + p_1(x)(u(t, 1) - \hat{u}(t, 1)) \\ \hat{v}_t(t, x) - \lambda_2 \hat{v}_x(t, x) &= c_2(x) \hat{u}(t, x) + p_2(x)(u(t, 1) - \hat{u}(t, 1)) \\ \hat{u}(t, 0) &= q \hat{v}(t, 0) \\ \hat{v}(t, 1) &= \rho u(t, 1) + U(t) \end{aligned} \quad (1.85)$$

where $\hat{u}, \hat{v} : \mathbb{R}_+^0 \times [0, 1] \rightarrow \mathbb{R}$ are the states of the observer. Furthermore, the output injections terms of the observer (1.85) were given as follows:

$$\begin{aligned} p_1(x) &= -\lambda_1 P^{uu}(x, 1) \\ p_2(x) &= -\lambda_1 P^{vu}(x, 1) \end{aligned} \quad (1.86)$$

where P^{uu} and P^{vu} are obtained from the solutions of the linear hyperbolic kernel equations in [29].

As a part of the perspectives in [30], an observer-based event-triggered boundary control to stabilize a 2×2 coupled linear hyperbolic system subject to reflection terms at the boundaries was proposed in this work. The results in this paper may be extended to $m+n$ hyperbolic equations and coupled **PDE-ODEs** (inspired by e.g. [54]). Applications to hydraulic networks and to traffic congestion control (inspired e.g. by [73, 136]) using ramp metering strategies are expected.

1.4.3 Distributed sampled-data control

In the past, sampled-data observers/controllers under the sampled in space and time measurements were suggested for parabolic systems. In the following sections, we present some recent results that consider this problem for the case of hyperbolic systems.

1.4.3.1 A semilinear damped wave equation

Many applications are arisen from semilinear damped wave equation, including nonlinear elasticity as a model of a vibrating string in a viscous medium, where the semilinear term corresponds to the elastic force [94]. In [121], a sampled-data observer was constructed for a hyperbolic system governed by 1-D semilinear wave equation with either viscous or boundary damping. The considered class of systems includes damped sine-Gordon equations that model the dynamics of a current driven coupled Josephson junctions with applications in superconducting single-electron transistors [25, 76]. The semilinear damped wave equation under consideration was given by:

$$z_{tt}(x, t) = z_{xx}(x, t) - \beta z_t(x, t) + f(z(x, t), x, t), \quad x \in (0, \pi), \quad t \geq t_0 \quad (1.87)$$

under the Dirichlet

$$z(0, t) = z(\pi, t) = 0 \quad (1.88)$$

or Neumann

$$z_x(0, t) = z_x(\pi, t) = 0 \quad (1.89)$$

or mixed

$$z(0, t) = z_x(\pi, t) = 0, \quad \text{or} \quad z_x(0, t) = z(\pi, t) = 0 \quad (1.90)$$

boundary conditions. Here $z(x, t) \in \mathbb{R}$ is the state, $\beta > 0$ is the damping coefficient and f is a function of class \mathcal{C}^1 . It was assumed that the derivative f_z is uniformly bounded by a constant $g_1 > 0$:

$$|f_z(z, x, t)| \leq g_1 \quad \forall (z, x, t) \in \mathbb{R} \times [0, \pi] \times [t_0, \infty). \quad (1.91)$$

The initial conditions were given by

$$z(x, t_0) = z_0(x), \quad z_t(x, t_0) = z_1(x). \quad (1.92)$$

Similar to [37], an observer for (1.87) under the appropriate boundary conditions (1.88) based on sampled in space and in time measurements (1.36) was considered:

$$\begin{aligned} \hat{z}_{tt}(x, t) &= \hat{z}_{xx}(x, t) - \beta \hat{z}_t(x, t) + f(\hat{z}, x, t) + L \sum_{j=0}^{N-1} \chi_j(x) [y_{jk} - \hat{z}(\bar{x}_j, t_k)] \\ x &\in (0, \pi), t \in [t_k, t_{k+1}), k = 0, 1, 2, \dots \end{aligned} \quad (1.93)$$

under the boundary conditions

$$\hat{z}(0, t) = \hat{z}(\pi, t) = 0 \quad (1.94)$$

and the initial conditions

$$[\hat{z}(\cdot, t_0), \hat{z}_t(\cdot, t_0)]^\top \in H,$$

where

$$H = H_0^1 \times L^2(0, \pi)$$

and

$$H_0^1 = \{z_0 \in H^1(0, \pi) \mid z_0(0) = z_0(\pi) = 0\}.$$

Here L is a scalar observer gain. The measurements are applied after multiplication by the characteristic functions $\chi_j(x)$, defined by

$$\chi_j(x) = \begin{cases} 1, & \text{if } x \in [x_j, x_{j+1}); \\ 0, & \text{else.} \end{cases} \quad (1.95)$$

Note that

$$\sum_{j=0}^{N-1} \chi_j(x) [y(\bar{x}_j, t) - \hat{z}(\bar{x}_j, t)] \approx [y(x, t) - \hat{z}(x, t)],$$

when $\Delta \rightarrow 0$.

Sufficient conditions for the exponential stability of the estimation error $e = z - \hat{z}$ governed by

$$\begin{aligned} e_{tt}(x, t) &= e_{xx}(x, t) - \beta e_t(x, t) + ge(x, t) - L \sum_{j=0}^{N-1} \chi_j(x) e(\bar{x}_j, t_k), \\ x &\in (0, \pi), t \in [t_k, t_{k+1}), k \in \mathbb{N} \end{aligned} \quad (1.96)$$

with $ge = f(z, x, t) - f(z - e, x, t)$ under the Dirichlet boundary conditions

$$e(0, t) = e(\pi, t) = 0 \quad (1.97)$$

were derived by using the time-delay approach to sampled-data control and appropriate Lyapunov-Krasovskii functionals of the form

$$V(t) = V_0(t) + V_r(t), \quad t \in [t_k, t_{k+1}) \quad (1.98)$$

where $V_0(t)$ was given by

$$V_0(t) = p_3 \int_0^\pi e_x^2 dx + \int_0^\pi [e \ e_t] P_0 [e \ e_t]^\top dx, \quad (1.99)$$

with

$$P_0 \triangleq \begin{bmatrix} p_1 & p_2 \\ * & p_3 \end{bmatrix} > 0, \quad (1.100)$$

and

$$V_r(t) = r \int_0^\pi (t_{k+1} - t) \int_{t_k}^t \exp(2\alpha_0(s - t)) e_s^2(\zeta, s) ds d\zeta, \quad r > 0, \quad \alpha_0 > 0. \quad (1.101)$$

The global boundedness assumption (1.91) is restrictive. This assumption can be relaxed to regional boundedness as considered in Section 4 of [38] that, for the case of control, should lead to regional stabilization and should be applicable to Klein-Gordon equation. Regional stabilization under regional boundedness in (1.91) may be a topic for future research. Moreover, the presented method might be developed for event-triggered control under discrete time measurements.

1.4.3.2 A damped semilinear beam equation

Vibration analysis of beam-like structures subjected to moving loads provides a wide range of engineering applications, including railway bridges with crossing vehicles [41, 92]. Since the moving load has a significant impact on the dynamic response of the beam-like structure, especially at high speeds, people are interested in effectively and reliably solving the moving loads problem. Based on some initial studies on the vibration of moving load beams [68, 124], more complex beam models with general boundary conditions, discontinuities, multiple spans and flexible connections are analyzed to consider for more general structural categories [53, 8, 23, 28, 49, 114]. In the case of sampled-data controllers, distributed static output-feedback stabilization of a damped semilinear beam equation was considered in [119]. The system under consideration was shown as follows:

$$z_{tt}(x, t) = -z_{xxxx}(x, t) - \beta z_t(x, t) + \rho(z(x, t), x, t) + \sum_{j=1}^N \chi_j(x) u_{jk}(t),$$

$$t \geq t_0, x \in (0, \pi), \quad (1.102)$$

under the boundary conditions

$$\begin{aligned} z(0, t) &= z_x(0, t) = 0, \\ z(\pi, t) &= z_x(\pi, t) = 0, \end{aligned} \quad (1.103)$$

or

$$\begin{aligned} z(0, t) &= z_{xx}(0, t) = 0, \\ z_x(\pi, t) &= z_{xxx}(\pi, t) = 0, \end{aligned} \quad (1.104)$$

and the initial conditions

$$z(x, 0) = z_1(x), \quad (1.105)$$

$$z_t(x, 0) = z_2(x). \quad (1.106)$$

Here $z(x, t) \in \mathbb{R}$ is the state (modeling the beam height position), $u_{jk}(t)$ is the control input, $\beta > 0$ is the damping coefficient (the damping is proportional to an angle of inclination of the center of the beam). It was assumed that ρ is of class \mathcal{C}^2 and satisfies

$$\rho(0, x, t) \equiv 0.$$

The authors first considered the case of globally Lipschitz in z nonlinearity ρ . It was assumed that

$$\phi_m \leq \rho_z(z, x, t) \leq \phi_M \quad \forall z, x, t. \quad (1.107)$$

Then

$$\rho(z, x, t) = \phi(z, x, t)z, \quad \phi = \int_0^1 \rho_z(\theta z, x, t) d\theta. \quad (1.108)$$

The later implies

$$\phi_m \leq \phi \leq \phi_M \quad (1.109)$$

for all z, x, t , where ϕ_m and ϕ_M are known bounds.

Distributed in space measurements were either point or pointlike, similar to the distribution and boundedness of sampling points in (1.33)-(1.35). The sampling intervals in time may be variable, but have a known bound

$$0 \leq s_{k+1} - s_k \leq \mathbf{MATI}, k \in \mathbb{N},$$

where **MATI** is *Maximum Allowable Transmission Interval*. The total round-trip transmission time-varying delays from sensors to actuators is $\eta_k \leq \mathbf{MAD}$, where **MAD** is the *Maximum Allowable Delay*. The updating time of the actuators is $t_k = s_k + \eta_k$ and the authors assume that $t_k < t_{k+1}, k \in \mathbb{N}$. For all $j \in \{1, \dots, N\}, k \in \mathbb{N}$ pointlike measurements were described by:

$$y_j = \int_{\Omega_j} c_j(\xi) z(\xi, s_k) d\xi, \quad (1.110)$$

$$c_j(x) = \begin{cases} \varepsilon_j^{-1}, & x \in \Omega_j^y \\ 0, & \text{elsewhere} \end{cases} \quad (1.111)$$

where Ω_j^y is subinterval of Ω_j of the length ε_j and $\varepsilon_j \leq \varepsilon$ with some $\varepsilon \in (0, \Delta)$. Point measurements in the middle of Ω_j were given by:

$$y_j = z(\hat{x}_j, s_k), \quad \hat{x}_j = \frac{x_{j-1} + x_j}{2}. \quad (1.112)$$

The network-based implementation of the control law which enters the **PDE** through shape functions was studied. Variable sampling intervals and transmission delays were taken into account. A static output-feedback controller was considered

$$u_j(t) = -Ky_j(t_k - \eta_k), \quad t \in [t_k, t_{k+1}) \quad (1.113)$$

based on the measurements y_j given by (1.110) or (1.112). By using the time-delay approach to networked control systems [39, 43], the resulting control input was modeled as a delayed one:

$$u_j(t) = -Ky_j(t - \tau(t)), \quad t \geq t_0$$

where

$$\tau(t) = t - t_k + \eta_k, \quad t \in [t_k, t_{k+1})$$

and

$$\tau(t) \leq t_{k+1} - t_k + \eta_k \leq \mathbf{MATI} + \mathbf{MAD} \triangleq \tau_M$$

For brevity, in what follows the time argument of τ is omitted.

The control input was remodelled as:

$$u_j(t) = -K[z(x, t - \tau) - f_j - \vartheta_j], \quad t \geq t_0 \quad (1.114)$$

Here

$$f_j = z(x, t) - z(x_j^\top, t), \quad \vartheta_j = \int_{\Omega_j} c_j(x)[z(x, t) - z(x, t - \tau)]dx$$

for the pointlike measurements (1.110) and

$$f_j = z(x, t - \tau) - z(\hat{x}_j, t - \tau), \quad \vartheta_j = z(x, t) - z(x, t - \tau)$$

for the point measurements (1.112). Then the closed-loop system has the form

$$z_{tt} = -z_{xxxx} - \beta z_t + [\phi(z, x, t) - K]z + K \sum_{j=1}^N \chi_j [f_j + \vartheta_j], \quad t \geq t_0 \quad (1.115)$$

In order to derive stability conditions for (1.115) the following Lyapunov-Krasovskii functional was used

$$V(t) = V_0(t) + V_s(t) + V_r(t), \quad t \in [t_k, t_{k+1}) \quad (1.116)$$

where

$$V_0(t) = p_3 \int_0^\pi z_{xx}^2 dx + \int_0^\pi [zz_t] P [zz_t]^\top dx \quad (1.117)$$

$$V_s(t) = s \sum_{j=1}^N \int_{\Omega_j} \int_{t-\tau_M}^t e^{2\delta_0(s-t)} \kappa^2(x, s) ds dx$$

$$V_r(t) = r\tau_M \sum_{j=1}^N \int_{\Omega_j} \int_{-\tau_M}^0 \int_{t+\theta}^t e^{2\delta_0(s-t)} \kappa_s^2(x, s) ds d\theta dx \quad (1.118)$$

with some scalars $p_1, p_2, p_3 > 0$, $s \geq 0$, $r \geq 0$, $P = \begin{bmatrix} p_1 & p_2 \\ * & p_3 \end{bmatrix}$ and κ given by

$$\kappa(x, s) = \begin{cases} \int_{\Omega_j} c_j(\xi) z(\xi, s) d\xi, & \text{pointlike measurements,} \\ z(x, s), & \text{point measurements.} \end{cases} \quad (1.119)$$

The results were derived and compared under both types of measurements in terms of the upper bound on the delays and sampling intervals that preserve the stability for the same (as small as possible) number of sensors/actuators. For locally Lipschitz nonlinearities, regional stabilization was achieved. Numerical results have shown that the pointlike measurements lead to larger delays and samplings, provided the subdomains where these measurements are averaged, are not too small. Some preliminary results for network-based control of damped beam equation with globally Lipschitz nonlinearities were presented in [120].

1.5 Goal of the thesis

In summary, the analysis of sampled-data hyperbolic **PDEs** is still a wide open problem. In this thesis we aim at studying the stability properties of sampled-data controlled linear hyperbolic systems:

$$\begin{aligned} \partial_t y(t, x) + \Lambda \partial_x y(t, x) + \Upsilon y(t, x) + u(t, x) &= 0, \\ y(t, 0) &= G y(t, L), \forall t \geq 0, \\ y(0, x) &= y_0(x), \forall x \in [0, L], \end{aligned} \quad (1.120)$$

where $y : [0, +\infty) \times [0, L] \rightarrow \mathbb{R}^n$, $t \in \mathbb{R}_+$, $x \in [0, L]$, $\Lambda = \text{diag} \{ \lambda_1, \dots, \lambda_m, \dots, \lambda_n \}$ with $\lambda_1, \dots, \lambda_m > 0 > \lambda_{m+1}, \dots, \lambda_n$, G and Υ are real $n \times n$ matrices. $u(t, x)$ is the control law which will be represented later. This class of systems is known to be relevant in several practical problems [4, 6] related to fluid dynamics, road traffic control, etc. - see also [24, 104, 117] for some recent theoretical developments. Although distributed sampled-data

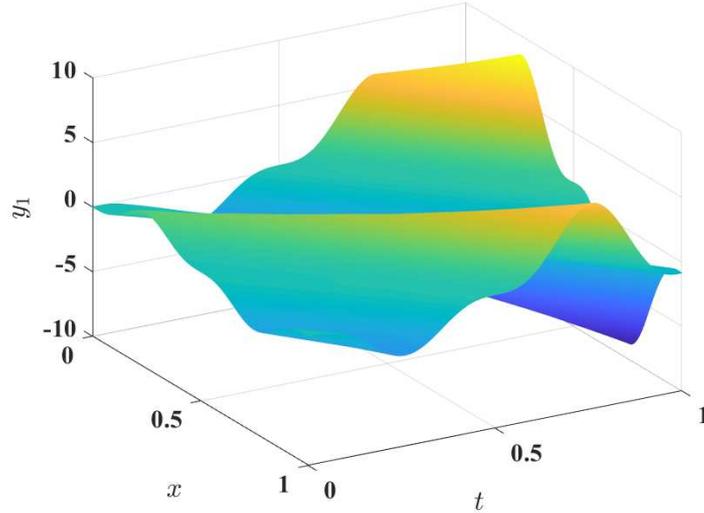


Figure 1.1: Response of state y_1 for the open-loop system.

controllers often appear in practical applications, the results in this case are scarce and only a few papers discussed the special case of hyperbolic systems [121, 119].

In the following, we use an example to motivate our study: an open-loop unstable system is stabilized with a continuous time controller. However, the stability can be lost under some sampled-data control implementations.

Example 1:

Consider a system of the form (1.120) with

$$L = 1, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} -1.5 & 1 \\ -3 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The open-loop system is unstable. This is illustrated in Figures 1.1 and 1.2 where the time evolution of states y_1 and y_2 for the open-loop system with the initial condition

$$y_0(x) = \begin{bmatrix} (1 - \cos 4\pi x) \sin 2\pi x \\ (\cos 2\pi x - 1) \sin 4\pi x \end{bmatrix}$$

is presented.

For closed-loop system (1.120) with the following continuous-time controller

$$u(t, x) = Ky(t, x), \tag{1.121}$$

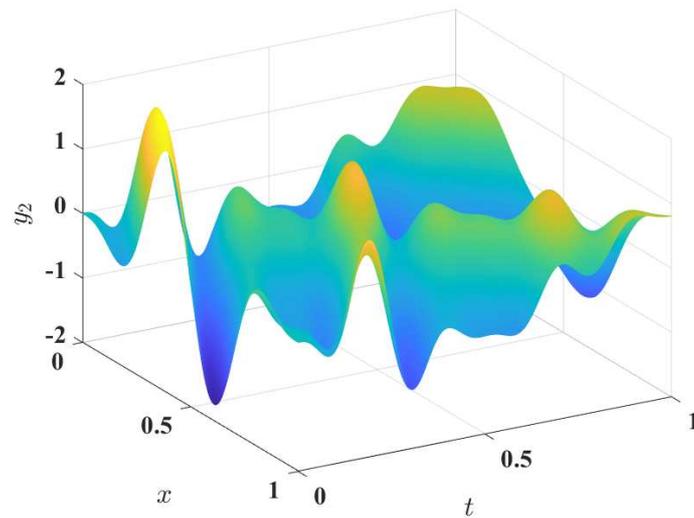


Figure 1.2: Response of state y_2 for the open-loop system.

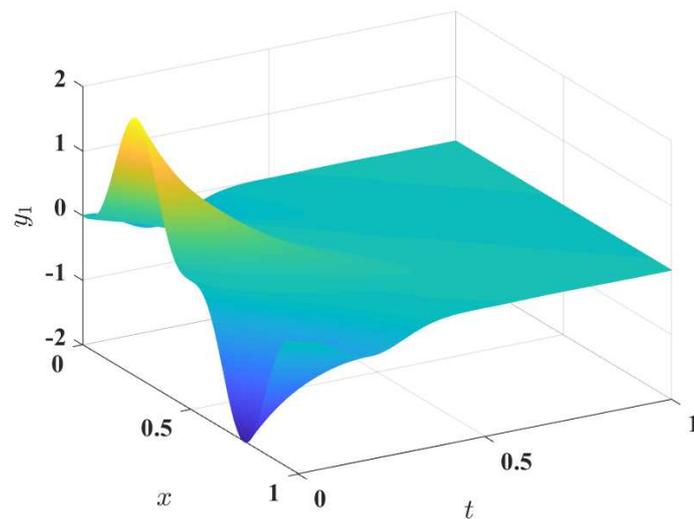


Figure 1.3: Response of state y_1 for the closed-loop system with continuous-time controller.

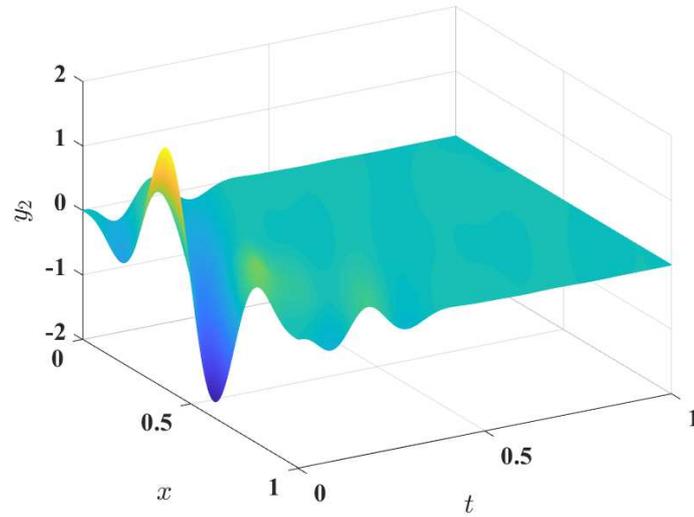


Figure 1.4: Response of state y_2 for the closed-loop system with continuous-time controller.

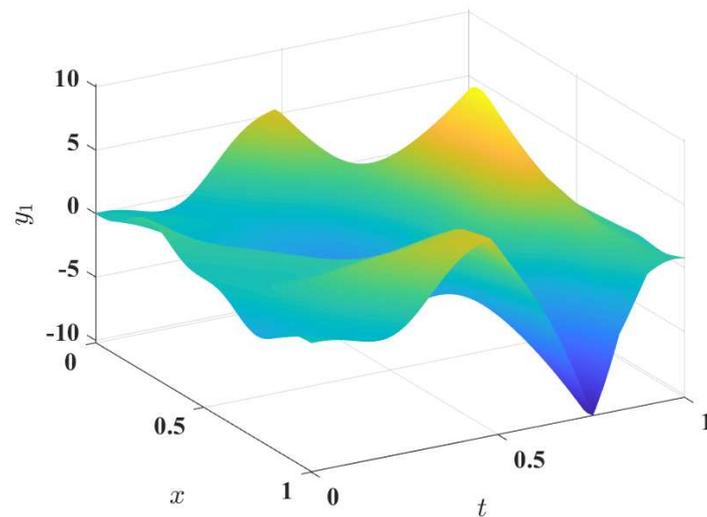


Figure 1.5: Response of state y_1 for the closed-loop system with sampled-data controller.

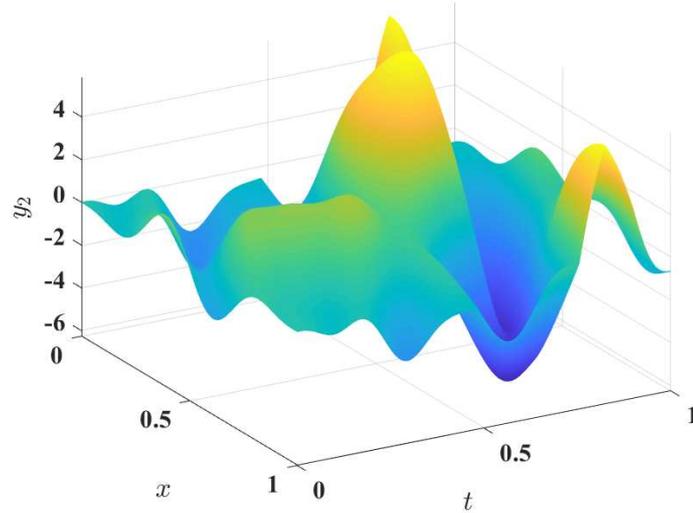


Figure 1.6: Response of state y_2 for the closed-loop system with sampled-data controller.

the stabilizing control gain

$$K = \begin{bmatrix} 9 & 0 \\ 7 & 6 \end{bmatrix}$$

was designed using classical continuous-time methods for hyperbolic systems. Then, the time evolution of states y_1 and y_2 for the closed-loop system with controller (1.121) are shown in Figures 1.3 and 1.4.

However, the sampled-data implementation may lead to an unstable behavior. This is the case for a sampling interval $\bar{h} = 0.5$. In order to illustrate the response of the sampled-data system with $\bar{h} = 0.5$, we choose the following controller

$$u(t, x) = Ky(t_k, x), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \quad (1.122)$$

where the sampling sequence in (1.122) satisfies

$$t_0 = 0, t_{k+1} - t_k \in (0, \bar{h}]. \quad (1.123)$$

We can see from Figures 1.5 and 1.6 that the system states are divergent with a sampling interval $\bar{h} = 0.5$ for controller (1.122). ■

It can be concluded from the above example that, in practice, it is necessary to analyze the stability of the hyperbolic system under sampled-data control. The following questions arise naturally when thinking about the stability of such dynamical systems:

- Is it possible to guarantee the stability of the linear hyperbolic system of balance laws with spatio-temporal sampled controllers?

- What are the conditions that have to be checked to guarantee the stability of the system?

This thesis aims at answering the above question. Let us present briefly the main contributions of this thesis.

1.6 Contributions and structure of the thesis

The rest of the thesis is structured as follows:

- In Chapter 2, we provide methods for analysing the stability of a class of linear hyperbolic systems with distributed sampled-data controllers. Sufficient conditions based on Lyapunov techniques for the local practical stability ($R\epsilon$ -stability) of the system (expressed as **LMIs**) are obtained.
- In Chapter 3, we extend the results from Chapter 2 for controllers with both time and the space discretization based on the Lyapunov-Razumikhin method. We introduce sufficient conditions for the local practical stability ($R\epsilon$ -stability) for the proposed linear hyperbolic systems of balance laws by recasting the original system into an equivalent input-output form with a continuous time control loop and operators representing the discretization errors (spatio-temporal sampling errors). In addition, the special case where the space is not discretized is also considered.
- In Chapter 4, we deal with the global exponential stability problem for a class of sampled linear hyperbolic systems. By employing the impulsive approach, the relation between the sampling interval, the system state and its sampled vector is characterized by an **IQC**. The acquired **IQC** is utilized to derive numerically tractable stability criteria.

Publications

Peer reviewed international journals

- Xinyong Wang, Ying Tang, Christophe Fiter, Laurentiu Hetel. "*Sampled-data Distributed Control for Homo-directional Linear Hyperbolic System with Spatially Sampled State Measurements*", **Accepted by Automatica, 2021.**
- Xinyong Wang, Ying Tang, Christophe Fiter, Laurentiu Hetel. "*Stability analysis for a class of sampled-data linear hyperbolic systems via a hybrid approach*", **Submitted, 2021.**

Peer reviewed international conferences

- Xinyong Wang, Christophe Fiter, Ying Tang, Laurentiu Hetel. "*Sampled-data control for a class of linear hyperbolic system via the Lyapunov-Razumikhin technique*". In European Control Conference, Rotterdam, Netherlands. hal-02566290v2, 2021.
- Xinyong Wang, Ying Tang, Christophe Fiter, Laurentiu Hetel. "*A hybrid system approach to exponential stability with sampled-data control for a class of linear hyperbolic systems*". In 7th IFAC Conference on Analysis and Design of Hybrid Systems, Brussels, Belgium. hal-03178668, 2021.
- Xinyong Wang, Ying Tang, Christophe Fiter, Laurentiu Hetel. "*Stability Analysis for A Class of Linear Hyperbolic System of Balance Laws with Sampled-data Control*". In IFAC World Congress, Berlin, Germany. hal-02491857, 2020.

Stability analysis for a class of linear hyperbolic system of balance laws with sampled-data control

Contents

2.1	Introduction	50
2.2	System description and problem formulation	50
2.2.1	System description	50
2.2.2	Problem formulation	52
2.3	Main result	53
2.3.1	System remodeling	53
2.3.2	Stability analysis	56
2.4	Numerical simulation	63
2.5	Conclusion	66
2.6	Appendix of Chapter 2	67

2.1 Introduction

In this chapter, we address the stability problem for sampled-data hyperbolic **PDEs**. Differently from the existing works where boundary control has been considered, here we study distributed sampled-data controllers. The problem is studied from an input-output point of view, extending the approaches for finite dimensional systems ([42, 86, 90, 62]). The main idea is to represent the sampling induced error as a perturbation for a continuous time hyperbolic system. By converting the original sampled-data system into an interconnection of a continuous-time **PDE** and a reset-integral operator, we construct an appropriate Lyapunov functional and obtain sufficient conditions for the R_ε - stability of the system based on **LMIs**.

The remainder of this chapter is organized as follows. Section 2.2 is dedicated to the system description and the problem formulation. In Section 2.3, we present the main results which include two parts. The first part is the system remodeling where an equivalent continuous hyperbolic **PDE** is introduced and the stability criteria based on the provided model are given in the second part. A numerical example illustrating our results is shown in 2.4. The conclusion of the whole chapter and a useful lemma are given in section 2.5 and section 2.6 respectively.

The main results of this chapter have been included in the accepted work [129].

2.2 System description and problem formulation

In this section, we first introduce the hyperbolic system and the sampled-data controller under consideration. The existence and uniqueness of the solution are also discussed. Then we state the stability property of the closed-loop system.

2.2.1 System description

We consider a class of 1-D hyperbolic systems with a control in domain. The system is represented as follows:

$$\begin{cases} \partial_t y(t, x) + \Lambda \partial_x y(t, x) + \Upsilon y(t, x) + u(t, x) = 0, & (2.1a) \\ y(t, 0) = 0, \forall t \geq 0, & (2.1b) \\ y(0, x) = y_0(x), \forall x \in [0, L], & (2.1c) \end{cases}$$

In system (2.1), the time variable $t \in [0, +\infty)$, the space variable $x \in [0, L]$. The state of the system $y : [0, +\infty) \times [0, L] \rightarrow \mathbb{R}^n$ depends on both time and space variables. The

partial derivative with respect to time and with respect to space are presented respectively by ∂_t and ∂_x . $\Lambda = \text{diag} \{ \lambda_1, \lambda_2, \dots, \lambda_n \}$ with $\lambda_1, \lambda_2, \dots, \lambda_n > 0$, Υ is a real $n \times n$ constant matrix. $u(t, x)$ is the controller. Since the execution of control in practical applications is usually a digital signal, we consider its sampled form as shown below:

$$u(t, x) = Ky(t_k, x), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \quad (2.2)$$

where K is a real $n \times n$ constant matrix, the sampling sequence in (2.2) is defined as $v = \{t_k\}_{k \in \mathbb{N}}$ where

$$t_0 = 0, t_{k+1} - t_k \in (0, \bar{h}], \quad (2.3)$$

and the maximum sampling interval $\bar{h} > 0$. There is no lower bound of \bar{h} and it can be any small positive number. The boundary condition and the initial condition are given by (2.1b) and (2.1c) respectively.

We discuss the solution of the system (2.1) with the sampled-data control (2.2) satisfying (2.3) in the following remarks.

Remark 2.1

The compatibility condition required for the well-posedness of the Cauchy problem depends on the functional space to which the solution belongs [4]. In this chapter, the solution of the closed-loop system (2.1)-(2.3) is considered in H^1 -norm and the compatibility condition:

$$y_0(0) = 0 \quad (2.4)$$

is guaranteed by (2.1b) and (2.1c).

Remark 2.2

Let us discuss the notion of the solution used in the present work. The system (2.1)-(2.3) can be rewritten as a first order system

$$\begin{cases} \frac{dy(t)}{dt} = \mathfrak{A}y(t) + f(y(t_k)), t \in [t_k, t_{k+1}), k \in \mathbb{N}, \\ y(0) = y_0, \end{cases}$$

where $f(y(t_k)) = -Ky(t_k)$ and the operator \mathfrak{A} is defined by

$$\mathfrak{A}y = -\Lambda \partial_x y(t, x) - \Upsilon y(t, x), \quad (2.5)$$

with domain

$$D(\mathfrak{A}) = \{y \in H^1(0, L; \mathbb{R}^n) \mid y(0) = 0. \} \quad (2.6)$$

The operator \mathfrak{A} generates a stable C_0 semigroup (see the proof of theorem A.1. in [4]). Moreover, we note that $f_k : H^1(0, L) \rightarrow H^1(0, L)$ is continuously differentiable for $t \in [t_k, t_{k+1})$. If $y_0 \in D(\mathfrak{A})$, then according to Theorem 6.1.5 of [96], there exists a classical solution for each $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Therefore, we can construct a solution by choosing the last value of the previous sampling interval as the initial condition of the following sampling interval such that it is continuous at each sampling instant.

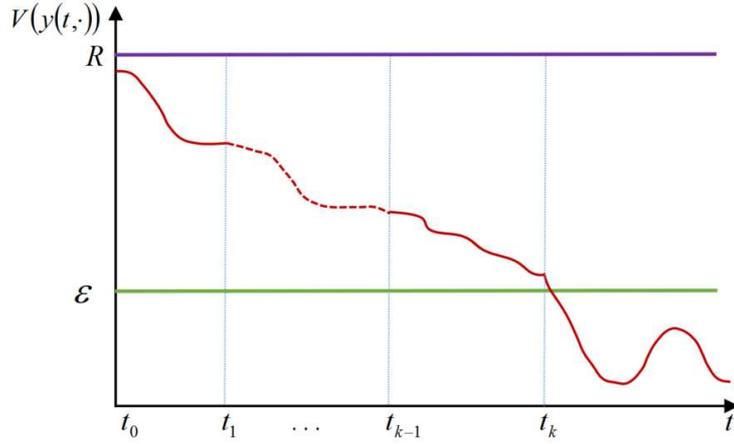


Figure 2.1: Schematic diagram of $R\varepsilon$ -stability under Lyapunov method.

2.2.2 Problem formulation

In this subsection, we first introduce the stability notion used in this chapter. It consists of local practical stability notion [98]. Given a function $V : H^1([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}_+$ and a positive constant C , we use the notation $\mathfrak{L}_{V \leq C}$ to define the set

$$\mathfrak{L}_{V \leq C} = \{y \in H^1([0, L]; \mathbb{R}^n) : V(y) \leq C\}.$$

In the following definition, we introduce the $R\varepsilon$ -stability notion.

Definition 2.1 (adopted from Polyakov 2008 [98])

(Definition of $R\varepsilon$ -stability): Consider positive scalars R and ε , such that $\varepsilon < R$ and a candidate Lyapunov function $V : H^1([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}_+$. If for all solutions of system (2.1)-(2.3) with initial condition $y_0(x) \in \mathfrak{L}_{V \leq R}$, the trajectory of the state $y(t, x)$ converges to $\mathfrak{L}_{V \leq \varepsilon}$ as t goes to infinity and the set $\mathfrak{L}_{V \leq \varepsilon}$ is positive invariant, then, system (2.1)-(2.3) is said to be $R\varepsilon$ -stable from $\mathfrak{L}_{V \leq R}$ to $\mathfrak{L}_{V \leq \varepsilon}$.

Remark 2.3

Intuitively, Definition 2.1 means that for a given controller (2.2) and an arbitrary initial condition satisfying $y_0(0) = 0$ in the domain where $V < R$, the solution of the system (2.1)-(2.3) converges from the attraction domain (R -neighborhood) to a steady motion domain (ε -neighborhood) and will never go out. A diagram of $R\varepsilon$ -stability under the Lyapunov method can be found in Figure 2.1.

In this work, our main goal is to guarantee the $R\varepsilon$ -stability of the closed loop system (2.1)-(2.3). We will use an input-output modeling approach which allow us to clearly identify the perturbation induced by sampling.

2.3 Main result

In order to achieve the goal for the previously proposed system and controller, we introduce the input-output modeling method and the main steps of the stability analysis in this section. The rest of this section is divided into two parts. First, we represent the sampled-data system as an equivalent continuous hyperbolic **PDE** where the sampling induced error appears in the input, as a disturbance. Secondly, based on the provided model, constructive $R\varepsilon$ -stability criteria are provided.

2.3.1 System remodeling

In order to remodel the system, we mainly divide it into four parts: the nominal closed-loop system, the introduction of the sampling error, the properties of the sampling error and the interconnection of the operators.

A. Nominal closed-loop system

We first consider the system represented in (2.1), the ideal continuous time control would have been considered as

$$u_c(t, x) = Ky(t, x). \quad (2.7)$$

Under the above controller (2.7), the system (2.1) is represented as the following nominal closed-loop system:

$$\begin{cases} \partial_t y(t, x) + \Lambda \partial_x y(t, x) + (\Upsilon + K)y(t, x) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & (2.8a) \\ y(t, 0) = 0, \forall t \geq 0, & (2.8b) \\ y(0, x) = y_0(x), \forall x \in [0, L]. & (2.8c) \end{cases}$$

Consider the following Lyapunov functional

$$V(y) = \int_0^L y^\top e^{-2\mu x} Q y dx, \quad (2.9)$$

with $\mu > 0$ and the matrix $Q \in \mathbb{R}^{n \times n}$ is symmetric positive. The system (2.8) is exponential stable [4] if the following condition satisfies

$$(K + \Upsilon)^\top Q + Q(K + \Upsilon) \succeq 0. \quad (2.10)$$

B. Sampling error model

Almost all controllers are now implemented in the form of digital algorithms on a computer. Hence the sampled version of the continuous signal (2.7) is obtained under the sampling trigger signal at discrete sampling instants t_k

$$u(t, x) = Ky(t_k, x), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad (2.11)$$

Let us define the error between $y(t, x)$ and its sampled version $y(t_k, x)$ as follows:

$$\omega(t, x) = y(t_k, x) - y(t, x). \quad (2.12)$$

Then the system (2.1) can be equivalently re-expressed as a perturbed version of the nominal system:

$$\begin{cases} \partial_t y(t, x) + \Lambda \partial_x y(t, x) + (\Upsilon + K)y(t, x) + K\omega(t, x) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & (2.13a) \\ y(t, 0) = 0, \forall t \geq 0, & (2.13b) \\ y(0, x) = y_0(x), \forall x \in [0, L]. & (2.13c) \end{cases}$$

where ω is the (sampling induced) perturbation parameter.

C. Properties of the sampling error

In the light of the equation defined by the sampling induced error (2.12), we can realize that the sampling error satisfies the following relation:

$$\omega(t, x) = - \int_{t_k}^t \frac{\partial y(\theta, x)}{\partial \theta} d\theta, k \in \mathbb{N}, x \in [0, L]. \quad (2.14)$$

Consider the notion:

$$\varphi(t, x) = \frac{\partial y(t, x)}{\partial t}, \forall t \geq 0, x \in [0, L]. \quad (2.15)$$

This parameter represents the derivative of the state between two sampling intervals.

According to (2.14) and (2.15), we have the following expression

$$\omega(t, x) = - \int_{t_k}^t \varphi(\theta, x) d\theta, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, x \in [0, L]. \quad (2.16)$$

This can be captured by the following operator J_v :

$$J_v : \begin{cases} \omega(t, x) = (J_v \varphi)(t, x) := - \int_{t_k}^t \varphi(\theta, x) d\theta, \\ \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, x \in [0, L]. \end{cases} \quad (2.17)$$

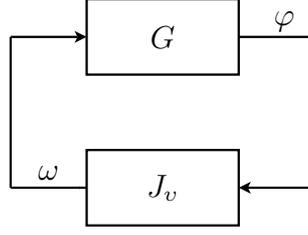


Figure 2.2: Alternative representation of the closed-loop system where G is a perturbed nominal control loop, φ is an auxiliary output representing the time derivative of the state between two sampling intervals, J_v is an integral operator and ω is the sampling error.

The operator J_v can also be re-expressed as the following infinite dimensional reset integrator:

$$J_v : \begin{cases} \frac{\partial \phi}{\partial t}(t, x) = \varphi(t, x), \forall t \in [t_k, t_{k+1}), \\ \omega(t, x) = -\phi(t, x), \\ \phi(t_k, x) = 0, k \in \mathbb{N}, x \in [0, L]. \end{cases} \quad (2.18)$$

D. Interconnection model

We consider system (2.13a) and define an auxiliary output $\varphi(t, x)$ as in (2.15). Then the sampled closed-loop system can be regarded as the interconnection of two systems G and J_v shown in Figure 2.2, where the operator G is defined by

$$G : \begin{cases} \partial_t y(t, x) = -\Lambda \partial_x y(t, x) - (K + \Upsilon) y(t, x) - K\omega(t, x), \\ y(t, 0) = 0, \forall t \geq 0, \\ y(0, x) = y_0(x), \forall x \in [0, L], \\ \varphi(t, x) = -\Lambda \partial_x y(t, x) - (K + \Upsilon) y(t, x) - K\omega(t, x) = \partial_t y(t, x). \end{cases} \quad (2.19)$$

Remark 2.4

Inspired by the existing works [42, 12, 91, 86, 123, 122] in the case of finite dimension systems, in the remodeling system (2.18)-(2.19), the operator J_v captures the effect of the sampling in the system. It has input (the derivative of the state φ) and the output (the sampling error ω).

This description is further used in the following subsection in order to derive our main analysis result.

2.3.2 Stability analysis

In this subsection, we provide our main results. It proposes constructive conditions for the analysis of $R\varepsilon$ -stability of sampled-data system (2.1)-(2.3) using the model (2.18) and (2.19).

Theorem 2.1

Consider systems (2.18)-(2.19) with (2.3) (or equivalently (2.1)-(2.3)) and the initial condition satisfying the compatibility condition (2.4):

(i) Let $\underline{\lambda} = \min_{i \in \{1, \dots, n\}} \lambda_i$. Assume that there exist $\mu, \gamma > 0$ and symmetric positive matrices $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{n \times n}$ satisfying the commutativity conditions: $\Lambda\Theta_1 = \Theta_1\Lambda$, $\Lambda\Theta_2 = \Theta_2\Lambda$ and

$$M(0) \preceq 0, \quad M(L) \preceq 0, \quad (2.20)$$

with $M(x)$ defined for all $x \in [0, L]$ as

$$M(x) = \begin{bmatrix} \Omega(x) & -e^{-2\mu x} Q_1 K & 0 & 0 \\ * & -\gamma I & 0 & 0 \\ * & * & E(x) & -e^{-2\mu x} Q_2 K \\ * & * & * & -\gamma I \end{bmatrix}, \quad (2.21)$$

where

$$\begin{aligned} \Omega(x) &= -e^{-2\mu x} \left[(K + \Upsilon)^\top Q_1 + Q_1 (K + \Upsilon) \right], \\ E(x) &= -e^{-2\mu x} \left[\Upsilon^\top Q_2 + Q_2 \Upsilon + \beta Q_2 \right]. \end{aligned} \quad (2.22)$$

(ii) Let there exist $\varepsilon \in \mathbb{R}_+$; $R \in \mathbb{R}_+$ such that $\varepsilon < R$ and

$$\gamma 3\bar{h} (|\Lambda|^2 \Phi_2 + (|\Upsilon|^2 + |K|^2) \Phi_1) + \gamma \Phi_2 < (2\nu - \beta)\varepsilon, \quad (2.23)$$

with

$$\begin{aligned} \Phi_1 &= \frac{R}{\lambda_{\min}(Q_1) e^{-2\mu L}}, \\ \Phi_2 &= \frac{R}{\lambda_{\min}(Q_2) e^{-2\mu L}}, \end{aligned} \quad (2.24)$$

where $\nu = \mu\underline{\lambda}$, $0 < \beta < 2\nu$.

Then the considered system (2.18)-(2.19) with (2.3) (or equivalently (2.1)-(2.3)) is $R\varepsilon$ -stable from $\mathfrak{L}_{V \leq R}$ to $\mathfrak{L}_{V \leq \varepsilon}$ with a Lyapunov functional defined as

$$V(y) = V_1(y) + V_2(y), \quad (2.25)$$

where

$$\mathfrak{L}_{V \leq R} = \{y \in H^1([0, L]; \mathbb{R}^n) : V(y) \leq R\},$$

$$\mathfrak{L}_{V \leq \varepsilon} = \{y \in H^1([0, L]; \mathbb{R}^n) : V(y) \leq \varepsilon\},$$

and

$$V_1(y) = \int_0^L y^\top e^{-2\mu x} Q_1 y dx, \quad (2.26)$$

$$V_2(y) = \int_0^L y_x^\top e^{-2\mu x} Q_2 y_x dx. \quad (2.27)$$

Proof of Theorem 2.1:

Before giving the proof of Theorem 2.1, we first state a temporary assumption that the solutions y are of class \mathcal{C}^2 (see Comment 4.6 in Bastin 2016 [4]). This assumption can be relaxed as shown in the proof of Lemma 4.5 in [4]. Indeed, using a density argument similar to Comment 4.6 in [4], the estimates of $(\dot{V}_i, i = 1, 2)$ given below remain valid, in the distribution sense with $y \in \mathcal{C}^0([0, +\infty], H^1([0, L]; \mathbb{R}^n))$ (see the statement of Theorem 10.1 in [97]).

Consider the Lyapunov functional (2.25)-(2.27). It can be bounded as follows:

$$\Theta \|y(t, \cdot)\|_{H^1([0, L]; \mathbb{R}^n)}^2 \leq V(y(t, \cdot)) \leq \Xi \|y(t, \cdot)\|_{H^1([0, L]; \mathbb{R}^n)}^2, \quad (2.28)$$

where

$$\begin{aligned} \Theta &= \min(\lambda_{\min}(Q_1), \lambda_{\min}(Q_2))e^{-2\mu L}, \\ \Xi &= \max(\lambda_{\max}(Q_1), \lambda_{\max}(Q_2)). \end{aligned}$$

The rest of the proof is divided into three steps. In step 1, we investigate the continuity of the Lyapunov functional $V(y)$ by constructive method. In step 2, we intend to study the time derivative of the Lyapunov functional $V(y)$ defined in (2.25). The time derivative of $V_1(y)$ defined in (2.26) is first derived, then with the similar computation of \dot{V}_1 , the time derivative of V_2 defined in (2.27) is obtained. Combining \dot{V}_1 and \dot{V}_2 , we get that the time derivative of $V(y)$ is bounded. In step 3, we discuss the negative properties of function $\dot{V}(y)$ with the assumption (2.23), which proves the $R\varepsilon$ -stability.

Step 1: In this step we study the continuity of the function V defined in (2.25).

1. Since $y(t, x)$ is continuous with respect to t for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$ and continuous at sampling instants by construction (see Remark 2.2), then V_1 is continuous for all $t \geq 0$.

2. From system (2.1), we can get

$$y_x(t, x) = \Lambda^{-1}(-y_t(t, x) - \Upsilon y(t, x) - Ky(t_k, x)), \quad (2.29)$$

for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. Since all the terms on the right of the equation (2.29) are continuous in t on (t_k, t_{k+1}) , $\forall k \in \mathbb{N}$, then $y_x(t, x)$ and thus V_2 are also continuous in t for all (t_k, t_{k+1}) , $k \in \mathbb{N}$.

Now we consider the time interval $[t_k, t_{k+1})$, for some $k \in \mathbb{N}$ and an initial condition $y_k(x)$. The solution of (2.1) is defined as $y(t, x)$ on the time interval $[t_k, t_{k+1})$ and is such that y and y_x are both \mathcal{C}^0 in $t \in [t_k, t_{k+1})$.

Next, we prolong the solution to \mathcal{C}^1 in t on $[t_k, t_{k+1}]$. We denote $z(t, x)$ the solution on $[t_k, t_{k+1}]$ with initial condition $y_k(x)$. $z(t, x)$ and $z_x(t, x)$ are \mathcal{C}^0 in t on $[t_k, t_{k+1}]$. We get the following properties on $[t_k, t_{k+1})$

$$\begin{cases} y(t, x) = z(t, x), \\ y_x(t, x) = z_x(t, x). \end{cases} \quad (2.30)$$

Then the left limit can be calculated as

$$\lim_{t \rightarrow t_{k+1}^-} y_x(t, x) = \lim_{t \rightarrow t_{k+1}^-} z_x(t, x) = z_x(t_{k+1}, x). \quad (2.31)$$

For the next time interval $[t_{k+1}, t_{k+2})$, we set the initial condition $y_{k+1}(x) = z(t_{k+1}, x)$. Then the solution $y(t, x)$ of system (2.1) on $[t_{k+1}, t_{k+2})$ satisfies $y_x(t, x)$ is \mathcal{C}^0 in t on $[t_{k+1}, t_{k+2})$. Therefore, we have the right limit property

$$\lim_{t \rightarrow t_{k+1}^+} y_x(t, x) = y_x(t_{k+1}, x) = z_x(t_{k+1}, x). \quad (2.32)$$

According to (2.31) and (2.32), we can see that by construction, $y_x(t, x)$ is continuous in t at time instant t_{k+1} . Similarly, we can show that the function $y_x(t, x)$ is continuous at all sampling instants, which shows both the continuity of $y_x(t, x)$ with respect to time for all $t \geq 0$ and the continuity of V_2 .

Step 2: In this step we study the time derivative of the function of $V(y)$ defined in (2.25). Thanks to commutativity condition: $\Lambda Q_1 = Q_1 \Lambda$, we first compute the time derivative of $V_1(y)$ along the solutions to (2.18)-(2.19), $\forall t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$,

$$\begin{aligned} \dot{V}_1(y) &= \int_0^L (\partial_t y^\top e^{-2\mu x} Q_1 y + y^\top e^{-2\mu x} Q_1 \partial_t y) dx \\ &= \int_0^L \left((-\Lambda \partial_x y - (K + \Upsilon)y - K\omega)^\top e^{-2\mu x} Q_1 y \right. \\ &\quad \left. + y^\top e^{-2\mu x} Q_1 (-\Lambda \partial_x y - (K + \Upsilon)y - K\omega) \right) dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^L -\partial_x [y^\top \Lambda e^{-2\mu x} Q_1 y] dx \\
&\quad + \int_0^L \left(-y^\top (K + \Upsilon)^\top e^{-2\mu x} Q_1 y - y^\top e^{-2\mu x} Q_1 (K + \Upsilon) y \right. \\
&\quad \left. - y^\top e^{-2\mu x} Q_1 K \omega - 2\mu y^\top \Lambda e^{-2\mu x} Q_1 y - \omega^\top K^\top e^{-2\mu x} Q_1 y \right) dx \\
&= - [y^\top \Lambda e^{-2\mu x} Q_1 y]_0^L \\
&\quad + \int_0^L \left(-y^\top \left((K + \Upsilon)^\top e^{-2\mu x} Q_1 + e^{-2\mu x} Q_1 (K + \Upsilon) \right) y \right. \\
&\quad \left. - \omega^\top K^\top e^{-2\mu x} Q_1 y - y^\top e^{-2\mu x} Q_1 K \omega \right) dx \\
&\quad - 2\mu \int_0^L y^\top \Lambda e^{-2\mu x} Q_1 y dx. \tag{2.33}
\end{aligned}$$

Since we have the assumption that y is of class \mathcal{C}^2 on $[0, +\infty) \times [0, L]$, then along the solutions of the system (2.1)-(2.2), we can obtain

$$\begin{aligned}
\partial_{xt} y(t, x) &= \partial_{tx} y(t, x) = -\Lambda \partial_{xx} y(t, x) - \Upsilon \partial_x y(t, x) - K \partial_x y(t_k, x), \\
\forall t &\in (t_k, t_{k+1}), k \in \mathbb{N}. \tag{2.34}
\end{aligned}$$

For the next calculation of the time derivative of V_2 , we use Lemma 2.1 in the appendix. According to (2.34) and Lemma 2.1, we have

$$\begin{cases} \partial_x y(t, 0) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & (2.35a) \\ y_0(0) = 0, \quad \partial_x y_0(0) = 0. & (2.35b) \end{cases}$$

Similarly to the computation of \dot{V}_1 , by using the commutativity condition: $\Lambda Q_2 = Q_2 \Lambda$, the time derivative of $V_2(y)$ along the solutions to (2.34)-(2.35), $\forall t \in (t_k, t_{k+1}), k \in \mathbb{N}$ is shown as follows

$$\begin{aligned}
\dot{V}_2(y) &= - [\partial_x y^\top \Lambda e^{-2\mu x} Q_2 \partial_x y]_0^L \\
&\quad + \int_0^L \left(-\partial_x y^\top (\Upsilon^\top e^{-2\mu x} Q_2 + e^{-2\mu x} Q_2 \Upsilon) \partial_x y \right. \\
&\quad \left. - \partial_x y^\top (t_k, \cdot) K^\top e^{-2\mu x} Q_2 \partial_x y \right. \\
&\quad \left. - \partial_x y^\top e^{-2\mu x} Q_2 K \partial_x y^\top (t_k, \cdot) \right) dx \\
&\quad - 2\mu \int_0^L \partial_x y^\top \Lambda e^{-2\mu x} Q_2 \partial_x y dx. \tag{2.36}
\end{aligned}$$

Recalling the definition of ω in (2.12) and adding $\gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 - \gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2$ to (2.33) and

$$\begin{aligned}
&\gamma \|\partial_x y(t_k, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 - \gamma \|\partial_x y(t_k, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2, \\
&\beta \int_0^L y_x^\top e^{-2\mu x} Q_2 y_x dx - \beta \int_0^L y_x^\top e^{-2\mu x} Q_2 y_x dx
\end{aligned}$$

to (2.36) for some $\gamma > 0, \beta > 0$, we have

$$\begin{aligned}
 \dot{V}(y) &= \dot{V}_1(y) + \dot{V}_2(y) \\
 &= - [y^\top \Lambda e^{-2\mu x} Q_1 y]_0^L \\
 &\quad + \int_0^L (y^\top \Omega(x) y - \omega^\top K^\top e^{-2\mu x} Q_1 y - y^\top e^{-2\mu x} Q_1 K \omega - \gamma \omega^\top \omega) dx \\
 &\quad - 2\mu \int_0^L y^\top \Lambda e^{-2\mu x} Q_1 y dx + \gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\
 &\quad - [\partial_x y^\top \Lambda e^{-2\mu x} Q_2 \partial_x y]_0^L \\
 &\quad + \int_0^L (\partial_x y^\top E(x) \partial_x y - \partial_x y^\top(t_k, \cdot) K^\top e^{-2\mu x} Q_2 \partial_x y \\
 &\quad - \partial_x y^\top e^{-2\mu x} Q_2 K \partial_x y(t_k, \cdot) - \gamma \partial_x y^\top(t_k, \cdot) \partial_x y(t_k, \cdot)) dx \\
 &\quad - 2\mu \int_0^L \partial_x y^\top \Lambda e^{-2\mu x} Q_2 \partial_x y dx \\
 &\quad + \gamma \|\partial_x y(t_k, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 + \beta \int_0^L y_x^\top Q_2 y_x dx. \tag{2.37}
 \end{aligned}$$

According to the boundary conditions (2.13b) and (2.35a), the two terms $- [y^\top \Lambda e^{-2\mu x} Q_1 y]_0^L$ and $- [\partial_x y^\top \Lambda e^{-2\mu x} Q_2 \partial_x y]_0^L$ in (2.37) are both negative. Then the following inequality can be obtain:

$$\begin{aligned}
 \dot{V}(y) &\leq \int_0^L (y^\top \Omega(x) y - \omega^\top K^\top e^{-2\mu x} Q_1 y - y^\top e^{-2\mu x} Q_1 K \omega - \gamma \omega^\top \omega) dx \\
 &\quad - 2\nu V_1(y) + \gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\
 &\quad + \int_0^L (\partial_x y^\top E(x) \partial_x y - \partial_x y^\top(t_k, \cdot) K^\top e^{-2\mu x} Q_2 \partial_x y \\
 &\quad - \partial_x y^\top e^{-2\mu x} Q_2 K \partial_x y(t_k, \cdot) - \gamma \partial_x y^\top(t_k, \cdot) \partial_x y(t_k, \cdot)) dx \\
 &\quad - 2\nu V_2(y_x) + \gamma \|\partial_x y(t_k, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\
 &\quad + \beta \int_0^L y_x^\top Q_2 y_x dx. \tag{2.38}
 \end{aligned}$$

with $\nu = \mu \underline{\lambda}$ and $\Omega(x), E(x)$ defined in (2.22).

In the light of $M(x)$ defined in (2.21), the inequality (2.38) can be further written as follows:

$$\begin{aligned}
 \dot{V}(y) &\leq - 2\nu V_1(y) - (2\nu - \beta) V_2(y) + \int_0^L \eta^\top M(x) \eta dx \\
 &\quad + \gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 + \gamma \|\partial_x y(t_k, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2. \tag{2.39}
 \end{aligned}$$

with

$$\eta = [y^\top, \omega^\top, (\partial_x y)^\top, (\partial_x y)^\top(t_k, \cdot)]^\top. \quad (2.40)$$

Since $M(x)$ satisfies **LMIs** (2.20), by convexity we have $\eta^\top M(x)\eta \leq 0, \forall x \in [0, L]$ and thus

$$\int_0^L \eta^\top M(x)\eta dx \leq 0, \quad (2.41)$$

we deduce from (2.39)

$$\begin{aligned} \dot{V}(y) &\leq -2\nu V_1(y) - (2\nu - \beta)V_2(y) + \gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\ &\quad + \gamma \|\partial_x y(t_k, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2. \end{aligned} \quad (2.42)$$

Step 3: The negative properties of function $\dot{V}(y)$ for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ will be discussed in this step. Consider some $k \in \mathbb{N}$ and let us first assume that $y(t, \cdot) \in \mathfrak{L}_{V \leq R} \setminus \mathfrak{L}_{V \leq \varepsilon}$ and $\forall \theta \in [t_k, t], y(\theta, \cdot) \in \mathfrak{L}_{V \leq R}$, we have

$$\begin{aligned} \int_0^L y^\top e^{-2\mu x} Q_1 y dx &\leq R, \forall y(\theta, \cdot) \in \mathfrak{L}_{V \leq R}, \\ \int_0^L \partial_x y^\top e^{-2\mu x} Q_2 \partial_x y dx &\leq R, \forall \partial_x y(\theta, \cdot) \in \mathfrak{L}_{V \leq R}. \end{aligned} \quad (2.43)$$

Then the following inequalities are further derived

$$\|y(\theta, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \leq \Phi_1, \forall \theta \in [t_k, t], \quad (2.44)$$

$$\|\partial_x y(\theta, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \leq \Phi_2, \forall \theta \in [t_k, t]. \quad (2.45)$$

with Φ_1 and Φ_2 defined in (2.24).

Recalling $y(t, \cdot) \in \mathfrak{L}_{V \leq R} \setminus \mathfrak{L}_{V \leq \varepsilon}$ and $\forall \theta \in [t_k, t], y(\theta, \cdot) \in \mathfrak{L}_{V \leq R}$, for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. The bound of $\|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2$ can be calculated under the definition of L^2 -norm

$$\|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 = \int_0^L |\omega(t, x)|^2 dx. \quad (2.46)$$

Let us recall the definition of ω in (2.12) as well as the operator J_v in (2.17), we have

$$\|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 = \int_0^L |(J_v \varphi)(t, x)|^2 dx. \quad (2.47)$$

By employing the definition of φ in (2.15) to (2.47), we have

$$\|\omega(t, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 = \int_0^L \left| \int_{t_k}^t \frac{\partial y(\theta, x)}{\partial \theta} d\theta \right|^2 dx \quad (2.48)$$

According to (2.1) and (2.2), (2.48) is equivalent to the following form

$$\|\omega(t, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 = \int_0^L \left| \int_{t_k}^t (\Lambda \partial_x y(\theta, x) + \Upsilon y(\theta, x) + Ky(t_k, x)) d\theta \right|^2 dx \quad (2.49)$$

Using the absolute value inequality and the inequality of arithmetic and geometric means for (2.49), it yields

$$\|\omega(t, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 \leq 3 \int_0^L \int_{t_k}^t (|\Lambda|^2 |\partial_x y(\theta, x)|^2 + |\Upsilon|^2 |y(\theta, x)|^2 + |K|^2 |y(t_k, x)|^2) d\theta dx. \quad (2.50)$$

Considering the definition of L^2 -norm to (2.50), we obtain

$$\begin{aligned} \|\omega(t, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 &\leq 3 \int_{t_k}^t \left(|\Lambda|^2 \|\partial_x y(\theta, x)\|_{L^2([0,L]; \mathbb{R}^n)}^2 + |\Upsilon|^2 \|y(\theta, x)\|_{L^2([0,L]; \mathbb{R}^n)}^2 \right. \\ &\quad \left. + |K|^2 \|y(t_k, x)\|_{L^2([0,L]; \mathbb{R}^n)}^2 \right) d\theta. \end{aligned} \quad (2.51)$$

Based on equation (2.3) and inequalities (2.44) and (2.45), we finally have

$$\|\omega(t, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 \leq 3\bar{h} (|\Lambda|^2 \Phi_2 + (|\Upsilon|^2 + |K|^2) \Phi_1) \quad (2.52)$$

In addition, since $y(t, \cdot), \partial_x y(t, \cdot) \notin \mathfrak{L}_{V \leq \varepsilon}$, we have

$$\begin{aligned} -2\nu V_1(y) - (2\nu - \beta)V_2(y) &\leq -(2\nu - \beta)(V_1(y) + V_2(y)) \\ &< -(2\nu - \beta)\varepsilon. \end{aligned} \quad (2.53)$$

Therefore, introducing (2.52) and (2.53) into (2.42), we have that for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$,

$$\dot{V}(y) < -(2\nu_2 - \beta)\varepsilon + \gamma 3\bar{h} (|\Lambda|^2 \Phi_2 + (|\Upsilon|^2 + |K|^2) \Phi_1) + \gamma \Phi_2, \quad (2.54)$$

and since condition (2.23) satisfies, we have shown that if $y(t, \cdot) \in \mathfrak{L}_{V \leq R} \setminus \mathfrak{L}_{V \leq \varepsilon}$ and $\forall \theta \in [t_k, t], y(\theta, \cdot) \in \mathfrak{L}_{V \leq R}$ for $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$

$$\dot{V}(y) < 0, \quad (2.55)$$

which means that since V is continuous, that y will remain in $\mathfrak{L}_{V \leq R}$ during the whole sampling interval $[t_k, t_{k+1}]$, and by recursion, we can see that it will always remains there. As a consequence, $\mathfrak{L}_{V \leq R}$ is positively invariant. Furthermore, since $\dot{V} < 0$ wherever $y \notin \mathfrak{L}_{V \leq \varepsilon}$, that means that $\mathfrak{L}_{V \leq \varepsilon}$ is attractive, which ends the proof of the $R\varepsilon$ -stability. \blacksquare

Remark 2.5

Theorem 2.1 provides sufficient conditions for the analysis of the $R\varepsilon$ -stability of the system (2.1)-(2.3). If the conditions are satisfied, then the system is $R\varepsilon$ -stable with respect to the sampling interval $h \in (0, \bar{h}]$. Note that if the conditions are satisfied for some other $\bar{h} = h^*$, then the system is $R\varepsilon$ -stable also for any other selection of sampling interval $h \in (0, h^*]$. Here the system is stable in the set where $V \in (0, \varepsilon]$.

Remark 2.6

The Lyapunov functional $V(y)$ chosen in this chapter consists of two parts V_1 and V_2 . In fact, if it is under continuous control, the function V_1 is sufficient to find an upper bound of y [4]. However, in our case, we study distributed sampled-data control, which increases the complexity of the calculation. Due to the influence of sampling signal, the term y_x is introduced in the derivative of V_1 . In order to bound the term y_x when calculating the upper bound of the sampling error ω , we have to introduce function V_2 .

Remark 2.7

The selection of the parameters in Theorem 2.1 is interpreted as follows. For $R\varepsilon$ -stability, R is the domain of attraction for a given Lyapunov function, ε specifies the positive invariant level set of V . They satisfy $0 < \varepsilon < R$. In this work, we can fix R then compute ε or vice versa. μ is related to the decay rate of V_1, V_2 . γ and β are found by linear search to satisfy the conditions given in Theorem 2.1. Q_1, Q_2 can be found by solving the **LMIs** in (2.20).

2.4 Numerical simulation

In this section, we present a numerical example to illustrate the stability we proposed in section 2.3. Consider the following system

$$\begin{cases} \partial_t y(t, x) + \Lambda \partial_x y(t, x) + \Upsilon y(t, x) + u(t, x) = 0, \\ u(t, x) = Ky(t_k, x), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \\ y(t, 0) = 0, \forall t \geq 0, \\ y(0, x) = y_0(x), \forall x \in [0, L], \end{cases}$$

where

$$\Lambda = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} 20 & 15 \\ 20 & 25 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}, \quad L = 1, \quad \bar{h} = 10^{-4}.$$

According to Remark 2.7, the parameters in Theorem 2.1 are as follows: the constants $\beta = 0.01$, $\gamma = 0.001$, the decay rates $\mu = 0.18$, the domain of attraction $R = 20$, the positive invariant level set $\varepsilon = 4.51$ and the Lyapunov matrices

$$Q_1 = 10^{-3} \times \begin{bmatrix} 5.3 & 1.2 \\ * & 2.8 \end{bmatrix}, \quad Q_2 = 10^{-3} \times \begin{bmatrix} 6.8 & 2.1 \\ * & 3.5 \end{bmatrix},$$

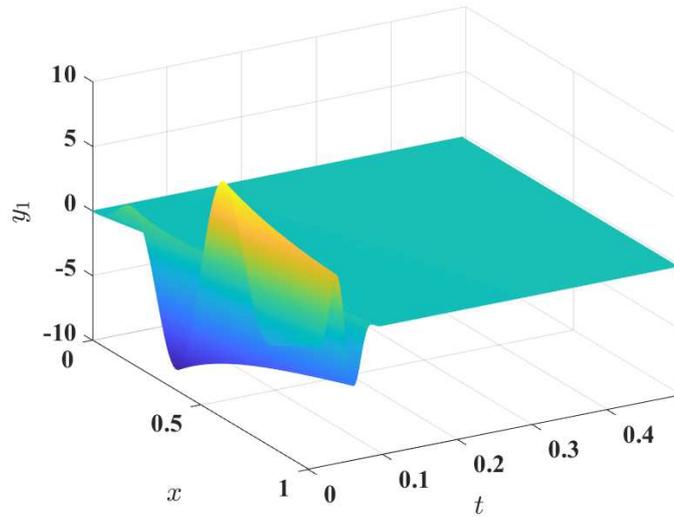


Figure 2.3: Response of state y_1 with initial condition (2.56).

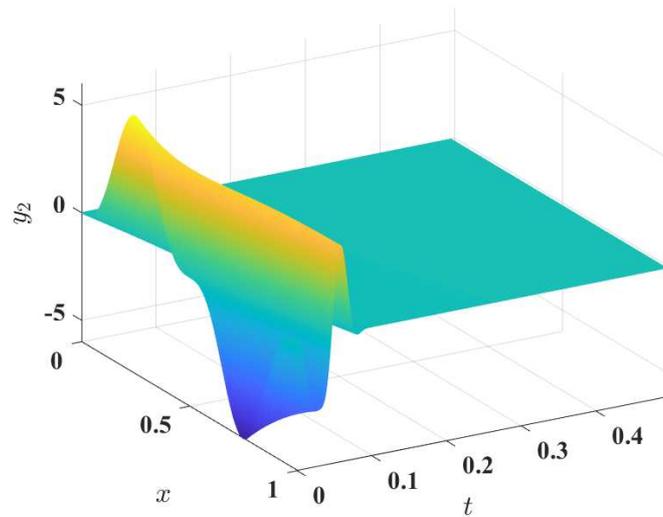


Figure 2.4: Response of state y_2 with initial condition (2.56).

which satisfy the conditions (2.20), (2.21) and (2.23).

The results of numerical simulations from the initial condition

$$y_0(x) = \begin{bmatrix} 4.8(1 - \cos 2\pi x) \sin 4\pi x \\ 3(1 - \cos 4\pi x) \sin 2\pi x \end{bmatrix} \quad (2.56)$$

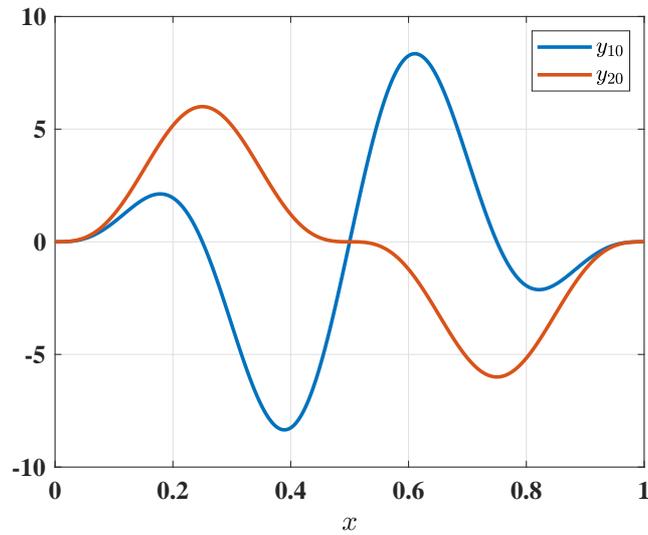


Figure 2.5: Initial data of y_1 and y_2 represented by equation (2.56).

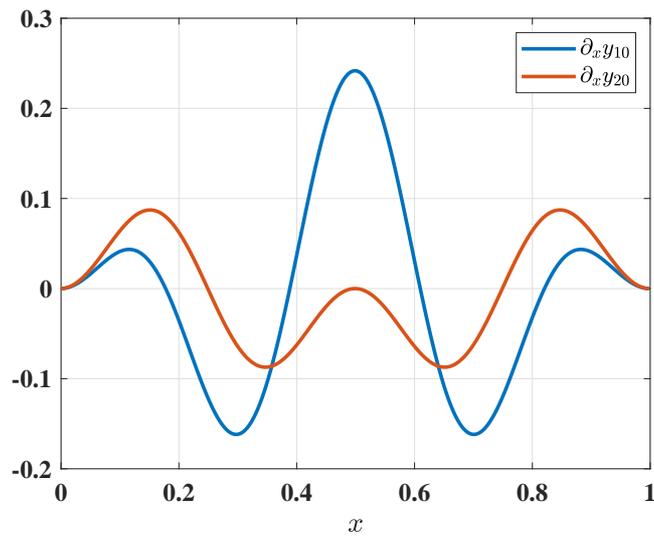


Figure 2.6: Initial data of $\partial_x y_1$ and $\partial_x y_2$.

are presented in Figures 2.3-2.7. Figures 2.3-2.4 show that the states converge to the vicinity of the origin with the sampled-data controller. The initial conditions satisfying the compatibility condition are given in Figures 2.5-2.6. The time-evolution of the Lyapunov functional V is shown in Figure 2.7 and the corresponding semi-log plot is shown in Figure 2.8. One can see that the Lyapunov functional V decreases when $\varepsilon < V(y) < R$.

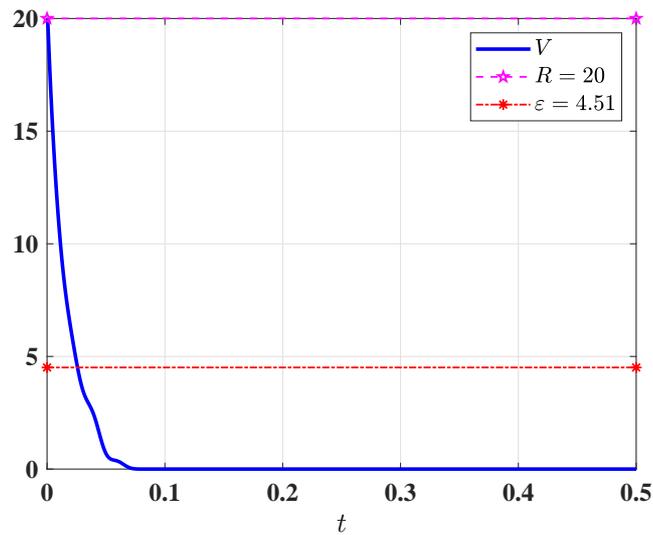


Figure 2.7: Time-evolution of function V for the initial condition (2.56) with a fixed maximum sampling interval $\bar{h} = 10^{-4}$.

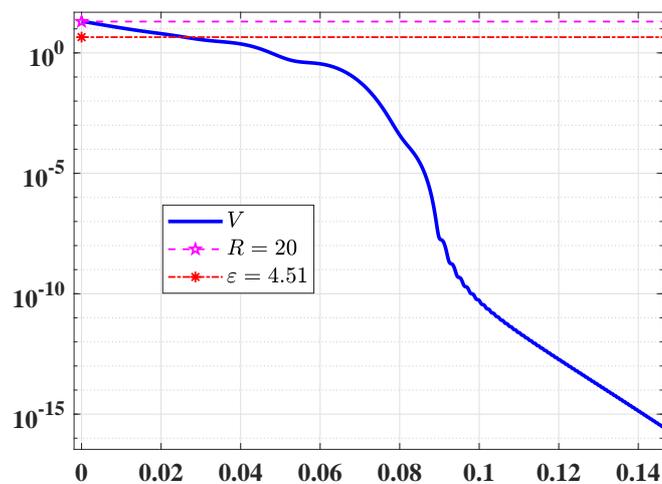


Figure 2.8: Zoom of time-evolution of function V in semi-log scale.

2.5 Conclusion

The main goal of this chapter was to provide methods for the analysis of linear hyperbolic systems with distributed sampled-data control. The closed-loop system was reformulated

from an input-output point of view. We provided methods for checking the local practical stability of the system by means of the Lyapunov method. This result will be used next in the following chapters as a stepping stone for a more complex analysis method. In Chapter 3, we will extend the proposed approach by the virtue of the Lyapunov-Razumikhin method and provide a new result that is able to handle the controller discretized both in space and time in a more efficient manner.

2.6 Appendix of Chapter 2

In this section, we give a lemma which is necessary in the proof of Theorem 2.1. More precisely, it is needed when deriving the time derivative of V_2 , so as to prove the negative definiteness of the Lyapunov functional $V(y)$.

Lemma 2.1

Consider the system (2.1)-(2.3) with initial condition y_0 satisfying the compatibility condition (2.4). Then we have

$$\partial_x y(t, 0) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}.$$

Proof of Lemma 2.1: We recall system (2.1)-(2.3)

$$\begin{cases} \partial_t y(t, x) + \Lambda \partial_x y(t, x) + \Upsilon y(t, x) + u(t, x) = 0, & (2.57a) \\ u(t, x) = Ky(t_k, x), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & (2.57b) \\ y(t, 0) = 0, \forall t \geq 0, & (2.57c) \\ y(0, x) = y_0(x), \forall x \in [0, L], & (2.57d) \end{cases}$$

The time derivative of the boundary condition leads to

$$\partial_t y(t, 0) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad (2.58)$$

Combining (2.57a) with (2.58), we obtain

$$0 = \partial_t y(t, 0) = -\Lambda \partial_x y(t, 0) - \Upsilon y(t, 0) - Ky(t_k, 0). \quad (2.59)$$

Since $y(t, 0) = 0, \forall t \geq 0$, we have

$$\partial_x y(t, 0) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad (2.60)$$

■

Sampled-data distributed control for a class of linear hyperbolic system with spatially sampled state measurements

Contents

3.1	Introduction	70
3.2	System description and problem formulation	70
3.2.1	System description	70
3.2.2	Problem formulation	72
3.3	Main result	72
3.3.1	System remodeling	73
3.3.2	Stability analysis	74
3.4	Numerical examples	85
3.5	Conclusion	90
3.6	Appendix of Chapter 3	92
3.6.1	Proof of Theorem 3.1	93
3.6.2	Proof of Step 2 in Theorem 3.2	95

This chapter addresses the stability analysis problem for a class of linear hyperbolic systems with distributed controllers sampled in both space and time. First, the considered system is recast in an equivalent form with a continuous time control loop and operators representing the discretization errors (spatio-temporal sampling errors). Then with the help of the Lyapunov Razumikhin approach, $\mathcal{R}\varepsilon$ -stability of the linear hyperbolic systems of balance laws is guaranteed via sufficient conditions. At last, the proposed method is illustrated numerically.

3.1 Introduction

In the present chapter, we aim at studying the stability properties of sampled-data controlled linear hyperbolic systems with discrete-space measurements. More precisely, we study the case of first-order hyperbolic system, which is different from the works [121, 119]. In those works, higher-order systems have been studied. We consider that the state-space is divided into several sub-domains, where sensors provide point state measurements to the controller. By generalizing the input-output approach [90] used for finite dimensional systems, an equivalent system with two sampling errors is deduced. We derive sufficient **LMI** conditions for the $\mathcal{R}\varepsilon$ -stability by utilizing appropriate Lyapunov-Razumikhin technique [67]. This chapter is a continuation of our preliminary results in [129] and Chapter 2, in which we used a simplified version of the current method for the case of hyperbolic systems where the controller is discretized only in time. Here we extend the approach by using the Lyapunov-Razumikhin function and we also consider a more general case of the controller with both time and space discretizations.

3.2 System description and problem formulation

3.2.1 System description

We consider the linear hyperbolic system (3.1) given below

$$\begin{cases} \partial_t y(t, x) + \Lambda \partial_x y(t, x) + \Upsilon y(t, x) + u(t, x) = 0, & (3.1a) \\ y(t, 0) = Gy(t, L), \forall t \geq 0, & (3.1b) \\ y(0, x) = y_0(x), \forall x \in [0, L], & (3.1c) \end{cases}$$

where $y : [0, +\infty) \times [0, L] \rightarrow \mathbb{R}^n$, $t \in \mathbb{R}_+$, $x \in [0, L]$, $\Lambda = \text{diag} \{ \lambda_1, \lambda_2, \dots, \lambda_n \}$ with $\lambda_1, \lambda_2, \dots, \lambda_n > 0$, G and Υ are real $n \times n$ matrices. $u(t, x)$ is the control law represented

as the following

$$u(t, x) = K \sum_{i=0}^{N-1} d_i(x) y(t_k, \bar{x}_i), t \in [t_k, t_{k+1}), \quad (3.2)$$

where K is a real $n \times n$ matrix.

Following [37], we assume that N sensors are uniformly distributed over the space interval $[0, L]$. The location of the sensors is denoted by

$$\bar{x}_i, i \in \{0, \dots, N-1\},$$

such that

$$\bar{x}_0 = 0, \bar{x}_i = \bar{x}_{i-1} + \bar{b}, i \in \{1, \dots, N-1\}$$

where

$$\bar{b} = \frac{L}{N-1}.$$

Each sensor is in charge of an interval

$$\Xi_i = [x_i, x_{i+1}), i \in \{0, \dots, N-1\}$$

where

$$\begin{cases} x_i = \frac{\bar{x}_{i-1} + \bar{x}_i}{2}, i \in \{1, \dots, N-1\}, \\ x_0 = 0, x_N = L. \end{cases} \quad (3.3a)$$

$$(3.3b)$$

We consider the sampling time instants

$$0 = t_0 < t_1 < \dots < t_k \dots, \quad \lim_{k \rightarrow \infty} t_k = \infty,$$

the sampling sequence is defined as

$$v = \{t_k\}_{k \in \mathbb{N}}.$$

The sampling intervals in time is bounded as

$$t_{k+1} - t_k \in [\underline{h}, \bar{h}]$$

and $\bar{h} \geq \underline{h} > 0$ are the corresponding bounds. The control setup is schematically presented in Figure 3.1. The plant is a linear hyperbolic system. Each sub-domain Ξ_i provides discrete time point measurements of the state. Then the sampling state $y(t_k, \bar{x}_i)$ is transferred to the controller and the resulting feedback with some constant gains is further implemented to the hyperbolic system through a **ZOH**. We consider that a **ZOH** control is applied using the shape function

$$\begin{cases} d_i(x) = 1, & x \in \Xi_i, \\ d_i(x) = 0, & \text{otherwise,} \end{cases} \quad i \in \{0, \dots, N-1\}. \quad (3.4)$$

The shape function is used to obtain a linear combination of controllers, each controller is responsible for the control of a region.

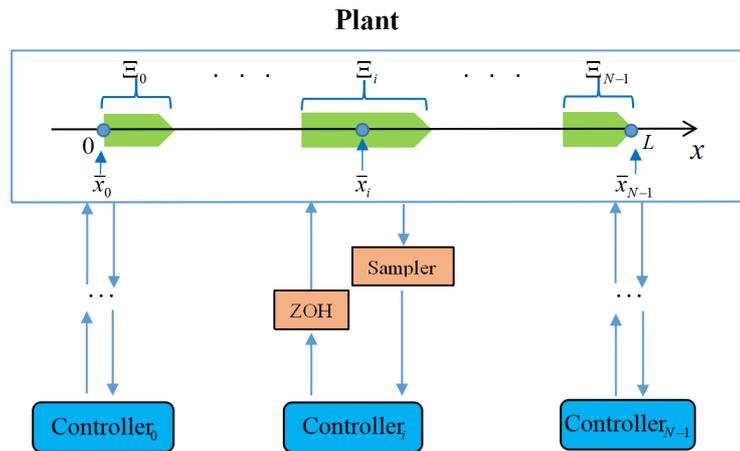


Figure 3.1: The system control setup.

Remark 3.1

To analyze the stability of the closed-loop system (3.1)-(3.2), the compatibility condition: $y_0(0) = Gy_0(L)$ is guaranteed by (3.1b) and (3.1c). In order to establish the existence and uniqueness of solutions in this section, we use the notion of piecewise continuous solutions as in Remark 2.2 of Chapter 2.

3.2.2 Problem formulation

In this chapter, we prove $R\varepsilon$ -stability defined as Definition 2.1 in Chapter 2. Our goal is to provide numerical tools for analysis of the $R\varepsilon$ -stability of the system (3.1)-(3.2), while ensuring some performances in terms of the convergence.

3.3 Main result

This section is divided into two parts. First, we represent the sampled-data system as an continuous time hyperbolic PDE. In the equivalent system, spatio-temporal sampling errors appear in the input as the disturbances. Secondly, based on the provided model, constructive $R\varepsilon$ -stability criteria are proposed.

3.3.1 System remodeling

1. We first define two parameters ω and ϑ corresponding to the induced errors. The time sampling error is

$$\omega(t, x) = y(t_k, x) - y(t, x), \quad (3.5)$$

and ϑ is the space discretization error

$$\vartheta(t_k, x) = y(t_k, x) - \sum_{i=0}^{N-1} d_i(x) y(t_k, \bar{x}_i). \quad (3.6)$$

Adding and subtracting $Ky(t, x)$ and $Ky(t_k, x)$ to (3.2), we can rewrite $u(t, x)$, $\forall t \in [t_k, t_{k+1})$ as

$$\begin{aligned} u(t, x) = & Ky(t, x) + K(y(t_k, x) - y(t, x)) \\ & - K \left(y(t_k, x) - \sum_{i=0}^{N-1} d_i(x) y(t_k, \bar{x}_i) \right). \end{aligned} \quad (3.7)$$

Combining (3.5) and (3.6) with (3.7), we have

$$u(t, x) = Ky(t, x) + K\omega(t, x) - K\vartheta(t_k, x). \quad (3.8)$$

According to (3.8), the closed-loop system (3.1)-(3.2) can be equivalently re-expressed as

$$\begin{cases} \partial_t y(t, x) + \Lambda \partial_x y(t, x) + (\Upsilon + K)y(t, x) + K\omega(t, x) - K\vartheta(t_k, x) = 0, \\ \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \end{cases} \quad (3.9a)$$

$$y(t, 0) = Gy(t, L), \forall t \geq 0, \quad (3.9b)$$

$$y(0, x) = y_0(x), \forall x \in [0, L]. \quad (3.9c)$$

In (3.9) the parameter ω , as a perturbation input, is the time sampling induced error satisfying

$$\omega(t, x) = - \int_{t_k}^t \frac{\partial y(\theta, x)}{\partial \theta} d\theta, k \in \mathbb{N}, x \in [0, L]. \quad (3.10)$$

The function φ is an auxiliary output for system (3.9) defined as follows

$$\varphi(t, x) = \frac{\partial y(t, x)}{\partial t}, \forall t \in (t_k, t_{k+1}), x \in [0, L]. \quad (3.11)$$

According to (3.5) and (3.11), we have the following expression

$$\omega(t, x) = - \int_{t_k}^t \varphi(\theta, x) d\theta, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, x \in [0, L]. \quad (3.12)$$

Then the perturbation input ω can be expressed by a sampling ν -dependent operator J_ν :

$$J_\nu : \begin{cases} \omega(t, x) = (J_\nu \varphi)(t, x) = - \int_{t_k}^t \varphi(\theta, x) d\theta, \\ \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, x \in [0, L]. \end{cases} \quad (3.13)$$

The parameter ϑ in (3.9) is another disturbance input, which is an error caused by space discretization, satisfying

$$\vartheta(t_k, x) = \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x \frac{\partial y(t_k, \varsigma)}{\partial \varsigma} d\varsigma. \quad (3.14)$$

We use the function ϕ as another auxiliary output for system (3.9), $\forall t \geq 0, x \in (x_i, x_{i+1}), i \in \{0, \dots, N-1\}$,

$$\phi(t, x) = \frac{\partial y(t, x)}{\partial x}. \quad (3.15)$$

The space discretization error ϑ can be handled by the operator E :

$$E : \begin{cases} \vartheta(t_k, x) = (E\phi)(t_k, x) = \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x \phi(t_k, \varsigma) d\varsigma, \\ \forall x \in \Xi_i, i \in \{0, \dots, N-1\}. \end{cases} \quad (3.16)$$

3.3.2 Stability analysis

In this subsection, before giving our main results and $\mathcal{R}\varepsilon$ -stability conditions, we first introduce the Lyapunov-Razumikhin approach which is adapted in the following theorems.

Compared with the case of **ODE**, it is much more difficult to find a functional whose time derivative is non-positive or strictly negative for *Retarded Functional Differential Equations* (**RFDEs**). In 1956, B.S. Razumikhin proposed a method to study the stability of solutions of time-delay systems [102]. Razumikhin's method essentially extends the stability theorem in the sense of Lyapunov. His basic idea is to constrain the derivative of the Lyapunov function only when the solution is about to exit the steady-state sphere. The method proposed by B.S. Razumikhin can make the Lyapunov function applicable in **RFDE** to a considerable extent. In some cases, this application is simpler and more intuitive than the application of general functions [55]. Although this method is dedicated to the stability analysis of time-delay system, we adapt it for the sampled-data hyperbolic systems.

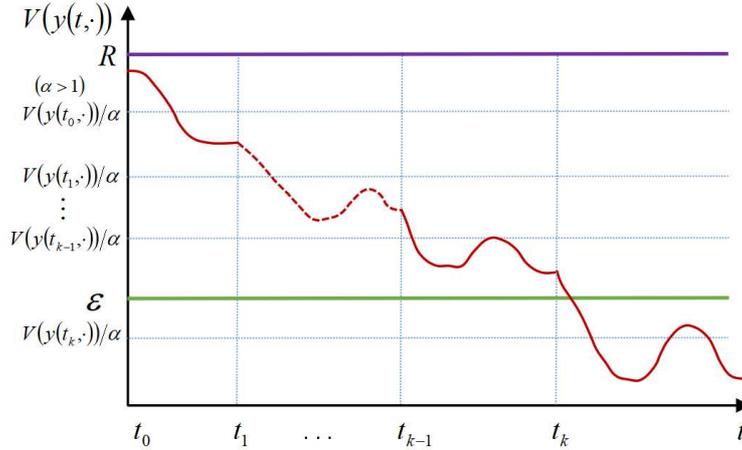


Figure 3.2: Schematic diagram of $R\varepsilon$ -stability under Lyapunov-Razumikhin method.

Theorem 3.1

Consider systems (3.9)-(3.16) with (3.3)-(3.4) and a candidate Lyapunov functional

$$V : H^1([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}_+$$

which is differentiable with respect to its argument and $V(0) = 0$, $V(\varrho) > 0, \forall \varrho \in \mathcal{O} \setminus \{0\}$, where \mathcal{O} is the neighborhood of the region. Given constants $\alpha > 1$, $\delta > 0$, $0 < \varepsilon < R$, suppose that along the trajectories of the system (3.9)-(3.16), the corresponding solution $y(t, \cdot)$ satisfies

$$\dot{V}(y) + 2\delta V(y) \leq 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$$

whenever

$$(i) R > V(y(t, \cdot)) \geq \max \left\{ \varepsilon, \frac{V(y(t_k, \cdot))}{\alpha} \right\},$$

$$(ii) y(t_k, \cdot) \in \mathfrak{L}_{V < R}, k \in \mathbb{N}.$$

Then the system is $R\varepsilon$ -stable from $\mathfrak{L}_{V < R}$ to $\mathfrak{L}_{V \leq \varepsilon}$.

The proof of Theorem 3.1 can be found in the appendix.

Remark 3.2

In the Lyapunov-Razumikhin approach, the main idea is that it is not necessary to ensure the negative definiteness of $\dot{V}(y(t, \cdot))$ along all the trajectories of the system. In fact, it is sufficient to guarantee its negative definiteness only for the solutions that tend to escape the neighborhood of $V(y(t, \cdot)) \leq \frac{V(y(t_k, \cdot))}{\alpha}$ of the equilibrium (see Figure 3.2). Furthermore, the Razumikhin method is also adapted here to local, practical stability from one large open

76 Chapter 3. Hyperbolic system with spatially sampled state measurements

set, to a smaller one, with guaranteed exponential decay. Theorem 3.1 is a generic result concerning the $R\varepsilon$ -stability properties of Lyapunov-Razumikhin functionals. It represents a stepping stone for Theorem 3.2 which is our main theoretical result.

In the following theorem, we provide sufficient conditions for the $R\varepsilon$ -stability of systems (3.9)-(3.16). In order to guarantee the negativeness of the time derivative of the Lyapunov function, we give four **LMIs** in condition (i). (3.17) and (3.18) are related to boundary control and (3.19) is derived by using Schur complement [10] (see Theorem A.5 in Appendix A) to eliminate some other terms in the derivative. Condition (ii) is an inequality with respect to the sampling interval \bar{h} and positive invariant set ε . Combining condition (ii) with condition (i), the stability of the systems can be proved.

Theorem 3.2

Consider systems (3.9)-(3.16) with (3.3)-(3.4) and an initial condition satisfying $y_0(0) = Gy_0(L)$:

(i) Let $\underline{\lambda} = \min_{i \in \{1, \dots, n\}} \lambda_i$. Assume that there exist constants $\mu, \gamma, \kappa > 0$, $\alpha > 1$ and symmetric positive definite matrices $\Theta_1 \in \mathbb{R}^{n \times n}$, $\Theta_2 \in \mathbb{R}^{n \times n}$ satisfying the commutativity conditions: $\Lambda\Theta_1 = \Theta_1\Lambda$, $\Lambda\Theta_2 = \Theta_2\Lambda$ and

$$-\Lambda e^{-2\mu L}\Theta_1 + G^\top \Lambda \Theta_1 G \preceq 0, \quad (3.17)$$

$$P := U^\top \Lambda \Theta_2 U - \chi \preceq 0 \quad (3.18)$$

with

$$U = [\Lambda^{-1}G\Lambda, \Lambda^{-1}G\Upsilon - \Lambda^{-1}\Upsilon G, \Lambda^{-1}GK - \Lambda^{-1}KG],$$

$$\chi = \left[\begin{array}{c|c} \Lambda e^{-2\mu L}\Theta_2 & \mathbf{0}_{n \times 2n} \\ \hline * & \mathbf{0}_{2n \times 2n} \end{array} \right]$$

and

$$M(0) \succeq 0, \quad M(L) \succeq 0, \quad (3.19)$$

with $M(x)$ defined for all $x \in [0, L]$ as

$$M(x) = \left[\begin{array}{c|c} Q(x) & S(x) \\ \hline * & I \end{array} \right] \succeq 0, \quad (3.20)$$

where

$$Q(x) = \begin{bmatrix} \Omega_1(x) & e^{-2\mu x} \Theta_1 K + \kappa e^{-2\mu x} \Theta_1 & 0 & 0 \\ * & \gamma I + \kappa e^{-2\mu x} \Theta_1 & 0 & 0 \\ * & * & \Omega_2(x) & 0 \\ * & * & * & \Omega_3(x) \end{bmatrix}, \quad (3.21)$$

$$S(x) = \begin{bmatrix} e^{-2\mu x} \Theta_1^\top \\ \mathbf{0}_{3n \times n} \end{bmatrix}, \quad (3.22)$$

and

$$\begin{aligned} \Omega_1(x) &= e^{-2\mu x} \left[(K + \Upsilon)^\top \Theta_1 + \Theta_1 (K + \Upsilon) - \kappa(\alpha - 1) \Theta_1 \right], \\ \Omega_2(x) &= e^{-2\mu x} \left[\Upsilon^\top \Theta_2 + \Theta_2 \Upsilon + \beta \Theta_2 - \kappa \alpha \Theta_2 \right], \\ \Omega_3(x) &= -\frac{\bar{b}^2}{\pi^2} K^\top K + \kappa e^{-2\mu x} \Theta_2. \end{aligned} \quad (3.23)$$

(ii) Assume that there exist $\varepsilon, R \in \mathbb{R}_+$ s.t. $0 < \varepsilon < R$ and

$$\gamma 3\bar{h} \left((|\Lambda|^2 + 2|K|^2 L^2) \Phi_2 + (|\Upsilon|^2 + 2|K|^2 L|G|^2) \Phi_1 \right) \leq (2\sigma - \beta)\varepsilon - 2\delta R, \quad (3.24)$$

where $\sigma = \mu \underline{\lambda}$, $0 < \beta < 2\sigma$, $\delta > 0$ and

$$\Phi_1 = \frac{R}{\lambda_{\min}(\Theta_1) e^{-2\mu L}}, \quad \Phi_2 = \frac{R}{\lambda_{\min}(\Theta_2) e^{-2\mu L}}.$$

Then the considered system is $R\varepsilon$ -stable from $\mathfrak{L}_{V < R}$ to $\mathfrak{L}_{V \leq \varepsilon}$ for any sampling sequence satisfying $t_{k+1} - t_k \in [\underline{h}, \bar{h}]$, with the Lyapunov functional defined as

$$V(y) = V_1(y) + V_2(y), \quad (3.25)$$

with

$$\mathfrak{L}_{V \leq R} = \{y \in H^1([0, L]; \mathbb{R}^n) : V(y) \leq R\},$$

$$\mathfrak{L}_{V \leq \varepsilon} = \{y \in H^1([0, L]; \mathbb{R}^n) : V(y) \leq \varepsilon\},$$

and

$$V_1(y) = \int_0^L y^\top e^{-2\mu x} \Theta_1 y dx, \quad (3.26)$$

$$V_2(y) = \int_0^L y_x^\top e^{-2\mu x} \Theta_2 y_x dx. \quad (3.27)$$

Proof of Theorem 3.2: Similar to the proof of Theorem 2.1, we assume that the solutions y are of class \mathcal{C}^2 (see Comment 4.6 in Bastin 2016 [4]).

Consider the Lyapunov functional (3.25)-(3.27). It can be bounded as follows:

$$\Psi_1 \|y(t, \cdot)\|_{H^1([0,L]; \mathbb{R}^n)}^2 \leq V(y(t, \cdot)) \leq \Psi_2 \|y(t, \cdot)\|_{H^1([0,L]; \mathbb{R}^n)}^2, \quad (3.28)$$

where $\Psi_1 = \min\{\lambda_{\min}(\Theta_1), \lambda_{\min}(\Theta_2)\}e^{-2\mu L}$, $\Psi_2 = \max\{\lambda_{\max}(\Theta_1), \lambda_{\max}(\Theta_2)\}$.

Step 1: Adopt to Step 1 in the proof of Theorem 2.1, the Lyapunov functional defined in (3.25) is continuous.

Step 2: In this step we study the time derivative of the function of $V(y)$ defined in (3.25) and its upper bound is defined by the following inequality:

$$\dot{V}(y) \leq -2\sigma V_1(y) - (2\sigma - \beta)V_2(y) + \int_0^L \eta^\top W(x)\eta dx + \gamma \|\omega(t, \cdot)\|_{L^2([0,L]; \mathbb{R}^n)}^2 \quad (3.29)$$

with

$$\begin{aligned} \sigma &= \mu\lambda, \\ \eta &= [y^\top, \omega^\top, (\partial_x y)^\top, (\partial_x y(t_k, \cdot))^\top]^\top, \\ \mathcal{A}_1(x) &= e^{-2\mu x} \left[(K + \Upsilon)^\top \Theta_1 + \Theta_1 (K + \Upsilon) \right], \\ \mathcal{A}_2(x) &= e^{-2\mu x} \left[\Upsilon^\top \Theta_2 + \Theta_2 \Upsilon + \beta \Theta_2 \right], \\ \mathcal{A}_3(x) &= -\mathcal{A}_1(x) + e^{-4\mu x} \Theta_1^\top \Theta_1 \end{aligned}$$

and

$$W(x) = \begin{bmatrix} \mathcal{A}_3(x) & -e^{-2\mu x} \Theta_1 K & 0 & 0 \\ * & -\gamma I & 0 & 0 \\ * & * & -\mathcal{A}_2(x) & 0 \\ * & * & * & \frac{\bar{b}^2}{\pi^2} K^\top K \end{bmatrix}.$$

A detailed proof of (3.29) is given in the Appendix of Chapter 3.

Step 3: In this step, we show that under the conditions of Theorem 3.2, for each sampling interval, V is decaying with a decay rate 2δ whenever it is greater than a target level set $\frac{V(y(t_k, \cdot))}{\alpha}$ and a positive invariant level set ε . This can be more intuitively observed through Figure 3.2 and can be expressed as

$$\dot{V}(y) + 2\delta V(y) \leq 0,$$

whenever

$$\left\{ R > V(y(t, \cdot)) \geq \max \left\{ \varepsilon, \frac{V(y(t_k, \cdot))}{\alpha} \right\}, \right. \quad (3.30a)$$

$$\left. y(\theta, \cdot) \in \mathfrak{L}_{V < R}, \forall \theta \in [t_k, t], k \in \mathbb{N}. \right\} \quad (3.30b)$$

Let us assume that conditions (3.30) hold. Since condition (3.19) holds, by convexity, we have

$$M(x) \succeq 0, \text{ for } x \in [0, L].$$

Moreover, by using Lemma 3.2 in the appendix, we have

$$W(x) + \kappa N(x) \preceq 0, \text{ for } x \in [0, L].$$

Therefore, we get

$$\int_0^L \eta^\top (W(x) + \kappa N(x)) \eta dx \leq 0, \quad (3.31)$$

with $N(x)$ defined as below

$$N(x) = e^{-2\mu x} \begin{bmatrix} (\alpha - 1)\Theta_1 & -\Theta_1 & 0 & 0 \\ * & -\Theta_1 & 0 & 0 \\ * & * & \alpha\Theta_2 & 0 \\ * & * & * & -\Theta_2 \end{bmatrix}. \quad (3.32)$$

Now, consider $t \in [t_k, t_{k+1})$ and a trajectory y satisfying (3.30). Since condition (3.30a) is satisfied, we have

$$V(y(t, \cdot)) \geq \frac{V(y(t_k, \cdot))}{\alpha} \text{ with some } \alpha > 1,$$

which can be rewritten as

$$\alpha V(y(t, \cdot)) - V(y(t_k, \cdot)) \geq 0.$$

According to (3.25)-(3.27), by simple manipulation we get

$$\int_0^L \eta^\top N(x) \eta dx \geq 0. \quad (3.33)$$

By Integral S-procedure [126] (see Lemma A.2 in Appendix A), (3.31) and (3.33) implies that

$$\int_0^L \eta^\top W(x) \eta dx \leq 0. \quad (3.34)$$

And because condition (3.30b) is satisfied, we have

$$\begin{aligned} \int_0^L y^\top e^{-2\mu x} \Theta_1 y dx &< R, \forall y(\theta, \cdot) \in \mathfrak{L}_{V < R}, \\ \int_0^L \partial_x y^\top e^{-2\mu x} \Theta_2 \partial_x y dx &< R, \forall \partial_x y(\theta, \cdot) \in \mathfrak{L}_{V < R}. \end{aligned}$$

The following inequalities are further derived

$$\|y(\theta, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 < \Phi_1, \|\partial_x y(\theta, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 < \Phi_2, \forall \theta \in [t_k, t]. \quad (3.35)$$

Using the operator J_v in (3.13) with (3.11), we can compute the upper bound

$$\|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2$$

$$\begin{aligned}
 &= \int_0^L |\omega(t, x)|^2 dx = \int_0^L \left| \int_{t_k}^t \frac{\partial y(\theta, x)}{\partial \theta} d\theta \right|^2 dx \\
 &= \int_0^L \left| \int_{t_k}^t \left(\Lambda \partial_x y(\theta, x) + \Upsilon y(\theta, x) + K \sum_{i=0}^{N-1} d_i(x) y(t_k, \bar{x}_i) \right) d\theta \right|^2 dx \\
 &\leq 3 \int_0^L \int_{t_k}^t \left(|\Lambda|^2 |\partial_x y(\theta, x)|^2 + |\Upsilon|^2 |y(\theta, x)|^2 + |K|^2 \left| \sum_{i=0}^{N-1} d_i(x) y(t_k, \bar{x}_i) \right|^2 \right) d\theta dx, \quad (3.36)
 \end{aligned}$$

According to (3.9b),

$$\begin{aligned}
 y(t_k, \bar{x}_i) &= y(t_k, \bar{x}_i) - y(t_k, 0) + y(t_k, 0) \\
 &= \int_0^{\bar{x}_i} \partial_x y(t_k, x) dx + Gy(t_k, L) \\
 &\leq \int_0^L |\partial_x y(t_k, x)| dx + Gy(t_k, L). \quad (3.37)
 \end{aligned}$$

This inequality corresponds to an upper bound on the control error due to space discretization. Cauchy-Schwarz inequality [14] (see Theorem A.2 in Appendix A) yields

$$\begin{aligned}
 &\left| \sum_{i=0}^{N-1} d_i(x) y(t_k, \bar{x}_i) \right|^2 \\
 &\leq L \left| \int_0^L |\partial_x y(t_k, x)| dx + Gy(t_k, L) \right|^2 \\
 &\leq 2L \left(\|\partial_x y(t_k, x)\|_{L^2([0, L]; \mathbb{R}^n)}^2 + |G|^2 |y(t_k, L)|^2 \right). \quad (3.38)
 \end{aligned}$$

Then by substituting (3.38) into (3.36) and by using (3.35) we have

$$\begin{aligned}
 &\|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\
 &\leq 3 \int_{t_k}^t \left(|\Lambda|^2 \|\partial_x y(\theta, x)\|_{L^2([0, L]; \mathbb{R}^n)}^2 + |\Upsilon|^2 \|y(\theta, x)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \right. \\
 &\quad \left. + |K|^2 2L^2 \|\partial_x y(t_k, x)\|_{L^2([0, L]; \mathbb{R}^n)}^2 + 2L|K|^2 |G|^2 \|y(t_k, L)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \right) d\theta \\
 &\leq 3\bar{h} \left((|\Lambda|^2 + 2|K|^2 L^2) \Phi_2 + (|\Upsilon|^2 + 2L|K|^2 |G|^2) \Phi_1 \right) = \varpi. \quad (3.39)
 \end{aligned}$$

This upper bound corresponds to an estimate of the maximum time sampling error. In addition, since condition (3.30a) is satisfied, we have

$$-2\sigma V_1(y) - (2\sigma - \beta)V_2(y) \leq -(2\sigma - \beta)(V_1(y) + V_2(y)) < -(2\sigma - \beta)\varepsilon. \quad (3.40)$$

The inequality (3.40) corresponds to an estimate of the domain of attraction. Therefore, instituting (3.34), (3.39) and (3.40) into (3.29), we have

$$\dot{V}(y) < -(2\sigma - \beta)\varepsilon + \gamma\varpi, \forall t \in [t_k, t_{k+1}).$$

Since (3.24) holds, we have

$$\dot{V}(y) < -2\delta R \leq -2\delta V(y).$$

Therefore, we have shown that

$$\dot{V}(y) + 2\delta V(y) \leq 0,$$

whenever conditions (3.30) are satisfied.

Step 4: In this step, we show that if

$$y(t_k, \cdot) \in \mathfrak{L}_{V < R},$$

then

$$y(t, \cdot) \in \mathfrak{L}_{V < R}, \forall t \in [t_k, t_{k+1}).$$

First, we consider y such that

$$y(t_k, \cdot) \in \mathfrak{L}_{V < R},$$

assume that

$$\exists t^\circ \in (t_k, t_{k+1}) \text{ such that } V(y(t^\circ, \cdot)) \geq R.$$

Let us then call T° the minimum of such t° , then

$$V(y(t, \cdot)) < R, \forall t \in [t_k, T^\circ).$$

Therefore conditions (3.30) are going to be satisfied for any $t \in [t_k, T^\circ)$. From step 3, we know that V is going to decrease during that time interval, either continuously, or until V reaches below $\max\left\{\varepsilon, \frac{V(y(t_k, \cdot))}{\alpha}\right\}$ and when it reaches that region, it never gets back out. Therefore, we have

$$V(y(T^\circ, \cdot)) < V(y(t_k, \cdot)) < R,$$

which contradicts the assumption that there exists $t^\circ \in (t_k, t_{k+1})$ such that $V(y(t^\circ, \cdot)) \geq R$.

Summary: From step 3 and step 4, it is clear that $\dot{V}(y) + 2\delta V(y) \leq 0$ wherever

$$\left\{ R > V(y(t, \cdot)) \geq \max\left\{\varepsilon, \frac{V(y(t_k, \cdot))}{\alpha}\right\}, \right. \quad (3.41a)$$

$$\left. y(t_k, \cdot) \in \mathfrak{L}_{V < R}, \right. \quad (3.41b)$$

and therefore, the conditions of Theorem 3.1 are satisfied, which concludes the proof of $R\varepsilon$ -stability. \blacksquare

Theorem 3.2 provides constructive conditions for the analysis of the sampled-data hyperbolic system (3.9)-(3.16). On the premise of ensuring that V is continuous, we first calculate the time derivative of V and then use the Lyapunov-Razumikhin technique to prove that it is negative. Finally, it is proved that as long as the state at the sampling instant is in the attraction domain, the state will not escape the domain $\mathfrak{L}_{V < R}$ during the entire time sampling interval.

Remark 3.3

We explain as follows the selection of parameters in the previous theorem. For $R\varepsilon$ -stability, R is the domain of attraction for a given Lyapunov function, ε specifies the positive invariant level set of V . They satisfy $0 < \varepsilon < R$. In this chapter, we can fix R then compute ε or vice versa. α is a parameter introduced in the Lyapunov-Razumikhin method to define level set in which the time derivative of $V(y(t, \cdot))$ should be negative between two sampling interval, we use a trial-and-error approach to choose it to be greater than 1 but as small as possible to reduce the conservativeness of conditions. 2μ is related to the decay rate of V_1, V_2 and δ is related to the decay rate of V . γ, κ, \bar{h} and \bar{b} are found by linear search to satisfy the conditions given in Theorem 3.1. Θ_1, Θ_2 can be found by solving the **LMIs** in (3.19) and (3.20). Due to (3.24), we adjust γ, β to be the smallest possible and μ to be the largest possible.

Theorem 3.2 is a general case. Next, we will give a special case in the following theorem where only time sampling is considered. Their difference is also reflected in the boundary condition. In Theorem 3.3, the boundary condition is also a special case, that is, $y(t, 0) = 0$.

Theorem 3.3

Consider systems (2.13)-(2.18) and an initial condition satisfying $y_0(0) = 0$:

(i) Let $\underline{\lambda} = \min_{i \in \{1, \dots, n\}} \lambda_i$. Assume that there exist $\mu, \gamma, \kappa > 0, \alpha > 1$ and symmetric positive matrices $\Theta_1 \in \mathbb{R}^{n \times n}, \Theta_2 \in \mathbb{R}^{n \times n}$ satisfying the commutativity conditions: $\Lambda\Theta_1 = \Theta_1\Lambda, \Lambda\Theta_2 = \Theta_2\Lambda$ and

$$W(0) + \kappa N(0) \preceq 0, \quad W(L) + \kappa N(L) \preceq 0, \quad (3.42)$$

with $W(x)$ and $N(x)$ defined for all $x \in [0, L]$ as follows:

$$W(x) = \begin{bmatrix} B_1(x) & -e^{-2\mu x}\Theta_1 K & 0 & 0 \\ * & -\gamma I & 0 & 0 \\ * & * & B_2(x) & -e^{-2\mu x}\Theta_2 K \\ * & * & * & -\gamma I \end{bmatrix}, \quad (3.43)$$

$$N(x) = e^{-2\mu x} \begin{bmatrix} (\alpha - 1)\Theta_1 & -\Theta_1 & 0 & 0 \\ * & -\Theta_1 & 0 & 0 \\ * & * & \alpha\Theta_2 & 0 \\ * & * & * & -\Theta_2 \end{bmatrix} \quad (3.44)$$

where

$$B_1(x) = -e^{-2\mu x} \left[(K + \Upsilon)^\top \Theta_1 + \Theta_1 (K + \Upsilon) \right],$$

$$B_2(x) = -e^{-2\mu x} \left[\Upsilon^\top \Theta_2 + \Theta_2 \Upsilon + \beta \Theta_2 \right].$$

(ii) If $\exists \varepsilon \in \mathbb{R}_+$, $R \in \mathbb{R}_+$ s.t. $0 < \varepsilon < R$ and

$$\gamma 3\bar{h} (|\Lambda|^2 \Phi_2 + (|\Upsilon|^2 + |K|^2) \Phi_1) + \gamma \Phi_2 \leq (2\sigma - \beta)\varepsilon - 2\delta R, \quad (3.45)$$

where $\sigma = \mu\lambda$, $0 < \beta < 2\sigma$, $\delta > 0$ and

$$\Phi_1 = \frac{R}{\lambda_{\min}(\Theta_1) e^{-2\mu L}}, \quad \Phi_2 = \frac{R}{\lambda_{\min}(\Theta_2) e^{-2\mu L}}, \quad (3.46)$$

Then the considered system (2.1) is $R\varepsilon$ -stable from $\mathfrak{L}_{V < R}$ to $\mathfrak{L}_{V \leq \varepsilon}$ for any sampling sequence satisfying $y_0(0) = 0$, with the Lyapunov functional defined by

$$V(y) = V_1(y) + V_2(y), \quad (3.47)$$

with

$$\begin{aligned} \mathfrak{L}_{V \leq R} &= \{y \in H^1([0, L]; \mathbb{R}^n) : V(y) \leq R\}, \\ \mathfrak{L}_{V \leq \varepsilon} &= \{y \in H^1([0, L]; \mathbb{R}^n) : V(y) \leq \varepsilon\}, \end{aligned}$$

and

$$V_1(y) = \int_0^L y^\top e^{-2\mu x} \Theta_1 y dx, \quad (3.48)$$

$$V_2(y) = \int_0^L y_x^\top e^{-2\mu x} \Theta_2 y_x dx. \quad (3.49)$$

Proof of Theorem 3.3:

The proof is obtained using similar arguments as in Theorem 2.1 and using an appropriate Lyapunov-Razumikhin technique as in Theorem 3.2.

Step 1: The time derivative of $V(y)$ defined in (3.47) is

$$\begin{aligned} \dot{V}(y) &= \dot{V}_1(y) + \dot{V}_2(y) \leq -2\sigma V_1(y) - (2\sigma - \beta)V_2(y) \\ &\quad + \int_0^L \eta^\top W(x) \eta dx + \gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\ &\quad + \gamma \|\partial_x y(t_k, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2. \end{aligned} \quad (3.50)$$

with

$$\begin{aligned} \sigma &= \mu\lambda, \\ \eta &= [y^\top, \omega^\top, (\partial_x y)^\top, (\partial_x y(t_k, \cdot))^\top]^\top \end{aligned}$$

and $W(x)$ defined in (3.43).

84 Chapter 3. Hyperbolic system with spatially sampled state measurements

For detailed proof, please refer to (2.33)-(2.39) in Chapter 2.

Step 2: In this step, we show that

$$\dot{V}(y) + 2\delta V(y) \leq 0,$$

whenever

$$\left\{ \begin{array}{l} R > V(y(t, \cdot)) \geq \max \left\{ \varepsilon, \frac{V(y(t_k, \cdot))}{\alpha} \right\}, \\ y(\theta, \cdot) \in \mathfrak{L}_{V < R}, \forall \theta \in [t_k, t], k \in \mathbb{N}. \end{array} \right. \quad (3.51a)$$

$$(3.51b)$$

Let us assume that conditions (3.51) hold. Since condition (3.42) is linear in $e^{-2\mu x}$ and $0 \leq x \leq L$, by convexity, we have

$$W(x) + \kappa N(x) \leq 0, \text{ for } x \in [0, L].$$

Therefore, we get

$$\int_0^L \eta^\top (W(x) + \kappa N(x)) \eta dx \leq 0, \quad (3.52)$$

with $W(x)$ and $N(x)$ given in (3.43) and (3.44).

Now, consider $t \in [t_k, t_{k+1})$ and a trajectory y satisfying (3.51). Since condition (3.51a) is satisfied, we have

$$V(y(t, \cdot)) \geq \frac{V(y(t_k, \cdot))}{\alpha} \text{ with some } \alpha > 1,$$

which can be rewritten as

$$\int_0^L \eta^\top N(x) \eta dx \geq 0. \quad (3.53)$$

In view of (3.52), (3.53) and $\kappa > 0$, it implies

$$\int_0^L \eta^\top W(x) \eta dx \leq 0. \quad (3.54)$$

According to conditions (3.51) and the upper bound of $\|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2$ defined as (2.52):

$$\|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \leq 3\bar{h} (|\Lambda|^2 \Phi_2 + (|\Upsilon|^2 + |K|^2) \Phi_1) = \varpi, \quad (3.55)$$

we have

$$\begin{aligned} -2\sigma V_1(y) - (2\sigma - \beta)V_2(y) &\leq -(2\sigma - \beta)(V_1(y) + V_2(y)) \\ &< -(2\sigma - \beta)\varepsilon. \end{aligned} \quad (3.56)$$

Therefore, substituting (3.54), (3.55) and (3.56) into (3.50), we have for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$,

$$\dot{V}(y) < -(2\sigma - \beta)\varepsilon + \gamma\varpi + \gamma\Phi_2. \quad (3.57)$$

Since (3.45) holds, we deduce from (3.57),

$$\dot{V}(y) < -2\delta R \leq -2\delta V(y). \quad (3.58)$$

Therefore, we have shown that $\dot{V}(y) + 2\delta V(y) \leq 0$, whenever conditions (3.51) are satisfied. ■

3.4 Numerical examples

In this section, we use two examples to verify Theorem 3.2 and Theorem 3.3. First, Example 1 corresponds to Theorem 3.2, which illustrates that the reaching time of the closed-loop system is less than the open-loop system case.

Example 1:

Consider system (3.1)-(3.2) with

$$\Lambda = \begin{bmatrix} 1.4 & 0 \\ 0 & 1.4 \end{bmatrix}, \Upsilon = \begin{bmatrix} 2 & 0.5 \\ -0.5 & 1 \end{bmatrix}, K = \begin{bmatrix} 9 & -0.1 \\ -0.5 & 8 \end{bmatrix},$$

$$L = 1, G = \begin{bmatrix} 0.1 & 0.02 \\ -0.1 & -0.01 \end{bmatrix}.$$

The system is open-loop stable, however, the use of control is required in practice to ensure some performances (reaching time, peak response, disturbance rejection, etc.). For example in [Section 5.1, [4]] the control is used for a stable chemical plug flow reactor to avoid the risk of peaks in the temperature profile. In our context, the use of control can improve some convergence performances.

In the following, we first provide the comparison of the state responses of the open-loop and closed-loop system. We can see from Figures 3.3-3.6 that the system states practically converge much faster in closed-loop. Figure 3.7 illustrates the time-evolutions of $\|y\|_{H^1}$ for open-loop and closed-loop systems. From Figure 3.7 we can see that the reaching time is $t_1 = 0.55s$ for the state with the sampled-data controller ($\|y_{close}\|_{H^1}$) to the 5% neighborhood of the origin ($\Delta = 0.62$), which is smaller than $t_2 = 0.85s$ in open-loop

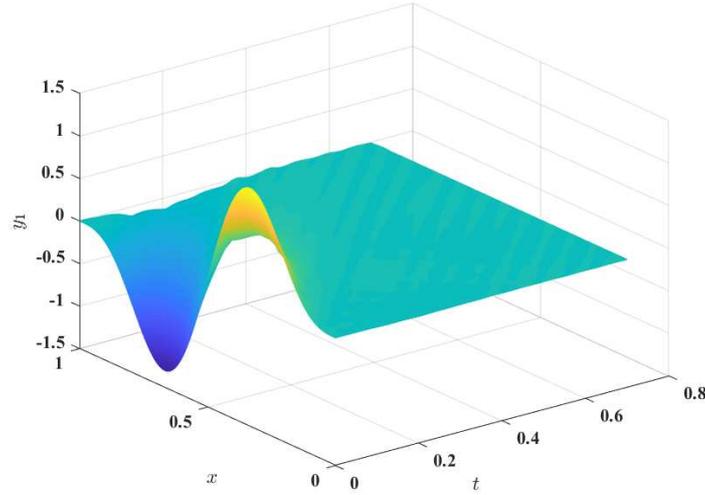


Figure 3.3: Response of state y_1 for the closed-loop system.

system case. The figures clearly demonstrate that the sampled-data controller improves system's performance in terms of reaching time.

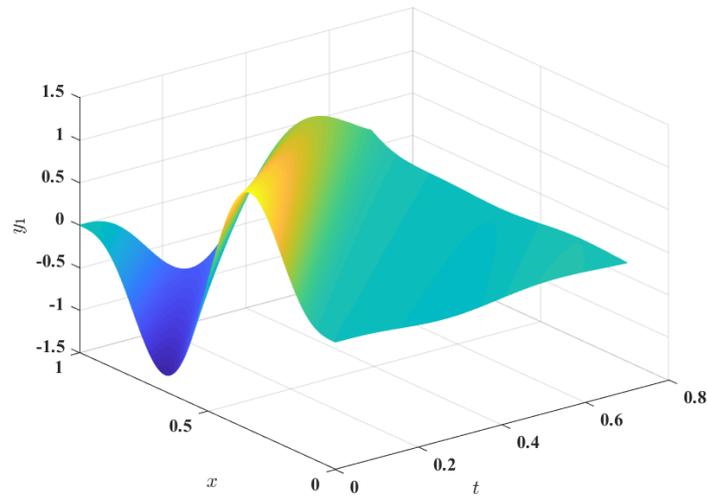
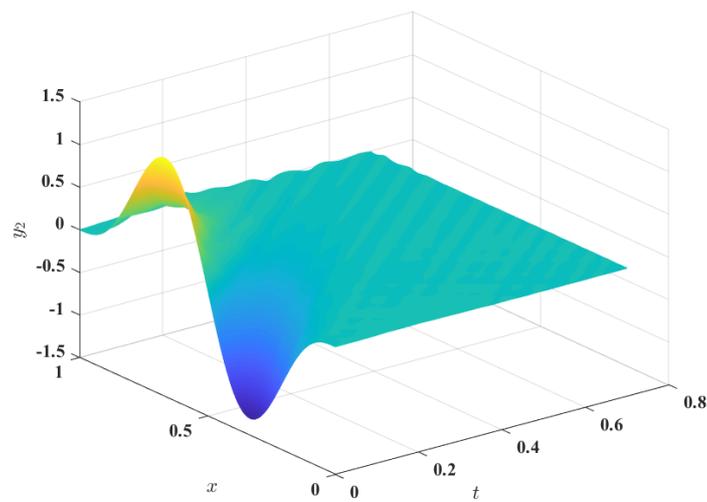
We apply Theorem 3.2 in order to verify the $R\varepsilon$ -stability of the closed-loop system with a sampled-data control for several values of the maximum sampling interval \bar{h} with a fixed decay rate $2\delta = 0.002$ and a fixed space discretization step $\bar{b} = 0.1$. Table 3.1 summarizes the results obtained based on the same Lyapunov-Razumikhin functional V defined in (3.47)-(3.49) with a fixed $R = 30$ and

$$\Theta_1 = 10^{-3} \times \begin{bmatrix} 4.7 & 0.24 \\ * & 4.8 \end{bmatrix}, \quad \Theta_2 = \begin{bmatrix} 1.24 & 0.74 \\ * & 0.91 \end{bmatrix}.$$

The parameters $\beta = 0.01$, $\mu = 0.7$, $\kappa = 1$, $\gamma = 0.02$, $\alpha = 1.3$ are selected according to Remark 3.3. The system can be shown to be $R\varepsilon$ -stable up to $\bar{h} = 0.0035$. It can be seen that ε linearly depends on \bar{h} which is consistent with condition (ii) of Theorem 3.2. Fig. 3.8 illustrates the time evolution of V in open-loop and closed-loop for $\bar{h} = 0.0001$ respectively with

$$y_0(x) = \begin{bmatrix} (1 - \cos 2\pi x) \sin 2\pi x \\ (\cos 2\pi x - 1) \sin 2\pi x \end{bmatrix}.$$

The closed-loop dynamics (V_{close} in Fig. 3.8) has a reaching time of $t = 0.17s$ (convergence time to the level set $\varepsilon = 0.87$), while the reaching time in open-loop is $t = 0.43s$.

Figure 3.4: Response of state y_1 for the open-loop system.Figure 3.5: Response of state y_2 for the closed-loop system.Table 3.1: Evaluation of ε for different values of \bar{h} when $R = 30$.

$\bar{h} _{\bar{b}=0.1}$	0.0001	0.0005	0.0015	0.0025	0.0035
ε	0.87	4.23	12.64	21.04	29.45

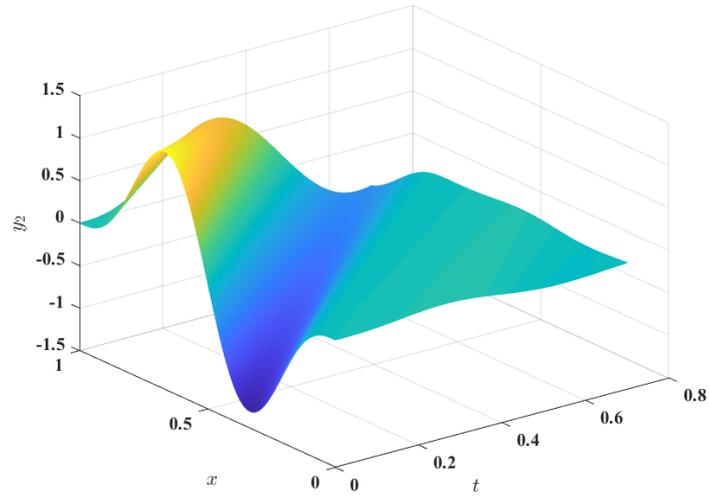


Figure 3.6: Response of state y_2 for the open-loop system.

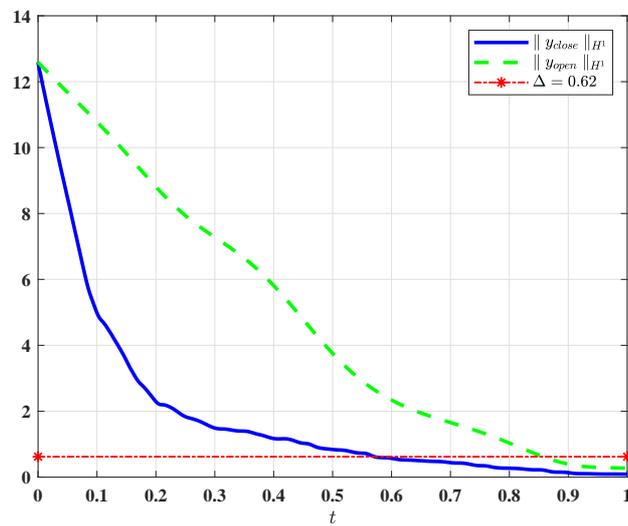


Figure 3.7: Time-evolutions of $\|y\|_{H^1}$ in open-loop system (green) and closed-loop system under sampled-data control (blue) with $\bar{h} = 0.0001$ and $\bar{b} = 0.1$.

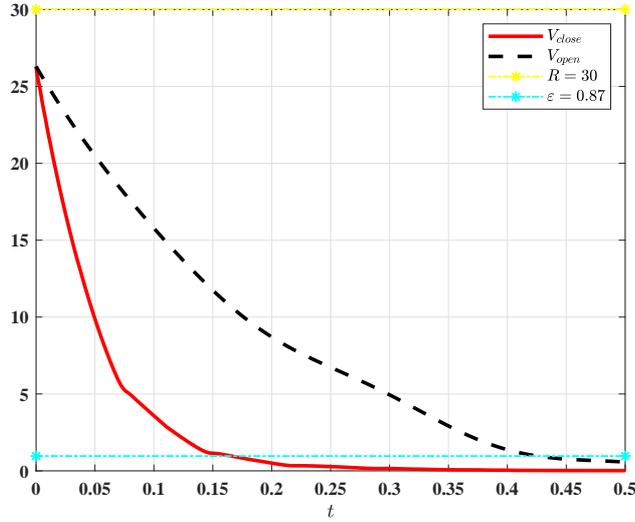


Figure 3.8: Time-evolutions of V in open-loop (black dashed line) and closed-loop (red real line) systems for $\bar{h} = 0.0001$ and $\bar{b} = 0.1$.

Next, we verify Theorem 3.3 in the following example. The advantages compared with Theorem 2.1 in Chapter 2 is also given.

Example 2:

In this numerical example, we illustrate the method we proposed in Theorem 3.3. Consider systems (2.13)-(2.18) and an initial condition satisfying $y_0(0) = 0$. The system parameters are the same as those chosen in Theorem 2.1 in Chapter 2, where

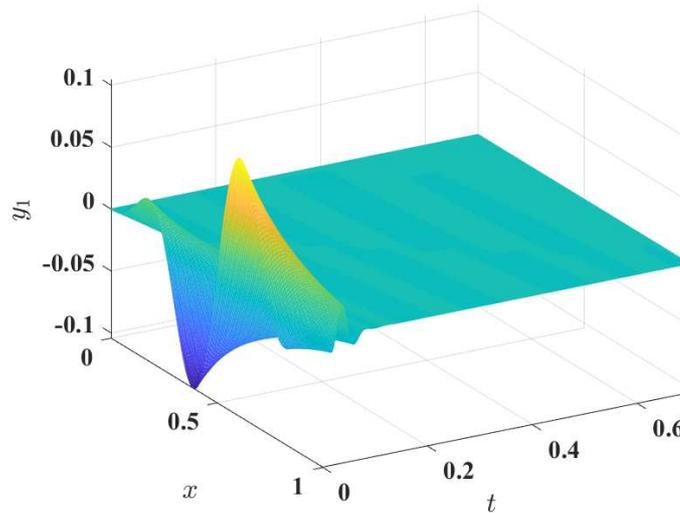
$$\Lambda = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}, \Upsilon = \begin{bmatrix} 20 & 15 \\ 20 & 25 \end{bmatrix}, K = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}, L = 1.$$

According to Remark 3.3, the parameters in condition (3.42)-(3.46) are selected as: $\mu = 0.18$, $\kappa = 1.8$, $\gamma = 0.001$, $\alpha = 1.001$, $\beta = 0.01$, $\delta = 0.001$. We fix $R = 20$, $\bar{h} = 0.1$, The Θ_1 , Θ_2 are satisfying the conditions of Theorem 3.3:

$$\Theta_1 = \begin{bmatrix} 22.86 & -9.88 \\ * & 13.76 \end{bmatrix}, \Theta_2 = \begin{bmatrix} 26.48 & -12.89 \\ * & 17.97 \end{bmatrix}.$$

The simulation results are presented in Figures 3.9-3.11 for $\bar{h} = 0.1$ with the initial conditions

$$y_0(x) = \begin{bmatrix} 0.06(1 - \cos 2\pi x) \sin 4\pi x \\ 0.1(1 - \cos 4\pi x) \sin 2\pi x \end{bmatrix}.$$

Figure 3.9: Response of state y_1 for $\bar{h} = 0.1$.

Figures 3.9-3.10 present that both state trajectories converge to near the origin with the controller and the initial conditions satisfying the compatibility condition. As can be seen from Figure 3.11, the time-evolution of Lyapunov functional $V(y(t, \cdot))$ decreases when $R > V(y(t, \cdot)) \geq \max \left\{ \varepsilon, \frac{V(y(t_k, \cdot))}{\alpha} \right\}, \alpha > 1$.

Remark 3.4

For the controller gain and sampling interval used here, the results in Chapter 2 are not feasible. The maximum sampling interval obtained based on results in Chapter 2 is 10^{-3} (to be compared to $\bar{h} = 0.1$ obtained here).

3.5 Conclusion

This chapter provided methods for checking the stability of linear hyperbolic balance laws under distributed sampled-data control with space discretization. By using the input-output approach to sampled-data control and Lyapunov-Razumikhin method in stability analysis, sufficient numerical conditions for the $R\varepsilon$ -stability were derived. In the next chapter we will work on providing a method for a global exponential stability analysis of such control systems.

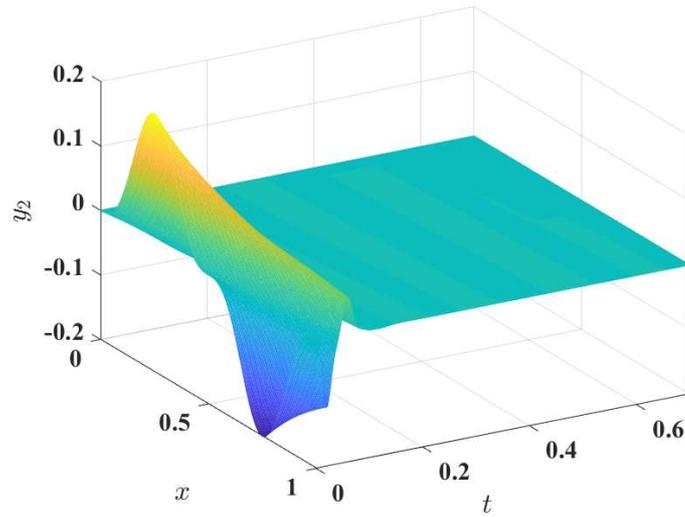


Figure 3.10: Response of state y_2 for $\bar{h} = 0.1$.

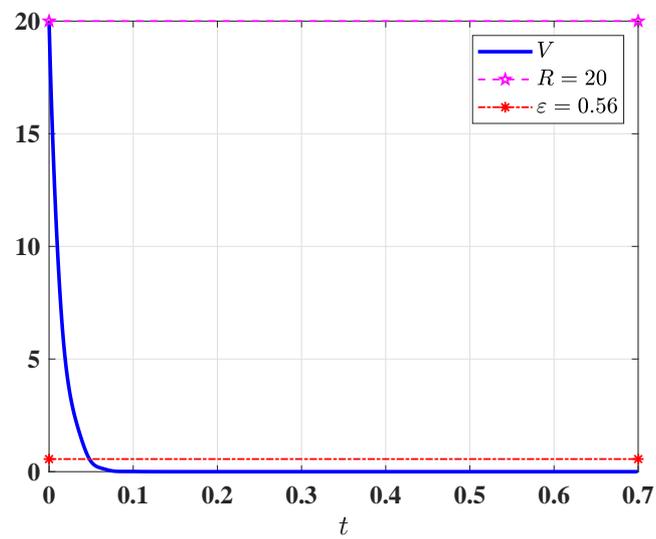


Figure 3.11: Time-evolution of function V for $\bar{h} = 0.1$.

3.6 Appendix of Chapter 3

Lemma 3.1

Consider system (3.1)-(3.4) with an initial condition y_0 satisfying $y_0(0) = Gy_0(L)$. Then $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$, we have

$$\begin{aligned} \partial_x y(t, 0) &= \Lambda^{-1} G \Lambda \partial_x y(t, L) + (\Lambda^{-1} G \Upsilon - \Lambda^{-1} \Upsilon G) y(t, L) \\ &\quad + (\Lambda^{-1} G K - \Lambda^{-1} K G) y(t_k, L). \end{aligned}$$

Proof of Lemma 3.1: From (3.1), the time derivative of the boundary condition is

$$\partial_t y(t, 0) = G \partial_t y(t, L), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad (3.59)$$

Combining (3.1a), (3.59) with (3.3b), we obtain

$$\begin{aligned} \partial_x y(t, 0) &= \Lambda^{-1} (-\Upsilon y(t, 0) - Ky(t_k, 0) - \partial_t y(t, 0)) \\ &= \Lambda^{-1} (-\Upsilon Gy(t, L) - KGy(t_k, L) \\ &\quad - G(-\Lambda \partial_x y(t, L) - \Upsilon y(t, L) - Ky(t_k, L))) \\ &= \Lambda^{-1} G \Lambda \partial_x y(t, L) + (\Lambda^{-1} G \Upsilon - \Lambda^{-1} \Upsilon G) y(t, L) \\ &\quad + (\Lambda^{-1} G K - \Lambda^{-1} K G) y(t_k, L) \end{aligned} \quad (3.60)$$

■

Lemma 3.2

Consider the condition (3.19) is satisfied. Then

$$W(x) + \kappa N(x) \preceq 0, \forall x \in [0, L]. \quad (3.61)$$

Proof of Lemma 3.2: According to Schur complement [10] (see Theorem A.5 in Appendix A), (3.20) is equivalent to

$$Q(x) - S_{4n \times n}(x) I_{n \times n}^{-1} S_{4n \times n}^\top(x) \succeq 0 \quad (3.62)$$

where $Q(x)$ and $S(x)$ are defined as (3.21) and (3.22). If (3.62) holds, we have

$$Q(x) - \bar{S}_{4n \times 4n}(x) I_{4n \times 4n}^{-1} \bar{S}_{4n \times 4n}^\top(x) \succeq 0 \quad (3.63)$$

with

$$\bar{S}_{4n \times 4n}(x) = \left[S_{4n \times n}(x) \mid \mathbf{0}_{4n \times 3n} \right].$$

Then (3.62) can be re-expressed for all $x \in [0, L]$ as

$$W(x) + \kappa N(x) \preceq 0, \quad (3.64)$$

with $W(x)$ defined in step 2 and $N(x)$ given in (3.32). ■

3.6.1 Proof of Theorem 3.1

During a sampling interval $[t_k, t_{k+1})$ with initial state $y(t_k, \cdot)$:

1. If $V(y(t_k, \cdot)) \leq \varepsilon$, $V(y(t, \cdot))$ will remain in ε during $[t_k, t_{k+1})$. Otherwise, we will have

$$\dot{V}(y) > 0 > -2\delta V(y)$$

at some point when

$$V(y(t, \cdot)) \geq \varepsilon \geq \frac{V(y(t_k, \cdot))}{\alpha},$$

which would contradict the proposition in Theorem 3.1.

2. If $R > V(y(t_k, \cdot)) \geq \varepsilon$:

(a) We have $V(y(t, \cdot)) \leq V(y(t_k, \cdot))$ during $[t_k, t_{k+1})$. Otherwise, we will have

$$\dot{V}(y) > 0 > -2\delta V(y)$$

at some point when

$$V(y(t, \cdot)) \geq V(y(t_k, \cdot)) \geq \frac{V(y(t_k, \cdot))}{\alpha},$$

which would contradict the proposition in Theorem 3.1.

(b) We can further show that

$$V(y(t, \cdot)) \leq \max \left\{ \varepsilon, \frac{V(y(t_k, \cdot))}{\alpha}, e^{-2\delta(t-t_k)} V(y(t_k, \cdot)) \right\} \quad (3.65)$$

during $[t_k, t_{k+1})$. In the following, we will discuss two possibilities in case (b):

(b1) If there exists $t' \in [t_k, t_{k+1})$ such that

$$V(y(t', \cdot)) = \max \left\{ \varepsilon, \frac{V(y(t_k, \cdot))}{\alpha} \right\}.$$

If

$$t \in [t_k, t'), \dot{V}(y) + 2\delta V(y) \leq 0$$

holds and we have

$$V(y(t, \cdot)) \leq e^{-2\delta(t-t_k)} V(y(t_k, \cdot)), \forall t \in [t_k, t').$$

Otherwise, we will have

$$\dot{V}(y) > 0 > -2\delta V(y)$$

at some point when

$$V(y(t, \cdot)) \geq \frac{V(y(t_k, \cdot))}{\alpha},$$

which would contradict the proposition in Theorem 3.1. If $t \in [t', t_{k+1})$, $V(y(t, \cdot))$ cannot go back above $\max\left\{\varepsilon, \frac{V(y(t_k, \cdot))}{\alpha}\right\}$, otherwise, according to the same principle, it would contradict the proposition in Theorem 3.1, then we have

$$V(y(t, \cdot)) \leq \max\left\{\varepsilon, \frac{V(y(t_k, \cdot))}{\alpha}\right\}, \quad \forall t \in [t', t_{k+1}).$$

So, over the whole sampling interval $t \in [t_k, t_{k+1})$, we can get inequality (3.65).

(b2) When

$$V(y(t, \cdot)) > \max\left\{\varepsilon, \frac{V(y(t_k, \cdot))}{\alpha}\right\}, \quad \forall t \in [t_k, t_{k+1}),$$

since

$$\dot{V}(y) + 2\delta V(y) \leq 0,$$

we have

$$V(y(t, \cdot)) \leq e^{-2\delta(t-t_k)} V(y(t_k, \cdot)).$$

Then it is not hard to get (3.65).

Now, let us consider

$$y(t_k, \cdot) \in \mathfrak{L}_{V < R}, \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}.$$

We have

$$\begin{aligned} V(y(t, \cdot)) &\leq \max\left\{\varepsilon, \frac{V(y(t_k, \cdot))}{\alpha}, e^{-2\delta(t-t_k)} V(y(t_k, \cdot))\right\} \\ &= \max\{\varepsilon, \xi V(y(t_k, \cdot))\}, \end{aligned} \quad (3.66)$$

with $\xi = \max\left\{\frac{1}{\alpha}, e^{-2\delta(t-t_k)}\right\} \leq 1$, then

$$V(y(t_k, \cdot)) \leq \max\{\varepsilon, \zeta V(y(t_{k-1}, \cdot))\}, \quad (3.67)$$

with $\zeta = \max\left\{\frac{1}{\alpha}, e^{-2\delta h}\right\} < 1$, $\forall k \in \mathbb{N} \setminus \{0\}$, where \underline{h} is the lower bound of the sampling interval.

By recursion, if $y(t_0, \cdot) \in \mathfrak{L}_{V < R}$, $\forall k \in \mathbb{N}$ we have

$$\begin{aligned} V(y(t_k, \cdot)) &\leq \max\{\varepsilon, \zeta \max\{\varepsilon, \zeta V(y(t_{k-2}, \cdot))\}\} \\ &\leq \max\{\varepsilon, \zeta \varepsilon, \zeta^2 V(y(t_{k-2}, \cdot))\} \\ &\leq \max\{\varepsilon, \zeta^2 V(y(t_{k-2}, \cdot))\} \\ &\leq \dots \\ &\leq \max\{\varepsilon, \zeta^k V(y(t_0, \cdot))\}. \end{aligned} \quad (3.68)$$

Then combining (3.66) and (3.68), we get that

$$\begin{aligned} V(y(t, \cdot)) &\leq \max\{\varepsilon, \xi \max\{\varepsilon, \zeta^k V(y(t_0, \cdot))\}\} \\ &= \max\{\varepsilon, \xi \varepsilon, \xi \zeta^k V(y(t_0, \cdot))\} \\ &= \max\{\varepsilon, \zeta^k V(y(t_0, \cdot))\} = \varepsilon \end{aligned} \quad (3.69)$$

when k is large enough. To prove the minimum value of k that guarantees (3.69) holds, we calculate the following equation

$$\zeta^k V(y(t_0, \cdot)) = \varepsilon$$

then we get

$$k = \left\lceil \log_{\zeta} \frac{\varepsilon}{V(y(t_0, \cdot))} \right\rceil. \quad (3.70)$$

Therefore,

$$\exists t_k \geq 0, y(t, \cdot) \in \mathfrak{L}_{V \leq \varepsilon}, \forall t \geq t_k,$$

with k defined in (3.70), which concludes the proof of $R\varepsilon$ -stability. \blacksquare

3.6.2 Proof of Step 2 in Theorem 3.2

Using the commutativity condition: $\Lambda \Theta_1 = \Theta_1 \Lambda$, we first compute the time derivative of $V_1(y)$ along the solutions to (3.9)-(3.15), $\forall t \in [t_k, t_{k+1})$,

$$\begin{aligned} \dot{V}_1(y) &= \int_0^L (\partial_t y^\top e^{-2\mu x} \Theta_1 y + y^\top e^{-2\mu x} \Theta_1 \partial_t y) dx \\ &= \int_0^L \left((-\Lambda \partial_x y - (K + \Upsilon)y - K\omega)^\top e^{-2\mu x} \Theta_1 y \right. \\ &\quad \left. + (K\vartheta)^\top e^{-2\mu x} \Theta_1 y + y^\top e^{-2\mu x} \Theta_1 K\vartheta \right. \\ &\quad \left. + y^\top e^{-2\mu x} \Theta_1 (-\Lambda \partial_x y - (K + \Upsilon)y - K\omega) \right) dx \\ &= \int_0^L -\partial_x [y^\top \Lambda e^{-2\mu x} \Theta_1 y] dx \\ &\quad + \int_0^L \left(-y^\top (K + \Upsilon)^\top e^{-2\mu x} \Theta_1 y + (K\vartheta)^\top e^{-2\mu x} \Theta_1 y \right. \\ &\quad \left. + y^\top e^{-2\mu x} \Theta_1 K\vartheta - y^\top e^{-2\mu x} \Theta_1 (K + \Upsilon)y \right. \\ &\quad \left. - y^\top e^{-2\mu x} \Theta_1 K\omega - 2\mu y^\top \Lambda e^{-2\mu x} \Theta_1 y - \omega^\top K^\top e^{-2\mu x} \Theta_1 y \right) dx \\ &= y^\top (\cdot, L) (-\Lambda e^{-2\mu L} \Theta_1 + G^\top \Lambda \Theta_1 G) y (\cdot, L) \\ &\quad + \int_0^L \left(-y^\top \left((K + \Upsilon)^\top e^{-2\mu x} \Theta_1 \right. \right. \end{aligned}$$

$$\begin{aligned}
 & +e^{-2\mu x}\Theta_1(K+\Upsilon)y+(K\vartheta)^\top e^{-2\mu x}\Theta_1y+y^\top e^{-2\mu x}\Theta_1K\vartheta \\
 & -\omega^\top K^\top e^{-2\mu x}\Theta_1y-y^\top e^{-2\mu x}\Theta_1K\omega)dx-2\mu\int_0^Ly^\top\Lambda e^{-2\mu x}\Theta_1ydx. \quad (3.71)
 \end{aligned}$$

Since we have the assumption that y is of class \mathcal{C}^2 on $[0, +\infty) \times [0, L]$, then along the solutions of the system (3.1), we can obtain

$$\partial_{xt}y(t, x) = \partial_{tx}y(t, x) = -\Lambda\partial_{xx}y(t, x) - \Upsilon\partial_xy(t, x), \forall t \in (t_k, t_{k+1}). \quad (3.72)$$

Using Lemma 3.1 in the appendix and similarly to the computation of \dot{V}_1 , the time derivative of $V_2(y)$ along the solutions to (3.60)-(3.72), $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$ is shown as follows

$$\begin{aligned}
 \dot{V}_2(y) & = Z^\top PZ + \int_0^L (-\partial_xy^\top (\Upsilon^\top e^{-2\mu x}\Theta_2 + e^{-2\mu x}\Theta_2\Upsilon) \partial_xy) dx \\
 & \quad - 2\mu \int_0^L \partial_xy^\top \Lambda e^{-2\mu x}\Theta_2\partial_xy dx. \quad (3.73)
 \end{aligned}$$

where P is defined in (3.18) and

$$Z = \left[(\partial_xy(\cdot, L))^\top, y^\top(\cdot, L), y^\top(t_k, L) \right]^\top.$$

Adding and subtracting $\gamma\|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2$ to (3.71) and $\beta\int_0^Ly_x^\top e^{-2\mu x}\Theta_2y_xdx$ to (3.73) for some $\Upsilon > 0, \beta > 0$, we have

$$\begin{aligned}
 \dot{V}(y) & = \dot{V}_1(y) + \dot{V}_2(y) \\
 & = y^\top(\cdot, L) (-\Lambda e^{-2\mu L}\Theta_1 + G^\top\Lambda\Theta_1G) y(\cdot, L) \\
 & \quad + \int_0^L \left(-y^\top\mathcal{A}_1(x)y + (K\vartheta)^\top e^{-2\mu x}\Theta_1y + y^\top e^{-2\mu x}\Theta_1K\vartheta \right. \\
 & \quad \left. -\omega^\top K^\top e^{-2\mu x}\Theta_1y - y^\top e^{-2\mu x}\Theta_1K\omega - \gamma\omega^\top\omega \right) dx \\
 & \quad - 2\mu \int_0^L y^\top\Lambda e^{-2\mu x}\Theta_1ydx + \gamma\|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\
 & \quad + Z^\top PZ + \int_0^L (-\partial_xy^\top\mathcal{A}_2(x)\partial_xy) dx \\
 & \quad - 2\mu \int_0^L \partial_xy^\top\Lambda e^{-2\mu x}\Theta_2\partial_xydx + \beta \int_0^L y_x^\top e^{-2\mu x}\Theta_2y_xdx. \quad (3.74)
 \end{aligned}$$

with

$$\begin{aligned}
 \mathcal{A}_1(x) & = e^{-2\mu x} \left[(K+\Upsilon)^\top\Theta_1 + \Theta_1(K+\Upsilon) \right], \\
 \mathcal{A}_2(x) & = e^{-2\mu x} \left[\Upsilon^\top\Theta_2 + \Theta_2\Upsilon + \beta\Theta_2 \right].
 \end{aligned}$$

Then by using conditions (3.17) and (3.18) we have

$$\dot{V}(y) = \dot{V}_1(y) + \dot{V}_2(y)$$

$$\begin{aligned}
&\leq \int_0^L (-y^\top \mathcal{A}_1(x)y - \omega^\top K^\top T e^{-2\mu x} \Theta_1 y + (K\vartheta)^\top e^{-2\mu x} \Theta_1 y + y^\top e^{-2\mu x} \Theta_1 K\vartheta \\
&\quad - y^\top e^{-2\mu x} \Theta_1 K\omega - \gamma \omega^\top \omega) dx - 2\sigma V_1(y) + \gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \\
&\quad + \int_0^L (-\partial_x y^\top \mathcal{A}_2(x) \partial_x y) dx - 2\sigma V_2(y) + \beta \int_0^L y_x^\top e^{-2\mu x} \Theta_2 y_x dx. \tag{3.75}
\end{aligned}$$

with $\sigma = \mu\lambda$. Then by using Young's inequality [135] (see Theorem A.3 in Appendix A) to $\int_0^L (K\vartheta)^\top e^{-2\mu x} \Theta_1 y dx$, we get

$$\begin{aligned}
&\int_0^L (K\vartheta)^\top e^{-2\mu x} \Theta_1 y dx \\
&\leq \frac{1}{2} \int_0^L (K\vartheta)^\top (K\vartheta) dx + \frac{1}{2} \int_0^L (e^{-2\mu x} \Theta_1 y)^\top (e^{-2\mu x} \Theta_1 y) dx \\
&= \frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (K\vartheta)^\top (K\vartheta) dx + \frac{1}{2} \int_0^L (e^{-2\mu x} \Theta_1 y)^\top (e^{-2\mu x} \Theta_1 y) dx \tag{3.76}
\end{aligned}$$

Let us recall the space discretization error (3.16) with (3.15)

$$\vartheta(t_k, x) = \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x y_\varsigma(t_k, \varsigma) d\varsigma, x \in [0, L]$$

and $\vartheta(t_k, \bar{x}_i) = 0$, for $\bar{x}_0 = 0, \bar{x}_{i+1} = \bar{x}_i + \bar{b}, \bar{b} = \frac{L}{N-1}, i \in \{0, \dots, N-2\}$.

By using the operator E in (3.16) with (3.15), we rewrite the first term in (3.76) for $x \in \Xi_i, i \in \{0, \dots, N-1\}$ as

$$\begin{aligned}
&\frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (K\vartheta)^\top (K\vartheta) dx \\
&= \frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{\bar{x}_i} \left(K \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x y_\varsigma(t_k, \varsigma) d\varsigma \right)^\top \left(K \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x y_\varsigma(t_k, \varsigma) d\varsigma \right) dx \\
&\quad + \frac{1}{2} \sum_{i=0}^{N-1} \int_{\bar{x}_i}^{x_{i+1}} \left(K \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x y_\varsigma(t_k, \varsigma) d\varsigma \right)^\top \left(K \sum_{i=0}^{N-1} d_i(x) \int_{\bar{x}_i}^x y_\varsigma(t_k, \varsigma) d\varsigma \right) dx \\
&= \frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{\bar{x}_i} (y(t_k, x) - y(t_k, \bar{x}_i))^\top K^\top K (y(t_k, x) - y(t_k, \bar{x}_i)) dx \\
&\quad + \frac{1}{2} \sum_{i=0}^{N-1} \int_{\bar{x}_i}^{x_{i+1}} (y(t_k, x) - y(t_k, \bar{x}_i))^\top K^\top K (y(t_k, x) - y(t_k, \bar{x}_i)) dx \tag{3.77}
\end{aligned}$$

98 Chapter 3. Hyperbolic system with spatially sampled state measurements

Using Wirtinger's inequality [77] (see Lemma A.1 in Appendix A) with $d - c \leq \frac{\bar{b}}{2}$ on each integral term, the above (3.77) becomes

$$\begin{aligned}
 & \frac{1}{2} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (K\vartheta)^\top (K\vartheta) dx \\
 & \leq \frac{1}{2} \frac{\bar{b}^2}{\pi^2} \sum_{i=0}^{N-1} \left[\int_{x_i}^{\bar{x}_i} y_x^\top(t_k, x) K^\top K y_x(t_k, x) + \int_{\bar{x}_i}^{x_{i+1}} y_x^\top(t_k, x) K^\top K y_x(t_k, x) dx \right] \\
 & = \frac{\bar{b}^2}{2\pi^2} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} y_x^\top(t_k, x) K^\top K y_x(t_k, x) dx \\
 & = \frac{\bar{b}^2}{2\pi^2} \int_0^L y_x^\top(t_k, x) K^\top K y_x(t_k, x) dx \tag{3.78}
 \end{aligned}$$

The above inequality (3.78) involves the upper bound of the space discretization error. Combining (3.76) and (3.78), we get

$$\begin{aligned}
 & \int_0^L (K\vartheta)^\top e^{-2\mu x} \Theta_1 y dx \\
 & \leq \frac{\bar{b}^2}{2\pi^2} \int_0^L y_x^\top(t_k, x) K^\top K y_x(t_k, x) dx + \frac{1}{2} \int_0^L e^{-4\mu x} y^\top \Theta_1^\top \Theta_1 y dx \tag{3.79}
 \end{aligned}$$

Then substituting (3.79) into (3.75), we get

$$\begin{aligned}
 \dot{V}(y) & \leq -2\sigma V_1(y) - 2\sigma V_2(y) + \int_0^L \eta^\top W(x) \eta dx \\
 & \quad + \gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 + \beta \int_0^L y_x^\top e^{-2\mu x} \Theta_2 y_x dx \\
 & = -2\sigma V_1(y) - (2\sigma - \beta) V_2(y) + \int_0^L \eta^\top W(x) \eta dx + \gamma \|\omega(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \tag{3.80}
 \end{aligned}$$

with $\sigma, \eta, W(x)$ defined below (3.29).

The proof of step 2 is complete. ■

Stability analysis for a class of sampled-data linear hyperbolic systems via an impulsive approach

Contents

4.1	Introduction	100
4.2	Problem statement	100
4.3	System remodeling	101
4.4	Stability analysis	102
4.4.1	IQC characterization of the sampling effect	102
4.4.2	Main stability result	106
4.5	Numerical example	111
4.6	Conclusion	114

This chapter deals with the global exponential stability problem for a class of linear hyperbolic systems by using distributed sampled-data control. First, the original system is transformed into a impulsive infinite dimensional model through an augmented system. By employing this model, the relation between the sampling interval, the system state and its sampled vector is characterized by an **IQC**. The acquired **IQC** is utilized to derive numerically tractable stability criteria. Finally, a numerical example illustrates the main results.

4.1 Introduction

In this chapter, we focus on the global exponential stability analysis problem for a class of linear sampled-data hyperbolic system by using impulsive system method. The state measurements can be aperiodically sampled in time. An approach based on impulsive models [9] and an adaptation of the **IQC** method [85] is proposed. The general idea of this chapter is to use an augmented state model with an impulsive form (see [44, 88] for basic results for **ODEs**) to derive an **IQC** that characterizes the sampling effect. While in the sampled-data control literature for **ODEs**, such **IQCs** are usually derived analytically [12, 13, 42, 61, 122, 3], here we proposed simple matrix inequalities where the parameters of the **IQC** are derived numerically. Using the obtained **IQCs**, numerical criteria for analyzing stability of the state-space model are provided. Differently from our previous works where local practical ($R\epsilon$) stability conditions are obtained [127, 129, 128], here global exponential stability conditions are proposed. A preliminary version of this work [126] was presented at the virtual ADHS 2021 Congress.

4.2 Problem statement

We consider the following hyperbolic systems:

$$\begin{cases} \partial_t y(t, x) + \Phi \partial_x y(t, x) + \Gamma y(t, x) + u(t, x) = 0, & (4.1a) \\ y(0, x) = y_0(x), \forall x \in [0, L], & (4.1b) \end{cases}$$

where $y : [0, +\infty) \times [0, L] \rightarrow \mathbb{R}^n$ is the system state, Γ is a real $n \times n$ constant matrix, $\Phi = \text{diag}\{\Phi_+, \Phi_-\}$, $\Phi_+ = \text{diag}\{\phi_1, \dots, \phi_m\}$, $\Phi_- = \text{diag}\{\phi_{m+1}, \dots, \phi_n\}$ with $\phi_1 > \dots > \phi_m > 0 > \phi_{m+1} > \dots > \phi_n$, $u(t, x) \in \mathbb{R}^n$ is the sampled-data control input. We use the notation $y = \begin{bmatrix} y^+ \\ y^- \end{bmatrix}$, where $y^+ : \mathbb{R}_+ \times [0, L] \rightarrow \mathbb{R}^m$ and $y^- : \mathbb{R}_+ \times [0, L] \rightarrow \mathbb{R}^{n-m}$.

Then, we consider the following boundary conditions:

$$\begin{bmatrix} y^+(t, 0) \\ y^-(t, L) \end{bmatrix} = K \begin{bmatrix} y^+(t, L) \\ y^-(t, 0) \end{bmatrix}, \forall t \geq 0 \quad (4.2)$$

where K is a real $n \times n$ constant matrix such that $K = \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}$ with K_{00} in $\mathbb{R}^{m \times m}$, K_{01} in $\mathbb{R}^{m \times (n-m)}$, K_{10} in $\mathbb{R}^{(n-m) \times m}$ and K_{11} in $\mathbb{R}^{(n-m) \times (n-m)}$.

At times t_k , $k \in \mathbb{N}$, the system state $y(\cdot, x)$ is sampled and used for computing a control law which is implemented via a **ZOH**

$$u(t, x) = Gy(t_k, x), \quad \forall t \in [t_k, t_{k+1}). \quad (4.3)$$

where G is a constant $n \times n$ matrix. We assume that the sampling sequence $\{t_k\}_{k \in \mathbb{N}}$ satisfies

$$t_0 = 0, \quad \lim_{k \rightarrow \infty} t_k = \infty, \quad t_{k+1} - t_k \in (0, \bar{h}]. \quad (4.4)$$

where $\bar{h} > 0$ is the maximum sampling interval.

In order to establish the existence and uniqueness of solutions in this section, we use the notion of piecewise continuous solutions as in Remark 2.2 of Chapter 2.

The goal of this chapter is to provide numerically tractable conditions for checking the exponential stability of system (4.1)-(4.4).

4.3 System remodeling

To facilitate the stability analysis, we first recast the above system (4.1)-(4.4) into an impulsive model. Let \hat{y} indicate a piecewise constant signal representing the latest state measurement of the plant available at the controller,

$$\hat{y}(t, x) = y(t_k, x), \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}.$$

Using the augmented system state $\varphi = [y^\top(t, x), \hat{y}^\top(t, x)]^\top$, leads to an impulsive system with following structure:

$$\begin{cases} \partial_t \varphi(t, x) + A \partial_x \varphi(t, x) + B \varphi(t, x) = 0, \quad \forall t \in (t_k, t_{k+1}), k \in \mathbb{N}, & (4.5a) \\ \varphi(t_k, x) = D \varphi(t_k^-, x), \quad k \in \mathbb{N}, & (4.5b) \\ \varphi(0, x) = \varphi_0(x) = [y_0(x), y_0(x)]^\top, \quad \forall x \in [0, L]. & (4.5c) \end{cases}$$

with

$$A = \begin{bmatrix} \Phi & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \end{bmatrix}, B = \begin{bmatrix} \Gamma & G \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \end{bmatrix}, D = \begin{bmatrix} \mathbf{I}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{I}_{n \times n} & \mathbf{0}_{n \times n} \end{bmatrix}$$

and

$$\varphi(t_k^-, \cdot) = \lim_{t \uparrow t_k} \varphi(t, \cdot).$$

The notation $\varphi = \left[(y^+)^\top, (y^-)^\top, (\hat{y}^+)^\top, (\hat{y}^-)^\top \right]^\top$ and the boundary condition is defined as

$$\begin{bmatrix} y^+(t, 0) \\ y^-(t, L) \\ \hat{y}^+(t, 0) \\ \hat{y}^-(t, L) \end{bmatrix} = \begin{bmatrix} K & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & K \end{bmatrix} \begin{bmatrix} y^+(t, L) \\ y^-(t, 0) \\ \hat{y}^+(t, L) \\ \hat{y}^-(t, 0) \end{bmatrix}, \forall t \geq 0. \quad (4.6)$$

Note that the initial condition (4.5c) is chosen based on condition (4.1b). Therefore, all the solution of system (4.1) are characterized by the first n components of the augmented state $\varphi(t, x)$. As a result, the closed-loop system can be regarded as an augmented impulsive system where the $y(t, x)$ state variable is continuous and where the $\hat{y}(t, x)$ variable is subject to a reset at impulse (i.e. sampling) times. In the following sections we will see how this new impulsive model is useful in deriving constructive stability conditions.

4.4 Stability analysis

In this section, we focus on providing constructive methods for the analysis of the sampled-data system (4.1)-(4.4) using the augmented impulsive system (4.5). Before presenting the main result, we first provide a technical lemma based on model (4.5). This lemma allows us to characterize the link between the system state y of system (4.1)-(4.4) and its sampled version \hat{y} by an **IQC**. Then, the stability conditions for system (4.1)-(4.4) are derived using the obtained **IQC**.

4.4.1 IQC characterization of the sampling effect

The main idea is to use a functional $\Upsilon(\varphi)$, $\Upsilon : L^2([0, L]; \mathbb{R}^{2n}) \rightarrow \mathbb{R}$, depending on the augmented system state φ and study the solution of impulsive system (4.5) during one sampling interval $[t_k, t_{k+1})$. Then, we over-bound the functional Υ using an exponential function with bounded growth rate. Taking into account this bound, an **IQC** capturing the effect of sampling is obtained. Intuitively, the **IQC** characterizes how much system (4.1) state deviates without control update.

Lemma 4.1

Consider sampled-data linear hyperbolic system (4.1)-(4.4), the augmented impulsive system (4.5) and a given $\alpha > 0$, assume that there exists a diagonal matrix $\Lambda \in \mathbb{R}^{2n \times 2n}$, satisfying

$$\Theta = 2\alpha\Lambda - B^\top\Lambda - \Lambda B \preceq 0, \quad (4.7)$$

$$\begin{aligned} & \begin{bmatrix} K_{00} & K_{01} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & K_{00} & K_{01} \\ 0 & 0 & 0 & I \end{bmatrix}^\top A\Lambda \begin{bmatrix} K_{00} & K_{01} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & K_{00} & K_{01} \\ 0 & 0 & 0 & I \end{bmatrix} \\ & \prec \begin{bmatrix} I & 0 & 0 & 0 \\ K_{10} & K_{11} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & K_{10} & K_{11} \end{bmatrix}^\top A\Lambda \begin{bmatrix} I & 0 & 0 & 0 \\ K_{10} & K_{11} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & K_{10} & K_{11} \end{bmatrix}. \end{aligned} \quad (4.8)$$

Then the following **IQC** is satisfied

$$\int_0^L \varphi^\top(t, x) N(t - t_k) \varphi(t, x) dx \geq 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \quad (4.9)$$

along the solutions $\varphi \in L^2([0, L]; \mathbb{R}^{2n})$ of (4.5), where

$$N(h) = e^{-2\alpha h} R^\top \Lambda R - \Lambda, \quad (4.10)$$

$$\text{with } R = \begin{bmatrix} \mathbf{0}_{n \times n} & \mathbf{I}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{I}_{n \times n} \end{bmatrix}.$$

Then, the **IQC** (4.9) holds along the solutions $\varphi \in L^2([0, L]; \mathbb{R}^{2n})$ of (4.5):

Proof. In the following, for the sake of readability, the arguments (t, x) of the different functions are dropped by an abuse of notations when we think that this would not lead to any confusion. Let us consider the following functional $\Upsilon : L^2([0, L]; \mathbb{R}^{2n}) \rightarrow \mathbb{R}$ defined by:

$$\Upsilon(\varphi(t, x)) = \int_0^L \varphi^\top(t, x) \Lambda \varphi(t, x) dx, \quad (4.11)$$

with $\varphi \in L^2([0, L]; \mathbb{R}^{2n})$ the solution of (4.5).

Computing the time derivative of Υ along the solutions of (4.5) leads to

$$\begin{aligned}
 \dot{\Upsilon}(\varphi) &= \int_0^L (\varphi_t^\top \Lambda \varphi + \varphi^\top \Lambda \varphi_t) dx \\
 &= \int_0^L \left((-A\partial_x \varphi - B\varphi)^\top \Lambda \varphi + \varphi^\top \Lambda (-A\partial_x \varphi - B\varphi) \right) dx \\
 &= \int_0^L -\partial_x [\varphi^\top A \Lambda \varphi] dx + \int_0^L (-\varphi^\top (B^\top \Lambda + \Lambda B) \varphi) dx \\
 &= \varphi^\top(t, 0) A \Lambda \varphi(t, 0) - \varphi^\top(t, L) A \Lambda \varphi(t, L) + \int_0^L (-\varphi^\top (B^\top \Lambda + \Lambda B) \varphi) dx. \quad (4.12)
 \end{aligned}$$

Using the boundary condition (4.6), we get

$$\begin{aligned}
 \dot{\Upsilon}(\varphi) &= \begin{bmatrix} y^+(t, L) \\ y^-(t, 0) \\ \hat{y}^+(t, L) \\ \hat{y}^-(t, 0) \end{bmatrix}^\top \left(\begin{bmatrix} K_{00} & K_{01} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & K_{00} & K_{01} \\ 0 & 0 & 0 & I \end{bmatrix}^\top A \Lambda \begin{bmatrix} K_{00} & K_{01} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & K_{00} & K_{01} \\ 0 & 0 & 0 & I \end{bmatrix} \right. \\
 &\quad \left. - \begin{bmatrix} I & 0 & 0 & 0 \\ K_{10} & K_{11} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & K_{10} & K_{11} \end{bmatrix}^\top A \Lambda \begin{bmatrix} I & 0 & 0 & 0 \\ K_{10} & K_{11} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & K_{10} & K_{11} \end{bmatrix} \right) \begin{bmatrix} y^+(t, L) \\ y^-(t, 0) \\ \hat{y}^+(t, L) \\ \hat{y}^-(t, 0) \end{bmatrix} \\
 &\quad + \int_0^L (\varphi^\top (-B^\top \Lambda - \Lambda B) \varphi) dx. \quad (4.13)
 \end{aligned}$$

Next, since (4.7) and (4.8) hold, we have

$$\dot{\Upsilon}(\varphi) + 2\alpha \Upsilon(\varphi) < \int_0^L (\varphi^\top (2\alpha \Lambda - B^\top \Lambda - \Lambda B) \varphi) dx \leq 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad (4.14)$$

Consequently, according to the comparison lemma we have

$$\Upsilon(\varphi(t, \cdot)) \leq e^{-2\alpha(t-t_k)} \Upsilon(\varphi(t_k, \cdot)), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \quad (4.15)$$

Then, inequality (4.15) can be re-expressed as:

$$e^{-2\alpha(t-t_k)} \Upsilon(\varphi(t_k, \cdot)) - \Upsilon(\varphi(t, \cdot)) = \int_0^L \varphi^\top(t, x) N(t-t_k) \varphi(t, x) dx \geq 0, \quad (4.16)$$

with $N(t-t_k)$ defined in (4.10). This concludes the proof of Lemma 4.1.

Remark 4.1

In Lemma 4.1, the IQC (4.9), with

$$\varphi(t, x) = [y^\top(t, x), \hat{y}^\top(t, x)]^\top,$$

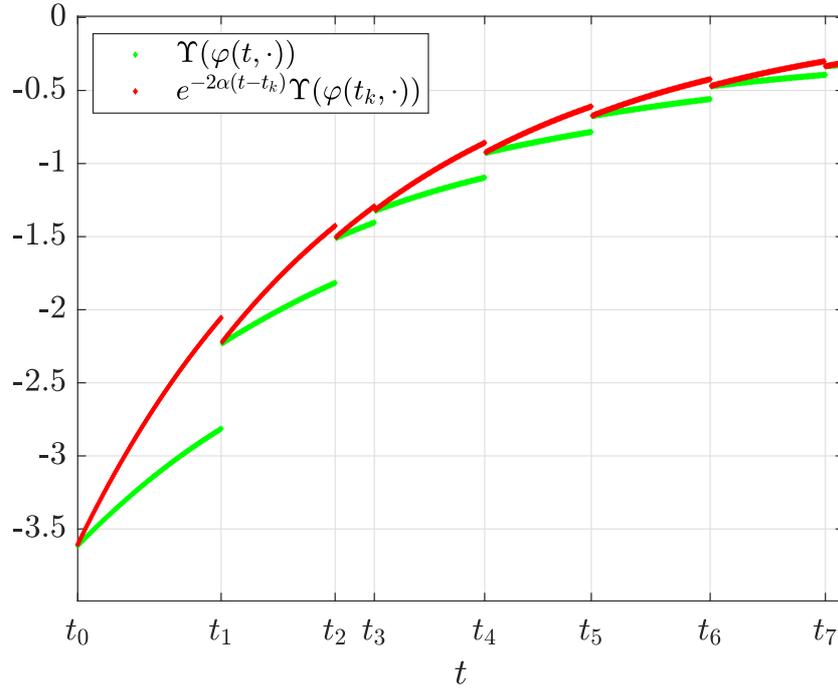


Figure 4.1: Time-evolution of the function $\Upsilon(\varphi)$ along a trajectory of system (4.5) for the case described in Example 1 (green) and its upper bound in each sampling interval (red).

allows to characterize the link between the state y of system (4.1) and its sampled version \hat{y} over the sampling interval. Note that the **IQC** notion used here is different from the one usually obtained in the sampled-data literature for **ODEs**. Here the integration parameter corresponds to domain variable x and the matrix $N(\cdot)$ depends on the clock variable $t - t_k$ counting the time since the last sampling instant. While **IQCs** are usually derived analytically [122, 42, 61], here, in Lemma 4.1, we propose simple matrix inequalities (4.7) and (4.8) where the parameters of the **IQC** are derived numerically. In the perturbation free case, by tuning the parameter α and solving the linear matrix inequality (4.7), one can derive the matrix Λ , which characterizes the **IQC** (4.9). In order to give some intuition about the tuning of α , note that α corresponds to an upper bound on the growth rate of the functional Υ , i.e. an indicator of the deviation of the system state in open loop. Clearly, the tighter the estimation of the growth rate α is, the better the obtained **IQC** (4.9) captures the effect of sampling on the system. The following example provides a numerical illustration.

Example 1. Consider a system of the form (4.1)-(4.4) with $L = 1$,

$$\Phi = \begin{bmatrix} 1.2 & 0 \\ 0 & -0.8 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1.3 & 1.5 \\ -0.5 & 2.5 \end{bmatrix}, \quad G = \begin{bmatrix} 1.1 & 0.5 \\ 1 & 0.9 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

a maximum sampling interval $\bar{h} = 0.1$. Using the state augmentation leads to an impulsive system (4.5) with

$$A = \begin{bmatrix} 1.2 & 0 & 0 & 0 \\ * & -0.8 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1.3 & 1.5 & 1.1 & 0.5 \\ -0.5 & 2.5 & 1 & 0.9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying Lemma 4.1 with $\alpha = 3$, we can derive an IQC of the form (4.9) and $N(h)$ in (4.10) is characterized by

$$\Lambda = \text{diag} \{-0.25, -0.39, -0.19, -0.19\}.$$

Note that the matrix Λ also allows to parameterize the function Υ in (4.11) which leads to the IQC of the form (4.9). The evolution of the functional Υ along the trajectories of system (4.1) with an initial condition

$$y_0(x) = \begin{bmatrix} 2.5 - 1.5 \cos(4\pi x) \\ -\sin(6\pi x) \end{bmatrix}$$

and an arbitrary sequence of sampling intervals is shown in Figure 4.1. The figure illustrates for each sampling interval $[t_k, t_{k+1}), k \in \{0, \dots, 7\}$, the deviation of the state $\varphi(t, x) = [y^\top(t, x), \hat{y}^\top(t, x)]^\top$ without control update (captured by the deviation of function $\Upsilon(\varphi(t, \cdot))$ with respect to its value at the sampling times $\Upsilon(\varphi(t_k, \cdot))$) and the upper bound $e^{-2\alpha(t-t_k)}\Upsilon(\varphi(t_k, \cdot))$ used for computing the IQC of the form (4.9) – see steps (4.15)-(4.16) in the proof of Lemma 4.1.

In the following subsection we show how IQCs of the form (4.9) can be used in order to derive tractable stability criteria for the sampled-data system (4.1).

4.4.2 Main stability result

Before presenting the main results, we first introduce the stability notion in the following proposition.

Proposition 4.1 (Wang 2021 [126])

Consider the systems (4.1)-(4.4) and a candidate Lyapunov functional $V : L^2([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}_+$ which is differentiable for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ w.r.t. its argument $b \in L^2([0, L]; \mathbb{R}^n)$ and there exist $0 < a_1 < a_2$, such that:

$$a_1 \|b\|_{L^2([0, L]; \mathbb{R}^n)}^2 \leq V(b) \leq a_2 \|b\|_{L^2([0, L]; \mathbb{R}^n)}^2. \quad (4.17)$$

Assume that along the trajectories of the system (4.1)-(4.4), the corresponding solution $y(t, \cdot)$ satisfies

$$\dot{V}(y(t, \cdot)) + 2\lambda V(y(t, \cdot)) \leq 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \quad (4.18)$$

for some $\lambda > 0$. Then the system is exponentially stable in L^2 -norm with a decay-rate larger than λ , that is for any initial condition $y_0 \in L^2(0, L)$ for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$

$$\|y(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \leq \frac{a_2}{a_1} e^{-2\lambda t} \|y_0\|_{L^2([0, L]; \mathbb{R}^n)}^2. \quad (4.19)$$

Proof. First, we declare that V defined in (4.17) is continuous (One can consult [4]). Since $y(t, x)$ is continuous with respect to t for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ and continuous at sampling instants by choosing the last value of the previous sampling interval as the initial condition of the following sampling interval, then V is continuous for all $t \geq 0$.

Then, we consider the differentiable Lyapunov functional $V : L^2([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}_+$ for which

$$\dot{V}(y(t, \cdot)) + 2\lambda V(y(t, \cdot)) \leq 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \quad (4.20)$$

for some $\lambda > 0$.

Applying the comparison lemma, we have

$$V(y(t, \cdot)) \leq e^{-2\lambda(t-t_k)} V(y(t_k, \cdot)), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \quad (4.21)$$

then we can derive

$$V(y(t_k, \cdot)) \leq e^{-2\lambda(t_k-t_{k-1})} V(y(t_{k-1}, \cdot)), \forall k \in \mathbb{N} \setminus \{0\}, \quad (4.22)$$

by recursion, the following inequality holds

$$\begin{aligned} V(y(t_k, \cdot)) &\leq e^{-2\lambda(t_k-t_{k-1})} e^{-2\lambda(t_{k-1}-t_{k-2})} V(y(t_{k-2}, \cdot)) \\ &\leq \dots \\ &\leq e^{-2\lambda(t_k-t_{k-1})} e^{-2\lambda(t_{k-1}-t_{k-2})} \\ &\quad \dots e^{-2\lambda(t_1-t_0)} V(y(t_0, \cdot)), \end{aligned} \quad (4.23)$$

Then instituting (4.23) into (4.21), we obtain

$$\begin{aligned} V(y(t, \cdot)) &\leq e^{-2\lambda(t-t_k)} V(y(t_k, \cdot)) \\ &\leq e^{-2\lambda(t-t_k)} e^{-2\lambda(t_k-t_{k-1})} e^{-2\lambda(t_{k-1}-t_{k-2})} \dots \\ &\quad e^{-2\lambda(t_1-t_0)} V(y(t_0, \cdot)) \\ &= e^{-2\lambda(t-t_0)} V(y(t_0, \cdot)) \\ &= e^{-2\lambda t} V(y_0). \end{aligned} \quad (4.24)$$

Combining (4.17) and (4.24), we get

$$\|y(t, \cdot)\|_{L^2([0,L];\mathbb{R}^n)}^2 \leq \frac{a_2}{a_1} e^{-2\lambda t} \|y_0\|_{L^2([0,L];\mathbb{R}^n)}^2, \\ \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}.$$

This concludes the proof of Proposition 4.4.2. \blacksquare

Theorem 4.1

Consider the sampled-data system (4.1) and the impulsive model (4.5). Given a prescribed decay rate μ and a tuning parameter $\alpha > 0$, let there exist a diagonal positive definite matrix $M \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{2n \times 2n}$ satisfying the following set of inequalities

$$\Theta = 2\alpha\Lambda - B^\top \Lambda - \Lambda B \preceq 0, \quad (4.25)$$

$$\begin{aligned} & \begin{bmatrix} K_{00} & K_{01} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & K_{00} & K_{01} \\ 0 & 0 & 0 & I \end{bmatrix}^\top A\Lambda \begin{bmatrix} K_{00} & K_{01} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & K_{00} & K_{01} \\ 0 & 0 & 0 & I \end{bmatrix} \\ & \prec \begin{bmatrix} I & 0 & 0 & 0 \\ K_{10} & K_{11} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & K_{10} & K_{11} \end{bmatrix}^\top A\Lambda \begin{bmatrix} I & 0 & 0 & 0 \\ K_{10} & K_{11} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & K_{10} & K_{11} \end{bmatrix}. \end{aligned} \quad (4.26)$$

and

$$Q + \sigma N(0) \preceq 0, \quad Q + \sigma N(\bar{h}) \preceq 0, \quad (4.27)$$

$$\begin{bmatrix} K_{00} & K_{01} \\ 0 & I \end{bmatrix}^\top \Phi M \begin{bmatrix} K_{00} & K_{01} \\ 0 & I \end{bmatrix} \prec \begin{bmatrix} I & 0 \\ K_{10} & K_{11} \end{bmatrix}^\top \Phi M \begin{bmatrix} I & 0 \\ K_{10} & K_{11} \end{bmatrix} \quad (4.28)$$

with

$$Q = \begin{bmatrix} 2\mu M - \Gamma^\top M - M\Gamma & -MG \\ * & \mathbf{0}_{n \times n} \end{bmatrix}, \quad (4.29)$$

and $N(h) = e^{-2\alpha h} R^\top \Lambda R - \Lambda$ as defined as in (4.10). Then system (4.1) is exponentially stable in L^2 -norm with a decay-rate larger than μ .

Proof. The proof is based on a Lyapunov functional $V : L^2([0, L]; \mathbb{R}^n) \rightarrow \mathbb{R}_+$ defined by

$$V(y(t, x)) = \int_0^L y^\top(t, x) M y(t, x) dx. \quad (4.30)$$

In this step we first show that the functional V defined in (4.30) is continuous. Since $y(t, x)$ is continuous with respect to t for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ and continuous at sampling instants by construction (see Remark 2.2 of Chapter 2), then V is continuous for all $t \geq 0$. Now consider system (4.1). Let us study the time derivative of the Lyapunov functional $V(y(t, x))$ as shown in (4.30). For simplicity, by an abuse of notations, we use the notation y instead of $y(t, x)$ when we think that no confusion can be made. For $t \in [t_k, t_{k+1}), k \in \mathbb{N}$, the time derivative of $V(y)$ along the trajectories of (4.1) is

$$\begin{aligned}\dot{V}(y) &= \int_0^L (y_t^\top M y + y^\top M y_t) dx \\ &= \int_0^L \left((-\Phi \partial_x y - \Gamma y - G \hat{y})^\top M y + y^\top M (-\Phi \partial_x y - \Gamma y - G \hat{y}) \right) dx \\ &= -[y^\top \Phi M y]_0^L + \int_0^L (-y^\top (\Gamma^\top M + M \Gamma) y - \hat{y}^\top G^\top M y - y^\top M G \hat{y}) dx. \quad (4.31)\end{aligned}$$

By using the boundary condition (4.2) and adding the term $2\mu V(y)$ to both sides of (4.31), then, we have

$$\begin{aligned}\dot{V}(y) + 2\mu V(y) &= \begin{bmatrix} y^+(t, L) \\ y^-(t, 0) \end{bmatrix}^\top \left(\begin{bmatrix} K_{00} & K_{01} \\ 0 & I \end{bmatrix}^\top \Phi M \begin{bmatrix} K_{00} & K_{01} \\ 0 & I \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} I & 0 \\ K_{10} & K_{11} \end{bmatrix}^\top \Phi M \begin{bmatrix} I & 0 \\ K_{10} & K_{11} \end{bmatrix} \right) \begin{bmatrix} y^+(t, L) \\ y^-(t, 0) \end{bmatrix} \\ &\quad + \int_0^L (y^\top (2\mu M - \Gamma^\top M - M \Gamma) y - \hat{y}^\top G^\top M y - y^\top M G \hat{y}) dx \quad (4.32)\end{aligned}$$

By using (4.28), the above inequality (4.32) is equivalent to

$$\dot{V}(y) + 2\mu V(y) < \int_0^L \xi^\top Q \xi dx \quad (4.33)$$

with

$$\xi = [y^\top(t, x), \hat{y}^\top(t, x)]^\top, \quad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}$$

and Q defined as (4.29).

Since condition (4.27) is linear in $e^{-2\alpha h}$ and $t - t_k \leq \bar{h}$, by using a convexity argument, we have $Q + \sigma N(t - t_k) \preceq 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Therefore, we get

$$\int_0^L \xi^\top (Q + \sigma N(t - t_k)) \xi dx \leq 0. \quad (4.34)$$

where $\xi = [y^\top(t, x), \hat{y}^\top(t, x)]^\top, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$.

By recalling (4.9)

$$\int_0^L \varphi^\top(t, x) N(t - t_k) \varphi(t, x) dx \geq 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N},$$

from (4.25)-(4.26) and Lemma 4.1 we have

$$\int_0^L \xi^\top Q \xi dx \leq 0. \quad (4.35)$$

In view of (4.32)-(4.35),

$$\dot{V}(y) + 2\mu V(y) \leq 0 \quad (4.36)$$

for $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$, which means that the following inequality is obtained:

$$V(y(t, \cdot)) \leq e^{-2\mu(t-t_k)} V(y(t_k, \cdot)), t \in [t_k, t_{k+1}), k \in \mathbb{N},$$

by recursion, we get

$$V(y(t, \cdot)) \leq e^{-2\mu t} V(y_0),$$

i.e. $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$, we have

$$\|y(t, \cdot)\|_{L^2([0, L]; \mathbb{R}^n)}^2 \leq \frac{\lambda_{\max}(M)}{\lambda_{\min}(M)} e^{-2\mu t} \|y_0\|_{L^2([0, L]; \mathbb{R}^n)}^2.$$

Hence, according to proposition , we can conclude that the system is exponentially stable.

■

Remark 4.2

Theorem 4.1 provides a simple method for checking the stability of system (4.1). In order to ease the applicability of the results, we summarize as follows each tuning parameter in detail. Λ and M are the main variables of the set of inequalities to be solved. M is a Lyapunov matrix corresponding to the Lyapunov functional

$$V(y(t, x)) = \int_0^L y^\top(t, x) M y(t, x) dx.$$

μ is a positive scalar corresponding to a lower bound on the system decay rate. Λ allows to characterize the IQC (4.9) in Lemma 4.1. As mentioned in Remark 4.1, α is a scalar corresponding to an estimation of the growth rate for the system without control update in between sampling times via the functional

$$\Upsilon(\varphi(t, x)) = \int_0^L \varphi^\top(t, x) \Lambda \varphi(t, x) dx.$$

If (4.25)-(4.26) are found feasible, then the IQC (4.9) is satisfied along the solutions of system (4.1). Condition (4.27)-(4.28) ensure that while the IQC (4.9) is satisfied, the Lyapunov functional V decrease with a decay rate μ along the solutions of system (4.1).

4.5 Numerical example

In this section we illustrate how the proposed conditions can be checked numerically and used to establish quantitative tradeoff regions between maximum allowable sampling intervals and performance (in terms of decay rate).

Consider a system of the form (4.1) with

$$L = 1, \quad \Phi = \begin{bmatrix} 1.2 & 0 \\ 0 & -0.8 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1.3 & 1.5 \\ -0.5 & 2.5 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the initial condition

$$y_0(x) = \begin{bmatrix} 2.5 - 1.5 \cos(4\pi x) \\ -\sin(6\pi x) \end{bmatrix}.$$

The stabilizing control gain

$$G = \begin{bmatrix} 1.1 & 0.5 \\ 1 & 0.9 \end{bmatrix}$$

was designed using classical continuous-time methods for hyperbolic systems. We apply Theorem 4.1 for several values of the maximum sampling interval \bar{h} and of the decay rate μ . The results obtained using Matlab **LMI** solvers are illustrated in Figure 4.2. Such plots can be used in order to make design tradeoffs between system performance (in terms of decay rate) and robustness to the sampled-data implementation (in terms of maximum allowable sampling interval). For instance, by increasing the requirements on the system performance (increasing the decay rate μ), the maximum admissible sampling interval \bar{h} becomes smaller (i.e. more constraints should be imposed on the sampled-data control implementation in terms of sampling and control computation frequency). The method also allows to find the maximum \bar{h} for which stability is granted. The best value of \bar{h} for which the conditions of Theorem 4.1 were found feasible is $\bar{h} = 0.024$ (derived with $\mu = 10^{-5}$, $\alpha = 3$, $\sigma = 1$). In order to illustrate the influence of the open-loop growth rate estimate α on the computation of the maximum allowable sampling interval \bar{h} , Theorem 4.1 was tested for several values of α and \bar{h} . The results are illustrated in Figure 4.3. As expected, the tighter the estimation of the growth rate α is, the better the obtained **IQC** allows to compute the maximum allowable sampling interval \bar{h} . This is consistent with the theoretical remarks concerning Lemma 4.1.

In order to illustrate the response of the sampled-data system, we choose one point corresponding to a maximum sampling interval $\bar{h} = 0.01$ and a decay rate $\mu = 0.1$. The conditions (4.25) and (4.27) in Theorem 4.1 were found feasible, leading to a Lyapunov functional characterized by

$$M = \begin{bmatrix} 5.06 & 0 \\ * & 4.54 \end{bmatrix}.$$

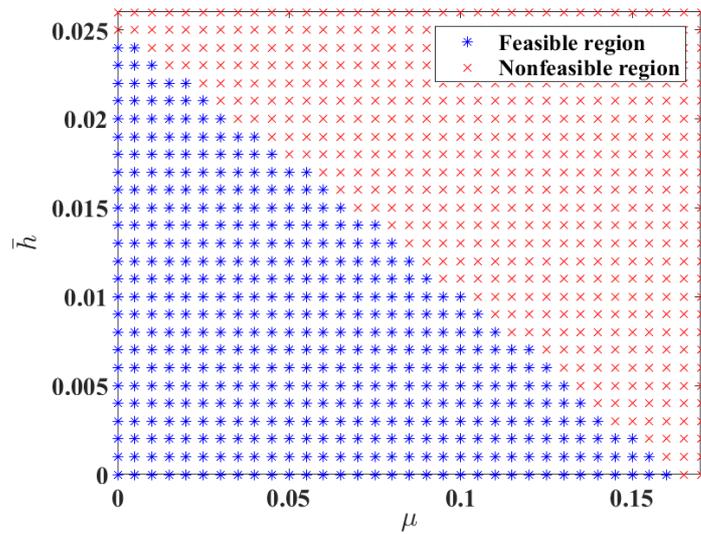


Figure 4.2: Tradeoff regions between system performance and maximum allowable sampling. Feasible region (blue area) and non feasible region (red area) guaranteed by Theorem 4.1.

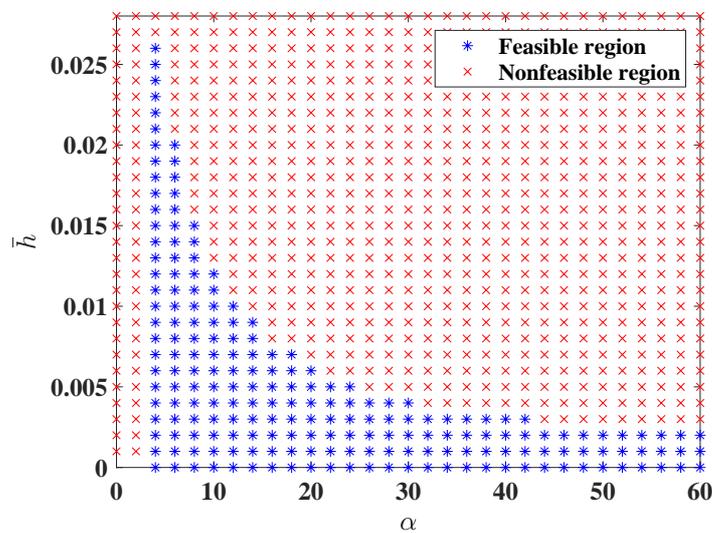


Figure 4.3: Influence of the estimate of the open-loop growth rate α on the computation of the maximum allowable sampling interval \bar{h} . Feasible region (blue area) and non feasible region (red area) guaranteed by Theorem 4.1 with $\mu = 10^{-5}$.

The evolution of the Lyapunov functional V along the solutions of the sampled-data system

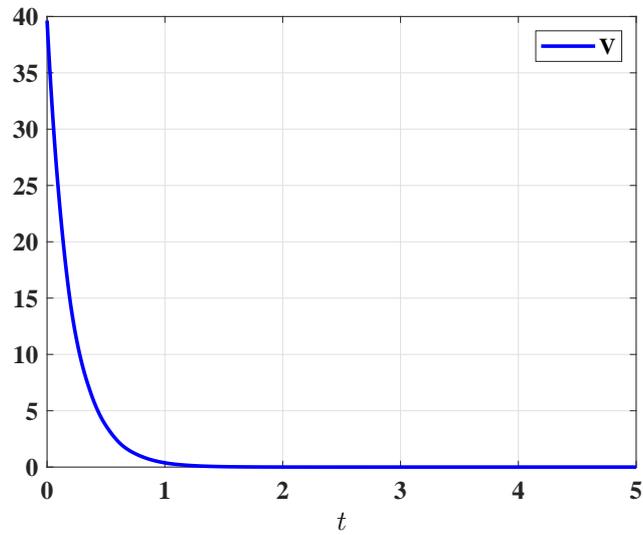


Figure 4.4: Time-evolution of functional V in the sampled-data system.

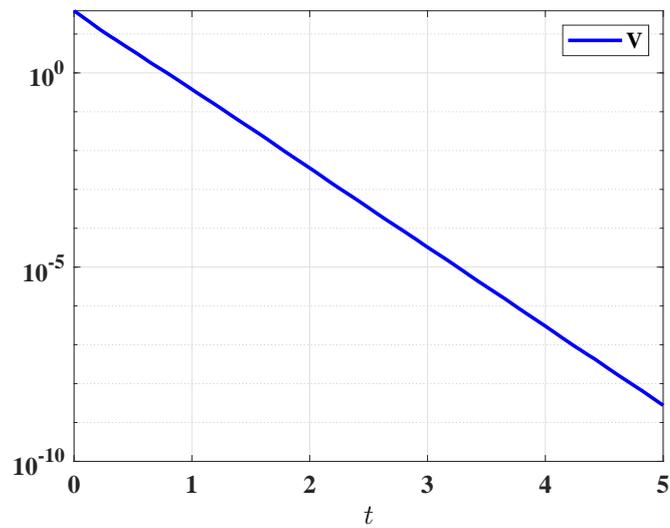


Figure 4.5: Time-evolution of functional V in semi-log scale.

is given in Figure 4.4 and its semi-log plot is given in Figure 4.5. The figures clearly illustrates the exponential stability of the system. As expected, Figures 4.6-4.7 illustrates the states convergence to the origin.

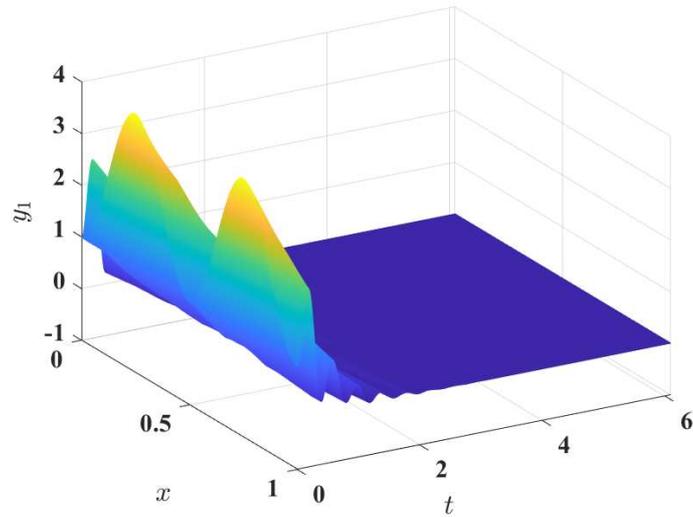


Figure 4.6: Response of state y_1 for the closed-loop system.

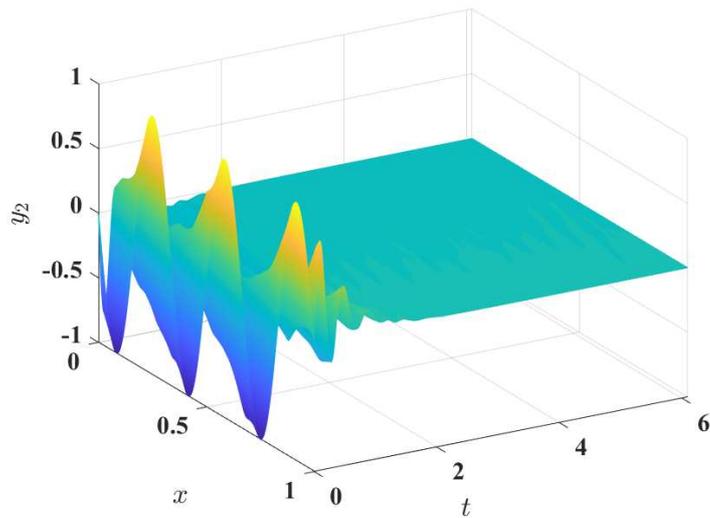


Figure 4.7: Response of state y_2 for the closed-loop system.

4.6 Conclusion

This chapter studied the distributed sampled-data control for a class of hyperbolic systems using impulsive system method. The closed-loop system is first represented as an augmented impulsive system. Moreover, the stability of the system has been established

via a method based on **IQCs**. The global exponential stability of the system can be checked. Finally, a numerical example verifies our results. In the future, we will focus on sampled-data control using space and time discretization for different types of hyperbolic systems.

Conclusion and perspectives

Conclusion

The problem of stability analysis for a class of linear hyperbolic systems with distributed sampled-data control laws was investigated in this thesis. We used the Lyapunov method to derive local stability conditions for 1-D hyperbolic systems of balance laws. With the help of the Lyapunov-Razumikhin technique, the spatio-temporal sampling controller was also addressed as a more general case. In addition, we have provided methods for checking the global stability of the system by rewriting the original system as an augmented impulsive model.

The main contributions of this thesis are introduced in Chapters 2-4.

- In Chapter 2, we mainly studied the stability conditions of the hyperbolic **PDE** with in-domain sampled-data controller. Inspired by the input-output approach on **ODE** and boundary control on **PDE**, we provided a method to check the local stability ($R\epsilon$ -stability) of the system through the Lyapunov technique. The method is based on two ingredients. First, we recast the sampled-data system into an equivalent continuous hyperbolic **PDE** (including four steps: the nominal closed-loop system, the introduction of sampling error, the properties of sampling error and the interconnection of operators). Then, based on the new model, we constructed a suitable Lyapunov functional and derived sufficient conditions for the stability of the system by means of **LMIs**. This method is the basis for the following chapters.
- In Chapter 3, we extended the results in Chapter 2 to the case where the controller is spatio-temporally sampled. Compared with Chapter 2, in addition to the time sampling error, it was also necessary to consider the spacial discretization error when remodeling the system. Based on the ideal continuous-time control loop and the operators representing the discretization errors, we derived sufficient conditions to ensure the $R\epsilon$ -stability of linear hyperbolic systems of balance laws by the Lyapunov-Razumikhin method. In a special case (no sampling in space), a larger estimation of the maximum allowable sampling interval was observed (in comparison with the results in Chapter 2), which reflects the advantages of the Lyapunov-Razumikhin method.
- In Chapter 4, inspired on basic results for sampled-data **ODEs**, conditions for checking the global exponential stability of sampled-data hyperbolic systems of

balance laws are proposed. An augmented state model with an impulsive form was employed to derive an **IQC** that characterizes the relation between the sampling interval, the system state and its sampled vector.

Perspectives

Even though many results have been achieved in the stability analysis of hyperbolic systems, there are still many open problems worth exploring. According to the work in this thesis, we list several directions that can be further studied in the future.

- In the light of the results obtained in Chapter 4, it is natural to consider extending the global exponential stability to the case where the controller is spatio-temporally sampled. Moreover, inspired by the work in [100], we can consider time-varying hyperbolic systems, which are more general in physical networks (such as in irrigation or road networks).
- Since the states of the control systems are not all directly measurable and the cost is high, output feedback control is widely used in practical engineering systems. In this context, we have to consider the design of sampled-data observer based controllers.
- In the real world, all systems are non-linear, which motivates us to extend the methods in this thesis to non-linear systems. The study of a distributed spatio-temporally sampled controllers for system with Lipschitz nonlinear dynamics would be valuable.

In short, the conditions provided by using the Lyapunov functionals for analyzing the stability of hyperbolic systems with distributed sampled-data control have potential. However, further exploration of the proposed methods may promote the development of Lyapunov methods in application fields.

Appendix A

Some useful theorems and lemmas

Theorem A.1 (Schwarz's theorem [56]) For a function $f : \Theta \rightarrow \mathbb{R}$ defined on a set $\Theta \subset \mathbb{R}^n$, if $\mathbf{a} \in \mathbb{R}^n$ is a point such that some neighborhood of \mathbf{a} is contained in Θ and f has continuous second partial derivatives at the point \mathbf{a} , then $\forall i, j \in \{1, 2, \dots, n\}$,

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{a}) = \frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{a})$$

The partial derivatives of this function commute at that point.

Theorem A.2 (Cauchy-Schwarz inequality [14]) If $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\bar{\beta} = (\beta_1, \dots, \beta_n)$ are sequences of real numbers, then

$$\left(\sum_{i=1}^n \alpha_i \beta_i \right)^2 \leq \sum_{i=1}^n \alpha_i^2 \sum_{i=1}^n \beta_i^2$$

with equality if and only if the sequences $\bar{\alpha}$ and $\bar{\beta}$ are proportional, i.e., there is a $r \in \mathbb{R}$ such that $\alpha_i = r\beta_i$ for each $i \in \{1, \dots, n\}$.

Theorem A.3 (Young's inequality [135]) If $a \geq 0$ and $b \geq 0$ are nonnegative real numbers and if $m > 1$ and $n > 1$ are real numbers such that $\frac{1}{m} + \frac{1}{n} = 1$, then

$$ab \leq \frac{a^m}{m} + \frac{b^n}{n}$$

Equality holds if and only if $a^m = b^n$.

Theorem A.4 (Wirtinger's inequality [46]) Let $y \in W_1[c, d]$ be a scalar function with $y(c) = 0$. Then,

$$\int_c^d y^2(\phi) d\phi \leq \frac{4(d-c)^2}{\pi^2} \int_c^d \dot{y}^2(\phi) d\phi$$

This inequality is trivially extended to the vector case.

Lemma A.1 (Liu 2010 [77]) Let $y \in W_m[c, d]$. Assume that $y(c) = 0$. Then, for any $m \times m$ -matrix $\Lambda \succ 0$, the following inequality holds:

$$\int_c^d y^\top(\phi) \Lambda y(\phi) d\phi \leq \frac{4(d-c)^2}{\pi^2} \int_c^d \dot{y}^\top(\phi) \Lambda \dot{y}(\phi) d\phi.$$

Theorem A.5 (Schur complement [10]) Let A and D be symmetric matrices. Then, the following are equivalent:

1. $\begin{bmatrix} A & B \\ * & D \end{bmatrix} \succ 0$ (resp. $\begin{bmatrix} A & B \\ * & D \end{bmatrix} \succeq 0$)
2. $D \succ 0, A - BD^{-1}B^\top \succ 0$ (resp. $D \succeq 0, A - BD^+B^\top \succeq 0, B(I - DD^+) = 0$)

where D^+ is the pseudo-inverse of D .

Theorem A.6 (S-procedure [132]) Let $Z_k \in \mathcal{M}_n(\mathbb{R}), k \in \{0, \dots, q\}$. Then, if

1. there exist scalars $\gamma_k \geq 0, k \in \{1, \dots, q\}$, such that $Z_0 - \sum_{k=1}^q \gamma_k Z_k > 0$ (resp. $Z_0 - \sum_{k=1}^q \gamma_k Z_k \geq 0$)
then
2. $\eta^T Z_0 \eta > 0$ (resp. $\eta^T Z_0 \eta \geq 0$) for any $\eta \in \mathbb{R}^n$ satisfying $\eta^T Z_k \eta \geq 0$ for all $k \in \{1, \dots, q\}$

For $q = 1$, these two statements are equivalent.

Lemma A.2 (Integral S-procedure [126]) Let F and G be symmetric matrices. Assume that the strict integral inequality

$$\int_0^L y^T(x) G y(x) dx \geq 0,$$

holds. Then the implication

$$\int_0^L y^T(x) F y(x) dx \leq 0$$

holds if there exists some nonnegative number ϖ such that

$$F + \varpi G \preceq 0$$

Proof: It is immediately proved by rewriting the inequality above as $F \preceq -\varpi G$ and multiplying on the left by $y^T(x)$ and on the right by $y(x)$, then taking the integral operation on both sides of the inequality. We have

$$\int_0^L y^T(x) F y(x) dx \leq \int_0^L y^T(x) (-\varpi G) y(x) dx.$$

Since $\int_0^L y^T(x) G y(x) dx \geq 0$ and $\varpi \geq 0$ hold, one can obtain

$$\int_0^L y^T(x) (-\varpi G) y(x) dx \leq 0,$$

which leads to

$$\int_0^L y^T(x) F y(x) dx \leq 0.$$

Hence, this lemma is proved.

Lemma A.3 (Halanay's inequality [45]) Assume that $\tau \geq 0$ and $V(t)$ is a positive function defined on $[t_0 - \tau, +\infty)$, with derivative $\dot{V}(t)$ on $[t_0, +\infty)$. If

$$\dot{V}(t) \leq -\alpha V(t) + \beta \sup_{t-\tau \leq s \leq t} V(s), \quad t \geq t_0$$

where $\alpha > \beta > 0$, then there exist $\gamma > 0$ and $k > 0$ such that

$$V(t) \leq k e^{(-\gamma(t-t_0))}, \quad t \geq t_0.$$

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Titre : Contributions à l'étude des systèmes hyperboliques à données échantillonnées

Résumé : Ce manuscrit présente de nouveaux résultats sur l'analyse de stabilité pour une classe d'Équations aux Dérivées Partielles (en anglais Partial Differential Equations-PDEs) hyperboliques linéaires avec commande distribuée à données échantillonnées. Nous utilisons d'abord la méthode de Lyapunov pour dériver des conditions de stabilité locale pour les systèmes hyperboliques 1-D de lois d'équilibre. Le système en boucle fermée est reformulé d'un point de vue entrée-sortie. Ensuite, nous étendons l'approche proposée en utilisant la méthode de Lyapunov-Razumikhin et nous fournissons un nouveau résultat capable de gérer le contrôleur discrétisé à la fois dans l'espace et dans le temps. De plus, nous proposons des méthodes pour vérifier la stabilité du système en réécrivant le système d'origine comme un système hybride impulsif augmenté. En outre, la stabilité du système a été établie via une méthode basée sur les Contraintes Quadratiques Intégrales (en anglais Integral Quadratic Constraints - IQCs). Dans ce cas, la stabilité exponentielle globale du système peut être vérifiée.

Mots-clefs : PDEs, commande à données échantillonnées, analyse de stabilité.

Title : Contributions to the study of sampled-data hyperbolic systems

Abstract : This manuscript presents new results on the stability analysis for a class of linear hyperbolic Partial Differential Equations (PDEs) with distributed sampled-data control. We first use the Lyapunov method to derive local stability conditions for 1-D hyperbolic systems of balance laws. The closed-loop system is reformulated from an input-output point of view. Then, we extend the proposed approach by the virtue of the Lyapunov-Razumikhin method and provide a new result that is able to handle the controller discretized both in space and time. In addition, we provide methods for checking the stability of the system by rewriting the original system as an augmented impulsive hybrid system. Moreover, the stability of the system is established via a method based on Integral Quadratic Constraints (IQCs). In this case, the global exponential stability of the system can be checked.

Keywords : PDEs, sampled-data control, stability analysis.