Real enumerative geometry of Fano varieties of dimension 3 and index 2

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Abstract

Enumerative geometry tends to enumerate solutions of geometrical problems. Since 1990s, many classical problems in enumerative geometry have been solved by the appearance of Gromov-Witten invariants. Enumerative Gromov-Witten invariants are known as the enumeration of complex curves passing through a generic constraint in projective varieties. In 2003, J-Y Welschinger developed a new real enumerative invariant, which is known to be the real counterpart of Gromov-Witten invariants, concerning to the enumeration of real curves with a consistent choice of sign ± 1 such that this count is independent of the choice of configuration.

Since 2000s, following the founding work of J-Y. Welschinger ([Wel05b], [Wel05c]) in symplectic geometry and G. Mikhalkin [Mik05] in tropical geometry, real enumerative geometry has experienced tremendous growth. Low-dimensional spaces (i.e. complex dimensions 2 and 3) turn out to have particularly rich real enumerative geometry. While Gromov-Witten and Welschinger invariants of 2-dimensional varieties have been systematically studied in recent years ([KM94], [GP96]; [BM07], [IKS09], [HS12], [CZ21b]), many studies in the 3-dimensional case remain to be explored, in particular the real enumeration. Up to now, several explicit computations have been performed ([KM94], [BG16], [Zah18]; [GZ17], [Din19], [CZ21a]).

In 2016, E. Brugallé and P. Georgieva [BG16] proposed a new strategy (based on an idea of J. Kollár [Kol15]) to compute Gromov-Witten and Welschinger invariants of complex projective space $\mathbb{C}P^3$ using pencils of quadrics. This thesis is inspired from their work, in which we use pencils of surfaces in the linear system $|-\frac{1}{2}K_X|$ to reduce the 3-dimensional problem into two independent ones: 1-dimensional problem on elliptic curves and 2-dimensional problem on del Pezzo surfaces. By this strategy, we obtain the formulas for computing genus-0 Gromov-Witten and Welschinger invariants of 3-dimensional Fano varieties, whose first Chern class is even, with emphasis on the cases of degree $6, 7$ and 8. We also generalize a result of Kollár [Kol15] to the optimality of the genus zero Welschinger invariants on such spaces. Based on these obtained results, we believe that this strategy has potential for some new research directions on the relationship between complex and real enumerative geometry, as well as between those of varieties of dimension 2 and 3.

Introduction (en français)

Histoire

La géométrie énumérative est une branche de la géométrie algébrique, dont le but est de déterminer le nombre de solutions de problèmes géométriques. L'un des plus simples problèmes d'énumération est :

Problème 0. *Dans le plan réel, combien de droites passent par deux points distincts ?*

La réponse à ce problème est triviale : il existe une unique telle droite. Cette affirmation est connue comme le cinquième axiome de la géométrie euclidienne datant des mathématiques grecques antiques. Il existe une extension culturelle de ce problème :

Problème 1. *Dans le plan projectif complexe, combien de courbes de degré d et de genre g passent par $3d-1+g$ points en position générale ?*

En 1994, **Problème 1** a été résolu pour le cas $g=0$ par la formule récursive de M. Kontsevich [KM94], pour laquelle la seule donnée initiale requise est le fait qu'il existe une unique droite passant par deux points distincts. Quatre ans plus tard, ce problème a été totalement résolu par L. Caporaso et J. Harris [CH98]. Dans ce qui suit, nous listons quelques directions de généralisation du **Problème 1**.

1. Changer de point de vue.

   Si nous remplaçons le point de vue algébrique par celui symplectique, alors le problème correspondant s'avère être l'énumération de courbes pseudo-holomorphes dans des variétés symplectiques. Ainsi, le problème énumératif des courbes dans les variétés algébriques revêt l'apparence des *invariants de Gromov-Witten*. Dans certains cas, lorsque les invariants de Gromov-Witten sont entiers, ils coïncident avec les réponses aux problèmes énumératifs algébriques ([LT98], [Sie],[Ful97]). Par conséquent, les invariants de Gromov-Witten deviennent un outil puissant pour étudier la géométrie énumérative.

2. Modifier l'espace ambiant en augmentant sa dimension, ou en ajoutant plus de structure.

   On peut étendre ce problème énumératif à d'autres variétés algébriques comme les espaces projectifs complexes n-dimensionnels $\mathbb{C}P^n$, les variétés de Fano n-dimensionnels ($n \geq 2$). De plus, si l'on équipe les variétés d'une *structure réelle* (c'est-à-dire une involution anti-holomorphe) et on compte les *courbes réelles* (c'est-à-dire les courbes conservées par la structure réelle), on enrichit grandement la *géométrie énumérative réelle*. La géométrie énumérative réelle a connu un formidable essor depuis près de vingt ans, notamment suite aux travaux fondateurs de J-Y. Welschinger en géométrie symplectique et G. Mikhalkin en géométrie tropicale. Dans [Wel05b] et [Wel05c], Welschinger a prouvé que dans certaines variétés de dimension deux et trois, on peut faire un choix cohérent de signe ± 1 tel que le nombre de courbes réelles de genre zéro comptées avec ce signe ne dépend que de la classe d'homologie des courbes (et pas du choix générique de la configuration par laquelle ils passent). En particulier, cette façon de compter donne des bornes inférieures (non-triviales) en géométrie énumérative réelle. Ces nombres sont appelés *invariants de Welschinger*. De plus, lorsqu'on a commencé à travailler avec les semi-corps des nombres tropicaux, la géométrie énumérative tropicale a été développée avec de nombreuses techniques inédites et les problèmes énumératifs correspondants, comme les correspondances de Mikhalkin, les multiplicités de Block-Göttsche, les invariants raffinés...

Bien sûr, il existe encore bien d'autres manières de généraliser, de modifier ou d'appliquer ces problèmes énumératifs originaux, que nous ne pouvons pas toutes mentionner ici. Nous nous intéressons aux généralisations suivantes du **Problème 1** :

Problème 2 (GW). *Dans une variété de del Pezzo non-singulière X de dimension trois, combien de courbes irréductibles de genre zéro représentent une classe d'homologie donnée $d \in H_2(X;\mathbb{Z})$ et passent par une configuration générique de $\frac{1}{2}c_1(X) \cdot d$ points ?*

Nous appelons ce nombre *l'invariant de Gromov-Witten de genre zéro* de X, noté $GW_X(d)$.

Problème 2 (W). *Dans une variété de del Pezzo non-singulière réelle X de dimension trois munie d'une structure réelle τ_X, combien de courbes irréductibles réelles de genre zéro, comptées avec le signe de Welschinger, représentent une classe d'homologie donnée $d \in H_2^{-\tau_X}(X;\mathbb{Z})$ et passent par une configuration générique réelle, constituée de r points réels et l paires de points complexes conjugués, où $r + 2l = \frac{1}{2}c_1(X) \cdot d$?*

Nous appelons ce nombre *l'invariant de Welschinger de genre zéro* de X, noté $W_{\mathbb{R}X}(d,l)$.

Motivations

Alors que la géométrie énumérative complexe et réelle dans le cas des variétés algébriques de dimension 2 a été systématiquement étudiée ces dernières années, de nombreux problèmes énumératifs non résolus subsistent encore dans le cas des variétés de dimension 3, en particulier concernant l'énumération réelle. Nous sommes motivés par les problèmes énumératifs (complexes et réels) de dimension 2 suivants :

Problème 3 (GW)**.** *Dans une surface de del Pezzo non-singulière Y, combien de courbes irréductibles de genre zéro représentent une classe d'homologie donnée $D \in H_2(Y;\mathbb{Z})$ et passent par une configuration générique de $c_1(Y)\cdot D - 1$ points ?*

En d'autres termes, on veut calculer l'invariant $GW_Y(D)$.

Problème 3 (W)**.** *Dans une surface réelle non-singulière de del Pezzo Y munie d'une structure réelle τ_Y, combien de courbes réelles de genre zéro irréductibles, comptées avec le signe de Welschinger, représentent une classe d'homologie donnée $D \in H_2^{-\tau_Y}(Y;\mathbb{Z})$ et passent par une configuration générique réelle, constituée de r points réels et l paires de points complexes conjugués, où $r + 2l = c_1(Y)\cdot D - 1$?*

En d'autres termes, on veut calculer l'invariant $W_{\mathbb{R}Y}(D, l)$.

Par classification, les surfaces de del Pezzo non-singulières sont soit $\mathbb{C}P^1 \times \mathbb{C}P^1$, le double produit de la droite projective, soit $\mathbb{C}P^2_r$, les éclatés du plan projectif en r points en position générale ($r \leq 8$). Depuis les années 1990, des calculs explicites pour le **Problème 3** (GW) ont été entièrement effectués : par M. Kontsevich et Y. Manin ([KM94], [KV07]) pour $\mathbb{C}P^2$ et $\mathbb{C}P^1 \times \mathbb{C}P^1$, en utilisant les formules de récurrence de Kontsevich ainsi que par L. Göttsche et R. Pandharipande [GP96] pour $\mathbb{C}P^2_r$ avec r quelconque.

Les surfaces de del Pezzo réelles ont également une classification complète due à l'étude de Comessatti depuis 1913 [Com13]. Le signe de Welschinger d'une courbe réelle dans de telles surfaces réelles est déterminé par la parité du nombre de ses *points doubles elliptiques* (c'est-à-dire les points réels dont l'équation locale de la courbe est donnée par $x^2 + y^2 = 0$ sur $\mathbb{R}$). Quelques calculs explicites des invariants de Welschinger du **Problème 3** (W) ont été effectués, par exemple :

- par J-Y. Welschinger [Wel07] lui-même, en utilisant la théorie des champs symplectiques dans le cas du plan projectif et de l'ellipsoïde quadrique ;
- par E. Brugallé et G. Mikhalkin ([BM07], [BM08]), en utilisant des diagrammes d'étage en géométrie tropicale, pour $\mathbb{C}P^2_{r,s}$, les éclatés du plan projectif en r points réels et s paires de points complexes conjugués ($r + 2s \leq 3$); puis par E. Brugallé [Bru15] pour $\mathbb{C}P^2_{r,s}$ ($r + 2s \leq 8$) avec plus de structures réelles ;
- par I. Itenberg, V. Kharlamov et E. Shustin ([IKS09], [IKS13b], [IKS13a], [IKS15]), en utilisant la géométrie tropicale et les formules de type Caporaso-Harris, pour $\mathbb{C}P^2_{r,s}$ ($r + 2s \leq 7$) avec des contraintes de points purement réels et d'autres structures réelles ;
- par J. Solomon et A. Horev ([Sol07], [HS12]), en utilisant les relations de type WDVV de Solomon, pour $\mathbb{C}P^2_{r,s}$ avec r, s quelconques.

Revenons maintenant à nos problèmes énumératifs tridimensionnels, d'après V.A. Iskovskikh [Isk77], les variétés de del Pezzo de dimension 3 sont totalement classifiées par leur degré $\deg$. On a $\deg(X) \in \{1, \ldots, 8\}$. Les calculs pour le **Problème 2** (GW) ont été effectués pour presque tous les cas. Plus précisément, la classe de degré 8, qui est l'espace projectif $\mathbb{C}P^3$, a été étudiée depuis les années 1990, en utilisant la relation WDVV de la théorie de Gromov-Witten [1] [KM94]; puis par A. Gathmann [Gat98b] et récemment par E. Brugallé et P. Georgieva [BG16] grâce à une nouvelle stratégie utilisant des pinceaux de quadriques. En fait, cette méthode a été mentionnée pour la première fois par Kollár en 2015 [Kol15] pour fournir des configurations qui simulent l'optimisation de certains problèmes enumératifs. Par la même stratégie d'utilisation de pinceaux de surfaces algériques, les classes de degrés $2, 3, 4, 5$ ont été étudiées par A. Zahariuc [Zah18] et la classe de degré 7 a été étudiée par Y. Ding [Din19]. Une classe de degré 6, qui est le triple produit de la droite projective, a été étudiée par X. Chen et A. Zinger [CZ21a], en utilisant les relations de type WDVV de Solomon.

La classification des variétés de del Pezzo réelles tridimensionnels n'est pas très connue (voir [Man20]). Le cas de degré 3, c'est-à-dire le cubique tridimensionnel, a été classifié par V.A. Krasnov [Kra09]. Les études sur les structures réelles sur les variétés réelles tridimensionnels se trouvent dans la série de Kollár [Kol97], [Kol98], [Kol99], [Kol+00], [Kol00]; ou par V. Kharlamov [Kha02]. Le signe de Welschinger d'une courbe réelle dans les variétés tridimensionnelles réelles, qui est aussi appelé *état de spineur*, est déterminé par les structures $Pin^{\pm}/Spin$ sur la partie réelle de cette variété. Pour le moment, quelques calculs explicites pour les invariants de Welschinger dans le **Problème 2** (W) ont été effectués, par exemple :

- l'espace projectif (le cas le plus étudié avec diverses approches) : par Brugallé-Mikhalkin [BM07], en utilisant des diagrammes d'étage ; par Georgieva-Zinger [GZ17], en utilisant des relations de type WDVV et par Brugallé-Georgieva [BG16], en utilisant des pinceaux de quadriques ;
- l'espace projectif éclaté en un point avec une structure réelle standard : par Ding [Din19] pour certains types de courbes également, en utilisant aussi des pinceaux de quadriques ;
- le triple produit de la droite projective muni de deux structures réelles : par Chen-Zinger ([CZ21b], [CZ21a]), en utilisant les relations de type WDVV de Solomon.

À ma connaissance, le calcul des invariants de Welschinger pour les variétés de del Pezzo réelles de dimension 3 de petit degré (i.e. degré de 1 à 5) n'est pas connu dans la littérature.

Résultats

Cette thèse s'inspire de la stratégie proposée par Brugallé-Georgieva [BG16] dans le cas de $\mathbb{C}P^3$, dans laquelle ils ont étudié des pinceaux de quadriques comme mentionné ci-dessus. Nous étendons leurs résultats aux variétés de Fano de dimension trois et d'indice deux, c'est-à-dire les variétés de del Pezzo de dimension 3. Nous obtenons de nouvelles formules pour résoudre le **Problème 2** (GW) et le **Problème 2** (W) pour la plupart des cas de degré $6, 7$ et 8.

1. En fait, la relation WDVV de la théorie de Gromov-Witten a résolu ce problème énumératif classique pour les espaces projectifs complexes en toute dimension.

Invariants de Gromov-Witten - Problème 2 (GW)

Soit X une variété de del Pezzo de dimension trois. Soit $E \subset X$ le lieu de base d'un pinceau $\mathcal{Q}$ de surfaces dans le système linéaire $|-\frac{1}{2}K_X|$. Nous prouvons d'abord que E est une courbe elliptique de classe d'homologie $\frac{1}{4}K_X^2$. Soit $\mathcal{C}_d(\underline{x}_d)$ l'ensemble des courbes algébriques connexes de genre arithmétique zéro représentant la classe d'homologie $d \in H_2(X;\mathbb{Z})$ et passant par $\underline{x}_d$, une configuration générique de $\frac{1}{2}c_1(X) \cdot d$ points dans E.

Le premier résultat est une généralisation de la Proposition 3 de Kollár [Kol15]. Nous prouvons que :

Proposition 2.3.2. *Toute courbe connexe rationnelle dans l'ensemble $\mathcal{C}_d(\underline{x}_d)$ est irréductible et contenue dans une surface du pinceau $\mathcal{Q}$.*

Dans le deuxième résultat, nous étendons la stratégie de Brugallé-Georgieva à des cas plus généraux, dans lesquels nous réduisons le **Problème 2** (GW) à deux problèmes indépendants : le premier s'appelle *problème unidimensionnel* sur la courbe elliptique E, et le second s'appelle *problème énumératif bidimensionnel* sur les surfaces del Pezzo dans $|-\frac{1}{2}K_X|$. Soit $\Sigma \in |-\frac{1}{2}K_X|$ une surface de del Pezzo non-singulière. Le *problème énumératif bidimensionnel* nous donne exactement les invariants de Gromov-Witten de genre zéro des surfaces de del Pezzo non-singulières, c'est-à-dire le **Problème 3** (GW), qui ont été calculé comme nous l'avons mentionné dans la section des motivations. Par conséquent, nous nous concentrons sur le *problème unidimensionnel*. Si l'application surjective $\psi : H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, (qui est induite par l'inclusion $\Sigma \hookrightarrow X$), satisfait $\ker(\psi) \cong \mathbb{Z}$, nous prouvons qu'il y a $(D.S)^2$ surfaces de del Pezzo dans le pinceau $\mathcal{Q}$ contenant une courbe dans l'ensemble $\mathcal{C}_d(\underline{x}_d)$, où la classe d'homologie $D \in H_2(\Sigma;\mathbb{Z})$ est dans $\psi^{-1}(d)$ et la classe d'homologie $S \in H_2(\Sigma;\mathbb{Z})$ est un générateur du groupe $\ker(\psi)$.

Nous résumons ce résultat dans le théorème suivant.

Théorème 2.1.1. *Soit X une variété de del Pezzo de dimension 3 telle que $\ker(\psi) \cong \mathbb{Z}$, où la surface $\Sigma \in |-\frac{1}{2}K_X|$, l'application ψ et la classe S sont comme décrits ci-dessus. Pour toute classe d'homologie $d \in H_2(X;\mathbb{Z})$, on a*

$$GW_X(d) = \frac{1}{2} \sum_{D \in \psi^{-1}(d)} (D.S)^2 GW_\Sigma(D).$$

Invariants de Welschinger - Problème 2 (W)

Supposons maintenant que le pinceau $\mathcal{Q}$ et la courbe elliptique E sont réels et ses parties réelles sont non-vides. Soit $\mathcal{R}_d(\underline{x}_d)$ l'ensemble des courbes rationnelles irréductibles réelles représentant la classe d'homologie $d \in H^{-\tau_X}(X;\mathbb{Z})$ et passant par $\underline{x}_d$, une configuration générique réelle de k_d points dans E dont l paires de points complexes conjugués. Nous adaptons l'idée de la preuve du **Problème 2** (GW) au **Problème 2** (W). Plus précisément, nous réduisons ce problème à deux problèmes indépendants : le premier s'appelle *problème unidimensionnel réel* sur la courbe elliptique réelle E, et le second s'appelle *problème énumératif réel bidimensionnel* sur des surfaces de del Pezzo réelles dans $|-\frac{1}{2}K_X|$. Dans ce cas, le *problème énumératif réel bidimensionnel* nous donne des invariants de Welschinger de genre zéro des surfaces réelles de del Pezzo, c'est-à-dire le **Problème 3** (W), qui ont été calculé comme nous l'avons mentionné dans la section des motivations, mais au signe près. Ainsi, à la différence du cas de Gromov-Witten, il y a un autre problème à résoudre : le *problème du signe*.

Soit $\Sigma \in |-\frac{1}{2}K_X|$ une surface réelle non-singulière telle que la classe du cycle d'évanescence S soit $\tau|_\Sigma$-anti-invariante. En résolvant le *problème unidimensionnel réel* sur la courbe elliptique réelle E, si $\ker(\psi) \cong \mathbb{Z}$, nous prouvons qu'il y a $|D.S|$ telles surfaces sur le pinceau réel $\mathcal{Q}$ contenant une courbe dans l'ensemble $\mathcal{R}_d(\underline{x}_d)$, où les classes d'homologie D et S sont $\tau|_\Sigma$-anti-invariantes.

Le *problème du signe* vient de la différence entre deux définitions du signe de Welschinger pour les courbes dans les variétés de dimension deux et trois, comme nous l'avons mentionné dans la section des motivations. D'une part, le signe de Welschinger pour les courbes sur la surface réelle Σ dépend du nombre de ses points doubles elliptiques. D'autre part, le signe de Welschinger pour les courbes sur la variété tridimensionnelle réelle X, (connu sous le nom d'état de spineur), dépend du choix de la structure $Pin_3^\pm$ sur $\mathbb{R}X$ (ou la structure $Spin_3$ sur $\mathbb{R}X$ si $\mathbb{R}X$ est orientable), et du choix de la classe d'isotopie du sous-fibré holomorphe réel en droites de degré (k_d-2) dans le fibré normal holomorphe en plans TX/TC. Désormais, si $\mathbb{R}X$ est orientable d'orientation $\mathfrak{o}_X$ et muni d'une structure $Spin_3$ $\mathfrak{s}_X$, on note $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)$ l'invariant de Welschinger de genre zéro correspondant. Par conséquent, pour résoudre ce *problème de signe*, nous devons interpréter ces deux paramètres dépendants des états des spineurs.

Pour le côté "structure $Pin^\pm/Spin$", on observe d'abord que la structure $Spin_3$ $\mathfrak{s}_X$ sur $\mathbb{R}X$ peut être restreinte à une structure Pin_2^- sur $\mathbb{R}\Sigma$, notée $\mathfrak{s}_{X|\Sigma}$. La deuxième observation est qu'il existe une correspondance bijective canonique entre le structure Pin_2^- $\mathfrak{s}_{X|\Sigma}$ sur $\mathbb{R}\Sigma$ et la fonction quasi-quadratique $s^{\mathfrak{s}_{X|\Sigma}} : H_1(\mathbb{R}\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_2$. Par la correspondance de Horev-Solomon entre la fonction quasi-quadratique et le signe de Welschinger de la courbe sur les surfaces, on résout la première partie du *problème du signe*, dans lequel on obtient le terme $(-1)^{g(D)+s^{\mathfrak{s}_{X|\Sigma}}(\rho D)}$, où $g(D)=\frac{1}{2}(K_\Sigma.D+D^2+2)$ et $\rho : H_2^{-\tau|_\Sigma}(\Sigma;\mathbb{Z}) \to H_1(\mathbb{R}\Sigma;\mathbb{Z}_2)$ est une application donnée par la partie réelle de classe représentative $\tau|_\Sigma$-anti-invariante.

Le côté "classe d'isotopie" est plus délicat. On remarque d'abord que le fibré normal holomorphe en droites $T\Sigma/TC$ est de degré (k_d-2), et est un sous-fibré naturel du fibré normal holomorphe en plans TX/TC. On construit l'application $\epsilon : H_2(\Sigma;\mathbb{Z}) \to \mathbb{Z}_2$, caractérisée par la classe d'isotopie du sous-fibré en droites $\mathbb{R}(T\Sigma/TC)$ dans $\mathbb{R}(TX/TC)$. Par conséquent, cela contribue à une différence de signe de $(-1)^{\epsilon(D)}$.

Nous résumons ce résultat dans le théorème suivant.

Théorème 3.1.1. *Soit (X,τ) une variété de del Pezzo réelle de dimension 3 telle que $\ker(\psi) \cong \mathbb{Z}$, où $\Sigma \in |-\frac{1}{2}K_X|$, les applications $\psi : H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, ρ, $s^{\mathfrak{s}_{X|\Sigma}}$, ϵ et $g(D)$ sont comme décrits ci-dessus. Soit S un générateur de $\ker(\psi)$. Supposons que*

$(\Sigma, \tau|_\Sigma)$ *est telle que* $\tau|_{\Sigma *}(S) = -S$. *Pour toute classe* $d \in H_2^{-\tau}(X;\mathbb{Z})$ *et* $0 \leq l < \frac{k_d}{2}$*, on a*

$$W_{\mathbb{R}X}^{\mathfrak{s}_X, \mathfrak{o}_X}(d,l) = \frac{1}{2} \sum_{D \in \psi^{-1}(d)} (-1)^{\epsilon(D) + g(D) + s^{\mathfrak{s}_X|_\Sigma}(\rho D)} |D.S| W_{\mathbb{R}\Sigma}(D,l).$$

Par les propriétés symétriques, les invariants de Welschinger de genre zéro $W_{\mathbb{R}X}^{\mathfrak{s}_X, \mathfrak{o}_X}(d,l)$ peut s'annuler. Dans le résultat suivant, nous montrons qu'il existe certaines configurations qui simulent l'optimalité des invariants de Welschinger de genre zéro lorsqu'ils sont nuls.

Théorème 3.1.2. *Soit* (X, τ) *une variété de del Pezzo réelle de dimension 3 telle que* $\ker(\psi) \cong \mathbb{Z}$*, où* $\Sigma \in |-\frac{1}{2}K_X|$ *et* ψ *sont comme dans le Théorème 3.1.1. Soit une classe* $d \in H_2^{-\tau}(X;\mathbb{Z})$ *tel que* $D.S$ *soit pair pour tout* $D \in \psi^{-1}(d)$*. Supposons que* E *est une courbe elliptique réelle maximale réalisant la classe d'homologie* $\frac{1}{4}K_X^2$*.*

Alors, il existe une configuration réelle de $\frac{1}{2}c_1(X) \cdot d$ *points dans* E*, constituée d'au moins un point réel, telle qu'il n'existe pas de courbes rationnelles irréductibles réelles de classe* d *qui la traverse.*

Organisation du manuscrit

Dans le Chapitre 1, nous introduisons les notions fondamentales que nous utiliserons tout au long de ce texte. Dans la première section, nous décrivons la classification des variétés de del Pezzo complexes et réelles en basse dimension. Dans les deux sections suivantes, nous rappelons les définitions et quelques propriétés des invariants de Gromov-Witten et de Welschinger dans les variétés de del Pezzo de dimension deux et trois. Dans la dernière section, nous présentons la relation entre les structures $Pin^{\pm}/Spin$ et la géométrie énumérative réelle.

Dans le Chapitre 2, nous présentons nos résultats principaux concernant les invariants de Gromov-Witten dans les variétés de del Pezzo de dimension trois. La première section est consacrée à énoncer notre théorème principal (Théorème 2.1.1). Dans les sections 2.2 et 2.3, en étudiant des pinceaux de surfaces dans le système linéaire $|-\frac{1}{2}K_X|$, nous résolvons le *problème unidimensionnel* sur la courbe elliptique (Proposition 2.2.16), et nous généralisons la Proposition 3 de [Kol15] (Proposition 2.3.2). La Section 2.4 est consacrée à une preuve du Théorème 2.1.1. Enfin, nous appliquons le Théorème 2.1.1 dans quelques cas particuliers (Théorèmes 2.5.1, 2.5.2, 2.5.4). Et, nous exposons des tables de calcul explicites des invariants de Gromov-Witten de genre zéro dans l'espace $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, en utilisant le programme *Maple*.

Dans le Chapitre 3, nous présentons nos résultats principaux concernant les invariants de Welschinger dans les variétés de del Pezzo réelles de dimension trois. La première section est consacrée à énoncer nos théorèmes principaux (Théorème 3.1.1 et Théorème 3.1.2). Dans la Section 3.2, nous étudions les pinceaux des surfaces de del Pezzo réelles dans le système linéaire $|-\frac{1}{2}K_X|$, et nous résolvons le *problème unidimensionnel réel* sur la courbe elliptique réelle, puis prouvons le Théorème 3.1.2 (Proposition 3.2.5, Corollaire 3.2.6). Dans les sections 3.3 et 3.4, nous construisons l'application ϵ (Définition 3.3.2) et nous résolvons le *problème du signe* (Proposition 3.4.4). Dans la Section 3.5, nous donnons une preuve du Théorème 3.1.1. À la fin de ce chapitre, nous appliquons le Théorème 3.1.1 dans quelques cas particuliers (Théorèmes 3.6.1, 3.6.5, 3.6.9, 3.6.14). Et, nous exposons des tables de calcul explicites des invariants de Welschinger de genre zéro dans les cas $\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}$ avec la structure réelle standard et $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ avec deux structures réelles distinctes, en utilisant à nouveau le programme *Maple*.

Introduction (in English)

0.1 History

Enumerative geometry is a branch of algebraic geometry which concerns to enumeration of solutions of geometrical problems. One of the simplest enumerative problems is:

Problem 0. *In the real plane, how many lines passing through two distinct points?*

The answer for this problem is trivial: there exists a unique such line. This statement is known as the fifth axiom of Euclidean geometry dating back to ancient Greek mathematics. A cultural extension of **Problem 0** is:

Problem 1. *In the complex projective plane, how many curves of degree d and genus g passing through $3d - 1 + g$ points in general position?*

In 1994, **Problem 1** had a significant answer for the case $g = 0$ by Kontsevich's recursive formula. In which, the only initial data required is the fact that there is a unique line passing through two distinct points (see [KM94]). Four years later, this problem was totally solved by L. Caporaso and J. Harris [CH98]. In the following, we list some directions of generalization for **Problem 1**.

1. Change the setting.

   If we replace the algebraic setting by the symplectic one, then the corresponding problem turns out to be enumeration of pseudo-holomorphic curves on symplectic manifolds. Thus, the enumerative problem of counting curves in algebraic varieties has a new appearance of *Gromov-Witten invariants*. In some cases, when Gromov-Witten invariants are integers, they agree with the answers to algebraic enumerative problems ([LT98], [Sie],[Ful97]). As a consequence, Gromov-Witten invariants have became a very powerful tool to study enumerative geometry.

2. Change the ambient space by increasing its dimension or by adding to it more structure.

   We can extend this enumerative problem to other algebraic varieties such as $\mathbb{C}P^n$ n-dimensional complex projective spaces, n-dimensional Fano varieties ($n \geq 2$). Moreover, if we equip these varieties with a *real structure* (i.e. an anti-holomorphic involution) and count *real curves* (i.e. curves preserved by the real structure), we greatly deepen the *real enumerative geometry*. Real enumerative geometry has experienced tremendous growth for over almost twenty years, particularly following the founding work of J-Y. Welschinger in symplectic geometry and G. Mikhalkin in tropical geometry. In [Wel05b] and [Wel05c], Welschinger proved that in some varieties of dimension two and three, we can make a consistent choice of sign ± 1 such that the number of real curves of genus zero, counted with this sign, depends only on the homology class of the curves (and not on the generic choice of the configuration through which they pass), in particular this count gives (non-trivial) lower bounds in real enumerative geometry. These numbers are known as *Welschinger invariants*. Moreover, since the semi-field of tropical numbers appeared, tropical enumerative geometry has been developed with a lots of brand-new techniques and corresponding enumerative problems, such as Mikhalkin's correspondences, Block-Göttsche's multiplicities, refined invariants...

Certainly, there are still many other ways to generalize, to modify or to apply these original enumerative problems that we could not mention them all here. We are interested in the following generalizations of **Problem 1**:

Problem 2 (GW)**.** *In a non-singular del Pezzo variety X of dimension three, how many genus-0 irreducible curves which represent a given homology class $d \in H_2(X;\mathbb{Z})$ and pass through a generic configuration of $\frac{1}{2}c_1(X) \cdot d$ points?*

We call this count the *genus-0 Gromov-Witten invariant* of X and denote it by $GW_X(d)$.

Problem 2 (W)**.** *In a real non-singular del Pezzo variety X of dimension three equipped with a real structure τ_X, how many real genus-0 irreducible curves, counted with Welschinger's sign, which represent a given τ_X-anti-invariant class $d \in H_2^{-\tau_X}(X;\mathbb{Z})$ and pass through a real generic configuration, consisting of r real points and l pairs of complex conjugated points such that $r + 2l = \frac{1}{2}c_1(X) \cdot d$?*

We call this count the *genus-0 Welschinger invariant* of X and denote it by $W_{\mathbb{R}X}(d,l)$.

0.2 Motivations

While complex and real enumerative geometry in the case of 2-dimensional algebraic varieties has been systematically studied in recent years, there are still many enumerative problems in the 3-dimensional case to be explored, especially the real enumeration. We are motivated by the following 2-dimensional complex and real enumerative problems:

Problem 3 (GW)**.** *In a non-singular del Pezzo surface Y, how many genus-0 irreducible curves which represent a given homology class $D \in H_2(Y;\mathbb{Z})$ and pass through a generic configuration of $c_1(Y) \cdot D - 1$ points?*

In this problem, we compute the value of $GW_Y(D)$.

Problem 3 (W). *In a real non-singular del Pezzo surface Y equipped with a real structure τ_Y, how many real genus-0 irreducible curves, counted with Welschinger's sign, which represent a given τ_Y-anti-invariant class $D \in H_2^{-\tau_Y}(Y;\mathbb{Z})$ and pass through a real generic configuration, consisting of r real points and l pairs of complex conjugated points such that $r + 2l = c_1(Y) \cdot D - 1$?*

In this problem, we compute the value of $W_{\mathbb{R}Y}(D, l)$.

It is known that non-singular del Pezzo surfaces are classified. They are either projective plane $\mathbb{C}P^2$ blown up in at most 8 points in general position, or $\mathbb{C}P^1 \times \mathbb{C}P^1$ twofold product of the projective line. Since 1990s, explicit calculations for **Problem 3** (GW) have been completely performed: by M. Kontsevich and Y. Manin ([KM94], [KV07]) for $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ using Kontsevich's recursive formulas and then by L. Göttsche and R. Pandharipande [GP96] for blown-ups of $\mathbb{C}P^2$ at arbitrary general points.

Due to the study since 1913 of Comessatti [Com13], there is also a complete classification of real del Pezzo surfaces. Welschinger's sign of a real curve in such real surfaces is determined by the parity of the number of its *elliptic nodes* (i.e. real points whose local equation of the curve is given by $x^2 + y^2 = 0$ over $\mathbb{R}$). Many explicit calculations for Welschinger invariants in **Problem 3** (W) have been performed, for example:

- by J-Y. Welschinger [Wel07] himself, using symplectic field theory, for the case projective plane and quadric ellipsoid;
- by E. Brugallé and G. Mikhalkin ([BM07], [BM08]), by using floor diagrams in tropical geometry, for $\mathbb{C}P^2_{r,s}$ blown-ups of projective plane at r real points and s pairs of complex conjugated points ($r + 2s \leq 3$); and then by E. Brugallé [Bru15] for $\mathbb{C}P^2_{r,s}$ ($r + 2s \leq 8$) with some other real structures;
- by I. Itenberg, V. Kharlamov and E. Shustin ([IKS09], [IKS13b], [IKS13a], [IKS15]), by using tropical geometry and the Caporaso-Harris type formulas, for the case $\mathbb{C}P^2_{r,s}$ ($r + 2s \leq 7$) with purely real points constraints and with some other real structures;
- by J. Solomon and A. Horev ([Sol07], [HS12]), by using the Solomon's WDVV-type relations, for $\mathbb{C}P^2_{r,s}$ with arbitrary r, s.

Now let us get back to the 3-dimensional enumerative problems. According to V.A. Iskovskikh [Isk77], 3-dimensional del Pezzo varieties are totally classified by their up-to-8 degree. The computations for **Problem 2** (GW) have been made for almost all cases. More precisely, the class of degree 8, which is the projective 3-space $\mathbb{C}P^3$, was studied in 1990s by Kontsevich-Manin [KM94] using the WDVV relation of Gromov-Witten theory [2], then by A. Gathmann [Gat98b] and recently by E. Brugallé and P. Georgieva [BG16] by a new strategy using pencils of quadrics. In fact, this method was first mentioned by Kollár in 2015 [Kol15] to provide configurations that simulate the optimization of some enumerative problems. Via the same strategy, the classes of degree $2, 3, 4, 5$ were

2. In fact, the WDVV relation of Gromov-Witten theory solved this classical enumerative problem for complex projective n-space for any n.

studied by A. Zahariuc [Zah18] and the class of degree 7 was studied by Y. Ding [Din19]. The class of degree 6 was partially studied by X. Chen and A. Zinger [CZ21a] by using Solomon's WDVV-type relations.

The classification of real 3-dimensional del Pezzo varieties has not much been known (see [Man20]). The degree 3 case, i.e. the cubic threefold, was classified by V.A. Krasnov [Kra09]. The studies about real structures on the real 3-dimensional varieties could be found in the series by Kollár [Kol97], [Kol98], [Kol99], [Kol+00], [Kol00]; or by V. Kharlamov [Kha02]. Welschinger's sign of a real curve in a real 3-dimensional variety, which is also called *spinor state*, is determined by (roughly speaking) the $Pin^{\pm}/Spin$-structures on the real part of this variety. Up to now, some explicit calculations for Welschinger invariants in **Problem 2** (W) have been performed; for example

- projective 3-space (the mostly studied case with diverse approaches): by Brugallé-Mikhalkin [BM07] using floor diagrams; by Georgieva-Zinger [GZ17] using WDVV-type relations and by Brugallé-Georgieva [BG16] using pencils of quadrics;
- projective 3-space blown-up at one point with standard real structure: by Ding [Din19] for certain types of curve also using pencils of quadrics;
- threefold product of the projective line equipped with two real structures: recently by Chen-Zinger ([CZ21b], [CZ21a]) using Solomon's WDVV-type relations.

To the best of author's knowledge, the computation for Welschinger invariants in real 3-dimensional del Pezzo varieties of low degree (i.e. degree from 1 to 5) is not known in the literature.

0.3 Results

This thesis is inspired by the strategy proposed by Brugallé-Georgieva [BG16] to compute genus-0 Gromov-Witten and Welschinger invariants of $\mathbb{C}P^3$, in which they studied pencils of quadrics as described above. We extend their results to Fano varieties of dimension three and index two, in other words the 3-dimensional del Pezzo varieties. We obtain new formulas to solve **Problem 2** (GW) and **Problem 2** (W) mostly in the cases where $\deg(X) \in \{6, 7, 8\}$.

Gromov-Witten invariants - Problem 2 (GW)

Let X be a three-dimensional del Pezzo variety. Let $E \subset X$ be the base locus of a pencil $\mathcal{Q}$ of surfaces in the linear system $|-\frac{1}{2}K_X|$. We first prove that E is a non-singular elliptic curve of homology class $\frac{1}{4}K_X^2$. Let $\mathcal{C}(\underline{x}_d)$ be the set of connected algebraic curves of arithmetic genus zero representing the given homology class $d \in H_2(X;\mathbb{Z})$ and passing through $\underline{x}_d$ a generic configuration of $\frac{1}{2}c_1(X) \cdot d$ points in E.

In the first result, we prove a generalization of Proposition 3 by Kollár [Kol15].

Proposition 2.3.2. *Every rational connected curve in the set $\mathcal{C}(\underline{x}_d)$ is irreducible and contained in a surface of the pencil $\mathcal{Q}$.*

In the second result, we extend the strategy of Brugallé-Georgieva to more general cases, in which we reduce **Problem 2** (GW) into two independent problems: the first one called 1-*dimensional problem* on the elliptic curve E, and the second one called 2-*dimensional enumerative problem* on del Pezzo surfaces in $|-\frac{1}{2}K_X|$. Let $\Sigma \in |-\frac{1}{2}K_X|$ be a non-singular del Pezzo surface. The 2-*dimensional enumerative problem* exactly gives us genus-0 Gromov-Witten invariants of non-singular del Pezzo surfaces, i.e. **Problem 3** (GW), which have been computed as we mentioned in the motivations section. Therefore, we focus on the 1-*dimensional problem*. If the surjective map $\psi : H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, (which is induced by the inclusion $\Sigma \hookrightarrow X$), satisfies $\ker(\psi) \cong \mathbb{Z}$, then we prove that there are $(D.S)^2$ surfaces on the pencil containing a curve in the set $\mathcal{C}(\underline{x}_d)$, where the homology class $D \in H_2(\Sigma;\mathbb{Z})$ is in $\psi^{-1}(d)$ and the homology class $S \in H_2(\Sigma;\mathbb{Z})$ is a generator of the group $\ker(\psi)$. We summarize this result in the following theorem.

Theorem 2.1.1. *Let X be a 3-dimensional del Pezzo variety such that $\ker(\psi) \cong \mathbb{Z}$, where the surface $\Sigma \in |-\frac{1}{2}K_X|$ and the map ψ are as described above. For every homology class $d \in H_2(X;\mathbb{Z})$, one has*

$$GW_X(d) = \frac{1}{2}\sum_{D\in\psi^{-1}(d)} (D.S)^2 GW_\Sigma(D).$$

Welschinger invariants - Problem 2 (W)

Suppose now that the pencil $\mathcal{Q}$ and the elliptic curve E are real and their real parts are non-empty. Let $\mathcal{R}_d(\underline{x}_d)$ be the set of real irreducible rational curves representing the homology class $d \in H^{-\tau_X}(X;\mathbb{Z})$ and passing through $\underline{x}_d$ a real generic configuration of k_d points in E including l pairs of complex conjugated points. We adapt the idea of the proof of **Problem 2** (GW). More precisely, we reduce **Problem 2** (W) into two independent problems: the first one called 1-*dimensional real problem* on the real elliptic curve E, and the second one called 2-*dimensional real enumerative problem* on real del Pezzo surfaces in $|-\frac{1}{2}K_X|$. In this case, the 2-*dimensional real enumerative problem* gives us genus-0 Welschinger invariants of real del Pezzo surfaces, i.e. **Problem 3** (W), which have been computed as we mentioned in the motivations section, but up to some sign. Thus, different from the Gromov-Witten case, there is one more problem to be concerned: the *sign problem*.

Let $\Sigma \in |-\frac{1}{2}K_X|$ be a real non-singular surface such that a real Lagrangian sphere in Σ is $\tau|_\Sigma$-anti-invariant. Solving the 1-*dimensional real problem* on the real elliptic curve E, we prove that there are $|D.S|$ such real surfaces on the real pencil $\mathcal{Q}$ containing a curve in the set $\mathcal{R}_d(\underline{x}_d)$ if $\ker(\psi) \cong \mathbb{Z}$, where $\psi : H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, and the homology classes D and S are $\tau|_\Sigma$-anti-invariant.

The *sign problem* comes from the difference between two definitions of Welschinger's sign for real curves in real varieties of dimension two and three, as we mentioned in the motivations section. On the one hand, the Welschinger's sign for a real curve on the real surface Σ depends on the number of its elliptic real nodes. On the other hand, that for a real curve on the real 3-dimensional variety X, (known as spinor state), depends on the choice of $Pin_3^{\pm}$-structure on $\mathbb{R}X$ (or $Spin_3$-structure on $\mathbb{R}X$ if $\mathbb{R}X$ is orientable), and on the choice of the isotopy class of the degree $(k_d - 2)$ real holomorphic line subbundle in the holomorphic normal bundle of rank two TX/TC. Henceforth, if $\mathbb{R}X$ is orientable with orientation $\mathfrak{o}_X$ and $Spin_3$-structure $\mathfrak{s}_X$, we denote by $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)$ the corresponding genus-0 Welschinger invariant. Therefore, to solve this *sign problem*, we need to interpret these two depending parameters of spinor states.

For the "$Pin^{\pm}/Spin$-structure" side, we first observe that the $Spin_3$-structure $\mathfrak{s}_X$ on $\mathbb{R}X$ can be restricted to a Pin_2^--structure on $\mathbb{R}\Sigma$, denoted by $\mathfrak{s}_{X|\Sigma}$. The second observation is that there is an one-to-one canonical correspondence between the Pin_2^--structure $\mathfrak{s}_{X|\Sigma}$ on $\mathbb{R}\Sigma$ and the quasi-quadratic enhancement $s^{\mathfrak{s}_{X|\Sigma}} : H_1(\mathbb{R}\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_2$. By Horev-Solomon's correspondence between quasi-quadratic enhancement and Welschinger's sign of curve on surfaces, we solve the first part of the *sign problem*, in which we obtain the term $(-1)^{g(D)+s^{\mathfrak{s}_{X|\Sigma}}(\rho D)}$ where

- $g(D) = \frac{1}{2}(K_\Sigma.D + D^2 + 2)$ and
- $\rho : H_2^{-\tau|_\Sigma}(\Sigma;\mathbb{Z}) \to H_1(\mathbb{R}\Sigma;\mathbb{Z}_2)$ is the map given by the real part of $\tau|_\Sigma$-anti-invariant representative class.

The "isotopy class" side is more delicate. We first remark that the holomorphic normal line bundle $T\Sigma/TC$ is of degree (k_d-2), and it is a natural subbundle of the holomorphic normal plane bundle TX/TC. We then construct the map $\epsilon : H_2(\Sigma;\mathbb{Z}) \to \mathbb{Z}_2$, characterised by the isotopy class of the real line subbundle $\mathbb{R}(T\Sigma/TC)$ in $\mathbb{R}(TX/TC)$. Hence, the difference of sign by $(-1)^{\epsilon(D)}$ is followed.

We summarize this result in the following theorem.

Theorem 3.1.1. *Let (X,τ) be a real 3-dimensional del Pezzo variety such that one has $\ker(\psi) \cong \mathbb{Z}$, where $\Sigma \in |-\frac{1}{2}K_X|$, the maps $\psi : H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, $s^{\mathfrak{s}_{X|\Sigma}}$, ρ, ϵ and $g(D)$ are as described above. Let S be a generator of $\ker(\psi)$. Suppose that $(\Sigma, \tau|_\Sigma)$ is such that $\tau|_{\Sigma *}(S) = -S$. For every $d \in H_2^{-\tau}(X;\mathbb{Z})$ and $0 \leq 2l < k_d$, one has*

$$W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l) = \frac{1}{2}\sum_{D\in\psi^{-1}(d)} (-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_X|\Sigma}(\rho D)}|D.S|W_{\mathbb{R}\Sigma}(D,l).$$

By means of symmetrical properties, the genus-0 Welschinger invariants $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)$ can vanish. In the following result, we show that there exists certain configurations that simulate the optimality of the vanishing genus-0 Welschinger invariants.

Theorem 3.1.2. *Let (X,τ) be a real 3-dimensional del Pezzo variety such that one has $\ker(\psi) \cong \mathbb{Z}$, where $\Sigma \in |-\frac{1}{2}K_X|$ and ψ are as in Theorem 3.1.1. Let $d \in H_2^{-\tau}(X;\mathbb{Z})$ such that $D.S$ is even for any $D \in \psi^{-1}(d)$. Suppose that E is a maximal real elliptic curve realizing homology class $\frac{1}{4}K_X^2$.*

Then, there exists a real configuration of $\frac{1}{2}c_1(X)\cdot d$ points on E, consisting of at least one real point, through which no real irreducible rational curve of homology class d passes.

0.4 Plan of manuscript

In Chapter 1, we introduce the fundamental notions which we will use throughout this text. The first section is dedicated to the classification of low dimensional complex and real del Pezzo varieties in literature. In the next two sections, we recall about definitions and some properties of Gromov-Witten and Welschinger invariants in two and three-dimensional del Pezzo varieties. In the last section, we present the relationship between $Pin^{\pm}/Spin$-structures and real enumerative geometry.

In Chapter 2, we present our main results concerning Gromov-Witten invariants in del Pezzo varieties of dimension three. The first section is devoted to stating our main theorem (Theorem 2.1.1). In Sections 2.2 and 2.3, by studying pencils of surfaces in the linear system $|-\frac{1}{2}K_X|$, we solve the 1-*dimensional problem* on the elliptic curve (Proposition 2.2.16), and we generalize Proposition 3 in [Kol15] (Proposition 2.3.2). Section 2.4 is dedicated to a proof of Theorem 2.1.1. In the last section, we apply Theorem 2.1.1 in some particular cases (Theorems 2.5.1, 2.5.2, 2.5.4) and we exhibit the explicit calculation tables of Gromov-Witten invariants in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, using *Maple program*.

In Chapter 3, we present our main results concerning Welschinger invariants in real del Pezzo varieties of dimension three. The first section is devoted to stating our main theorems (Theorem 3.1.1 and Theorem 3.1.2). In Section 3.2, we study about pencils of real del Pezzo surfaces in the linear system $|-\frac{1}{2}K_X|$, and we solve the 1-*dimensional real problem* on the real elliptic curve, then we prove Theorem 3.1.2 (Proposition 3.2.5, Corollary 3.2.6). In Sections 3.3 and 3.4, we construct the map ϵ (Definition 3.3.2) and solve the *sign problem* (Proposition 3.4.4). In Section 3.5, we give a proof of Theorem 3.1.1. In the end of this chapter, we apply Theorem 3.1.1 in some particular cases (Theorems 3.6.1, 3.6.5, 3.6.9, 3.6.14) and we exhibit the explicit calculation tables of genus-0 Welschinger invariants in the cases $\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}$ with the standard real structure and $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ with two distinct real structures, also using *Maple program*.

1

Preliminaries

Notations

In this text, we often use the following convention of notations:

— the lower point . stands for intersection between two homology classes;

— the upper point $\cdot$ stands for the cup product (or the dual pairing) between a cohomology class and a homology class;

— $X\sharp n\overline{Y}$ the blow-up at n points of variety X, whose exceptional divisor Y is orientable;

— $X\sharp nY$ the blow-up at n points of variety X, whose exceptional divisor Y is non-orientable.

In this introductory chapter, we review the enumerative genus-0 Gromov-Witten invariants (resp. Welschinger invariants) which are related to counting complex rational curves (resp. counting signed real rational curves), on del Pezzo varieties (resp. real del Pezzo varieties) of dimension two and three. We emphasize on some main properties which will be useful for the study about the relationship between these invariants in the next chapters.

1.1 Del Pezzo varieties

1.1.1 Definitions and examples

Del Pezzo varieties can be considered as particular classes of Fano varieties which were first introduced by Gino Fano in 1930s.

Definition 1.1.1. A Fano variety X of dimension n is a non-singular complex projective variety X of complex dimension n whose anti-canonical divisor $-K_X$ is ample.

The maximal integer $r > 0$ such that $-K_X = rH$, where H is a primitive divisor in $Pic(X)$, is called *index of* X, denoted by $ind(X)$. Note that H is determined up to a r-torsion element in $Pic(X)$ and one has $Pic(X) \cong H^2(X;\mathbb{Z})$ is torsion-free, so the index of X is well-defined, see [Isk77].

— If $ind(X) = n+1$ then X is a n-dimensional projective space $\mathbb{C}P^n$.

— If $ind(X) = n$ then X is a non-singular quadric hypersurface in $\mathbb{C}P^{n+1}$.

Non-singular n-dimensional Fano varieties of index $(n-1)$ are completely classified in 1980s by V. A. Iskovskikh (for $n = 3$) and T. Fujita (for any n) (see Chapter 3 and Chapter 12 in [Isk+10]).

Definition 1.1.2. A **n-dimensional del Pezzo variety** is a n-dimensional Fano variety of index divisible by $(n-1)$ for any $n \geq 2$.

The *degree* of a n-dimensional del Pezzo variety X, denoted by $\deg(X)$, is the integer given by $\frac{1}{(n-1)^n}(-K_X)^n$. The degree of a del Pezzo surface is no more than 9 and the degree of a three-dimensional del Pezzo variety is no more than 8 (see Chapter 2 and Chapter 3 in [Isk+10]). The Iskovskikh-Fujita's classification of del Pezzo varieties is based on their degree.

Here are some examples of low dimensional Fano and del Pezzo varieties.

Example 1.1.3. There is an unique (up to isomorphism) Fano line, which is the projective line $\mathbb{C}P^1$.

Example 1.1.4. Fano surfaces are del Pezzo surfaces. In particular, del Pezzo surface X of index 3 is the projective plane $\mathbb{C}P^2$, and that of index 2 is a non-singular quadric in $\mathbb{C}P^3$.

If a del Pezzo surface X has index 1 and $-K_X$ is very ample then the degree of X defined as above, i.e. $\deg(X) = K_X^2$, is compatible with the usual degree of X with respect to the anti-canonical embedding $\varphi_{|-K_X|} : X \hookrightarrow \mathbb{C}P^{\dim|-K_X|}$.

Example 1.1.5. A three-dimensional Fano variety can have index 4 (as $\mathbb{C}P^3$), index 3 (as a quadric hypersurface in $\mathbb{C}P^4$), index 2 or index 1. Those of index either 2 or 4 are three-dimensional del Pezzo varieties.

1.1.2 Classification of low dimensional del Pezzo varieties

For del Pezzo surfaces, see Theorem 24.4 in [Man86]; for del Pezzo varieties of higher dimension, see [Isk77] or [Isk+10].

$\deg(X)$	X	$Pic(X)$
1	$\mathbb{C}P^2\sharp 8\overline{\mathbb{C}P^2}$ (hypersurface of degree 6 in $\mathbb{P}(3,2,1,1)$)	$\mathbb{Z}^9$
2	$\mathbb{C}P^2\sharp 7\overline{\mathbb{C}P^2}$ (hypersurface of degree 4 in $\mathbb{P}(2,1,1,1)$)	$\mathbb{Z}^8$
3	$\mathbb{C}P^2\sharp 6\overline{\mathbb{C}P^2}$ (cubic surfaces in $\mathbb{C}P^3$)	$\mathbb{Z}^7$
4	$\mathbb{C}P^2\sharp 5\overline{\mathbb{C}P^2}$ (intersection of two quadrics in $\mathbb{C}P^4$)	$\mathbb{Z}^6$
5	$\mathbb{C}P^2\sharp 4\overline{\mathbb{C}P^2}$	$\mathbb{Z}^5$
6	$\mathbb{C}P^2\sharp 3\overline{\mathbb{C}P^2}$	$\mathbb{Z}^4$
7	$\mathbb{C}P^2\sharp 2\overline{\mathbb{C}P^2}$	$\mathbb{Z}^3$
8	$\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$ or $\mathbb{C}P^1\times\mathbb{C}P^1$	$\mathbb{Z}^2$
9	$\mathbb{C}P^2$	$\mathbb{Z}$

Table 1.1 – Classification of del Pezzo surfaces (up to isomorphism) and their Picard groups.

$\deg(X)$	X	$Pic(X)$
1	hypersurface of degree 6 in $\mathbb{P}(3,2,1,1,1)$	$\mathbb{Z}$
2	hypersurface of degree 4 in $\mathbb{P}(2,1,1,1,1)$	$\mathbb{Z}$
3	smooth cubic 3-fold	$\mathbb{Z}$
4	smooth intersection of two quadrics in $\mathbb{C}P^5$	$\mathbb{Z}$
5	linear section of $Gr(2,5)$ in $\mathbb{C}P^9$ by Plucker embedding	$\mathbb{Z}$
6	linear section of $\mathbb{C}P^2\times\mathbb{C}P^2$ in $\mathbb{C}P^8$ by Segre embedding	$\mathbb{Z}^2$
6	$\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1$	$\mathbb{Z}^3$
7	$\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}$	$\mathbb{Z}^2$
8	$\mathbb{C}P^3$	$\mathbb{Z}$

Table 1.2 – Classification of three-dimensional del Pezzo varieties (up to isomorphism) and their Picard groups.

Classification of del Pezzo varieties of dimension 2 and dimension 3

The classification of del Pezzo varieties of dimension 2 and dimension 3 is shown in Table 1.1 and 1.2.

Remarks about relations between del Pezzo varieties of dimension 2 and dimension 3

The interest of studying surfaces in the linear system $|-\frac{1}{ind(X)}K_X|$, where X is a Fano variety, is shown in [Isk77] or Section 3.2 [Isk+10].

As a consequence of Corollary 1.11 in [Isk77], one has the following first remark.

Remark 1.1.6. For every three-dimensional del Pezzo variety X and $\Sigma\in|-\frac{1}{2}K_X|$ a non-singular surface, one has Σ is a non-singular del Pezzo surface.

By the relationship between families of algebraic surfaces with the Picard-Lefschetz theorem (see [Sei08]), one has another remark as follows.

Remark 1.1.7. For every three-dimensional del Pezzo variety X, a general pencil of del Pezzo surfaces in the linear system $|-\frac{1}{2}K_X|$ is a Lefschetz pencil.

Note also that in a three-dimensional del Pezzo variety, the linear system $|-\frac{1}{2}K_X|$ is base point free (see [Isk+10]). Not to be confused with the base locus of a general pencil of surfaces in this linear system, which is an elliptic curve (see Proposition 2.2.1).

1.1.3 Classification of low dimensional real del Pezzo varieties

Definition 1.1.8. A **real structure** on an algebraic variety X is an anti-holomorphic involution on X.

Definition 1.1.9. A **real n-dimensional del Pezzo variety**, denoted by (X, τ), is a n-dimensional del Pezzo variety X equipped with a real structure τ.

The fixed locus of this involution is the *real part* (or the *set of real points*) of X, denoted by $\mathbb{R}X$. An algebraic curve in (X, τ) is called *real* if it is preserved by the real structure τ. A configuration of points $\underline{x} = \{x_1, \ldots, x_k\}$ in (X, τ), which is preserved by the real structure, is called *real*. Hence, a real configuration of points contains real points and pairs of complex conjugated points.

Let us fix the notation

$$\tau_n : \mathbb{C}P^n \to \mathbb{C}P^n, \quad \tau_n([z_0 : z_1 : \ldots : z_n]) = [\overline{z_0} : \overline{z_1} : \ldots : \overline{z_n}]$$

as the standard conjugation on the projective space $\mathbb{C}P^n$ ($n \geq 1$).

Real del Pezzo surfaces

A classification of real del Pezzo surfaces (up to $\mathbb{R}$-birational equivalence) was known since $1910s$ by A. Comessatti ([Com13]). This is restated in the modern literature in many points of view, see for example [Kol00] or [Rus08].

Here, we resume some main points of the classification of real del Pezzo surfaces of degree $6, 7$ and 8 as in the following examples.

Example 1.1.10. The degree 8 real del Pezzo surface (X, τ), which is isomorphic to the product of two projective lines, has four real structures and two of them make its real part $\mathbb{R}X$ to be non-empty. More precisely, for any $(u, v) \in X$, one has

- $\mathbb{R}X \cong \mathbb{R}P^1 \times \mathbb{R}P^1$, if $\tau(u, v) = (\tau_1(u), \tau_1(v))$;
- $\mathbb{R}X \cong \mathbb{S}^2$, if $\tau(u, v) = (\tau_1(v), \tau_1(u))$.

Example 1.1.11. Let (X, τ) be a real non-singular del Pezzo surface of degree $9 - r$, where $1 \leq r \leq 3$, such that $X \cong \mathbb{C}P^2 \sharp r \overline{\mathbb{C}P^2}$. The classification of such real del Pezzo surfaces is included in Table 1.3.

Note that the blow-up at complex conjugated points on a real surface has no effect on the set of its real points. This means that if we blow up a pair of complex conjugated points $(p_1, \tau(p_1))$ on $X \setminus \mathbb{R}X$, then $\mathbb{R}(Bl_{p_1,\tau(p_1)}X)$ is homeomorphic to $\mathbb{R}X$. If we blow up at a real point p on a connected component $\mathbb{R}X_i$ of $\mathbb{R}X$ then $\mathbb{R}(Bl_p X_i)$ is homeomorphic to $\mathbb{R}X_i \sharp \mathbb{R}P^2$.

We denote by $X_{a,b}$ the blow-up of a real points and b pairs of complex conjugated points on X. We denote by $Q^{r,s} \subset \mathbb{C}P^{r+s-1}$ a quadric hypersurface of equation

$$(x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2 = 0).$$

With these notations, we have for example $Q^{2,2}_{1,0} \cong \mathbb{C}P^2_{2,0}$ and $Q^{3,1}_{1,0} \cong \mathbb{C}P^2_{0,1}$. Note also that there is a homeomorphism between $(\mathbb{R}P^1 \times \mathbb{R}P^1)\sharp\mathbb{R}P^2$ and $\mathbb{R}P^2\sharp 2\mathbb{R}P^2$.

$\deg(X)$	X	$\mathbb{R}X$	$\rho(X)$
6	$Q^{4,0}_{0,1} \cong (Q^{3,0} \times Q^{2,1})_{0,1}$	$\emptyset$	3
6	$Q^{2,2}_{0,1}$	$\mathbb{R}P^1 \times \mathbb{R}P^1$	3
6	$Q^{3,1}_{0,1}$	$\mathbb{S}^2$	2
6	$\mathbb{C}P^2_{1,1}$	$\mathbb{R}P^2\sharp\mathbb{R}P^2$	3
6	$\mathbb{C}P^2_{3,0}$	$\mathbb{R}P^2\sharp 3\mathbb{R}P^2$	4
7	$\mathbb{C}P^2_{0,1}$	$\mathbb{R}P^2$	2
7	$\mathbb{C}P^2_{2,0}$	$\mathbb{R}P^2\sharp 2\mathbb{R}P^2$	3
8	$\mathbb{C}P^2_{1,0}$	$\mathbb{R}P^2\sharp\mathbb{R}P^2$	2
8	$Q^{3,1}$	$\mathbb{S}^2$	1
8	$Q^{2,2} \cong Q^{2,1} \times Q^{2,1}$	$\mathbb{R}P^1 \times \mathbb{R}P^1$	2
8	$Q^{4,0} \cong Q^{3,0} \times Q^{3,0}$ or $Q^{3,0} \times \mathbb{C}P^1$	$\emptyset$	2

Table 1.3 – Classification of real del Pezzo surfaces (X, τ) (up to $\mathbb{R}$-isomorphisms) with their real parts and real Picard numbers.

Real del Pezzo varieties of dimension three

The classification of real 3-dimensional del Pezzo varieties has not much been known (see [Man20]). The studies about real structures on the real 3-dimensional varieties (Fano and del Pezzo cases are included) could be found in the series of articles by Kollár in the late 1990s, which is summarized in [Kol00] and [Kol01], or by Kharlamov [Kha02].

We consider some following examples of real structures on real 3-dimensional del Pezzo varieties.

Example 1.1.12. On the complex projective 3-space $\mathbb{C}P^3$, the standard real structure τ_3 gives the corresponding fixed locus is the real projective 3-space $\mathbb{R}P^3$.

Example 1.1.13. On $\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}$ the blown-up at a real point of $(\mathbb{C}P^3,\tau_3)$, a natural real structure τ is induced from the standard one on $\mathbb{C}P^3$ above. This structure gives the corresponding fixed locus is the connected sum of two real projective 3-spaces $\mathbb{R}P^3\sharp\overline{\mathbb{R}P^3}$.

Example 1.1.14. By Borel-Serre's theorem (see [Ben16]), the equivalence classes of real structures on $\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1$ are in bijection with the finite cohomology group

$$H^1(\mathbb{Z}_2,(PGL_2(\mathbb{C}))^3\rtimes\mathfrak{S}_3).$$

In particular, we consider two real structures, whose corresponding real parts are non-empty, on $\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1$, as follows.

a) The *standard real structure*, denoted by $\tau_{1,1}$, is given by

$$\tau_{1,1}([x_0:x_1],[y_0:y_1],[z_0:z_1])=(\tau_1([x_0:x_1]),\tau_1[y_0:y_1],\tau_1[z_0:z_1])$$

then the fixed locus of $\tau_{1,1}$ is $\mathbb{R}P^1\times\mathbb{R}P^1\times\mathbb{R}P^1$.

b) The *twisted real structure*, denoted by $\tilde{\tau}_{1,1}$, is given by

$$\tilde{\tau}_{1,1}([x_0:x_1],[y_0:y_1],[z_0:z_1])=(\tau_1([y_0:y_1]),\tau_1[x_0:x_1],\tau_1[z_0:z_1])$$

then the fixed locus of $\tilde{\tau}_{1,1}$ is $\mathbb{S}^2\times\mathbb{R}P^1$.

1.2 Gromov-Witten invariants and moduli spaces of stable maps

1.2.1 Gromov-Witten invariants

For an algebro-geometric approach, see for examples [KV07], [FP97] or [LT95]; for a symplectic approach, see [RT95]. Let X be a non-singular complex projective variety of dimension n.

In our setting, a *rational curve representing a homology class* $\beta\in H_2(X;\mathbb{Z})$ is the image of an algebraic morphism $f:\mathbb{C}P^1\to X$ such that $f_*[\mathbb{C}P^1]=\beta$.

If $(n-1)$ divides $\big(c_1(X)\cdot\beta+n-3\big)$, where $c_1(X)\in H^2(X;\mathbb{Z})$ is the first Chern class of the tangent bundle over X, then we denote by k_β the integer $\frac{1}{n-1}\big(c_1(X)\cdot\beta+n-3\big)$, and call k_β the *associated integer* of class β. Let $\underline{x}_\beta$ be a generic configuration of k_β points in X where generic is in the sense that this configuration varies in a dense open subset of the set of all configurations of k_β points in X (see Definition 1.2.8).

Definition 1.2.1. The **genus-0 Gromov-Witten invariant of a n-dimensional del Pezzo variety** X ($n\in\{2,3\}$) **of homology class** $\beta\in H_2(X;\mathbb{Z})$, denoted by $GW_X(\beta)$, is the number of irreducible rational curves representing homology class β and passing through $\underline{x}_\beta$.

Remark 1.2.2. In general, the genus-0 Gromov-Witten invariants are defined as intersection numbers in some moduli spaces of stable maps. These invariants are indeed enumerative in the case of complex projective spaces, del Pezzo surfaces (see [RT95]), convex varieties (see Section 7 in [FP97]) and some blow-ups of projective spaces including $\mathbb{C}P^3 \sharp \overline{\mathbb{C}P^3}$ (see [Gat98b]).

Remark 1.2.3. The genus-0 Gromov-Witten invariants (Definition 1.2.1) do not depend on the choice of the configuration of points as long as it is generic. The chosen configuration in general can contain generic choices of not only points but also higher dimensional subvarieties, such that the sum of their codimensions is compatible with the expected dimension of the corresponding moduli space of stable maps.

Now we explore some properties of genus-0 Gromov-Witten invariants of del Pezzo surfaces. Suppose that Σ is a non-singular del Pezzo surface. Let $S \in H_2(\Sigma;\mathbb{Z})$ is a divisor class satisfying $K_\Sigma.S = 0$ and $S^2 = -2$. It can be shown that S is a vanishing cycle of a nodal degeneration of non-singular del Pezzo surfaces into singular ones in a 1-parameter family of del Pezzo surfaces (see the third lecture in [DPT06]). Considering the Dehn twist along the vanishing cycle S, we have the following proposition, which we call *symmetry property*.

Proposition 1.2.4. *Let Σ be a non-singular del Pezzo surface. Let $S \in H_2(\Sigma;\mathbb{Z})$ be as described above. For every $D \in H_2(\Sigma;\mathbb{Z})$, one has*

$$GW_\Sigma(D) = GW_\Sigma(D + (D.S)S).$$

Proof. On a complex projective manifold, as non-singular del Pezzo surface, it is shown that the algebraic and symplectic genus-0 Gromov-Witten invariants coincide, see [LT98].

This proposition then follows from a similar symmetry property in symplectic geometry, see Remark 1.3 in [Bru16]. □

Remark 1.2.5. A purely algebraic proof for this proposition due to the degeneration of del Pezzo surfaces into weak del Pezzo surfaces, on which the class S is represented by a non-singular rational curve. The Abramovich-Bertram-Vakil's formula (briefly ABV formula) allows us to compute the Gromov-Witten invariants (of any genus) in terms of some enumerative invariants of the above weak del Pezzo surfaces. This proposition follows from the explicit genus-0 ABV formula in Section 4 in [IKS15].

As a consequence, we can define the following $\mathbb{Z}$-valued function

$$\begin{aligned} GW_\Sigma : H_2(\Sigma;\mathbb{Z})/_{D\sim D+(D.S)S} &\to \mathbb{Z} \\ \overline{D} &\mapsto GW_\Sigma(\overline{D}) := GW_\Sigma(D). \end{aligned}$$

1.2.2 Moduli spaces of stable maps

In this part, we remind about the structures and properties of the moduli spaces of stable maps for studying curve-counting problems in general setting.

Let X be a non-singular complex projective variety of dimension n.

Stable maps

Let $(C; p_1, \dots, p_k)$ be a connected k-pointed nodal curve, where $\{p_1, \dots, p_k\}$ are distinct marked points and are non-singular points on C, of arithmetic genus g_a.

The points of an irreducible component $C_0 \subset C$ are called *special* if they are the marked points or the component intersections of C that lie on C_0.

Let $f : C \to X$ be an algebraic map. A component $C_0 \subset C$ is called *contracted* if C_0 is mapped to a point by f.

Definition 1.2.6. The map f is called **stable** if the following conditions hold for every contracted component $C_0 \subset C$ of arithmetic genus 0 or 1:

— If $g_a(C_0) = 0$ then C_0 must contain at least three special points.

— If $g_a(C_0) = 1$ then C_0 must contain at least one special point.

Equivalently, one has $f : C \to X$ is a *stable map* if f has finite automorphism group. This means that $|Aut(f)| < \infty$ where an automorphism of f is an automorphism of C which commutes with f.

We call $f_*[C] \in H_2(X;\mathbb{Z})$ the *class of the stable map* f.

The *arithmetic genus of a stable map* is defined to be the arithmetic genus $g_a(C)$ of the domain nodal curve C.

A genus-0 stable map $f : \mathbb{C}P^1 \to X$ is called *simple* if it is non-multiple cover, i.e. it can not be written as $u \circ g$ where $u : \mathbb{C}P^1 \to X$ is a birational map onto its image and $g : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a non-trivial ramified covering.

A stable map $f : (C; p_1, \dots, p_k) \to X$ where $p_i \in C$ and $f(p_i)$ are all distinct ($i \in \{1, \dots, k\}$) is called *k-pointed stable map*.

Moduli spaces of genus-0 k_β-pointed stable maps

Suppose that X is a del Pezzo n-variety ($n \in \{2, 3\}$). For any $\beta \in H_2(X;\mathbb{Z})$, let $k_\beta \in \mathbb{Z}$ be its associated integer.

Definition 1.2.7. The space of genus-0 k_β-pointed stable maps of class β, defined up to reparametrization, is denoted by $\mathcal{M}_{0,k_\beta}(X,\beta)$ and is called **moduli space of genus-0 k_β-pointed stable maps**.

We denote by $\mathcal{M}^*_{0,k_\beta}(X,\beta)$ the subspace of $\mathcal{M}_{0,k_\beta}(X,\beta)$ consisting of simple maps. It can be shown that $\mathcal{M}^*_{0,k_\beta}(X,\beta)$ is a dense open subset of $\mathcal{M}_{0,k_\beta}(X,\beta)$.

We denote by $\overline{\mathcal{M}}_{0,k_\beta}(X,\beta)$ the compactification (due to M. Kontsevich) of the moduli space $\mathcal{M}_{0,k_\beta}(X,\beta)$.

It can be shown that the moduli space $\overline{\mathcal{M}}_{0,k_\beta}(X,\beta)$ is a projective variety (maybe reducible and singular) of expected dimension (resp. virtual dimension in the case X is non-convex) $v\dim(\overline{\mathcal{M}}_{0,k_\beta}(X,\beta))$, which is equal to $c_1(X)\cdot\beta + n - 3 + k_\beta$ (see section 10 in [RT95]).

The corresponding **total evaluation map** ev_X is defined as follows:

$$\begin{aligned} ev_X : \overline{\mathcal{M}}_{0,k_\beta}(X,\beta) &\longrightarrow X^{k_\beta} \\ [(\mathbb{C}P^1; p_1, \dots, p_{k_\beta}); f] &\longmapsto (f(p_1), \dots, f(p_{k_\beta})) \end{aligned}$$

It is shown that counting such genus-0 stable maps is the same as counting rational curves in our setting (see for example [KM94] or [Gat98a]). More precisely, there is an one-to-one correspondence between an irreducible and reduced rational curve C in X representing the homology class $\beta \in H_2(X;\mathbb{Z})$ and passing through a configuration $\underline{x}_\beta = \{x_1, \dots, x_{k_\beta}\}$ of k_β distinct points in X and an element $[(\mathbb{C}P^1; p_1, \dots, p_{k_\beta}); f]$ in $\mathcal{M}^*_{0,k_\beta}(X,\beta)$ such that $f(\mathbb{C}P^1) = C$ and $f(p_i) = x_i$ for all $i \in \{1, \dots, k_\beta\}$.

We consider all configurations of k_β points in X which are images of k_β marked points of the evaluation map ev_X. We define the genericity of such configuration as follows.

Definition 1.2.8. A configuration of k_β points $\underline{x}_\beta$ in X is said **generic** if every stable map $[(\mathbb{C}P^1; p_1, \dots, p_{k_\beta}); f] \in \overline{\mathcal{M}}_{0,k_\beta}(X,\beta)$ satisfying $\underline{x}_\beta = f(\{p_1, \dots, p_{k_\beta}\})$ is a regular point of the corresponding total evaluation map ev_X.

It turns out that the curves arose from the regularity of the evaluation map are balanced curves which we proceed to describe.

Balanced rational curves

Suppose now that $f : \mathbb{C}P^1 \to X$ is an immersion and $f_*[\mathbb{C}P^1] = \beta \in H_2(X;\mathbb{Z})$.

Let $\mathcal{N}_f := f^*TX/T\mathbb{C}P^1$ be the rank $(n-1)$ holomorphic normal vector bundle over $\mathbb{C}P^1$. One has the following short exact sequence of holomorphic vector bundles over $\mathbb{C}P^1$:

$$0 \longrightarrow T\mathbb{C}P^1 \longrightarrow f^*TX \longrightarrow \mathcal{N}_f \longrightarrow 0.$$

By a theorem of Grothendieck, the normal bundle $\mathcal{N}_f$ is isomorphic to the direct sum of $(n-1)$ holomorphic line bundles

$$\bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{C}P^1}(a_i)$$

and the ordered sequence $a_1 \geq a_2 \geq \dots \geq a_{n-1}$ is uniquely determined. Moreover, by adjunction formula, one has $a_1 + \dots + a_{n-1} = c_1(X) \cdot \beta - 2$.

Definition 1.2.9. The normal bundle $\mathcal{N}_f$ is called **balanced** if $a_1 = \dots = a_{n-1}$.

Example 1.2.10.

1. If $\dim_{\mathbb{C}}(X) = 2$ then the normal line bundle $\mathcal{N}_f$ is always balanced. It is isomorphic to $\mathcal{O}_{\mathbb{C}P^1}(c_1(X) \cdot \beta - 2)$.
2. If $\dim_{\mathbb{C}}(X) = 3$ and $c_1(X) \cdot \beta$ is even then the rank 2 normal bundle $\mathcal{N}_f$ is balanced if it is isomorphic to $\mathcal{O}_{\mathbb{C}P^1}(\frac{1}{2}c_1(X) \cdot \beta - 1) \oplus \mathcal{O}_{\mathbb{C}P^1}(\frac{1}{2}c_1(X) \cdot \beta - 1)$.

Definition 1.2.11. A rational curve in X (i.e. a curve parametrized by $f : \mathbb{C}P^1 \to X$) is **balanced** if f is an immersion and the normal bundle $\mathcal{N}_f$ is balanced.

The following lemma is induced directly from Lemma 1.2 and Lemma 4.5 in [Wel05c].

Lemma 1.2.12. *Let X be a three-dimensional del Pezzo variety and $\beta \in H_2(X;\mathbb{Z})$. Let ev_X be the total evaluation map corresponding to the moduli space $\overline{\mathcal{M}}_{0,k_\beta}(X,\beta)$. Then, an element $[(\mathbb{C}P^1; p_1, \dots, p_{k_\beta}); f]$ in $\overline{\mathcal{M}}_{0,k_\beta}(X,\beta)$ is a regular point of the map ev_X if and only if f is balanced.*

As a consequence, balanced rational curves play a crucial role in studying genus-0 Gromov-Witten invariants of three-dimensional del Pezzo varietes.

1.3 Welschinger invariants: definitions and properties

In this section, we first recall Welschinger's signs and Welschinger invariants on real del Pezzo surfaces, see [Wel05b]. In the next part, we study about the $Spin_n$ (resp. $Pin_n^{\pm}$) structures on principal SO_n-bundles (resp. principal O_n-bundles) where $n \geq 1$ (see for examples [LM16], [Mil63], [KT90], [Kir06] or [CZ19]). Afterwards, we introduce spinor states of balanced real rational curves and Welschinger invariants on real del Pezzo varieties of dimension three, see [Wel05c] or [Wel05a]. In the last part, some remarks of Pin_2^--structures on topological surfaces are established to prepare for Section 3.4, Chapter 3.

We denote by

$$H_2^{-\tau}(X;\mathbb{Z}) := \{D \in H_2(X;\mathbb{Z}) : \tau_*(D) = -D\}$$

the subset of $H_2(X;\mathbb{Z})$, consisting of τ-anti-invariant homology classes in (X,τ).

1.3.1 Welschinger invariants of real del Pezzo surfaces

Let (X,τ) be a real del Pezzo surface. Let $D \in H_2^{-\tau}(X;\mathbb{Z})$ and let $k_D = c_1(X)\cdot D - 1$ be its associated integer. Let C be an irreducible real rational curve in X representing the homology class D and nodal, i.e. having only ordinary real nodes as singularities. By adjunction formula, the total number of complex nodes of C, which consists of real nodes and pairs of complex conjugated nodes, is equal to

$$g(D) = \frac{1}{2}(-c_1(X)\cdot D + D^2 + 2).$$

A real node of C is either hyperbolic or elliptic in the following sense.

Definition 1.3.1. A **hyperbolic real node** of a curve is a real point whose local equation of the curve is given by $x^2 - y^2 = 0$ over $\mathbb{R}$ (i.e. it is the intersection of two real branches). An **elliptic real node** (or an isolated real node) of a curve is a real point whose local equation of the curve is given by $x^2 + y^2 = 0$ over $\mathbb{R}$ (i.e. it is the intersection of two complex conjugated branches).

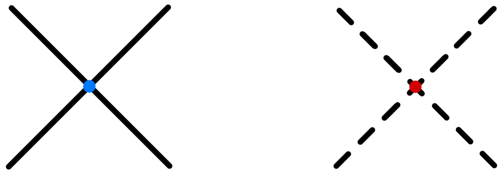

Figure 1.1 – Hyperbolic and elliptic real nodes.

Let $\delta_E(C)$ be the numbers of elliptic real nodes of C. Let $\delta_H(C)$ be the numbers of hyperbolic real nodes of C. And let $\delta_{\mathbb{C}}(C)$ be the numbers of pairs of complex conjugated nodes of C. As mentioned above, one has

$$\delta_E(C) + \delta_H(C) + 2\delta_{\mathbb{C}}(C) = g(D).$$

Definition 1.3.2. The **Welschinger's sign** of an irreducible real rational curve C in (X, τ), denoted by $s_{\mathbb{R}\Sigma}(C)$, is defined as

$$s_{\mathbb{R}\Sigma}(C) := (-1)^{\delta_E(C)}.$$

Let $\underline{x}_D$ be a real generic configuration (see Definition 1.2.8) of k_D points consisting of r real points and l pairs of complex conjugated points in X, such that $r + 2l = k_D$. Let $\mathcal{R}(\underline{x}_D)$ be a set of real rational curves in X representing the homology class D and passing through $\underline{x}_D$. One has moreover, the genericity of $\underline{x}_D$ implies that every real curve in $\mathcal{R}(\underline{x}_D)$ is nodal and irreducible.

Definition 1.3.3. Let (X, τ) be a real del Pezzo surface and $D \in H_2^{-\tau}(X; \mathbb{Z})$. We associate each real curve $C \in \mathcal{R}(\underline{x}_D)$ to its Welschinger's sign $s_{\mathbb{R}X}(C)$ as in Definition 1.3.2.

The **Welschinger invariant** of (X, τ) of homology class D is defined as

$$W_{\mathbb{R}X}(D, l) := \sum_{C \in \mathcal{R}(\underline{x}_D)} s_{\mathbb{R}X}(C).$$

Remark 1.3.4. If either $D \in H_2(X; \mathbb{Z}) \setminus H_2^{-\tau}(X; \mathbb{Z})$ or $k_D \leq 0$, then the corresponding Welschinger invariants $W_{\mathbb{R}X}(D, l)$ vanish.

Theorem 1.3.5. *(Welschinger[Wel05b], Brugallé[Bru21]) The Welschinger invariants in Definition 1.3.3 do not depend on the choice of a real generic configuration of points, provided its number of real points (or equivalently, its number of complex conjugated points).*

For a fixed partition of real points of $\underline{x}_D$ in the connected components of $\mathbb{R}X$, this theorem was proved partly by J-Y. Welschinger (Theorem 2.1, [Wel05b]). Then, E. Brugallé proved for every such partition, and hence the theorem (Theorem 1.3, [Bru21]).

One of the significant contribution of the Welschinger invariants is that these invariants give lower bounds in real enumerative geometry. More precisely, one has the following theorem.

Theorem 1.3.6. *(Welschinger[Wel05b]) For any real structure τ on a real algebraic surface X, for any homology class $D \in H^{-\tau}(X;\mathbb{Z})$, one has*

$$|W_{\mathbb{R}X}(D,l)| \leq |\mathcal{R}(\underline{x}_D)| \leq GW_X(D)$$

independantly of the choice of real configurations $\underline{x}_D \in X^{k_D}$.

Analogous to the case of genus-0 Gromov-Witten invariants, we explore some properties of genus-0 Welschinger invariants of real del Pezzo surfaces with related to the -2-curves. Let $S \in H_2^{-\tau}(X;\mathbb{Z})$ be a divisor class such that $K_X.S = 0$ and $S^2 = -2$. Considering the Dehn twist along the vanishing cycle S, we have the following *real symmetry property*, which is implied directly from Remark 1.3 in [Bru16].

Proposition 1.3.7. *Let (X,τ) be a non-sigular real del Pezzo surface of degree at least 2. Let $S \in H_2^{-\tau}(X;\mathbb{Z})$ be as described above. For any $D \in H_2^{-\tau}(X;\mathbb{Z})$, one has*

$$W_{\mathbb{R}X}(D,l) = W_{\mathbb{R}X}(D+(D.S)S,l).$$

As a consequence, we can define the following $\mathbb{Z}$-valued function

$$\begin{aligned} W_{\mathbb{R}\Sigma} : H_2(\Sigma;\mathbb{Z})/_{D\sim D+(D.S)S} \times \mathbb{Z} &\to \mathbb{Z} \\ (\overline{D},l) &\mapsto W_{\mathbb{R}\Sigma}(\overline{D},l) := W_{\mathbb{R}\Sigma}(D,l). \end{aligned}$$

On higher dimensional real algebraic varieties, in order to associate a sign $\{\pm 1\}$ to a real rational curve we need to equip some extra structures on the real part of the varieties, for example $Spin$ or $Pin^{\pm}$ structures which we proceed to describe.

1.3.2 $Spin_n$- and $Pin_n^{\pm}$-structures on principal bundles

Let G be a topological group, for instance the n-th orthogonal group $O_n(\mathbb{R})$ or the n-th special orthogonal group $SO_n(\mathbb{R})$. Let B be a paracompact space. Let V be a rank n real vector bundle over B ($n \geq 1$).

Definition 1.3.8. A **principal G-bundle over** B, denoted by $G \to O(G) \to B$, is a fiber bundle on whose fibers a free and transitive action by G is given.

We equip the vector bundle V over B with a Riemannian metric g. This metric determines an orthonormal frame bundle over B, whose fibers are ordered tuples of orthonormal basis on V. This orthonormal frame bundle is a principal O_n-bundle over B, which we will denote simply by $O(V)$ in the sequel.

If moreover, V is orientable and is equipped with an orientation $\mathfrak{o}_V$, one can determine an oriented orthonormal frame bundle over B. This is a principal SO_n-bundle over B, which we will denote simply by $SO(V, \mathfrak{o}_V)$ in the sequel.

For $n \geq 1$, we have the following double covers:

$$q_n : Spin_n(\mathbb{R}) \to SO_n(\mathbb{R})$$

and

$$q_n^{\pm} : Pin_n^{\pm}(\mathbb{R}) \to O_n(\mathbb{R})$$

where q_n^+ lifts a reflexion to an order two element, and q_n^- lifts a reflexion to an order four element. In particular, the groups $Spin_1(\mathbb{R})$, $Pin_1^+(\mathbb{R})$ and $Pin_1^-(\mathbb{R})$ are $\mathbb{Z}_2$, $\mathbb{Z}_2^2$ and $\mathbb{Z}_4$ respectively. We can moreover equip group structures on $Spin_n(\mathbb{R})$ (resp. $Pin_n^{\pm}(\mathbb{R})$) so as to make q_n (resp. $q_n^{\pm}$) become continuous homomorphisms.

Definition 1.3.9. A $Pin_n^{\pm}$**-structure** ($n \geq 1$) on a principal O_n-bundle $O(V)$ is a pair $(Pin^{\pm}(V), q_V^{\pm})$ consisting of

1. a principal $Pin_n^{\pm}$-bundle $Pin^{\pm}(V)$ over B,
2. a bundle map $q_V^{\pm} : Pin^{\pm}(V) \to O(V)$ which commutes with the projection to B and $q_n^{\pm}$-equivariant, meaning that the following diagram is commutative.

$$\begin{tikzcd} Pin^{\pm}(V) \times Pin_n^{\pm}(\mathbb{R}) \arrow[r] \arrow[d, "q_V^{\pm} \times q_n^{\pm}"] & Pin^{\pm}(V) \arrow[d, "q_V^{\pm}"] \\ O(V) \times O_n(\mathbb{R}) \arrow[r] & O(V) \end{tikzcd}$$

Definition 1.3.10. A $Spin_n$**-structure** ($n \geq 1$) on a principal SO_n-bundle $SO(V, \mathfrak{o}_V)$ is a pair $(Spin(V, \mathfrak{o}_V), q_V)$ consisting of

1. a principal $Spin_n$-bundle $Spin(V, \mathfrak{o})$ over B,
2. a bundle map $q_V : Spin(V, \mathfrak{o}_V) \to SO(V, \mathfrak{o}_V)$ which commutes with the projection to B and q_n-equivariant, meaning that the following diagram is commutative.

$$\begin{tikzcd} Spin(V, \mathfrak{o}_V) \times Spin_n(\mathbb{R}) \arrow[r] \arrow[d, "q_V \times q_n"] & Spin(V, \mathfrak{o}_V) \arrow[d, "q_V"] \\ SO(V, \mathfrak{o}_V) \times SO_n(\mathbb{R}) \arrow[r] & SO(V, \mathfrak{o}_V) \end{tikzcd}$$

The following definitions are equivalent to that above for $Pin_n^{\pm}$-structure and $Spin_n$-structure (see [Mil63] and [KT90]).

Definition 1.3.11. A $Pin_n^{\pm}$-structure ($n \geq 1$) on a principal O_n-bundle $O(V)$ over B is a cohomology class $\sigma \in H^1(O(V); \mathbb{Z}_2)$ whose restriction to each fiber $O(V)|_b = O_n(\mathbb{R})$ ($b \in B$) is a generator of the cyclic group $H^1(O_n(\mathbb{R}); \mathbb{Z}_2)$.

Definition 1.3.12. A $Spin_n$-structure ($n \geq 1$) on a principal SO_n-bundle $SO(V, \mathfrak{o}_V)$ over B is a cohomology class $\sigma \in H^1(SO(V, \mathfrak{o}_V); \mathbb{Z}_2)$ whose restriction to each fiber $SO(V, \mathfrak{o}_V)|_b = SO_n(\mathbb{R})$ ($b \in B$) is a generator of the cyclic group $H^1(SO_n(\mathbb{R}); \mathbb{Z}_2)$. (In the special case $n = 1$, the restriction assumption is omitted.)

By Definition 1.3.11 (resp. Definition 1.3.12), any such cohomology class σ determines a double cover of $O(V)$ (resp. $SO(V, \mathfrak{o}_V)$). Hence, this double covering space is to be taken as the total space $Pin^{\pm}(V)$ and $Spin(V, \mathfrak{o}_V)$ respectively.

Example 1.3.13. In the case $n = 1$, one has the double cover

$$q_1 : Spin_1(\mathbb{R}) = \mathbb{Z}_2 \to SO_1(\mathbb{R}) = \{1\}.$$

The tangent bundle of the circle $\mathbb{S}^1$ has two distinct $Spin_1$-structures. One $Spin_1$-structure on $T\mathbb{S}^1$ is the trivial $\mathbb{Z}_2$-bundle covering the trivial SO_1-bundle over $\mathbb{S}^1$, i.e. the disconnected double cover of $\mathbb{S}^1$, also known as the *bounding Spin structure* on $\mathbb{S}^1$. The other one is the connected double cover of $\mathbb{S}^1$, also known as the *Lie group Spin structure* on $\mathbb{S}^1$, see Figure 1.2.

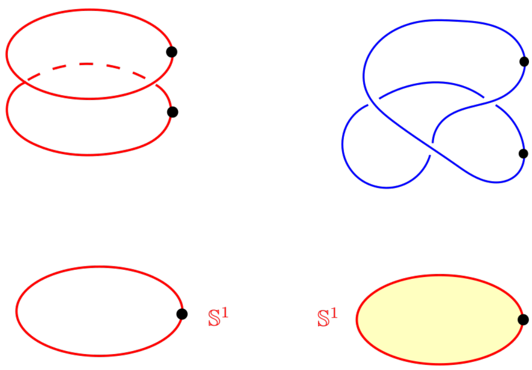

Figure 1.2 – Two distinct $Spin_1$-structures over $\mathbb{S}^1$.

It is known that a rank n real vector bundle V over B is orientable if and only if its first Stiefel-Whitney class $w_1(V)$ vanishes. If $w_1(V) = 0$ then the number of distinct orientations of V is equal to the cardinal of the cohomology group $H^0(B; \mathbb{Z}_2)$.

The obstruction to the existence of $Spin_n$-structures on $(V, \mathfrak{o}_V)$ and $Pin_n^{\pm}$-structures on V depending on its first and second Stiefel-Whitney classes, which is shown in the following proposition (see a proof in [Mil63], [KT90] or [LM16]).

Proposition 1.3.14.
1. *An oriented real bundle V admits a $Spin_n$-structure if and only if $w_2(V) = 0$.*
2. *A real bundle V admits a Pin_n^+-structure if and only if $w_2(V) = 0$, and admits a Pin_n^--structure if and only if $w_2(V) + w_1^2(V) = 0$.*

We denote by $\mathcal{P}_n^{\pm}(V)$ the set of $Pin_n^{\pm}$-structures on principal O_n-bundle $O(V)$. If V is orientable with orientation $\mathfrak{o}_V$, we denote by $\mathcal{SP}_n(V, \mathfrak{o}_V)$ the set of $Spin_n$-structures on principal SO_n-bundle $SO(V, \mathfrak{o}_V)$. We have the following property.

Proposition 1.3.15. *1. The action of the cohomology group $H^1(B;\mathbb{Z}_2)$ on the set $\mathcal{P}_n^{\pm}(V)$ is free and transitive.*

2. If V is orientable with orientation $\mathfrak{o}_V$, the action of the cohomology group $H^1(B;\mathbb{Z}_2)$ on the set $\mathcal{SP}_n^{\pm}(V, \mathfrak{o}_V)$ is free and transitive.

We denote this action by $(\eta, \mathfrak{s}_V) \mapsto \eta\mathfrak{s}_V$ (resp. $(\eta, \mathfrak{p}_V^{\pm}) \mapsto \eta\mathfrak{p}_V^{\pm}$) where $\eta \in H^1(B;\mathbb{Z}_2)$ and $\mathfrak{s}_V \in \mathcal{SP}_n(V, \mathfrak{o}_V)$ (resp. $\mathfrak{p}_V^{\pm} \in \mathcal{P}_n^{\pm}(V)$).

Example 1.3.16. There are four different $Spin_2$-structures on a principal SO_2-bundle over $\mathbb{R}P^1 \times \mathbb{R}P^1$, since $H^1(\mathbb{R}P^1 \times \mathbb{R}P^1;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

In particular, the structure corresponding to the trivial class in the cohomology group $H^1(\mathbb{R}P^1 \times \mathbb{R}P^1;\mathbb{Z}_2)$ is known as the *Lie group Spin structure* on a torus.

Example 1.3.17. There are eight different $Spin_3$-structures on a principal SO_3-bundle over $\mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}P^1$, since $H^1(\mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}P^1;\mathbb{Z}_2) = \mathbb{Z}_2^3$.

In particular, the structure corresponding to the trivial class in the cohomology group $H^1(\mathbb{R}P^1\times\mathbb{R}P^1\times\mathbb{R}P^1;\mathbb{Z}_2)$ is known as the *Lie group Spin structure* on a three-dimensional torus.

Remark 1.3.18. Suppose that the manifold M has a $Spin_n$- or $Pin_n^{\pm}$-structure and there is $N \subset M$ a codimension-one submanifold of M. If there is a "natural" map, in the sense that it is induced from the homomorphism $H^1(M;\mathbb{Z}_2) \to H^1(N;\mathbb{Z}_2)$, from $Spin_n$- or $Pin_n^{\pm}$-structures on M to that on N, we say that N inherits a *restricted structure* from the structure on M.

Example 1.3.19. There is a unique $Spin_2$-structure $\mathfrak{s}_{\mathbb{D}^2}$ on the two-dimensional disk $\mathbb{D}^2$ since one has $|H^1(SO(T\mathbb{D}^2, \mathfrak{o}_{T\mathbb{D}^2});\mathbb{Z}_2)| = 1$ (Definition 1.3.12). The restricted $Spin_1$-structure $\mathfrak{s}_{\mathbb{D}^2|\mathbb{S}^1}$ on its boundary $\partial\mathbb{D}^2 = \mathbb{S}^1$ is a non-zero cohomology class in $H^1(SO(T\mathbb{S}^1, \mathfrak{o}_{T\mathbb{S}^1});\mathbb{Z}_2) \cong \mathbb{Z}_2$. In fact, the restricted $Spin_1$-structure $\mathfrak{s}_{\mathbb{D}^2|\mathbb{S}^1}$ associates to the connected double cover of $\mathbb{S}^1$, which is described in Example 1.3.13.

1.3.3 Spinor states and Welschinger invariants of real del Pezzo varieties of dimension three

According to J-Y. Welschinger ([Wel05c], [Wel05a]), given a $Spin_n$- or $Pin_n^{\pm}$-structure on the real part of real non-singular projective Fano varieties [1], we can define a sign (also known as spinor state) ± 1 for each immersed balanced real rational curve on such varieties.

1. They are examples of strongly semipositive real symplectic manifolds as described in [Wel05a].

Spinor states of balanced real rational curves on a real three-dimensional del Pezzo variety

Let (X, τ) be a real three-dimensional del Pezzo variety with $\mathbb{R}X \neq \emptyset$. We equip the real part $\mathbb{R}X$ with a Riemannian metric g_X and consider the tangent vector bundle $T\mathbb{R}X$ over $\mathbb{R}X$. By definition, the first Chern class of X is even, hence X is spin (meaning that $w_1(X) = w_2(X) = 0 \mod 2$) (see Remark 2.2, [LM16]). Moreover, if either $H^1(X;\mathbb{Z}_2) = 0$ or there are real rational curves on X with non-empty real part (hence the real part of such curve is connected), then the real part $\mathbb{R}X$ is orientable and also spin ([Kol00], [Kha02]). Now we restrict attention to the cases $\mathbb{R}X$ is orientable and we equip $\mathbb{R}X$ with a $Spin_3$-structure, denoted by $\mathfrak{s}_X := (Spin(T\mathbb{R}X, \mathfrak{o}_X), q_X)$. With abuse of notation, here we use the notation $\mathfrak{o}_X$ for an orientation on $T\mathbb{R}X$.

Example 1.3.20. If (X, τ) is the blown-up of $(\mathbb{C}P^3, \tau_3)$ at a real point x_0, meaning that $X = \mathbb{C}P^3 \sharp \overline{\mathbb{C}P^3}$ whose real part is $\mathbb{R}X = \mathbb{R}P^3 \sharp \overline{\mathbb{R}P^3}$, it can be shown that the principal SO_3-bundle $SO(T\mathbb{R}X)$ over $\mathbb{R}X$ admits a $Spin_3$-structure $\mathfrak{s}_{\mathbb{R}P^3 \sharp \overline{\mathbb{R}P^3}} := \mathfrak{s}_{\mathbb{R}P^3} \sharp \overline{\mathfrak{s}}_{\mathbb{R}P^3}$ associating to a $Spin_3$-structure $\mathfrak{s}_{\mathbb{R}P^3}$ on $\mathbb{R}P^3$, see Section 4 in [Wel05c].

Let $d \in H_2(X;\mathbb{Z})$ and let $k_d = \frac{1}{2}c_1(X) \cdot d$ be its associated integer.

We consider a real balanced immersion $f : (\mathbb{C}P^1, \tau_1) \to (X, \tau)$ and its associated short exact sequence of holomorphic vector bundles over $\mathbb{C}P^1$ (see Section 1.2.2)

$$0 \longrightarrow T\mathbb{C}P^1 \longrightarrow f^*TX \longrightarrow \mathcal{N}_f \longrightarrow 0$$

where $\mathcal{N}_f = f^*TX/T\mathbb{C}P^1$ is a rank 2 holomorphic normal vector bundle over $\mathbb{C}P^1$. Since X and f are real, we also have the corresponding real bundles $T\mathbb{R}P^1$, $f|^*_{\mathbb{R}P^1}T\mathbb{R}X$, and $\mathbb{R}\mathcal{N}_f$ fitting in a split short exact sequence of vector bundles over $\mathbb{R}P^1$

$$0 \longrightarrow T\mathbb{R}P^1 \longrightarrow f|^*_{\mathbb{R}P^1}T\mathbb{R}X \longrightarrow \mathbb{R}\mathcal{N}_f \longrightarrow 0.$$

Up to homotopy, one has the decomposition $f|^*_{\mathbb{R}P^1}T\mathbb{R}X = T\mathbb{R}P^1 \oplus \mathbb{R}\mathcal{N}_f$. Suppose that $f_*[\mathbb{C}P^1] = d \in H_2^{-\tau}(X;\mathbb{Z})$. Let $C = Im(f)$ with $\mathbb{R}C \neq \emptyset$. One has $f(\mathbb{R}P^1) \subset \mathbb{R}X$ is an immersed knot. In order to define a spinor state for the curve C, we first show a consistent way to decompose the real vector bundle $f|^*_{\mathbb{R}P^1}T\mathbb{R}X$ into sum of real line subbundles, and then define a loop of direct orthogonal frames of this bundle as the following.

We denote by $\mathcal{O}_{\mathbb{R}P^1}(0)$ an orientable real line vector bundle and by $\mathcal{O}_{\mathbb{R}P^1}(1)$ a non-orientable one. Since $T\mathbb{R}P^1$ is an orientable real line bundle, one has $T\mathbb{R}P^1 = \mathcal{O}_{\mathbb{R}P^1}(0)$. Up to homotopy, one has

— if k_d is even then $\mathbb{R}\mathcal{N}_f = \mathcal{O}_{\mathbb{R}P^1}(1) \oplus \mathcal{O}_{\mathbb{R}P^1}(1)$,

— if k_d is odd then $\mathbb{R}\mathcal{N}_f = \mathcal{O}_{\mathbb{R}P^1}(0) \oplus \mathcal{O}_{\mathbb{R}P^1}(0)$.

Since the normal bundle $\mathcal{N}_f$ is balanced, there is a consistent way to choose a real line subbundle $L \subset \mathbb{R}\mathcal{N}_f$ as follows. One should notice that the balancing property of $\mathcal{N}_f$ implies its projectivization $\mathbb{P}(\mathcal{N}_f) \cong \mathbb{C}P^1 \times \mathbb{C}P^1$. Up to real isotopy, there are two holomorphic real line subbundles of $\mathcal{N}_f$ of degree $(k_d - 2)$ such that its projectivization in

$\mathbb{P}(\mathcal{N}_f)$ is a section of bidegree $(1,1)$ in the basis $(L_1, L_2) = ([\mathbb{C}P^1 \times \{p\}], [\{q\} \times \mathbb{C}P^1])$, where $p, q \in \mathbb{C}P^1$. Note that the orientation on $\mathbb{R}X$ and the orientation on $T\mathbb{R}P^1$ induces an orientation on $\mathbb{R}\mathcal{N}_f$. Then, these two real isotopy classes are characterised by the direction in which the real part of a fiber rotates in $\mathbb{R}\mathcal{N}_f$. We denote by E^+ the isotopy class of the real part of the real line subbundle of $\mathcal{N}_f$ whose real part of a fiber rotates positively in $\mathbb{R}\mathcal{N}_f$ and by E^- otherwise. The choice of L is as follows.

— In the case k_d **is even**, we choose L in the isotopy class of E^+.

— In the case k_d **is odd**, we choose L in the isotopy class of E^-.

In both cases, to be able to associate some framing of $f|^*_{\mathbb{R}P^1} T\mathbb{R}X$, we need to cut the ribbon of the knot $f(\mathbb{R}P^1)$ at some point, make half a twist in some direction and then glue it again. The choice of the real line subbundle $L \subset \mathbb{R}\mathcal{N}_f$ as above corresponds to the choice of such a direction. More precisely,

— In the case k_d **is even**, we glue the ribbon in the positive direction.

— In the case k_d **is odd**, we glue the ribbon in the negative direction.

Then, there exists a unique choice of a real line subbundle $\tilde{L}$ in $\mathbb{R}\mathcal{N}_f$ such that we have the decomposition $f|^*_{\mathbb{R}P^1} T\mathbb{R}X = T\mathbb{R}P^1 \oplus L \oplus \tilde{L}$.

Now we choose some trivialization of each factor of the above decomposition of $f|^*_{\mathbb{R}P^1} T\mathbb{R}X$ given by non-vanishing sections v_i $(i \in \{1,2,3\})$ and then define a loop of direct orthogonal frames of the vector bundle $f|^*_{\mathbb{R}P^1} T\mathbb{R}X$. Assuming that the vector $v_1(\xi) \in T\mathbb{R}P^1$ $(\xi \in \mathbb{R}P^1)$ and (v_1, v_2, v_3) forms a direct basis. In the case k_d is even, we can choose[2] a non-vanishing section $v_2(\xi) \in L$ such that $(v_1(\xi), v_2(\xi))_{\xi \in \mathbb{R}P^1}$ is an orthogonal section of $T\mathbb{R}P^1 \oplus L$. Then, the unique non-vanishing section $v_3(\xi) \in \tilde{L}$ forms a loop of direct orthogonal frames $(v_1(\xi), v_2(\xi), v_3(p))_{\xi \in \mathbb{R}P^1}$ of the vector bundle $f|^*_{\mathbb{R}P^1} T\mathbb{R}X$, see Figure 1.3. In the case k_d is odd, we first choose the non-vanishing sections $(v_1(\xi), v_2(\xi), v_3(\xi))_{\xi \in [0,\pi] \subset \mathbb{R}P^1}$, then we extend this trivialization over $\mathbb{R}P^1$ by applying the action of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\xi) & \sin(\xi) \\ 0 & -\sin(\xi) & \cos(\xi) \end{pmatrix} \in SO_3(\mathbb{R})$.

Note that the pull-back via the immersion f of a principal SO_3-bundle $SO(T\mathbb{R}X, \mathfrak{o}_X)$ over $\mathbb{R}X$ is a principal SO_3-bundle $f^*SO(T\mathbb{R}X, \mathfrak{o}_X)$ over $\mathbb{R}P^1$, and a $Spin_3$-structure on $SO(T\mathbb{R}X, \mathfrak{o}_X)$ induces a $Spin_3$-structure on this bundle. Hence, we call a loop in the above construction a *loop of the principal SO_3-bundle $f^*SO(T\mathbb{R}X, \mathfrak{o}_X)$*.

Then, we are ready to define spinor states of the curve C.

Definition 1.3.21. Let $((X,\tau), \mathfrak{s}_X, \mathfrak{o}_X))$ be a real three-dimensional del Pezzo variety whose real part is orientable and equipped with a $Spin_3$-structure $(Spin(T\mathbb{R}X, \mathfrak{o}_X), q_X)$, denoted by $\mathfrak{s}_X$. Let $C = Im(f)$ be a real balanced irreducible rational curve immersed in X. Let $(v_1(\xi), v_2(\xi), v_3(\xi))_{\xi \in \mathbb{R}P^1}$ be a loop of the principal SO_3-bundle $f^*SO(T\mathbb{R}X, \mathfrak{o}_X)$

2. This choice is not unique since we can also choose $-v_2(\xi) \in \mathbb{R}L$.

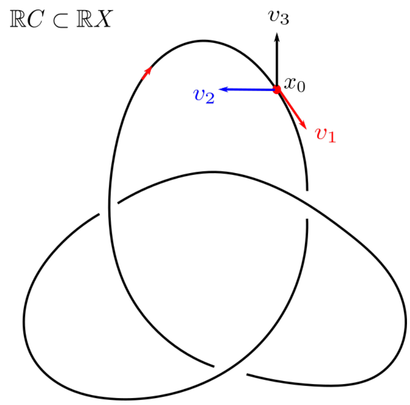

Figure 1.3 – A direct orthogonal frame $(v_1(x_0), v_2(x_0)), v_3(x_0)$ at $x_0 = f(\xi) \in \mathbb{R}C$ of the vector bundle $T\mathbb{R}X|_{\mathbb{R}C}$.

defined as above. The **spinor state** of C, denoted by $sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C)$, is given by

$$sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C) = \begin{cases} +1 & \text{if this loop lifts to a loop of } f^*Spin(T\mathbb{R}X, \mathfrak{o}_X) \\ -1 & \text{otherwise.} \end{cases}$$

Remark 1.3.22. The spinor state of a real rational curve in Definition 1.3.21 does not depend on the choice of neither the sections v_i $(i \in \{1, 2, 3\})$ nor the metric g_X or the parametrization f. This spinor state only depends on the choice of a $Spin_3$-structure on $\mathbb{R}X$ and of an orientation on $\mathbb{R}X$.

Geometrically, if we change the orientation of $\mathbb{R}X$ then we interchange the two isotopy classes E^+ and E^-. Since E^+ and E^- differ by exactly one full rotation, i.e. a generator of $\pi_1(SO_2(\mathbb{R}))$, the associating loops in $\pi_1(SO_3(\mathbb{R}))$ differ by a non-trivial element. Moreover, recall that the set of $Spin_3$-structures on the bundle $SO(T\mathbb{R}X, \mathfrak{o}_X)$ are in one-to-one correspondence with the group $H^1(\mathbb{R}X; \mathbb{Z}_2)$, which is determined by mapping in circles. We resume these properties in the following proposition.

Proposition 1.3.23. *Let $\eta \in H^1(\mathbb{R}X; \mathbb{Z}_2)$. One has*

$$sp_{\mathfrak{s}_X,-\mathfrak{o}_X}(C) = -sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C),$$

and

$$sp_{\eta\mathfrak{s}_X,\mathfrak{o}_X}(C) = (-1)^{\eta\cdot(f_*[\mathbb{R}P^1])} sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C).$$

Example 1.3.24. Let (X, τ) be $(\mathbb{C}P^3, \tau_3)$ with its real part $\mathbb{R}X = \mathbb{R}P^3$. We have $H_1(\mathbb{R}P^3; \mathbb{Z}_2) = \mathbb{Z}_2$. Then, there are two $Spin_3$-structures on $SO(T\mathbb{R}P^3, \mathfrak{o}_{\mathbb{R}P^3})$.

We can choose a $Spin_3$-structure $\mathfrak{s}_{\mathbb{R}P^3}$ on $SO(T\mathbb{R}P^3, \mathfrak{o}_{\mathbb{R}P^3})$ such that the spinor state of a line L in $\mathbb{C}P^3$ is $sp_{\mathfrak{s}_{\mathbb{R}P^3},\mathfrak{o}_{\mathbb{R}P^3}}(L) = 0$. This choice is the same as the choice of a trivialisation on $T\mathbb{R}P^3$ by Brugallé-Georgieva in [BG16].

Welschinger invariants of real three-dimensional del Pezzo varieties

Let $((X,\tau),\mathfrak{s}_X,\mathfrak{o}_X))$ and $C = Im(f)$ as in Definition 1.3.21.

Let $d \in H_2^{-\tau}(X;\mathbb{Z})$ and let k_d be its associated integer.

Let $\underline{x}_d$ be a real generic configuration (see Definition 1.2.8) of k_d points consisting of r real points and l pairs of complex conjugated points in X such that $r + 2l = k_d$.

Denote by $\mathcal{R}_d(\underline{x}_d)$ the set of balanced irreducible real rational curves in X representing the homology class d and passing through $\underline{x}_d$.

Definition 1.3.25. The **Welschinger invariant** of a real three-dimensional del Pezzo variety $((X,\tau),\mathfrak{s}_X,\mathfrak{o}_X))$ of class $d \in H_2^{-\tau}(X;\mathbb{Z})$, denoted by $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)$, is defined as

$$W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l) := \sum_{C\in\mathcal{R}_d(\underline{x}_d)} sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C)$$

Remark 1.3.26. If either $d \in H_2(X;\mathbb{Z}) \setminus H_2^{-\tau}(X;\mathbb{Z})$ or $k_d \leq 0$, then the corresponding Welschinger invariants $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)$ vanish.

The following theorem follows from Theorem 2.2 and Theorem 4.1 by J-Y. Welschinger in [Wel05c].

Theorem 1.3.27. *(Welschinger[Wel05c]) The Welschinger invariant in Definition 1.3.25 does not depend on the choice of a real generic configuration $\underline{x}_d$ up to isotopy, provided its number of real points.*

This invariant only depends on the $Spin_3$-structure $\mathfrak{s}_X$ on $SO(T\mathbb{R}X,\mathfrak{o}_X)$, on the homology class d and on the number l.

As in the two-dimensional case, the genus-0 Welschinger invariants in Definition 1.3.25 give lower bounds in the corresponding real enumerative problem.

1.4 Remarks about Pin_2^--structures on topological surfaces

For more details, see §3 [KT90], §11 [CZ19], [Sol06], [Sol07] and [HS12].

Let F be a real non-singular topological surface.

In this part, we recall some properties of quadric and quasi-quadric functions corresponding to Pin_2^--structures of $O(TF)$, which we call quadratic enhancements and quasi-quadratic enhancements respectively.

1.4.1 Quadratic enhancements

Definition 1.4.1. A function $q : H_1(F;\mathbb{Z}_2) \to \mathbb{Z}_4$ is called **a quadratic enhancement** on F if it satisfies

$$q(\alpha+\beta) = q(\alpha) + q(\beta) + 2\alpha.\beta$$

where $\alpha.\beta \in \mathbb{Z}_2$ is the homology intersection number of $\alpha, \beta \in H_1(F;\mathbb{Z}_2)$.

Furthermore, the group $H^1(F;\mathbb{Z}_2)$ acts freely and transitively on the set of quadratic enhancements. Precisely, this action is given by

$$(q_\eta)(\alpha) := q(\alpha) + 2\eta \cdot \alpha$$

where $\eta \cdot \alpha \in \mathbb{Z}_2$ is the cap product between $\eta \in H^1(F;\mathbb{Z}_2)$ and $\alpha \in H_1(F;\mathbb{Z}_2)$.

We have the following correspondence:

Proposition 1.4.2. *(Kirby-Taylor[KT90]) There is a canonical one-to-one correspondence between the set of Pin_2^--structures of $O(TF)$ and the set of quadratic enhancements on F.*

We denote the quadratic enhancement corresponding to a Pin_2^--structure $\mathfrak{p}^-$ by $q^{\mathfrak{p}^-}$.

To describe the correspondence in Proposition 1.4.2, one remarks the following commutative diagram

$$\begin{array}{ccc} Pin_2^-(\mathbb{R}) & \xrightarrow{\tilde{\iota}} & Spin_3(\mathbb{R}) \\ \downarrow{\scriptstyle q_2^-} & & \downarrow{\scriptstyle q_3} \\ O_2(\mathbb{R}) & \xrightarrow{\iota} & SO_3(\mathbb{R}) \end{array}$$

where ι is a Lie group homomorphism defined by $\iota(A) := \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix}$, for $A \in O_2(\mathbb{R})$; q_2^- and q_3 are covering maps; $\tilde{\iota}$ is a Lie group homomorphism which is the lifting of ι.

A function between sets of $Spin$ or $Pin^\pm$-structures on TF is called "natural" in the sense that it is equivariant with respect to the automorphism of $H^1(F;\mathbb{Z}_2)$.

Proposition 1.4.3. *(Kirby-Taylor[KT90]) There exists a natural bijection*

$$\gamma : \mathcal{P}_2^-(TF) \to \mathcal{SP}_3(TF \oplus \det TF, \mathfrak{o}_{TF\oplus \det TF})$$

where $\det TF$ is the determinant line bundle of TF.

A construction of this bijection is as follows. Given a Pin_2^--structure on F, we can write down transition data for its principal O_2-bundle and principal Pin_2^--bundle. Then, we can construct canonically transition data for principal SO_3-bundle and principal $Spin_3$-bundle over F via the above commutative diagram. Hence, we obtain an induced $Spin_3$-structure on the right. We are done by proving that such map is $H^1(F;\mathbb{Z}_2)$-equivariant.

Given a Pin_2^--structure $\mathfrak{p}_F^-$ on TF which is induced from a $Spin_3$-structure $\mathfrak{s}_V$ on $(V, \mathfrak{o}_V) := (TF \oplus \det TF, \mathfrak{o}_{TF\oplus\det TF})$, meaning that $\mathfrak{p}_F^- := \mathfrak{s}_{V|F} = \gamma^{-1}(\mathfrak{s}_V)$, we consider $C \subset F$ an embedded circle. The $Spin_3$-structure on $(V, \mathfrak{o}_V)$ induces a trivialization of $V|_C = T\mathbb{S}^1 \oplus \mathcal{N}_{C/F} \oplus \mathcal{N}_{F/V}$ where $\mathcal{N}_{X/Y}$ denotes the normal bundle of X in Y. Since the tangent bundle $T\mathbb{S}^1$ is trivial and $V|_C$ is oriented then $\mathcal{N}_{C/F} \oplus \mathcal{N}_{F/V}$ has a preferred framing (called "even").

Fix a framing on $\mathcal{N}_{C/F} \oplus \mathcal{N}_{F/V}$ which does not[3] come from the $Spin_3$-structure $\mathfrak{s}_V$ (i.e. an "odd" framing), one can define a $\mathbb{Z}_4$-valued function $\hat{q}$ from the set of embedded circles in F given by

$$\hat{q}(C) := \sharp\Big(\text{right half twists that } \mathcal{N}_{C/F} \text{ makes when going along } \mathbf{C}\Big) \mod 4.$$

If $C = C_1 \sqcup \cdots \sqcup C_n$ is a disjoint union of circles, then $\hat{q}(C) := \hat{q}(C_1) + \cdots + \hat{q}(C_n)$.

Moreover, the function $\hat{q}$ depends only on the homology class $[C]_{\mathbb{Z}_2} \in H_1(F;\mathbb{Z}_2)$ and satisfies properties in Definition 1.4.1. Hence, such function produces a quadratic enhancement $q^{\mathfrak{p}_F^-} : H_1(F;\mathbb{Z}_2) \to \mathbb{Z}_4$.

If we change the Pin_2^--structure $\mathfrak{p}_F^-$ into $\eta\mathfrak{p}_F^-$, where $\eta \in H^1(F;\mathbb{Z}_2)$, then one has $q^{\eta\mathfrak{p}_F^-}([C]) = q_\eta^{\mathfrak{p}_F^-}([C])$, or equivalently, the induced quadratic enhancement $q^{\mathfrak{p}_F^-}$ is equivariant for the $H^1(F;\mathbb{Z}_2)$ action. The one-to-one correspondence then follows.

In the particular case that the topological surface is spin, then there are many ways to associate a quadratic form (either $\mathbb{Z}_2$-valued or $\mathbb{Z}_4$-valued) to a $Spin_2$-structure on such surface, see [Kha02] or [Joh80].

1.4.2 Quasi-quadratic enhancements

Definition 1.4.4. A function $s : H_1(F;\mathbb{Z}_2) \to \mathbb{Z}_2$ is called **a quasi-quadratic enhancement** if it satisfies

$$s(\alpha+\beta) = s(\alpha) + s(\beta) + \alpha.\beta + (w_1(F)\cdot\alpha)(w_1(F)\cdot\beta)$$

where $.$ denotes the homology intersection number, and $\cdot$ denotes the dual pairing of $H^1(F;\mathbb{Z}_2)$ and $H_1(F;\mathbb{Z}_2)$.

Furthermore, $H^1(F;\mathbb{Z}_2)$ acts freely and transitively on the set of quasi-quadratic enhancements. Precisely, this action is given by

$$(s_\eta)(\alpha) := s(\alpha) + \eta\cdot\alpha.$$

By using differential equations for the open Gromov-Witten potential, Solomon proved the following proposition.

Proposition 1.4.5. *(Solomon [HS12]) There is a canonical one-to-one correspondence between the set of Pin_2^--structures of $O(TF)$ and the set of quasi-quadratic enhancements s.*

We denote the quasi-quadratic enhancement corresponding to Pin_2^--structure $\mathfrak{p}^-$ by $s^{\mathfrak{p}^-}$. Then, we have $s_\eta^{\mathfrak{p}^-} = s^{\eta\mathfrak{p}^-}$ for any $\eta \in H^1(F;\mathbb{Z}_2)$.

The relationship between quadratic and quasi-quadratic enhancements is shown in the following proposition.

3. This choice is due to Kirby and Taylor in [KT90].

Proposition 1.4.6. *If $s : H_1(F;\mathbb{Z}_2) \to \mathbb{Z}_2$ is a function such that*

$$q(\alpha) = 2s(\alpha) + w_1(F)\cdot\alpha \mod 4$$

where q is a quadratic enhancement, then s is a quasi-quadratic enhancement.

Proof. Remark that $w_1(F)\cdot x = x^2 \mod 2$ for any $x \in H_1(F;\mathbb{Z}_2)$. We have that

$$\begin{aligned} 2s(\alpha+\beta) &= q(\alpha+\beta) + w_1(F)\cdot(\alpha+\beta) \mod 4 \\ &= q(\alpha+\beta) + (\alpha+\beta)^2 \mod 4. \end{aligned}$$

We also have

$$\begin{aligned} &2s(\alpha) + 2s(\beta) + 2\alpha.\beta + 2(w_1(F)\cdot\alpha)(w_1(F)\cdot\beta) \\ &= \Big(q(\alpha) + w_1(F)\cdot\alpha\Big) + \Big(q(\beta) + w_1(F)\cdot\beta\Big) + 2\alpha.\beta + 2\alpha^2\beta^2 \mod 4 \\ &= \Big(q(\alpha) + q(\beta) + 2\alpha.\beta\Big) + \alpha^2 + \beta^2 + 2\alpha^2\beta^2 \mod 4 \\ &= q(\alpha+\beta) + (\alpha+\beta)^2 + 2\alpha.\beta(\alpha.\beta+1) \mod 4 \\ &= q(\alpha+\beta) + (\alpha+\beta)^2 \mod 4. \end{aligned}$$

Hence, we get

$$2s(\alpha+\beta) = 2s(\alpha) + 2s(\beta) + 2\alpha.\beta + 2(w_1(F)\cdot\alpha)(w_1(F)\cdot\beta) \mod 4.$$

The statement then follows. □

1.4.3 Construction of quasi-quadratic enhancement corresponding to a Pin_2^--structure on a topological surface

It is known that every non-singular topological surface F has a Pin_2^--structure on its tangent bundle TF. We first equip TF with a Pin_2^--structure $\mathfrak{p}_{TF} := (Pin^-(TF), q_{TF})$. Then, we proceed to construct a quasi-quadratic enhancement corresponding to such Pin_2^--structure.

Let $\alpha : \mathbb{S}^1 \to F$ be an immersion and let $\mathcal{N}_\alpha := \alpha^*TF/T\mathbb{S}^1$ be the normal bundle of this map. We also denote the induced Pin_2^--structure on α^*TF by $\alpha^*\mathfrak{p}_{TF}$. We consider the short exact sequence of real vector bundles over $\mathbb{S}^1$ determined by α

$$0 \longrightarrow T\mathbb{S}^1 \longrightarrow \alpha^*TF \longrightarrow \mathcal{N}_\alpha \longrightarrow 0, \quad (ses)_\alpha.$$

Up to homotopy, one has the splitting $\alpha^*TF = T\mathbb{S}^1 \oplus \mathcal{N}_\alpha$. We first choose the bounding $Spin_1$-structure on $T\mathbb{S}^1$, denoted by $\mathfrak{s}_0$, which corresponds to the disconnected double cover of $\mathbb{S}^1$. Then, on the normal line bundle $\mathcal{N}_\alpha$, there is a canonical Pin_1^--structure, denoted by $\mathfrak{p}_0$, which maps to $0 \in \mathbb{Z}_2$ by the natural bijection $\mathcal{P}^-(\mathcal{N}_\alpha) \to \mathbb{Z}_2$ (see Lemma 8.2 [Sol06] or Section 11 [CZ19]). We say that a given Pin_2^--structure on TF is *compatible* with the short exact sequence $(ses)_\alpha$ if $\alpha^*\mathfrak{p}_{TF}$ is mapped from $< \mathfrak{s}_0, \mathfrak{p}_0 >_\alpha$ via the natural $H^1(\mathbb{S}^1;\mathbb{Z}_2)$-biequivariant map

$$< .,. >_\alpha : \mathcal{SP}_1(T\mathbb{S}^1) \times \mathcal{P}_1^-(\mathcal{N}_\alpha) \to \mathcal{P}_2^-(\alpha^*TF).$$

Definition 1.4.7. Let F be a non-singular topological surface equipped with a Pin_2^--structure $\mathfrak{p}_{TF}$ on TF and $\alpha : \mathbb{S}^1 \to F$ be an immersion. We define the following function:

$$s_{\mathfrak{p}_{TF}} : \{\text{immersed circles in } F\} \to \mathbb{Z}_2$$
$$\alpha \mapsto \begin{cases} 1, & \text{if } \alpha^*\mathfrak{p}_{TF} =<\mathfrak{s}_0, \mathfrak{p}_0>_\alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1.4.8. Since a loop $\tilde{\alpha}$, which lifts α in the principal O_2-bundle $\alpha^*(O(TF))$ (see Figure 1.4), can be written as r ($r \in \mathbb{Z}$) times a generator of $\pi_1(O_2(\mathbb{R})) \cong \mathbb{Z}$, we can also interpreter $s_{\mathfrak{p}_{TF}}(\alpha) \in \{0, 1\}$ as

$$s_{\mathfrak{p}_{TF}}(\alpha) = 1 + r \mod 2.$$

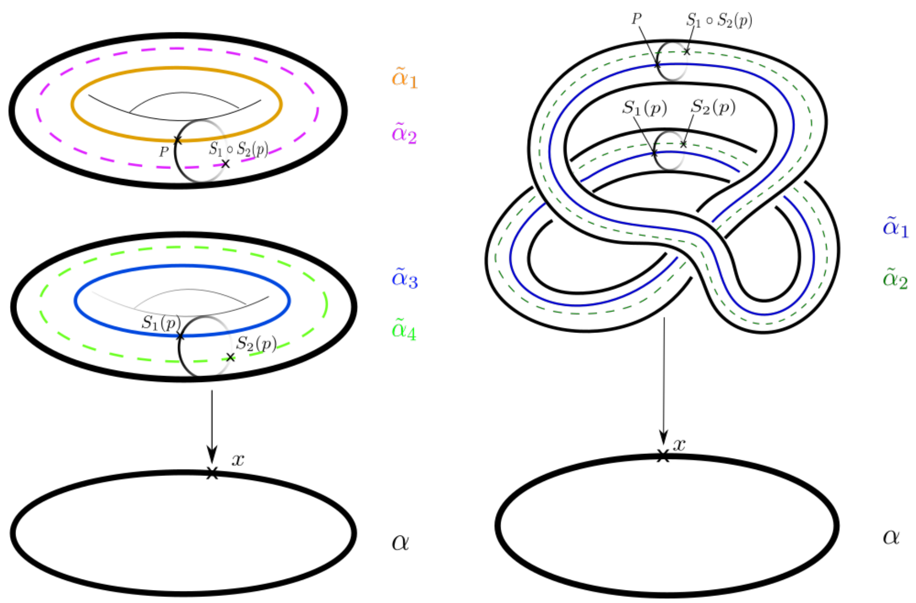

Figure 1.4 – Liftings of an immersed circle $C = Im(\alpha) \subset F$ to the principal O_2-bundle $O(TF)$ in two cases: $\mathcal{N}_\alpha$ is orientable (LHS) and $\mathcal{N}_\alpha$ is non-orientable (RHS).

Since $Pin_2^-(\mathbb{R})$ is a double cover of $O_2(\mathbb{R})$, then the liftings of α in the principal Pin_2^--bundle $\alpha^*(Pin^-(TF))$ are either all loops or all non-closed paths.

By definition, the function $s_{\mathfrak{p}_{TF}}$ is additive on disjoint unions of embedded circles, i.e. $s_{\mathfrak{p}_{TF}}(\alpha_1 + \alpha_2) = s_{\mathfrak{p}_{TF}}(\alpha_1) + s_{\mathfrak{p}_{TF}}(\alpha_2)$, where α_i are disjointed embeddings.

Remark 1.4.9. The function $s_{\mathfrak{p}_{TF}}$ does not depend on the orientation of α, where the inverse orientation of α is defined as $-\alpha(z) := \alpha(\bar{z})$, $\forall z \in \mathbb{S}^1 \subset \mathbb{C}$.

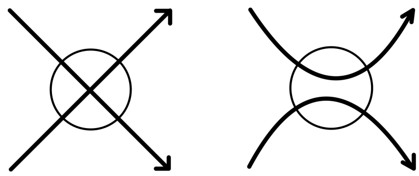

Figure 1.5 – Smoothing a transverse intersection.

We can deform an union of immersed circles in F such that all their intersections and self-intersections are transverse. The rule of smoothing a transverse intersection is shown in Figure 1.5.

After smoothing an union $\bigcup\limits_{i=1}^{k} C_i$ of k immersed circles in F we get a disjoint union $\bigsqcup\limits_{i=1}^{k'} C'_i$ of k' embedded circles in F. One has

$$\sum_{i=1}^{k}[C_i] = \sum_{i=1}^{k'}[C'_i] \in H_1(F;\mathbb{Z}_2).$$

The difference between k and k' is characterised by the number of intersection and self-intersection points of the immersed curves in the former union. More precisely, let $\delta(\alpha)$ be the number of self-intersection points of $C = Im(\alpha)$, one has

$$k' \equiv k + \sum_{i=1}^{k}\delta(\alpha_i) + \sum_{i<j}[C_i].[C_j] \mod 2.$$

Lemma 1.4.10. *The function $s_{\mathfrak{p}_{TF}}$ admits two following properties.*

1) $s_{\mathfrak{p}_{TF}}(\alpha) = 0$ *for any homologically trivial embedding α.*

2) If $\sum\limits_{i=1}^{k}[C_i] = 0 \in H_1(F;\mathbb{Z}_2)$ *then*

$$\sum_{i=1}^{k}\Big(s_{\mathfrak{p}_{TF}}(\alpha_i) + \delta(\alpha_i)\Big) \equiv \sum_{i<j}\Big([C_i].[C_j] + (w_1(F)\cdot[C_i])(w_1(F)\cdot[C_j])\Big) \mod 2. \tag{1.1}$$

Proof. Each part follows by its corresponding following claim.

Claim 1. Let $\mathfrak{s}_{\mathbb{D}^2} = (Spin(T\mathbb{D}^2, \mathfrak{o}_{T\mathbb{D}^2}), q_{T\mathbb{D}^2}))$ be the unique $Spin_2$-structure on a two dimensional disk $\mathbb{D}^2$. If $Im(\alpha) = \mathbb{S}^1$ bounds the disk $\mathbb{D}^2$ then $s_{\mathfrak{s}_{\mathbb{D}^2}}(\alpha) = 0$.

One has that the standard orientations on $\mathbb{S}^1$ and $\mathbb{D}^2$ induce that on $\mathcal{N}_\alpha$ via the short exact sequence of real vector bundles over $\mathbb{S}^1$:

$$0 \longrightarrow T\mathbb{S}^1 \longrightarrow \alpha^*T\mathbb{D}^2 \longrightarrow \mathcal{N}_\alpha \longrightarrow 0.$$

Hence, the $Spin_2$-structure on $\alpha^* T\mathbb{D}^2$ induced by the $Spin_1$-structure on $T\mathbb{S}^1$ and on $\mathcal{N}_\alpha$ via the above exact sequence is determined by the section

$$Id \times \tilde{\alpha} : \mathbb{S}^1 \to SO(T\mathbb{D}^2|_{\mathbb{S}^1}) = \mathbb{S}^1 \times SO_2(\mathbb{R})$$
$$e^{2i\pi t} \mapsto (e^{2i\pi t}, \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}).$$

This $Spin_2$-structure differs from that of the unique $Spin_2$-structure $\mathfrak{s}_{\mathbb{D}^2}$ restricted to its boundary $\mathbb{S}^1 = \partial\mathbb{D}^2$. In other words, $\tilde{\alpha}$ defines a loop generating $\pi_1(SO_2(\mathbb{R}))$. Hence, $s_{\mathfrak{s}_{\mathbb{D}^2}}(\alpha) = 0$.
One can deduce that $s_{\mathfrak{p}_{TF}}(\alpha) = 0$ for any Pin_2^--structure $\mathfrak{p}_{TF}$ on $O(TF)$ and α a homotopically trivial embedding.

Claim 2. If $\bigsqcup\limits_{i=1}^{k'} C_i'$ is a smoothing of $\bigcup\limits_{i=1}^{k} C_i$ then

$$\sum_{i=1}^{k'} s_{\mathfrak{p}_{TF}}(\alpha_i') + \sum_{i<j} (w_1(F) \cdot [C_i'])(w_1(F) \cdot [C_j'])$$
$$\equiv \sum_{i=1}^{k} s_{\mathfrak{p}_{TF}}(\alpha_i) + \sum_{i<j} (w_1(F) \cdot [C_i])(w_1(F) \cdot [C_j]) + k' - k \mod 2. \quad (1.2)$$

The equation (1.2) follows by a similar argument as in proof of Lemma 11.11 in [CZ19].

$\square$

Proposition 1.4.11. *The function $s_{\mathfrak{p}_{TF}}$ induces a quasi-quadratic enhancement*

$$s^{\mathfrak{p}_{TF}} : H_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$$

given by

$$s^{\mathfrak{p}_{TF}}([\alpha]) \equiv s_{\mathfrak{p}_{TF}}(\alpha) + \delta(\alpha) \mod 2.$$

Proof. The function $s^{\mathfrak{p}_{TF}}$ is well-defined. Indeed, suppose that C_1 and C_2 represent the same homology class, i.e. $[C_1] = [C_2] \in H_1(F; \mathbb{Z}_2)$. One has

$$\begin{aligned} & s^{\mathfrak{p}_{TF}}([\alpha_1]) + s^{\mathfrak{p}_{TF}}([\alpha_2]) \\ & \equiv (s_{\mathfrak{p}_{TF}}(\alpha_1) + \delta(\alpha_1)) + (s_{\mathfrak{p}_{TF}}(\alpha_1) + \delta(\alpha_1)) \mod 2 \\ & \equiv [C_1].[C_2] + (w_1(F) \cdot [C_1])(w_1(F) \cdot [C_2])\big) \mod 2 \\ & \equiv [C_1]^2 + [C_1]^2[C_1]^2 \mod 2 \\ & \equiv 0 \mod 2. \end{aligned}$$

Hence, $s^{\mathfrak{p}_{TF}}([\alpha_1]) \equiv s^{\mathfrak{p}_{TF}}([\alpha_2]) \mod 2$.

Moreover, the function $s^{\mathfrak{p}_{TF}}$ satisfies

$$\begin{aligned} & s^{\mathfrak{p}_{TF}}([\alpha_1]+[\alpha_2]) \\ = {} & s^{\mathfrak{p}_{TF}}([\alpha_1]) + s^{\mathfrak{p}_{TF}}([\alpha_2]) + [C_1].[C_2] + (w_1(F)\cdot[C_1])(w_1(F)\cdot[C_2]) \mod 2. \end{aligned} \tag{1.3}$$

Indeed, this is induced directly from Lemma 1.4.10.2) and proof of Theorem 11.1 in [CZ19]. □

Remark 1.4.12. Given any basis of $H_1(F;\mathbb{Z}_2)$, we can make $s^{\mathfrak{p}_{TF}}$ take any specified values on the elements of the basis by choosing an appropriate $\mathfrak{p}_{TF}$.

2

Gromov-Witten invariants

In 2016, E. Brugallé and P. Georgieva ([BG16]) proposed a new strategy to compute genus-0 Gromov-Witten and Welschinger invariants of complex projective space $\mathbb{C}P^3$ using pencils of quadrics, in which they related the enumerative problems in $\mathbb{C}P^3$ and in twofold product of the projective line $\mathbb{C}P^1 \times \mathbb{C}P^1$. In particular, they obtained a formula which relates genus-0 Gromov-Witten and Welschinger invariants in these two varieties. In this chapter, we focus on the genus-0 Gromov-Witten case. We first prove a generalization of Proposition 3 by Kollár ([Kol15]) and then we extend the strategy of Brugallé-Georgieva to del Pezzo varieties of dimension three. By doing this, we reduce the 3-dimensional enumerative problem into two independent ones: the first one is the 1-*dimensional problem* on a complex elliptic curve, and second one is the 2-*dimensional enumerative problem* on del Pezzo surfaces. As a result, we obtain a formula to compute genus-0 Gromov-Witten invariants in such varieties.

Throughout this chapter, we denote by X a 3-dimensional non-singular del Pezzo variety. We organise this chapter as follows. In the first section, we introduce our main theorem of this chapter (Theorem 2.1.1). In Section 2.2, we recall about properties of pencils of surfaces in the linear system of divisors $|-\frac{1}{2}K_X|$ and explain the monodromy action to such pencils. As a consequence, we prove the 1-*dimensional problem* (Proposition 2.2.16). In Section 2.3, we investigate properties of rational curves on variety X and that on a non-singular del Pezzo surface $\Sigma \in |-\frac{1}{2}K_X|$, then we prove a generalization of Proposition 3 by Kollár (Proposition 2.3.2). In Section 2.4, we combine the results from the previous sections to prove Theorem 2.1.1. The last section is followed by applications of Theorem 2.1.1 in particular cases (Theorems 2.5.1, 2.5.2, 2.5.4) and explicit computations of genus-0 Gromov-Witten invariants in the case $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ using *Maple program*.

2.1 Main results

Let $\Sigma \subset X$ be a non-singular surface realizing half of the anti-canonical class of X, i.e. a non-singular element in the linear system $|-\frac{1}{2}K_X|$.

Let $\psi : H_2(\Sigma;\mathbb{Z}) \longrightarrow H_2(X;\mathbb{Z})$ be the map induced by the inclusion of Σ in X.

We have the following theorem.

Theorem 2.1.1. *Let X be a 3-dimensional del Pezzo variety such that $\ker(\psi) \cong \mathbb{Z}$ where Σ and ψ are as described above. Let S be a generator of $\ker(\psi)$. For every homology class $d \in H_2(X;\mathbb{Z})$, one has*

$$GW_X(d) = \frac{1}{2} \sum_{D \in \psi^{-1}(d)} (D.S)^2 GW_\Sigma(D).$$

Theorem 2.1.1 is a generalization of Theorem 2 in [BG16].

This theorem allows us to compute the genus-0 Gromov-Witten invariants for almost all three-dimensional del Pezzo varieties of degree $6, 7$ and 8. In 2018, A. Zahariuc also adapted the strategy suggested by Kollár in [Kol15], as Brugallé-Georgieva, to study genus-0 Gromov-Witten invariants in three-dimensional del Pezzo varieties of degree $2, 3, 4$ and 5 (see [Zah18]).

2.2 Pencils of surfaces in the linear system $|-\frac{1}{2}K_X|$

Let $\mathcal{Q}$ be a pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$. Let E be the base locus and Σ be a general element of the pencil $\mathcal{Q}$. Firstly, we prove that the base locus E is an elliptic curve (Proposition 2.2.1) and Σ is a non-singular del Pezzo surface of the same degree as $\deg(X)$ (Proposition 2.2.2). We then explain the interest of studying three-dimensional del Pezzo varieties of degree $6, 7$ and 8 (Proposition 2.2.4). Afterwards, we present the role of monodromy action on the non-singular fibres of the (Lefschetz) pencil $\mathcal{Q}$ and on the corresponding genus-0 Gromov-Witten invariants. In the end of this section, we give the main ingredient in the proof the 1-*dimensional problem* of Theorem 2.1.1 (Proposition 2.2.16).

2.2.1 Properties of pencils of surfaces in $|-\frac{1}{2}K_X|$

Proposition 2.2.1. *If the pencil $\mathcal{Q}$ is generic enough, then its base locus is a non-singular elliptic curve. Moreover, this curve realises the homology class $\frac{1}{4}K_X^2$ in $H_2(X;\mathbb{Z})$.*

Proof. By Bertini's theorem (see [GH78]) and Theorem 1.2 in [Šok80], a general element $\Sigma \in |-\frac{1}{2}K_X|$ is a non-singular surface.

The base locus E of the pencil $\mathcal{Q}$ realises the homology class $[\Sigma].[\Sigma]$, that is the self-intersection product of homology class $[\Sigma] \in H_4(X;\mathbb{Z})$. By the adjunction formula, we have:

$$K_\Sigma = (K_X + [\Sigma]).[\Sigma] = -[\Sigma]^2.$$

Thus, the curve E realises the homology class $-K_\Sigma$, hence $[E] = \frac{1}{4}K_X^2$ in $H_2(X;\mathbb{Z})$. Since two general elements in pencil have traversal intersections, the curve E is in fact a non-singular representative of the class $-K_\Sigma$.

The canonical bundle of Σ restricting to E is $\mathcal{K}_\Sigma|_E = \mathcal{O}_\Sigma(-E)|_E = \mathcal{O}_E(-E)$. By the adjunction formula, the genus of the curve E is:

$$\begin{aligned} g(E) &= 1 + \frac{1}{2}\deg(\mathcal{K}_\Sigma \otimes \mathcal{O}_\Sigma(E)|_E) \\ &= 1 + \frac{1}{2}\deg(\mathcal{O}_E(-E) \otimes \mathcal{O}_E(E)) \\ &= 1. \end{aligned}$$

Hence, the proposition follows.

□

The interest of studying surfaces in the linear system $|-\frac{1}{2}K_X|$ is shown in the following proposition.

Proposition 2.2.2. *Every non-singular surface $\Sigma \in |-\frac{1}{2}K_X|$ is a non-singular del Pezzo surface whose degree is the same as* $\deg(X)$.

Proof. As we have seen in Remark 1.1.6, every non-singular surface $\Sigma \in |-\frac{1}{2}K_X|$ is a non-singular del Pezzo surface.

Moreover, by definition, the degree of del Pezzo surface Σ is equal to K_Σ^2 and degree of del Pezzo space X is equal to $\frac{1}{8}(-K_X)^3$. The adjunction formula implies that

$$K_\Sigma^2 = \frac{1}{4}K_X^2.[\Sigma].$$

Since $[\Sigma] = -\frac{1}{2}K_X$, we deduce that $\deg(\Sigma) = \deg(X)$.

□

We now show that one can construct a pencil of surfaces from a particular given elliptic curve.

Proposition 2.2.3. *Let E be a non-singular elliptic curve in X realising the homology class $\frac{1}{4}K_X^2$ in $H_2(X;\mathbb{Z})$ and contained in a non-singular surface $\Sigma \in |-\frac{1}{2}K_X|$.*

Then, E is the base locus of a pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$.

Proof. By adjunction formula, the elliptic curve E realizes the class $-K_\Sigma \in H_2(\Sigma;\mathbb{Z})$. It is sufficient to prove that E is the complete intersection of Σ with another non-singular surface $\Sigma' \in |-\frac{1}{2}K_X|$. In other words, we prove that

$$H^0(X, \mathcal{O}_X(-\frac{1}{2}K_X)|_\Sigma) = H^0(\Sigma, \mathcal{O}_\Sigma(-K_\Sigma)).$$

It is evident that $H^0(\Sigma, \mathcal{O}_\Sigma(-K_\Sigma)) \subset H^0(X, \mathcal{O}_X(-\frac{1}{2}K_X)|_\Sigma)$ as subspace. We then prove that these two vector spaces have the same dimension. We consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(0) \longrightarrow \mathcal{O}_X(-\frac{1}{2}K_X) \longrightarrow \mathcal{O}_X(-\frac{1}{2}K_X)|_\Sigma \longrightarrow 0$$

which induces the long exact sequence of cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{O}_X(0)) \arrow[r] & H^0(X, \mathcal{O}_X(-\frac{1}{2}K_X)) \arrow[r] & H^0(X, \mathcal{O}_X(-\frac{1}{2}K_X)|_\Sigma) \arrow[dll, "\delta"] \\ & H^1(X, \mathcal{O}_X(0)) \arrow[r] & \cdots & \end{tikzcd}$$

We have the following facts (see [Isk+10], Section 3.2).

- Firstly, we have that $\dim(H^0(X, \mathcal{O}_X(0))) = 1$ and $H^1(X, \mathcal{O}_X(0)) = 0$.
- Secondly, by Riemann-Roch theorem, we have

$$\dim H^0(X, \mathcal{O}_X(-\frac{1}{2}K_X)) = (-\frac{1}{2}K_X)^3 + 3 - 1 = \deg(X) + 2,$$

  and

$$\dim H^0(\Sigma, \mathcal{O}_\Sigma(-K_\Sigma)) = (-K_\Sigma)^2 + 2 - 1 = \deg(\Sigma) + 1.$$

Therefore, we deduce that

$$\dim H^0(X, \mathcal{O}_X(-\frac{1}{2}K_X)|_\Sigma) = \dim H^0(X, \mathcal{O}_X(-\frac{1}{2}K_X)) - 1 = \deg(X) + 1.$$

Remark that $\deg(X) = \deg(\Sigma)$. As a consequence, we obtain the identification $H^0(X, \mathcal{O}_X(-\frac{1}{2}K_X)|_\Sigma) = H^0(\Sigma, \mathcal{O}_\Sigma(-K_\Sigma))$.

Thus, the statement follows.

$\square$

Proposition 2.2.4. *For every $\Sigma \in |-\frac{1}{2}K_X|$ non-singular, one has a surjective map*

$$\psi : H_2(\Sigma; \mathbb{Z}) \to H_2(X; \mathbb{Z}),$$

which is induced by the inclusion of Σ in X.

In particular, if $X \in \{\mathbb{C}P^3, \mathbb{C}P^3 \sharp \overline{\mathbb{C}P^3}, \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1\}$ then $\ker(\psi) \cong \mathbb{Z}$.

Proof. One has the divisor $-\frac{1}{2}K_X$ is ample. Hence, the statement follows by Lefschetz hyperplane theorem (see for example Theorem C.8 in [EH16]).

In particular,

- if $X = \mathbb{C}P^3$ then $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$ and $H_2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^2$ for any $\Sigma \in |-\frac{1}{2}K_X|$ non-singular;
- if $X = \mathbb{C}P^3 \sharp \overline{\mathbb{C}P^3}$ then $H_2(X; \mathbb{Z}) \cong \mathbb{Z}^2$ and $H_2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^3$ for any $\Sigma \in |-\frac{1}{2}K_X|$ non-singular;

— if $X = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ then $H_2(X;\mathbb{Z}) \cong \mathbb{Z}^3$ and $H_2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^4$ for any $\Sigma \in |-\frac{1}{2}K_X|$ non-singular.

By the first isomorphism theorem for groups, we have $H_2(X;\mathbb{Z}) \cong H_2(\Sigma;\mathbb{Z})/\ker(\psi)$. Hence, if $X \in \{\mathbb{C}P^3, \mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}, \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1\}$ and $\Sigma \in |-\frac{1}{2}K_X|$ is a non-singular surface then $\ker(\psi) \cong \mathbb{Z}$. $\square$

In the next lemma, we show that the set of singular surfaces in a generic pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$ is finite and its cardinality is divisible by 4.

Lemma 2.2.5. *Let X be a three-dimensional del Pezzo variety. There are finitely many singular surfaces in a generic pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$.*

In particular, if $\deg(X) \in \{5,6,7,8\}$ then there are exactly four such singular surfaces.

Proof. Let $\mathcal{Q}$ be a generic pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$ whose base locus is denoted by E. We proved that E is an elliptic curve of class $\frac{1}{4}K_X^2$. Let N be the number of singular elements in this pencil.

Let $\tilde{X}$ be the blown-up of X along the elliptic curve E. We will compute N by comparing the Euler characteristic of $\tilde{X}$ in two equivalent ways.

1. On the one hand, we have $\chi(\tilde{X}) = \chi(X)$ since the Euler characteristic of the blown-up locus is $\chi(E) = 0$.
2. On the other hand, we consider $\tilde{X}$ as blown-up of the pencil $\mathcal{Q}$ along its base locus E. It is known that the space of all algebraic surfaces realising the homology class $-\frac{1}{2}K_X \in H_2(X;\mathbb{Z})$ forms a projective space $\mathbb{C}P^m$, where m is $\dim(H^0(X, \mathcal{O}_X(-\frac{1}{2}K_X)))$. And the set of singular elements in the linear system $|-\frac{1}{2}K_X|$, which we denote by Δ, forms a hypersurface in $\mathbb{C}P^m$ (also known as *projective discriminant hypersurface*). Therefore, the number N is also equal to the intersection numbers of the general projective line defined by $\mathcal{Q}$ and the discriminant hypersurface Δ in $\mathbb{C}P^m$. Let $\Sigma_{t_1}, \Sigma_{t_2}, \ldots, \Sigma_{t_N}$ be N singular surfaces of the pencil $\mathcal{Q}$. Then, $\tilde{X} - \{\bigcup_{i=1}^{N} \Sigma_{t_i}\}$ is a fibration over $\mathbb{C}P^1 - \{t_1, \ldots, t_N\}$, whose fibres are non-singular del Pezzo surfaces. Let Σ be one of such fibres. We have

$$\chi(\tilde{X} - \{\bigcup_{i=1}^{N} \Sigma_{t_i}\}) = \chi(\mathbb{C}P^1 - \{t_1, \ldots, t_N\}) \times \chi(\Sigma).$$

This implies that

$$\chi(\tilde{X}) = (2-N)\chi(\Sigma) + \sum_{i=1}^{N} \chi(\Sigma_{t_i}). \ (*)$$

Let $d := \deg(\Sigma)$ $(1 \leq d \leq 9)$. It is known that the Euler characteristic of Σ is $\chi(\Sigma) = 12 - d$, for $1 \leq d \leq 9$. Since $\mathcal{Q}$ is a Lefschetz pencil (Remark 1.1.7), then

singular del Pezzo surfaces have only one double point and no other singularities. Therefore, the Euler characteristic of these singular surfaces is $\chi(\Sigma_{t_i}) = 11 - d$, for $1 \le d \le 9$.
Hence, the equality $(*)$ becomes

$$\chi(\tilde{X}) = (2 - N)(12 - d) + N(11 - d).$$

As a consequence from two ways of computing $\chi(\tilde{X})$ as above, we have:

$$\chi(X) = 24 - 2d - N. \ (**)$$

The values of $\chi(X)$ and N inducing from the equality $(**)$ are shown in Table 2.1, for $d \in \{2, \ldots, 8\}$.

degree d	$\chi(X)$	N
2	-100	120
3	-6	24
4	0	16
5	10	4
6	8	4
7	6	4
8	4	4

Table 2.1 – Euler characteristic $\chi(X)$ of three-dimensional del Pezzo varieties of degree from 2 to 8 and the number of singular elements in a generic pencil of surfaces in $|-\frac{1}{2}K_X|$.

The statement then follows from the calculations in this table. □

Example 2.2.6. In the case case $X = \mathbb{C}P^3$ (see for example [BG16]), a non-singular element in the linear system $|-\frac{1}{2}K_{\mathbb{C}P^3}|$ is a non-singular quadric surface in $\mathbb{C}P^3$, hence is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. Moreover, there are exactly four singular elements in a generic pencil of quadrics in this linear system, which are quadratic cones in $\mathbb{C}P^3$.

2.2.2 Monodromy

We use the same notations as in Lemma 2.2.5. Let $\Sigma_{t_1}, \Sigma_{t_2}, \ldots, \Sigma_{t_N}$ be N singular elements of the Lefschetz pencil $\mathcal{Q}$ (whose base locus is E).

One has that blow-up of X along the base locus E of the Lefschetz pencil $\mathcal{Q}$ is a Lefschetz fibration over a Riemann sphere, whose non-singular fibres are non-singular del Pezzo surfaces. We denote this Lefschetz fibration by $f : \tilde{X} \to \mathbb{C}P^1$. We suppose that N critical points of f have corresponding images $t_1, \ldots, t_N$ in $\mathbb{C}P^1$.

Let $t \in \mathbb{C}P^1 - \{t_1, \ldots, t_N\}$. Let $u_i : [0, 1] \to \mathbb{C}P^1$ be a path which joins the regular value t with some critical value t_i, i.e. $u_i(0) = t, u_i(1) = t_i$, and does not pass through any other critical values of f. For each $i \in \{1, \ldots, N\}$, there is a map from a non-singular fibre $\Sigma = f^{-1}(t)$ to a singular one $\Sigma_{t_i} = f^{-1}(t_i)$ which is homeomorphic everywhere except along a 2-dimensional embedded (Lagrangian) sphere δ_i in Σ, which gets collapsed to the single node of Σ_{t_i}. Let $\gamma_i : H_2(\Sigma; \mathbb{Z}) \to H_2(\Sigma_{t_i}; \mathbb{Z})$ be a homomorphism arising on the homology groups of fibres of f (when we go along the path u_i).

Definition 2.2.7. A homology class $S_i \in H_2(\Sigma; \mathbb{Z})$ such that S_i is a generator of $\ker(\gamma_i)$ is called a **vanishing cycle** (along the path u_i) of the Lefschetz fibration f.

By definition, the class S_i is represented by the 2-dimensional embedded sphere δ_i.

A loop w_{u_i} in the fundamental group $\pi_1(\mathbb{C}P^1 - \{t_1, \ldots, t_N\}, t)$ is called **simple corresponding to the path** u_i if it is represented as a loop going along the path u_i from the point t to near the point t_i, going around the point t_i in the positive direction and returning to the point t along the path u_i (see Figure 2.1). It can be shown that the fundamental group $\pi_1(\mathbb{C}P^1 - \{t_1, \ldots, t_N\}, t)$ is a free group of $N-1$ generators. By Picard–Lefschetz repesentation, a generator w_{u_i} of the group $\pi_1(\mathbb{C}P^1 - \{t_1, \ldots, t_n\}, t)$ can map to a Dehn twist T_{S_i} along the vanishing cycle S_i of the mapping class group of the non-singular fibre Σ (see Proposition 2.3 in [AS08]).

Moreover, this Dehn twist acts on homology group $H_2(\Sigma; \mathbb{Z})$ by Picard–Lefschetz formula as follows:

$$\begin{aligned} T_{S_{i*}} : H_2(\Sigma; \mathbb{Z}) &\to H_2(\Sigma; \mathbb{Z}) \\ D &\mapsto D + (D.S_i)S_i. \end{aligned}$$

The homomorphism $T_{S_{i*}}$ is called the **monodromy transformation** on $H_2(\Sigma; \mathbb{Z})$ corresponding to the vanishing cycle S_i.

One can deduce that $T_{S_{i*}}(S_i) = -S_i$ and $S_i^2 = -2$, see Lemma 1.4 in [AVGZ12]. Moreover, the monodromy transformation does not depend on the choice of either S_i or $-S_i$, i.e. $T_{(-S_i)_*}(D) = T_{S_{i*}}(D)$.

Lemma 2.2.8. *Let X be a 3-dimensional del Pezzo variety such that*

$$\ker(\psi : H_2(\Sigma; \mathbb{Z}) \to H_2(X; \mathbb{Z})) \cong \mathbb{Z}$$

where $\Sigma \in |-\frac{1}{2}K_X|$ is a non-singular surface. Then, the group $\ker(\psi)$ is generated by a vanishing cycle S_i, for every $i \in \{1, 2, \ldots, N\}$.

Proof. By definition, a vanishing cycle $S_i \in H_2(\Sigma; \mathbb{Z})$ is such that $\gamma_i(S_i) = 0 \in H_2(\Sigma_{t_i}; \mathbb{Z})$, where γ_i is the homomorphism corresponding to the path u_i. Since $\Sigma_{t_i} \hookrightarrow X$, one can deduce that $\gamma_i(S_i) = 0 \in H_2(X; \mathbb{Z})$. Thus, $S_i \in \ker(\psi)$.

Suppose that S is a generator of $\ker(\psi) \cong \mathbb{Z}$, we can write $S_i = kS$ where $k \in \mathbb{Z}$. Since $S_i^2 = -2$, we obtain $k^2 = 1$ and hence $k \in \{\pm 1\}$. This implies that $S_i = \pm S$ for any $i \in \{1, 2, \ldots, N\}$, and then completes the proof. ∎

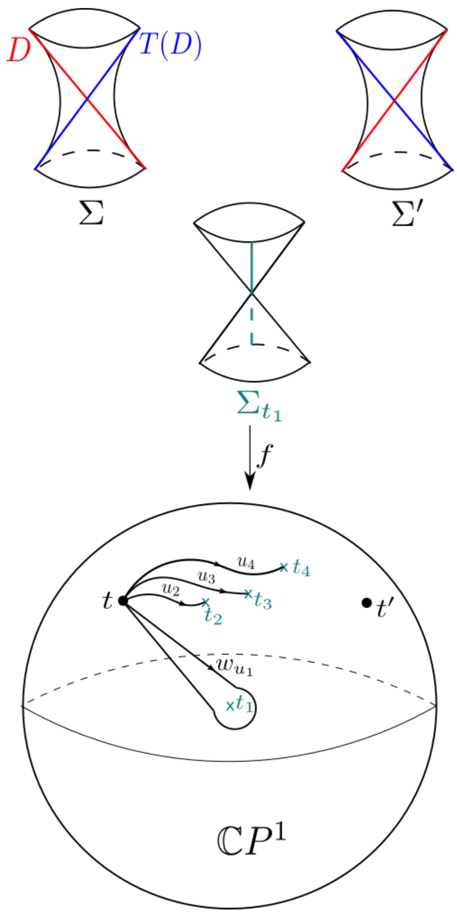

Figure 2.1 – Example of monodromy transformation where the general fibres Σ are quadric surfaces and the singular ones are quadric cones in $\mathbb{C}P^3$.

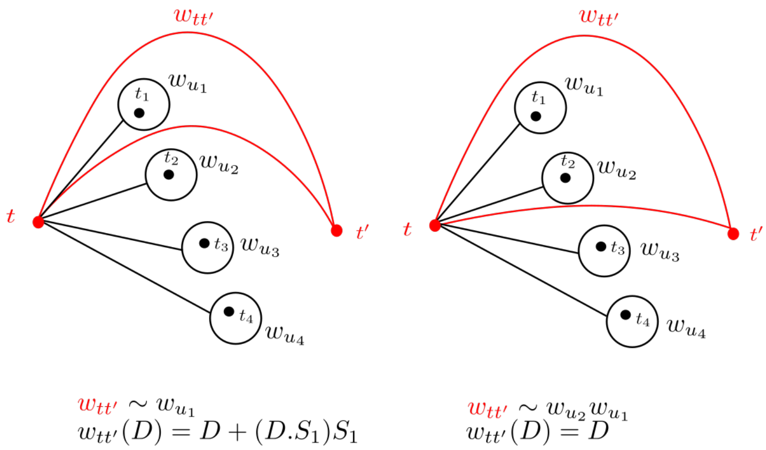

Figure 2.2 – Two homotopy classes of a loop $w_{tt'} \in \pi_1(\mathbb{C}P^1 - \{t_1, \dots, t_4\}, t)$ and their corresponding actions on $H_2(\Sigma; \mathbb{Z})$.

Suppose now that X is as in Lemma 2.2.8. In this case, we can identify S_i with a generator S of $\ker(\psi)$, for any $i \in \{1, 2, \dots, N\}$, when studying about monodromy transformations. In the sequel, we simply denote by T instead of $T_{S_{i*}}$ for the monodromy transformation on $H_2(\Sigma; \mathbb{Z})$, i.e. $T(D) := D + (D.S)S$.

We define the following equivalent relation $\sim$ on $H_2(\Sigma; \mathbb{Z})$ (where $\Sigma \in \mathcal{Q}$ non-singular):

$$D_1 \sim D_2 \text{ if } D_2 = D_1 + (D_1.S)S.$$

We denote by $\overline{D}$ the equivalence class of D in $H_2(\Sigma; \mathbb{Z})/_\sim$.

Lemma 2.2.9. *Let Σ and Σ' be non-singular elements in the pencil $\mathcal{Q}$. Then the Lefschetz fibration f (see the beginning of this section) induces the following canonical isomorphism*

$$H_2(\Sigma; \mathbb{Z})/_\sim \;\cong H_2(\Sigma'; \mathbb{Z})/_\sim.$$

Proof. Let $t, t' \in \mathbb{C}P^1 - \{t_1, \dots, t_N\}$ such that $\Sigma = f^{-1}(t)$ and $\Sigma' = f^{-1}(t')$.

Let $w_{tt'} \in \pi_1(\mathbb{C}P^1 - \{t_1, \dots, t_N\}, t)$ be a loop passing through the regular value t'.

We first show that the automorphisms of $H_2(\Sigma; \mathbb{Z})$, which arise when we go along the loop $w_{tt'}$, depend on the homotopy class of this loop. Indeed, since $N \equiv 0 \mod 2$ (see Table 2.1), there are two homotopy classes of loops in $\pi_1(\mathbb{C}P^1 - \{t_1, \dots, t_N\}, t)$ characterised by the parity of critical values on both sides of each loop: one corresponds to the automorphism given by $T(D) = D \pm (D.S)S$, the other corresponds to the identity automorphism $T(D) = D$ of $H_2(\Sigma; \mathbb{Z})$ (see Figure 2.2).

As a consequence, the map arising (when we go from t to t' along the loop $w_{tt'}$) on the homology groups of the corresponding fibres implies a canonical isomorphism $H_2(\Sigma; \mathbb{Z})/_\sim \cong H_2(\Sigma'; \mathbb{Z})/_\sim$.

□

Example 2.2.10. In the case $X = \mathbb{C}P^3$, we consider the Lefschetz fibration on $\mathbb{C}P^1$ whose non-singular fibres are isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Let $(L_1, L_2) = ([\mathbb{C}P^1 \times \{pt\}], [\{pt\} \times \mathbb{C}P^1])$ be a basis of $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$. Let the vanishing class $S = L_1 - L_2$ in $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$. It is straightforward that two homology classes $aL_1 + bL_2$ and $bL_1 + aL_2$ differ up to monodromy transformation, for every $(a, b) \in \mathbb{Z}^2$. In other words, one has $bL_1 + aL_2 = T(aL_1 + bL_2)$.

Therefore, for any non-singular fibre Σ in this fibration, we can identity $H_2(\Sigma; \mathbb{Z})/_{\sim}$ with $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})/_{aL_1+bL_2 \sim bL_1+aL_2}$.

In the sequel, we will use interchangeably the notations of homology class in $H_2(\Sigma; \mathbb{Z})$ and divisor class in $Pic(\Sigma)$, where Σ is a del Pezzo surface. We denote the restriction of a divisor D in $Pic(\Sigma)$ to $Pic(E)$ by $D|_{E,\Sigma} \in Pic_{D.E}(E)$, where $D.E \in \mathbb{Z}$ is the intersection product of homology classes D and $[E]$ in $H_2(\Sigma; \mathbb{Z})$.

Lemma 2.2.11. *The restriction of the vanishing cycle to E is in $Pic_0(E)$.*

Proof. Since the base locus E of the pencil $\mathcal{Q}$ satisfies $[E] = -K_\Sigma$ for any $\Sigma \in \mathcal{Q}$ non-singular, it is sufficient to prove that $K_\Sigma.S = 0$. Indeed, there exists a degeneration of del Pezzo surfaces into weak ones (i.e. non-singular surfaces with nef and big anti-canonical divisor, on which curve has self intersection number at least -2), on which the class S is represented by a non-singular rational curve of self-intersection -2. By definition of -2-curve, the equality $K_\Sigma.S = 0$ then follows (see Chapter III.6 in [Deg+00]). □

Since $S.E = 0$, for every divisor $D \in Pic(\Sigma)$, the restrictions $D|_{E,\Sigma}$ and $T(D)|_{E,\Sigma}$ are in $Pic_{D.E}(E)$. Given any divisor D and its monodromy transformation $T(D)$, we can construct the following holomorphic map:

$$\begin{aligned} \phi : \mathcal{Q} &\longrightarrow Pic_{2D.E}(E) \\ \Sigma_t &\longmapsto (D + T(D))|_{E,\Sigma_t} \end{aligned}$$

Lemma 2.2.12. *One has ϕ is a constant map.*

Proof. This lemma follows directly from the fact that compact Riemann surfaces can be mapped holomorphically to surfaces of lower genus, but not to higher genus, except as constant maps. □

We denote the image of the constant map ϕ in Lemma 2.2.12 by

$$\phi(\overline{D}) := (D + T(D))|_{E,\Sigma_t} = 2D|_{E,\Sigma_t} + (D.S)S|_{E,\Sigma_t}.$$

With abuse of notation, we also use $\sim$ for an equivalent relation on $Pic_{D.E}(E)$, which is given by $x \sim \phi(\overline{D}) - x$. We denote by $\overline{x}$ the equivalence class of x in $Pic_{D.E}(E)/_{x \sim \phi(\overline{D}) - x}$. The equality $D|_{E,\Sigma_t} = \phi(\overline{D}) - T(D)|_{E,\Sigma_t}$ then implies that $\overline{D|_{E,\Sigma_t}} = \overline{T(D)|_{E,\Sigma_t}}$. Hence, we can define the following holomorphic map between Riemann spheres:

$$\begin{aligned} \phi_{\overline{D}} : \mathcal{Q} &\longrightarrow Pic_{D.E}(E)/_{x \sim \phi(\overline{D}) - x} \\ \Sigma_t &\longmapsto \overline{D|_{E,\Sigma_t}} \end{aligned}$$

Lemma 2.2.13. *There exists $\{L, T(L)\} \subset H_2(\Sigma;\mathbb{Z})$ such that $(L.S)^2 = 1$.*
In particular, classes L and $T(L)$ can be chosen to be both effective.

Proof. According to Lemma 2.2.11 and Claim (6) in page 26 of [DPT06], we deduce that the vanishing cycle S can be written as $L_1 - L_2$ where $(L_1, L_2, L_3, \ldots, L_k)$ is a k-uplet of effective generators of $H_2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^k$. The surjection $\psi : H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z}) =< \tilde{L}_{12}, \tilde{L}_3, \ldots, \tilde{L}_k >$ is then given by $\psi(L_1 - L_2) = 0;\ \psi(L_1 + L_2) = \tilde{L}_{12};\ \psi(L_i) = \tilde{L}_i$ for $i \in \{3, \ldots, k\}$.

We first prove that there exists a class L such that $L.S = 1$. Let $L = \sum_{i=1}^{k} a_i L_i$. One has $L^2 = (L+S)^2$ since $S^2 = -2$. This implies that

$$(a_1 L_1 + a_2 L_2)^2 \equiv ((a_1+1)L_1 + (a_2-1)L_2)^2 \mod 2.$$

Hence, a_1 and a_2 satisfy the following equation

$$a_1 L_1^2 - a_2 L_2^2 + (a_2 - a_1) L_1.L_2 \equiv 1 \mod 2.$$

The intersection form on $H_2(\Sigma;\mathbb{Z})$ implies that $a_2 \equiv a_1 + 1 \mod 2$. Thus, the class L is of the form $a_1 L_1 + (a_1+1)L_2 + a_3 L_3 + \ldots + a_k L_k$, where $a_i \in \mathbb{Z}$.

In particular, if $a_i = 0$ for all i, then $L = L_2$ is effective, and so is $T(L) = L_1$. $\square$

Lemma 2.2.14. *One has $\deg(\phi_{\overline{S}}) = 4$.*

Proof. By Lemma 2.2.13, a couple of effective divisors $L|_E$ and $T(L)|_E$ determines a surface (non-singular if $L|_E \neq T(L)|_E$ and singular if $L|_E = T(L)|_E$) in a pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$ whose base locus is E. We have the following commutative diagram:

$$\begin{tikzcd}
Pic_{L.E}(E) \arrow[r, "\pi:\, x \mapsto \Sigma_x"] \arrow[d, "2:1"] & \mathcal{Q} \arrow[dl, "\cong"] \\
Pic_{L.E}(E)/_{x \sim \phi(\overline{L}) - x} &
\end{tikzcd}$$

where π is a ramified double covering map. Note that $\Sigma_{\phi(\overline{L})-x} = \Sigma_x$. By computing the ramification values of the map π, we get exactly 4 singular surfaces in the pencil $\mathcal{Q}$. (This is compatible with Lemma 2.2.5).

Then, we have the following commutative diagram

$$\begin{tikzcd}
 & \mathcal{Q} \arrow[dl, "\cong"'] \arrow[d, "\phi_{\overline{S}}"] \\
Pic_{L.E}(E)/_{x \sim \phi(\overline{L}) - x} \arrow[r, "\overline{x} \mapsto \phi(\overline{L}) - 2x"'] & Pic_0(E)/_{y \sim -y}
\end{tikzcd}$$

From these diagrams, we deduce that each singular surface in the pencil $\mathcal{Q}$ corresponds to an element of the set $\{x \in Pic_{L.E}(E) : x = \phi(\overline{L}) - x\}$, which is sent to $\overline{0} \in Pic_0(E)/_{y\sim -y}$. Hence, the degree of the map $\phi_{\overline{S}}$ is exactly the number of singular surfaces in the pencil $\mathcal{Q}$, which is equal to 4.

□

Corollary 2.2.15. *Let $\{L, T(L)\} \subset H_2(\Sigma;\mathbb{Z})$ be such that $(L.S)^2 = 1$. Then, the corresponding map $\phi_{\overline{L}}$ is an isomorphism.*

The following proposition is the main point for a proof of the 1-*dimensional problem* that we mentioned in the beginning of this chapter.

Proposition 2.2.16. *Let X be a 3-dimensional del Pezzo variety such that*

$$\ker(\psi : H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z})) \cong \mathbb{Z}$$

where Σ is a non-singular surface in $\mathcal{Q}$, which is a pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$ whose base locus is E. Let $S \in \ker(\psi)$ be a generator.
Then, the map $\phi_{\overline{D}}$ has degree $\deg(\phi_{\overline{D}}) = (D.S)^2$.

Proof. We consider the following diagram of holomorphic maps between Riemann spheres:

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "\phi_{\overline{D}}"] \arrow[d, "\phi_{\overline{S}}"] & Pic_{D.E}(E)/_{x\sim\phi(\overline{D})-x} \arrow[d, "\theta_1"] \\
Pic_0(E)/_{y\sim -y} \arrow[r, "\theta_2"] & Pic_0(E)/_{y\sim -y}
\end{tikzcd} \qquad (I)$$

where

$$\theta_1(\overline{x}) = \overline{\phi(\overline{D}) - 2x}\,;$$
$$\theta_2(\overline{y}) = (D.S)\overline{y}.$$

The diagram (I) is commutative. Indeed, given $\Sigma_t \in \mathcal{Q}$, one has:

$$\theta_1(\phi_{\overline{D}}(\Sigma_t)) = \theta_1(\overline{D|_{E,\Sigma_t}}) = \overline{\phi(\overline{D}) - 2D|_{E,\Sigma_t}} = (D.S)\overline{S|_{E,\Sigma_t}};$$
$$\theta_2(\phi_{\overline{S}}(\Sigma_t)) = \theta_2(\overline{S|_{E,\Sigma_t}}) = (D.S)\overline{S|_{E,\Sigma_t}}.$$

Thus, $\theta_1(\phi_{\overline{D}}(\Sigma_t)) = \theta_2(\phi_{\overline{S}}(\Sigma_t))$ for every $\Sigma_t \in \mathcal{Q}$.
Therefore, we have the following equality

$$\deg(\phi_{\overline{D}}) \times \deg(\theta_1) = \deg(\phi_{\overline{S}}) \times \deg(\theta_2).$$

Claim 1. One has $\deg(\theta_2) = (D.S)^2$.

Proof. Let $\overline{y_1} \in Pic_0(E)/_{y\sim -y}$ (the quotient which is in the corner of Diagram (I)). One has

$$\deg(\theta_2) = |\{\overline{y} \in Pic_0(E)/_{y\sim -y} \;:\; (D.S)\overline{y} = \overline{y_1}\}|.$$

The following commutative diagram

$$\begin{array}{ccc} Pic_0(E) & \xrightarrow{y\mapsto (D.S)y} & Pic_0(E) \\ \Big\downarrow{\scriptstyle 2:1} & & \Big\downarrow{\scriptstyle 2:1} \\ Pic_0(E)/_{y\sim -y} & \xrightarrow{\theta_2} & Pic_0(E)/_{y\sim -y} \end{array}$$

implies that $\deg(\theta_2)$ is exactly the number of solutions $y \in Pic_0(E)$ of the equation $(D.S)y = y_1$ in $Pic_0(E)$ where y_1 is a given divisor. Any two solutions y of this equation differ by a torsion point of order $D.S$, then this equation has exactly $(D.S)^2$ solutions y. Hence, the degree of the map θ_2 is equal to $(D.S)^2$. □

Claim 2. One has $\deg(\theta_1) = 4$.

Proof. We apply the same argument as in the previous claim.

Indeed, let $\overline{y_1} \in Pic_0(E)/_{y\sim -y}$ (the quotient which is in the corner of Diagram (I)). One has

$$\deg(\theta_1) = |\{\overline{x} \in Pic_{D.E}(E)/_{x\sim\phi(\overline{D})-x} \;:\; \overline{\phi(\overline{D}) - 2x} = \overline{y_1}\}|.$$

The following commutative diagram

$$\begin{array}{ccc} Pic_{D.E}(E) & \xrightarrow{x\mapsto \phi(\overline{D})-2x} & Pic_0(E) \\ \Big\downarrow{\scriptstyle 2:1} & & \Big\downarrow{\scriptstyle 2:1} \\ Pic_{D.E}(E)/_{x\sim\phi(\overline{D})-x} & \xrightarrow{\theta_1} & Pic_0(E)/_{y\sim -y} \end{array}$$

implies that $\deg(\theta_1)$ is exactly the number of solutions $x \in Pic_{D.E}(E)$ of the equation $\phi(\overline{D}) - 2x = y_1$ in $Pic_0(E)$ where y_1 and $\phi(\overline{D})$ are given divisors. Any two solutions x of this equation differ by a torsion point of order 2, then this equation has exactly 2^2 solutions x. Hence, the degree of the map θ_1 is equal to 4. □

Claim 3. One has $\deg(\phi_{\overline{S}}) = 4$, see Lemma 2.2.14.

Conclusion: **Claim 1**, **Claim 2** and **Claim 3** imply that the degree of the map $\phi_{\overline{D}}$ is exactly equal to $(D.S)^2$. □

2.3 Rational curves on del Pezzo varieties of dimension three and two: relationship

Recall from the last section that a non-singular element $\Sigma \in |-\frac{1}{2}K_X|$ is a non-singular del Pezzo surface where X is a del Pezzo variety of dimension three. Recall also that $\psi : H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z})$ is a surjective map induced by the inclusion $\Sigma \hookrightarrow X$.

In this section, we also use the notation $\mathcal{Q}$ for a pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$, whose base locus is an elliptic curve E of class $\frac{1}{4}K_X^2$. Let $d \in H_2(X;\mathbb{Z})$ and $k_d = \frac{1}{2}c_1(X)\cdot d \in \mathbb{Z}$. Let $D \in H_2(\Sigma;\mathbb{Z})$ and $k_D = c_1(\Sigma)\cdot D - 1 \in \mathbb{Z}$. Throughout this section, we study a degenerate situation of configuration of points, which is contained in the elliptic curve E. We first prove a generalization of a result of Kollár (Proposition 2.3.2). Then, we prove that this is a "good" degeneration, meaning that this actually gives us genus-0 Gromov-Witten invariants of X (Proposition 2.3.6, Corollary 2.3.7 and Proposition 2.3.8) and allows us to relate $GW_X(d)$ to $GW_\Sigma(D)$.

2.3.1 Degeneration of configuration of points

Lemma 2.3.1. *For every 3-dimensional del Pezzo variety X, if $\Sigma \in |-\frac{1}{2}K_X|$ non-singular and $D \in \psi^{-1}(d)$ then one has $k_D = k_d - 1$.*

Proof. By the Poincaré duality $\frac{1}{2}c_1(X) = PD(-\frac{1}{2}K_X)$ and by hypothesis $\Sigma \in |-\frac{1}{2}K_X|$ is non-singular, one has

$$k_d = \frac{1}{2}c_1(X) \cdot d = [\Sigma].d.$$

By the adjunction formula, by hypothesis $D \in \psi^{-1}(d)$ and by the Poincaré duality $c_1(\Sigma) = PD(-K_\Sigma)$, one has

$$[\Sigma].d = -K_\Sigma.D = k_D + 1.$$

Hence, the lemma follows. □

Let $\underline{x}_d$ be a configuration of k_d distinct points in the base locus E of pencil $\mathcal{Q}$. Let $\mathcal{C}(\underline{x}_d)$ be the set of *connected algebraic curves of arithmetic genus zero representing the homology class* d *and passing through* $\underline{x}_d$.

The following proposition generalizes Proposition 3 by J. Kollár in [Kol15] for the case $\mathbb{C}P^3$.

Proposition 2.3.2. *Suppose that the k_d points of $\underline{x}_d$ are in general position in E. Then every curve in the set $\mathcal{C}(\underline{x}_d)$ is irreducible and is contained in some non-singular surface of the pencil $\mathcal{Q}$.*

Proof. Let C_d be a curve in the set $\mathcal{C}(\underline{x}_d)$. We then prove the following claims.

Claim 1: *Suppose that C_d is irreducible. Then, the whole curve C_d has to be contained in some non-singular surface of the pencil $\mathcal{Q}$.*

Proof. The first observation is that any point of $C_d \setminus \underline{x}_d$ is contained in some surface, which is denoted by Σ_t, of the pencil $\mathcal{Q}$. Then, the number of intersection points between C_d and Σ_t is $\sharp(C_d \cap \Sigma_t) > \frac{1}{2}c_1(X) \cdot d$. By Bézout's theorem, the whole curve C_d has to be contained in this surface.

We show that such surface Σ_t is non-singular if the chosen points of $\underline{x}_d$ are generic enough. Indeed, suppose that the curve C_d (passing through the configuration $\underline{x}_d$ in E) is contained in a singular surface Σ_{sing} of the pencil $\mathcal{Q}$. Let $rank(Pic(\Sigma_{sing})) = k - 1$, $k \geq 2$. Note that $rank(Pic(\Sigma_{sing})) = rank(Pic(\Sigma)) - 1$, where Σ is a non-singular surface in $\mathcal{Q}$. Let $(L_1, L_2 \ldots, L_k)$ be a basis of $H_2(\Sigma; \mathbb{Z})$, such that the vanishing cycle $S = L_1 - L_2 \in H_2(\Sigma; \mathbb{Z})$ and $T(L_1) = L_2, T(L_2) = L_1, T(L_i) = L_i \ \forall i \in \{3, \ldots, k\}$

for T is the monodromy transformation. Let $(\tilde{L_{12}}, \tilde{L_3} \ldots, \tilde{L_k})$ be a basis of $H_2(\Sigma_{sing}; \mathbb{Z})$. Consider a surjective homomorphism given by

$$\begin{aligned} \gamma : H_2(\Sigma; \mathbb{Z}_2) &\to H_2(\Sigma_{sing}; \mathbb{Z}) \\ L_1 - L_2 &\mapsto 0 \\ L_1 + L_2 &\mapsto \tilde{L_{12}} \\ L_i &\mapsto \tilde{L_i}, \forall i \in \{3, \ldots, k\}. \end{aligned}$$

Suppose that the curve C_d realises homology class

$$D = \alpha_1 L_1 + \alpha_1 L_2 + \alpha_3 L_3 \ldots + \alpha_k L_k \in H_2(\Sigma; \mathbb{Z}).$$

One has $\gamma(D) = \alpha_1 \tilde{L_{12}} + \alpha_3 \tilde{L_3} + \ldots + \alpha_k \tilde{L_k} \in H_2(\Sigma_{sing}; \mathbb{Z})$. And the configuration $\underline{x}_d = C_d \cap E$ realises the divisor class $\gamma(D)|_E$ in $Pic_{k_d}(E)$. If we fix $k_d - 1$ first points on $\underline{x}_d \subset E^{k_d}$ and move the last point (still in E), then we get a new configuration $\underline{x}'_d$, such that $[\underline{x}'_d] \neq [\underline{x}_d]$. On the contrary, the divisor class $\gamma(D)|_E$ stays fixed. In other words, we can choose a new configuration $\underline{x}'_d \subset E^{k_d}$ in an open subset of E^{k_d} such that it represents a divisor $[\underline{x}'_d] \neq \gamma(D)|_E$ in $Pic_{k_d}(E)$. Thus, if $C_d \subset \Sigma_{sing}$ then the chosen configuration $\underline{x}_d$, through which C_d passes, is not generic. Moreover, there are only finitely many singular surfaces in the pencil $\mathcal{Q}$ (see Lemma 2.2.5) on which the curve C_d can not be contained. □

We now prove that C_d can not be reducible via four following claims. Assume that C_d can be decomposed into n ($n \geq 0$) irreducible rational curves C_{d_i}, i.e. $C_d = \bigcup\limits_{i=1}^{n} C_{d_i}$, where a curve C_{d_i} realises a homology class $d_i \in H_2(X; \mathbb{Z})$ such that $\sum\limits_{i=1}^{n} d_i = d$ and passes through $\underline{x}_{d_i}$ such that $\bigcup\limits_{i=1}^{n} \underline{x}_{d_i} = \underline{x}_d$.

Claim 2A: *Each component curve C_{d_i} passes though exactly $k_{d_i} = \frac{1}{2} c_1(X) \cdot d_i$ points of configuration $\underline{x}_d$.*

Proof. Suppose that there exists a component of C_d, noted by C_{d_1}, passing through more than $\frac{1}{2} c_1(X) \cdot d_1$ points of $\underline{x}_d$. Then the curve C_{d_1} intersects with every surface of the pencil $\mathcal{Q}$ at more than $\frac{1}{2} c_1(X) \cdot d_1$ points. Therefore, the curve C_{d_1} must be contained in every surface of the pencil $\mathcal{Q}$ (because of Bézout's theorem). This means that $C_{d_1} = E$, then C_d contains E, this is impossible since E is of higher genus than C_d. Hence, every component curve C_{d_i} passes through a configuration $\underline{x}_{d_i}$ of exactly k_{d_i} points of $\underline{x}_d$. As a consequence, different C_{d_i} passes though different points of $\underline{x}_d$ for all i, i.e. $\bigsqcup\limits_{i=1}^{n} \underline{x}_{d_i} = \underline{x}_d$ and $\sum\limits_{i=1}^{n} k_{d_i} = k_d$ where k_d, k_{d_i} are the associated integers of classes d and d_i. □

Claim 2B: *The reducible curve $C_d = \bigcup\limits_{i=1}^{n} C_{d_i}$ is contained in one non-singular surface of the pencil $\mathcal{Q}$.*

Proof. Note that the curve C_d is connected. If there exists the case where two component curves C_{d_i} and C_{d_j}, whose intersection is not empty, are contained in two different surfaces $\Sigma_i \neq \Sigma_j$ of the pencil $\mathcal{Q}$, then $\emptyset \neq C_{d_i} \cap C_{d_j} \subset \Sigma_i \cap \Sigma_j = E$. This implies that two different curves C_{d_i} and C_{d_j} have some points of $\underline{x}_d$ in common, contradiction to **Claim 2A**. Hence the reducible curve C_d is contained in only one surface of the pencil $\mathcal{Q}$. Moreover, such surface is non-singular by the genericity of the chosen configuration. One can suppose that $C_d \subset \Sigma_t \in \mathcal{Q}$ and one can write its homology class as $[C_d] = \sum\limits_{i=1}^{n} [C_{d_i}] \in H_2(\Sigma_t; \mathbb{Z})$. □

Claim 2C: *Let* $\Sigma \in \mathcal{Q}$*. If* $D \in H_2(\Sigma; \mathbb{Z}) \cong Pic(\Sigma)$ *is an effective divisor class then there exists only countably many decompositions of* D *into a sum of classes of effective divisors.*

Proof. Since Σ is a del Pezzo surface, one has $Pic(\Sigma) \cong \mathbb{Z}^k$ (i.e. torsion free) for some k ($1 \leq k \leq 9$). Suppose that D can be decomposed into a sum of classes of effective divisors as $\sum\limits_{i=1}^{N} D_i$ where $D_i \in \mathbb{Z}^k$ for all $i \geq 1$ and $N \in \mathbb{N}^*$. Then, the set of such decompositions, i.e. the set $\{(D_1, D_2, \ldots, D_N), D_i \in \mathbb{Z}^k, i \geq 1$ such that $\sum\limits_{i=1}^{N} D_i = D\}$, is contained in $\{(D_1, D_2, \ldots, D_N), D_i \in \mathbb{Z}^k, i \geq 1\} \subset \bigcup\limits_{N=1}^{\infty} (\mathbb{Z}^k)^N$. The later is countable then the statement follows. □

Claim 2D: *If the configuration* $\underline{x}_d$ *is generic then the curve* C_d *is irreducible.*

Proof. We prove this claim by contradiction. Fix an integer k ($0 < k < k_d$). Let $\underline{x}_{d_1} \subsetneq \underline{x}_d$ be a strict sub-configuration consisting of points in E such that its cardinality is equal to k. The reducibility and genericity of the configuration $\underline{x}_d$ implies that we can vary $\underline{x}_d$ in an open subset of E^{k_d} such that we get another configuration $\underline{x}'_d$ satisfying $[\underline{x}'_d] \sim [\underline{x}_d]$ and there exists $\underline{x}'_{d_1} \subsetneq \underline{x}'_d$ and $[\underline{x}_{d_1}] \sim [\underline{x}_{d_1}]$. By the Riemann-Roch theorem, every effective divisor linearly equivalent to $[\underline{x}_d]$ is determined by $k_d - 1$ points and that to $[\underline{x}_{d_1}]$ is determined by $k - 1$ points. By avoiding a set of configurations of $k_d - 1$ points in E which forms a subvariety in $E^{k_d - 1}$, we can vary $k_d - 1$ points in E until we get a new configuration $\underline{x}'_d$ and its sub-configuration $\underline{x}'_{d_1}$, such that $[\underline{x}'_d] \sim [\underline{x}_d]$ in $Pic_{k_d-1}(E)$ but $[\underline{x}'_{d_1}] \neq [\underline{x}_{d_1}]$ in $Pic_{k-1}(E)$. By **Claim 2B** and **Claim 2C**, there exists a non-singular surface $\Sigma_t \in \mathcal{Q}$, which contains the reducible curve C_d, and the effective divisor $[C_d]$ in $H_2(\Sigma_t; \mathbb{Z})$ has countably many decompositions into effective divisors. Therefore, the configurations satisfying the above variation can be chosen in an open subset of E^{k_d-1}. As a result, if the curve C_d is reducible, then we can only choose the configurations, through which C_d passes, in a measure zero subset, hence such configurations are not generic. □

The last claim finishes our proof of this proposition. □

Corollary 2.3.3. *If Σ_t is a non-singular surface of the pencil $\mathcal{Q}$ containing a curve in the set $\mathcal{C}(\underline{x}_d)$, then every irreducible rational curve on Σ_t which realizes homology class D in $H_2(\Sigma_t;\mathbb{Z})$ such that $D \in \psi_t^{-1}(d)$, where $\psi_t : H_2(\Sigma_t;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, and contains $(k_d - 1)$ points of $\underline{x}_d \subset E^{k_d}$ is an element of the set $\mathcal{C}(\underline{x}_d)$, (i.e. contains the whole $\underline{x}_d$).*

Proof. This statement follows directly from the Riemann-Roch theorem that every effective divisor linearly equivalent to divisor $[\underline{x}_d]$ is determined by $k_d - 1$ points in E. □

2.3.2 Moduli spaces and enumeration of balanced rational curves

In this technical part, we show that we can recover the genus-0 Gromov-Witten invariants of a three-dimensional del Pezzo variety X, and of a non-singular del Pezzo surface in $\Sigma \in |-\frac{1}{2}K_X|$ after specializing our configuration of points, see Section 5 in [BG16].

Moduli spaces of stable maps and balanced rational curves

Most of notations in this section are found in Chapter 1, Section 1.2.2.

We consider the moduli space $\mathcal{M}^*_{0,k_d}(X,d)$ of irreducible and simple genus-0 k_d-pointed stable maps $f : (\mathbb{C}P^1; p_1,\dots,p_{k_d}) \to X$ of homology class $d \in H_2(X;\mathbb{Z})$, and its corresponding total evaluation map ev_X.

Recall that, by the genericity of the configuration of points $\underline{x}_d = f(\{p_1,\dots,p_{k_d}\})$ in X, the genus-0 Gromov-Witten invariant $GW_X(d)$ can be identified with the enumeration of elements $[(\mathbb{C}P^1; p_1,\dots,p_{k_d}); f]$ in $\mathcal{M}^*_{0,k_d}(X,d)$ which are regular points of ev_X.

Lemma 2.3.4. *Let $f : \mathbb{C}P^1 \to X$ be an immersion such that $f(\mathbb{C}P^1) \subset \Sigma$. If $f(\mathbb{C}P^1)$ is not a complete intersection in X, then f is balanced.*

Proof. Let $\mathcal{N}_f = f^*TX/T\mathbb{C}P^1$ and $\mathcal{N}'_f = f^*T\Sigma/T\mathbb{C}P^1$ be normal bundles. We consider the following short exact sequence of holomorphic vector bundles over $\mathbb{C}P^1$:

$$0 \longrightarrow \mathcal{N}'_f \longrightarrow \mathcal{N}_f \longrightarrow f^*TX/T\Sigma \longrightarrow 0. \ (*)$$

Claim 1. The map f is balanced if and only if the short exact sequence $(*)$ does not split.

Suppose that $f_*[\mathbb{C}P^1] = D \in H_2(\Sigma;\mathbb{Z})$. We have seen that the rank 2 normal bundle $\mathcal{N}_f$ has degree $c_1(X)\cdot d - 2 = 2k_d - 2$ where $d = \psi(D) \in H_2(X;\mathbb{Z})$. By definition, we have f is balanced if the normal bundle $\mathcal{N}_f$ is isomorphic to $\mathcal{O}_{\mathbb{C}P^1}(k_d - 1) \oplus \mathcal{O}_{\mathbb{C}P^1}(k_d - 1)$. We also have an isomorphism $\mathcal{N}'_f \cong \mathcal{O}_{\mathbb{C}P^1}(c_1(\Sigma)\cdot D - 2)$ (see Example 1.2.10). This implies that the normal line bundle $\mathcal{N}'_f$ has degree $c_1(\Sigma)\cdot D - 2 = k_d - 2$. Hence, the normal line bundle $f^*TX/T\Sigma$ has degree $(2k_d - 2) - (k_d - 2) = k_d$. Therefore, the map f is balanced implies that the short exact sequence $(*)$ does not split. Conversely, if the short exact sequence $(*)$ does not split then f is balanced. Indeed, we suppose that

$\mathcal{N}_f \cong \mathcal{O}_{\mathbb{C}P^1}(k_1) \oplus \mathcal{O}_{\mathbb{C}P^1}(k_2)$ such that $k_1 > k_2$, i.e. $k_1 > c_1(\mathcal{N}'_f)$. This implies the null-morphism $\mathcal{O}_{\mathbb{C}P^1}(k_1) \to \mathcal{N}'_f$. Then, there is an isomorphism $\mathcal{O}_{\mathbb{C}P^1}(k_1) \to \mathcal{N}_f/\mathcal{N}'_f$, where $\mathcal{N}_f/\mathcal{N}'_f \cong \mathcal{O}_{\mathbb{C}P^1}(k_d)$. Hence, the short exact sequence $(*)$ splits (see page 252 [GH83]).

Claim 2. The exact sequence $(*)$ splits if and only if $f(\mathbb{C}P^1)$ is a complete intersection in X.

An idea of a proof is using the first-order deformation of the surface Σ in a pencil $\mathcal{Q}$ generated by an elliptic curve E which realizes the homology class $\frac{1}{4}K_X^2$; and considering the divisor classes restricted on E by the first-order deformation of $f(\mathbb{C}P^1)$ in Σ. By extending a result in the proof of Theorem $(4.f.3)$ in [GH83], the following two statements hold.

1. If the exact sequence $(*)$ splits then the section of $\mathcal{N}_f$ which vanishes at $D.E$ points in the intersection $f(\mathbb{C}P^1) \cap E$ implies that restricted divisor class $[f(\mathbb{C}P^1)]|_E$ is constant.
2. If $f(\mathbb{C}P^1)$ is not a complete intersection with Σ in X, then the restricted divisor class $[f(\mathbb{C}P^1)]|_E \in Pic(E)$ must vary as we deform $f(\mathbb{C}P^1)$. Indeed, let $f_\epsilon(\mathbb{C}P^1)$ be a first-deformation of $f(\mathbb{C}P^1)$ in the pencil, i.e. $f_\epsilon(\mathbb{C}P^1) \subset \Sigma_\epsilon \in \mathcal{Q}$. One has

$$2[f_\epsilon(\mathbb{C}P^1)]|_{E,\Sigma_\epsilon} \sim \phi(\overline{D}) - (D.S)S|_{E,\Sigma_\epsilon} = \phi(\overline{D}) - (D.S)\phi(\overline{L}) + 2(D.S)L|_{E,\Sigma_\epsilon}.$$

   By Lemma 2.2.12, Lemma 2.2.13 and the fact that $L|_{E,\Sigma_\epsilon}$ is not a constant as we deform Σ_ϵ, the statement follows.

The proposition follows by the sufficient condition of **Claim 1** and necessary condition of **Claim 2**. In other words, if $f(\mathbb{C}P^1)$ is not a complete intersection, then the exact sequence $(*)$ does not split, and hence f is balanced.

□

Remark 2.3.5. If an irreducible curve C is a complete intersection of a non-singular surface of the pencil $\mathcal{Q}$ in X then $[C].S = 0$.

Enumeration of balanced rational curves

In this part, we show that we can choose a configuration of points on E, so that every irreducible balanced rational curve passing though such configuration is a regular point of the corresponding total evaluation map.

We denote by V_{k_d} the set of configurations of k_d distinct points on E.

Recall that for each homology class $D \in H_2(\Sigma;\mathbb{Z})$ satisfying $D \in \psi^{-1}(d)$, where $\psi : H_2(\Sigma;\mathbb{Z}) \twoheadrightarrow H_2(X;\mathbb{Z})$, then we have $k_D = k_d - 1$ (Lemma 2.3.1). Let $\underline{y}_D \subset E^{k_D}$ be a configuration of $k_d - 1$ distinct points on E.

For every non-singular surface Σ_t in the pencil $\mathcal{Q}$, let $\mathcal{C}_{\Sigma_t,D}(\underline{y}_D)$ be the set of k_D-*pointed stable maps* $f : (C_0; p_1, \dots, p_{k_D}) \to \Sigma_t$*, where* C_0 *is a connected nodal curve of arithmetic genus* 0*, such that* $f(C_0)$ *represents the homology class* $D \in H_2(\Sigma_t;\mathbb{Z})$ *and* $f(\{p_1, \dots, p_{k_D}\}) = \underline{y}_D$.

Proposition 2.3.6. *There exists a dense open subset U_{k_D} in the set V_{k_D}, such that for every configuration $\underline{y}_D$ in U_{k_D} and for every element f in $\mathcal{C}_{\Sigma_t,D}(\underline{y}_D)$, one has $C_0 = \mathbb{C}P^1$ and f is an immersion.*

A proof of this proposition is similar to the one of Proposition 5.2 in [BG16].

Corollary 2.3.7. *Let $\underline{y}_D \in U_{k_D}$ be a configuration of points as in Proposition 2.3.6. Then, every non-singular surface Σ_t in the pencil $\mathcal{Q}$, which contains a curve of class D of the set $\mathcal{C}(\underline{x}_d)$, contains exactly $GW_{\Sigma_t}(D)$ elements of $\mathcal{C}(\underline{x}_d)$.*

Proof. We consider the moduli space of stable maps $\mathcal{M}^*_{0,k_D}(\Sigma_t, D)$ and its corresponding total evaluation map ev_{Σ_t}. Proposition 2.3.6 and Lemma 1.2.12 imply that for every map $f \in \mathcal{C}_{\Sigma_t,D}(\underline{y}_D)$ such that $\underline{y}_D \in U_{k_D}$, one has $[(\mathbb{C}P^1; p_1, \ldots, p_{k_D}); f]$ is a regular point of ev_{Σ_t} if f is an immersion. Hence, we get $|\mathcal{C}_{\Sigma_t,D}(\underline{y}_D)| = GW_{\Sigma_t}(D)$. $\square$

Proposition 2.3.8. *There exists a dense open subset U_{k_d} in the set V_{k_d} such that for every $\underline{x}_d \in U_{k_d}$, the total evaluation map ev_X is regular at every $[(\mathbb{C}P^1; p_1, \ldots, p_{k_d}); f]$ such that $f(\{p_1, \ldots, p_{k_d}\}) = \underline{x}_d$.*

Moreover, the cardinality of the set $\mathcal{C}_d(\underline{x}_d)$ is equal to $GW_X(d)$.

Proof. Let $\underline{x}_d$ be a configuration of k_d distinct points on E for which the conclusion of Proposition 2.3.2 holds, i.e. every curve in $\mathcal{C}(\underline{x}_d)$ is irreducible of arithmetic genus 0 and is contained in some non-singular surface Σ of the pencil $\mathcal{Q}$.

By Lemma 1.2.12, an element $[(\mathbb{C}P^1; p_1, \ldots, p_{k_d}); f]$ is a regular point of ev_X if and only if f is a balanced immersion.

Note that if we replace $\underline{x}_d$ by another linearly equivalent configuration of points $\underline{\tilde{x}}_d$ on E, then the set of surfaces of the pencil $\mathcal{Q}$ containing a curve in $\mathcal{C}(\underline{\tilde{x}}_d)$ is the same as that containing a curve in $\mathcal{C}(\underline{x}_d)$.

By the Riemann-Roch theorem and by Proposition 2.3.6, we can choose $\underline{x}_d$ generically such that every curve in $\mathcal{C}(\underline{x}_d)$ is parametrized by an algebraic immersion (see also the proof of Proposition 5.3 in [BG16]).

It remains to show that f is balanced. Indeed, since the configuration of points $\underline{x}_d$ is generic, we can chose such configuration that is outside an algebraic subset with related to complete intersections of Σ restricted to E. Hence, the balancity of f follows from Lemma 2.3.4. $\square$

2.4 A proof of Theorem 2.1.1

Remark 2.4.1. By Proposition 2.2.4 and Proposition 2.3.2, we can degenerate the problem of computing genus-0 Gromov-Witten invariants in del Pezzo varieties of dimension three into that of dimension two.

Remind about our hypotheses and notations:

— X is a del Pezzo variety of dimension three such that $\ker(\psi) \cong \mathbb{Z}$, where surface $\Sigma \in |-\frac{1}{2}K_X|$ is non-singular and the surjection $\psi : H_2(\Sigma;\mathbb{Z}) \twoheadrightarrow H_2(X;\mathbb{Z})$ is induced by the inclusion $\Sigma \hookrightarrow X$;

— $\mathcal{Q}$ is a pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$ whose base locus is E;

— $E \subset X$ is an elliptic curve realising the homology class $\frac{1}{4}K_X^2$.

Let $d \in H_2(X;\mathbb{Z})$ and let k_d be its associated integer. Recall that $\mathcal{C}(\underline{x}_d)$ denotes the set of connected algebraic curves of arithmetic genus 0 representing the given homology class d and passing through a generic configuration $\underline{x}_d$ of k_d points in E (see Proposition 2.3.2).

Let $U_{k_d} \subset E^{k_d}$ be as in Proposition 2.3.8. Choose $\underline{x}_d \in U_{k_d}$ and a subconfiguration $\underline{y}_D \subset \underline{x}_d$ of $k_D = k_d - 1$ points in $\underline{x}_d$. By Proposition 2.3.2 and Corollary 2.3.3, every irreducible rational curve, which is contained in a non-singular surface of the pencil $\mathcal{Q}$ and passes through $\underline{y}_D$, intersects the elliptic curve E at the k_d^{th}-point x_{k_d}, which is exactly $\underline{x}_d \backslash \underline{y}_D$. Let Σ be a surface of $\mathcal{Q}$ which contains a curve in $\mathcal{C}(\underline{x}_d)$. As mentioned in Section 2.2.2, up to monodromy, we can identify canonically $H_2(\Sigma_t;\mathbb{Z})/\sim$ with $H_2(\Sigma;\mathbb{Z})/\sim$ for every non-singular surface Σ_t in the pencil $\mathcal{Q}$.

Let S be a generator of $\ker(\psi)$. Note that S can be identified with the vanishing cycle in the monodromy transformation $T(D) = D + (D.S)S$. Let $\overline{D} \in H_2(\Sigma;\mathbb{Z})/_{D\sim T(D)}$. The surjectivity of ψ implies a surjective map $\overline{\psi} : H_2(\Sigma;\mathbb{Z})/_{D\sim T(D)} \twoheadrightarrow H_2(X;\mathbb{Z})$.

By Proposition 2.3.2, we deduce the following equality

$$|\mathcal{C}(\underline{x}_d)| = \sum_{\overline{D}\in\overline{\psi}^{-1}(d)} \sum_{\substack{\Sigma_t\in\mathcal{Q}: \\ \overline{D}|_{E,\Sigma_t}\sim[\underline{y}_D\sqcup\{x_{k_d}\}]}} |\{C \in \mathcal{C}(\underline{x}_d) : C \in \Sigma_t \text{ and } \overline{[C]} = \overline{D}\}| \quad (\star)$$

By Proposition 2.3.8, we have

$$|\mathcal{C}(\underline{x}_d)| = GW_X(d).$$

Moreover, the total evaluation map ev_X is regular at every $[(\mathbb{C}P^1; p_1, \ldots, p_{k_d}); f]$ such that $f(\{p_1, \ldots, p_{k_d}\}) = \underline{x}_d$, then every curve in $\mathcal{C}(\underline{x}_d)$ is parametrized by an algebraic immersion $f : \mathbb{C}P^1 \to X$.

Remind about a consequence of the symmetry property of genus-0 Gromov-Witten invariants via monodromy transformation (see Proposition 1.2.4), we denoted by

$$GW_{\Sigma_t}(\overline{D}) := GW_{\Sigma_t}(D) = GW_{\Sigma_t}(T(D)).$$

Hence, the cardinality of the set $\{C \in \mathcal{C}(\underline{x}_d) : C \in \Sigma_t \text{ and } \overline{[C]} = \overline{D}\}$ is exactly the cardinality of the set $\mathcal{C}_{\Sigma_t,D}(\underline{y}_D)$, and hence is equal to $GW_{\Sigma_t}(\overline{D})$ (see Proposition 2.3.6 and Corollary 2.3.7).

Moreover, $\overline{D}|_{E,\Sigma_t} = \overline{D|_{E,\Sigma_t}} \in Pic_{D.E}(E)/_{x\sim\phi(\overline{D})-x}$. Then, the cardinality of the set $\{\Sigma_t \in \mathcal{Q} : \overline{D}|_{E,\Sigma_t} \sim [\underline{y}_D \sqcup \{x_{k_d}\}]\}$ is exactly the degree of the map $\phi_{\overline{D}}$ (see Proposition

2.2.16). Since we are considering the cases where a generator of $\ker(\psi)$ can be identified with the vanishing cycle (Lemma 2.2.8), the degree of the corresponding map $\phi_{\overline{D}}$ is exactly $(D.S)^2$. Therefore, one has

$$\begin{aligned} GW_X(d) &= \sum_{\overline{D}\in\overline{\psi}^{-1}(d)} GW_{\Sigma_t}(\overline{D}) \times |\{\Sigma_t \in \mathcal{Q} : \overline{D}|_{E,\Sigma_t} \sim [\underline{y}_D \sqcup \{x_{k_d}\}]\}| \\ &= \sum_{\overline{D}\in\overline{\psi}^{-1}(d)} (D.S)^2 GW_{\Sigma_t}(\overline{D}) \quad (\star\star). \end{aligned}$$

By an easy calculation, one has $(D.S)^2 = (T(D).S)^2$. Then, the equality $(\star\star)$ becomes

$$GW_X(d) = \sum_{\substack{\{D,T(D)\}\subset H_2(\Sigma;\mathbb{Z}):\\ D\in\psi^{-1}(d)}} (D.S)^2 GW_\Sigma(D).$$

Thus,

$$GW_X(d) = \frac{1}{2} \sum_{\substack{D\in H_2(\Sigma;\mathbb{Z}):\\ D\in\psi^{-1}(d)}} (D.S)^2 GW_\Sigma(D).$$

2.5 Some applications

In this section, we apply Theorem 2.1.1 to obtain explicit formulas of computing genus-0 Gromov-Witten invariants for some known cases in [BG16] and [Din19], as well as for the case $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ (see Section 2.5.2).

2.5.1 Three-dimensional del Pezzo varieties of degree 7 and 8

Applying Theorem 2.1.1, we recover the same results which have been studied for three-dimensional del Pezzo varieties X of degree 7 and 8 as the following.

Three-dimensional projective space: $X = \mathbb{C}P^3$.

In this case, we have seen that a non-singular element $\Sigma \in |-\frac{1}{2}K_{\mathbb{C}P^3}|$ is a non-singular quadric surface in $\mathbb{C}P^3$ and Σ is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$.

We identify $H_2(\Sigma;\mathbb{Z})$ with $\mathbb{Z}^2$ by considering the standard basis (L_1, L_2) of $H_2(\Sigma;\mathbb{Z})$ given by

- $L_1 = [\mathbb{C}P^1 \times \{p\}]$,
- $L_2 = [\{q\} \times \mathbb{C}P^1]$,

where p, q are points in $\mathbb{C}P^1$. Then, the class $aL_1 + bL_2 \in H_2(\Sigma;\mathbb{Z})$ can be identified with the pair $(a, b) \in \mathbb{Z}^2$.

We identify $H_2(X;\mathbb{Z})$ with $\mathbb{Z}$ by considering the basis $\{H\}$ of $H_2(X;\mathbb{Z})$ which is the class of a line in $\mathbb{C}P^3$.

With these identifications and the natural inclusion of a quadric surface into projective space, the corresponding surjective map ψ is given by:

$$\begin{aligned} \psi : H_2(\Sigma;\mathbb{Z}) &\longrightarrow H_2(X;\mathbb{Z}) \\ (a,b) &\longmapsto a+b. \end{aligned}$$

This implies that $\ker(\psi) = \mathbb{Z}(1,-1)$.

Hence, Theorem 2.1.1 implies the following theorem.

Theorem 2.5.1. *For every positive integer d, one has*

$$GW_{\mathbb{C}P^3}(d) = \frac{1}{2}\sum_{a\in\mathbb{Z}}(d-2a)^2 GW_{\mathbb{C}P^1\times\mathbb{C}P^1}(a,d-a).$$

This formula is in agreement with Theorem 2 in [BG16].

Three-dimensional projective space blown-up at a point: $X = \mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}$

Let us denote the blowing-up point by x. We have seen that a non-singular surface Σ in $|-\frac{1}{2}K_X|$ is a non-singular del Pezzo surface of degree 7, and moreover Σ is isomorphic to the blow-up of a quadric surface at x. One can write $\Sigma \cong \mathbb{C}P^1\times\mathbb{C}P^1\sharp\overline{\mathbb{C}P^2}$.

We identify $H_2(\Sigma;\mathbb{Z})$ with $\mathbb{Z}^3$ by considering a basis $(L_1,L_2;E)$ of $H_2(\Sigma;\mathbb{Z})$ given by

- $L_1 = [\mathbb{C}P^1\times\{p\}]$,
- $L_2 = [\{q\}\times\mathbb{C}P^1]$,
- E is the exceptional divisor class

where p,q are points in $\mathbb{C}P^1$. Then, the class $aL_1+bL_2-kE \in H_2(\Sigma;\mathbb{Z})$ can be identified with the triplet $(a,b;k)\in\mathbb{Z}^3$.

We identify $H_2(X;\mathbb{Z})$ with $\mathbb{Z}^2$ by considering a basis $(H;E)$ of $H_2(X;\mathbb{Z})$ given by

- H is the class of a line in $\mathbb{C}P^3$,
- E is the class of a line in the exceptional divisor.

Then, the class $dH-kE\in H_2(X;\mathbb{Z})$ can be identified with the point $(d;k)\in\mathbb{Z}^2$.

With these identifications and the natural inclusion of a quadric surface into projective space, the corresponding surjective map ψ is given by:

$$\begin{aligned} \psi : H_2(\Sigma;\mathbb{Z}) &\longrightarrow H_2(X;\mathbb{Z}) \\ (a,b;k) &\longmapsto (a+b;k). \end{aligned}$$

This implies that $\ker(\psi) = \mathbb{Z}(1,-1;0)$.

Hence, Theorem 2.1.1 implies the following theorem.

Theorem 2.5.2. *For every positive integer d, for every non-negative integer k such that $k \leq d$, one has*

$$GW_{\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}}(d;k) = \frac{1}{2}\sum_{a\in\mathbb{Z}}(d-2a)^2 GW_{\mathbb{C}P^1\times\mathbb{C}P^1\sharp\overline{\mathbb{C}P^2}}(a,d-a;k).$$

This formula is in agreement with Theorem 1.2 in [Din19].

2.5.2 Three-dimensional del Pezzo varieties of degree 6

We consider the threefold product of the projective line: $X = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$. We have seen that a non-singular surface $\Sigma \in |-\frac{1}{2}K_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}|$, i.e. a non-singular surface of tridegree $(1,1,1)$ in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, is a non-singular del Pezzo surface of degree 6.

It can be shown that a non-singular $(1,1,1)$-surface Σ in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ is isomorphic to the blow-up of a quadric surface at two distinct points, denoted by x_1 and x_2. One can write $\Sigma \cong \mathbb{C}P^1 \times \mathbb{C}P^1\sharp 2\overline{\mathbb{C}P^2}$.

We identify $H_2(\Sigma;\mathbb{Z})$ with $\mathbb{Z}^4$ by considering a basis $(L_1, L_2; E_1, E_2)$ of $H_2(\Sigma;\mathbb{Z})$ given by

- $L_1 = [\mathbb{C}P^1 \times \{p\}]$,
- $L_2 = [\{q\} \times \mathbb{C}P^1]$,
- E_1 is the exceptional divisor class over x_1,
- E_2 is the exceptional divisor class over x_2

where p, q are points in $\mathbb{C}P^1$. Then, the class $aL_1 + bL_2 - \alpha E_1 - \beta E_2 \in H_2(\Sigma;\mathbb{Z})$ can be identified with the quadruple $(a, b; \alpha, \beta) \in \mathbb{Z}^4$.

We identify $H_2(X;\mathbb{Z})$ with $\mathbb{Z}^3$ by considering a basis (M_1, M_2, M_3) of $H_2(X;\mathbb{Z})$ given by

- $M_1 = [\mathbb{C}P^1 \times \{p_1\} \times \{p_2\}]$,
- $M_2 = [\{q_1\} \times \mathbb{C}P^1 \times \{q_2\}]$,
- $M_3 = [\{r_1\} \times \{r_2\} \times \mathbb{C}P^1]$

where $p_i, q_i, r_i \in \mathbb{C}P^1$. Then, the class $a_1M_1 + a_2M_2 + a_3M_3 \in H_2(X;\mathbb{Z})$ can be identified with the point $(a_1, a_2, a_3) \in \mathbb{Z}^3$.

Lemma 2.5.3. *The surjective map ψ induced from the inclusion of Σ in X is given by:*

$$\begin{aligned} \psi : H_2(\Sigma;\mathbb{Z}) &\longrightarrow H_2(X;\mathbb{Z}) \\ (a,b;\alpha,\beta) &\longmapsto (a, b, a+b-\alpha-\beta) \end{aligned}$$

Proof. A $(1,1,1)$-surface in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ is given by a polynomial

$$u_3P(u_1,v_1,u_2,v_2) + v_3Q(u_1,v_1,u_2,v_2) = 0$$

where P and Q are two homogeneous polynomials of degree 1 in the variables (u_i, v_i), for $i \in \{1,2\}$. Since Σ is a non-singular surface, $\{P = 0\}$ and $\{Q = 0\}$ intersects transversely, i.e. $\{P = 0\} \cap \{Q = 0\} = \{x_1, x_2\}$ and $x_1 \neq x_2$.

The projection π from $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ onto the first two factors identifies Σ with the blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ at two points x_1 and x_2. Indeed, considering the fiber of π at a point $x \in \mathbb{C}P^1 \times \mathbb{C}P^1$, we have

— if $x \in \{P = 0\} \cap \{Q = 0\}$ then $\pi^{-1}(x) = \mathbb{C}P^1$;

— if $x \notin \{P = 0\} \cap \{Q = 0\}$ then $\pi^{-1}(x) = [Q(x) : -P(x)]$ is a point in $\mathbb{C}P^1$.

Hence, the inclusion of Σ in X implies the following identifications:

- $L_1 = M_1 + M_3$,
- $L_2 = M_2 + M_3$,
- $E_1 = M_3$,
- $E_2 = M_3$

Then, the map ψ sends a class

$$aL_1 + bL_2 - \alpha E_1 - \beta E_2 \in H_2(\Sigma; \mathbb{Z})$$

to the class

$$a(M_1+M_3)+b(M_2+M_3)-\alpha M_3-\beta M_3 = aM_1+bM_2+(a+b-\alpha-\beta)M_3 \in H_2(X;\mathbb{Z}).$$

Hence, the statement follows by identifying $H_2(\Sigma; \mathbb{Z})$ with $\mathbb{Z}^4$ and $H_2(X; \mathbb{Z})$ with $\mathbb{Z}^3$. $\square$

We have $\psi((0,0;\alpha,-\alpha)) = (0,0,0)$ then $\ker(\psi) = \mathbb{Z}(0,0;1,-1)$.

Hence, Theorem 2.1.1 implies the following theorem.

Theorem 2.5.4. *For every triplet of non-negative integers (a,b,c), one has*

$$\begin{aligned} &GW_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}(a,b,c) \\ &= \frac{1}{2}\sum_{\alpha\in\mathbb{Z}}(a+b-c-2\alpha)^2 GW_{\mathbb{C}P^1\times\mathbb{C}P^1\sharp 2\overline{\mathbb{C}P^2}}(a,b;\alpha,a+b-c-\alpha) \end{aligned} \tag{2.1}$$

In the next section, we will show some precise values of Formula (2.1) (see Table 2.2).

2.5.3 Computations and comments

Remark 2.5.5. By L. Gottsche and R. Pandharipande [GP96], we can compute the genus-0 Gromov-Witten invariants of $\mathbb{C}P^2\sharp 3\overline{\mathbb{C}P^2}$ by recursive formulas.

In our formula, Formula (2.1), we need to compute the genus-0 Gromov-Witten invariants of $\mathbb{C}P^1 \times \mathbb{C}P^1\sharp 2\overline{\mathbb{C}P^2}$. In the following, we first precise how to compare these two genus-0 Gromov-Witten invariants. Then, we deduce some vanishing results for the genus-0 Gromov-Witten invariants $GW_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}(a,b,c)$.

We identify $H_2(\mathbb{C}P^2\sharp 3\overline{\mathbb{C}P^2}; \mathbb{Z})$ with $\mathbb{Z}^4$ by considering a basis $(L; F_1, F_2, F_3)$ of $H_2(\mathbb{C}P^2\sharp 3\overline{\mathbb{C}P^2}; \mathbb{Z})$ given by

- L is the class of a line in $\mathbb{C}P^2$,
- F_1 is the exceptional divisor class over x_1,
- F_2 is the exceptional divisor class over x_2,
- F_3 is the exceptional divisor class over x_3,

where x_1, x_2, x_3 are in $\mathbb{C}P^2$.

We consider the map $\kappa_* : H_2(\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp 2\overline{\mathbb{C}P^2}; \mathbb{Z}) \to H_2(\mathbb{C}P^2 \sharp 3\overline{\mathbb{C}P^2}; \mathbb{Z})$ sending

- L_1 to $L - F_1$,
- L_2 to $L - F_2$,
- E_1 to $L - F_1 - F_2$,
- E_2 to F_3.

Hence, one can deduce that κ_* sends the class $(a, b; \alpha, \beta)$ to $(a+b-\alpha; a-\alpha, b-\alpha, \beta)$. One has the following equality

$$GW_{\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp 2\overline{\mathbb{C}P^2}}(a, b; \alpha, \beta) = GW_{\mathbb{C}P^2 \sharp 3\overline{\mathbb{C}P^2}}(a + b - \alpha; a - \alpha, b - \alpha, \beta). \tag{2.2}$$

Remark 2.5.6. Theorem 2.1.1 implies that if either $|D.S| = 0$ or $GW_\Sigma(D) = 0$ for all $D \in \psi^{-1}(d)$ then $GW_X(d)$ vanishes.

In particular, we have

— if $|D.S| = 0$, one has $|(a, b; \alpha, a + b - c - \alpha)(0, 0; 1, -1)| = 0$ or equivalently one has $2\alpha = a + b - c$. Thus, the class D is $(a, b; \alpha, \alpha)$ in $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp 2\overline{\mathbb{C}P^2}; \mathbb{Z})$;

— if $GW_\Sigma(D) = GW_\Sigma(T(D)) = 0$, then Equation (2.2) implies that

$$\begin{cases} GW_{\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp 2\overline{\mathbb{C}P^2}}(a, b; \alpha, a + b - c - \alpha) = 0 \\ GW_{\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp 2\overline{\mathbb{C}P^2}}(a, b; a + b - c - \alpha, \alpha) = 0 \end{cases}$$

and hence

$$\begin{cases} GW_{\mathbb{C}P^2 \sharp 3\overline{\mathbb{C}P^2}}(a + b - \alpha; a - \alpha, b - \alpha, a + b - c - \alpha) = 0 \\ GW_{\mathbb{C}P^2 \sharp 3\overline{\mathbb{C}P^2}}(c + \alpha; c - b + \alpha, c - a + \alpha, \alpha) = 0. \end{cases} \tag{2.3}$$

Moreover, it is known that $GW_{\mathbb{C}P^2 \sharp 3\overline{\mathbb{C}P^2}}(d; a_1, a_2, a_3) = 0$ if there exists a_i, a_j such that $a_i + a_j > d$, unless $(d; a_1, a_2, a_3) \in \{(1; 1, 1, 0), (1; 1, 0, 1), (1; 0, 1, 1)\}$ (see [GP96]).

Therefore, Equations (2.3) holds if α satisfies

$$\begin{cases} \alpha < max\{0, b - c, a - c\}, and \\ \alpha > min\{a, b, a + b - c\}. \end{cases}$$

Together with the symmetrical property

$$GW_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}(a,b,c) = GW_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}(\sigma(a),\sigma(b),\sigma(c))$$

where σ is any permutation of the set $\{a,b,c\}$, one has the following proposition.

Proposition 2.5.7. *One has $GW_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}(a,b,c) = 0$ if $a+b+c > 1$ and either one of the following cases holds:*

- $a \geq b+c$,
- $b \geq c+a$,
- $c \geq a+b$.

In Table 2.2, we show the first values of non-vanishing genus-0 Gromov-Witten invariants $GW_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}(a,b,c)$, i.e. either $a+b+c = 1$ or $a < b+c$, where $4 \geq a \geq b \geq c$ and $a+b+c$ increases up to 12.

Remark 2.5.8. The values of $GW_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}(a,b,c)$ in this computation are compatible with that given by X. Chen and A. Zinger using Solomon's WDVV-type relations in [CZ21b].

d (a,b,c)	$GW_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}(d)$	D $(a,b;\alpha,a+b-c-\alpha)$	$\lvert D.S\rvert$	$GW_{\mathbb{C}P^1\times\mathbb{C}P^1\sharp 2\overline{\mathbb{C}P^2}}(D)$
$(1,0,0)^*$	1	(1,0;0,1)	1	1
(1,1,1)	1	(1,1;0,1)	1	1
(2,2,1)	1	(2,2;0,3)	3	0
		(2,2;1,2)	1	1
(2,2,2)	4	(2,2;0,2)	2	1
(3,2,2)	12	(3,2;0,3)	3	0
		(3,2;1,2)	1	12
(3,3,1)	1	(3,3;0,5)	5	0
		(3,3;1,4)	3	0
		(3,3;2,3)	1	1
(3,3,2)	48	(3,3;0,4)	4	0
		(3,3;1,3)	2	12
(3,3,3)	728	(3,3;0,3)	3	12
		(3,3;1,2)	1	620
(4,3,2)	96	(4,3;0,5)	5	0
		(4,3;1,4)	3	0
		(4,3;2,3)	1	96
(4,3,3)	2480	(4,3;0,4)	4	0
		(4,3;1,3)	2	620
(4,4,1)	1	(4,4;0,7)	7	0
		(4,4;1,6)	5	0
		(4,4;2,5)	3	0
		(4,4;3,4)	1	1
(4,4,2)	384	(4,4;0,6)	6	0
		(4,4;1,5)	4	0
		(4,4;2,4)	2	96
(4,4,3)	23712	(4,4;0,5)	5	0
		(4,4;1,4)	3	620
		(4,4;2,3)	1	18132
(4,4,4)	359136	(4,4;0,4)	4	620
		(4,4;1,3)	2	87304

Table 2.2 – Genus-0 Gromov-Witten invariants of $\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1$ via Formula (2.1) with $a+b+c\leq 12$.

3

Welschinger invariants

In this chapter, we focus on the real counterpart of genus-0 Gromov-Witten invariants studied in the previous chapter. In particular, we study about an extension of the Brugallé-Georgieva's strategy in [BG16] to compute genus-0 Welschinger invariants of real del Pezzo varieties of dimension three. More precisely, we reduce the 3-dimensional real enumerative problem into two independent ones: the first one is the 1-*dimensional real problem* on a real elliptic curve and the second one is the 2-*dimensional enumerative problem* on real del Pezzo surfaces. Moreover, different from the Gromov-Witten case, there is a *sign problem* which comes from the difference between two definitions of Welschinger's sign for curves in varieties of dimension two and three. By solving these problems, we obtain a formula to compute genus-0 Welschinger invariants for most of real three-dimensional del Pezzo varieties of degree 6, 7 and 8. Moreover, we show that in certain cases the lower bounds of genus-0 Welschinger invariants are optimal.

We organise this chapter as follows. In the first section, we introduce our main theorems of this chapter (Theorem 3.1.1 and Theorem 3.1.2). In Section 3.2, by studying pencils of real del Pezzo surfaces in the linear system $|-\frac{1}{2}K_X|$ (where X is a real three-dimensional del Pezzo variety), we solve the 1-*dimensional real problem* on the real elliptic curve and then prove Theorem 3.1.2 (Proposition 3.2.5, Corollary 3.2.6). In Section 3.3 and Section 3.4, we construct the relation between Welschinger's signs of real rational curves in varieties of dimension three and that in surfaces, and then we solve the *sign problem* (Proposition 3.4.4). In Section 3.5, we combine the results from the previous sections to give a proof of Theorem 3.1.1. In the end of this chapter, we apply Theorem 3.1.1 and Theorem 3.1.2 in some particular cases (Theorems 3.6.1, 3.6.5, 3.6.9, 3.6.14), and we exhibit the explicit computations of genus-0 Welschinger invariants in the cases $\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}$ with standard real structure and $\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1$ with two real structures using *Maple program*.

The first remark about real structures on real pencils of surfaces

Let (X, τ) be a real three-dimensional del Pezzo variety equipped with a real structure τ, unless otherwise stated.

We consider a real pencil of del Pezzo surfaces in the linear system $|-\frac{1}{2}K_X|$. Suppose that the real part of the pencil is non-empty. Let $\tau|_\Sigma$ be the real structure on Σ restricted from the one on X. Let $(\Sigma_1, \tau|_{\Sigma_1})$ and $(\Sigma_2, \tau|_{\Sigma_2})$ be two real non-singular surfaces on the real part of this pencil, such that Σ_2 is a *surgery* of Σ_1 along a real Lagrangian sphere in Σ_1, i.e. a Lagrangian sphere globally invariant under $\tau|_{\Sigma_1}$. Moreover, the two real surfaces $(\Sigma_1, \tau|_{\Sigma_1})$ and $(\Sigma_2, \tau|_{\Sigma_2})$ can be seen as two different real fibers of a real Lefschetz fibration, having a unique singular fiber for which a real Lagrangian sphere realizes the vanishing cycle. One can also say that Σ_2 is a surgery of Σ_1 along the vanishing cycle. It can be shown that the Euler characteristics of the real parts of these two surfaces differ by two. Suppose that $\chi(\mathbb{R}\Sigma_2) = \chi(\mathbb{R}\Sigma_1) + 2$. One has the following remark (see [Bru16]).

Remark 3.0.1. The vanishing cycle S in $H_2(\Sigma_1; \mathbb{Z})$ is $\tau|_{\Sigma_1}$-anti-invariant if and only if it is $\tau|_{\Sigma_2}$-invariant.

3.1 Main results

Let $(\Sigma, \tau|_\Sigma) \in |-\frac{1}{2}K_X|$ be a non-singular real del Pezzo surface.

Let $d \in H_2^{-\tau}(X; \mathbb{Z})$ be a given homology class and let $k_d = \frac{1}{2}c_1(X)\cdot d$ be its associated integer. Suppose that $\mathbb{R}X$ is orientable with orientation $\mathfrak{o}_X$ and is equipped with a $Spin_3$-structure $\mathfrak{s}_X$. Let $W_{\mathbb{R}X}^{\mathfrak{s}_X, \mathfrak{o}_X}(d, l)$ be the genus-0 Welschinger invariant of the real del Pezzo variety $((X, \tau), \mathfrak{s}_X, \mathfrak{o}_X)$ of class d (see Definition 1.3.25).

Let $D \in H_2^{-\tau|_\Sigma}(\Sigma; \mathbb{Z})$ and let $k_D = c_1(\Sigma) \cdot D - 1$ be its associated integer. Let $W_{\mathbb{R}\Sigma}(D, l)$ be the genus-0 Welschinger invariant of $(\Sigma, \tau|_\Sigma)$ of class D (see Definition 1.3.3).

Let $\psi : H_2(\Sigma; \mathbb{Z}) \to H_2(X; \mathbb{Z})$ be the surjective map induced by the inclusion from Σ to X.

Let $g(D) = \frac{1}{2}(K_\Sigma.D + D^2 + 2) \in \mathbb{Z}$.

Let $\epsilon : H_2(\Sigma; \mathbb{Z}) \to \mathbb{Z}_2$ be a map given by

$$\epsilon(D) = \begin{cases} k_D + 1 \mod 2, \text{ if } \mathbb{R}f^*T\Sigma/T\mathbb{C}P^1 = E^+, \\ k_D \mod 2, \text{ if } \mathbb{R}f^*T\Sigma/T\mathbb{C}P^1 = E^- \end{cases}$$

where $f : \mathbb{C}P^1 \to \Sigma \subset X$ is a real immersion such that $f_*[\mathbb{C}P^1] = D \in H_2(\Sigma; \mathbb{Z})$, and $E^\pm$ are two isotopy classes of the real part of a holomorphic line subbundle of degree $(k_D - 1)$ of the rank 2 holomorphic normal bundle $f^*TX/T\mathbb{C}P^1$ (see Section 1.3.3 and Definition 3.3.2).

Let $s^{\mathfrak{s}_X|_\Sigma} : H_1(\mathbb{R}\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$ be the quasi-quadratic enhancement in one-to-one correspondence to the Pin_2^--structure $\mathfrak{s}_{X|\Sigma}$ on $O(T\mathbb{R}\Sigma)$, which is restricted from the $Spin_3$-structure $\mathfrak{s}_X$ over $\mathbb{R}X$ (see Proposition 1.4.11 and Equation (3.3)).

Let $\rho : H_2^{-\tau}(\Sigma;\mathbb{Z}) \to H_1(\mathbb{R}\Sigma;\mathbb{Z}_2)$ be the homomorphism given by the homology class realized by the real part of the τ-anti-invariant representative of D.

We have the following theorem.

Theorem 3.1.1. *Let (X,τ) be a real three-dimensional del Pezzo variety such that one has $\ker(\psi) \cong \mathbb{Z}$, for any $\Sigma \in |-\frac{1}{2}K_X|$ non-singular surface, and for ψ a map as described above. Let S be a generator of $\ker(\psi)$. Suppose $(\Sigma, \tau|_\Sigma)$ is such that S is $\tau|_\Sigma$-anti-invariant in $H_2(\Sigma;\mathbb{Z})$. For every $d \in H_2^{-\tau}(X;\mathbb{Z})$ and $0 \leq l < \frac{k_d}{2}$, one has*

$$W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l) = \frac{1}{2}\sum_{D\in\psi^{-1}(d)} (-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_X|_\Sigma}(\rho D)}|D.S|W_{\mathbb{R}\Sigma}(D,l).$$

Theorem 3.1.1 is a generalization of Theorem 1 in [BG16].

As a result, this theorem allows us to compute the genus-0 Welschinger invariants of almost all real three-dimensional del Pezzo varieties of degree $6, 7$ and 8.

Since the Welschinger invariants count real curves with some sign ± 1, in the case where these invariants vanish, it is not clear that there exists no such real curves. This gives rise to one of the optimal problems in real enumerative geometry: the sharpness of the Welschinger invariants, in which one shows that the lower bounds given by these invariants can be obtained. The first examples of vanishing genus-0 Welschinger invariants in projective space was explored by G. Mikhalkin for rational curves of degree 4 and then by J. Kollár for any even degree, see [Kol15]. We extend Kollár 's results for some classes of del Pezzo varieties of dimension three, as shown in the following theorem.

Theorem 3.1.2. *Let (X,τ) be a real three-dimensional del Pezzo variety such that one has $\ker(\psi) \cong \mathbb{Z}$, where Σ and ψ are as in Theorem 3.1.1. Let $d \in H_2^{-\tau}(X;\mathbb{Z})$ such that $D.S$ is even for any $D \in \psi^{-1}(d)$. Let k_d be the associated integer of d. Suppose that $E \subset X$ is a maximal real elliptic curve realizing homology class $\frac{1}{4}K_X^2$.*

Then, there exists a real configuration of k_d points on E, consisting of at least one real point, through which no real irreducible rational curve of class d passes.

3.2 Pencils of real del Pezzo surfaces: properties and enumerations - A proof of Theorem 3.1.2

Throughout this section, we restrict attention to (X,τ) a real three-dimensional del Pezzo variety such that $\ker(\psi) \cong \mathbb{Z}$, for any $\Sigma \in |-\frac{1}{2}K_X|$ non-singular surface and for surjection $\psi : H_2(\Sigma;\mathbb{Z}) \twoheadrightarrow H_2(X;\mathbb{Z})$ induced by the inclusion $\Sigma \hookrightarrow X$. Henceforth, the notations of vanishing cycle and generator of $\ker(\psi)$ might be used interchangeably (see Lemma 2.2.8). Let $\mathcal{Q}$ be a pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$. Let E be its base locus. Note that E is an elliptic curve realising the homology class $\frac{1}{4}K_X^2$ (see Proposition 2.2.1). Suppose that the curve E and the pencil $\mathcal{Q}$ are real with non-empty real parts.

In this section, we first study some properties of the real pencil $\mathcal{Q}$, which make them distinct from the complex case. Then, we give the main ingredient to solve the *1-dimensional real problem* of Theorem 3.1.1, and we prove Theorem 3.1.2 (see Proposition 3.2.5 and Corollary 3.2.6).

3.2.1 More properties of real pencils of surfaces

It is known that the choice of a point $x_0 \in E$ induces an isomorphism $E \cong Pic_k(E)$ for any $k \in \mathbb{N}$. Since E is real and $\mathbb{R}E \neq \emptyset$, the real parts $\mathbb{R}E$ and $\mathbb{R}Pic(E)$ have the same topological type, we then have a homeomorphism $\mathbb{R}E \cong \mathbb{R}Pic_k(E)$ for any $k \in \mathbb{N}$. Suppose that $x_0 \in \mathbb{R}E$. The real part $\mathbb{R}E$ is either connected or disconnected with two connected components. In the former case, we write $\mathbb{R}E = \mathbb{S}^1$. In the later case, we write $\mathbb{R}E = \mathbb{S}^1_0 \sqcup \mathbb{S}^1_1$ and suppose that $x_0 \in \mathbb{S}^1_0$. We call $\mathbb{S}^1_0$ the **pointed component** and $\mathbb{S}^1_1$ the **non-pointed component** of $\mathbb{R}E$.

Remark 3.2.1. As we have seen in the previous chapter, if D is a homology class such that $D.S = 0$ then there no surface in the pencil $\mathcal{Q}$ which contains a curve realizing such class. Henceforth, in the sequel the classes D, to which we will mention, satisfy implicitly $D.S \neq 0$.

Lemma 3.2.2. *The surfaces in the real pencil $\mathcal{Q}$ containing a real rational curve in (X, τ) are in the real part of this pencil.*

Proof. Let C be a real rational curve in (X, τ). Suppose that $\Sigma \in \mathcal{Q}$ contains C. Since $\mathcal{Q}$ is a real pencil, we have $\tau(\Sigma) \in \mathcal{Q}$. Since C is real, we have $\tau(C) = C$. If $\Sigma \notin \mathbb{R}\mathcal{Q}$, i.e. $\tau(\Sigma) \neq \Sigma$, then $C \subset (\Sigma \cap \tau(\Sigma)) = E$ which is not possible since C is rational. Thus, $\tau(\Sigma) = \Sigma$. In other words, $\Sigma \in \mathbb{R}\mathcal{Q}$. □

Let $(\Sigma_t, \tau|_{\Sigma_t}) \in \mathbb{R}\mathcal{Q}$ be a real surface containing a real rational curve. We denote the surjective map $\psi_t : H_2(\Sigma_t; \mathbb{Z}) \to H_2(X; \mathbb{Z})$ induced from the inclusion of Σ_t in X.

Proposition 3.2.3. *Let $d \in H^{-\tau}(X; \mathbb{Z})$. Let $\alpha \in \{D, T(D)\} \subset \psi_t^{-1}(d)$ such that $D.S$ does not vanish. In a real surface $(\Sigma_t, \tau|_{\Sigma_t})$, one has that the vanishing cycle is $\tau|_{\Sigma_t}$-anti-invariant if and only if α is $\tau|_{\Sigma_t}$-anti-invariant.*

Proof. The sufficient condition is direct. Indeed, if

$$\begin{cases} \tau|_{\Sigma_t *}(D) = -D \\ \tau|_{\Sigma_t *}(T(D)) = -(D + (D.S)S) \end{cases}$$

then

$$(D.S)\tau|_{\Sigma_t *}(S) = \tau|_{\Sigma_t *}(T(D) - D) = -(D.S)\tau|_{\Sigma_t *}(S).$$

Hence, S is $\tau|_{\Sigma_t}$-anti-invariant.

We prove the necessary condition by two steps.

In the first step, we prove that $\tau|_{\Sigma_t *}(\{D, T(D)\}) = \{-D, -T(D)\}$. Suppose that $\alpha = D$. It is sufficient to prove that if D satisfying $\psi_t(D) = d \in H^{-\tau}(X; \mathbb{Z})$ then

— either $\tau|_{\Sigma_t *}(D) = -D$, i.e. D is $\tau|_{\Sigma_t}$-anti-invariant,

— or $\tau|_{\Sigma_t *}(D) = -(D + (D.S)S)$ where S is a generator of $\ker(\psi_t)$.

Indeed, we have $\tau_*(d) = -d$ then $\tau_*(\psi_t(D)) = -\psi_t(D)$. Since $\psi_t \circ \tau|_{\Sigma_t *} = \tau_* \circ \psi_t$, we obtain $\psi_t(\tau|_{\Sigma_t *}(D) + D) = 0$. Moreover, $\ker(\psi_t) = \mathbb{Z}S$ then there exists an integer $k \in \mathbb{Z}$ such that $\tau|_{\Sigma_t *}(D) + D = kS$. Since the intersection form is preserved via $\tau|_{\Sigma_t *}$, i.e. $(\tau|_{\Sigma_t *}(D))^2 = D^2$, we deduce that k is equal to either 0 or $-D.S$. Then, the first step is done.

In the second step, we prove that if $(\Sigma_t, \tau|_{\Sigma_t})$ is such that the vanishing cycle S is $\tau|_{\Sigma_t}$-anti-invariant in $H_2(\Sigma_t;\mathbb{Z})$, then D and $T(D)$ have to be $\tau|_{\Sigma_t}$-anti-invariant. Suppose that $\tau|_{\Sigma_t *}(D) = -T(D)$. One has $\tau|_{\Sigma_t *}(D) = -D - (D.S)S = \tau|_{\Sigma_t *}(T(D)) - (D.S)S$. This implies that

$$\tau|_{\Sigma_t *}(T(D)) - \tau|_{\Sigma_t *}(D) = (D.S)S.$$

Moreover,

$$\tau|_{\Sigma_t *}(T(D)) = \tau|_{\Sigma_t *}(D + (D.S)S) = \tau|_{\Sigma_t *}(D) + (D.S)\tau|_{\Sigma_t *}(S).$$

Hence, $S = \tau|_{\Sigma_t *}(S)$. This means that S is not $\tau|_{\Sigma_t}$-anti-invariant. Thus, the second step is done, and the proposition follows. □

As a consequence, the enumeration of real curves representing τ-anti-invariant classes in X can degenerate to the enumeration of real curves representing $\tau|_{\Sigma_t}$-anti-invariant classes in such Σ_t.

Let $\underline{x}_d$ be a real configuration of $D.E$ points in E. Suppose that $\alpha|_{E,\Sigma_t} = [\underline{x}_d]$ in $Pic_{D.E}(E)$ where α is as in the above proposition. We consider the following maps

$$\begin{aligned} \mu_t : Pic(\Sigma_t) &\to Pic(\Sigma_t) \\ \sum_i n_i[C_i] &\mapsto \sum_i n_i[\tau|_{\Sigma_t}(C_i)] \end{aligned}$$

and

$$\begin{aligned} \mu_E : Pic(E) &\to Pic(E) \\ \sum_i n_i[P_i] &\mapsto \sum_i n_i[\tau|_E(P_i)]. \end{aligned}$$

The restriction of $Pic(\Sigma_t)$ to $Pic(E)$ induces the following commutative diagram:

$$\begin{array}{ccc} Pic(\Sigma_t) & \xrightarrow{\mu_t} & Pic(\Sigma_t) \\ \downarrow & & \downarrow \\ Pic(E) & \xrightarrow{\mu_E} & Pic(E) \end{array}$$

One can show that if α is $\tau|_{\Sigma_t}$-anti-invariant then $\mu_E(\alpha|_{E,\Sigma_t}) = \alpha|_{E,\Sigma_t}$. Suppose that $\alpha = D$. Since $\mu_t = -\tau|_{\Sigma_t *}$, one has $D|_{E,\Sigma_t} = \mu_t(D)|_{E,\Sigma_t}$. The above commutative diagram implies that $\mu_E(D|_{E,\Sigma_t}) = \mu_t(D)|_{E,\Sigma_t}$. Hence, we get $D|_{E,\Sigma_t} = \mu_E(D|_{E,\Sigma_t})$.

As a consequence, if α is $\tau|_{\Sigma_t}$-anti-invariant then $\alpha|_{E,\Sigma_t} \in \mathbb{R}Pic(E)$.

We consider the following ramified double covering map

$$\pi : Pic(E) \to Pic(E)/_{x\sim\phi(\overline{D})-x}\ ,\ x \mapsto \overline{x}.$$

The critical points are the solutions of equation $\{x \in Pic(E) :\ 2x = \phi(\overline{D})\}$. One has a homeomorphism $\mathbb{R}(Pic(E)/_{x\sim\phi(\overline{D})-x}) \cong \mathbb{S}^1$, and moreover

$$\begin{aligned}\pi^{-1}(\mathbb{R}(Pic(E)/_{x\sim\phi(\overline{D})-x})) = &\Big(\{x, \phi(\overline{D}) - x\} \subset \mathbb{R}Pic(E)\Big)\\ &\cup \Big(\{y, \tau|_E(y)\} : y + \tau|_E(y) = \phi(\overline{D})\Big).\end{aligned}$$

These two types of real divisors on $\mathbb{R}(Pic(E)/_{x\sim\phi(\overline{D})-x})$ are as illustrated in Figure 3.1.

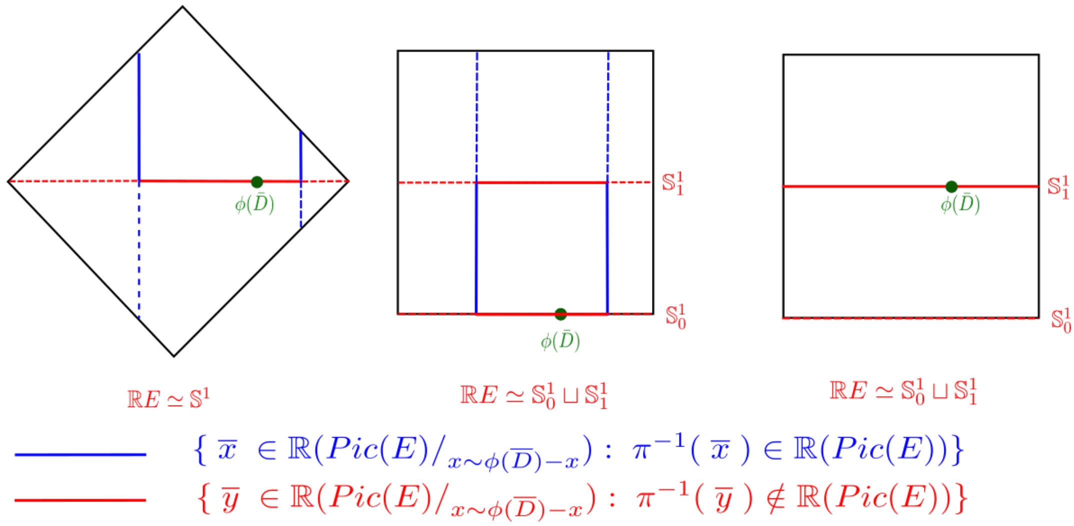

Figure 3.1 – Real divisors in the Picard group $\mathbb{R}(Pic(E)/_{x\sim\phi(\overline{D})-x})$ of a real elliptic curve with non-empty real part.

3.2.2 Enumerations on pencils of real del Pezzo surfaces

We keep the same notations as in Section 2.2: the vanishing cycle S of the Lefschetz fibration induced from the pencil $\mathcal{Q}$; the constant map ϕ whose image is the divisor $\phi(\overline{D}) = (2D + (D.S)S)|_{E,\Sigma_t}$ in $Pic_{2D.E}(E)$ which is independent of $\Sigma_t \in \mathcal{Q}$; and the degree $(D.S)^2$ map $\phi_{\overline{D}}$. Recall that the map $\phi_{\overline{D}}$ is given as

$$\begin{aligned}\phi_{\overline{D}} : \mathcal{Q} &\longrightarrow Pic_{D.E}(E)/_{x\sim\phi(\overline{D})-x}\\ \Sigma_t &\longmapsto \overline{D|_{E,\Sigma_t}}\end{aligned}$$

Suppose that $\{L, T(L)\} \subset H_2(\Sigma_t;\mathbb{Z})$ is as in Lemma 2.2.13. Recall in particular that $\phi(\overline{L})$ is in $Pic_{2L.E}(E)$ and $\phi_{\overline{L}}$ is an isomorphism.

We have seen that the divisor $\phi(\overline{D})$ is real, for any $D \in H^{-\tau}(\Sigma_t;\mathbb{Z})$ (see Proposition 3.2.3). In particular, the divisor $\phi(\overline{L})$ is real, and we have $\phi_{\overline{D}}$ and $\phi_{\overline{L}}$ are real maps.

Let $\mathcal{R}_d(\underline{x}_d)$ be a set of real irreducible rational curves in X representing the given homology class $d \in H^{-\tau}(X;\mathbb{Z})$ and passing through a real configuration $\underline{x}_d$ of k_d points in E.

Lemma 3.2.4. *The divisor $\Big(D-|D.S|L\Big)|_{E,\Sigma_t}$ of degree $|(D.E)-|D.S|(L.E)|$ in $Pic(E)$, denoted by $\phi(\overline{L},\overline{D})$, does not depend on the choice of surface Σ_t in the pencil $\mathcal{Q}$.*

Proof. Clearly that the divisor $\Big(D - |D.S|L\Big)|_{E,\Sigma_t}$ has degree $|(D.E) - |D.S|(L.E)|$. Without loss of generality, suppose that $D.S > 0$. We have

$$\begin{aligned}2\Big(D-(D.S)L\Big)|_{E,\Sigma_t} &= \Big(2D-2(D.S)L\Big)|_{E,\Sigma_t}\\ &= \Big((2D+(D.S)S)-(D.S)(2L+S)\Big)|_{E,\Sigma_t}.\end{aligned}$$

Since $(2L + S)|_{E,\Sigma_t} = \phi(\overline{L})$ and $(2D + (D.S)S)|_{E,\Sigma_t} = \phi(\overline{D})$ are both constant with respect to Σ_t, one has $2\Big(D-(D.S)L\Big)|_{E,\Sigma_t}$ is constant with respect to Σ_t. Thus, by continuity, the divisor $\Big(D-(D.S)L\Big)|_{E,\Sigma_t}$ does not depend on Σ_t neither. □

Proposition 3.2.5. *Fix $\Sigma \in \mathcal{Q}$ a real non-singular surface. Let $d \in H^{-\tau}(X;\mathbb{Z})$ and $\alpha \in \{D, T(D)\} \subset \psi^{-1}(d)$. Then, the real surfaces $(\Sigma_t, \tau|_{\Sigma_t}) \in \mathbb{R}\mathcal{Q}$, which contain a curve in the set $\mathcal{R}_d(\underline{x}_d)$, satisfy that the vanishing cycle is $\tau|_{\Sigma_t}$-anti-invariant.*

Furthermore, such real surfaces correspond to the real solutions $l \in \mathbb{R}Pic(E)$ of the equation

$$|D.S|l = [\underline{x}_d] - \phi(\overline{L},\overline{D}). \tag{3.1}$$

Proof. The first part is followed directly from the fact that if the vanishing cycle is $\tau|_{\Sigma_t}$-invariant then $D \notin H_2^{-\tau}(\Sigma_t;\mathbb{Z})$, for any $D \in H_2(\Sigma_t;\mathbb{Z})$.

We now prove the second part. Since the equality $D.S = -T(D).S$ holds, one can suppose that $D.S > 0$. Then, one has

$$\phi(\overline{L},\overline{D}) = \Big(D-(D.S)L\Big)|_{E,\Sigma_t} = \Big(D+(T(D).S)L\Big)|_{E,\Sigma_t}.$$

For every $\Sigma_t \in \mathcal{Q}$ that contains a curve in $\mathcal{R}_d(\underline{x}_d)$, suppose that this curve represents homology class $D \in H_2^{-\tau}(\Sigma_t;\mathbb{Z})$. Then one has $[\underline{x}_d] = D|_{E,\Sigma_t}$ is a real divisor in $Pic_{D.E}(E)$.

One can rewrite $D = (D.S)L + D - (D.S)L$. Restricting to E, with a remark that L is also $\tau|_{\Sigma_t}$-anti-invariant, one gets

$$[\underline{x}_d] = (D.S)L|_{E,\Sigma_t} + \Big(D - (D.S)L\Big)|_{E,\Sigma_t}$$

is an equation in $Pic(E)$.

By Lemma 3.2.4, the divisor $\Big(D - (D.S)L\Big)|_{E,\Sigma_t} = \phi(\overline{L}, \overline{D})$ does not depend on the choice of surface Σ_t in the pencil. Thus, every surface $\Sigma_t \in \mathcal{Q}$, which contains a curve in $\mathcal{R}_d(\underline{x}_d)$ of homology class $D \in H_2^{-\tau}(\Sigma_t; \mathbb{Z})$ where $D.S > 0$, satisfies the following equation

$$(D.S)L|_{E,\Sigma_t} = [\underline{x}_d] - \phi(\overline{L}, \overline{D})$$

in $Pic(E)$.

Therefore, a real surface $\Sigma_t \in \mathbb{R}\mathcal{Q}$, on which a curve in $\mathcal{R}_d(\underline{x}_d)$ represents a $\tau|_{\Sigma_t}$-anti-invariant class $\alpha \in \{D, T(D)\}$, satisfies the equation

$$|D.S|L|_{E,\Sigma_t} = [\underline{x}_d] - \phi(\overline{L}, \overline{D}) \tag{3.2}$$

in $Pic(E)$.

Moreover, the following isomorphism is real

$$\phi_{\overline{L}} : \mathcal{Q} \xrightarrow{\cong} Pic_{L.E}(E)/_{x \sim \phi(\overline{L}) - x}, \quad \Sigma_t \mapsto \overline{L|_{E,\Sigma_t}}.$$

Together with the remarks about real structures on the pencil $\mathcal{Q}$ and that on the quotient of the Picard group of its base locus, one deduces that such real surfaces $\Sigma_t \in \mathbb{R}\mathcal{Q}$ are exactly the solutions in $\mathbb{R}Pic(E)$ of Equation (3.2). The proposition then follows. □

As a consequence, the set of such real surfaces in the pencil $\mathcal{Q}$ is in one-to-one correspondence to the set of real solutions of Equation (3.1). Hence, the number of such real surfaces depends on the number of real $|D.S|$-torsion points on $\mathbb{R}Pic(E)$. This means that this number depends on the real structure on E and the position of the real divisors $\phi(\overline{L}, \overline{D})$ and $[\underline{x}_d]$ on $\mathbb{R}Pic(E)$ if $\mathbb{R}E$ is not connected. More precisely, this number is described in the following corollary, see also Figure 3.2.

Corollary 3.2.6. *The set of real surfaces $(\Sigma_t, \tau|_{\Sigma_t})$ in $\mathbb{R}\mathcal{Q}$, in which each one contains a real curve of class $\alpha \in \{D, T(D)\} \subset \psi^{-1}(d)$ in $\mathcal{R}_d(\underline{x}_d)$, contains exactly*

$a)$ $|D.S|$ *elements if either $\mathbb{R}E$ is connected or $\mathbb{R}E$ is disconnected and $D.S$ is odd;*

$b)$ $2|D.S|$ *elements if $\mathbb{R}E$ is disconnected and $D.S$ is even and both $[\underline{x}_d]$ and $\phi(\overline{L}, \overline{D})$ are on the same component of $\mathbb{R}E$;*

$c)$ 0 *element if $\mathbb{R}E$ is disconnected and $D.S$ is even and $[\underline{x}_d]$ and $\phi(\overline{L}, \overline{D})$ are on the different components of $\mathbb{R}E$.*

Proof. Firstly, we count the real $|D.S|$-torsion points, i.e. the real solutions of the equation $|D.S|l = 0$, which depend on the real structure on E. More precisely,

1. if $\mathbb{R}E$ is connected then there are exactly $|D.S|$ real $|D.S|$-torsion points;
2. if $\mathbb{R}E = \mathbb{S}^1_0 \sqcup \mathbb{S}^1_1$ disconnected then
   — if $|D.S|$ is odd, then there are exactly $|D.S|$ real $|D.S|$-torsion points which are in the pointed component $\mathbb{S}^1_0$,
   — if $|D.S|$ is even, then there are exactly $2|D.S|$ real $|D.S|$-torsion points such that half of which are in each connected component of $\mathbb{R}E$.

Then, we count the real solutions l of Equation (3.1), which depend on the position of the real divisors $\phi(\overline{L}, \overline{D})$ and $[\underline{x}_d]$ on $\mathbb{R}E$ when $\mathbb{R}E$ is disconnected. More precisely,

1. if $\mathbb{R}E = \mathbb{S}^1$ then there are exactly $|D.S|$ real solutions of Equation (3.1);
2. if $\mathbb{R}E = \mathbb{S}^1_0 \sqcup \mathbb{S}^1_1$ then
   - if $|D.S|$ is odd, then there are exactly $|D.S|$ real solutions of Equation (3.1) whether $\phi(\overline{L}, \overline{D}) - [\underline{x}_d]$ is in the pointed component $\mathbb{S}^1_0$ or in $\mathbb{S}^1_1$;
   - if $|D.S|$ is even then
     — there are exactly $2|D.S|$ real solutions of Equation (3.1) if $[\underline{x}_d] - \phi(\overline{L}, \overline{D})$ is in the pointed component $\mathbb{S}^1_0$ (or equivalently, if $\phi(\overline{L}, \overline{D})$ and $[\underline{x}_d]$ are in the same connected component of $\mathbb{R}E$), and
     — there is no real solution of Equation (3.1) if $[\underline{x}_d] - \phi(\overline{L}, \overline{D})$ is in $\mathbb{S}^1_1$ (or equivalently, if $\phi(\overline{L}, \overline{D})$ and $[\underline{x}_d]$ are in the different connected components of $\mathbb{R}E$), see Figure 3.2.

Thus, the corollary follows. □

3.2.3 A proof of Theorem 3.1.2

Let $\phi(\overline{L}, \overline{D}) \in Pic_{|D.E - |D.S| L.E|}(E)$ be a divisor as in Lemma 3.2.4.

We can choose the two real surfaces generated a pencil of surfaces in the linear system $|-\frac{1}{2}K_X|$ such that their real parts intersect in a maximal elliptic real curve. Note that a real elliptic curve E is maximal is equivalent to the real part $\mathbb{R}E$ is disconnected.

Proposition 3.2.5 and Corollary 3.2.6.c) imply that if the divisor of real configurations of points $\underline{x}_d$ and the divisor $\phi(\overline{L}, \overline{D})$ do not belong to the same component of $\mathbb{R}E$, then there is no real irreducible rational curve of class d passing through $\underline{x}_d$.

A choice of such a configuration $\underline{x}_d$ is described as follows. Suppose that the divisor $\phi(\overline{L}, \overline{D}) \in \mathbb{S}^1_0$ is in the pointed component of $\mathbb{R}E$, then we choose the configuration $\underline{x}_d$ such that it has an odd number of real points on the non-pointed component of $\mathbb{R}E$. If the divisor $\phi(\overline{L}, \overline{D}) \in \mathbb{S}^1_1$ is in the non-pointed component of $\mathbb{R}E$, then we choose the configuration $\underline{x}_d$ such that it has at least one real point and that an even number (possible be zero) of its real points is on the non-pointed component of $\mathbb{R}E$. Moreover, in both cases, such real configurations of points are contained in a semi-algebraic open set in $\mathbb{R}(Sym^{k_d}(X))$ the space of real points of the symmetric power of X. Indeed, this follows by a natural extension for the case $X = \mathbb{C}P^3$ (Theorem 10 in [Kol15]).

Hence, Theorem 3.1.2 follows.

One can consult some explicit applications of Theorem 3.1.2 in the Section 3.6.

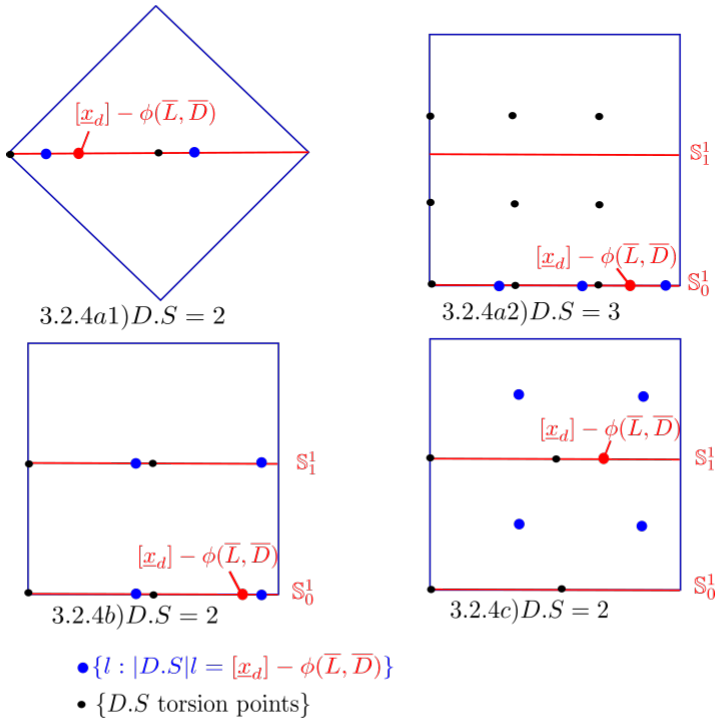

Figure 3.2 – The real solutions of Equation (3.1) associate to three cases of Corollary 3.2.6.

3.3 Spinor states of real rational curves on real del Pezzo varieties of dimension three - revisited

In Section 1.3.3, we distinguished the real isotopy class of a real degree $(k_d - 2)$ holomorphic line subbundle of the rank 2 holomorphic normal bundle $\mathcal{N}_f = f^*TX/T\mathbb{C}P^1$. And, we defined spinor states of a real rational curve $C = Im(f : (\mathbb{C}P^1, \tau_1) \to (X, \tau))$ where f is a real balanced immersion and $f_*[\mathbb{C}P^1] = d \in H^{-\tau}(X; \mathbb{Z})$.

Throughout this section, we suppose that $C = f(\mathbb{C}P^1)$ is contained in a surface Σ, where $\Sigma \in | - \frac{1}{2}K_X|$ a real non-singular surface. Suppose also that C represents a homology class $D \in H_2^{-\tau}(\Sigma; \mathbb{Z})$, where $D \in \psi^{-1}(d)$. We will study some properties of the holomorphic line subbundle $\mathcal{N}'_f = f^*T\Sigma/T\mathbb{C}P^1$, which is a holomorphic line subbundle of $\mathcal{N}_f$, and then relate such subbundle to the definition of spinor states of curves.

Remark 3.3.1. The isotopy class of the normal bundle of C in Σ does not depend on the curve C but on homology class D of C in $H_2(\Sigma; \mathbb{Z})$, then we can write it as $\mathcal{N}_{D/\Sigma}$ instead of $\mathcal{N}_{C/X}$. With this notation, we have $\mathcal{N}'_f = f^*\mathcal{N}_{D/\Sigma}$.

By the adjunction formula, one has

$$\deg(\mathcal{N}_{D/\Sigma}) = c_1(\Sigma) \cdot D - 2 = k_D - 1.$$

This implies that $\deg(\mathcal{N}'_f) = k_d - 2$. Hence, the real line bundle $\mathbb{R}\mathcal{N}'_f$ is orientable if k_D is odd and is non-orientable otherwise.

Recall in Section 1.3.3 that there are two isotopy classes, denoted by E^+ and E^-, of the real part of a holomorphic real line subbundle of degree $(k_d - 2)$ of $\mathcal{N}_f$ which are characterised by the direction in which the real part of a fiber rotates in $\mathbb{R}\mathcal{N}_f$.

Note that $k_D = k_{T(D)}$ for any $\{D, T(D)\} \subset H_2(\Sigma;\mathbb{Z})$, where T is the monodromy transformation $T(D) = D + (D.S)S$.

Definition 3.3.2. Let $f : \mathbb{C}P^1 \to \Sigma \subset X$ be a real immersion such that $f_*[\mathbb{C}P^1] = D$ in $H_2^{-\tau}(\Sigma;\mathbb{Z})$ and let $\mathbb{R}\mathcal{N}'_f = \mathbb{R}f^*\mathcal{N}_{D/\Sigma}$ be as described above. We define the following homomorphism:

$$\epsilon : H_2(\Sigma;\mathbb{Z}) \to \mathbb{Z}_2$$
$$D \mapsto \begin{cases} k_D + 1 \mod 2, \text{ if } \mathbb{R}\mathcal{N}'_f = E^+ \\ k_D \mod 2, \text{ if } \mathbb{R}\mathcal{N}'_f = E^-. \end{cases}$$

Lemma 3.3.3. *Let $\{L, T(L)\} \subset H_2(\Sigma;\mathbb{Z})$ be such that $|L.S| = 1$. One has*

$$\epsilon(T(L)) \neq \epsilon(L).$$

Proof. We have seen that a couple of divisors $L_t|_E \neq T(L_t)|_E$ such that $|L_t.S| = 1$ determines a unique non-singular surface Σ_t in the pencil $\mathcal{Q}$ (Corollary 2.2.15).

Let $\{L_0, T(L_0)\} \subset H_2^{-\tau}(\Sigma_0;\mathbb{Z})$. Let $\mathbb{R}\mathcal{N}'_a = \mathbb{R}f^*\mathcal{N}_{L_0/\Sigma}$ and $\mathbb{R}\mathcal{N}'_b = \mathbb{R}f^*\mathcal{N}_{T(L_0)/\Sigma}$ be the real part of the normal bundles of L_0 and $T(L_0)$ respectively. By monodromy phenomenon, the role of L_0 and $T(L_0)$ in $H_2^{-\tau}(\Sigma_0;\mathbb{Z})$ interchanges when we make a loop around a singular fiber. In particular, L_0 and $T(L_0)$ rotate in the different direction in Σ_0 and also when we go along the pencil from Σ_0 to itself around a singular surface. Hence, we deduce that $\mathbb{R}\mathcal{N}'_a$ and $\mathbb{R}\mathcal{N}'_b$ rotate in different direction in $\mathbb{R}X$, see Figure 3.3.

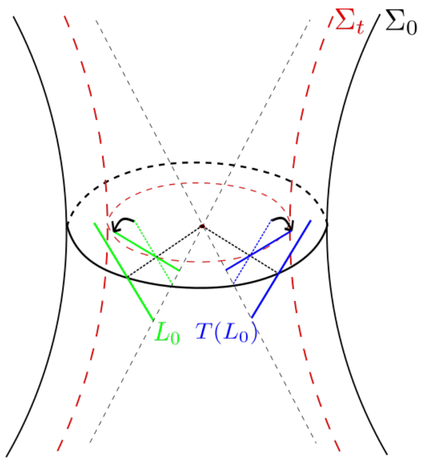

Figure 3.3 – L_0 and $T(L_0)$ rotate in different direction in $\mathbb{R}X$.

$\square$

Now, we can suppose L and $T(L)$ are such that $L.S = -1$ and $\mathbb{R}\mathcal{N}'_{L/\Sigma} = E^+$. As a consequence, $T(L).S = 1$ and $\mathbb{R}\mathcal{N}'_{T(L)/\Sigma} = E^-$.

Proposition 3.3.4. *For any* $\{D, T(D)\} \subset H_2^{-\tau}(\Sigma;\mathbb{Z})$ *such that* $D.S \neq 0$*, one has*

$$\epsilon(T(D)) = \epsilon(D) + 1.$$

Proof. By Lemma 3.3.3, this proposition is true for $D = L$.

Let Σ_0 be a real non-singular surface in the pencil $\mathcal{Q}$. Let Σ_t be the first order real deformation of Σ_0 in $\mathcal{Q}$.

Claim 1: The isotopy class of the real line subbundle $\mathbb{R}f^*\mathcal{N}_{D/\Sigma_0}$ in $\mathbb{R}\mathcal{N}_f$ is determined by the direction of the vector $\frac{dD|_{E,\Sigma_t}}{dt}\Big|_{t=0}$.

A proof of Claim 1 is similar to the proof of Proposition 3.4 in [BG16], we stress on some main points that might be useful later.

We fix a point x_0 in the transverse intersection of $f(\mathbb{C}P^1)$ and the base locus E of the pencil $\mathcal{Q}$ such that $p_0 = f^{-1}(x_0) \in \mathbb{R}P^1$. By considering the first order real deformation f_t of f in the pencil $\mathcal{Q}$ such that $f_t(\mathbb{C}P^1)$ passes through all points of intersection $f(\mathbb{C}P^1) \cap E$ except x_0 for all t, we define a degree $(k_d - 1)$ holomorphic line subbundle M in $\mathcal{N}_f$ such that the fibre $M|_{p_0} = \sigma(p_0)$, where $\sigma : \mathbb{C}P^1 \to \mathcal{N}_f$ is a non-null holomorphic section vanishing on $f^{-1}(E \setminus \{x_0\})$. Since this deformation stays in the base locus E, one has $\sigma(p_0) \in f^*\mathcal{N}_{D/\Sigma_0}$. Then, we can compare the mutual position between fibres $\mathbb{R}M|_{p_t}$ and $\mathbb{R}f^*\mathcal{N}_{D/\Sigma_0}|_{p_t}$, where $p_t \in \mathbb{R}P^1$ in the neighborhood of p_0, by proving that the isotopy class of $\mathbb{R}f^*\mathcal{N}_{D/\Sigma_0}$ is determined by the direction of the vector $\sigma(p_0)$. Moreover, the deformation $x_t = f(p_t)$ of x_0 as an intersection point of $f_t(\mathbb{C}P^1) \cap E$ is determined by the deformation of the divisor class $D|_{E,\Sigma_t}$ when t approaches to 0. Hence, Claim 1 follows.

We now consider the deformation of divisors $D|_{E,\Sigma_t}$ and $T(D)|_{E,\Sigma_t}$ on $Pic_{D.E}(E)$ when t approaches to 0.

Claim 2: The direction of vectors $\frac{dD|_{E,\Sigma_t}}{dt}\Big|_{t=0}$ and $\frac{dT(D)|_{E,\Sigma_t}}{dt}\Big|_{t=0}$ are opposite.

Indeed, this claim follows directly from the fact that $D|_{E,\Sigma_t} + T(D)|_{E,\Sigma_t} = \phi(\overline{D})$ is a constant divisor class for all t (see Figure 3.4).

Claim 3: The class D is in the same isotopy class as class L if and only if $T(D)$ is not.

This claim follows by Lemma 3.2.4. Indeed, one has

$$\frac{dD|_{E,\Sigma_t}}{dt}\Big|_{t=0} = |D.S|\frac{dL|_{E,\Sigma_t}}{dt}\Big|_{t=0}.$$

Since L and $T(L)$ are not in the same isotopy class, then so are D and $T(D)$. Hence, the statement of the proposition holds.

□

Proposition 3.3.5. *For any* $D, D' \in H_2(\Sigma;\mathbb{Z})$ *such that* $D.S \neq 0$ *and* $D'.S \neq 0$*, one has*

$$\epsilon(D) = \epsilon(D') \Leftrightarrow (D.S)(D'.S) > 0.$$

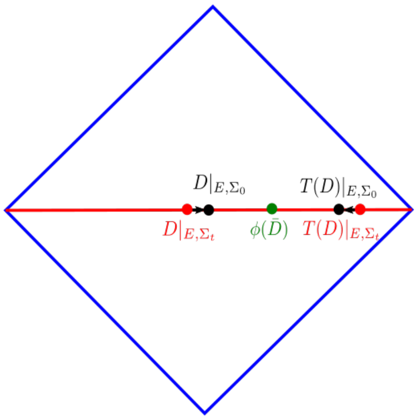

Figure 3.4 – Two isotopy classes of a real line subbundle of $\mathbb{R}\mathcal{N}_f$ are represented in $Pic(E)$ in the case $\mathbb{R}E$ is connected.

Proof. On the pencil $\mathcal{Q}$, we suppose that Σ_0 is the fiber at $t_0 \in \mathbb{C}P^1$ and Σ_t is the fiber at $t \in \mathbb{C}P^1$ in the neighborhood of t_0. We have two following claims.

Claim 1: $\frac{dD|_{E,\Sigma_t}}{dt}\Big|_{t=0} = \frac{-D.S}{2}\frac{dS|_{E,\Sigma_t}}{dt}\Big|_{t=0}$.

Indeed, this claim is directly implied from the fact that

$$2D|_{E,\Sigma_0} + (D.S)S|_{E,\Sigma_0} = 2D|_{E,\Sigma_t} + (D.S)S|_{E,\Sigma_t}(= \phi(\overline{D})).$$

Claim 2: $\frac{dD|_{E,\Sigma_t}}{dt}\Big|_{t=0} \times \frac{dD'|_{E,\Sigma_t}}{dt}\Big|_{t=0} > 0 \Leftrightarrow (D.S)(D'.S) > 0.$

Indeed, **Claim 1** implies that

$$\frac{dD|_{E,\Sigma_t}}{dt}\Big|_{t=0} \times \frac{dD'|_{E,\Sigma_t}}{dt}\Big|_{t=0} = \Big(\frac{-D.S}{2}\frac{dS|_{E,\Sigma_t}}{dt}\Big|_{t=0}\Big) \times \Big(\frac{-D'.S}{2}\frac{dS|_{E,\Sigma_t}}{dt}\Big|_{t=0}\Big).$$

Hence,

$$\frac{dD|_{E,\Sigma_t}}{dt}\Big|_{t=0} \times \frac{dD'|_{E,\Sigma_t}}{dt}\Big|_{t=0} = \frac{(D.S)(D'.S)}{4}\Big(\frac{dS|_{E,\Sigma_t}}{dt}\Big|_{t=0}\Big)^2.$$

Then, the statement follows. □

3.4 Welschinger's signs of real rational curves on real del Pezzo varieties of dimension two and three

In this section, we study the properties of the Pin_2^--structure on $O(T\mathbb{R}\Sigma)$ (where $\Sigma \in |-\frac{1}{2}K_X|$ is a real non-singular surface), which is restricted from a $Spin_3$-structure on $SO(T\mathbb{R}X)$ (Proposition 3.4.2). The corresponding quasi-quadratic enhancement allows us to relate spinor states of rational curves in (X, τ) and Welschinger's signs of these curves in $(\Sigma, \tau|_\Sigma)$ (Proposition 3.4.4).

3.4.1 Quasi-quadratic enhancements and restricted Pin_2^--structures

Let $\mathcal{N}_{\Sigma/X}$ be the holomorphic normal line bundle of Σ in X. The Riemannian metric on $\mathbb{R}X$ allows us to identify the real bundles $\mathbb{R}\mathcal{N}_{\Sigma/X}$ and $\mathcal{N}_{\mathbb{R}\Sigma/\mathbb{R}X}$.

Lemma 3.4.1. *One has $\mathcal{N}_{\mathbb{R}\Sigma/\mathbb{R}X} \cong \det T\mathbb{R}\Sigma$.*

Proof. On the one hand, by the adjunction formula, we have

$$\Sigma|_\Sigma = K_\Sigma - K_X|_\Sigma.$$

One has $\Sigma \in |-\frac{1}{2}K_X|$ implies that $\Sigma|_\Sigma = -K_\Sigma$.

Moreover, $c_1(\mathcal{N}_{\Sigma/X}) = \Sigma|_\Sigma$. Thus, $c_1(\mathcal{N}_{\Sigma/X}) = -K_\Sigma$.

On the other hand, $c_1(\det T\Sigma) = c_1(T\Sigma) = -K_\Sigma$. Therefore, $c_1(\mathcal{N}_{\Sigma/X}) = c_1(\det T\Sigma)$.

We consider two real line bundles over $\mathbb{R}\Sigma$: $\mathcal{N}_{\mathbb{R}\Sigma/\mathbb{R}X}$ and $\det T\mathbb{R}\Sigma$. The above calculations imply that

$$w_1(\mathcal{N}_{\mathbb{R}\Sigma/\mathbb{R}X}) = w_1(\det T\mathbb{R}\Sigma).$$

This induces the line bundle isomorphism $\mathcal{N}_{\mathbb{R}\Sigma/\mathbb{R}X} \cong \det T\mathbb{R}\Sigma$. □

Proposition 3.4.2. *A $Spin_3$-structure on $SO(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X})$ can be restricted naturally to a Pin_2^--structure on $O(T\mathbb{R}\Sigma)$.*

In particular, if $\mathbb{R}X$ and $\mathbb{R}\Sigma$ are oriented then $SO(T\mathbb{R}\Sigma, \mathfrak{o}_{T\mathbb{R}\Sigma})$ inherits a $Spin_2$-structure from a $Spin_3$-structure on $SO(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X})$.

Proof. It is sufficient to prove that $O(T\mathbb{R}\Sigma)$ can be embedded as a subspace in the restriction $SO(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X})|_{\mathbb{R}\Sigma}$. Proposition 1.4.3 implies a natural bijection

$$\mathcal{P}_2^-(T\mathbb{R}\Sigma) \cong \mathcal{SP}_3(T\mathbb{R}\Sigma \oplus \det T\mathbb{R}\Sigma, \mathfrak{o}_{T\mathbb{R}\Sigma\oplus\det T\mathbb{R}\Sigma}).$$

Up to homotopy, Lemma 3.4.1 implies the decomposition

$$T\mathbb{R}X = T\mathbb{R}\Sigma \oplus \det T\mathbb{R}\Sigma.$$

Let $(x, (v_1, v_2)) \in O(T\mathbb{R}\Sigma)$. Then one can construct a unique vector $v_3 := v_1 \wedge v_2$ in $\det T\mathbb{R}\Sigma$ such that $(x, (v_1, v_2, v_3))$ forms an element in $SO(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X})|_{\mathbb{R}\Sigma}$, whose orientation is agree with that coming from the $Spin_3$-structure on $SO(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X})$. □

We denote the above bundle map by $\iota_\Sigma : O(T\mathbb{R}\Sigma) \hookrightarrow SO(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X})$.

Let $\mathfrak{s}_X := (Spin(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X}), q_{T\mathbb{R}X})$ be a $Spin_3$-structure on $SO(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X})$. Let $\mathfrak{s}_{X|\Sigma} = (Pin^-(X|\Sigma), q_{X|\Sigma})$ be the restricted Pin_2^--structure to $O(T\mathbb{R}\Sigma)$ from the $Spin_3$-structure $\mathfrak{s}_X$. We can describe this structure as follows. The commutativity of diagram in Section 1.4.1 implies the following commutative diagram of principal bundle maps

$$\begin{tikzcd} Pin^-(X|\Sigma) \arrow[r] \arrow[d, "q_{X|\Sigma}"] & Spin(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X}) \arrow[d, "q_{T\mathbb{R}X}"] \\ O(T\mathbb{R}\Sigma) \arrow[r, hook, "\iota_\Sigma"] & SO(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X}) \end{tikzcd}$$

One has

— $Pin^-(X|\Sigma) \cong \iota_\Sigma^*(Spin(T\mathbb{R}X, \mathfrak{o}_{T\mathbb{R}X})|_\Sigma)$ as a Pin_2^--principal bundle isomorphism,

— $q_{X|\Sigma} = \iota_\Sigma^*(q_{T\mathbb{R}X})$.

Recall that we have a natural homomorphism

$$\rho : H_2^{-\tau}(\Sigma;\mathbb{Z}) \to H_1(\mathbb{R}\Sigma;\mathbb{Z}_2), \quad D \mapsto \rho D.$$

Hence, for every $D \in H_2^{-\tau}(\Sigma;\mathbb{Z})$ and every Pin_2^--structure $\mathfrak{p}_{T\mathbb{R}\Sigma}$ on $O(T\mathbb{R}\Sigma)$, we can associate a quasi-quadratic enhancement $s^{\mathfrak{p}_{T\mathbb{R}\Sigma}}$ valuating ρD (see Section 1.4). In particular, the Pin_2^--structure $\mathfrak{s}_{X|\Sigma}$ on $O(T\mathbb{R}\Sigma)$ implies the following properties on $s^{\mathfrak{s}_{X|\Sigma}}$.

Proposition 3.4.3. *Suppose that* $\ker(\psi : H_2(\Sigma;\mathbb{Z}) \to H_2(X;\mathbb{Z})) \cong \mathbb{Z}$.
Let $S \in H_2^{-\tau}(\Sigma;\mathbb{Z})$ *be a generator of* $\ker(\psi)$. *One has*

1. $s^{\mathfrak{s}_{X|\Sigma}}(\rho S) = 0$.
2. *Furthermore, for all* $D \in H_2^{-\tau}(\Sigma;\mathbb{Z})$, *one has*

$$s^{\mathfrak{s}_{X|\Sigma}}(\rho(D + (D.S)S)) \equiv s^{\mathfrak{s}_{X|\Sigma}}(\rho D) + (D.S) \mod 2.$$

Proof. 1. One has $\psi(S) = 0 \in H_2^{-\tau}(X;\mathbb{Z})$ and a natural homomorphism

$$\rho : H_2^{-\tau}(X;\mathbb{Z}) \to H_1(\mathbb{R}X;\mathbb{Z}_2), \quad d \mapsto \rho d.$$

Hence, $\rho\psi(S) = 0 \in H_1(\mathbb{R}X;\mathbb{Z}_2)$. Since S is also the vanishing cycle of a Lefschetz fibration, $\rho\psi(S)$ bounds a two dimensional disk $\mathbb{D}^2$ in $\mathbb{R}X$. Then, the statement follows by applying the same argument as in proof of Lemma 1.4.10(1) with a remark that the $Spin_3$-structure over $\mathbb{D}^2$ is induced canonically from the $Spin_2$-one.

2. Claim **1** and the properties of a quasi-quadratic enhancement imply that (with abuse of notation, in this proof, we write $\omega_1(\rho D)$ for the dual pairing $\omega_1(\mathbb{R}\Sigma) \cdot (\rho D)$)

$$\begin{aligned}
&s^{\mathfrak{s}_{X|\Sigma}}(\rho(D + (D.S)S)\\
=&s^{\mathfrak{s}_{X|\Sigma}}(\rho D + (D.S)\rho S)\\
=&s^{\mathfrak{s}_{X|\Sigma}}(\rho D) + (D.S)s^{\mathfrak{s}_{X|\Sigma}}(\rho S) + (D.S)(\rho D.\rho S) + (D.S)\omega_1(\rho D)\omega_1(\rho S)\\
\equiv&s^{\mathfrak{s}_{X|\Sigma}}(\rho D) + 0 + (D.S)^2 + (D.S)(D)^2(-2) \mod 2\\
\equiv&s^{\mathfrak{s}_{X|\Sigma}}(\rho D) + (D.S)^2 \mod 2\\
\equiv&s^{\mathfrak{s}_{X|\Sigma}}(\rho D) + (D.S) \mod 2.
\end{aligned}$$

Thus, the proposition follows. □

Let $C = Im(f : (\mathbb{C}P^1, \tau_1) \to (\Sigma, \tau))$ be a rational immersed real curve in Σ representing homology class $D \in H_2^{-\tau}(\Sigma;\mathbb{Z})$. Note that $f_*[\mathbb{R}P^1]_{\mathbb{Z}_2} = \rho D$.

For every Pin_2^--structure $\mathfrak{p}_{T\mathbb{R}\Sigma}$ on $O(T\mathbb{R}\Sigma)$, we define

$$s_{\mathfrak{p}_{T\mathbb{R}\Sigma}}(C) := s_{\mathfrak{p}_{T\mathbb{R}\Sigma}}(f|_{\mathbb{R}P^1}) \in \{0, 1\}.$$

Note that $\delta_H(C) = \delta(f|_{\mathbb{R}P^1})$, where $\delta_H(C)$ is the number of hyperbolic real nodes of C.

From Section 1.3.1, we deduce that

$$\delta_H(C) \equiv g(D) + \delta_E(C) \mod 2,$$

where $\delta_E(C)$ is the number of elliptic real nodes of C and $g(D) = \frac{1}{2}(K_\Sigma.D + D^2) + 1$. Then, from Proposition 1.4.11, we deduce a quasi-quadratic enhancement

$$s^{\mathfrak{p}_{T\mathbb{R}\Sigma}} : H_1(\mathbb{R}\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$$

satisfying

$$s^{\mathfrak{p}_{T\mathbb{R}\Sigma}}(\rho D) = g(D) + \delta_E(C) + s_{\mathfrak{p}_{T\mathbb{R}\Sigma}}(C) \mod 2. \tag{3.3}$$

3.4.2 Quasi-quadratic enhancements and spinor states

Let $C \subset ((X,\tau), \mathfrak{s}_X, \mathfrak{o}_X)$ be a real rational curve representing a given homology class $d \in H_2^{-\tau}(X;\mathbb{Z})$. Suppose that $C \subset (\Sigma, \tau)$ where Σ is a real surface of the pencil $\mathcal{Q}$ and C represents homology class $D \in H_2^{-\tau}(\Sigma;\mathbb{Z})$.

Let k_d and k_D be the associated integers of classes d and D respectively. Recall that if $\Sigma \in |-\frac{1}{2}K_X|$ then $k_D = k_d - 1$.

In the next proposition, we show how the quasi-quadratic enhancement $s^{\mathfrak{p}_{T\mathbb{R}\Sigma}}$ defined by Equation (3.3) allows us to relate spinor state $sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C)$ in Definition 1.3.21 to Welschinger's sign $s_{\mathbb{R}\Sigma}(C)$ in Definition 1.3.3.

Proposition 3.4.4. *Given $\mathfrak{s}_X$ a $Spin_3$-structure on $SO(T\mathbb{R}X, \mathfrak{o}_X)$ and $\mathfrak{s}_{X|\Sigma}$ the restricted Pin_2^--structure on $O(T\mathbb{R}\Sigma)$, one has*

$$sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C) = (-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{X|\Sigma}}(\rho D)} s_{\mathbb{R}\Sigma}(C) \tag{3.4}$$

where ϵ is the homomorphism defined in Definition 3.3.2.

Proof. At the left hand side of (3.4), the spinor state $sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C)$ is determined by two parameters: one is the $Spin_3$-structure $\mathfrak{s}_X$ and the other is the isotopy class of the real line subbundle $L \subset \mathbb{R}\mathcal{N}_f$.

On the one hand, the $Spin_3$-structure $\mathfrak{s}_X$ induces a canonical Pin_2^--structure $\mathfrak{s}_{X|\Sigma}$ on $O(T\mathbb{R}\Sigma)$ and hence the corresponding quasi-quadratic enhancement $s^{\mathfrak{s}_{X|\Sigma}}$. As a consequence, this enhancement contributes the difference of $(-1)^{g(D)+s^{\mathfrak{s}_{X|\Sigma}}(\rho D)}$ in Equation (3.4) (see also Equation (3.3)).

On the other hand, recall that the holomorphic normal line bundle $\mathcal{N}'_f$ is of the same degree as the one, whose real part is L, and also is a subbundle of the holomorphic normal plane bundle $\mathcal{N}_f$. The map ϵ (Definition 3.3.2) allows us to compare the isotopy class of two real line subbundles $\mathbb{R}\mathcal{N}'_f$ and L in $\mathbb{R}\mathcal{N}_f$. Hence, the difference of sign by $(-1)^{\epsilon(D)}$ in Equation (3.4) is deduced.

Therefore, Equation (3.4) follows.

□

Remark 3.4.5. 1. The signs' relation shown in Formula (3.4) is true for any three-dimensional del Pezzo variety (of any degree) whose real part is orientable.

2. If we change the $Spin_3$-structure $\mathfrak{s}_X$ to $\mathfrak{s}'_X = \eta\mathfrak{s}_X$ on $SO(T\mathbb{R}X)$ by an action of $\eta \in H^1(\mathbb{R}X;\mathbb{Z}_2)$, then we have (see Proposition 1.3.23)

$$sp_{\eta\mathfrak{s}_X,\mathfrak{o}_X}(C) = (-1)^{\eta\cdot\rho(\psi D)} sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C).$$

By the universal coefficient theorem, one may notice the following isomorphisms:

$$H^1(\mathbb{R}X;\mathbb{Z}_2) \cong Hom_{\mathbb{Z}_2}(H_1(\mathbb{R}X;\mathbb{Z}_2)),$$

and

$$H^1(\mathbb{R}\Sigma;\mathbb{Z}_2) \cong Hom_{\mathbb{Z}_2}(H_1(\mathbb{R}\Sigma;\mathbb{Z}_2)).$$

And, one has the following commutative diagram.

$$\begin{tikzcd}
H_2(\Sigma;\mathbb{Z}) \arrow[r, "\psi"] \arrow[d, "\rho"] & H_2(X;\mathbb{Z}) \arrow[d, "\rho"] \\
H_1(\mathbb{R}\Sigma;\mathbb{Z}_2) \arrow[r, "\psi_{\mathbb{R}}"] \arrow[dr] & H_1(\mathbb{R}X;\mathbb{Z}_2) \arrow[d, "\eta"] \\
 & \mathbb{Z}_2
\end{tikzcd}$$

Moreover, by quasi-quadratic enhancements' properties, one also has

$$s^{(\eta\circ\psi_{\mathbb{R}})(\mathfrak{s}_{X|\Sigma})}(\rho D) = s^{\mathfrak{s}_{X|\Sigma}}_{\eta\circ\psi_{\mathbb{R}}}(\rho D) = s^{\mathfrak{s}_{X|\Sigma}}(\rho D) + (\eta\circ\psi_{\mathbb{R}})\cdot\rho D.$$

The commutativity of the above diagram implies that

$$\eta\cdot\rho(\psi D) = (\eta\circ\psi_{\mathbb{R}})\cdot\rho D(\in\mathbb{Z}_2).$$

3. If we change the orientation $\mathfrak{o}_X$ to $\mathfrak{o}'_X = -\mathfrak{o}_X$ then we have

$$sp_{\mathfrak{s}_X,-\mathfrak{o}_X}(C) = -sp_{\mathfrak{s}_X,\mathfrak{o}_X}(C).$$

Moreover, the orientation on $\mathbb{R}\mathcal{N}_f$ changes, so E^+ and E^- interchanges. Thus, the function ϵ changes its parity.

3.5 Proof and consequence of Theorem 3.1.1

Remark 3.5.1. By Proposition 2.3.2 and Proposition 3.2.3, we can degenerate the problem of computing genus-0 Welschinger invariants in real del Pezzo varieties of dimension three into that of dimension two. In addition to the Gromov-Witten case, we have a *sign problem* which comes from the difference between two definitions of Welschinger's sign for curves in those varieties.

Remind about our hypotheses and notations:

— (X, τ) is a real del Pezzo variety of dimension three such that $\ker(\psi) \cong \mathbb{Z}$, where $\Sigma \in |-\frac{1}{2}K_X|$ is a non-singular surface and $\psi : H_2(\Sigma;\mathbb{Z}) \twoheadrightarrow H_2(X;\mathbb{Z})$ is induced by the inclusion $\Sigma \hookrightarrow X$;

— $\mathcal{Q}$ is a real pencil of surfaces, whose real part is $\mathbb{R}\mathcal{Q} \cong \mathbb{R}P^1$, in the linear system $|-\frac{1}{2}K_X|$, whose base locus is E;

— $E \subset X$ is a real elliptic curve, whose real part is $\mathbb{R}E \neq \emptyset$, realising the homology class $\frac{1}{4}K_X^2$.

3.5.1 A proof of Theorem 3.1.1

Let $d \in H_2^{-\tau}(X;\mathbb{Z})$ and let k_d be its associated integer. Let $\mathcal{R}_d(\underline{x}_d)$ be the set of real irreducible rational curves representing the homology class d and passing through a real generic configuration $\underline{x}_d$ of k_d points in E consisting of l pairs of complex conjugated points. Suppose that $\underline{x}_d$ has at least one real point.

Let $U_{k_d} \subset E^{k_d}$ be as in Proposition 2.3.8. Choose $\underline{x}_d \in U_{k_d}$ and a subconfiguration $\underline{y}_D \subset \underline{x}_d$ of $k_D = k_d - 1$ points in $\underline{x}_d$. By Proposition 2.3.8 and Lemma 3.2.2, there exists $(\Sigma, \tau|_\Sigma) \in \mathbb{R}\mathcal{Q}$ a real non-singular del Pezzo surface containing a real curve in $\mathcal{R}_d(\underline{x}_d)$.

Let S be a generator of $\ker(\psi)$. By Proposition 3.2.3, in such real surface the vanishing cycle realises a $\tau|_\Sigma$-anti-invariant class. An analogous remark about monodromy transformation as in the proof of Theorem 2.1.1 is also applied. Let $\overline{D} \in H_2^{-\tau}(\Sigma;\mathbb{Z})/_{D\sim T(D)}$. One consider the surjective map $\overline{\psi} : H_2^{-\tau}(\Sigma;\mathbb{Z})/_{D\sim T(D)} \twoheadrightarrow H_2^{-\tau}(X;\mathbb{Z})$.

Remind about a consequence of the real symmetry property of genus-0 Welschinger invariants via monodromy transformation (see Proposition 1.3.7), we denoted by

$$W_{\mathbb{R}\Sigma}(\overline{D}, l) := W_{\mathbb{R}\Sigma}(D, l) = W_{\mathbb{R}\Sigma}(T(D), l).$$

From Proposition 3.4.4, we deduce that: for every anti-invariant class $D \in H_2^{-\tau}(\Sigma;\mathbb{Z})$ such that $\psi(D) = d$, the contribution to $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d, l)$ of elements of $\mathcal{R}_d(\underline{x}_d)$ which are contained in Σ is

$$(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{X|\Sigma}}(\rho D)} W_{\mathbb{R}\Sigma}(\overline{D}, l).$$

Suppose that $D.S$ is odd. From Corollary 3.2.6, we deduce that there are exactly $|D.S|$ such real surfaces $(\Sigma, \tau|_\Sigma)$ in $\mathbb{R}\mathcal{Q}$.

Therefore, we have

$$W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d, l) = \sum_{\overline{D}\in\overline{\psi}^{-1}(d)} (-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{X|\Sigma}}(\rho D)} |D.S| W_{\mathbb{R}\Sigma}(\overline{D}, l).$$

Moreover, we also have

- $\epsilon(T(D)) = \epsilon(D) + 1$ (see Proposition 3.3.4);

- $s^{\mathfrak{s}_{X|\Sigma}}(\rho T(D)) \equiv s^{\mathfrak{s}_{X|\Sigma}}(\rho D)+(D.S) \mod 2$. Thus, $s^{\mathfrak{s}_{X|\Sigma}}(\rho T(D)) \equiv s^{\mathfrak{s}_{X|\Sigma}}(\rho D)+1 \mod 2$ since $D.S$ is odd (see Proposition 3.4.3.2);
- $g(T(D)) = g(D)$ by a straightforward calculation;
- $T(D).S = -D.S$ by a straightforward calculation.

Thus,

$$\begin{aligned}
&(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_X|\Sigma}(\rho D)}|D.S|W_{\mathbb{R}\Sigma}(D,l)\\
&\quad+(-1)^{\epsilon(T(D))+g(T(D))+s^{\mathfrak{s}_X|\Sigma}(\rho T(D))}|T(D).S|W_{\mathbb{R}\Sigma}(T(D),l)\\
=&\Big((-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_X|\Sigma}(\rho D)}+(-1)^{\epsilon(D)+1+g(D)+s_{\mathfrak{s}_X|\Sigma}(\rho D)+1}\Big)|D.S|W_{\mathbb{R}\Sigma}(D,l)\\
=&2\times(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_X|\Sigma}(\rho D)}|D.S|W_{\mathbb{R}\Sigma}(D,l).
\end{aligned}$$

Then,

$$W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)=\sum_{\substack{\{D,T(D)\}\subset H_2^{-\tau}(\Sigma;\mathbb{Z}):\\ D\in\psi^{-1}(d)}}(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_X|\Sigma}(\rho D)}|D.S|W_{\mathbb{R}\Sigma}(D,l).$$

In other words, we obtain

$$W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)=\frac{1}{2}\sum_{\substack{D\in H_2^{-\tau}(\Sigma;\mathbb{Z}):\\ D\in\psi^{-1}(d)}}(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_X|\Sigma}(\rho D)}|D.S|W_{\mathbb{R}\Sigma}(D,l).$$

If $D.S$ is even then, by symmetry property we deduce that $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)=0$. Indeed, from Corollary 3.2.6, we deduce that there are $\gamma_{(D,S)}\in\{0,2|D.S|\}$ real surfaces $(\Sigma,\tau|_\Sigma)$ in $\mathbb{R}\mathcal{Q}$ which contain a curve in the set $\mathcal{R}_d(\underline{x}_d)$ and in which the vanishing cycle is $\tau|_\Sigma$-anti-invariant. As mentioned above, we have

$$W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)=\frac{1}{2}\sum_{D\in\psi^{-1}(d)}(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_X|\Sigma}(\rho D)}\gamma_{(D,S)}W_{\mathbb{R}\Sigma}(D,l).$$

By Proposition 3.4.3.2, we have

$$s^{\mathfrak{s}_{X|\Sigma}}(\rho T(D))\equiv s^{\mathfrak{s}_{X|\Sigma}}(\rho D)+D.S\equiv s^{\mathfrak{s}_{X|\Sigma}}(\rho D)\mod 2.$$

Then $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)$ becomes the sum of

$$\Big((-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_X|\Sigma}(\rho D)}+(-1)^{\epsilon(D)+1+g(D)+s^{\mathfrak{s}_X|\Sigma}(\rho D)}\Big)\gamma_{(D,S)}W_{\mathbb{R}\Sigma}(D,l)=0$$

for all $D\in\psi^{-1}(d)$. Thus, the vanishing of $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)$ follows.

3.5.2 A consequence of Theorem 3.1.1

From the proof of Theorem 3.1.1, we obtain directly the following consequence.

Corollary 3.5.2. *If $d \in H_2^{-\tau}(X;\mathbb{Z})$ such that $D.S$ is even, where $D \in \psi^{-1}(d)$, then the genus-0 Welschinger invariants $W_{\mathbb{R}X}^{\mathfrak{s}_X,\mathfrak{o}_X}(d,l)$ vanish for any $Spin_3$-structure $\mathfrak{s}_X$ on $SO(T\mathbb{R}X,\mathfrak{o}_X)$.*

3.6 Applications

Recall that we use the notation

$$\tau_n : \mathbb{C}P^n \to \mathbb{C}P^n, \quad \tau_n([z_0 : z_1 : \ldots : z_n]) = [\overline{z_0} : \overline{z_1} : \ldots : \overline{z_n}]$$

for the standard conjugation on the projective space $\mathbb{C}P^n$ ($n \geq 1$).

In this section, we consider some applications of Theorem 3.1.1 and Theorem 3.1.2 in the cases of real del Pezzo varieties of dimension three and of degree $6, 7$ or 8 with real structures described in Section 1.1.3.

3.6.1 Real three-dimensional projective space

We consider the real three-dimensional projective space with the standard real structure: $X = (\mathbb{C}P^3, \tau_3)$. We also consider the non-singular real quadric surface Σ in X with non-empty real part and real Lagrangian sphere realising a $\tau_3|_\Sigma$-anti-invariant class. In other words, Σ is isomorphic to $(\mathbb{C}P^1 \times \mathbb{C}P^1, \tau_3|_\Sigma)$, where

$$\tau_3|_\Sigma(p,q) = (\tau_1(p), \tau_1(q))$$

for all $(p,q) \in \mathbb{C}P^1 \times \mathbb{C}P^1$.

Hence, the fixed loci of τ_3 and $\tau_3|_\Sigma$ are the real projective space $\mathbb{R}P^3$ and the 2-torus $\mathbb{R}P^1 \times \mathbb{R}P^1$ respectively.

The group $H_2^{-\tau_3}(\mathbb{C}P^3;\mathbb{Z})$ is freely generated by L the homology class of a line embedded in $\mathbb{C}P^3$. The group $H_1(\mathbb{R}P^3;\mathbb{Z}_2)$ is generated by ρL.

We choose a basis for the group $H_2^{-\tau_3}(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z})$ as

$$(L_1, L_2) = ([\mathbb{C}P^1 \times \{p\}], [\{q\} \times \mathbb{C}P^1]), \quad (p, q \in \mathbb{C}P^1)$$

and a basis for the group $H_1(\mathbb{R}P^1 \times \mathbb{R}P^1;\mathbb{Z}_2)$ as

$$(\rho L_1, \rho L_2) = ([\mathbb{R}P^1 \times \{p_\mathbb{R}\}], [\{q_\mathbb{R}\} \times \mathbb{R}P^1]), \quad (p_\mathbb{R}, q_\mathbb{R} \in \mathbb{R}P^1).$$

We choose the $Spin_3$-structure $\mathfrak{s}_{\mathbb{R}P^3}$ on $SO(T\mathbb{R}P^3, \mathfrak{o}_{\mathbb{R}P^3})$ such that the spinor state of a real line in $\mathbb{C}P^3$ is $+1$. In other words, the $Spin_3$-structure $\mathfrak{s}_{\mathbb{R}P^3}$ is chosen such that $sp_{\mathfrak{s}_{\mathbb{R}P^3},\mathfrak{o}_{\mathbb{R}P^3}}(f(\mathbb{C}P^1)) = +1$ where $f : (\mathbb{C}P^1, \tau_1) \to (\mathbb{C}P^3, \tau_3)$ is a real immersion and $f_*[\mathbb{C}P^1] = L$.

One has the following theorem.

Theorem 3.6.1. *For every odd positive integer d and $0 \leq l \leq d-1$ one has*

$$W_{\mathbb{R}P^3}^{\mathfrak{s}_{\mathbb{R}P^3},\mathfrak{o}_{\mathbb{R}P^3}}(d,l) = \frac{1}{2}\sum_{a\in\mathbb{Z}}(-1)^a(d-2a)W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a,d-a),l). \tag{3.5}$$

The Formula (3.5) can be reduced to Theorem 1 in [BG16].

Proof. For any $D=(a,b)\in H_2^{-\tau_3}(\mathbb{C}P^1\times\mathbb{C}P^1;\mathbb{Z}_2)$, one has

$$k_D = (2,2).(a,b)-1 \equiv 1 \mod 2,$$

and

$$g(D) = \frac{1}{2}((2,2).(a,b)+(a,b)^2+2) \equiv (a+1)(b+1) \mod 2.$$

We consider a real balanced immersion $f:(\mathbb{C}P^1,\tau_1)\to(\mathbb{C}P^3,\tau_3)$ such that

$$Im(f)\subset \mathbb{C}P^1\times\mathbb{C}P^1, \text{ and } f_*[\mathbb{C}P^1]\in H_2^{-\tau_3}(\mathbb{C}P^1\times\mathbb{C}P^1;\mathbb{Z}).$$

Remark that the homology class $L\in H_2^{-\tau_3}(\mathbb{C}P^3;\mathbb{Z})$ has two effective classes [1] in its preimages via the surjective homomorphism $\psi : H_2(\mathbb{C}P^1\times\mathbb{C}P^1;\mathbb{Z})\to H_2(\mathbb{C}P^3;\mathbb{Z})$, they are classes $(1,0)$ and $(0,1)$ in $H_2^{-\tau_3}(\mathbb{C}P^1\times\mathbb{C}P^1;\mathbb{Z})$.

Lemma 3.6.2. *Let $L=(1,0)\in H_2^{-\tau_3}(\mathbb{C}P^1\times\mathbb{C}P^1;\mathbb{Z})$.*

Suppose that $\mathbb{R}f^*\mathcal{N}_{L/\mathbb{C}P^1\times\mathbb{C}P^1}=E^+$. Then, the function $\epsilon: H_2(\mathbb{C}P^1\times\mathbb{C}P^1;\mathbb{Z})\to\mathbb{Z}_2$ (see Definition 3.3.2) satisfies $\epsilon(a,b)=0$ if and only if $a>b$.

Proof. Recall that $S\in\mathbb{Z}(1,-1)=\ker(\psi)$.

By an easy calculation one has $(D.S)(L.S)=a-b$. By definition, one has

$$\epsilon(L)=k_L+1 \mod 2.$$

Hence, $\epsilon(L)=0 \mod 2$. Thus, the lemma follows by Proposition 3.3.5. □

Moreover, the restricted Pin_2^--structure on $O(T(\mathbb{R}P^1\times\mathbb{R}P^1))$ makes

$$s^{\mathfrak{s}_{\mathbb{R}P^3|\mathbb{R}P^1\times\mathbb{R}P^1}}(1,0)=0,$$

and then

$$s^{\mathfrak{s}_{\mathbb{R}P^3|\mathbb{R}P^1\times\mathbb{R}P^1}}(0,1)=1.$$

Hence,

$$s^{\mathfrak{s}_{\mathbb{R}P^3|\mathbb{R}P^1\times\mathbb{R}P^1}}(a,b)\equiv b+ab \mod 2.$$

Thus,

1. Recall: they are homology classes can be realized by algebraic curves.

— if $D = (a, b)$ such that $a > b$ then

$$(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{\mathbb{R}P^3|\mathbb{R}P^1\times\mathbb{R}P^1}}(\rho D)} = (-1)^{0+(a+1)(b+1)+b+ab};$$

by simplifying, this sign is equal to $(-1)^{a+1}$;

— if $D = (a, b)$ such that $a < b$ then

$$(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{\mathbb{R}P^3|\mathbb{R}P^1\times\mathbb{R}P^1}}(\rho D)} = (-1)^{1+(a+1)(b+1)+b+ab};$$

by simplifying, this sign is equal to $(-1)^{a}$;

In conclusion, if $a + b \equiv 1 \mod 2$ then

$$\begin{aligned} W_{\mathbb{R}P^3}^{\mathfrak{s}_{\mathbb{R}P^3},\mathfrak{o}_{\mathbb{R}P^3}}(d,l) =& \frac{1}{2}\Big(\sum_{0\leq a<\frac{d}{2}} (-1)^a(d-2a)W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a,d-a),l) \\ &+ \sum_{d\geq a>\frac{d}{2}} (-1)^{a+1}(-(d-2a))W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a,d-a),l)\Big) \\ =& \sum_{0\leq a<\frac{d}{2}} (-1)^a(d-2a)W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a,d-a),l). \end{aligned}$$

Since the genus-0 Welschinger invariants $W_{\mathbb{R}P^1\times\mathbb{R}P^1}((a, d-a), l)$ vanish when the integer $a \notin [0, d]$, the Formula (3.5) follows. □

Remark 3.6.3. Directly from Corollary 3.5.2 and the fact that

$$|D.S| = |(a, d-a).(1, -1)| = |d - 2a| \equiv d \mod 2,$$

one deduces that the invariants $W_{\mathbb{R}P^3}^{\mathfrak{s}_{\mathbb{R}P^3},\mathfrak{o}_{\mathbb{R}P^3}}(d,l)$ vanish if d is even for any $Spin_3$-structure $\mathfrak{s}_{\mathbb{R}P^3}$ over $\mathbb{R}P^3$. This result was obtained by G. Mikhalkin [BM07] by symmetry reasons.

Proposition 3.6.4. *Suppose that $\mathbb{R}E$ is disconnected and the hyperplane class is on its pointed component. If d is even and if the configuration $\underline{x}_d$ has an odd number of real points on each component of $\mathbb{R}E$, then there is no real irreducible rational degree d curve in $\mathbb{C}P^3$ passing through $\underline{x}_d$.*

Proof. This is a consequence of Theorem 3.1.2 and the following remarks.

— The base locus of the pencil of surfaces in the linear system $|-\frac{1}{2}K_{\mathbb{C}P^3}|$ is an elliptic curve E of degree 4 in $\mathbb{C}P^3$. Moreover, E is of homology class

$$(2, 2) \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z}).$$

— By the intersection form on $\mathbb{C}P^3$ and the surjective morphism ψ, one has the hyperplane section is of class

$$H = (1, 1) \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z}) \cong Pic(\mathbb{C}P^1 \times \mathbb{C}P^1),$$

Moreover, one has $H = L + T(L) = L_1 + L_2$.

— Let $h := H|_E \in Pic_4(E)$ be the hyperplane class on E. One can precise the degree $|D.E - |D.S|(L.E)| = |4(d-a)|$ divisor $\phi(\overline{L}, \overline{D})$ as

$$\begin{aligned}\phi(\overline{L}, \overline{D}) &= (a, d-a)|_E - (d-2a)(1,0)|_E \\ &= (d-a)h.\end{aligned}$$

Hence, the hyperplane class on E and the divisor $\phi(\overline{L}, \overline{D})$ are in the same connected component of $\mathbb{R}E$, i.e. in the pointed component $\mathbb{S}^1_0$ of $\mathbb{R}E$ by hypothesis.

If the configuration $\underline{x}_d$ has an odd number of real points on each component of $\mathbb{R}E$ then its divisor class $[\underline{x}_d]$ is in the non-pointed component $\mathbb{S}^1_1$ of $\mathbb{R}E$. Thus, by Corollary 3.2.6, the statement follows.

□

This vanishing result was obtained by J. Kollár in [Kol15].

3.6.2 Real three-dimensional projective space blown-up at a point

We consider $(X, \tau) = (\mathbb{C}P^3 \sharp \overline{\mathbb{C}P^3}, \tau)$ real three-dimensional projective space blown-up at a real point of $(\mathbb{C}P^3, \tau_3)$. Recall that the real structure τ is chosen such that its fixed locus is the blow-up at a real point of the real projective space $\mathbb{R}P^3 \sharp \overline{\mathbb{R}P^3}$.

We consider the immersed non-singular real del Pezzo surfaces $(\Sigma, \tau|_\Sigma) \subset (X, \tau)$ of degree 7 which are isomorphic to the blow-ups at a real point of $(\mathbb{C}P^1 \times \mathbb{C}P^1, \tau_3|_\Sigma)$. Moreover, the fixed locus of $\tau|_\Sigma$ is the blow-up at a real point of a 2-torus $\mathbb{R}P^1 \times \mathbb{R}P^1 \sharp \mathbb{R}P^2$.

We can choose a basis for the group $H_2^{-\tau}(\mathbb{C}P^3 \sharp \overline{\mathbb{C}P^3}; \mathbb{Z})$ as

- $H :=$ the homology class of a line embedded in $\mathbb{C}P^3$,
- F is the class of a line in the exceptional divisor,

and a basis for the group $H_1(\mathbb{R}P^3 \sharp \overline{\mathbb{R}P^3}; \mathbb{Z}_2)$ as

- $\rho H :=$ the real part of homology class of a line embedded in $\mathbb{C}P^3$,
- ρF is the real part of homology class of a line in the exceptional divisor.

We can choose a basis for the group $H_2^{-\tau}(\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp \overline{\mathbb{C}P^2}; \mathbb{Z})$ as

- $L_1 = [\mathbb{C}P^1 \times \{p\}]$,
- $L_2 = [\{q\} \times \mathbb{C}P^1]$,
- $\hat{E}$ is the exceptional divisor class,

where $p, q \in \mathbb{C}P^1$, and a basis for the group $H_1(\mathbb{R}P^1 \times \mathbb{R}P^1 \sharp \mathbb{R}P^2; \mathbb{Z}_2)$ as

- $\rho L_1 = [\mathbb{R}P^1 \times \{p_\mathbb{R}\}]$,
- $\rho L_2 = [\{q_\mathbb{R}\} \times \mathbb{R}P^1]$,
- $\rho\hat{E}$ is the real part of the exceptional divisor class,

where $p_{\mathbb{R}}, q_{\mathbb{R}} \in \mathbb{R}P^1$.

We choose the $Spin_3$-structure $\mathfrak{s}_{\mathbb{R}P^3\sharp\overline{\mathbb{R}P^3}}$ on $SO(T(\mathbb{R}P^3\sharp\overline{\mathbb{R}P^3}), \mathfrak{o}_{\mathbb{R}P^3\sharp\overline{\mathbb{R}P^3}})$ such that $sp_{\mathfrak{s}_{\mathbb{R}P^3\sharp\overline{\mathbb{R}P^3}}, \mathfrak{o}_{\mathbb{R}P^3\sharp\overline{\mathbb{R}P^3}}}(f(\mathbb{C}P^1)) = +1$ where $f : (\mathbb{C}P^1, \tau_1) \to (\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}, \tau)$ is a real immersion and $f_*[\mathbb{C}P^1] = H$, i.e. $f_*[\mathbb{C}P^1] = (1;0) \in H_2^{-\tau}(\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3};\mathbb{Z})$.

To simplify the notations, in the following we denote by

$$(\mathbb{R}P^3)_1 := \mathbb{R}P^3\sharp\overline{\mathbb{R}P^3}$$

and by

$$(\mathbb{R}P^1)^2_1 := \mathbb{R}P^1 \times \mathbb{R}P^1\sharp\mathbb{R}P^2.$$

One has the following theorem.

Theorem 3.6.5. *For every odd positive integer d, every non-negative integer $k \leq d$ and $0 \leq l \leq \frac{1}{2}(2d-1-k)$, one has*

$$W^{\mathfrak{s}_{(\mathbb{R}P^3)_1}, \mathfrak{o}_{(\mathbb{R}P^3)_1}}_{(\mathbb{R}P^3)_1}((d;k), l) = \frac{1}{2}\sum_{a\in\mathbb{Z}}(-1)^{a+\frac{1}{2}(k+k^2)}(d-2a)W_{(\mathbb{R}P^1)^2_1}((a, d-a;k), l). \quad (3.6)$$

If k is *even* then the Formula (3.6) can be reduced to Theorem 1.1 in [Din19].

Proof. A similar proof as the one of Theorem 3.6.1 can be applied. We emphasize some main points as follows.

For every $D = (a, b; k) \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1\sharp\overline{\mathbb{C}P^2};\mathbb{Z})$, one has the following topological invariants

$$k_D = (2,2;1).(a,b;k) - 1 \equiv k+1 \mod 2;$$

and

$$\begin{aligned} g(D) &= \frac{1}{2}\Big((2,2;1).(a,b;k) + (a,b;k)^2 + 2\Big) \\ &\equiv a + b + ab + \frac{1}{2}(k+k^2) + 1 \mod 2. \end{aligned}$$

Recall that $S \in \mathbb{Z}(1,-1;0)$.

We consider a real balanced immersion $f : (\mathbb{C}P^1, \tau_1) \to (\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}, \tau)$ such that $Im(f) \subset \mathbb{C}P^1 \times \mathbb{C}P^1\sharp\overline{\mathbb{C}P^2}$ and $f_*[\mathbb{C}P^1] \in H_2^{-\tau}(\mathbb{C}P^1 \times \mathbb{C}P^1\sharp\overline{\mathbb{C}P^2};\mathbb{Z})$.

Remark that the homology class $(1;0) \in H_2^{-\tau}(\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3};\mathbb{Z})$ has two effective classes in its preimages via the surjective homomorphism

$$\psi : H_2(\mathbb{C}P^1 \times \mathbb{C}P^1\sharp\overline{\mathbb{C}P^2};\mathbb{Z}) \to H_2(\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3};\mathbb{Z}).$$

These are classes $(1,0;0)$ and $(0,1;0)$ in $H_2^{-\tau}(\mathbb{C}P^1 \times \mathbb{C}P^1\sharp\overline{\mathbb{C}P^2};\mathbb{Z})$.

Lemma 3.6.2 implies directly the following lemma.

Lemma 3.6.6. *Let $L = (1, 0; 0) \in H_2^{-\tau}(\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp \overline{\mathbb{C}P^2}; \mathbb{Z})$.*
Suppose that $\mathbb{R}f^*\mathcal{N}_{L/\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp \overline{\mathbb{C}P^2}} = E^+$. Then, the function

$$\epsilon : H_2(\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp \overline{\mathbb{C}P^2}; \mathbb{Z}) \to \mathbb{Z}_2$$

(see Definition 3.3.2) satisfies $\epsilon(a, b; k) = 0$ if and only if $a > b$.

Note that $\epsilon(L) = k_L + 1 \equiv 0 \mod 2$.
Moreover, the restricted Pin_2^--structure on $O(T((\mathbb{R}P^1)^2_1))$ makes

$$s^{\mathfrak{s}_{(\mathbb{R}P^3)_1|(\mathbb{R}P^1)^2_1}}(1, 0; 0) = 0,$$

and then

$$s^{\mathfrak{s}_{(\mathbb{R}P^3)_1|(\mathbb{R}P^1)^2_1}}(0, 1; 0) = 1.$$

By properties of quasi-quadratic enhancements, we also have $s^{\mathfrak{s}_{(\mathbb{R}P^3)_1|(\mathbb{R}P^1)^2_1}}(0, 0; 1) = 0$. Hence, for all $a, b, k \in \mathbb{Z}$ one has

$$s^{\mathfrak{s}_{(\mathbb{R}P^3)_1|(\mathbb{R}P^1)^2_1}}(a, b; k) \equiv b + ab \mod 2.$$

Thus,

— if $D = (a, b; k)$ such that $a > b$ then

$$(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{(\mathbb{R}P^3)_1|(\mathbb{R}P^1)^2_1}}(\rho D)} = (-1)^{0+(a+b+ab+\frac{1}{2}(k+k^2)+1)+(b+ab)};$$

by simplifying, this sign is equal to $(-1)^{a+1+\frac{1}{2}(k+k^2)}$;

— if $D = (a, b; k)$ such that $a < b$ then

$$(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{(\mathbb{R}P^3)_1|(\mathbb{R}P^1)^2_1}}(\rho D)} = (-1)^{1+(a+b+ab+\frac{1}{2}(k+k^2)+1)+(b+ab)};$$

by simplifying, this sign is equal to $(-1)^{a+\frac{1}{2}(k+k^2)}$;

In conclusion, if $a + b = 1 \mod 2$ then

$$\begin{aligned} W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;k), l) =& \frac{1}{2}\Big(\sum_{0 \le a < \frac{d}{2}} (-1)^{a+\frac{1}{2}(k+k^2)}(d-2a) W_{(\mathbb{R}P^1)^2_1}((a, d-a; k), l) \\ &+ \sum_{d \ge a > \frac{d}{2}} (-1)^{a+1+\frac{1}{2}(k+k^2)}(-(d-2a)) W_{(\mathbb{R}P^1)^2_1}((a, d-a; k), l) \Big) \\ =& \sum_{0 \le a < \frac{d}{2}} (-1)^{a+\frac{1}{2}(k+k^2)}(d-2a) W_{(\mathbb{R}P^1)^2_1}((a, d-a; k), l). \end{aligned}$$

Since the genus-0 Welschinger invariants $W_{(\mathbb{R}P^1)^2_1}((a, d-a; k), l)$ vanish when $a \notin [0, d]$, the Formula (3.6) follows. □

Remark 3.6.7. Directly from Corollary 3.5.2 and the fact that

$$|D.S| = |(a, d-a; k).(1, -1; 0)| = |d - 2a| \equiv d \mod 2,$$

one deduces that the invariants $W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1}, \mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;k), l)$ vanish if d is even for any $Spin_3$-structure $\mathfrak{s}_{(\mathbb{R}P^3)_1}$ over $(\mathbb{R}P^3)_1$.

Proposition 3.6.8. *Suppose that $\mathbb{R}E$ is disconnected and that the hyperplane class and the exceptional divisor class are on its pointed component. If d is even and if the configuration $\underline{x}_d$ has an odd number of real points on the non-pointed component of $\mathbb{R}E$, then there is no real irreducible rational curve of class $(d;k)$ in $\mathbb{C}P^3 \sharp \overline{\mathbb{C}P^3}$ passing through $\underline{x}_d$.*

Proof. This is a consequence of Theorem 3.1.2 and the following remarks.

— The base locus of the pencil of surfaces in the linear system $|-\frac{1}{2}K_{\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}}|$ is an elliptic curve E of homology class

$$\frac{1}{4}K^2_{\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}} = (4;1) \in H_2(\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3};\mathbb{Z}).$$

Moreover, E is of homology class

$$(2,2;1) \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1 \sharp \overline{\mathbb{C}P^2};\mathbb{Z}).$$

— By the intersection form on $\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3}$ and the surjective morphism ψ, one has the hyperplane section is of class

$$\tilde{H} = (1,1;0) \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1\sharp\overline{\mathbb{C}P^2};\mathbb{Z}) \cong Pic(\mathbb{C}P^1 \times \mathbb{C}P^1\sharp\overline{\mathbb{C}P^2}).$$

Moreover, one has $\tilde{H} = L + T(L) = L_1 + L_2$.

— Let $\tilde{h} := \tilde{H}|_E \in Pic_4(E)$ be the hyperplane class on E. One can precise the degree $|D.E + (D.S)(L.E)| = |4(d-a) - k|$ divisor $\phi(\overline{L}, \overline{D})$ as

$$\begin{aligned} \phi(\overline{L}, \overline{D}) &= (a, d-a; k)|_E - (d-2a)(1,0;0)|_E \\ &= (d-a)\tilde{h} - k\hat{E}|_E. \end{aligned}$$

Hence, the position of the divisor $\phi(\overline{L}, \overline{D})$ in $\mathbb{R}E$ depends on that of the hyperplane class $\tilde{h}$ and the exceptional divisor class $\hat{E}|_E$ on E. Moreover, if $\underline{x}_d$ has an odd number of real points on the non-pointed component $\mathbb{S}^1_1$ of $\mathbb{R}E$, then $[\underline{x}_d] \in \mathbb{S}^1_1$. Thus, by Corollary 3.2.6.c), the statement follows.

□

3.6.3 Real threefold product of the projective line

We have seen in Section 2.5 that non-singular surfaces $\Sigma \in |-\frac{1}{2}K_{\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1}|$ are non-singular del Pezzo surfaces of degree 6. In particular, they are non-singular tri-degree $(1,1,1)$-surfaces in $\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1$. Moreover, they isomorphic to the blown-ups of non-singular quadric surfaces in $\mathbb{C}P^3$ at two distinct points, i.e. $\Sigma \cong \mathbb{C}P^1\times\mathbb{C}P^1\sharp 2\overline{\mathbb{C}P^2}$.

To simply the notations, in this part we denote by

$$(\mathbb{C}P^1)^3 := \mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1$$

and by

$$(\mathbb{C}P^1)^2_2 := \mathbb{C}P^1\times\mathbb{C}P^1\sharp 2\overline{\mathbb{C}P^2}.$$

We recall the following identifications (see Section 2.5.2).

- $H_2((\mathbb{C}P^1)^2_2;\mathbb{Z}) =< (L_1,L_2;E_1,E_2) >$ with $\mathbb{Z}^4$ and a homology class in $H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})$ of the form $aL_1+bL_2-\alpha E_1-\beta E_2$ is written as $(a,b;\alpha,\beta)$. Note that $(a,b;\alpha,\beta)$ is an effective class if $(a,b)\in\mathbb{N}^2$ and $(\alpha,\beta)\in\mathbb{Z}^2$ such that if $a=b=0$ then both α and β are negative; otherwise both α and β are non-negative.
- $H_2((\mathbb{C}P^1)^3;\mathbb{Z}) =< (M_1,M_2,M_3) >$ with $\mathbb{Z}^3$ and a homology class in $H_2((\mathbb{C}P^1)^3;\mathbb{Z})$ of the form $a_1M_1+a_2M_2+a_3M_3$ is written as (a_1,a_2,a_3). Note that a class (a_1,a_2,a_3) is effective if $a_i\in\mathbb{N}$ for all $i\in\{1,2,3\}$ and at least one $a_i>0$.

Moreover, the inclusion of $(\mathbb{C}P^1)^2_2$ in $(\mathbb{C}P^1)^3$ implies the surjective map

$$\begin{aligned} \psi : H_2((\mathbb{C}P^1)^2_2;\mathbb{Z}) &\longrightarrow H_2((\mathbb{C}P^1)^3;\mathbb{Z}) \\ (a,b;\alpha,\beta) &\longmapsto (a,b,a+b-\alpha-\beta). \end{aligned}$$

We have $\psi(0,0;\alpha,-\alpha)=(0,0,0)$ then $\ker(\psi)=\mathbb{Z}(0,0;1,-1)$.

For every $D=(a,b;\alpha,\beta)\in H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})$, one has the following topological invariants:

$$k_D = (2,2;1,1).(a,b;\alpha,\beta)-1 \equiv \alpha+\beta+1 \mod 2;$$

and

$$\begin{aligned} g(D) &= \frac{1}{2}\Big((2,2;1,1).(a,b;\alpha,\beta)+(a,b;\alpha,\beta)^2+2\Big) \\ &= a+b+ab+\frac{1}{2}(\alpha+\alpha^2)+\frac{1}{2}(\beta+\beta^2)+1 \mod 2. \end{aligned}$$

We consider two real structures on $(\mathbb{C}P^1)^3$: the standard one, denoted by $\tau_{1,1}$ with the fixed locus $\mathbb{R}P^1\times\mathbb{R}P^1\times\mathbb{R}P^1$, and the twisted one, denoted by $\tilde{\tau}_{1,1}$ with the fixed locus $\mathbb{S}^2\times\mathbb{R}P^1$ (see Section 1.1.3).

Since we are working on the real $(1,1,1)$-surfaces on $(\mathbb{C}P^1)^3$ such that S is $\tau_{1,1}$-anti-invariant and $\tilde{\tau}_{1,1}$-anti-invariant respectively, two blowing up points of the non-singular quadric are real. Thus, these two real structures restricted to $(\mathbb{C}P^1)^2_2$ gives the associating real parts are $\mathbb{R}P^1\times\mathbb{R}P^1\sharp 2\mathbb{R}P^2$ and $\mathbb{S}^2\sharp 2\mathbb{R}P^2$ respectively.

Case 1: The standard real structure

To simply the notations, we denote by

$$(\mathbb{R}P^1)^3 := \mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}P^1$$

and by

$$(\mathbb{R}P^1)^2_2 := \mathbb{R}P^1 \times \mathbb{R}P^1 \sharp 2\mathbb{R}P^2.$$

We can choose a basis for the group $H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^3;\mathbb{Z})$ as the basis for the group $H_2((\mathbb{C}P^1)^3;\mathbb{Z})$, i.e.

$$(M_1, M_2, M_3) = ([\mathbb{C}P^1 \times \{p_1\} \times \{p_2\}], [\{q_1\} \times \mathbb{C}P^1 \times \{q_2\}], [\{r_1\} \times \{r_2\} \times \mathbb{C}P^1]),$$

where $p_i, q_i, r_i \in \mathbb{C}P^1$; and a basis for the group $H_1((\mathbb{R}P^1)^3;\mathbb{Z}_2)$ as

$$(\rho M_1, \rho M_2, \rho M_3) = ([\mathbb{R}P^1 \times \{p_{1\mathbb{R}}\} \times \{p_{2\mathbb{R}}\}], [\{q_{1\mathbb{R}}\} \times \mathbb{R}P^1 \times \{q_{2\mathbb{R}}\}], [\{r_{1\mathbb{R}}\} \times \{r_{2\mathbb{R}}\} \times \mathbb{R}P^1]),$$

where $p_{i\mathbb{R}}, q_{i\mathbb{R}}, r_{i\mathbb{R}} \in \mathbb{R}P^1$ ($i \in \{1,2\}$).

Suppose that two blowing up points are $x_1 \neq x_2 \in \mathbb{R}P^1 \times \mathbb{R}P^1$.

We can choose a basis for the group $H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z})$ as the basis for the group $H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})$, i.e.

- $L_1 = [\mathbb{C}P^1 \times \{p\}]$,
- $L_2 = [\{q\} \times \mathbb{C}P^1]$,
- E_1 is the exceptional divisor class over x_1,
- E_2 is the exceptional divisor class over x_2,

where $p, q \in \mathbb{C}P^1$, and a basis for the group $H_1((\mathbb{R}P^1)^2_2;\mathbb{Z}_2)$ as

- $\rho L_1 = [\mathbb{R}P^1 \times \{p_\mathbb{R}\}]$,
- $\rho L_2 = [\{q_\mathbb{R}\} \times \mathbb{R}P^1]$,
- ρE_1 is the real part of the exceptional divisor class over x_1,
- ρE_2 is the real part of the exceptional divisor class over x_2,

where $p_\mathbb{R}, q_\mathbb{R} \in \mathbb{R}P^1$.

We choose the $Spin_3$-structure $\mathfrak{s}_{(\mathbb{R}P^1)^3}$ on $SO(T((\mathbb{R}P^1)^3), \mathfrak{o}_{(\mathbb{R}P^1)^3})$ such that

$$sp_{\mathfrak{s}_{(\mathbb{R}P^1)^3}, \mathfrak{o}_{(\mathbb{R}P^1)^3}}(f(\mathbb{C}P^1)) = +1,$$

where $f : (\mathbb{C}P^1, \tau_1) \to ((\mathbb{C}P^1)^3, \tau_{1,1})$ is a real immersion and

$$f_*[\mathbb{C}P^1] \in \{(0,0,1), (0,1,0), (0,0,1)\} \in H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^3;\mathbb{Z}).$$

The formula induced from Theorem 3.1.1 in this case is given as in the following theorem.

Theorem 3.6.9. *For every triplet of non-negative integers* (a,b,c) *such that* $a+b+c$ *is odd, and* $0 \leq l \leq \frac{1}{2}(a+b+c-1)$*, one has*

$$\begin{aligned} & W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l) \\ &= \sum_{0\leq\alpha<\frac{|a+b-c|}{2}} (-1)^{\alpha+\frac{a+b-c-1}{2}}(a+b-c-2\alpha)W_{(\mathbb{R}P^1)^2_2}((a,b;\alpha,a+b-c-\alpha),l) \end{aligned} \tag{3.7}$$

Proof. Let $D=(a,b;\alpha,a+b-c-\alpha)\in H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z})$. One has

$$k_D \equiv a+b+c+1 \mod 2;$$

and

$$\begin{aligned} g(D) &= a+b+ab+\alpha^2+\alpha(a+b-c)+\frac{1}{2}(a+b-c)(a+b-c-1)+1 \\ &\equiv a+b+1+ab+\frac{1}{2}(a+b-c-1) \mod 2, \quad \text{if } a+b+c=1 \mod 2. \end{aligned}$$

We consider a real balanced immersion $f:(\mathbb{C}P^1,\tau_1)\to((\mathbb{C}P^1)^3,\tau_{1,1})$ such that $Im(f)\subset(\mathbb{C}P^1)^2_2$ and $f_*[\mathbb{C}P^1]\in H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z})$.

Remark that the homology class $(0,0,1)\in H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^3;\mathbb{Z})$ has two effective classes in its preimages via the surjective homomorphism

$$\psi: H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})\to H_2((\mathbb{C}P^1)^3;\mathbb{Z}),$$

that are classes $(0,0;0,-1)$ and $(0,0;-1,0)$ in $H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z})$.

Lemma 3.6.10. *Let* $L=(0,0;0,-1)\in H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z})$.

Suppose that $\mathbb{R}f^*\mathcal{N}_{L/(\mathbb{C}P^1)^2_2}=E^+$*. Then, the function* $\epsilon: H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})\to\mathbb{Z}_2$ *(see Definition 3.3.2) satisfies* $\epsilon(a,b;\alpha,\beta)=0$ *if and only if* $\alpha>\beta$.

Proof. By definition, one has $\epsilon(L)=k_L+1\equiv 0+0+1+1\equiv 0 \mod 2$.

Recall that $S\in\mathbb{Z}(0,0;1,-1)=\ker\Big(H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})\to H_2((\mathbb{C}P^1)^3;\mathbb{Z})\Big)$. By an easy calculation, one has

$$(D.S)(L.S)=\alpha-\beta.$$

Thus, the lemma follows by Proposition 3.3.5. □

Moreover, the restricted Pin_2^--structure on $O(T(\mathbb{R}P^1)^2_2))$ makes

$$s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,0;0,-1)=0,$$

and then

$$s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,0;-1,0)=1.$$

Since the spinor states of a curve of class either $(1,0,0)$ or $(0,1,0)$ or $(0,0,1)$ in $(\mathbb{C}P^1)^3$ are identified, i.e.

$$sp_{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}(0,0,1) = sp_{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}(1,0,0) = sp_{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}(0,1,0),$$

one deduces that

$$s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(1,0;1,0) \equiv s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,1;1,0) \equiv 0.$$

Hence,

$$s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(1,0;0,0) \equiv s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(1,0;1,0) + s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,0;-1,0) \equiv 1,$$

and

$$s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,1;0,0) \equiv s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,1;1,0) + s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,0;-1,0) \equiv 1.$$

Thus,

$$\begin{aligned}
&s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(a,b;\alpha,\beta)\\
\equiv& a s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(1,0;0,0) + b s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,1;0,0)\\
&- \alpha s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,0;-1,0) - \beta s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(0,0;0,-1) + ab \mod 2\\
\equiv& a+b+ab+\alpha \mod 2.
\end{aligned}$$

Therefore,

— if $D=(a,b;\alpha,\beta)$ such that $\alpha>\beta$ and $\alpha+\beta=a+b-c\equiv 1 \mod 2$ then

$$(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(\rho D)} = (-1)^{0+(a+b+1+ab+\frac{1}{2}(a+b-c-1))+(a+b+ab+\alpha)};$$

by simplifying, this sign becomes $(-1)^{\alpha+1+\frac{1}{2}(a+b-c-1)}$;

— if $D=(a,b;\alpha,\beta)$ such that $\alpha<\beta$ and $\alpha+\beta=a+b-c\equiv 1 \mod 2$ then

$$(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{(\mathbb{R}P^1)^3|(\mathbb{R}P^1)^2_2}}(\rho D)} = (-1)^{1+(a+b+1+ab+\frac{1}{2}(a+b-c-1))+(a+b+ab+\alpha)};$$

by simplifying, this sign becomes $(-1)^{\alpha+\frac{1}{2}(a+b-c-1)}$.

In conclusion, if $a+b+c=1 \mod 2$ then

$$\begin{aligned}
&W^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}_{(\mathbb{R}P^1)^3}((a,b,c),l)\\
&=\frac{1}{2}\Big(\sum_{0\le\alpha<\frac{|a+b-c|}{2}} (-1)^{\alpha+\frac{a+b-c-1}{2}}(a+b-c-2\alpha)W_{(\mathbb{R}P^1)^2_2}((a,b;\alpha,a+b-c-\alpha),l)\\
&+\sum_{|a+b-c|\ge\alpha>\frac{|a+b-c|}{2}} (-1)^{\alpha+1+\frac{a+b-c-1}{2}}(-(a+b-c-2\alpha))W_{(\mathbb{R}P^1)^2_2}((a,b;\alpha,a+b-c-\alpha),l)\Big)\\
&=\sum_{0\le\alpha<\frac{|a+b-c|}{2}} (-1)^{\alpha+\frac{a+b-c-1}{2}}(a+b-c-2\alpha)W_{(\mathbb{R}P^1)^2_2}((a,b;\alpha,a+b-c-\alpha),l).
\end{aligned}$$

Since the genus-0 Welschinger invariants $W_{(\mathbb{R}P^1)^2_2}((a,b;\alpha,a+b-c-\alpha),l)$ vanish when $\alpha \notin [0,|a+b-c|]$, the Formula (3.7) follows.

□

Definition 3.6.11. A surface in $(\mathbb{C}P^1)^3$ is said of **tri-degree** (x,y,z) if it is of class $xH_1+yH_2+zH_3$ in $H_4((\mathbb{C}P^1)^3;\mathbb{Z})$ with $(x,y,z)\in\mathbb{N}^3$ and (H_1,H_2,H_3) is the basis of $H_4((\mathbb{C}P^1)^3;\mathbb{Z})\cong\mathbb{Z}^3$ defined as

$$(H_1,H_2,H_3)=([\{p\}\times\mathbb{C}P^1\times\mathbb{C}P^1],[\mathbb{C}P^1\times\{q\}\times\mathbb{C}P^1],[\mathbb{C}P^1\times\mathbb{C}P^1\times\{r\}])$$

where $p,q,r\in\mathbb{C}P^1$.

We also can see that a surface in $(\mathbb{C}P^1)^3$ is of tri-degree (x,y,z) if it has the number of intersection points with the basis (M_1,M_2,M_3) of $H_2((\mathbb{C}P^1)^3;\mathbb{Z})$ are x,y,z respectively.

We consider the **intersection forms** [2] **of** $(\mathbb{C}P^1)^3$

$$\begin{aligned}&H_4((\mathbb{C}P^1)^3;\mathbb{Z})\times H_4((\mathbb{C}P^1)^3;\mathbb{Z})\to H_2((\mathbb{C}P^1)^3;\mathbb{Z})\\&\qquad((a_1,a_2,a_3),(b_1,b_2,b_3))\mapsto(a_2b_3+a_3b_2,a_1b_3+a_3b_1,a_1b_2+a_2b_1),\end{aligned}$$

and

$$\begin{aligned}&H_2((\mathbb{C}P^1)^3,\mathbb{Z})\times H_4((\mathbb{C}P^1)^3,\mathbb{Z})\to\mathbb{Z}\\&\qquad((a,b,c),(x,y,z))\mapsto ax+by+cz.\end{aligned}$$

We also consider the **intersection form of** $(\mathbb{C}P^1)^2_2$

$$\begin{aligned}&H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})\times H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})\to\mathbb{Z}\\&\qquad((a,b;\alpha,\beta),(a',b';\alpha',\beta'))\mapsto ab'+ba'-\alpha\alpha'-\beta\beta'.\end{aligned}$$

We now describe some particular examples of (x,y,z)**-surface classes on elliptic curve** E.

1. The $(0,0,1)$-surface class on E.

   One has a tri-degree $(0,0,1)$ surface in $(\mathbb{C}P^1)^3$ intersects a tri-degree $(1,1,1)$ surface in $(\mathbb{C}P^1)^3$ at a curve of class $(1,1,0)\in H_2((\mathbb{C}P^1)^3;\mathbb{Z})$. The surjective map ψ implies that the $(1,1,1)$-surface section is of class $(1,1;1,1)\in H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})$. Let $h_3:=(1,1;1,1)|_E\in Pic_2(E)$ be the $(0,0,1)$-surface class on E. Thus, one has the following linear equivalence [3]

$$h_3=L_1|_E+L_2|_E-E_1|_E-E_2|_E.$$

2. These intersection forms are mentioned in order to clarify the calculations of the degree of intersections in the sequel.

3. With abuse of notation, here we use $=$ for a linearly equivalent relation in the Picard group of an elliptic curve.

2. The $(b,a,0)$-surface class on E.
   One has a tri-degree $(b,a,0)$ surface in $(\mathbb{C}P^1)^3$ intersects a tri-degree $(1,1,1)$ surface in $(\mathbb{C}P^1)^3$ at a curve of class $(a,b,a+b) \in H_2((\mathbb{C}P^1)^3;\mathbb{Z})$. The surjective map ψ implies that the $(b,a,0)$-surface section is of class $(a,b;0,0) \in H_2((\mathbb{C}P^1)^2_2;\mathbb{Z})$. Let $h_{b,a} := (a,b;0,0)|_E \in Pic_{2(a+b)}(E)$ be the $(b,a,0)$-surface class on E. Thus, one can write
   $$h_{b,a} = aL_1|_E + bL_2|_E.$$

Note that
$$L_1|_E, L_2|_E \in Pic_2(E), \quad E_1|_E, E_2|_E \in Pic_1(E),$$
and
$$E_1|_E + E_2|_E = h_{1,1} - h_3.$$

Remark 3.6.12. Directly from Corollary 3.5.2 and the fact that
$$|D.S| = |(a,b;\alpha,a+b-c-\alpha).(0,0;1,-1)| = |a+b-c-2\alpha| \equiv a+b+c \mod 2,$$
one deduces that the invariants $W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l)$ vanish if $a+b+c$ is even for any $Spin_3$-structure $\mathfrak{s}_{(\mathbb{R}P^1)^3}$ over $(\mathbb{R}P^1)^3$.

Proposition 3.6.13. *Suppose that $\mathbb{R}E$ is disconnected. Suppose also that the $(0,0,1)$-surface class, the $(1,1,0)$-surface class and the $(b,a,0)$-surface class ($a,b \in \mathbb{N}^*$) are on the pointed component of $\mathbb{R}E$. If $a+b+c$ is even and if the real configuration $\underline{x}_d$ of $k_d = a+b+c$ points in E has an odd number of real points on each component of $\mathbb{R}E$, then there is no real rational curve of class $(a,b,c) \in H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^3;\mathbb{Z})$ passing through $\underline{x}_d$.*

Proof. This is a consequence of Theorem 3.1.2 and the following remarks.

— The base locus of the pencil of surfaces in the linear system $|-\frac{1}{2}K_{(\mathbb{C}P^1)^3}|$ is an elliptic curve E of homology class
$$\frac{1}{4}K^2_{(\mathbb{C}P^1)^3} = (2,2,2) \in H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^3;\mathbb{Z}).$$
Moreover, E is of homology class
$$(2,2;1,1) \in H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z}).$$

— By the intersection forms of $(\mathbb{C}P^1)^3$ and of $(\mathbb{C}P^1)^2_2$ described as above, one has the class
$$L + T(L) = E_1 + E_2 = (0,0;-1,-1) \in H_2((\mathbb{C}P^1)^2_2;\mathbb{Z}) \cong Pic((\mathbb{C}P^1)^2_2).$$
Taking its restriction to E, we get the divisor
$$\phi(\overline{L}) = h_{1,1} - h_3 \in Pic_2(E)$$
which is determined by the $(0,0,1)$-surface class and the $(1,1,0)$-surface class on E.

— Moreover, if $0 \leq \alpha \leq \frac{1}{2}(a+b-c)$ then the degree $|D.E-|D.S|(L.E)| = 2(c+\alpha)$ divisor $\phi(\overline{L}, \overline{D})$ in this case is

$$\begin{aligned}\phi(\overline{L}, \overline{D}) &= (a, b; a+b-c-\alpha, \alpha)|_E - (a+b-c-2\alpha)(0,0;0,-1)|_E \\ &= (a, b; 0, 0)|_E - (a+b-c-\alpha)(0,0;-1,-1)|_E \\ &= h_{b,a} - (a+b-c-\alpha)(h_{1,1} - h_3)\end{aligned}$$

Hence, Equation (3.1) becomes

$$(a+b-c-2\alpha)l = [\underline{x}_d] - h_{b,a} + (a+b-c-\alpha)(h_{1,1} - h_3). \tag{3.8}$$

Thus, by hypothesis and by Corollary 3.2.6.c), the statement follows. $\square$

Case 2: The twisted real structure

We can choose a basis for the group $H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^3; \mathbb{Z})$ as

$$(M_1 + M_2, M_3) = ([\mathbb{C}P^1 \times \{p_1\} \times \{p_2\}] + [\{q_1\} \times \mathbb{C}P^1 \times \{q_2\}], [\{r_1\} \times \{r_2\} \times \mathbb{C}P^1])$$

where $p_i, q_i, r_i \in \mathbb{C}P^1$.

Hence, the group $H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^3; \mathbb{Z})$ can be identified with $\mathbb{Z}^2$.

Note that the group $H_1(\mathbb{S}^2 \times \mathbb{R}P^1; \mathbb{Z}_2)$ has a generator $\rho M_3 := [\{r_\mathbb{R}\} \times \mathbb{R}P^1]$, where $r_\mathbb{R} \in \mathbb{S}^2$.

Note also that two blowing up points are $x_1 \neq x_2 \in \mathbb{S}^2$.

We can choose a basis for the group $H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^2_2; \mathbb{Z})$ as

- $L_1 + L_2 = [\mathbb{C}P^1 \times \{p\}] + [\{q\} \times \mathbb{C}P^1]$,
- E_1 is the exceptional divisor class over x_1,
- E_2 is the exceptional divisor class over x_2,

where $p, q \in \mathbb{C}P^1$.

Hence, the group $H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^2_2; \mathbb{Z})$ can be identified with $\mathbb{Z}^3$.

Note that the group $H_1(\mathbb{S}^2 \sharp 2\mathbb{R}P^2; \mathbb{Z}_2)$ has two generators given as

- ρE_1 is the real part of the exceptional divisor class over x_1,
- ρE_2 is the real part of the exceptional divisor class over x_2,

One can write $H_1(\mathbb{S}^2 \sharp 2\mathbb{R}P^2; \mathbb{Z}_2) =< (0; -1, 0), (0; 0, -1) >$.

We choose the $Spin_3$-structure $\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}$ on $SO(T(\mathbb{S}^2 \times \mathbb{R}P^1), \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1})$ such that $sp_{\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}, \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1}}(f(\mathbb{C}P^1)) = +1$ where $f : (\mathbb{C}P^1, \tau_1) \to ((\mathbb{C}P^1)^3, \tilde{\tau}_{1,1})$ is a real immersion and $f_*[\mathbb{C}P^1] = (0; 1) \in H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^3; \mathbb{Z})$.

The formula induced from Theorem 3.1.1 in this case is given in the following theorem.

Theorem 3.6.14. *For every non-negative integer a, every odd positive integer c and integer l such that $0 \leq l \leq \frac{1}{2}(2a+c-1)$, one has*

$$W^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1},\mathfrak{o}_{\mathbb{S}^2\times\mathbb{R}P^1}}_{\mathbb{S}^2\times\mathbb{R}P^1}((a;c),l) = \sum_{0\leq\alpha<\frac{|2a-c|}{2}} (-1)^{\alpha+\frac{c+1}{2}}(2a-c-2\alpha)W_{\mathbb{S}^2\sharp 2\mathbb{R}P^2}((a;\alpha,2a-c-\alpha),l). \tag{3.9}$$

Proof. Similar proof as the one of Theorem 3.6.9 can be applied. We emphasize some distinguished points coming from the changing of real structures on $(\mathbb{C}P^1)^3$ as follows.

Let $D = (a;\alpha,2a-c-\alpha) \in H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z})$. One has

$$k_D \equiv c+1 \mod 2;$$

and

$$\begin{aligned} g(D) &= 2a + a^2 + \alpha^2 + (2a-c)\alpha + \frac{1}{2}(2a-c)(2a-c-1) + 1 \\ &\equiv \alpha^2 + c\alpha + 1 + \frac{1}{2}c(c+1) \mod 2 \\ &\equiv \frac{1}{2}(c-1) \mod 2, \quad \text{if } c \equiv 1 \mod 2. \end{aligned}$$

We consider a real balanced immersion $f : (\mathbb{C}P^1,\tau_1) \to ((\mathbb{C}P^1)^3,\tilde{\tau}_{1,1})$ such that $Im(f) \subset (\mathbb{C}P^1)^2_2$ and $f_*[\mathbb{C}P^1] \in H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z})$.

Remark that the homology class $(0;1) \in H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^3;\mathbb{Z})$ has two effective classes in its preimages via the surjective homomorphism $\psi : H_2((\mathbb{C}P^1)^2_2;\mathbb{Z}) \to H_2((\mathbb{C}P^1)^3;\mathbb{Z})$, they are classes $(0;0,-1)$ and $(0;-1,0)$ in $H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z})$.

Lemma 3.6.10 implies directly the following lemma.

Lemma 3.6.15. *Let $L = (0;0,-1) \in H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z})$.*

Suppose that $\mathbb{R}f^*\mathcal{N}_{L/(\mathbb{C}P^1)^2_2} = E^+$. Then, the function $\epsilon : H_2((\mathbb{C}P^1)^2_2;\mathbb{Z}) \to \mathbb{Z}_2$ (see Definition 3.3.2) satisfies $\epsilon(a;\alpha,\beta) = 0$ if and only if $\alpha > \beta$.

Moreover, the restricted Pin_2^--structure on $O(T\mathbb{S}^2 \times \mathbb{R}P^1)$ makes

$$s^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1}|_{\mathbb{S}^2\times\mathbb{R}P^1}}(0;0,-1) = 0,$$

and then

$$s^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1}|_{\mathbb{S}^2\times\mathbb{R}P^1}}(0;-1,0) = 1.$$

Since $H_1(\mathbb{S}^2\sharp 2\mathbb{R}P^2;\mathbb{Z}_2) =< (0;-1,0),(0;0,-1) >$, one has

$$\begin{aligned} &s^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1}|_{\mathbb{S}^2\times\mathbb{R}P^1}}(0;-\alpha,-\beta) \\ \equiv&\alpha s^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1}|_{\mathbb{S}^2\times\mathbb{R}P^1}}(0;-1,0) + \beta s^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1}|_{\mathbb{S}^2\times\mathbb{R}P^1}}(0;0,-1) \mod 2 \\ \equiv&\alpha \mod 2. \end{aligned}$$

Therefore,

— if $D = (a; \alpha, \beta)$ such that $\alpha > \beta$ and $\alpha + \beta = 2a - c \equiv 1 \mod 2$ then

$$(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1}|\mathbb{S}^2\times\mathbb{R}P^1}(\rho D)} = (-1)^{0+\frac{1}{2}(c-1)+\alpha};$$

— if $D = (a; \alpha, \beta)$ such that $\alpha < \beta$ and $\alpha + \beta = 2a - c \equiv 1 \mod 2$ then

$$(-1)^{\epsilon(D)+g(D)+s^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1}|\mathbb{S}^2\times\mathbb{R}P^1}(\rho D)} = (-1)^{1+\frac{1}{2}(c-1)+\alpha}.$$

In conclusion, if $c = 1 \mod 2$ then

$$\begin{aligned}
&W_{\mathbb{S}^2\times\mathbb{R}P^1}^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1},\mathfrak{o}_{\mathbb{S}^2\times\mathbb{R}P^1}}((a;c),l)\\
&= \frac{1}{2}\Big(\sum_{0\leq\alpha<\frac{|2a-c|}{2}}(-1)^{\alpha+\frac{c+1}{2}}(2a-c-2\alpha)W_{\mathbb{S}^2\sharp 2\mathbb{R}P^2}((a;\alpha,2a-c-\alpha),l)\\
&+\sum_{|2a-c|\geq\alpha>\frac{|2a-c|}{2}}(-1)^{\alpha+1+\frac{c+1}{2}}(-(2a-c-2\alpha))W_{\mathbb{S}^2\sharp 2\mathbb{R}P^2}((a;\alpha,2a-c-\alpha),l)\Big)\\
&= \sum_{0\leq\alpha<\frac{|2a-c|}{2}}(-1)^{\alpha+\frac{c+1}{2}}(2a-c-2\alpha)W_{\mathbb{S}^2\sharp 2\mathbb{R}P^2}((a;\alpha,2a-c-\alpha),l).
\end{aligned}$$

Since the genus-0 Welschinger invariants $W_{\mathbb{S}^2\sharp 2\mathbb{R}P^2}((a;\alpha,2a-c-\alpha),l)$ vanish when $\alpha \notin [0, |2a-c|]$, the Formula (3.9) follows.

$\square$

Recall that a surface in $(\mathbb{C}P^1)^3$ is of tri-degree (x, y, z) if it can be represented in $H_4((\mathbb{C}P^1)^3;\mathbb{Z})$ as $xH_1 + yH_2 + zH_3$.

Recall also that we can define (x, y, z)-surface classes on elliptic curve E. In particular, if restricting to the real structure $\tilde{\tau}_{1,1}$ then we have

1. The $(0, 0, 1)$-surface class on E:

$$h_3 := (1; 1, 1)|_E = (L_1 + L_2)|_E - E_1|_E - E_2|_E \in Pic_2(E).$$

2. The $(a, a, 0)$-surface class on E:

$$h_{a,a} := (a; 0, 0)|_E = a(L_1 + L_2)|_E \in Pic_{4a}(E).$$

Remark 3.6.16. Directly from Corollary 3.5.2 and the fact that

$$|D.S| = |(a; \alpha, 2a-c-\alpha).(0; 1, -1)| = |2a-c-2\alpha| \equiv c \mod 2,$$

one has the invariants $W_{\mathbb{S}^2\times\mathbb{R}P^1}^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1},\mathfrak{o}_{\mathbb{S}^2\times\mathbb{R}P^1}}((a;c),l)$ vanish if c is even for any $Spin_3$-structure $\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1}$ over $\mathbb{S}^2\times\mathbb{R}P^1$.

Proposition 3.6.17. *Suppose that $\mathbb{R}E$ is disconnected and that the $(0,0,1)$-surface class and the $(a,a,0)$-surface class ($a \in \mathbb{N}^*$) are on the pointed component of $\mathbb{R}E$. If c is even and if the real configuration $\underline{x}_d$ of $k_d = 2a + c$ points in E has an odd number of real points on each component of $\mathbb{R}E$, then there is no real rational curve of class $(a;c)$ in $H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^3;\mathbb{Z})$ passing through $\underline{x}_d$.*

Proof. This is a consequence of Theorem 3.1.2, Proposition 3.6.13 and the following remarks.

— The base locus of the pencil of surfaces in the linear system $|-\frac{1}{2}K_{(\mathbb{C}P^1)^3}|$ is an elliptic curve E of homology class

$$\frac{1}{4}K^2_{(\mathbb{C}P^1)^3} = (2;2) \in H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^3;\mathbb{Z}).$$

Moreover, E is of homology class

$$(2;1,1) \in H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z}).$$

— By the intersection form on $H_4((\mathbb{C}P^1)^3;\mathbb{Z})$, we deduce that the class

$$L + T(L) = E_1 + E_2 = (0;-1,-1) \in H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^2_2;\mathbb{Z}) \cong Pic((\mathbb{C}P^1)^2_2).$$

Taking its restriction to E, we get the divisor

$$\phi(\overline{L}) = h_{1,1} - h_3 \in Pic_2(E)$$

which is formed by the $(0,0,1)$-surface class and the $(1,1,0)$-surface class on E.

— Moreover, if $0 \le \alpha \le \frac{1}{2}(2a-c)$ then the degree $|D.E - |D.S|(L.E)| = 2(c+\alpha)$ divisor $\phi(\overline{L},\overline{D})$ in this case is

$$\begin{aligned}\phi(\overline{L},\overline{D}) &= (a;2a-c-\alpha,\alpha)|_E - (2a-c-2\alpha)(0;0,-1)|_E \\ &= (a;0,0)|_E - (2a-c-\alpha)(0;-1,-1)|_E \\ &= h_{a,a} - (2a-c-\alpha)(h_{1,1}-h_3)\end{aligned}$$

Hence, Equation (3.1) becomes

$$(2a-c-2\alpha)l = \underline{x}_d - h_{a,a} + (2a-c-\alpha)(h_{1,1}-h_3). \tag{3.10}$$

Thus, by hypothesis and by Corollary 3.2.6.c), the statement follows. □

In the next section, we will show some precise values of the Formula (3.6), Formula (3.7) and Formula (3.9) in Table 3.1, Table 3.2 and Table 3.3 respectively.

3.7 Computations and further comments

Remark 3.7.1. The explicit computations of genus-0 Welschinger invariants in real del Pezzo surfaces has been studied by E. Brugallé and G. Mikhalkin by using floor diagrams in tropical geometry ([BM07]); I. Itenberg, V. Kharlamov, E. Shustin by using another tropical techniques ([IKS13b]); X. Chen and A. Zinger by using WDVV-type relations ([CZ21b]).

We list the first values of $W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;k),l)$, $W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l)$ and $W_{\mathbb{S}^2\times\mathbb{R}P^1}^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1},\mathfrak{o}_{\mathbb{S}^2\times\mathbb{R}P^1}}((a;c),l)$ in the Tables 3.1, 3.2 and 3.3 respectively. These computations have been made using *Maple program*. We thank Erwan and Duc for their support in programming. We thank Xujia for her support in comparing many calculations.

3.7.1 Real three-dimensional projective space blown-up at a real point

We provide in Table 3.1 the values of $W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;k),l)$ for small values of d.

Note that $0 \le l \le d - \frac{1}{2}(2d-1-k)$ and $k \le d$. Note also that $W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;k),l)$ vanishes if d is even.

Table 3.1 shows computations of genus-0 Welschinger invariants $W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;k),l)$ for the $Spin_3$-structure $\mathfrak{s}_{(\mathbb{R}P^3)_1}$ on $SO(T((\mathbb{R}P^3)_1),\mathfrak{o}_{(\mathbb{R}P^3)_1})$ such that the spinor state $sp_{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}(f(\mathbb{C}P^1)) = +1$ where

$$f_*[\mathbb{C}P^1] = (1;0) \in H_2^{-\tau}(\mathbb{C}P^3\sharp\overline{\mathbb{C}P^3};\mathbb{Z}).$$

Note that in this table, $0 \le k \le d \le 9$.

Observations

1. For every $Spin_3$-structure on $(\mathbb{R}P^3)_1$, for every $d \in \mathbb{N}^*$ and for every integer l such that $l \in \{0,\dots,\dfrac{1}{2}(2d-1-k)\}$, one has
$$W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;0),l) = -W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;1),l).$$

2. For every $Spin_3$-structure on $(\mathbb{R}P^3)_1$, for every $d \in \mathbb{N}^*$, for every integer k and l such that $k \in \{\dfrac{1}{2}(d+1),\dots,d\}$ and $l \in \{0,\dots,\dfrac{1}{2}(3k+1)\}$, one has
$$W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;k),l) = 0.$$

3. For every $Spin_3$-structure on $(\mathbb{R}P^3)_1$, for every $k \in \mathbb{N}$ and for every integer l such that $l \in \{0,\dots,\dfrac{1}{2}(3k+1)\}$, one has
$$W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((2k+1;k),l) = (-1)^{\frac{1}{2}(k^2-k)}W_{\mathbb{R}P^2}(k+1,l).$$

These observations can be proven by geometrical explanations or deduced directly from Theorem 3.6.5.

Remark 3.7.2. The values of $W_{(\mathbb{R}P^3)_1}^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}((d;0),l)$ in these computations as shown in Table 3.1 agree with that given by E. Brugallé and P. Georgieva ([BG16]).

3.7.2 Real threefold product of the projective line: the standard real structure

One has

$$W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l) = W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((\sigma(a),\sigma(b),\sigma(c)),l),$$

where σ is a permutation of the set $\{a,b,c\}$. Without lose of generality, we can suppose that $a \geq b \geq c$ when computing of $W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l)$.

As a direct consequence of Proposition 2.5.7 and the remark after Theorem 3.6.9, one has the following proposition.

Proposition 3.7.3. *Suppose that $a \geq b \geq c > 1$. Then, for every $Spin_3$-structure $\mathfrak{s}_{(\mathbb{R}P^1)^3}$ on $SO(T((\mathbb{R}P^1)^3),\mathfrak{o}_{(\mathbb{R}P^1)^3})$, one has $W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l) = 0$ if one of the following cases holds:*

- $a > b + c$,
- $a + b + c$ *is even.*

Table 3.2 shows computations of genus-0 Welschinger invariants $W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l)$ for the $Spin_3$-structure $\mathfrak{s}_{(\mathbb{R}P^1)^3}$ on $SO(T((\mathbb{R}P^1)^3),\mathfrak{o}_{(\mathbb{R}P^1)^3})$ such that the spinor state $sp_{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}(f(\mathbb{C}P^1)) = +1$ where

$$f_*[\mathbb{C}P^1] \in \{(0,0,1),(0,1,0),(0,0,1)\} \in H_2^{-\tau_{1,1}}((\mathbb{C}P^1)^3;\mathbb{Z}).$$

Note that in this table, $7 \geq a \geq b \geq c \geq 0$ and $a+b+c$ increases up to 15. Note also that we show only the non-vanishing genus-0 Welschinger invariants $W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l)$, i.e. the cases where $a \leq b+c$ and $a+b+c$ odd.

Observation:

The following is a natural observation of $GW_{(\mathbb{C}P^1)^3}(a,a,1) = 1$.

For every $Spin_3$-structure on $(\mathbb{R}P^1)^3$, for every positive integer a and integer l, one has

$$|W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,a,1),l)| = 1.$$

Computations from Table 3.2 lead us to make the following conjecture about the positivity of the Welschinger invariants.

Conjecture 1. *Let $\mathfrak{s}_{(\mathbb{R}P^1)^3}$ be the $Spin_3$-structure on $(\mathbb{R}P^1)^3$ as in Theorem 3.1.1. Then for all $(a,b,c) \in \mathbb{N}^* \times \mathbb{N} \times \mathbb{N}$ and for all $l \in \{0, \ldots, \frac{1}{2}(a+b+c-1)\}$, one has*

$$W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3}, \mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l) \geq 0.$$

Moreover, the equality happens exactly if either $a+b+c$ is even or $a > b+c$. In other words, the statement of Proposition 3.7.3 becomes $W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3}, \mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l) = 0$ if and only if either $a+b+c \equiv 0 \mod 2$ or $a > b+c$.

3.7.3 Real threefold product of the projective line: the twisted real structure

As a direct consequence of Proposition 2.5.7 and the remark after Theorem 3.6.14, one has the following proposition.

Proposition 3.7.4. *For every $Spin_3$-structure $\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}$ on $SO(T(\mathbb{S}^2 \times \mathbb{R}P^1), \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1})$, one has $W_{\mathbb{S}^2 \times \mathbb{R}P^1}^{\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}, \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1}}((a;c),l) = 0$ if one of the following cases holds:*

- *$c > 2a$,*
- *c is even.*

Table 3.3 shows computations of genus-0 Welschinger invariants $W_{\mathbb{S}^2 \times \mathbb{R}P^1}^{\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}, \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1}}((a;c),l)$ for the $Spin_3$-structure $\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}$ on $SO(T(\mathbb{S}^2 \times \mathbb{R}P^1), \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1})$ such that the spinor state $sp_{\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}, \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1}}(f(\mathbb{C}P^1)) = +1$ where $f_*[\mathbb{C}P^1] = (0;1)$ in $H_2^{-\tilde{\tau}_{1,1}}((\mathbb{C}P^1)^3; \mathbb{Z})$.

Note that in this table, $a \leq 5$ and $2a+c$ increases up to 19. Note also that we show only the genus-0 Welschinger invariants $W_{\mathbb{S}^2 \times \mathbb{R}P^1}^{\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}, \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1}}((a;c),l)$ in the cases where $c < 2a$ and c odd (since in these cases, the Welschinger invariants are "almost" non-zeros by Proposition 3.7.4).

Observation:

The following is also a natural observation coming from $GW_{(\mathbb{C}P^1)^3}(a,a,1) = 1$.

For every $Spin_3$-structure on $\mathbb{S}^2 \times \mathbb{R}P^1$, for every positive integer a and integer l, one has

$$|W_{\mathbb{S}^2 \times \mathbb{R}P^1}^{\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}, \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1}}((a;1),l)| = 1.$$

Remark 3.7.5. The values of $W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3}, \mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l)$ and $W_{\mathbb{S}^2 \times \mathbb{R}P^1}^{\mathfrak{s}_{\mathbb{S}^2 \times \mathbb{R}P^1}, \mathfrak{o}_{\mathbb{S}^2 \times \mathbb{R}P^1}}((a;c),l)$ in these computations as shown in Table 3.2 and Table 3.3 agree with that given by X. Chen and A. Zinger ([CZ21b], [Che]).

l \ $(d;k)$	$(1;0)$	$(1;1)$	$(3;0)$	$(3;1)$	$(3;2)$	$(3;3)$
0	1	-1	-1	1	0	0
1			-1	1	0	0
2			-1	1		

l \ $(d;k)$	$(5;0)$	$(5;1)$	$(5;2)$	$(5;3)$	$(5;4)$	$(5;5)$
0	45	-45	-8	0	0	0
1	29	-29	-6	0	0	0
2	17	-17	-4	0	0	0
3	9	-9	-2	0		
4	5	-5				

l \ $(d;k)$	$(7;0)$	$(7;1)$	$(7;2)$	$(7;3)$	$(7;4)$	$(7;5)$	$(7;6)$	$(7;7)$
0	-14589	14589	3816	-240	0	0	0	0
1	-6957	6957	1932	-144	0	0	0	0
2	-3093	3093	912	-80	0	0	0	0
3	-1269	1269	396	-40	0	0	0	0
4	-477	477	152	-16	0	0		
5	-173	173	44	0				
6	-85	85						

	$(9;0)$	$(9;1)$	$(9;2)$	$(9;3)$	$(9;4)$	$(9;5)$	$(9;6)$	$(9;7)$	$(9;8)$	$(9;9)$
0	17756793	-17756793	-5519664	603840	18264	0	0	0	0	0
1	6717465	-6717465	-2155050	256080	9096	0	0	0	0	0
2	2407365	-2407365	-797604	102912	4272	0	0	0	0	0
3	812157	-812157	-278046	38880	1872	0	0	0	0	0
4	256065	-256065	-90392	13568	744	0	0	0	0	0
5	75281	-75281	-27058	4208	248	0	0	0		
6	21165	-21165	-7500	1088	64	0				
7	6165	-6165	-2086	256						
8	1993	-1993								

Table 3.1 – Genus-0 Welschinger invariants $W^{\mathfrak{s}_{(\mathbb{R}P^3)_1},\mathfrak{o}_{(\mathbb{R}P^3)_1}}_{(\mathbb{R}P^3)_1}((d;k),l)$ of real three-dimensional del Pezzo varieties of degree 7 via Formula (3.6) with $d \leq 9$.

l \ (a,b,c)	$(1,0,0)$	$(1,1,1)$	$(2,2,1)$	$(3,2,2)$	$(3,3,1)$
0	1	1	1	8	1
1		1	1	6	1
2			1	4	1
3				2	1

l \ (a,b,c)	$(3,3,3)$	$(4,3,2)$	$(4,4,1)$	$(4,4,3)$	$(5,3,3)$	$(5,4,2)$	$(5,5,1)$
0	216	48	1	3864	1086	256	1
1	126	32	1	1980	606	160	1
2	68	20	1	960	318	96	1
3	34	12	1	444	158	56	1
4	16	8	1	200	78	32	1
5				92	46	16	1

l \ (a,b,c)	$(5,4,4)$	$(5,5,3)$	$(6,4,3)$	$(6,5,2)$	$(6,6,1)$
0	174360	57608	18424	1280	1
1	77064	27276	9256	768	1
2	32496	12400	4432	448	1
3	13136	5444	2032	256	1
4	5160	2328	904	144	1
5	2040	988	408	80	1
6	896	448	224	48	1

l \ (a,b,c)	$(5,5,5)$	$(6,5,4)$	$(6,6,3)$	$(7,4,4)$	$(7,5,3)$	$(7,6,2)$	$(7,7,1)$
0	15253434	5998848	773808	819200	268575	6144	1
1	5840298	2410496	347202	360896	125855	3584	1
2	2148978	931712	151028	152192	56831	2048	1
3	762450	347520	63958	61568	24831	1152	1
4	262842	125952	26488	24064	10559	640	1
5	89418	45056	10794	9280	4415	352	1
6	31122	16512	4412	3712	1887	192	1
7	12690	6784	1982	1536	991	96	1

Table 3.2 – Non-vanishing genus-0 Welschinger invariants $W_{(\mathbb{R}P^1)^3}^{\mathfrak{s}_{(\mathbb{R}P^1)^3},\mathfrak{o}_{(\mathbb{R}P^1)^3}}((a,b,c),l)$ of real three-dimensional del Pezzo varieties of degree 6 with real part is $\mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}P^1$ via Formula (3.7) with $a+b+c \leq 15$.

l \ $(a;c)$	$(1;1)$	$(2;1)$	$(2;3)$	$(3;1)$	$(3;3)$	$(3;5)$
0	-1	1	6	-1	-120	-576
1	-1	1	4	-1	-62	-288
2		1	2	-1	-28	-128
3			0	-1	-10	-48
4					0	-16
5						-16

l \ $(a;c)$	$(4;1)$	$(4;3)$	$(4;5)$	$(4;7)$
0	1	1692	61704	294336
1	1	768	24048	116352
2	1	324	8592	42624
3	1	128	2760	14208
4	1	44	792	4224
5		0	224	1152
6			96	320
7				-256

l \ $(a;c)$	$(5;1)$	$(5;3)$	$(5;5)$	$(5;7)$	$(5;9)$
0	-1	-20760	-3792888	-97970688	-493848576
1	-1	-8764	-1271016	-31204224	-160966656
2	-1	-3536	-399216	-9331200	-49582080
3	-1	-1380	-117072	-2593536	-14303232
4	-1	-520	-32056	-663936	-3821568
5	-1	-172	-8136	-157344	-938496
6		0	-1872	-35776	-215040
7			-1104	-7840	-47872
8				-4096	-10496
9					-26880

Table 3.3 – Genus-0 Welschinger invariants $W^{\mathfrak{s}_{\mathbb{S}^2\times\mathbb{R}P^1},\mathfrak{o}_{\mathbb{S}^2\times\mathbb{R}P^1}}_{\mathbb{S}^2\times\mathbb{R}P^1}((a;c),l)$ of real three-dimensional del Pezzo varieties of degree 6 with real part is $\mathbb{S}^2\times\mathbb{R}P^1$ via Formula (3.9) with $a\leq 5$.

Conclusions and perspectives

We end this thesis by summarizing the main contributions and some perspectives.

Conclusion

In this thesis, I first extended a theorem of Kollár to make a rational irreducible curve in 3-varieties X be contained in a non-singular surfaces in the linear system $|-\frac{1}{2}K_X|$, by specializing the position of configurations through which it passes; then I reduced the 3-dimensional enumerative problem into two independent ones:

— the 1-*dimensional problem* on a complex elliptic curve,

— the 2-*dimensional enumerative problem* on del Pezzo surfaces.

For the real case, I first adapted the Kollár's Theorem and then reduced the **real** enumerative problem on **real** 3-varieties into three independent ones:

— the *real* 1-*dimensional problem* on a **real** elliptic curve,

— the *real* 2-*dimensional enumerative problem* on **real** del Pezzo surfaces,

— the *sign problem*.

By this strategy, I proved the formulas for computing genus-0 Gromov-Witten and Welschinger invariants of 3-dimensional del Pezzo varieties mostly for the cases of degree $\{6, 7, 8\}$, by using the relationship to that of del Pezzo surfaces. More precisely, I showed

1. a generalization of Theorem 2 [BG16] with respect to genus zero Gromov-Witten invariants'relation; and
2. a generalization of Theorem 1 [BG16] with respect to genus zero Welschinger invariants'relation.

Moreover, I obtained a generalization of result of Kollár [Kol15] to the optimality of the genus zero Welschinger invariants on such spaces. As an application, I exhibited the explicit computations of genus-0 Gromov-Witten and Welschinger invariants in some del Pezzo 3-varieties using these formulas and Maple program.

Based on these obtained results, I believe that this strategy has potential for some new research directions on the relationship between complex and real enumerative geometry, as well as between those of varieties of dimension 2 and 3.

Perspectives

Since the computation of genus-0 Gromov-Witten invariants in del Pezzo varieties of dimension three was studied by Brugallé-Georgieva, Zahariuc, Ding and myself by using the same strategy: properties of pencils of surfaces, which was first proposed by Kollár. Zahariuc's formula is applied for cases of degree $2, 3, 4, 5$ and our formula is applied for almost cases of degree $6, 7, 8$. Moreover, our approach gives rise to the computation of genus-0 Welschinger invariants in the corresponding spaces.

Some proposed research directions

We suggest the following research directions that I put as question marks $(?)$ in the Table 3.4.

del Pezzo 3-varieties	**Gromov-Witten invariants' relation**	**Welschinger invariants' relation**
$\mathbb{C}P^3$	Brugallé-Georgieva [BG16]	Brugallé-Georgieva [BG16]
$\deg 2, 3, 4, 5$	Zahariuc [Zah18]	(?)
$\mathbb{C}P^3 \sharp \overline{\mathbb{C}P^3}$	Ding [Din19]	Ding[Din19]
$\mathbb{C}P^3, \mathbb{C}P^3 \sharp \overline{\mathbb{C}P^3}$, $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$	my thesis' work	my thesis'work
$\mathbb{P}T_{\mathbb{C}P^2}$	(?)	(?)
$\deg 1$	(??)	(??)

Table 3.4 – Some related results about enumerative geometry on del Pezzo 3-varieties

There are other points of view to generalize the idea and strategy which are used in my thesis and the related works as follows.

1. Relate the genus-0 Gromov-Witten and Welschinger invariants of del Pezzo varieties of dimension three and of any degree to that of del Pezzo surfaces using pencils of surfaces.
2. Relate the genus-0 Gromov-Witten and Welschinger invariants of Fano 3-varieties of index divisible by k ($k \in \{3, 2, 1\}$) to that of Fano (i.e. del Pezzo) surfaces.
3. Relate the genus-0 Gromov-Witten and Welschinger invariants of Calaubi-Yau 3-varieties (whose important property is having vanishing first Chern class) with that of K3 surfaces.

We can study the sign of real curves in two-dimensional varieties and three-dimensional varieties proposed by J. Solomon and A. Horev [HS12] and then fullfilled by X. Chen and A. Zinger [CZ19] to compare the associating real enumerative invariants of these two kinds of varieties.

Moreover, we have one observation that the logarithmic asymptotic properties of genus-0 Gromov-Witten and Welschinger invariants of del Pezzo surfaces are well-studied, for examples in a serie of articles by I. Itenberg, V. Kharlamov, E. Shustin [IKS13b] [IKS13a][IKS15]. These studies and our results give rise to the question about the logarithmic asymptotic properties of genus-0 Gromov-Witten and Welschinger invariants of del Pezzo 3-varieties.

Another observation is that, the del Pezzo varieties which we consider in this thesis are all *toric*. Furthermore, it is possible to compute Gromov-Witten and Welschinger invariants in toric varieties using tropical tools, for examples floor diagrams, or Caporaso-Harris formulas, throughout the work of many mathematiciens. Also, one can compute tropical refined invariants in these varieties. Therefore, there might exist a combinatorical approach to refine the relationship between tropical refined invariants of toric del Pezzo 3-varieties and that of toric del Pezzo surfaces.

Some remarks

We remark that, in this study we consider pencils of surfaces which realise half of the anti-canonical divisor of the 3-varieties X, for example pencils of quadrics if X is the projective 3-space. Doing this, we succeeded to deal with the enumeration of rational (i.e. genus 0) curves by studying the properties of an elliptic (i.e. genus 1) curve, which is the base locus of such pencils. Remark that if we want to do enumeration of genus g curves ($g \geq 0$) on 3-varieties, we have to consider pencils of surfaces such that its base locus is of genus at least $g + 1$. In particular, by the lack of definition for the sign of real curves of higher genus, we could progress to hope only for the relation between genus-g Gromov-Witten invariants of some 3-varieties (not Fano) and surfaces.

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Titre : Géométrie énumérative réelle des variétés de Fano de dimension 3 et d'indice 2

Mot clés : Géométrie énumérative, invariants de Gromov-Witten et de Welschinger.

Resumé : En géométrie énumérative, les invariants de Gromov-Witten sont connus comme l'un des problèmes classiques concernant le compte de courbes complexes passant par une contrainte générique dans les variétés projectives. Depuis 2003, les invariants de Welschinger sont connus comme la contrepartie réelle des invariants de Gromov-Witten, c'est-à-dire l'énumération de courbes réelles avec un choix cohérent de signe ± 1, tel que le compte se fait indépendamment du choix de contrainte réelle générique.

Cette thèse s'inspire des travaux sur les calculs des invariants de Gromov-Witten et de Welschinger de l'espace projectif complexe par E. Brugallé et P. Georgieva en 2016. Nous appliquons leur stratégie pour obtenir les formules calculant ces invariants des variétés de Fano de dimension 3, dont la première classe de Chern est paire, en mettant l'accent sur les cas des degrés $6, 7$ et 8. Nous généralisons également un résultat de J. Kollár sur l'optimalité des invariants de Welschinger de genre zéro sur de tels espaces.

Title: Real enumerative geometry in Fano varieties of dimension 3 and index 2

Keywords : Enumerative geometry, Gromov-Witten and Welschinger invariants.

Abstract: In enumerative geometry, Gromov-Witten invariants are known as one of the classical problems concerned with counting complex curves passing through a generic constraint in projective varieties. Since 2003, Welschinger invariants have been known as the real counterpart of Gromov-Witten invariants, i.e. the enumeration of real curves with a consistent choice of sign ± 1, such that this count is independent of the choice of a generic real configuration.

This thesis is inspired by the work on the computations of Gromov-Witten and Welschinger invariants of complex projective space by E. Brugallé and P. Georgieva in 2016. We apply their strategy to obtain the formulas for computing these invariants of 3-dimensional Fano varieties, whose first Chern class is even, with emphasis on the cases of degree $6, 7$ and 8. We also generalize a result of J. Kollár about the optimality of the genus zero Welschinger invariants on such spaces.