Smoothness of companion points on the trianguline variety

Seginus Mowlavi

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Singularité des points compagnons sur la variété trianguline

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Résumé: Cette thèse porte sur des aspects du programme de Langlands local p-adique. Nous étudions la géométrie locale de la variété trianguline en un point cristallin générique. En un tel point, lorsqu'il est de plus classique, Breuil-Hellmann-Schraen ont calculé la dimension de l'espace tangent à la variété trianguline, qui dépend notamment d'une certaine permutation w_{sat} attachée au point classique en question. Ils conjecturent aussi une formule générale pour le cas non-classique. En général, les points cristallins génériques sont paramétrés par la donnée d'un point classique et d'une permutation w telle que $w \succeq w_{\text{sat}}$ pour l'ordre de Bruhat. Nous mettons en évidence l'importance d'une certaine propriété combinatoire de la paire (w_{sat}, w). Nous étudions d'abord cette propriété pour le groupe de Weyl d'un systèmes de racines général. En particulier, nous montrons que l'ensemble des paires ayant cette propriété (que nous appelons bonnes paires) est lié au « pattern avoidance ». Nous prouvons alors la conjecture de Breuil-Hellmann-Schraen sur la dimension de l'espace tangent à la variété trianguline en tout point cristallin générique automorphe tel que (w_{sat}, w) est une bonne paire. Ensuite, nous prouvons en partie une conjecture sur la géométrie des cellules de Schubert d'un schéma provenant de la théorie géométrique des représentations, pour des paires de cellules paramétrées par des bonnes paires. Ce schéma servant de modèle local pour la variété trianguline, ceci conduit à une preuve de la formule pour la dimension de l'espace tangent en un point cristallin générique associé à une bonne paire, qu'il soit automorphe ou non. Enfin, nous montrons que la conjecture ci-dessus est fausse pour une famille infinie de paires de cellules.

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Keywords: trianguline variety, p-adic Hodge theory, Schubert cell

Abstract: This thesis is concerned with aspects of the p-adic local Langlands programme. We study the local geometry of the trianguline variety at a crystalline generic point. At such a point, when it is classical, Breuil-Hellmann-Schraen have computed the dimension of the tangent space to the trianguline variety, which notably depends on a certain permutation w_{sat} attached to the classical point in question. They also conjecture a general formula for the non-classical case. In general, crystalline generic points are known to be parametrised by the data of a classical point and a permutation w such that $w \succeq w_{\text{sat}}$ for the Bruhat order. We highlight the importance of a certain combinatorial property of the pair (w_{sat}, w). We first study this property in the context of Weyl groups of root systems. In particular, we show that the set of pairs having this property (which we call good pairs) is related to "pattern avoidance". We then improve on a method of Breuil-Hellmann-Schraen to get a partial proof of their conjecture about the dimension of the tangent space to the trianguline variety, at all automorphic crystalline generic points such that (w_{sat}, w) is a good pair. Next, we prove parts of a conjecture about the geometry of Schubert cells of a scheme coming from geometric representation theory, for pairs of cells parametrised by good pairs. This scheme being a local model for the trianguline variety, we get a different proof of the formula for the dimension of the tangent space at crystalline generic points associated to good pairs, whether it is automorphic or not. Finally, we show that the above conjecture is false for an infinite family of pairs of cells.

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Contents

Chapter 1

Introduction

1.1 Version française

Soit p un nombre premier. Tirant inspiration dans l'étude par Kisin [Kis03] des formes propres p-adiques surconvergentes et en se servant de la notion de représentation trianguline de Colmez [Col08], Hellmann [Hel12] puis Hellmann-Schraen [HS16] ont introduit une variété rigide analytique qui paramétrise les déformations cadrées d'une représentation résiduelle $\bar{r}$ fixée, appelée la variété trianguline $X_{\mathrm{tri}}^{\square}(\bar{r})$. Hellmann a utilisé cette construction pour poursuivre l'étude par Bellaïche-Chenevier [BC09] des variétés de Hecke. Dans une série de papiers [BHS17b], [BHS17a], [BHS19], Breuil-Hellmann-Schraen se sont intéressés à la géométrie de la variété trianguline et surtout à sa géométrie locale aux points cristallins génériques. Ils en ont tiré un certain nombre de conséquences, parmi lesquelles des résultats concernant la classicalité des formes surconvergentes, l'existence de certaines formes propres appelées formes propres compagnons, ainsi que l'existence de constituants compagnons (nous ne nous concentrons pas sur les représentations automorphes dans ce travail, mais ceci joue un rôle notable dans le programme de Langlands local p-adique ; voir par exemple [BE10], [Ber14], [CD14] et [CDP14]).

En particulier, Breuil-Hellmann-Schraen obtiennent une majoration de la dimension de l'espace tangent à $X_{\mathrm{tri}}^{\square}(\bar{r})$ en un point cristallin générique, et conjecturent que cette majoration est exacte. Ils prouvent aussi, pour des points strictements dominants (*i.e.* des points « classiques »), que cette conjecture est une conséquence des conjectures de modularité dans le cas cristallin. En particulier, ils soulignent l'importance d'un couple (w_{sat}, w) de permutations attaché à un point cristallin générique. L'objet de notre travail est d'investiguer leur formule conjecturale sur la dimension de l'espace tangent à la variété trianguline en un tel point cristallin générique (non nécessairement classique).

Nous faisons la découverte d'une propriété importante de (w_{sat}, w) que nous appelons « propriété de bonne paire ». Dans le Chapitre 2, nous développons une théorie combinatoire des bonnes paires dans le contexte des groupes de Weyl de systèmes de racines et établissons différents critères qui détectent cette propriété. La plupart des paires sont bonnes, et nous montrons que l'ensemble des bonnes paires est lié au « pattern avoidance ». Bien que nous n'ayons pas trouvé de références à cette notion dans la littérature, nous nous demandons si cette notion ne se retrouve pas naturellement dans d'autres contextes que celui de la variété trianguline.

Revenant à la théorie des représentations p-adiques, notre résultat principal est une preuve de la formule conjecturée par Breuil-Hellmann-Schraen sur la dimension de l'espace tangent à la variété trianguline en n'importe quel point cristallin générique tel que (w_{sat}, w) est une bonne paire.

Dans le Chapitre 3, nous généralisons les idées de [BHS17a] pour montrer que ce résultat est une conséquence des conjectures de modularité dans le cas cristallin, non seulement pour les points strictement dominants mais aussi pour n'importe quel point companion (dont la paire de permutations associée est bonne).

Dans le Chapitre 4, nous donnons une preuve différente, cette fois inconditionnelle, de notre théorème en utilisant le modèle local de [BHS19]. Ce modèle local décrit la géométrie de $X_{\mathrm{tri}}^{\square}(\bar{r})$ aux points cristallins génériques en termes de la géométrie locale d'un certain schéma provenant de la théorie géométrique des représentations. Dans *loc. cit.*, il est démontré que la formule pour la dimension de l'espace tangent à $X_{\mathrm{tri}}^{\square}(\bar{r})$ en ces points est une conséquence d'une conjecture concernant ce schéma géométrique énoncée en termes de paires d'éléments du groupe de Weyl. Nous prouvons cette dernière conjecture dans le cas des bonnes paires en utilisant des méthodes purement géométriques.

Enfin, nous exhibons dans le Chapitre 5 une famille de contre-exemples à cette conjecture géométrique (lorsque la paire d'éléments du groupe de Weyl est mauvaise). Ceci suggère que la géométrie locale de $X_{\mathrm{tri}}^{\square}(\bar{r})$ en un point compagnon « à mauvaise paire » renferme sans doute plus de richesse que ce que l'on pensait jusqu'à présent.

1.1.1 La variété trianguline

Soient K, L des extensions finies de $\mathbb{Q}_p$ telles que $\Sigma := \mathrm{Hom}_{\mathbb{Q}_p}(K, L)$ est de cardinal $[K : \mathbb{Q}_p]$. Notons $\mathcal{G}_K$ le groupe de Galois absolu de K et k_L le corps résiduel de L. Fixons un nombre entier $n \in \mathbb{Z}_{>0}$ ainsi qu'une représentation continue $\bar{r} \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k_L)$.

Notre objet d'étude principal est la variété trianguline $X_{\mathrm{tri}}^{\square}(\bar{r})$; c'est une variété rigide analytique réduite sur L définie de la manière suivante. Soit $R_{\bar{r}}^{\square}$ l'anneau de déformations cadrées de $\bar{r}$ construit par Mazur [Maz89, §1.2] et Kisin [Kis09]. En prenant son spectre formel $\mathrm{Spf}(R_{\bar{r}}^{\square})$ puis en appliquant le foncteur de rigidification de Berthelot [Ber96, §0.2], l'on obtient un espace analytique rigide $\mathfrak{X}_{\bar{r}}^{\square} := \mathrm{Spf}(R_{\bar{r}}^{\square})^{\mathrm{rig}}$ sur L. Soit $\mathcal{T}_L := \widehat{K^\times} \times_{\mathbb{Q}_p} L$ l'espace analytique rigide sur L qui paramétrise les caractères continus de $K^\times$. Étant donné $\mathbf{k} = (k_\tau) \in \mathbb{Z}^\Sigma$, nous notons $z^{\mathbf{k}} \in \mathcal{T}_L(L)$ le caractère $z \longmapsto \prod_{\tau \in \Sigma} \tau(z)^{k_\tau}$ et $|z|_K \in \mathcal{T}_L(L)$ la valeur absolue p-adique normalisée par $|p|_K = p^{-[K:\mathbb{Q}_p]}$. Un caractère $\underline{\delta} = (\delta_i) \in \mathcal{T}_L^n$ est dit régulier lorsque pour tout $1 \leq i \neq j \leq n$, le caractère $\delta_i \delta_j^{-1}$ n'est pas de la forme $z^{-\mathbf{k}}$ ou $N_{K/\mathbb{Q}_p}(z) z^{\mathbf{k}} |z|_K$ pour un certain $\mathbf{k} \in \mathbb{Z}_{\geq 0}^\Sigma$, où $N_{K/\mathbb{Q}_p}$ est l'application norme. Notons $\mathcal{T}_{\mathrm{reg}}^n \subset \mathcal{T}_L^n$ le sous-ensemble de caractères réguliers ; c'est un ouvert de Zariski. Soit $U^{\square}_{\mathrm{tri}}(\bar{r})$ l'ensemble des points $x = (r, \underline{\delta}) \in \mathfrak{X}^{\square}_{\bar{r}} \times \mathcal{T}^n_{\mathrm{reg}}$ tels que la représentation $r\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k(x))$ est trianguline au sens de Colmez avec paramètre $\underline{\delta}\colon (K^\times)^n \longrightarrow k(x)^\times$ (voir *e.g.* [BHS19, Dfn. 3.3.8]). La variété trianguline $X^{\square}_{\mathrm{tri}}(\bar{r})$ est alors définie comme l'adhérence de Zariski de $U^{\square}_{\mathrm{tri}}(\bar{r})$ dans $\mathfrak{X}^{\square}_{\bar{r}} \times \mathcal{T}^n_L$. Cette définition est expliquée de manière plus détaillée au § 3.3.

Soit $x = (r, \underline{\delta}) \in X^{\square}_{\mathrm{tri}}(\bar{r})$ un point cristallin, *i.e.* un point tel que r est une représentation cristalline. Nous supposons que r est générique, ce qui signifie que (i) pour chaque $\tau \in \Sigma$, les τ-poids de Hodge-Tate de r sont tous distincts, et (ii) $\varphi/\varphi' \notin \{1, p^{[K_0:\mathbb{Q}_p]}\}$ pour n'importe quelle paire de valeurs propres (φ, φ') du Frobenius linéarisé Φ sur $\mathbf{D}_{\mathrm{crys}}(r)$ (avec multiplicités). Pour chaque $\tau \in \Sigma$, nous notons les τ-poids de Hodge-Tate de r par ordre croissant $h_{\tau,1} < \ldots < h_{\tau,n}$.

Sous ces conditions, $\underline{\delta}$ est déterminé par la donnée de (i) un ordre $\underline{\varphi} = (\varphi_1, \ldots, \varphi_n)$ sur les valeurs propres de Φ et (ii) pour chaque $\tau \in \Sigma$, une permutation w_τ dans le groupe symétrique $\mathcal{S}_n$. Plus précisément, l'on peut écrire $\underline{\delta} = z^{w(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})$, où le caractère $\mathrm{unr}(\underline{\varphi}) = (\mathrm{unr}(\varphi_i))_i$ de $(K^\times)^n$ est défini par $\mathrm{unr}(\varphi_i)(p) = \varphi_i$ et $\mathrm{unr}(\varphi_i)|_{\mathcal{O}_K^\times} = 1$ et où nous notons $w \coloneqq (w_\tau) \in (\mathcal{S}_n)^\Sigma$ et $w(\mathbf{h}) \coloneqq \big(h_{\tau, w_\tau^{-1}(i)}\big)_{\tau,i} \in (\mathbb{Z}^n)^\Sigma$. De plus, il existe une unique permutation $w_{\mathrm{sat}} \in (\mathcal{S}_n)^\Sigma$ telle que le point $x_{\mathrm{sat}} \coloneqq \big(r, z^{w_{\mathrm{sat}}(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})\big) \in \mathfrak{X}^{\square}_{\bar{r}} \times \mathcal{T}^n_L$ se trouve sur $U^{\square}_{\mathrm{tri}}(\bar{r})$. La paire $(w_{\mathrm{sat}}, w) \in \big((\mathcal{S}_n)^\Sigma\big)^2$ attachée à x a la jolie propriété suivante : x existe en tant que point de $X^{\square}_{\mathrm{tri}}(\bar{r})$ si et seulement si $w_{\mathrm{sat}} \preceq w$ pour l'ordre de Bruhat sur $(\mathcal{S}_n)^\Sigma$ [BHS19, Thm. 1.8]. (Notons que w_{sat} ne dépend pas simplement de r ; un choix d'ordre sur les valeurs propres de Φ a été fait.) Le point x est dit dominant (ou classique) lorsque $w = w_0$ est l'unique élément maximal pour l'ordre de Bruhat ; dans les autres cas où $w_{\mathrm{sat}} \preceq w \prec w_0$, on appelle x un point compagnon.

L'objectif principal de cette thèse est de comprendre la géométrie locale de l'espace rigide analytique réduit $X^{\square}_{\mathrm{tri}}(\bar{r})$ en un tel point compagnon x. Cela s'inscrit dans la lignée des travaux de Breuil-Hellmann-Schraen, qui ont obtenu de nombreuses avancées ([BHS17b], [BHS17a], [BHS19]). Premièrement, la géométrie de $U^{\square}_{\mathrm{tri}}(\bar{r})$ est très bien comprise : $U^{\square}_{\mathrm{tri}}(\bar{r})$ est une partie Zariski-dense et Zariski-ouverte de $X^{\square}_{\mathrm{tri}}(\bar{r})$ qui est lisse de dimension $n^2 + [K:\mathbb{Q}_p]\frac{n(n+1)}{2}$. En particulier, $X^{\square}_{\mathrm{tri}}(\bar{r})$ est également équidimensionnelle de dimension $n^2 + [K:\mathbb{Q}_p]\frac{n(n+1)}{2}$. Ensuite, $X^{\square}_{\mathrm{tri}}(\bar{r})$ est normale et Cohen-Macaulay en n'importe quel point x comme ci-dessus ; voir [BHS19, Thm. 1.4]. En particulier, elle est localement irréductible en x. Dès lors, la principale question restante est de déterminer si x est un point lisse ou singulier. Il est donc intéressant de calculer la dimension de l'espace tangent de $X^{\square}_{\mathrm{tri}}(\bar{r})$ en x. Une formule explicite pour un majorant de $\dim_{k(x)} T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x}$ est donnée dans [BHS19, Prop. 4.1.5(ii)] : c'est le membre de droite de (1.1) ci-dessous. Cette majoration implique déjà la lissité dans le cas où $w = w_0$ et w_{sat} est un produit de réflexions simples distinctes ; voir [BHS19, Rmk. 4.1.6(ii)]. Dans le cas général, les auteurs conjecturent implicitement dans [BHS19, Rmk. 4.1.6(iii)] que cette majoration est toujours une égalité. Dans le cas strictement dominant $w = w_0$ sans condition sur w_{sat}, il est démontré [BHS17a, Prop. 5.17] que c'est une conséquence des conjectures de modularité dans le cas cristallin (voir [BHS17b, Conj. 1.3, Thm. 1.4]) et d'une conjecture de globalisation qui a récemment été prouvée par Emerton-Gee (voir [EG19, Thm. 1.2.3]).

Notre résultat principal est que cette majoration de $\dim_{k(x)} T_{X_{\mathrm{tri}}^{\square}(\bar{r}),x}$ est bien une égalité pour une classe substantielle de points compagnons qui contient le cas strictement dominant $w = w_0$. Cette classe est définie par la condition suivante portant sur la paire (w_{sat}, w). Une telle paire est dans la classe lorsque pour chaque $\tau \in \Sigma$, il existe des réflexions $s_1, \dots, s_r \in \mathcal{S}_n$ dont les supports sont tous inclus dans le support d'un cycle de la décomposition de $w_{\mathrm{sat},\tau} w_\tau^{-1}$ en cycles à support disjoints, et qui donnent lieu à une chaîne ascendante $w_{\mathrm{sat},\tau} \prec w_{\mathrm{sat},\tau} s_1 \prec \dots \prec w_{\mathrm{sat},\tau} s_1 \dots s_r = w_\tau$ pour l'ordre de Bruhat. Nous appelons *bonne paire* une paire (w_{sat}, w) qui satisfait cette condition. Notre résultat se formule alors comme suit :

Théorème 1.1.1 (Thm. 3.5.4, Thm. 4.2.4)**.** *Si (w_{sat}, w) est une bonne paire, alors*

$$\dim_{k(x)} T_{X_{\mathrm{tri}}^{\square}(\bar{r}),x} = \dim X_{\mathrm{tri}}^{\square}(\bar{r}) - d_{ww_{\mathrm{sat}}^{-1}} + \sum_{\tau \in \Sigma} \dim_{k(x)} T_{\overline{(Bw_\tau B/B)}^{\mathrm{rig}}_{k(x)}, x_\tau} - \lg(w_{\mathrm{sat}}) \quad (1.1)$$

(nous renvoyons au § 1.1.2 pour la définition de $d_{ww_{\mathrm{sat}}^{-1}}$ et à l'énoncé du Thm. 3.5.4 pour celle de x_τ).

Cela résout ainsi la question de la lissité pour les cas considérés dans [BHS19] : lorsque x est strictement dominant, $X_{\mathrm{tri}}^{\square}(\bar{r})$ est lisse en x si et seulement si w_{sat} est un produit de réflexions simples distinctes (nous rappelons que le sens « si » était déjà connu). Pour un point compagnon x tel que (w_{sat}, w) est une bonne paire, nous obtenons grâce au Théorème 1.1.1 et à une estimée de $d_{ww_{\mathrm{sat}}^{-1}}$ (voir Prop. 2.3.4) l'énoncé suivant (Cor. 4.2.5) : la lissité de $\overline{(Bw_\tau B/B)}$ en x_τ est une condition nécessaire à la lissité de $X_{\mathrm{tri}}^{\square}(\bar{r})$ en x.

Pour un point compagnon x tel que (w_{sat}, w) est une mauvaise paire, nous montrons de manière analogue que x est un point singulier de $X_{\mathrm{tri}}^{\square}(\bar{r})$ dès lors que (1.1) est vérifiée. Cependant, nous avons des raisons de penser que (1.1) n'est pas vraie en général pour les mauvaises paires (bien que nous n'ayons pas de contre-exemple étayant cette assertion) ; voir § 1.1.5 et Chapitre 5.

La preuve du Théorème 1.1.1, ainsi que notre analyse des cas où (w_{sat}, w) n'est pas une bonne paire, font intervenir le modèle local découvert par Breuil-Hellmann-Schraen dans [BHS19]. Nous donnons également une autre démonstration conditionnelle (qui repose sur les conjectures de modularité mais qui n'utilise pas le modèle local) qui part de la même idée initiale que [BHS17a, Prop. 5.17]. De manière inattendue, la condition de bonne paire se révèle être un élément fondamental de ces deux preuves, et elle y apparaît à travers deux énoncés différents (mais équivalents). (Cette condition n'est pas apparue dans les travaux de Breuil-Hellmann-Schraen car elle est automatiquement satisfaite lorsque $w = w_0$). Avant d'esquisser la stratégie de chaque preuve de manière plus détaillée, nous explorons donc brièvement le contexte naturel dans lequel cette condition s'exprime, à savoir celui des groupes de Weyl de systèmes de racines.

1.1.2 Bonnes paires dans un groupe de Weyl/Coxeter

Nous souhaitons tout d'abord reformuler la définition de bonne paire donnée au § 1.1.1. Considérons le groupe réductif déployé connexe $G = \prod_{\tau\in\Sigma} \mathrm{GL}_{n,L}$ sur L, et choisissons comme tore maximal déployé T le produit des groupes de matrices diagonales. Alors le système de racines $\Phi := \Phi(G, T)$ est isomorphe à $\coprod_{\tau\in\Sigma} \Phi_\tau$ où chaque Φ_τ est une copie du système de racines de type A_{n-1}, et son groupe de Weyl $W := W(G, T)$ est isomorphe à $(\mathcal{S}_n)^\Sigma$. Il semble donc raisonnable de voir w_{sat} et w en tant qu'éléments de W. Dans ce contexte, considérons le sous-tore $\big(T^{w_{\mathrm{sat}}w^{-1}}\big)^\circ$ fixé par $w_{\mathrm{sat}}w^{-1}$ sous l'action naturelle de W sur T. Alors le centralisateur connexe $C_G\big(\big(T^{w_{\mathrm{sat}}w^{-1}}\big)^\circ\big)$ est un groupe réductif déployé connexe sur L dont T est un tore maximal déployé ; nous notons $\Phi_{w_{\mathrm{sat}}w^{-1}} := \Phi\big(C_G\big(\big(T^{w_{\mathrm{sat}}w^{-1}}\big)^\circ\big), T\big)$ son système de racines associé. De plus, $\Phi_{w_{\mathrm{sat}}w^{-1}} \subseteq \coprod_{\tau\in\Sigma} \Phi_\tau$ est l'ensemble des racines $\alpha \in \Phi_\tau$ pour un certain $\tau \in \Sigma$ telles que la réflexion s_α a son support inclus dans un des supports disjoints de $w_{\mathrm{sat},\tau}w_\tau^{-1}$. Par conséquent, en utilisant l'ordre de Bruhat $\prec$ sur $(\mathcal{S}_n)^\Sigma$ (qui provient du choix d'un sous-groupe de Borel de G contenant T), la condition de bonne paire peut être reformulée de la manière suivante : (w_{sat}, w) est bonne si et seulement s'il existe des racines $\alpha_1, \dots, \alpha_r \in \Phi_{w_{\mathrm{sat}}w^{-1}}$ qui donnent lieu à une chaîne ascendante $w_{\mathrm{sat}} \prec w_{\mathrm{sat}}s_{\alpha_1} \prec \dots \prec w_{\mathrm{sat}}s_{\alpha_1}\dots s_{\alpha_r} = w$. Cette notion se généralise naturellement à n'importe quel groupe réductif déployé connexe.

En fait, le cadre naturel pour cette notion est celui des systèmes de racines abstraits : puisque n'importe quel système de racines abstrait Φ provient d'un groupe réductif déployé connexe G, il est raisonnable de s'attendre à ce que Φ_w, pour $w \in W := W(\Phi)$, puisse être défini uniquement grâce à Φ, sans mettre en jeu G. Il se trouve que c'est effectivement le cas, et notre définition « définitive » de Φ_w est la suivante : soit $\Gamma_w := \sum_{\alpha\in\Phi} \mathbb{Z}(w(\alpha) - \alpha)$, soit $E_w := \Gamma_w \otimes_{\mathbb{Z}} \mathbb{Q}$ et soit enfin $\Phi_w := \Phi \cap E_w$. L'équivalence de cette définition avec celle du paragraphe ci-dessus pour les groupes réductifs déployés est l'énoncé de la Prop. 2.2.2. Il ne reste plus qu'à fixer une base I de Φ, qui donne une fonction de longueur lg et un ordre de Bruhat $\prec$ sur W, et l'on peut définir une bonne paire (w_1, w_2) grâce à une condition de chaîne qui utilise $\Phi_{w_1w_2^{-1}}$ comme dans le paragraphe précédent. Nous disons également qu'une paire (w_1, w_2) est mauvaise lorsqu'elle n'est pas bonne et $w_1 \preceq w_2$.

Dans le § 2.1, nous établissons quelques résultats combinatoires sur Φ_w. Le plus important est le suivant :

Proposition 1.1.2 (Cor. 2.1.5)**.** *Le sous-système de racines $\Phi_w \subseteq \Phi$ est l'ensemble des $\alpha \in \Phi$ tels que s_α apparaît dans la décomposition de w sur l'alphabet $R := \{s_\beta \mid \beta \in \Phi\}$ qui est de longueur minimale, et cette longueur minimale est $d_w := \mathrm{rk}_{\mathbb{Z}}\,\Gamma_w$.*

Remarquons que ceci permet de définir la notion de bonne paire dans un groupe de Coxeter quelconque ; cependant nous n'y avons pas trouvé d'intérêt, car nous ne savons pas si les résultats obtenus au Chapitre 2 restent vrais dans ce cadre général.

La démonstration de la Prop. 1.1.2 repose sur un argument de récurrence sur d_w. Avec une variante de cette récurrence, nous prouvons que $d_w \leq \lg(w)$ avec égalité si et seulement si w est un produit de réflexions simples distinctes (Cor. 2.1.7). Plus généralement, pour n'importe quels $w_1, w_2 \in W$, l'on a $d_{w_2 w_1^{-1}} \leq \lg(w_2) - \lg(w_1)$ avec égalité seulement si (w_1, w_2) est une bonne paire (Prop. 2.3.4). Ces estimations sont notamment utiles à la compréhension du membre de droite de (1.1) : par exemple, Breuil-Hellmann-Schraen se servent de la majoration $d_w \leq \lg(w)$ dans le cas particulier d'un système de type A pour étudier les point strictement dominants de $X_{\mathrm{tri}}^{\square}(\bar{r})$.

Bien que les développements du § 2.1 fournissent une bonne compréhension théorique des sous-systèmes Φ_w, ils ne sont pas facilement applicables en pratique. Ce point est rattrapé au § 2.3, où nous établissons un critère pratique pour les bonnes paires. Celui-ci trouve sa source dans le fait que, pour n'importe quel $w \in W$, le sous-système Φ_w est conjugué à un sous-système standard, *i.e.* un sous-système de racines $\Phi_J \subseteq \Phi$ engendré par un sous-ensemble $J \subseteq I$ de racines simples. Plus précisément, étant donné $J \subseteq I$, soit W_J le sous-groupe de W engendré par $\{s_\alpha \mid \alpha \in J\}$, et soit W^J l'ensemble des représentants de longueur minimale de $W_J \backslash W$. Alors l'on peut écrire $\Phi_w = u^J(\Phi_J)$ pour un certain $J \subseteq I$ et un certain $u^J \in W^J$.

Proposition 1.1.3 (Prop. 2.3.7)**.** *Soit* $(w_1, w_2) \in W^2$*. Écrivons* $\Phi_{w_2 w_1^{-1}} = u^J(\Phi_J)$ *pour un certain* $J \subseteq I$ *et un certain* $u^J \in W^J$*. Alors il existe* $v^J \in W^J$ *et* $w_{J,1}, w_{J,2} \in W_J$ *tels que*

$$w_1 = u^J w_{J,1} (v^J)^{-1}, \quad w_2 = u^J w_{J,2} (v^J)^{-1},$$

et la paire (w_1, w_2) *est bonne si et seulement si* $w_{J,1} \preceq w_{J,2}$*.*

Nous concluons le § 2.6 par une jolie application de ce dernier critère. Il se formule en termes de « pattern avoidance » (ou « évitement de motifs ») : l'idée est de caractériser certaines classes de permutations w dans $\bigcup_{n \in \mathbb{Z}_{>0}} \mathcal{S}_n$ par la propriété que w ne contient aucun motif parmi une liste fixée. Nous entendons ici, par « contenir un motif », que $w \in \mathcal{S}_n$ contient le motif $f \in \mathcal{S}_m$ lorsqu'il existe $1 \leq i_1 < \ldots < i_m \leq n$ tel que les m-uplets $(w(i_1), \ldots, w(i_m))$ et $(f(1), \ldots, f(m))$ sont dans le même ordre ; voir Def. 2.6.1. L'on peut en fait formuler ceci dans le cadre général d'un système de racines abstrait, et de nombreuses propriétés telles que la lissité des variétés de Schubert sont connues pour être des exemples de « pattern avoidance » ; voir par exemple [BP05, §2, §4]. Nous montrons que, au moins pour les systèmes de racines de type A, la propriété de « ne pas apparaître dans une mauvaise paire » est également un cas de « pattern avoidance » ; plus précisément :

Théorème 1.1.4 (Thm. 2.6.3)**.** *Soit* $w \in \mathcal{S}_n$*.*

(1) *La paire* (w', w) *est bonne pour tout* $w' \in \mathcal{S}_n$ *si et seulement si* w *évite les quatre motifs* [4231], [42513], [35142] *et* [351624].

(2) *La paire* (w, w'') *est bonne pour tout* $w'' \in \mathcal{S}_n$ *si et seulement si* w *évite les quatre motifs* [1324], [24153], [31524] *et* [426153].

En faisant le lien entre ces motifs et ceux qui déterminent la lissité des variétés de Schubert, nous obtenons le Cor. 2.6.5 (voir § 4.1 pour plus de détails sur la décomposition de Schubert). Ce corollaire affirme que si $(w, w') \in (\mathcal{S}_n)^2$ est une mauvaise paire, alors la cellule de Schubert fermée $\overline{BwB/B}$ dans la variété de drapeaux G/B est singulière (mais pas nécessairement au point $w'B \in \overline{BwB/B}$).

1.1.3 Démonstration arithmétique du Théorème 1.1.1

Nous esquissons ici notre preuve, dans le Chapitre 3, que les conjectures de « modularité » [BHS17b, Conj. 1.3] impliquent Thm. 1.1.1. Elle est basée sur une généralisation des idées de [BHS17a] (où les auteurs se restreignent au cas strictement dominant).

L'idée grossière est, pour un point compagnon x comme défini au § 1.1.1, de trouver deux sous-espaces $T_1, T_2 \subseteq T_{X_{\mathrm{tri}}^{\square}(\bar{r}),x}$ à propos desquels nous pouvons prouver que $\dim(T_1+T_2)$ est égal au membre de droite de (1.1). Cela fournit une minoration de $\dim T_{X_{\mathrm{tri}}^{\square}(\bar{r}),x}$ par $\dim(T_1+T_2)$ qui, mise bout-à-bout avec la majoration de [BHS19, Prop. 4.1.5(ii)], fournit le résultat voulu.

Directions tangentes cristallines La notion centrale qui intervient dans la construction du premier sous-espace T_1 est celle des raffinements de représentations cristallines. Nous en donnons ici un aperçu ; nous renvoyons le lecteur au § 3.1 pour les notations et une revue détaillée des parties de la théorie de Hodge p-adique qui nous sont nécessaires, et au § 3.3 pour la construction du sous-espace T_1.

Soit $r\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(L')$ une représentation cristalline, où L' est une extension finie de L. Un raffinement de r est un ordre $(\varphi_1, \dots, \varphi_n)$ sur les valeurs propres du Frobenius linéarisé Φ de r. Lorsque ces valeurs propres sont distinctes, un raffinement est équivalent à la donnée d'un drapeau Φ-stable complet $\mathcal{F}_\bullet$ du $(K_0 \otimes_{\mathbb{Q}_p} L')$-module $\mathbf{D}_{\mathrm{crys}}(r)$, où K_0 est l'extension maximale non-ramifiée de $\mathbb{Q}_p$ contenue dans K. Par changement de base, un raffinement fournit un drapeau complet $K \otimes_{K_0} \mathcal{F}_\bullet$ sur le module $\mathbf{D}_{\mathrm{dR}}(r)$ sur $K \otimes_{\mathbb{Q}_p} L' \simeq \prod_{\tau\in\Sigma} L'$. Notons que ce dernier drapeau est équivalent à la donnée d'un drapeau complet sur chaque L'-espace vectoriel $\mathbf{D}_{\mathrm{dR}}(r) \otimes_{K,\tau} L'$ pour $\tau \in \Sigma$. D'un autre côté, lorsque les τ-poids de Hodge-Tate de r sont distincts pour tout $\tau \in \Sigma$, la filtration de Hodge sur $\mathbf{D}_{\mathrm{dR}}(r)$ induit un autre drapeau complet $\mathrm{Fil}_\bullet\, \mathbf{D}_{\mathrm{dR}}(r)$ sur le $(K \otimes_{\mathbb{Q}_p} L')$-module $\mathbf{D}_{\mathrm{dR}}(r)$. En fixant une base de $\mathbf{D}_{\mathrm{dR}}(r)$ compatible au drapeau $K \otimes_{K_0} \mathcal{F}_\bullet$, l'on peut considérer le drapeau $\mathrm{Fil}_\bullet\, \mathbf{D}_{\mathrm{dR}}(r)$ comme un L'-point de la variété de drapeaux G/B, où G est le groupe réductif déployé connexe $G := \mathrm{Res}_{K/\mathbb{Q}_p}(\mathrm{GL}_{n,K}) \otimes_{\mathbb{Q}_p} L \simeq \prod_{\tau\in\Sigma} \mathrm{GL}_{n,L}$ et B est le sous-groupe de Borel de G des matrices triangulaires supérieures. Nous appelons ce point de G/B la position relative de $\mathrm{Fil}_\bullet\, \mathbf{D}_{\mathrm{dR}}(r)$ par rapport à $K \otimes_{K_0} \mathcal{F}_\bullet$.

À présent, fixons un uplet d'entiers $\mathbf{h} = (h_{\tau,1} < \dots < h_{\tau,n})_\tau \in (\mathbb{Z}^n)^\Sigma$; nous avions également fixé une représentation continue $\bar{r}\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k_L)$ (qui est utilisée dans la définition de $X_{\mathrm{tri}}^{\square}(\bar{r})$). En utilisant les travaux de Kisin [Kis08], Breuil-Hellmann-Schraen [BHS17a, §2.2] construisent un espace analytique rigide réduit

$\widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}}$ sur L qui paramétrise la donnée d'une déformation cadrée r de $\bar{r}$ qui est cristalline générique dans le sens du § 1.1.1 et de poids de Hodge-Tate $\mathbf{h}$, ainsi que d'un raffinement $\underline{\varphi} = (\varphi_1, \ldots, \varphi_n)$ de r. Il existe aussi un morphisme lisse d'espaces L-analytiques rigides $h\colon \widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}} \longrightarrow (G/B)^{\mathrm{rig}}$ qui envoie un point $(r, \underline{\varphi})$ de $\widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}}$ sur la position relative du drapeau de Hodge sur $\mathbf{D}_{\mathrm{dR}}(r)$ par rapport au raffinement $\underline{\varphi}$. Le plongement de $W(G) = (\mathcal{S}_n)^\Sigma$ dans G via les matrices de permutations donne la décomposition de Schubert de la variété de drapeaux $G/B = \coprod_{w\in(\mathcal{S}_n)^\Sigma} BwB/B$. Pour $w \in (\mathcal{S}_n)^\Sigma$, nous considérons ensuite le sous-espace rigide fermé $\widetilde{W}_{\bar{r},w}^{\mathbf{h}-\mathrm{cr}}$ de $\widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}}$ défini par $\widetilde{W}_{\bar{r},w}^{\mathbf{h}-\mathrm{cr}} \coloneqq h^{-1}\big(\overline{BwB/B}^{\mathrm{rig}}\big)$. Alors il existe une immersion fermée d'espace analytiques rigides $\iota_{\mathbf{h},w}\colon \widetilde{W}_{\bar{r},w}^{\mathbf{h}-\mathrm{cr}} \longrightarrow X_{\mathrm{tri}}^{\square}(\bar{r})$ définie par $\iota_{\mathbf{h},w}(r, \underline{\varphi}) = \big(r, z^{w(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})\big)$.

Étant donné un point compagnon $x = (r, \underline{\delta}) \in X_{\mathrm{tri}}^{\square}(\bar{r})$ comme dans le § 1.1.1, nous prenons $\mathbf{h}$ comme l'uplet ordonné de poids de Hodge-Tate de r. Soient $\underline{\varphi} = (\varphi_i)_i$ un raffinement de r et $w \in (\mathcal{S}_n)^\Sigma$ une permutation telle que $\underline{\delta} = z^{w(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})$. Alors le point $(r, \underline{\varphi})$ de $\widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}}$ est en fait dans $\widetilde{W}_{\bar{r},w}^{\mathbf{h}-\mathrm{cr}}$, et $\iota_{\mathbf{h},w}$ envoie $(r, \underline{\varphi})$ sur x. Par conséquent, $\iota_{\mathbf{h},w}$ induit une injection $T_{\widetilde{W}_{\bar{r},w}^{\mathbf{h}-\mathrm{cr}},(r,\underline{\varphi})} \hookrightarrow T_{X_{\mathrm{tri}}^{\square}(\bar{r}),x}$, et nous définissons T_1 comme l'image de cette injection.

Directions tangentes triangulines L'idée derrière la construction de T_2, que nous effectuons au § 3.5, est d'utiliser l'automorphisme rigide $\mathcal{T}_L^n \xrightarrow{\sim} \mathcal{T}_L^n$ défini par $\underline{\eta} \longmapsto z^{w(\mathbf{h})-w_{\mathrm{sat}}(\mathbf{h})}\underline{\eta}$, qui induit l'automorphisme $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}\colon \mathfrak{X}_{\bar{r}}^{\square} \times \mathcal{T}_L^n \xrightarrow{\sim} \mathfrak{X}_{\bar{r}}^{\square} \times \mathcal{T}_L^n$ d'espaces analytiques rigides sur L. Par définition, $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$ envoie x_{sat} sur x ; pour obtenir des vecteurs tangents à $X_{\mathrm{tri}}^{\square}(\bar{r})$ en x, nous voudrions que $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$ envoie une sous-variété localement fermée $U_{\mathrm{tri}}^{\square}(\bar{r})$ contenant x_{sat} dans $X_{\mathrm{tri}}^{\square}(\bar{r})$. C'est ici que la Conj. 1.3 de [BHS17b] et la propriété de bonne paire de (w_{sat}, w) sont nécessaires.

Plus précisément, nous utilisons le morphisme de poids $\omega\colon X_{\mathrm{tri}}^{\square}(\bar{r}) \longrightarrow \mathcal{W}_L^n$, où $\mathcal{W}_L$ est l'espace analytique rigide sur L qui paramétrise les caractères continus de $\mathcal{O}_K^\times$, défini par $\omega(r, \underline{\delta}) = \underline{\delta}|_{(\mathcal{O}_K^\times)^n}$. Notons qu'un caractère $\eta\colon \mathcal{O}_K^\times \longrightarrow (L')^\times$, où L' est une extension finie de L, est un morphisme de groupes de Lie p-adiques ; il induit ainsi une action différentielle $\mathrm{d}\eta \in \mathrm{Hom}_{\mathbb{Q}_p}(\mathfrak{t}, L') \simeq \prod_{\tau\in\Sigma} L'$, où $\mathfrak{t} \simeq K$ est l'algèbre de Lie de $\mathcal{O}_K^\times$. Le τ-poids $\mathrm{wt}_\tau(\eta) \in L'$ de η est alors défini comme $\mathrm{d}\eta = (\mathrm{wt}_\tau(\eta))$. Nous considérons alors la sous-variété de caractères $\mathcal{W}_{w,w_{\mathrm{sat}},\mathbf{h},L}^n \subset \mathcal{W}_L^n$ définie par les équations $\mathrm{wt}_\tau(\eta_{w_{\mathrm{sat},\tau}(i)}\eta_{w_\tau(i)}^{-1}) = k_{\tau,i} - k_{\tau,w_{\mathrm{sat},\tau}^{-1}w_\tau(i)}$.

Proposition 1.1.5 (Prop. 3.5.3)**.** *Supposons que la Conj. 1.3 de [BHS17b] est vérifiée. Si la paire (w_{sat}, w) de $(\mathcal{S}_n)^\Sigma$ est bonne, alors il existe un voisinage ouvert $U_{x_{\mathrm{sat}}}$ de x_{sat} dans $U_{\mathrm{tri}}^{\square}(\bar{r})$ tel que $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$ induit un plongement Zariski-fermé d'espaces rigides analytiques réduits sur L*

$$\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}\colon \overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}_{w,w_{\mathrm{sat}},\mathbf{h},L}^n} \hookrightarrow X_{\mathrm{tri}}^{\square}(\bar{r})\,. \tag{1.2}$$

Ainsi $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$ induit une injection $T_{U_{x_{\mathrm{sat}}}\times_{\mathcal{W}_L^n}\mathcal{W}_{w,w_{\mathrm{sat}},\mathbf{h},L}^n,x_{\mathrm{sat}}} \hookrightarrow T_{X_{\mathrm{tri}}^{\square}(\bar{r}),x}$, et nous définissons T_2 comme l'image de cette injection. Le calcul $\dim T_2 = \dim X_{\mathrm{tri}}^{\square}(\bar{r}) -$

$d_{ww_{\mathrm{sat}}^{-1}}$ est assez direct ; la difficulté majeure consiste à démontrer la Prop. 1.1.5.

À cette fin, nous avons besoin de globaliser nos objets. Nous donnons ici un bref aperçu de ce cadre global, et renvoyons au § 3.4 pour plus de détails. Soit F/F^+ une extension quadratique totalement imaginaire, où F^+ est un corps de nombres totalement réel ; en notant S_p l'ensemble des places finies de F^+ divisant p, nous supposons que toutes les places $v \in S_p$ sont déployées dans F et satisfont $F_v^+ \simeq K$. Soit $\bar{\rho}\colon \mathcal{G}_{F^+} \longrightarrow \mathrm{GL}_n(k_L)$ une représentation continue irréductible qui est automorphe [EG14, Def. 5.3.1] ; elle provient d'une certaine forme automorphe sur $G(\mathbb{A}_{F^+})$ d'un niveau modéré $U^p = \prod_{v \notin S_p \text{ finite}} U_v \subset G(\mathbb{A}_{F^+}^{p\infty})$, où G est un groupe unitaire en n variables sur F^+. Nous supposons également les « hypothèses standard de Taylor-Wiles » ; voir [BHS17b, §2.4]. Un résultat récent d'Emerton-Gee [EG19, Thm. 1.2.3] donne l'existence de ce cadre global, avec la propriété additionnelle que pour toute place $v \in S_p$, il existe une place $\tilde{v} \mid v$ de F telle que $\bar{\rho}_{\tilde{v}} := \bar{\rho}|_{\mathcal{G}_{F_{\tilde{v}}}}$ est isomorphe à $\bar{r}$ via $F_{\tilde{v}} \simeq F_v^+ \simeq K$.

Dans ce cadre, Breuil-Hellmann-Schraen ont construit dans [BHS17b, §3] la « variété patchée » $X_p(\bar{\rho})$ en utilisant le patching de [CEG+16]. C'est un sous-espace L-analytique rigide de $\mathfrak{X}_{\bar{\rho}^p} \times \prod_{v \in S_p} (\mathfrak{X}_{\bar{\rho}_v} \times \mathcal{T}_L^n) \times \mathbb{U}^g$, où $\mathfrak{X}_{\bar{\rho}^p} := \prod_{v \notin S_p} \mathfrak{X}_{\bar{\rho}_v}$, $\mathbb{U}$ est le disque unité rigide sur L et $g \in \mathbb{Z} > 0$ est un entier. L'on a un automorphisme rigide ι de $\prod_{v \in S_p} \mathcal{T}_L^n$ qui induit un plongement fermé $\iota^{-1}\colon X_p(\bar{\rho}) \hookrightarrow \mathfrak{X}_{\bar{\rho}^p} \times \prod_{v \in S_p} X_{\mathrm{tri}}^{\square}(\bar{\rho}_{\tilde{v}}) \times \mathbb{U}^g$.

La Conj. 3.4.6 donne une description conjecturale de l'image de ι^{-1}. Une conséquence en est l'existence d'un voisinage $U_{x_{\mathrm{sat}}}$ de x_{sat} dans $U_{\mathrm{tri}}^{\square}(\bar{r})$ tel que $X_p(\bar{\rho}) \hookrightarrow \mathfrak{X}_{\bar{\rho}^p} \times \prod_{v \in S_p} U_{x_{\mathrm{sat}}} \times \mathbb{U}^g$ est dans l'image de ι^{-1} ; ici nous voyons $U_{x_{\mathrm{sat}}}$ comme un sous-espace de chaque $X_{\mathrm{tri}}^{\square}(\bar{\rho}_{\tilde{v}})$, $v \in S_p$, via l'isomorphisme $\bar{\rho}_{\tilde{v}} \simeq \bar{r}$. Alors (1.2) peut être vérifié à travers une assertion analogue à (1.2) sur $X_p(\bar{\rho})$, cette dernière étant une conséquence de : (i) le Thm. 5.5 de [BHS17a], qui réduit le problème à la vérification du fait que, pour n'importe quel $\underline{\delta}' \in \mathcal{T}_L^n \times_{\mathcal{W}_L^n} \mathcal{W}_{w,w_{\mathrm{sat}},\mathbf{h},L}^n$, le caractère $\iota\left((\underline{\delta}')_{v \in S_p}\right)$ est « strongly linked » à $\iota\left(\left(z^{w(\mathbf{h})-w_{\mathrm{sat}}(\mathbf{h})}\underline{\delta}'\right)_{v \in S_p}\right)$ au sens de [Hum08, §5.1] ; et (ii) un calcul direct qui montre que ce « strong linkage » est une conséquence de la chaîne ascendante entre w_{sat} et w donnée par la condition de bonne paire. Notons que c'est ici que l'hypothèse de bonne paire est nécessaire.

Dimension de $T_1 + T_2$ Nous calculons $\dim(T_1 + T_2)$ en utilisant le diagramme commutatif d'espaces L-analytiques rigides

$$\begin{tikzcd}
\widetilde{W}_{x_{\mathrm{sat}}} \arrow[r, hook, "\iota_{\mathbf{h},w_{\mathrm{sat}}}"] \arrow[d, hook, "\subseteq"] & \overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}_{w,w_{\mathrm{sat}},\mathbf{h},L}^n} \arrow[d, hook, "\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}"] \\
\widetilde{W}_{\bar{r},w}^{\mathbf{h}-\mathrm{cr}} \arrow[r, hook, "\iota_{\mathbf{h},w}"] & X_{\mathrm{tri}}^{\square}(\bar{r})\,,
\end{tikzcd}$$

où $\widetilde{W}_{x_{\mathrm{sat}}} := \iota_{\mathbf{h},w_{\mathrm{sat}}}^{-1}\left(\overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}_{w,w_{\mathrm{sat}},\mathbf{h},L}^n}\right) \subseteq \widetilde{W}_{\bar{r},w_{\mathrm{sat}}}^{\mathbf{h}-\mathrm{cr}}$, qui induit un diagramme commutatif d'espaces tangents

$$\begin{tikzcd}
T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}},(r,\underline{\varphi})} \arrow[r, hook] \arrow[d, hook] & T_{\overline{U_{x_{\mathrm{sat}}}\times_{\mathcal{W}^n_L}\mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}},x_{\mathrm{sat}}} \arrow[d, hook] \\
T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w},(r,\underline{\varphi})} \arrow[r, hook] & T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x}\,.
\end{tikzcd} \tag{1.3}$$

En déroulant les définitions, nous montrons que la proposition suivante, appliquée à l'algèbre $A = k(x)[\varepsilon]/(\varepsilon^2)$ des nombres infinitésimaux sur le corps résiduel local de x, implique que (1.3) est un carré cartésien.

Proposition 1.1.6 (Prop. 3.2.13)**.** *Soit A une L-algèbre locale artinienne de corps résiduel L. Soit $\mathcal{G}_K \longrightarrow \mathrm{GL}(V)$ une représentation cristalline régulière de rang $n \in \mathbb{N}$ sur A, de poids de Hodge-Tate $\mathbf{h} \in (\mathbb{Z}^n)^\Sigma$. Supposons qu'il existe $\mathbf{k} \in \mathbb{Z}^\Sigma$ et $\underline{\varphi} \in (A^\times)^n$, où les réductions de φ_i dans L sont deux-à-deux distinctes, tels que V est trianguline de paramètre $z^{\mathbf{k}}\,\mathrm{unr}(\underline{\varphi})$.*

Alors il existe $w \in (\mathcal{S}_n)^\Sigma$ tel que $\mathbf{k} = w(\mathbf{h})$, et le Frobenius linéarisé Φ sur $\mathbf{D}_{\mathrm{crys}}(V)$ admet $\underline{\varphi}$ comme n-uplet de valeurs propres. De plus, en notant $\mathcal{F}_\bullet$ le drapeau $\mathbf{D}_{\mathrm{crys}}(V)$ induit par le raffinement $\underline{\varphi}$, la position relative $x \in \prod_{\tau\in\Sigma}(\mathrm{GL}_n/B)(A)$ de $\mathrm{Fil}_\bullet\,\mathbf{D}_{\mathrm{dR}}(V)$ par rapport à $K \otimes_{K_0} \mathcal{F}_\bullet$ se situe dans la cellule $\prod_{\tau\in\Sigma}(Bw_\tau B/B)(A)$.

Nous précisons que la Prop. 1.1.6 est une généralisation de [BC09, Prop. 2.4.1], qui s'occupe du cas $K = \mathbb{Q}_p$ and $A = L$. Notre démonstration utilise (i) le dictionnaire de Berger entre les (φ, Γ_K)-modules cristallins sur l'anneau de Robba et les φ-modules filtrés ([Ber08, Thm. A] ; voir aussi [Ber02, Thm. 3.6]), et (ii) la description des (φ, Γ_K)-modules de rang 1 à travers ce dictionnaire (voir *e.g.* [KPX14, Ex. 6.2.6(3)]). Les éléments nécessaires de la théorie des (φ, Γ_K)-modules sont passés en revue au § 3.2.

De là, nous obtenons $\dim(T_1 + T_2)$ sans grande difficulté, en utilisant le fait que (1.3) et le diagramme suivant

$$\begin{tikzcd}
\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}} \arrow[r, "h"] \arrow[d, "\subseteq"] & \overline{(Bw_{\mathrm{sat}}B/B)}^{\mathrm{rig}}_L \arrow[d, "\subseteq"] \\
\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w} \arrow[r, "h"] & \overline{(BwB/B)}^{\mathrm{rig}}_L
\end{tikzcd}$$

sont des carrés cartésiens, ainsi que la lissité de h.

1.1.4 Démonstration géométrique du Théorème 1.1.1

Dans le Chapitre 4, nous étudions une version simplifiée du modèle local de $X^{\square}_{\mathrm{tri}}(\bar{r})$ introduit dans [BHS19]. Il s'agit d'un schéma $\widetilde{X}$ avec une décomposition en cellules indexées par $(\mathcal{S}_n)^\Sigma$, tel que la géométrie locale de $X^{\square}_{\mathrm{tri}}(\bar{r})$ en un point cristallin générique avec permutations associées (w_{sat}, w) est décrite par la géométrie locale de la cellule fermée de $\widetilde{X}$ d'indice w en un certain point de son bord dans la cellule ouverte d'indice w_{sat}. Cette étude aboutit à une preuve inconditionnelle du Thm.

1.1.1 : contrairement à celle expliquée au § 1.1.3 et au Chapitre 3, elle n'utilise pas les conjectures de modularité dans le cas cristallin. Nous donnons ici quelques détails sur $\widetilde{X}$ et son utilisation dans la démonstration du Thm. 1.1.1.

Soit k un corps de caractéristique 0. Soient G groupe réductif déployé connexe sur k, T un tore maximal déployé dans G, B un sous-groupe de Borel de G contenant T et U le sous-groupe unipotent de B. Notons $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{b}$ et $\mathfrak{u}$ leurs algèbres de Lie respectives, et $W := N_G(T)/T$ le groupe de Weyl de (G, T). Nous rappelons brièvement la décomposition de Schubert sur G/B (voir aussi § 4.1). Pour $w \in W$, choisissons un relevé w de $N_G(T)$ que nous notons encore w ; considérons l'orbite BwB/B de la classe à gauche wB/B par l'action de multiplication à gauche de B sur G/B. L'on a alors une décomposition $G/B = \coprod_{w\in W} BwB/B$ de G/B en strates localement fermées BwB/B, et une cellule ouverte $Bw'B/B$ intersecte une cellule fermée $\overline{BwB/B}$ si et seulement si $w' \preceq w$ pour l'ordre de Bruhat sur W, auquel cas $Bw'B/B \subseteq \overline{BwB/B}$.

Soit $G \times^B \mathfrak{g}$ le quotient de $G \times \mathfrak{g}$ par l'action à gauche de B donnée par $(g, \psi)b = (gb, \mathrm{Ad}(b^{-1})\psi)$. Définissons les morphismes $q\colon G \times^B \mathfrak{g} \longrightarrow \mathfrak{g}$ et $\widetilde{\pi}\colon G \times^B \mathfrak{g} \longrightarrow G/B$ par $q(g, \psi) = \mathrm{Ad}(g)\psi$ et $\widetilde{\pi}(g, \psi) = gB$. Le modèle local simplifié de $X^{\square}_{\mathrm{tri}}(\bar{r})$ est alors la variété $\widetilde{X}$ sur k définie par $\widetilde{X} := q^{-1}(\mathfrak{b})$. Pour $w \in W$, considérons la sous-variété localement fermée $\widetilde{V}_w := \widetilde{X} \cap \widetilde{\pi}^{-1}(BwB/B)$ de $\widetilde{X}$ ainsi que son adhérence de Zariski $\widetilde{X}_w$ dans $\widetilde{X}$; ceci donne une décomposition cellulaire de $\widetilde{X}$.

Notons que dans cette partie et dans le Chapitre 4, contrairement au § 1.1.3 et au Chapitre 3, nous désignons par la lettre L un groupe réductif déployé connexe sur k. Nous ajoutons la lettre L en indice aux notations $q, \widetilde{\pi}, \widetilde{X}, \widetilde{V}_w, \widetilde{X}_w$ (pour $w \in W(L)$) pour désigner les objets définis comme précédemment pour le groupe réductif L au lieu du groupe réductif G ; par défaut, ces notations s'entendent pour G lorsque rien n'est précisé.

De manière semblable au cas des cellules de Schubert dans G/B, l'intersection $\widetilde{V}_{w'} \cap \widetilde{X}_w$ est non-vide si et seulement si $w' \preceq w$. La différence est que dans ce cas, $\widetilde{V}_{w'}$ n'est pas inclus dans $\widetilde{X}_w$. En fait, Breuil-Hellmann-Schraen montrent dans [BHS19, Lem. 2.3.4] que $\widetilde{V}_{w'} \cap \widetilde{X}_w \subseteq \widetilde{V}_{w'} \cap q^{-1}\left(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u}\right)$, et conjecturent que c'est une égalité :

Conjecture 1.1.7 (Conj. 2.3.7 de [BHS19])**.** *Soient $w, w' \in W$. Si $w' \preceq w$, alors*

$$\widetilde{V}_{w'} \cap \widetilde{X}_w = \widetilde{V}_{w'} \cap q^{-1}\left(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u}\right).$$

L'on peut montrer que la formule (1.1) pour la dimension $T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x}$ est une conséquence de la Conj. 1.1.7 pour $w_{\mathrm{sat}} \preceq w$; voir Prop. 4.1.2 (voir aussi la preuve [BHS19, Thm. 4.1.5] lorsque $w = w_0$). Il se trouve que la Conj. 1.1.7 est fausse en général (mais il n'est pas clair que (1.1) soit aussi fausse) ; voir le § 1.1.5 et le Chapitre 5. Nous la démontrons néanmoins dans le cas des bonnes paires, ce qui implique le Thm. 1.1.1 :

Théorème 1.1.8 (Thm. 4.2.3). *Soient $w, w' \in W$ tels que $w' \preceq w$ et tels que (w', w) est une bonne paire. Alors, pour tout $t \in \mathfrak{t}$ fixé par ww'^{-1}, l'on a*

$$\widetilde{V}_{w'} \cap q^{-1}(t+\mathfrak{u}) \subset \widetilde{X}_w \,. \tag{1.4}$$

Ainsi la Conj. 1.1.7 pour (w', w) est vraie.

Avant d'esquisser la preuve, nous remarquons qu'il suffit de montrer (1.4) pour tout point $t \in \mathfrak{t}^{ww'^{-1}}$ « suffisamment générique » : voir Prop. 4.3.1 et Lem. 4.3.2. Sans perte de généralité, nous pouvons aussi supposer que k est algébriquement clos ; voir § 4.4. La preuve du Thm. 1.1.8 se déroule alors en trois étapes.

Étape 1 Puisque (w', w) est une bonne paire, la Prop. 1.1.3 fournit un sous-ensemble $J \subseteq I$ de racines simples de $\Phi(G, T)$ ainsi que $u^J, v^J \in W^J$, $w_J, w'_J \in W_J$, tels que $w = u^J w_J (v^J)^{-1}$, $w' = u^J w'_J (v^J)^{-1}$ et $w'_J \preceq w_J$ dans W_J. Soit L le sous-groupe de Levi du sous-groupe parabolique standard P_J de G ; alors $L = C_G\big(\big((u^J)^{-1} T^{ww'^{-1}} u^J\big)^{\circ}\big)$.

Il est démontré dans [BHS19, §2.4], au moyen notamment de la localisation de Beilinson-Bernstein, que (1.4) est valide dans le cas $t = 0$ pour tous $w' \preceq w$ (même pour des mauvaises paires). Comme $w'_J \preceq w_J$, on peut appliquer ceci au groupe réductif déployé connexe L muni de son tore maximal déployé T, de son sous-groupe de Borel $B_L \coloneqq B \cap L$ et du sous-groupe unipotent $U \coloneqq U \cap L$. Pour n'importe quel $t \in \mathfrak{t}_J = \mathrm{Ad}(u^J)^{-1} \mathfrak{t}^{ww'^{-1}}$, nous déduisons en translatant par t que $\widetilde{V}_{L,w'_J} \cap q^{-1}(t+\mathfrak{u}_L) \subset \widetilde{X}_{L,w_J}$.

Étape 2 Le résultat suivant montre la pertinence de $\widetilde{X}_L$ pour l'étude de $\widetilde{X}$.

Proposition 1.1.9 (Prop. 4.4.2). *Soient $u^J, v^J \in W^J$. Il existe un morphisme $\iota_{t,u^J,v^J} \colon L \times^{L\cap B} \mathfrak{l} \cap \mathfrak{b} \longrightarrow G \times^B \mathfrak{b}$ défini par $\iota_{t,u^J,v^J}(l, \psi_L)) = \big(u^J l (v^J)^{-1}, \mathrm{Ad}(v^J)\psi_L\big)$. Pour n'importe quel $t \in \mathrm{Ad}(u^J)^{-1} \mathfrak{t}^{ww'^{-1}}$ suffisamment générique et n'importe quel $w_J \in W_J$, ce morphisme ι_{t,u^J,v^J} induit un isomorphisme de variétés algébriques*

$$\iota_{t,u^J,w_J,v^J} \colon \widetilde{V}_{L,w_J} \cap q_L^{-1}(t+\mathfrak{u}_L) \xrightarrow{\sim} \widetilde{V}_{u^J w_J (v^J)^{-1}} \cap q^{-1}\left(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\right) \,.$$

La preuve utilise quelques lemmes techniques qui constituent l'essentiel du § 4.3.

Étape 3 Nous démontrons le lemme suivant grâce à des résultats techniques sur les décompositions de Jordan dans $\mathfrak{g}$ que nous passons en revue dans l'appendice 4.A du Chapitre 4.

Lemme 1.1.10 (Lem. 4.4.3). *Soit $t \in \mathrm{Ad}(u^J)^{-1} \mathfrak{t}^{ww'^{-1}}$ suffisamment générique, soit $u^J \in W^J$. Alors le morphisme suivant est surjectif :*

$$f \colon \begin{array}{ccc} U \times \big(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\big) & \twoheadrightarrow & \mathrm{Ad}(u^J)t + \mathfrak{u} \\ (u, x) & \longmapsto & \mathrm{Ad}(u)x \,. \end{array}$$

Ce lemme permet de passer de $q^{-1}\left(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\right)$ à $q^{-1}\left(\mathrm{Ad}(u^J)t+\mathfrak{u}\right)$. Plus précisément, le morphisme surjectif f du Lem. 1.1.10 induit des morphismes de variétés algébriques $U \times \left(\widetilde{V}_{L,w_J} \cap q_L^{-1}(t+\mathfrak{u}_L)\right) \twoheadrightarrow \widetilde{V}_w \cap q^{-1}\left(\mathrm{Ad}(u^J)t+\mathfrak{u}\right)$ et $U \times \widetilde{X}_{L,w_J} \longrightarrow \widetilde{X}_w$, qui se retrouvent dans un diagramme commutatif

$$\begin{tikzcd}
U \times \left(\widetilde{V}_{L,w'_J} \cap q_L^{-1}(t+\mathfrak{u}_L)\right) \arrow[r, "\subseteq"] \arrow[dd, two heads] & U \times \widetilde{X}_{L,w_J} \arrow[d] \\
 & \widetilde{X}_w \arrow[d, hook, "\subseteq"] \\
\widetilde{V}_{w'} \cap q^{-1}(\mathrm{Ad}(u^J)t+\mathfrak{u}) \arrow[r, "\subseteq"] & \widetilde{X}
\end{tikzcd}$$

où la flèche horizontale du haut est donnée par l'étape 1. La surjectivité de la flèche verticale de gauche prouve alors que l'image de la flèche horizontale du bas est contenue dans $\widetilde{X}_w$, ce qui est l'assertion de (1.4).

1.1.5 Contre-exemples à la Conjecture 1.1.7

Nous concluons ce travail dans le Chapitre 5, où nous trouvons des équations explicites pour les variétés projectives $\widetilde{X}$ et $\widetilde{V}_w$ du § 1.1.4, dans le cas $G = \mathrm{GL}_n$. Bien que nous n'ayons pas de description explicite similaire des variétés $\widetilde{X}_w$, cela nous suffit pour trouver une famille de contre-exemples à la Conj. 1.1.7.

Dans ce but, nous modifions légèrement notre point de vue et nous utilisons une définition équivalente de $\widetilde{X}$: nous la considérons alors comme la sous-variété $\{(gB,\psi) \in G/B \times \mathfrak{b} \mid \mathrm{Ad}(g^{-1})\psi \in \mathfrak{b}\}$ de $G/B \times \mathfrak{b}$, voir (5.5). Cela rend la tâche plus aisée que pour une variété quotient. En effet, $\mathfrak{b}$ est un espace affine et la variété de drapeaux G/B est un object bien connu. Spécifiquement, G/B est une sous-variété d'un produit de Grassmanniennes définie par des relations d'incidence ; de même, les Grassmanniennes peuvent être définies comme des sous-variétés de l'espace projectif via le plongement de Plücker et les équations les définissant sont bien connues. Par conséquent, $G/B \times \mathfrak{b}$ est bien comprise en tant que sous-variété de $\prod_{d=1}^{n-1} \mathbb{P}(\bigwedge^d k^n) \times \mathbb{A}^{n(n+1)/2}$. Les détails sont fournis au § 5.1.

Nous établissons ensuite un ensemble d'équations définissant $\widetilde{X}$ dans $G/B \times \mathfrak{b}$; c'est le Lem. 5.2.3. L'étape suivante est d'utiliser les (in)équations définissant chaque cellule de Schubert BwB/B dans G/B pour trouver un ensemble $\{F_i = 0, G_j \neq 0\}_{i\in I, j\in J}$ d'(in)équations définissant $\widetilde{V}_w$ dans $\prod_{d=1}^{n-1} \mathbb{P}(\bigwedge^d k^n) \times \mathbb{A}^{n(n+1)/2}$. Il n'est pas très ardu d'y arriver par force brute ; mais l'objectif est d'avoir suffisamment d'équations pour que, on l'espère, l'ensemble d'équations $\{F_i = 0\}_{i\in I}$ décrive l'adhérence de Zariski $\widetilde{X}_w$ de $\widetilde{V}_w$ dans $\prod_{d=1}^{n-1} \mathbb{P}(\bigwedge^d k^n) \times \mathbb{A}^{n(n+1)/2}$. Nous atteignons dans Prop. 5.2.4 un résultat assez satisfaisant, dans le sens où la Conj. 1.1.7 est équivalente à ce que $\widetilde{X}_w$ soit le sous-ensemble fermé $\{F_i = 0\}_{i\in I}$; voir la Prop. 5.3.1.

D'un autre côté, grâce à un travail technique de simplifications au § 5.3, nous améliorons la Prop. 5.2.4 en trouvant des équations additionnelles et inattendues satisfaites sur $\widetilde{V}_w$. En conséquence, nous obtenons, pour une certaine famille de couples $(w, w') \in (\mathcal{S}_n)^2$, une équation remarquablement simple satisfaite sur $\widetilde{X}_w \cap \widetilde{V}_{w'}$ mais pas sur $\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})$, d'où le résultat suivant :

Théorème 1.1.11 (Thm. 5.3.7, Rmk. 5.3.8, Rmk. 5.3.9)**.** *Pour tout $n \in \mathbb{Z}_{\geq 4}$, il existe une mauvaise paire (w, w') dans $\mathcal{S}_n$ telle que*

$$\dim\left(\widetilde{X}_w \cap \widetilde{V}_{w'}\right) < \dim\left(\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})\right) .$$

En particulier, la Conj. 1.1.7 est fausse.

À notre connaissance, la question de savoir si la Conj. 1.1.7 est fausse pour toutes les mauvaises paires ou non reste ouverte (nous savons au moins que c'est le cas pour GL_4 et GL_5). Une autre piste d'approfondissements est la recherche d'estimations précises pour le déficit de dimension $\dim\left(\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})\right) - \dim\left(\widetilde{X}_w \cap \widetilde{V}_{w'}\right)$. En dernier lieu, nos résultats suggèrent que la formule (1.1) pour la dimension de l'espace tangent à $X_{\mathrm{tri}}^{\square}(\bar{r})$ en un point compagnon est possiblement fausse lorsque (w_{sat}, w) est une mauvaise paire, ou tout du moins lorsque (w_{sat}, w) satisfait les hypothèses du Thm. 5.3.7. Si tel est bien le cas, il serait intéressant de trouver un contre-exemple à la formule (1.1).

1.2 English version

Let p be a prime number. Inspired by Kisin's study [Kis03] of p-adic overconvergent eigenforms, and building on the notion of trianguline representation from Colmez [Col08], Hellmann [Hel12] followed by Hellmann-Schraen [HS16] introduced a rigid analytic variety parametrising triangulations of framed deformations of a fixed residual representation $\bar{r}$, called the trianguline variety $X_{\mathrm{tri}}^{\square}(\bar{r})$. Hellmann used this construction to advance the study of eigenvarieties by Bellaïche-Chenevier [BC09]. Breuil-Hellmann-Schraen focused on the geometry of the trianguline variety in a series of papers [BHS17b], [BHS17a], [BHS19], especially the local geometry at crystalline generic points. They derived a number of consequences, among which results concerning classicality of overconvergent forms, existence of certain eigenforms called companion eigenforms, and the existence of companion constituents (we do not focus on automorphic representations in this work, but this features prominently in the p-adic local Langlands program; see for instance [BE10], [Ber14], [CD14] and [CDP14]).

In particular, Breuil-Hellmann-Schraen prove an upper bound for the dimension of the tangent space of $X_{\mathrm{tri}}^{\square}(\bar{r})$ at a crystalline generic point, and conjecture that it is exact. They also show, for strictly dominant points (*i.e.* "classical" points), that this conjecture is implied by classical modularity lifting conjectures. In particular, they highlight the importance of a couple (w_{sat}, w) of permutations attached to a crystalline generic point. The aim of our work is to investigate their conjectural formula for the dimension of the tangent space of the trianguline variety at such a crystalline generic (not necessarily classical) point.

We discover along the way an important property of (w_{sat}, w). We call it the "good pair property", and in Chapter 2 we develop a combinatorial theory of good pairs in the context of Weyl groups of root systems. We establish different criteria for this property. Most pairs are good, and we show that the set of good pairs is related to pattern avoidance. We couldn't find references to the notion of good pairs in the literature, but we still wonder wether it is naturally encountered in other contexts than that of the trianguline variety.

Coming back to p-adic representation theory, our main result is a proof of the formula conjectured by Breuil-Hellmann-Schraen for the dimension of the tangent space of $X_{\mathrm{tri}}^{\square}(\bar{r})$ at any crystalline generic point such that (w_{sat}, w) is a good pair.

In Chapter 3, we first generalise the ideas of [BHS17a] to show that this is a consequence of a version of the modularity lifting conjectures, not only for strictly dominant points but also for any companion point (whose attached pair of permutations is good).

In Chapter 4, we provide an unconditional different proof of our theorem using the local model of [BHS19]. This local model describes the local geometry of $X_{\mathrm{tri}}^{\square}(\bar{r})$ at crystalline generic points in terms of the local geometry of a certain scheme coming from geometric representation theory. In *loc. cit.*, it is proved that the formula for the dimension of the tangent space of $X_{\mathrm{tri}}^{\square}(\bar{r})$ at those points is a consequence of a conjecture concerning this geometric scheme stated in terms of pairs of Weyl group elements. We proof this conjecture in the case of good pairs using purely geometric methods.

Finally, we find in Chapter 5 a family counter-examples to this geometric conjecture (when the pair of Weyl group elements is bad). This suggests that the local geometry of $X_{\mathrm{tri}}^{\square}(\bar{r})$ at a "bad pair" companion point may be richer than previously expected.

1.2.1 The trianguline variety

Let K, L be finite extensions of $\mathbb{Q}_p$ such that $\Sigma := \mathrm{Hom}_{\mathbb{Q}_p}(K, L)$ has $[K : \mathbb{Q}_p]$ elements. We write $\mathcal{G}_K$ for the absolute Galois group of K and k_L for the residual field of L. We fix an integer $n \in \mathbb{Z}_{>0}$ and a continuous representation $\bar{r}\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k_L)$.

Our main object of study is the trianguline variety $X_{\mathrm{tri}}^{\square}(\bar{r})$; it is a reduced rigid analytic space over L defined as follows. Let $R_{\bar{r}}^{\square}$ be the framed deformation ring of $\bar{r}$ constructed by Mazur [Maz89, §1.2] and Kisin [Kis09]. Taking its formal spectrum $\mathrm{Spf}(R_{\bar{r}}^{\square})$ and then applying the rigidification functor of Berthelot [Ber96, §0.2], gives a rigid analytic space $\mathfrak{X}_{\bar{r}}^{\square} := \mathrm{Spf}(R_{\bar{r}}^{\square})^{\mathrm{rig}}$ over L. Let $\mathcal{T}_L := \widehat{K^{\times}} \times_{\mathbb{Q}_p} L$ be the rigid analytic space over L parametrising the continuous characters of $K^{\times}$. For any $\mathbf{k} = (k_\tau) \in \mathbb{Z}^{\Sigma}$, we write $z^{\mathbf{k}} \in \mathcal{T}_L(L)$ for the character $z \longmapsto \prod_{\tau\in\Sigma} \tau(z)^{k_\tau}$, and $|z|_K \in \mathcal{T}_L(L)$ for the p-adic absolute value normalised by $|p|_K = p^{-[K:\mathbb{Q}_p]}$. A character

$\underline{\delta} = (\delta_i) \in \mathcal{T}_L^n$ is called regular if, for all $1 \leq i \neq j \leq n$, the character $\delta_i\delta_j^{-1}$ is not of the form $z^{-\mathbf{k}}$ or $N_{K/\mathbb{Q}_p}(z)z^{\mathbf{k}}|z|_K$ for some $\mathbf{k} \in \mathbb{Z}_{\geq 0}^{\Sigma}$, where $N_{K/\mathbb{Q}_p}$ is the norm map. We write $\mathcal{T}_{\mathrm{reg}}^n \subset \mathcal{T}_L^n$ for the subset of regular characters, which is Zariski-open. Let $U_{\mathrm{tri}}^{\square}(\bar{r})$ be the set of points $x = (r, \underline{\delta}) \in \mathfrak{X}_{\bar{r}}^{\square} \times \mathcal{T}_{\mathrm{reg}}^n$ such that the representation $r \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k(x))$ is trianguline in the sense of Colmez with parameter $\underline{\delta} \colon (K^\times)^n \longrightarrow k(x)^\times$ (see *e.g.* [BHS19, Dfn. 3.3.8]). Then the trianguline variety $X_{\mathrm{tri}}^{\square}(\bar{r})$ is defined as the Zariski-closure of $U_{\mathrm{tri}}^{\square}(\bar{r})$ in $\mathfrak{X}_{\bar{r}}^{\square} \times \mathcal{T}_L^n$. More details about this definition are given in §3.3.

Let $x = (r, \underline{\delta}) \in X_{\mathrm{tri}}^{\square}(\bar{r})$ be a crystalline point, *i.e.* a point such that r is a crystalline representation. We assume that r is generic, which means that (i) for each $\tau \in \Sigma$, the τ-Hodge-Tate weights of r are all distinct, and (ii) $\varphi/\varphi' \notin \{1, p^{[K_0:\mathbb{Q}_p]}\}$ for any two eigenvalues φ, φ' of the linearised Frobenius Φ on $\mathbf{D}_{\mathrm{crys}}(r)$ (with multiplicities). For each $\tau \in \Sigma$, we write the τ-Hodge-Tate weights of r in increasing order $h_{\tau,1} < \ldots < h_{\tau,n}$.

Then $\underline{\delta}$ is determined by the data of (i) an ordering $\underline{\varphi} = (\varphi_1, \ldots, \varphi_n)$ of the eigenvalues of Φ and (ii) for each $\tau \in \Sigma$, a permutation w_τ in the symmetric group $\mathcal{S}_n$. More precisely, one can write $\underline{\delta} = z^{w(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})$, where the character $\mathrm{unr}(\underline{\varphi}) = (\mathrm{unr}(\varphi_i))_i$ of $(K^\times)^n$ is defined by $\mathrm{unr}(\varphi_i)(p) = \varphi_i$ and $\mathrm{unr}(\varphi_i)|_{\mathcal{O}_K^\times} = 1$, and where $w := (w_\tau) \in (\mathcal{S}_n)^\Sigma$ and $w(\mathbf{h}) := \left(h_{\tau, w_\tau^{-1}(i)}\right)_{\tau,i} \in (\mathbb{Z}^n)^\Sigma$. Furthermore, there is a unique $w_{\mathrm{sat}} \in (\mathcal{S}_n)^\Sigma$ such that the point $x_{\mathrm{sat}} := \left(r, z^{w_{\mathrm{sat}}(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})\right) \in \mathfrak{X}_{\bar{r}}^{\square} \times \mathcal{T}_L^n$ lies in $U_{\mathrm{tri}}^{\square}(\bar{r})$. The pair $(w_{\mathrm{sat}}, w) \in \left((\mathcal{S}_n)^\Sigma\right)^2$ attached to x has the beautiful property that x exists as a point of $X_{\mathrm{tri}}^{\square}(\bar{r})$ if and only if $w_{\mathrm{sat}} \preceq w$ for the product Bruhat order on $(\mathcal{S}_n)^\Sigma$ [BHS19, Thm. 1.8]. (Note that w_{sat} does not depend solely on r; indeed a choice of ordering of the eigenvalues of Φ has been made.) The point x is called strictly dominant (or classical) in the case where $w = w_0$ is the unique maximal element for the Bruhat order; in the other cases where $w_{\mathrm{sat}} \preceq w \prec w_0$, we call x a companion point.

The main object of this thesis is to understand the local geometry of the reduced rigid analytic space $X_{\mathrm{tri}}^{\square}(\bar{r})$ at such a companion point x. This follows in the step of Breuil-Hellmann-Schraen, who have made a number of advances in the series of papers [BHS17b], [BHS17a], [BHS19]. First, the geometry of $U_{\mathrm{tri}}^{\square}(\bar{r})$ is very well understood: $U_{\mathrm{tri}}^{\square}(\bar{r})$ is a Zariski-dense and Zariski-open subset of $X_{\mathrm{tri}}^{\square}(\bar{r})$ which is smooth of dimension $n^2 + [K : \mathbb{Q}_p]\frac{n(n+1)}{2}$; see [BHS17b, Thm. 2.6]. In particular, $X_{\mathrm{tri}}^{\square}(\bar{r})$ is also equidimensional of dimension $n^2 + [K : \mathbb{Q}_p]\frac{n(n+1)}{2}$. Secondly, $X_{\mathrm{tri}}^{\square}(\bar{r})$ is normal and Cohen-Macaulay at any point x as above; see [BHS19, Thm. 1.4]. In particular it is locally irreducible at x. Thus the main question that remains is whether x is a smooth or singular point, thus it is interesting to compute the dimension of the tangent space of $X_{\mathrm{tri}}^{\square}(\bar{r})$ at x. An explicit formula for an upper bound of $\dim_{k(x)} T_{X_{\mathrm{tri}}^{\square}(\bar{r}),x}$ is given in [BHS19, Prop. 4.1.5(ii)]: it is the right-hand side of (1.5) below. This upper bound already proves smoothness in the case where $w = w_0$ and w_{sat} is a product of distinct simple reflections; see [BHS19, Rmk. 4.1.6(ii)]. In general, the authors implicitly conjecture in [BHS19, Rmk. 4.1.6(iii)] that the upper bound is always an equality. In the strictly dominant case $w = w_0$

and without condition on w_{sat}, it is known from [BHS17a, Prop. 5.17] that this is implied by the classical modularity lifting conjectures (see [BHS17b, Conj. 1.3, Thm. 1.4]), together with a globalisation conjecture that has recently been proved by Emerton-Gee (see [EG19, Thm. 1.2.3]).

Our main result is that the upper bound for $\dim_{k(x)} T_{X^\square_{\mathrm{tri}}(\bar{r}),x}$ of [BHS19] is indeed an equality for a wide class of companion points, which comprises the strictly dominant case $w = w_0$. This class is defined by the following condition on the pair (w_{sat}, w). Such a pair belongs in the class if for each $\tau \in \Sigma$, there are reflections $s_1, \dots, s_r \in \mathcal{S}_n$ whose support are all included in the support of some cycle in the cycle decomposition of $w_{\mathrm{sat},\tau} w_\tau^{-1}$, which give rise to an ascending sequence $w_{\mathrm{sat},\tau} \prec w_{\mathrm{sat},\tau} s_1 \prec \dots \prec w_{\mathrm{sat},\tau} s_1 \dots s_r = w_\tau$ for the Bruhat order. We call (w_{sat}, w) a *good pair* when this condition is satisfied. The result we prove can then be stated as follows:

Theorem 1.2.1 (Thm. 3.5.4, Thm. 4.2.4)**.** *If (w_{sat}, w) is a good pair, then*

$$\dim_{k(x)} T_{X^\square_{\mathrm{tri}}(\bar{r}),x} = \dim X^\square_{\mathrm{tri}}(\bar{r}) - d_{ww_{\mathrm{sat}}^{-1}} + \sum_{\tau \in \Sigma} \dim_{k(x)} T_{\overline{(Bw_\tau B/B)}^{\mathrm{rig}}_{k(x)}, x_\tau} - \lg(w_{\mathrm{sat}}) \quad (1.5)$$

(we refer to §1.2.2 for the definition of $d_{ww_{\mathrm{sat}}^{-1}}$ and to the statement of Thm. 3.5.4 for the definition of x_τ).

This therefore settles the question of smoothness for the cases considered in [BHS19]: when x is strictly dominant, $X^\square_{\mathrm{tri}}(\bar{r})$ is smooth at x if and only if w_{sat} is a product of distinct simple reflections (recall that the "if" part was already known). For a companion point x such that (w_{sat}, w) is a good pair, with an estimation of $d_{ww_{\mathrm{sat}}^{-1}}$ (see Prop. 2.3.4), we get from Theorem 1.2.1 that smoothness of $\overline{(Bw_\tau B/B)}$ at x_τ is a necessary condition for smoothness of $X^\square_{\mathrm{tri}}(\bar{r})$ at x: see Cor. 4.2.5.

For a companion point x such that (w_{sat}, w) is a bad pair, we show similarly that if (1.5) holds, then x is a singular point of $X^\square_{\mathrm{tri}}(\bar{r})$. However we have reasons to believe that (1.5) does not hold in general for bad pairs (though we do not have a counter-example supporting this statement); see §1.2.5 and Chapter 5.

The proof of Theorem 1.2.1, as well as the analysis of cases where (w_{sat}, w) is not a good pair, involve the so-called "local model" discovered by Breuil-Hellmann-Schraen in [BHS19]. We also give another conditional proof (which assumes the modularity lifting conjectures but does not use the local model) using the same starting point as [BHS17a, Prop. 5.17]. Strikingly, the good pair condition appears as a fundamental component of both proofs, albeit in quite a different formulation for the two proofs. (This condition did not appear in the work of Breuil-Hellmann-Schraen as it is automatically satisfied when $w = w_0$.) Thus, before outlining the strategies of each proof in more detail, we briefly explore how this condition may arise in its natural context, namely Weyl groups of root systems.

1.2.2 Good pairs in a Weyl/Coxeter group

We first want to rephrase the definition of a good pair given in §1.2.1. Consider the connected split reductive group $G = \prod_{\tau\in\Sigma} \mathrm{GL}_{n,L}$ over L, with the product of diagonal matrix groups as the given choice of split maximal torus T. Then its root system $\Phi := \Phi(G,T)$ is isomorphic to $\coprod_{\tau\in\Sigma} \Phi_\tau$ where each Φ_τ is a copy of the root system of type A_{n-1}, and its Weyl group $W := W(G,T)$ is isomorphic to $(\mathcal{S}_n)^\Sigma$. Thus it feels reasonable to view w_{sat} and w as elements of W. In this context, consider the subtorus $\big(T^{w_{\mathrm{sat}}w^{-1}}\big)^\circ$ fixed by $w_{\mathrm{sat}}w^{-1}$ under the natural action of W on T. Then the connected centraliser $C_G\big(\big(T^{w_{\mathrm{sat}}w^{-1}}\big)^\circ\big)$ is a connected split reductive group over L with T as a split maximal torus; we write $\Phi_{w_{\mathrm{sat}}w^{-1}} := \Phi\big(C_G\big(\big(T^{w_{\mathrm{sat}}w^{-1}}\big)^\circ\big), T\big)$ for its associated root system. Further, $\Phi_{w_{\mathrm{sat}}w^{-1}} \subseteq \coprod_{\tau\in\Sigma}\Phi_\tau$ is the set of roots $\alpha \in \Phi_\tau$ for some $\tau\in\Sigma$ such that the reflection s_α is supported in the cycle decomposition of $w_{\mathrm{sat},\tau}w_\tau^{-1}$. Therefore, using the product Bruhat order $\prec$ on $(\mathcal{S}_n)^\Sigma$ (which comes from a choice of Borel subgroup of G containing T), the good pair condition can be rephrased as follows: (w_{sat}, w) is good if and only if there are roots $\alpha_1,\dots,\alpha_r \in \Phi_{w_{\mathrm{sat}}w^{-1}}$ which give rise to an ascending sequence $w_{\mathrm{sat}} \prec w_{\mathrm{sat}}s_{\alpha_1} \prec \ldots \prec w_{\mathrm{sat}}s_{\alpha_1}\ldots s_{\alpha_r} = w$. This notion naturally generalises to any connected split reductive group.

In fact, the correct setting is that of abstract root systems: as any abstract root system Φ arises from a connected split reductive group G, it is reasonable to expect that Φ_w, for $w\in W := W(\Phi)$ can be defined simply in terms of Φ, without involving G. This is indeed the case, and our "definitive" definition of Φ_w is the following: let $\Gamma_w := \sum_{\alpha\in\Phi}\mathbb{Z}(w(\alpha)-\alpha)$, let $E_w := \Gamma_w\otimes_{\mathbb{Z}}\mathbb{Q}$ and finally let $\Phi_w := \Phi\cap E_w$. That this definition of Φ_w is equivalent to the previous one for split reductive groups is Prop. 2.2.2. Fixing a basis I of Φ then gives a length function lg and a Bruhat order $\prec$ on W, and a good pair (w_1,w_2) is defined by a chain condition using $\Phi_{w_1w_2^{-1}}$ as in the previous paragraph. We also call (w_1,w_2) a bad pair if it is not good and $w_1\preceq w_2$.

In §2.1, we establish a few combinatorial results about Φ_w. The most important one is:

Proposition 1.2.2 (Cor. 2.1.5)**.** *The root subsystem $\Phi_w\subseteq\Phi$ is the set of $\alpha\in\Phi$ such that s_α appears in a decomposition of w on the alphabet $R := \{s_\beta \mid \beta\in\Phi\}$ which is of minimal length, and this minimal length is $d_w := \mathrm{rk}_{\mathbb{Z}}\,\Gamma_w$.*

Note that this allows the notion of good pair to be defined in any Coxeter group; however we have not found it to be satisfying, as we do not know if the results we obtain in Chapter 2 remain true in this general context.

The proof of Prop. 1.2.2 rests on an inductive argument on d_w. Using a variant of this inductive argument, we prove that $d_w\le \lg(w)$ with equality if and only if w is a product of distinct simple reflections (Cor. 2.1.7). More generally, for any $w_1,w_2\in W$, we have $d_{w_2w_1^{-1}}\le \lg(w_2)-\lg(w_1)$ with equality only if (w_1,w_2) is a good pair (Prop. 2.3.4). These are useful for understanding the right-hand side of (1.5): for instance the bound $d_w\le\lg(w)$ in the particular case of a root system of type A is used by Breuil-Hellmann-Schraen to study strictly dominant points on $X_{\mathrm{tri}}^{\square}(\bar{r})$.

While the developments of §2.1 provide a good theoretical understanding of the subsystems Φ_w, they are not readily useful for practical applications. We address this issue in §2.3, where we prove a criterion for good pairs. It is motivated by the fact that, for any $w \in W$, the subsystem Φ_w is conjugated to a standard subsystem, *i.e.* a root subsystem $\Phi_J \subseteq \Phi$ generated by some subset $J \subseteq I$ of simple roots. More precisely, for $J \subseteq I$, let W_J be the subgroup of W generated by $\{s_\alpha \mid \alpha \in J\}$, and let W^J be the set of representatives of minimal length for $W_J \backslash W$. Then one can write $\Phi_w = u^J(\Phi_J)$ for some $J \subseteq I$ and $u^J \in W^J$. The criterion is:

Proposition 1.2.3 (Prop. 2.3.7). *Let $(w_1, w_2) \in W^2$. Write $\Phi_{w_2 w_1^{-1}} = u^J(\Phi_J)$ for some $J \subseteq I$ and $u^J \in W^J$. Then there exist $v^J \in W^J$ and $w_{J,1}, w_{J,2} \in W_J$ such that*

$$w_1 = u^J w_{J,1} (v^J)^{-1}, \quad w_2 = u^J w_{J,2} (v^J)^{-1},$$

and the pair (w_1, w_2) is good if and only if $w_{J,1} \preceq w_{J,2}$.

We conclude in §2.6 with a lovely application of this last criterion. It is stated in terms of pattern avoidance: the idea is that some classes of permutations w in $\bigcup_{n \in \mathbb{Z}_{>0}} \mathcal{S}_n$ can be characterised by the property that w does not contain any of a fixed set of patterns. Here we say that a permutation $w \in \mathcal{S}_n$ has a pattern $f \in \mathcal{S}_m$ when there exist $1 \leq i_1 < \ldots < i_m \leq n$ such that the m-uples $(w(i_1), \ldots, w(i_m))$ and $(f(1), \ldots, f(m))$ are in the same order; see Def. 2.6.1. This can in fact be stated in the general setting of an abstract root system, and many properties such as smoothness of Schubert varieties have been found to be instances of pattern avoidance; see for example [BP05, §2, §4]. We prove that, at least for root systems of type A, the property of "only occuring in good pairs" is also an instance of pattern avoidance; more precisely, we prove:

Theorem 1.2.4 (Thm. 2.6.3). *Let $w \in \mathcal{S}_n$.*

(1) *The pair (w', w) is good for any $w' \in \mathcal{S}_n$ if and only if w avoids the four patterns* [4231], [42513], [35142] *and* [351624].

(2) *The pair (w, w'') is good for any $w'' \in \mathcal{S}_n$ if and only if w avoids the four patterns* [1324], [24153], [31524] *and* [426153].

Linking these patterns with those deciding smoothness of Schubert varieties, we get Cor. 2.6.5 (see §4.1 for more details about the Schubert decomposition). It states that if $(w, w') \in (\mathcal{S}_n)^2$ is a bad pair, then the Schubert closed cell $\overline{BwB/B}$ in the flag variety G/B is singular (but not necessarily at the point $w'B \in \overline{BwB/B}$).

1.2.3 Arithmetic proof of Theorem 1.2.1

We outline here our proof, in Chapter 3, that the "modularity" conjecture [BHS17b, Conj. 1.3] implies Thm. 1.2.1. It is based on a generalisation of the ideas of [BHS17a] (who only study the strictly dominant case).

The rough idea is to find, for a companion point x as in §1.2.1, two subspaces $T_1, T_2 \subseteq T_{X^\square_{\mathrm{tri}}(\bar{r}),x}$, for which we can prove that $\dim(T_1 + T_2)$ is equal to the right-hand side of (1.5). This gives a lower bound for $\dim T_{X^\square_{\mathrm{tri}}(\bar{r}),x}$ by $\dim(T_1+T_2)$, which together with the upper bound of [BHS19, Prop. 4.1.5(ii)] yields the result.

Crystalline tangent directions The central notion involved in our construction of the first subspace T_1 is that of refinements of crystalline representations. We sketch the ideas here; we refer the reader to §3.1 for the notation and a detailed review of the parts of p-adic Hodge theory we need, and to §3.3 for the construction of the subspace T_1.

Let $r\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(L')$ be a crystalline representation, where L' is a finite extension of L. A refinement of r is an ordering $(\varphi_1, \dots, \varphi_n)$ of the eigenvalues of the linearised Frobenius Φ of r. When these eigenvalues are distinct, a refinement is equivalent to the datum of a Φ-stable complete flag $\mathcal{F}_\bullet$ of the $(K_0 \otimes_{\mathbb{Q}_p} L')$-module $\mathbf{D}_{\mathrm{crys}}(r)$, where K_0 is the maximal unramified extension of $\mathbb{Q}_p$ contained in K. By base change, a refinement gives a complete flag $K \otimes_{K_0} \mathcal{F}_\bullet$ on the module $\mathbf{D}_{\mathrm{dR}}(r)$ over $K \otimes_{\mathbb{Q}_p} L' \simeq \prod_{\tau\in\Sigma} L'$. Note that the latter is equivalent to the datum of a complete flag on each L'-vector space $\mathbf{D}_{\mathrm{dR}}(r) \otimes_{K,\tau} L'$ for $\tau \in \Sigma$. On the other hand, when the τ-Hodge-Tate weights of r are distinct for all $\tau \in \Sigma$, the Hodge filtration on $\mathbf{D}_{\mathrm{dR}}(r)$ induces another complete flag $\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{dR}}(r)$ of the $(K \otimes_{\mathbb{Q}_p} L')$-module $\mathbf{D}_{\mathrm{dR}}(r)$. Fixing a basis of $\mathbf{D}_{\mathrm{dR}}(r)$ compatible with the flag $K \otimes_{K_0} \mathcal{F}_\bullet$, we can then consider the flag $\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{dR}}(r)$ as an L'-point of the flag variety G/B, where G is the connected split reductive group $G \coloneqq \mathrm{Res}_{K/\mathbb{Q}_p}(\mathrm{GL}_{n,K}) \otimes_{\mathbb{Q}_p} L \simeq \prod_{\tau\in\Sigma} \mathrm{GL}_{n,L}$ and B is the Borel subgroup in G of upper triangular matrices. We call this point of G/B the relative position of $\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{dR}}(r)$ with respect to $K \otimes_{K_0} \mathcal{F}_\bullet$.

Now fix a tuple of integers $\mathbf{h} = (h_{\tau,1} < \ldots < h_{\tau,n})_\tau \in (\mathbb{Z}^n)^\Sigma$; recall that we have also fixed a continuous representation $\bar{r}\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k_L)$ (used to define $X^\square_{\mathrm{tri}}(\bar{r})$). Using the work of Kisin [Kis08], Breuil-Hellmann-Schraen [BHS17a, §2.2] construct a reduced rigid analytic space $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r}}$ over L which parametrises the data of a framed deformation r of $\bar{r}$ which is crystalline generic in the sense of §1.2.1 and with Hodge-Tate weights $\mathbf{h}$, together with a refinement $\underline{\varphi} = (\varphi_1, \dots, \varphi_n)$ of r. There is also a smooth map of L-rigid analytic spaces $h\colon \widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r}} \longrightarrow (G/B)^{\mathrm{rig}}$ which sends a point $(r, \underline{\varphi})$ of $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r}}$ to the relative position of the Hodge flag on $\mathbf{D}_{\mathrm{dR}}(r)$ with respect to the refinement $\underline{\varphi}$. The embedding of $W(G) = (\mathcal{S}_n)^\Sigma$ into G via matrices of permutation gives the Schubert decomposition of the flag variety $G/B = \coprod_{w\in(\mathcal{S}_n)^\Sigma} BwB/B$. For $w \in (\mathcal{S}_n)^\Sigma$, we then consider the closed rigid subspace $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w}$ of $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r}}$ defined by $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w} \coloneqq h^{-1}\big(\overline{BwB/B}^{\mathrm{rig}}\big)$. Then there is a closed immersion of rigid analytic spaces $\iota_{\mathbf{h},w}\colon \widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w} \longrightarrow X^\square_{\mathrm{tri}}(\bar{r})$ defined by $\iota_{\mathbf{h},w}(r, \underline{\varphi}) = \big(r, z^{w(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})\big)$.

Given a companion point $x = (r, \underline{\delta}) \in X^\square_{\mathrm{tri}}(\bar{r})$ as in §1.2.1, we take $\mathbf{h}$ to be the ordered tuple of Hodge-Tate weights of r. Also, let $\underline{\varphi} = (\varphi_i)_i$ be the refinement of r and let $w \in (\mathcal{S}_n)^\Sigma$ be the permutation such that $\underline{\delta} = z^{w(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})$. Then the point $(r, \underline{\varphi})$ of $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r}}$ is in fact in $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w}$, and $\iota_{\mathbf{h},w}$ maps $(r, \underline{\varphi})$ to x. Therefore $\iota_{\mathbf{h},w}$

induces an injection $T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w},(r,\underline{\varphi})} \hookrightarrow T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x}$, and we take T_1 to be the image of this injection.

Trianguline tangent directions The idea behind the construction of T_2, which we carry out in §3.5, is to use the rigid automorphism $\mathcal{T}_L^n \xrightarrow{\sim} \mathcal{T}_L^n$ defined by $\underline{\eta} \longmapsto z^{w(\mathbf{h})-w_{\mathrm{sat}}(\mathbf{h})}\underline{\eta}$, which induces the automorphism $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}} \colon \mathfrak{X}^{\square}_{\bar{r}} \times \mathcal{T}_L^n \xrightarrow{\sim} \mathfrak{X}^{\square}_{\bar{r}} \times \mathcal{T}_L^n$ of rigid analytic spaces over L. By definition, $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$ maps x_{sat} to x; in order to get tangent vectors to $X^{\square}_{\mathrm{tri}}(\bar{r})$ at x, we want $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$ to map a locally closed subvariety of $U^{\square}_{\mathrm{tri}}(\bar{r})$ containing x_{sat} into $X^{\square}_{\mathrm{tri}}(\bar{r})$. This is where Conj. 1.3 of [BHS17b] and the goodness of (w_{sat}, w) are required.

More precisely, we use the weight morphism $\omega\colon X^{\square}_{\mathrm{tri}}(\bar{r}) \longrightarrow \mathcal{W}_L^n$, where $\mathcal{W}_L$ is the rigid analytic space over L parametrising continuous characters of $\mathcal{O}_K^{\times}$, defined by $\omega(r,\underline{\delta}) = \underline{\delta}|_{(\mathcal{O}_K^{\times})^n}$. Note that a character $\eta\colon \mathcal{O}_K^{\times} \longrightarrow (L')^{\times}$, where L' is a finite extension of L, is a morphism of p-adic Lie groups; hence it induces a differential action $\mathrm{d}\eta \in \mathrm{Hom}_{\mathbb{Q}_p}(\mathfrak{t}, L') \simeq \prod_{\tau\in\Sigma} L'$, where $\mathfrak{t} \simeq K$ is the Lie algebra of $\mathcal{O}_K^{\times}$. The τ-weight $\mathrm{wt}_\tau(\eta) \in L'$ of η is then defined by $\mathrm{d}\eta = (\mathrm{wt}_\tau(\eta))$. We cut out a closed subspace of characters $\mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L} \subset \mathcal{W}_L^n$ with the equations $\mathrm{wt}_\tau(\eta_{w_{\mathrm{sat},\tau}(i)}\eta^{-1}_{w_\tau(i)}) = k_{\tau,i} - k_{\tau,w^{-1}_{\mathrm{sat},\tau}w_\tau(i)}$.

Proposition 1.2.5 (Prop. 3.5.3). *Assume Conj. 1.3 of [BHS17b]. If the pair (w_{sat}, w) in $(\mathcal{S}_n)^{\Sigma}$ is good, then there exists an open neighborhood $U_{x_{\mathrm{sat}}}$ of x_{sat} in $U^{\square}_{\mathrm{tri}}(\bar{r})$ such that $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$ induces a Zariski-closed embedding of reduced rigid analytic spaces over L*

$$\jmath_{w,w_{\mathrm{sat}},\mathbf{h}} \colon \overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}} \hookrightarrow X^{\square}_{\mathrm{tri}}(\bar{r})\,. \tag{1.6}$$

Therefore $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$ induces an injection $T_{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}, x_{\mathrm{sat}}} \hookrightarrow T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x}$, and we take T_2 to be the image of this injection. It is quite straightforward to compute $\dim T_2 = \dim X^{\square}_{\mathrm{tri}}(\bar{r}) - d_{ww^{-1}_{\mathrm{sat}}}$. The main difficulty rests in proving Prop. 1.2.5.

To achieve this, we need to globalise the setting. We give a quick outline of this global setting here, and refer to §3.4 for more details. Let F/F^+ be a quadratic totally imaginary extension, where F^+ is a totally real number field; writing S_p for the set of finite places of F^+ dividing p, we assume that all places $v \in S_p$ split in F and satisfy $F_v^+ \simeq K$. Let $\bar{\rho}\colon \mathcal{G}_{F^+} \longrightarrow \mathrm{GL}_n(k_L)$ be a continuous irreducible representation which is automorphic [EG14, Def. 5.3.1]; it comes from some automorphic form on $G(\mathbb{A}_{F^+})$ of a tame level $U^p = \prod_{v\notin S_p \text{ finite}} U_v \subset G(\mathbb{A}^{p\infty}_{F^+})$, where G is a unitary group in n variables over F^+. We also assume the "standard Taylor-Wiles hypotheses"; see [BHS17b, §2.4]. A recent result of Emerton-Gee [EG19, Thm. 1.2.3] asserts the existence of this setting, and that we can moreover ensure that for each $v \in S_p$, there exists a place $\tilde{v} \mid v$ of F such that $\bar{\rho}_{\tilde{v}} := \bar{\rho}|_{\mathcal{G}_{F_{\tilde{v}}}}$ is isomorphic to $\bar{r}$ via $F_{\tilde{v}} \simeq F_v^+ \simeq K$.

In this setting, Breuil-Hellmann-Schraen constructed in [BHS17b, §3], using the patching construction of [CEG+16], the so-called patched eigenvariety $X_p(\bar{\rho})$. It is a reduced L-rigid analytic subspace of $\mathfrak{X}_{\bar{\rho}^p} \times \prod_{v\in S_p}(\mathfrak{X}_{\bar{\rho}_v} \times \mathcal{T}_L^n) \times \mathbb{U}^g$, where $\mathfrak{X}_{\bar{\rho}^p} := \prod_{v\notin S_p} \mathfrak{X}_{\bar{\rho}_v}$, $\mathbb{U}$ is the rigid unit open disk over L and $g \in \mathbb{Z} > 0$ is an integer.

There is a rigid automorphism ι of $\prod_{v\in S_p}\mathcal{T}_L^n$ which induces a closed embedding $\iota^{-1}\colon X_p(\bar\rho)\longhookrightarrow \mathfrak{X}_{\bar\rho^p}\times\prod_{v\in S_p}X_{\mathrm{tri}}^{\square}(\bar\rho_{\tilde v})\times\mathbb{U}^g$.

Conj. 3.4.6 gives a conjectural description of the image of ι^{-1}. A consequence is the existence of a neighbourhood $U_{x_{\mathrm{sat}}}$ of x_{sat} in $U_{\mathrm{tri}}^{\square}(\bar r)$ such that $X_p(\bar\rho)\longhookrightarrow \mathfrak{X}_{\bar\rho^p}\times\prod_{v\in S_p}U_{x_{\mathrm{sat}}}\times\mathbb{U}^g$ is in the image of ι^{-1}; here we view $U_{x_{\mathrm{sat}}}$ as a subspace of each $X_{\mathrm{tri}}^{\square}(\bar\rho_{\tilde v})$, $v\in S_p$, via the isomorphism $\bar\rho_{\tilde v}\simeq\bar r$. Then (1.6) can be checked by a statement analogous to (1.6) on $X_p(\bar\rho)$, which is a consequence of: (i) Thm. 5.5 of [BHS17a], which reduces the problem to checking that, for any $\underline{\delta}'\in\mathcal{T}_L^n\times_{\mathcal{W}_L^n}\mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}$, the character $\iota\left((\underline{\delta}')_{v\in S_p}\right)$ is strongly linked to $\iota\left(\left(z^{w(\mathbf{h})-w_{\mathrm{sat}}(\mathbf{h})}\underline{\delta}'\right)_{v\in S_p}\right)$ in the sense of [Hum08, §5.1]; and (ii) a direct computation that show that this strong linkage is a consequence of the chain condition of good pairs between w_{sat} and w. Note that this is where the hypothesis of a good pair appears.

Dimension of T_1+T_2 We compute $\dim(T_1+T_2)$ using the commutative diagram of rigid analytic spaces over L

$$\begin{tikzcd}
\widetilde{W}_{x_{\mathrm{sat}}} \arrow[r, hook, "\iota_{\mathbf{h},w_{\mathrm{sat}}}"] \arrow[d, hook, "\subseteq"] & \overline{U_{x_{\mathrm{sat}}}\times_{\mathcal{W}_L^n}\mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}} \arrow[d, hook, "\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}"] \\
\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar r,w} \arrow[r, hook, "\iota_{\mathbf{h},w}"] & X_{\mathrm{tri}}^{\square}(\bar r)\,,
\end{tikzcd}$$

where $\widetilde{W}_{x_{\mathrm{sat}}}:=\iota^{-1}_{\mathbf{h},w_{\mathrm{sat}}}\left(\overline{U_{x_{\mathrm{sat}}}\times_{\mathcal{W}_L^n}\mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}}\right)\subseteq\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar r,w_{\mathrm{sat}}}$, which induces a commutative diagram of tangent spaces

$$\begin{tikzcd}
T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar r,w_{\mathrm{sat}}},(r,\underline{\varphi})} \arrow[r, hook] \arrow[d, hook] & T_{\overline{U_{x_{\mathrm{sat}}}\times_{\mathcal{W}_L^n}\mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}},x_{\mathrm{sat}}} \arrow[d, hook] \\
T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar r,w},(r,\underline{\varphi})} \arrow[r, hook] & T_{X_{\mathrm{tri}}^{\square}(\bar r),x}\,.
\end{tikzcd} \tag{1.7}$$

Unraveling the definitions, we find that (1.7) is a cartesian square as a consequence of the following proposition, applied to the algebra $A=k(x)[\varepsilon]/(\varepsilon^2)$ of infinitesimal numbers over the local residue field of x:

Proposition 1.2.6 (Prop. 3.2.13)**.** *Let A be a local artinian L-algebra with residue field L. Let $\mathcal{G}_K\longrightarrow\mathrm{GL}(V)$ be a regular crystalline representation of rank $n\in\mathbb{N}$ over A, with Hodge-Tate weights $\mathbf{h}\in(\mathbb{Z}^n)^\Sigma$. Assume that there exist $\mathbf{k}\in\mathbb{Z}^\Sigma$ and $\underline{\varphi}\in(A^\times)^n$, with the φ_i having pairwise distinct reductions in L, such that V is trianguline of parameter $z^{\mathbf{k}}\,\mathrm{unr}(\underline{\varphi})$.*

Then there exists $w\in(\mathcal{S}_n)^\Sigma$ such that $\mathbf{k}=w(\mathbf{h})$, and the linearised Frobenius Φ on $\mathbf{D}_{\mathrm{crys}}(V)$ admits $\underline{\varphi}$ as a set of eigenvalues. Furthermore, writing $\mathcal{F}_\bullet$ for the flag on $\mathbf{D}_{\mathrm{crys}}(V)$ induced by the refinement $\underline{\varphi}$, the position $x\in\prod_{\tau\in\Sigma}(\mathrm{GL}_n/B)(A)$ of $\mathrm{Fil}_\bullet\,\mathbf{D}_{\mathrm{dR}}(V)$ relative to $K\otimes_{K_0}\mathcal{F}_\bullet$ lies in the cell $\prod_{\tau\in\Sigma}(Bw_\tau B/B)(A)$.

We point out that Prop. 1.2.6 generalises [BC09, Prop. 2.4.1], which dealt with the case where $K = \mathbb{Q}_p$ and $A = L$. Our proof uses (i) Berger's dictionary between cristalline (φ, Γ_K)-modules over the Robba ring and filtered φ-modules ([Ber08, Thm. A]; see also [Ber02, Thm. 3.6]), and (ii) the description of rank 1 (φ, Γ_K)-modules under this dictionary (see *e.g.* [KPX14, Ex. 6.2.6(3)]). The necessary parts from the theory of (φ, Γ_K)-modules are reviewed in §3.2.

From there, we get $\dim(T_1 + T_2)$ without difficulty, using the fact that (1.7) and the following diagram

$$\begin{array}{ccc} \widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}} & \xrightarrow{h} & \overline{(Bw_{\mathrm{sat}}B/B)}^{\mathrm{rig}}_L \\ \downarrow\subseteq & & \downarrow\subseteq \\ \widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w} & \xrightarrow{h} & \overline{(BwB/B)}^{\mathrm{rig}}_L \end{array}$$

are cartesian squares, as well as the smoothness of h.

1.2.4 Geometric proof of Theorem 1.2.1

In Chapter 4, we study a simplified version of the "local model" for $X^{\square}_{\mathrm{tri}}(\bar{r})$ introduced in [BHS19]. It is a scheme $\widetilde{X}$ with a decomposition into cells indexed by $(\mathcal{S}_n)^{\Sigma}$, such that the local geometry of $X^{\square}_{\mathrm{tri}}(\bar{r})$ at a crystalline generic point with associated permutations (w_{sat}, w) is described by the local geometry of the closed cell of $\widetilde{X}$ of index w at a particular point of its boundary in the open cell of index w_{sat}. Our study yields an unconditional proof of Thm. 1.2.1: unlike the one laid out in §1.2.3 and Chapter 3, it does not use the "modularity" conjecture. Here, we provide a few details about $\widetilde{X}$ and its use in proving Thm. 1.2.1.

Let k be a field of characteristic 0. Let G be a connected split reductive group over k, T a split maximal torus inside G, B a Borel subgroup of G containing T and U the unipotent subgroup of B. We write $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{b}$ and $\mathfrak{u}$ for their respective Lie algebras, and $W \coloneqq N_G(T)/T$ for the Weyl group of (G, T). Recall the Schubert decomposition of G/B (see also §4.1). For $w \in W$, we choose a lifting of w in $N_G(T)$ which we still call w; consider the orbit BwB/B of the coset wB/B under the left-action by multiplication of B on G/B. Then there is a decomposition $G/B = \coprod_{w\in W} BwB/B$ of G/B in locally closed strata BwB/B, and an open cell $Bw'B/B$ is intersects a closed cell $\overline{BwB/B}$ if and only if $w' \preceq w$ for the Bruhat order on W, in which case $Bw'B/B \subseteq \overline{BwB/B}$.

Let $G \times^B \mathfrak{g}$ be the quotient of $G \times \mathfrak{g}$ by the right-action of B given by $(g, \psi)b = (gb, \mathrm{Ad}(b^{-1})\psi)$. We define the morphisms $q\colon G \times^B \mathfrak{g} \longrightarrow \mathfrak{g}$ and $\widetilde{\pi}\colon G \times^B \mathfrak{g} \longrightarrow G/B$ by $q(g, \psi) = \mathrm{Ad}(g)\psi$ and $\widetilde{\pi}(g, \psi) = gB$. The simplified local model of $X^{\square}_{\mathrm{tri}}(\bar{r})$ is then the variety $\widetilde{X}$ over k defined by $\widetilde{X} \coloneqq q^{-1}(\mathfrak{b})$. For $w \in W$, we consider the locally closed subvariety $\widetilde{V}_w \coloneqq \widetilde{X} \cap \widetilde{\pi}^{-1}(BwB/B)$ of $\widetilde{X}$ as well as its Zariski-closure $\widetilde{X}_w$ in $\widetilde{X}$; they give a cell decomposition of $\widetilde{X}$.

Note that in this part and in Chapter 4, contrary to §1.2.3 and Chapter 3, we use the letter L to denote a split reductive group over k. We then add the subscript

L to $q, \widetilde{\pi}, \widetilde{X}, \widetilde{V}_w, \widetilde{X}_w$ (for $w \in W(L)$) to mean the same definition with the reductive group L in place of the reductive group G; when nothing is specified, we mean G by default.

Similarly to Schubert cells in G/B, the intersection $\widetilde{V}_{w'} \cap \widetilde{X}_w$ is non-empty if and only if $w' \preceq w$. The difference is that in this case $\widetilde{V}_{w'}$ is not contained in $\widetilde{X}_w$. In fact, Breuil-Hellmann-Schraen show in [BHS19, Lem. 2.3.4] that $\widetilde{V}_{w'} \cap \widetilde{X}_w \subseteq \widetilde{V}_{w'} \cap q^{-1}\left(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u}\right)$, and conjecture this it is an equality:

Conjecture 1.2.7 (Conj. 2.3.7 of [BHS19])**.** *Let $w, w' \in W$. If $w' \preceq w$, then*

$$\widetilde{V}_{w'} \cap \widetilde{X}_w = \widetilde{V}_{w'} \cap q^{-1}\left(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u}\right).$$

One can prove that the formula (1.5) for the dimension of $T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x}$ is a consequence of Conj. 1.2.7 for $w_{\mathrm{sat}} \preceq w$; see Prop. 4.1.2 (see also the proof of [BHS19, Thm. 4.1.5] when $w = w_0$). As it turns out, we show that Conj. 1.2.7 is false in general (but it is not clear if (1.5) is false); see §1.2.5 and Chapter 5. We nonetheless prove it in the case of good pairs, which proves Thm. 1.2.1:

Theorem 1.2.8 (Thm. 4.2.3)**.** *Let $w, w' \in W$ such that $w' \preceq w$ and (w', w) is a good pair. Then, for all $t \in \mathfrak{t}$ fixed by ww'^{-1}, we have*

$$\widetilde{V}_{w'} \cap q^{-1}(t + \mathfrak{u}) \subset \widetilde{X}_w\,. \tag{1.8}$$

Hence Conj. 1.2.7 for (w', w) is true.

Before outlining the proof, we remark that it suffices to prove (1.8) for all "sufficiently generic" points $t \in \mathfrak{t}^{ww'^{-1}}$: see Prop. 4.3.1 and Lem. 4.3.2. Without loss of generality, we also assume k to be algebraically closed; see §4.4. The proof of Thm. 1.2.8 then proceeds in three steps.

Step 1 As (w', w) is a good pair, Prop. 1.2.3 gives a subset $J \subseteq I$ of simple roots of $\Phi(G, T)$ and $u^J, v^J \in W^J$, $w_J, w'_J \in W_J$, such that $w = u^J w_J (v^J)^{-1}$, $w' = u^J w'_J (v^J)^{-1}$ and $w'_J \preceq w_J$ in W_J. Let L be the Levi subgroup of the standard parabolic subgroup P_J of G; then $L = C_G\big(\big((u^J)^{-1} T^{ww'^{-1}} u^J\big)^{\circ}\big)$.

It is proved in [BHS19, §2.4] using the Beilinson-Bernstein localisation that (1.8) holds for $t = 0$ for all $w' \preceq w$ (even in the case of bad pairs). Since $w'_J \preceq w_J$, we can apply this to the connected split reductive group L with split maximal torus T, Borel $B_L := B \cap L$ and unipotent subgroup $U := U \cap L$. For any $t \in \mathfrak{t}_J = \mathrm{Ad}(u^J)^{-1} \mathfrak{t}^{ww'^{-1}}$, we deduce by translating by t that $\widetilde{V}_{L, w'_J} \cap q^{-1}(t + \mathfrak{u}_L) \subset \widetilde{X}_{L, w_J}$.

Step 2 The following result shows how the picture on $\widetilde{X}_L$ is relevant to the study of $\widetilde{X}$.

Proposition 1.2.9 (Prop. 4.4.2)**.** *Let $u^J, v^J \in W^J$. One can define a morphism $\iota_{t,u^J,v^J}\colon L \times^{L \cap B} \mathfrak{l} \cap \mathfrak{b} \longrightarrow G \times^B \mathfrak{b}$ by $\iota_{t,u^J,v^J}(l, \psi_L)) = \big(u^J l (v^J)^{-1}, \mathrm{Ad}(v^J)\psi_L\big)$. For* *any* $t \in \mathrm{Ad}(u^J)^{-1}\mathfrak{t}^{ww'^{-1}}$ *sufficiently generic and any* $w_J \in W_J$*, this morphism* ι_{t,u^J,v^J} *induces an isomorphism of algebraic varieties*

$$\iota_{t,u^J,w_J,v^J} \colon \widetilde{V}_{L,w_J} \cap q_L^{-1}(t+\mathfrak{u}_L) \xrightarrow{\sim} \widetilde{V}_{u^J w_J (v^J)^{-1}} \cap q^{-1}\left(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\right) .$$

The proof uses a few technical lemmas which make up §4.3.

Step 3 We prove the following lemma using technical results about Jordan decompositions in $\mathfrak{g}$ which are reviewed in the appendix 4.A to Chapter 4.

Lemma 1.2.10 (Lem. 4.4.3)**.** *Let* $t \in \mathrm{Ad}(u^J)^{-1}\mathfrak{t}^{ww'^{-1}}$ *be sufficiently generic, let* $u^J \in W^J$*. Then there is a surjective morphism given by*

$$f \colon \begin{array}{ccc} U \times \left(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\right) & \twoheadrightarrow & \mathrm{Ad}(u^J)t + \mathfrak{u} \\ (u,x) & \longmapsto & \mathrm{Ad}(u)x \,. \end{array}$$

This lemma allows us to recover the entirety of $q^{-1}\left(\mathrm{Ad}(u^J)t+\mathfrak{u}\right)$ from its subset $q^{-1}\left(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\right)$. More precisely, the surjective morphism f of Lem. 1.2.10 induces morphisms of algebraic varieties $U \times \left(\widetilde{V}_{L,w_J} \cap q_L^{-1}(t+\mathfrak{u}_L)\right) \twoheadrightarrow \widetilde{V}_w \cap q^{-1}\left(\mathrm{Ad}(u^J)t+\mathfrak{u}\right)$ and $U \times \widetilde{X}_{L,w_J} \longrightarrow \widetilde{X}_w$, which fit into a commutative diagram

$$\begin{tikzcd}
U \times \left(\widetilde{V}_{L,w'_J} \cap q_L^{-1}(t+\mathfrak{u}_L)\right) \arrow[r, "\subseteq"] \arrow[dd, two heads] & U \times \widetilde{X}_{L,w_J} \arrow[d] \\
 & \widetilde{X}_w \arrow[d, hook, "\subseteq"] \\
\widetilde{V}_{w'} \cap q^{-1}(\mathrm{Ad}(u^J)t+\mathfrak{u}) \arrow[r, "\subseteq"] & \widetilde{X}
\end{tikzcd}$$

where the top horizontal arrow is given by step 1. The surjectivity of the left vertical arrow then proves that the image of the bottom horizontal arrow is contained in $\widetilde{X}_w$, which is the statement of (1.8).

1.2.5 Counter-examples to Conjecture 1.2.7

We conclude this work in Chapter 5, where we give explicit equations and inequations for the projective varieties $\widetilde{X}$ and $\widetilde{V}_w$ of §1.2.4, in the case $G = \mathrm{GL}_n$. Even though we do not have such an explicit description for the varieties $\widetilde{X}_w$, this is enough for us to find a family of counter-examples to Conj. 1.2.7.

To this end, we slightly shift the point of view and use an equivalent definition of $\widetilde{X}$ as the subvariety $\{(gB,\psi) \in G/B \times \mathfrak{b} \mid \mathrm{Ad}(g^{-1})\psi \in \mathfrak{b}\}$ of $G/B \times \mathfrak{b}$; see (5.5). This is easier to handle than a quotient variety. Indeed $\mathfrak{b}$ is an affine space, and the flag variety G/B is very well-understood. Specifically, G/B is a subvariety of a product of Grassmannians defined by incidence relations; and there are well-known equations defining the Grassmannian as a subvariety of a projective space *via* the Plücker embedding. Therefore $G/B \times \mathfrak{b}$ is well-understood as a subvariety of $\prod_{d=1}^{n-1} \mathbb{P}(\bigwedge^d k^n) \times \mathbb{A}^{n(n+1)/2}$. Details are spelled out in §5.1.

We then work out a set of equations defining $\widetilde{X}$ inside $G/B \times \mathfrak{b}$; this is Lem. 5.2.3. The next step is to use the (in)equations defining each Schubert cell BwB/B inside G/B, in order to find a set $\{F_i = 0, G_j \neq 0\}_{i \in I, j \in J}$ of (in)equations defining $\widetilde{V}_w$ inside $\prod_{d=1}^{n-1} \mathbb{P}(\bigwedge^d k^n) \times \mathbb{A}^{n(n+1)/2}$. Doing so by brute force is not too difficult; however the challenge is to have enough equations so that, hopefully, the set of equations $\{F_i = 0\}_{i \in I}$ describes the Zariski-closure $\widetilde{X}_w$ of $\widetilde{V}_w$ inside $\prod_{d=1}^{n-1} \mathbb{P}(\bigwedge^d k^n) \times \mathbb{A}^{n(n+1)/2}$. The result we achieve in Prop. 5.2.4 is quite satisfying, as Conj. 1.2.7 is equivalent to $\widetilde{X}_w$ being the closed subset $\{F_i = 0\}_{i \in I}$; see Prop. 5.3.1.

On the other hand, through technical simplifications in §5.3, we improve on Prop. 5.2.4 by finding unexpected additional equations satisfied on $\widetilde{V}_w$. As a consequence, we get for a certain family of couples $(w, w') \in (\mathcal{S}_n)^2$, a beautifully simple equation satisfied on $\widetilde{X}_w \cap \widetilde{V}_{w'}$ but not on $\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})$, hence the following result:

Theorem 1.2.11 (Thm. 5.3.7, Rmk. 5.3.8, Rmk. 5.3.9)**.** *For any $n \in \mathbb{Z}_{\geq 4}$, there exist a bad pair (w, w') in $\mathcal{S}_n$ such that*

$$\dim\left(\widetilde{X}_w \cap \widetilde{V}_{w'}\right) < \dim\left(\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})\right) .$$

In particular, Conj. 1.2.7 is false.

To our knowledge, the question of deciding whether or not Conj. 1.2.7 is false for all bad pairs remains open (we know that this holds at least for GL_4 and GL_5). Another area of further investigation is to find accurate estimates for the dimension deficit $\dim\left(\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})\right) - \dim\left(\widetilde{X}_w \cap \widetilde{V}_{w'}\right)$. Finally, our results suggest that formula (1.5) for the dimension of the tangent space of $X_{\mathrm{tri}}^{\square}(\bar{r})$ at a companion point is possibly false when (w_{sat}, w) is a bad pair, or at the very least when (w_{sat}, w) satisfies the hypotheses of Thm. 5.3.7. If so, it would be interesting to find a counterexample to formula (1.5).

Chapter 2

Some combinatorics in a Weyl group

In this section, we study particular subsystems of a root system Φ associated with an element of the Weyl group $W(\Phi)$. We then define the notions of *good pairs* and *bad pairs* in $W(\Phi)$, seen as a Coxeter group. Finally, we give a concrete description of the picture for systems of type A.

2.1 Minimal generating root subsystems

Let E be a finite dimensional a vector space over $\mathbb{Q}$ and let Φ be a root system inside E (see [Bou02, Chapter VI, §1]). Write the (finite) Weyl group of Φ as $W \subset \mathrm{End}_{\mathbb{Q}}(V)$. For a root $\alpha \in \Phi$, we denote by $s_\alpha \in W$ its associated reflection.

Definition 2.1.1. Let $w \in W$, Γ_w the $\mathbb{Z}$-submodule of E generated by $w(\alpha) - \alpha$ for $\alpha \in \Phi$, $E_w \coloneqq \Gamma_w \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{im}(w - \mathrm{id}_E)$ and $\Phi_w \coloneqq \Phi \cap E_w$. We call Φ_w the *minimal generating (root) subsystem* of w.

We also define $d_w \coloneqq \mathrm{rk}_{\mathbb{Z}} \Gamma_w = \dim_{\mathbb{Q}} E_w$.

Being the intersection of Φ with a subspace E_w of E, the set Φ_w is a root subsystem in Φ, *i.e.* it is a root system in the subspace of E that it generates (see the Corollary to Proposition 4 of [Bou02, Chapter VI, §1.1]). Note that this latter subspace may be strictly smaller than E_w; however we will see later in Corollary 2.1.5 that this never occurs.

It therefore makes sense to consider the Weyl group $W(\Phi_w)$ of Φ_w as a subgroup of W, with the convention $W(\emptyset) = \{1\}$.

Proposition 2.1.2. *Let $w \in W$. Then w is in the Weyl group $W(\Phi_w)$ of its minimal generating subsystem.*

Proof. When Φ is the root system of some connected split reductive group G over $\mathbb{Q}$, the statement $w \in W(\Phi_w)$ is Corollary 2.2.3 below (which is obtained independently from the results of this section). However this is always the case, see for example [Hum75, 33.6]. □

We equip E with a non-degenerate symmetric bilinear form $(\cdot \mid \cdot)$ invariant by W (see Proposition 3 of [Bou02, Chapter VI, §1.1]). Then every root $\alpha \in \Phi$ satisfies $(\alpha \mid \alpha) \neq 0$, and $s_\alpha(x) = x - 2\frac{(x|\alpha)}{(\alpha|\alpha)}\alpha$ for all $x \in E$. For any $x \in E$, we also note $x^\perp \coloneqq \ker(\cdot \mid x)$ the orthogonal subspace of x.

Lemma 2.1.3. *Let $w \in W$ and $x \in E$. Then $x \in E_w$ if and only if $\ker(w - \mathrm{id}_E) \subseteq x^\perp$.*

Proof. Suppose $x \in E_w$; take $y \in E$ such that $x = w(y) - y$. Then, for all $z \in \ker(w - \mathrm{id}) = \ker(w^{-1} - \mathrm{id})$, we have

$$(z \mid x) = (z \mid w(y) - y) = (z \mid w(y)) - (z \mid y) = (w^{-1}(z) - z \mid y) = 0$$

hence $z \in x^\perp$.

Conversely, suppose that $\ker(w - \mathrm{id}) \subseteq x^\perp$. Then the linear form $(\cdot \mid x)$ factors through a linear form l on $E/\ker(w - \mathrm{id})$. The map $w - \mathrm{id}$ also factors through an isomorphism $f\colon E/\ker(w - \mathrm{id}) \xrightarrow{\sim} \mathrm{im}(w - \mathrm{id})$. The linear form $l \circ f^{-1}$ on $\mathrm{im}(w - \mathrm{id})$ can be extended arbitrarily to a linear form $\widetilde{l}$ on E. Since $(\cdot \mid \cdot)$ is non-degenerate, there exists $y \in E$ such that $\widetilde{l} = (\cdot \mid y)$. Then, for all $z \in E$,

$$(z \mid x) = (w(z) - z \mid y) = (w(z) \mid y) - (z \mid y) = (z \mid w^{-1}(y) - y)$$

thus, by non-degeneracy, $x = w^{-1}(y) - y = (w - \mathrm{id})(-w^{-1}(y)) \in E_w$. ☐

Proposition 2.1.4. *Let $w \in W$ and $\alpha \in \Phi$. Let w' be either $s_\alpha w$ or $w s_\alpha$.*

(1) *If $\alpha \notin \Phi_w$, then $E_{w'} = \mathbb{Q}\alpha \oplus E_w$. In particular, $d_{w'} = d_w + 1$.*

(2) *If $\alpha \in \Phi_w$, then $E_w = \mathbb{Q}\alpha \oplus E_{w'}$. In particular, $d_{w'} = d_w - 1$.*

Proof. Assume that $w' = w s_\alpha$. Using the decomposition $E = \mathbb{Q}\alpha \oplus \alpha^\perp$, one can write

$$E_w = \mathbb{Q}(w(\alpha) - \alpha) + (w - \mathrm{id})\alpha^\perp \tag{2.1}$$

and $E_w = (w - \mathrm{id})\alpha^\perp$ if and only if $\ker(w - \mathrm{id}) \not\subseteq \alpha^\perp$. From Lemma 2.1.3, this happens if and only if $\alpha \notin E_w$. Similarly, because $w'(\alpha) = -w(\alpha)$ and $w'|_{\alpha^\perp} = w|_{\alpha^\perp}$,

$$E_{w'} = \mathbb{Q}(w(\alpha) + \alpha) + (w - \mathrm{id})\alpha^\perp \tag{2.2}$$

and $E_{w'} = (w - \mathrm{id})\alpha^\perp$ if and only if $\alpha \notin E_{w'}$.

Suppose first that $\alpha \notin \Phi_w$, or equivalently $\alpha \notin E_w$. Then

$$w(\alpha) - \alpha \in E_w = (w - \mathrm{id})\alpha^\perp \subseteq E_{w'}\,.$$

Since $w(\alpha) + \alpha \in E_{w'}$, we get $\alpha = \frac{1}{2}(w(\alpha) + \alpha) - \frac{1}{2}(w(\alpha) - \alpha) \in E_{w'}$. We also know from (2.2) that $E_w = (w - \mathrm{id})\alpha^\perp \subseteq E_{w'}$ and that $\dim E_{w'} \leq \dim E_w + 1$. Altogether, this yields $E_{w'} = \mathbb{Q}\alpha \oplus E_w$, so (1) is proved.

Now suppose that $\alpha \in \Phi_w$, or equivalently $\alpha \in E_w$. Take $x \in E$ such that $\alpha = w(x) - x$. Then

$$(w(x) + x \mid \alpha) = (w(x) + x \mid w(x) - x) = (w(x) \mid w(x)) - (x \mid x) = 0$$

so $w(x) + x \in \alpha^\perp$, hence $x = \frac{1}{2}(-\alpha + w(x) + x) \in -\frac{1}{2}\alpha + \alpha^\perp$. Therefore

$$2\alpha = 2(w(x) - x) \in \alpha - w(\alpha) + (w - \mathrm{id})\alpha^\perp$$

which can be rewritten as $w(\alpha) + \alpha \in (w - \mathrm{id})\alpha^\perp$. From (2.2), this means $E_{w'} = (w - \mathrm{id})\alpha^\perp$, so $\alpha \notin E_{w'}$. As before, this also implies, using (2.1), that $E_{w'} \subseteq E_w$ and $\dim E_w \leq \dim E_{w'} + 1$. We deduce that $E_w = \mathbb{Q}\alpha \oplus E_{w'}$, so (2) is proved.

If $w' = s_\alpha w$, then one can also write $w' = ww^{-1}s_\alpha w = ws_{w^{-1}(\alpha)}$. Since $\alpha \in \Phi_w$ if and only if $w^{-1}(\alpha) \in \Phi_w$, and similarly for Φ'_w, the problem reduces to the previous case. □

Let $R \coloneqq \{s_\alpha \mid \alpha \in \Phi\}$ be the set of reflections of W; it generates W as a group, and in particular as a monoid. We can then view any element $w \in W$ as a word on the alphabet R (in a non-unique way); we call *length of w on R* the minimal length of such a word, and *reduced decomposition of w on R* any such word of minimal length. The following corollary to Proposition 2.1.4 interprets minimal generating subsystems using reduced decompositions on R.

Corollary 2.1.5. *Let $w \in W$.*

(1) *The number d_w is the length of w on the alphabet R.*

(2) *The subsystem Φ_w is the set of roots $\alpha \in \Phi$ such that s_α appears in a reduced decomposition of w on the alphabet R.*

(3) *The subsystem Φ_w spans E_w as a vector space.*

Proof. If $w = 1$, then all claims are true because $E_w = \mathrm{im}(\mathrm{id}_E - \mathrm{id}_E) = \{0\}$. We now suppose that $w \neq 1$.

Let $w = s_{\alpha_r} \dots s_{\alpha_1}$ be a decomposition of w on R, with $\alpha_i \in \Phi$ for all $1 \leq i \leq r$. By Proposition 2.1.4, we know that $d_{s_{\alpha_i} \cdots s_{\alpha_1}} \leq d_{s_{\alpha_{i-1}} \cdots s_{\alpha_1}} + 1$ for all $1 \leq i \leq d$, with equality if and only if $E_{s_{\alpha_i} \cdots s_{\alpha_1}} = \mathbb{Q}\alpha_i \oplus E_{s_{\alpha_{i-1}} \cdots s_{\alpha_1}}$. An induction on i gives $d_w = d_{s_{\alpha_r} \cdots s_{\alpha_1}} \leq r$, with equality if and only if

$$E_w = \mathbb{Q}\alpha_r \oplus \ldots \oplus \mathbb{Q}\alpha_1 \,. \tag{2.3}$$

Therefore the three claims will be proved if it can be shown that Φ_w is non-empty and that for any $\alpha \in \Phi_w$, there is a decomposition of w on R of length d_w containing s_α.

The non-emptiness of Φ_w is a consequence of Proposition 2.1.2. Now take $\alpha \in \Phi_w$. The wanted decomposition can be obtained by induction on d_w: Proposition

2.1.4 implies that $d_{s_\alpha w} = d_w - 1$, and if $s_\alpha w = s_{\alpha_r} \dots s_{\alpha_1}$ is a decomposition of length $r \coloneqq d_{s_\alpha w}$, then $w = s_\alpha s_{\alpha_r} \dots s_{\alpha_1}$ is a decomposition of length d_w. $\square$

The following corollary explains the terminology of minimal generating subsystem introduced in Definition 2.1.1.

Corollary 2.1.6. *Let $w \in W$ and let E' be a subspace of E. Let $\Phi' \coloneqq \Phi \cap E'$; it is a root system in E', we can thus consider its Weyl group $W(\Phi') \subseteq W$.*

Then $w \in W(\Phi')$ if and only if $\Phi_w \subseteq \Phi'$, if and only if $E_w \subseteq E'$.

Proof. The implication $E_w \subseteq E' \implies \Phi_w \subseteq \Phi'$ is a mere consequence of the definitions, and the implication $\Phi_w \subseteq \Phi' \implies w \in W(\Phi')$ is Proposition 2.1.2.

Now suppose that $w \in W(\Phi')$. Let Φ'_w be the minimal generating subsystem of w in Φ' and let $E'_w \coloneqq \mathrm{im}(w|_{E'} - \mathrm{id}_{E'}) \subseteq E' \cap E_w$. From Corollary 2.1.5(1) applied to Φ', w has a reduced decomposition on $R' \coloneqq \{s_\alpha \mid \alpha \in \Phi'\}$ of length $\dim E'_w$. Since $R' \subseteq R$, the same result applied to Φ shows that $\dim E'_w \geq \dim E_w$. Therefore, the chain of inclusions $E'_w \subseteq E' \cap E_w \subseteq E_w$ is a chain of equalities. In particular, $E' \cap E_w = E_w$; or equivalently $E_w \subseteq E'$. $\square$

Fix $I \subset \Phi$ a basis of simple roots of Φ, and write $S \coloneqq \{s_\alpha \mid \alpha \in I\} \subseteq W$ for the set of simple reflections corresponding to the simple roots. These define a length function lg on W.

The following result is a generalisation of [BHS17a, Lemma 2.7] where it was proved for root systems Φ of type A.

Corollary 2.1.7. *Let $w \in W$. Then*

$$d_w \leq \lg(w)$$

with equality if and only if w is a product of distinct simple reflections.

Proof. Let $w = s_{\alpha_{\lg(w)}} \dots s_{\alpha_1}$ be a reduced decomposition of w on S, with $\alpha_i \in I$ for all $1 \leq i \leq \lg(w)$. Corollary 2.1.5(1) and its proof give $d_w \leq \lg(w)$, with equality if and only if $E_w = \mathbb{Q}\alpha_{\lg(w)} \oplus \dots \oplus \mathbb{Q}\alpha_1$ (see (2.3)). Therefore, in the equality case, the α_i must be distinct.

Conversely, suppose that w is a product of distinct simple reflections. Write $w = s_{\alpha_r} \dots s_{\alpha_1}$ with α_i for $1 \leq i \leq r$ being distinct elements of I; then $r \geq \lg(w)$ by definition. Also, since the elements of I are linearly independent, the sum $\mathbb{Q}\alpha_r + \dots + \mathbb{Q}\alpha_1$ is direct. The proof of Corollary 2.1.5 implies that $r = d_w$; hence $d_w \geq \lg(w)$, so we have equality. $\square$

2.2 Interpretation for a split reductive group

We establish in this part an interpretation of the minimal generating subsystem $\Phi_w \subseteq \Phi$, for any $w \in W(\Phi)$, in the case when Φ arises from a connected split reductive group, in terms of Levi groups.

It is used crucially for Proposition 2.1.2; we therefore keep this section independent from the results of section 2.1

Let k be a field of characteristic 0. We use the following notation.

- (G, T) is a connected split reductive algebraic group over k.
- B is a Borel subgroup of G containing T.
- $X(T) \coloneqq \mathrm{Hom}(T, \mathbb{G}_\mathrm{m})$ is the group of algebraic characters of T, viewed as a $\mathbb{Z}$-module.
- $\Phi \subset X(T)$ is the set of roots of (G, T), *i.e.* the set of $\alpha \in X(T) \setminus \{0\}$ such that the eigenspace $\mathfrak{g}_\alpha \coloneqq \{x \in \mathfrak{g} \mid \mathrm{Ad}(t)x = \alpha(t)x \ \forall t \in T\}$ is non-zero (where $\mathfrak{g}$ is the Lie algebra of G). For any root α, we have $\dim \mathfrak{g}_\alpha = 1$ (see [Mil17, Theorem 21.11(b)]).
- $\Gamma \subseteq X(T)$ is the $\mathbb{Z}$-submodule generated by Φ , and $E \coloneqq \Gamma \otimes_\mathbb{Z} \mathbb{Q}$.
- I is the basis of the root system Φ (inside E) defined by the Borel B.
- Φ_+ is the set of positive roots defined by I.
- $W \coloneqq W(\Phi)$ is the Weyl group of Φ. Recall that $W(G, T) \coloneqq N_G(T)/T$ is a finite group with a natural faithful action on T and hence on $X(T)$, which makes $W(G, T)$ equal to W as groups of automorphisms of $X(T)$ (see [Mil17, Corollary 21.38]). We therefore make the identification $W = N_G(T)/T$. For $w \in W$, we write $\mathrm{Ad}(w)$ for the conjugation morphism on T by any $\dot{w} \in N_G(T)$ which lifts w; meaning $\mathrm{Ad}(w)t = \dot{w}t\dot{w}^{-1}$ for $t \in T$. Then the action of W on Φ can be written as
$$(w\alpha)(t) = \alpha(\mathrm{Ad}(w^{-1})t) \in k^\times$$
for all $\alpha \in \Phi$, $w \in W$ and $t \in T$.
- $T^w \coloneqq \{t \in T \mid \mathrm{Ad}(w)t = t\}$ is the subgroup of T of invariants under $\mathrm{Ad}(w)$. The identity component $(T^w)^\circ$ of T^w is a subtorus of T (see the discussion in [Mil17, Notation 12.29]; note that T^w is always reduced in characteristic 0).

We start by recalling a classical lemma.

Lemma 2.2.1. *Let $N \geq 1$ be an integer and M be a $\mathbb{Z}[\frac{1}{N}]$-module equipped with a linear action of the group $\mathbb{Z}/N\mathbb{Z}$. Then $H^1(\mathbb{Z}/N\mathbb{Z}, M) = 0$.*

Proof. The $\mathbb{Z}[\frac{1}{N}]$-module $H^1(\mathbb{Z}/N\mathbb{Z}, M)$ is killed by N (see [Ser79, §VII.7, Proposition 6]), hence it must be 0. □

Proposition 2.2.2. *Let $w \in W$. Then the centraliser $(C_G((T^w)^\circ), T)$ is a connected split reductive group whose root system is the minimal generating subsystem Φ_w of w.*

Proof. $(T^w)^\circ$ is a torus in G, hence $C_G((T^w)^\circ)$ is connected and split reductive by [Mil17, Corollary 17.59]. Note that it always contains T, so $(C_G((T^w)^\circ), T)$ is split reductive.

Let $\alpha \in \Phi$ be a root of (G,T). The eigenspace $\mathfrak{g}_\alpha$ has dimension 1, so α is a root of $(C_G((T^w)^\circ), T)$ if and only if $\mathfrak{g}_\alpha$ is in the Lie group of $C_G((T^w)^\circ)$. By [Mil17, Theorem 21.11(c)], we thus have

$$\Phi' \coloneqq \Phi\left(C_G((T^w)^\circ), T\right) = \left\{\alpha \in \Phi \;\middle|\; \alpha|_{(T^w)^\circ} = 1\right\} .$$

Let N be the order of w in W. The morphism of algebraic varieties $T \longrightarrow T$, $t \longmapsto \prod_{k=0}^{N-1} \mathrm{Ad}(w^{-k})t$ has a connected image (because T is connected) included in T^w. Hence, $\prod_{k=0}^{N-1} \mathrm{Ad}(w^{-k})t \in (T^w)^\circ$ for all $t \in T$. Then, for $\alpha \in \Phi'$ and $t \in T$, we have

$$\left(\sum_{k=0}^{N-1} w^k\alpha\right)(t) = \alpha\left(\prod_{k=0}^{N-1} \mathrm{Ad}(w^{-k})t\right) = 1$$

hence $(\sum_{k=0}^{N-1} w^k)\alpha = 0$. Conversely, if $\alpha \in \ker(\sum_{k=0}^{N-1} w^k)$, then for all $t \in (T^w)^\circ$,

$$\alpha(t^N) = \alpha\left(\prod_{k=0}^{N-1} \mathrm{Ad}(w^{-k})t\right) = 1$$

so, since the group of characters on the torus $(T^w)^\circ$ has no torsion, we necessarily have $\alpha(t) = 1$. Hence $\alpha|_{(T^w)^\circ} = 1$. Therefore $\Phi' = \Phi \cap \ker(\sum_{k=0}^{N-1} w^k)$. Since $H^1(\langle w\rangle, E) = 0$ by Lemma 2.2, we have $\Phi' = \Phi \cap \mathrm{im}(w - \mathrm{id}_E) = \Phi_w$. □

Corollary 2.2.3. *Let $w \in W$. Then $w \in W(\Phi_w)$.*

Proof. Seeing w as an element of $N_G(T)/T$, we have $w \in C_G((T^w)^\circ)/T$, so that actually $w \in N_{C_G((T^w)^\circ)}(T)/T = W(C_G((T^w)^\circ), T) = W(\Phi_w)$ where the last equality comes from Proposition 2.2.2. □

2.3 Pairs of elements in the Weyl group

As before, fix $I \subset \Phi$ a basis of simple roots of Φ, and write Φ_+ for the subset of positive roots that it generates. Write $S \coloneqq \{s_\alpha \mid \alpha \in I\} \subseteq W$ for the set of simple reflections corresponding to the simple roots. They define a length function lg and a Bruhat order $\preceq$ on W.

Definition 2.3.1. Let $(w_1, w_2) \in W^2$ such that $w_1 \preceq w_2$. Let $\Phi_{w_1 w_2^{-1}} \subseteq \Phi$ be the minimal generating subsystem of $w_1 w_2^{-1}$.

We say that (w_1, w_2) is a *good pair* if there exists a sequence $(\alpha_1, \dots, \alpha_r)$ of roots in $\Phi_{w_1 w_2^{-1}}$ such that $w_2 = s_{\alpha_r} s_{\alpha_{r-1}} \dots s_{\alpha_1} w_1$ and that for each $1 \leq i \leq r$, the relation $s_{\alpha_i} \dots s_{\alpha_1} w_1 \succ s_{\alpha_{i-1}} \dots s_{\alpha_1} w_1$ holds.

If such a sequence does not exist, we say that (w_1, w_2) is a *bad pair*.

Remark 2.3.2. The set of reflections $R = \{s_\alpha \mid \alpha \in \Phi\}$ of W can be rewritten as $R = \bigcup_{w \in W} wSw^{-1}$. On the other hand, using (2) of Corollary 2.1.5, one sees that the subset $\left\{s_\alpha \;\middle|\; \alpha \in \Phi_{w_1 w_2^{-1}}\right\}$ of R is the set of reflections that appear in some reduced decomposition of w on the alphabet R. Hence the notion of good pair is intrinsic to the Coxeter system (W, S).

In section 2.5 below, we provide a few interesting bad pairs for root systems of type A.

Proposition 2.3.3. *Let $(w_1, w_2) \in W^2$, and let $w_0 \in W$ be the maximal element for the Bruhat order.*

Then (w_1, w_2) is a good pair (resp. bad pair) if and only if $(w_2 w_0, w_1 w_0)$ is a good pair (resp. bad pair). Similarly, (w_1, w_2) is a good pair (resp. bad pair) if and only if $(w_0 w_2, w_0 w_1)$ is a good pair (resp. bad pair).

Proof. It is well known that $w \longmapsto w w_0$ and $w \longmapsto w_0 w$ are antiautomorphisms of W for the Bruhat order (see [BB06, Proposition 2.3.4]):

$$w_1 \preceq w_2 \quad \Longleftrightarrow \quad w_2 w_0 \preceq w_1 w_0 \quad \Longleftrightarrow \quad w_0 w_2 \preceq w_0 w_1 \,.$$

Therefore the statement for bad pairs is equivalent to the one for good pairs. We now assume that (w_1, w_2) is good, and take s_{α_i} for $1 \leq i \leq r$ as in Definition 2.3.1.

Then, for all $1 \leq i \leq r$, we have $s_{\alpha_i} \in \Phi_{w_1 w_2^{-1}} = \Phi_{(w_1 w_0)(w_2 w_0)^{-1}}$ and since the s_{α_i} are involutions, $s_{\alpha_{r+1-i}} \dots s_{\alpha_r} w_2 = s_{\alpha_{r-i}} \dots s_{\alpha_1} w_1$. We then get

$$s_{\alpha_{r+1-i}} \dots s_{\alpha_r} w_2 w_0 \succ s_{\alpha_{r-i}} \dots s_{\alpha_r} w_2 w_0$$

by the antiautomorphism property. The s_{α_i} in reverse order hence give the desired chain, and $(w_2 w_0, w_1 w_0)$ is good.

If we instead apply the antiautomorphism $w \longmapsto w_0 w$, then we get the relations

$$w_0 s_{\alpha_{r+1-i}} \dots s_{\alpha_r} w_2 \succ w_0 s_{\alpha_{r-i}} \dots s_{\alpha_r} w_2 \,. \tag{2.4}$$

If we set $\beta_i = w_0(\alpha_i)$ for all $1 \leq i \leq r$, then $s_{\beta_i} = w_0 s_{\alpha_i} w_0$, so (2.4) is rephrased as

$$s_{\beta_{r+1-i}} \dots s_{\beta_r} w_0 w_2 \succ s_{\beta_{r-i}} \dots s_{\beta_r} w_0 w_2 \,.$$

Since $\beta_i = w_0(\alpha_i) \in w_0(\Phi_{w_1 w_2^{-1}}) = \Phi_{(w_0 w_1)(w_0 w_2)^{-1}}$, the s_{β_i} in reverse order give the desired chain from $w_0 w_2$ to $w_0 w_1$.

Conversely, this shows that if $(w_2 w_0, w_1 w_0)$ is good, then $(w_1, w_2) = (w_1 w_0^2, w_2 w_0^2)$ is good as well; and similarly $(w_0 w_2, w_0 w_1)$ good implies $(w_1, w_2) = (w_0^2 w_1, w_0^2 w_2)$ good. $\square$

Proposition 2.3.4. *Let $w_1, w_2 \in W$ with $w_1 \preceq w_2$. Then*

$$d_{w_2 w_1^{-1}} \leq \lg(w_2) - \lg(w_1)$$

and if there is equality, then the pair (w_1, w_2) is good.

Proof. By the chain property in W (see [BB06, Theorem 2.2.6]), there is a sequence of size $r = \lg(w_2) - \lg(w_1)$ of roots $(\alpha_1, \dots, \alpha_r)$ in Φ with $w_2 = s_{\alpha_r} s_{\alpha_{r-1}} \dots s_{\alpha_1} w_1$ and such that for each $1 \leq i \leq r$, we have $\lg(s_{\alpha_i} \dots s_{\alpha_1} w_1) = \lg(s_{\alpha_{i-1}} \dots s_{\alpha_1} w_1) + 1$ (hence automatically $s_{\alpha_i} \dots s_{\alpha_1} w_1 \succ s_{\alpha_{i-1}} \dots s_{\alpha_1} w_1$).

Then $w_2 w_1^{-1} = s_{\alpha_r} s_{\alpha_{r-1}} \dots s_{\alpha_1}$ is a decomposition of $w_2 w_1^{-1}$ on $R \coloneqq \{s_\alpha \mid \alpha \in \Phi\}$. Corollary 2.1.5(1) then yields $d_{w_2 w_1^{-1}} \leq r = \lg(w_2) - \lg(w_1)$. If there is equality, 2.1.5(2) gives $\alpha_i \in \Phi_{w_2 w_1^{-1}}$ for all $1 \leq i \leq r$, which implies that (w_1, w_2) is a good pair. $\square$

Let $J \subseteq I$ be a set of simple roots. We use the following notation.

- $S_J \coloneqq \{s_\alpha \mid \alpha \in J\}$ is the set of simple reflections associated to the elements of J.
- $W_J \coloneqq \langle s \mid s \in S_J \rangle$ is the subgroup of W it spans. Note that $W_J = W(\Phi_J)$, where Φ_J is the root subsystem of Φ generated by J, and the Bruhat order on W_J is the same as the restriction of the Bruhat order on W.
- $W^J \coloneqq \{w \in W \mid ws \succ w \ \ \forall s \in S_J\}$. It is known that W^J is the set of representatives of minimal length for the left cosets representatives W/W_J, and that for all $w \in W$, the decomposition $w = w^J w_J$ with $w^J \in W^J$ and $w_J \in W_J$ satisfies $\lg(w) = \lg(w^J) + \lg(w_J)$ (see [BB06, §2.4]).

Lemma 2.3.5. *Let $J \subseteq I$ be a subset of the simple roots, let u^J and v^J be elements of W^J, let w_J be an element of W_J and let $s \in W_J$ be a reflection,* i.e. *an element of $\bigcup_{w \in W_J} w S_J w^{-1}$. Then $s w_J \succ w_J$ if and only if $u^J s w_J (v^J)^{-1} \succ u^J w_J (v^J)^{-1}$.*

Proof. Consider the set R of reflections of W, *i.e.* $R \coloneqq \{wSw^{-1} \mid w \in W\}$. Define a map

$$\eta\colon \begin{array}{rcl} W \times R & \longrightarrow & \{-1, 1\} \\ (w, t) & \longmapsto & \begin{cases} 1 & \text{if } tw \succ w \\ -1 & \text{if } tw \prec w \end{cases} \end{array}$$

(see Theorem 1.4.3 of [BB06] and its proof). Our goal is then to prove that

$$\eta(u^J w_J (v^J)^{-1}, u^J s (u^J)^{-1}) = \eta(w_J, s).$$

For any $w \in W$, $t \in R$ and $\varepsilon \in \{-1, 1\}$, define

$$\pi_w(t, \varepsilon) \coloneqq (wtw^{-1}, \varepsilon \eta(w^{-1}, t)) \in R \times \{-1, 1\}\,.$$

This defines a mapping π from W to the finite group of bijections from $R \times \{-1, 1\}$ to itself, and π is a group morphism (*loc. cit.*, Theorem 1.3.2(i)). Therefore, for any $v, w \in W$ and $t \in R$, we can write $\pi_{w^{-1}v^{-1}}(t, 1) = \pi_{w^{-1}}(\pi_{v^{-1}}(t, 1))$ and get the formula

$$\eta(vw, t) = \eta(v, t)\eta(w, v^{-1}tv)\,.$$

First apply this formula to $v = u^J$, $w = w_J(v^J)^{-1}$ and $t = u^J s(u^J)^{-1}$:

$$\eta(u^J w_J(v^J)^{-1}, u^J s(u^J)^{-1}) = \eta(u^J, u^J s(u^J)^{-1})\eta(w_J(v^J)^{-1}, s)\,. \tag{2.5}$$

Now, since $u^J \in W^J$ and $w_J^{-1}, s \in W_J$ and using the fact that the map $w \longmapsto w^{-1}$ preserves the length on W, we have

$$\lg(sw_J(v^J)^{-1}) = \lg(v^J w_J^{-1} s) = \lg(v^J) + \lg(w_J^{-1} s) = \lg(v^J) + \lg(sw_J)$$

and

$$\lg(w_J(v^J)^{-1}) = \lg(v^J w_J^{-1}) = \lg(v^J) + \lg(w_J^{-1}) = \lg(v^J) + \lg(w_J)\,,$$

therefore $\eta(w_J(v^J)^{-1}, s) = \eta(w_J, s)$. Similarly, $u^J \in W^J$ and $s \in W_J$, therefore

$$\lg(u^J s(u^J)^{-1} u^J) = \lg(u^J s) = \lg(u^J) + \lg(s) > \lg(u^J)\,,$$

which means that $\eta(u^J, u^J s(u^J)^{-1}) = 1$. By (2.5), this completes the proof. □

Lemma 2.3.6. *Let $w \in W$. Then Φ_w is in the W-orbit of Φ_J for some subset J of I,* i.e. *there exists $u \in W$ and $J \subseteq I$ such that $\Phi_w = u(\Phi_J)$.*

Proof. Given any fixed basis J' of Φ_w, there is a basis I' of Φ containing J' (see Proposition 24 in §1.8 of [Bou02, Chapter VI]). The sets I and I' are in the same W-orbit (see Remark 4 in §1.5 of *loc. cit.*), so we get $u \in W$ such that $I' = u(I)$ and $J := u^{-1}(J)$ is the subset of I we are looking for. □

The following proposition provides a direct way of finding if a given pair is good or bad.

Proposition 2.3.7. *Let $(w_1, w_2) \in W^2$. Let $\Phi_{w_1 w_2^{-1}} \subseteq \Phi$ be the minimal generating subsystem of $w_1 w_2^{-1}$, and let $J \subseteq I$ be such that $\Phi_{w_1 w_2^{-1}}$ is in the W-orbit of the root subsystem $\Phi_J \subseteq \Phi$ spanned by J (see Lemma 2.3.6). Then we can write*

$$w_1 = u^J w_{J,1}(v^J)^{-1}\,, \quad w_2 = u^J w_{J,2}(v^J)^{-1}$$

for some $u^J, v^J \in W^J$ and $w_{J,1}, w_{J,2} \in W_J$. Such a decomposition of w_1 and w_2, as well as the choice of J, may not be unique.

Furthermore, for one such (or equivalently any such) decomposition of w_1 and w_2, we have $w_{J,1} \preceq w_{J,2}$ if and only if the pair (w_1, w_2) is good.

Proof. Take $u \in W$ such that $\Phi_{w_1 w_2^{-1}} = u(\Phi_J)$. Let $u^J \in W^J$ be the representative of uW_J, then $\Phi_{w_1 w_2^{-1}} = u^J(\Phi_J)$. We know from Proposition 2.1.2 that $w_1 w_2^{-1} \in W(\Phi_{w_1 w_2^{-1}}) = u^J W_J (u^J)^{-1}$. Therefore the cosets $w_1^{-1} u^J W_J$ and $w_2^{-1} u^J W_J$ are the same. Let $v^J \in W^J$ be the representative for that coset; we then have elements $w_{J,1}$ and $w_{J,2}$ of W_J such that $w_1^{-1} u^J = v^J w_{J,1}^{-1}$ and $w_2^{-1} u^J = v^J w_{J,2}^{-1}$. Rewriting this, we get the decompositions

$$w_1 = u^J w_{J,1}(v^J)^{-1}\,, \quad w_2 = u^J w_{J,2}(v^J)^{-1}\,.$$

Now assume that $w_{J,1} \preceq w_{J,2}$. Then, by the chain property in W_J (see [BB06, Theorem 2.2.6]) there is a sequence $(\alpha_1, \ldots, \alpha_r)$ of roots in Φ_J such that $w_{J,2} = s_{\alpha_r} s_{\alpha_{r-1}} \ldots s_{\alpha_1} w_{J,1}$ and such that for each $1 \leq i \leq r$, the relation $s_{\alpha_i} \ldots s_{\alpha_1} w_1 \succ s_{\alpha_{i-1}} \ldots s_{\alpha_1} w_{J,1}$ holds. Now, let

$$\beta_i = u^J(\alpha_i) \in u^J(\Phi_J) = \Phi_{w_1 w_2^{-1}}$$

for each i. Then $s_{\beta_i} = u^J s_{\alpha_i} (u^J)^{-1}$ for each i, we have

$$w_2 = u^J w_{J,2} (v^J)^{-1} = s_{\beta_r} \ldots s_{\beta_1} u^J w_{J,1} (v^J)^{-1} = s_{\beta_r} \ldots s_{\beta_1} w_1 \,,$$

and for each i the relation

$$s_{\beta_i} \ldots s_{\beta_1} w_1 = u^J s_{\alpha_i} \ldots s_{\alpha_1} w_{J,1} (v^J)^{-1} \succ u^J s_{\alpha_{i-1}} \ldots s_{\alpha_1} w_{J,1} (v^J)^{-1} = s_{\beta_{i-1}} \ldots s_{\beta_1} w_1$$

holds as a consequence of Lemma 2.3.5. Since this chain implies in particular that $w_1 \preceq w_2$, we have shown all the conditions for the pair (w_1, w_2) to be good. Conversely, going through all the steps backwards and using Lemma 2.3.5, we see that (w_1, w_2) being a good pair implies that $w_{J,1} \preceq w_{J,2}$. □

2.4 Systems of type A

In this part, we provide a concrete description of the notions of minimal generating subsystems (Proposition 2.4.1) and good pairs (Proposition 2.4.5 and Corollary 2.4.7) for root systems of type A.

Let $n > 1$ be an integer and $\mathcal{S}_n$ be the symmetric group, *i.e.* the group of bijections from $\{1, \ldots, n\}$ to itself; its elements are called *permutations*. We will sometimes use the *line* (or *bracket*) notation for permutations: we write $w \in \mathcal{S}_n$ as $[w(1), \ldots, w(n)]$. We will also use the *cycle* notation: for a subset $A = \{a_1, \ldots, a_k\}$ of $\{1, \ldots, n\}$, we define $(a_1, \ldots, a_k)$ to be the permutation that takes any a_i to a_{i+1} (with the convention $a_{k+1} = a_1$), and fixes the other values. The commas will be dropped where there is no ambiguity.

In this part, Φ is a root system of type A_{n-1} with $n > 1$. Let $(e_i)_{1 \leq i \leq n}$ be the standard basis of $\mathbb{Q}^n$. We can assume that $\Phi = \{e_i - e_j \mid 1 \leq i, j \leq n \,,\ i \neq j\} \subset \mathbb{Q}^n$. The Weyl group W is $\mathcal{S}_n$, and its action on Φ is induced by the embedding $\mathcal{S}_n \longrightarrow \mathrm{GL}(\mathbb{Q}^n)$ that sets $w(e_i) = e_{w(i)}$ for $w \in \mathcal{S}_n$ and $1 \leq i \leq n$. Given a root $\alpha = e_i - e_j$ with $i \neq j$, the associated reflection is $s_\alpha = (i, j)$.

We set the standard basis $I \coloneqq \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$. It generates the set $\Phi_+ \coloneqq \{e_i - e_j \mid 1 \leq i < j \leq n\}$ of positive roots, and the set of simple reflections is $S = \{s_i = (i, i+1) \mid 1 \leq i \leq n-1\}$.

Proposition 2.4.1. *Let $w \in \mathcal{S}_n$. Write $\coprod_{1 \leq k \leq r} \Omega_k$ for the partition of $\{1, \ldots, n\}$*

into orbits under the action of w. For $1 \leq k \leq r$, let

$$f_k \colon \begin{array}{ccc} \mathbb{Q}^n & \longrightarrow & \mathbb{Q} \\ (x_i)_{1\leq i\leq n} & \longmapsto & \displaystyle\sum_{i\in\Omega_k} x_i \end{array}.$$

Then $E_w = \bigcap_{1\leq k\leq r} \ker(f_k)$ and $\Phi_w = \bigcup_{1\leq k\leq r} \{e_i - e_j \mid i,j \in \Omega_k\ ,\ i \neq j\}$.

Proof. For $1 \leq i, j \leq n$ with $i \neq j$, we have

$$w(e_i - e_j) - (e_i - e_j) = (e_{w(i)} - e_i) - (e_{w(j)} - e_j) \in \bigcap_{1\leq k\leq r} \ker(f_k)\,.$$

This proves $E_w \subseteq \bigcap_{1\leq k\leq r} \ker(f_k)$.

Conversely, since Φ is the root system of (GL_n, T) where T is the group of diagonal matrices, we can use Proposition 2.2.2. This gives

$$\Phi_w = \Phi(C_{\mathrm{GL}_n}((T^w)^\circ)) = \bigcup_{1\leq k\leq r} \{e_i - e_j \mid i,j \in \Omega_k\ ,\ i \neq j\}\ .$$

Since $\Phi_w \subset E_w$ and $\bigcup_{1\leq k\leq r} \{e_i - e_j \mid i,j \in \Omega_k\ ,\ i \neq j\}$ generates $\bigcap_{1\leq k\leq r} \ker(f_k)$, we get $\bigcap_{1\leq k\leq r} \ker(f_k) \subseteq E_w$. $\square$

Definition 2.4.2. Let d be an integer such that $1 \leq d \leq n$. We put a partial order on the set of size d subsets of $\{1, \ldots, n\}$ as follows: for any such subsets A and A', we say that $A \preceq A'$ if and only if

$$|A \cap \{1, \ldots, m\}| \geq |A' \cap \{1, \ldots, m\}|$$

for all $1 \leq m \leq n$. Equivalently, $A \preceq A'$ if and only if

$$|A \cap \{m, \ldots, n\}| \leq |A' \cap \{m, \ldots, n\}|$$

for all $1 \leq m \leq n$.

The relation $A \preceq A'$ can be seen in more concrete terms in the following way: if we write $A = \{a_1 < \ldots < a_d\}$ and $A' = \{a'_1 < \ldots < a'_d\}$, then $A \preceq A'$ if and only if $a_k \leq a'_k$ for all $k \in \{1, \ldots, d\}$.

Definition 2.4.3. Let $w \in \mathcal{S}_n$. For $1 \leq i, j \leq n$, we write

$$w[i,j] \coloneqq |w(\{1, \ldots, i\}) \cap \{j, \ldots, n\}|\ .$$

Additionally, for a subset Σ of $\{1, \ldots, n\}$, we write

$$w[i,j]_\Sigma \coloneqq |w(\{1, \ldots, i\}) \cap \{j, \ldots, n\} \cap \Sigma|\ .$$

Informally, if w is plotted on a graph as a function from $\{1, \ldots, n\}$ to itself, then

$w[i,j]_\Sigma$ counts the number of points $(k, w(k))$ that lie to the northwest of the point (i,j), with $w(k) \in \Sigma$.

We will also sometimes use a variant of this notation. Consider that the point (i,j) partitions the grid $\{1,\dots,n\} \times \{1,\dots,n\}$ in four quadrants (with (i,j) *inside* the northeast quadrant). Then, for any direction $* \in \{\mathrm{NW}, \mathrm{SW}, \mathrm{SE}, \mathrm{NE}\}$, we denote by $w[i,j]_{\Sigma,*}$ the number of points $(k, w(k))$ that lie in the quadrant $*$, with $w(k) \in \Sigma$. For instance, $w[i,j]_{\Sigma,\mathrm{NE}} \coloneqq w[i,j]_\Sigma$ and

$$w[i,j]_{\Sigma,\mathrm{SE}} \coloneqq |w(\{i+1,\dots,n\}) \cap \{1,\dots,j-1\} \cap \Sigma| \ .$$

Example 2.4.4. A classic criterion of the Bruhat order on $\mathcal{S}_n$ is the following: for any $w, w' \in \mathcal{S}_n$, we have $w' \preceq w$ if and only if $w'[i,j] \leq w[i,j]$ for all $1 \leq i,j \leq n$ (see for example [BB06, Theorem 2.1.5]). Equivalently, $w' \preceq w$ if and only if $\{w(1),\dots,w(d)\} \preceq \{w'(1),\dots,w'(d)\}$ for all $1 \leq d \leq n$.

The following proposition is an application of Proposition 2.3.7 to the type A_{n-1}.

Proposition 2.4.5. *Let w_1, w_2 be elements of $\mathcal{S}_n$. Then the pair (w_1, w_2) is good if and only if, for any orbit Ω of $w_1 w_2^{-1}$ under the action of $\mathcal{S}_n$ on $\{1,\dots,n\}$ and any $1 \leq d \leq n$, we have $\Omega \cap w_1(\{1,\dots,d\}) \succeq \Omega \cap w_2(\{1,\dots,d\})$ (as subsets of $\{1,\dots,n\}$ in the sense of Definition 2.4.2).*

Equivalently, the pair (w_1, w_2) is good if and only if, for any orbit Ω of $w_1 w_2^{-1}$ and any $1 \leq i,j \leq n$, we have $w_1[i,j]_\Omega \leq w_2[i,j]_\Omega$.

Proof. Recall that $S = \{s_i = (i, i+1) \mid 1 \leq i \leq n-1\}$ is the standard set of simple reflections that generates $\mathcal{S}_n$. Let r be a positive integer, and let

$$\{1,\dots,n\} = \coprod_{1 \leq k \leq r} \Sigma_k$$

be a partition of $\{1,\dots,n\}$ with the property that for any $1 \leq k \leq r-1$ and any $x \in \Sigma_k, y \in \Sigma_{k+1}$, we have $x < y$. We say that a partition with such a property is ordered. To an ordered partition, we associate the subset $S_J \subset S$ of those s_i such that there exists a $k \in \{1,\dots,r\}$ with $i, i+1 \in \Sigma_k$. This defines a bijection between the ordered partitions of indefinite size and the subsets of S, or equivalently the sub-bases J of I.

We thus identify those sub-bases of roots J with ordered partitions $(\Sigma_1,\dots,\Sigma_r)$. Under this identification, it is easy to see that W_J is the set of permutations that keeps the partition stable, that is to say

$$W_J = \{w \in \mathcal{S}_n \mid \forall\, 1 \leq k \leq r,\ w(\Sigma_k) = \Sigma_k\} \ ,$$

and W^J is the set of permutations that are increasing on the partition, that is to say

$$W^J = \{w \in \mathcal{S}_n \mid \forall\, 1 \leq k \leq r, x, y \in \Sigma_k \text{ with } x < y \implies w(x) < w(y)\} \ . \tag{2.6}$$

Now decompose the permutations $w_1, w_2 \in W$ as in Proposition 2.3.7:

$$w_1 = u^J w_{J,1} (v^J)^{-1}, \quad w_2 = u^J w_{J,2} (v^J)^{-1}$$

for some J corresponding to $(\Sigma_1, \ldots, \Sigma_r)$. Then, by Example 2.4.4 (w_1, w_2) is a good pair if and only if, for any $1 \leq d \leq n$, we have $\{w_{J,1}(1), \ldots, w_{J,1}(d)\} \succeq \{w_{J,2}(1), \ldots, w_{J,2}(d)\}$, where the order $\succeq$ is the one of Definition 2.4.2. Since $w_{J,1}, w_{J,2} \in W_J$, this is equivalent to checking that for all $1 \leq k \leq r$, we have

$$w_{J,1}(\Sigma_k \cap \{1, \ldots, d\}) \succeq w_{J,2}(\Sigma_k \cap \{1, \ldots, d\}) \tag{2.7}$$

for all $1 \leq d \leq n$. Since $v^J \in W^J$, by (2.6), the family of $v^J(\Sigma_k \cap \{1, \ldots, d\})$ when d varies from 1 to n is the same as the family of $v^J(\Sigma_k) \cap \{1, \ldots, d\}$. Applying $(v^J)^{-1}$, we get that the family $\{\Sigma_k \cap \{1, \ldots, d\}\}$ is the same as the family $\{\Sigma_k \cap (v^J)^{-1}(\{1, \ldots, d\})\}$. Hence, because $w_{J,1}(\Sigma_k) = w_{J,2}(\Sigma_k)$, the condition (2.7) can be formulated as

$$\Sigma_k \cap w_{J,1}(v^J)^{-1}(\{1, \ldots, d\}) \succeq \Sigma_k \cap w_{J,2}(v^J)^{-1}(\{1, \ldots, d\}) \quad \forall\, 1 \leq d \leq n\,. \tag{2.8}$$

Now, for any $1 \leq k \leq r$, write $\Omega_k = u^J(\Sigma_k)$. Since $u^J \in W^J$, we see by (2.6) that (2.8) is equivalent to

$$\Omega_k \cap w_1(\{1, \ldots, d\}) \succeq \Omega_k \cap w_2(\{1, \ldots, d\}) \quad \forall\, 1 \leq d \leq n\,.$$

Finally, when J is as in Proposition 2.3.7, the Ω_k are the orbits of the action of $w_1 w_2^{-1}$ on $\{1, \ldots, n\}$. This completes the proof of the first criterion.

From this, it is straightforward to deduce the formulation of the criterion as $w_1[i,j]_\Omega \leq w_2[i,j]_\Omega$, by laying out the definition of $\Omega_k \cap w_1(\{1, \ldots, d\}) \succeq \Omega_k \cap w_2(\{1, \ldots, d\})$. $\square$

This result gives a way to recognise good and bad pairs visually through flattenings, which we now define.

Definition 2.4.6. Let m, n be integers with $0 < m \leq n$, and let $\Sigma = \{i_1 < \ldots < i_m\}$ be a subset of $\{1, \ldots, n\}$ of size m. We define the *flattening map* $\mathrm{fl}_\Sigma \colon \mathcal{S}_n \longrightarrow \mathcal{S}_m$ as follows. For any $w \in \mathcal{S}_n$, we take $\mathrm{fl}_\Sigma(w)$ as the unique $f \in \mathcal{S}_m$ such that $(w(i_1)\ldots, w(i_m))$ is in the same relative order as $(f(1), \ldots, f(m))$.

This means that $w \in \mathcal{S}_n$ flattens to $f \in \mathcal{S}_m$ through Σ if and only if for any $1 \leq k < l \leq m$, we have $w(i_k) < w(i_l)$ if and only if $f(k) < f(l)$.

Corollary 2.4.7. *Let $w_1, w_2 \in \mathcal{S}_n$. Then the pair (w_1, w_2) is good if and only if for any orbit Ω of $w_1^{-1} w_2$, we have $\mathrm{fl}_\Omega(w_1) \preceq \mathrm{fl}_\Omega(w_2)$.*

Proof. For any $w \in \mathcal{S}_n$, one can read the plot graph of $\mathrm{fl}_\Omega(w)$ on the intersection of $\Omega \times w(\Omega)$ with the plot graph of w. Therefore, since $w_1(\Omega) = w_2(\Omega)$, the set of couples

$$(w_1[i,j]_{w_1(\Omega)}, w_2[i,j]_{w_2(\Omega)})$$

for $1 \leq i, j \leq n$ is the same as the set of couples

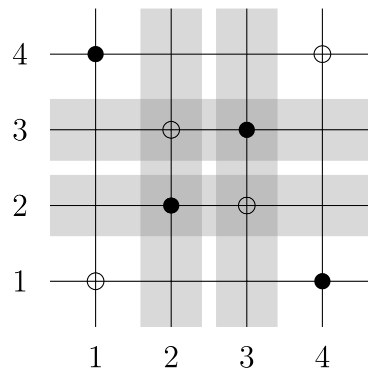
$$(\mathrm{fl}_\Omega(w_1)[i',j'], \mathrm{fl}_\Omega(w_2)[i',j'])$$

for $1 \leq i', j' \leq m$. Since w_1 (or w_2) maps the orbits of $w_1^{-1}w_2$ to the orbits of $w_1w_2^{-1}$, Proposition 2.4.5 means that the pair (w_1, w_2) is good if and only if $\mathrm{fl}_\Omega(w_1)[i',j'] \leq \mathrm{fl}_\Omega(w_2)[i',j'])$ for all $1 \leq i', j' \leq n$. This is the criterion for $\mathrm{fl}_\Omega(w_1) \preceq \mathrm{fl}_\Omega(w_2)$ (see Example 2.4.4). $\square$

2.5 Examples of bad pairs

We provide in this part a few examples of bad pairs that carry a particular interest. For each of these examples (w_1, w_2), we plot in white (resp. black) the graph $j = w_1(i)$ (resp. $j = w_2(i)$) on a grid with horizontal i-axis and vertical j-axis; we also highlight the subgraph that shows a bad flattening of the pair (given by Corollary 2.4.7).

1. $w_1 = [1324]$ and $w_2 = [4231]$ in $\mathcal{S}_4$.



The orbits of $w_1w_2^{-1} = (14)(23)$ are $\{1,4\}$ and $\{2,3\}$, hence the minimal generating system of $w_1w_2^{-1}$ is $\Phi_{w_1w_2^{-1}} = \{\pm(e_1 - e_4), \pm(e_2 - e_3)\}$ with Weyl group $W(\Phi_{w_1w_2^{-1}}) = \{\mathrm{id}, (14), (23), (14)(23)\}$.

The chain $w_1 = [1324] \prec [1423] \prec [4123] \prec [4213] \prec [4231] = w_2$ shows that $w_1 \prec w_2$. Note that in the first step of the chain, $[1423] = s_{e_3-e_4}[1324]$ and $e_3 - e_4 \notin \Phi_{w_1w_2^{-1}}$; so this chain does not imply that (w_1, w_2) is good.

On the contrary, we can see with Proposition 2.3.7 that (w_1, w_2) is bad. Take the sub-basis $J = \{e_1 - e_2, e_3 - e_4\}$ of I. Then $W_J = \{\mathrm{id}, (12), (34), (12)(34)\}$ and $W^J = \{[1234], [1324], [1423], [2314], [2413], [3412]\}$. Setting $u^J = [1423]$, we see that $\Phi_{w_1w_2^{-1}} = u^J(\Phi_J)$; setting also $v^J = [1423]$, we then have

$$w_1 = u^J(34)(v^J)^{-1}, \quad w_2 = u^J(12)(v^J)^{-1}.$$

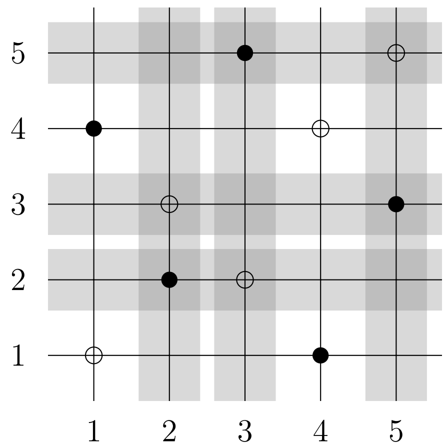
Then (w_1, w_2) is a bad pair because $(34) \not\preceq (12)$

This can also be seen with the flattening criterion of Corollary 2.4.7. Taking the orbit $\Omega = \{2,3\}$ of $w_1^{-1}w_2 = (14)(23)$, we have

$$\mathrm{fl}_\Omega(w_1) = \mathrm{fl}_{\{2,3\}}([1324]) = [21], \quad \mathrm{fl}_\Omega(w_2) = \mathrm{fl}_{\{2,3\}}([4231]) = [12]$$

Then (w_1, w_2) is a bad pair because $[21] \not\preceq [12]$.

2. $w_1 = [13245]$ and $w_2 = [42513]$ in $\mathcal{S}_5$.



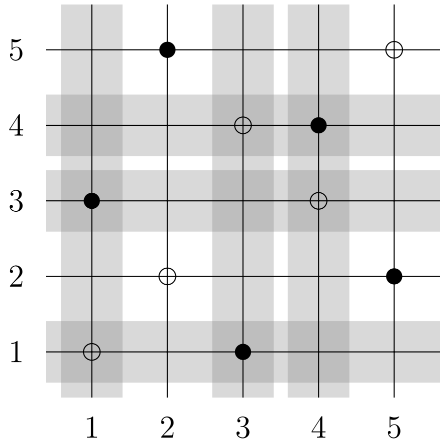
Consider the orthogonal embedding $\mathbb{Q}^4 \hookrightarrow \mathbb{Q}^5$ given by $e_i \longmapsto e_i$; it sends a basis of A_3 to a sub-basis of A_4, therefore induces an injective group morphism $\mathcal{S}_4 = W(A_3) \hookrightarrow \mathcal{S}_5 = W(A_4)$ that preserves the Bruhat order. This morphism sends $[1324]$ to $[13245] = w_1$ and $[4231]$ to $[42315]$, hence $w_1 \prec [42315]$ by the previous example. Since $[42315] \prec [42513] = w_2$, we get $w_1 \prec w_2$.

The orbits of $w_1^{-1}w_2 = (14)(235)$ are $\{1,4\}$ and $\{2,3,5\}$. Flattening through $\{2,3,5\}$ yields

$$\mathrm{fl}_{\{2,3,5\}}(w_1) = [213], \quad \mathrm{fl}_{\{2,3,5\}}(w_2) = [132].$$

Seeing that $[213] \not\preceq [132]$ (both are of length 1), the pair (w_1, w_2) is bad.

3. $w_1 = [12435]$ and $w_2 = [35142]$ in $\mathcal{S}_5$.



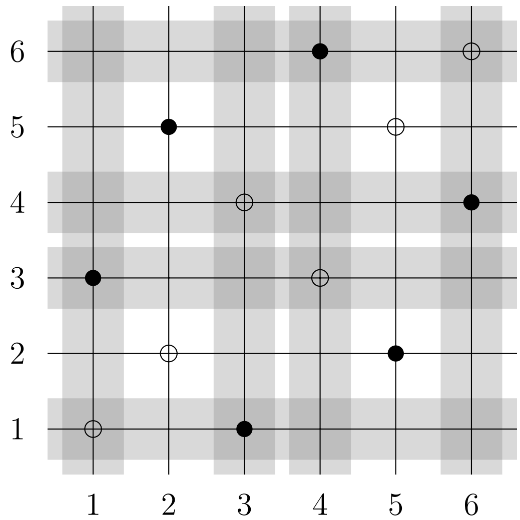
As with the previous example, the embedding $\mathbb{Q}^4 \hookrightarrow \mathbb{Q}^5$ given by $e_i \longmapsto e_{i+1}$ induces an injective group morphism $\mathcal{S}_4 \hookrightarrow \mathcal{S}_5$ that preserves the Bruhat order. This morphism sends [1324] to $[12435] = w_1$ and [4231] to [15342], hence $w_1 \prec [15342]$ by the first example. Since $[15342] \prec [35142] = w_2$, we get $w_1 \prec w_2$.

The orbits of $w_1^{-1}w_2 = (143)(25)$ are $\{1,3,4\}$ and $\{2,5\}$. Flattening through $\{1,3,4\}$ yields

$$\mathrm{fl}_{\{1,3,4\}}(w_1) = [132]\,, \quad \mathrm{fl}_{\{1,3,4\}}(w_2) = [213]\,.$$

Seeing that $[132] \not\preceq [213]$, the pair (w_1, w_2) is bad.

4. $w_1 = [124356]$ and $w_2 = [351624]$ in $\mathcal{S}_6$.



As previously, the embedding $\mathbb{Q}^4 \hookrightarrow \mathbb{Q}^6$ given by $e_i \longmapsto e_{i+1}$ induces a morphism $\mathcal{S}_4 \hookrightarrow \mathcal{S}_5$ that sends the relation $[1324] \prec [4231]$ to $w_1 = [124356] \prec [153426]$. Since $[153426] \prec [351426] \prec [351624] = w_2$, we get $w_1 \prec w_2$.

The orbits of $w_1^{-1}w_2 = (1463)(25)$ are $\{1,3,4,6\}$ and $\{2,5\}$. Flattening through $\{1,3,4,6\}$ yields

$$\mathrm{fl}_{\{1,3,4,6\}}(w_1) = [1324]\,, \quad \mathrm{fl}_{\{1,3,4,6\}}(w_2) = [2143]\,.$$

Seeing that $[1324] \not\preceq [2143]$ (because the reduced word decomposition $[2143] = s_1s_3$ does not contain a reduced word for $[1324] = s_2$), the pair (w_1, w_2) is bad.

5. $w_1 = [\dots a \dots c \dots b \dots d \dots]$ and $w_2 = [\dots d \dots b \dots c \dots a \dots]$ in $\mathcal{S}_n$ for some $1 \leq a < b < c < d \leq n$, where the dots are the same for w_1 and w_2. More precisely, we assume that there exist $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ such that $w_1(i_1, i_2, i_3, i_4) = (a, c, b, d)$ and $w_2(i_1, i_2, i_3, i_4) = (d, b, c, a)$, and that $w_1(i) = w_2(i)$ for $i \notin \{i_1, i_2, i_3, i_4\}$.

   In example 1, multiplying [1324] to the left by $s_{e_3-e_4}$, $s_{e_1-e_4}$, $s_{e_1-e_2}$ and $s_{e_1-e_3}$ successively gave an ascending chain from [1324] to [4231]. Similarly here, multiplying w_1 to the left by $s_{e_c-e_d}$, $s_{e_a-e_d}$, $s_{e_a-e_b}$ and $s_{e_a-e_c}$ successively gives an ascending chain from w_1 to w_2, so $w_1 \prec w_2$. Now $w_1^{-1}w_2$ has orbits $\{i_1, i_4\}$ and $\{i_2, i_3\}$. The flattenings of w_1 and w_2 through $\{i_2, i_3\}$ are [21] and [12] respectively, hence (w_1, w_2) is a bad pair.

6. Generalising the previous example, we see that given $w_1, w_2 \in \mathcal{S}_n$, if we set $\Sigma = \{i \mid w_1(i) \neq w_2(i)\}$, then (w_1, w_2) is good (resp. bad) if and only if $(\mathrm{fl}_\Sigma(w_1), \mathrm{fl}_\Sigma(w_2))$ is bad.

2.6 Pattern avoidance

In this section, we give a criterion that relates good pairs to pattern avoidance for Coxeter groups of type A.

Definition 2.6.1. Let $m, n \leq 1$ be integers. For $w \in \mathcal{S}_n$ and $f \in \mathcal{S}_m$, we say that *w has the pattern f* if and only if w flattens to f (through some $\Sigma \subseteq \{1, \dots, n\}$ of size m), *i.e.* there exists a flattening map $\mathrm{fl}_\Sigma \colon \mathcal{S}_n \longrightarrow \mathcal{S}_m$ (see Definition 2.4.6) such that $f = \mathrm{fl}_\Sigma(w)$.

Example 2.6.2. If $f = [3412] \in \mathcal{S}_4$ (resp. $f = [4231]$), the permutation $w \in \mathcal{S}_n$ has the pattern f if and only if there exist $1 \leq a < b < c < d \leq n$ such that $w(c) < w(d) < w(a) < w(b)$ (resp. $w(d) < w(b) < w(c) < w(a)$).

For example, $w = (14)(35) = [42513] \in \mathcal{S}_5$, $w' = (13)(25) = [35142] \in \mathcal{S}_5$ and $w'' = (13)(25)(46) = [351624] \in \mathcal{S}_6$ have the pattern [3412] but don't have the pattern [4231].

Theorem 2.6.3. *Let $w \in \mathcal{S}_n$.*

(1) *There exists $w' \in \mathcal{S}_n$ such that (w', w) is a bad pair if and only if w has at least one of the four patterns* [4231], [42513], [35142] *or* [351624].

(2) *There exists* $w'' \in \mathcal{S}_n$ *such that* (w, w'') *is a bad pair if and only if* w *has at least one of the four patterns* [1324], [24153], [31524] *or* [426153].

Proof. The second statement is equivalent to the first by applying Proposition 2.3.3. Also, if w has one of the four patterns [4231], [42513], [35142] and [351624], then the examples of section 2.5 provide a w' such that (w', w) is a bad pair. We thus only have to prove that for any bad pair (w', w), the permutation w has one of the four patterns.

Let (w', w) be such a bad pair. From Corollary 2.4.7, there exist an orbit Ω of ww'^{-1} and $(i, j) \in \{1, \dots, n\}^2$ such that $w[i, j]_\Omega < w'[i, j]_\Omega$; we fix one such Ω for the rest of the proof, and set $\Sigma = \{1, \dots, n\} \setminus \Omega$. The strategy will be to find the patterns on the plot graph of w_1 using this fact. The different elements of the proof are represented in Figure 2.1.

The first step is to divide the graph into a few regions. First consider i_1 minimal with the property that there exists some j such that $w[i_1, j]_\Omega < w'[i_1, j]_\Omega$ (its existence is guaranteed by Corollary 2.4.7). Then take j_1 maximal such that $w[i_1, j_1]_\Omega < w'[i_1, j_1]_\Omega$, and j_2 minimal such that $w[i_1, j_2]_\Omega < w'[i_1, j_2]_\Omega$. Finally take i_2 maximal such that $w[i_2, j_2]_\Omega < w'[i_2, j_2]_\Omega$. We know that $w(i_1) \in \Omega$, or equivalently $w'(i_1) \in \Omega$ (because Ω is an orbit of $ww'{-}1$). Indeed if it were not, then we would have $w[i_1 - 1, j_1]_\Omega = w[i_1, j_1]_\Omega$ and $w'[i_1 - 1, j_1]_\Omega = w'[i_1, j_1]_\Omega$, and this would contradict the minimality of i_1. By similar arguments, $w(i_2 + 1) \in \Omega$ and $j_1, j_2 - 1 \in \Omega$.

The second step will be to identify up to six points of interest on the graph of w:

A. Since $w[i_1, j_1] = w[i_1, j_1]_\Omega + w[i_1, j_1]_\Sigma$ and similarly with w', we know that

$$w[i_1, j_1]_\Sigma > w'[i_1, j_1]_\Sigma\,; \tag{2.9}$$

in particular $w[i_1, j_1]_\Sigma \geq 1$. There must then exist a point $A = (i_A, j_A) \in w^{-1}(\Sigma) \times \Sigma$ with $i_A \leq i_1$ and $j_A \geq j_1$ on the graph of w, *i.e.* such that $j_A = w(i_A)$. Since $(i_1, j_1) \in w^{-1}(\Omega) \times \Omega$, we must have $i_A < i_1$ and $j_A > j_1$.

B. Let $B = (i_B, j_B)$ be defined by $i_B = w^{-1}(j_2 - 1)$ and $j_B = j_2 - 1$. Then $B \in w^{-1}(\Omega) \times \Omega$. Also, we have $i_B \leq i_1$: the contrary would mean that

$$w[i_1, j_2 - 1]_\Omega = w[i_1, j_2]_\Omega < w'[i_1, j_2]_\Omega \leq w'[i_1, j_2 - 1]_\Omega$$

which contradicts the minimality of j_2.

C. Let $C = (i_C, j_C)$ be defined by $i_C = i_1$ and $j_C = w(i_1)$. Then $C \in w^{-1}(\Omega) \times \Omega$. Also, we have $j_C < j_2$: the contrary would mean that

$$w[i_1 - 1, j_2]_\Omega = w[i_1, j_2]_\Omega - 1 < w'[i_1, j_2]_\Omega - 1 \leq w'[i_1 - 1, j_2]_\Omega$$

which contradicts the minimality of i_1.

D. Let $D = (i_D, j_D)$ be defined by $i_D = i_2 + 1$ and $j_D = w(i_2 + 1)$. Then $D \in w^{-1}(\Omega) \times \Omega$. Also, we have $j_D \geq j_2$: the contrary would mean that

$$w[i_2 + 1, j_2]_\Omega = w[i_2, j_2]_\Omega < w'[i_2, j_2]_\Omega \leq w'[i_2 + 1, j_2]_\Omega$$

which contradicts the maximality of i_2.

E. Let $E = (i_E, j_E)$ be defined by $i_E = w^{-1}(j_1)$ and $j_E = j_1$. Then $E \in w^{-1}(\Omega) \times \Omega$. Also, we have $i_E > i_1$: the contrary would mean that

$$w[i_1, j_1 + 1]_\Omega = w[i_1, j_1]_\Omega - 1 < w'[i_1, j_1]_\Omega - 1 \leq w'[i_1, j_1 + 1]_\Omega$$

which contradicts the maximality of j_1.

F. By the same argument that led to (2.9), we have

$$w[i_2, j_2]_\Sigma > w'[i_2, j_2]_\Sigma \,. \tag{2.10}$$

Now, remark that $w[i_2, j_2]_\Sigma + w[i_2, j_2]_{\Sigma,\mathrm{SW}}$ is the number of points $(i, w(i))$ with $1 \leq i \leq i_2$; that is precisely i_2. Similarly, $w[i_2, j_2]_{\Sigma,\mathrm{SW}} + w[i_2, j_2]_{\Sigma,\text{ffl}}$ is the number of points $(i, w(i))$ with $1 \leq w(i) < j_2$; that is precisely $j_2 - 1$. Put together, we get

$$w[i_2, j_2]_\Sigma - w[i_2, j_2]_{\Sigma,\mathrm{SE}} = i_2 - j_2 + 1$$

and similarly for w',

$$w'[i_2, j_2]_\Sigma - w'[i_2, j_2]_{\Sigma,\mathrm{SE}} = i_2 - j_2 + 1 \,.$$

Taking the difference of the latter two equations gives

$$w[i_2, j_2]_\Sigma - w'[i_2, j_2]_\Sigma = w[i_2, j_2]_{\Sigma,\mathrm{SE}} - w'[i_2, j_2]_{\Sigma,\mathrm{SE}} \,. \tag{2.11}$$

Finally, (2.10) and (2.11) together give

$$w[i_2, j_2]_{\Sigma,\mathrm{SE}} > w'[i_2, j_2]_{\Sigma,\mathrm{SE}} \,;$$

in particular $w[i_2, j_2]_{\Sigma,\mathrm{SE}} \geq 1$. There must then exist a point $F = (i_F, j_F) \in w^{-1}(\Sigma) \times \Sigma$ with $i_F > i_2$ and $j_F < j_2$ on the graph of w, *i.e.* such that $j_F = w(i_F)$. Since $(i_2 + 1, j_2 + 1) \in w^{-1}(\Omega) \times \Omega$, we must have $i_F > i_2 + 1$ and $j_F > j_2 + 1$.

Note that the sets of points $\{A\}$, $\{B, C\}$, $\{D, E\}$, $\{F\}$ lie in distinct quadrants delimited by i_1 and j_2. Hence $B = C$ and $D = E$ are the only equalities that may happen between those six points. Also note that since those points are all on the graph of w, unless two of those are equal, both of their coordinates are distinct (for example $i_A \neq i_B$ and $j_A \neq j_B$).

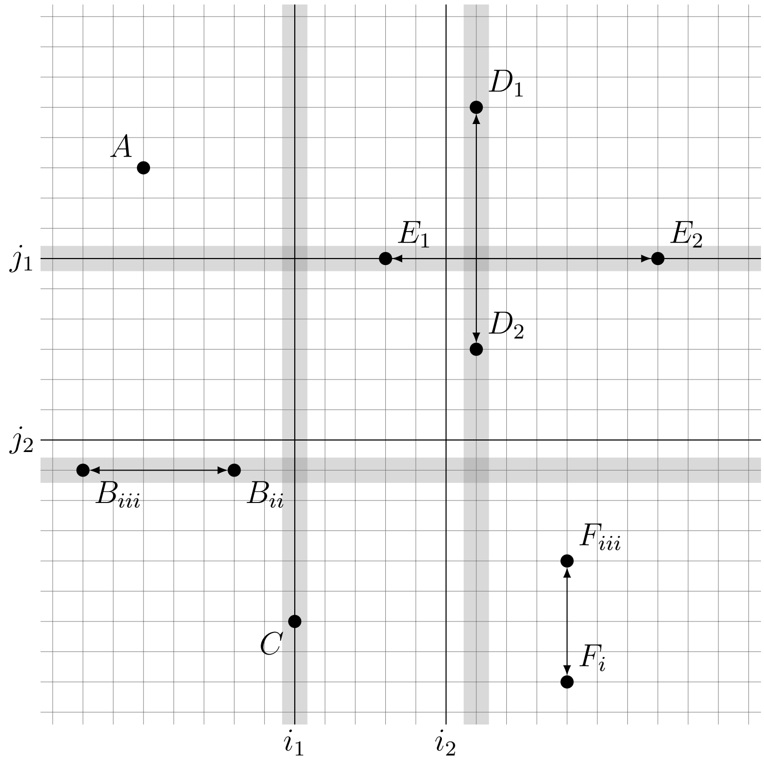


Figure 2.1 – The black dots are on the graph $j = w(i)$; different possibilities for the position of a given point are connected by an arrow. Gray bands are either of i-coordinate in $w^{-1}(\Omega)$ or of j-coordinate in Ω.

The third and final step is to distinguish cases according to the relative positions of the points.

Case 1: $j_D > j_A$ and $i_E > i_F$. Necessarily $D \neq E$ since $j_D > j_A = j_E$.

Case 1.i: $j_F < j_C$. $ACDFE$ gives the pattern [42513]; that is to say, w flattens to [42513] through $\{i_A, i_C, i_D, i_F, i_E\}$.

Case 1.ii: $i_B > i_A$. $ABDFE$ gives the pattern [42513].

Case 1.iii: $j_F > j_C$ and $i_B < i_A$. In this case $B \neq C$ because $i_B < i_A < i_1 = i_C$, and $BACDFE$ gives the pattern [351624].

Case 2: $j_D < j_A$ or $i_E < i_F$. Then, D in the case $j_D < j_A$, or E in the case $i_E < i_F$, have the same relative position with respect to the other four points: that is to say, between C and F along the i-axis and between B and A along the j-axis. We can therefore just assume that $j_D < j_A$.

Case 2.i: $j_F < j_C$. $ACDF$ gives the pattern [4231].

Case 2.ii: $i_B > i_A$. $ABDF$ gives the pattern [4231].

Case 2.iii: $j_F > j_C$ and $i_B < i_A$. As in case 1.iii, we have $B \neq C$; and here $BACDF$ gives the pattern [35142]. □

Remark 2.6.4. Theorem 2.6.3 connects the study of bad pairs to the techniques associated to pattern avoidance. One of the key early results about pattern avoidance is the Lakshmibai-Sandhya theorem [LS90] about Schubert varieties. More will be said about Schubert varieties at the beginning of section 4.1, so we only go over them quickly for now. Let G be the reductive group SL_n over $\mathbb{C}$, of type A_{n-1}. Using the notation of section 2.2, B is the subgroup of upper triangular matrices, and each element of $W = \mathcal{S}_n$ can be lifted to G. The projective variety G/B admits a cell decomposition $G/B = \coprod_{w \in \mathcal{S}_n} BwB/B$. The Lakshmibai-Sandhya theorem states that the Zariksi closure of a cell BwB/B is smooth if and only if w avoids the patterns [3412] and [4231].

Corollary 2.6.5. *Let $w, w' \in \mathcal{S}_n$ such that (w', w) is a bad pair. Then, keeping the notation of Remark 2.6.4, the Schubert cell $\overline{BwB/B}$ is singular.*

Proof. Since the patterns [42513], [35142] and [351624] all contain the pattern [3412], this is a direct consequence of Theorem 2.6.3 and the Lakshmibai-Sandhya theorem [LS90, Theorem 1]. □

Remark 2.6.6. Given a bad pair (w', w) in $\mathcal{S}_n$, the point $w'B/B$ has no reason to be in the singular locus of $\overline{BwB/B}$. For instance, when $n = 4$, the pair ([1324], [4231]) is bad but $[1324]B/B$ is smooth in $\overline{B[4231]B/B}$; in fact the singular locus of $\overline{B[4231]B/B}$ is $\overline{B[2143]B/B}$ (see [LS90, §3]).

Chapter 3

Tangent spaces on the trianguline variety

Let p be a prime number and let K be a finite extension of $\mathbb{Q}_p$, equipped with the p-adic norm $|\cdot|_K$ normalised so that $|p|_K = p^{-[K:\mathbb{Q}_p]}$. Let $K^{\mathrm{sep}} = \overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$, equipped with the p-adic norm $|\cdot|$ such that $|p| = p^{-1}$, let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$. Let $\mathcal{G}_{\mathbb{Q}_p} \coloneqq \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathcal{G}_K \coloneqq \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Let $\chi\colon \mathcal{G}_{\mathbb{Q}_p} \twoheadrightarrow \mathbb{Z}_p^\times$ be the cyclotomic character on $\mathbb{Q}_p$, let $\chi_K = \chi|_{\mathcal{G}_K}$, $\mathcal{H}_K \coloneqq \ker \chi_K$ and $\Gamma_K \coloneqq \mathcal{G}_K/\mathcal{H}_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$. Let K_0 be the maximal unramified extension of $\mathbb{Q}_p$ inside K; there is a unique endomorphism φ of K_0, called the Frobenius endomorphism, which extends the morphism $x \longmapsto x^p$ on the residue field of K_0.

Let $\mathbf{Ar}_{\mathbb{Q}_p}$ be the category of local artinian $\mathbb{Q}_p$-algebras whose residue field is a finite extension of $\mathbb{Q}_p$ (*i.e.* finite-dimensional local $\mathbb{Q}_p$-algebras). Any object A of $\mathbf{Ar}_{\mathbb{Q}_p}$ is equipped with its Banach topology of $\mathbb{Q}_p$-algebra; it is induced by a norm, unique up to a constant factor, that we denote $|\cdot|_A$ or $|\cdot|$ when there is no ambiguity. When given an $A \in \mathbf{Ar}_{\mathbb{Q}_p}$, we usually write $\mathfrak{m}$ its maximal ideal and $L \coloneqq A/\mathfrak{m}$ its residue field. We often assume that L splits K, meaning that $|\mathrm{Hom}_{\mathbb{Q}_p}(K, L)| = [K : \mathbb{Q}_p]$; in this case we write $\Sigma \coloneqq \mathrm{Hom}_{\mathbb{Q}_p}(K, L)$.

3.1 Crystalline representations over a local artinian algebra

We review in this part the basics of p-adic Hodge theory, in particular crystalline and de Rham representations and some of their associated objects, such as the Hodge filtration and refinements. The theory is well-known; however we want it established for representations not only over $\mathbb{Q}_p$ but over a more general algebra $A \in \mathbf{Ar}_{\mathbb{Q}_p}$. Since proofs in this case are not always easily found in the literature, we provide most of them here.

B-admissibility in p-adic Hodge theory We recall Fontaine's formalism of B-admissible representations.

Let E be a topological field, let B be a topological E-algebra and let G be a topological group acting continuously and E-linearly on B in such a way that B is *G-regular* (see [Fon94b, §1.4] for the definition). In particular, the algebra of invariants $F := B^G$ is a field; we make the further assumption that F/E is a finite separable extension. (We will always take $E = \mathbb{Q}_p$ and $G = \mathcal{G}_K$ in the cases we consider.)

Given a finite-dimensional E-algebra A (with its natural topology), we call $\mathbf{Rep}_A(G)$ the category of free A-modules of finite type equipped with a continuous linear action of G, with morphisms being the G-equivariant A-linear maps.

Definition 3.1.1 (Fontaine)**.** For any $V \in \mathbf{Rep}_E(G)$, we write

$$\mathbf{D}_B(V) := (B \otimes_E V)^G \ ;$$

it is a vector space over F, with a functorial injective morphism of B-modules

$$\alpha_V \colon B \otimes_F \mathbf{D}_B(V) \hookrightarrow B \otimes_E V$$

(the injectivity of α_V is part of the condition of G-regularity of B).

When A is an E-algebra and $V \in \mathbf{Rep}_A(G)$, we can also write

$$\mathbf{D}_B(V) = \big((B \otimes_E A) \otimes_A V\big)^G$$

which is a module over $(B \otimes_E A)^G = (F \otimes_E A)$. We can also see

$$\alpha_V \colon (B \otimes_E A) \otimes_{(F \otimes_E A)} \mathbf{D}_B(V) \hookrightarrow (B \otimes_E A) \otimes_A V$$

as a morphism of $(B \otimes_E A)$-modules.

Definition 3.1.2 (Fontaine)**.** Let $A \in \mathbf{Ar}_E$. We say that $V \in \mathbf{Rep}_A(G)$ is *B-admissible* when the map α_V is an isomorphism.

We denote by $\mathbf{Rep}_{A,B}(G)$ the full subcategory of B-admissible representations. Note that a representation $V \in \mathbf{Rep}_A(G)$ is admissible if and only if it is admissible as a representation of $\mathbf{Rep}_E(G)$.

Lemma 3.1.3 (Fontaine)**.** *Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$. For any exact sequence*

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

in $\mathbf{Rep}_A(G)$, if $V \in \mathbf{Rep}_{A,B}(G)$, then $V', V'' \in \mathbf{Rep}_{A,B}(G)$ and the sequence of $(F \otimes_E A)$-modules

$$0 \longrightarrow \mathbf{D}_B(V') \longrightarrow \mathbf{D}_B(V) \longrightarrow \mathbf{D}_B(V'') \longrightarrow 0$$

is exact.

Proof. We reproduce the proof of [Fon94b, Proposition 1.5.2]. The exactness can be checked seeing $\mathbf{D}_B$ as a functor to vector spaces over E. The left-exactness of $\mathbf{D}_B$ is clear from the definition. Then

$$\dim_F \mathbf{D}_B(V) \leq \dim_F \mathbf{D}_B(V') + \dim_F \mathbf{D}_B(V'') \leq \dim_E V' + \dim_E V'' = \dim_E V$$

is a actually chain of equalities when V is B-admissible; then the first equality gives the exactness (already knowing left-exactness) and the second one gives the B-admissibility of V' and V''. □

Proposition 3.1.4. *Let $A \in \mathbf{Ar}_E$. Then $V \in \mathbf{Rep}_A(G)$ is B-admissible if and only if $\mathbf{D}_B(V)$ is free of rank $\mathrm{rk}_A(V)$ over $F \otimes_E A$.*

Proof. Write $n := \mathrm{rk}_A(V)$ and $d := \dim_E(A)$.

First assume that $\mathbf{D}_B(V)$ is free over $F \otimes_E A$ of rank n. Then $\dim_F(\mathbf{D}_B(V)) = nd = \dim_E(V)$, so V is B-admissible by [Fon94b, Proposition 1.4.2].

Conversely, let $V \in \mathbf{Rep}_A(G)$ be B-admissible. First assume the existence of a surjective morphism

$$p\colon (F \otimes_E A)^n \twoheadrightarrow \mathbf{D}_B(V)$$

of $(F \otimes_E A)$-modules. Then $B \otimes_F \mathbf{D}_B(V)$ is a free B-module of rank

$$\mathrm{rk}_B\left(B \otimes_F \mathbf{D}_B(V)\right) = \dim_F \mathbf{D}_B(V) \leq \dim_F\left((F \otimes_E A)^n\right) = nd\,. \tag{3.1}$$

Since $\alpha_V\colon B \otimes_F \mathbf{D}_B(V) \longrightarrow B \otimes_E V$ is an isomorphism and $B \otimes_E V$ is a free B-module of rank $\dim_E(V) = nd$, the inequality in (3.1) is an equality. Therefore p is bijective and $\mathbf{D}_B(V)$ is free of rank n over $F \otimes_E A$.

We thus only need to show that $\mathbf{D}_B(V)$ is generated, as an $(F \otimes_E A)$-module, by n elements. We first assume that $A = L$ is a field; enlarging it if necessary, we can assume that $|\mathrm{Hom}_E(F, L)| = [F, E]$. In this case, for any vector space M over F, the natural map $L \otimes_E M \longrightarrow \bigoplus_{\tau \in \mathrm{Hom}_E(F,L)} M_\tau$ is an isomorphism of $(L \otimes_E F)$-modules, where $M_\tau := L \otimes_{\tau,F} M$. We therefore make the identifications

$$F \otimes_E L \simeq \bigoplus_{\tau \in \mathrm{Hom}_E(F,L)} L_\tau\,, \quad B \otimes_E L \simeq \bigoplus_{\tau \in \mathrm{Hom}_E(F,L)} B_\tau$$

of L-algebras, where $L_\tau = L \otimes_{\tau,F} F \simeq L$. For each $\tau \in \mathrm{Hom}_E(K, L)$, we write $e_\tau = 1 \in L_\tau \subseteq \bigoplus L_\tau$, so that $e_\tau^2 = e_\tau$ in $F \otimes_E L$; and $D_\tau := e_\tau \mathbf{D}_B(V) \subseteq \mathbf{D}_B(V)$. Then D_τ is a vector space over L and

$$\mathbf{D}_B(V) \simeq \bigoplus_{\tau \in \mathrm{Hom}_E(F,L)} D_\tau \tag{3.2}$$

as $\left(\bigoplus_{\tau \in \mathrm{Hom}_E(F,L)} L\right)$-modules, and

$$\alpha_V\colon (B \otimes_E L) \otimes_{(F \otimes_E L)} \mathbf{D}_B(V) = \bigoplus_\tau (B_\tau \otimes_L D_\tau) \xrightarrow{\sim} (B \otimes_E L) \otimes_L V = \bigoplus_\tau (B_\tau \otimes_L V)$$

decomposes through morphisms $\alpha_{V,\tau}\colon B_\tau \otimes_L D_\tau \longrightarrow B_\tau \otimes_L V$ of B_τ-modules on each coordinate. As α_V is assumed to be an isomorphism, so are the $\alpha_{V,\tau}$; since $B_\tau \otimes_L D_\tau$ and $B_\tau \otimes_L V$ are free modules of ranks $\dim_L D_\tau$ and n respectively, this shows that D_τ is of dimension n over L. Combining with (3.2), we get $\mathbf{D}_B(V)$ free of rank n over $F \otimes_E L$.

In general, let $\mathfrak{m}$ be the maximal ideal of A and let $L := A/\mathfrak{m}A$. The exact sequence $0 \longrightarrow \mathfrak{m}V \longrightarrow V \longrightarrow V \otimes_A L \longrightarrow 0$ gives rise to an exact sequence

$$0 \longrightarrow \mathbf{D}_B(\mathfrak{m}V) \longrightarrow \mathbf{D}_B(V) \longrightarrow \mathbf{D}_B(V \otimes_A L) \longrightarrow 0$$

where $V \otimes_A L$ is B-admissible, by applying Proposition 3.1.3 (here we see the terms of the sequence as objects of $\mathbf{Rep}_E(G)$). We easily see from the definition that $\mathbf{D}_B(\mathfrak{m}V) = \mathfrak{m}\mathbf{D}_B(V)$, so that $\mathbf{D}_B(V)/\mathfrak{m}\mathbf{D}_B(V) \simeq \mathbf{D}_B(V \otimes_A L)$. Since $V \otimes_A L \in \mathbf{Rep}_{L,B}(G)$, we know that $\mathbf{D}_B(V \otimes_A L)$ is generated by n elements over $F \otimes_E L$. Then by Nakayama's lemma (as $F \otimes_E \mathfrak{m}$ is a nilpotent ideal of $F \otimes_E A$ with quotient $F \otimes_E L$), $\mathbf{D}_B(V)$ is generated by n elements over $F \otimes_E A$. □

In [Fon82] (see also [Fon94a]), Fontaine constructs different rings over $E = \mathbb{Q}_p$ that are $\mathcal{G}_K$-regular, and studies the corresponding categories of admissible representations. We recall in particular the two following.

– The field B_{dR} which is a $\overline{\mathbb{Q}}_p$-algebra equipped with a semi-linear action of $\mathcal{G}_K$ (meaning that $g(kb) = g(k)g(b)$ for $g \in \mathcal{G}_K$, $k \in \overline{\mathbb{Q}}_p$ and $b \in B_{\mathrm{dR}}$), for which B_{dR} is $\mathcal{G}_K$-regular and its field of invariants is $(B_{\mathrm{dR}})^{\mathcal{G}_K} = K$. There is a valuation v_{dR} invariant under the actions of $\overline{\mathbb{Q}}_p$ and $\mathcal{G}_K$, for which B_{dR} is complete with residue field isomorphic to $\mathbb{C}_p$. The valuation induces a decreasing filtration $\mathrm{Fil}^\bullet(B_{\mathrm{dR}})$ of B_{dR} by $\overline{\mathbb{Q}}_p$-algebras given by

  $$\mathrm{Fil}^k(B_{\mathrm{dR}}) := \{b \in B_{\mathrm{dR}} \mid v_{\mathrm{dR}}(b) \geq k\} , \quad k \in \mathbb{Z}$$

  (where v_{dR} is normalised in such a way that $v_{\mathrm{dR}}(B_{\mathrm{dR}}) = \mathbb{Z}$), and the elements of the filtration are stable under the action of $\mathcal{G}_K$.

– The K_0-subalgebra $B_{\mathrm{crys}} \subset B_{\mathrm{dR}}$ stable by the action of $\mathcal{G}_K$, which is also $\mathcal{G}_K$-regular with fields of invariants $(B_{\mathrm{crys}})^{\mathcal{G}}_K = K_0$. The morphism $K \otimes_{K_0} B_{\mathrm{crys}} \longrightarrow B_{\mathrm{dR}}$ is an injection. The Frobenius φ on K_0 extends to an operator on B_{crys}, still called Frobenius and denoted by φ, which is semi-linear with respect to the Frobenius on K_0.

De Rham representations We recall the basic notions about B_{dR}-admissible representations, called *de Rham* representations. In particular we define the Hodge-Tate weights of a de Rham representation over some sufficiently large $A \in \mathbf{Ar}_{\mathbb{Q}_p}$.

Definition 3.1.5. Let R be ring. Write $\mathbf{Fil}^\bullet_R$ for the category of *filtered R-modules*, *i.e.* of finitely generated R-modules M equipped with a decreasing filtration $\mathrm{Fil}^\bullet M =$

$(\mathrm{Fil}^k M)_{k\in\mathbb{Z}}$ by R-submodules which is exhaustive (*i.e.* $\bigcup_{k\in\mathbb{Z}} \mathrm{Fil}^k M = M$) and separated (*i.e.* $\bigcap_{k\in\mathbb{Z}} \mathrm{Fil}^k M = 0$). The morphisms are morphisms of R-modules $f\colon M \longrightarrow M'$ such that $f(\mathrm{Fil}^k M) \subseteq \mathrm{Fil}^k M'$ for all $k \in \mathbb{Z}$.

A sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathbf{Fil}^\bullet_R$ is said to be exact if it is exact as a sequence of R-modules and $0 \longrightarrow \mathrm{Fil}^k M' \longrightarrow \mathrm{Fil}^k M \longrightarrow \mathrm{Fil}^k M'' \longrightarrow 0$ are exact for all $k \in \mathbb{Z}$.

Remark 3.1.6. For an object $(M, \mathrm{Fil}^\bullet M)$ of $\mathbf{Fil}^\bullet_R$, write

$$\mathrm{gr}^k M \coloneqq \mathrm{Fil}^k M / \mathrm{Fil}^{k+1} M\,, \quad k \in \mathbb{Z}$$

for the graded pieces of the filtration. Note that $\mathrm{Fil}^k M = M$ for $k \ll 0$ since $\mathrm{Fil}^\bullet M$ is exhaustive and M is finitely generated. Therefore, we see that a sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathbf{Fil}^\bullet_R$ is exact if and only if

$$0 \longrightarrow \mathrm{gr}^k M' \longrightarrow \mathrm{gr}^k M \longrightarrow \mathrm{gr}^k M'' \longrightarrow 0$$

is exact for all $k \in \mathbb{Z}$, by an induction on k using a variant of the snake lemma.

Write $\mathbf{D}_{\mathrm{dR}}$ for $\mathbf{D}_{B_{\mathrm{dR}}}$. Given $A \in \mathbf{Ar}_{\mathbb{Q}_p}$ and $V \in \mathbf{Rep}_A(G)$, the filtration on B_{dR} induces a decreasing filtration $\mathrm{Fil}^\bullet \mathbf{D}_{\mathrm{dR}}(V)$ of $\mathbf{D}_{\mathrm{dR}}(V)$ by $K \otimes_{\mathbb{Q}_p} A$-modules given by

$$\mathrm{Fil}^k \mathbf{D}_{\mathrm{dR}}(V) \coloneqq \left(\mathrm{Fil}^k(B_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} V\right)^{\mathcal{G}_k}\,, \quad k \in \mathbb{Z}\,,$$

which is exhaustive and separated; it is called the *Hodge filtration.* This makes $\mathbf{D}_{\mathrm{dR}}$ into a functor from $\mathbf{Rep}_A(\mathcal{G}_K)$ to $\mathbf{Fil}^\bullet_{K\otimes_{\mathbb{Q}_p} A}$.

Definition 3.1.7 (Fontaine)**.** Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$. A representation $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ is said to be *de Rham* when it is B_{dR}-admissible.

The category of de Rham representations is written $\mathbf{Rep}_{A,\mathrm{dR}}(\mathcal{G}_K)$.

Lemma 3.1.8 (Fontaine)**.** *Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$. The functor $\mathbf{D}_{\mathrm{dR}}$ from $\mathbf{Rep}_{A,\mathrm{dR}}(\mathcal{G}_K)$ to $\mathbf{Fil}^\bullet_{K\otimes_{\mathbb{Q}_p} A}$ is exact.*

Proof. Write $\mathrm{gr}^k B_{\mathrm{dR}} \coloneqq \mathrm{Fil}^k B_{\mathrm{dR}} / \mathrm{Fil}^{k+1} B_{\mathrm{dR}}$. For $V \in \mathbf{Rep}_{\mathrm{dR}}$, there is a natural isomorphism

$$\mathrm{gr}^k \mathbf{D}_{\mathrm{dR}}(V) \xrightarrow{\sim} \left((\mathrm{gr}^k B_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} V\right)^{\mathcal{G}_K}\,.$$

Indeed the left-exactness of $(- \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$ implies injectivity. From there, bijectivity comes from computing the dimensions and the fact that $B_{\mathrm{HT}} \coloneqq \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}^k B_{\mathrm{dR}}$ is $\mathcal{G}_K$-regular with invariants $(B_{\mathrm{HT}})^{\mathcal{G}_K} = K$ ([Fon94b, Proposition 3.6]; see also [Fon82, 3.7]), using the same argument as in the proof of 3.1.3.

Hence the functor $\mathrm{gr}^k \mathbf{D}_{\mathrm{dR}}(-)$ from de Rham representations to K-vector spaces is left-exact. Let $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ be an exact sequence in $\mathbf{Rep}_{\mathrm{dR}}(\mathcal{G}_K)$. Knowing that $\sum_{k\in\mathbb{Z}} \dim_K(\mathrm{gr}^k \mathbf{D}_{\mathrm{dR}}(V)) = \dim_K \mathbf{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$, and similarly for V' and V'', the argument of the proof of 3.1.3 again gives the exactness of

$$0 \longrightarrow \operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V') \longrightarrow \operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V) \longrightarrow \operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V'') \longrightarrow 0$$

for all $k \in \mathbb{Z}$. The conclusion follows from Remark 3.1.6. $\square$

For the remainder of this part, we fix an $A \in \mathbf{Ar}_{\mathbb{Q}_p}$ which is an algebra over its residue field $L := A/\mathfrak{m}A$ and assume that L splits K. Then we can write

$$K \otimes_{\mathbb{Q}_p} A \simeq \bigoplus_{\tau \in \Sigma} A_\tau$$

as L-algebras, where $A_\tau = A \otimes_{\tau, K} K \simeq A$. For $\tau \in \Sigma$, write $e_\tau = 1 \in A_\tau \subseteq K \otimes_{\mathbb{Q}_p} A$, so that $e_\tau^2 = e_\tau$ and $A_\tau = (k \otimes_{\mathbb{Q}_p} A)e_\tau$. For any $V \in \mathbf{Rep}_A(\mathcal{G}_K)$, write

$$\mathbf{D}_{\mathrm{dR}}(V)_\tau := e_\tau \mathbf{D}_{\mathrm{dR}}(V)\,, \quad \tau \in \Sigma\,,$$

so that $\mathbf{D}_{\mathrm{dR}}(V) = \bigoplus_{\tau \in \Sigma} \mathbf{D}_{\mathrm{dR}}(V)_\tau$. Then each $\mathbf{D}_{\mathrm{dR}}(V)_\tau$ has an exhaustive separated decreasing filtration $\operatorname{Fil}^\bullet \mathbf{D}_{\mathrm{dR}}(V)_\tau$ by A-modules given by

$$\operatorname{Fil}^k \mathbf{D}_{\mathrm{dR}}(V)_\tau := e_\tau \operatorname{Fil}^k \mathbf{D}_{\mathrm{dR}}(V)\,, \quad k \in \mathbb{Z}\,,$$

so that $\operatorname{Fil}^k \mathbf{D}_{\mathrm{dR}}(V) = \bigoplus_{\tau \in \Sigma} \operatorname{Fil}^k \mathbf{D}_{\mathrm{dR}}(V)_\tau$. There is an isomorphism

$$\operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V)_\tau \xrightarrow{\sim} e_\tau \operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V) \quad \forall\, k \in \mathbb{Z}\,,$$

on the graded pieces, and as always $\operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V) = \bigoplus_{\tau \in \Sigma} \operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V)_\tau$.

Proposition 3.1.9. *Let $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ be a de Rham representation. Then, for all $\tau \in \Sigma$ and $k \in \mathbb{Z}$, the A-module $\operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V)_\tau$ is free of rank*

$$\operatorname{rk}_A \left(\operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V)_\tau\right) = \dim_L \left(\operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V \otimes_A L)_\tau\right)\,.$$

Proof. As in the proof of Proposition 3.1.4, Lemma 3.1.8 applied to the exact sequence $0 \longrightarrow \mathfrak{m}V \longrightarrow V \longrightarrow V \otimes_A L \longrightarrow 0$ gives an isomorphism

$$\left(\operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V)_\tau\right) \otimes_A L \xrightarrow{\sim} \operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V \otimes_A L)_\tau\,.$$

Let $d_k = \dim_L \left(\operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V \otimes_A L)_\tau\right)$; then by Nakayama's Lemma, there is a surjective morphism $f\colon A^{d_k} \longrightarrow \operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V)_\tau$ of A-modules. We therefore have inequalities $\dim_L \left(\operatorname{gr}^k \mathbf{D}_{\mathrm{dR}}(V)_\tau\right) \leq d_k \dim_L A$, whose summation over $k \in \mathbb{Z}$ is

$$\dim_L \mathbf{D}_{\mathrm{dR}}(V)_\tau \leq \dim_L \mathbf{D}_{\mathrm{dR}}(V \otimes_A L)_\tau \dim_L A\,.$$

This last inequality is an equality, hence there is equality everywhere and f is an isomorphism. $\square$

Definition 3.1.10. Let $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ be a de Rham representation of rank n over A. For each $\tau \in \Sigma$, the τ-*Hodge-Tate weights* of V are the integers $h_{\tau,1} \leq \ldots \leq h_{\tau,n}$ such that

$$\mathrm{gr}^{-h_{\tau,i}} \mathbf{D}_{\mathrm{dR}}(V)_\tau \neq 0\,, \quad 1 \leq i \leq n$$

(where an integer h_τ is counted with multiplicity $\mathrm{rk}_A\left(\mathrm{gr}^{-h_\tau} \mathbf{D}_{\mathrm{dR}}(V)_\tau\right)$).

A consequence of Proposition 3.1.9 is that the Hodge-Tate weights of any $V \in \mathbf{Rep}_{A,\mathrm{dR}}(\mathcal{G}_K)$ are the same, up to multiplicity, as those of $V \otimes_A L \in \mathbf{Rep}_{L,\mathrm{dR}}(\mathcal{G}_K)$.

Definition 3.1.11. A de Rham representation $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ is said to be *regular* if for each $\tau \in \Sigma$, the τ-Hodge-Tate weights $h_{\tau,i}$ are pairwise distinct.

Crystalline representations We recall the basics of B_{crys}-admissible representations, called *crystalline* representations. In particular we define the notion of refinement for some crystalline representations. The functor $\mathbf{D}_{B_{\mathrm{crys}}}$ is denoted $\mathbf{D}_{\mathrm{crys}}$. For $A \in \mathbf{Ar}_{\mathbb{Q}_p}$, it is a functor from $\mathbf{Rep}_A(\mathcal{G}_K)$ to modules over $K_0 \otimes_{\mathbb{Q}_p} A$.

For each $V \in \mathbf{Rep}_A(G)$, the Frobenius on B_{crys} induces a Frobenius action φ on $B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V$ which stabilises $\mathbf{D}_{\mathrm{crys}}(V) = (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$; this action on $\mathbf{D}_{\mathrm{crys}}(V)$ is of course φ-semilinear, which means that $\varphi((k \otimes a)d) = (\varphi(k) \otimes a)\varphi(d)$ for $k \in K_0$, $a \in A$ and $d \in \mathbf{D}_{\mathrm{crys}}(V)$. Since $\varphi^{[K_0:\mathbb{Q}_p]}$ is the identity on K_0, the operator $\Phi \coloneqq \varphi^{[K_0:\mathbb{Q}_p]}$ on $\mathbf{D}_{\mathrm{crys}}$ is $(K_0 \otimes_{\mathbb{Q}_p} A)$-linear.

In the same way as with B_{dR}, the $\mathcal{G}_K$-regularity of B_{crys} allows us to consider the following definition.

Definition 3.1.12 (Fontaine)**.** Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$. A representation $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ is said to be *crystalline* when it is B_{crys}-admissible.

For any $V \in \mathbf{Rep}_A(G)$, the functor $(- \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$ is left-exact, so the injection $K \otimes_{K_0} B_{\mathrm{crys}} \longrightarrow B_{\mathrm{dR}}$ induces an injective morphism

$$K \otimes_{K_0} \mathbf{D}_{\mathrm{crys}}(V) = (K \otimes_{\mathbb{Q}_p} A) \otimes_{K_0 \otimes_{\mathbb{Q}_p} A} \mathbf{D}_{\mathrm{crys}}(V) \hookrightarrow \mathbf{D}_{\mathrm{dR}}(V)$$

of $(K \otimes_{\mathbb{Q}_p} A)$-modules. Also, we know from [Fon94b, Proposition 1.4.2] that we always have $\dim_K \mathbf{D}_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$ (resp. $\dim_{K_0} \mathbf{D}_{\mathrm{crys}}(V) \leq \dim_{\mathbb{Q}_p} V$) with equality if and only if V is de Rham (resp. crystalline). Therefore being crystalline implies being de Rham.

We next recall a classical result of linear algebra that will be used in Proposition 3.2.13.

Lemma 3.1.13. *Let A be a commutative ring with unity, let M be an A-module and $f \in \mathrm{End}_A(M)$.*

(1) *Assume that there are $\lambda_1, \ldots, \lambda_m \in A$ for some $m \in \mathbb{N}$, with $\lambda_i - \lambda_j \in A^\times$ for all $1 \leq i < j \leq m$, such that*

$$(f - \lambda_1) \cdots (f - \lambda_m) = 0 \in \mathrm{End}_A(M)\,.$$

Then M is the direct sum of eigenspaces

$$M = \ker(f - \lambda_1) \oplus \cdots \oplus \ker(f - \lambda_m). \tag{3.3}$$

(2) *Assume that A is local with maximal ideal $\mathfrak{m}$, and that M is free of rank $n \in \mathbb{Z}_{>0}$. Also assume that the characteristic polynomial of f can be written $P_f(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i) \in A[\lambda]$ with the λ_i having pairwise distinct reductions $\overline{\lambda_i}$ modulo $\mathfrak{m}$.*

Then the eigenspaces $\ker(f - \lambda_i)$, $1 \leq i \leq n$, are free of rank 1 over A and any choice of generators of these eigenspaces form a basis of M.

Proof. We first prove (1). By induction on m, one can find polynomials $P_1, \dots, P_m \in A[\lambda]$ such that

$$\sum_{1\leq i\leq m} P_i(\lambda) \prod_{j\neq i}(\lambda - \lambda_j) = 1 \in A[\lambda] \tag{3.4}$$

Indeed, for $m = 2$ one can take constant polynomials $P_1 = (\lambda_1 - \lambda_2)^{-1}$ and $P_2 = (\lambda_2 - \lambda_1)^{-1}$. In general, let $Q_1, \dots, Q_{m-1} \in A[\lambda]$ satisfy

$$\sum_{1\leq i\leq m-1} Q_i(\lambda) \prod_{j\neq i,m} (\lambda - \lambda_j) = 1 \in A[\lambda]\,.$$

For $l \coloneqq \prod_{1\leq i<m}(\lambda_m - \lambda_i) \in A^\times$, there is $Q \in A[\lambda]$ such that $\prod_{1\leq i<m}(\lambda - \lambda_i) = Q(\lambda)(\lambda - \lambda_m) + l$. Then taking $P_i \coloneqq -l^{-1}Q_iQ \in A[\lambda]$ for $1 \leq i < m$ and $P_m \coloneqq l^{-1}$ gives (3.4).

Applying (3.4) to $\lambda = f$, we see that any $v \in M$ decomposes as $v = v_1 + \ldots + v_m$, where $v_i \coloneqq \prod_{j\neq i}(f - \lambda_j)p_i(f)v$ for $1 \leq i \leq m$, and obviously $v_i \in \ker(f - \lambda_i)$ by the assumption. Now to check that the sum $M = \ker(f - \lambda_1) + \ldots + \ker(f - \lambda_m)$ is direct, let $v_1 + \ldots + v_m = 0$, with $v_i \in \ker(f - \lambda_i)$. We show by induction on the number of nonzero v_i that $v_i = 0$ for all i. Applying $f - \lambda_m$ gives $(\lambda_1 - \lambda_m)v_1 + \ldots + (\lambda_{m-1} - \lambda_m)v_{m-1} = 0$, so by induction $(\lambda_i - \lambda_m)v_i = 0$ for all $1 \leq i < m$. Since the $\lambda_i - \lambda_m$ are units, we get $v_i = 0$ for $1 \leq i < m$, and finally $v_m = 0$. This proves (1).

We now prove (2). The reduction of f to the $(A/\mathfrak{m})$-vector space $M/\mathfrak{m}M$ has pairwise distinct eigenvalues $\overline{\lambda_i}$; choose a corresponding diagonalisation basis $(\overline{v_i})_{1\leq i\leq n}$. Choose a lift $w_i \in M$ of each $\overline{v_i}$. By (1) and Cayley-Hamilton's theorem, we can write $w_i = v_{i,1} + \ldots + v_{i,n}$ with $v_{i,j} \in \ker(f - \lambda_j)$ for all $1 \leq i, j \leq n$. Then, in $M/\mathfrak{m}M$,

$$\sum_{1\leq j\leq n} (\overline{\lambda_j} - \overline{\lambda_i})\overline{v_{i,j}} = f(\overline{v_i}) - \overline{\lambda_i}\overline{v_i} = 0$$

for all $1 \leq i \leq n$. Applying (1) to $M/\mathfrak{m}M$, we see that the eigenvectors $(\overline{\lambda_j} - \overline{\lambda_i})\overline{v_{i,j}}$ are zero for all $1 \leq i, j \leq n$, which means, since the $\overline{\lambda_i}$ are pairwise distinct, that $v_{i,j} \in \mathfrak{m}$ for all $i \neq j$. Therefore $v_i \coloneqq v_{i,i} = w_i - \sum_{j\neq i} v_{i,j}$ lifts $\overline{v_i}$. By Nakayama's lemma, the v_i, $1 \leq i \leq n$, generate M. Since M is free of rank n, the v_i actually form a basis of M. Since $Av_i \subseteq \ker(f - \lambda_i)$ for each i, (3.3) gives $Av_i = \ker(f - \lambda_i)$. This proves (2). □

Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$ which is an algebra over its residue field $L := A/\mathfrak{m}A$ such that L splits K. Consider a crystalline representation $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ on which the linearised Frobenius $\Phi := \varphi^{[K_0:\mathbb{Q}_p]}$ has eigenvalues $\varphi_i \in A$, $1 \leq i \leq n$ (meaning that $\varphi_i = 1 \otimes \varphi_i \in K_0 \otimes_{\mathbb{Q}_p} A$) with pairwise distinct reductions in L.

Then as before, we can write $K_0 \otimes_{\mathbb{Q}_p} A = \bigoplus_{\tau \in \Sigma_0} A e_\tau$ with $e_\tau^2 = e_\tau$ and $\mathbf{D}_{\text{crys}}(V) = \bigoplus_{\tau \in \Sigma_0} \mathbf{D}_{\text{crys}}(V)_\tau$ with $\mathbf{D}_{\text{crys}}(V)_\tau := e_\tau \mathbf{D}_{\text{crys}}(V)$ a free A-module of rank n. Each $\mathbf{D}_{\text{crys}}(V)_\tau$ is stabilised by Φ, and the induced Φ_τ is an endomorphism of A-module with eigenvalues φ_i, $1 \leq i \leq n$. Then each $\mathbf{D}_{\text{crys}}(V)_\tau$ has a diagonalisation basis $(d_{\tau,i})$ for (φ_i), and $(\sum_{\tau \in \Sigma_0} d_{\tau,i})$ is a diagonalisation basis of the $(K_0 \otimes_{\mathbb{Q}_p} A)$-module $\mathbf{D}_{\text{crys}}(V)$ for (φ_i).

Refinements and the flag variety

Definition 3.1.14. Let A be a local artinian ring, let M be a free A-module of finite type. A *complete flag* on M is an increasing sequence $\operatorname{Fil}_\bullet M$ of A-submodules of M

$$\operatorname{Fil}_0 M := \{0\} \subset \ldots \subset \operatorname{Fil}_i M \subset \ldots \subset \operatorname{Fil}_n M := M$$

with $n = \operatorname{rk}_A(M)$, such that $\operatorname{Fil}_i M / \operatorname{Fil}_{i-1} M$ is free of rank 1 over A for all $1 \leq i \leq n$.

Let L be a field and A be a local artinian L-algebra. Consider the following situation: M is a free module of rank $n \in \mathbb{Z}_{>0}$ over A, equipped with a complete flag $\operatorname{Fil}_\bullet M = (\operatorname{Fil}_i(M))_{0 \leq i \leq n}$. Let $(e_i)_{1 \leq i \leq n}$ be a basis of $\operatorname{Fil}_\bullet M$, *i.e.* a family of elements of M such that $e_i \in \operatorname{Fil}_i M$ and $(e_i \bmod \operatorname{Fil}_{i-1} M)$ is a basis of $\operatorname{Fil}_i M / \operatorname{Fil}_{i-1} M$ for all $1 \leq i \leq n$; it is obviously a basis of M.

Let $G := \mathrm{GL}_{n,L}$ be the general linear group over L and let B be its Borel subgroup of upper triangular matrices; the Weyl group of G is the group $\mathcal{S}_n$ of bijections of $\{1, \ldots, n\}$. The map

$$G(A) \longrightarrow \{\text{complete flags of } M\}$$

defined by sending a matrix $(a_{ij})_{1 \leq i,j \leq n} \in G(A)$ to the flag $\left(\bigoplus_{j \leq k} A b_j\right)_{1 \leq k \leq n}$, where $b_j = \sum_{1 \leq i \leq n} a_{ij} e_i$, factors through $B(A)$ and induces an bijection

$$(G/B)(A) = G(A)/B(A) \xrightarrow{\sim} \{\text{complete flags of } M\} \tag{3.5}$$

(note that we have an injective morphism $G(A)/B(A) \hookrightarrow (G/B)(A)$ in general, see [Jan03, §5.4, §5.5], which is an isomorphism in our case since A is local artinian). Hence G/B is called the *flag variety*; in this text, we sometimes call the point of $(G/B)(A)$ corresponding to a given flag the *position of the flag relative to* $\operatorname{Fil}_\bullet M$.

We embed $\mathcal{S}_n$ in $G(\mathbb{Q}_p)$ by viewing each $w \in \mathcal{S}_n$ as the matrix $(w_{ij})_{1 \leq i,j \leq n}$ defined by $w_{ij} = 1$ if $i = w(j)$ and $w_{ij} = 0$ if $i \neq w(j)$. Then G/B has a cell decomposition

$$G/B = \coprod_{w \in \mathcal{S}_n} BwB/B$$

called the *Schubert decomposition*; we give more details in §4.1 and §5.1. The points of the Schubert cells are easily described as such: a point $x \in (G/B)(A)$ is in the closed cell $(\overline{BwB/B})(A)$ if and only if x can be represented by a matrix $(a_{ij})_{1\le i,j\le n} \in G(A)$ such that $a_{ij} = 0$ for all $i > w(j)$; and x is in the open cell $(BwB/B)(A)$ if and only if we can impose the additional condition $a_{w(j),j} \in A^\times$ for all j. (Thus we see that the Schubert decomposition induces a partition of the A-points if and only if A is a field).

Lemma 3.1.15. *Let A be a local artinian algebra over a field L, let M be a free A-module of rank $n \in \mathbb{N}$ equipped with a complete flag $\operatorname{Fil}_\bullet M$. Let $x \in (G/B)(A)$ be a point corresponding to a flag $(\mathcal{F}_j)_{1\le j\le n}$ of M under the bijection* (3.5)*, where $G = \mathrm{GL}_{n,L}$ with subgroup B of upper triangular matrices.*

Then $x \in (BwB/B)(A)$ for some $w \in \mathcal{S}_n$ if and only if $(\mathcal{F}_j \cap \operatorname{Fil}_{w(j)})/(\mathcal{F}_j \cap \operatorname{Fil}_{w(j)-1})$ is free of rank 1 over A for all $1 \le j \le n$.

Proof. Let $(e_i)_{1\le i\le n}$ be a basis of $\operatorname{Fil}_\bullet M$.

If $x \in (BwB/B)(A)$, then there is a basis $(b_j)_{1\le j\le n}$ of $\mathcal{F}_\bullet$ such that each b_j can be written $b_j = \sum_{i\le w(j)} a_{ij}e_i$ with $a_{w(j),j} \in A^\times$. Then, for all $1 \le j \le n$, the injective morphism

$$(\mathcal{F}_j \cap \operatorname{Fil}_{w(j)})/(\mathcal{F}_j \cap \operatorname{Fil}_{w(j)-1}) \longrightarrow \operatorname{Fil}_{w(j)} / \operatorname{Fil}_{w(j)-1}$$

sends $a_{w(j),j}^{-1} b_j \bmod (\mathcal{F}_j \cap \operatorname{Fil}_{w(j)-1})$ to $e_{w(j)} \bmod \operatorname{Fil}_{w(j)-1}$, so it is an isomorphism.

Conversely, assume that $(\mathcal{F}_j \cap \operatorname{Fil}_{w(j)})/(\mathcal{F}_j \cap \operatorname{Fil}_{w(j)-1})$ is free of rank 1 over A for all $1 \le j \le n$; it has a basis element that lifts to some $b_j \in \mathcal{F}_j \cap \operatorname{Fil}_{w(j)}$. Then each b_j can be written $b_j = \sum_{i\le w(j)} a_{ij}e_i$ in $\mathcal{F}_j$ with $a_{w(j),j} \in A^\times$. Therefore it is enough to prove that $(b_j)_{1\le j\le n}$ forms a basis of $\mathcal{F}_\bullet$, *i.e.* that $b_j \bmod \mathcal{F}_{j-1}$ is a basis of $\mathcal{F}_j/\mathcal{F}_{j-1}$ for all $1 \le j \le n$. We proceed by induction on j. Let $b = \sum_{1\le i\le n} a_i e_i \in \mathcal{F}_j$ such that $b \bmod \mathcal{F}_{j-1}$ is a basis of $\mathcal{F}_j/\mathcal{F}_{j-1}$. As $b_j \in \mathcal{F}_j$, one can write

$$b_j = \sum_{k<j} \lambda_k b_k + \lambda b \tag{3.6}$$

for some $\lambda_k, \lambda \in A$. Let I be the ideal of A generated by λ. If $\lambda_k \notin I$ for some $k < j$, take such a λ_k with $w(k)$ maximal. Then taking the coordinate along $e_{w(k)}$ of (3.6) gives

$$a_{w(k),j} = \lambda_k a_{w(k),k} + \sum_{k'<j \text{ s.t. } w(k')>w(k)} \lambda_{k'} a_{w(k),k'} + \lambda a_{w(k)} \notin I$$

since $a_{w(k),k} \in A^\times$. In particular $a_{w(k),j} \neq 0$, so $w(k) \le w(j)$. This proves that $\lambda_k \in I$ for all $k < j$ such that $w(k) > w(j)$. Then (3.6) along $e_{w(j)}$ gives

$$a_{w(j),j} = \sum_{k<j \text{ s.t. } w(k)>w(j)} \lambda_k a_{w(j),k} + \lambda a_{w(j)} \in I\,.$$

As $a_{w(j),j} \in A^\times$, we deduce $I = A$. Hence $\lambda \in A^\times$, so $b_j \bmod \mathcal{F}_{j-1}$ is a basis of $\mathcal{F}_j/\mathcal{F}_{j-1}$; this concludes the proof. □

For the remainder of this part, we fix an $A \in \mathbf{Ar}_{\mathbb{Q}_p}$ which is an algebra over its residue field $L := A/\mathfrak{m}A$ and assume that L splits K.

Given a regular de Rham representation $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ of rank n over A (see Definition 3.1.11), we define a complete flag $\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{dR}}(V)$ of the free $(K \otimes_{\mathbb{Q}_p} A)$-module $\mathbf{D}_{\mathrm{dR}}(V)$ by $\mathrm{Fil}_0 \mathbf{D}_{\mathrm{dR}}(V) := 0$ and

$$\mathrm{Fil}_i \mathbf{D}_{\mathrm{dR}}(V) := \bigoplus_{\tau \in \Sigma} \mathrm{Fil}^{-h_{\tau,i}} \mathbf{D}_{\mathrm{dR}}(V)_\tau \,, \quad 1 \leq i \leq n$$

where $(h_{\tau,1} < \ldots < h_{\tau,n})$ are the τ-Hodge-Tate weights of V. We see indeed that $\mathrm{Fil}_n \mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}_{\mathrm{dR}}(V)$ and, with Proposition 3.1.9, that each $(K \otimes_{\mathbb{Q}_p} A)$-module $\mathrm{Fil}_i \mathbf{D}_{\mathrm{dR}}(V) / \mathrm{Fil}_{i-1} \mathbf{D}_{\mathrm{dR}}(V)$ for $1 \leq i \leq n$ is free of rank 1.

Definition 3.1.16. Let $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ be a crystalline representation. A *refinement* of V is a complete flag on $\mathbf{D}_{\mathrm{crys}}(V)$ which is stable by the linearised Frobenius Φ.

A refinement induces an ordered sequence $(\varphi_1, \ldots, \varphi_n)$ of eigenvalues of Φ. Conversely, if Φ has eigenvalues $\varphi_i \in A$, $1 \leq i \leq n$ (meaning that $\varphi_i = 1 \otimes \varphi_i \in K_0 \otimes_{\mathbb{Q}_p} A$) with pairwise distinct reductions in L, then an ordering of the φ_i determines a refinement; see Lemma 3.1.13(2) and the discussion following it.

When $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ is a regular crystalline representation, a refinement $\mathcal{F}_\bullet$ determines a complete flag $K \otimes_{K_0} \mathcal{F}_\bullet$ on $\mathbf{D}_{\mathrm{dR}}(V)$, and the position of $\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{dR}}(V)$ relative to $K \otimes_{K_0} \mathcal{F}_\bullet$ determines an A-point of

$$\big(\mathrm{Res}_{K/\mathbb{Q}_p}(G/B)_K\big) \times_{\mathbb{Q}_p} L \simeq \prod_{\tau \in \Sigma} (G/B)_L$$

where $G = \mathrm{GL}_{n,\mathbb{Q}_p}$ and B is the Borel subgroup of upper triangular matrices.

3.2 (φ, Γ_K)-modules over the Robba ring

In this part we recall the theory of (φ, Γ_K)-modules over the relative Robba ring $\mathcal{R}_{A,K}$, in particular the notion of triangular (φ, Γ_K)-module, and we prove a generalisation of a result of Bellaïche-Chenevier concerning refinements of a trianguline crystalline representation (which will be used in the proof of Theorem 3.5.4).

The relative Robba ring We give the definition of the relative Robba ring $\mathcal{R}_{A,K}$ for $A \in \mathbf{Ar}_{\mathbb{Q}_p}$; while there exist much more general versions (for example $\mathcal{R}_{X,K}$ defined for a rigid analytic space X over $\mathbb{Q}_p$, see [KPX14]), we don't need such a degree of generality here. We also recall the nice algebraic properties obtained by Lazard and Berger for the case where A is a field and $K = \mathbb{Q}_p$.

Definition 3.2.1. Let K_0' be the maximal unramified subextension of $\mathbb{Q}_p$ (or K_0) inside $K(\mu_{p^\infty})$. The *Robba ring* is the ring $\mathcal{R}_K$ of Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n\,, \quad a_n \in K_0'$$

such that $|a_n|_{K_0} r^n \longrightarrow 0$ as $n \longrightarrow \pm\infty$ for all $r(f) \leq r < 1$, given some $r(f) \in \mathbb{R}_{<1}$ (we say that f converges on the annulus $r(f) \leq |z| < 1$ in $\mathbb{C}_p$). For any $r < 1$, the subspace $\mathcal{R}_K^r$ of power series f satisfying $r(f) \leq r$ is naturally a Fréchet space; we then equip $\mathcal{R}_K = \bigcup_{r<1} \mathcal{R}_K^r$ with its LF topology.

For $A \in \mathbf{Ar}_{\mathbb{Q}_p}$, the *Robba ring with coefficients in* A (also called relative Robba ring) is the ring $\mathcal{R}_{A,K} \coloneqq \mathcal{R}_K \otimes_{\mathbb{Q}_p} A$. We also write $\mathcal{R} \coloneqq \mathcal{R}_{\mathbb{Q}_p}$ when $K = \mathbb{Q}_p$.

The Robba ring $\mathcal{R}_K$ is equipped with an operator which acts continuously and K_0-semilinearly with respect to the Frobenius φ on K_0, which we still call the Frobenius φ. There is also a continuous K_0 linear action of Γ_K on $\mathcal{R}_K$. The φ- and Γ_K-actions commute and extend A-linearly to $\mathcal{R}_{A,K}$. For more details on these actions, see [Ber02, §2.6] (where $\mathcal{R}_K$ is written $\mathbf{B}^\dagger_{\mathrm{rig},K}$) or [KPX14, Definition 2.2.2] (where $\mathcal{R}_{A,K}$ is written $\mathcal{R}_A(\pi_K)$).

There is an element $t \in \mathcal{R}_K$ such that $\varphi(t) = pt$ and $\gamma(t) = \chi(\gamma)t$ for all $\gamma \in \Gamma_K$; see [Ber02, §1.2] or [Ber08, §I.2]. It is sometimes called the "$2i\pi$-element".

Remark 3.2.2. Note that this definition only depends on the unramified field K_0' if one forgets the (φ, Γ_K)-action: as $\mathbb{Q}_p$-algebras, $\mathcal{R}_{A,K} \simeq \mathcal{R} \otimes_{\mathbb{Q}_p} (K_0' \otimes_{\mathbb{Q}_p} A)$ for all $A \in \mathbf{Ar}_{\mathbb{Q}_p}$. This algebraic isomorphism is sometimes quite convenient; however it is in general not compatible with the (φ, Γ_K)-action.

For instance, when K is unramified over $\mathbb{Q}_p$ (*i.e.* $K = K_0$), the (φ, Γ_K)-action is described by the simple formulas

$$\varphi(z) = (1+z)^p - 1\,, \qquad \gamma(z) = (1+z)^{\chi(\gamma)} - 1 \quad \forall\, \gamma \in \Gamma_K$$

and, for general $f \in \mathcal{R}_K$, by defining $\varphi(f)$ (resp. $\gamma(f)$) using semilinearity (resp. linearity) and continuity. (Note that since $|(1+z)^p - 1| = |z|^p$ for all $z \in \mathbb{C}_p$ such that $p^{-\frac{1}{p-1}} < |z| < 1$, the series $\varphi(f)$ converges on the annulus $\max\left(r(f)^{\frac{1}{p}}, p^{-\frac{1}{p-1}}\right) < |z| < 1$; similarly $|(1+z)^a - 1| = |z|$ for all $z \in \mathbb{C}_p$ such that $|z| < 1$ and $a \in \mathbb{Z}_p^\times$, so $\gamma(f)$ converges on $r(f) \leq |z| < 1$). In this case, the $2i\pi$-element is just the power series $t = \log(z)$. When K has ramification over $\mathbb{Q}_p$, the (φ, Γ_K)-action on $\mathcal{R}_K$ does not have an easy description in term of power series, nor does t.

(φ, Γ_K)-modules and representations We recall the notion of (φ, Γ_K)-module over the Robba ring, the functor $\mathbf{D}_{\mathrm{rig}}$ and the functors $\mathcal{D}_{\mathrm{crys}}, \mathcal{D}_{\mathrm{dR}}$.

Definition 3.2.3. Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$. A (φ, Γ_K)-*module over* $\mathcal{R}_{A,K}$ is a $\mathcal{R}_{A,K}$-module D which is finitely-generated over $\mathcal{R}_{A,K}$ and free over $\mathcal{R}_K$, equipped with actions of

φ and Γ_K which commute and are continuous (as actions of free $\mathcal{R}_K$-modules), and such that $\mathcal{R}_K\varphi(D) = D$.

We write $\mathbf{Mod}_A^{\varphi,\Gamma_K}$ for the additive category whose objects are the (φ, Γ_K)-modules over $\mathcal{R}_{A,K}$ and arrows are $\mathcal{R}_{A,K}$-linear maps that commute with the respective actions of φ and Γ_K.

Theorem 3.2.4 (Fontaine, Cherbonnier-Colmez [CC98], Berger)**.** *There is a ring B equipped with commuting φ- and $\mathcal{G}_K$-actions satisfying $B^{\mathcal{G}_K} = \mathcal{R}_K$, and such that the construction*

$$\mathbf{D}_{\mathrm{rig}}(V) \coloneqq \left(B \otimes_{\mathbb{Q}_p} V\right)^{\mathcal{H}_K}$$

defines a fully faithful and exact functor $\mathbf{D}_{\mathrm{rig}}$ from $\mathbf{Rep}_{\mathbb{Q}_p}(\mathcal{G}_K)$ to $\mathbf{Mod}^{\varphi,\Gamma_K}$.

For $A \in \mathbf{Ar}_{\mathbb{Q}_p}$ and $V \in \mathbf{Rep}_A(\mathcal{G}_K)$, the (φ, Γ_K)-module $\mathbf{D}_{\mathrm{rig}}(V) \in \mathbf{Mod}_A^{\varphi,\Gamma_K}$ is free of rank $\mathrm{rk}_A(V)$ over $\mathcal{R}_{A,K}$.

Proof. See §3.2 and §3.4 of [Ber02] for the first part (the rings B and $\mathcal{R}_K$ are noted $\mathbf{B}^{\dagger}_{\mathrm{rig}}$ and $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ there); the exactness of $\mathbf{D}_{\mathrm{rig}}$ can be checked in the same way as in Lemma 3.1.3. For the second part, see [BC09, Lemma 2.2.7]. □

Remark 3.2.5.
1. When $K = \mathbb{Q}_p$, the functor $\mathbf{D}_{\mathrm{rig}}$ induces an equivalence between $\mathbf{Rep}_{\mathbb{Q}_p}(\mathcal{G}_K)$ and a subcategory of $\mathbf{Mod}^{\varphi,\Gamma_K}$ which is well understood since the work of Kedlaya [Ked04]; see [Col08, Proposition 1.7].
2. A generalisation of Theorem 3.2.4 to the case where A is an affinoid algebra over $\mathbb{Q}_p$ has been obtained by Berger-Colmez [BC08] and Kedlaya-Liu [KL10]; see [Liu15, §1.1] or [KPX14, Theorem 2.2.17].

Berger showed in [Ber02] and [Ber08] how to recover the functors of p-adic Hodge theory from $\mathbf{D}_{\mathrm{rig}}$; we summarize the results we need in the following two theorems.

Theorem 3.2.6 (Berger)**.** *Let $\mathcal{D}_{\mathrm{crys}}$ be the functor from $\mathbf{Mod}^{\varphi,\Gamma_K}$ to the category of K_0-modules equipped with a φ-semilinear endomorphism, defined by*

$$\mathcal{D}_{\mathrm{crys}}(D) \coloneqq D[1/t]^{\Gamma_K}$$

for all (φ, Γ_K)-module D.

(1) *For all $D \in \mathbf{Mod}^{\varphi,\Gamma_K}$,*

$$\dim_{K_0} \mathcal{D}_{\mathrm{crys}}(D) \leq \mathrm{rk}_{\mathcal{R}_K} D.$$

(2) *Write $\mathbf{Mod}_{\mathrm{crys}}^{\varphi,\Gamma_K}$ for the full subcategory of $\mathbf{Mod}^{\varphi,\Gamma_K}$ of (φ, Γ_K)-modules D such that $\dim_{K_0} \mathcal{D}_{\mathrm{crys}}(D) = \mathrm{rk}_{\mathcal{R}_K} D$; its objects are called* crystalline (φ, Γ_K)-modules. *Then $\mathbf{Mod}_{\mathrm{crys}}^{\varphi,\Gamma_K}$ is stable by subobjects and quotients, and the restriction of $\mathcal{D}_{\mathrm{crys}}$ to $\mathbf{Mod}_{\mathrm{crys}}^{\varphi,\Gamma_K}$ is exact.*

(3) *For all* $V \in \mathbf{Rep}_{\mathbb{Q}_p}(\mathcal{G}_K)$,

$$\mathcal{D}_{\mathrm{crys}}(\mathbf{D}_{\mathrm{rig}}(V)) = \mathbf{D}_{\mathrm{crys}}(V).$$

Proof. Statement (3) is [Ber02, Théorème 3.6]. In particular $(\mathcal{R}_K[1/t])^{\mathcal{G}_K} = K_0$, so (1) and (2) are easily seen by the method used for Lemma 3.1.3. □

Theorem 3.2.7 (Berger)**.** *There exists a functor* $D \longmapsto (\mathcal{D}_{\mathrm{dR}}(D), \mathrm{Fil}^\bullet \mathcal{D}_{\mathrm{dR}}(D))$ *from* $\mathbf{Mod}_{\mathrm{crys}}^{\varphi,\Gamma_K}$ *to* $\mathbf{Fil}_K^\bullet$ *which is exact and such that:*

(1) *For all* $D \in \mathbf{Mod}_{\mathrm{crys}}^{\varphi,\Gamma_K}$,

$$\mathcal{D}_{\mathrm{dR}}(D) = K \otimes_{K_0} \mathcal{D}_{\mathrm{crys}}(D).$$

(2) *For all* $V \in \mathbf{Rep}_{\mathrm{crys}}(\mathcal{G}_K)$,

$$\mathcal{D}_{\mathrm{dR}}(\mathbf{D}_{\mathrm{rig}}(V)) = \mathbf{D}_{\mathrm{dR}}(V)$$

as filtered vector spaces over K.

Proof. Let $D \in \mathbf{Mod}_{\mathrm{crys}}^{\varphi,\Gamma_K}$. In the langage of [Ber08], the connexion ∇_D associated to D is locally trivial. Indeed, arguing as in [Ber02, Proposition 3.7], one sees that $D[1/t] = \mathcal{D}_{\mathrm{crys}}(D) \otimes_{K_0} \mathcal{R}_K[1/t]$; therefore $D[1/t]^{\nabla_D=0} = \mathcal{D}_{\mathrm{crys}}(D)$, hence $D[1/t] = D[1/t]^{\nabla_D=0} \otimes_{K_0} \mathcal{R}_K[1/t]$. This shows that ∇_D is even "globally trivial" on D.

Therefore, one can define a functor $\mathcal{D}_{\mathrm{dR}}$ from $\mathbf{Mod}_{\mathrm{crys}}^{\varphi,\Gamma_K}$ to $\mathbf{Fil}_K^\bullet$ by $D \longmapsto K \otimes_{K_0} M(D)$, where $D \longmapsto M(D)$ is the inverse of the equivalence of categories $\mathcal{M}$ of [Ber08, Théorème A] (beware that contrary to us, Berger write his (φ, Γ_K)-module as $\mathbf{M}$ and his filtered vector space as D); note that $M(D)$ is indeed a vector space over K_0 as D is crystalline hence semistable. Then $\mathcal{D}_{\mathrm{dR}}$ is exact because $D \longmapsto M(D)$ is. For any $D \in \mathbf{Mod}_{\mathrm{crys}}^{\varphi,\Gamma_K}$, we know from [*ibid.*, §II.2] that

$$M(D) \otimes_{K_0} \mathcal{R}_K[1/t, \log z] = D \otimes_{\mathcal{R}_K} \mathcal{R}_K[1/t, \log z]\,.$$

We also know from [Ber02, Théorème 3.6] that

$$\mathcal{D}_{\mathrm{crys}}(D) = (D \otimes_{\mathcal{R}_K} \mathcal{R}_K[1/t, \log z])^{\Gamma_K}$$

(as crystalline implies semistable) and that $(\mathcal{R}_K[1/t, \log z])^{\Gamma_K} = K$. (Note that $\mathcal{R}_K[\log z]$ is written $\mathbf{B}^\dagger_{\log,K}$ in [Ber02] and $\mathbf{B}^\dagger_{\mathrm{rig},K}[\ell_X]$ in [Ber08]). Therefore $M(D) = \mathcal{D}_{\mathrm{crys}}(D)$ and $\mathcal{D}_{\mathrm{dR}}(D) = K \otimes_{K_0} \mathcal{D}_{\mathrm{crys}}(D)$. This proves (1). Finally, (2) is [Ber08, Théorème C.2]. □

Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$ be an algebra over its residue field $L \coloneqq A/\mathfrak{m}A$ such that L splits K. Let $D \in \mathbf{Mod}_{\mathrm{crys}}^{\varphi,\Gamma_K}$. As with de Rham representations, by functoriality $\mathcal{D}_{\mathrm{dR}}(D)$ is an object of $\mathrm{Fil}^\bullet_{K\otimes_{\mathbb{Q}_p} A}$ which is a free $K \otimes_{\mathbb{Q}_p} A$-module of rank $\mathrm{rk}_{\mathcal{R}_{A,K}} D$. We then have, for each $\tau \in \Sigma$, a filtered A-module $(\mathcal{D}_{\mathrm{dR}}(D)_\tau, \mathrm{Fil}^\bullet \mathcal{D}_{\mathrm{dR}}(D)_\tau)$ which is free of rank n over A, and an isomorphism on the graded pieces

$$\left(\mathrm{gr}^k(\mathcal{D}_{\mathrm{dR}}(D)_\tau\right) \otimes_A L \xrightarrow{\sim} \mathrm{gr}^k \mathcal{D}_{\mathrm{dR}}(D \otimes_A L)_\tau$$

in the same fashion as Proposition 3.1.9. This allows us to define the Hodge-Tate weights of D.

Triangular (φ, Γ_K)-modules We recall the classification of (φ, Γ_K)-modules of rank 1 and the closely related notion of triangulation of a (φ, Γ_K)-module. This was developed by Colmez [Col08] over $\mathcal{R}_{L,\mathbb{Q}_p}$, generalised by Bellaïche-Chenevier [BC09] over $\mathcal{R}_{A,\mathbb{Q}_p}$, and then by Liu [Liu15] and Kedlaya-Pottharst-Xiao [KPX14] over $\mathcal{R}_{X,K}$ for a rigid analytic space X. Aside from this review, the main purpose of this part is to prove Proposition 3.2.13, a technical result which we will crucially need in the proof of our main Theorem 3.5.4.

Let $\delta\colon K^\times \longrightarrow A^\times$ be a continuous character. Choosing a uniformiser ϖ of K gives a decomposition $K^\times \xrightarrow{\sim} \mathcal{O}_K^\times \times \mathbb{Z}$ (with inverse $(u,k) \longmapsto \varpi^k u$). Then δ is uniquely determined by $\delta(\varpi) \in A^\times$ and $\delta|_{\mathcal{O}_K^\times}$; let $\widetilde{\delta}_\varpi\colon K^\times \longrightarrow A^\times$ be the unique character such that $\widetilde{\delta}_\varpi(\varpi) = 1$ and $\widetilde{\delta}_\varpi|_{\mathcal{O}_K^\times} = \delta|_{\mathcal{O}_K^\times}$. Let $W_K \subset \mathcal{G}_K$ be the Weil group of K and let $\theta\colon W_K^{\mathrm{ab}} \xrightarrow{\sim} K^\times$ be the local reciprocity isomorphism, normalised so that geometric Frobeniuses are sent to uniformisers. Then $\widetilde{\delta}_\varpi \circ \theta\colon W_K^{\mathrm{ab}} \longrightarrow A^\times$ extends to $\mathcal{G}_K^{\mathrm{ab}}$ and induces a 1-dimensional A-representation $\widehat{\delta}_\varpi\colon \mathcal{G}_K \longrightarrow A^\times$.

Definition 3.2.8. Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$ be a K_0-algebra, let $\delta\colon \mathbb{Q}_p^\times \longrightarrow A^\times$ be a continuous character. We define

$$\mathcal{R}_{A,K}(\delta) \coloneqq \mathbf{D}_{\mathrm{rig}}\big(\widehat{\delta}_\varpi\big) \otimes_{K_0 \otimes_{\mathbb{Q}_p} A} D_{\delta(\varpi)}$$

where $D_{\delta(\varpi)}$ is the unique free $K_0 \otimes_{\mathbb{Q}_p} A$-module of rank one equipped with a $(\varphi \otimes 1)$-semilinear operator φ such that $\varphi^{[K_0:\mathbb{Q}_p]} = 1 \otimes \delta(\varpi)$ (see [KPX14, Lemma 6.2.3]).

Then $\mathcal{R}_{A,K}(\delta)$, equipped with the Γ_K-action induced from the one on $\mathbf{D}_{\mathrm{rig}}\big(\widehat{\delta}_\varpi\big)$ and with the diagonal φ-action, is a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module which is free of rank 1 over $\mathcal{R}_{A,K}$ and this construction does not depend on the choice of the uniformiser ϖ (see [KPX14, Construction 6.2.4]).

Example 3.2.9. 1. When $K = \mathbb{Q}_p$, the composition $\theta\colon W_{\mathbb{Q}_p}^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{Q}_p^\times = p^{\mathbb{Z}} \times \mathbb{Z}_p^\times \twoheadrightarrow \mathbb{Z}_p^\times$ is just the cyclotomic character $\chi|_{W_{\mathbb{Q}_p}^{\mathrm{ab}}}\colon W_{\mathbb{Q}_p}^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{Z}_p^\times$. Therefore $\widehat{\delta}_p = \delta \circ \chi$, and

$$\mathbf{D}_{\mathrm{rig}}(\widehat{\delta}_p) = \big(B \otimes_{\mathbb{Q}_p} A(\delta \circ \chi)\big)^{\ker \chi}$$

(where B is the $\mathcal{G}_{\mathbb{Q}_p}$-algebra of Theorem 3.2.4 and $A(\delta \circ \chi)$ is the 1-dimensional representation $\delta \circ \chi\colon \mathcal{G}_{\mathbb{Q}_p} \longrightarrow A^\times$) identifies to

$$B^{\ker \chi} \otimes_{\mathbb{Q}_p} A(\delta \circ \chi) = \mathcal{R}_{A,\mathbb{Q}_p} \otimes_A A(\delta \circ \chi)$$

as $(\varphi,\Gamma_{\mathbb{Q}_p})$-module. Also, $D_{\delta(p)}$ is just A equipped with the A-linear operator φ which is the multiplication by $\delta(p)$. We conclude that $\mathcal{R}_{A,\mathbb{Q}_p}(\delta)$ is the free rank 1 module over $\mathcal{R}_{A,\mathbb{Q}_p}$ with a basis (e) such that the $(\varphi,\Gamma_{\mathbb{Q}_p})$-action is given by

$$\varphi(e)=\delta(p)e\,,\quad \gamma(e)=\delta\left(\chi(\gamma)\right)e\quad \forall\,\gamma\in\Gamma_{\mathbb{Q}_p}\,.$$

2. Let $A\in\mathbf{Ar}_{\mathbb{Q}_p}$ be an algebra over its residue field $L\coloneqq A/\mathfrak{m}A$ such that L splits K. For $a\in A^\times$ and $\mathbf{k}=(k_\tau)_{\tau\in\Sigma}\in\mathbb{Z}^\Sigma$, define the following two continuous characters:

$$\mathrm{unr}(a)\colon\ \begin{array}{rcl} K^\times & \longrightarrow & A^\times \\ z & \longmapsto & a^{v_K(z)} \end{array},\qquad z^{\mathbf{k}}\colon\ \begin{array}{rcl} K^\times & \longrightarrow & A^\times \\ z & \longmapsto & \displaystyle\prod_{\tau\in\Sigma}\tau(z)^{k_\tau} \end{array}$$

where v_K is the valuation on K normalised to have image $\mathbb{Z}$ (for instance, $\mathrm{unr}(p^{[K_0:\mathbb{Q}_p]})(p)=p^{[K:\mathbb{Q}_p]}$).

Then for $\delta=z^{\mathbf{k}}\,\mathrm{unr}(a)$, we have $\mathcal{D}_{\mathrm{crys}}(\mathcal{R}_{A,K}(\delta))=D_a$ hence $\mathcal{R}_{A,K}(\delta)$ is crystalline, and the unique τ-Hodge-Tate weight of $\mathcal{R}_{A,K}(\delta)$ is k_τ; see [KPX14, Example 6.2.6].

Proposition 3.2.10 (Colmez, Kedlaya-Pottharst-Xiao)**.** *Let $A\in\mathbf{Ar}_{\mathbb{Q}_p}$ be a K_0-algebra. Given any (φ,Γ_K)-module D over $\mathcal{R}_{A,K}$ which is free of rank 1 as $\mathcal{R}_{A,K}$-module, there is a unique continuous character $\delta\colon K^\times\longrightarrow A^\times$ such that $D\simeq\mathcal{R}_{A,K}(\delta)$ as (φ,Γ_K)-modules.*

Proof. This is [KPX14, Lemma 6.2.13]; see also [Col08, Proposition 3.1] over $\mathcal{R}_{L,\mathbb{Q}_p}$ and [BC09, Proposition 2.3.1] over $\mathcal{R}_{A,\mathbb{Q}_p}$. □

We recall the following definition from [BC09, Definition 2.3.2] and [BHS19, Definition 3.3.8].

Definition 3.2.11. Let $A\in\mathbf{Ar}_{\mathbb{Q}_p}$. Let D be a (φ,Γ_K)-module over $\mathcal{R}_{A,K}$ which is free as an $\mathcal{R}_{A,K}$-module. A *triangulation* of D is a strictly increasing sequence $\mathrm{Fil}_\bullet D$ of (φ,Γ_K)-submodules over $\mathcal{R}_{A,K}$

$$\mathrm{Fil}_0 D\coloneqq\{0\}\subsetneq\ldots\subsetneq\mathrm{Fil}_i D\subsetneq\ldots\subsetneq\mathrm{Fil}_d D\coloneqq D$$

with $d=\mathrm{rk}_{\mathcal{R}_{A,K}}(D)$, such that each $\mathrm{Fil}_i(D)$ is a direct summand of D as $\mathcal{R}_{A,K}$-modules.

We say that D is *triangulable* if it can be equipped with a triangulation, and the data of D with a triangulation is called a *triangular* (φ,Γ_K)-module. A representation $V\in\mathbf{Rep}_A(\mathcal{G}_K)$ is called *trianguline* if $\mathbf{D}_{\mathrm{rig}}(V)$ is triangulable.

Given a triangular (φ,Γ_K)-module $(D,\mathrm{Fil}_\bullet D)$ over $\mathcal{R}_{A,K}$, the graded pieces

$$\mathrm{gr}_i D\coloneqq\mathrm{Fil}_i D/\mathrm{Fil}_{i-1} D\,,\quad 1\le i\le d$$

of the triangulation are free of rank 1 as $\mathcal{R}_{A,K}$-modules. When A is a K_0-algebra, according to Proposition 3.2.10, for each $1 \leq i \leq d$ there is a unique continuous character $\delta_i \colon K^\times \longrightarrow A^\times$ such that $\mathrm{gr}_i D \simeq \mathcal{R}_L(\delta_i)$.

Definition 3.2.12. Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$ be a K_0-algebra. Let $(D, \mathrm{Fil}_\bullet D)$ be a triangular (φ, Γ_K)-module of rank d over $\mathcal{R}_{A,K}$. The *parameter* of the triangulation $\mathrm{Fil}_\bullet D$ is the unique continuous character

$$\underline{\delta} = (\delta_1, \ldots, \delta_d) \colon (K^\times)^d \longrightarrow A^\times$$

such that $\mathrm{gr}_i D \simeq \mathcal{R}_{A,K}(\delta_i)$ for all $1 \leq i \leq d$.

The following result somewhat generalises [BC09, Proposition 2.4.1].

Proposition 3.2.13. *Let $A \in \mathbf{Ar}_{\mathbb{Q}_p}$ be an algebra over its residue field $L \coloneqq A/\mathfrak{m}A$ such that L splits K. Let $V \in \mathbf{Rep}_A(\mathcal{G}_K)$ be a regular crystalline representation of rank $n \in \mathbb{N}$ and of Hodge-Tate weights $(h_{\tau,1} < \ldots < h_{\tau,n})_{\tau \in \Sigma}$.*

Assume that there exist $\mathbf{k} = (k_{\tau,i})_{\tau \in \Sigma, 1 \leq i \leq n} \in (\mathbb{Z}^n)^\Sigma$ and $\varphi_i \in A^\times$ for $1 \leq i \leq n$ with pairwise distinct reductions in L, such that V is trianguline of parameter

$$\underline{\delta} = z^{\mathbf{k}} \operatorname{unr}(\underline{\varphi}) \; \coloneqq \left(z^{\mathbf{k}_1} \operatorname{unr}(\varphi_1), \ldots, z^{\mathbf{k}_n} \operatorname{unr}(\varphi_n)\right)$$

(we use the notation of Example 3.2.9). Then:

(1) *There exists $w = (w_\tau)_{\tau \in \Sigma} \in (\mathcal{S}_n)^\Sigma$ such that $k_{\tau,i} = h_{\tau, w_\tau^{-1}(i)}$ for all $\tau \in \Sigma$ and $1 \leq i \leq n$.*

(2) *By the discussion following Definition 3.1.16, the linearised Frobenius Φ on $\mathbf{D}_{\mathrm{crys}}(V)$ admits $(1 \otimes \varphi_i)_{1 \leq i \leq n}$ as eigenvalues, and this ordering determines a refinement $\mathcal{F}_\bullet$ of V.*

(3) *The position $x \in \prod_{\tau \in \Sigma} (\mathrm{GL}_n / B)(A)$ of $\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{dR}}(V)$ relative to $\mathcal{F}_{\mathrm{dR},\bullet} \coloneqq K \otimes_{K_0} \mathcal{F}_\bullet$ lies in the cell $\prod_{\tau \in \Sigma} (B w_\tau B / B)(A)$.*

Proof. Let $(D_i)_{0 \leq i \leq n}$ be a triangulation of $\mathbf{D}_{\mathrm{rig}}(V)$ of parameter $\underline{\delta}$. For $0 \leq i \leq n$, write $\mathcal{F}_i \coloneqq \mathcal{D}_{\mathrm{crys}}(D_i) = D_i[1/t]^{\Gamma_K}$. The D_i are crystalline (as subobjects of $\mathbf{D}_{\mathrm{rig}}(V)$), hence by exactness of $\mathcal{D}_{\mathrm{crys}}$ each exact sequence

$$0 \longrightarrow D_{i-1} \longrightarrow D_i \longrightarrow \mathcal{R}_{A,K}\left(z^{\mathbf{k}_i} \operatorname{unr}(\varphi_i)\right) \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow D_{\varphi_i} \longrightarrow 0$$

of $K_0 \otimes_{\mathbb{Q}_p} A(\varphi)$-modules. Therefore Φ acts on $\mathcal{F}_i/\mathcal{F}_{i-1}$ by multiplication by $1 \otimes \varphi_i$, thus $(\mathcal{F}_i)$ is the unique (see Lemma 3.1.13) refinement of V which has a basis of eigenvectors of $(1 \otimes \varphi_i)_{1 \leq i \leq n}$. This proves (2).

Similarly, by Theorem 3.2.7, the exact functor $\mathcal{D}_{\mathrm{dR}}$ induces for each $1 \le i \le n$ an exact sequence

$$0 \longrightarrow \mathcal{F}_{\mathrm{dR},i-1} \longrightarrow \mathcal{F}_{\mathrm{dR},i} \longrightarrow \mathcal{D}_{\mathrm{dR}}\left(\mathcal{R}_{A,K}\left(z^{\mathbf{k}_i}\,\mathrm{unr}(\varphi_i)\right)\right) \longrightarrow 0$$

in $\mathbf{Fil}^{\bullet}_{K\otimes_{\mathbb{Q}_p}A}$. It induces, for each $1 \le i \le n$, each $\tau \in \Sigma$ and each $k \in \mathbb{Z}$, an isomorphism

$$\begin{aligned}\mathrm{Fil}^k\left(\mathcal{F}_{\mathrm{dR},i}\right)_\tau / \mathrm{Fil}^k\left(\mathcal{F}_{\mathrm{dR},i-1}\right)_\tau \\ \xrightarrow{\sim} \mathrm{Fil}^k\,\mathcal{D}_{\mathrm{dR}}\left(\mathcal{R}_{A,K}\left(z^{\mathbf{k}_i}\,\mathrm{unr}(\varphi_i)\right)\right)_\tau \simeq \begin{cases} 0 & \text{if } k \neq k_{\tau,i} \\ A & \text{if } k = k_{\tau,i}\end{cases}\end{aligned} \tag{3.7}$$

(see Example 3.2.9.2. for the last isomorphism). Applying (3.7) to $k = k_{\tau,i}+1$, we get $\mathrm{Fil}^{k_{\tau,i}+1}\left(\mathcal{F}_{\mathrm{dR},i}\right)_\tau / \mathrm{Fil}^{k_{\tau,i}+1}\left(\mathcal{F}_{\mathrm{dR},i-1}\right)_\tau \xrightarrow{\sim} 0$, thus

$$\begin{aligned}\mathrm{Fil}^{k_{\tau,i}}\left(\mathcal{F}_{\mathrm{dR},i}\right)_\tau / \mathrm{Fil}^{k_{\tau,i}+1}\left(\mathcal{F}_{\mathrm{dR},i}\right)_\tau = \mathrm{Fil}^{k_{\tau,i}}\left(\mathcal{F}_{\mathrm{dR},i}\right)_\tau / \mathrm{Fil}^{k_{\tau,i}+1}\left(\mathcal{F}_{\mathrm{dR},i-1}\right)_\tau \\ \twoheadrightarrow \mathrm{Fil}^{k_{\tau,i}}\left(\mathcal{F}_{\mathrm{dR},i}\right)_\tau / \mathrm{Fil}^{k_{\tau,i}}\left(\mathcal{F}_{\mathrm{dR},i-1}\right)_\tau\end{aligned} \tag{3.8}$$

The exactness of $\mathbf{D}_{\mathrm{dR}}$ also shows that $\mathrm{Fil}^k\,\mathcal{F}_{\mathrm{dR},i} = \mathcal{F}_{\mathrm{dR},i} \cap \mathrm{Fil}^k\,\mathbf{D}_{\mathrm{dR}}(V)$ for all $k \in \mathbb{Z}$, therefore (3.8) rephrases as

$$\begin{aligned}\left(\mathcal{F}_{\mathrm{dR},i} \cap \mathrm{Fil}^{k_{\tau,i}}\,\mathbf{D}_{\mathrm{dR}}(V)\right)_\tau / \left(\mathcal{F}_{\mathrm{dR},i} \cap \mathrm{Fil}^{k_{\tau,i}+1}\,\mathbf{D}_{\mathrm{dR}}(V)\right)_\tau \\ \twoheadrightarrow \mathrm{Fil}^{k_{\tau,i}}\left(\mathcal{F}_{\mathrm{dR},i}\right)_\tau / \mathrm{Fil}^{k_{\tau,i}}\left(\mathcal{F}_{\mathrm{dR},i-1}\right)_\tau .\end{aligned} \tag{3.9}$$

On the other hand, there is an obvious injective morphism

$$\begin{aligned}\left(\mathcal{F}_{\mathrm{dR},i} \cap \mathrm{Fil}^{k_{\tau,i}}\,\mathbf{D}_{\mathrm{dR}}(V)\right)_\tau / \left(\mathcal{F}_{\mathrm{dR},i} \cap \mathrm{Fil}^{k_{\tau,i}+1}\,\mathbf{D}_{\mathrm{dR}}(V)\right)_\tau \\ \hookrightarrow \mathrm{Fil}^{k_{\tau,i}}\,\mathbf{D}_{\mathrm{dR}}(V)_\tau / \mathrm{Fil}^{k_{\tau,i}+1}\,\mathbf{D}_{\mathrm{dR}}(V)_\tau .\end{aligned} \tag{3.10}$$

The right-hand side of (3.9) (resp. (3.10)) is free of rank one over A by (3.7) applied to $k = k_{\tau,i}$ (resp. by Proposition 3.1.9). We deduce, by comparing dimensions over L, that (3.9) and (3.10) are isomorphisms, hence the A-module $\left(\mathcal{F}_{\mathrm{dR},i} \cap \mathrm{Fil}^{k_{\tau,i}}\,\mathbf{D}_{\mathrm{dR}}(V)\right)_\tau / \left(\mathcal{F}_{\mathrm{dR},i} \cap \mathrm{Fil}^{k_{\tau,i}+1}\,\mathbf{D}_{\mathrm{dR}}(V)\right)_\tau$ is free of rank 1. This means that the jumps of the Hodge filtration of V are in degrees $(k_{\tau,i})_{\tau\in\Sigma,1\le i\le n}$, which proves (1). Furthermore, (3) then follows from Lemma 3.1.15 (note that our computations give the position of $\mathcal{F}_{\mathrm{dR},\bullet}$ relative to $\mathrm{Fil}_\bullet\,\mathbf{D}_{\mathrm{dR}}(V)$, which is inverse of the position of the position of $\mathrm{Fil}_\bullet\,\mathbf{D}_{\mathrm{dR}}(V)$ relative to $\mathcal{F}_{\mathrm{dR},\bullet}$). □

3.3 Deformation varieties

In this part, we provide the definitions and basic facts about the trianguline variety $X^{\square}_{\mathrm{tri}}(\bar{r})$, and about an auxiliary space $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r}}$ which is useful to the study of crystalline points on the trianguline variety. (The rigid analytic spaces we deal with are to be considered in the sense of Tate exclusively).

For the rest of §3, we let L be a finite extension of $\mathbb{Q}_p$ which splits K, with integer ring $\mathcal{O}_L$ and residue field k_L, and we write $\Sigma := \mathrm{Hom}_{\mathbb{Q}_p}(K, L)$. We fix $n \in \mathbb{N}$ and a continuous representation

$$\bar{r}\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k_L)$$

(where k_L has the discrete topology).

We write $\mathbf{Ar}_L$ for the category of local artinian L-algebras, $\mathbf{Ar}_{\mathcal{O}_L}$ for the category of local artinian $\mathcal{O}_L$-algebras with residue field k_L, and $\widehat{\mathbf{Ar}}_{\mathcal{O}_L}$ for the category of complete local noetherian $\mathcal{O}_L$-algebras with residue field k_L.

The trianguline variety We introduce the trianguline variety, first studied by Hellmann [Hel12] and subsequently with Breuil and Schraen in [HS16], [BHS17b], [BHS17a], [BHS19].

When $r\colon G \longrightarrow \mathrm{GL}_n(A)$ is a representation of a group G over a ring A and $A \longrightarrow A'$ is a morphism of ring, we write $r \otimes_A A'$ for the composition of r with the natural morphism $\mathrm{GL}_n(A) \longrightarrow \mathrm{GL}_n(A')$.

Definition 3.3.1. Let $A \in \widehat{\mathbf{Ar}}_{\mathcal{O}_L}$. A *framed deformation of* $\bar{r}$ *over* A is a continuous representations $r\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(A)$ such that $r \otimes_A k_L = \bar{r}$.

The notion of framed deformation is due to the work of Mazur [Maz89] and Kisin [Kis09], who show the existence of $R_{\bar{r}}^{\square} \in \widehat{\mathbf{Ar}}_{\mathcal{O}_L}$ with isomorphisms

$$\mathrm{Hom}_{\text{local } \mathcal{O}_L\text{-alg.}}(R_{\bar{r}}^{\square}, A) \simeq \{\text{framed deformations of } \bar{r} \text{ over } A\}$$

functorial in $A \in \widehat{\mathbf{Ar}}_{\mathcal{O}_L}$; we call $R_{\bar{r}}^{\square}$ is called the *framed deformation ring of* $\bar{r}$. The *framed deformation variety of* $\bar{r}$ is the rigid analytic space over L

$$\mathfrak{X}_{\bar{r}}^{\square} := \mathrm{Spf}(R_{\bar{r}}^{\square})^{\mathrm{rig}}$$

where Spf is the formal spectrum and $(-)^{\mathrm{rig}}$ is the rigidification functor of Berthelot [Ber96, §0.2] (see also [dJ95, §7.1]). By [dJ95, Proposition 7.1.7], for any $B \in \mathbf{Ar}_L$ with residue field E,

$$\mathfrak{X}_{\bar{r}}^{\square}(B) \simeq \varinjlim_{A \in \mathrm{Int}(B)} \mathrm{Hom}_{\text{local } \mathcal{O}_E\text{-alg.}}(R_{\bar{r}}^{\square}, A)$$

functorially in B, where $\mathrm{Int}(B)$ is the category of models of B, *i.e.* rings $A \in \mathbf{Ar}_{\mathcal{O}_L}$ flat of topologically finite type over $\mathcal{O}_E$ with an isomorphism $A \otimes_{\mathcal{O}_E} E \xrightarrow{\sim} B$. Note that any two objects of $\mathrm{Int}(B)$ are contained in a third, and any object $A \in \mathrm{Int}(B)$ is a subring of B (by flatness) which is finite over $\mathcal{O}_E$ (since B is artinian). In particular, the composition $A \longrightarrow B \twoheadrightarrow E$, has finite image over $\mathcal{O}_E$, so is contained in $\mathcal{O}_E$; hence A is canonically equipped with a morphism $A \longrightarrow \mathcal{O}_E$. Therefore, using [Kis09, Proposition 2.3.5], we see that

$$\mathfrak{X}_{\bar{r}}^{\square}(B) \simeq \left\{ (r_B, r_{\mathcal{O}_E}) \;\middle|\; \begin{array}{l} r_{\mathcal{O}_E} \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(\mathcal{O}_E) \text{ framed deformation of } \bar{r} \otimes_{k_L} k_E, \\ r_B \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(B) \text{ framed deformation of } r_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} E \end{array} \right\}.$$

Alternatively, one can see an element of $\mathfrak{X}_{\bar{r}}^{\square}(B)$ as a representation $r_B \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(B)$ such that $r_B \otimes_B E$ has image in $\mathrm{GL}_n(\mathcal{O}_E)$ and is a framed deformation of $\bar{r} \otimes_{k_L} k_E$.

Let $\mathcal{W} \coloneqq \widehat{\mathcal{O}_K^\times}$ be the rigid analytic space over $\mathbb{Q}_p$ that parametrises continuous characters of $\mathcal{O}_K^\times$, *i.e.* such that $\mathcal{W}(X) = \mathrm{Hom}_{\mathrm{cont}}\left(\mathcal{O}_K^\times, \Gamma(X, \mathcal{O}_X)^\times\right)$ for all rigid analytic space X over $\mathbb{Q}_p$; the existence and smoothness of $\mathcal{W}$ can be seen using a decomposition $\mathcal{O}_K^\times \simeq H \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$ of topological groups, where H is a finite torsion abelian group (see [Neu99, Proposition II.5.7]). Similarly, let $\mathcal{T} \coloneqq \widehat{K^\times}$ be the rigid analytic space over $\mathbb{Q}_p$ that parametrises continuous characters of $K^\times$; it is also smooth and any decomposition $K^\times \simeq \mathbb{Z} \times \mathcal{O}_K^\times$ induces an isomorphism $\mathcal{T} \simeq \mathbb{G}_{\mathrm{m}}^{\mathrm{rig}} \times_{\mathbb{Q}_p} \mathcal{W}$ of rigid analytic spaces over $\mathbb{Q}_p$. The restriction of a $K^\times$-character to $\mathcal{O}_K^\times$ induces a morphism $\mathcal{T} \longrightarrow \mathcal{W}$ which corresponds to the projection $\mathbb{G}_{\mathrm{m}}^{\mathrm{rig}} \times_{\mathbb{Q}_p} \mathcal{W} \twoheadrightarrow \mathcal{W}$.

Write $\mathcal{T}_L \coloneqq \mathcal{T} \times_{\mathbb{Q}_p} L$. The set

$$\left\{ (\delta_i)_{1 \le i \le n} \in \mathcal{T}_L^n(L) \;\middle|\; \begin{array}{l} \delta_i \delta_j^{-1} = z^{-\mathbf{k}} \text{ or } \delta_i \delta_j^{-1} = N_{K/\mathbb{Q}_p}(z) z^{\mathbf{k}} |z|_K \\ \text{for some } i \neq j, \mathbf{k} \in \mathbb{Z}_{\ge 0}^{\Sigma} \end{array} \right\}$$

is an analytic subset of $\mathcal{T}_L^n$ (in the sense of [BGR84, §9.5.2]), so its complement is a Zariski-open subset of $\mathcal{T}_L^n$, which we call $\mathcal{T}_{\mathrm{reg}}^n$.

Definition 3.3.2 (Hellmann)**.** Let $U_{\mathrm{tri}}^{\square}(\bar{r})$ be the subset

$$U_{\mathrm{tri}}^{\square}(\bar{r}) \coloneqq \left\{ (r, \underline{\delta}) \in \mathfrak{X}_{\bar{r}}^{\square} \times_L \mathcal{T}_{\mathrm{reg}}^n \;\middle|\; r \text{ is trianguline of parameter } \underline{\delta} \right\}$$

of $\mathfrak{X}_{\bar{r}}^{\square} \times_L \mathcal{T}_{\mathrm{reg}}^n$. The *trianguline variety* is the Zariski-closure $X_{\mathrm{tri}}^{\square}(\bar{r})$ of $U_{\mathrm{tri}}^{\square}(\bar{r})$ in $\mathfrak{X}_{\bar{r}}^{\square} \times_L \mathcal{T}_L^n$, with its structure of reduced rigid analytic space over L.

The composition of the inclusion $X_{\mathrm{tri}}^{\square}(\bar{r}) \hookrightarrow \mathfrak{X}_{\bar{r}}^{\square} \times_L \mathcal{T}_L^n$, the projection $\mathfrak{X}_{\bar{r}}^{\square} \times_L \mathcal{T}_L^n \twoheadrightarrow \mathcal{T}_L^n$ and the restriction morphism $\mathcal{T}_L^n \longrightarrow \mathcal{W}_L^n$ is a morphism of rigid analytic varieties over L

$$\omega \colon X_{\mathrm{tri}}^{\square}(\bar{r}) \longrightarrow \mathcal{W}_L^n$$

called the *weight morphism*.

A point $x = (r, \underline{\delta}) \in X_{\mathrm{tri}}^{\square}(\bar{r})$ is said to be *crystalline* if the representation $r \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k(x))$ is crystalline ($k(x)$ is the local residue field of $X_{\mathrm{tri}}^{\square}(\bar{r})$ at x).

Note that $U_{\mathrm{tri}}^{\square}(\bar{r})$ (resp. $X_{\mathrm{tri}}^{\square}(\bar{r})$) is also noted $U_{\mathrm{tri}}^{\square}(\bar{r})^{\mathrm{reg}}$ or $U_{\mathrm{tri}}(\bar{r})$ (resp. $X_{\mathrm{tri}}(\bar{r})$) in the literature.

Theorem 3.3.3 (Breuil-Hellmann-Schraen)**.** (1) *The rigid space $X_{\mathrm{tri}}^{\square}(\bar{r})$ is equidimensional of dimension $n^2 + [K : \mathbb{Q}_p]\frac{n(n+1)}{2}$.*

(2) *The subset $U_{\mathrm{tri}}^{\square}(\bar{r})$ of $X_{\mathrm{tri}}^{\square}(\bar{r})$ is Zariski-open and Zariski-dense.*

(3) *The open set $U_{\mathrm{tri}}^{\square}(\bar{r})$ is smooth over $\mathbb{Q}_p$, and the restriction of ω to $U_{\mathrm{tri}}^{\square}(\bar{r})$ is a smooth morphism.*

Proof. This is [BHS17b, Théorème 2.6]. $\square$

Lemma 3.3.4 (Breuil-Hellmann-Schraen)**.** *Let $x = (r, \underline{\delta}) \in X_{\mathrm{tri}}^{\square}(\bar{r})$ be a crystalline point. Then there exist $\mathbf{h} = (h_{\tau,i}) \in (\mathbb{Z}^{\Sigma})^n$ and $\underline{\varphi} = (\varphi_1, \dots, \varphi_n) \in (k(x)^{\times})^n$ such that*

$$\underline{\delta} = z^{\mathbf{h}} \operatorname{unr}(\underline{\varphi}) \ := \left(z^{\mathbf{h}_1} \operatorname{unr}(\varphi_1), \dots, z^{\mathbf{h}_n} \operatorname{unr}(\varphi_n)\right) .$$

Moreover, the $1 \otimes \varphi_i \in K_0 \otimes_{\mathbb{Q}_p} k(x)$ are the eigenvalues of the linearised Frobenius Φ on $\mathbf{D}_{\mathrm{crys}}(r)$ and, for each $\tau \in \Sigma$, the multiset of τ-Hodge-Tate weights of r is $\{h_{\tau,i} \mid 1 \leq i \leq n\}$.

If $\varphi_i \varphi_j^{-1} \notin \{1, p^{[K_0:\mathbb{Q}_p]}\}$ for $i \neq j$, then there also exists $\mathbf{h}' \in (\mathbb{Z}^{\Sigma})^n$ such that $(r, z^{\mathbf{h}'} \operatorname{unr}(\underline{\varphi})) \in U_{\mathrm{tri}}^{\square}(\bar{r})$.

Proof. This is [BHS17a, Lemma 2.1]; the second part is a consequence of its proof. Note that necessarily $\{h_{\tau,i} \mid 1 \leq i \leq n\} = \left\{h'_{\tau,i} \;\middle|\; 1 \leq i \leq n\right\}$ for all $\tau \in \Sigma$. $\square$

A crystalline deformation space We introduce a rigid analytic space over L parametrising crystalline deformations of given Hodge-Tate weights together with an ordering of the eigenvalues of the Frobenius. We follow the exposition of [BHS17a, §2.2] and [BHS19, §4.2].

Throughout this paragraph, we fix $\mathbf{h} = (h_{\tau,1} < \ldots < h_{\tau,n})_{\tau \in \Sigma} \in (\mathbb{Z}^n)^{\Sigma}$.

The work of Kisin [Kis08, Corollary 2.6.2] shows the existence of a quotient $R_{\bar{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ of $R_{\bar{r}}^{\square}$, reduced and flat over $\mathbb{Z}_p$, which represents framed deformations of $\bar{r}$ which are crystalline of Hodge-Tate weights $\mathbf{h}$. Namely, for $A \in \widehat{\mathbf{Ar}}_{\mathcal{O}_L}$, a morphism $R_{\bar{r}}^{\square} \longrightarrow A$ factors through $R_{\bar{r}}^{\square,\mathbf{k}-\mathrm{cr}}$ if and only if the corresponding deformation $r \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(A)$ is crystalline with Hodge-Tate weights $\mathbf{h}$. Write

$$\mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}} := \operatorname{Spf}(R_{\bar{r}}^{\square,\mathbf{k}-\mathrm{cr}})^{\mathrm{rig}}$$

for the associated rigid analytic space over L. Similarly to the case of $\mathfrak{X}_{\bar{r}}^{\square}$, arguing as in [Kis09, Proposition 2.3.5], we see that the set of B-points $x \in \mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}}(B)$ for $B \in \mathbf{Ar}_L$ is the set of points $x \in \mathfrak{X}_{\bar{r}}^{\square}(B)$ such that the associated representation $r_x \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(B)$ is crystalline with Hodge-Tate weights $\mathbf{h}$.

According to [Kis08, Theorem 2.5.5] (see also [BC08, Corollaire 6.3.3]), there is a coherent locally free module $\mathcal{D}$ over $K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}}}$ equipped with a φ-semilinear automorphism φ_{crys} such that, for all $x \in \mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}}$ corresponding to a deformation $r_x \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k(x))$, there is an isomorphism

$$\mathcal{D} \otimes_{\mathcal{O}_{\mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}}}} k(x) \simeq \mathbf{D}_{\mathrm{crys}}(r_x)$$

such that $\varphi_{\mathrm{crys}} \otimes_{\mathcal{O}_{\mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}}}} k(x)$ induces the φ-action on $\mathbf{D}_{\mathrm{crys}}(r_x)$. Furthermore, for each $\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(K_0, L)$, the linearised Frobenius $\Phi_{\mathrm{crys}} := (\varphi_{\mathrm{crys}})^{[K_0:\mathbb{Q}_p]}$ induces an invertible linear action on $\mathbf{D}_{\mathrm{crys}}(r_x) \otimes_{K_0,\tau} L$ which is independent, up to isomorphism, on the choice of τ. Therefore, taking the coefficients of the characteristic polynomial of this action gives a morphism of rigid analytic spaces over L

$$\mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}} \longrightarrow \big((\mathbb{G}_{\mathrm{a}}^{\mathrm{rig}})^{n-1} \times \mathbb{G}_{\mathrm{m}}^{\mathrm{rig}}\big) \times_{\mathbb{Q}_p} L \tag{3.11}$$

where the image $(a_{n-1}, \ldots, a_1, a) \in k(x)^{n-1} \times k(x)^{\times}$ of a point $x \in \mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}}$ is such that the Frobenius on $\mathbf{D}_{\mathrm{crys}}(r_x)$ has eigenvalues $1 \otimes \varphi_i \in K_0 \otimes_{\mathbb{Q}_p} k(x)$, where the $\varphi_i, 1 \leq i \leq n$ are the roots of the polynomial $t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a$. See also [BHS17a, §2.2].

As in the paragraph following Definition 3.1.14, let $G := \mathrm{GL}_{n,\mathbb{Q}_p}$, $B \subset G$ be the Borel of upper triangular matrices, $T \subset B$ be the maximal torus inside B; the Weyl group of (G, T) is the symmetric group $\mathcal{S}_n$. The isomorphism $L[e_1, \ldots, e_n, f]/(e_n f - 1) \xrightarrow{\sim} L[t_1, \ldots, t_n, u]^{\mathcal{S}_n}/(u \prod t_i - 1)$ of the fundamental theorem of symmetric polynomials gives an isomorphism of schemes

$$T_L/\mathcal{S}^n \xrightarrow{\sim} \big(\mathbb{G}_{\mathrm{a}}^{n-1} \times \mathbb{G}_{\mathrm{m}}\big) \times_{\mathbb{Q}_p} L$$

where the left-hand side parametrises a polynomial by its roots and the right-hand side parametrises it by its coefficients. Therefore (3.11) is a morphism $\mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}} \longrightarrow T_L^{\mathrm{rig}}/\mathcal{S}^n$ of rigid analytic spaces over L, which we use to define

$$\widetilde{\mathfrak{X}}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}} := \mathfrak{X}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}} \times_{T_L^{\mathrm{rig}}/\mathcal{S}^n} T_L^{\mathrm{rig}}$$

which is a reduced rigid analytic space over L by [BHS17a, Lemma 2.2].

From the proof of [BHS19, Theorem 4.2.3] (see also [BHS17a, Lemma 2.4]), the set

$$\widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}} := \Big\{(r, \underline{\varphi}) \in \widetilde{\mathfrak{X}}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}} \;\Big|\; \varphi_i \varphi_j^{-1} \notin \{1, p^{[K_0:\mathbb{Q}_p]}\} \ \forall\, i \neq j\Big\}$$

is a Zariski-open dense subset of $\widetilde{\mathfrak{X}}_{\bar{r}}^{\square,\mathbf{h}-\mathrm{cr}}$, and there is a smooth morphism

$$h : \widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}} \longrightarrow \Big(\mathrm{Res}_{K/\mathbb{Q}_p}(G/B)_K^{\mathrm{rig}}\Big) \times_{\mathbb{Q}_p} L \simeq \prod_{\tau \in \Sigma} (G/B)_L^{\mathrm{rig}} \tag{3.12}$$

of rigid analytic spaces over L sending a point $(r, \underline{\varphi}) \in \widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}}$ to the position of the Hodge flag $\mathrm{Fil}_\bullet \mathbf{D}_{\mathrm{dR}}(r)$ relative to the refinement on $K \otimes_{K_0} \mathbf{D}_{\mathrm{crys}}(r)$ induced by the ordering $\underline{\varphi}$ (see the discussion following Definition 3.1.16).

Fix a permutation $w = (w_\tau) \in \mathcal{S}_n^{\Sigma}$ and define the analytic subset

$$\widetilde{W}_{\bar{r},w}^{\mathbf{h}-\mathrm{cr}} := h^{-1}\left(\prod_{\tau \in \Sigma} (\overline{B w_\tau B/B})_L^{\mathrm{rig}}\right) \subseteq \widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}}$$

of $\widetilde{W}_{\bar{r}}^{\mathbf{h}-\mathrm{cr}}$ (note that it is written $\overline{\widetilde{W}_{\bar{r},w}^{\mathbf{h}-\mathrm{cr}}}$ in [BHS19, §4.2]; we drop the bar for clarity). Then the morphism of rigid analytic spaces over L

$$\iota_{\mathbf{h},w}\colon \begin{array}{ccc} \widetilde{W}_{\bar{r},w}^{\mathbf{h}-\mathrm{cr}} & \longrightarrow & \mathfrak{X}_{\bar{r}}^{\square} \times_L \mathcal{T}_L^n \\ (r, \underline{\varphi}) & \longmapsto & (r, z^{w(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})) \end{array}$$

(where $w(\mathbf{h}) \coloneqq (h_{\tau, w_\tau^{-1}(i)})_{\tau,i} \in (\mathbb{Z}^{\Sigma})^n$) factors through the inclusion $X_{\mathrm{tri}}^{\square}(\bar{r}) \subset \mathfrak{X}_{\bar{r}}^{\square} \times_L \mathcal{T}_L^n$: see the proof of [BHS19, Theorem 4.2.3].

3.4 The patched eigenvariety

In this part we review the *patched eigenvariety*. It is a rigid analytic variety $X_p(\bar{\rho})$ over L that was introduced and studied by Breuil-Hellmann-Schraen in [BHS17b], [BHS17a] assuming the existence of a suitable globalisation $\bar{\rho}$ of $\bar{r}$; this implies in particular $p \nmid 2n$. In this case, this assumption is now known to always be satisfied since the recent work of Emerton-Gee [EG19]. The global aspects of this part are quite technical and rather unrelated to the rest of this text. Hence, for the sake of brevity we often omit details; we refer the reader to relevant parts of the literature whenever this happens.

For the rest of §3, we make the assumption $p \nmid 2n$ (which implies in particular $p > 2$). We also keep the representation $\bar{r}\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k_L)$ of §3.3.

We consider the following global setting.

- F is a CM field, *i.e.* a quadratic totally imaginary extension of a totally real number field F^+, and S_p is the set of places of F^+ dividing p. We assume that F/F^+ is unramified in all finite places, and that all places $v \in S_p$ split in F and satisfy $F_v^+ \simeq K$.

- G is a unitary group in n variables over F^+ quasi-split in all finite places of F^+, such that $G \times_{F^+} F \simeq \mathrm{GL}_{n,F}$ and $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact, and $U^p = \prod_{v \notin S_p \text{ finite}} U_v \subset G(\mathbb{A}_{F^+}^{p\infty})$ is a tame level, where each U_v is a compact subgroup of F_v^+ which is assumed to be hyperspecial whenever v is inert. We then fix a finite set S of finite places of F^+ containing S_p and all the places v such that U_v is not hyperspecial; in particular all places $v \in S$ split in F.

- $\bar{\rho}\colon \mathcal{G}_{F^+} \longrightarrow \mathrm{GL}_n(k_L)$ is a continuous irreducible representation which is automorphic (see for example [EG14, Definition 5.3.1]). In particular, it follows from [EG14, §5.1, §5.2, §5.3] or [CEG+16, §2.1, §2.3] that $\bar{\rho}$ "comes from" p-adic automorphic forms on $G(\mathbb{A}_{F^+})$ of tame level U^p; see [BHS17b, §2.4] for a detailed exposition (see also the beginning of [BHS17a, §3.1, §3.2]). Also, $\bar{\rho}\left(\mathrm{Gal}\left(\overline{F}/F(\zeta_p)\right)\right)$ is adequate in the sense of [Tho12, Def. 2.3].

- For each $v \in S_p$, there exists a place $\tilde{v} \mid v$ of F such that $\bar{\rho}_{\tilde{v}} \coloneqq \bar{\rho}|_{\mathcal{G}_{F_{\tilde{v}}}}$ is isomorphic to $\bar{r}$ via $F_{\tilde{v}} \simeq F_v^+ \simeq K$. We fix such a place $\tilde{v}$.

Enlarging L if necessary, the existence of such a globalisation has been proved by Emerton-Gee using the Emerton-Gee stack [EG19]; see [EG19, Theorem 1.2.2] and [EG14, Corollary A.7].

For $v \in S_p$, let $R^{\overline{\square}}_{\bar{\rho}_{\tilde{v}}}$ be the maximal reduced quotient of $R^{\square}_{\bar{\rho}_{\tilde{v}}}$ without p-torsion. By [dJ95, Lemma 7.1.4.a], the rigid analytic space $\mathfrak{X}_{\bar{\rho}_{\tilde{v}}} := \mathrm{Spf}(R^{\overline{\square}}_{\bar{\rho}_{\tilde{v}}})^{\mathrm{rig}}$ is reduced and has the same underlying topological space as $\mathrm{Spf}(R^{\square}_{\bar{\rho}_{\tilde{v}}})^{\mathrm{rig}}$. We also write

$$\mathfrak{X}_{\bar{\rho}^p} := \prod_{v \in S \setminus S_p} \mathfrak{X}_{\bar{\rho}_{\tilde{v}}}\,, \quad \mathfrak{X}_{\bar{\rho}_p} := \prod_{v \in S_p} \mathfrak{X}_{\bar{\rho}_{\tilde{v}}}\,.$$

Let $T_p := \prod_{v \in S_p} T_v$ where $T_v := (F_{\tilde{v}}^{\times})^n$ is the n-dimensional torus over $F_{\tilde{v}}$. Then the continuous characters $T_v \longrightarrow L^{\times}$ for each $v \in S_p$ (resp. $T_p \longrightarrow L^{\times}$) are parametrised by a rigid analytic space $\widehat{T}_{v,L}$ (resp. $\widehat{T}_{p,L} := \prod_{v \in S_p} \widehat{T}_{v,L}$) over L. Note that a character $T_v \longrightarrow L^{\times}$ can be seen as a character $(K^{\times})^n \longrightarrow L^{\times}$ through $F_{\tilde{v}} \simeq K$; this induces a rigid isomorphism $\widehat{T}_{v,L} \simeq \mathcal{T}_L^n$ over L. We define $X^{\square}_{\mathrm{tri}}(\bar{\rho}_{\tilde{v}})$ in the same way as $X^{\square}_{\mathrm{tri}}(\bar{r})$ with $\bar{\rho}_{\tilde{v}}$ instead of $\bar{r}$ for each $v \in S_p$, and $X^{\square}_{\mathrm{tri}}(\bar{\rho}_p) := \prod_{v \in S_p} X^{\square}_{\mathrm{tri}}(\bar{\rho}_{\tilde{v}})$; again the isomorphisms $F_{\tilde{v}} \simeq K$ induce isomorphisms $X^{\square}_{\mathrm{tri}}(\bar{\rho}_p) \simeq (X^{\square}_{\mathrm{tri}}(\bar{r}))^{|S_p|}$. Also, since $X^{\square}_{\mathrm{tri}}(\bar{\rho}_p)$ is by definition reduced, there is a Zariski-closed embedding

$$X^{\square}_{\mathrm{tri}}(\bar{\rho}_p) \longhookrightarrow \mathfrak{X}_{\bar{\rho}_p} \times_L \widehat{T}_{p,L}$$

of rigid analytic spaces over L.

We fix $g \in \mathbb{Z}_{\geq 1}$, and we let $R_\infty := R^{\overline{\square}}_{\bar{\rho}_S}[\![t_1, \ldots, t_g]\!]$, where $R^{\overline{\square}}_{\bar{\rho}_S} := \widehat{\bigotimes}_{v \in S} R^{\overline{\square}}_{\bar{\rho}_{\tilde{v}}}$, and $\mathfrak{X}_\infty := \mathrm{Spf}(R_\infty)^{\mathrm{rig}}$. Then, by [dJ95, Proposition 7.2.4.g], we have

$$\mathfrak{X}_\infty = \mathfrak{X}_{\bar{\rho}^p} \times_L \mathfrak{X}_{\bar{\rho}_p} \times_L \mathbb{U}^g$$

where $\mathbb{U} := \mathrm{Spf}(\mathcal{O}_L[\![t]\!])^{\mathrm{rig}}$ is the unit open disk over L. By adapting the patching construction of Caraiani-Emerton-Gee-Geraghty-Paškūnas-Shin [CEG+16, §2], Breuil-Hellmann-Schraen constructed for some $g \in \mathbb{Z}_{\geq 1}$ an R_∞-representation Π_∞ of $G_p := \prod_{v \in S_p} G(F_{\tilde{v}})$ over L called the "patched representation", satisfying different properties for which we refer to [BHS17b, Théorème 3.5]. They then consider the subspace $\Pi_\infty^{R_\infty - \mathrm{an}}$ of locally R_∞-analytic vectors (see [BHS17b, Définition 3.2]), and apply Emerton's Jacquet module functor J_{B_p} (see [Eme06, Definition 3.4.5]) associated to $B_p := \prod_{v \in S_p} B_{\tilde{v}}$, where $B_{\tilde{v}}$ is the Borel subgroup of upper triangular matrices in $\mathrm{GL}_n(F_{\tilde{v}})$. This gives a coherent $\mathcal{O}_{\mathfrak{X}_\infty \times_L \widehat{T}_{p,L}}$-module $\mathcal{M}_\infty$ whose module of global sections $\Gamma(\mathfrak{X}_\infty \times_L \widehat{T}_{p,L}, \mathcal{M}_\infty)$ is isomorphic to $J_{B_p}(\Pi_\infty^{R_\infty - \mathrm{an}})'$, where $(-)'$ is the continuous dual.

Definition 3.4.1 (Breuil-Hellmann-Schraen). The *patched eigenvariety* $X_p(\bar{\rho})$ is the support of $\mathcal{M}_\infty$ (in the sense of [BGR84, §9.5.2]), which is an analytic subset of $\mathfrak{X}_\infty \times_L \widehat{T}_{p,L}$ (see Proposition 4 of *ibid.*), equipped with its structure of reduced rigid analytic space over L. It is also called the *Hecke-Taylor-Wiles* variety in [BHS17b].

For more details about this construction, we refer to §3.2 and Corollaire 3.20 of [BHS17b].

We next recall a powerful tool from [BHS17a] to find points on $X_p(\bar\rho)$, which is related to the notion of strong linkage of characters which we now explain.

Let $\mathfrak{g}$ be a Lie algebra over a field k of characteristic 0, together with a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$. Let $\Phi \subseteq \mathfrak{t}^* := \operatorname{Hom}_k(\mathfrak{t}, \overline{k})$ be the root system of $(\mathfrak{g}, \mathfrak{t})$ (where $\overline{k}$ is an algebraic closure of k), let $W := W(\mathfrak{g}, \mathfrak{t})$ be the Weyl group which acts on $\mathfrak{t}^*$ and Φ. Let $E \subset \mathfrak{t}^*$ be the $\mathbb{Q}$-span of Φ; as in §2.1, the axioms of root system give a nondegenerate bilinear form $(\cdot \mid \cdot)$ on E which is invariant by W. Choose a basis I of Φ and call its elements the *simple roots*; this choice induces a partial order on $\mathfrak{t}^*$ by setting $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a $\mathbb{Z}_{\geq 0}$-linear combination of simple roots. For each root $\alpha \in \Phi$, let $\alpha^\vee := \frac{2\alpha}{(\alpha|\alpha)}$ be the coroot associated to α. Then $\{\alpha^\vee \mid \alpha \in I\}$ is a basis of E; it has an orthogonal dual under $(\cdot \mid \cdot)$, whose elements we call *fundamental weights*. We call *special weight* the sum of the fundamental weights; it is an element $\varpi \in E$ that satisfies in particular $s_\alpha \varpi = \varpi - \alpha$ for any simple root $\alpha \in I$, where $s_\alpha \in W$ is the reflection associated to α, defined by $s_\alpha \colon \lambda \mapsto \lambda - (\lambda \mid \alpha^\vee)\alpha$. Note that ϖ is usually noted ρ, *e.g.* [Hum08], but we wish to avoid confusion with global representations.

Definition 3.4.2. The *dot action* is the action of W on $\mathfrak{t}^*$ defined by

$$w \cdot \lambda := w(\lambda + \varpi) - \varpi$$

for all $w \in W$ and $\lambda \in \mathfrak{t}^*$.

Let $\lambda, \mu \in \mathfrak{t}^*$. We say that μ is *strongly linked* to λ, and write $\mu \uparrow \lambda$, if there exist roots $\alpha_1, \ldots, \alpha_r \in \Phi$ for some $r \in \mathbb{Z}_{\leq 0}$ such that $\mu = (s_{\alpha_r} \ldots s_{\alpha_1}) \cdot \lambda$ and $(s_{\alpha_{i+1}} \ldots s_{\alpha_1}) \cdot \lambda < (s_{\alpha_i} \ldots s_{\alpha_1}) \cdot \lambda$ for all $1 \leq i \leq r-1$.

We apply this, for each place $v \in S_p$, to the context of the split reductive group $G_{v,L} := \left(\operatorname{Res}_{/\mathbb{Q}_p} \mathrm{GL}_{n,F_{\tilde v}}\right) \times_{\mathbb{Q}_p} L \simeq \prod_{\tau \in \Sigma_v} \mathrm{GL}_{n,L}$ over L (where $\Sigma_v := \operatorname{Hom}_{\mathbb{Q}_p}(F_{\tilde v}, L) \simeq \Sigma$), with Lie algebra $\mathfrak{g}_{v,L} = \prod_{\tau \in \Sigma} \mathfrak{gl}_{n,L}$ and Cartan algebra $\mathfrak{t}_{v,L} = \prod_{\tau \in \Sigma_v} \mathfrak{t}_L$ (with $\mathfrak{t}_L \simeq L^n$). Then we have $\mathfrak{t}^*_{v,L} \simeq (\overline{L}^n)^{\Sigma_v}$ as vector spaces over $\overline{L}$, and the Weyl group $W = (\mathcal{S}_n)^{\Sigma_v}$ acts on $\mathfrak{t}^*_{v,L}$ in the obvious way. We choose the basis $I \subset \Phi$ corresponding to the Borel subgroup of upper triangular matrices; the simple roots are the $\alpha = (\alpha_{\tau',i'})_{\tau' \in \Sigma_v, 1 \leq i' \leq n} \in \mathfrak{t}^*_{v,L}$ such that, for some $\tau \in \Sigma_v$ and $1 \leq i \leq n-1$, we have $\alpha_{\tau,i} = -\alpha_{\tau,i+1} = 1$ and $\alpha_{\tau',i'} = 0$ for all $\tau' \neq \tau$ and $i' \notin \{i, i+1\}$. Then the special weight is $\varpi = \left(\frac{n+1}{2} - i\right)_{\tau \in \Sigma_v, 1 \leq i \leq n} \in \mathfrak{t}^*_{v,L}$, and the dot action expresses as

$$w \cdot \lambda = \left(\lambda_{\tau, w_\tau^{-1}(i)} + i - w_\tau^{-1}(i)\right)_{\tau \in \Sigma_v, 1 \leq i \leq n}$$

for $w = (w_\tau)_{\tau \in \Sigma_v} \in W$ and $\lambda = (\lambda_{\tau,i})_{\tau \in \Sigma_v, 1 \leq i \leq n} \in \mathfrak{t}^*_{v,L}$.

Let $\delta \colon K^\times \longrightarrow A^\times$ be a continuous character, where A is a finite L-algebra. Then δ is locally $\mathbb{Q}_p$-analytic, so it is a morphism of p-adic Lie groups and we can consider its differential $\mathrm{d}\delta \colon K \longrightarrow A$ which is a morphism of $\mathbb{Q}_p$-vector spaces. For

$\tau \in \Sigma$, we still write $\tau\colon K \longrightarrow A$ for the composition of $\tau\colon K \hookrightarrow L$ with $L \longrightarrow A$; then $\mathrm{Hom}_{\mathbb{Q}_p\text{-vect. sp.}}(K, A)$ is generated by the τ.

Definition 3.4.3. Let A be a finite L-algebra and $\delta\colon K^\times \longrightarrow A^\times$ be a continous character. The *weights* of δ are the $\mathrm{wt}_\tau(\delta) \in A$, $\tau \in \Sigma$, such that $\mathrm{d}\delta = \sum_{\tau\in\Sigma} \mathrm{wt}_\tau(\delta)\tau$. Equivalently, they are uniquely determined by the limit

$$\lim_{a\in\mathcal{O}_K, a\to 0} \frac{\left|\delta(1+a) - 1 - \sum_{\tau\in\Sigma} \mathrm{wt}_\tau(\delta)\tau(a)\right|_A}{|a|_K} = 0\,.$$

We also define the weights of a continous character $\delta'\colon \mathcal{O}_K^\times \longrightarrow A^\times$ by setting $\mathrm{wt}_\tau(\delta') \coloneqq \mathrm{wt}_\tau(\delta)$, where $\delta\colon K^\times \longrightarrow A^\times$ is any character such that $\delta|_{\mathcal{O}_K^\times} = \delta'$; this is well-defined since $K^\times$ is topologically isomorphic to the product of $\mathcal{O}_K^\times$ with the discrete group $\mathbb{Z}$.

If $\underline{\delta}_v = (\delta_{v,i})_{1\le i\le n}\colon T_{v,L} \longrightarrow L'^\times$ is a continuous character, for some place $v \in S_p$ and some finite extension L' of L, then one can see $\mathrm{d}\underline{\delta}_v = (\mathrm{wt}_\tau(\delta_{v,i}))_{\tau\in\Sigma_v, 1\le i\le n}$ as an element of $\mathfrak{t}^*_{v,L}$.

Definition 3.4.4. Let $\underline{\delta} = (\underline{\delta}_v)_{v\in S_p}, \underline{\epsilon} = (\underline{\epsilon}_v)_{v\in S_p} \in \widehat{T}_{p,L}$. We say that $\underline{\epsilon}$ is *strongly linked* to $\underline{\delta}$, if $\mathrm{d}\underline{\epsilon} \uparrow \mathrm{d}\underline{\delta}$ in $\mathfrak{t}^*_{v,L}$ for all $v \in S_p$, and if $\underline{\epsilon}\underline{\delta}^{-1}$ is algebraic in the sense that for all $v \in S_p$ there exists $\mathbf{k}_v \in \mathbb{Z}^{\Sigma_v}$ such that $\underline{\epsilon}_v\underline{\delta}_v^{-1} = z^{\mathbf{k}_v}$). In this case we write $\underline{\epsilon} \uparrow \underline{\delta}$.

Theorem 3.4.5 (Breuil-Hellmann-Schraen)**.** *Let $x = (y, \underline{\delta}) \in \mathfrak{X}_\infty \times_L \widehat{T}_{p,L}$ be a point lying in $X_p(\bar\rho)$, and let $\underline{\epsilon} \in \widehat{T}_{p,L}$ be a character satisfying $\underline{\epsilon} \uparrow \underline{\delta}$. Then the point $x' = (y, \underline{\epsilon}) \in \mathfrak{X}_\infty \times_L \widehat{T}_{p,L}$ also lies in $X_p(\bar\rho)$.*

Proof. This is [BHS17a, Theorem 5.5]; see (3.10) of *loc. cit.* □

We next recall the relation between $X_p(\bar\rho)$ and $X^\square_{\mathrm{tri}}(\bar\rho_p)$. For each $v \in S_p$, define a character $\underline{\delta}_{B_v} = (|\cdot|^{n+1-2i}_{F_{\tilde v}})_{1\le i\le n} \in \widehat{T}_{v,L}$, where $|\cdot|_{F_{\tilde v}}$ is normalised as usual by $|p|_{F_{\tilde v}} = p^{-[F_{\tilde v}:\mathbb{Q}_p]}$, and an automorphism ι_v of $\widehat{T}_{v,L}$ as follows:

$$\iota_v(\underline{\delta}) \coloneqq \underline{\delta}_{B_v} \cdot (\delta_i \cdot (\chi \circ \theta_{F_{\tilde v}}^{-1})^{i-1})_{1\le i\le n}$$

where χ is the cyclotomic character and $\theta_{F_{\tilde v}}\colon W^{\mathrm{ab}}_{F_{\tilde v}} \xrightarrow{\sim} F^\times_{\tilde v}$ is the local reciprocity isomorphism normalised so that geometric Frobeniuses are sent to uniformisers. The inverse of the automorphism

$$\iota\colon \begin{array}{ccc} \mathfrak{X}_\infty \times_L \widehat{T}_{p,L} & \longrightarrow & \mathfrak{X}_\infty \times_L \widehat{T}_{p,L} \\ \big(y, (\delta_v)_{v\in S_p}\big) & \longmapsto & \big(y, (\iota_v(\delta_v))_{v\in S_p}\big) \end{array}$$

induces a Zariski-closed embedding

$$\iota^{-1}\colon X_p(\bar\rho) \hookrightarrow \mathfrak{X}_{\bar\rho^p} \times_L \left(\mathfrak{X}_{\bar\rho_p} \times_L \widehat{T}_{p,L}\right) \times_L \times \mathbb{U}^g\,.$$

According to [BHS17b, Théorème 3.21], ι^{-1} induces an isomorphism of rigid analytic spaces over L between $X_p(\bar{\rho})$ and a union of irreducible components of $\mathfrak{X}_{\bar{\rho}^p} \times_L X_{\mathrm{tri}}^{\square}(\bar{\rho}_p) \times_L \mathbb{U}^g$. Such irreducible components are of the form $\mathfrak{X}^p \times_L Z \times_L \mathbb{U}^g$, where $\mathfrak{X}^p$ is an irreducible component of $\mathfrak{X}_{\bar{\rho}^p}$ and Z is an irreducible component of $X_{\mathrm{tri}}^{\square}(\bar{\rho}_p)$. Therefore, we have an isomorphism

$$\iota^{-1} \colon X_p(\bar{\rho}) \xrightarrow{\sim} \bigcup_{\mathfrak{X}^p \text{ irr. comp. of } \mathfrak{X}_{\bar{\rho}^p}} \left(\mathfrak{X}^p \times_L X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\bar{\rho}_p) \times_L \mathbb{U}^g\right) \tag{3.13}$$

where, for each $\mathfrak{X}^p$, $X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\bar{\rho}_p)$ is a union of irreducible components of $X_{\mathrm{tri}}^{\square}(\bar{\rho}_p)$.

The following conjecture is [BHS17b, Conjecture 3.23].

Conjecture 3.4.6. *For any irreducible component $\mathfrak{X}^p$ of $\mathfrak{X}_{\bar{\rho}^p}$, the rigid analytic subvariety $X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\bar{\rho}_p) \subseteq X_{\mathrm{tri}}^{\square}(\bar{\rho}_p)$ is equal to $\widetilde{X}_{\mathrm{tri}}^{\square}(\bar{\rho}_p) \coloneqq \prod_{v \in S_p} \widetilde{X}_{\mathrm{tri}}^{\square}(\bar{\rho}_{\tilde{v}})$, where $\widetilde{X}_{\mathrm{tri}}^{\square}(\bar{\rho}_{\tilde{v}})$ is the union of irreducible components Z of $X_{\mathrm{tri}}^{\square}(\bar{\rho}_{\tilde{v}})$ such that $Z \cap U_{\mathrm{tri}}^{\square}(\bar{\rho}_{\tilde{v}})$ contains a crystalline point. In particular, $X_{\mathrm{tri}}^{\mathfrak{X}^p-\mathrm{aut}}(\bar{\rho}_p)$ does not depend on $\mathfrak{X}^p$.*

3.5 Tangent vectors at a companion point

In this part, we prove the main result of §3: Theorem 3.5.4. It computes the dimension of $X_{\mathrm{tri}}^{\square}(\bar{r})$ at crystalline points satisfying certain technical conditions, generalising [BHS17a, Corollary 5.17] and [BHS19, Proposition 4.1.5] (see [BHS19, Remark 4.1.6(iii)] for the latter).

We fix a crystalline point $x = (r, \underline{\delta}) \in X_{\mathrm{tri}}^{\square}(\bar{r})$ such that r is regular in the sense of Definition 3.1.11. We write $\mathbf{h} = (h_{\tau,i})_{\tau \in \Sigma, 1 \leq i \leq n}$ for its Hodge-Tate weights, ordered such that $h_{\tau,1} < \ldots < h_{\tau,n}$ for all $\tau \in \Sigma$. We make the additional assumption that $\varphi_i \varphi_j^{-1} \notin \{1, p^{[K_0:\mathbb{Q}_p]}\}$ for $i \neq j$; then, by Lemma 3.3.4, we can attach two elements $w, w_{\mathrm{sat}} \in \mathcal{S}_n^{\Sigma}$ such that:

- The character $\underline{\delta}$ is of the form

  $$\underline{\delta} = z^{w(\mathbf{h})} \operatorname{unr}(\underline{\varphi})$$

  where $w(\mathbf{h}) \coloneqq (h_{\tau, w_\tau^{-1}(i)})_{\tau,i} \in (\mathbb{Z}^{\Sigma})^n$ and the $1 \otimes \varphi_i \in K_0 \otimes_{\mathbb{Q}_p} k(x)$ are the eigenvalues of the linearised Frobenius Φ on $\mathbf{D}_{\mathrm{crys}}(r)$.

- The point $x_{\mathrm{sat}} \coloneqq (r, \underline{\delta}_{\mathrm{sat}}) \in \mathfrak{X}_{\bar{r}}^{\square} \times_L \mathcal{T}_L^n$ defined by

  $$\underline{\delta}_{\mathrm{sat}} = z^{w_{\mathrm{sat}}(\mathbf{h})} \operatorname{unr}(\underline{\varphi})$$

  lies in $U_{\mathrm{tri}}^{\square}(\bar{r})$.

Remember that $(\mathcal{S}_n)^{\Sigma}$ is the Weyl group of the root system $\Phi = \coprod_{\tau \in \Sigma} \Phi_v$ attached to the split reductive group $\prod_{\tau \in \Sigma} \mathrm{GL}_{n,L}$ over L, as in the paragraph following Definition 3.4.2. We choose again the basis of Φ corresponding to the Borel of upper triangular matrices. Then, as in §2.3, we have a length function lg, a partial Bruhat order $\preceq$ and a notion of "good pairs" in $(\mathcal{S}_n)^\Sigma$. By [BHS19, Theorem 4.2.3], we have $w_{\mathrm{sat}} \preceq w$.

Write by $\mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L} \subset \mathcal{W}^n_L$ for the analytic subset of characters $\underline{\eta} = (\eta_1, \dots, \eta_n)$ defined by the equations:

$$\mathrm{wt}_\tau\left(\eta_{w_{\mathrm{sat},\tau}(i)}\eta^{-1}_{w_\tau(i)}\right) = h_{\tau,i} - h_{\tau,w^{-1}_{\mathrm{sat},\tau}w_\tau(i)}\,, \quad 1 \leq i \leq n\,, \quad \tau \in \Sigma$$

with its structure of reduced rigid analytic space over L. Such a character $\underline{\eta}$ is of the form $z^{w_{\mathrm{sat}}(\mathbf{h})}\underline{\eta}'$ where $\underline{\eta}' = (\eta'_1, \dots, \eta'_n) \in \mathcal{W}^n_L$ satisfies $\mathrm{wt}_\tau\left(\eta'_{w_\tau w^{-1}_{\mathrm{sat},\tau}(i)}\right) = \mathrm{wt}_\tau(\eta'_i)$ for all τ, i. In particular, this is the case of $\underline{\delta}_{\mathrm{sat}}$.

We define an automorphism $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}} \colon \mathcal{T}^n_L \xrightarrow{\sim} \mathcal{T}^n_L$ by

$$\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(\underline{\eta}) \coloneqq z^{w(\mathbf{h})-w_{\mathrm{sat}}(\mathbf{h})}\underline{\eta}$$

and still write $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$ for the automorphism $\mathrm{id}_{\mathfrak{X}^\square_{\bar{r}}} \otimes \jmath_{w,w_{\mathrm{sat}},\mathbf{h}} \colon \mathfrak{X}^\square_{\bar{r}} \times_L \mathcal{T}^n_L \xrightarrow{\sim} \mathfrak{X}^\square_{\bar{r}} \times_L \mathcal{T}^n_L$.

From now on we assume the following:

Hypothesis 3.5.1. There exists an irreducible component $\mathfrak{X}^p$ of $\mathfrak{X}_{\bar{\rho}^p}$ such that $(x_{\mathrm{sat},v})_{v\in S_p} \in X^{\mathfrak{X}^p-\mathrm{aut}}_{\mathrm{tri}}(\bar{\rho}_p)$, where $x_{\mathrm{sat},v} \in X^\square_{\mathrm{tri}}(\bar{\rho}_{\tilde{v}})$ corresponds to $x_{\mathrm{sat}} \in X^\square_{\mathrm{tri}}(\bar{r})$ under $\bar{\rho}_{\tilde{v}} \simeq \bar{r}$ for each $v \in S_p$.

Note that this is always the case if Conjecture 3.4.6 is true, since $x_{\mathrm{sat},v} \in \widetilde{X}^\square_{\mathrm{tri}}(\bar{\rho}_{\tilde{v}})$ by definition. By [BHS19, Corollary 3.7.10], the rigid variety $X^\square_{\mathrm{tri}}(\bar{r})$ is locally irreducible at x_{sat}, so we can identify $X^{\mathfrak{X}^p-\mathrm{aut}}_{\mathrm{tri}}(\bar{\rho}_p)$ to $X^\square_{\mathrm{tri}}(\bar{\rho}_p)$ locally at $(x_{\mathrm{sat},v})_{v\in S_p}$. Therefore, there exists a Zariski-open neighbourhood

$$U_{x_{\mathrm{sat}}} \subseteq U^\square_{\mathrm{tri}}(\bar{r})$$

of x_{sat} such that $\prod_{v\in S_p} U_{x_{\mathrm{sat},v}} \subseteq X^{\mathfrak{X}^p-\mathrm{aut}}_{\mathrm{tri}}(\bar{\rho}_p)$, where each $U_{x_{\mathrm{sat},v}} \subset X^\square_{\mathrm{tri}}(\bar{\rho}_{\tilde{v}})$ corresponds to $U_{x_{\mathrm{sat}}} \subset X^\square_{\mathrm{tri}}(\bar{r})$ under $\bar{\rho}_{\tilde{v}} \simeq \bar{r}$.

Since $U_{x_{\mathrm{sat}}}$ a Zariski-open subset of $U^\square_{\mathrm{tri}}(\bar{r})$, by Theorem 3.3.3 and by base change, $U_{x_{\mathrm{sat}}} \times_{\mathcal{W}^n_L} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}$ is smooth over $\mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}$ hence reduced, and it is a Zariski-open subset of $X^\square_{\mathrm{tri}}(\bar{r}) \times_{\mathcal{W}^n_L} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}$. Consider its Zariski-closure $\overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}^n_L} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}}$, with its structure of reduced rigid analytic space over L; it fits in a chain of Zariski-closed embeddings

$$\begin{aligned}\overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}^n_L} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}} &\hookrightarrow (X^\square_{\mathrm{tri}}(\bar{r}) \times_{\mathcal{W}^n_L} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L})^{\mathrm{red}} \\ &\hookrightarrow X^\square_{\mathrm{tri}}(\bar{r}) \times_{\mathcal{W}^n_L} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L} \hookrightarrow X^\square_{\mathrm{tri}}(\bar{r}) \subseteq \mathfrak{X}^\square_{\bar{r}} \times_L \mathcal{T}^n_L\end{aligned}$$

and the automorphism $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}} \colon \mathfrak{X}^\square_{\bar{r}} \times_L \mathcal{T}^n_L \xrightarrow{\sim} \mathfrak{X}^\square_{\bar{r}} \times_L \mathcal{T}^n_L$ induces a Zariski-closed embedding

$$\jmath_{w,w_{\mathrm{sat}},\mathbf{h}} \colon \overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}^n_L} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}} \hookrightarrow \mathfrak{X}^\square_{\bar{r}} \times_L \mathcal{T}^n_L\,. \tag{3.14}$$

Proposition 3.5.2. *We have*

$$\dim_{k(x)} T_{U_{x_{\mathrm{sat}}}\times_{\mathcal{W}_L^n}\mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L},x_{\mathrm{sat}}} = \dim X^\square_{\mathrm{tri}}(\bar{r}) - d_{ww_{\mathrm{sat}}^{-1}}$$

where $d_{ww_{\mathrm{sat}}^{-1}} \in \mathbb{Z}_{\geq 0}$ is defined in Definition 2.1.1.

Proof. This is the same as [BHS17a, Proposition 5.15]. □

The following result is adapted from [BHS17a, Proposition 5.9].

Proposition 3.5.3. *Assume that the pair (w_{sat}, w) in $(\mathcal{S}_n)^\Sigma$ is good in the sense of Definition 2.3.1. Then* (3.14) *induces a Zariski-closed embedding of reduced rigid analytic spaces over L*

$$\jmath_{w,w_{\mathrm{sat}},\mathbf{h}} \colon \overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}} \hookrightarrow X^\square_{\mathrm{tri}}(\bar{r})\,. \tag{3.15}$$

Proof. It is enough to prove that any point $x' = (r', \underline{\delta}') \in U_{x_{\mathrm{sat}}}$ such that $\underline{\delta}'|_{(\mathcal{O}_K^\times)^n} \in \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}$ satisfies $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(x') \in X^\square_{\mathrm{tri}}(\bar{r})$. By definition, the open subset $\prod_{v\in S_p} U_{x_{\mathrm{sat},v}}$ of $X^\square_{\mathrm{tri}}(\bar{\rho}_p)$ is in the image of the morphism

$$X_p(\bar{\rho}) \hookrightarrow \mathfrak{X}_{\bar{\rho}^p} \times_L X^\square_{\mathrm{tri}}(\bar{\rho}_p) \times_L \times \mathbb{U}^g \twoheadrightarrow X^\square_{\mathrm{tri}}(\bar{\rho}_p) \tag{3.16}$$

where the first arrow is (3.13). Let $y' = (s', \underline{\epsilon}') \in \mathfrak{X}_\infty \times \widehat{T}_{p,L}$ be a point of $X_p(\bar{\rho})$ whose image by (3.16) is $(x'_v)_{v\in S_p}$, where $x'_v \in U_{x_{\mathrm{sat},v}}$ corresponds to $x' \in U_{x_{\mathrm{sat}}}$ under $U_{x_{\mathrm{sat},v}} \simeq U_{x_{\mathrm{sat}}}$. We know in particular that $\underline{\epsilon}' = (\iota_v(\underline{\delta}'_v))_{v\in S_p}$, where $\underline{\delta}'_v$ is the composition of $(F_{\tilde{v}}^\times)^n \xrightarrow{\sim} (K^\times)^n$ with $\underline{\delta}'$. Consider the point

$$\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(y') \coloneqq (s', \jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(\underline{\epsilon}')) \in \mathfrak{X}_\infty \times \widehat{T}_{p,L}$$

where $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(\underline{\epsilon}') \coloneqq (\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(\iota_v(\underline{\delta}')))_{v\in S_p}$. It is enough to prove that $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(y') \in X_p(\bar{\rho})$, since $(\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(x'_v))_{v\in S_p}$ would then be the image of $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(y')$ under (3.16), hence a point of $X^\square_{\mathrm{tri}}(\bar{\rho}_p) = \prod_{v\in S_p} X^\square_{\mathrm{tri}}(\bar{\rho}_{\tilde{v}})$.

By Theorem 3.4.5, all we need to prove is that $\underline{\epsilon}' \uparrow \jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(\underline{\epsilon}')$ in the sense of Definition 3.4.4. Since $\underline{\epsilon}'\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(\underline{\epsilon}')^{-1} = z^{w_{\mathrm{sat}}(\mathbf{h})-w(\mathbf{h})}$ is algebraic by definition of $\jmath_{w,w_{\mathrm{sat}},\mathbf{h}}$, it is equivalent to check that

$$(\mathrm{wt}_\tau(\epsilon_{v,i}))_{\tau\in\Sigma_v,1\leq i\leq n} \uparrow \Big(h_{w_v^{-1}(i)} - h_{w_{\mathrm{sat},v}^{-1}}(i) + \mathrm{wt}_\tau(\epsilon_{v,i})\Big)_{\tau\in\Sigma_v,1\leq i\leq n} \tag{3.17}$$

as elements of $\mathfrak{t}^*_{v,L}$, for all $v \in S_p$. Viewing $\underline{\delta}'$ as its restriction to $(\mathcal{O}_K^\times)^n$, we can write:

$$\underline{\delta}' = (z^{\mathbf{h}_{w_{\mathrm{sat}}^{-1}(1)}}\chi_1, \ldots, z^{\mathbf{h}_{w_{\mathrm{sat}}^{-1}(n)}}\chi_n)$$

where $\underline{\chi} \in \mathcal{W}_L^n$ satisfies $\mathrm{wt}_\tau(\chi_{w_\tau w_{\mathrm{sat},\tau}^{-1}(i)}) = \mathrm{wt}_\tau(\chi_i)$ for all $1 \leq i \leq n$ and $\tau \in \Sigma$. Unpacking the definitions and using the fact that $\chi \circ \theta_K^{-1} = N_{K/\mathbb{Q}_p}|N_{K/\mathbb{Q}_p}|_{\mathbb{Q}_p}$ has the same restriction to $\mathcal{O}_K^\times$ as $N_{K/\mathbb{Q}_p} = z^{\mathbf{1}}$, we see that (3.17) can be reformulated as

$$\left(h_{w_{\mathrm{sat},\tau}^{-1}(i)} + i - 1 + \mathrm{wt}_\tau(\chi_i)\right)_{\tau\in\Sigma, 1\le i\le n} \uparrow \left(h_{w_\tau^{-1}(i)} + i - 1 + \mathrm{wt}_\tau(\chi_i)\right)_{\tau\in\Sigma, 1\le i\le n}.$$

Since $\mathrm{wt}_\tau(\chi_{w_\tau w_{\mathrm{sat},\tau}^{-1}(i)}) = \mathrm{wt}_\tau(\chi_i)$ for all i, this equation can also be written

$$w_{\mathrm{sat}} \cdot \lambda \uparrow w \cdot \lambda \quad \text{with} \quad \lambda := (h_i + i - 1 + \mathrm{wt}_\tau(\chi_{w_\tau(i)}))_{\tau\in\Sigma, 1\le i\le n} \tag{3.18}$$

where $\cdot$ is the dot action of Definition 3.4.2.

Since (w_{sat}, w) is a good pair in $(\mathcal{S}_n)^\Sigma$, there exists a chain $w_0 := w_{\mathrm{sat}} \prec w_1 \prec \ldots \prec w_r := w$ in $(\mathcal{S}_n)^\Sigma$ for some $r \in \mathbb{N}$, such that $w_k w_{k-1}^{-1}$ is a reflection in the minimal generating subsystem of ww_{sat}^{-1} in the sense of Definition 2.1.1, for each $1 \le k \le r$. By the condition on the weights of $\underline{\chi}$, this means in particular that $\mathrm{wt}_\tau(\chi_{w_{k,\tau}w_{k-1,\tau}^{-1}(i)}) = \mathrm{wt}_\tau(\chi_i)$ for each τ, k, i (see Proposition 2.4.1). Thus, by a descending recursion on k, we see that

$$\mathrm{wt}_\tau\left(\chi_{w_\tau w_{k,\tau}^{-1}(i)}\right) = \mathrm{wt}_\tau(\chi_i) = \mathrm{wt}_\tau\left(\chi_{w_\tau w_{k-1,\tau}^{-1}(i)}\right)$$

for each τ, k, i. We then get, for all $1 \le k \le n$,

$$w_k \cdot \lambda - w_{k-1} \cdot \lambda = \left(h_{w_{k,\tau}^{-1}(i)} - h_{w_{k-1,\tau}^{-1}(i)}\right)_{\tau\in\Sigma, 1\le i\le n}$$

thus $w_k \cdot \lambda > w_{k-1} \cdot \lambda$ (remember that $>$ is the partial order on $\mathfrak{t}_L^*$ induced by the lattice of simple roots) since $(h_{\tau,i})_{1\le i\le n}$ is strictly increasing for each $\tau \in \Sigma$ and $w_{k-1} \prec w_k$. This means that $w_{k-1} \cdot \lambda \uparrow w_k \cdot \lambda$ for each $1 \le k \le n$, hence by chaining these together $w_{\mathrm{sat}} \cdot \lambda \uparrow w \cdot \lambda$. This is exactly (3.18). $\square$

Theorem 3.5.4. *We keep the notation and assumptions of the beginning of §3.5. We also assume Hypothesis 3.5.1 (which is implied by Conjecture 3.4.6) and that the pair (w_{sat}, w) in $(\mathcal{S}_n)^\Sigma$ is good in the sense of Definition 2.3.1. Then*

$$\dim_{k(x)} T_{X_{\mathrm{tri}}^\square(\bar r), x} = \dim X_{\mathrm{tri}}^\square(\bar r) - d_{ww_{\mathrm{sat}}^{-1}} + \sum_{\tau\in\Sigma} \dim_{k(x)} T_{\overline{(Bw_\tau B/B)}_{k(x)}^{\mathrm{rig}}, x_\tau} - \lg(w_{\mathrm{sat}})$$

where $(x_\tau)_{\tau\in\Sigma} := h(r, \underline{\varphi}) \in \prod_{\tau\in\Sigma}(Bw_{\mathrm{sat},\tau}B/B)_L^{\mathrm{rig}}$.

Proof. The inequality (right-hand side) $\le$ (right-hand side) is [BHS19, Theorem 4.1.5(ii)] (where our w_{sat} is noted w_x). We prove the inequality in the other direction. Consider the following commutative diagram:

$$\begin{tikzcd} \widetilde{W}_{x_{\mathrm{sat}}} \arrow[r, hook, "\iota_{\mathbf{h}, w_{\mathrm{sat}}}"] \arrow[d, hook, "\subseteq"] & \overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w, w_{\mathrm{sat}}, \mathbf{h}, L}} \arrow[d, hook, "\jmath_{w, w_{\mathrm{sat}}, \mathbf{h}}"] \\ \widetilde{W}_{\bar r, w}^{\mathbf{h}-\mathrm{cr}} \arrow[r, hook, "\iota_{\mathbf{h}, w}"] & X_{\mathrm{tri}}^\square(\bar r) \end{tikzcd} \tag{3.19}$$

where $\widetilde{W}_{x_{\mathrm{sat}}} := \iota_{\mathbf{h},w_{\mathrm{sat}}}^{-1}\left(\overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}}\right)$ is an analytic subset of $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}}$, hence an analytic subset of $\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w}$. From the definitions, we see that $\iota_{\mathbf{h},w_{\mathrm{sat}}} : \widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}} \longrightarrow X^{\square}_{\mathrm{tri}}(\bar{r})$ factors through $X^{\square}_{\mathrm{tri}}(\bar{r}) \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L} \longrightarrow X^{\square}_{\mathrm{tri}}(\bar{r})$. Since $\iota_{\mathbf{h},w_{\mathrm{sat}}}(r,\underline{\varphi}) = x_{\mathrm{sat}} \in U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}$ and $U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}$ is a Zariski-open subset of $X^{\square}_{\mathrm{tri}}(\bar{r}) \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}$, it follows that $T_{\widetilde{W}_{x_{\mathrm{sat}}},(r,\underline{\varphi})} = T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}},(r,\underline{\varphi})}$. Therefore, (3.19) induces a commutative diagram of tangent spaces

$$\begin{tikzcd}
T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}},(r,\underline{\varphi})} \arrow[r, hook] \arrow[d, hook] & T_{\overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}},x_{\mathrm{sat}}} \arrow[d, hook] \\
T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w},(r,\underline{\varphi})} \arrow[r, hook] & T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x} .
\end{tikzcd} \tag{3.20}$$

Let $T_1, T_2 \subseteq T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x}$ be the respective images of the bottom and right arrows of (3.20). Then this diagram induces a chain of morphisms

$$T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}},(r,\underline{\varphi})} \longrightarrow T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w},(r,\underline{\varphi})} \times_{T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x}} T_{\overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}},x_{\mathrm{sat}}} \xrightarrow{\sim} T_1 \cap T_2 \tag{3.21}$$

of vector spaces.

The first morphism in (3.21) is injective since $T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}},(r,\underline{\varphi})} \longrightarrow T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w},(r,\underline{\varphi})}$ is injective; we then wish to prove its surjectivity. Let $A = k(x)[\varepsilon]/(\varepsilon^2)$ be the algebra of infinitesimal numbers over $k(x)$. An element $v_{\mathrm{sat}} \in T_{U_{x_{\mathrm{sat}}},x_{\mathrm{sat}}}$ is a pair $v_{\mathrm{sat}} = (r_A, \underline{\delta}_{\mathrm{sat},A})$ where $r_A \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(A)$ is a representation which reduces to r modulo (ε) and $\underline{\delta}_{\mathrm{sat},A}$ is a character $(K^\times)^n \longrightarrow A^\times$ which reduces to $\underline{\delta}_{\mathrm{sat}}$ modulo (ε). We know from [KPX14, Corollary 6.3.10] that the triangulation on r globalises in a neighbourhood of x_{sat} in $U^{\square}_{\mathrm{tri}}(\bar{r})$, hence r_A is triangular with parameter $\underline{\delta}_{\mathrm{sat},A}$; see the argument in the proof of [BHS17a, Proposition 5.16]. Therefore, an element v in the right-hand side of (3.21) is a triple $v = (r_A, \underline{\varphi}_A, \underline{\delta}_A)$ where $r_A \colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(A)$ is a crystalline representation of Hodge-Tate weights $\mathbf{h}$ which reduces to r modulo (ε), $\underline{\varphi}_A \in (A^\times)^n$ is the ordering of the eigenvalues of r_A which reduces to $\underline{\varphi}$ modulo (ε), and $\underline{\delta}_A$ is the parameter of a triangulation of r_A, such that

$$z^{w^{-1}(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi}_A) = \jmath_{w,w_{\mathrm{sat}},\mathbf{h}}(\underline{\delta}_A) = z^{w^{-1}(\mathbf{h}) - w_{\mathrm{sat}}^{-1}(\mathbf{h})}\underline{\delta}_A .$$

In particular, $\underline{\delta} = z^{w_{\mathrm{sat}}^{-1}(\mathbf{h})}\,\mathrm{unr}(\underline{\varphi})$; hence Proposition 3.2.13(3) implies that $(r_A, \underline{\varphi}_A) \in \widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}}$, which is to say that v is in the image of (3.21). This proves that (3.21) is a chain of isomorphisms of vector spaces.

Therefore,

$$\dim_{k(x)}(T_1 \cap T_2) = \dim_{k(x)} T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}},(r,\underline{\varphi})}$$

thus

$$\begin{aligned}\dim_{k(x)} T_{X^{\square}_{\mathrm{tri}}(\bar{r}),x} \geq \dim_{k(x)}(T_1 \cup T_2) = \dim_{k(x)} & T_{\overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}},x_{\mathrm{sat}}} \\ & + \dim_{k(x)} T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w},(r,\underline{\varphi})} - \dim_{k(x)} T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}},(r,\underline{\varphi})} .\end{aligned} \tag{3.22}$$

By Proposition 3.5.2, we have

$$\dim_{k(x)} T_{\overline{U_{x_{\mathrm{sat}}} \times_{\mathcal{W}_L^n} \mathcal{W}^n_{w,w_{\mathrm{sat}},\mathbf{h},L}}, x_{\mathrm{sat}}} = \dim X^{\square}_{\mathrm{tri}}(\bar{r}) - d_{ww_{\mathrm{sat}}^{-1}}\,. \tag{3.23}$$

Also, the cartesian diagram

$$\begin{array}{ccc} \widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}} & \xrightarrow{h} & \prod_{\tau\in\Sigma} \overline{(Bw_{\mathrm{sat},\tau}B/B)}^{\mathrm{rig}}_L \\ \big\downarrow \subseteq & & \big\downarrow \subseteq \\ \widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w} & \xrightarrow{h} & \prod_{\tau\in\Sigma} \overline{(Bw_{\tau}B/B)}^{\mathrm{rig}}_L \end{array}$$

and the smoothness of h yield

$$\begin{aligned} &\dim_{k(x)} T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w},(r,\underline{\varphi})} - \dim_{k(x)} T_{\widetilde{W}^{\mathbf{h}-\mathrm{cr}}_{\bar{r},w_{\mathrm{sat}}},(r,\underline{\varphi})} \\ &\qquad = \sum_{\tau\in\Sigma} \dim_{k(x)} T_{\overline{(Bw_\tau B/B)}^{\mathrm{rig}}_{k(x)},x_\tau} - \sum_{\tau\in\Sigma} \dim_{k(x)} T_{\overline{(Bw_{\mathrm{sat},\tau} B/B)}^{\mathrm{rig}}_{k(x)},x_\tau}\,. \end{aligned} \tag{3.24}$$

Finally, as (x_τ) lies in the smooth cell $\prod_{\tau\in\Sigma}(Bw_{\mathrm{sat},\tau}B/B)^{\mathrm{rig}}_L$, we have

$$\sum_{\tau\in\Sigma} \dim_{k(x)} T_{\overline{(Bw_{\mathrm{sat},\tau} B/B)}^{\mathrm{rig}}_{k(x)},x_\tau} = \dim \prod_{\tau\in\Sigma} (Bw_{\mathrm{sat},\tau}B/B)^{\mathrm{rig}}_L = \lg(w_{\mathrm{sat}})\,. \tag{3.25}$$

Putting together (3.22), (3.23), (3.24), (3.25) gives the desired inequality, which finishes the proof. □

Remark 3.5.5. We give in Chapter 4 another proof of Theorem 3.5.4 which eliminates the need for Hypothesis 3.5.1; see Theorem 4.2.4. We also state an consequence of Theorem 3.5.4 for smoothness of $X^{\square}_{\mathrm{tri}}(\bar{r})$ at x: see Corollary 4.2.5.

Chapter 4

A related conjecture on the local model

Let k be a field of characteristic 0 with algebraic closure k^{a}. As in section 2, we use the following notation.

- G is a connected split reductive algebraic group over k.
- T is a split maximal torus inside G.
- B is a Borel subgroup of G containing T.
- $U \subset B$ is the subgroup of unipotent elements in B.
- $\Phi := \Phi(G,T) \subset \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$ is the set of roots of (G,T).
- $U_\alpha \subset G$ is the root group of the root $\alpha \in \Phi$ (see [Mil17, 21.10, 21.11]).
- $\mathfrak{g}$ is the Lie algebra of G, *i.e.* the schematic affine space over k such that $\mathfrak{g}(k)$, as a finite-dimensional vector space over k, is naturally isomorphic to the tangent space T_eG of G at the neutral element $e \in G$ (note that this differs from the standard point of view where $\mathfrak{g}$ is simply defined as T_eG). Similarly, $\mathfrak{t}$, $\mathfrak{b}$, $\mathfrak{u}$, $\mathfrak{u}_\alpha$ are the Lie algebras of T, B, U, U_α respectively ($\mathfrak{u}_\alpha$ is sometimes also noted $\mathfrak{g}_\alpha$).
- $\mathrm{Ad}\colon G \longrightarrow \mathrm{GL}(\mathfrak{g})$ and $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ are the adjoint representations.
- I is the basis of the root system Φ defined by the Borel B.
- Φ_+ is the set of positive roots defined by I.
- $W := N_G(T)/T$ is the Weyl group of (G,T) (or equivalently, the Weyl group of Φ; see §2.2).

We deal with algebraic varieties over k, *i.e.* reduced separated schemes of finite type over k (see [Vak17, Definition 10.1.7]). Morphisms of varieties are then uniquely determined by their value on k^{a}-points, and their "image" (more precisely, surjectivity onto a subvariety of the target) can be checked on closed points (see [Vak17, Exercise 7.4.C]). Given two varieties X_1 and X_2 equipped with immersions into a third variety X, we write

$$X_1 \cap X_2 \coloneqq (X_1 \times_X X_2)^{\mathrm{red}}$$

when there is no ambiguity. By base change, the natural map $X_1 \cap X_2 \longrightarrow X_1$ is a closed (resp. locally closed) immersion when $X_2 \hookrightarrow X$ is closed (resp. locally closed), and similarly for $X_1 \cap X_2 \longrightarrow X_2$. In the same way, given a morphism of varieties $f \colon X \longrightarrow Y$ and a closed (resp. locally closed) immersion $Y' \hookrightarrow Y$, we write

$$f^{-1}(Y') \coloneqq (X \times_Y Y')^{\mathrm{red}} ,$$

and the natural map $f^{-1}(Y') \longrightarrow X$ is a closed (resp. locally closed) immersion.

4.1 A scheme related to the Springer resolution

Recall the Schubert decomposition on G/B. Let $w \in W$ and $\dot{w} \in N_G(T)(k)$ be any closed point lifting w. The left-action of B on G by multiplication induces an action on G/B, write BwB/B for the orbit of $\dot{w}B/B$ under this action. Then BwB/B is a smooth locally closed subvariety of G/B called a *Schubert cell*, and we have the decomposition

$$G/B = \coprod_{w \in W} BwB/B \tag{4.1}$$

(see [Mil17, Theorem 21.73]). The cells BwB/B are stable under the left action of B, thus their Zariski closure $\overline{BwB/B}$ too, and this action is transitive on BwB/B. Thus, a closed cell $\overline{BwB/B}$ intersects an open cell $Bw'B/B$ if and only if $Bw'B/B \subseteq \overline{BwB/B}$, which happens if and only if $w' \preceq w$ for the Bruhat order (see [BGG73, Theorem 2.11]). Therefore (4.1) gives a good stratification of G/B, and we have a more general decomposition into locally closed strata

$$\overline{BwB/B} = \coprod_{w' \preceq w} Bw'B/B$$

for all $w \in W$.

One can get a similar stratification on $G/B \times G/B$: consider the isomorphism

$$f \colon \begin{array}{ccc} G \times^B G/B & \xrightarrow{\sim} & G/B \times G/B \\ (g_1, g_2B) & \longmapsto & (g_1B, g_1g_2B) \end{array} \tag{4.2}$$

where $G \times^B G/B$ is the quotient of $G \times G/B$ by the right B-action defined by $(g_1, g_2B)b \coloneqq (g_1b, b^{-1}g_2B)$ (see [Slo80, §3.7]). For $w \in W$, the right B-action leaves $G \times (BwB/B)$ invariant, so we can write

$$U_w \coloneqq f(G \times^B (BwB/B)) \subseteq G/B \times G/B .$$

Equivalently, U_w is the orbit of $(B/B, \dot{w}B/B)$ under the left G-action on $G/B \times G/B$ given by diagonal left multiplication. Then $G/B \times G/B = \coprod_{w\in W} U_w$ and for any $w \in W$, the decomposition into locally closed strata

$$\overline{U_w} = \coprod_{w' \preceq w} U_{w'}$$

is a consequence of the analogous one in G/B.

As in [BHS19], let X be the following algebraic variety:

$$X \coloneqq \left\{(g_1B, g_2B, \psi) \in G/B \times G/B \times \mathfrak{g} \;\middle|\; \mathrm{Ad}(g_1^{-1})\psi \in \mathfrak{b}\,,\, \mathrm{Ad}(g_2^{-1})\psi \in \mathfrak{b}\right\}\,.$$

Let us write π for the composition $X \hookrightarrow G/B \times G/B \times \mathfrak{b} \twoheadrightarrow G/B \times G/B$. For $w \in W$, define

$$V_w \coloneqq \pi^{-1}(U_w) \subset X\,, \quad X_w \coloneqq \overline{V_w}\,.$$

Note that V_w is open in X_w but not in X, since it is only locally closed. We also define two morphisms

$$\kappa_1\colon \begin{array}{ccc} X & \longrightarrow & \mathfrak{t} \\ (g_1B, g_2B, \psi) & \longmapsto & \overline{\mathrm{Ad}(g_1^{-1})\psi} \end{array}, \quad \kappa_2\colon \begin{array}{ccc} X & \longrightarrow & \mathfrak{t} \\ (g_1B, g_2B, \psi) & \longmapsto & \overline{\mathrm{Ad}(g_2^{-1})\psi} \end{array}$$

where $\overline{\mathrm{Ad}(g_1^{-1})\psi}$ (resp. $\overline{\mathrm{Ad}(g_2^{-1})\psi}$) denotes the image of $\mathrm{Ad}(g_1^{-1})\psi$ (resp. $\mathrm{Ad}(g_2^{-1})\psi$) under the projection $\mathfrak{b} \twoheadrightarrow \mathfrak{b}/\mathfrak{u} = \mathfrak{t}$.

We have the left action of G on $G/B \times G/B$ by diagonal left multiplication, as well as the action of G on $\mathfrak{g}$ by adjunction. This gives a diagonal action on $G/B \times G/B \times \mathfrak{g}$ by $g \cdot (g_1B, g_2B, \psi) = (gg_1B, gg_2B, \mathrm{Ad}(g)\psi)$. For this action, X is stable and $\pi\colon X \longrightarrow G/B \times G/B$ is G-equivariant. It therefore induces an action on V_w for any given $w \in W$.

The cell decompositions $\overline{U_w} = \coprod_{w' \preceq w} U_{w'}$ for $w \in W$ imply that $X_w \cap V_{w'} = \emptyset$ unless $w' \preceq w$, and that (set theoretically)

$$X_w = \coprod_{w' \preceq w} X_w \cap V_{w'}\,.$$

The aim of this section is to prove some cases of the following conjecture on the cells $X_w \cap V_{w'}$, which appears as 2.3.7 in [BHS19].

Conjecture 4.1.1 (Breuil-Hellmann-Schraen). *Let $w, w' \in W$ such that $w' \preceq w$. Then*

$$X_w \cap V_{w'} = V_{w'} \cap {\kappa_1}^{-1}(\mathfrak{t}^{ww'^{-1}})$$

as reduced closed subschemes of X.

A consequence of this conjecture is the following proposition, that is mentioned in [BHS19, Remark 2.5.4] at least when $w = w_0$ is the element of maximal length in W. We provide here a proof for the general case.

Proposition 4.1.2. *Let $w, w' \in W$ and $x = (\pi(x), 0)$ be a closed point of $X_w \cap V_{w'}$. If Conjecture 4.1.1 is true for (w', w), then*

$$\dim_{k(x)} T_{X_w,x} = \dim_{k(\pi(x))} T_{\overline{U_w},\pi(x)} + \dim_{k(x)} \mathfrak{t}^{ww'^{-1}}(k(x)) + \lg(w'w_0)\,. \tag{4.3}$$

Proof. Since the action of G on $U_{w'}$ is transitive and π is G-equivariant, we can assume $\pi(x) = (B, w'B)$ (in particular the residue field $k(x)$ is k). Consider the composition $X_w \hookrightarrow X \hookrightarrow G/B \times G/B \times \mathfrak{g}$; it factors through a closed immersion $\iota := X_w \hookrightarrow \overline{U_w} \times \mathfrak{g}$. Because $\pi(x) = (B, w'B)$, the proof of [BHS19, Proposition 2.5.3] shows that the differential $\mathrm{d}_x\iota\colon T_{X_w,x} \hookrightarrow T_{\overline{U_w},\pi(x)} \oplus T_{\mathfrak{g},0} = T_{\overline{U_w},\pi(x)} \oplus \mathfrak{g}(k)$ factors through an injection of k-vector spaces

$$\mathrm{d}_x\iota\colon T_{X_w,x} \hookrightarrow T_{\overline{U_w},\pi(x)} \oplus \mathfrak{t}^{ww'^{-1}}(k) \oplus (\mathfrak{u}(k) \cap \mathrm{Ad}(w')\mathfrak{u}(k))\,. \tag{4.4}$$

Assuming Conjecture 4.1.1, we wish to prove the surjectivity of this map; comparing dimensions would then give the desired equality (4.3).

Consider the closed immersion $G/B \times G/B \hookrightarrow X$ given by $(g_1B, g_2B) \longmapsto (g_1B, g_2B, 0)$. Its restriction to $\overline{U_w}$ factors through a closed immersion $\iota_1 := \overline{U_w} \hookrightarrow X_w$. Since $\iota \circ \iota_1$ is the closed immersion $\overline{U_w} \longrightarrow \overline{U_w} \times \mathfrak{g}$ given by $u \longrightarrow (u, 0)$, the composition $\mathrm{d}_x\iota \circ \mathrm{d}_{\pi(x)}\iota_1$ is the inclusion $T_{\overline{U_w},\pi(x)} \hookrightarrow T_{\overline{U_w},\pi(x)} \oplus \mathfrak{g}(k)$. Hence, in (4.4), the first summand of the right-hand side is contained in the image of $\mathrm{d}_x\iota$.

Now, we know from Proposition 2.2.1 of [BHS19] that $\pi\colon V_{w'} \longrightarrow U_{w'}$ is a geometric vector bundle, and the fiber of $\pi(x)$ in X is

$$\pi^{-1}(\pi(x)) = (B, w'B) \times (\mathfrak{t} \oplus (\mathfrak{u} \cap \mathrm{Ad}(w')\mathfrak{u})) \hookrightarrow U_{w'} \times \mathfrak{g} \hookrightarrow G/B \times G/B \times \mathfrak{g}\,. \tag{4.5}$$

Take $Y := (B, w'B) \times (\mathfrak{t}^{ww'^{-1}} \oplus (\mathfrak{u} \cap \mathrm{Ad}(w')\mathfrak{u}))$. The closed immersion of reduced schemes $Y \hookrightarrow X$ factors through $Y \hookrightarrow V_{w'}$ and $Y \hookrightarrow \kappa_1^{-1}(\mathfrak{t}^{ww'^{-1}})$. This gives a closed immersion

$$Y \hookrightarrow \left(V_{w'} \times_X {\kappa_1}^{-1}(\mathfrak{t}^{ww'^{-1}})\right)^{\mathrm{red}} = V_{w'} \cap {\kappa_1}^{-1}(\mathfrak{t}^{ww'^{-1}})\,.$$

Conjecture 4.1.1 of [BHS19] gives a closed immersion $V_{w'} \cap {\kappa_1}^{-1}(\mathfrak{t}^{ww'^{-1}}) \hookrightarrow X_w$. The composition $\iota_2 := Y \hookrightarrow X_w$ is such that $\iota \circ \iota_2$ is the composition $Y \hookrightarrow U_{w'} \times \mathfrak{g} \hookrightarrow \overline{U_w} \times \mathfrak{g}$. Therefore, by definition of Y, the differential $\mathrm{d}_x\iota \circ \mathrm{d}_x\iota_2$ composed with the projection $T_{\overline{U_w} \times \mathfrak{g},x} \twoheadrightarrow T_{\mathfrak{g},0}$ has image $\mathfrak{t}^{ww'^{-1}} \oplus (\mathfrak{u} \cap \mathrm{Ad}(w')\mathfrak{u})$. This gives the second and third summands of (4.4). $\square$

4.2 Statement of the main result

In the following we work with a simplified version of X. Recall from [Slo80, §4.7] or [KW01, §VI.8] *Grothendieck's simultaneous resolution of singularities*

$$q\colon \begin{array}{rcl} G\times^B \mathfrak{b} & \longrightarrow & \mathfrak{g} \\ (g,\psi) & \longmapsto & \mathrm{Ad}(g)\psi \end{array} \tag{4.6}$$

where $G\times^B \mathfrak{b}$ is the quotient of $G\times\mathfrak{b}$ by the right B-action given by $(g,\psi)b := (gb,\mathrm{Ad}(b^{-1})\psi)$. There is also a map $\widetilde{\pi}\colon G\times^B\mathfrak{b}\longrightarrow G/B$ given by $\widetilde{\pi}(g,\psi) = gB$.

Define, for $w\in W$:

$$\widetilde{X} := q^{-1}(\mathfrak{b}), \quad \widetilde{V}_w := \widetilde{X}\cap\widetilde{\pi}^{-1}(BwB/B), \quad \widetilde{X}_w := \overline{\widetilde{V}_w}.$$

Similarly as for X, we have the decomposition $\widetilde{X}_w = \coprod_{w'\preceq w}\widetilde{X}_w\cap\widetilde{V}_{w'}$ for any $w\in W$, and $\widetilde{X}_w\cap\widetilde{V}_{w'}\neq\emptyset$ implies that $w'\preceq w$.

The goal of §4 is to prove the following reformulation of Conjecture 2.3.7 of [BHS19] in the case of good pairs:

Conjecture 4.2.1 (Breuil-Hellmann-Schraen)**.** *Let $w,w'\in W$. Consider the locally closed immersion $\widetilde{V}_{w'}\hookrightarrow\widetilde{X}$ which induces a locally closed immersion $\widetilde{V}_{w'}\cap q^{-1}(\mathfrak{t}^{ww'^{-1}}+\mathfrak{u})\hookrightarrow\widetilde{X}$.*

If $w'\preceq w$, then the latter factors through $\widetilde{X}_w\hookrightarrow\widetilde{X}$ to give a locally closed immersion

$$\widetilde{V}_{w'}\cap q^{-1}(\mathfrak{t}^{ww'^{-1}}+\mathfrak{u})\hookrightarrow\widetilde{X}_w.$$

Proposition 4.2.2. *Conjecture 4.1.1 and Conjecture 4.2.1 are equivalent.*

Proof. We have an isomorphism

$$G\times^B\widetilde{X}\xrightarrow{\sim}X,\quad (g_1,(g_2,\psi))\longmapsto(g_1B,g_1g_2B,\mathrm{Ad}(g_1g_2)\psi) \tag{4.7}$$

where B has a right-action on $G\times\widetilde{X}$ given by $(g_1,(g_2,\psi))b := (g_1b,(b^{-1}g_2,\psi))$. This isomorphism commutes with κ_1 and the composition of q with $\mathfrak{b}\longrightarrow\mathfrak{b}/\mathfrak{u}=\mathfrak{t}$, identifies $G\times^B\widetilde{V}_{w'}$ with $V_{w'}$ and identifies $G\times^B\widetilde{X}_w$ with X_w. Thus everything translates from $\widetilde{X}$ to X.

Therefore Conjecture 4.2.1 is equivalent to having a locally closed immersion $V_{w'}\cap\kappa_1^{-1}(\mathfrak{t}^{ww'^{-1}})\hookrightarrow X_w\cap V_{w'}$, and [BHS19, Lemma 2.3.4] shows that it has to be an isomorphism. □

Theorem 4.2.3. *Let (w',w) be a good pair in W. Then Conjecture 4.1.1 and Conjecture 4.2.1 are true for the pair (w',w),* i.e. *the locally closed immersion $\widetilde{V}_{w'}\hookrightarrow\widetilde{X}$ induces*

$$\widetilde{V}_{w'}\cap q^{-1}(\mathfrak{t}^{ww'^{-1}}+\mathfrak{u})\hookrightarrow\widetilde{X}_w. \tag{4.8}$$

We postpone the proof of Theorem 4.2.3 to section 4.4, and first state a major consequence: a proof of Theorem 3.5.4 which does not need Hypothesis 3.5.1.

Theorem 4.2.4. *Let K, L be finite extensions of $\mathbb{Q}_p$ such that L splits K, let $\bar{r}\colon \mathcal{G}_K \longrightarrow \mathrm{GL}_n(k_L)$ be a continuous representation, let $X_{\mathrm{tri}}^{\square}(\bar{r})$ be the trianguline variety (see Definition 3.3.2). Let $x \in X_{\mathrm{tri}}^{\square}(\bar{r})$ be a crystalline point satisfying the assumptions of §3.5 (except for Hypothesis 3.5.1); we have defined $w, w_{\mathrm{sat}} \in W = (\mathcal{S}_n)^{\mathrm{Hom}(K,L)}$ associated to x.*

If (w_{sat}, w) is a good pair in the sense of Definition 2.3.1 (see also the paragraph following Definition 3.4.2), then

$$\dim_{k(x)} T_{X_{\mathrm{tri}}^{\square}(\bar{r}),x} = \dim X_{\mathrm{tri}}^{\square}(\bar{r}) - d_{ww_{\mathrm{sat}}^{-1}} + \sum_{\tau\in\Sigma} \dim_{k(x)} T_{\overline{(Bw_\tau B/B)}^{\mathrm{rig}}_{k(x)},x_\tau} - \lg(w_{\mathrm{sat}}) \quad (4.9)$$

(see the statement of Theorem 3.5.4 for a definition of x_τ).

Proof. It is shown in the proof of [BHS19, Proposition 4.1.5] that this is a consequence of (4.3), which itself is a consequence of Theorem 4.2.3, Proposition 4.2.2 and Proposition 4.1.2. □

Corollary 4.2.5. *If formula (4.9) is true and $X_{\mathrm{tri}}^{\square}(\bar{r})$ is smooth at x, then (w_{sat}, w) is a good pair and $\overline{(Bw_\tau B/B)}^{\mathrm{rig}}$ is smooth at x_τ for each $\tau \in \mathrm{Hom}(K, L)$.*

Proof. Since $\dim \overline{(Bw_\tau B/B)}^{\mathrm{rig}} = \lg(w_\tau)$ and $\sum_{\tau\in\mathrm{Hom}(K,L)} \lg(w_\tau) = \lg(w)$, this is a direct consequence of Proposition 2.3.4. □

4.3 Levi subgroups and the adjoint action

This section establishes a few technical lemmas about Levi subgroups and the derived adjoint action, which are used in the proof of Theorem 4.2.3.

Let $J \subseteq I$ be a subset of simple roots of G. We introduce the following notation.

- $\Phi_J \subseteq \Phi$ is the root subsystem generated by J.
- $T_J \coloneqq \left(\bigcap_{\alpha\in\Phi_J}(\ker\alpha)\right)^\circ \subseteq T$ is the largest subtorus of T such that $\alpha(T_J) = 1$ for all $\alpha \in \Phi_J$ (or simply $\alpha \in J$). It is indeed a torus, as it is a reduced (the characteristic is 0) and connected diagonalisable group; see the discussion in [Mil17, Notation 12.29].
- $L \coloneqq C_G(T_J)$ is the standard Levi subgroup of G associated to J. According to [Mil17, Proposition 21.90], L is a connected reductive group with T as a maximal torus and root system Φ_J. Therefore $L = \langle T, U_\alpha \mid \alpha \in \Phi_J\rangle$, *i.e.* L is the subgroup of G generated by T and the U_α for $\alpha \in \Phi_J$.
- $U_L \coloneqq \langle U_\alpha \mid \alpha \in \Phi_J \cap \Phi_+\rangle$ is the unipotent radical of $L \cap B$.
- $\mathfrak{t}_J$, $\mathfrak{l}$ and $\mathfrak{u}_L$ are the Lie algebras of T_J, L and U_L respectively.

For a root $\alpha\colon T \longrightarrow \mathbb{G}_{\mathrm{m}}$, consider its differential $\mathrm{d}_e\alpha\colon \mathfrak{t} \longrightarrow k$; we can see it as $\mathrm{Lie}(\alpha)$, where Lie is the Lie algebra functor from algebraic groups over k to Lie algebras over k. Since Lie commutes with finite limits, and in particular with kernels and fibred products (see [Mil17, 10.14]), we have

$$\mathfrak{t}_J = \bigcap_{\alpha\in\Phi_J} (\ker \mathrm{d}_e\alpha)\,.$$

Proposition 4.3.1. *There is a Zariski-dense open subscheme $\mathfrak{t}_J^{\mathrm{gen}}$ of the affine scheme $\mathfrak{t}_J$ such that, for any closed point $t \in \mathfrak{t}_J^{\mathrm{gen}}$:*

(1) $L = C_G(t)^\circ \coloneqq \{g \in G \mid \mathrm{Ad}(g)t = t\}^\circ$*;*

(2) $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}}(t) \coloneqq \{x \in \mathfrak{g} \mid [x,t] = 0\}$*.*

Proof. For any closed point $t \in \mathfrak{t}$, the centraliser $C_G(t)$ contains T and is normalised by T, therefore the same is true of the identity component $C_G(t)^\circ$ since T is connected. By [Mil17, 21.66], we then have

$$C_G(t)^\circ = \langle T, U_\alpha \mid \alpha\in\Phi,\, U_\alpha \subseteq C_G(t)^\circ\rangle = \langle T, U_\alpha \mid \alpha\in\Phi,\, U_\alpha \subseteq C_G(t)\rangle \tag{4.10}$$

since each U_α is connected.

Let $\alpha \in \Phi$ be a root and $t \in \mathfrak{t}(k^{\mathrm{a}})$ be a k^{a}-point such that $U_\alpha \subseteq C_G(t)$. Then the morphism

$$\phi\colon U_\alpha \longrightarrow \mathfrak{g}\,, \quad u \longmapsto \mathrm{Ad}(u)t$$

is constant, so its differential

$$\mathrm{d}_e\colon \mathfrak{u}_\alpha \longrightarrow \mathfrak{g}\,, \quad u_\alpha \longmapsto \mathrm{ad}(u_\alpha)t$$

is zero. But for any $u_\alpha \in \mathfrak{u}_\alpha(k^{\mathrm{a}})$, we have $\mathrm{ad}(u_\alpha)t = [u_\alpha, t] = -\mathrm{d}_e\alpha(t)u_\alpha$. This shows that $t \in \ker \mathrm{d}_e\alpha$.

Not let $\alpha \in \Phi\setminus\Phi_J$. Since Φ_J is the root system of L, necessarily $U_\alpha \not\subseteq L = C_G(T_J)$. Therefore there exist $u_\alpha \in U_\alpha(k^{\mathrm{a}})$ and $t \in T_J(k^{\mathrm{a}})$ such that $u_\alpha t \neq t u_\alpha$, which means $u_\alpha \neq t u_\alpha t^{-1}$. This implies $\alpha(t) \neq 1$ (see [Mil17, 21.19]). Hence $\ker(\alpha)\cap T_J \neq T_J$. Applying the Lie functor (which preserves finite limits), we get $\ker(\mathrm{d}_e\alpha)\cap\mathfrak{t}_J \neq \mathfrak{t}_J$, so $\mathfrak{t}_J \setminus (\ker(\mathrm{d}_e\alpha)\cap\mathfrak{t}_J)$ is dense open in $\mathfrak{t}_J$. We thus define

$$\mathfrak{t}_J^{\mathrm{gen}} \coloneqq \mathfrak{t}_J \setminus \left(\bigcup_{\alpha\in\Phi\setminus\Phi_J} \ker(\mathrm{d}_e\alpha)\cap\mathfrak{t}_J\right) = \bigcap_{\alpha\in\Phi\setminus\Phi_J} (\mathfrak{t}_J\setminus(\ker(\mathrm{d}_e\alpha)\cap\mathfrak{t}_J))$$

which is dense open in $\mathfrak{t}_J$.

Fix a closed point $t \in \mathfrak{t}_J^{\mathrm{gen}}$. For any root $\alpha\in\Phi$, the previous discussion shows that $U_\alpha \subseteq C_G(t)$ implies $\alpha \in \Phi_J$. By (4.10), we have

$$C_G(t) \subseteq \langle T, U_\alpha \mid \alpha \in \Phi_J\rangle = L\,.$$

The converse $L = C_G(T_J) \subseteq C_G(t)$ is true for any $t \in \mathfrak{t}_J$, so (1) is proved.

Statement (2) is a consequence of (1) and [Mil17, Proposition 17.76]. $\square$

The following lemma explains the relevance of Proposition 4.3.1.

Lemma 4.3.2. *Let $\mathfrak{t}' \subseteq \mathfrak{t}^{ww'^{-1}}$ be a Zariski-dense subscheme. Then the k-algebraic variety*

$$\widetilde{V}(w', \mathfrak{t}') \coloneqq \widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}' + \mathfrak{u})$$

is a dense subvariety of

$$\widetilde{V}(w', \mathfrak{t}^{ww'^{-1}}) \coloneqq \widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})\,.$$

Proof. Recall the morphism $\widetilde{\pi}\colon G \times^B \mathfrak{b} \longrightarrow G/B$, which restricts to two morphisms

$$\begin{aligned} \widetilde{\pi}_1 \colon \quad & q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u}) \longrightarrow G/B\,, \\ \widetilde{\pi}'_1 \colon \quad & q^{-1}(\mathfrak{t}' + \mathfrak{u}) \longrightarrow G/B\,. \end{aligned}$$

Then $\widetilde{V}(w', \mathfrak{t}^{ww'^{-1}})$ (resp. $\widetilde{V}(w', \mathfrak{t}')$) is just the fibre of $\widetilde{\pi}_1$ (resp. $\widetilde{\pi}'_1$) at $Bw'B/B$.

It is thus enough to check density on fibres at closed points: let $gB \in Bw'B/B \subset G/B$ with $g \in G$, we want to show that $\widetilde{\pi}'^{-1}_1(gB)$ is dense in $\widetilde{\pi}^{-1}_1(gB)$. Now the left-action of B on $G \times \mathfrak{b}$ given by $b(g,\psi) = (bg, \psi)$ induces a left-action on $G \times^B \mathfrak{b}$. This action stabilises $q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})$ and $q^{-1}(\mathfrak{t}' + \mathfrak{u})$ because $\mathrm{Ad}(b)$ stabilises $t + \mathfrak{u}$ for any $b \in B(k^{\mathrm{a}})$ and $t \in \mathfrak{t}(k^{\mathrm{a}})$. Moreover, this action induces an action on the set of fibres of $\widetilde{\pi}_1$ that is continuous in the sense that for any $b \in B$ and $gB \in G/B$, we have an isomorphism of k-algebraic varieties $\widetilde{\pi}^{-1}_1(gB) \longrightarrow \widetilde{\pi}^{-1}_1(bgB)$, $x \longmapsto bx$; and similarly for $\widetilde{\pi}'_1$. Also, this action on the fibres is obviously transitive on the subset of fibres at points of $Bw'B/B$.

Therefore it suffices to consider the fibre at $w'B \in G/B$: we want to check that $\widetilde{\pi}'^{-1}_1(ww'B)$ is dense in $\widetilde{\pi}^{-1}_1(w'B)$. Combining the formula (4.5) from [BHS19, Proposition 2.2.1] with the isomorphism (4.7) gives

$$\widetilde{\pi}'^{-1}_1(ww'B) = \big\{(w', \psi) \in G \times^B \mathfrak{b} \;\big|\; \psi \in \mathfrak{t}' \oplus (\mathfrak{u} \cap \mathrm{Ad}(w')\mathfrak{u})\big\}$$

and

$$\widetilde{\pi}^{-1}_1(ww'B) = \left\{(w', \psi) \in G \times^B \mathfrak{b} \;\middle|\; \psi \in \mathfrak{t}^{ww'^{-1}} \oplus (\mathfrak{u} \cap \mathrm{Ad}(w')\mathfrak{u})\right\}\,,$$

from which we easily deduce the desired density statement. $\square$

We introduce other objects associated to J.

- $P \coloneqq P_J = \langle U_\alpha \mid \alpha \in \Phi_J \cup \Phi_+ \rangle$ be the parabolic subgroup associated to J.
- $U_P \coloneqq \langle U_\alpha \mid \alpha \in \Phi_+ \setminus \Phi_J \rangle$ is the unipotent radical of P.
- $W_J \coloneqq W(\Phi_J) \subseteq W$ is the Weyl group of Φ_J; therefore $W_J = N_L(T)/T$ is also the Weyl group of L.

– $W^J \subset W$ is the set of $w \in W$ such that $ws_\alpha \succ w$ for all $\alpha \in \Phi_J$, *i.e.* such that $w(\alpha) \in \Phi_+$ for all $\alpha \in \Phi_J \cap \Phi_+$.

The reductive group L is in fact the Levi factor of P, and we have the decomposition $P = LU_P$ (see [Mil17, Theorem 21.91]). Also recall from §2.3 the decomposition $W = W^J W_J$.

Lemma 4.3.3. *Let $l \in L$ and $w^J \in W^J$ be such that $w^J l (w^J)^{-1} \in B$. Then $l \in L \cap B$.*

Proof. From the Bruhat decomposition in L, let $w_J \in W_J$ be such that $l \in (L \cap B) w_J (L \cap B)$. The hypothesis $w^J \in W^J$ can be reformulated as $w^J (L \cap B)(w^J)^{-1} \subseteq B$. Thus we have $w^J l (w^J)^{-1} \in B w^J w_J (w^J)^{-1} B$. The Bruhat decomposition in G and our hypothesis then give $w^J w_J (w^J)^{-1} = 1$, which is $w_J = 1$, hence $l \in B$. $\square$

Lemma 4.3.4. *Let $t \in \mathfrak{t}_J^{\mathrm{gen}}(k^{\mathrm{a}})$ be a k^{a}-point. Let $w^J \in W^J$, $u_P \in U_P(k^{\mathrm{a}})$, $l \in L(k^{\mathrm{a}})$ and $x \in \mathfrak{u}_L(k^{\mathrm{a}})$ be such that $\mathrm{Ad}(w^J u_P l)(t + x) \in \mathfrak{b}(k^{\mathrm{a}})$. Then $\mathrm{Ad}(l)x \in \mathfrak{u}_L(k^{\mathrm{a}})$.*

Proof. Since U_P is normal in P, the action of L on P by conjugation (resp. the Lie algebra of P by adjunction) keeps U_P (resp. $\mathfrak{u}_P$) stable. Also, the action of U_P by conjugation on B induces a trivial action on B/U_P, hence on $\mathfrak{b}/\mathfrak{u}_P(k^{\mathrm{a}})$ as well. Therefore, setting $u'_P \coloneqq l^{-1} u_P l \in U_P(k^{\mathrm{a}})$, since $t + x \in \mathfrak{b}(k^{\mathrm{a}})$:

$$y \coloneqq \mathrm{Ad}(u'_P)(t + x) - (t + x) \in \mathfrak{u}_P(k^{\mathrm{a}})\,.$$

Hence, setting $z \coloneqq \mathrm{Ad}(l)y \in \mathfrak{u}_P(k^{\mathrm{a}})$, we have using $\mathrm{Ad}(l)t = t$:

$$\mathrm{Ad}(w^J u_P l)(t + x) = \mathrm{Ad}(w^J l u'_P)(t + x) = \mathrm{Ad}(w^J)(t + \mathrm{Ad}(l)x + z)$$

which is in $\mathfrak{b}(k^{\mathrm{a}}) = \mathfrak{t}(k^{\mathrm{a}}) \oplus \bigoplus_{\alpha \in \Phi_+} \mathfrak{u}_\alpha(k^{\mathrm{a}})$ by assumption.

Now $\mathrm{Ad}(l)x \in \bigoplus_{\alpha \in \Phi_J} \mathfrak{u}_\alpha(k^{\mathrm{a}})$ and $z \in \mathfrak{u}_P(k^{\mathrm{a}}) \subset \bigoplus_{\alpha \in \Phi \setminus \Phi_J} \mathfrak{u}_\alpha(k^{\mathrm{a}})$, therefore $\mathrm{Ad}(w^J)\,\mathrm{Ad}(l)x$ and $\mathrm{Ad}(w^J)z$ have disjoint components in the sum $\bigoplus_{\alpha \in \Phi} \mathfrak{u}_\alpha(k^{\mathrm{a}})$. In particular they each cannot have any component corresponding to some $\alpha \notin \Phi_+$, so

$$\mathrm{Ad}(w^J)\,\mathrm{Ad}(l)x \in \mathfrak{u}(k^{\mathrm{a}})\,.$$

However, as $w^J \in W^J$, it sends negative roots in Φ_J to negative roots in Φ. Because $\mathrm{Ad}(l)x \in \bigoplus_{\alpha \in \Phi_J} \mathfrak{u}_\alpha(k^{\mathrm{a}})$, this necessarily means that $\mathrm{Ad}(l)x \in \bigoplus_{\alpha \in \Phi_J \cap \Phi_+} \mathfrak{u}_\alpha(k^{\mathrm{a}}) = \mathfrak{u}_L(k^{\mathrm{a}})$. $\square$

Lemma 4.3.5. *With the setting of Lemma 4.3.4, we have $w^J u_P (w^J)^{-1} \in U(k^{\mathrm{a}})$.*

Proof. We know that $\mathrm{Ad}(l)x \in \mathfrak{u}_L(k^{\mathrm{a}})$ and that $\mathrm{Ad}(l)t = t$, so replacing x by $\mathrm{Ad}(l)x$ if necessary we can assume that $\mathrm{Ad}(w^J u_P)(t + x) \in \mathfrak{b}(k^{\mathrm{a}})$.

Partition $\Phi_+ \setminus (\Phi_J \cap \Phi_+)$ into the two sets

$$\Phi_1 \coloneqq \left\{\alpha \in \Phi_+ \setminus (\Phi_J \cap \Phi_+) \;\middle|\; w^J(\alpha) > 0\right\}\,,$$
$$\Phi_2 \coloneqq \left\{\alpha \in \Phi_+ \setminus (\Phi_J \cap \Phi_+) \;\middle|\; w^J(\alpha) < 0\right\}$$

and put a total order $\prec$ on Φ_2 such that $\alpha \preceq \beta$ whenever $\alpha > \beta$ for any $\alpha, \beta \in \Phi_2$ (recall that by definition $\alpha > \beta$ if and only if $\alpha - \beta$ is in the $\mathbb{Z}_{\geq 0}$-span of I). Then, for any total order on Φ_1, there is an isomorphism of schemes

$$\begin{array}{ccc} \left(\prod_{\alpha\in\Phi_1} U_\alpha\right) \times \left(\prod_{\alpha\in\Phi_2} U_\alpha\right) & \xrightarrow{\sim} & U_P \\ \left(u_{\alpha_1}, \ldots, u_{\alpha_{|\Phi_1|+|\Phi_2|}}\right) & \longmapsto & u_{\alpha_1} \ldots u_{\alpha_{|\Phi_1|+|\Phi_2|}} \end{array};$$

this can be seen by applying [Mil17, Theorem 21.68] to U and to U_L, and using the Levi decomposition $U = U_L U_P$. Let $u_{P,1}$ be the component of u_P in $\prod_{\alpha\in\Phi_1} U_\alpha(k^{\mathrm{a}})$; by definition of Φ_1, we have $w^J u_{P,1}(w^J)^{-1} \in U(k^{\mathrm{a}})$. In particular,

$$\mathrm{Ad}\left(w^J u_{P,1}^{-1} u_P\right)(t+x) = \mathrm{Ad}\left(\left(w^J u_{P,1}(w^J)^{-1}\right)^{-1}\right) \mathrm{Ad}\left(w^J u_P\right)(t+x) \in \mathfrak{b}(k^{\mathrm{a}})\,.$$

This means that replacing u_P by $u_{P,1}^{-1}u_P$ changes neither our hypothesis nor our aim.

Therefore, we can assume that $u_P = \prod_{\alpha\in\Phi_2} u_\alpha$ for some uniquely determined $u_\alpha \in U_\alpha(k^{\mathrm{a}})$. Write also $x = \sum_{\beta\in\Phi_J\cap\Phi_+} x_\beta$, for uniquely determined $x_\beta \in \mathfrak{u}_\beta(k^{\mathrm{a}})$. Fix some $\alpha \in \Phi_2$. For any root $\beta \in \Phi_J \cap \Phi_+$, we have $\alpha + \beta \neq 0$ (because both are positive roots), so [Spr98, Proposition 8.2.3] implies that

$$\mathrm{Ad}(u_\alpha)x_\beta - x_\beta \in \bigoplus_{i,j>0} \mathfrak{u}_{i\alpha+j\beta}(k^{\mathrm{a}})\,.$$

This is indeed the case with $\alpha = \beta$ as well since $\mathrm{Ad}(u_\alpha)x_\alpha - x_\alpha = 0$. As $i\alpha + j\beta > \alpha$ for $i, j > 0$, we can sum over the x_β and get

$$\mathrm{Ad}(u_\alpha)x - x \in \bigoplus_{\gamma>\alpha} \mathfrak{u}_\gamma(k^{\mathrm{a}})\,. \tag{4.11}$$

Now suppose that $u_P \neq 1$; write $u_P = u_{\alpha_r} u_{\alpha_{r-1}} \ldots u_{\alpha_0}$ for uniquely determined $r \geq 0$ and $u_{\alpha_i} \neq 1$ in $\mathfrak{u}_{\alpha_i}$. Remember that the ordering on Φ_2 forbids (in particular) to have $\alpha_0 > \alpha_i$ for $i > 0$. Applying (4.11), we get, for each $1 \leq i \leq r$

$$\mathrm{Ad}(u_{\alpha_i})\,\mathrm{Ad}(u_{\alpha_{i-1}} \ldots u_{\alpha_0})x - \mathrm{Ad}(u_{\alpha_{i-1}} \ldots u_{\alpha_0})x \in \bigoplus_{\gamma>\alpha_i} u_\gamma(k^{\mathrm{a}})$$

and summing over i yields

$$\mathrm{Ad}(u_P)x - x \in \bigoplus_{\gamma>\alpha_r} \mathfrak{u}_\gamma(k^{\mathrm{a}}) \oplus \ldots \oplus \bigoplus_{\gamma>\alpha_0} \mathfrak{u}_\gamma(k^{\mathrm{a}}) \subseteq \bigoplus_{\gamma\neq\alpha_0} \mathfrak{u}_\gamma(k^{\mathrm{a}})\,. \tag{4.12}$$

Since $(u_{\alpha_0} t' u_{\alpha_0}^{-1})t'^{-1} = u_{\alpha_0}(t' u_{\alpha_0}^{-1} t'^{-1}) \in U_{\alpha_0}(k^{\mathrm{a}})$ for $t' \in T(k^{\mathrm{a}})$, we must have $\mathrm{Ad}(u_{\alpha_0})t - t \in \mathfrak{u}_{\alpha_0}(k^{\mathrm{a}})$, and this last quantity is non-zero because $u_{\alpha_0} \notin \mathfrak{z}_{\mathfrak{g}}(t)(k^{\mathrm{a}})$ by Proposition 4.3.1(2). Applying again (4.11) repeatedly and summing as in (4.12) gives

$$\mathrm{Ad}(u_P)t - t \in (\mathfrak{u}_{\alpha_0}(k^{\mathrm{a}}) \setminus \{0\}) \oplus \bigoplus_{\gamma\neq\alpha_0} \mathfrak{u}_\gamma(k^{\mathrm{a}})\,. \tag{4.13}$$

As $x \in \mathfrak{u}_L(k^{\mathrm{a}})$, we have in particular obviously $t + x \in \mathfrak{t}(k^{\mathrm{a}}) \oplus \bigoplus_{\gamma \neq \alpha_0} \mathfrak{u}_\gamma(k^{\mathrm{a}})$. Combining with (4.12) and (4.13), we see that $\mathrm{Ad}(u_P)(t+x)$ has a non-zero component in $\mathfrak{u}_{\alpha_0}(k^{\mathrm{a}})$. Therefore, $\mathrm{Ad}(w^J)(u_P)(t+x)$ has a non-zero component in $\mathfrak{u}_{w^J(\alpha_0)}(k^{\mathrm{a}})$. This contradicts the hypothesis that $\mathrm{Ad}(w^J)(u_P)(t+x) \in \mathfrak{b}(k^{\mathrm{a}})$ because $\alpha_0 \in \Phi_2$ means that $w^J(\alpha_0) < 0$. As a conclusion, $u_P = 1$ so our claim is true. □

The two lemmas 4.3.4 and 4.3.5 can be nicely summarised as:

Lemma 4.3.6. *Let $t \in \mathfrak{t}_J^{\mathrm{gen}}(k^{\mathrm{a}})$ be a k^{a}-point. Then $\mathrm{Ad}(w^J u_P l)(t+x) \in \mathfrak{b}(k^{\mathrm{a}})$ if and only if $w^J u_P (w^J)^{-1} \in U(k^{\mathrm{a}})$ and $\mathrm{Ad}(l)x \in \mathfrak{u}_L(k^{\mathrm{a}})$.*

Proof. The "only" way is the combination of the two previous lemmas.

As for the "if" way, assume that $w^J u_P (w^J)^{-1} \in U(k^{\mathrm{a}})$ and $\mathrm{Ad}(l)x \in \mathfrak{u}_L(k^{\mathrm{a}})$. Then since $w^J \in W^J$, we have $\mathrm{Ad}(w^J)\,\mathrm{Ad}(l)x \in \mathfrak{b}(k^{\mathrm{a}})$, so

$$\mathrm{Ad}(w^J l)(t+x) = \mathrm{Ad}(w^J)(t + \mathrm{Ad}(l)x) \in \mathfrak{b}(k^{\mathrm{a}})\,,$$

hence $\mathrm{Ad}(w^J u_P (w^J)^{-1})\,\mathrm{Ad}(w^J l)(t+x) \in \mathfrak{b}(k^{\mathrm{a}})$ which is what we want to prove. □

4.4 Proof of Theorem 4.2.3

This part is entirely dedicated to proving Theorem 4.2.3. Before beginning the proof, we make the following two observations about base change to the algebraic closure k^{a}.

1. If S is a scheme over k, then $(S_{k^{\mathrm{a}}})^{\mathrm{red}} = (S^{\mathrm{red}})_{k^{\mathrm{a}}}$; indeed $(S^{\mathrm{red}})_{k^{\mathrm{a}}}$ is reduced by [Sta18, Lemma 020I], hence it satisfies the universal property defining $(S_{k^{\mathrm{a}}})^{\mathrm{red}}$. Therefore our definition of intersections of varieties and inverse image of morphism of varieties commute with base change. In particular, starting from the reductive group $G_{k^{\mathrm{a}}}$ over k^{a} with Borel subgroup $B_{k^{\mathrm{a}}}$ and respective Lie algebras $\mathfrak{g}_{k^{\mathrm{a}}}$ and $\mathfrak{b}_{k^{\mathrm{a}}}$, one can define the morphism $q_{k^{\mathrm{a}}} \colon G_{k^{\mathrm{a}}} \times^{B_{k^{\mathrm{a}}}} \mathfrak{b}_{k^{\mathrm{a}}} \longrightarrow \mathfrak{g}_{k^{\mathrm{a}}}$ and the varieties $\widetilde{X}_{k^{\mathrm{a}}}$, $\widetilde{V}_{w,k^{\mathrm{a}}}$ and $\widetilde{X}_{w,k^{\mathrm{a}}}$ for $w \in W$, similarly as for the split reductive group G over k. We then easily see that these constructions commute with base change, and that there is a natural isomorhism
$$\widetilde{V}_{w',k^{\mathrm{a}}} \cap q_{k^{\mathrm{a}}}^{-1}(\mathfrak{t}_{k^{\mathrm{a}}}^{ww'^{-1}} + \mathfrak{u}_{k^{\mathrm{a}}}) \xrightarrow{\sim} \left(\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})\right) \times_k k^{\mathrm{a}}$$
for any $w' \in W$.

2. If $Z \hookrightarrow S$ is a closed immersion of k-schemes, then a morphism $S' \longrightarrow S$ of k-schemes factors through Z if and only if the base change $S'_{k^{\mathrm{a}}} \longrightarrow S_{k^{\mathrm{a}}}$ factors through $Z_{k^{\mathrm{a}}}$. Indeed, it suffices to check this for affine schemes, *i.e.* to check that if I is an ideal of a k-algebra A, then any morphism of k-algebras $A \longrightarrow B$ factors through A/I if and only if the extension of scalars $A \otimes_k k^{\mathrm{a}} \longrightarrow B \otimes_k k^{\mathrm{a}}$ factors through $(A/I) \otimes_k k^{\mathrm{a}} = (A \otimes_k k^{\mathrm{a}})/(I \otimes_k k^{\mathrm{a}})$.

Therefore we can, and do, assume that k is algebraically closed for the rest of §4. In particular, the set of closed points of any k-scheme S is in natural bijection with the set of k-points of S. Hence, in the following, we liberally write "$s \in S$ is a closed point" to mean "$s \in S(k)$".

Theorem 4.2.3 can be stated as follows: for all good pair (w', w) in W and closed points $t \in \mathfrak{t}^{ww'^{-1}}$, we have the inclusion

$$\widetilde{V}_{w'} \cap q^{-1}(t + \mathfrak{u}) \hookrightarrow \widetilde{X}_w \,. \tag{4.14}$$

The proof of this will proceed in three steps. We first note that if L is the Levi subgroup of a standard parabolic subgroup P of G (*i.e.* such that $B \subseteq P$), then L is a connected reductive closed subgroup of G and $L \cap B$ is a Borel subgroup of L. Therefore one can define $q_L \colon L \times^{L\cap B} \mathfrak{l} \cap \mathfrak{b} \longrightarrow \mathfrak{l}$ and $\widetilde{X}_L$ in the same fashion as for G, as well as $\widetilde{V}_{L,w}$ and $\widetilde{X}_{L,w}$ for any $w \in W_J$. These objects are always assumed to be defined for G by default when L is not written in subscript.

The first step is a result of [BHS19] stating that (4.14) is true when $t = 0$, regardless of whether (w', w) is a good pair or not. We then apply this to a conjugate L of the Levi subgroup $C_G((T^{ww'^{-1}})^\circ)$ of G, to get the inclusion $\widetilde{V}_{L,w'_L} \cap q_L^{-1}(\mathfrak{u}_L) \hookrightarrow \widetilde{X}_{L,w_L}$ for any w'_L, w_L in the Weyl group of L. It is of particular interest because we can then translate by any suitable closed point $t \in \mathfrak{t}^{ww'^{-1}}$ when such a t is centralised by L; therefore we have, in fact, $\widetilde{V}_{L,w'_L} \cap q_L^{-1}(t + \mathfrak{u}_L) \hookrightarrow \widetilde{X}_{L,w_L}$.

The second step establishes a connection between X_L and X. More precisely, we construct an isomorphism between $\widetilde{V}_{L,w_L} \cap q_L^{-1}(\mathfrak{u}_L)$ (or equivalently, by the aforementioned remark, $\widetilde{V}_{L,w_L} \cap q_L^{-1}(t' + \mathfrak{u}_L)$ for any t' centralised by L and conjugate to some $t \in \mathfrak{t}^{ww'^{-1}}$) and a closed subvariety of $\widetilde{V}_w \cap q^{-1}(t + \mathfrak{u})$ for some w_L in the Weyl group of L and $w \in W$. We momentarily call this subvariety $\widetilde{V}'_w$.

The third step bridges the last gap from $\widetilde{V}'_w$ to $\widetilde{V}_w \cap q^{-1}(t + \mathfrak{u})$: an action of U on $\widetilde{V}_w \cap q^{-1}(t + \mathfrak{u})$ is defined such that all the orbits intersect $\widetilde{V}'_w$.

In the end, the good pair condition on some given $(w' \preceq w)$ allows us to find a standard Levi subgroup L of G and an ordered pair $(w'_L \preceq w_L)$ of elements in the Weyl group of L, such that the second and third steps can reduce the problem to the situation of the first.

Step 1

Theorem 4.4.1 (Breuil-Hellmann-Schraen)**.** *Let $w, w' \in W$ such that $w' \preceq w$. Then the locally closed immersion $\widetilde{V}_{w'} \hookrightarrow \widetilde{X}$ induces*

$$\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{u}) \hookrightarrow \widetilde{X}_w \,.$$

Proof. By the proof of Proposition 4.2.2, the theorem is equivalently stated as:

$$V_{w'} \cap \kappa_1^{-1}(0) \hookrightarrow X_w \tag{4.15}$$

for all $w' \preceq w$. This statement is proved in [BHS19, §2.4]. We provide here a brief summary of the ideas involved, and refer to *loc. cit.* for the details of the proofs and notation.

Let $Z \coloneqq (\kappa_1^{-1}(0))^{\mathrm{red}} \subset X$ be the Steinberg variety (see [Ste76]) and, for $w \in W$, let $Z_w \subseteq Z$ be the Zariski closure of $V_w \cap Z = V_w \cap \kappa_1^{-1}(0)$. Then the irreducible components of Z are the Z_w (see [BHS19, Proposition 2.4.1]). Let $Z^0(Z)$ be the free abelian group generated by these irreducible components; when viewed in $Z^0(Z)$, the component Z_w is denoted $[Z_w]$. Also, for $w \in W$, let $\overline{X}_w \coloneqq \kappa_{1,w}^{-1}(0)$ in the usual sense (we do not take the reduced scheme here, so $\overline{X}_w$ is not reduced in general), where $\kappa_{1,w} \coloneqq \kappa_1|_{X_w}$. Then the irreducible components of $\overline{X}_w^{\mathrm{red}}$ are some of the $Z_{w'}$, so we can define the class

$$[\overline{X}_w] \coloneqq \sum_{w' \in W} m(Z_{w'}, \overline{X}_w)[Z_{w'}] \quad \in Z^0(Z)$$

where $m(Z_{w'}, \overline{X}_w)$ is the multiplicity of the component $Z_{w'}$ in $\overline{X}_w$.

The inclusion (4.15) is equivalent to having $Z_{w'} \subset \overline{X}_w^{\mathrm{red}}$ for any $w' \preceq w$. The strategy will be to compute the class $[\overline{X}_w]$ and show that $m(Z_{w'}, \overline{X}_w) \neq 0$ for $w' \preceq w$. As we sketch in the following paragraphs, this can be reduced through the Beilinson-Bernstein correspondence to the theory of representations of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

Let $\mathcal{O}$ be the BGG-category of representations of $U(\mathfrak{g})$ (see [Hum08, §1.1]). Let $\mathcal{O}(0)$ be the full subcategory of $\mathcal{O}$ whose objects are those of trivial infinitesimal character (see *loc. cit.*, §1.12). For a k-linear morphism $\mu\colon \mathfrak{t} \longrightarrow k$, *i.e.* a weight, we note $M(\mu)$ the Verma module of highest weight μ, and $L(\mu)$ the unique irreducible quotient of $M(\mu)$ (see *loc. cit.*, §1.2). For $x, y \in W$, we note $P_{x,y}(T) \in \mathbb{Z}_{\geq 0}[T]$ the Kazhdan-Lusztig polynomial associated to x and y (see *loc. cit*, §8.2). Then, for $w \in W$, the irreducible constituents of $M(ww_0 \cdot 0)$, where $\cdot$ is the dot-action of §1.8 of *loc. cit.*, are the $L(w'w_0 \cdot 0)$ for $w' \in W$; and the constituent $L(w'w_0)$ has multiplicity $P_{w_0w, w_0w'}(1)$ (see *loc. cit.*, §8.4).

Let $\mathcal{D} - \mathrm{Mod}^{\mathrm{rh}}_{G/B \times G/B}$ be the category of regular holonomic G-equivariant $\mathcal{D}$-modules on $G/B \times G/B$ (see [HTT08, §6.1]; these objects are in particular coherent $\mathcal{O}_{G/B \times G/B}$-modules). The Beilinson-Bernstein correspondence is an exact functor

$$\mathrm{BB}_G\colon \mathcal{O}(0) \xrightarrow{\sim} \mathcal{D} - \mathrm{Mod}^{\mathrm{rh}}_{G/B \times G/B}$$

which is an equivalence of artinian categories (see [HTT08, §11] and [BHS19, Remark 2.4.3]). Let

$$\pi\colon T^*(G/B \times G/B) \longrightarrow G/B \times G/B$$

be the cotangent bundle. To a coherent $\mathcal{D}$-module $\mathfrak{M}$ on $G/B \times G/B$ can be associated a closed subscheme $\mathrm{Ch}(\mathfrak{M})$ of $T^*(G/B \times G/B)$, called its characteristic variety, as follows. There is a filtration $F = (F_i\mathfrak{M})_{i \in \mathbb{Z}}$ of $\mathfrak{M}$ by quasicoherent

$\mathcal{O}_{G/B\times G/B}$-submodules such that its graded module

$$\mathrm{gr}^F\mathfrak{M} := \bigoplus_{i\in\mathbb{Z}} F_i\mathfrak{M}/F_{i-1}\mathfrak{M}$$

is a coherent $\pi_*\mathcal{O}_{T^*(G/B\times G/B)}$-module (see [HTT08, §2.1]). We then define $\mathrm{Ch}(\mathfrak{M})$ as the schematic support of the coherent $\mathcal{O}_{T^*(G/B\times G/B)}$-module

$$\mathcal{O}_{T^*(G/B\times G/B)} \otimes_{\pi^{-1}\pi_*\mathcal{O}_{T^*(G/B\times G/B)}} \mathrm{gr}^F\mathfrak{M}$$

which does not depend on the choice of the filtration F in the filtered $\mathcal{D}$-module $(\mathfrak{M}, F)$ (see [HTT08, §2.2]). If $\mathfrak{M}$ is in $\mathcal{D}-\mathrm{Mod}^{\mathrm{rh}}_{G/B\times G/B}$, then $\mathrm{Ch}(\mathfrak{M})$ has the property that $\mathrm{Ch}(\mathfrak{M})^{\mathrm{red}} \subseteq Z \subseteq T^*(G/B\times G/B)$ (see [BHS19, Proposition 2.4.4]). Therefore, we can define

$$[\mathfrak{M}] := [\mathrm{Ch}(\mathfrak{M})] \in Z^0(Z)$$

and the map $\mathfrak{M} \longmapsto [\mathfrak{M}]$ is additive.

For a weight μ, write $\mathfrak{M}(\mu) := \mathrm{BB}_G(M(\mu))$ and $\mathfrak{L}(\mu) := \mathrm{BB}_G(L(\mu))$. Then, from [BR12, Proposition 2.14.2] (or [Gin86, (6.2.3)]: see [BR12, Remark 2.14.3]) and the decomposition of $M(ww_0\cdot 0)$, we have

$$[\overline{X}_w] = [\mathfrak{M}(ww_0\cdot 0)] = \sum_{w'\in W} P_{w_0w,w_0w'}(1)[\mathfrak{L}(w'w_0\cdot 0)]$$

for all $w\in W$ (see [BHS19, Proposition 2.4.6]). From there we conclude as follows (see [BHS19, Theorem 2.4.7]). Write

$$[\mathfrak{L}(ww_0\cdot 0)] = \sum_{w'\in W} a_{w,w'}[Z_{w'}], \quad [\overline{X}_w] = \sum_{w'\in W} b_{w,w'}[Z_{w'}]$$

for some $a_{w,w'}, b_{w,w'} \in \mathbb{Z}_{\geq 0}$. Then, as matrices indexed by $(w,w')\in W^2$, we have $(b_{w,w'}) = (P_{w_0w,w_0w'}(1))(a_{w,w'})$. For any w, the coefficient $b_{w,w}$ is the multiplicity of the closure Z_w of $(\kappa_1|_{V_w}^{-1}(0))^{\mathrm{red}}$ in $\overline{X}_w = \kappa_1|_{\overline{X}_w}^{-1}(0)$. Since V_w is open in X_w, for any $x\in Z_w$, we have $\mathcal{O}_{Z_w,x} = (\mathcal{O}_{\overline{X}_w,x})^{\mathrm{red}}$; but $\kappa_{1,w}|_{V_w}$ is smooth (see [BHS19, Lemma 2.3.5]), so $\mathcal{O}_{\overline{X}_w,x}$ is reduced (see [Sta18, Lemma 056S]). Hence $\mathcal{O}_{Z_w,x} = \mathcal{O}_{\overline{X}_w,x}$ and $b_{w,w} = 1$. Also, we already know that $Z_{w'} \subseteq \overline{X}_w^{\mathrm{red}}$ only if $w' \preceq w$; therefore the matrix $(b_{w,w'})$ is lower triangular with entries 1 on the diagonal. We also know from the properties of the Kazhdan-Lusztig polynomials that $(P_{w_0w,w_0w'}(1))$ is lower triangular with entries 1 on the diagonal. It follows that the matrix $(a_{w,w'})$ is also lower triangular with entries 1 on the diagonal. Hence:

$$b_{w,w'} = \sum_{w''\in W} P_{w_0w,w_0w''}(1)a_{w'',w'} \geq P_{w_0w,w_0w'}(1)a_{w',w'} = P_{w_0w,w_0w'}(1) > 0$$

for $w'\preceq w$, which means that $Z_{w'}$ is an irreducible component of X_w (with multiplicity at least $P_{w_0w,w_0w'}(1)$). $\square$

Step 2

Proposition 4.4.2. *Let $t \in \mathfrak{t}^{\mathrm{gen}}$ be a closed point, and let $u^J, v^J \in W^J$.*

(1) *There is a morphism ι_{t,u^J,v^J} given by*

$$\iota_{t,u^J,v^J} : \begin{array}{rcl} L \times^{L\cap B} \mathfrak{l} \cap \mathfrak{b} & \longrightarrow & G \times^B \mathfrak{b} \\ (l, \psi_L) & \longmapsto & \big(u^J l (v^J)^{-1}, \mathrm{Ad}(v^J)(t + \psi_L)\big) \end{array}$$

and it is a closed immersion.

(2) *The closed immersion ι_{t,u^J,v^J} induces, for any $w_J \in W_J$, an isomorphism*

$$\iota_{t,u^J,w_J,v^J} : \widetilde{V}_{L,w_J} \cap q_L^{-1}(\mathfrak{u}_L) \xrightarrow{\sim} \widetilde{V}_{u^J w_J (v^J)^{-1}} \cap q^{-1}\left(\mathrm{Ad}(u^J)(t + \mathfrak{u}_L)\right)$$

(remember that the symbol $\cap$ denotes the reduced subscheme associated to the fiber product over $\widetilde{X}_L$ in the left-hand side and over $\widetilde{X}$ in the right-hand side, respectively).

Proof. We first prove statement (1). We begin by checking that ι_{t,u^J,v^J} is well defined. Let $b \in L \cap B$. Since $L \cap B = T \prod_{\alpha \in \Phi_J \cap \Phi_+} U_\alpha$ and $v^J \in W^J$, we have

$$b' := v^J b (v^J)^{-1} \in B\,.$$

Thus, for any $l \in L$ and $\psi_L \in \mathfrak{l} \cap \mathfrak{b}$, we have

$$\big(u^J l b (v^J)^{-1}, \mathrm{Ad}(v^J)(t + \mathrm{Ad}(b)\psi_L)\big) = \big(u^J l (v^J)^{-1} b', \mathrm{Ad}(b')\,\mathrm{Ad}(v^J)(t + \psi_L)\big)$$

(note that b fixes t as $b \in L$), which shows that ι_{t,u^J,v^J} is well defined.

Now we check that ι_{t,u^J,v^J} is a closed immersion. First consider the morphism

$$\begin{array}{ccc} L \times \mathfrak{l} \cap \mathfrak{b} & \longrightarrow & (u^J L (v^J)^{-1} B)/B \times \mathfrak{l} \\ (l, \psi_L) & \longmapsto & (u^J l (v^J)^{-1} B, \mathrm{Ad}(l)\psi_L) \end{array}. \tag{4.16}$$

From $u^J l b (v^J)^{-1} = u^J l (v^J)^{-1} b'$, we know that for any $b \in L \cap B$ and $(l, \psi_L) \in L \times \mathfrak{l} \cap \mathfrak{b}$, the images of (l, ψ_L) and $(lb, \mathrm{Ad}(b^{-1})\psi_L)$ under (4.16) are the same. Conversely, assume that (l, ψ_L) and (l', ψ'_L) have the same image. This means that

$$\mathrm{Ad}(l)\psi_L = \mathrm{Ad}(l')\psi'_L$$

and that there is $b \in B$ such that

$$u^J l (v^J)^{-1} = u^J l' (v^J)^{-1} b\,.$$

Write $b' = l'^{-1} l$. Then $v^J b' (v^J)^{-1} = b$, so $b' \in L \cap B$ according to Lemma 4.3.3. We also have $l = l' b'$, so $\psi_L = \mathrm{Ad}(b'^{-1})\psi'_L$ and $(l, \psi_L) = (l', \psi'_L)$ in $L \times^{L\cap B} \mathfrak{l} \cap \mathfrak{b}$. Therefore, by definition of the quotient space $L \times^{L\cap B} \mathfrak{l} \cap \mathfrak{b}$, the morphism (4.16) factorises as

$$f_L\colon \begin{array}{ccc} L\times^{L\cap B}\mathfrak{l}\cap\mathfrak{b} & \longrightarrow & \big(u^J L(v^J)^{-1}B\big)/B\times\mathfrak{l} \\ (l,\psi_L) & \longmapsto & \big(u^J l(v^J)^{-1}B, \operatorname{Ad}(l)\psi_L\big) \end{array}.$$

Together with the morphism

$$f_G\colon \begin{array}{ccc} G\times^B\mathfrak{b} & \longrightarrow & G/B\times\mathfrak{g} \\ (g,\psi) & \longmapsto & (gB,\operatorname{Ad}(g)\psi) \end{array},$$

the morphism ι_{t,u^J,v^J} and the closed immersion

$$\iota\colon \begin{array}{ccc} (u^J L(v^J)^{-1}B)/B\times\mathfrak{l} & \hookrightarrow & G/B\times\mathfrak{g} \\ (gB,\psi_L) & \longmapsto & \big(gB,\operatorname{Ad}(u^J)(t+\psi_L)\big) \end{array},$$

we get a commutative diagram

$$\begin{tikzcd} L\times^{L\cap B}\mathfrak{l}\cap\mathfrak{b} \arrow[r, "\iota_{t,u^J,v^J}"] \arrow[d, "f_L"] & G\times^B\mathfrak{b} \arrow[d, "f_G"] \\ (u^J L(v^J)^{-1}B)/B\times\mathfrak{l} \arrow[r, hook, "\iota"] & G/B\times\mathfrak{g}\,. \end{tikzcd} \tag{4.17}$$

This diagram is a cartesian square. Indeed, let $(g,\psi)\in G\times^B\mathfrak{b}$ be such that $(gB,\operatorname{Ad}(g)\psi)$ is in the image of ι. Then, modifying $(g,\psi)\in G\times\mathfrak{b}$ if necessary (so that its image in $G\times^B\mathfrak{b}$ is the same), there is $l\in L$ such that $g=u^J l(v^J)^{-1}$, and $\psi_L\in\mathfrak{l}$ such that $\operatorname{Ad}(g)\psi=\operatorname{Ad}(u^J l(v^J)^{-1})\psi=\operatorname{Ad}(u^J)(t+\psi_L)$. From this last equality we get

$$\psi=\operatorname{Ad}(v^J l^{-1})(t+\psi_L)=\operatorname{Ad}(v^J)(t+\operatorname{Ad}(l^{-1})\psi_L)$$

with $\operatorname{Ad}(l^{-1})\psi_L\in\mathfrak{l}=\mathfrak{t}\oplus\bigoplus_{\alpha\in\Phi_J}\mathfrak{u}_\alpha$. Furthermore, $v^J\in W^J$ implies that

$$\operatorname{Ad}(v^J)\left(\bigoplus_{\alpha\in\Phi_J\cap\Phi_-}\mathfrak{u}_\alpha\right)\subseteq\bigoplus_{\alpha\in\Phi_-}\mathfrak{u}_\alpha\,,$$

hence, as $\operatorname{Ad}(v^J)\operatorname{Ad}(l^{-1})\psi_L=\psi-\operatorname{Ad}(v^J l^{-1})t\in\mathfrak{b}$, we necessarily have

$$\operatorname{Ad}(l^{-1})\psi_L\in\mathfrak{t}\oplus\bigoplus_{\alpha\in\Phi_J\cap\Phi_+}\mathfrak{u}_\alpha=\mathfrak{l}\cap\mathfrak{b}\,.$$

This gives $(g,\psi)=\iota_{t,u^J,v^J}\big(l,\operatorname{Ad}(l^{-1})\psi_L\big)$ as we wanted. Since the diagram (4.17) is indeed a cartesian square, the arrow ι being a closed immersion proves that ι_{t,u^J,v^J} is one as well by base change.

As for statement (2), we had already established the inclusion $u^J(L\cap B)(u^J)^{-1}\subseteq B$ as a consequence of $u^J\in W^J$, and similarly for v^J. From this we easily see that

ι_{t,u^J,v^J} restricts to a closed immersion of algebraic varieties

$$\widetilde{V}_{L,w_J} \cap q_L^{-1}(\mathfrak{u}_L) \hookrightarrow \widetilde{V}_{u^J w_J (v^J)^{-1}} \cap q^{-1}\left(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\right)$$

which we want to be an isomorphism. For this, let $(g,\psi) \in q^{-1}\left(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\right)$. We know that

$$G = \coprod_{\substack{w^J \in W^J \\ w_J \in W_J}} B w^J w_J B = \coprod_{w^J \in W^J} B w^J P = \coprod_{w^J \in W^J} B w^J U_P L\,,$$

where the first equality comes from the Bruhat decomposition of G, the second from the Bruhat decomposition of $P = P_J$ and the third from the Levi decomposition of P (see [Mil17, Theorem 21.91] for the latter two). Hence, there are $b \in B$, $w^J \in W^J$, $u_P \in U_P$ and $l \in L$ such that $g^{-1}u^J = b w^J u_P l$. Choosing another lift for (g,ψ) in $G \times \mathfrak{b}$ if necessary, we can assume $b = 1$, and by assumption there is $x \in \mathfrak{u}_L$ such that

$$(g,\psi) = \left(u^J l^{-1} u_P^{-1} (w^J)^{-1}, \mathrm{Ad}(w^J u_P l)(t+x)\right)\,.$$

Since $\psi \in \mathfrak{b}$, Lemma 4.3.6 implies that $w^J u_P (w^J)^{-1} \in U \subset B$ and that

$$\psi_L \coloneqq \mathrm{Ad}(l)x \in \mathfrak{u}_L\,,$$

so we actually have in $G \times^B \mathfrak{b}$

$$(g,\psi) = \left(u^J l^{-1} (w^J)^{-1}, \mathrm{Ad}(w^J l)(t+x)\right) = \left(u^J l^{-1} (w^J)^{-1}, \mathrm{Ad}(w^J)(t+\psi_L)\right)\,. \quad (4.18)$$

Finally, $l^{-1} \in (L\cap B) w'_J (L \cap B)$ for some $w'_J \in W_J$, which gives as usual $g = u^J l^{-1} (w^J)^{-1} \in B u^J w'_J (w^J)^{-1} B$. If we add the additional condition that $(g,\psi) \in \widetilde{V}_{u^J w_J (v^J)^{-1}}$, or equivalently that $g \in B u^J w_J (v^J)^{-1} B$, then we have $w'_J (w^J)^{-1} = w_J (v^J)^{-1}$. This forces $w'_J = w_J$ and $w^J = v^J$, thus (4.18) translates to

$$(g,\psi) = \iota_{t,u^J,v^J}(l^{-1},\psi_L)$$

with $(l^{-1},\psi_L) \in \widetilde{V}_{L,w_J}$. This finishes the proof of Proposition 4.4.2. □

Step 3 The following lemma uses a few results about Jordan decompositions, which we summarise in §4.A.

Lemma 4.4.3. *Let $t \in \mathfrak{t}_J^{\mathrm{gen}}$ be a closed point (*i.e. *a k-point), and let $u^J \in W^J$. Then there is a surjective morphism given by*

$$f\colon \begin{array}{ccc} U \times \left(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\right) & \twoheadrightarrow & \mathrm{Ad}(u^J)t + \mathfrak{u} \\ (u,x) & \longmapsto & \mathrm{Ad}(u)x\,. \end{array}$$

Proof. The existence of the map f is a direct consequence of the fact that $\mathrm{Ad}(u)$ sends any $t' \in \mathfrak{t}$ to $t' + \mathfrak{u}$ when $u \in U$. Therefore we only have to prove the surjectivity of f.

Let then $y \in \mathrm{Ad}(u^J)t + \mathfrak{u}$, and consider its Jordan decomposition $y = y_\mathrm{s} + y_\mathrm{n}$ in $\mathfrak{g}$ (see Theorem 4.A.2). From Proposition 4.A.9, there is $g \in G$ such that

$$y_\mathrm{s} = \mathrm{Ad}(g)\,\mathrm{Ad}(u^J)t\,.$$

We use the decomposition

$$G = \coprod_{w^J \in W^J} B w^J P = \coprod_{w^J \in W^J} U T w^J P = \coprod_{w^J \in W^J} U w^J U_P L\,.$$

Let $u \in U$, $w^J \in W^J$ and $u_P \in U_P$ such that $gu^J \in uw^J u_P L$, from the fact that t is centralised by L we get:

$$y_\mathrm{s} = \mathrm{Ad}(uw^J u_P)t\,.$$

Now, from Proposition 4.A.9(1), we have

$$y_\mathrm{n} \in \mathfrak{u}, \quad y_\mathrm{s} \in y + \mathfrak{u} \subseteq \mathfrak{b}\,. \tag{4.19}$$

Hence, $\mathrm{Ad}(w^J u_P)t = \mathrm{Ad}(u^{-1})y_\mathrm{s} \in \mathfrak{b}$, so Lemma 4.3.5 implies that $w^J u_P (w^J)^{-1} \in U$. Set $u' := uw^J u_P (w^J)^{-1} \in U$. We can now write

$$y_\mathrm{s} = \mathrm{Ad}(u'w^J)t \in \mathrm{Ad}(w^J)t + \mathfrak{u}\,. \tag{4.20}$$

Also, (4.19) and the assumption on y give $y_\mathrm{s} \in \mathrm{Ad}(u^J)t + \mathfrak{u}$. Together with (4.20), we deduce that

$$\mathrm{Ad}(w^J)t = \mathrm{Ad}(u^J)t\,,$$

hence $(u^J)^{-1}w^J \in C_G(t)$. If $(u^J)^{-1}w^J \notin W_J$, then there exists $\alpha \in \Phi_J$ such that $\beta := (u^J)^{-1}w^J(\alpha) \notin \Phi_J$. Since $U_\alpha \subseteq C_G(t)$, we would have

$$U_\beta = (u^J)^{-1} w^J U_\alpha (w^J)^{-1} u^J \subseteq C_G(t)$$

so $U_\beta \subseteq C_G(t)^\circ = L$ because U_β is connected, which is a contradiction as β is not a root of L. Hence $(u^J)^{-1}w^J \in W_J$, from which we deduce $w^J = u^J$ as W^J is a set of representatives of W/W_J.

Moreover, the commutation $[y_\mathrm{s}, y_\mathrm{n}] = 0$ together with $y_\mathrm{s} = \mathrm{Ad}(u'w^J)t$ from (4.20) and Proposition 4.3.1(2) give $y_\mathrm{n} \in \mathrm{Ad}(u'w^J)\mathfrak{l}$. Since $y_\mathrm{n} \in \mathfrak{u}$ and $w^J \in W^J$, we necessarily get $y_\mathrm{n} \in \mathrm{Ad}(u'w^J)\mathfrak{u}_L$. Thus,

$$y = y_\mathrm{s} + y_\mathrm{n} \in \mathrm{Ad}(u'w^J)(t + \mathfrak{u}_L) = \mathrm{Ad}(u')\,\mathrm{Ad}(u^J)(t + \mathfrak{u}_L)$$

which concludes the proof. $\square$

Proposition 4.4.4. *Let $t \in \mathfrak{t}_J^{\mathrm{gen}}$ be a closed point, let $u^J, v^J \in W^J$ and let $w_J \in W_J$.*

Write $w := u^J w_J (v^J)^{-1}$*. Then there is a surjective morphism of algebraic varieties*

$$\mu_{t,u^J,w_J,v^J} : \begin{array}{ccc} U \times \left(\widetilde{V}_{L,w_J} \cap q_L^{-1}(\mathfrak{u}_L)\right) & \twoheadrightarrow & \widetilde{V}_w \cap q^{-1}\left(\mathrm{Ad}(u^J)t + \mathfrak{u}\right) \\ (u,(l,\psi_L)) & \longmapsto & \left(uu^J l (v^J)^{-1}B, \mathrm{Ad}(v^J)(t+\psi_L)\right). \end{array}$$

Proof. Remember that the surjectivity of a morphism of algebraic varieties can be checked on closed points ([Vak17, Exercise 7.4.C]). The morphism of algebraic varieties

$$\begin{array}{ccc} U \times q^{-1}(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)) & \longrightarrow & q^{-1}(\mathrm{Ad}(u^J)t + \mathfrak{u}) \\ (u,(g,x)) & \longmapsto & (ug,x) \end{array} \tag{4.21}$$

is surjective as a consequence of Lemma 4.4.3. Since $U \subseteq B$, for any $u \in U$ and $g \in G$, we have $g \in BwB$ if and only if $ug \in BwB$. Therefore (4.21) induces a surjective morphism of algebraic varieties

$$\begin{array}{ccc} U \times \left(\widetilde{V}_w \cap q^{-1}\left(\mathrm{Ad}(u^J)(t+\mathfrak{u}_L)\right)\right) & \longrightarrow & \widetilde{V}_w \cap q^{-1}\left(\mathrm{Ad}(u^J)t + \mathfrak{u}\right) \\ (u,(g,x)) & \longmapsto & (ug,x) \end{array}.$$

Composing the latter with the bijective morphism $\mathrm{id} \times \iota_{t,u^J,w_J,v^J}$ of Proposition 4.4.2(2) gives the desired morphism μ_{t,u^J,w_J,v^J}. This concludes the proof. $\square$

We can finally achieve our goal of proving Theorem 4.2.3, which states that for all good pairs (w',w) in W, we have the inclusion

$$\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u}) \hookrightarrow \widetilde{X}_w .$$

Proof of Theorem 4.2.3. Let Φ be the root system of G with basis I, let $\Phi_{ww'^{-1}} \subseteq \Phi$ be the minimal generating subsystem of ww'^{-1} in the sense of Definition 2.1.1. According to Proposition 2.2.2, it is the root system of the Levi subgroup $C_G((T^{ww'^{-1}})^\circ)$. Proposition 2.3.7 gives a subset J of the basis I of roots, and elements $u^J, v^J \in W^J$, $w'_J, w_J \in W$ satisfying

$$w = u^J w_J (v^J)^{-1}, \quad w' = u^J w'_J (v^J)^{-1}, \quad w'_J \preceq w_J$$

and such that $u^J L (u^J)^{-1} = C_G((T^{ww'^{-1}})^\circ)$, where $L := L_J$ is the standard Levi subgroup associated to J. Then $L = C_G((T^{w_J(w'_J)^{-1}})^\circ)$, with $(T^{w_J(w'_J)^{-1}})^\circ = T_J$ and $\mathfrak{t}^{w_J(w'_J)^{-1}} = \mathfrak{t}_J$ (using the notation of section 4.3).

The variety $\mathfrak{t}_J^{\mathrm{gen}}$ is Zariski-dense in $\mathfrak{t}_J = \mathfrak{t}^{w_J(w'_J)^{-1}}$, therefore $\mathrm{Ad}(u^J)\mathfrak{t}_J^{\mathrm{gen}}$ is Zariski-dense in $\mathrm{Ad}(u^J)\mathfrak{t}_J = \mathfrak{t}^{ww'^{-1}}$. According to Lemma 4.3.2, we thus only need to prove that $\widetilde{V}_{w'} \hookrightarrow \widetilde{X}$ induces

$$\widetilde{V}_{w'} \cap q^{-1}(\mathrm{Ad}(u^J)t + \mathfrak{u}) \hookrightarrow \widetilde{X}_w$$

for all closed points $t \in \mathfrak{t}_J^{\mathrm{gen}}$. In the following, fix one such t.

The closed immersion ι_{t,u^J,v^J} of Proposition 4.4.2 restricts to a closed immersion

$$\iota_{t,u^J,v^J} \colon \widetilde{X}_L = q_L^{-1}(\mathfrak{l} \cap \mathfrak{b}) \hookrightarrow \widetilde{X} = q^{-1}(\mathfrak{b})\,.$$

Indeed, it is easily checked that this amounts to having, for all $(l, \psi_L) \in L \times (\mathfrak{l} \cap \mathfrak{b})$ such that $\mathrm{Ad}(l)\psi_L \in \mathfrak{l} \cap \mathfrak{b}$, the inclusion $\mathrm{Ad}(u^J)\,\mathrm{Ad}(l)\psi_L \in \mathfrak{b}$, which is true because $u^J \in W^J$ means that $\mathrm{Ad}(u^J)(\mathfrak{l} \cap \mathfrak{b}) \subseteq \mathfrak{b}$.

In the same way, we can further restrict to

$$\iota_{t,u^J,v^J} \colon \widetilde{V}_{L,w_J} \hookrightarrow \widetilde{V}_w$$

because $u^J \in W^J$ means that $u^J(L \cap B)(u^J)^{-1} \subseteq B$ and similarly for v^J. Taking the Zarisk closures gives a closed immersion $\iota_{t,u^J,v^J} \colon \widetilde{X}_{L,w_J} \hookrightarrow \widetilde{X}_w$. There is also a morphism

$$\begin{aligned} U \times \widetilde{X}_w &\longrightarrow \widetilde{X}_w \\ (u, (g, x)) &\longmapsto (ug, x) \end{aligned}$$

which, when composed with $\mathrm{id} \times \iota_{t,u^J,v^J} \colon U \times \widetilde{X}_{L,w_J} \longrightarrow U \times \widetilde{X}_w$, gives a morphism

$$\bar{\mu}_{t,u^J,w_J,v^J} \colon U \times \widetilde{X}_{L,w_J} \longrightarrow \widetilde{X}_w$$

which can be defined by a similar formula to the one for the surjective morphism μ_{t,u^J,w_J,v^J} of Proposition 4.4.4.

We also have an obvious locally closed immersion

$$\varphi_G \colon \widetilde{V}_{w'} \cap q^{-1}(\mathrm{Ad}(u^J)t + \mathfrak{u}) \hookrightarrow \widetilde{X}$$

as well as a less obvious locally closed immersion

$$\varphi_L \colon \widetilde{V}_{L,w'_J} \cap q_L^{-1}(\mathfrak{u}_L) \hookrightarrow \widetilde{X}_{L,w_J}$$

given by theorem 4.4.1 applied to the reductive group L together with the good pair condition $w'_J \preceq w_J$.

Altogether, these morphisms fit into a commutative diagram of algebraic varieties

$$\begin{tikzcd} U \times \left(\widetilde{V}_{L,w'_J} \cap q_L^{-1}(\mathfrak{u}_L)\right) \arrow[r, "\mathrm{id} \times \varphi_L"] \arrow[dd, two heads, "\mu_{t,u^J,w'_J,v^J}"] & U \times \widetilde{X}_{L,w_J} \arrow[d, "\bar{\mu}_{t,u^J,w_J,v^J}"] \\ & \widetilde{X}_w \arrow[d, hook] \\ \widetilde{V}_{w'} \cap q^{-1}(\mathrm{Ad}(u^J)t + \mathfrak{u}) \arrow[r, "\varphi_G"] & \widetilde{X} \end{tikzcd}$$

where the surjectivity of the left arrow μ_{t,u^J,w'_J,v^J} (Proposition 4.4.4) proves that the image of the bottom arrow φ_G lands in $\widetilde{X}_w$. This finishes the proof. $\square$

Appendix 4.A Jordan decomposition and orbits in $\mathfrak{g}$

We recall here a few well-known results about Jordan decompositions in a Lie algebra. In particular, Proposition 4.A.9 is needed in §4.4; we thank Anne Moreau for its proof, which we couldn't find precisely in the literature.

In the following, k is an algebraically closed field of characteristic 0 (this is the setting of many reference books, however it is stronger than really necessary). This allows us in particular, for any k-scheme, to use interchangeably "k-points" and "closed points" as the same notion. For instance, there is no distinction between viewing a Lie algebra as a k-scheme (as we do in §4) or as a k-vector space.

Definition 4.A.1. Let G be an algebraic group over k with Lie algebra $\mathfrak{g}$. A closed point $x \in \mathfrak{g}$ is called *semisimple* (resp. *nilpotent*) if the following property holds. For any morphism of algebraic groups $\phi\colon G \longrightarrow \mathrm{GL}(V)$ inducing $\mathrm{d}\phi\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, where V is a finite-dimensional vector space over k, the endomorphism $\mathrm{d}\phi(x) \in \mathfrak{gl}(V) \simeq \mathrm{End}(V)$ is diagonalisable (resp. nilpotent).

Theorem 4.A.2 (Jordan decomposition)**.** (1) *Let G be an algebraic group over k with Lie algebra $\mathfrak{g}$. For any closed point $x \in \mathfrak{g}$, there is a unique couple $(x_{\mathrm{s}}, x_{\mathrm{n}}) \in \mathfrak{g}^2$ with x_{s} semisimple and x_{n} nilpotent such that $[x_{\mathrm{s}}, x_{\mathrm{n}}] = 0$ and $x = x_{\mathrm{s}} + x_{\mathrm{n}}$. We call this the* Jordan decomposition *of x.*

(2) *The Jordan decomposition is functorial: for any morphism $\phi\colon G \longrightarrow H$ of algebraic groups over k inducing $\mathrm{d}\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ and any closed point $x \in \mathfrak{g}$, we have*

$$\mathrm{d}\phi(x_{\mathrm{s}}) = (\mathrm{d}\phi(x))_{\mathrm{s}}, \quad \mathrm{d}\phi(x_{\mathrm{n}}) = (\mathrm{d}\phi(x))_{\mathrm{n}}\,.$$

Proof. The existence of a functorial Jordan decomposition is a combination of 23.6.2, 23.6.3 and 23.6.4 of [TY06]. The uniqueness property in (1) is given by the fact that for any algebraic group G, there exists a finite-dimensional vector space V with a morphism of algebraic groups $\phi\colon G \longrightarrow \mathrm{GL}(V)$ such that $\mathrm{d}\phi\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is injective (see 22.1.5 and 24.4.1 of *loc. cit.*). □

Proposition 4.A.3. *Let G be a connected reductive group over k, let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{z}$ be the centre of $\mathfrak{g}$. There is a decomposition of k-vector spaces*

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z} \tag{4.22}$$

and for any closed point $x \in \mathfrak{g}$, the following conditions are equivalent:

(i) *x is semisimple (resp. nilpotent).*

(ii) $\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ *is semisimple (resp.* $\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ *is nilpotent and the component of x in $\mathfrak{z}$ is 0).*

(iii) *For any finite-dimensional representation* $\sigma\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, *the endomorphism* $\sigma(x) \in \mathfrak{gl}(V) \simeq \mathrm{End}(V)$ *is diagonalisable (resp. nilpotent).*

Proof. The decomposition (4.22) is 27.2.2 and 20.5.5 of [TY06]. Note that $\mathfrak{z} = 0$ if and only if G is semisimple (see 20.1.2 and 20.5.4 of *loc. cit.*).

The implication (iii) $\Longrightarrow$ (i) is a consequence of the definitions. We now prove (i) $\Longrightarrow$ (ii). Since $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is the differential of $\mathrm{Ad}\colon G \longrightarrow \mathrm{GL}(\mathfrak{g})$ (see 23.5.5 of *loc. cit.*), we only need to prove the following statement (the rest follows from the definitions): if $x \in \mathfrak{g}$ decomposes as $x = y + z$ via (4.22) and x is nilpotent, then $z = 0$. Since $[\mathfrak{g}, \mathfrak{g}]$ is the Lie algebra of the derived group (G, G) (see 27.2.3 of *loc. cit.*), we can consider the Jordan decomposition $y = y_{\mathrm{s}} + y_{\mathrm{n}}$ of y in $[\mathfrak{g}, \mathfrak{g}]$. Applying Theorem 4.A.2(2) to the inclusion $(G, G) \hookrightarrow G$, we see that y_{s} (resp. y_{n}) is semisimple (resp. nilpotent) in $\mathfrak{g}$. As z is semisimple in $\mathfrak{g}$ (see 27.2.2(ii) of *loc. cit.*) and commutes with everyone, we see that $y_{\mathrm{s}} + z$ is semisimple and $x = (y_{\mathrm{s}} + z) + y_{\mathrm{n}}$ is the Jordan decomposition of x. By uniqueness, $y_{\mathrm{s}} + z = 0$ hence $z = 0$. This finishes the proof of (i) $\Longrightarrow$ (ii). Finally, (ii) $\Longrightarrow$ (iii) is given by 20.5.7 and 20.5.8 of *loc. cit.* (note that their definition of semisimple and nilpotent elements differs from ours). □

Lemma 4.A.4. *Let G be a semisimple algebraic group over k, with semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}, \mathfrak{h}'$ be Cartan subalgebras of $\mathfrak{g}$. Then there exists a closed point $g \in G$ such that $\mathfrak{h}' = \mathrm{Ad}_G(g)\mathfrak{h}$.*

Proof. We proceed exactly as in [TY06, 29.2.3(ii)]. We first recall the notation of 19.1.1 of *loc. cit.* For a closed point $x \in \mathfrak{g}$, we write

$$\mathfrak{g}^0(x) \coloneqq \cap_{n \in \mathbb{Z}_{\geq 0}} \ker((\mathrm{ad}\, x)^n)$$

and we say that x is *generic* if $\dim \mathfrak{g}^0(x)$ (which is the multiplicity of 0 as a root of the characteristic polynomial of $\mathrm{ad}(x)$) is minimal among all $x \in \mathfrak{g}$. Call $\mathfrak{g}_{\mathrm{gen}}$ the set of generic elements; it is a Zariski-open subset of $\mathfrak{g}$ since the coefficients of the characteristic polynomial are a polynomial expression in $x \in \mathfrak{g}$. We know from 29.2.3(i) of *loc. cit.* that a Lie subalgebra of $\mathfrak{g}$ is Cartan if and only if it is of the form $\mathfrak{g}^0(x)$ for some $x \in \mathfrak{g}_{\mathrm{gen}}$.

Hence choose $x \in \mathfrak{g}_{\mathrm{gen}}$ such that $\mathfrak{h} = \mathfrak{g}^0(x)$. Consider the morphism

$$\theta\colon \begin{array}{rcl} G \times \mathfrak{h} & \longrightarrow & \mathfrak{g} \\ (g, y) & \longmapsto & \mathrm{Ad}(g)y \end{array}$$

whose differential at the point $(e, x) \in G \times \mathfrak{h}$ (where e is the neutral element in G) is

$$\mathrm{d}\theta_{(e,x)}\colon \begin{array}{rcl} \mathfrak{g} \times \mathfrak{h} & \longrightarrow & \mathfrak{g} \\ (z, y) & \longmapsto & [z, x] + y \end{array}$$

(see 29.1.4(i) and 23.5.5 of *loc. cit.*). By definition, $\mathfrak{h} = \mathfrak{g}^0(x)$ is a subspace of $\mathfrak{g}$ stable by $\mathrm{ad}(x)$, and $\mathrm{ad}(x)$ induces an automorphism on $\mathfrak{g}/\mathfrak{g}^0(x) = \mathfrak{g}/\mathfrak{h}$. Therefore, as in 29.2.1 of *loc. cit.*, $\mathrm{d}\theta_{(e,x)}$ is surjective, so θ is dominant by 16.5.7 of *loc. cit.* Hence, from 15.4.2 of *loc. cit.*, the image $\mathrm{Ad}(G)\mathfrak{h}$ contains an open dense subset of $\mathfrak{g}$.

Since the same can be said of $\mathrm{Ad}(G)\mathfrak{h}'$, we have

$$\mathrm{Ad}(G)\mathfrak{h} \cap \mathrm{Ad}(G)\mathfrak{h}' \cap \mathfrak{g}_{\mathrm{gen}} \neq \emptyset .$$

Therefore there is $t \in \mathfrak{g}_{\mathrm{gen}}$ and $g, g' \in G$ such that $\mathrm{Ad}(g)t \in \mathfrak{h}$ and $\mathrm{Ad}(g')t \in \mathfrak{h}'$. Since $\mathrm{Ad}(g)t$ and $\mathrm{Ad}(g')t$ are generic as well, 19.8.5 of *loc. cit.* gives

$$\mathfrak{h} = \mathfrak{g}^0(\mathrm{Ad}(g)t), \quad \mathfrak{h}' = \mathfrak{g}^0(\mathrm{Ad}(g')t)$$

hence $\mathfrak{h}' = \mathrm{Ad}(g'g^{-1})\mathfrak{h}$. □

For any vector space V over any field, write V^* for its dual, $S(V)$ for the symmetric algebra of V and $S^n(V)$ for the graded piece of $S(V)$ of degree n (where $n \in \mathbb{Z}_{\geq 0}$). Recall that V is finite dimensional, $S(V^*)$ can be interpreted as the algebra of polynomial functions on V.

If $\mathfrak{g}$ is a semisimple Lie algebra, then the action of $\mathfrak{g}$ on itself by adjunction induces an action on $S(\mathfrak{g}^*)$. We can then consider the algebra $S(\mathfrak{g}^*)^{\mathfrak{g}}$ of invariants under this action. Similarly, given a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ acts on $\mathfrak{h}$ (see [TY06, §19.8]), so we can consider $S(\mathfrak{h}^*)^W$ as well.

Theorem 4.A.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra over k, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and W be the Weyl group of the root system associated to $(\mathfrak{g}, \mathfrak{h})$.*

(1) *The restriction morphism $S(\mathfrak{g}^*) \longrightarrow S(\mathfrak{h}^*)$ induces a surjection $S(\mathfrak{g}^*)^{\mathfrak{g}} \twoheadrightarrow S(\mathfrak{h}^*)^W$.*

(2) *For $n \in \mathbb{Z}_{\geq 0}$, the vector space $S^n(\mathfrak{g}^*)$ is spanned by functions of the form $x \longmapsto \mathrm{tr}(\sigma(x)^n)$, where σ is a finite-dimensional representation of $\mathfrak{g}$ over k.*

Proof. This is [TY06, 31.2.6]. □

Lemma 4.A.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra over k, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and W be the Weyl group of the root system associated with $(\mathfrak{g}, \mathfrak{h})$. Let $x, y \in \mathfrak{h}$ be closed points such that $f(x) = f(y)$ for all $f \in S(\mathfrak{h}^*)^W$. Then x and y are in the same orbit for the action of W on $\mathfrak{h}$.*

Proof. This is [TY06, 34.2.1]. □

Proposition 4.A.7. *Let G be a semisimple algebraic group with Lie algebra $\mathfrak{g}$, let $x, y \in \mathfrak{g}$ be semisimple elements.*

Suppose that $\mathrm{tr}(\sigma(x)^n) = \mathrm{tr}(\sigma(y)^n)$ for all finite-dimensional representation σ of $\mathfrak{g}$ and all $n \in \mathbb{Z}_{\geq 0}$. Then x and y are in the same orbit for the action of G on $\mathfrak{g}$ by adjunction.

Proof. We know from [TY06, 20.6.3(ii)] that x and y are each contained in a Cartan subalgebra of $\mathfrak{g}$. According to Lemma 4.A.4, we can assume that x and y lie in the same Cartan subalgebra $\mathfrak{h}$. Let W be the Weyl group associated to the root system of $(\mathfrak{g}, \mathfrak{h})$. It follows from Theorem 4.A.5 and Lemma 4.A.6 that x and y are in the same orbit of the action of W on $\mathfrak{h}$.

There is a maximal torus T of G such that $\mathfrak{h}$ is the Lie algebra of T (see 29.2.5 of *loc. cit.* and [Hum75, 26.2.A]). The action of the Weyl group $W(G,T) = N_G(T)/T$ on T induces, through differentiation, an action of $\mathfrak{h}$; this identifies $W(G,T)$ with $W = W(\mathfrak{g}, \mathfrak{h})$. Hence W is isomorphic to $N_G(T)/T$, and the corresponding action of $w \in N_G(T)/T$ on $\mathfrak{h}$ is given by the adjunction by a lifting of w in $N_G(T)$ (see the discussion in §2.2). Therefore there is an element $n \in N_G(T) \subset G$ such that $x = \mathrm{Ad}_G(n)y$. □

Lemma 4.A.8. *Let $\mathfrak{g}$ be a semisimple Lie algebra over k, let $\sigma\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a finite-dimensional representation and let $n \in \mathbb{Z}_{\geq 0}$.*

(1) *Let $x \in \mathfrak{g}$ be a closed point with Jordan decomposition $x = x_{\mathrm{s}} + x_{\mathrm{n}}$. Then $\mathrm{tr}(\sigma(x)^n) = \mathrm{tr}(\sigma(x_{\mathrm{s}})^n)$.*

(2) *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, let $x, t \in \mathfrak{b}$ be closed points such that $x - t$ is nilpotent. Then $\mathrm{tr}(\sigma(x)^n) = \mathrm{tr}(\sigma(t)^n)$.*

Proof. The assertion (1) is [TY06, 34.2.3]. As for (2), note that by semisimplicity $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, so $\sigma(\mathfrak{b})$ is a solvable subalgebra of $\mathfrak{sl}(V) = [\mathfrak{gl}(V), \mathfrak{gl}(V)]$. Hence $\sigma(\mathfrak{b})$ is contained in a Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{sl}(V)$. Using 29.4.7 of *loc. cit.*, one can find a basis of V such that $\mathfrak{b}'$ is the set of upper triangular matrices in $\mathfrak{sl}(V)$. Since $x - t$ is nilpotent, by Proposition 4.A.3 the matrix $\sigma(x) - \sigma(t)$ is nilpotent and in $\mathfrak{b}'$ hence is strictly upper triangular. Therefore $\sigma(x)$ and $\sigma(t)$ are upper triangular with the same diagonal in $\mathfrak{b}'$. Thus so are $\sigma(x)^n$ and $\sigma(t)^n$, which therefore have the same trace on V. □

Proposition 4.A.9. *Let G be a connected reductive group over k with a maximal torus T and a Borel subgroup B containing T, let $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{b}$ be their respective Lie algebra. Let $x \in \mathfrak{b}$ be a closed point with Jordan decomposition $x = x_{\mathrm{s}} + x_{\mathrm{n}}$.*

(1) *The components x_{s} and x_{n} are in $\mathfrak{b}$.*

(2) *Let $t \in \mathfrak{t}$ and $u \in \mathfrak{b}$ such that u is nilpotent and $x = t + u$. Then there exists $g \in (G, G)$ such that $x_{\mathrm{s}} = \mathrm{Ad}_G(g)t$.*

Proof. The inclusion $\mathfrak{b} \subset \mathfrak{g}$ is the differential of the inclusion $B \subset G$, hence by Theorem 4.A.2(2), the Jordan decomposition of x viewed as an element of $\mathfrak{b}$ is the same as that of x viewed as an element of $\mathfrak{g}$. This proves (1).

For (2), the decomposition (4.22) gives $x = y + z$ with $y \in [\mathfrak{g}, \mathfrak{g}]$ and $z \in \mathfrak{z}$ where $\mathfrak{z}$ is the centre of $\mathfrak{g}$. Proposition 4.A.3 gives $u \in [\mathfrak{g}, \mathfrak{g}]$, hence $t = s + z$ for some semisimple element $s \in [\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{z}$ is the Lie algebra of the center $Z(G)$ of G (see

[TY06, 24.4.2(ii)]) and $Z(G) \subset T \subset B$ (see [Hum75, 26.2.A]), we have $\mathfrak{z} \subset \mathfrak{t} \subset \mathfrak{b}$, hence $y = x - z \in \mathfrak{b}$ and $s = t - z \in \mathfrak{b}$, thus $y, s \in \mathfrak{b}' := \mathfrak{b} \cap [\mathfrak{g}, \mathfrak{g}]$.

Note that $\mathfrak{b} = \mathfrak{b}' \oplus \mathfrak{z}$ since $\mathfrak{z} \subset \mathfrak{b}$. Let $\mathfrak{b}'_1$ be a solvable Lie subalgebra of $[\mathfrak{g}, \mathfrak{g}]$ containing $\mathfrak{b}'$, let $\mathfrak{b}_1 := \mathfrak{b}'_1 \oplus \mathfrak{z} \subseteq \mathfrak{g}$. Then $\mathfrak{b}_1$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and it is solvable since $[\mathfrak{b}_1, \mathfrak{b}_1] = [\mathfrak{b}'_1, \mathfrak{b}'_1]$. Therefore $\mathfrak{b}_1 = \mathfrak{b}$, and $\mathfrak{b}'_1 = \mathfrak{b}'$. This shows that $\mathfrak{b}'$ is a Borel subalgebra of $[\mathfrak{g}, \mathfrak{g}]$. Consequently, since $y = x - z = t + u - z = s + u$, we can apply Lemma 4.A.8 and Proposition 4.A.7 to the semisimple group (G, G) and the elements y, s of its Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ (see [TY06, 27.2.3]). This gives $g \in (G, G) \subset G$ such that

$$y_{\mathrm{s}} = \mathrm{Ad}_{(G,G)}(g)s = \mathrm{Ad}_G(g)s\,.$$

From the proof of Proposition 4.A.3, we know that $x_{\mathrm{s}} = y_{\mathrm{s}} + z$. Since the conjugation by g is the identity on $Z(G)$, its differential $\mathrm{Ad}_G(g)$ is trivial on $\mathfrak{z}$, so

$$x_{\mathrm{s}} = y_{\mathrm{s}} + z = \mathrm{Ad}_G(g)(s + z) = \mathrm{Ad}_G(g)t$$

which finishes the proof. □

Chapter 5

Computations on the local model

In §4, we proved Conjecture 4.1.1 in the case of good pairs. The goal of the present section is to explore the case of bad pairs, more specifically for the split reductive group $G = \mathrm{GL}_{n/k}$ where $n \in \mathbb{Z}_{>0}$ and k is any field of characteristic 0.

We fix $G = \mathrm{GL}_{n/k}$ and keep the notation of §4, with B the subgroup of upper triangular matrices in G, T the subgroup of diagonal matrices in G, U the subgroup of upper triangular matrices in G whose diagonal coordinates are 1, and $W = \mathcal{S}_n$ the Weyl group of G.

5.1 Grassmannians, the flag variety and Schubert cells

In this part, we recall a few well-know facts about the flag variety and its decomposition in Schubert cells.

Definition 5.1.1. For $1 \leq d \leq n$, the *Grassmannian* over k of d-planes in the n-space is defined to be the set of linear subspaces of dimension d in k^n. It is noted $\mathrm{Gr}_{d,n/k}$.

For $V \in \mathrm{Gr}_{d,n/k}$, let $(v_1, \ldots, v_d)$ be a basis of V. Then $v_1 \wedge \ldots \wedge v_d$ is a non-zero element of $\bigwedge^d k^n$. Choosing a different basis would change this alternate product by a multiplicative factor, which is the determinant of one basis in the other. Therefore the line spanned by $v_1 \wedge \ldots \wedge v_d$ in $\bigwedge^d k^n$ is a well-defined element of $\mathbb{P}(\bigwedge^d k^n)$. One can also recover V from this line by the property that a vector v is in V if and only if $v \wedge v_1 \wedge \ldots \wedge v_d = 0$ in $\bigwedge^{d+1} k^n$.

This discussion allows the following definition.

Definition 5.1.2. The *Plücker embedding* is the map

$$\begin{aligned} \mathrm{Gr}_{d,n/k} &\longrightarrow \mathbb{P}\Big(\bigwedge^d k^n\Big) \\ \langle v_1, \ldots, v_d \rangle &\longmapsto k \cdot (v_1 \wedge \ldots \wedge v_d) \end{aligned}$$

where $(v_1, \ldots, v_d)$ is any linearly independent family in k^n.

Remark 5.1.3. Proposition 5.1.5 below together with the discussion above Definition 5.1.2 show the Grassmannian is a projective variety with the Plücker embedding being a closed immersion.

We write any point $x \in \mathbb{P}(\bigwedge^d k^n)$ as

$$x = [x_{i_1 \ldots i_d}]_{1 \leq i_1 < \ldots < i_d \leq n}$$

using the projective coordinates along the standard basis $(e_{i_1} \wedge \ldots \wedge e_{i_d})_{1 \leq i_1 < \ldots < i_d \leq n}$ of $\bigwedge^d k^n$, where $(e_i)_{1 \leq i \leq n}$ is the standard basis of k^n. It is sometimes convenient to use the variables $x_{i_1 \ldots i_d}$ for any sequence $(i_1, \ldots, i_d)$ of integers between 1 and n; they are subject to the relations $x_{i_{\sigma(1)} \ldots i_{\sigma(d)}} = \operatorname{sgn}(\sigma) x_{i_1 \ldots i_d}$ for any $\sigma \in \mathcal{S}_d$ and $x_{i_1, \ldots, i_d} = 0$ whenever there are $1 \leq k < l \leq d$ such that $i_k = i_l$.

Definition 5.1.4. The *Plücker coordinates* of a point in $\mathrm{Gr}_{d,n/k}$ are the projective coordinates $[x_{i_1 \ldots i_d}]_{1 \leq i_1 < \ldots < i_d \leq n}$ of its image under the Plücker embedding.

Let $(v_1, \ldots, v_d)$ be a linearly independent family in k^n, and for any $1 \leq j \leq d$ write the coordinates of v_j along the standard basis as $(v_{ij})_{1 \leq i \leq n}$. Then the Plücker coordinate $x_{i_1 \ldots i_d}$ of the span $\langle v_1, \ldots, v_d \rangle$ can be computed as the rank d minor of the matrix $(v_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ along the rows $i_1, \ldots, i_d$ and columns $1, \ldots, d$.

Proposition 5.1.5. *The Plücker coordinates satisfy the following quadratic relations, called the* Plücker relations in dimension *d:*

$$\sum_{k=1}^{d+1} (-1)^k x_{i_1 \ldots i_{d-1} j_k} x_{j_1 \ldots \widehat{j_k} \ldots j_{d+1}} = 0 \tag{5.1}$$

where $(i_1, \ldots, i_{d-1})$ and $(j_1, \ldots, j_{d+1})$ are any increasing sequences of indices. The notation $\widehat{j}$ indicates that the index j is omitted.

These relations define precisely the points in $\mathbb{P}(\bigwedge^d k^n)$ that arise from the Grassmannian.

Proof. See for example [KL72, Theorem 1] or [Ful97, §9.1 Lemma 1]. □

In the same vein as Proposition 5.1.5, one can express the property that a subspace of k^n is included into another.

Proposition 5.1.6. *Let $1 \leq d < d' \leq n$ be two integers. The set of pairs $(V, V') \in \mathrm{Gr}_{d,n/k} \times \mathrm{Gr}_{d',n/k}$ such that $V \subset V'$ is identified with the set of points*

$$\left([x_{i_1, \ldots, i_d}]_{1 \leq i_1 < \ldots < i_d \leq n}, [x'_{i_1, \ldots, i_{d'}}]_{1 \leq i_1 < \ldots < i_{d'} \leq n} \right) \in \mathbb{P}\Big(\bigwedge^d k^n \Big) \times \mathbb{P}\Big(\bigwedge^{d'} k^n \Big)$$

which satisfy both sets of Plücker relations and the following additional quadratic relations:

$$\sum_{k=1}^{d'+1} (-1)^k x_{i_1 \dots i_{d-1} j_k} x'_{j_1 \dots \widehat{j_k} \dots j_{d'+1}} = 0 \tag{5.2}$$

where $(i_1, \dots, i_{d-1})$ and $(j_1, \dots, j_{d'+1})$ are any increasing sequences of indices.

These relations are called the incidence relations in dimensions (d, d').

Proof. See [Ful97, §9.1 Lemma 2]. □

Definition 5.1.7. A *flag* in k^n is a strictly increasing sequence $(V_1, \dots, V_{n-1})$ of proper subspaces of k^n. The *flag variety over* k is the set of flags and it is noted $\mathrm{Flag}_{n/k}$.

There is an obvious map $\mathrm{Flag}_{n/k} \hookrightarrow \prod_{d=1}^{n-1} \mathrm{Gr}_{d,n/k}$ which, when composed with the product of the Plücker embeddings, gives the embedding

$$\mathrm{Flag}_{n/k} \hookrightarrow \prod_{d=1}^{n-1} \mathbb{P}\Big(\bigwedge^d k^n\Big). \tag{5.3}$$

This realises the flag variety as the projective subvariety of $\prod_{d=1}^{n-1} \mathbb{P}(\bigwedge^d k^n)$ defined by the Plücker relations and the incidence relations (note that the Segre embedding [Sta18, Lemma 01WD] indeed makes $\prod_{d=1}^{n-1} \mathbb{P}(\bigwedge^d k^n)$ projective).

Example 5.1.8. For $n = 4$, the variety $\mathrm{Flag}_{4/k}$ is given by:

$$\begin{aligned}
\{ \quad & ([a_1, a_2, a_3, a_4], [b_{12}, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}], [c_{123}, c_{124}, c_{134}, c_{234}]) \\
& \in \mathbb{P}^4(k) \times \mathbb{P}^6(k) \times \mathbb{P}^4(k) \mid \\
& b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23} = 0 \\
& a_1 b_{23} - a_2 b_{13} + a_3 b_{12} = 0 \\
& a_1 b_{24} - a_2 b_{14} + a_4 b_{12} = 0 \\
& a_1 b_{34} - a_3 b_{14} + a_4 b_{13} = 0 \\
& a_2 b_{34} - a_3 b_{24} + a_4 b_{23} = 0 \\
& a_1 c_{234} - a_2 c_{134} + a_3 c_{124} - a_4 c_{123} = 0 \\
& b_{12} c_{134} - b_{13} c_{124} + b_{14} c_{123} = 0 \\
& b_{12} c_{234} - b_{23} c_{124} + b_{24} c_{123} = 0 \\
& b_{13} c_{234} - b_{23} c_{134} + b_{34} c_{123} = 0 \\
& b_{14} c_{234} - b_{24} c_{134} + b_{34} c_{124} = 0 \quad \}.
\end{aligned}$$

Here we have changed the names of the variables from x to a, b or c according to the dimension for the sake of readability.

Remark 5.1.9. The flag variety $\mathrm{Flag}_{n/k}$ can be identified with the algebraic group G/B in the following way. The left action of the group $G = \mathrm{GL}_{n/k}$ on k^n induces a left action of G on the flag variety that is transitive. Indeed, to any flag $(V_1, \ldots, V_{n-1})$ one can attach a sequence of linearly independent vectors $(v_1, \ldots, v_n)$ such that each V_d is the span of $(v_1, \ldots, v_d)$. This flag can then be obtained from the standard flag spanned by the standard basis, by the action of the matrix whose columns are the v_i. Finally, the stabiliser of the standard flag is precisely the Borel subgroup B.

As in the first paragraph of §4.1, for a permutation $w \in \mathcal{S}_n$ (the Weyl group of G), write BwB/B for the corresponding Schubert cell of G/B, so that we can write the decomposition $G/B = \coprod_{w \in \mathcal{S}_n} BwB/B$.

The following is well-known (see for example [Ful97, §10.5 Exercise 11]):

Proposition 5.1.10. *The cell BwB/B is the set of points in G/B that satisfy the following equations on Plücker coordinates, for all $d \in \{1, \ldots, n-1\}$:*

$$x_{w(1)\ldots w(d)} \neq 0 \quad \text{and} \quad x_{i_1 \ldots i_d} = 0 \quad \text{if} \quad \{i_1, \ldots, i_d\} \not\preceq \{w(1), \ldots, w(d)\} \tag{5.4}$$

where $\preceq$ is the order defined in Definition 2.4.2.

5.2 Explicit equations for $\widetilde{X}$ and $\widetilde{V}_w$

As in §4, we write $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$ and $\mathfrak{u}$ for the Lie algebras of G, B, T and U respectively, and $\mathrm{Ad}\colon G \mapsto \mathrm{Aut}(\mathfrak{g})$ for the adjoint representation (note that $\mathrm{Ad}(g)\psi = g\psi g^{-1}$ in matrix notation). The goal of this part is to give a description of the algebraic k-varieties $\widetilde{X}$ and $\widetilde{V}_w$ introduced in §4.2, as locally closed subvarieties of a projective space defined by explicit equations.

Consider the morphism

$$\begin{aligned} G \times \mathfrak{b} &\longrightarrow G/B \times \mathfrak{g} \\ (g, \psi) &\longmapsto (gB, \mathrm{Ad}(g)\psi). \end{aligned}$$

It induces an injection $G \times^B \mathfrak{b} \hookrightarrow G/B \times \mathfrak{g}$, which restricts to the isomorphism

$$\widetilde{X} \xrightarrow{\sim} \left\{(gB, \psi) \in G/B \times \mathfrak{b} \;\middle|\; \mathrm{Ad}(g^{-1})\psi \in \mathfrak{b}\right\} \tag{5.5}$$

between $\widetilde{X}$ and a closed subvariety of $G/B \times \mathfrak{b}$.

From now on, we use this isomorphism (5.5) as the definition of $\widetilde{X}$. This way, the maps $\widetilde{\pi}\colon \widetilde{X} \longrightarrow G/B$ and $q\colon \widetilde{X} \longrightarrow \mathfrak{b}$ also introduced in §4.2, are the projections on the first and second coordinate respectively. Also remember, for $w \in \mathcal{S}_n$ the cell

$$\widetilde{V}_w = \left\{(gB, \psi) \in \widetilde{X} \;\middle|\; g \in BwB/B\right\}$$

and its Zariski-closure $\widetilde{X}_w$, which give the cell decomposition $\widetilde{X} = \coprod_{w \in \mathcal{S}_n} \widetilde{V}_w$.

A point $(gB, \psi) \in \widetilde{X}$ will be identified by the projective coordinates $x_{i_1 \ldots i_d}$ of the flag gB as seen inside $\prod_{d=1}^{n-1} \mathbb{P}(\bigwedge^d k^n)$ by (5.3), and by the affine coordinates $(u_{ij})_{i \le j}$ given by the matrix coefficients of ψ. We sometimes write t_i instead of u_{ii} to increase readability.

Let us view ψ as the matrix of an endomorphism u of k^n, written relatively to the standard basis; recall that u is upper triangular. Let us also view g as a flag $(V_1, \dots, V_{n-1})$, so that V_d is the span of the first d columns of g.

Lemma 5.2.1. *Suppose that u is an isomorphism. Then $\mathrm{Ad}(g^{-1})\psi \in \mathfrak{b}$ if and only if, for each $1 \le d \le n-1$, the point $x_d = (x_{i_1 \dots i_d})_{i_1 < \dots < i_d} \in k^{\binom{n}{d}}$ is colinear to*

$$u(x_d) \coloneqq \left(\sum_{j_1 < \dots < j_d} \Delta_{i_1 \dots i_d}^{j_1 \dots j_d}(u) x_{j_1 \dots j_d} \right)_{i_1 < \dots < i_d} \in k^{\binom{n}{d}}$$

where $\Delta_{i_1 \dots i_d}^{j_1 \dots j_d}(u)$ is the minor of u associated to rows $i_1, \dots, i_d$ and columns $j_1, \dots, j_d$. This amounts to $\binom{n}{d}\left(\binom{n}{d} - 1\right)/2$ equations.

Proof. $\mathrm{Ad}(g^{-1})\psi$ is the matrix of u relative to the basis given by the columns of g. It is upper triangular if and only if each V_d is stable by u. Since u is an isomorphism, $u(V_d)$ is of dimension d and we can consider its Plücker coordinates: we specifically want that $u(V_d)$ and V_d have the same image in $\bigwedge^d k^n$ under the Plücker embedding. The image of V_d is the line spanned by the vector

$$x_d = \sum_{i_1 < \dots < i_d} x_{i_1 \dots i_d} e_{i_1} \wedge \dots \wedge e_{i_d} \quad \in \bigwedge^d k^n$$

and a quick calculation shows that the image of $u(V_d)$ is the line spanned by

$$u(x_d) = \sum_{i_1 < \dots < i_d} \left(\sum_{j_1 < \dots < j_d} \Delta_{i_1 \dots i_d}^{j_1 \dots j_d}(u) x_{j_1 \dots j_d} \right) e_{i_1} \wedge \dots \wedge e_{i_d} \quad \in \bigwedge^d k^n . \qquad \square$$

Remark 5.2.2. Note that since u is upper triangular, we have $\Delta_{i_1 \dots i_d}^{j_1 \dots j_d}(u) = 0$ unless $\{i_1, \dots, i_d\} \preceq \{j_1, \dots, j_d\}$ for the partial order of Definition 2.4.2. Also note that $\Delta_{i_1 \dots i_d}^{i_1 \dots i_d}(u) = t_{i_1} \dots t_{i_d}$.

Lemma 5.2.3. *Let A be the polynomial ring in the coordinates of $(gB/B, \psi)$, that is $A = \mathbb{Z}[x_{i_1 \dots i_d}, u_{kl}, t_m]_{1 \le d < n,\, 1 \le i_1 < \dots < i_d \le n,\, 1 \le k < l \le n,\, 1 \le m \le n}$. Let λ be an indeterminate variable.*

(1) *Keep the notation of Lemma 5.2.1. The condition*

$$x_d = (x_{i_1 \dots i_d})_{i_1 < \dots < i_d} \quad \textit{is colinear to}$$
$$(u + \lambda\, \mathrm{id})(x_d) = \left(\sum_{j_1 < \dots < j_d} \Delta_{i_1 \dots i_d}^{j_1 \dots j_d}(u + \lambda\, \mathrm{id}) x_{j_1 \dots j_d} \right)_{i_1 < \dots < i_d} \tag{5.6}$$

translates to $d\binom{n}{d}\left(\binom{n}{d}-1\right)/2$ *equations in* A.

(2) *The collection of all these equations in* A, *for* $1 \leq d \leq n-1$, *defines* $\widetilde{X}$ *inside* $G/B \times \mathfrak{b}$.

Proof. For all $i_1 < \ldots < i_d$ and $j_1 < \ldots < j_d$, the degree in λ of $\Delta_{i_1\ldots i_d}^{j_1\ldots j_d}(u+\lambda\,\mathrm{id})$ is at most $d-1$, unless $(j_1,\ldots,j_d)=(i_1,\ldots,i_d)$ where the degree is d with leading term λ^d. Hence the degree of $\sum_{j_1<\ldots<j_d}\Delta_{i_1\ldots i_d}^{j_1\ldots j_d}(u+\lambda\,\mathrm{id})x_{j_1\ldots j_d}$ in λ is d, with leading term $x_{i_1\ldots i_d}\lambda^d$. Therefore the $\binom{n}{d}\left(\binom{n}{d}-1\right)/2$ "cross-product" equations in $A[\lambda]$ that express the condition (5.6) are of degree $d-1$ in λ. This proves (1).

As for (2), the equations defining $\widetilde{X}$ are the ones that express the condition $\mathrm{Ad}(g^{-1})\psi \in \mathfrak{b}$. They are equivalent to saying that

$$\mathrm{Ad}(g^{-1})(\psi+\lambda\,\mathrm{id}) \in \mathfrak{b}$$

for all $\lambda \in k$. Enlarging k if necessary, we can assume that it is algebraically closed, in particular infinite. Therefore there is a $\lambda \in k$ such that $\psi+\lambda\,\mathrm{id}$ is invertible. Applying Lemma 5.2.1, our condition is equivalent to (5.6) for any such λ.

Note that (5.6) still holds true for any value of λ. Indeed, if $(u+\lambda)(V_d)$ is of dimension strictly $< d$, then $(u+\lambda)(\sum x_{i_1\ldots i_d}e_{i_1}\wedge\ldots\wedge e_{i_d})=0$, *i.e.* the Plücker coordinates of $(u+\lambda)(V_d)$ are zero. Hence their vector is colinear to any vector, in particular to the vector of the Plücker coordinates of V_d.

In the end, $\mathrm{Ad}(g^{-1})\psi \in \mathfrak{b}$ if and only if (5.6) is true for all $\lambda \in k$. Since k is algebraically closed, this amounts to the corresponding polynomials in $A[\lambda]$ being identically zero. □

Proposition 5.2.4. *Let* $w \in \mathcal{S}_n$. *Fix an integer* $1 \leq d \leq n-1$, *let* A *be the ring* $\mathbb{Z}[x_{i_1\ldots i_d}, u_{kl}, t_m]_{1\leq i_1<\ldots<i_d\leq n,\,1\leq k<l\leq n,\,1\leq m\leq n}$, *let* λ *be an indeterminate variable. For a sequence* $1 \leq i_1 < \ldots < i_d \leq n$, *let* $P_{w,i_1\ldots i_d} \in A[\lambda]$ *be the polynomial*

$$P_{w,i_1\ldots i_d}(\lambda) := \sum_{j_1<\ldots<j_d}\Delta_{i_1\ldots i_d}^{j_1\ldots j_d}(u+\lambda\,\mathrm{id})x_{j_1\ldots j_d} - \left(\prod_{k=1}^{d}(t_{w(k)}+\lambda)\right)x_{i_1\ldots i_d}$$

and for $0 \leq s \leq d-1$, *let* $P_{w,i_1\ldots i_d,s} \in A$ *be the unique homogeneous polynomials such that*

$$P_{w,i_1\ldots i_d}(\lambda) = \sum_{s=0}^{d-1} P_{w,i_1\ldots i_d,s}\lambda^s\,.$$

Then, using the Plücker coordinates $(x_{i_1\ldots i_d})_{1\leq d<n,\,1\leq i_1<\ldots<i_d\leq n}$ *on* G/B *and the affine coordinates* $(u_{kl}, t_m)_{1\leq k<l\leq n,\,1\leq m\leq n}$ *on* $\mathfrak{b}=\mathfrak{u}\oplus\mathfrak{t}$, *the subvariety* $\widetilde{V}_w$ *of* $G/B\times\mathfrak{b}$ *is defined by:*

- *the Plücker relations in dimension* d, *given in Proposition 5.1.5, for each* $1 \leq d \leq n-1$
- *the incidence relations in dimension* $(d, d+1)$, *given in Proposition 5.1.6, for each* $1 \leq d \leq n-2$

– *the equations for* $(x_{i_1\dots i_d})_{1\le d<n,\,1\le i_1<\dots<i_d\le n}$ *to be in the cell* BwB/B *of* G/B, *given in Proposition 5.1.10*

– *the equations* $P_{w,i_1\dots i_d,s}=0$ *for all* $1\le d\le n-1$, $1\le i_1<\dots<i_d\le n$ *and* $0\le s\le d-1$.

Proof. Lemma 5.2.3 gives a desciption of $\widetilde{X}$ as a subvariety of $G/B\times\mathfrak{b}$ defined by $\sum_{d=1}^{n-1} d\binom{n}{d}\left(\binom{n}{d}-1\right)/2$ equations (which are not necessarily independent). To find the equations for $\widetilde{V}_w$, we add the ones given by Proposition 5.1.10.

Now, $x_{j_1\dots j_d}=0$ for $\{j_1,\dots,j_d\}\not\preceq\{w(1),\dots,w(d)\}$ and $\Delta^{j_1\dots j_d}_{w(1)\dots w(d)}(u+\lambda\,\mathrm{id})=0$ for $\{j_1,\dots,j_d\}\not\succeq\{w(1),\dots,w(d)\}$. Hence the $(w(1),\dots,w(d))$-th term of the vector $(u+\lambda\,\mathrm{id})(x)$, as expressed in (5.6), is

$$\left(\prod_{k=1}^{d}(t_{w(k)}+\lambda)\right)x_{w(1)\dots w(d)}\,.$$

Knowing that $x_{w(1)\dots w(d)}\neq 0$ and that $x_{j_1\dots j_d}=0$ for $\{j_1,\dots,j_d\}\not\preceq\{w(1),\dots,w(d)\}$, the colinearity condition (5.6) is reduced to the following equations:

$$\left(\prod_{k=1}^{d}(t_{w(k)}+\lambda)\right)x_{i_1\dots i_d}=\sum_{j_1<\dots<j_d}\Delta^{j_1\dots j_d}_{i_1\dots i_d}(u+\lambda\,\mathrm{id})x_{j_1\dots j_d}$$

where $i_1<\dots<i_d$ is any strictly increasing sequence of indices. These are the announced equations. □

Remark 5.2.5. We have also seen that for each $w\in\mathcal{S}_n$, $1\le d\le n-1$ and $1\le i_1<\dots<i_d\le n$, the polynomial $P_{w,i_1\dots i_d}$ is a sum of terms

$$a_{j_1\dots j_d}x_{j_1\dots j_d},\quad \{j_1,\dots,j_d\}\succeq\{i_1,\dots,i_d\}\,,$$

where $a_{j_1\dots j_d}$ is an element of $\mathbb{Z}[u_{kl},t_m,\lambda]_{1\le k<l\le n,1\le m\le n}$. Also, the particular "first term coefficient" $a_{i_1\dots i_d}$ is equal to $\prod_{k=1}^{d}(t_{i_k}+\lambda)-\prod_{k=1}^{d}(t_{w(k)}+\lambda)$.

In particular, for $\{i_1,\dots,i_d\}\succeq\{w(1),\dots,w(d)\}$ the polynomial $P_{w,i_1\dots i_d}$ is in the ideal generated by the $x_{j_1\dots j_d}$ for $\{j_1,\dots,j_d\}\succ\{w(1),\dots,w(d)\}$, which are all zero on the cell BwB/B of G/B by Proposition 5.1.10. Therefore the associated equation is tautological on $\widetilde{V}_w$.

5.3 Taking the Zariski closure

For $w\in\mathcal{S}_n$, let $\widetilde{X}'_w$ be the subvariety of $\prod_{d=1}^{n-1}\mathbb{P}(\bigwedge^d k^n)\times\mathbb{A}^{n(n+1)/2}$, with coordinate ring $A\coloneqq\mathbb{Z}[x_{j_1\dots j_d},u_{kl},t_m]$, defined by all the equations listed in Proposition 5.2.4 except for the inequations $x_{w(1)\dots w(d)}\neq 0$ for $1\le d\le n-1$.

It is therefore the subvariety of $G/B\times\mathfrak{b}$ that satisfies the equations $P_{w,i_1\dots i_d,s}=0$ for all $1\le d\le n-1$, $1\le i_1<\dots<i_d\le n$ and $0\le s\le d-1$.

In particular, note that $\widetilde{X}'_w$ contains $\widetilde{V}_w$ by Proposition 5.2.4, and it is Zariski-closed. Therefore $\widetilde{X}_w \subseteq \widetilde{X}'_w$.

Proposition 5.3.1. *Let $w, w' \in \mathcal{S}_n$ such that $w' \preceq w$. Then, as algebraic varieties over k,*

$$\widetilde{X}'_w \cap \widetilde{V}_{w'} = \widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})\,.$$

Proof. Let $(gB/B, \psi) \in G/B \times \mathfrak{b}$ be a point in the intersection $\widetilde{X}'_w \cap \widetilde{V}_{w'}$. Its coordinates $(x_{j_1 \dots j_d}, u_{kl}, t_m)$ satisfy in particular $P_{w,i_1 \dots i_d}(\lambda) = 0$ in $k[\lambda]$ for any $\{i_1, \dots, i_d\}$ and $x_{i_1,\dots,i_d} = 0$ if $\{i_1, \dots, i_d\} \not\preceq \{w'(1), \dots, w'(d)\}$. Under these latter conditions, almost all the terms in $P_{w,w'(1)\dots w'(d)}(\lambda)$ become zero, and the equation $P_{w,w'(1)\dots w'(d)}(\lambda) = 0$ becomes

$$\left(\prod_{k=1}^{d}(t_{w'(k)} + \lambda) - \prod_{k=1}^{d}(t_{w(k)} + \lambda)\right) x_{w'(1)\dots w'(d)} = 0$$

on $\widetilde{X}'_w$. Since we also have $x_{w'(1)\dots w'(d)} \neq 0$, we then must have $\prod_{k=1}^{d}(t_{w'(k)} + \lambda) = \prod_{k=1}^{d}(t_{w(k)} + \lambda)$ as polynomials in $k[\lambda]$.

Also, we have by definition, for all $\{i_1, \dots, i_d\}$,

$$P_{w,i_1 \dots i_d}(\lambda) - P_{w',i_1 \dots i_d}(\lambda) = \prod_{k=1}^{d}(t_{w(k)} + \lambda) - \prod_{k=1}^{d}(t_{w'(k)} + \lambda)$$

in $A[\lambda]$. As a consequence, $P_{w,i_1 \dots i_d,s}$ and $P_{w',i_1 \dots i_d,s}$ are equal for all s and $\{i_1, \dots, i_d\}$ on $\widetilde{X}'_w$. Therefore $\widetilde{X}'_w \cap \widetilde{V}_{w'}$ is the subvariety of $\widetilde{V}_{w'}$ where $\prod_{k=1}^{d}(t_{w'(k)} + \lambda) = \prod_{k=1}^{d}(t_{w(k)} + \lambda)$ for all $1 \leq d \leq n-1$. This last equality of polynomials in $k[\lambda]$ means that the multisets $\{t_{w'(1)}, \dots, t_{w'(d)}\}$ and $\{t_{w(1)}, \dots, t_{w(d)}\}$ are equal for all $1 \leq d \leq n-1$. A recursion on d shows that this amounts to saying $t_{w'(k)} = t_{w(k)}$ for all $1 \leq k \leq d$. For $d = n-1$, this is equivalently to having our point lie in $q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})$. □

Replacing $\widetilde{X}'_w$ by $\widetilde{X}_w$ in Proposition 5.3.1 would yield Conjecture 4.2.1. However we shall see in Theorem 5.3.7 that this is not always true.

The strategy to find counterexamples to Conjecture 4.2.1 is to use the inequations $x_{w(1)\dots w(d)} \neq 0$ satisfied on $\widetilde{V}_w$ to simplify other equations satisfied on $\widetilde{V}_w$, by "removing" nonzero factors. We then get an equation that is satisfied on $\widetilde{V}_w$, hence on $\widetilde{X}_w$ as well, but not necessarily on $\widetilde{X}'_w$. This is the object of the following two lemmas. Once this is done, this equation, when considered on $\widetilde{V}_{w'}$, gives a condition that turns out to be strictly more restrictive that simply being in $q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})$; hence we are done.

Lemma 5.3.2. *Let $w \in \mathcal{S}_n$, let $q \in \{1, \dots, n-2\}$ and $b \in \{w(1), \dots, w(q+1)\}$; write $\{w(1), \dots, w(q+1)\} = \{i_1, \dots, i_q, b\}$. Let $\{j_1, \dots, j_q\}$ be a subset of $\{1, \dots, n\}$ that does not contain b.*

Assume that for any $1 \leq l \leq q$ such that $j_l < b$, we have $j_l \in \{w(1), \dots, w(q+1)\}$.

Then, for any $a \in \{1, \ldots, n\}$, the equation

$$x_{i_1 \ldots i_q a} x_{j_1 \ldots j_q b} = \pm x_{i_1 \ldots i_q b} x_{j_1 \ldots j_q a}$$

is satisfied on $\widetilde{V}_w$ for a certain choice of sign.

Proof. Consider the Plücker relation in dimension $q+1$ with indices $\{i_1, \ldots, i_q\}$ and $\{j_1, \ldots, j_q, a, b\}$ (see Proposition 5.1.6), which is satisfied on $\widetilde{V}_w$. Up to signs, it is of the form

$$\pm x_{i_1 \ldots i_q a} x_{j_1 \ldots j_q b} \pm x_{i_1 \ldots i_q b} x_{j_1 \ldots j_q a} + \sum_{1 \leq l \leq q} \pm x_{i_1 \ldots i_q k_l} x_{j_1 \ldots \hat{j_l} \ldots j_q a b} = 0 \tag{5.7}$$

where the hat indicates that an index is removed.

Let $1 \leq l \leq q$. If $j_l < b$, then our hypothesis implies that $x_{i_1 \ldots i_q j_l} = 0$ because the indices are redundant. Else we have $j_l > b$, so that $\{i_1, \ldots, i_q, j_l\} \succ \{i_1, \ldots, i_q, b\} = \{w(1), \ldots, w(q+1)\}$, hence $x_{i_1 \ldots i_q j_l} = 0$ on $\widetilde{V}_w$ by virtue of Proposition 5.1.10. Eliminating all those null variables from (5.7) gives the desired equation. □

Lemma 5.3.3. *Keep the notation and assumption of Lemma 5.3.2.*

Assume further that $\{j_1, \ldots, j_q, a\} \preceq \{i_1, \ldots, i_q, b\}$.

Then the equation

$$\sum_{\{j_1, \ldots, j_q\} \preceq \{k_1, \ldots, k_q\} \preceq \{i_1, \ldots, i_q\}} \pm \Delta^{k_1 \ldots k_q b}_{j_1 \ldots j_q b}(u + \lambda \,\mathrm{id}) x_{k_1 \ldots k_q a} - \left(\prod_{i=1}^{p+1} (t_{w(i)} + \lambda) \right) x_{j_1 \ldots j_q a} = 0 \tag{5.8}$$

is satisfied on $\widetilde{V}_w$ for a certain choice of sign.

Proof. We can assume that the sequences $(i_l)_{1 \leq l \leq q}$ and $(j_l)_{1 \leq l \leq q}$ are in increasing order. Let $1 \leq r \leq q$ be such that $j_r < b < j_{r+1}$ (setting $j_{q+1} = \infty$ if necessary). Then $\{j_1, \ldots, j_q, a\}$ has either r or $r+1$ elements smaller than or equal to b, depending on wether $a > b$ or $a \leq b$. Because $\{j_1, \ldots, j_q, a\} \preceq \{i_1, \ldots, i_q, b\}$, this implies that $\{i_1, \ldots, i_q, b\}$ has at most $r+1$ elements smaller than or equal to b. However the hypothesis of Lemma 5.3.2 implies that $\{j_1, \ldots, j_r, b\} \subseteq \{i_1, \ldots, i_q, b\}$. This forces $j_1, \ldots, j_r$ to be the r smallest elements of $\{i_1, \ldots, i_q\}$, hence $i_l = j_l$ for all $l \leq r$.

In particular, we have $\{j_1, \ldots, j_q, b\} \preceq \{i_1, \ldots, i_q, b\}$ and any set of integers that sits between those two is of the form $\{k_1, \ldots, k_q, b\}$ with $k_l = i_l$ for $l \leq r$ and $k_l > b$ for $l > r$. We can therefore apply Lemma 5.3.2 replacing $\{j_1, \ldots, j_q\}$ by $\{k_1, \ldots, k_q\}$; this yields, up to sign,

$$x_{i_1 \ldots i_q a} x_{k_1 \ldots k_q b} = \pm x_{i_1 \ldots i_q b} x_{k_1 \ldots k_q a} \,. \tag{5.9}$$

Now, keeping the notation of Proposition 5.2.4, we multiply by $x_{i_1 \ldots i_q a}$ the equation $P_{w, j_1 \ldots j_q b}(\lambda) = 0$ (in $A[\lambda]$). From the discussion of Remark 5.2.5, the resulting equation only has terms in $x_{k_1 \ldots k_q b} x_{i_1 \ldots i_q a}$ for $\{j_1, \ldots, j_q\} \preceq \{k_1, \ldots, k_q\}$. Also, because of Proposition 5.1.10, the $x_{k_1 \ldots k_q b}$ are nonzero on $\widetilde{V}_w$ only for $\{k_1, \ldots, k_q\} \preceq$

$\{i_1,\dots,i_q\}$. With all these simplifications, we get on $\widetilde{V}_w$ the equation

$$\sum_{\{j_1,\dots,j_q\}\preceq\{k_1,\dots,k_q\}\preceq\{i_1,\dots,i_q\}} \Delta^{k_1\dots k_q b}_{j_1\dots j_q b}(u+\lambda\,\mathrm{id})x_{k_1\dots k_q b}x_{i_1\dots i_q a} \\ - \left(\prod_{i=1}^{q+1}(t_{w(i)}+\lambda)\right)x_{j_1\dots j_q b}x_{i_1\dots i_q a}=0\,.$$

Using the equations (5.9), we get

$$\sum_{\{j_1,\dots,j_q\}\preceq\{k_1,\dots,k_q\}\preceq\{i_1,\dots,i_q\}} \pm\Delta^{k_1\dots k_q b}_{j_1\dots j_q b}(u+\lambda\,\mathrm{id})x_{k_1\dots k_q a}x_{i_1\dots i_q b} \\ - \left(\prod_{i=1}^{q+1}(t_{w(i)}+\lambda)\right)x_{j_1\dots j_q a}x_{i_1\dots i_q b}=0$$

which, after factorisation and simplification by $x_{i_1\dots i_q b}$ (the simplification is valid because $x_{i_1\dots i_q b}=x_{w(1)\dots w(q+1)}\neq 0$ on $\widetilde{V}_w$), gives the equation we wanted. $\square$

Proposition 5.3.4. *Let $w,w'\in\mathcal{S}_n$ with $w'\preceq w$. Assume that there exist integers $a<b$ and $q\in\{1,\dots,n-2\}$ such that:*

- *$a\notin\{w(1),\dots,w(q+1)\}$ and $a\in\{w'(1),\dots,w'(q+1)\}$,*
- *$b\in\{w(1),\dots,w(q+1)\}$ and $b\notin\{w'(1),\dots,w'(q+1)\}$,*
- *for any $i\in\{w'(1),\dots,w'(q+1)\}$ such that $i\neq a$ and $i<b$, we have $i\in\{w(1),\dots,w(q+1)\}$.*

Then the equation $t_a=t_b$ is satisfied on $\widetilde{X}_w\cap\widetilde{V}_{w'}$.

Example 5.3.5. Take $n=4$, $w=[4231]$ and $w'=[1324]$.

Then $q=1$ and $(a,b)=(1,2)$ meets the condition of Proposition 5.3.4, hence $\widetilde{X}_w\cap\widetilde{V}_{w'}$ satisfies $t_1=t_2$. This is stronger than for $\widetilde{X}'_w\cap\widetilde{V}_{w'}$, which by Proposition 5.3.1 only satisfies $t_1=t_4$ and $t_2=t_3$ (the equations defining $\mathfrak{t}^{ww'^{-1}}$).

One could also take $(a,b)=(3,4)$; this gives the equation $t_3=t_4$ on $\widetilde{X}_w\cap\widetilde{V}_{w'}$, which is equivalent to the already found $t_1=t_2$, since $t_1=t_4$ and $t_2=t_3$ are true.

Proof. Write $\{j_1,\dots,j_q\}=\{w'(1),\dots,w'(q+1)\}\setminus\{a\}$; then the hypotheses of Lemma 5.3.2 are satisfied. From Example 2.4.4 applied to $w'\preceq w$, we also have $\{j_1,\dots,j_q,a\}\preceq\{i_1,\dots,i_q,b\}$. We can therefore apply Lemma 5.3.3 to get

$$\sum_{\{j_1,\dots,j_q\}\preceq\{k_1,\dots,k_q\}\preceq\{i_1,\dots,i_q\}} \pm\Delta^{k_1\dots k_q b}_{j_1\dots j_q b}(u+\lambda\,\mathrm{id})x_{k_1\dots k_q a} - \left(\prod_{i=1}^{p+1}(t_{w(i)}+\lambda)\right)x_{j_1\dots j_q a}=0$$

which is satisfied on $\widetilde{V}_w$, hence on the Zariski closure $\widetilde{X}_w$ as well.

Now consider this equation on $\widetilde{X}_w \cap \widetilde{V}_{w'}$. From Proposition 5.1.10, all variables $x_{k_1\dots k_q a}$ are zero on $\widetilde{V}_{w'}$ for $\{k_1,\dots,k_q\} \succ \{j_1,\dots,j_q\}$, so it reduces to one term:

$$\left(\Delta^{j_1\dots j_q b}_{j_1\dots j_q b}(u+\lambda\,\mathrm{id}) - \prod_{i=1}^{p+1}(t_{w(i)}+\lambda)\right) x_{j_1\dots j_q a} = 0\,.$$

Again on $\widetilde{V}_{w'}$ we can simplify by $x_{j_1\dots j_q a} = x_{w'(1)\dots w'(q+1)} \neq 0$. Moreover, u being upper triangular gives $\Delta^{j_1\dots j_q b}_{j_1\dots j_q b}(u+\lambda\,\mathrm{id}) = (t_b+\lambda)\prod_{l=1}^{p}(t_{j_l}+\lambda) = \frac{t_b+\lambda}{t_a+\lambda}\prod_{i=1}^{p+1}(t_{w'(i)}+\lambda)$, so the equation becomes

$$\frac{t_b+\lambda}{t_a+\lambda}\prod_{i=1}^{p+1}(t_{w'(i)}+\lambda) = \prod_{i=1}^{p+1}(t_{w(i)}+\lambda)\,.$$

From the proof of Proposition 5.3.1, we also know that $\prod_{i=1}^{p+1}(t_{w'(i)}+\lambda) = \prod_{i=1}^{p+1}(t_{w(i)}+\lambda)$; so we finally get $t_a+\lambda = t_b+\lambda$, which implies $t_a = t_b$. $\square$

Remark 5.3.6. The same result holds when the last condition on a, b in Proposition 5.3.4 is replaced by the following variant: for any $j \in \{w(1),\dots,w(q+1)\} \setminus \{b\}$ such that $j > a$, we have $j \in \{w'(1),\dots,w'(q+1)\}$. The proof carries out in almost identical fashion, with the difference that, in the proof of Lemma 5.3.2, instead of using the Plücker relation with indices $\{i_1,\dots,i_q\}$ and $\{k_1,\dots,k_q,a,b\}$ for equation (5.7), we use the Plücker relation with indices $\{k_1,\dots,k_q\}$ and $\{i_1,\dots,i_q,a,b\}$.

Theorem 5.3.7. *Let $w, w' \in \mathcal{S}$ with $w' \preceq w$. Assume that there exist a, b as in Proposition 5.3.4 such that a and b are not in the same orbit under ww'^{-1}.*

Then Conjecture 4.1.1 is false for the pair (w, w'), that is:

$$\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u}) \not\subseteq \widetilde{X}_w\,.$$

Proof. Consider any point $x = (x_{i_1\dots i_d}, t_i, u_{ij}) \in \prod_{d=1}^{n-1}\mathbb{P}(\bigwedge^d k^n) \times \mathbb{A}^{n(n+1)/2}$ where, for $1 \le d \le n-1$ and $1 \le i_1 < \dots < i_d < n$, $x_{i_1\dots i_d} \neq 0$ if and only if $\{i_1,\dots,i_d\} = \{w'(1),\dots,w'(d)\}$, and where $u_{ij} = 0$ for all $1 \le i < j \le n$. It is easily checked that $x \in \widetilde{V}_{w'}$, either from the equations of Proposition 5.2.4, or from the fact that such a point x corresponds to $(gB/B, \psi) \in G/B \times \mathfrak{b}$ where g is an invertible matrix with zeroes everywhere but in positions $(w'(j), j)$ for $1 \le j \le n$, and where $\psi \in \mathfrak{t}$.

When a and b are not in the same orbit under ww'^{-1}, it is possible to set $\psi = (t_1,\dots,t_n) \in \mathfrak{t}^{ww'^{-1}}$ with $t_a \neq t_b$. This gives $x \in q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})$, while Proposition 5.3.4 forces $x \notin \widetilde{X}_w \cap \widetilde{V}_{w'}$, so $x \notin \widetilde{X}_w$. $\square$

Remark 5.3.8. In the situation of Theorem 5.3.7, we actually have a strict inclusion

$$\widetilde{X}_w \cap \widetilde{V}_{w'} \subsetneq \widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})$$

since the (non strict) inclusion is already known (see Proposition 5.3.1 or the proof of Proposition 4.2.2).

It is also known that $\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})$ is irreducible: by an argument similar to the proof of Proposition 4.2.2, this statement is equivalent to the analogous statement with $V_{w'}$ (defined in §4.1) in place of $\widetilde{V}_{w'}$, which is [BHS19, Lemma 2.3.5]. Therefore $\widetilde{X}_w \cap \widetilde{V}_{w'}$ is a union of irreducible component which have dimension strictly smaller than the dimension of $\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})$, thus

$$\dim\left(\widetilde{X}_w \cap \widetilde{V}_{w'}\right) < \dim\left(\widetilde{V}_{w'} \cap q^{-1}(\mathfrak{t}^{ww'^{-1}} + \mathfrak{u})\right) .$$

Remark 5.3.9. The easiest example of w, w' meeting the conditions of Theorem 5.3.7 is when there exist $1 \leq i < j < k < l \leq n$ and $1 \leq a < b < c < d \leq n$ such that:

1. $(w(i), w(j), w(k), w(l)) = (d, b, c, a)$.
2. $(w'(i), w'(j), w'(k), w'(l)) = (a, c, b, d)$.
3. for all $m \in \{1, \dots, n\} \setminus \{i, j, k, l\}$, $w(m) = w'(m)$.

The smallest such example is when (w, w') is the first bad pair, for $n = 4$ (see Theorem 2.6.3). In fact, all cases of interest (*e.g.* when (w, w') is a bad pair, see Lemma 5.3.10 below) for $n \leq 5$ meet the conditions of Theorem 5.3.7.

For $n = 6$, a few cases arise where neither those conditions nor the variant of Remark 5.3.6 are met, for example when $w = (16)(25)$ and $w' = (34)$ (using the cycles notation for elements of the symmetric group $\mathcal{S}_6$). In this specific case, the approach of the proof of Theorem 5.3.7 is not adapted. Indeed, the only equations defining the Schubert cell BwB/B in G/B are $x_6 \neq 0$, $x_{56} \neq 0$, $x_{356} \neq 0$, $x_{3456} \neq 0$, $x_{23456} \neq 0$ and $x_{456} = 0$. Because x_{456} is the only zero variable, there are too few Plücker relations and incidence relations that have only two nonzero terms on BwB/B (in the same way equation (5.7) gives Lemma 5.3.2). Furthermore those few relations cannot be used to get new equations satisfied on $\widetilde{V}_w$ (and hence $\widetilde{X}_w$) that are nontrivial on $\widetilde{V}_{w'}$ (the same way the equation $t_a = t_b$ of Proposition 5.3.4 is), because all Plücker coordinates but $x_1, x_{12}, x_{123}, x_{124}, x_{1234}, x_{12345}$ are zero on $\widetilde{V}_{w'}$. The status of Conjecture 4.1.1 remains unknown for this pair.

The following lemma confirms that Theorem 5.3.7 and Theorem 4.2.3 are consistent with each other.

Lemma 5.3.10. *Let $(w', w) \in W$ satisfy the conditions of Theorem 5.3.7. Then it is a bad pair in the sense of Chapter 2.*

The same conclusion holds with the conditions of Remark 5.3.6.

Proof. If the pair (w, w') satisfies the conditions of Theorem 5.3.7, including the one that states that a and b are not in the same orbit under ww'^{-1}, then it is a bad pair in the sense of section 2. Indeed, let Ω the orbit of b under ww'^{-1}. By assumption, $a \notin \Omega$, so the last condition of Proposition 5.3.4 implies that

$$(\Omega \cap w(\{1, \dots, q+1\})) \cap \{1, \dots, b\} \supsetneq (\Omega \cap w'(\{1, \dots, q+1\})) \cap \{1, \dots, b\}$$

with the inequality stemming from the presence of b in the left-hand side and not in the right-hand side. Therefore $\Omega \cap w(\{1, \ldots, q+1\}) \succeq \Omega \cap w'(\{1, \ldots, q+1\})$ is false (see the second characterisation in Definition 2.4.2), so the pair (w, w') is bad according to Proposition 2.4.5.

In the situation of Remark 5.3.6, take Ω to be the orbit of a under ww'^{-1}. Again, $b \notin \Omega$, so the condition spelled out in Remark 5.3.6 implies that

$$(\Omega \cap w(\{1, \ldots, q+1\})) \cap \{a, \ldots, n\} \subsetneq (\Omega \cap w'(\{1, \ldots, q+1\})) \cap \{a, \ldots, n\}$$

from which we get $|(\Omega \cap w(\{1, \ldots, q+1\})) \cap \{a, \ldots, n\}| < |(\Omega \cap w'(\{1, \ldots, q+1\})) \cap \{a, \ldots, n\}|$, or equivalently $|(\Omega \cap w(\{1, \ldots, q+1\})) \cap \{1, \ldots, a-1\}| > |(\Omega \cap w'(\{1, \ldots, q+1\})) \cap \{1, \ldots, a-1\}|$. As before, this means that $\Omega \cap w(\{1, \ldots, q+1\}) \succeq \Omega \cap w'(\{1, \ldots, q+1\})$ is false and hence (w, w') is a bad pair. $\square$

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