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Robust optimization for a lot-sizing problems under production yield uncertainty

Paula Metzker Soares

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Par

Paula METZKER SOARES

**L'optimisation robuste pour des problèmes de lotissement dans
un contexte de rendement incertain**

Robust optimization for lot-sizing problems under production yield uncertainty

Thèse présentée et soutenue à IMT Atlantique - Campus Nantes , le 25 novembre 2022

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Rapporteurs avant soutenance :

El-Houssaine AGHEZZAF Professor at Ghent University
Safia KEDAD-SIDHOUM Professor at Universités au Conservatoire National des Arts et Métiers

Composition du Jury :

Président :	Ola JABALI	Associate Professor at Politecnico di Milano
Examineurs :	El-Houssaine AGHEZZAF	Professor at Ghent University
	Ioanis FRAGKOS	Associate Professor at Rotterdam School of Management
	Philippe MAHEY	Emeritus Professor at Université Clermont Auvergne
	Safia KEDAD-SIDHOUM	Professor at Universités au Conservatoire National des Arts et Métiers
Dir. de thèse :	Alexandre DOLGUI	Professor at IMT Atlantique
Encadrant de thèse :	Simon THEVENIN	Associate Professor à IMT Atlantique
	Yossiri ADULYASAK	Professor at HEC Montréal

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“

Even the darkest night will end and the sun will rise. [Hugo, 1863]

”

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RÉSUMÉ EN FRANÇAIS

Les fabricants doivent gérer leurs capacités de production et leur performance pour satisfaire la demande des clients avec des produits de qualité. De plus, ils ont constamment été mis au défi d'optimiser l'utilisation de leurs ressources et leur performance de production dans un contexte de marché dynamique et volatil, et ce, de manière rentable. Pour atteindre cet objectif commercial, un défi tactique à relever parmi d'autres, les fabricants doivent minutieusement prendre des décisions de dimensionnement de lots de production afin de déterminer la configuration et les quantités de production répondant aux demandes avec des produits de qualité, tout en minimisant les coûts globaux de production et de gestion de stocks [Murthy and Ma, 1991].

La qualité des produits est un enjeu fondamental qui impacte directement les coûts et la rentabilité de l'entreprise. Les planificateurs de production s'appuient généralement sur le rendement de la production pour mesurer la quantité attendue des produits de qualité résultant d'un lot de production. Ensuite, les décideurs définissent les décisions de dimensionnement de lots de production en se basant sur les taux de rendement de la production attendus. Le rendement de la production est estimé à partir des données historiques et des spécifications de contrôle-qualité. La qualité de la production peut être influencée par des facteurs exogènes (e.g., le changement climatique, le retard ou les perturbations dans l'approvisionnement et l'évolution de la qualité des matières premières) et endogènes (e.g., l'entretien des machines, la disponibilité et la capacité de la main-d'œuvre, une défaillance dans le déroulement du processus de production, la température et la variation d'humidité) [Grosfeld-Nir and Gerchak, 2004].

Les problèmes de dimensionnement de lots de production (*lot-sizing problems* - LSPs - en anglais) déterminent sur un horizon de planification la configuration et les lots de production qui minimisent les coûts globaux, satisfont la demande et maintiennent un niveau de service satisfaisant par des produits de qualité. En raison de son importance en pratique, les LSPs ont attiré un large spectre de secteurs d'activité, de celui de la fabrication à celui de la recherche en optimisation mathématique. En effet, les systèmes de distribution et de production sont installés dans un environnement complexe où les départements de production, de qualité, de vente, d'achat, de logistique, de l'entreprise,

des services techniques, de comptabilité et de marketing sont constamment affectés par des événements imprévisibles. Ainsi, les LSPs deviennent inadéquats pour répondre aux besoins de l'industrie s'ils ne s'adaptent pas de manière suffisamment simple à l'évolution de l'environnement.

Les systèmes de production et de distribution sont confrontés à diverses sources d'incertitude (demande, délai d'exécution, rendement de la production, entre autres) qui affectent les coûts et le niveau de service associés aux lots de production. Traditionnellement, ces systèmes atténuent ces incertitudes en modifiant les paramètres des systèmes de planification, tels que le stock de sûreté, le délai de sûreté et la fréquence de replanification. Les avancées technologiques de calcul et l'accès aux données massives ont conduit à la conception de méthodes d'optimisation axées sur les données afin d'incorporer directement les incertitudes dans le LSP, comme la programmation stochastique (SP) [Birge and Louveaux, 2011], l'optimisation robuste (RO) [Ben-Tal et al., 2009] ou encore l'optimisation robuste distributionnelle (DRO) [Wiesemann et al., 2014].

En raison de la complexité et du grand nombre de facteurs susceptibles d'affecter la qualité de production, les estimations du rendement de la production sont souvent inexactes, les pertes durant le processus de production sont difficiles à prédire et leur impact endommagent grandement le système [Inderfurth and Vogelgesang, 2013]. En ce qui concerne les produits à cycle de vie court, de tels impacts peuvent s'avérer considérables, par exemple la rétrogradation de certains articles pour répondre aux demandes dans l'industrie électronique [e.g., Bitran and Gilbert, 1994] ou le large stock et la périssabilité dans les industries biologiques et agricoles [e.g., Chick et al., 2008, Hyytiäinen et al., 2011, Eskandarzadeh et al., 2014]. L'incertitude du rendement de la production dans la production de vaccins a un impact direct sur le bien-être social, car cela peut conduire à l'affaiblissement de l'immunité collective aux nouvelles souches ou mutations virales [Cho, 2010]. Pour finir, l'incertitude du rendement de la production peut aussi impacter les accords contractuels dans les chaînes d'approvisionnement, dans lesquelles le faible rendement de la production implique une indemnité d'assurance plus coûteuse, un investissement en capital réduit, ou des difficultés à obtenir un crédit et des subventions [Anderson and Monjardino, 2019]. Ces exemples illustrent l'importance de développer des approches d'optimisation qui couvrent l'ensemble des incertitudes de manière systématique, efficace et efficiente.

A notre connaissance, Kazemi Zanjani et al. [2010] et Quezada et al. [2020] sont les seuls travaux portant sur la programmation stochastique pour un LSP sous contraintes d'incertitudes du rendement de la production; et il n'existe à ce jour aucune

étude portant sur l'optimisation robuste, robuste adaptative ou robuste distributionnelle pour du problème du dimensionnement de lots de production sous contraintes d'incertitudes du rendement dans un contexte de planification de production. La carence d'études sur l'application de ces méthodologies aux LSPs avec incertitude du rendement est la motivation première de ce travail. Nous proposons différents modèles et stratégies méthodologiques pour calculer des plans de production optimaux robustes à conservation moindre. Nous fournissons également une analyse de risque pour aider les décideurs à adapter leurs plans décisionnels à leurs objectifs stratégiques et opérationnels.

Les contributions de cette thèse sont organisées comme suit, le :

Chapitre 1 présente un état de l'art des problèmes de dimensionnement de lots de production sous contraintes d'incertitudes. Il fournit les méthodologies de résolution les plus courantes appliquées à ces problèmes, et il introduit également les études de problèmes de dimensionnement de lots de production sous contraintes d'incertitudes du rendement et de la rupture de stock.

Chapitre 2 étudie l'optimisation robuste appliquée au problème du dimensionnement de lots de production pour un seul produit sous contraintes d'incertitudes du rendement et la possibilité de délai dans la production. Nous introduisons le modèle linéaire en nombres entiers mixtes robuste ainsi que la définition de la politique règle optimale pour le cas particulier du problème de gestion de stocks sous contraintes d'incertitudes du rendement. Nous fournissons également la structure de la solution optimale robuste, et développons un programme dynamique à complexité polynomiale pour calculer un plan de production robuste. Le résultat des calculs et des perspectives de gestion sont présentés.

Chapitre 3 étudie l'utilisation de l'optimisation robuste adaptative pour immuniser le plan de production contre les incertitudes du rendement, et pour réagir en conséquence lorsque des données sont actualisées. Nous proposons une règle robuste adaptative myope pour le problème de gestion de stocks, et nous montrons que cette règle est optimale sous hypothèses modérées. Ensuite, nous formulons un modèle robuste adaptatif qui exploite une approximation linéaire pour transformer les contraintes quadratiques en un programme linéaire à nombres entiers mixtes. Pour obtenir une solution exacte via ce modèle, nous proposons un algorithme de génération de colonnes et de contraintes. Nous fournissons aussi une technique d'horizon mobile pour mesurer la performance des

modèles proposés dans un contexte décisionnel dynamique. Des expériences numériques approfondies sont fournies.

Chapitre 4 traite le problème du dimensionnement de lots de production à produits et périodes multiples sous contraintes d'incertitudes du rendement et du délai de la production par l'optimisation robuste distributionnelle. Nous fournissons une formulation robuste distributionnelle qui peut être résolue par des solveurs commerciaux. Ensuite, nous développons un modèle robuste en nombres entiers mixtes pour calculer des plans de production robustes distributionnels suffisamment satisfaisants. Nous comparons les plans robustes distributionnels avec les plans de production proposés par d'autres méthodologies d'optimisation afin de mesurer la performance des modèles robustes distributionnels et ainsi, couvrir l'incertitude de rendement de la production et proposer des plans de production plus proches des applications du monde réel.

Ces chapitres ont donné lieu aux quatre papiers scientifiques ci-dessous:

1. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. (2023). Robust optimization for lot-sizing problems under yield uncertainty. *Computers & Operations Research*, 149, 106025.
2. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. Adaptive robust optimization for lot-sizing under yield uncertainty. *European Journal of Operational Research* (1st revision), November 2022.
3. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. Distributionally robust optimization for the multi-period multi-item lot-sizing problems under yield uncertainty. Target journal: *Naval Research Logistics*, November 2022.
4. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. A state of the art on the LSPs under uncertainty: trends on optimization methodologies and approaches. Working draft, November 2022.

Nous détaillons les parties qui proviennent des papiers précédemment mentionnés dans le préambule des chapitres correspondants. Ce travail a donné également lieu à deux actes de conférence ci-dessous :

1. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. Optimization for Lot-Sizing Problems Under Uncertainty: A Data-Driven Perspective. In *IFIP International Conference on Advances in Production Management Systems* (pp. 703-709). Springer, Cham, September 2021.
2. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. The adaptive robust lot-sizing problem with backorders under demand uncertainty. In *2021 IEEE 17th International Conference on Automation Science and Engineering (CASE)* (pp. 997-1001). IEEE, August 2021.

Cette thèse de doctorat est organisée comme suit : 1 passe en revue les études des problèmes de *lot-sizing* non déterministes, et met en évidence le manque de connaissances sur les approches pour traiter les problèmes de dimensionnement de lots de production sous contraintes d'incertitudes de rendement. Le chapitre 2 présente une méthodologie statique robuste pour définir un plan de production. Le chapitre 3 présente des approches robustes adaptatives pour calculer un plan de production adaptatif. Le chapitre 4 fournit des programmes robustes distributionnels pour concevoir un plan de production robuste axé sur les données. Enfin, 5 conclut ce travail et propose des orientations pour les travaux futurs.

INTRODUCTION

Manufacturers must efficiently manage their production capacities and their performances to satisfy customer demands with quality goods. They also have been constantly challenged to optimize resource usage and production performance in a dynamic and volatile market context in a cost effective manner. To achieve this business objective, among other tactical problems, manufacturers must carefully make lot-sizing decisions to determine the production setups and quantities that meet demands with quality goods, while minimizing the overall production and inventory management costs [Murthy and Ma, 1991].

Product quality is a fundamental issue which directly impacts costs and profitability of the firm. Production planners usually rely on the production yield to measure the expected quantity of quality goods resulting from a lot size. Then, the decision makers define the lot-sizing decisions based on the estimated production yield rates. The production yield is estimated based on historical data and quality control specifications. The quality of the production can be influenced by exogenous factors (e.g., climate changes, supply disruption or delay, and change in the quality of raw materials) and endogenous factors (e.g., machinery maintenance, workforce ability and availability, failure in the production process flow, temperature, and humidity variation) [Grosfeld-Nir and Gerchak, 2004].

The lot-sizing problem (LSP) determines the setups and lot sizes over a planning horizon that minimize overall costs, meet the demands and maintains a satisfactory level of service with quality goods [Wagner and Whitin, 1958]. Due to its practical importance, the LSPs attracted a wide range of research from the manufacturing and mathematical optimization communities. In fact, production and distribution systems are settled in a complex environment where production, quality, sales, purchasing, logistics, corporate, technical, accounting and marketing department are constantly affected by unexpected events. Thus, LSPs become inadequate to meet the needs of the industry if they are not simple enough to be adapted to changes in the environment [Dolgui and Prodhon, 2007].

Production and distribution systems face various sources of uncertainties (demand, lead time, production yield, among others) that affect the costs and service level associated with the lot-sizes. Traditionally, these systems dampen these uncertainties by changing

parameters of the planning systems, such as the safety stock, safety lead-time, and re-planning frequency. Advances in computing technologies and the massive availability of data led to the design of data-driven optimizations to directly incorporate the uncertainties within the LSP, such as stochastic programming (SP)[Birge and Louveaux, 2011], robust optimization (RO) [Ben-Tal et al., 2009], and distributionally robust optimization (DRO) [Wiesemann et al., 2014].

Due to the complexity and a large number of factors that can affect the quality of the production, production yield estimations are often inaccurate, the losses in the production process are difficult to predict and their impact are highly damaging to the system [Inderfurth and Vogelgesang, 2013]. For short life cycle products, such impacts can be significant, for example, downgrading some items to meet the customers demand in the electronics industry [e.g., Bitran and Gilbert, 1994] or large inventory and perishability in the biological and agricultural industries [e.g., Chick et al., 2008, Hyytiäinen et al., 2011, Eskandarzadeh et al., 2014]. Yield uncertainty in vaccine production has a direct impact on social welfare, as it can lead to weak immunity to new virus strain or mutation [Cho, 2010]. Finally, yield uncertainty can also impact the contractual arrangement in supply chains, where low production yield implies a more expensive insurance payout, reduced capital investment, or difficulties to get credit and subsidies [Anderson and Monjardino, 2019]. These examples illustrate the importance of developing optimization approaches that hedge against uncertainties in a systematic, effective and efficient way.

To the best of our knowledge, Kazemi Zanjani et al. [2010] and Quezada et al. [2020] are the only works on stochastic programming for an LSP under yield uncertainty; and there is no existing study on robust, adaptive robust or distributionally robust optimization for lot-sizing under yield uncertainty in a production planning context. The lack of studies on the application of these methodologies to the LSPs with yield uncertainty has motivated this work. We propose different models and methodological strategies to compute optimal robust and less conservative production plans. We also provide a risk analysis to help decision-makers to tailor their decision plans to their strategic and operational goals.

The contributions of this thesis are organized as follows:

Chapter 1 presents a state of the art on lot-sizing problems under uncertainties. It provides the most common resolution methodologies applied to the problems, and it also introduces the studies on the lot-sizing problems with backorder and uncertain production yield.

Chapter 2 studies the robust optimization methodology applied to the single-item lot-sizing problem with backorders under yield uncertainty. We introduce the robust mixed-integer linear model as well as the definition of the optimal robust policy for the special case of the inventory management problem under yield uncertainty. We also provide the structure of the optimal robust solution, and we develop a dynamic program with polynomial complexity to compute a robust production plan. Computational results and managerial insights are provided.

Chapter 3 investigates the use of adaptive robust optimization to immunize the production plan against the yield uncertainty, and to react properly when updated information unfolds. We propose a myopic adaptive robust policy for the inventory management problem, and we show that the policy is optimal under mild assumptions. Then, we formulate an adaptive robust model that exploits a linear approximation to transform the quadratic constraints into a mixed-integer linear program. To solve this model exactly, we propose a column and constraint generation algorithm. We also provide a rolling horizon technique to measure the performance of the proposed models in a dynamic decision context. Extensive numerical experiments are provided.

Chapter 4 addresses the multi-period multi-item lot-sizing problem with backorder under yield uncertainty through the distributionally robust optimization. We provide a distributionally robust formulation that can be solved by commercial solvers. Then, we develop a mixed-integer robust model to compute sufficiently good distributionally robust production plans. We compare the distributionally robust plans with production plans proposed by other optimization methodologies to measure the performance of the distributional robust models to hedge against the production yield uncertainty and propose a production plan closer to real world applications.

The chapters produced four scientific articles, namely:

1. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. (2023). Robust optimization for lot-sizing problems under yield uncertainty. *Computers & Operations Research*, 149, 106025.
2. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. Adaptive robust optimization for lot-sizing under yield uncertainty. *European Journal of Operational Research* (1st revision), November 2022.

3. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. Distributionally robust optimization for the multi-period multi-item lot-sizing problems under yield uncertainty. Target journal: *Naval Research Logistics*, November 2022.
4. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. A state of the art on the LSPs under uncertainty: trends on optimization methodologies and approaches. Working draft, November 2022.

At the beginning of the corresponding chapters, we provide details on the parts that were from the papers. This work has also yielded two following proceedings papers:

1. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. Optimization for Lot-Sizing Problems Under Uncertainty: A Data-Driven Perspective. In *IFIP International Conference on Advances in Production Management Systems* (pp. 703-709). Springer, Cham, September 2021.
2. Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. The adaptive robust lot-sizing problem with backorders under demand uncertainty. In *2021 IEEE 17th International Conference on Automation Science and Engineering (CASE)* (pp. 997-1001). IEEE, August 2021.

This dissertation is organized as follows. Chapter 1 reviews the studies on non-deterministic lot-sizing problems, and it highlights the gap of knowledge on approaches to handle lot-sizing problems with uncertain production yield. Chapter 2 presents a static robust methodology to define a production plan. Chapter 3 gives adaptive robust approaches to compute an adaptive production plan. Chapter 4 provides distributionally robust programs to design a data-driven robust production plan. Finally, Chapter 5 concludes this work and provides some direction for future work.

NON-DETERMINISTIC LOT-SIZING PROBLEMS

This chapter is based on the following article:

Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. A state of the art on the LSPs under uncertainty: trends on optimization methodologies and approaches. Working draft, November 2022.

The research interest in non-deterministic LSPs has grown over the last decades. To the best of our knowledge, no paper presents the state-of-the-art on optimization methodologies to deal with LSPs under uncertainty, and we aim to fill this gap. In addition, we discuss a framework for data-driven approach that link the data available in the decision environment to the decision systems. This chapter provides the following contributions: 1) it summarizes the state-of-the-art of the LSP under uncertain environment via stochastic programming, robust optimization, and the distributionally robust optimization methodology; 2) it identifies the gap of knowledge of the considered methodologies to deal with different uncertain parameters of the LSPs; and, finally, 3) it introduces the data-driven interpretation of the optimization under uncertainties methodologies to deal with non-deterministic LSPs.

1.1 Introduction

Supply Chain Management is “the task of integrating organizational units along a supply chain and coordinating material, information and financial flows in order to fulfill (ultimate) customer demands with the aim of improving the competitiveness of a supply chain as a whole” [Stadtler, 2005]. As a result, a production planning activity is designed to make the best use of resources and satisfy the production requirements over a planning horizon. It encompasses three process scheduling, which are: *long-term*, that focuses on the main strategic decisions, *medium-term*, which defines the production planning (e. g., production quantity and scheduling) according to the needs and the strategy adopted, and *short-term* dealing with day-to-day planning and decisions [Karimi et al., 2003].

Focuses on the medium-term scheduling, the Manufacturing Resource Planning (MRP) supports decision makers to manage manufacturing processes and determines a replenishment schedule over the production horizon. This system determines the quantity and the availability of materials and components needed to meet a specific demand [Murthy and Ma, 1991]. While MRP assists exclusively the production processes, a system focused on supporting the material flow across a supply chain and related areas is required. Consequently, Advanced Planning Systems (APS) have been developed to fulfill this request. Based on the hierarchical planning, the APS incorporate models and solution algorithms to define a production plan that considers the supply chain as a whole [Stadtler, 2005]. We are interested in the master planning to achieve the most efficient way to meet the demands for both, MRP and APS, environments. Furthermore, we intend to adjust the production planning over the production horizon to respect the environmental restrictions, such as the available capacity, processing or lead time, demands, costs, etc.

The lot-sizing problem (LSP) emerges as an optimization problem that determines the setups and production quantity to meet the demands over the planning horizon. Introduced by Wagner and Whitin [1958], the LSP decisions can be incorporated into production planning activities optimizing system performance and results [Barany et al., 1984a]. Lot-sizing decisions face various sources of uncertainties (demand, lead time, production yield, among others) that affect the costs and service level associated with the lot sizes, and then influence the quality of the production plan. Traditionally, these uncertainties are controlled by some additional parameters of the planning systems, such as the safety stock, safety lead-time, and re-planning frequency [Dolgui and Prodhon, 2007]. However, these classical techniques are not sufficient to efficiently mitigate uncertainties.

To overcome these limitations, some optimization methodologies can be applied to deal with these non-deterministic problem more efficiently. Here, three common approaches to support the production planning are stochastic programming (SP) [Birge and Louveaux, 2011], robust optimization (RO) [Ben-Tal et al., 2009], and distributionally robust optimization (DRO) [Wiesemann et al., 2014].

The ease of access to information in the era of big data motivates us to take into account any data available to improve decision systems [Shang and You, 2019]. There is an increasing amount of data generated from the shop floor that is available and that can support decision-making processes to be more assertive. For lot-sizing decisions, the available data can improve the quality of the production plan if exploited appropriately. Advances in computing technologies and the massive availability of data led to the design of optimization methods that directly incorporate the uncertainties in the models, and achieve better solutions. In this context, the interest in data-driven optimization (DDO) has grown during the last decade [Shang and You, 2019].

Stadtler and Kilger [2002] provide a review of the LSPs in the scope of the SCM environment. A number of studies propose models and solution approaches for the deterministic LSPs. We refer readers to [e.g., Wagner and Whitin, 1958, Barany et al., 1984a, Pochet and Wolsey, 1988, Karimi et al., 2003, Brahimi et al., 2006, 2017] for further information on the LSPs. Some studies on the non-deterministic perspective of the SCM were proposed in [e.g., Murthy and Ma, 1991, Dolgui and Prodhon, 2007]. Readers are referred to Barany et al. [1984a], Pochet and Wolsey [2006] for further information about the general single-item LSPs. The general multi-item version is presented in [e.g., Barany et al., 1984b, Pochet and Wolsey, 1988, 2006, Absi and Kedad-Sidhoum, 2006]. A survey on the capacitated LSPs is presented in Karimi et al. [2003]. A recent survey of single-item dynamic lot-sizing problems is presented in Brahimi et al. [2006], and it was updated and extended by the authors in Brahimi et al. [2017].

The interest in non-deterministic LSPs is growing as reported on the review covering publications on these problems from 2000 to 2013 [Aloulou et al., 2014]. Brahimi et al. [2006, 2017] indicate that most studies about optimization under uncertainty for LSPs consider stochastic programs, robust models and dynamic programs. While there exist state of the art on the different optimization under uncertainty approaches (we refer interested readers to a review of the SP is presented in Birge [1997], an overview of the RO is presented in Ben-Tal et al. [2009], and an introduction of the DRO is proposed in Wiesemann et al. [2014]), and state of the art on lot-sizing under uncertainty (see Brahimi

et al. [2006] and Brahimi et al. [2017]), to the best of our knowledge, there is no review on the state of the art on the application of optimization under uncertainty approach to the lot-sizing problem.

This work seeks to review a data-driven perspective of non-deterministic LSPs. Supported by the literature review proposed by Aloulou et al. [2014] and by our updated survey of the published works from the past 60 years on non-deterministic LSPs, we present a data-driven application of some of the most applied optimization under uncertainty methods to deal with LSPs. We introduce a data-based perspective of the SP and the RO methodologies to handle different uncertain parameters on LSPs. In addition, we present the DRO methodology applied to LSPs, a new resolution trend for solving non-deterministic problems that integrate the probabilistic aspect from SP and the robustness from RO. Note that we are not attempting to give an exhaustive bibliography based on a systematic review. Instead, we select the papers for their quality and their relevance, considering the following key dimensions: journal quality, number of citations, innovation, practical applications, reference.

The remainder of this chapter is organized as follows. Section 1.2 introduces the LSPs, its major characteristics, and its interpretation under uncertain environments. Section 1.3 indicates the more common optimization approaches to deal with the non-deterministic LSP. The three next sections survey three optimization methodologies to handle the LSPs under uncertainty with a data-based perspective. Section 1.4 review the SP methodology. Section 1.5 outlines the RO and its variations. Section 1.6 presents the joint perspective of the stochastic programming and the robust optimization methodologies, also called DRO. Section 1.7 provides a comparison between the three optimization methodologies to handle LSPs. Finally, Section 1.9 highlights some remarks and discusses future research directions, and it also summarizes the main findings of this work.

1.2 The lot-sizing problems

The LSP aims to meet customers' needs while minimizing the overall costs, and maintaining the satisfactory level of service [Dolgui and Proth, 2010]. The uncapacitated single-item LSP can be described in production management terminology as follows. Given the need for one or more goods, one or several resources should be coordinated on the production horizon to ensure the manufacture a sufficient quantity of goods that meets demand and satisfies production strategies (e.g., quality of the produced items, lead-time, processing time, resource availability, etc). Several authors have proposed studies on the deterministic LSPs in the last decades. Strong formulations for deterministic uncapacitated and capacitated versions of the LSPs were proposed by Barany et al. [1984a] and Barany et al. [1984b], respectively. Exact methods to the deterministic LSPs were proposed by Belvaux and Wolsey [2000, 2001] and Leung and Wu [2004]. Pochet and Wolsey [2006] present a study on mixed-integer programming (MIP) for the LSPs, and Brahimi et al. [2017] present an exhaustive state-of-the-art on the single item LSP with a focus on the modeling and resolution methods. As decision-makers deal with the LSPs in various production contexts, multiple extensions of the general LSP exist, such as capacitated resource availability, single or multi-items, single or multi-levels, single or multi-echelons or machinery, etc. In this section, we introduce the general LSP and its main features. Next, we focus our study on LSPs under uncertainty and on a data-driven perspective of some optimization methods to deal with non-deterministic LSPs.

1.2.1 Problem statement

The uncapacitated single-item LSP without backorder or lost sale determines the production setups and lot sizes in the finite planning horizon $T = \{1, \dots, |T|\}$ that minimize the overall costs and meet demands. For each period $t \in T$, we have setup cost s_t , unit production cost v_t , inventory holding cost h_t and demands d_t . We also have the following decision variables: lot size X_t , inventory level I_t at the end of the period, and setup Y_t . Although the problem is assumed to have uncapacitated availability of resources, the uncapacitated single-item LSP has a natural capacity given by the upper bound of the production quantity.

The mathematical formulation of the general LSP is given as follows:

$$\min \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t \quad (1.1)$$

s.t.

$$I_t = \sum_{j=1}^t (X_j - d_j) \quad \forall t \in T \quad (1.2)$$

$$X_t \leq MY_t \quad \forall t \in T \quad (1.3)$$

$$X_t, I_t \geq 0 \quad \forall t \in T$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

Without loss of generality, we assume that there is no stock at the beginning of the production horizon ($I_0 = 0$). The objective function (1.1) minimizes the total cost composed of setup, unit production and inventory costs. The inventory balance constraints (1.2) compute the inventory level in period t from the demand, the produced goods at period t , and the stock kept in the previous period (I_{t-1}). Here, the production in the current period t and the inventory from the previous period $t - 1$ are used to meet the demand d_t . Any remaining amount of goods after satisfying the demand, if any, is kept in stock (I_t). The Constraints (1.3) are setup-forcing constraints that relate the production quantities (X_t) to the setup decisions (Y_t). These constraints set the setup variable Y_t to 1 if any production occurs in period t , and the setup remains inactive otherwise ($Y_t = 0$). A natural upper bound on the production quantity is the total amount of demands to be met over the entire production horizon, and is given by $M = \sum_{t \in T} d_t$. Note that in the capacitated version of the problem M is replaced by a given capacity C .

subsectionCharacteristics of lot-sizing models

Different features can be taken into account by an LSP leading to a wide range of lot-sizing models. The main characteristics of an LSP are the planning horizon, the number of products, the number of levels, capacity constraints, the structure of the demand, the structure of the setups, the structure of the inventory shortages, and the stochastic nature [Karimi et al., 2003].

Planning horizon

The planning horizon $|T|$ may be *finite* or *infinite*, and it may be observed *continuously* or at *discrete time*. In term of period terminology, the problem may be either *big bucket* (allows to produce multiple items at the time) or *small bucket* (time period involves the production of a single item).

Demand

The demand is said *static* when it is a constant value over time ($d_t = d$), and *dynamic* otherwise.

Number of items

The LSP may consider a *single-item* or *multi-item*. In the multi-item case, a set of N items is produced, and the lot size decision becomes X_{it} goods produced for item $i \in N$ in period $t \in T$.

Number of echelons

Production process may be *single-echelon* or *multi-echelon*. On the one hand, a single-echelon system produces an end-item directly from raw materials or purchased materials with no intermediate sub-assemblies. On the other hand, in a multi-echelon system, the production process has multiple steps, and there is a parent-component relationship between the items. Having E the set of echelons, the lot size decision for a multi-level LSP becomes X_{int} goods produced for item $i \in N$ at echelon $n \in E$ and period $t \in T$. Note that a single-echelon LSP can have a single or multiple end-items, whereas a multi-echelon problem is automatically a multi-item problem.

Capacity constraints

When there is no restriction or constraint on resources then the problem is said to be *uncapacitated*, otherwise the problem is named *capacitated*. The capacity can be represented in Constraints (1.3) when M is replaced by the capacity C . Constraint capacity can be defined in terms of material capacity (e.g.: number of goods that can be processed at a time), or other measures for available resources, such as time (e.g.: available time to produce goods).

Setup structure

Production changeover between different items often incurs setup time and setup cost. In this context, there are two setup structure: a *simple setup*, where setup time and cost in a period are independent of the sequence and the decisions made in previous periods, and a *complex setup* when setup cost and time are dependent on the sequence or

previous periods. The complex setup has three configurations: i) *setup carry-over* when it is possible to continue the production run between two subsequent periods without the need for additional setup, ii) *family or major setup* that is caused by similarities in manufacturing process and design of a group of items, and finally iii) *sequence-dependent setup*, setup cost and time would depend on the production sequence.

Inventory shortage

Inventory shortage is represented with the delay or the loss of a demand. When it is possible to meet the demand for a period in the current period or the next period, then we have a backordering, if the demand is accumulated to be met in future periods, we have a backorder B_t . Considering a context where backorders are accepted, Constraints (1.2) become $I_t - B_t = I_{t-1} - B_{t-1} + X_t - d_t$. If the demand cannot be satisfied at all, we have a *lost sale* LS_t , and Constraints (1.2) become $I_t = I_{t-1} - LS_t + X_t - d_t$. A mixed situation occurs when backorder and lost sale are considered simultaneously [e.g., Absi et al., 2011, Grubbström and Ding, 1993].

Nature of the problem

The problem is said to be *deterministic* when it is not subject to any uncertainty, and *non-deterministic* when at least one parameter is uncertain. As an example, when the demand is known in advance, we have a *deterministic* demand, otherwise we have an *uncertain* demand whose value is unknown and can be estimated or predicted but subject to estimation errors.

Complexity

While the complexity of the uncapacitated LSPs is polynomial (e.g., $O(T^2)$ [Wagner and Whitin, 1958] and $O(T \log T)$ [Federgruen and Tzur, 1991]), the single item capacitated LSPs are polynomial (e.g., $O(T^4)$, $O(T^3)$, $O(T \log T)$, $O(T)$ [Bitran and Yanasse, 1982] and $O(T \log T)$ [Wagelmans et al., 1992]), and the general case is NP-Hard [Bitran and Yanasse, 1982].

LSPs with backordering

Several studies have been done on LSPs. For the sake of clarity, we present the method for optimization under uncertainty on the simplest LSPs. That is, the single-item non-

deterministic lot-sizing problems over a discrete and finite planning horizon with backordering and a simple setup environment. However, the models can easily be extended to other cases such as the capacitated and/or multi-item versions.

The mixed-integer formulation of the LSP can be difficult to be solved. Although for small and medium-sized instances either commercial solvers or some optimization approaches are able to compute an optimal solution, hard medium-sized and the large-sized instances may be addressed by some approximations that can be solved with reasonable computational effort. Pochet and Wolsey [1988] introduce an approximation of the LSP to the network flow problem as the most efficient reformulation technique applied to the LSPs, leading to a stronger model efficiently computed in polynomial time via shortest-path formulation. A review on deterministic LSPs modeled for industrial applications is presented by Jans and Degraeve [2008]. Other formulations of the problem and further information on the LSPs is presented in Pochet and Wolsey [2006].

The modeling of LSPs depends on the context and decision environment. Therefore, we limit ourselves to some illustrative examples that are more common in real world applications.

$$\min \quad \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t + b_t B_t \quad (1.4)$$

s.t. :

$$I_t - B_t = I_{t-1} - B_{t-1} + X_t - d_t \quad t \in T \quad (1.5)$$

$$X_t \leq M Y_t \quad t \in T \quad (1.6)$$

$$X_t, I_t, B_t \geq 0 \quad t \in T$$

$$Y_t \in \{0, 1\} \quad t \in T$$

This paper focuses on the non-deterministic lot sizing problems. The more natural strategy to better deal with uncertainties is to allow the problem to backorder any demand that could not be satisfied. For this reason, in the next sections we consider the version of the problem given by (1.4).

LSPs with lead time

The lead time refers to the time from the order to the delivery of the goods. Assuming that Q_t goods are ordered and lot size X_t of goods delivered in period t , and having the lead time l_t for goods ordered in period t , problem (1.4) should be modified to minimize

costs while respecting the lead process constraint.

LSPs with processing time

The processing time composes the lead time as the time required to manufacture a good. Assuming the processing time of p_t to produce the lot size X_t of goods delivered in period t , and that the capacity is given in terms of available time (ct), problem (1.4) should be modified to consider the processing time as a resource constraint in Constraints (1.6).

LSPs with setup time

The setup time also composes the lead time. It indicated the time required to setup the machinery and release the production. Let st_t be the setup time and p_t be the processing time of the lot size X_t in period t , such that available time is constrained and given by CT . Problem (1.4) should be modified to consider the setup time to release a production in Constraints (1.6).

LSPs with production yield

The production yield integrates the quality factors into the lot-sizing models by measuring the expected quantity of non-defective items resulting from a lot size. Considering the production yield rate ρ as the quantity of quality goods expected from a lot size X_t , problem (1.4) may be modified to take into account the quality outputs on the optimization in Constraints (1.5).

1.2.2 Lot-sizing problems under uncertainty

The optimal solution for problem (1.4) may be computed in polynomial time based on the properties proposed in Wagner and Whitin [1958]. However, multiple parameters of the lot-sizing problems are not known with certainty when the decisions are made, and these uncertainties affect the costs and service level of a company. Uncertainties emerge from different sources such as: partial information, inaccurate forecast, misinformation and lack of expert insights. They can lead to the most common uncertain parameters, which are: demand, production yield, lead time, processing time, setup time, capacity and cost. We successively present below the main sources of uncertainties, and the classical

approach to hedge against uncertainties, before to introduce non-deterministic lot-sizing models.

Demand uncertainty

The case of demand uncertainty is the more natural uncertainty for LSPs, as the demand can be highly volatile and influenced by multiple factors such as age, lifecycle, economic context, reference groups, culture, festive season. As a result, most applications of optimization under uncertainties to the LSP consider the case of demand uncertainty. Some models addressing the simplest LSP with uncertain demand have achieved tractable models and efficient algorithms to solve the small and medium-sized instances [Brahimi et al., 2006]. For the large-sized instances, there are methodologies suited to the chosen optimization approach to compute a solution with reasonable computational effort [Brahimi et al., 2017, Brandimarte, 2006], but further work still required to develop new strategies to solve the problem in reasonable computational time, even if an exponential behaviour is expected.

Lead-time uncertainty

Machine breakdowns, quality or transportation issues can lead to an uncertain lead-time. The solution to a multi-echelon LSP may become unfeasible if the lead-time deviates from its planned value, because the production of an item requires the delivery of its components. Some modification is then needed to represent the lead-time in a constraint-wise way. The results obtained for the static formulations addressing the uncertain lead-time are promising [Thorsen and Yao, 2017, Hnaien and Afsar, 2017], but no studies were found handling the adaptive problem.

Production yield uncertainty

When the amount of quality goods from the lot size differs from the expected output, we have an uncertain production yield rate. Although the production yield has not been much investigated for non-deterministic LSPs, this parameter directly indicates the quality of the lot size. The production yield rate can be easily handled within a static strategy. However, within an adaptive strategy, the production yield uncertainty leads to the same quadratic issue observed for the uncertain lead-time because the production yield is a coefficient of the lot size. As a result, the static-dynamic and dynamic strategies are

intractable in general [Ben-Tal et al., 2004]. Further work is required to handle the yield uncertainty for LSPs models via tractable models, and to propose a production plan with reasonable calculation time.

Capacity uncertainty

We face an uncertain capacity when the availability of resources is unknown or inaccurate. Such an uncertainty makes the problem infeasible for the case of available resource insufficient to meet the demand. Further investigation is required not only to modeling LSPs under capacity uncertainty, but also to handle the static-dynamic and dynamic decision strategies in a tractable and scalable way.

Processing time uncertainty

We have an uncertain processing time when machine speed, manpower, resource consumption or availability unexpectedly vary. Although often ignored in the models, the processing time approximates the model to realistic cases in which the production is not immediate or follows different production stages. As the processing time directly impacts the production of a lot size, this uncertainty should be handled similarly to the approaches used to deal with the reformulated lead-time, the production yield and the setup time uncertainties. No additional limitations to the challenges already cited for similar uncertainties are found by the processing time uncertainty.

Setup time uncertainty

Setup time uncertainty occurs when the setup of the machinery varies unexpectedly due to the non-standardization of the process, or when some other issues delay the start of the production process. The setup duration also accounts for the first items in the lot that are scrapped until the machines are correctly tuned, and this duration may vary significantly. In addition, lot-sizing models often approximate the reality, and the setup time is often modeled as a fixed value at the production planning level, even when they depend on the production sequence at the scheduling level. In this setting, the actual setup duration is unknown at the lot-sizing level, and it is observed after scheduling. This uncertainty also directly impacts the production of the lot sizes, and it may be addressed using approaches similar to those used with lead-time and production yield uncertainties.

Cost uncertainty

Cost uncertainty occurs when the costs incurred from the production cannot be predicted accurately. The cost uncertainty is a column wise uncertainty as it is a coefficient of the decision variables in the objective function. In addition, classical optimization methods are usually tractable for models whose uncertainty is expressed on the left size of the constraints, and do not often are able to handle uncertainty on the objective function [Ben-Tal et al., 2009]. Therefore, a reformulation should be done to represent the objective function under uncertainty by using additional variables. Moreover, since the costs directly impact the definition of the lot sizes, some approach similar to those proposed to handle uncertainties that directly modify the production plan should be considered.

Table 1.1 reports the references considered in this work to handle LSPs under at least one uncertainty cited above via the SP, RO, ARO and DRO methodologies. The most studied uncertainty is the demand, followed by the lead-time, the cost, the production yield, the processing time and the setup time uncertainty. No study was found dealing with capacity uncertainty via the considered optimizations.

	demand	lead time	yields	cost	processing time	setup time
SP	Alem et al. [2018], Augusto et al. [2016], Bansal et al. [2018], Beraldi and Ruszczyński [2002], Bollapragada and Rao [2006], Bookbinder and Tan [1988], Brandimarte [2006], De Saint Germain et al. [2017], Guan and Miller [2008], Guan [2011], Jiang and Guan [2016, 2018], Kang and Lee [2013], Martel et al. [1995], Minoux [2018], Quezada et al. [2019], Raa and Aghezzaf [2005], Rahmani et al. [2013], Sahebjamnia and Torabi [2011], Tarim and Kingsman [2004], Tempelmeier [2007, 2013], Wong et al. [2009]	Huang and KüçüKyavuz [2008], Jiang and Guan [2011]	Zanjani et al. [2007]	Bansal et al. [2018], Şen et al. [2014], Minoux [2018], Zhou and Guan [2010, 2013], Quezada et al. [2019]	Augusto et al. [2016]	Taş et al. [2019]
RO	Agra et al. [2018], Aharon et al. [2009], Alem et al. [2018], Ben-Tal et al. [2004, 2005], Bertsimas and Thiele [2006a,b], Bertsimas et al. [2010], Bertsimas and Bidkhori [2015], Bertsimas and de Ruiter [2016], Bertsimas and Dunning [2016], Bertsimas et al. [2018b], Coniglio et al. [2018], Guo and Liu [2018], Kaganova [2013], Klabjan et al. [2013], Leung and Wu [2004], Melamed et al. [2016], Santos et al., Shapiro [2011], Solyah et al. [2015], Thorsen and Yao [2017], van Pelt and Fransoo [2015], Wagner [2017], Zhu et al. [2012]	Hnaien and Afsar [2017], Thorsen and Yao [2017]	Vayanos et al. [2012]			
DRO	Analui and Pflug [2014], Bertsimas et al. [2019], Coniglio et al. [2018], Postek et al. [2018], Yang [2017], Zhang et al. [2016]					

Table 1.1 – References of optimization methods dealing the non-deterministic LSPs

1.3 Optimization under uncertainties for the LSPs: methods and concepts

To propose satisfactory decisions in an uncertain environment, the decision-makers have to considerate and even anticipate these uncertainties. The optimization under uncertainties seeks for an optimal solution when some parameters are unknown. For that, different decision frameworks can support the decision according to the available information and the computation approach or applied methodology.

1.3.1 Decision framework

Each optimization approach can model a specific control strategy to deal with the uncertain parameter. Bookbinder and Tan [1988] differentiate these strategies in three groups, that are: static, dynamic and static-dynamic strategy. The decisions are then partitioned into two groups: static (here-and-now) and adjustable (wait-and-see) decisions. The adaptive decisions are modified by the revealed values of the uncertain parameter.

Static strategy Within a static strategy (single-stage or two-stage frameworks), the decision maker determines both setup of production and lot sizes a priori and no modification is allowed (the decisions are frozen).

Dynamic strategy In a dynamic strategy (or multistage framework), the setup of production, the lot size and the inventory control variables are updated at the beginning of every period t of the production horizon.

Static-dynamic strategy The static-dynamic strategy (multistage framework) is a hybrid strategy where the setup decisions are fixed a priori, while recourse actions can update the lot sizes and the inventory levels when uncertainties are revealed.

Figure 1.1 indicates, for each approach, the model that deals with each of the three control strategies. Even if the dynamic strategy is the most suitable to describe real applications taking into account the dynamic system and decision environment, the resulting models are computationally hard to solve [Dyer and Stougie, 2006]. Few studies address the dynamic strategy of LSP.

These decision frameworks can handle two groups of variables that are: *here-and-now* and *wait-and-see* Ben-Tal et al. [2004]. Here-and-now variables are a set of variables whose

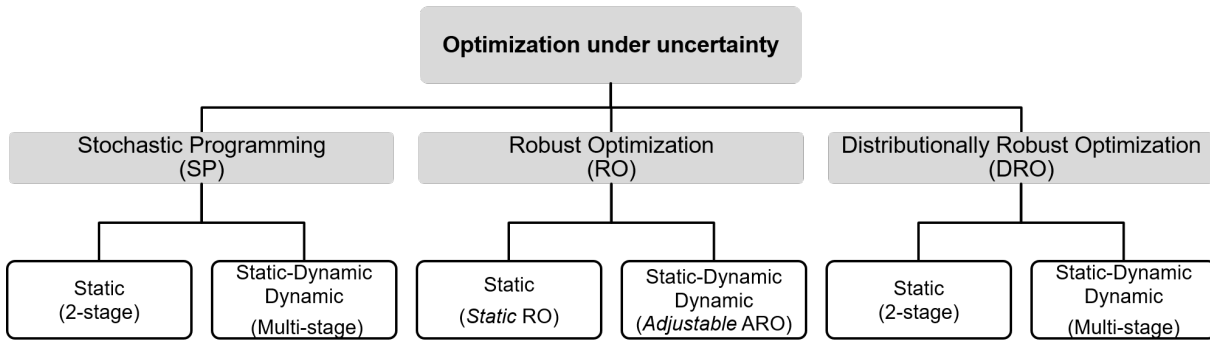


Figure 1.1 – Overview of the optimization under uncertainty

the values are fixed before the uncertainty event. wait-and-see variables are the set of the remaining variables that can adjust themselves according to the values taken by certain parameters after some uncertain event is revealed. In this manner, even if the here-and-now variables are fixed and no recourse action is permitted, an uncertain event could affect their implications for the problem. Therefore, the *wait-and-see* variables are taken as a recourse decision, such a way the decision is adjusted to compensate any effect that might be revealed as a result of the uncertain event and could not be taken into consideration for the *here-and-now* decisions.

From the decision framework, there are three main models defined by the recourse action and the revelation of uncertainty [Gabrel et al., 2014]. A *single-stage model* fix all decisions, as here-and-now variables, before the realization of any uncertainty and no recourse action is allowed in all the planning horizon. In a *two-stage model*, the here-an-now variables are *first-stage* decisions fixed before the realization of any uncertainty and the *second-stage* decisions are the wait-and-see variables fixed after the realization of the uncertainty. Finally, the *multistage model*, considers the first-stage decision as the here-and-now variables fixed at the beginning of the time horizon of the decision problem and the wait-and-see variables are the n^{th} -stage variables associated to the n^{th} period following the realization of some uncertainty.

We focus on the static–static and the static-dynamic strategies such that the modeling approaches are respectively a two-stage and a multistage problem. Among these approaches and strategies, we are interested in the LSP consideration within the ro-

bust optimization and the stochastic programming methodologies. Furthermore, we intend to highlight the advantages that robust optimization perspective of the stochastic programming brings to the robustness of the solution and the scalability, computation time, tractability and flexibility of the problem in handling unforeseen events. Hence, the next three sections introduce the considered optimization methods and present the LSP within the mentioned frameworks.

1.3.2 Most common methods to deal with the non-deterministic LSPs

The production planning considers and even anticipates uncertainties to propose satisfactory decisions. Under an uncertain environment, some parameters should be considered to preserve the feasibility of a solution. Dolgui and Prodhon [2007] briefly describes some classical approaches to control or mitigate the effects of these uncertainties for the MRP environment. They are:

safety stocks, aiming to ensure a given service level and ward off the random factors, by reducing the risk of shortages and increasing the holding costs

safety lead-time, whose principle is similar to that applied to the safety stocks, but acting from the perspective of time

lot-sizing rules, that combine the net requirements for more than one time period or bucket aiming to reduce the overall costs or optimize the inventory control

freezing the production planning, that aims to compromise between the need of information updating and the nervousness produced by too many changes of the available data by freezing some time periods over the planning horizon and thus any modification is forbidden during the frozen periods, even if a rescheduling occurs.

These alternatives, however, require separate computation to deal with the uncertainty thought some strong, or even unrealistic, assumptions. To increase the feasibility of a production plan over exogenous uncertainties (such as demand and supply delivery lead time) and endogenous uncertainties (such as production yield) safety stocks are commonly used [Dolgui and Prodhon, 2007]. Although the safety stock approach helps mitigate the impact of uncertain yield, safety stock calculations rely on separate calculations and strong assumptions. This results in sub-optimal solutions as shown in Thevenin et al. [2020]. The authors indicate that the safety stock levels are computed either manually to meet a given service level in the master production schedule, or based on strict assumptions (e.g., static

demand, base stock policy, lot-for-lot policy, critical stock policy). Since the lot sizes and the safety stocks are not defined simultaneously, this often results in sub-optimal decisions.

Safety stock can be integrated into the LSPs as a lower bound on the inventory level [De Bodt et al., 1982]. Zhao et al. [2001] report different approaches to define the safety stock level at the master production schedule with the following parameters: safety coverage, service level and standard deviation of the forecast errors. The newsboy model, as presented in Khouja [1999], helps determine safety stocks to balance inventory and backorder cost, rather than to satisfy a given service level. To improve the quality of non-deterministic decisions, and to hedge against the uncertainties that affect the LSPs, models are usually formulated based on approximations or optimization methodologies that make the original problems easier to solve. Therefore, reformulating the problems as mixed-integer linear programs (MILP) or other mathematical formulations has emerged as a promising approach.

A common approach to deal with the case where the setup costs can be omitted is relied on a newsboy model [Khouja, 1999]. Focused on the static strategy, this approach minimizes the expected costs of overestimating or underestimating the uncertain parameters via a single period problem. For the multi-period problems or the adaptive decision framework, some heuristics should be integrated to the approximation. Some adversarial approach can also be applied to improve the quality of the proposed solution. Although the approximation of the LSP models to this model can lead to easier calculation, this approach is based on strong assumptions, such as the single-period problem, that are not consistent or frequent in real cases. In addition, to address dynamic decision context or to improve the quality of the proposed plan, some heuristics may be integrated with these classical techniques.

Newsboy models The decomposition of the LSPs by the periods of the production horizon leads to an inventory management based on the Newsboy model [Khouja, 1999]. It considers each period of time independently, and it minimizes the expected value of the overall costs π to meet the demands with respect to an optimal lot size. Here, we define a production plan via the LSPs with uncertain demand δ by solving for each period $t \in T$ the Newsboy-based problem given in (1.7). Applying the Leibniz's rule to this equation we can show that $E(\pi)$ is concave through the first and second derivatives. In addition, we can demonstrate that the solution proposed by the Newsboy-based model is optimal for a lot size X_t^* defined by the formula given in (1.8).

$$\min \pi = v_t X_t + b_t \int_0^{X_t} (\delta_t - X_t) f(\delta) d\delta + h_t \int_{X_t}^{\infty} (X_t - \delta_t) f(\delta) d\delta \quad (1.7)$$

$$F(X_t^*) = \frac{b_t - v_t}{h_t + b_t} \quad (1.8)$$

Although it is not the objective of this work to list the studies on the newsboy-based LSP models, we can cite some work addressing uncertainties on demand [e.g., Henig and Gerchak, 1990, Gallego and Moon, 1993, Moon and Choi, 1995, Wang and Gerchak, 1996, Grosfeld-Nir and Gerchak, 1996], lead-time [e.g., Grosfeld-Nir and Gerchak, 2004, Jian et al., 2015, Zeng et al., 2015], production yield [e.g., Agnihotri et al., 2000, Yano, 1989, Dada et al., 2007, Abdel-Malek et al., 2008], capacity [Wu et al., 2013] and, finally, costs [e.g., Pantumsinchai and Knowles, 1991, Khouja, 1996, Lin and Kroll, 1997].

Optimization methodologies

The optimization under uncertainty methods overcome the limitations faced by the classical techniques and approximations through approaches that are adaptable to various production environments. Based on mathematical models, the optimization under uncertainties handles different parameters, uncertain or not, simultaneously. Aloulou et al. [2014] present a literature review where a five-field notation is proposed to classify the LSP models. This notation, (T, n, m, unc, mod) , indicates respectively the number of periods T , number of products n , number of levels m , uncertain parameters $unc = \{dmd \text{ (= demand), yid \text{ (= production yield), lead \text{ (= lead time), proc \text{ (= processing time), setup \text{ (= setup time), cost \text{ (= cost), capa \text{ (= capacity)}\}$ and modeling approaches for the LSPs $mod = \{stochasticprogramming, fuzzy, gametheory, on-linedecision, queuingtheory, intervalarithmetic, simulation\}$. We intend to extend this classification to consider the capacity constraint $cap_cst = \{uncap, cap\}$, the inventory shortage approach $inv_ctrl = \{backorder, lostSales\}$ (if any), the decision strategy $frame = \{st, st - dyn, dyn\}$ and the optimization methodology $opt = \{SP, RO, DRO\}$ applied. For that, we focus on the publications from the 1990s to 2020s addressing the non-deterministic LSPs via the stochastic programming, robust optimization or the distributionally robust optimization methodologies.

Stochastic programming The stochastic programming (SP) represents the uncertain parameters with their probability distributions, and it seeks to make the decisions minimizing the expected costs within any of the decision strategies [Birge and Louveaux,

2011]. Although this methodology is very efficient for small size instances and a limited number of scenarios, its main downside is its poor scalability for the large instances and large number of scenarios, notably within the dynamic contexts [Brandimarte, 2006].

Robust optimization The robust optimization (RO) is a common approach to mitigate uncertainties within the static framework [Ben-Tal et al., 2009]. The main idea is to describe the uncertainty in a set and then optimize against the worst-case realization within this set [Bertsimas et al., 2011]. However, this approach deals with static problems and leads to a extremely conservative solutions [Ben-Tal et al., 2009]. The adjustable robust optimization (ARO) is a robust methodology to static-dynamic and dynamic strategies [Ben-Tal et al., 2004]. The formulation considering the dependence of the recourse decisions on the uncertain parameters can lead to difficult and even intractable models. To overcome this difficulty, the methodology often relies on the definition of affine function to represent the adjustable variables [Aharon et al., 2009]. Under these affine policies, the recourse decisions become linearly dependent on the realization of the uncertain parameters up to the current period of time [Ouorou, 2011]. As a result, the reformulated problem can be modeled as convex optimization problems such as linear, quadratic, conic or semi-definite optimization problem. Usually classical approaches to solve the robust model are applied [Bertsimas and Bidkhori, 2015]. When these techniques are not efficient, some adversarial approaches should be applied to compute a solution.

Distributionally robust optimization Distributionally robust optimization (DRO) describes uncertainties by an ambiguity set, which is a set of probability distributions with specific properties. The DRO methodology seeks to optimize the worst-case expected cost with respect to the probability distributions of the uncertain parameter described on an ambiguity set [Delage and Ye, 2010]. Therefore, DRO aims to protect against the worst-case probability distribution in this set. As a result, DRO can avoid over-conservative solutions by incorporating partial distributional information in a robust model.

The three optimization methodologies introduced above are promising to hedge against uncertainties in LSP environment. We demonstrate how to apply each one of them for the scenario-based modeling of non-deterministic LSPs within a data-based perspective. Note that a scenario-based modeling indicates the uncertain parameters by scenarios that represent the possible values that these parameters can assume. Here, the scenarios of

the SP model are constructed by the probability distributions, while the scenarios for the RO models are given by values included in the uncertainty set. For the DRO models, the scenarios are represented by the ambiguity set, that is itself scenarios of possible scenarios. Therefore, we are interested in the scenario-based model of the multi-period single-item LSPs with backorder under uncertainties via the SP, RO and DRO methodologies within all decision strategies. Assuming the classification presented in Section 1.3.2, we focus on the LSPs classified by $LSPs = (T, 1, unc, sce, backorder, frame, opt)$.

1.4 Stochastic Programming for LSPs

SP is one of the most used optimization methods to deal with LSPs under uncertainty [Brahimi et al., 2006]. SP represents the uncertain parameters with their probability distributions, and it seeks to make the decisions minimizing the expected costs. In general, the expected value and the chance-constrained approaches are used to formulate SP models Tempelmeier [2013]. The most applied techniques to solve the SP models for LSPs are: i) *approximations* to other known problems whose solution is easily computed; ii) *direct methods* such as extreme or interior-points; iii) *dualization*; iv) *decomposition approaches* as column splitting, column generation or progressive hedging algorithm; v) *Lagrangian approaches*; and vi) *sampling approaches* Birge [1997].

Representing the uncertain parameters with their probability distributions via a scenario-based formulation and expected value approach, the SP models handle uncertainty through mathematical programming techniques whose main idea is to minimize the expected cost from the decision problem according to its random variables Birge and Louveaux [2011].

First, we give a brief presentation of the chance-constrained and the expected cost approaches to handle LSPs via SP. Then, we concentrate on the expected cost modeling approach for the single-item LSP under demand uncertainty, within the static-static and static-dynamic decision strategies.

1.4.1 Chance-constrained approach

A chance-constrained LSP model requires to respect the validity of some constraints with regard to a given probability [Charnes and Cooper, 1959]. Readers are referred to Bookbinder and Tan [1988], Tarim and Kingsman [2004], Tempelmeier [2007] for further references.

Assuming a probability $\mathcal{P}\nabla$ and confidence level ϵ that the inventory balance constraints are respected for the LSP under demand uncertainty, the chance-constrained model can be formulated as follows:

$$\min \quad E \left[\sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t + b_t B_t \right] \quad (1.9)$$

s.t. :

$$\mathcal{P}\nabla \{ I_t - B_t = I_{t-1} - B_{t-1} + X_t - d_t \} \geq \epsilon \quad t \in T \quad (1.10)$$

$$\begin{aligned}
 X_t &\leq MY_t & t \in T & \quad (1.11) \\
 X_t, I_t, B_t &\geq 0 & t \in T & \\
 Y_t &\in \{0, 1\} & t \in T &
 \end{aligned}$$

1.4.2 Expected cost approach

Considering the LSP under demand uncertainty via a scenario-based formulation, we can represent the uncertain demand d_t^ω in a scenario $\omega \in \Omega$, such that Ω includes all possible scenarios of realization of the demand according to the given distribution. Each scenario $\omega \in \Omega$ has a probability $\Pr_\omega \mid \Pr_\omega > 0; \sum_{\omega \in \Omega} \Pr_\omega = 1$ of realization of a demand d_t^ω in the period t . Within static strategy, the setup and the lot sizes are fixed, and the inventory and backorder levels can change according to the scenario ω .

The two-stage SP for an LSP with uncertain demand presented by Raa and Aghezzaf [2005] is given as follows:

$$\min \sum_{\omega \in \Omega} \sum_{t \in T} \Pr_\omega (s_t Y_t + v_t X_t + h_t I_t^\omega + b_t B_t^\omega) \quad (1.12)$$

s.t.

$$\begin{aligned}
 I_t^\omega - B_t^\omega &= I_{t-1}^\omega - B_{t-1}^\omega + X_t - d_t^\omega & t \in T; \omega \in \Omega & \quad (1.13) \\
 X_t &\leq M_t \cdot Y_t & t \in T & \\
 X_t &\geq 0 & t \in T & \\
 I_t^\omega, B_t^\omega &\geq 0 & t \in T; \omega \in \Omega & \\
 Y_t &\in 0, 1 & t \in T &
 \end{aligned}$$

An immediate difficulty that arises when solving such a program is the computation of the expected costs. While for continuous distributions, the exact computation involves multiple integrals, for discrete distributions it involves a prohibitive number of individual problems related to the realization of each possible realization of the uncertainty. As a consequence, solving stochastic problems is often hard. Some techniques can be applied to reduce the number of scenarios and to make the resolution computationally tractable. Even if a sufficiently large quantity of scenarios is needed to estimate a good solution, approaches as scenario sampling can adequately address these issues and propose good stochastic lot size decisions.

The most known sampling method for a scenario-based modeling is the Monte Carlo sampling [Shapiro, 2003]. Other methods such as Latin Hypercube sampling have also

been used to solve large instances. The computation of a stochastic solution often relies on methods as the Sample Average Approximation (SAA). Readers are referred to Shapiro [2003], Englund and Rackwitz [1993], Schuëller and Pradlwarter [2007] for further information on the sampling techniques.

The use of the SP method for the other uncertain parameters presented in Section 1.2.2 is similar to the case of the demand. We consider the different scenarios of the realization of the uncertain parameter, and we propose a production plan for each one of these scenarios. Then we compute an expected cost taking into account every scenario with regard to its probability of realization.

1.4.3 Multistage SP for LSP with backordering

To overcome the limitation of the static strategy of adapting the lot size according to the realization of the uncertainty, an adaptive strategy is needed. Here, the definition of the scenario structure differs from the two-stage to the multistage model. In the one hand, the two-stage model assumes the scenario as a vector expressing possible realization of the uncertain parameter according to its index in the original problem. On the other hand, the multistage model considers the scenario as a tree structure, decomposing the scenarios for each stage while preserving the knowledge about the revealed uncertainty in previous stages. We will introduce the scenario tree generation in the scope of the multistage LSP presented below.

Within the multistage static-dynamic strategy, the first-stage decisions are the setup decisions (Y_t), whereas the lot size (X_t^ω), inventory level (I_t^ω) and backorder level (B_t^ω) compose the t^{th} -stage decisions updated according to the realization of the uncertain parameter for scenario ω in period t . As the t^{th} -stage decisions may depend only on the uncertainty revealed up to the period t , *non-anticipativity constraints* must be introduced to ensure that the lot sizes X_t^ω have the same values over different scenarios that do not differ up to the period $t - 1$.

A scenario tree is the most common representation of the scenarios [Dupačová et al., 2000]. It expresses the scenarios as an oriented graph starting with a root (the only node before the planning horizon) and branches into nodes at level $t = 1$ to $t = |T|$. Each node represents a realization of the uncertainty for the scenario ω at the stage t . As a result, the scenario tree represents the paths of possible realization of the uncertainties. Figure 1.2 presents an example of a 3-stage scenario tree where the nodes filled in gray represent a possible realization of the uncertainty. We can notice that all scenarios going through

some node have the same realization of the preceding stages.

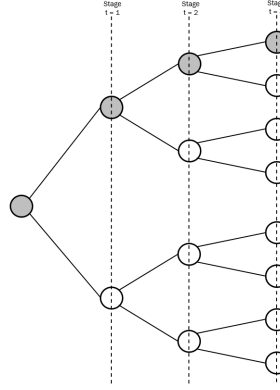


Figure 1.2 – A 3-stage scenario tree

Let us consider a scenario tree \mathcal{N} with t levels, such that for any node n in the level $T(n)$, a sub-set $\Omega(n)$ of scenarios cross this node. Assuming that R_n is the value of $X_t^{\omega'}$ for all $\omega' \in \Omega(n)$, the adaptive multistage SP for the LSP under uncertain demand is given as follows:

$$\min \sum_{\omega \in \Omega} \sum_{t \in T} \Pr_{\omega}(s_t Y_t + v_t X_t^{\omega} + h_t I_t^{\omega} + \pi_t B_t^{\omega}) \quad (1.14)$$

s.t.

$$I_t^{\omega} - B_t^{\omega} = I_{t-1}^{\omega} - B_{t-1}^{\omega} + X_t^{\omega} - d_t^{\omega} \quad t \in T \quad (1.15)$$

$$X_t^{\omega} \leq M_t Y_t \quad t \in T; \omega \in \Omega$$

$$X_{T(n)}^{\omega} = R_n \quad n \in \mathcal{N}; \omega \in \Omega(n) \quad (1.16)$$

$$X_t^{\omega}, I_t^{\omega}, B_t^{\omega} \geq 0 \quad t \in T; \omega \in \Omega$$

$$Y_t \in \{0, 1\} \quad t \in T$$

Similar to the static model, the multistage model aims to minimize the expected cost over all the considered scenarios. However, the inventory balance constraints (1.15) set lot size, inventory and backorder levels according to the realization of the uncertain demand, meeting the demand from each period t for each scenario ω . This model faces the difficulty of dependence of the decisions in given stage t on the uncertainties revealed in the previous stage. Even if the stages correspond to the steps in the decision process, for the general LSPs we can assume that each step refers to several periods in the production horizon. Although this assumption can reduce the size of the scenario tree, the model remains computationally heavy, as the number of nodes in the scenario tree grows expo-

nentially with the number of periods considered in the problem. To address the different uncertain parameters cited in Section 1.2.2, we may consider a formulation similar to those introduced for the static strategy, replacing X_t by X_t^ω .

Addressing the dynamic strategy, binary recourse variables should be considered. In other words, a well-structured multistage approach should be applied to the LSPs. Since the number of binaries would be exponential in the number of time period, the resulting deterministic models is itself hard to solve. While prohibitive computational efforts to solve the deterministic LSPs can be overcome by heuristic techniques, good scenario generation strategies and sampling approaches are required to control scenario and scalability issues. Brandimarte [2006] indicate the Latin Hypercube sampling combined with an optimized scenario generation as a promising method to address the multistage LSPs. Some decomposition, simulation or heuristic approach is often used to optimize the problem and compute a solution to large-size instances in reasonable time. A recent work proposed an efficient decomposition method applying the dual dynamic programming [Quezada et al., 2019]. De Saint Germain et al. [2017] propose a simulation method based on the SAA to handle real-world instances, though this approach suffers scalability issues. Şen et al. [2014] propose some heuristics to handle both static and dynamic strategies, and Kang and Lee [2013] propose a heuristic dynamic programming to compute a solution.

1.4.4 Literature review of scenario-based stochastic LSPs

The majority of the studies of the non-deterministic LSPs were addressed with SP models. Table 1.2 lists studies published from 1995 to 2020 addressing single and multi-item stochastic program LSP via a scenario-based modeling. It reports that the SP methodology reached a mature literature, since it has been relevant in the research community for a longer time than the other methods. The studies focus primarily on the single-item problem, even if some recent works investigate the multi-item problem. Besides that, the uncapacitated version and the uncertain demand are much more analysed than the capacitated version and other uncertain parameters.

The main issues faced by the stochastic LSPs, that are: i) the complexity of the mixed-integer LSPs is hard and some strong formulations and heuristics are needed to handle the non-deterministic problem, ii) the scenario generation issues and the difficulty of sampling the uncertainties, and, finally, iii) the complexity of representing practical applications with multistage models. Such difficulties raised with the adaptive models justify the preference for the stochastic modeling within a static strategy [Brandimarte,

Reference	Period	Item	Echelon	Capacity	Uncertainty	Decision framework	Inventory shortage
Alem et al. [2018]	multi	multi	multi	capacitated	demand	st-st	B
Augusto et al. [2016]	multi	multi	single	capacitated	demand, processing time and perishability	st-st	LS
Bansal et al. [2018]	multi	single	single	capacitated	demand and cost	st-st	B
Beraldi and Ruszczyński [2002]	multi	single	single	capacitated	demand	st-st	B - SL
Bollapragada and Rao [2006]	multi	single	single	capacitated	demand	st-st	B - SL
Bookbinder and Tan [1988]	multi	single	single	uncapacitated	demand	st	SL
Brandimarte [2006]	multi	multi	single	capacitated	demand	st-st ; st-dyn	B
Sen et al. [2014]	multi	multi	single	capacitated	cost	st-dyn	
De Saint Germain et al. [2017]	multi	multi	single	capacitated	demand	st-st ; st-dyn	B
Guan and Miller [2008]	multi	single	single	uncapacitated	demand	st-dyn	
Guan [2011]	multi	single	single	(un)capacitated	demand	st-dyn	B
Huang and Küçükyavuz [2008]	multi	single	s	uncapacitated	lead time	st-dyn	
Jiang and Guan [2011]	multi	single	single	uncapacitated	lead time	st-dyn	
Jiang and Guan [2016]	multi	single	single	uncapacitated	demand	st-st	
Jiang and Guan [2018]	multi	single	single	capacitated	demand	st-st	
Kang and Lee [2013]	multi	multi	multi	capacitated	demand	st-dyn	
Martel et al. [1995]	multi	multi	single	uncapacitated	demand	st-st	B - SL
Minoux [2018]	multi	single	single	capacitated	demand and cost	st-dyn	B
Quezada et al. [2019]	multi	multi	multi	uncapacitated	demand	st-dyn	LS
Raa and Aghezzaf [2005]	multi	single	single	capacitated	demand	st-st ; st-dyn	B
Rahmani et al. [2013]	multi	multi	multi	capacitated	demand	st-st	B
Sahebjamnia and Torabi [2011]	multi	multi	multi	capacitated	demand	st-dyn	B
Taş et al. [2019]	multi	multi	single	capacitated	setup time	st-st	
Tarim and Kingsman [2004]	multi	single	single	uncapacitated	demand	st	SL
Tempelmeier [2007]	multi	single	single	uncapacitated	demand	st-st	B - SL
Tempelmeier [2013]	multi	single ; multi	single ; multi	(un)capacitated	demand	st-dyn	
Wong et al. [2009]	multi	single	single	capacitated	demand	st-st	B
Zanjani et al. [2007]	multi	multi	multi	capacitated	yields	st-st	B
Zhou and Guan [2010]	multi	single	single	uncapacitated	cost	st-st ; st-dyn	
Zhou and Guan [2013]	multi	single	single	uncapacitated	cost	st-st	B

Table 1.2 – Some references from 1995 to 2019 that handle the LSP via SP

2006]. While decomposition methods efficiently handle the two-stage stochastic models [Sen, 2005], further investigation is need to address the multistage model with reasonable computational effort. The scalability issue is extended to any uncertain parameter or coefficient associated to the model.

1.5 Robust Optimization for LSPs

RO is a common approach to optimize under uncertainty [Ben-Tal and Nemirovski, 2000]. RO describes the uncertainty in a set, and to optimize against the worst-case realization of the uncertainty within this set [Bertsimas et al., 2011]. For that reason, the choice of the uncertainty set is crucial to achieve a good trade-off between system performance and protection against uncertainty [Bertsimas and Brown, 2009]. The RO decisions are determined before the realization of the uncertainties, and no recourse action is possible after revealing the uncertain parameters [Sözüer and Thiele, 2016].

The robust LSP has gained interest from the research community over the last decades, and different approaches exist to solve robust problems [Gabrel et al., 2014]. Robust problems are often handled through reformulation per constraint and dualization, or adversarial approaches. Note that the adversarial approach [Yanikoğlu et al., 2019] refers to the type of approaches which does not rely on a tractable reformulation of the robust counterpart but typically by iteratively solving a restricted robust model with a limited set of uncertainty realizations to determine a possible robust solution, and then finding a scenario realization which makes this solution candidate infeasible. If such a scenario is found, this scenario realization is then added to the restricted robust model and the process terminates when a possible robust solution is guaranteed to be feasible. The reformulation per constraint and dualization leverages the optimality condition of the duality theory. In fact, the constraint-wise formulation, where the uncertain parameters appear only in the constraints, allows us to directly exploit the duality technique of Beck and Ben-Tal [2009], for which the best dual in robust models is proved to be equal to the worst primal. Thus, a robust counterpart model can be defined by reformulating all the constraints subject to uncertainty by a worst case perspective of the uncertain parameter under a predetermined uncertainty set.

Construction of the uncertainty set The uncertainty sets are often rooted in some statistical considerations as presented by Gabrel et al. [2014]. A prescriptive methodology for constructing uncertainty sets within the robust optimization framework for linear problems with uncertain data is presented by Bertsimas and Brown [2009]. Most researches have focused on uncertainty sets whose shape can be expressed as a well-known formulation as linear programs, second-order cone programs due to their tractability and their geometry favorable for some calculations over incomplete data or lack of forecast expertise.

The first uncertainty set was presented by Soyster [1973]. Also called *box or interval uncertainty set*, this set contains all possible values that the uncertain parameters can assume within a range limited by their maximum and minimum values. Even if this first approach is easily computed, it leads to an extremely conservative solution [Ben-Tal and Nemirovski, 2000]. To overcome this conservatism, Bertsimas and Sim [2004] present a linear approach to control the size of the interval proposed by Soyster [1973] with a budget Γ , constructing then a polyhedral uncertainty set. This set, also called a *budgeted uncertainty set*, limits the deviation of the uncertain parameters from its mean with a budget of uncertainty controlled by the decision maker. This reformulation allows decision makers to determine the *price of robustness* of the model according to the decision context needs and strategies.

Since these seminal works, various authors proposed uncertainty set, with the objective to design tractable models, where the conservatism is controllable. Some examples of these uncertainty sets are: *box uncertainty set* Qiu et al. [2017], (*closed, convex, pointed or second-order*) *cone uncertainty set* Ben-Tal et al. [2004], *ellipsoid uncertainty set, polyhedral uncertainty set* Bertsimas and de Ruiter [2016]. For further details on the uncertainty sets, the reader is referred to Bertsimas and Brown [2009], Thiele [2010], Gorissen et al. [2012].

Uncertainty	\mathcal{Z}	Robust counterpart	Tractability
Box	$\ \xi\ _\infty \leq 1$	$\mathbf{a}^\top \mathbf{x} + \ \mathbf{P}^\top \mathbf{x}\ _1 \leq d$	LP
Ellipsoidal	$\ \xi\ _2 \leq 1$	$\mathbf{a}^\top \mathbf{x} + \ \mathbf{P}^\top \mathbf{x}\ _2 \leq d$	CQP
Polyhedral	$\mathbf{D}\xi + \mathbf{q} \geq \mathbf{0}$	$\begin{cases} \mathbf{a}^\top \mathbf{x} + \mathbf{q}^\top \mathbf{w} \leq d \\ \mathbf{D}^\top \mathbf{w} = -\mathbf{P}^\top \mathbf{x} \\ \mathbf{w} \geq \mathbf{0} \end{cases}$	LP
Cone (closed, convex, pointed)	$\mathbf{D}\xi + \mathbf{q} \in K$	$\begin{cases} \mathbf{a}^\top \mathbf{x} + \mathbf{q}^\top \mathbf{w} \leq d \\ \mathbf{D}^\top \mathbf{w} = -\mathbf{P}^\top \mathbf{x} \\ \mathbf{w} \in K^* \end{cases}$	Conic Opt.
Convex cons.	$h_k(\xi) \leq 0 \quad \forall k$	$\begin{cases} \mathbf{a}^\top \mathbf{x} + \sum_k u_k h_k^* \left(\frac{\mathbf{w}^k}{u_k} \right) \leq d \\ \sum_k \mathbf{w}^k = \mathbf{P}^\top \mathbf{x} \\ \mathbf{u} \geq \mathbf{0} \end{cases}$	Convex Opt.

Figure 1.3 – Tractable reformulations for the uncertainty constraint $aX + bX \leq d$ for most common uncertainty sets. Table taken from Yanıkoğlu et al. [2019]

Figure 1.3 introduces some uncertainty sets and their representation in the RC models. For the static RO, these uncertainty sets often lead to tractable robust models.

1.5.1 Approaches to solve robust reformulations

The RO method reformulates the non-deterministic problems as a robust counterpart (RC) model in which the uncertainty is taken into account under an uncertainty set. As

a result, the reformulated model considers an infinite number of constraints representing each possible of the uncertain parameter included on the uncertainty set. There are two approaches to deal with the infinity constraints and to solve the RC models [Gorissen et al., 2015]. First, a reformulation technique can be applied to turn the RC model tractable. If this reformulation is complex or intractable, then an adversarial approach should be developed. Usually, the reformulation techniques efficiently deal with the majority of the robust models within a static strategy. However, the complexity of worst case perspective of the adaptive robust models leads to very hard adjustable robust counterpart (ARC) models. Therefore, an adversarial approach is more required to solve a robust model within an adaptive strategy.

Reformulation per constraint and dualization *Reformulation per constraint and dualization approach* reformulates into models with a finite numbers of constraints in three steps: 1) reformulation of each constraint subject to the uncertainty as a worst-case reformulation, 2) modeling of the dual of the worst-case reformulation, and finally () reformulation of the RC model by dropping all dualized reformulation into the initial RC without the inner optimization term [Bertsimas et al., 2011]. In fact, the duality overcomes the prohibitive quantifier of the uncertainty supported by the proof demonstrated by Beck and Ben-Tal [2009] that indicates that “primal worst equals to dual best”. As a result, the infinite number of constraints from the original robust problem is replaced by a dualized model whose constraints are limited to a finite amount. Hence, the reformulated RC model, if tractable, can be solved by a commercial solver.

Adversarial approach An *adversarial approach* addresses the cases when the reformulated model cannot be solved. Therefore, the RC model may be solved via heuristic algorithms or approximations to a tractable model. Different adversarial techniques or methodologies can be applied to solve the complex robust models. The choice of the technique depends on the structure and the hardness of the non-deterministic problem. Among the general existing adversarial approaches, we can cite the decomposition techniques (e.g., column generation) and the sampling techniques (e.g., sample average approximation).

Tractable robust models requires a convex optimization formulation which is are often handled by dualization [Bertsimas and de Ruiter, 2016], reformulation of the robust counterpart [e.g., See and Sim, 2010, Guillaume et al., 2017], Lagrangian duality [Rodrigues et al., 2021], dynamic programs [e.g., Bienstock and Özbay, 2008, Agra et al., 2016] and

decomposition algorithms [e.g., Bienstock and Özbay, 2008, Attila et al., 2021]. Although these techniques often yield feasible formulations that are easy to solve, they can lead to conservative solutions [Thiele, 2010]. Gorissen and Den Hertog [2013] discuss different ways to alleviate the conservatism of the formulations. For further information on recent advances in robust optimization, we refer the interested readers to Gabrel et al. [2014].

Bertsimas and Thiele [2004] proposed a first reformulation per constraint and dualization to solve the robust LSP and its variations or simplifications (e.g., the inventory management problem). The authors consider the inventory management problem under a budgeted uncertainty set and uncertain demand, which results in a MILP formulation that can be solved by a commercial solver. The authors suggest carrying out a simulation to define a budget of uncertainty that best mitigates the uncertainties. The objective is to find a suitable compromise between cost and robustness [Thiele, 2010]. In the same dualization vein, Zhang [2011] propose a formulation for the uncapacitated LSP with uncertain demand, where the inner maximization is reformulated as a longest path problem. The authors take the dual of the longest path problem to obtain the final robust model as an MILP, and this final model can also be solved by a commercial solver. Considering an example of adversarial approach, Bienstock and Özbay [2008] propose some decomposition techniques to iteratively constrain the space of the realization of the uncertainty. The resulting approach provides plans that are sufficiently robust, and it scales well to solve large instances.

Within the static strategy, the reformulation per constraint and dualization can be applied for the RC LSP models under any uncertain parameter. However, for the adaptive LSPs, since the uncertain parameters directly modify the adjustable decisions, the ARC models usually become intractable and should be addressed via an adversarial approach. We introduce in the next sections the robust methodology for the non-deterministic LSPs.

1.5.2 Static models

We recall the static LSP under demand uncertainty proposed by Thorsen and Yao [2017], where the RC is solved via reformulation per constraint and dualization approach. For that, we assume an uncertain demand \tilde{d} included in budgeted uncertainty set \mathcal{U} given by $\mathcal{U} = \{\tilde{d} \in \mathbb{R}^t : \tilde{d}_i = \bar{d}_i + Z_i \hat{d}_i \ i \in T; -1 \leq Z_i \leq 1 \ i \in T ; \sum_{i=1}^t |Z_i| \leq \Gamma_t\}$, where Γ_t is the budget, \bar{d} as the demand average, \hat{d} the demand standard deviation, and Z the disturbance of worst case realization of the demand from the mean.

The inventory and the backorder costs are piecewise cost functions that can be ex-

exploited by their convexity. Thus, we can formulate a linearized RC reformulation of the original problem as a mixed-integer problem [Bertsimas and Thiele, 2004]. Therefore, the RC for the LSP under uncertain demand is given as follows:

$$\min \quad \sum_{t \in T} s_t Y_t + v_t X_t + H_t \quad (1.17)$$

s.t. :

$$H_t \geq h_t \max_{\tilde{d} \in \mathcal{U}} \left[\sum_{\tau=1}^t (X_\tau - \tilde{d}_\tau) \right] \quad t \in T \quad (1.18)$$

$$H_t \geq -b_t \max_{\tilde{d} \in \mathcal{U}} \left[\sum_{\tau=1}^t (X_\tau - \tilde{d}_\tau) \right] \quad t \in T \quad (1.19)$$

$$X_t \leq MY_t \quad t \in T$$

$$X_t, H_t \geq 0 \quad t \in T$$

$$Y_t \in \{0, 1\} \quad t \in T$$

Within a static strategy, the use of the RO methodology for the other uncertain parameters presented in Section 1.2.2 is similar to the case of the demand. We consider the uncertain parameter within the uncertainty set, and we propose a production plan that respects the worst-case realization of the uncertainty within the considered set. No additional difficulty is expected to solve the different uncertainties.

Solyalı et al. [2015] propose some reformulations leading to better performances of the robust methodology for LSPs. Within the static framework, a MIP model solved by any commercial MIP solver is often applied to solve the LSP due to its tractability and optimality as presented in Hnaien and Afsar [2017]. Qiu et al. [2017] propose a tractable linear and second-order cone program to deal with the solution of the LSP under demand uncertainty.

When we extend this optimization approach to the different uncertain parameters defined in Section 1.2.2 we notice that all the uncertainty can be dealt in the similar way, since they meet the assumptions of the RO. Focusing on real-world production planning applications, Alem et al. [2018] have proposed the integration of lot sizing and scheduling model to determine a production plan that satisfies uncertain demand and uses efficiently the available resources and the system.

1.5.3 Adaptive models

ARO extends RO methodology to make some variables adjustable to the realization of the uncertainties. ARO supposes that uncertainty is revealed as time goes, and one can adjust the values of some of the decision variables according to the current knowledge of the uncertain parameters [Ben-Tal et al., 2004]. The coefficients associated with these variables denote the recourse actions applied to the adaptive framework. A fixed recourse indicates that the coefficients associated with the adaptive variables are not dependent on the uncertain parameter. On the other hand, a non-fixed recourse denotes that the coefficient of the adaptive decisions changes with the realization of the uncertainty.

The adjustable problem is by definition a complex problem whose formulation would lead to intractable models [Yanikoğlu et al., 2019]. To restrict the adaptive model to a tractable formulation, the ARO community often relies on affine decision functions to compute the value of the adaptive variables based on the values of uncertain parameters [Ben-Tal et al., 2004]. Under these affine policies, the adaptive decisions become linearly dependent on the uncertain parameter [Ouorou, 2011]. For fixed recourse models, the adaptive robust counterpart (ARC) models can be efficiently solved via some well-known optimization problems, such as linear, quadratic, conic or semi-definite optimization problem [Bertsimas and Bidkhori, 2015]. For the non-fixed recourse models, the adaptive formulation where both coefficient and variable are modified by the uncertain parameter yields to an indefinite quadratic representation of the uncertainty. The tractability of these models via the reformulation per constraint approach is guaranteed by a semi-definite representation of the constraints under an ellipsoidal uncertainty set [Ben-Tal et al., 2009]. Otherwise, an adversarial approach should be envisaged. A survey of the ARO methodology is presented by Yanikoğlu et al. [2019]. We refer the reader to Ben-Tal et al. [2009] for further information on the robust methodology.

Although most robust optimization model can be reformulated as tractable convex models [e.g., Ben-Tal et al., 2004, Ben-Tal and Den Hertog, 2014, Delage and Iancu, 2015], the reformulation of adaptive models as convex problem (e.g., a linear, quadratic, or semidefinite optimization problem) may not always be possible and it often requires high-level mathematical theory [Yanikoğlu et al., 2019]. Adaptive robust counterpart (ARC) models can have a fixed or non-fixed recourse [Yanikoğlu et al., 2019]. In ARC with fixed recourse (e.g., the adaptive models with uncertain demand), the coefficients of the adaptive variables are not subject to uncertainties. In ARC with a non-fixed recourse, the coefficients of some of the adaptive variables are uncertain parameters. The former case

leads to a convex formulation that can be solved using efficient algorithms or commercial solvers. However, the models with non-fixed recourse have constraints that are indefinite quadratic in the uncertainty, which yield in intractable formulations [Yanikoğlu et al., 2019]. The ARC of the LSP under uncertain yield has non-fixed recourse since the uncertain parameter (the yield) is a coefficient of the recourse action (the production quantity). The gap of knowledge on the adjustable robust models with non-fixed recourse has motivated us to further investigate the advantages of this approach on the adaptive LSP with uncertain production yield.

For general adaptive robust formulations, Bertsimas et al. [2010] proved the optimality of affine decision rules for adjustable robust models, Bertsimas and de Ruiter [2016] derive some properties on the dualized formulations for adaptive linear problems, and Chuong and Jeyakumar [2020] establish the strong duality between the affinely adjustable robust counterpart models and their dual semidefinite programs. A convex adaptive model can be solved by the same approaches applied to robust models [Yanikoğlu et al., 2019]. These resolution approaches include reformulation and dualization [e.g., See and Sim, 2010, Guillaume et al., 2017, Rodrigues et al., 2021] and adversarial approaches such as decomposition or dynamic programs [e.g., Bienstock and Özbay, 2008, Agra et al., 2016, Attila et al., 2021]. Since the tractability of robust models often relies on bounded uncertainty sets, and so does the optimality of these models, in this work we consider only the robust optimization literature applied to bounded uncertainty sets.

For the LSPs, the ARO methodology denotes a min-max formulation targeting to minimize the worst-case overall costs over the uncertainty set. We present the ARO method for the LSPs based on reformulation per constraint and dualization approach with the affinely adjustable robust counterpart (AARC) model. As the production quantity X_t , and so inventory/backorder cost H_t , have their values updated with the realization of the uncertain parameter ζ , they are assumed adaptive decisions. The parameterization of these adjustable variables by the affinely linear decision rules (ADR) with regard to the realization of the uncertainty for the periods up to t are given as follows:

$$X_t(\tilde{\zeta}) = X_t^0 + \sum_{\tau=1}^{t-1} X_t^\tau \tilde{\zeta}_\tau \quad t \in T \quad (1.20)$$

$$H_t(\tilde{\zeta}) = H_t^0 + \sum_{\tau=1}^{t-1} H_t^\tau \tilde{\zeta}_\tau \quad t \in T \quad (1.21)$$

LSPs with fixed recourse

Most studies on the ARO methodology address fixed recourse models, the case for the LSPs under demand or setup time uncertainty [e.g., Bertsimas and de Ruiter, 2016, Melamed et al., 2016, Bertsimas et al., 2019]. To illustrate the LSPs with uncertain demand under budgeted uncertainty set and model as a AARC model, we recall the model proposed by Metzker et al. [2023b]. Let us assume that the demand is uncertain and defined within the budgeted uncertainty set \mathcal{U} introduced in 1.5.2. The LSP aims decide the adaptive lot size quantities to minimize the overall costs over the planning horizon. The AARC model is given as follows:

$$\min_Y \sum_{t \in T} \left[s_t Y_t + \max_{\tilde{d} \in \mathcal{U}_t} \left(v_t X_t(\tilde{d}) + H_t(\tilde{d}) \right) \right] \quad (1.22)$$

s.t. :

$$H_t(\tilde{d}) \geq h_t \sum_{\tau=1}^t (X_\tau(\tilde{d}) - \tilde{d}_\tau) \quad \forall t \in T; \tilde{d} \in \mathcal{U}_t \quad (1.23)$$

$$H_t(\tilde{d}) \geq -b_t \sum_{\tau=1}^t (X_\tau(\tilde{d}) - \tilde{d}_\tau) \quad \forall t \in T; \tilde{d} \in \mathcal{U}_t \quad (1.24)$$

$$X_t(\tilde{d}) \leq M_t Y_t \quad \forall t \in T; \tilde{d} \in \mathcal{U}_t$$

$$X_t(\tilde{d}), H_t(\tilde{d}) \geq 0 \quad \forall t \in T; \tilde{d} \in \mathcal{U}_t$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

where \tilde{d} is the uncertain demand under the uncertainty set \mathcal{U} , $X_t(\tilde{d})$ represents the affine decision rule for the adaptive lot size, and $H_t(\tilde{d})$ is the affine decision rule for the adaptive inventory/backorder costs.

Applying the reformulation per constraint and dualization approach, the authors obtain the final AARC model that is tractable, as all constraints and the objective function are linear with regard to the realization of the uncertain demand. Solving the AARC before period $t = 1$, we obtain the optimal vectors X_t^{0*} and $X_t^{\tau*}$ (resp. H_t^{0*} and $H_t^{\tau*}$) that are components from the $H_t(\tilde{d})$ (resp. $H_t(\tilde{d})$) functions. Then, as time goes, at the beginning of each period t , the optimal production X_t^* and inventory/backorder H_t^* variables can be computed with regard to the affine linear decision rules (1.20) and (1.21), and with the revealed demand.

As well as demand, setup time uncertainty is not directly associated with any decision variable. The ARO methodology applied to LSPs under these uncertain parameters in

a static-dynamic or dynamic strategy remains tractable when affine decision rules are defined. Few modifications may be done to the AARC model presented above to model the uncertain setup time.

In a dynamic framework, an adaptive setup of production would require an adversarial approach or other techniques to deal with the adaptive binary decision. One possibility is to manage adaptive robust LSPs without setup, and compute the setup costs when a production is held at a specific time.

LSPs with non-fixed recourse

For the non-fixed recourse models, as the LSPs under yield, lead-time, processing-time, setup time or cost uncertainty, the coefficients of the adjustable variable are themselves uncertain. It yields quadratic constraints in the uncertainty. As a result, the adaptive model may be difficult to solve or even intractable [Ben-Tal et al., 2004]. Very few studies exist on ARC problems with non-fixed recourse. Even the most recent literature published about the adjustable robust methodology limits their analysis to the complexity to handle those problems, and some mathematical consideration to applied mathematical approaches that could overcome the quadratic difficulty via convex tractable approximations [e.g., Ben-Tal et al., 2004, Yanıkoğlu et al., 2019]. Ben-Tal et al. [2005] indicate that the tractability of these problems, when possible, should be guaranteed via an ellipsoidal uncertainty that approximates the problem via convex semi-defined programming. Otherwise, an approximation or an adversarial approach may be applied to handle the adaptive problems.

We demonstrate a non-fixed recourse model with the adaptive LSP with uncertain production yield presented by Metzker et al. [2023b]. Assuming a budgeted uncertainty set \mathcal{U} based on the mean and the standard deviation of the production yield rate, the authors parameterize the uncertain production yield by $\tilde{\rho} = \bar{\rho} + Z\hat{\rho}$, and we estimate the budgeted uncertainty set \mathcal{U} by $\mathcal{U} = \{\mathbf{Z} \in \mathbb{R}^T : \sum_{\tau=1}^t |Z_\tau| \leq \Gamma_t \ t \in T\}$. Applying the reformulation approach, the non-fixed AARC model with uncertain production yield is given as follows:

$$\min_Y \sum_{t \in T} \left[s_t Y_t + \max_{\rho \in \mathcal{U}_t} \left(v_t X_t(\rho) + H_t(\rho) \right) \right] \quad (1.25)$$

s.t. :

$$H_t(\rho) \geq h_t \sum_{\tau=1}^t (\rho_\tau X_\tau(\rho) - d_\tau) \quad \forall t \in T; \rho \in \mathcal{U}_t \quad (1.26)$$

$$H_t(\rho) \geq -b_t \sum_{\tau=1}^t (\rho_\tau X_\tau(\rho) - d_\tau) \quad \forall t \in T; \rho \in \mathcal{U}_t \quad (1.27)$$

$$X_t(\rho) \leq M_t Y_t \quad \forall t \in T; \rho \in \mathcal{U}_t$$

$$X_t(\rho), H_t(\rho) \geq 0 \quad \forall t \in T; \rho \in \mathcal{U}_t$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

The non-fixed recourse are represented in Constraints (1.26) and Constraints (1.27). The uncertain production yield $\tilde{\rho}$ is the coefficient for the adaptive lot size $X(\rho)$. This leads to the quadratic constraints in the uncertainty, such that at each period t the disturbance for this period multiplies the disturbances for the periods up to t that composes the affine rule of the adaptive lot size. Similar to the non-fixed model for the LSPs with uncertain yield, the capacity, lead-time and processing time has non-fixed recourse. In addition, the modeling should take into account the structure of the problems. For instance, the lead time and processing time are integer parameters and, thus, cannot be dualized. As a result, modeling such uncertainties can be very challenging.

1.5.4 Literature review of robust LSPs

Since the RO is a topic under development in the scientific community, the number of researches is small and limited to specific research groups. This also justifies why studies focus on uncertain demand, as this is more natural to the uncertainties. Table 1.3 lists the references addressing single and multi-item LSP within a static and an adaptive strategy. While the RO studies refer to the static strategy, the ARO studies relate to the static-dynamic strategy. Any study for the dynamic framework was found, which helps to proof the difficulty to address such problems. The dynamic robust models are non-convex models. Since the setup decisions are integer and binary variables, we cannot describe them through linear decision rules. In addition, the difficulty of defining the integer and continuous adaptive decisions as decision rules valid within a bounded uncertainty set prevents the use of dualization techniques. As a result, no direct reformulation is possible, and some adversarial approach may be developed to compute a dynamic solution. In addition, there is a gap of knowledge of the robust perspective for multi-item LSPs under uncertainties, and further studies for other uncertainties than demand in the LSPs environment.

Reference	Period	Item	Echelon	Capacity	Uncertainty	Decision framework	Inventory shortage	Optimization method
Agra et al. [2018]	multi	multi	single	capacitated	demand	st-st	B	RO
Aharon et al. [2009]	multi	multi	multi	uncapacitated	demand	st-dyn	B	ARO
Alem et al. [2018]	multi	multi	multi	capacitated	demand	st-st	B	RO
Ben-Tal et al. [2004]	multi	single	single	capacitated	demand	st-dyn	x	ARO
Ben-Tal et al. [2005]	multi	single	multi	capacitated	demand	st-dyn	x	ARO
Bertsimas and Thiele [2006a]	multi	multi	multi	(un)capacitated	demand	st-st	B	RO
Bertsimas and Thiele [2006b]	multi	single	single	uncapacitated	demand	st-st	B	RO
Bertsimas et al. [2010]	multi	single	single	capacitated	demand	st-dyn	B	ARO
Bertsimas and Bidkhori [2015]	single	multi	single	uncapacitated	demand	st-st	B	ARO
Bertsimas and de Ruiter [2016]	single	multi	single	capacitated	demand	st-st	x	ARO
Bertsimas and Dunning [2016]	multi	single	single	uncapacitated	demand	st-st ; st-dyn		RO ; ARO
Bertsimas et al. [2018b]	multi	single	single	uncapacitated	demand	st-dyn	B	ARO
Coniglio et al. [2018]	multi	single	single	uncapacitated	demand	st-st	B	RO
Guo and Liu [2018]	single	multi	single	uncapacitated	demand	st-st	x	RO
Hnaen and Afsar [2017]	multi	single	single	capacitated	lead time	st-st	x	RO
Kaganova [2013]	multi	multi	single	capacitated	demand	st-st; st-dyn	LS - B	RO - ARO
Kang and Lee [2013]	multi	multi	multi	capacitated	demand	st-dyn	x	SP
Klabjan et al. [2013]	multi	single	single	uncapacitated	demand	st-st	B	RO
Leung and Wu [2004]	multi	multi	multi	capacitated	demand	st-st	x	RO
Mehdizadeh et al. [2018]	multi	multi	multi	capacitated	demand and lead time	st-st	B	ARO
Melamed et al. [2016]	multi	single	single	uncapacitated	demand	st-st	x	ARO
Minoux [2018]	multi	single	single	capacitated	demand and cost	st-dyn	B	ARO
Santos et al.	multi	single	single	capacitated	demand	st-st ; st-dyn	B	RO - ARO
Shapiro [2011]	multi	single	single	uncapacitated	demand	st-dyn		ARO
Solyah et al. [2015]	multi	single	single	uncapacitated	demand	st-st ; st-dyn	B	RO - ARO
Thorsen and Yao [2017]	multi	single	single	uncapacitated	demand and lead time	st-st	B	RO
van Pelt and Fransoo [2015]	multi	(single/multi)	single	capacitated	demand	st-st ; st-dyn	B	RO - ARO
Vayanos et al. [2012]	multi	single	single	capacitated	demand and yields	st-dyn	B	ARO
Wagner [2017]	multi	single	single	uncapacitated	demand	st-st ; st-dyn	x	RO
Zhu et al. [2012]	multi	single	single	capacitated	demand	st-dyn	x	ARO

Table 1.3 – References handling the LSP via RO and ARO

1.6 Distributionally Robust Optimization for LSPs

DRO seeks to optimize against the worst-case expected cost over a set of probability distributions described by an ambiguity set [Delage and Ye, 2010]. The ambiguity set contains a family of probability distributions characterized by the properties of the unknown distribution of the uncertain parameter [Delage and Ye, 2010].

Although the DRO community questions whether some studies from the 1950s propose concepts equivalent to those known today as DRO methodology, most advances in the field began in the 2000s. Delage and Ye [2010] is among the first to tackle the distributionally robust optimization problem by considering an ambiguity set based on moment information. They propose a generic model for this case and a semi-infinite programming approach to solve it. Wiesemann et al. [2014] provide an extensive review of convex DRO methodology and introduce standardized forms for convex ambiguity sets. The authors also present important conditions which ensure tractability of the final distributionally robust models. Esfahani and Kuhn [2018] present the Wasserstein ambiguity set, an ambiguity set that is convex, ensures tractability of the model and leads to powerful performance. Kuhn et al. [2019] extend this work to provide more theory about the Wasserstein ambiguity set, and show the conceptual and computational benefits of this approach. The authors also report the promising results when combining Wasserstein ambiguity set with machine learning applications. Bertsimas et al. [2019] propose a tractable adaptive DRO formulation based on second-order conic representable ambiguity sets. The authors provide tools to reformulate the distributionally robust models as MILP that can be easily solved by commercial solvers. In the same vein, Chen et al. [2020] introduce the event-wise formulation for the ambiguity sets, and provide a new modeling package, RSOME, to help modelers to reformulate and solve distributionally robust models. RSOME is an open-source Python library that makes it easier to formulate and to solve distributionally robust problems, through pre-defined reformulation steps. RSOME also allows us to directly call commercial solvers to compute a solution. An extensive review on DRO, we refer readers to Rahimian and Mehrotra [2019].

In a general framework, the DRO assumes an ambiguity set known a priori and aims to hedge against the worst-case probability distribution in the ambiguity set. Therefore, DRO combines the expected cost computation from the SP with the conservatism of the RO under a worst-case perspective. The different probability distributions are considered over their worst case and the DRO model can be translated into an RO model based on

distributional information. Hence, DRO avoids over-conservative solutions by incorporating partial stochastic information into the robust model. For more information on DRO, readers are referred to Wiesemann et al. [2014], and the recent literature review given by Rahimian and Mehrotra [2019].

The DRO aims to reformulate the objective function of a decision problem under worse case perspective over the possible distributions of the uncertain parameter in the ambiguity set [Delage and Ye, 2010]. For the LSP, this optimization holds as a min-max-min approach targeting to minimize the worst-case expected cost maximized over the ambiguity set. In terms of solution strategies for a DRO, MILP, approximations and heuristic approaches are the most common methods used to solve the LSP under uncertainty [Rahimian and Mehrotra, 2019].

Analui and Pflug [2014] propose a successive programming for finding the solution for a reformulation of the LSP as a min max problem. A mixed-integer conic quadratic program with mean-risk functions based on the shortest-path reformulation of the LSP is proposed and solved by a branch-and-bound interval search method in Zhang et al. [2016]. A DRO model for the LSP with random demand and no lead-time is presented in Zhang et al. [2016]. Under the demand uncertainty described by an ambiguity set, robust solutions were derived as mixed 0-1 conic quadratic programs (CQPs) by exploiting shortest path reformulations of the LSP. A parametric optimization method was also proposed to solve the mixed 0-1 CQPs, provided then an approximate solution for the original problem. Jiang and Guan [2018] solve the LSP under demand uncertainty based on sample average approximation solution algorithm (a data-driven approach is reviewed in Bertsimas et al. [2018a]).

The rest of this section successively introduces the two-stage DRO model for the static-static LSP, and the multistage DRO approaches for the static-dynamic decision strategy. No studies on dynamic distributionally robust LSP models were found.

1.6.1 Static models

Focusing on the static strategy, Zhang et al. [2016] present a two-stage distributionally robust LSP model with uncertain demand and no lead-time. We focus on the case when backordering is allowed. The setup variables (Y_t) are the first-stage decisions and the production quantities (X_t), the inventory level (I_t) and the backorders (B_t). Considering a joint probability distribution function \mathcal{F} of the uncertain demand, even if F is unknown, it belongs to a known ambiguity set described with specific properties

$\mathcal{D} = \{F : (\mathbb{E}_F[\tilde{d}] - \mu)^T \Sigma^{-1} (\mathbb{E}_F[\tilde{d}] - \mu) \leq \epsilon^2\}$, where Σ is a positive definite estimation of the covariance matrix of the uncertainty and ϵ controls the size of the ambiguity set. They describe the ambiguity set \mathcal{D} as an ellipsoid centered at the estimated mean μ and its shape is controlled by Σ , the covariance matrix.

The two-stage LSP formulation based on the DRO method is modeled as follows:

$$\min_{Y \in \{0,1\}^T, X \geq 0} \sum_{t \in T} s_t Y_t + v_t X_t + \sup_{F \in \mathcal{D}} \mathbb{E}_F [f(Y, X, \tilde{d})] \quad (1.28)$$

where $f(Y, X, \tilde{d})$ is the second-stage cost for the fixed setup, lots size and a given demand realization.

Considering backorder variables to control the uncertainty, the second-stage cost function is given as follows:

$$f(Y, X, \tilde{d}) = \min \sum_{t \in T} h_t I_t + b_t B_t \quad (1.29)$$

s.t.

$$\begin{aligned} I_t - B_t &= I_{t-1} - B_{t-1} + X_t - \tilde{d}_t & t \in T \\ X_t &\leq M \cdot Y_t & t \in T \\ B_t, I_t &\geq 0 & t \in T \end{aligned}$$

where b_t is a penalty cost for backorders and M is a sufficiently large number to express the unrestricted resource capacity.

Similar to SP and RO for the static framework, the DRO is tractable in general, if the ambiguity set is tractable and the reformulation is convex. As this subject is under development, the tractability is ensured under strong hypothesis. Further work is needed to investigate the performance and limits of the method in the static decision framework.

Hanasusanto et al. [2015] propose a risk-averse distributionally robust multi-item newsvendor problem with uncertain demand. The authors approximate the distributionally robust model to a quadratic problem, and it yields a conservative but tractable formulation. Huang et al. [2021] give a reformulation approach based on the conditional value-at-risk (CVAR) for solving the multi-product assembly and the portfolio selection problems with a cutting plane algorithm. Qiu et al. [2021] develop a reformulation by Lagrange multipliers for solving the multi-product inventory problem with demand and

supply uncertainties. The authors also propose decomposition methods to compute solutions for the case of reliable and unreliable suppliers. Finally, Wang and Delage [2021] reformulate the multi-item newsvendor problem with the help of event-wise affine decision rules and propose a column generation algorithm to tackle scalability issues and compute a good solution.

1.6.2 Adaptive models

Dealing with the static-dynamic framework, Bertsimas et al. [2019] propose a tractable formulation of an adaptive DRO. The model minimizes the worst-case expected cost over an ambiguity set. Similar to the two-stage model presented above, the tractability is ensured by a second-order conic (SOC) representation of the ambiguity set and the affine linear decision rule techniques to express the adjustable variables.

Assuming the demand is represented with an integrated moving average process of order (0,0,1) expressed as $\tilde{d}_t(\tilde{z}) = \tilde{z}_t + \alpha\tilde{z}_{t-1} + \alpha\tilde{z}_{t-2} + \dots + \alpha\tilde{z}_1 + \mu = \tilde{d}_{t-1}(\tilde{z}) - (1-\alpha)\tilde{z}_{t-1} + \tilde{z}_t$ for $t \in T$ where $\alpha \in [0, 1]$, the uncertain factor \tilde{z}_t . We observe that when $\alpha = 0$ then the demands are uncorrelated, and the correlation increase with the growth of α .

The multistage DRO LSP is given as:

$$\begin{aligned} & \min \sup_{\mathcal{P} \in \mathcal{F}} \mathbb{E}_{\mathcal{P}} \sum_{t \in T} (v_t X_t(\tilde{z}_{t-1}) + H_t(\tilde{z}_t)) & (1.30) \\ & s.t. \\ & I_{t+1}(\tilde{z}_t) = I_t(\tilde{z}_{t-1}) + X_t(\tilde{z}_{t-1}) - \tilde{d}_t(\tilde{z}_t) & t \in T; \tilde{z} \in \mathcal{W} \\ & H_t(\tilde{z}_t) \geq h_t I_{t+1}(\tilde{z}_t) & t \in T; \tilde{z} \in \mathcal{W} \\ & H_t(\tilde{z}_t) \geq -b_t I_{t+1}(\tilde{z}_t) & t \in T; \tilde{z} \in \mathcal{W} \\ & 0 \leq X_t(\tilde{z}_{t-1}) \leq \bar{X}_t & t \in T; \tilde{z} \in \mathcal{W} \\ & X_t \in \mathcal{R}^{t-1,1}; I_t \in \mathcal{R}^{t,1}; H_t \in \mathcal{R}^{t,1} & t \in T \end{aligned}$$

where \bar{X}_t is an upper bound for the ordered quantity. Let us consider \mathcal{F}_{PCM} , \mathcal{G}_{PCM} and $\bar{\mathcal{W}}_{PCM}$ be respectively the ambiguity set, the lifted ambiguity set and the joint lifted support set partial cross moment. We can observe that the variances of the factors leading to the period t are incorporated.

Based on the SOC representation, the authors describe the SOC ambiguity set as a tractable set supported by commercial solvers. By definition, a SOC is an ambiguity set of probability distributions. A generalization to conic representation and its main classes is

introduced by Wiesemann et al. [2014]. Bertsimas et al. [2019] approximate its ambiguity set to the cross moment ambiguity set, a class of conic ambiguity set, such that the semidefinite constraints functions $\mathbb{E}_{\mathbb{P}}[(\tilde{z} - \mu)(\tilde{z} - \mu)'] \leq \Sigma$ defining the representation are approximated to some semi-infinity quadratic constraints $\mathbb{E}_{\mathbb{P}}[(f'(\tilde{z} - \mu))^2] \leq f' \Sigma f \quad \forall f \in \mathbb{R}^D$, being the uncertainty D -dimensional.

Hence, a partial cross moment ambiguity set \mathcal{F}_{PCM} is given by:

$$\mathcal{F}_{PCM} = \left\{ \begin{array}{l} \tilde{z} \sim \mathbb{P} \\ \mathbb{E}_{\mathbb{P}}[\tilde{z}] = 0 \\ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^T) \quad \mathbb{E}_{\mathbb{P}}\left[\left(\sum_{r=s}^t \tilde{z}_r\right)^2\right] \leq \phi_{st}^2 \quad \forall s \leq t; s, t \in T \\ \mathbb{E}_{\mathbb{P}}[\tilde{z} \in [\underline{z}, \bar{z}]] = 1 \end{array} \right.$$

A lifted ambiguity set \mathcal{G}_{PCM} is defined to cover the primary uncertain variables \tilde{z} and their auxiliary uncertain variables as the lifted variables \tilde{u} is given as follows:

$$\mathcal{G}_{PCM} = \left\{ \begin{array}{l} (\tilde{z}, \tilde{u}) \sim \mathbb{P} \\ \mathbb{E}_{\mathbb{P}}[\tilde{z}] = 0 \\ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^T \times \mathbb{R}^{(T+1)T/2}) \quad \mathbb{E}_{\mathbb{P}}[\tilde{u}_{st}] \leq \phi_{st}^2 \quad \forall s \leq t; s, t \in T \\ \mathbb{E}_{\mathbb{P}}[(\tilde{z}, \tilde{u})] \in \bar{\mathcal{W}}_{PCM} = 1 \end{array} \right.$$

where the lifted support set $\bar{\mathcal{W}}_{PCM}$ is assumed to be as follows:

$$\bar{\mathcal{W}}_{PCM} = \left\{ \begin{array}{l} (z, u) \in \mathbb{R}^T \times \mathbb{R}^{(T+1)T/2} \quad u_{st} \geq \left(\sum_{r=s}^t z_r\right)^2 \quad \forall s \leq t; s, t \in T \\ z \in [\underline{z}, \bar{z}] \end{array} \right.$$

Recent studies for the dynamic framework have sought to improve the performance of the method to deal with uncertain parameters ensuring the tractability and the trade-off between quality of the solution and computational effort. No work has been done to extend the DRO methodology to uncertainties others than the demand.

1.6.3 Literature review of distributional robust LSPs

DRO hedges against the conservatism of robust solutions by incorporating distributional information while remaining computationally tractable. The optimization community has investigated this methodology as a promising and tractable method leading to satisfactory solutions according to quality and calculation time. Although this section

deals only with the standard case, where the ambiguity set is assumed to be known a priori.

Table 1.4 lists some references addressing single and multi-item LSP via DRO. This is the most recent concept in development, whose studies started in the mid-2000s and have been intensified since 2010. Similarly to the RO, the few studies already published are mostly focused on the uncertain demand. The studies of this methodology are concentrated in a restricted group of researchers. Besides that, its current stage requires strong and restrictive initial hypothesis to guarantee the tractability. Many perspectives of work are encouraged to mature the development of this method. We will address some perspectives in section 1.8.

Reference	Period	Item	Echelon	Capacity	Uncertainty	Decision framework	Inventory shortage
Analui and Pflug [2014]	multi	multi	single	capacitated	demand	st-dyn	
Bertsimas et al. [2019]	multi	single	single	uncapacitated	demand	st-st ; st-dyn	B
Coniglio et al. [2018]	multi	single	single	uncapacitated	demand	st-st	B
Postek et al. [2018]	multi	single	single	uncapacitated	demand	st-st	B
Yang [2017]	multi	single	single	uncapacitated	demand	st-dyn	
Zhang et al. [2016]	multi	single	single	uncapacitated	demand	st-st	LS - B

Table 1.4 – References handling the LSP via DRO

1.7 Comparison of the optimization methodologies

The LSP has been studied for decades, but still has room for improvement and further investigation. Bringing the problem closer to real applications, it is essential to consider uncertainties and their implications in solving the problem. Dealing with uncertainties, three optimization methods have stood out either for the quality of the solution, or for the ease of computing. These methods are: SP, RO and recently the DRO, whose objective is to combine the strengths of the SP with the RO and thus easily calculate an appropriate solution for the decision context. In this section we analyse each approach in terms of strengths, downsides, challenges and applications for handling the LSPs.

The choice between the methods depends on the decision makers' preferences, the instance structures (and size), available information, and the expected trade-off in terms of solution quality and solving time. Based on the existing literature, we analyse the performance of each approach in terms of tractability, conservatism and applicability. We refer to Bertsimas et al. [2011] to define these performance criteria. The *tractability* concerns the possibility of computing the solution with some known algorithm running in a reasonable time. The tractability is directly associated with *scalability* that aims to guarantee reasonable computational effort for large size instances. Finally, the *conservatism* concerns the immunity of the obtained solution to the uncertainties, and to which the ability obtains a feasible solution even if the proposed model is applied to unforeseen events is measured in terms of *flexibility*. These unforeseen events comprise cases of estimation errors, disruptions in the decision context and any other scenarios that were not anticipated.

1.7.1 Principle and maturity stage

SP is at a much more mature stage in the research community, as it has studies that have been widely developed since the 1900s and its features are well-known and understood. RO has been actively investigated since the 2000s, and its studies have been enhanced by Beck and Ben-Tal [2009] study proving that the primal worst case equals dual best. Hence, the tractability is achieved to handle the uncertain decision problems. The static RO reached a theoretical stage that is considered mature or at least well understood as presented in Ben-Tal et al. [2009]. However, even if the formulations were tractable, the solutions were often too much conservative. Therefore, the advances in the static RO have encouraged the research community to address dynamic environment with the ARO [Ben-

Tal et al., 2004], and to consider some distributional information to soften conservatism in the light of the DRO [e.g., Goh and Sim, 2010, Delage and Ye, 2010]. Hence, there are more literature covering the LSP via SP, an increasing rate of studies addressing the problem via RO and since the last decade few studies have investigated the LSP via the DRO. This behaviour has expected as the RO theory still concentrated in a few groups of research while the SP has been largely developed for either the optimization or the mathematical research community. Hence, we can distinguish the same stage of maturity from the methods, both in theory and to deal with the LSP.

1.7.2 Tractability and scalability

The LSP is naturally hard to solve due to its binary decisions on the setup of the production periods. Computing an optimal solution under an SP framework requires full knowledge of the probability distribution of uncertainty. Therefore, SP models often suffer from scalability issues because an appropriate representation of the uncertainty distribution requires a large number of scenarios. Specially, multistage SP is restricted to short planning horizon within a small number of scenarios, since the size of the model grows exponentially with the time horizon and the scenario size [Birge, 1997]. The SP-based models that deal with the LSP assume an exponential behaviour in the number of scenarios and periods to be considered. Even if the static-static decision models can be addressed by reasonable computational time methods, Brandimarte [2006] indicates that the static-dynamic models cannot compute an optimal solution even for medium-sized instances. Besides the hard complexity of the LSP, the multistage formulation is computationally hard even impossible. As a result, the exponential behaviour implies scalability issues that may be addressed via heuristic methods and approaches. As stated in Brandimarte [2006] conclusions, while efficient methods address the static SP framework in reasonable time, further investigation is needed to handle dynamic framework with reasonable computational efforts.

On the contrary, the main advantage of the RO is being computationally tractable for many cases [Chen et al., 2007]. The RO methodology relies on the duality (where primal worst equals dual best) or adversarial approaches to handle the problem as a model solvable in reasonable time even for the large size instances [Beck and Ben-Tal, 2009]. Solyali et al. [2015] claim that most of the RO-based methods are polynomially solvable even via the classical robust methodologies or by reformulating the problem via network design reformulations. This implies that the method is tractable for general instances.

Even if the resolution becomes computationally hard with the growth of the instances, some reformulations manage to compute solutions for large instances and complex models of the problem in a reasonable time.

Finally, the LSP via DRO remain tractable for some known and well-structured ambiguity sets and formulations of the model subject to strong hypothesis as the full knowledge of the structure and geometry of the ambiguity set [Wiesemann et al., 2014]. Even if the uncertainty is in the probability distribution that represent an uncertain parameter, for the medium sized instances the computational effort is reasonable. The computation may become heavier with the growth of the size and when dynamic context is taken into account. Although the published results seem to be promising, further investigation is needed to ensure the performance and the limits of the DRO especially for large-sized and dynamic framework.

1.7.3 Conservatism

As presented in Section 1.5, the RO seeks a solution feasible for the worst case realization of uncertainty. Such an approach leads to over conservative solutions [Bertsimas and Sim, 2004] and, thus, to large costs [Thiele, 2010]. However, the uncertainty set may be adjusted to control the degree of conservatism of the robust solution. Bertsimas et al. Although SP does not suffer from conservatism as the solutions are computed with complete information of a known probability distribution, SP is highly dependent on the quality of the estimation of the distribution. Finally, the DRO stands between SP and RO. On the one hand, DRO assumes partial distributional information of uncertainty and may be not as conservative as RO. On the other hand, DRO stills ignoring some distributional information, and then it cannot be reduced to SP. In fact, DRO maintains a sufficiently high number of possible distributions that represent uncertainty, so that both the usual scenarios and the worst-case scenario can be covered by the ambiguity set.

Dealing with the LSP, as RO disregard the distributional information about the uncertainty and construct its uncertainty set based on some moment information (e.g. mean and standard deviation), the robust solutions are often over-conservative. The conservatism is due to the decomposition of the problem for each period in different constraints that allows the realization of the worst-case of the uncertainty for each constraint. As a result, direct reformulations explore the worst-scenario for each computational step and thus obtain an overly-conservative. However, this conservatism can be controlled by some parameters composing the uncertainty set, such as the budget value into the budget uncertainty set

[Thorsen and Yao, 2017]. In such cases, the parameters controlling the uncertainty can be modified to increase the freedom of realization of the uncertainty and so the size of the uncertainty set, while feasibility stills guaranteed.

The SP-based models assume to fully know the probability distribution of the uncertainty to compute a solution. As a result, the solution is totally dependent on the realization of the uncertainty and there is no conservatism to be considered. Even if the conservatism is not applied to this optimization modeling, we may consider the robustness of the proposed solution in the light of the proper simulation of the scenarios describing the uncertainty [Brandimarte, 2006]. While accurate representation of the uncertainty leads to optimal solutions, if the distribution does not correctly capture the distributive behaviour of uncertainty, a solution far from the nominal problem or even the infeasibility can be obtained.

The DRO is less conservative than the RO, as partial distributional information is taken into account, but it is still conservative with regard to the SP. In fact, the conservatism of the method depends on the modeling proposed and on the quality and accuracy of the available data used to compute a solution. An investigation of the DRO for the LSP may be performed to measure the conservatism of the methodology to the problem, but Shang and You [2018] present some computational evaluations of the conservatism of the DRO in comparison to the ARO methodology. Even if the DRO takes longer computational time than ARO models, they conclude that DRO overcomes the ARO for a large magnitude of uncertainties. DRO often computes sub-optimal solutions based on the affine decision rule approximations while ARO achieves a global optimum. Finally, comparing the first and second decisions determined via both of the methods, they showed a numerical example of scheduling that the solutions proposed by the DRO are susceptible to greater investments that can lead to greater profits. Hence, they demonstrate less conservatism on the part of the DRO method compared to the ARO that optimizes over the worst case scenario. As a matter of the fact, the distributional information taken into account by the DRO, even that partial, achieves better performance of the optimization under uncertainty to approximate the decision to a real case application.

1.7.4 Flexibility

The flexibility of the solution here is highly correlated with the conservatism of the method. Assuming the flexibility as the capability of the solution to be suitable to the realization of an unforeseen uncertainty, we are interested in the performance of the meth-

ods to compute a valid solution even if the revealed uncertainty differs from the set of possible realizations considered for the calculation.

As the RO methodology requires only the support of the uncertain parameter to compute a solution, the method provides the decision makers more flexibility to deal with such unforeseen events. On the contrary, the SP solutions are based on the realization of the uncertainties and their probability distribution, and so are poorly capable to propose a solution that suits to unforeseen or misrepresented events. Although the solution proposed by the SP is often an accurate approximation of the true realization of uncertainty, in case the scenarios do not represent well the behaviour of the uncertain parameter, SP can yield poor decisions, the problem with multistage scenario based SP is that if the realization of the unknown parameter is not part of the sampled scenario, the solution to the multistage SP does not tell how to react. Thus, the decision maker must resolve from scratch. Affine decision rules from RO do not suffer from this issue. At the intersection of previous methods, the DRO incorporate a robust behaviour into the problem against the unknown true distribution. Specifically, the DRO incorporate partial distributional information to approximate the robust perspective to the nominal problem while the formulation remains conservative to unforeseen events.

We illustrate the flexibility of a method to deal with the LSP as the ability of computation of a feasible solution suitable for scenarios out of the sampled or considered ones. As presented by Thorsen and Yao [2017], the RO computes a more stable solution even if the true realization of the uncertainty is out of the considered distribution of the uncertainty. As a result, the flexibility stands as the calculation of a feasible solution that better suits for different possible realizations of the uncertainty with regard to the standard deviation and worst-case scenarios. When we consider the SP, once the solution is based on the distribution of the uncertainty, if the realization is not approximated by the representation, a poor solution can be obtained and the SP performance decreased.

The DRO present promising results in terms of the flexibility of the solution to be feasible or near-optimal to the realization of the uncertainty out-of-sample possibilities in the computation. As presented by Bertsimas et al. [2019], the out-of-sample solutions, when the actual demand is far from the simulated one, are robust and mitigate the impact of the wrong estimation.

1.8 Discussion and future works

As the more mature theory, the SP handle the problem in a more realistic context as its solutions are based on the probability distribution of the uncertain parameter. Although for the static case the solution is easy to calculate, a complete knowledge of the probability distribution is necessary. Thus, the biggest challenge is to tune out the forecasting errors or hypotheses that mislead to solutions. Even if the dynamic framework allows an effective adjustment of the decisions over time, its computation is based on a fully known probability distribution of the uncertainty, what can be difficult even impossible. In addition, the large size instances have an exponential behaviour in the number of scenarios or possible realization of the uncertainty. The multistage LSP model can lead to a prohibitive computational effort. Therefore, the main issue of the dynamic SP is the scalability for the large instances and big number of scenarios for the uncertainty.

Even if the SP is the most developed concept, there are few references dealing with the multi-item problem and little development for uncertainties in parameters other than demand. The multistage framework may be developed not only for the dynamic-dynamic case (which currently has few studies) but also to complete the studies in the static-dynamic case under uncertainty in parameters such as capacity. Further work is envisaged to deal with LSPs under uncertainty in several parameters simultaneously and also to propose heuristics or approaches that turn tractable those formulations via SP for large instances or that have exponential behaviour. Some alternative to mitigate non-optimal solutions due to forecasting errors is needed to incorporate factors that smooth the result in the face of poor representation of uncertainty. A promising technique is to validate the probability distribution of the uncertainty with data-driven approaches based on available data from the decision environment (e.g. historical data, market information). In addition, the development of heuristic or adversarial approaches is required to address the scalability issue and compute satisfactory solutions for large instances in a reasonable time.

RO is an interesting method not only in tractability but also in the feasibility of the solution. Within the static-static framework, the RO allows the computation of a feasible solution for the decision problem whose information about the uncertainty is partial or missing. Not only the tractability is ensured but also the models via RO remain scalable for any instance. Even if this method is highly tractable, it handles only the static decision problems. Besides that, the construction of the uncertainty set can disregard any distributional information about the uncertainty, and so the tractability is often guaranteed at

the expense of the over-conservative solution. In a matter of fact, less conservative solutions can be computed with regard to a non-linear uncertainty set, but this formulation is often computationally hard (even intractable). The main challenges of the RO are then the reduction of the conservatism and the transformation of the intractable uncertainty sets into tractable ones. As the method is in a theoretical maturation stage, little variation of uncertain parameters has been addressed to handle the LSP. Hence, it is necessary to handle different uncertain parameters individually or simultaneously to improve the robust optimization theory.

The ARO is a more flexible version of the RO that handles static-dynamic and perhaps the dynamic framework (whose studies are absent). ARO models alleviate the conservatism updating knowledge about the uncertainty in each stage. The formulations via this method are usually intractable due to the infinite quantity of constraints added to the model when the recourse is affected by the uncertainty. Currently, tractability can be guaranteed by affine linear functions. However, this approach is well structured to continuous variables and only tractable for non-linear formulations addressed with ellipsoidal uncertainty set. Therefore, the main challenge of the ARO methodology is the tractability to handle non-linear modeling to describe the recourse and also the scalability and tractability of the modeling of integer adjustable variables. Further work is demanded to propose tractable formulations for adjustable integer variables, whether dynamic framework is expected. Besides that, tractable approximations to address the intractable formulations is required and the investigation of other geometrical representations of the uncertainty set can be promising perspectives of research. Hence, to handle the LSP, similarly to the RO, tractable formulations dealing with different sources of uncertainty are envisaged as well as the investigation of cases in which different sources of uncertainty occur simultaneously. In addition, the uncertainty representation fine tuned to the decision context can boost the quality of the solution.

Table 1.5 summarizes the advantages and downsides for each method within different decision frameworks when addressing decision problems and in particular the LSP. Most of the research is now focused on developing the robust theory in order to boost the optimization systems with respect to the trade-off between cost, computation time, achievements and requirements.

The joint perspective of the SP and the RO seems to be an interesting trade-off between the intractability of SP for large size problems and the conservatism of the solution obtained by the RO. The DRO modelling is sufficiently flexible to take advantage

Table 1.5 – Overview of the SP, RO and DRO within different decision frameworks

Optimization methodology	Decision framework	Advantages	Downsides	Challenges	Further research
SP	<i>Static-Static</i>	Addresses the problem in a more realistic static context	Require full knowledge on the probability distribution of the uncertain information	(1) Flexibility to handle unforeseen events; (2) Attenuate assumptions or forecast errors	Incorporate into the model alternatives to mitigate unforeseen events or forecasting and assumption errors
	<i>Static-Dynamic</i>	Dynamically addresses the problem allowing effective adjustment of the decisions over time	(1) Require full knowledge on the probability distribution of the uncertain information; (2) Exponential behaviour in the number of considered scenarios or different possibilities of realization of uncertainty	Scalability for a large number of scenarios	Develop heuristics or adversarial formulations to address the scalability issue
RO	<i>Dynamic-Dynamic</i>	Not considered in the scope of this work			
	<i>Static-Static</i>	(1) Computes a feasible solution even for problem whose information about uncertain parameter is partial or missing; (2) Scalable for any problem and computationally tractable for structured uncertainty sets	(1) Addresses problems only within a static context; Conservative solutions are computed with respect to the uncertainty sets; (2) Less conservative in some cases	(1) Reduction of conservatism; (2) Transformation of recently developed intractable uncertainty set into tractable ones	Construction of tractable and less conservative uncertainty sets
ARO	<i>Static-Dynamic</i>	(1) Robust approach addressing problems within a dynamic context; (2) Alternate conservatism updating knowledge about uncertainty in each stage	(1) Intractability for non-linear problems; (Few studies handling non-linear problems);	(1) Tractability to handle non-linear uncertainty sets; (2) Scalability to address integer adjustable	(1) Propose tractable approximations to address intractable uncertainty; (2) Further investigation on non-linear formulations; (3) Further investigation of different uncertainty geometrical representations
	<i>Dynamic-Dynamic</i>	Not considered in the scope of this work			
DRO	<i>Static-Static</i>	Statistically hedges against the conservatism of robust solutions by incorporating distributional information while remains computationally tractable	(1) Computation over strong and restrictive initial assumptions (probability distributions properties are assumed as known even if the distributional information is missing or partial); (2) Computationally hard and even intractable depending on the formulation	(1) Modeling problems for a large variety of ambiguity sets; (2) Ensure the tractability; (3) Further comprehension; (4) Incorporate distributional information over time into the model; (5) Bound or update the ambiguity set according to revealed data	(1) Investigate other ambiguity sets to handle the problem in a tractable way; (2) Data-driven modeling
	<i>Static-Dynamic</i>	Addresses dynamic problem considering a tractable robust computation with respect to distributional information updated over time	(1) Behaviour known only with respect to distributional information updated over time	Dynamic development of the concept in parallel with the theoretical development of one of its pillars (ARO)	Complete investigation to explore the challenges
	<i>Dynamic-Dynamic</i>	Not considered in the scope of this work			

of the behaviour and best performance of each of its base methods according to the context and the decision environment. This means that some distributional information about the available data is introduced into the model through reformulations within robust optimization theory that is tractable in general. Hence, the exponential behaviour of SP is mitigated while the robust counterpart model remains tractable in the light of affine policies and dualization. The downsides of this method are the strong and even restrictive initial assumptions to compute a tractable solution. Besides that, the computational effort can be hard and even intractable depending on the formulation. Therefore, the main challenges of the DRO are: formulate a tractable model, model decision problems for a wide range of ambiguity sets and fine tune the available distributional information into the uncertainty representation.

Further work is needed to address the construction of different ambiguity sets to handle distinct LSP uncertainties in a tractable way. Within the multistage framework, further investigation is required to update the problem model with respect to the revealed uncertainty. In general, further comprehension of this methodology is required to properly investigate any decision problem via the DRO methodology. As a new methodology, its behaviour and performance so far are guaranteed under many strong assumptions, then an exploration of its capabilities and limitations is necessary not only via the methodology itself but also applied to some decision problems such as the LSP.

We notice that further investigation is needed to address the dynamic-dynamic decision framework via the three presented methods. Even if it adds difficulty to the computing of the solution, the dynamic-dynamic strategy is the framework the closest to the real decision context and environment. It is expected that a solution via this framework is more suitable for an optimal system performance. In addition, further investigation is needed to incorporate the distributional information revealed over time into the optimization models. A promising approach consists of updating and bounding the uncertainty representation within the optimization methodology according to processing and analysis data-driven techniques applied to the revealed data to boost the computation of a solution. Hence, the data-driven optimization should be more explored to improve the model efficiently and over time and complete work on the dynamic framework is envisaged to address the LSP in a more practical way via the three addressed optimization methodology.

The data-driven optimization concerns a promising manipulation, processing and analysis techniques aiming to integrate knowledge about data into the optimization models.

Hence, a less conservative solution is aimed while the other features (specifically tractability, flexibility and scalability of the model) should remain stable. The data-driven approaches can help excavating rich information from available data. By leveraging data information, these approaches can effectively hedge against uncertainty since the conservatism of solutions can be reduced and better solutions can be computed. In the light of the data-driven approaches, the optimization methodology can learn the distribution of through data processing and then formulate the uncertainty modeling. As a continuation of this research direction, one may be interested in anticipating scenarios or even the realization of the uncertainty through data-driven techniques.

1.9 Contributions and Final remarks

To understand this manuscript contribution, we should understand the importance of the industrial quality control activity. In fact, the production yield rate is one of the key performance index that indicates the performance and efficiency of the production system. A satisfactory production yield rate gives us a lot size with a sufficient quantity of products with good quality, which is also called compliant products. This later results in maintaining contracts with the same customers. Thus, it increases customer loyalty. It also helps to reduce the liability risk, for which the vulnerability of the system to loss performance, quality or profits is reduced since a good control and safe maintenance of the production system is established. Finally, its improve the competitiveness of the company in the market supported by positive customer reviews.

This manuscript provides the first studies on robust, robust adaptive and distributionally robust optimization methodologies to solve the LSP under yield uncertainty. The contributions are the following: first this present chapter presented a state of the art whose complexity is to gather studies and notations on the lot-sizing problems under uncertainties and to give the recent trends on the methodologies to handle them in a data- driven perspective. A second contribution, presented in Chapter 2, is the development of a first robust lot-sizing problem under yield uncertainty. The complexity there is to understand the impact of the production yield on the lot-size in order to obtain optimal robust plans. The third contribution, given in Chapter 3 consists is the extension of the robust model to the adaptive decision strategy. The complexity there is a problem quadratically uncertain and some alternative approach should be developed to compute a sufficiently robust adaptive plan. Finally, Chapter 4 indicates how to directly integrate real data in the decision process with the application of the distributionally robust optimization to deal with the problem. The complexity there was to tackle with the uncertain production yield whose distributions can be discrete and continuously estimated.

ROBUST LOT-SIZING PROBLEMS UNDER YIELD UNCERTAINTY

This chapter is based on the following article:

Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. (2023). Robust optimization for lot-sizing problems under yield uncertainty. *Computers & Operations Research*, 149, 106025.

Production yield can be highly volatile and uncertain, especially in industries where exogenous and environmental factors such as the climate or raw material quality can impact the manufacturing process. To address this issue, this chapter proposes multi-period, single-item lot-sizing problem with backorder under yield uncertainty and static strategy via a robust optimization methodology. First, we formulate a robust model under a budgeted uncertainty set, which is optimized under the worst case perspective to ensure the feasibility of the proposed plan for any realization of the yield described by the uncertainty set. Second, we analyze the structure of the optimal lot-sizing solution, and we derive the optimal robust policy for the special case of the inventory management problem under a box uncertainty. These results help us develop a dynamic program with polynomial complexity for the lot-sizing problem with stationary yield rate. Finally, extensive computational experiments show the robustness and effectiveness of the proposed model through an average and worst case analysis. The results demonstrate that the robust approach immunizes the system against uncertainty. In addition, a comparison of the robust model with the nominal model, the deterministic model with safety stock, and the stochastic model shows that the robust model balances the costs better by reducing the backorders at the expense of more often producing a larger amount of goods.

2.1 Introduction

Many industries constantly face exogenous factors that can affect the quality of their products. In addition, the new products have become increasingly more complex with shorter life cycles, and product customization breaks the regularity of the production process and increases the number of failure sources [Duncan et al., 2021]. It becomes particularly challenging to precisely estimate the production yields that are necessary in the production process. In this context, accounting for yield uncertainty is crucial because an underestimation of the production yield leads to excess inventory, whereas an overestimation creates significant stock-outs [Yano and Lee, 1995, Khouja, 1999].

The production yield incorporates quality factors into the lot-sizing model by measuring the expected quantity of non-defective items resulting from the release of a given production lot. Traditionally, this production yield is estimated based on historical data or machine specifications, but such estimations can be imprecise. In practice, the production yield is subject to multiple sources of uncertainty, such as deviations from standard operating procedures, environmental factors such as temperature and humidity, failures in the machinery of the system, a lack of a quality control system, material imperfections, process dysfunction, workforce inefficiency [Grosfeld-Nir and Gerchak, 2004]. The yield uncertainty concerns the inability to predict precisely the output quantities associated with a lot size. This uncertainty occurs in many industries, such as electronics [e.g., Akella et al., 1992, Schemeleva et al., 2018], pharmaceuticals [Cho, 2010], food [Kazaz, 2004], agriculture [e.g., Jones et al., 2001, Anderson and Monjardino, 2019, Roell et al., 2020], steel and metallurgical industries [Lalpoor et al., 2011], as well as in remanufacturing processes [Panagiotidou et al., 2013].

Lot size decisions are a crucial step in production planning when aiming to meet customers' needs and minimize the overall costs [Dolgui and Proth, 2010]. Although modelers often rely on deterministic variants of lot-sizing problems (LSPs) based on the hypothesis that all data is known or can be correctly predicted, in practice, many parameters are uncertain [Jans and Degraeve, 2008]. The discrepancy between data estimation and their actual values can have a critical impact on the quality of the lot-sizing decision. There is a need to develop LSPs that take these uncertainties into account. In static strategy, the production decisions are fixed for the entire horizon [Bookbinder and Tan, 1988]. Therefore, this work provides the first in-depth investigation of the static strategy for robust LSPs under yield uncertainty. Since the static solution is an upper bound of the adaptive

solution, our contributions will serve as a basis for future works that address the adaptive strategy.

A static strategy has a practical relevance to reduce system nervousness at minimal computational effort [Blackburn et al., 1986]. Furthermore, Sereshti et al. [2021] indicates that static lot-sizing plans do not experience nervousness, as the amount of assembly and production remains the same despite the realization of uncertainty. The authors also report that the static strategy combined with the receding horizon is a good approximation of the robust adaptive strategy where some decisions are updated to react to the realization of the uncertainty in the previous periods. Thus, the static solution can be executed in a rolling horizon framework to tackle the dynamic strategy [Blackburn et al., 1986, Sereshti et al., 2021].

The yield uncertainty may have different impacts depending on the situation, such as an increase in the production costs, processing duration, or lead times, and it often results in a waste of materials and available resources. The consequences of these losses can be highly damaging to the system [Inderfurth and Vogelgesang, 2013]. Most studies on LSPs with uncertain yield consider the single-item single-period problem [Yano and Lee, 1995, Khouja, 1999]. In this simple setting, the optimal lot size can be derived through a mathematical analysis based on the newsboy inventory management model [Khouja, 1999]. This technique, however, cannot be applied in a more general context of multi-period lot-sizing, and it can lead to poor solutions [Yano and Lee, 1995].

This chapter proposes a methodology based on robust optimization for the non-stationary multi-period LSP under yield uncertainty, and we analyze it in terms of its applicability, optimality, and efficiency. To the best of our knowledge, we are the first to consider robust optimization for an LSP under yield uncertainty in a non-stationary production context, where the production parameters such as costs, demands, and production yield rates may change at each production period. The contribution of this chapter is fourfold. First, we derive an optimal policy for the stationary case of the nominal and maximum deviation values of the uncertain yield. This special case considers the box uncertainty set without setup, much like the case for the demand uncertainty presented by Bertsimas and Thiele [2006a]. Second, we propose a polynomial-time dynamic programming algorithm for the special case of the lot-sizing problem with stationary yield rate, non-stationary costs and demands, and box uncertainty set. Third, we propose a robust optimization formulation for a non-stationary multi-period LSP under budgeted uncertainty set and yield uncertainty. Finally, we perform an in-depth analysis of the re-

sulting methods in terms of the quality of the solution, scalability, stability, robustness, and flexibility. In particular, we compare the production plans resulting from the robust, nominal, deterministic with safety stock in, and stochastic models. Although the robust models guarantee the robustness and feasibility of the proposed plan, the stochastic programs seek the production plan with the best expected costs. In addition, the deterministic problem with safety stock aims to reduce the risk of shortages. Thus, we intend to analyze when each technique is best suited to deal with production yield uncertainty and to avoid nervousness in the process.

The current chapter is organized as follows: Section 2.2 gives a review of previous work on non-deterministic lot-sizing problems, with a focus on uncertain production yield. Section 2.3 formally describes the considered problem and introduces the robust optimization methodology, more specifically, the mixed-integer linear formulation for the non-stationary case of the problem, the optimal robust properties and policies, and a dynamic programming formulation for the problem with stationary yield and non-stationary costs and demands. Section 2.4 presents the instances and simulation framework used in our experiments, as well as the experimental results. Finally, Section 2.5 concludes this work and provides some future research directions.

2.2 Literature Review

The production yield was first introduced by Bowman [1955], and it was initially measured as the proportion of items that were accepted, and that reached a high enough quality to be sent to costumers. We use the same notion to capture the proportion of good quality items in a lot. Studies on robust LSPs under yield uncertainty are scarce. Unlike the case of uncertain demand for which the quantity of quality goods obtained from the production is known, yield uncertainty affects the quantity of quality items obtained from a production lot. The dependence of the optimal amount of quality goods on the realization of the production yield increases the complexity of the adaptive formulation for the LSP [Ben-Tal et al., 2005]. Since the production yield multiplies the decision variable, the adaptive robust model is intractable. Therefore, it is important to derive solutions within a static decision strategy to understand the impact of the uncertain yield on the lot-sizing decision in a context with low or no nervousness, and to solve it with tractable models. We refer the reader to Yano and Lee [1995] for more on an LSP under yield uncertainty.

To the best of our knowledge, Kazemi Zanjani et al. [2010] and Quezada et al. [2020] are the only works on stochastic programming for an LSP under yield uncertainty; and there is no existing study on robust optimization for lot-sizing under yield uncertainty in a production planning context. While Kazemi Zanjani et al. [2010] address the uncertain yield in terms of quality of raw materials, Quezada et al. [2020] formulate the production yield of refurbished items in a remanufacturing context. These show that the stochastic method is efficient within a static strategy because it minimizes the occurrence of the backorder. However, the stochastic models require the use of a sufficiently large scenario set to approximate properly the underlying distributions, and they may not scale well. Considering the robust optimization approach, Vayanos et al. [2012] tackle the stationary inventory management model with an uncertain production yield and fixed inventory and backorder costs, but they ignore the setup decisions and production costs. The authors propose a constraint sampling approximation to mitigate over conservative solutions. In a similar approach, Chu et al. [2019] propose a robust model for the procurement perspective of the stationary inventory management problem. They restrict the maximum value of the production yield to its nominal value, and they show that the problem can be formulated as a nominal problem with modified deterministic demand in terms of the accumulated deviation of both the uncertain demand and uncertain yield. The authors analyze the

impact of the budget by controlling the uncertainty and average and standard deviation of the uncertain parameters on the average performance of the robust models. Even though Chu et al. [2019] propose an insightful analysis of the inventory management problem with uncertain yield, they perform their studies in a procurement perspective, for which the production yield is set to its nominal value. Thus, Chu et al. [2019] do not allow obtaining an amount of quality goods larger than the nominal ordered quantities to be obtained. Hence, we note a lack of studies on the application of the robust optimization methodology to the LSPs with yield uncertainty. In addition, contrary to our study, Chu et al. [2019] do not compare the performance of the robust optimization approach with the stochastic programming and deterministic models.

This work aims to fill the knowledge gap on the impact of the uncertain production yield on a multi-period lot-sizing problem within a robust perspective for a non-stationary case of production yield, costs and demands. Our work differs from the aforementioned literature in several aspects. First, to the best of our knowledge, this is the first work that formulates the non-stationary case of single-item and multi-period LSPs under yield uncertainty via robust optimization. Second, we derive the optimal robust policies for the single-period and multi-period LSPs with stationary nominal value and maximum deviation of the yield rate, with non-stationary inventory/backorder/production costs and demands, and with zero setup costs. Third, we propose a polynomial-time dynamic programming algorithm based on optimal robust policies to solve the considered problem for the special case where setup costs are considered. This helps us provide simple tools to compute a sufficiently robust plan, better suited for small decision settings or when an efficient commercial solver is not available. Fourth, we give an in-depth analysis of the impact of robust optimization for production planning based on numerical experiments. The results show that robust optimization is highly efficient and produces production plans that are more robust to different yield scenarios when compared to other decision-making methods (e.g., stochastic program, nominal problem, deterministic problem with safety stock) in terms of quality of the solution, cost savings, and robustness to changes or uncertainties in the system.

2.3 Robust Optimization

This section introduces the considered model and the solution approaches. For the sake of clarity, we present the nominal lot-sizing problem prior to introducing the robust optimization models under an uncertainty set. The robust optimization methodology immunizes the problem from uncertainties by computing a solution that remains feasible for any realization of the uncertainty within the uncertainty set [Bertsimas et al., 2011]. First, we provide the robust optimization model for the LSP under yield uncertainty. Second, we derive a closed-form solution for the single-period model with the box uncertainty set. Next, we show how this closed-form solution can be extended to the multi-period inventory management problem, where we provide an optimal policy for the inventory management problem. Based on the optimal policy for the inventory management problem, we derive properties on the optimal solution of the lot-sizing problem. Finally, the aforementioned properties allow us to develop dynamic programming algorithm to compute optimal solutions for the LSP with uncertain yield, with non-stationary costs and demands, and with stationary nominal value and maximum deviation of the production yield.

2.3.1 Nominal problem

A single-item multi-period uncapacitated LSP with backorder and production yield determines the quantity to produce in each period of the finite planning horizon $T = \{1, \dots, |T|\}$. The objective is to meet demands efficiently and with quality goods while minimizing the overall costs. For each period $t \in T$, we are given the setup cost s_t , the unit production cost v_t , the inventory holding cost h_t , the backorder cost b_t , and the demand d_t . The model comprises the following decision variables: the lot size X_t to be produced, the inventory level I_t and the backorder level B_t at the end of the period, and the setup decision Y_t , such that $Y_t = 1$ if a setup occurs in t ($X_t > 0$) and $Y_t = 0$ otherwise. We define ρ_t as the strictly positive production yield in period t , such that $0 < \rho_t \leq 1$.

The formulation of the deterministic LSP with production yield is as follows:

$$\min \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t + b_t B_t \quad (2.1)$$

s.t. :

$$I_t - B_t = I_{t-1} - B_{t-1} + \rho_t X_t - d_t \quad \forall t \in T \quad (2.2)$$

$$\begin{aligned} X_t &\leq M_t Y_t && \forall t \in T && (2.3) \\ X_t, I_t, B_t &\geq 0 && \forall t \in T \\ Y_t &\in \{0, 1\} && \forall t \in T \end{aligned}$$

Without a loss of generality, we assume that there is no stock or backorder at the beginning of the planning horizon. The objective function (2.1) minimizes the total cost comprising the setup, unit production, inventory, and backorder costs. The inventory balance constraints (2.2) compute the level of backorder and inventory in period t from the demand, the produced goods at period t , and the inventory and backorder levels in period $t - 1$. The constraints (2.3) are setup-forcing constraints that relate the production quantities (X_t) to the setup decisions (Y_t), whereas $M_t = \frac{\sum_{t \in T} d_t}{\min_{\tau < t} \rho_\tau}$. These constraints set the setup variable Y_t to 1 if any production occurs in period t , and the setup remains inactive otherwise ($Y_t = 0$). In addition, constraints (2.3) can represent the capacity constraint by setting $M_t = \min\{C_t, \frac{\sum_{t \in T} d_t}{\min_{\tau < t} \rho_\tau}\}$, where C_t is the available capacity in period t .

2.3.2 Definition of the uncertainty set

The robust optimization considers that the uncertain parameter belongs to a bounded and convex uncertainty set. However, the tractability of the robust counterpart model depends on the tractability of the uncertainty set [Bertsimas and Brown, 2009]. Modelers often rely on some statistical consideration of historical data or previous knowledge of the studied system. The first robust optimization models used the *box uncertainty set*, introduced by Soyster [1973], which describes the uncertainty within an interval of possible values and which is bounded by its minimal and maximal acceptable realizations. To alleviate the over-conservatism of this approach, Bertsimas and Sim [2004] propose the *box polyhedral uncertainty set*, which is also known as the *budgeted uncertainty set*, where the uncertain parameter takes values within a range of values whose size is controlled by the decision-maker through a budget of uncertainty Γ . This budget of uncertainty reduces the size of the uncertainty set and reflects the degree of risk aversion of the decision-maker. The budget is a threshold for the number of uncertain parameters that can take their worst value [Thiele, 2010]. This budget indicates a degree of acceptable variance of the uncertainty from its nominal value, where the larger the budget, the more averse to risk the decision-maker is. Note that the box uncertainty corresponds to the budgeted uncertainty set with the largest Γ value.

In the present work, we consider the widely adopted budgeted uncertainty set that

is given by $\mathcal{U}_t = \{-1 \leq Z_\tau^t \leq 1 : \sum_{\tau=1}^t |Z_\tau^t| \leq \Gamma_t; t \in T; \tau \leq t\}$. This set is based on the nominal value and maximum deviation of the uncertain yield because these values are largely used in statistical quality control to bound the quality in terms of the key performance indicators [Godfrey, 1999]. These values are easily obtained from historical data, and they reflect the basic characteristics of the uncertain parameter. We estimate the uncertain production yield $\tilde{\rho}$ through a natural parameterization $\tilde{\rho}(Z) = \bar{\rho} + Z\hat{\rho}$, with $Z \in [-1, 1]$. Here, the uncertain yield belongs to a range centered on its nominal value $\bar{\rho}$ and spread by its maximum deviation $\hat{\rho}$. The disturbance arising from the nominal value is given by the term Z . Therefore, we replace the production yield ρ_t in constraints (2.2) by the uncertain production yield $\tilde{\rho}_t$.

2.3.3 A robust counterpart formulation for the LSP with uncertain yield

In similar manner to Bertsimas and Thiele [2004], we rely on the reformulation per constraint approach and robust formulation under a budgeted uncertainty set to determine the production plan that minimizes the total costs for the LSP under yield uncertainty. The robust model is similar to the nominal model, but the inventory balance constraints are replaced by a pair of inequalities based on the convexity and the piecewise linearity of the inventory and backorder cost functions. Because the backorder corresponds to a negative stock level, these costs are complementary [Bertsimas and Thiele, 2004]. We can then drop the inventory and backorder variables, so we directly compute the inventory and backorder costs according to the difference between the number of quality goods and demand. Thus, constraints (2.5) and (2.6) replace constraints (2.2), and H_t represents either the inventory or backorder cost in period t , and constraints (2.5) (resp. (2.6)) compute the worst case inventory (resp. backorder) costs. In addition, the inventory and backorder costs constraints are subject to yield uncertainty, and they are optimized under the worst case realization of the uncertain parameter in the uncertainty set. Since the reformulation per constraint and dualization approach is applied, H_t is independently defined for each period t as the highest cost between the worst inventory cost and the worst backorder cost under the uncertainty set \mathcal{U}_t . While the adversarial approach usually requires more computation time than the dualization method, it yields less conservative lot-sizing solutions than the reformulation per constraints and dualization approach. In the reformulation per constraint and dualization approach, the worst case is computed independently in each period (the computation of the inventory/backorder costs in each period may rely on different yield realizations), whereas the adversarial approach finds

a unique worst case yield vector for all time periods. As a result, the total cost of the robust model includes the setup cost, the production costs, and the maximum periodic inventory and backorder costs for any production yield value in the uncertainty set \mathcal{U}_t for each period t . Therefore, we can model a robust counterpart for the LSP with uncertain yield as follows:

$$\min \sum_{t \in T} (s_t Y_t + v_t X_t + H_t) \quad (2.4)$$

s.t. :

$$H_t \geq \max_{\tilde{\rho} \in \mathcal{U}_t} \left[h_t \sum_{\tau=1}^t (\tilde{\rho}_\tau X_\tau - d_\tau) \right] \quad \forall t \in T \quad (2.5)$$

$$H_t \geq \max_{\tilde{\rho} \in \mathcal{U}_t} \left[-b_t \sum_{\tau=1}^t (\tilde{\rho}_\tau X_\tau - d_\tau) \right] \quad \forall t \in T \quad (2.6)$$

$$X_t \leq M_t Y_t \quad \forall t \in T$$

$$X_t, H_t \geq 0 \quad \forall t \in T$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

where the upper bound on the production quantity M_t can be set based on the lowest possible value of the production yield, that is, $\min_{\tau \leq t} (\bar{\rho}_\tau - \hat{\rho}_\tau)$. Thus, for an LSP under yield uncertainty, M_t is set to $M_t = \frac{\sum_{t \in T} d_t}{\min_{\tau \leq t} (\bar{\rho}_\tau - \hat{\rho}_\tau)}$.

2.3.4 A robust mixed integer linear formulation for the LSP with uncertain yield reformulation

Since our robust counterpart is a constraint-wise formulation, for which the uncertain yield occurs on the right side of the inventory and backorder cost constraints, it can be addressed via the reformulation per constraint and dualization approach. In addition, we rely on the budgeted uncertainty set to alleviate the conservatism with a sufficiently good budget of uncertainty. We only show the detailed steps of reformulation per constraint for inventory inequalities (2.5), because the application for backorder inequalities is analogous. In fact, the inequalities differ only by the sign and the costs associated with the inventory level. Thus, the inventory level is negative in case of a backorder, and positive if the production exceeds the demand. As a result, within the worst case perspective of the robust approach, the optimal plan corresponds to a decision leading to a higher cost among these two groups of constraints.

Constraints (2.5) are nonlinear because of the maximization function. Therefore, the reformulation per constraint allows us to transform the non-linear constraints into linear ones. To avoid the inclusion of all possible quantifiers of the uncertainty in the uncertainty set and to linearize the constraints, the reformulation per constraint and dualization approach consists of three steps: [1] reformulation of the constraints subject to uncertainty as a worst case reformulation; [2] dualization of the reformulation problem; and [3] dropping the dualized reformulation into the initial formulation without the inner optimization term [Ben-Tal and Nemirovski, 2000].

The first step is the worst case reformulation of Equation (2.5) that becomes:

$$H_t \geq h_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau) + \max_{\mathbf{Z} \in \mathcal{U}_t} \sum_{\tau=1}^t \hat{\rho}_\tau X_\tau Z_\tau^t \right] \quad \forall t \in T \quad (2.7)$$

For the inventory cost constraints, the worst case scenario occurs only when the deviation is positive, that is, when $Z_\tau^t \geq 0$ for $t \in T$; $\tau \leq t$. Then, we perform the second step to obtain a dual for the worst case reformulation. Assuming that λ and μ are the dual variables, by following the dualization technique presented by Beck and Ben-Tal [2009], we obtain the follow primal-dual problem:

$$\begin{array}{ccc} \max \sum_{\tau=1}^t \hat{\rho}_\tau X_\tau Z_\tau^t & & \min \Gamma_t \lambda_t + \sum_{\tau=1}^t \mu_\tau^t \\ \sum_{\tau=1}^t Z_\tau^t \leq \Gamma_t & \xrightarrow{\text{dualized}} & \lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau \quad \forall \tau \leq t \\ 0 \leq Z_\tau^t \leq 1 \quad \forall \tau \leq t & & \lambda_t, \mu_\tau^t \geq 0 \quad \forall \tau \leq t \end{array}$$

We can finally perform the third step, and replace the worst case reformulation with its dual formulation. Therefore, constraints (2.7) are reformulated as follows:

$$H_t \geq h_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau) + \min_{\lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau} \left(\Gamma_t \lambda_t + \sum_{\tau=1}^t \mu_\tau^t \right) \right] \quad \forall t \in T$$

Similarly, the application of the reformulation per constraint and dualization in constraints (2.6) leads to the following reformulation:

$$H_t \geq -b_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau) - \min_{\lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau} \left(\Gamma_t \lambda_t + \sum_{\tau=1}^t \mu_\tau^t \right) \right] \quad \forall t \in T$$

Note that for each period t , the computation of the worst case inventory cost in Equation (2.5) maximizes $\sum_{\tau \leq t} \hat{\rho}_\tau Z_\tau^t$, while the computation of the worst case backorder cost in Equation (2.6) seeks to minimize this quantity. The uncertainty set \mathcal{U}_t is symmetric in Z_τ^t

for the inventory and backorder cost constraints. This symmetry is due to the convexity and the piecewise linearity exploited in the inventory balance constraints. As a result, the worst realization of $Z_\tau^t \in [-1, 1] \forall \tau \leq t$ for constraint (2.6) are the opposite of the values obtained for constraint (2.5), and no feasible production yield in the uncertainty set realizes both constraints at once. Thus, we can use the same dual variables (λ_t and μ_τ^t) for both cases, but they have a positive coefficient in constraints (2.9) and negative in constraints (2.10).

From the duality theory, since the primal worst equals dual best holds [Beck and Ben-Tal, 2009], it only takes at least one pair of feasible dual variables in the inner minimization to ensure that the solution holds for the outer minimization. Therefore, we can drop the minimization terms of the reformulated constraints. Hence, we obtain the final reformulation of the robust counterpart under budgeted uncertainty set, which is given as follows:

$$\min \sum_{t \in T} s_t Y_t + v_t X_t + H_t \quad (2.8)$$

s.t. :

$$H_t \geq h_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau + \mu_\tau^t) + \lambda_t \Gamma_t \right] \quad \forall t \in T \quad (2.9)$$

$$H_t \geq -b_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau - \mu_\tau^t) - \lambda_t \Gamma_t \right] \quad \forall t \in T \quad (2.10)$$

$$\lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau \quad \forall t \in T; \tau \leq t \quad (2.11)$$

$$X_t \leq M_t Y_t \quad \forall t \in T$$

$$X_t, H_t, \lambda_t \geq 0 \quad \forall t \in T$$

$$\mu_\tau^t \geq 0 \quad \forall t \in T; \tau \leq t$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

Our final robust reformulation shares some similarities with the model for the inventory management problem with uncertain demand proposed by Bertsimas and Thiele [2004]. However, in the model for uncertain demand, the deviation affects the uncertain parameter (demand) only. On the contrary, in our model, the deviation corresponds to the worst case impact of the uncertain parameter on the resulting production quantity.

2.3.5 Properties of an optimal robust policy for the inventory management problem

This section derives a closed-form solution and the optimal robust policy for special cases of the considered problem. The closed-form solution can be determined for the inventory management problem under the following assumptions. First, we consider the inventory management problem where the setup and the unit production cost are equal to zero, while the inventory and backorder costs are non-stationary. Second, we consider a stationary case of the nominal value and maximum deviation of the yield. Third, our policy is derived from the inventory management problem under a box uncertainty set, where the box uncertainty set is given by $\mathcal{U}_t = \{-1 \leq Z_\tau^t \leq 1 : \sum_{\tau=1}^t |Z_\tau^t| \leq t, t \in T, \forall \tau \leq t\}$. Fourth, we assume that the unit inventory cost h_t is lower than the unit backorder cost b_t , and that demands are positive to obtain a positive production quantity X_t . The first two assumptions are common in the inventory management literature, and they are also often encountered in practice. Furthermore, although the worst case perspective from a model under a box uncertainty set is a conservative approach to handle the robust LSP with uncertain yield, it helps us derive the property of an optimal policy to compute a fully immunized plan.

This section is organized as follows. First, we prove that the optimal policy for the inventory management problem under yield uncertainty sets the production quantity such that the worst case inventory cost equals the worst case backorder cost. Second, we derive the optimal policy for the single period problem with uncertain yield. Third, we extend the single period policy for the multi-period inventory management problem. Fourth, we show that these policies remain valid for the LSP that accounts for unit production costs under mild assumptions. Finally, we indicate the need to anticipate the impact of uncertain yield on future costs in order to extend the policy to the budgeted-based model and the problem with non-stationary yield information.

Based on Proposition 2.3.1 we obtain the property for an optimal solution for the inventory management problem with uncertain yield, Proposition 2.3.2 gives the optimal policy for the single-period problem.

Proposition 2.3.1. *The optimal production quantity for the inventory management problem is at the point where the worst inventory cost is equal to the worst backorder cost.*

Proof. See A.1 □

Proposition 2.3.2. *The optimal robust policy for the stationary case of a single-period inventory management problem under box uncertainty set and yield uncertainty is as follows:*

$$X = \frac{d}{\bar{\rho} + \hat{\rho} \left(\frac{h-b}{h+b} \right)}$$

Proof. See A.2 □

We extend our analysis to the multi-period inventory management problem under box uncertainty set and yield uncertainty, with stationary nominal value and maximum deviation of the yield, with non-stationary costs, and with no setup costs. It leads to Proposition 2.3.3.

Proposition 2.3.3. *The optimal robust policy for the multi-period inventory management problem under box uncertainty set and yield uncertainty, with no setup costs, with stationary nominal value and maximum deviation of the production yield, non-stationary inventory, backorder and unit production costs is given as follows:*

$$X_t = \frac{\bar{\rho}d_t + \hat{\rho} \left[\left(\frac{h_{t-1}-b_{t-1}}{h_{t-1}+b_{t-1}} \right) \bar{D}_t - \left(\frac{h_t-b_t}{h_t+b_t} \right) \bar{D}_{t-1} \right]}{\left(\bar{\rho} + \hat{\rho} \left(\frac{h_t-b_t}{h_t+b_t} \right) \right) \left(\bar{\rho} + \hat{\rho} \left(\frac{h_{t-1}-b_{t-1}}{h_{t-1}+b_{t-1}} \right) \right)}$$

if the following condition is respected: $v_t \leq (\bar{\rho} - \hat{\rho})b_t, \forall t \in T$.

Proof. See A.3 □

Note that for the special case of the multi-period inventory management problem under box uncertainty set and yield uncertainty, with stationary production, inventory and backorder costs (i.e., $v_t = v_{t-1} = v$, $h_t = h_{t-1} = h$ and $b_t = b_{t-1} = b$) and without setup cost, the optimal robust policy can be simplified as follows:

$$X_t = \frac{d_t}{\bar{\rho} + \hat{\rho} \left(\frac{h-b}{h+b} \right)}$$

Note also that this formula is a direct extension from Proposition 2.3.2, if we replace d_t with d . In addition, if $v = 0$, then the condition $v \leq (\bar{\rho} - \hat{\rho})b$ is always valid.

Proposition 2.3.3 does not hold for the case with non-stationary nominal value and maximum deviation of the production yield. The policy described in Proposition 2.3.3 is myopic in the sense that the computation of X_t does not consider parameter values at period later than t (i.e., $v_\tau, h_\tau, b_\tau, d_\tau$, for $\tau > t$). We explain with an example that

a myopic policy cannot be optimal if the nominal value and maximum deviation of the yield are not constant. Let us consider an extreme case for three periods, with $h = 1$, $b = 10$, $\bar{\rho} = (0.55, 1, 0.6)$, $\hat{\rho} = (0.45, 0, 0.4)$, $d = (15, 10, 25)$, $s = v = 0$. The robust model returns an optimal value equal to 175 with a lot size $X = (0, 50.0, 0)$, where the production for the entire production horizon is done in the second period for which the yield is maximum. A myopic policy cannot obtain the aforementioned optimal solution, because this solution requires not performing the setup in period 1 to favor production in period 2, where the deviation of the production yield from its nominal value is zero. The policy should anticipate the occurrence of better production yields in later periods for the definition of an optimal production quantity that respects the inventory balance constraints. However, it is not trivial to impose such condition to obtain an analytical solution.

While Proposition 2.3.2 can be adapted to take into account the budgeted uncertainty set, the extension of Proposition 2.3.3 becomes more complicated. For the single period problem, the budgeted set restricts only the range of the maximum deviation, which leads to a worst case scenario that changes from $|Z| = 1$ to $|Z| = \min\{\Gamma, 1\}$. If the optimal policy for the budgeted uncertainty set in the single-period model can be obtained by replacing $\hat{\rho}$ by $\min\{\hat{\rho}; \hat{\rho}\Gamma\}$, that is not the case for the Proposition 2.3.3. Returning to the example from the previous paragraph, we set the nominal value to $\bar{\rho} = 0.55$ and the maximum deviation to $\hat{\rho} = 0.05$. To transform the box-based formulation into a budgeted-based formulation, we replace $\hat{\rho}_t$ by $\hat{\rho}_t\Gamma_t$ in the policy given in Proposition 2.3.3. Assuming $\Gamma_t = 0.5t$, our policy gives a lot size $X = (28.33, 19.64, 51.16)$ which leads to a budgeted-based optimal value equal to 12, while the robust model returns a budgeted-based optimal value equal to 10 with a lot size $X = (28.33, 19.24, 47.97)$. The budget of uncertainty prevents setting all the values of Z_t to 1 or -1 . In each period, the value of Z_t changes such that it takes the highest value for the period with largest production quantity. However, the disturbance values are period independent. As the value of Z_t changes depending on the decided production quantity, the anticipation of the occurrence of better production yields in later periods is also needed to compute an optimal lot. Similarly to the case of non-stationary nominal value and maximum deviation of the yield rate, it is complex to define an analytical solution.

Our myopic policy gives an optimal solution for the case of stationary nominal value and maximum deviation under a box uncertainty set. In Section 2.3.6 we provide a dynamic program based on the optimal policy to solve the stationary LSP under yield

uncertainty with non-stationary demands and costs (including the setup and production costs). As our myopic policy does not optimize within a global perspective of the planning horizon, it does not anticipate the occurrence of better production yields for later periods, and so it gives a sub-optimal solution for the case of non-stationary nominal value and maximum deviation of the yield rate, and also for the budgeted-based robust solution.

2.3.6 Dynamic programming for the uncapacitated robust LSP with uncertain yield and static costs

In this section, we propose a dynamic programming algorithm to address the lot-sizing problem with non-stationary costs and demands, and with a stationary nominal value and maximum deviation of the yield. The proposed dynamic programming algorithm extends the method of Zangwill [1969] that computes a solution from a succession of regeneration intervals. The approach of Zangwill [1969] is based on the property that between any two periods with production there is a regeneration period where the inventory level is 0. This property is not true in the robust LSP under yield uncertainty, since we consider backorders and uncertain yield. However, Proposition 2.3.4 below shows that between two periods with production there is a point where the worst inventory cost is equal to the worst backorder cost. We define such a period as a *regeneration period*. In addition, the approach of Zangwill [1969] must be adjusted because our extension for the case of uncertain yield allows the backorder at the end of the production horizon. This section successively introduces the main concepts that support our reasoning, the property of an optimal solution for the LSP with uncertain yield and the optimal policy in Section 2.3.6, and finally the dynamic programming algorithm in Section 2.3.6.

Structure of the optimal solution to the LSP under yield uncertainty

Let us define the cumulative demand as $\bar{D}_t = \sum_{\tau=1}^t d_\tau$ and the cumulative production quantity up to period t as $\bar{X}_t = \sum_{\tau=1}^t X_\tau$. The worst inventory cost ($\hat{I}_t(\bar{X}_t)$) and the worst backorder cost ($\hat{B}_t(\bar{X}_t)$) in period t can be written as follows:

We assume $\bar{D}_t = \sum_{\tau=1}^t d_\tau$ to be the cumulative demand, while $\bar{X}_t = \sum_{\tau=1}^t X_\tau$ is the cumulative production quantity in period t . Similarly to the proof from Proposition 2.3.1, $\hat{I}_t(\bar{X}_t)$ and $\hat{B}_t(\bar{X}_t)$ can be given as follows:

$$\hat{I}_t(\bar{X}_t) = \max \left\{ h_t \left[\bar{X}_t(\bar{\rho} + \hat{\rho}) - \bar{D}_t \right] ; 0 \right\}$$

$$\hat{B}_t(\bar{X}_t) = \max \left\{ b_t \left[\bar{D}_t - \bar{X}_t(\bar{\rho} - \hat{\rho}) \right] ; 0 \right\}$$

Definition 2.3.1 (Regeneration period). A regeneration period is a period r where the worst inventory cost ($\hat{I}_r(\bar{X}_r)$) equals to the worst backorder cost ($\hat{B}_r(\bar{X}_r)$). Therefore, the regeneration period r is such that: $\hat{I}_r(\bar{X}_r) = \hat{B}_r(\bar{X}_r)$.

Definition 2.3.2 (Regeneration interval). A regeneration interval $[m, n]$ is a set of consecutive periods such that $\hat{I}_m(\bar{X}_m) = \hat{B}_m(\bar{X}_m)$, $\hat{I}_n(\bar{X}_n) = \hat{B}_n(\bar{X}_n)$ and $\hat{I}_t(\bar{X}_t) \neq \hat{B}_t(\bar{X}_t)$ for all $t \in [m + 1, n - 1]$. In addition, the production is strictly positive in the setup period¹ β in the interval $[m, n]$, and it is 0 for all periods t , $t \neq \beta$ in the interval $[m + 1, n - 1]$.

Proposition 2.3.4. *The solution to the robust LSP under yield uncertainty with stationary nominal value and maximal deviation of the yield is a succession of regeneration intervals. For each regeneration interval, there is a setup period, beta, that minimizes the total cost over the interval.*

Proof. Given two consecutive periods with production β and γ (with no production in the interval $[\beta + 1, \gamma - 1]$), the worst case inventory and backorder cost H_τ in the interval $[\beta, \gamma]$ depends only on the cumulative production \bar{X}_β in period β . Note that the costs after period γ will depend on the production quantity of period γ . Therefore, the production quantity in period β only impacts the costs in the interval $[\beta, \gamma]$. We redefine the worst case inventory or backorder cost in period τ in terms of the cumulative production in t by $H_\tau(\bar{X}_t)$ that is given as follows:

$$H_\tau(\bar{X}_t) = \max \left\{ \begin{array}{l} \hat{I}_\tau(\bar{X}_t) \\ \hat{B}_\tau(\bar{X}_t) \end{array} \right\} = \max \left\{ \begin{array}{l} h \left[\bar{X}_t(\bar{\rho} + \hat{\rho}) - \bar{D}_\tau \right] \\ b \left[\bar{D}_\tau - \bar{X}_t(\bar{\rho} - \hat{\rho}) \right] \end{array} \right\}$$

where \bar{D}_t is the cumulative demand up to period t .

Figure 2.1 illustrates the total cost $G_{[\beta, \gamma]}(\bar{X}_\beta)$ over the interval $[\beta, \gamma]$ as a function of the cumulative production \bar{X}_β in period β . \bar{X}_β must be chosen to minimize the sum of the inventory and backorder cost in the interval $[\beta, \gamma]$: $G_{[\beta, \gamma]}(\bar{X}_\beta) = \sum_{\tau \in [\beta, \gamma]} H_\tau(\bar{X}_\beta)$. If the production quantity is low in the period with production β (e.g., $\bar{X}_\beta = \bar{X}_\alpha$, where α is the period with production before β), the function $\sum_{\tau=\beta}^\gamma H_\tau(\bar{X}_\beta)$ corresponds to the sum of the backorder $\sum_{\tau=\beta}^\gamma \hat{B}_\tau(\bar{X}_\beta)$ in all periods, because the production is too low to cover the demand in any period of the interval $[\beta, \gamma]$, and the worst case cost corresponds to backorder over the entire interval $[\beta, \gamma]$. If the quantity \bar{X}_β increases to meet the demand

1. A setup period can be a regeneration period for a regeneration interval of only one period.

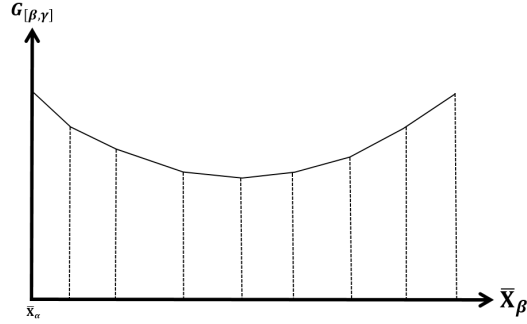


Figure 2.1 – Piecewise inventory and backorder cost functions

in period β , the worst case cost corresponds to inventory in period β and to backorder afterwards $\sum_{\tau=\beta}^{\gamma} H_{\tau}(\bar{X}_{\beta}) = \hat{I}_{\beta}(\bar{X}_{\beta}) + \sum_{\tau=\beta+1}^{\gamma} \hat{B}_{\tau}(\bar{X}_{\beta})$. In the last segment, where \bar{X}_{β} is large enough to meet the demand until period γ , the worst case cost corresponds to $\sum_{\tau=\beta}^{\gamma} \hat{I}_{\tau}(\bar{X}_{\beta})$. Note that backorder costs are decreasing with the production quantity, whereas the inventory costs are increasing with \bar{X}_{β} . Therefore, $G_{[\beta, \gamma]}$ is a piecewise linear and convex function, such that the minimum is at a breakpoint. Each breakpoint in the function $G_{[\beta, \gamma]}$ corresponds to the case where the worst case backorder cost equals the worst case inventory cost in a period. As X_{β} is chosen to minimize $G_{[\beta, \gamma]}$, it is chosen such that the worst case backorder cost equals the worst case inventory cost in a period $n \in [\beta, \gamma]$. We call such a period a regeneration period, and a regeneration interval is a set of period $[m, n]$ where m and n are regeneration periods. As there is a regeneration period between each successive period with setup, each regeneration interval contains one period with setup. In the case the last regeneration period r occurs before the end of the production horizon, that is $r < T$, then we cannot define a complete regeneration interval and the minimum cost in the interval $[r, T]$ corresponds to $G_{[r, T]} = \sum_{\tau=r+1}^T H_{\tau}(\bar{X}_r)$. In this case, all demands are backordered from the last regeneration period until the end of the production horizon. \square

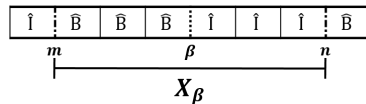


Figure 2.2 – Representation of the regeneration period and regeneration interval concepts

Let m be the last regeneration period before starting production in the period β , such that n is the next regeneration period after β and $m \leq \beta \leq n$. Figure 2.2 illustrates the

regeneration interval concept. Here, \hat{I} (resp. \hat{B}) indicates that the worst case cost at each period corresponds to the inventory (resp. backorder) costs. As previously mentioned, the production quantity in period β in the regeneration interval $[m, n]$ is chosen such that, at the regeneration periods m and n , the worst inventory cost is equal to the worst backorder cost. In addition, the worst cost corresponds to inventory costs (resp. backorder cost) from period β to $n - 1$ (resp. $m + 1$ to $\beta - 1$).

Proposition 2.3.4 indicates that the solution to LSP under yield uncertainty is a succession of regeneration periods, where each lot size covers the demand over the regeneration interval in which the production quantity is defined. Proposition 2.3.5 gives the optimal lot size to cover each regeneration interval.

Proposition 2.3.5. *The amount to produce in the setup period β to cover the regeneration interval $[m, n]$ (with $\beta \in [m, n]$ and $m < n$) is given by X_β , and it is calculated as follows:*

$$X_\beta = \bar{X}_n - \bar{X}_m$$

Proof. The previous proposition (Proposition 2.3.4) show that the optimal quantity to produce to fulfil demands over a regeneration interval $[m; n]$ is hold in period β where the cost is minimal. From Equation A.1, the cumulative production quantity \bar{X}_m (resp. \bar{X}_n) at regeneration period m and n (respectively) is $\bar{X}_m = \frac{\sum_{\tau=1}^m d_\tau}{\bar{\rho} + \hat{\rho} \left(\frac{h_m - b_m}{h_m + b_m} \right)}$ (resp. $\bar{X}_n = \frac{\sum_{\tau=1}^n d_\tau}{\bar{\rho} + \hat{\rho} \left(\frac{h_n - b_n}{h_n + b_n} \right)}$). Note that Equation A.1 applies here because the production in period β to cover a regeneration interval $[m; n]$ equals the difference between the cumulative production quantity in n (the end of the interval) and the cumulative production quantity in m (the beginning of the interval). Since there is no production in interval $[\beta + 1, n]$, the cumulative production \bar{X}_β in period β is equal to \bar{X}_n . In addition, there is no production in interval $[m + 1, \beta - 1]$ and the cumulative production in this interval corresponds to \bar{X}_m . The production quantity in period β (with $m < n$) corresponds to the difference between the cumulative production at the end of the regeneration interval and the cumulative production at its beginning. Therefore, $X_\beta = \bar{X}_n - \bar{X}_m$. Note that if a setup period is a regeneration period, then the lot size in period β should cover only the demand from period β . For this special case, the lot size is directly computed by the policy $X_\beta = \frac{d_\beta}{\bar{\rho} + \hat{\rho} \left(\frac{h_\beta - b_\beta}{h_\beta + b_\beta} \right)}$ given in Proposition 2.3.3. \square

The dynamic program for the LSP with uncertain yield

Our dynamic program for the LSP under yield uncertainty iteratively defines successive regeneration intervals from the first production period until the end of the production horizon. For each interval, it defines the setup period that minimizes the costs over the interval. The dynamic programming recursion requires low computational effort because it works with the optimal cumulative policies that are easily computed, and it exploits forward recursion to avoid recalculating decision values that have already been defined.

The function $MCI(m, \beta, n)$ gives the minimal cost over the regeneration interval $[m, n]$ for a production setup in period $\beta \in [m, n]$. $MCI(m, \beta, n)$ includes a setup cost, the production cost associated with the optimal production quantity for the regeneration interval $[m, n]$ (computed according to Proposition 2.3.5) and the inventory balance costs over the regeneration interval. From Proposition 2.3.4, we know that the cumulative production in n covers all the demands up to n and that the cumulative production in m covers all demands up to m . Thus, for the $MCI(m, \beta, n)$'s inventory management cost calculation is given by $\sum_{\tau=m+1}^{\beta-1} H_{\tau}(\bar{X}_m)$ in $[m+1, \beta-1]$ and by $\sum_{\tau=\beta}^n H_{\tau}(\bar{X}_n)$ in $[\beta, n]$. Thus, $MCI(m, \beta, n)$ is then given by $MCI(m, \beta, n) = \left\{ s_{\beta} + v_{\beta} X_{\beta} + \sum_{\tau=m+1}^{\beta-1} H_{\tau}(\bar{X}_m) + \sum_{\tau=\beta}^n H_{\tau}(\bar{X}_n) \right\}$, where $X_{\beta} = \bar{X}_n - \bar{X}_m$ if $m < n$, and $X_{\beta} = \frac{d_{\beta}}{\bar{\rho} + \hat{\rho} \left(\frac{h_{\beta} - b_{\beta}}{h_{\beta} + b_{\beta}} \right)}$ otherwise. Contrary to the deterministic case, it is possible to obtain a plan for which the backorder level at the end of the production horizon is not zero. Thus, we denote by $MCR(r, T)$ the minimal cost from the last regeneration period r to T , the end of the planning horizon, when only the backorder levels are considered. $MCR(r, T) = \sum_{\tau=r+1}^T H_{\tau}(\bar{X}_r)$.

Finally, we define the dynamic program recursive function by $F(t)$, which gives the optimal cost from period 0 to t . The costs up to period t correspond to the cost of the last regeneration period m plus the cost up to period t (that is in the interval $[m, t]$). Therefore, $F(t)$ can be computed recursively. Given the optimal cost $F(m)$ up to period $m < t$, the computation of $F(t)$ enumerates all possible values for the last regeneration period m and the period with setup $\beta \in [m, t]$. Thus, $F(t)$ is given by

$$F(t) = \min_{m \leq \beta \leq t} \{F(m) + MCI(m, \beta, t)\} \quad \forall t \leq T - 1$$

Since the last regeneration period can occur before period T , the end of the production horizon, we should modify $F(T)$ to take into account the possibility of backordering all unmet demands from the last regeneration period r until T . Therefore, the optimal cost

at the period T is given by

$$F(T) = \min \left\{ \min_{m \leq \beta \leq t} \{F(m) + MCI(m, \beta, T)\} ; \min_{m \leq T} \{F(m) + MCR(m, T)\} \right\}$$

Proposition 2.3.6. *The dynamic program for solving the LSP under yield uncertainty runs in $O(T^3)$.*

Proof. See Appendix A.4 □

Proposition 2.3.6 indicates that our dynamic program performs in $O(T^3)$. The problem becomes NP-Hard when extended to the capacitated context, and this can be verified with a reduction from the deterministic single-period capacitated LSP [Florian and Klein, 1971, Bitran and Yanasse, 1982].

2.4 Results and discussions

This section presents the experimental study, and its objective is threefold: (1) to demonstrate the robustness of the presented approaches in coping with a non-deterministic LSP; (2) to report an in-depth investigation on the robust LSP with uncertain yield, its performance, the quality of the solutions, and the computational efficiency, and (3) to evaluate and compare the performance of the different optimization approaches in terms of the average and worst case quality of the solution.

The experiments consider the following solution approaches: DET_{SS} , the deterministic model with safety stock, as presented in A.6; DET , the nominal model with $\rho_t = \bar{\rho}_t$, as presented in Section 2.3.1; SP , the stochastic program, as presented in A.7; RO_{Γ} , the LSP under yield uncertainty, with uncertain yield and budget Γ , as presented in Section 2.3.3; and finally DP , the dynamic program for computing an optimal robust plan for the stationary LSP under box uncertainty set and with uncertain yield, as presented in Section 2.3.6.

Note that DET and SP are natural benchmarks to compare solution approaches that cope with uncertainties. DET_{SS} corresponds to the approach commonly used in practice, where a safety stock is computed separately from the lot sizes. This section is organized as follows: Section 2.4.1 presents the instance generation method. Section 2.4.2 introduces the simulation framework used to compare the methods. Finally, Section 2.4.3 presents an analysis of the developed models. We discuss the performance of the dynamic program in Section 2.4.3, the investigation of the price of robustness in terms of the budget of uncertainty in Section 2.4.3, and the performance and quality of the plans resulting from different optimization approaches in Section 2.4.3. We also highlight the advantages of using robust optimization to hedge against uncertainties in a highly uncertain context.

2.4.1 Instance generation

We generate each parameter of the instances using uniform distributions. We set the value of the support of these uniform distributions in a similar manner to Brandimarte [2006] to generate setup and inventory costs. As Brandimarte [2006] considers demand uncertainty, we generate the demand similarly to Alem et al. [2018]). The production costs, the inventory costs, the demands, the nominal values of the production yield rate, and the maximum deviations of the production yield were randomly generated using an uniform distribution within the following intervals: $v_t \in U(10, 20)$, $h_t \in U(1, 10)$, $d_t \in U(140, 480)$,

$\bar{\rho}_t \in U(0.5, 0.7)$, and $\hat{\rho}_t \in U(0.1, 0.3)$, respectively. The setup costs are computed with the time between order formula: $s_t = \frac{\bar{D}_t \cdot TBO^2 \cdot h_t}{2}$, where \bar{D}_t represents the average demand up to period t . The capacity can be computed as $C_t = \Phi \frac{\sum_{t \in T} d_t}{\min_{\tau \leq t} (\bar{\rho}_\tau - \hat{\rho}_\tau)}$, where Φ is a factor to control the tightness of the capacity. To adapt these instances for the LSP under yield uncertainty, we generate the nominal and maximum deviation of the yield with uniform distributions with means in the range (0.5; 0.7) and (0.1; 0.3), respectively.

For instance, Chu et al. [2019] consider the stationary case, where they set the mean and standard deviation of the production yield to 0,95 and 0,05, respectively. As we aim to study the impact of yield uncertainty in the lot-sizing model, we consider here the non-stationary uncertain production quality, where the quantity of quality goods varies significantly between periods. This setting relates to complex production systems in which factors affecting production yield are difficult to measure or estimate precisely due to lack of data. We consider instances with 4, 12, and 24 periods, a time between orders of 1, 2, or 4, and a backorder cost that equals 2, 5, or 10 times the holding cost for each period t . In addition, we consider a capacity factor Φ of 25%, 50%, and 75% for the capacitated model, and we assume a capacity factor of 100% for the non-capacitated model. We generate the instances with a full factorial design for these four parameters, which leads to 108 instances. We set the inventory and backorder levels to zero at the beginning of the horizon. Because the optimal policies are valid only for the uncapacitated models and the special case with non-stationary costs and demands but with constant nominal value and maximum deviation of the production yield, we generate 81 additional instances resulting from the factorial design of the following parameters: 6, 12, 24, 30, 36, 48, 96, 192, 384 periods, time between orders of 1, 2 or 4, and backorder costs equaling $2h_t$, $5h_t$, or $10h_t$. We increase the instance size compared to the other instances to measure in terms of computational time the performance of the dynamic program for larger size problems.

2.4.2 Simulation

We analyze the quality of the production plans through a simulation with $|\Omega| = 5000$ scenarios generated with Monte Carlo sampling, where each scenario represents a possible realization of the production yield over the horizon. We simulate the yield with a uniform distribution with support $[\bar{\rho}_t - \hat{\rho}_t; \bar{\rho}_t + \hat{\rho}_t]$ in period t . We note $EVPI$, the expected value of perfect information, the average cost of the perfect information solutions, where $EVPI_\omega$ is the cost of the solution computed with the deterministic model for scenario ω .

To evaluate each optimization method, we fix X_t and Y_t obtained from the optimization step in the deterministic model for each scenario ω .

2.4.3 Experimental results

This section presents an average cost analysis for the uncapacitated and capacitated problems. The algorithms were implemented in Python 3.6, and the mathematical models are solved with CPLEX version 12.10. The experiments were run on Intel(R) Xeon Broadwell 2683/2.1GHz processors with 125GB of RAM. All the models for all the instances were solved until optimality. We compare the methods based on the objective value provided by each optimization approach (e.g., the objective function given in (2.4) for RO_T), the average computational time (in seconds), the expected value (Exp. Cost) of each solution approach evaluated in the simulation, along with the worst case cost in the simulation, and the 95th and 99th percentile cost (p.c.). We also report the relative difference between the expected value of perfect information $EVPI$ and the simulated expected value of each method given by $GAP_{EVPI} = 100 \times \frac{\text{Exp. Cost} - EVPI}{EVPI}$. In addition, we define by $GAP_{OPT} = 100 \times \frac{\text{Exp. Cost} - \text{Obj. Value}}{\text{Obj. Value}}$ the relative difference between the objective value of a solution approach and its simulated expected cost. Finally, we report the coefficient of variation CV , an index that indicates a high variability of the costs in the simulation. Thus, CV is the ratio of the standard deviation to the mean, such that the higher CV is, the more widely dispersed the values are from the mean.

Performance of the dynamic program

First, we analyze the performance of the dynamic program to solve the multi-period LSP under box uncertainty set and yield uncertainty, with constant nominal value and maximum deviation of the production yield, and with non-stationary costs and demands. Table 2.1 reports the computational times (in seconds) for DP and RO on the 81 instances generated for the special case. Each column corresponds to 9 instances with the same planning horizon, and we indicate the average time to solve them.

Table 2.1 shows that the robust dynamic program is computationally less demanding than the robust model for instances with at most 30 periods, while DP becomes more computationally demanding than RO for a production horizon greater than 30 periods. In fact, by taking the overall run time to solve instances with at most 30 periods, the dynamic program takes approximately 0.12 seconds, whereas the robust model needs approximately

Table 2.1 – Computation time of the dynamic programming approach and the reformulated MILP for the uncapacitated problem

Approach	Planning horizon size								
	6	12	24	30	36	48	96	192	384
DP	0.002s	0.01s	0.15s	0.28s	0.54s	1.65s	22.05s	320.63s	5245.16s
RO	0.183s	0.21s	0.37s	0.35s	0.30s	0.48s	2.30s	26.41s	2182.94s

0.28 seconds to solve the same instances. However, when we consider the instances with production horizon larger than 30 periods, *DP* requires around 10.35 minutes to determine a solution for instances with a planning horizon up to 384 periods, while *RO* takes only 4.05 minutes on average to compute the same solution. These results show that the *DP* approach is better suited for small instances or when an efficient commercial solver is not available. However, for medium and large sized instances, the computational effort required by *DP* becomes too expensive compared to the solution computed with the *MILP* robust model. Moreover, *RO* can deal with more general LSPs, which makes its performance more competitive with a problem-specific approach. Therefore, the MILP model can be extended to include different practical constraints while it can still remain efficient in terms of speed in computing an optimal solution.

Price of robustness

To analyze the impact of the budget Γ , we consider different budgets to represent the decision-maker's risk aversion. For each instance, the budget varies from low aversion ($\Gamma = 0.1T$) to extreme aversion $\Gamma = T$, going through a progression with the size of the production planning. The budget represents the maximum number of uncertain parameters that can take the worst case value [Thiele, 2010]. Since Γ indicates the number of periods where the production yield can take its worst case realization, it is convenient to express Γ as a proportion of T . Table 2.2 indicates the impact of the budget of uncertainty on the costs for all the considered instances. While the Obj Value column gives the average of the objective function values computed in the optimization step, the remaining columns report the average value for the features obtained in the simulation step.

As reported in Table 2.2, for both the uncapacitated and capacitated model, the best solutions in terms of objective value and expected cost are obtained for a very tiny budget (Γ equals $0.1T$ and $0.2T$), so the decision-maker should be willing to accept a high degree of risk resulting from uncertainties. Considering the 95th percentile costs, the lowest costs

Table 2.2 – Impact of the budget of uncertainty on the robust lot-sizing decision

Γ	Uncapacitated						Capacitated					
	Exp. Cost	95 th p.c.	99 th p.c.	Worst Cost	Obj Value	CV	Exp. Cost	95 th p.c.	99 th p.c.	Worst Cost	Obj Value	CV
0.1T	124,759	154,848	185,884	243,203	132,004	10.8%	132,974	163,341	195,299	246,471	143,413	10.9%
0.2T	137,315	154,685	165,599	195,557	169,567	7.2%	145,625	164,536	175,202	201,651	184,442	7.5%
0.3T	146,202	163,330	170,991	188,427	192,411	6.6%	156,249	174,874	182,625	200,529	210,890	6.7%
0.4T	155,134	174,025	181,590	191,107	218,111	7.0%	166,748	187,231	195,255	208,007	237,618	6.9%
0.5T	157,734	178,326	186,106	195,401	227,836	7.4%	169,338	190,815	199,027	210,768	247,834	7.1%
0.6T	159,507	181,895	190,214	197,804	236,085	7.9%	172,066	194,805	203,491	213,806	256,740	7.7%
0.7T	159,353	182,381	190,537	197,941	237,278	8.3%	171,205	194,614	203,547	213,448	258,356	7.9%
0.8T	158,777	182,053	190,164	197,690	237,573	8.4%	170,770	194,318	203,299	213,112	258,566	7.9%
0.9T	158,883	182,202	190,322	197,836	237,732	8.4%	170,782	194,377	203,370	213,232	258,805	8.0%
1.0T	158,883	182,202	190,322	197,836	237,732	8.4%	170,782	194,377	203,370	213,232	258,805	8.0%

for the uncapacitated (resp. capacitated) model are obtained for Γ between $0.1T$ and $0.3T$, while the lowest 99th percentile costs are given by a budget equaling $0.2T$ and $0.3T$ for both variants. In addition, the lowest worst case costs are obtained for Γ between $0.2T$ and $0.5T$ for both uncapacitated and capacitated model, and the lowest coefficient of variation CV can be achieved for a budget that equals $0.3T$, $0.4T$ (resp. $0.3T$, $0.4T$ or $0.5T$). However, for Γ lower than $0.1T$, the robust objective value fails to cover the 95th percentile cost (and, consequently, the worst case cost). Therefore, with such a low Γ value, the decision-maker has no guarantee that the plan will be sufficiently immunized from uncertainties. On the other hand, although the solutions obtained for Γ greater than $0.4T$ achieve a robustness level that covers the worst case scenarios, they are overly conservative. In fact, for Γ between $0.5T$ and T , the objective value is much higher than the worst case cost. Thus, even if for the largest value of Γ the solutions are robust, they do not result in the best option in terms of the expected costs and stability. Therefore, the budget of uncertainty in the interval from $0.2T$ to $0.4T$ offers better production plans, since the decisions are sufficiently stable because of a low CV , the expected cost is relatively low, and the robust objective value covers at least the 99th percentile costs (and even the worst case costs for when Γ equals $0.3T$ or $0.4T$).

Comparison of the LSP with uncertain yield resolution approaches

We now compare the production plans resulting from RO , SP , DET and DET_{SS} . We consider RO with a budget that is equal to $0.2T$, $0.3T$, and $0.4T$ because we have previously shown that these values result in better trade-offs in average costs, conservatism and robustness. We also consider the extreme case of the robust approach with a more conservative solution for $\Gamma = T$. All results reported in these sections are an average over

all the considered instances. In addition, all values are computed during the simulation step except for the optimal values and computational time which are obtained from the optimization. Note that from the non-deterministic literature, nominal models ignore the occurrence of uncertainties, and they tend to have higher costs compared to other optimization approaches that take uncertainties into account. We also emphasize that the safety stock is a classic complementary approach to control uncertainties in a non-deterministic context, and such a safety stock can be incorporated into the deterministic model. The safety stock calculation, which relies on different assumptions, can be done separately as preprocessing step.

Table 2.3 shows the simulation results for the uncapacitated LSP. Here, the computation times required by DET (about 0.01s) and DET_{SS} (about 0.03s) are extremely low. Next, RO takes on average 0.38s to compute a solution, followed by SP , the most demanding approach, which takes about 19.6s to propose a production plan. DET is the less robust method because its GAP_{OPT} and its CV are the highest among all the methods tested, which indicates a costly solution highly impacted by yield disruptions. DET is followed by DET_{SS} , SP and RO solutions. Considering the quality of the solution in terms of GAP_{EVPI} , SP proposes the best expected cost followed respectively by the DET_{SS} , DET and RO models. However, RO is less impacted by disruptions of the production yield than SP , since the robust CV is about 7% on average, whereas SP gives a production plan whose CV is 15% on average. In addition, although DET_{SS} and DET give better expected costs than RO , these models lead to large costs for more adverse production yield values (since they have larger worst case, 99th and 95th percentile costs). Moreover, DET_{SS} and DET are much more impacted by yield uncertainty than RO . While, on average, DET_{SS} has a CV of around 19%, and DET has a CV of 25%.

Table 2.3 – Performance of the uncapacitated models in terms of the average cost and worst case simulated costs

Model	Exp. Cost	95 th p.c.	99 th p.c.	Worst Cost	Comp. Time	CV	GAP _{EVPI}	GAP _{OPT}
$EVPI$	89,454	98,728	102,679	107,073		6%		
DET_{SS}	129,241	191,368	241,716	301,517	0.03	19%	36%	2%
DET	131,284	222,968	287,950	350,106	0.01	25%	27%	26%
SP	115,624	152,899	194,764	258,646	19.60	15%	20%	-4%
$RO_{\Gamma=0.2T}$	137,315	154,685	165,599	195,557	0.63	7%	44%	-18%
$RO_{\Gamma=0.3T}$	146,202	163,330	170,991	188,427	0.52	7%	53%	-23%
$RO_{\Gamma=0.4T}$	155,134	174,025	181,590	191,107	0.34	7%	60%	-27%
$RO_{\Gamma=T}$	158,883	182,202	190,322	197,836	0.04	8%	62%	-32%

Since DET and DET_{SS} do not offer good solutions in terms of robustness, we fo-

cus on the comparison between RO and SP . Although SP provides lower expected costs (115,624) than RO (149,384, on average), the robust model leads to a lower worst case cost (about 193,232 on average over the different budgets) than the worst cost obtained with the stochastic plan (258,646). In addition, the robust models have a coefficient of variation of 7% on average, while SP has a CV equal to 15%. This value confirms that SP is more impacted by disturbances on the production yield than the robust plans. Nevertheless, we stress that the SP and RO methodologies have fundamentally different objectives. Although the stochastic program seeks the minimum expected costs, the robust optimization method aims the minimum objective value that covers the costs incurred even for the worst realization of the uncertain yield. In the same vein, the relative difference between $EVPI$ and RO_{Γ} is greater than the GAP_{EVPI} between $EVPI$ and SP because the robust models propose a production plan that remains cost effective, even for the worst realization of the uncertain yield for a well-chosen budget of uncertainty, while SP may be ineffective in case of adverse events. The robust strategy leads to more conservative solutions than the production plan proposed by the stochastic program, for which the strategy is defined regarding the probability of the realization of the uncertainty. While SP is known to be prone to changes in the underlying uncertainty (e.g., if the distribution changes), the RO remains stable and robust for different and unrelated uncertainty realizations. This can be verified in the column 99th percentile and worst case average costs, for which robust models are much less impacted by the uncertain parameter, leading to lower costs.

Table 2.4 – Performance of the capacitated models in terms of the average cost and worst case simulated costs

Model	Exp. Cost	95 th p.c.	99 th p.c.	Worst Cost	Comp. Time	CV	GAP_{EVPI}	GAP_{OPT}
$EVPI$	99,655	112,824	118,132	124,212		8%		
DET_{SS}	136,003	188,261	232,546	289,263	0.03	17%	30%	-4%
DET	137,722	222,418	281,401	346,124	0.01	24%	22%	23%
SP	124,029	162,394	204,674	263,150	15.64	16%	17%	-4%
$RO_{\Gamma=0.2T}$	145,625	164,536	175,202	201,651	0.31	7%	38%	-20%
$RO_{\Gamma=0.3T}$	156,249	174,874	182,625	200,529	0.37	7%	48%	-25%
$RO_{\Gamma=0.4T}$	166,748	187,231	195,255	208,007	0.24	7%	56%	-28%
$RO_{\Gamma=T}$	170,782	194,377	203,370	213,232	0.03	8%	59%	-32%

Table 2.4 presents the results for the capacitated version of the LSP. Much like the uncapacitated model, DET has the lowest computational time of 0.01s, followed by DET_{SS} (about 0.03s), RO (about 0.24s on average), and finally SP (around 15.64s). The capacitated DET is not robust enough in comparison with other approaches, since it has the

highest CV and it gives the highest costs in terms of 95th and 99th percentile, and worst case costs. In addition, although the capacitated DET_{SS} model has a GAP_{OPT} as good as SP , its 95th and 99th percentile costs, as well as its worst case cost, are higher than the respective SP costs. DET is also less robust since its CV is around 24%. Therefore, DET and DET_{SS} are not competitive in terms of performance to mitigate uncertainties when compared to SP and RO models. While the lowest expected cost is given by SP , the lowest 99th percentile and worst case costs are given by the robust models. RO not only gives the lowest costs, especially for Γ equals $0.3T$ or $0.4T$, but also fully immunizes the problem from uncertainties with a production plan whose objective value covers any realization of the uncertain yield. The CV of RO models remains the same in comparison with the uncapacitated version of the problem, and the values are still lower than the CV for the SP solutions (that decreases to 24%). This indicates that the robust plans are more stable than the stochastic ones for different realizations of the uncertain yield. In addition, the relative difference between the optimization methods and $EVPI$ becomes lower, and the optimality gap for RO also decreases. The results show that the robust approach is efficient when mitigating uncertainties because it offers a good, stable and robust production plan. Although the expected costs from the robust models (159,851 on average) are higher than the respective SP costs (about 124,029), the robust 99th percentile cost (on average 189,113) is lower than the stochastic one (about 204,674). Similarly, the robust worst case cost (about 205,855 on average) is much lower than the stochastic one (around 263,150). In addition, for a budget of uncertainty greater than or equal to $0.3T$, RO worst case cost is covered by the objective value. As a result, the SP model is less efficient when uncertainty information is relatively limited or if we want to limit a downside risk due to the realization of uncertainties.

The RO framework provides a systematic approach to determine a robust production plan that mitigates uncertainties with a conservatism partially controlled by the budget of uncertainty. Similarly to Hnaien and Afsar [2017], where the authors conclude that a robust capacitated LSP is easier to solve than the uncapacitated version, we also observe the same pattern. Our capacitated RO model is generally easier to solve, which is not usually the case for the deterministic model where the capacitated model is more difficult to solve (e.g., Brahimi et al. [2006]). Hnaien and Afsar [2017] indicate that the capacitated model has a bound on the lot size lower than the natural bound, and as the linear relaxation of the capacitated version may be less fractional, it may lead to faster calculations. However, there may be other explanations. There are some capacitated problems

in the literature where the instances are designed to be the most challenging, and so time consuming. As a result, the computational effort to solve the capacitated problem may change depending on the instance configuration in size, structure (e.g., multi level), costs, setup features (e.g., setup times, setup carryover, high setup costs), resource availability (e.g., tight capacity), and resolution approach [Buschkühl et al., 2010]. On the other hand, robust models search for solutions that are immunized from uncertainties, which leads to a large number of feasible solutions. As a consequence, capacitated *RO* models have a reduced amount of feasible solutions in the worst case perspective, since limited resources lead to more restrictions defining the solution space [Zhanga et al., 2020]. Hence, the robust models with capacity constraints potentially require less computing effort. This has also been observed in other literature (e.g., Hnaien and Afsar [2017], Zhanga et al. [2020]).

To analyze the cost components incurred in the simulated production system, Figure 2.3 indicates the setup frequency for the uncapacitated and capacitated models, Figure 2.4 reports the proportion of the average expected costs imputable to the setup, production, inventory, and backorder costs for the uncapacitated and capacitated models. Finally, Figure 2.5 reports the average lot size $\|\mathbf{X}\|$, inventory $\|\mathbf{I}\|$, and backorders $\|\mathbf{B}\|$ levels accumulated over the entire production horizon, and the inventory and backorder levels at the end of the production planning for the uncapacitated and capacitated models.

Figure 2.3 shows the setup frequency for different capacity levels. As expected, the frequency of setup increases when the capacity decreases. For the uncapacitated models, *RO* has the highest setup frequency (with setup frequency of 62% on average). It is followed by *SP* with a setup frequency of 46%, *DET_{SS}* with a setup frequency of 44%, and finally *DET* with a setup frequency of 43%. The *RO* models adopt a strategy with more frequent production setups to reduce the total costs and to avoid backorder costs when the problem faces low-yield values. Figure 2.3 also reports that the robust production plans for Γ from $0.2T$ to $0.4T$ not only mitigate uncertainties better, they also provide a configuration setup that is still efficient even if resource availability is disrupted. As shown, for the different capacity factors, the setup frequency for these robust models does not suffer major disturbances.

Figure 2.4 gives an overview of the cost distributions for all considered approaches. *RO* leads to fewer backorders (6% of the cost for *RO* when $\Gamma = 0.3T$ versus 20% for *SP*) at the expense of higher setup, production and inventory costs. *RO* tends to exceed the costs from *DET*, *DET_{SS}* and *SP* for all components except the backorder costs. However, the

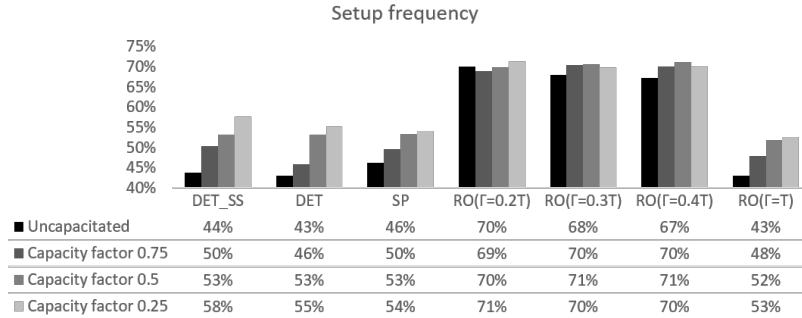
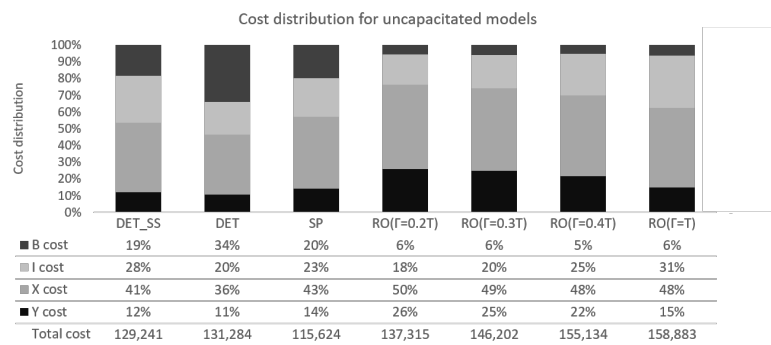


Figure 2.3 – Characteristics of the solutions in terms of setup frequency

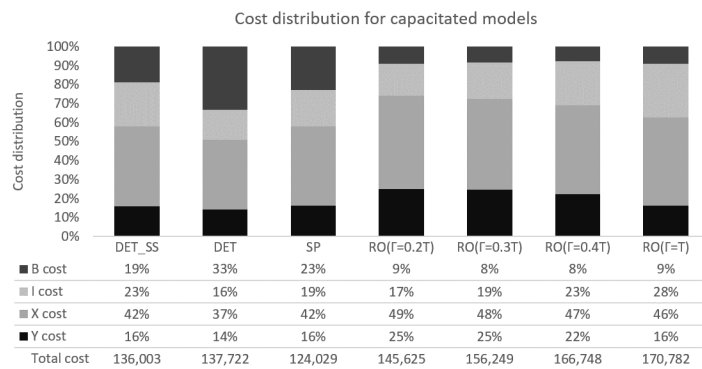
robust backorder costs are much lower than the respective DET , DET_{SS} and SP costs, which compensate for the highest setup, production and inventory costs. Therefore, the robust plan offers more possibilities for the decision-maker to take advantage of the available resources, while also reducing the impact of the uncertain events on the production plan. For instance, when the availability of the resources is more restricted, backordering becomes more frequent. RO manages to control (and even reduce) the backorder cost by increasing the lot size and the frequency of production, yet lot size and inventory levels remain acceptable. As a result, the robust model favors a large production level to meet demands, DET_{SS} relies on large stock level, and DET and SP take the risk of backordering goods.

Figure 2.5 shows that all approaches keep as many goods in stock over the entire production planning as possible. Although this strategy reduces the backorder levels (with little increase for the capacitated models), it requires larger lot sizes to keep enough quality goods to meet demands. The production quantity becomes even higher for the DET_{SS} than for DET to ensure that the safety stock is respected. DET and SP have quite similar production plans, with a more important difference in terms of the inventory level. For the uncapacitated problem, RO produces an amount of goods relatively close to that proposed by SP (6,411 units on average when considering all Γ values versus 6,033 units produced on average with SP and 6,229 units produced with DET), yet it leads to a low backorder level (an average of 507 units backordered with the robust plan versus 1,021 units backordered units with SP and 1,450 units with DET). Besides, the robust models have a larger inventory level on average of 192 units (resp. 241 units), versus 53 units (resp. 52 units) for SP and 115 units (resp. 111 units) for DET .

To conclude, the RO provides effective support for decision-makers. Contrary to DET_{SS} , DET and SP , RO provides an objective value that is larger than the expected



(a) Uncapacitated problem



(b) Capacitated problem

Figure 2.4 – Characteristics of the solutions in terms of the cost distribution

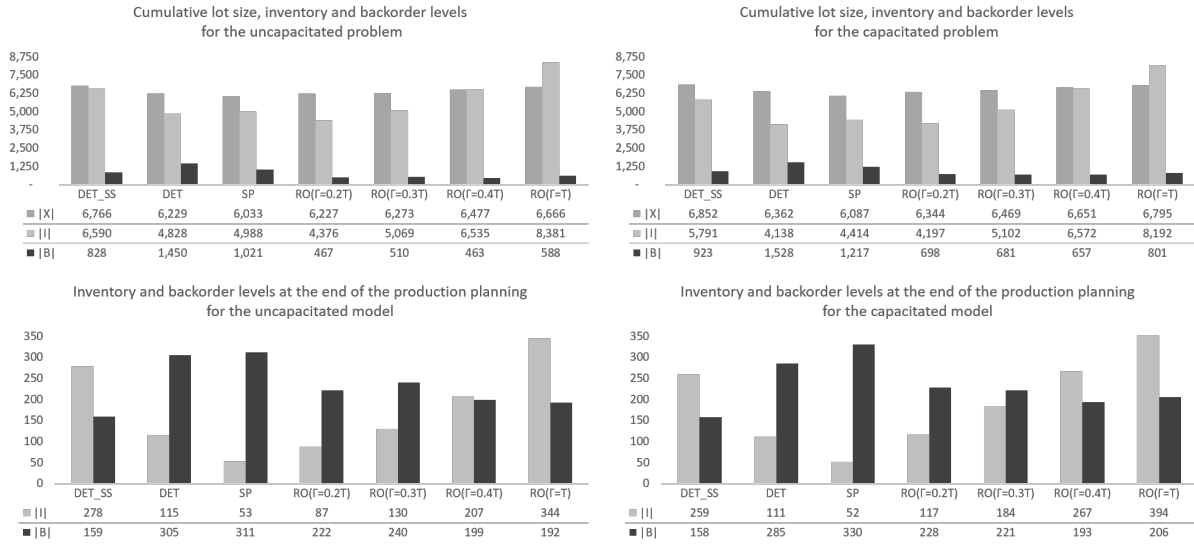


Figure 2.5 – Characteristics of the solutions in terms of production, inventory and back-order levels

simulated costs, and this can reassure the decision-maker. In addition, unlike other approaches, the robust plan covers even the most pessimistic scenario. When we investigate the stability and robustness of the proposed plans, the robust approaches provide the production plan that copes better with uncertainties because it tends to offer greater cost savings with a low impact of yield disturbance on the production plan. In addition, contrarily to *SP*, *DET* and *DET_{SS}* which adopt a strategy that places backorders more often to reduce the inventory and production costs, *RO* relies on a sufficiently low stock level that satisfies demands and which is supported by sparse production setups that minimize costs with sporadic production backorders. Therefore, the robust models mitigate the impact of the realization of unknown and pessimistic scenarios on the production plan better.

2.5 Conclusion

In this chapter, we have introduced the robust formulation for lot-sizing under yield uncertainty. We show that the multi-period problem under box uncertainty set, where the average and standard deviation of the production yield are constant over the planning horizon while costs and demands are not, can be solved in polynomial time with a dynamic programming approach. The chapter also proposes a mixed-integer linear program for the non-stationary LSP with uncertain yield, and it provides insights into robust production plans. Our results show that with a proper budget of uncertainty, the robust model mitigates uncertainties with a balance between production quantities, setup costs, and inventory management costs. In addition, the robust optimization method requires less computational effort than stochastic programming, and its solutions are less conservative, yet more robust, compared to the classical approaches to dealing with uncertainties on LSPs (represented here by the nominal problem and the deterministic problem with safety stock). Other major advantages of robust optimization over the other proposed approaches are that the robust approach requires little information about the uncertainty factors and no strong assumption on the uncertain parameter characteristic. Further investigation is still needed to propose an adaptive framework to cope with uncertainties within a static-dynamic, or even a dynamic, decision framework. The present work could also be extended to deal with multi-echelon systems.

ADAPTIVE ROBUST LOT-SIZING PROBLEMS UNDER YIELD UNCERTAINTY

This chapter is based on the following article:

Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. Adaptive robust optimization for lot-sizing under yield uncertainty. *European Journal of Operational Research* (1st revision), August 2022.

In manufacturing environments, uncertain production yield directly impacts the quality and feasibility of the production planning decisions. This chapter investigates the use of adaptive robust optimization to hedge against uncertain yield when determining a production plan, and to react properly when updated information unfolds. We first derive a myopic adaptive robust policy for the inventory management problem, a special case of the lot-sizing problem where the setup and the production costs are omitted. We show that the policy is optimal under mild assumptions. Second, we address a multi-period single-item lot-sizing problem with backorder and uncertain yield via the adaptive robust optimization. We formulate an adaptive robust model based on the budgeted uncertainty set, where we exploit a linear approximation to transform the quadratic constraints into a mixed-integer linear program. We also propose a column and constraint generation algorithm to solve the adaptive model exactly. Finally, we demonstrate the performances of the proposed approaches and the value of the adaptive robust solutions through extensive numerical experiments.

3.1 Introduction

Manufacturers must efficiently manage their production capacities and their performances to comply with contractual agreements and to satisfy customer demands with quality goods. To achieve this business objective, among other tactical problems, manufacturers must make lot-sizing decisions carefully. These decisions determine the production setups and quantities to meet the demands with quality goods, while minimizing the overall production and inventory management costs [Murthy and Ma, 1991]. Product quality is a fundamental issue which directly impacts costs and the profitability of the firm. Production planners often rely on the production yield to measure the expected quantity of quality goods resulting from a lot size.

Recent trends foster the agility of the manufacturing shop floor, and they include reconfigurable manufacturing systems [Koren and Shpitalni, 2010] and supply chains [Dolgui et al., 2020]. The resulting shop floors can cope with an adaptive planning strategy, since they can easily adjust process planning, workforce availability, raw material deliveries, and the production schedule. Adaptive plans are then possible, and such plans can lead to better performance by taking advantage of resource and process flexibility. The adjustable robust optimization (ARO), also denoted as adaptive robust optimization, emerges as a robust methodology applied to static-dynamic and dynamic strategies where decisions can be updated over time [Ben-Tal et al., 2004]. Adaptive robust planning decisions also enhance the system responsiveness the better to account for shop floor volatility [Mittal and Jain, 2014].

The static robust lot-sizing problem (LSP) under uncertain yield has been studied by Chu et al. [2019] and Metzker et al. [2023a]. Chu et al. [2019]’s work partially addresses the LSP problem, since the authors investigate the inventory management problem with demand and yield uncertainties under the budgeted uncertainty set for the case with stationary costs and constant nominal value and maximum deviation of the uncertainties. The authors study the performance of the model under different degrees of uncertainty controlled by the budget of uncertainty. Metzker et al. [2023a] consider the robust LSP with non-stationary costs, demands, and nominal value and maximum deviation of the production yield. Furthermore, the authors provide extensive numerical experiments to measure the performance of the robust optimization against other approaches, namely stochastic programming, the nominal model and the deterministic model with safety stock.

Some flexibility and adaptability are necessary to make the decision process closer

to real applications. They are crucial in reacting to variations in demand with changes in consumption trends, changes in resources or workforce that may impact production performance, and so on. In addition, the acquisition of new machines, or modifications due to quality control requirements would impact production performance, for example improvements to obtain a quality label. Therefore, being able to take these changes into account and react directly to their realization would potentially improve the quality of robust decisions.

To the best of our knowledge, no study considers adaptive lot-sizing decisions with uncertain production yield, where the actual production yield unfolds at the end of each period, and the next lot size computation reacts to this information. ARO often uses decision rules to express the dependence of adaptive decisions to the uncertain parameters [Ben-Tal et al., 2009]. In our case, the decision rules map the realization on the uncertain yield to the adaptive lot size decisions.

Over recent decades, various approaches have been proposed to handle robust LSPs [Metzker et al., 2023a]. Among the approaches to handling adaptive models, Zeng and Zhao [2013] propose a decomposition technique, called a column & constraint generation (C&CG) algorithm, that seeks an optimal solution by decomposing the problem into smaller sub-problems that are easier to solve. This approach was applied to various other problems, such as the facility location problem [Zeng and Zhao, 2013, An et al., 2014], and power grid problems [e.g., An and Zeng, 2014, Jabr et al., 2014, Lee et al., 2013, Wei et al., 2014, Zugno and Conejo, 2015], etc.

Contrary to uncertain demand, whose adaptive model has an uncertain parameter indirectly impacting the adaptive variable (that is, we have a fixed recourse action to modify the adaptive variables), the uncertain production yield becomes a non fixed coefficient of the adaptive lot size decisions [Ben-Tal et al., 2004, Yanıkoğlu et al., 2019]. As a result, we have a non-fixed recourse modifying the adaptive variables. More specifically, the adaptive robust model contains quadratically uncertain terms to define the quantity of quality goods in the inventory balance constraints. As a consequence, the model is intractable [Ben-Tal et al., 2005]. In this chapter, we deal with the adaptive robust model through a linear approximation and propose a decomposition approach to determine an optimal solution. In addition, we demonstrate the properties of a myopic optimal adaptive policy for the inventory management problem. To the best of our knowledge, we are the first to consider ARO for LSP under yield uncertainty, and our contribution is threefold: 1) development of an ARO model to deal with uncertain production yield, 2) modeling a

linear approximation to handle the adaptive problem, and 3) design of a C&CG algorithm to compute an optimal adaptive solution. We also present numerical tests to evaluate the performance of the methods we have developed. Our results indicate that the adaptive robust approach adequately mitigates uncertainty since it protects the production plan better against the most adverse yield realization, and it provides better cost savings and less conservative plans than the classical robust model. Our numerical results also show that ignoring the quadratic term in the adaptive robust model leads to less conservative solutions, while the production plans remain sufficiently robust. This chapter also provides risk analysis to help decision-makers to tailor their decision plans to their strategic and operational goals.

This chapter is organized as follows. Section 3.2 reviews the literature on adaptive robust lot-sizing with a focus on the production yield uncertainty. Section 3.3 formally describes the problem being considered. Section 3.4 presents the adaptive robust model, an approximation, a myopic policy, and the proposed C&CG algorithm. Section 3.5 reports the experimental results, and finally Section 3.6 concludes this work, and provides future research directions.

Notation: We use boldface (e.g., \mathbf{X}) to denote vectors.

3.2 Literature review

Although there exist lots of studies on the LSPs, this section only reviews the literature on the LSPs under yield uncertainty focusing on the adaptive (static-dynamic) strategy, and other studies on the optimization theory relevant to our work.

Solyali et al. [2016] report the impact of the RO on the inventory management problem under uncertain demand. The authors show that the adjustable robust models are solvable in polynomial time, and they can yield better solutions than static models. A study of the LSP under demand and lead-time uncertainty was presented in Mehdizadeh et al. [2018], where the authors use a branch and bound to compute an optimal solution. Ben-Tal et al. [2005] propose an ARC model for the LSP under demand uncertainty. The authors model the adaptive variables (the production quantity) by affine functions of the uncertain demand, and they assume the demand takes a value in bounded uncertainty sets. As a result, their ARC is a MILP that can be easily solved. Melamed et al. [2016] extend the previous study and measure the average performance of the adjustable solution. Zhu et al. [2012] also consider the affine decision rules for the LSP under uncertain demand. They propose an ARC using an ellipsoidal uncertainty set to reduce the conservatism with some predefined risk preferences.

Postek and Hertog [2016] suggest an iterative splitting method to build the uncertainty set for the LSP with uncertain demand. This approach seeks the most likely demand scenarios to reduce the worst case costs. Following the same logic, Chuong et al. [2021] through the sum of squares convex program, present an exact dual semi-definite formulation to deal with the adaptive robust LSP under uncertain demand. Then, Woolnough et al. [2021] extend the affine decision rules to a quadratic form to improve the performance of the ARO method. While few studies consider adaptive robust production planning approaches, most of them address uncertain demand [e.g., Ben-Tal et al., 2005, Melamed et al., 2016, Zhu et al., 2012], although, to the best of our knowledge, none of them addresses the uncertain production yield.

Bowman [1955] present the production yield as a proportion of quality items in a lot. A review on LSPs with uncertain yield is presented by Yano and Lee [1995]. Metzker et al. [2023a] gives an updated review on robust LSPs with uncertain yield. Often, the stochastic programming approach is used. The adaptive stochastic programs could be addressed by a scenario-based formulation, where a tree structure scenario represents the possible realizations of the unknown data [Brandimarte, 2006]. However, such a scenario-based

formulation would be computationally cumbersome as the number of scenarios grows exponentially with the number of periods. ARO can potentially be more scalable. Vayanos et al. [2012] propose a resolution approach for ARO models based on the constraint sampling technique and a finite linear combination of adaptive decision rules.

To the best of our knowledge, no investigation has been carried out on adaptive LSPs with uncertain production yield, and we intend to address this issue and analyse the impact of uncertain production yield on adaptive lot-sizing decisions. Therefore, in this chapter we investigate the ARO methodology to handle LSPs under yield uncertainty. Our research can be considered as an extension to the adaptive case of Metzker et al. [2023a], where the authors investigate the static robust model based on the budgeted uncertainty set.

Our work differs from the aforementioned literature in several aspects:

- First, this chapter is the first work that formulates the LSPs under yield uncertainty via ARO. The resulting adaptive robust lot-sizing problem updates the lot size, inventory and backorder levels once the production yields from previous production periods are known. To introduce the recourse actions into the model, we represent the dependence of adaptive decisions on uncertain parameters via affine decision rules in function of the realization of the uncertain yield for periods prior to the decision variable period. However, the consideration of these affine policies in the LSP model leads to quadratic terms on the uncertain production yield, and the model becomes intractable. To overcome this difficulty, we propose a linear approximation for the static-dynamic strategy that simplifies the quadratic term. As a result, we can apply the reformulation per constraint and dualization approach to solve this linear model.
- Second, we derive a closed-form myopic solution for the stationary case of the single and multi-period inventory management problem, where we assume zero setup and zero production costs, and the yield uncertainty is handled by a static-dynamic strategy.
- Third, we develop a C&CG algorithm to solve optimally the adaptive lot-sizing problems under yield uncertainty within a static-dynamic strategy.
- Finally, we provide an in-depth analysis of the quality and performance of the developed adaptive robust models and a policy for production planning based on numerical experiments. We also present some observations about the optimality of our approaches within a dynamic strategy through a rolling horizon simulation.

The results show that our adaptive robust approximation is efficient, and it protects the system against uncertainties. We found that ARO leads to production plans that are more robust, yet still adjustable to changes in the production system.

3.3 Problem statement

3.3.1 Nominal problem

The single-level single-item LSP with backorder and production yield determines the production setups and lot sizes which minimize the overall costs, and meet the demands with quality goods in a finite horizon $T = \{1, \dots, |T|\}$. For each period t , the following parameters are given: setup cost s_t , unit production cost v_t , unit inventory cost h_t , unit backorder cost b_t , demand d_t , and finally production yield ρ_t . ρ_t is assumed to be strictly positive ($0 < \rho_t \leq 1$), where $\rho_t = 1$ indicates a perfect production yield where the entire lot size is made up of quality goods. The LSP aims to decide for each period t the quantity to produce X_t , the inventory level I_t , the backorder level B_t at the end of the period, and the setup decision Y_t . The deterministic formulation of the LSP is as follows:

$$\min \sum_{t \in T} (s_t Y_t + v_t X_t + h_t I_t + b_t B_t) \quad (3.1a)$$

s.t. :

$$I_t - B_t = I_{t-1} - B_{t-1} + \rho_t X_t - d_t \quad \forall t \in T \quad (3.1b)$$

$$X_t \leq M_t Y_t \quad \forall t \in T \quad (3.1c)$$

$$X_t, I_t, B_t \geq 0 \quad \forall t \in T$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

Without loss of generality, we assume that the inventory and the backorder levels at the beginning of the production planning horizon are zero. The objective function (3.1a) minimizes overall costs over the entire production horizon. Constraints (3.1b) are the inventory balance equations, where the production yield rate ρ_t allows us to compute the quantity of the quality goods ($\rho_t X_t$) obtained from the lot size X_t produced in the period t . Therefore, the total amount of quality goods and the inventory from the previous period ($t - 1$) fulfill the demand d_t from the current period t plus all backordered demand from the past, if any. If the amount of quality goods available is insufficient to meet the demand, then the quantity of missing goods is supplied by the backorder level (B_t). After satisfying both the demand and the backordered items, if any, the remaining amount of quality goods is kept in stock (I_t). Constraints (3.1c) are setup-forcing constraints that indicate a setup by $Y_t = 1$ whenever a lot size (X_t) is produced. This chapter considers the uncapacitated and capacitated problems. Here, Constraints (3.1c) can also represent capacity constraint

when the upper bound of the production quantity given by $M_t = \frac{\sum_{\tau \in T} d_\tau}{\min_{\tau \leq t} \rho_\tau}$ is replaced by the capacity C_t such that $M_t = \min \left\{ C_t, \frac{\sum_{\tau \in T} d_\tau}{\min_{\tau \leq t} \rho_\tau} \right\}$. This deterministic formulation does not consider any uncertainty in the model, and the production yield rate is assumed to be known. Thus, the model leads to suboptimal or even unrealistic plans in an uncertain context. We describe below the adaptive robust optimization approach that accounts for production yield uncertainty.

3.3.2 Analytical solution for the inventory management problem under static yield uncertainty

In this subsection, we analytically show that there is an optimal adaptive policy for the special case of the inventory management problem with no production cost, static uncertain yields (i.e., a constant nominal value and a maximum deviation), non-stationary demand, and a box uncertainty set. This special case extends the inventory management problem with the multiplicative random yield considered by Keren [2009] to the multi-period problem, and it minimizes the worst-case cost rather than the expected cost. The proposed policy also extends the results of Bertsimas and Thiele [2004], who consider the robust inventory management problem under uncertain demand (with fixed recourse on the adaptive variables), and also the case with uncertain production yield (and non-fixed recourse on the adaptive variables).

Since the adaptive problem reacts to the realization of the uncertain parameter in the previous period, an optimal policy must take into account the revealed information. However, the production yield can only be calculated after the end of production in a certain period. Thus, it becomes unfeasible to calculate the direct impact of the realization of uncertainty on the cost of periods subsequent to the current one. Therefore, we seek to find an optimal policy that disregards future costs. Proposition 3.3.1 describes the optimal myopic policy that computes the optimal production in period t for the inventory management problem by ignoring the future costs from period $t + 1$ to T . Proposition 3.3.2 indicates the optimality of the myopic policy under mild assumptions.

Proposition 3.3.1. *For the stationary inventory management problem with no setup and no unit production costs, with constant inventory unit cost, backorder unit cost, nominal value ($\bar{\rho}$) and maximum deviation ($\hat{\rho}$) of the production yield, with non-stationary demand, and under box uncertainty set, the optimal myopic adaptive robust policy is given by:*

$$X_t = \frac{\bar{D}_t - \sum_{\tau=1}^{t-1} \tilde{\rho}_\tau X_\tau}{\bar{\rho} + \hat{\rho} \left(\frac{h-b}{h+b} \right)} \quad (3.2)$$

Proof. The optimal adaptive robust policy provides the production quantity in period t after observing the production yield ρ_τ for all $\tau \in \{1, \dots, t-1\}$. Since the demand is non-stationary, we represent the cumulative demand up to period t by \bar{D}_t . Considering the box uncertainty set with static unit costs of inventory and backorder (respectively given by h and b), constant nominal value ($\bar{\rho}$) and constant maximum deviation of the production yield ($\hat{\rho}$), ρ_τ takes any value in the interval $[\bar{\rho} - \hat{\rho}; \bar{\rho} + \hat{\rho}]$. We assume δ_{t-1} to be the global production yield for the cumulative production up to period $t-1$. The amount of quality goods $\sum_{\tau=1}^{t-1} \tilde{\rho}_\tau X_\tau$ produced in periods prior to t (with $\tilde{\rho} \in [\bar{\rho} - \hat{\rho}, \bar{\rho} + \hat{\rho}]$) is equal to $\delta_{t-1} \sum_{\tau=1}^{t-1} X_\tau = \delta_{t-1} \bar{X}_{t-1}$ for some $\delta_{t-1} \in [\bar{\rho} - \hat{\rho}, \bar{\rho} + \hat{\rho}]$, where \bar{X}_{t-1} is the cumulative production up to period $t-1$.

Since X_t is defined only for period t , the production quantity from period t does not impact the cost in the previous periods (up to $t-1$). We seek a myopic policy that ignores the future costs of periods $t+1$ to T . Thus, X_t is chosen specifically only to minimize the cost up to period t . The production yield rate is computed only after having produced something in a given production period. Therefore, one should be satisfied on measuring the impact of the uncertainty on the total cost. As the backorder correspond to a negative inventory, the backorder costs are complementary to the inventory costs. Thus, these costs can be defined by a piecewise cost function. In a robust perspective, the impact should cover the worst realization of the uncertainty, leading to the worst feasible cost. For the inventory management problem, this cost is represented by $H_t(X_t)$, that is, the highest cost between the worst inventory cost and the worst backorder cost. $H_t(X_t)$ is given as follows:

$$H_t(X_t) = \max \left\{ \begin{array}{l} h [\delta_{t-1} \bar{X}_{t-1} + (\bar{\rho} + \hat{\rho}) X_t - \bar{D}_t] \\ b [\bar{D}_t - \delta_{t-1} \bar{X}_{t-1} - (\bar{\rho} - \hat{\rho}) X_t] \end{array} \right\}. \quad (3.3)$$

$H_t(X_t)$ is a piecewise linear convex cost function that reaches its minimum when the worst case inventory costs are equal to the worst case backorder costs [Metzker et al., 2023a]. Therefore, we have:

$$h [\delta_{t-1} \bar{X}_{t-1} + (\bar{\rho} + \hat{\rho}) X_t - \bar{D}_t] = b [\bar{D}_t - \delta_{t-1} \bar{X}_{t-1} - (\bar{\rho} - \hat{\rho}) X_t].$$

With a little algebra to isolate our production quantity X_t , we obtain:

$$X_t [h(\bar{\rho} + \hat{\rho}) + b(\bar{\rho} - \hat{\rho})] = (\bar{D}_t - \delta_{t-1}\bar{X}_{t-1}) (b + h),$$

which results in the myopic policy (3.2). □

Proposition 3.3.1 is optimal for the case where the future costs after period t that may be derived from the realization of the production yield in period t do not impact the validity of the policy. Proposition 3.3.2 indicates the conditions under which the policy is valid and optimal.

Proposition 3.3.2. *For the special case of the stationary inventory management problem under box uncertainty set, with no setup cost and no unit production costs, with constant nominal value and maximum deviation of the uncertain production yield, and with non-stationary demand, the myopic adaptive robust policy is optimal if:*

$$\frac{(\bar{\rho} - \hat{\rho})(\bar{\rho} + \hat{\rho})}{K} \leq 2\bar{\rho}, \quad (3.4)$$

with $K = \bar{\rho} + \hat{\rho}\left(\frac{h-b}{h+b}\right)$.

Proof. The myopic policy is optimal if the optimal production quantity in period t does not depend on the future costs (from period $t + 1$ to T). The proof is done recursively. First, we show that under condition (3.4), the future costs from period T do not impact the optimal cumulative production \bar{X}_{T-1} (up to period $T - 1$). We conclude this proof by recursively showing that future costs do not impact the optimality of the myopic policy for the previous periods.

Since T is the last production period, the policy given in (3.2) yields the optimal production quantity that should be produced in T . To demonstrate the validity and optimality of the policy, first we derive the cost in period T as a function of \bar{X}_{T-1} in function (3.3) with the optimal production quantity X_T given by (3.2). As the inventory and back-order costs are complementary, we choose arbitrarily the inventory costs to demonstrate our proof. Later, we show that accounting for these future costs to compute \bar{X}_{T-1} has no impact if condition (3.4) holds.

Following the myopic policy, the inventory cost for period T is given as follows:

$$H_T = h \left[\delta_{T-1}\bar{X}_{T-1} + (\bar{\rho} + \hat{\rho}) \frac{\bar{D}_T - \delta_{T-1}\bar{X}_{T-1}}{K} - \bar{D}_T \right]$$

$$H_T = \delta_{T-1} \bar{X}_{T-1} \left(1 - \frac{\bar{\rho} + \hat{\rho}}{K} \right) h + \frac{\bar{D}_T(\bar{\rho} + \hat{\rho} - K)}{K} h$$

The worst case value of δ_{T-1} does not depend on the cumulative production \bar{X}_{T-1} , and δ_{T-1} can take either one of the following two values:

$$\begin{aligned} \delta_{T-1} &= \bar{\rho} + \hat{\rho} & | & \frac{\bar{\rho} + \hat{\rho}}{K} \leq 1 \\ \delta_{T-1} &= \bar{\rho} - \hat{\rho} & | & \frac{\bar{\rho} + \hat{\rho}}{K} \geq 1 \end{aligned}$$

Note that when computing \bar{X}_{T-1} , δ_{T-2} is known, but δ_{T-1} is still unknown. Nevertheless, the values above characterize the bound on the impact of \bar{X}_{T-1} on the cost of period T . Condition $\frac{\bar{\rho} + \hat{\rho}}{K} \leq 1$ is valid only if $\frac{h-b}{h+b} \geq 1$, which is impossible for a positive b . Since the costs cannot be negative, the only realistic worst case scenario corresponds to $\delta = \bar{\rho} - \hat{\rho}$. Replacing δ_{T-1} by $\bar{\rho} - \hat{\rho}$ in H_T , and with a little algebra, we find that the cumulative quantity \bar{X}_{T-1} computed in period $T - 1$ impact the future cost F_T in period T as follows:

$$F_T(\bar{X}_{T-1}) = h \left[(\bar{\rho} - \hat{\rho}) - \frac{(\bar{\rho} - \hat{\rho})(\bar{\rho} + \hat{\rho})}{K} \right] \bar{X}_{T-1} \quad (3.5)$$

However, \bar{X}_{T-1} must be chosen to minimize the total cost in $T - 1$, which is given by the sum of the worst inventory or backorder cost and the future cost of \bar{X}_{T-1} in period T . The function representing this sum is given as follows:

$$H_{T-1}(\bar{X}_{T-1}) + F_T(\bar{X}_{T-1}) = \max \left\{ \begin{aligned} &h [\delta_{T-2} \bar{X}_{T-2} + (\bar{\rho} + \hat{\rho}) \bar{X}_{T-1} - \bar{D}_{T-1}] + F_t(\bar{X}_{T-1}) \\ &b [\bar{D}_{T-1} - \delta_{T-2} \bar{X}_{T-2} - (\bar{\rho} - \hat{\rho}) \bar{X}_{T-1}] + F_t(\bar{X}_{T-1}) \end{aligned} \right\}$$

This function ($H_{T-1}(\bar{X}_{T-1}) + F_T(\bar{X}_{T-1})$) is a piecewise linear convex function on the cumulative production \bar{X}_{T-1} , and its minimum remains in \bar{X}_{T-1} unless $F_T(\bar{X}_{T-1})$ changes the sign of the slope. This direction changes if $h \left[(\bar{\rho} + \hat{\rho}) + (\bar{\rho} - \hat{\rho}) - \frac{(\bar{\rho} - \hat{\rho})(\bar{\rho} + \hat{\rho})}{K} \right]$ becomes negative, that is, if condition (3.4) is not respected.

Finally, the policy relies on the observed cumulative production $\delta_{T-1} \bar{X}_{T-1}$ to compute \bar{X}_T (see the proof of Proposition 3.3.1). The value of \bar{X}_{T-2} has no impact on the costs in period T , because \bar{X}_{T-1} can take any value larger than \bar{X}_{T-2} , and ρ_{T-1} can take any value in $[\bar{\rho} - \hat{\rho}, \bar{\rho} + \hat{\rho}]$. The impact of \bar{X}_{T-2} on the future costs concern only the costs in period $T - 1$, and it is similar to (3.5). Thus, the optimal production \bar{X}_{T-2} does not depend on the future costs if condition (3.4) holds. Hence, the myopic policy is optimal in period

$T - 1$ if we follow the myopic policy in period T . As the myopic policy is valid in the last period T , we can demonstrate by recursion (from period T to period 1) that, under assumption (3.4), the myopic adaptive robust policy is optimal. \square

3.3.3 The adaptive robust model for LSP under yield uncertainty

This section introduces the adaptive multi-period lot-sizing problem with uncertain production yield and non-stationary costs. The setups are fixed at the beginning of the horizon, but the lot sizes, inventory and backorder levels are computed based on some decision rules, and the adaptive decisions react to the production yield observed in previous periods. We use the convexity and piecewise linearity of the inventory and backorder cost functions to translate the inventory balance constraints in terms of the inventory/backorder costs, where the backorders represent a negative stock [Bertsimas and Thiele, 2004]. The inventory balance constraints (3.1b) are then respectively replaced by the pair of inequalities given by (3.6b) and (3.6c) below. Thus, we can directly compute the worst inventory and backorder costs and give the highest worst cost using $H_t(\rho)$. Since the level of stocks or backorders cannot be precisely defined before the realization of uncertain yield, from a robust perspective, we might be satisfied to know the impact of realization of the uncertainty on costs. Therefore, considering the worst feasible cost resulting from the inventory management activity (here represented by $H_t(\rho)$) should be sufficient.

We model the problem as an adaptive robust LSP with uncertain production yield, and we rely on the budgeted uncertainty set introduced in [Bertsimas and Sim, 2004]. This set is based on the nominal value ($\bar{\rho}_t$) and the maximum deviation ($\hat{\rho}_t$) of the production yield rate. The uncertain production yield $\tilde{\rho}_t$ is given by $\tilde{\rho}_t = \bar{\rho}_t + Z_t \hat{\rho}_t$, where Z_t is the deviation of the production yield from its nominal value. As a result, the uncertain yield belongs to a range centred on its $\bar{\rho}$ and spread by its maximum deviation $\hat{\rho}$. To reduce the conservatism of the uncertainty set, the vector of disturbance \mathbf{Z} is controlled by the budget $\mathbf{\Gamma}$. More precisely, for each period t , \mathbf{Z}_t takes values in the budgeted uncertainty set $\mathcal{U}_t = \{\mathbf{Z}_t \in [0; 1]^T : \sum_{\tau=1}^t |Z_\tau^\tau| \leq \Gamma_t\}$, where the budget indicates the decision maker's risk acceptance. The larger the budget, the more averse to risk the decision-maker is. Note that if Γ_t is large enough (e.g., $\Gamma_t = T$), the budgeted uncertainty set reduces to the box uncertainty set.

We consider some affine decision rules to parameterize the adaptive variables as a function of the revealed production yield [Aharon et al., 2009]. We represent the adaptive lot size and inventory/backorder costs in t by functions that depend on the production yield observed in periods up to the last observation of the uncertain parameter (in period $t - 1$), as follows:

$$X_t(\rho) = X_t^0 + \sum_{\tau=1}^{t-1} X_t^\tau \rho_\tau \quad H_t(\rho) = H_t^0 + \sum_{\tau=1}^{t-1} H_t^\tau \rho_\tau$$

Considering the adaptive lot size and the adaptive inventory/backorder costs, we obtain the following ARC LSP model under production yield uncertainty:

$$\min_Y \sum_{t \in T} \left[s_t Y_t + \max_{\rho \in \mathcal{U}_t} \left\{ v_t X_t(\rho) + H_t(\rho) \right\} \right] \quad (3.6a)$$

s.t. :

$$\max_{\rho \in \mathcal{U}_t} \left\{ H_t(\rho) \right\} \geq h_t \max_{\rho \in \mathcal{U}_t} \left\{ \sum_{\tau=1}^t (\rho_\tau X_\tau(\rho) - d_\tau) \right\} \quad \forall t \in T \quad (3.6b)$$

$$\max_{\rho \in \mathcal{U}_t} \left\{ H_t(\rho) \right\} \geq -b_t \max_{\rho \in \mathcal{U}_t} \left\{ \sum_{\tau=1}^t (\rho_\tau X_\tau(\rho) - d_\tau) \right\} \quad \forall t \in T \quad (3.6c)$$

$$\max_{\rho \in \mathcal{U}_t} \left\{ X_t(\rho) \right\} \leq M_t Y_t \quad \forall t \in T$$

$$\max_{\rho \in \mathcal{U}_t} \left\{ X_t(\rho) \right\} \geq 0 \quad \forall t \in T$$

$$\max_{\rho \in \mathcal{U}_t} \left\{ H_t(\rho) \right\} \geq 0 \quad \forall t \in T$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

Replacing the uncertain yield and the adjustable variables according to their respective decision rules, we can reformulate the ARC model as follows:

$$\min \sum_{t \in T} \left(s_t Y_t + v_t X_t^0 + H_t^0 + \sum_{\tau=1}^{t-1} \bar{\rho}_\tau (v_t X_t^\tau + H_t^\tau) \right) + F \quad (3.7a)$$

s.t. :

$$F \geq \max_{\mathbf{Z}_t \in \mathcal{U}_t} \sum_{t \in T} \sum_{\tau=1}^{t-1} \hat{\rho}_\tau (v_t X_t^\tau + H_t^\tau) Z_t^\tau \quad (3.7b)$$

$$\begin{aligned}
 H_t^0 \geq \max_{\mathbf{Z}_t \in \mathcal{U}_t} \left\{ \sum_{\tau=1}^t \left(h_t(\bar{\rho}_\tau X_\tau^0 - d_\tau) + h_t \hat{\rho}_\tau X_\tau^0 Z_t^\tau + h_t \left[\sum_{r=1}^{\tau-1} (\bar{\rho}_\tau \bar{\rho}_r X_\tau^r + \hat{\rho}_\tau \bar{\rho}_r X_\tau^r Z_t^\tau + \right. \right. \right. \\
 \left. \left. \left. \bar{\rho}_\tau \hat{\rho}_r X_\tau^r Z_t^\tau + \hat{\rho}_\tau \hat{\rho}_r Z_t^\tau X_\tau^r Z_t^\tau) \right] \right) - \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau H_t^\tau + \hat{\rho}_\tau H_t^\tau Z_t^\tau) \right\} \quad \forall t \in T
 \end{aligned} \tag{3.7c}$$

$$\begin{aligned}
 H_t^0 \geq \max_{\mathbf{Z}_t \in \mathcal{U}_t} \left\{ \sum_{\tau=1}^t \left(b_t(d_\tau - \bar{\rho}_\tau X_\tau^0) - b_t \hat{\rho}_\tau X_\tau^0 Z_t^\tau - b_t \left[\sum_{r=1}^{\tau-1} (\bar{\rho}_\tau \bar{\rho}_r X_\tau^r + \hat{\rho}_\tau \bar{\rho}_r X_\tau^r Z_t^\tau + \right. \right. \right. \\
 \left. \left. \left. \bar{\rho}_\tau \hat{\rho}_r X_\tau^r Z_t^\tau + \hat{\rho}_\tau \hat{\rho}_r Z_t^\tau X_\tau^r Z_t^\tau) \right] \right) - \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau H_t^\tau + \hat{\rho}_\tau H_t^\tau Z_t^\tau) \right\} \quad \forall t \in T
 \end{aligned} \tag{3.7d}$$

$$\max_{\mathbf{Z}_t \in \mathcal{U}_t} \left\{ X_t^0 + \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau X_t^\tau + \hat{\rho}_\tau X_t^\tau Z_t^\tau) \right\} \leq M_t Y_t \quad \forall t \in T \tag{3.7e}$$

$$\min_{\mathbf{Z}_t \in \mathcal{U}_t} \left\{ X_t^0 + \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau X_t^\tau + \hat{\rho}_\tau X_t^\tau Z_t^\tau) \right\} \geq 0 \quad \forall t \in T \tag{3.7f}$$

$$\min_{\mathbf{Z}_t \in \mathcal{U}_t} \left\{ H_t^0 + \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau H_t^\tau + \hat{\rho}_\tau H_t^\tau Z_t^\tau) \right\} \geq 0 \quad \forall t \in T \tag{3.7g}$$

$$\begin{aligned}
 F, X_t^0, H_t^0 &\geq 0 & \forall t \in T \\
 X_t^\tau, H_t^\tau &\in \mathbb{R} & \forall t \in T; \tau < t \\
 Y_t &\in \{0, 1\} & \forall t \in T
 \end{aligned}$$

To apply the constraint-wise RO technique, the part of the objective function, given by (3.6a), which contains uncertain parameters is replaced by an epigraph represented by an arbitrary variable F . As a result, the objective function is given by (3.7b), and its uncertain part is represented in Constraint (3.7b). Constraints (3.7c) and (3.7d) describe the inventory and backorder cost inequalities, respectively. Constraints (3.7e) are the setup forcing constraints. Finally, Constraints (3.7f) and (3.7g) ensure that the adaptive lot size and inventory/backorder costs are non-negative.

3.4 Solution approaches

In this section, first, we propose an adaptive linear robust approximation (ARO_Γ) for the LSP under uncertain yield. Then, we suggest a $C\&CG_\Gamma$ algorithm to compute an optimal adaptive robust plan.

3.4.1 Adaptive robust approximation: ARO_Γ^0

Our ARC model has a non-fixed recourse in Constraints (3.7c) and Constraints (3.7d) because the uncertain production yield is a coefficient of the adaptive lot size decisions. Since the reformulation of an adaptive robust LSP model with a non-fixed recourse is complex, we propose a linear approximation (denoted by ARO_Γ^0) that ignores the quadratic term in Constraints (3.7c) and Constraints (3.7d). As the disturbance Z_t^τ is lower than 1, the quadratic disturbance has an upper bound equal to 1 (e.g., $Z_3^1 Z_3^2 \leq 1 \times 1 \leq 1$). Therefore, the quadratic term leads to a low variance in costs. As a result, Constraints (3.7c) and Constraints (3.7d) are replaced by Constraints (3.8) and Constraints (3.9), respectively. In the new constraints, the terms with quadratic uncertainty are neglected.

$$\begin{aligned}
 H_t^0 \geq & h_t \left[\sum_{\tau=1}^t \left(\bar{\rho}_\tau X_\tau^0 - d_\tau + \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \bar{\rho}_r X_\tau^r \right) \right] + \left[\sum_{\tau=1}^{t-1} \left(h_t \hat{\rho}_\tau X_\tau^0 - \hat{\rho}_\tau H_t^\tau + \sum_{r=1}^{\tau-1} h_t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r \right) Z_t^\tau \right] \\
 & + h_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau \right) Z_t^t + h_t \left[\sum_{\tau=1}^t \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \hat{\rho}_r X_\tau^r Z_t^\tau \right] - \sum_{\tau=1}^{t-1} \bar{\rho}_\tau H_t^\tau \quad \forall t \in T
 \end{aligned} \tag{3.8}$$

$$\begin{aligned}
 H_t^0 \geq & -b_t \left[\sum_{\tau=1}^t \left(\bar{\rho}_\tau X_\tau^0 - d_\tau + \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \bar{\rho}_r X_\tau^r \right) \right] - \left[\sum_{\tau=1}^{t-1} \left(b_t \hat{\rho}_\tau X_\tau^0 + \hat{\rho}_\tau H_t^\tau + \sum_{r=1}^{\tau-1} b_t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r \right) Z_t^\tau \right] \\
 & - b_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau \right) Z_t^t - b_t \left[\sum_{\tau=1}^t \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \hat{\rho}_r X_\tau^r Z_t^\tau \right] - \sum_{\tau=1}^{t-1} \bar{\rho}_\tau H_t^\tau \quad \forall t \in T
 \end{aligned} \tag{3.9}$$

The resulting ARO_Γ model is equivalent to a robust model in which we can apply the reformulation per constraint and dualization approach. The details of the reformulation are available in B.1. The final ARO_Γ approximation is given as follows:

$$\min \sum_{t \in T} \left[s_t Y_t + v_t X_t^0 + H_t^0 + \Gamma_t \gamma_t + \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau (v_t X_t^\tau + H_t^\tau) + \alpha_t^\tau + \delta_t^\tau) \right] \tag{3.10a}$$

s.t. :

$$H_t^0 \geq \Gamma_t \lambda_t - \sum_{\tau=1}^{t-1} \bar{\rho}_\tau H_t^\tau + \sum_{\tau=1}^t \left[h_t (\bar{\rho}_\tau X_\tau^0 - d_\tau) + \mu_t^\tau + \varepsilon_t^\tau + \sum_{r=1}^{\tau-1} h_t \bar{\rho}_\tau \bar{\rho}_r X_\tau^r \right] \quad \forall t \in T \quad (3.10b)$$

$$H_t^0 \geq \Gamma_t \psi_t - \sum_{\tau=1}^{t-1} \bar{\rho}_\tau H_t^\tau + \sum_{\tau=1}^t \left[b_t (d_\tau - \bar{\rho}_\tau X_\tau^0) + \varpi_t^\tau + \xi_t^\tau - \sum_{r=1}^{\tau-1} b_t \bar{\rho}_\tau \bar{\rho}_r X_\tau^r \right] \quad \forall t \in T \quad (3.10c)$$

$$X_t^0 + \Gamma_t \eta_t + \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau X_t^\tau + \beta_t^\tau + \theta_t^\tau) \leq M_t Y_t \quad \forall t \in T \quad (3.10d)$$

$$X_t^0 - \Gamma_t \phi_t + \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau X_t^\tau - \pi_t^\tau - \chi_t^\tau) \geq 0 \quad \forall t \in T \quad (3.10e)$$

$$H_t^0 - \Gamma_t \epsilon_t + \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau H_t^\tau - \sigma_t^\tau - \kappa_t^\tau) \geq 0 \quad \forall t \in T \quad (3.10f)$$

$$\mu_t^t - \varepsilon_t^t + \mu_t'^\tau - \varepsilon_t'^\tau = h_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau \right) \quad \forall t \in T \quad (3.10g)$$

$$\lambda_t - \mu_t'^\tau - \varepsilon_t'^\tau \geq 0 \quad \forall t \in T; \tau < t \quad (3.10h)$$

$$\varpi_t^t - \xi_t^t + \varpi_t'^t - \xi_t'^t = -b_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau \right) \quad \forall t \in T \quad (3.10i)$$

$$\psi_t - \varpi_t'^t - \xi_t'^t \geq 0 \quad \forall t \in T; \tau < t \quad (3.10j)$$

$$\alpha_t^\tau - \delta_t^\tau + \alpha_t'^\tau - \delta_t'^\tau = \hat{\rho}_\tau (v_t X_t^\tau + H_t^\tau) \quad \forall t \in T; \tau < t \quad (3.10k)$$

$$\gamma_t - \alpha_t'^\tau - \delta_t'^\tau \geq 0 \quad \forall t \in T; \tau < t \quad (3.10l)$$

$$\beta_t^\tau - \theta_t^\tau + \beta_t'^\tau - \theta_t'^\tau = \hat{\rho}_\tau X_t^\tau \quad \forall t \in T; \tau < t \quad (3.10m)$$

$$\eta_t - \beta_t'^\tau - \theta_t'^\tau \geq 0 \quad \forall t \in T; \tau < t \quad (3.10n)$$

$$-\pi_t^\tau + \chi_t^\tau - \pi_t'^\tau + \chi_t'^\tau = \hat{\rho}_\tau X_t^\tau \quad \forall t \in T; \tau < t \quad (3.10o)$$

$$\phi_t - \pi_t'^\tau - \chi_t'^\tau \geq 0 \quad \forall t \in T; \tau < t \quad (3.10p)$$

$$-\sigma_t^\tau + \kappa_t^\tau - \sigma_t'^\tau + \kappa_t'^\tau = \hat{\rho}_\tau H_t^\tau \quad \forall t \in T; \tau < t \quad (3.10q)$$

$$\epsilon_t - \sigma_t'^\tau - \kappa_t'^\tau \geq 0 \quad \forall t \in T; \tau < t \quad (3.10r)$$

$$\mu_t^\tau - \varepsilon_t^\tau + \mu_t'^\tau - \varepsilon_t'^\tau = \hat{\rho}_\tau (h_t X_\tau^0 - H_t^\tau) + \sum_{r=1}^{\tau-1} h_t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r + \sum_{r=\tau+1}^t h_t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r \quad \forall t \in T; \tau < t \quad (3.10s)$$

$$\lambda_t - \mu_t'^\tau - \varepsilon_t'^\tau \geq 0 \quad \forall t \in T; \tau < t \quad (3.10t)$$

$$\varpi_t^\tau - \xi_t^\tau + \varpi_t'^\tau - \xi_t'^\tau = -\hat{\rho}_\tau (b_t X_\tau^0 + H_t^\tau) - \sum_{r=1}^{\tau-1} b_t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r - \sum_{r=\tau+1}^t b_t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r \quad \forall t \in T; \tau < t \quad (3.10u)$$

$$\psi_t - \varpi_t'^\tau - \xi_t'^\tau \geq 0 \quad \forall t \in T; \tau < t \quad (3.10v)$$

$$X_t^0, H_t^0, \gamma_t, \lambda_t, \psi_t, \eta_t, \epsilon_t, \phi_t \geq 0 \quad \forall t \in T \quad (3.10w)$$

$$\alpha_t^\tau, \alpha_t'^\tau, \beta_t^\tau, \beta_t'^\tau, \delta_t^\tau, \delta_t'^\tau, \kappa_t^\tau, \kappa_t'^\tau, \mu_t^\tau, \mu_t'^\tau, \chi_t^\tau, \chi_t'^\tau \geq 0 \quad \forall t \in T; \tau < t \quad (3.10x)$$

$$\xi_t^\tau, \xi_t'^\tau, \theta_t^\tau, \theta_t'^\tau, \varepsilon_t^\tau, \varepsilon_t'^\tau, \varpi_t^\tau, \varpi_t'^\tau, \sigma_t^\tau, \sigma_t'^\tau \geq 0 \quad \forall t \in T; \tau < t \quad (3.10y)$$

$$X_t^\tau, H_t^\tau \in \mathbb{R} \quad \forall t \in T; \tau < t \quad (3.10z)$$

$$Y_t \in \{0, 1\} \quad \forall t \in T \quad (3.10)$$

3.4.2 Adaptive robust linearization: ARO_{Γ}^1

Since the reformulation ARO_{Γ}^0 neglects part of the inventory management costs, we propose another linear approximation (denoted by ARO_{Γ}^1) where the quadratic term is replaced by its upper bound value. In fact, the disturbance Z_t^r is bounded in the interval $[-1, 1]$, the quadratic disturbance has an upper bound equal to 1 (e.g., $|Z_3^1| |Z_3^2| \leq 1 \times 1 \leq 1$). Therefore, the quadratic term leads to a low variance in costs. In fact, we have an increase of at most one unit of cost on the inventory management costs when the quadratic term is considered. As a result, Constraints (3.7c) and Constraints (3.7d) are replaced by Constraints (3.8) and Constraints (3.9), respectively. In the new constraints, the terms with quadratic uncertainty are replaced by their upper bound.

$$\begin{aligned} H_t^0 \geq & h_t \left[\sum_{\tau=1}^t \left(\bar{\rho}_{\tau} X_{\tau}^0 - d_{\tau} + \sum_{r=1}^{\tau-1} (\bar{\rho}_{\tau} \bar{\rho}_r + \hat{\rho}_{\tau} \hat{\rho}_r) X_{\tau}^r \right) \right] + \left[\sum_{\tau=1}^{t-1} \left(h_t \hat{\rho}_{\tau} X_{\tau}^0 - \hat{\rho}_{\tau} H_t^{\tau} + \sum_{r=1}^{\tau-1} h_t \hat{\rho}_{\tau} \bar{\rho}_r X_{\tau}^r \right) Z_t^{\tau} \right] \\ & + h_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_{\tau} X_t^{\tau} \right) Z_t^t + h_t \left[\sum_{\tau=1}^t \sum_{r=1}^{\tau-1} \bar{\rho}_{\tau} \hat{\rho}_r X_{\tau}^r Z_t^{\tau} \right] - \sum_{\tau=1}^{t-1} \bar{\rho}_{\tau} H_t^{\tau} \end{aligned} \quad \forall t \in T \quad (3.11)$$

$$\begin{aligned} H_t^0 \geq & -b_t \left[\sum_{\tau=1}^t \left(\bar{\rho}_{\tau} X_{\tau}^0 - d_{\tau} + \sum_{r=1}^{\tau-1} (\bar{\rho}_{\tau} \bar{\rho}_r - \hat{\rho}_{\tau} \hat{\rho}_r) X_{\tau}^r \right) \right] - \left[\sum_{\tau=1}^{t-1} \left(b_t \hat{\rho}_{\tau} X_{\tau}^0 + \hat{\rho}_{\tau} H_t^{\tau} + \sum_{r=1}^{\tau-1} b_t \hat{\rho}_{\tau} \bar{\rho}_r X_{\tau}^r \right) Z_t^{\tau} \right] \\ & - b_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_{\tau} X_t^{\tau} \right) Z_t^t - b_t \left[\sum_{\tau=1}^t \sum_{r=1}^{\tau-1} \bar{\rho}_{\tau} \hat{\rho}_r X_{\tau}^r Z_t^{\tau} \right] - \sum_{\tau=1}^{t-1} \bar{\rho}_{\tau} H_t^{\tau} \end{aligned} \quad \forall t \in T \quad (3.12)$$

As a result, the final ARO_{Γ} model given in 3.10 should becomes:

$$\min \sum_{t \in T} \left[s_t Y_t + v_t X_t^0 + H_t^0 + \Gamma_t \gamma_t + \sum_{\tau=1}^{t-1} (\bar{\rho}_{\tau} (v_t X_t^{\tau} + H_t^{\tau}) + \alpha_t^{\tau} + \delta_t^{\tau}) \right] \quad (3.13a)$$

s.t. :

$$(3.10d), (3.10e), (3.10f), (3.10g), (3.10h), (3.10i), (3.10j), (3.10k), (3.10l), (3.10m), (3.10n), (3.10o) \quad (3.13b)$$

$$(3.10p), (3.10q), (3.10r), (3.10s), (3.10t), (3.10u), (3.10v), (3.10w), (3.10x), (3.10y), (3.10z) \quad (3.10) \quad (3.13c)$$

$$H_t^0 \geq \Gamma_t \lambda_t - \sum_{\tau=1}^{t-1} \bar{\rho}_\tau H_t^\tau + \sum_{\tau=1}^t \left[h_t(\bar{\rho}_\tau X_\tau^0 - d_\tau) + \mu_t^\tau + \varepsilon_t^\tau + \sum_{r=1}^{\tau-1} h_t(\bar{\rho}_\tau \bar{\rho}_r + \hat{\rho}_\tau \hat{\rho}_r) X_\tau^r \right] \quad \forall t \in T \quad (3.13d)$$

$$H_t^0 \geq \Gamma_t \psi_t - \sum_{\tau=1}^{t-1} \bar{\rho}_\tau H_t^\tau + \sum_{\tau=1}^t \left[b_t(d_\tau - \bar{\rho}_\tau X_\tau^0) + \varpi_t^\tau + \xi_t^\tau - \sum_{r=1}^{\tau-1} b_t(\bar{\rho}_\tau \bar{\rho}_r - \hat{\rho}_\tau \hat{\rho}_r) X_\tau^r \right] \quad \forall t \in T \quad (3.13e)$$

3.4.3 A column-and-constraint generation algorithm

As presented in Section 3.4.1, ARO_Γ underestimates the cost since it neglects the quadratic term in the inventory and backorder cost constraints. This section proposes a new approach that relies on the *C&CG* framework introduced by Zeng and Zhao [2013] to iteratively add a set of inventory and backorder cost constraints (with the quadratic term). More precisely, the master problem (*MP*) is the ARO_Γ model presented in Section 3.4.1, where Constraints (3.8) and Constraints (3.9) partially account for the uncertain production yield, and Constraints (3.7c) and Constraints (3.7d) accounting for the quadratic terms are added iteratively with the help of (3.14) and (3.15), respectively.

At each iteration, we add Constraints (3.7c) (resp. Constraints (3.7d)) associated with the disturbances given by the vector \mathbf{Z} that lead to the largest inventory costs (resp. backorder costs). We denote as \mathbf{U}^{INV} (resp. \mathbf{U}^{BACK}) the partial set of the considered disturbance. To find the worst case disturbances, we solve sub-problems SP_t^{INV} and SP_t^{BACK} for each period $t \in T$, as follows:

$$SP_t^{INV} : \max_{\mathbf{Z}_t \in \mathbf{U}^{INV}} \sum_{\tau=1}^t [h_t(\bar{\rho}_\tau X_\tau^0 - d_\tau) + h_t \hat{\rho}_\tau X_\tau^0 Z_t^\tau + h_t [\sum_{r=1}^{\tau-1} (\bar{\rho}_\tau \bar{\rho}_r X_\tau^r + \hat{\rho}_\tau \bar{\rho}_r X_\tau^r Z_t^r + \bar{\rho}_\tau \hat{\rho}_r X_\tau^r Z_t^r + \hat{\rho}_\tau \hat{\rho}_r Z_t^r X_\tau^r Z_t^r)] - H_t^0 - \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau H_t^\tau + \hat{\rho}_\tau H_t^\tau Z_t^\tau)] \quad (3.14)$$

$$SP_t^{BACK} : \max_{\mathbf{Z}_t \in \mathbf{U}^{BACK}} \sum_{\tau=1}^t [b_t(d_\tau - \bar{\rho}_\tau X_\tau^0) - b_t \hat{\rho}_\tau X_\tau^0 Z_t^\tau - b_t [\sum_{r=1}^{\tau-1} (\bar{\rho}_\tau \bar{\rho}_r X_\tau^r + \hat{\rho}_\tau \bar{\rho}_r X_\tau^r Z_t^r + \bar{\rho}_\tau \hat{\rho}_r X_\tau^r Z_t^r + \hat{\rho}_\tau \hat{\rho}_r Z_t^r X_\tau^r Z_t^r)] - H_t^0 - \sum_{\tau=1}^{t-1} (\bar{\rho}_\tau H_t^\tau + \hat{\rho}_\tau H_t^\tau Z_t^\tau)] \quad (3.15)$$

The sub-problems seek the worst case disturbances of the uncertain production yield for each period t under the uncertainty set \mathcal{U}_t with respect to the feasible solution (S^*) obtained from *MP* of the current iteration. Therefore, we fix the values of the X_t^0 and X_t^τ variables in the sub-problems to their values in the current solution S^* . As a result, we obtain the worst case disturbances \mathbf{Z}_t^{INV} (resp. \mathbf{Z}_t^{BACK}) to include them in the set \mathbf{U}^{INV}

(resp. \mathbf{u}^{BACK}) of the MP from the next iteration.

Algorithm 1 $C\&CG_\Gamma$ for the adaptive robust LSP under yield uncertainty

```

1: procedure CALCULATION OF AN OPTIMAL ADAPTIVE ROBUST SOLUTION
2:    $MP = ARO_\Gamma$ 
3:    $LB = 0$ ;  $UB = \infty$ ;  $nbIter = 0$ ; time = 0 : 00
4:    $\mathcal{U}_t^{INV} = \{ \}$ ;  $\mathcal{U}_t^{BACK} = \{ \}$ ;  $S^* = \emptyset$ 
5:   while  $\frac{UB-LB}{LB} \geq 10^{-5}$  &  $time < 1h$  do
6:     Solve  $MP$  to get the current solution  $S^*$  and to update the  $LB$ 
7:     for  $t \in T$  do
8:       Solve  $SP_t^{INV}$  (resp.  $SP_t^{BACK}$ ), by fixing  $\mathbf{X}$  to its value defined in  $S^*$  to get  $\mathbf{Z}_t^{INV}$  (resp.
        $\mathbf{Z}_t^{BACK}$ )
9:       Add  $\mathbf{Z}_t^{INV}$  (resp.  $\mathbf{Z}_t^{BACK}$ ) to  $\mathcal{U}_t^{INV}$  (resp.  $\mathcal{U}_t^{BACK}$ ) and update the  $MP$ 
10:    end for
11:    Solve  $MP$ , fixing  $Y_t, X_t^0, X_t^\tau$  by the respective values defined in  $S^*$  to update  $UB$ 
12:     $nbIter+ = 1$ 
13:  end while
14: return  $S^*$ 
15: end procedure

```

Algorithm 1 gives the steps of the $C\&CG_\Gamma$ to solve the adaptive LSP with uncertain production yield and a budgeted uncertainty set. Ideally, the algorithm stops when the lower bound (LB) is equal to the upper bound (UB). As the sets \mathcal{U}^{INV} and \mathcal{U}^{BACK} are built iteratively, MP underestimate the cost, and its solution yields a LB of the optimal solution. However, the solution S^* of MP is a feasible (but not necessarily optimal) solution. For simplicity, UB is computed with MP using with the worst case disturbance associated with S^* , and with the variables Y_t, X_t^0, X_t^τ set to their values in S^* .

At each iteration of algorithm 1, we solve MP to obtain LB and a feasible solution S^* . Then, we find the worst-case yield associated with S^* by solving (3.14) and (3.15), where $Y_t, X_t^0, X_t^\tau, H_t^0$ and H_t^τ correspond to their respective values in S^* . We extend MP to account for the worst case disturbance values \mathbf{Z} , where \mathbf{Z}_t^{INV} (resp. \mathbf{Z}_t^{BACK}) belong to \mathcal{U}_t^{INV} (resp. \mathcal{U}_t^{BACK}) for each period t . Finally, we compute UB with the worst case disturbances \mathbf{Z}_t^{INV} (resp. \mathbf{Z}_t^{BACK}) associated with a feasible solution S^* obtained in the current iteration. We repeat this process until the relative difference between LB and UB is insignificant (about 10^{-5}), or the computation time exceeds the time limit of 1 hour.

3.5 Numerical tests and discussions

The goals of the computational experiments are fourfold: i) to demonstrate the robustness of the adaptive robust models; ii) to present an in-depth investigation of the computation of adaptive robust plans, their performance, the quality of their solutions, and their computational efficiency; iii) to demonstrate that our ARO_Γ approximation provides good adaptive plans, even neglecting some costs on the inventory balance constraints; iv) to highlight the quality of our adaptive robust plans (i.e., ARO_Γ and $C\&CG_\Gamma$) to mitigate uncertainties in an adaptive context.

For this, we consider the following models with budgeted uncertainty set and budget Γ in the experiments:

1. RO_Γ , the static robust lot-sizing model presented in B.2
2. ARO_Γ^0 , our adaptive robust lot-sizing linear approximation, where the quadratic term is omitted
3. ARO_Γ^1 , our adaptive robust lot-sizing linear approximations, where the quadratic term is replaced by its upper bound, such that $Z_t^r Z_t^r = 1$ (resp. $Z_t^r Z_t^r = -1$) for the inventory cost constraints (resp. backorder cost constraints)
4. ARO_Γ , our adaptive robust lot-sizing linear approximations which yields the lowest costs in the simulation
5. $C\&CG_\Gamma$, the C&CG approach to computing an optimal adaptive lot-sizing solution, for which the initial MP is given by ARO_Γ
6. ARO^0 , the optimal myopic adaptive robust policy for the adaptive robust inventory management problem with box uncertainty set, where the future costs from periods $t + 1$ to T are neglected

In the experimental results, we assume the robust model as a natural benchmark for the adaptive robust plans because the static framework provides a feasible (but suboptimal) plan in the static-dynamic strategy [Metzker et al., 2023a, Tunc et al., 2013]. Section 3.5.1 presents the instances generation approach, and it introduces the simulation framework used to evaluate the models and algorithms proposed in this chapter. Section 3.5.2 discusses the quality of the solutions, and it concludes with the performance and quality of the proposed adaptive models. Finally, Section 3.5.2 evaluates adaptive solutions within a dynamic strategy through rolling horizon simulation. We refer interested readers to B.5 where some conclusions about the optimal myopic adaptive robust policy introduced in Section 3.3.2 is presented.

3.5.1 Instance generation and simulation approach

The experiments are performed with randomly generated instances following the standard approach in the literature for LSP problems. The value of each parameter was drawn from a uniform distribution, such that the production cost, the holding stock cost, the demand, the nominal value, and the maximum deviation of the uncertain production yield supports correspond to the following intervals: $v_t \in U(10, 30)$, $h_t \in U(1, 10)$, $d_t \in U(140, 480)$, $\bar{\rho}_t \in U(0.5, 0.7)$, and $\hat{\rho}_t \in U(0.1, 0.3)$, respectively. The setup costs (\mathbf{s}) are computed with the following formula based on the time between orders (TBO): $s_t = \frac{\bar{D}_t \cdot TBO^2 \cdot h_t}{2}$, where \bar{D}_t represents the average demand up to period t . We set the inventory and the backorder levels at the beginning of the horizon to zero, and we define the capacity in terms of a tightness control factor Φ , such that the capacity is given by $C_t = \Phi M_t$. We consider instances with 4, 12, and 24 periods, a time between orders of 1, 2, or 4, a backorder cost equaling 2, 5, or 10 times the holding cost for each period t , and a Φ of 25 %, 50%, 75 % or 100%. The case where $\Phi = 1$ (or 100% of \mathbf{M}) corresponds to the uncapacitated variant of the problem. In the uncapacitated case, the capacity allows the production of enough quality goods to meet all the demands even if the production yield is very low. Considering a full factorial design for these parameters leads to 108 instances. As the optimal policy is valid only for the stationary and uncapacitated problem, we generate 18 additional instances resulting from three replications of the full factorial design of the following parameters: 4, 12, and 24 periods, and the backorder cost equals 5 or 10 times the holding cost for each period t .

To investigate the impact of the proposed models, we evaluate their performance through a simulation with $|\Omega| = 1000$ Monte Carlo generated scenarios, where each scenario is a possible realization of the production yields over the planning horizon. The production yields were simulated with a uniform distribution with support $[\bar{\rho}_t - \hat{\rho}_t; \bar{\rho}_t + \hat{\rho}_t]$ in period t . In the simulation, the adaptive production quantity for each scenario ω is computed with the affine function presented in section 3.4.1, and the factors of the production variables for each period t (X_t^0 and X_t^τ) are obtained from the optimal solution given by ARO_Γ and $C\&CG_\Gamma$, such that $\mathbf{X}_t(\rho^\omega) = X_t^0 + \sum_{\tau=1}^{t-1} X_t^\tau \rho_\tau^\omega$.

3.5.2 Experimental results

This section reports the performance and the quality of the adaptive production plans created with the adaptive robust models.

Performance of the ARO_{Γ} linear approximations

To analyse the quality of our linear approximations, we investigate their performance with regard to the average and pessimistic perspective of their costs in a simulation. For that, we consider the uncapacitated version of the problem. We recall that ARO_{Γ}^0 refers to the linear approximation that omit the quadratic terms, whereas the ARO_{Γ}^1 model replaces the quadratic terms by 1 such a way the inventory management costs are as high as possible. Table 3.1 reports the results for ARO_{Γ}^0 , ARO_{Γ}^1 , and $C\&CG_{\Gamma}$ over all the instances. All models are solved to optimality, and we evaluate their costs depending on Γ , where the case with $\Gamma = T$ is the most conservative solution that corresponds to the box uncertainty set. The table gives the following outcomes from the simulation step: the average expected cost (Exp. Cost column), the 99th percentile costs (99th p.c. column), and the coefficient of variation (CV column). The coefficient of variation CV represents the variability of the simulated costs as the ratio between the standard deviation and the average costs of the production plans.

Table 3.1 – Result comparisons of the uncapacitated ARO_{Γ}^0 and ARO_{Γ}^1 models

Γ	ARO_{Γ}^0				ARO_{Γ}^1				$C\&CG_{\Gamma}$			
	Exp. Cost	99 th p.c.	Time	CV	Exp. Cost	99 th p.c.	Time	CV	Exp. Cost	99 th p.c.	Time	CV
0.1T	504616	580476	2.4	6.0%	550297	633024	0.9	11.10%	465344	552256	92	6.67%
0.2T	342222	388081	0.0	4.98%	369780	419332	1.7	9.79%	465723	520730	112	4.49%
0.3T	454873	494676	1.6	3.79%	498434	624473	1.9	7.64%	466358	513679	115	3.93%
0.4T	501938	537867	2.1	3.29%	544333	637526	4.2	6.43%	474310	511691	101	3.69%
0.5T	503743	541922	1.7	3.44%	510748	578632	3.8	5.22%	468084	503536	92	3.08%
0.6T	455330	489727	1.5	3.26%	466531	524356	3.3	4.66%	475156	509251	585	2.78%
0.7T	400891	435820	1.0	3.52%	369944	414344	1.6	4.45%	483046	512210	239	2.54%
0.8T	494220	529562	1.6	3.15%	506110	569825	2.9	4.96%	482165	523926	274	3.42%
0.9T	493520	527989	1.3	3.04%	465058	521511	2.5	4.61%	474514	511333	213	3.55%
T	454096	485666	1.2	3.02%	465040	521540	2.0	4.6%	481957	515872	311	3.31%

Table 3.1 indicates that ARO_{Γ}^0 gives the best simulated results. ARO_{Γ}^0 has the lowest average costs, about 460545 over all budgets of uncertainty compared to the average expected cost of 474627 for ARO_{Γ}^1 . Furthermore, ARO_{Γ}^0 has the lowest variation in costs with a coefficient of variation of about 3.8%, while ARO_{Γ}^1 has a CV of about 6.2% for all Γ . However, both models converge to the same optimal adaptive solution with the help of the decomposition method $C\&CG_{\Gamma}$, with average expected cost of 473665 over all Γ , and a coefficient of variation of 3.8%. As the two linear approximations lead to equivalent optimal adaptive planes given by $C\&CG_{\Gamma}$, for the other results reported in this work, ARO_{Γ} refers to ARO_{Γ}^0 linearization. We refer interested readers to B.3 where the conclusions for the introduced capacitated model are presented.

Average cost and solution quality

To validate the quality of the production plans obtained with the different models, we examine first the impact of the risk acceptance level given by the budget of uncertainty Γ on the average costs obtained from each considered model. Then, we analyse the setup, lot-size, inventory and backorder decisions resulting from the different models. As the uncapacitated and capacitated models have similar performances, we analyse the quality of the production plans based on the uncapacitated models. It is common knowledge that the robust model within a static strategy gives an upper bound on the optimal adaptive costs.

Figure 3.1 reports the impact of Γ on the average 95th percentile, 99th percentile and worst case costs from the simulation, in comparison to the objective values for all the instances for RO_Γ , ARO_Γ and $C\&CG_\Gamma$. Figure 3.1 confirms that RO_Γ remains very conservative, since it has the largest average costs. As the average costs computed with RO_Γ increase significantly with Γ , while its average worst case costs decrease slightly, RO_Γ is the most conservative model. The second most conservative approach is $C\&CG_\Gamma$, due to the marginal increase of its average costs from the simulation, even if its average costs increase with Γ . Since the quadratic terms in the inventory balance constraints are linearized in ARO_Γ , and due to the negligible impact of these quadratic terms on the overall costs, ARO_Γ has the smoothest impact on costs with the increase of Γ and the lowest simulated costs compared to the aforementioned models. This reinforces the conclusion that ARO_Γ is less conservative, although it remains robust.

From Figure 3.1, ARO_Γ provides the plans which are better than the optimal adaptive robust plans obtained with $C\&CG_\Gamma$. The relative difference between the average costs simulated with ARO_Γ and $C\&CG_\Gamma$ is not high. The difference is higher within high-risk decision contexts (scenarios with higher flexibility, that is, low Γ), and it significantly increases with the budget of uncertainty Γ . Low values of Γ reduce the possibility that several yield realizations take their worst values, reducing the impact of the quadratic term. In a more conservative decision context, risks must be avoided, and even if ARO_Γ does not consider the quadratic terms in all their complexity, the definition of the uncertainty set forces the solution calculated by the approximate model to protect itself against the worst case realization of uncertainty. As a result, ARO_Γ becomes closer to the optimal adaptive plan. Considering the fundamentals of ARO, it is counter intuitive to get an $ARO_{\Gamma=T}$ worst case cost lower than the cost obtained with the plan given by $C\&CG_{\Gamma=T}$. In fact, our approximation has a real risk aversion which is lower than Γ , because further risks

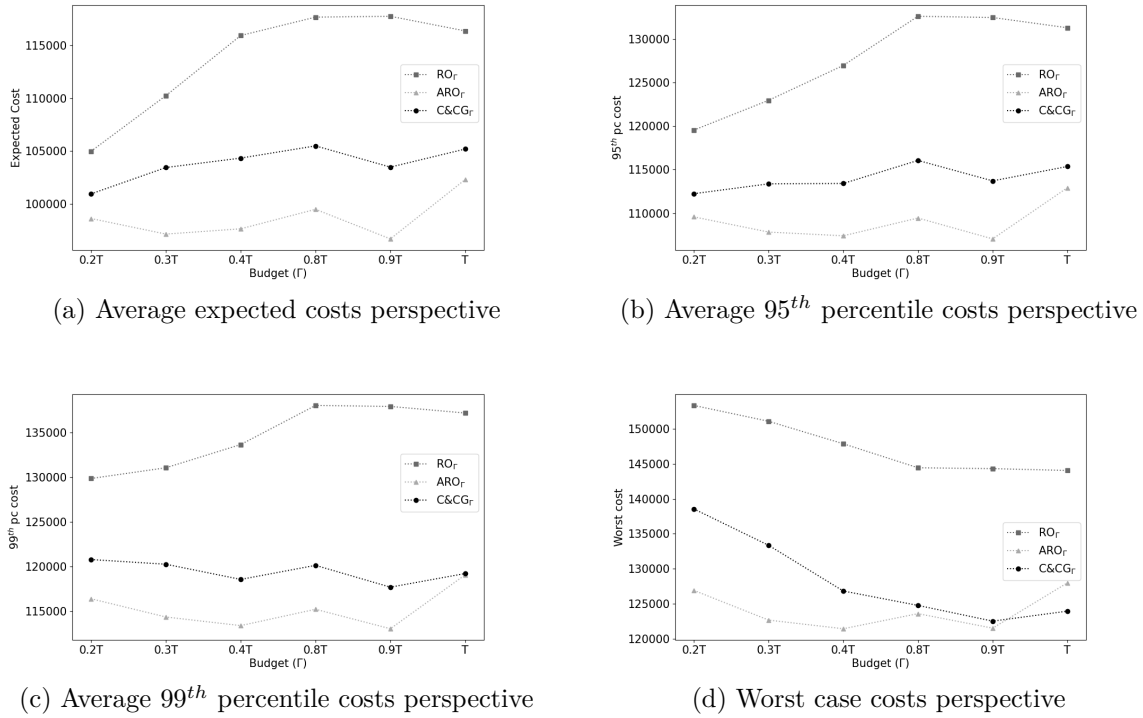


Figure 3.1 – Model’s performance according to different budget Γ

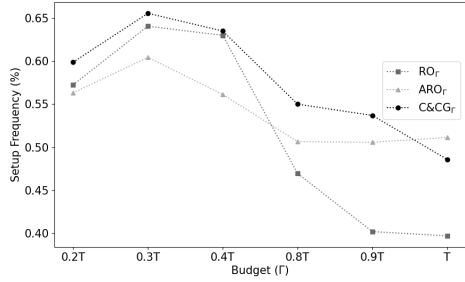
are also taken due to the ignored costs in the inventory and backorder cost constraints. Since the theoretical worst-case scenario is unknown and our experiments are based on a simulation of over 1000 scenarios, our results show that the relaxed costs have little impact on the robustness of the production plans. As the simulated worst case scenario does not necessarily represent the worst realization of the uncertainty in the uncertainty set, ARO_{Γ} neglects very pessimistic contexts. Therefore, ARO_{Γ} still gives a satisfactory trade-off between conservatism and robustness, as even for the worst case costs in our experiments, it results in the lowest costs.

Figure 3.2 shows the impact of the budget of uncertainty Γ on the setup frequency, production quantity, inventory level, and backorder level for the considered models. The figure shows that RO_{Γ} produces less often, yet gives the highest quantity of goods. RO_{Γ} keeps the largest amount of stock, which results in lower backorder levels. On the other hand, $C\&CG_{\Gamma}$ is the approach that produces most frequently, and each time it produces a median quantity at the time. $C\&CG_{\Gamma}$ yields the smallest amount of stocks on hand and a sufficiently low backorder level. ARO_{Γ} performs a median number of setups with smaller

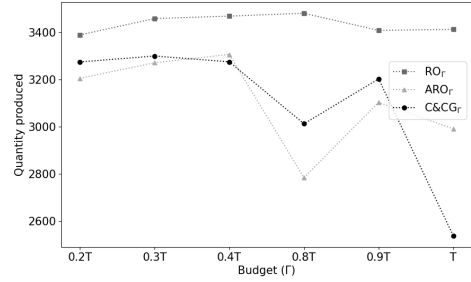
lot sizes, reducing (or even mitigating greater variation on) the stock and backorder levels.

Figure 3.2 also indicates that ARO_{Γ} and $C\&CG_{\Gamma}$ maintain a high setup frequency to update the lot sizes easily and to react to the realization of the production yield. The frequency reduces with Γ , since it can become cheaper to backorder goods than to produce for extremely adverse production yield realizations. Similarly, to ensure that the static plan remains robust even for the worst realization of the production yield, and since RO_{Γ} defines a production plan for the entire production horizon, the frequency of the setup decreases with Γ , which favors the occurrence of backorders.

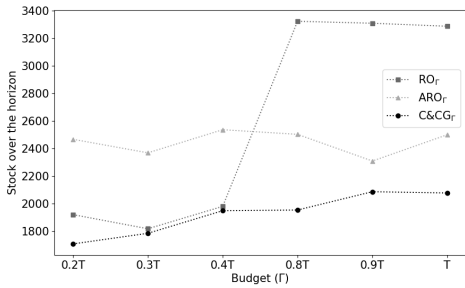
Regarding the number of items produced, due to the high setup frequency in the ARO_{Γ} plans, the lot sizes are relatively small, and their values are defined to react appropriately to the uncertainties and avoid backorders. On the other hand, $C\&CG_{\Gamma}$ produces larger lots which decrease slightly with the increase of Γ , avoiding unexpected costs due to expensive production in adverse cases. Thus, the optimal adaptive plan reacts to the uncertainties by maintaining a sufficiently large production and favoring the stocks with limited backorder levels. For RO_{Γ} , as the frequency of setup decreases with the increase of Γ , the lot sizes are the largest needed to guarantee the availability of quality goods to meet demands even when the worst realization of the yield is observed.



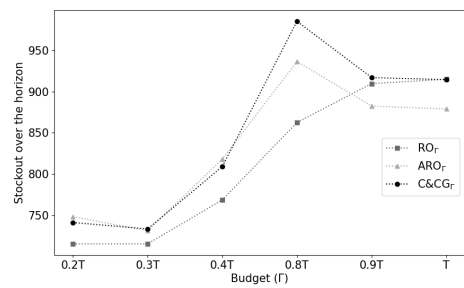
(a) Occurrence of setup of the production



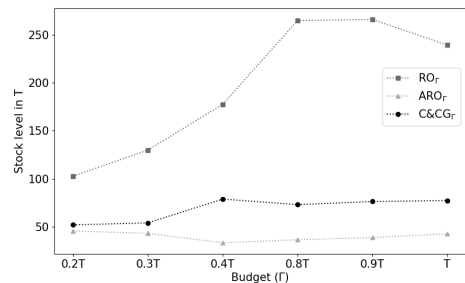
(b) Average quantity of produced goods



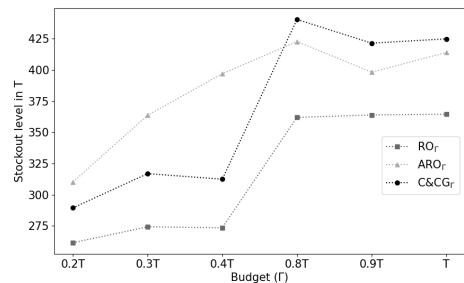
(c) Average inventory level over the production horizon



(d) Average backorder level over the production horizon



(e) Average inventory level at the end of the production



(f) Average backorder level at the end of the production

Figure 3.2 – Production plan structure according to different budgets of uncertainty Γ

Regarding the inventory level, the ARO_{Γ} inventory level remains low for any Γ , and it ensures the strict and necessary availability of quality goods to meet demands and to hedge against the realization of the worst production yield. As a result, ARO_{Γ} keeps a very low quantity of quality goods in stock at the end of the production horizon (as observed in Figure 3.2(e)). Similarly, $C\&CG_{\Gamma}$ maintains a low inventory level. However, as

$C\&CG_{\Gamma}$ reduces the production setup frequency for the highest values of Γ , its inventory level tends to increase with Γ . For RO_{Γ} , as the setup is not frequent, there is a very high inventory level which increases with Γ , to ensure the availability of quality goods. In addition, a larger quantity of quality goods is kept in stock at the last period, which confirms its more conservative production plan.

Finally, for the backorder level, ARO_{Γ} adopts a strategy of reacting to uncertainties with the adequate availability of quality goods, avoiding backorders as much as possible. However, the suboptimal robustness leads to the highest backorder rate. Since a backorder occurs when it is not profitable to produce at the last production period, Figure 3.2(f) shows that about half of backorders from ARO_{Γ} plans are performed on period T . Similarly, 1/2 of the backorder level for $C\&CG_{\Gamma}$ plans occurs in the last period. The remaining amount of backorders occur when the quality goods' availability is not sufficient to meet the demands over the production horizon. The increase of the backorder with Γ was expected, since the setup for the adaptive robust plans becomes less frequent with the increase of Γ , and so backorders become more important whenever the production and the inventory are not sufficient to satisfy orders. The static strategy avoids backorders as much as possible, which can be verified with RO_{Γ} having the lowest backorder level, even if it increases strongly with Γ . The static strategy has the lowest backorder rate at period T , since the cumulative lot sizes over the production horizon is very high.

$C\&CG_{\Gamma}$ takes into account the quadratic terms that are linearized by the ARO_{Γ} approximation. As a consequence, the $C\&CG_{\Gamma}$ objective value represents the optimal cost based on the worst case uncertainty under the budgeted uncertainty set. Due to the approximation performed in ARO_{Γ} , a certain impact of the uncertain production yield is not measured. For this reason, the adaptive approximation cannot guarantee an optimal solution under the budgeted uncertainty set. Nevertheless, from Figure 3.2, the average costs obtained from ARO_{Γ} are sufficiently close to the respective costs computed by the $C\&CG_{\Gamma}$, and both approaches provide similar adaptive plans. ARO_{Γ} gives an average 95th percentile costs about 5.6% lower than $C\&CG_{\Gamma}$, and its 99th percentile costs are 5.7% lower than the respective $C\&CG_{\Gamma}$ ones. However, the ARO_{Γ} objective value cannot cover the worst case cost computed with the approximated adaptive plan (from which we obtain a negative GAP_{ARO}), whereas $C\&CG_{\Gamma}$ achieves this robustness level even for the less conservative uncertainty set whose budget is greater than $0.3T$.

The decision-makers must adjust the budget of uncertainty to the strategic goals of the company (to hedge against extreme cases or get lower costs). While a frozen plan

computed with the static RO_{Γ} model provides an upper bound on the costs, it yields an over-conservative strategy that hedges against the worst possible realization of the production yields. This strategy should be adopted in rigid production environments where changes cannot be easily performed and little or no update on the production yield is available in each period. However, if the production system is flexible enough to modify the production quantity in each period, a static-dynamic strategy should be envisaged. When an exact comprehension of the impact of the uncertain production yield on the costs is necessary, decision-makers should rely on the $C\&CG_{\Gamma}$. Nevertheless, when the average performance of the production plan is enough for the decision, without requiring optimal information on the worst case perspective of the realization of the production yield rate, the ARO_{Γ} approximation proposes sufficiently robust solutions, with lower average costs and in short computation times.

Solution quality in a rolling horizon simulation

This section compares the static and adaptive robust models in a rolling horizon simulation. We consider the general instances with a box uncertainty set. A rolling horizon technique allows us to handle LSPs under a dynamic strategy. We develop a rolling horizon framework similar to Venkataraman [1996] and Meistering and Stadtler [2017], where the models are solved in period t with the horizon $[t; T]$ to compute the lot sizes for the next $\|F\|$ periods, where F is the frozen horizon. Since we intend to compare the static and static-dynamic strategies, we freeze the setup computed in period 1 for frozen horizon and we update lot-size and inventory/backorder decisions with the rolling horizon technique. The models are solved every F periods, for which we fix the lot size for the corresponding periods. Then we perform the production for the frozen horizon so that the lot size for the period $p + 1$ can be updated as soon as the quality inspection for the period p ends. Finally, after the production of the frozen horizon is complete, we repeat the process until the whole production horizon is complete.

Table 3.2 reports the results for the rolling horizon simulation (RH) for F equals 4. We add the subscript $F = 4$ to the method in the first column of Table 3.2 to denote the considered frozen horizon. Table 3.2 also report the results for $C\&CG_{\Gamma=T}$ with $F = T$ (as applied for the models in Section B.5). To better measure the performance of the robust models, we also consider the deterministic model where the perfect information on the production yield is known. This deterministic formulation is called the expected value of the perfect information ($EVPI$), and it represents the lower bound on the costs incurred

from the production system within a static-dynamic strategy.

Table 3.2 – Performance of the robust models through a rolling horizon simulation

Model	Exp. Cost	95 th p.c.	99 th p.c.	Worst Cost
<i>EVPI</i>	21,284	25,563	25,955	26,158
$RO_{\Gamma=T}^{F=4}$	22,334	26,807	27,441	27,695
$ARO_{\Gamma=T}^{F=4}$	22,256	26,854	27,437	27,680
$C\&CG_{\Gamma=T}^{F=4}$	22,179	26,448	27,095	27,370
$C\&CG_{\Gamma=T}^{F=T}$	22,115	26,494	27,114	27,385

As reported in Table 3.2, $C\&CG_{\Gamma=T}^{F=4}$ gives the best average expected cost with a relative difference from *EVPI* of approximately 4.2% on average, versus 4.6% for $ARO_{\Gamma=T}^{F=4}$ and 4.9% for $RO_{\Gamma=T}^{F=4}$. $C\&CG_{\Gamma=T}^{F=4}$ also provides the lowest costs within the rolling horizon approach, since its average expected cost is about 0.4% (resp. 1.6%) lower than the respective costs computed by $ARO_{\Gamma=T}^{F=4}$ (resp. $RO_{\Gamma=T}^{F=4}$). In addition, the worst-case costs of $C\&CG_{\Gamma=T}^{F=4}$ are on average 1.1% (resp. 1.2%) lower than the same cost for $ARO_{\Gamma=T}^{F=4}$ (resp. $RO_{\Gamma=T}^{F=4}$). As a result, $C\&CG_{\Gamma=T}^{F=4}$ better accounts for the adaptive decision, and it is followed by $ARO_{\Gamma=T}^{F=4}$. $RO_{\Gamma=T}^{F=4}$ gives the largest costs since it does not intend to react to the information revealed over time. Even if $C\&CG_{\Gamma=T}^{F=4}$ average expected cost is about 0.3% higher than $C\&CG_{\Gamma=T}^{F=T}$ average expected cost, $C\&CG_{\Gamma=T}^{F=4}$ achieves a reduction of 0.2% on the 95th percentile costs and 0.1% on both the 99th percentile costs and average worst case costs. In the rolling horizon framework, $C\&CG_{\Gamma}$ yields better average results, since the model is re-optimized after revealing the true value of the uncertain return for each period.

3.6 Conclusion

This chapter considers the adaptive robust single-item multi-period lot-sizing problem with backorders and uncertain production yield. The adaptive robust model is difficult to solve since it has a non-fixed recourse that relates the uncertain production yield to the adaptive lot size decisions. To solve the problem, we propose a *C&CG* approach, as well as an approximation of the model that can be solved with commercial solvers. In addition, we derive a myopic policy for a special case (with no setup and no unit production costs, with a stationary nominal value and a stationary maximum deviation of the production yield, and under the box uncertainty set). We show that this policy is optimal under mild assumptions. Our experiments show that the models presented offer a production plan that is robust and easily changeable in an adaptive decision context, while the production system remains protected against uncertain production yields. These results also indicate that the approximation of the adaptive robust model yields less conservative solutions than the optimal solution proposed by *C&CG*, while the approximation can be solved in a few seconds (versus around 4 minutes taken from the exact approach to compute a solution). Further studies should be carried on to extend the adaptive model to the distributionally robust methodology. This is a promising methodology that relies on a joint perspective of robust optimization and stochastic programming. In addition, an extension of this model to the multi-item case is envisaged to bring the proposed model closer to real-world applications. In this direction, chapter 4 introduces the distributionally robust model for the multi-item lot-sizing problem under yield uncertainty, and it gives some insights for solving DRO LSPs.

DISTRIBUTIONALLY ROBUST LOT-SIZING PROBLEMS UNDER YIELD UNCERTAINTY

This chapter is based on the following article:

Metzker, P., Thevenin, S., Adulyasak, Y., Dolgui, A. Distributionally robust optimization for the multi-period multi-item lot-sizing problems under yield uncertainty. Target journal: *Naval Research Logistics*, November 2022.

Production yields in various industries can be highly uncertain which result in unpredictable product availability. Estimating a good distribution can be difficult due to the various factors that must be taken into account in a highly volatile context. Distributionally robust optimization seeks to alleviate this difficulty with a robust solution optimized against the worst expected probability distribution. Thus, we formulate a mixed-integer distributionally robust multi-item lot-sizing model with uncertain production yield to determine a data-driven and sufficiently robust production plan. As various factors can influence changes in the production yield, we use a scenario-wise formulation that partitions the available data as scenarios that define different patterns influencing the quality of the goods. The experimental results show that distributionally robust production plans have better cost cutting strategy compared with robust or stochastic program solutions. In a case study, the distributionally robust model better tackles the uncertainty and increases the performance of the production plans. The results also show that distributionally robust solutions are also less sensitive to errors in predictions.

4.1 Introduction

Recent manufacturing environments are constantly challenged to optimize resource usage and production performance to satisfy demands for quality products at minimal cost in an extremely dynamic and volatile market context. The quality of the production is often influenced by exogenous factors (e.g., climate changes, supply disruption or delay, and change in the quality of raw materials) and endogenous factors (e.g., machinery maintenance, workforce ability and availability, and failure in the production process flow). Due to these reasons, the actual production yield rate can be highly unpredictable and directly impacts the amount of quality goods obtained from a production.

The production yield measures the performance and efficiency of the production system to meet specification with quality goods. It helps manufacturers to maintain the system under control, reducing its vulnerability to errors, malfunctions, irregularities and loss of profit. As a result, the company achieves its strategical and operational goals, which favors its positioning and relevance in the market scenario.

The production yield is often estimated from the historical data and quality control data associated with the production process. Due to the complexity and various uncontrollable factors that can affect the quality of the production, this estimation can be inaccurate. Therefore, the losses in the production process are difficult to predict and measure, and their impact are highly damaging to the system [Inderfurth and Vogelgesang, 2013]. Yield uncertainty in vaccine production has a direct impact on social welfare, as it can lead to weak immunity to new virus strain or mutation [Cho, 2010]. In the olive oil industry, yield uncertainty can increase overall costs as low production yields force producers to buy olives from other farmers to meet demands and satisfy the signed contracts [Kazaz, 2004]. Finally, yield uncertainty can also impact the contractual arrangement in supply chains, where low production yield implies a more expensive insurance payout, reduced capital investment or difficulties to get credit and subsidies [Anderson and Monjardino, 2019]. These examples illustrate the importance of optimization approaches that hedge against uncertainties in a systematic, effective and efficient way.

DRO optimizes under an ambiguity set that is a family of probability distributions of the uncertain parameter characterized through partial stochastic information obtained from estimation or historical data [Delage and Ye, 2010]. As DRO is a recent optimization methodology, some challenges remain to be addressed, such restrictive assumptions to compute a tractable solution, complex formulation, intractability of the proposed model

[Wiesemann et al., 2014]. The formulation of a tractable model requires well-defined ambiguity sets that accounts for the available distributional information using an effective uncertainty representation. Chen et al. [2020] propose an event-wise formulation for the ambiguity set. This formulation jointly considers the scenario-tree method from SP with the bounded representation of uncertainties under an uncertainty set from RO. The event-wise formulation generalizes the well-known ambiguity sets (e.g., generalized moment, mean absolute, Wasserstein and clustering) with the help of an additional random variable in the definition of the uncertain parameter [Chen et al., 2020].

This chapter proposes a distributionally robust model for multi-item lot-sizing problems under yield uncertainty. To the best of our knowledge, we are the first to consider the DRO methodology for LSPs under yield uncertainty, and our contribution is fourfold. First, we propose a distributionally robust formulation for the multi-item LSPs under production yield uncertainty represented in a scenario-wise ambiguity set (a reduction of the event-wise formulation presented by Chen et al. [2020]). Second, we reformulate the distributionally robust model as a MILP that can be easily solved by commercial solvers. Third, we compare the distributionally robust plans with production plans proposed by stochastic and robust models within the same decision settings. Fourth, we present a case study to highlight the advantages of using DRO to deal with production yield uncertainty. Our experimental results show that distributionally robust plans propose better cost-saving strategies that meet demands with quality products, avoiding stock outs as much as possible. Our results also indicate that DRO models are less sensitive to changes in the realization of the production yield, and distribution plans better integrate historical data into the decision system to respond to production process challenges more effectively.

This work is organized as follows: Section 4.2 reviews the literature on distributionally robust optimization with a focus on studies that address lot-sizing problems. Section 4.3 formally describes the considered problem, it introduces the scenario-wise distributionally robust optimization methodology, and it provides the MILP reformulations for our problem. Section 4.4 presents the experimental results on the quality of distributionally robust plans. Finally, Section 4.5 concludes this work and provides some future research directions.

4.2 Literature review

As the literature on lot-sizing problems (LSPs) is rich, this section only considers studies relevant to this work. This is the case of lot-sizing decisions subject to uncertain production yield. More precisely, the setups and lot sizes decisions made by manufacturers minimize the overall costs to meet the demands when the quantity of quality goods obtained from a released production may be less than the quantity of items produced. First, we present the stochastic and robust studies on LSPs under yield uncertainty. Then, we review the distributionally robust optimization theory and its application to the multi-item LSPs.

Although the DRO community questions whether some studies from the 1950s propose concepts equivalent to those known today as DRO methodology, most advances in the field began in the 2000s. Delage and Ye [2010] is among the first to tackle the distributionally robust optimization problem by considering an ambiguity set based on moment information. They propose a generic model for this case and a semi-infinite programming approach to solve it. Wieselmann et al. [2014] provide an extensive review of convex DRO methodology and introduce standardized forms for convex ambiguity sets. The authors also present important conditions which ensure tractability of the final distributionally robust models. Esfahani and Kuhn [2018] present the Wasserstein ambiguity set, an ambiguity set that is convex and ensure tractability to formulation with powerful performance. Kuhn et al. [2019] extend this work to provide more theory about the Wasserstein ambiguity set, and show the conceptual and computational benefits of this approach. The authors also report the promising results when combining Wasserstein ambiguity set with machine learning applications. Bertsimas et al. [2019] propose a tractable adaptive DRO formulation based on second-order conic representable ambiguity sets. The authors provide tools to reformulate the distributionally robust models as MILP that can be easily solved by commercial solvers. In the same vein, Chen et al. [2020] introduce the event-wise formulation for the ambiguity sets, and provide a new modeling package, RSOME, to help modelers to reformulate and solve distributionally robust models. For an extensive review on DRO, we refer readers to Rahimian and Mehrotra [2019].

Considering the application of the DRO methodology, Hanasusanto et al. [2015] propose a risk-averse distributionally robust multi-item newsvendor problem with uncertain demand. The authors approximate the distributionally robust model to a quadratic problem, and it yields a conservative but tractable formulation. Huang et al. [2021] give a

reformulation approach based on the conditional value-at-risk (CVAR) for solving the multi-product assembly and the portfolio selection problems with a cutting plane algorithm. Qiu et al. [2021] develop a reformulation by Lagrange multipliers for solving the multi-product inventory problem with demand and supply uncertainties. The authors also propose decomposition methods to compute solutions for the case of reliable and unreliable suppliers. Finally, Wang and Delage [2021] reformulate the multi-item newsvendor problem with the help of event-wise affine decision rules and propose a column generation algorithm to tackle scalability issues and compute a good solution.

This chapter investigates the distributionally robust multi-item LSPs under yield uncertainty. Since the production yield measurements can be impacted by different factors (e.g.: change in ambient temperature, change in raw material quality), an event-wise formulation seems very appropriate to reduce the conservatism of production plans and to better integrate the available data and the production context to the decision making process. Our work differs from the aforementioned literature in several aspects. To the best of our knowledge, this chapter is the first work that addresses the multi-item LSPs under yield uncertainty. First, we adapt the design of the event-wise ambiguity set to a scenario-wise formulation that represents different patterns that the yield uncertainty can follow. Second, we present the reformulations under a MILP reformulation scheme of Wang and Delage [2021] and RSOME [Chen et al., 2020] for the LSPs under yield uncertainty. Finally, we compare the distributionally robust plans with production plans proposed by the stochastic and robust models, and highlight the advantages of DRO modeling through a case study. Our experimental results show that although the distributionally robust models can suffer from scalability issues for large instances and large ambiguity sets, distributionally robust models provide sufficiently robust production plans that integrate a good cost cutting strategy and lower risk sensitivity to the realization of the production yield.

Our work differs from the aforementioned literature in several aspects. To the best of our knowledge, this study is the first to address the multi-item LSPs under yield uncertainty. First, we adapt the design of the event-wise ambiguity set to a scenario-wise formulation that represents different patterns that the yield uncertainty can follow. Second, we present the reformulations under a MILP reformulation scheme of Wang and Delage [2021] and [Chen et al., 2020]’s event-wise ambiguity set. Finally, we investigate the modeling and the computational effort from MILP reformulations, and then we compare the distributionally robust plans with production plans proposed by the stochastic

and robust models. Our experimental results show that although the distributionally robust models suffer from scalability issues for large instances and large ambiguity sets, distributionally robust models provide sufficiently robust production plans that integrate a good cost cutting strategy and lower risk sensitivity to the realization of the production yield.

Notation: We use boldface (e.g., \mathbf{X}) to denote vectors. We also need the following definitions: a) The support function of a convex set \mathcal{Q} describing a variable \mathbf{z} gives the bounds for the set, and it is defined as $\delta^*(z|\mathcal{Q}) = \sup_{\boldsymbol{\xi} \in \mathcal{Q}} \boldsymbol{\xi}^\top \mathbf{z}$. b) The perspective of a function often helps modelers to provide non-intuitive behaviours or to demonstrate some properties associated to optimization problems. The perspective of a function $g(x) : \mathbb{R}^n \Rightarrow \mathbb{R}$ is a function $g(x, t) : \mathbb{R}^n \times \mathbb{R} \Rightarrow \mathbb{R} = tg(\frac{x}{t})$, $\forall t > 0$. c) The conjugate of a function $g(x)$ is a convex function $g^*(y)$, even if $g(x)$ is not convex. The conjugate is given by $g^*(y) = \sup_{x \in \mathbf{dom}g(x)} y^\top x - g(x)$, where **dom**, the domain of a function, gives all the inputs for the function.

4.3 The distributionally robust multi-item multi-period lot-sizing problem under yield uncertainty

The multi-item multi-period lot-sizing problem (LSP) with backorder and uncertain production yield seeks to minimize the overall costs and efficiently meet demands with quality goods. LSP determines the production setups and lot sizes for a set of items I over a finite planning horizon T , which leads to a production plan at minimal cost. For each item $i \in N = \{1, \dots, |N|\}$ and each period $t \in T = \{1, \dots, |T|\}$, we have the following known parameters: setup cost s_{it} , unit production cost v_{it} , unit inventory cost h_{it} , unit backorder cost b_{it} , and demand d_{it} . We also have the uncertain production yield $\tilde{\rho}_{it}$, which we assume to be strictly positive (then $0 < \tilde{\rho}_{it} \leq 1$). To meet demands and minimize costs, LSP seeks the quantity to produce X_{it} and the setup decision Y_{it} . These decisions are made to minimize the worst-case expected inventory management cost with respect to a set of probability distributions \mathcal{F} . We denote by $H_{it}(\tilde{\boldsymbol{\rho}})$ the inventory or backorder cost for item i and period t depending on the uncertain yield $\tilde{\boldsymbol{\rho}}$. We assume that the decision-maker follows a static strategy, where the production plan is decided before to observe the realization of the uncertain production yield, and the decision are frozen. The distributionally robust multi-item lot-sizing problem (DRLSP) can be formulated as follows:

$$\min \left(\sum_{i \in N} \sum_{t \in T} (s_{it} Y_{it} + v_{it} X_{it}) + \sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}} \left[\sum_{i \in N} \sum_{t \in T} H_{it}(\tilde{\rho}_{it}) \right] \right) \quad (4.1a)$$

s.t.

$$H_{it}(\tilde{\boldsymbol{\rho}}) \geq h_{it} \left[\sum_{\tau=1}^t (\tilde{\rho}_{i\tau} X_{i\tau} - d_{i\tau}) \right] \quad \forall i \in N; t \in T; \tilde{\boldsymbol{\rho}} \in \mathbb{P} \quad (4.1b)$$

$$H_{it}(\tilde{\boldsymbol{\rho}}) \geq -b_{it} \left[\sum_{\tau=1}^{it} (\tilde{\rho}_{i\tau} X_{i\tau} - d_{i\tau}) \right] \quad \forall i \in N; t \in T; \tilde{\boldsymbol{\rho}} \in \mathbb{P} \quad (4.1c)$$

$$X_{it} \leq M_{it} Y_{it} \quad \forall i \in N; t \in T \quad (4.1d)$$

$$X_{it}, H_{it} \geq 0 \quad \forall i \in N; t \in T \quad (4.1e)$$

$$Y_{it} \in \{0, 1\} \quad \forall i \in N; t \in T \quad (4.1f)$$

where \mathcal{F} is the ambiguity set, and \mathbb{P} is a distribution that represents the uncertain yield $\tilde{\boldsymbol{\rho}}$. Without a loss of generality, we assume that there is no stock or backorder at the beginning of the planning horizon. The objective function (4.1a) minimizes the

total cost comprising the setup, unit production, and the worst-case expected inventory and backorder costs. Since the level of stocks or backorders depends on the realization of the uncertain yield, and due to the complementarity of these values, we should be interested in how uncertainty affects total cost. Therefore, we consider the inventory management activity in term of its impact on the overall cost, here represented by $H_t(\rho)$. Constraints (4.1b) compute the inventory cost for item i in period t with the help of the cumulative amount of quality goods obtained for periods up to t . Similarly, Constraints (4.1c) compute the backorder cost if the cumulative amount of quality goods is not enough to meet all demands up to t . The Constraints (4.1d) are setup-forcing constraints that relate the lot sizes (X_{it}) to the setup decisions (Y_{it}), where M_{it} is an upper bound on the production quantity. These constraints set the setup variable Y_{it} to 1 if any production for item i occurs in period t , and the setup remains inactive otherwise ($Y_{it} = 0$). The multi-item and multi-period model with no capacity could also be solved by solving the uncapacitated single-item problem separately. To address the capacitated version of the problem, Constraints (4.1d) should be modified to represent the resource availability by setting $M_{it} = \min\{C_{it}, M_{it}\}$, where C_{it} is the available capacity for item i in period t .

The $\sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}}[\cdot]$ represents the worst case expected cost over the ambiguity set \mathcal{F} . Problem (4.1) is not solvable by commercial solvers in the current form. Thus, we intend to develop a MILP distributionally robust model for LSP under yield uncertainty. This section provides our MILP reformulation. First, we define the scenario-wise ambiguity set. Second, we define the scenario-wise affine decision rule that computes the inventory management costs as a function of the realization of the random production yield. Finally, we rely on Slater’s condition and Sion’s minimax theorem techniques to reformulate our DRO model as a robust model [Chen et al., 2020, Wang and Delage, 2021].

4.3.1 Definition of the scenario-wise ambiguity set

The event-wise ambiguity set was first introduced by Chen et al. [2020], and it incorporates the scenario tree representation from the stochastic optimization with the affine decision rules related to the robust optimization to represent the uncertainties in the ambiguity set. For a formal definition of the event-wise ambiguity set and further information on its application, we refer the interested readers to Chen et al. [2020].

As we work in a static strategy, where all decisions are fixed before the uncertainties are realized, we assume that our event ambiguity set takes only one event with all possible scenarios that can describe the uncertainty. In fact, considering a static strategy, only the

inventory management costs are directly impacted by the realization of the production yield. Therefore, we reduced the event-wise ambiguity set to a scenario-wise ambiguity set that provides a pattern describing one possible behaviour of the uncertain yield per scenario.

Let us assume an historical data \mathcal{H} of the production yield, which contains various measurement of production performance PP from the past (e.g., $\mathcal{H} = \{PP_1, PP_2, PP_3, PP_4, PP_5, PP_6, PP_7, PP_8\}$, where 8 production performance are available). From \mathcal{H} , we can identify different scenarios s that influence the production yield (e.g, s_1 change in ambient temperature, s_2 change in raw material quality, s_3 change in machine performance). Therefore, we can partition \mathcal{H} in S exclusive scenarios that contains at least one measurement (e.g., if we obtain $s_1 \in S | s_1 = \{PP_1, PP_6, PP_7\}$, $s_2 \in S | s_2 = \{PP_2, PP_4, PP_5\}$, $s_3 \in S | s_3 = \{PP_3, PP_8\}$, such that $s_1 \cup s_2 \cup s_3 = S \subset \mathcal{H}$). Each scenario help us to estimate the true value of our random variable (here our production yield $\tilde{\rho}$). More precisely, from \mathcal{H} we obtain a set $S = s_1, \dots, s_S$ of scenarios, where each scenario s represents a conditional moment information that the uncertain production yield follows.

A confidence set of a distribution gives the range of values compatible with the data estimated with the distribution considered. Although each scenario individually can be estimated by a probability distribution, the definition of the appropriate confidence set can be complicated. To ensure the tractability and correctness of the confidence sets in a richer and unified manner, Wiesemann et al. [2014] redefine the random variable with the inclusion of an additional variable \mathbf{m} . This additional variable associates the outcomes from different probability distributions with their respective confidence sets without imposition a condition on the confidence set [Chen et al., 2020]. Therefore, the scenario-wise model remains valid for different settings and definitions of the ambiguity set.

We redefine the random production yield as $(\tilde{\rho}, \tilde{\mathbf{m}})$, where the primary random variable $\tilde{\rho} \in \mathbb{R}^{(N \times T)}$ gives the uncertain production yield, and the auxiliary random variable $\tilde{\mathbf{m}} \in \mathbb{R}^{(N \times T)_m}$ ensure that the scenario-wise model remains valid for different probability distributions without imposing additional or specific conditions on the confidence set. The space $\mathbb{R}^{(N \times T)_m}$ defines a sub-space from the space $\mathbb{R}^{(N \times T)}$ of the appropriated size to represent the auxiliary variable m .

For each scenario s , we define the convex sets \mathcal{Q}_s and \mathcal{W}_s . \mathcal{Q}_s represents the expected value from the estimation of the random variable, while the support set \mathcal{W}_s indicates the support of the random variable. \mathcal{W}_s , given by $\mathcal{W}_s = \{(\boldsymbol{\rho}, \mathbf{m}) \in \mathbb{R}^{(N \times T)} \times \mathbb{R}^{(N \times T)_m} | \underline{\boldsymbol{\rho}} \leq \boldsymbol{\rho} \leq$

$\bar{\rho}, \mathbf{g}_s(\boldsymbol{\rho}) \leq \mathbf{m}$, is the epigraph of \mathbf{g}_s and relates the random variable $\boldsymbol{\rho}$ to the auxiliary variable \mathbf{m} . \mathcal{W}_s also gives the lower bound $\underline{\boldsymbol{\rho}}$ (resp. upper bound $\bar{\boldsymbol{\rho}}$) of the production yield. We can define the scenario-wise ambiguity set \mathcal{F} as follows:

$$\mathcal{F} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{(N \times T) + (N \times T)_m} \times S) \left| \begin{array}{ll} (\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}, \tilde{s}) \sim \mathbb{P}, \mathbf{p} \in \mathbb{P} & \\ \mathbb{E}_{\mathbb{P}}[(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}) \mid \tilde{s} = s] \in \mathcal{Q}_s & \forall s \in S \\ \mathbb{P}[(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}) \in \mathcal{W}_s \mid \tilde{s} = s] = 1 & \forall s \in S \\ \mathbb{P}(\tilde{s} = s) = p_s & \forall s \in S \end{array} \right. \right\}$$

where $\mathcal{P}_0(\mathbb{R}^{(N \times T) + (N \times T)_m} \times S)$ is the set of the S distributions the random yield for item $i \in N$ and period $t \in T$, \mathbb{P} is one possible probability distribution for the random production yield, and p_s is the probability of scenario s , such that $\mathbf{p} > 0$, $\sum_{s \in S} p_s = 1$.

Before developing the MILP reformulation for our problem based on the scenario ambiguity set, we introduce the ambiguity sets we are interested in using. The mean absolute ambiguity set is considered one of the most conservative within the DRO community, as it only depends on the mean and absolute deviation of the uncertain parameter. On the other hand, the Wasserstein ambiguity set allows the decision maker to control the risk evaluation of the realization of the uncertainty (with respect to distribution uncertainty) based on a distance θ from the empirical distribution. As a result, the Wasserstein ambiguity set limits the conservatism of the decision through a proper definition of θ .

Mean absolute ambiguity set

The mean absolute ambiguity set \mathcal{F}_M [Esfahani and Kuhn, 2018, Wang and Delage, 2021] defines the set of possible distributions based on the mean ($\bar{\boldsymbol{\rho}}_s$) and standard deviation ($\hat{\boldsymbol{\rho}}_s$) of production yield from the lot sizes of each item i in a given period t and scenario $s \in S$. A distribution $\mathbb{P}_s \in \mathcal{P}_M$ can be represented by the support set $\mathcal{W}_s = \{(\boldsymbol{\rho}, \mathbf{m}) \in \mathbb{R}^{(N \times T)} \times \mathbb{R}^{(N \times T)_m} \mid \underline{\boldsymbol{\rho}} \leq \boldsymbol{\rho} \leq \bar{\boldsymbol{\rho}}, \|\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}\| \leq \mathbf{m}\} \forall s \in S$. In addition, the expectation for the random variables are given by the set $\mathcal{Q}_s = \{(\mathbf{q}_\rho, \mathbf{q}_m) \mid \mathbf{q}_\rho = \bar{\boldsymbol{\rho}}_s; \mathbf{q}_m = \hat{\boldsymbol{\rho}}_s\}$, and $p_s = \frac{1}{|S|}$. The mean absolute ambiguity set for the uncertain production yield is given as follows:

$$\mathcal{F}_M = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{(N \times T) + (N \times T)_m} \times S) \left| \begin{array}{ll} ((\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}), \tilde{s}) \sim \mathbb{P} & \\ \mathbb{E}_{\mathbb{P}}[\tilde{\boldsymbol{\rho}} \mid \tilde{s} = s] = \bar{\boldsymbol{\rho}}_s & \forall s \in S \\ \mathbb{E}_{\mathbb{P}}[\tilde{\mathbf{m}} \mid \tilde{s} = s] = \hat{\boldsymbol{\rho}}_s & \forall s \in S \\ \mathbb{P}[(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}) \in \mathcal{W}_s \mid \tilde{s} = s] = 1 & \forall s \in S \\ \mathbb{P}(\tilde{s} = s) = \frac{1}{|S|} & \forall s \in S \end{array} \right. \right\}$$

Wasserstein ambiguity set

A Wasserstein ambiguity set contains a collection of probability distributions that are at most θ distant from the available distributions representing past production yield behaviours. As explained by Esfahani and Kuhn [2018] and Wang and Delage [2021], the Wasserstein ambiguity set \mathcal{F}_W considers an empirical distribution $\check{\mathbb{P}}$ estimated from historical data, and \mathcal{F}_W contains all probability distribution \mathbb{P} whose Wasserstein distance with $\check{\mathbb{P}}$ is lower than θ . θ represents the radius of the Wasserstein ball centered at the empirical distribution.

The Wasserstein ambiguity set can be represented as a scenario-wise ambiguity set. For each scenario s , \mathcal{Q}_s is given by $\mathcal{Q}_s = \{(q_\rho, q_m) | q_m = \theta\}$, while \mathcal{W}_s is given by $\mathcal{W}_s = \{(\boldsymbol{\rho}, m) \mid \boldsymbol{\rho} \in [\underline{\boldsymbol{\rho}}, \bar{\boldsymbol{\rho}}]; \|\boldsymbol{\rho} - \check{\boldsymbol{\rho}}_s\|_p \leq m\}$. Note that p , in the Wasserstein support set, indicates the p-norm. For the uncertain production yield, the Wasserstein ambiguity set is given as follows:

$$\mathcal{F}_W = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{N \times T+1} \times S) \left| \begin{array}{l} ((\tilde{\boldsymbol{\rho}}, \tilde{m}), \tilde{s}) \sim \mathbb{P} \\ \mathbb{E}_{\mathbb{P}}[\tilde{m} \mid \tilde{s} \in S] = \theta \\ \mathbb{P}[(\tilde{\boldsymbol{\rho}}, \tilde{m}) \in \mathcal{W}_s] = 1 \quad \forall s \in S \\ \mathbb{P}(\tilde{s} = s) = \frac{1}{S} \quad \forall s \in S \end{array} \right. \right\}$$

4.3.2 Scenario-wise LSP under yield uncertainty reformulation

Under a given static production strategy, inventory and backorder quantities react to the realization of the production yield. Therefore, $H_{it}(\tilde{\boldsymbol{\rho}}, \tilde{m}, \tilde{s})$ represents the recourse decision that gives the inventory management cost decision H_{it} to the realization of the random variable $(\tilde{\boldsymbol{\rho}}, \tilde{m})$ and the scenario s for each item i in each period t . We represent $H_{it}(\tilde{\boldsymbol{\rho}}, \tilde{m}, \tilde{s})$ linearly dependent to the uncertain yield through the following scenario-wise decision rule:

$$H_{it}(\boldsymbol{\rho}, \mathbf{m}, s) = H_{its}^0 + \sum_{l \in T} \sum_{k \in N} H'_{itkls} \rho_{kl} + \sum_{j \in (N \times T)_{|m|}} H''_{itjs} m_j$$

where H_{its}^0 is the cost component free from disturbances due to uncertainty, \mathbf{H}'_{its} is the cost component that reacts to the realization of the random yield $\boldsymbol{\rho}$, and \mathbf{H}''_{its} is the cost component subject to changes in the auxiliary variable \mathbf{m} for item i in period t and for scenario s . We also have $\mathbf{H}^0 \in \mathbb{R}_+^{(N \times T \times S)}$; $\mathbf{H}' \in \mathbb{R}^{(N \times T \times (N \times T)_{|\rho|} \times S)}$; $\mathbf{H}'' \in \mathbb{R}^{(N \times T \times (N \times T)_{|m|} \times S)}$.

When we model problem (4.1) based on the scenario-wise ambiguity set and the scenario-wise decision rule, we obtain the following DRLSP problem under yield uncer-

tainty reformulation:

$$\min \left\{ \sum_{i \in N} \sum_{t \in T} (s_{it} Y_{it} + v_{it} X_{it}) + \sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}} \left[\sum_{i \in N} \sum_{t \in T} H_{it}(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}, \tilde{\mathbf{s}}) \right] \right\} \quad (4.2a)$$

s.t. (4.1d), (4.1f)

$$H_{it}(\boldsymbol{\rho}, \mathbf{m}, \mathbf{s}) \geq h_{it} \left[\sum_{\tau=1}^t (\rho_{i\tau} X_{i\tau} - d_{i\tau}) \right] \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; i \in N; t \in T; s \in S \quad (4.2b)$$

$$H_{it}(\boldsymbol{\rho}, \mathbf{m}, \mathbf{s}) \geq -b_{it} \left[\sum_{\tau=1}^{it} (\rho_{i\tau} X_{i\tau} - d_{i\tau}) \right] \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; i \in N; t \in T; s \in S \quad (4.2c)$$

$$H_{it}(\boldsymbol{\rho}, \mathbf{m}, \mathbf{s}) \geq 0 \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; i \in N; t \in T; s \in S \quad (4.2d)$$

$$X_{it} \geq 0 \quad \forall i \in N; t \in T \quad (4.2e)$$

which can be rewritten as:

$$\min \left\{ \sum_{i \in N} \sum_{t \in T} (s_{it} Y_{it} + v_{it} X_{it}) + \sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}} \left[\sum_{i \in N} \sum_{t \in T} \left(H_{it}^0 + \mathbf{H}'_{it}(\tilde{\mathbf{s}})^\top \boldsymbol{\rho} + \mathbf{H}''_{it}(\tilde{\mathbf{s}})^\top \mathbf{m} \right) \right] \right\} \quad (4.3a)$$

s.t. (4.1d), (4.1f), (4.2e)

$$H_{its}^0 + \mathbf{H}'_{its}{}^\top \boldsymbol{\rho} + \mathbf{H}''_{its}{}^\top \mathbf{m} \geq h_{it} \left[\sum_{\tau=1}^t (\rho_{i\tau} X_{i\tau} - d_{i\tau}) \right] \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; i \in N; t \in T; s \in S \quad (4.3b)$$

$$H_{its}^0 + \mathbf{H}'_{its}{}^\top \boldsymbol{\rho} + \mathbf{H}''_{its}{}^\top \mathbf{m} \geq -b_{it} \left[\sum_{\tau=1}^{it} (\rho_{i\tau} X_{i\tau} - d_{i\tau}) \right] \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; i \in N; t \in T; s \in S \quad (4.3c)$$

$$H_{its}^0 + \mathbf{H}'_{its}{}^\top \boldsymbol{\rho} + \mathbf{H}''_{its}{}^\top \mathbf{m} \geq 0 \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; i \in N; t \in T; s \in S \quad (4.3d)$$

Problem (4.3) can be solved by commercial solvers when implemented with the help of RSOME. However, RSOME is an open source library under development and can be quite challenging not only in terms of modeling but it also can be computationally exhausting. Therefore, we reformulate problem (4.3) as a MILP model that can be easily solved by any commercial solver and provides modelers more understanding of the impact of the ambiguity set on the final model.

To obtain a MILP model for problem (4.3) one can follow the Chen et al. [2020] first theorem to reformulate the worst expectation term as robust constraints. Then, Wang and Delage [2021] first theorem should be applied to model all constraints subject to the production yield uncertainty as a robust counterpart constraint.

From Chen et al. [2020] first theorem, if Slater's condition holds for the worst-case expectation problem, then the problem can be reformulated as a robust optimization problem. For our model, if Slater's condition is valid for $\sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}} [\sum_{i \in N} \sum_{t \in T} H_{it}(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}, \tilde{\mathbf{s}})]$, then the worst expectation for $\sum_{i \in N} \sum_{t \in T} H_{it}(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}, \tilde{\mathbf{s}})$ can be formulated in a robust fash-

ion as follows:

$$\begin{aligned}
 & \inf \gamma \\
 & \gamma \geq \sum_{s \in S} \left(p_s \alpha_s + \boldsymbol{\mu}'_s{}^\top \boldsymbol{\beta}'_s + \boldsymbol{\mu}''_s{}^\top \boldsymbol{\beta}''_s \right) \quad \forall s \in S \\
 & \alpha_s + \boldsymbol{\rho}^\top \boldsymbol{\beta}'_s + \mathbf{m}^\top \boldsymbol{\beta}''_s \geq \sum_{i \in N} \sum_{t \in T} \left(H_{its}^0 + \mathbf{H}'_{its}{}^\top \boldsymbol{\rho} + \mathbf{H}''_{its}{}^\top \mathbf{m} \right) \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; s \in S \\
 & \gamma \in \mathbb{R}, \boldsymbol{\alpha} \in \mathbb{R}^S, \boldsymbol{\beta}'_s \in \mathbb{R}^{(N \times T)}, \boldsymbol{\beta}''_s \in \mathbb{R}^{(N \times T)_m} \quad \forall s \in S
 \end{aligned}$$

Let us assume that $\boldsymbol{\mu}'_s$ and $\boldsymbol{\mu}''_s$ are the expected values of the random variables $\boldsymbol{\rho}$ and \mathbf{m} in the set \mathcal{Q}_s for a scenario s ($\boldsymbol{\mu}'_s, \boldsymbol{\mu}''_s \in \mathcal{Q}_s$). With the help of the law of total probability we can redefine the worst expectation $\sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}} [\sum_{i \in N} \sum_{t \in T} H_{it}(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}, \tilde{\mathbf{s}})] = \lambda(\mathbf{p}, \boldsymbol{\mu})$ as follows:

$$\begin{aligned}
 \lambda(\mathbf{p}, \boldsymbol{\mu}) &= \sup_{\mathbb{P} \in \mathcal{F}} \sum_{s \in S} p_s \mathbb{E}_{\mathbb{P}_s} \left[\sum_{i \in N} \sum_{t \in T} H_{it}(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}, s) \right] \\
 & \text{s.t.} \\
 & \mathbb{E}_{\mathbb{P}_s}[\tilde{\boldsymbol{\rho}}] = \boldsymbol{\mu}'_s \quad \forall s \in S \\
 & \mathbb{E}_{\mathbb{P}_s}[\tilde{\mathbf{m}}] = \boldsymbol{\mu}''_s \quad \forall s \in S \\
 & \mathbb{P}[(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}) \in \mathcal{W}_s] = 1 \quad \forall s \in S
 \end{aligned}$$

The min-max theorem helps us to obtain the following dual from the supremum:

$$\begin{aligned}
 \lambda'(\mathbf{p}, \boldsymbol{\mu}) &= \inf \sum_{s \in S} (\alpha_s + p_s \boldsymbol{\mu}'_s{}^\top \boldsymbol{\beta}'_s + p_s \boldsymbol{\mu}''_s{}^\top \boldsymbol{\beta}''_s) \\
 & \text{s.t.} \\
 & \alpha_s + p_s \boldsymbol{\rho}^\top \boldsymbol{\beta}'_s + p_s \mathbf{m}^\top \boldsymbol{\beta}''_s \geq H_{it}(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}, \tilde{\mathbf{s}}) \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; s \in S \\
 & \boldsymbol{\alpha} \in \mathbb{R}^S, \boldsymbol{\beta}'_s \in \mathbb{R}^{(N \times T)}, \boldsymbol{\beta}''_s \in \mathbb{R}^{(N \times T)_m} \quad \forall s \in S
 \end{aligned}$$

Since strong duality works, $\lambda^* = \lambda'^*$. As $\mathbf{p}\boldsymbol{\mu}$ is non convex, we can replace μ_s by $\frac{\mu_s}{p_s}$ to obtain a convex representation of the infimum problem. We also replace α_s by $p_s \alpha_s$ (as $p_s > 0$, $p \in \mathbb{P} \forall s \in S$). Then, we divide the λ' 's first constraint by p_s which leads to the following reformulation of the infimum:

$$\begin{aligned}
 \lambda'(\mathbf{p}, \boldsymbol{\mu}) &= \inf (\mathbf{p}^\top \boldsymbol{\alpha} + \boldsymbol{\mu}'^\top \boldsymbol{\beta}' + \boldsymbol{\mu}''^\top \boldsymbol{\beta}'') \\
 & \text{s.t.} \\
 & \alpha_s + \boldsymbol{\rho}^\top \boldsymbol{\beta}'_s + \mathbf{m}^\top \boldsymbol{\beta}''_s \geq \sum_{i \in N} \sum_{t \in T} H_{it}(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{m}}, \tilde{\mathbf{s}}) \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; s \in S \\
 & \boldsymbol{\alpha} \in \mathbb{R}^S, \boldsymbol{\beta}'_s \in \mathbb{R}^{(N \times T)}, \boldsymbol{\beta}''_s \in \mathbb{R}^{(N \times T)_m} \quad \forall s \in S
 \end{aligned}$$

Replacing λ by λ' in problem (4.3) leads to the following robust reformulation of our

DRLSP model:

$$\min_{Y, X} \sum_{i \in N} \sum_{t \in T} (s_{it} Y_{it} + v_{it} X_{it}) + \gamma \quad (4.4a)$$

s.t. (4.1d), (4.1f), (4.2e), (4.3b), (4.3c), (4.3d),

$$\gamma \geq \sum_{s \in S} (p_s \alpha_s + \boldsymbol{\mu}'_s{}^\top \boldsymbol{\beta}'_s + \boldsymbol{\mu}''_s{}^\top \boldsymbol{\beta}''_s) \quad \forall p_s \in \mathbb{P}_s; \frac{\mu_s}{p_s} \in \mathcal{Q}_s; s \in S \quad (4.4b)$$

$$\alpha_s + \boldsymbol{\rho}^\top \boldsymbol{\beta}'_s + \mathbf{m}^\top \boldsymbol{\beta}''_s \geq \sum_{i \in N} \sum_{t \in T} (H_{its}^0 + \mathbf{H}'_{its}{}^\top \boldsymbol{\rho} + \mathbf{H}''_{its}{}^\top \mathbf{m}) \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; s \in S \quad (4.4c)$$

$$\gamma \in \mathbb{R}, \boldsymbol{\alpha} \in \mathbb{R}^S, \boldsymbol{\beta}'_s \in \mathbb{R}^{(N \times T)}, \boldsymbol{\beta}''_s \in \mathbb{R}^{(N \times T)_m} \quad \forall s \in S \quad (4.4d)$$

Problem (4.4) is not a MILP due to the dependence of the constraints with yield uncertainty to the distributions supported by \mathcal{W}_s . To reformulate these constraints as linear robust counterpart constraints, we recall Wang and Delage [2021] first theorem. The authors demonstrate how to reformulate the worst expectation dual constraints in terms of the support functions of the ambiguity set. For that, they apply Sion's minimax theorem (see Sion [1958]) to reverse the order of a sup and inf problem with regard to a bounded distribution \mathbb{P} . Then, the authors indicate how to apply the strong duality to derive robust counterpart reformulations from a constraints subject to uncertainty in terms of the support set.

We derive the counterpart reformulation in terms of the support function of the ambiguity set for (4.4b), as shown with the equalities (4.5). In the second line we isolate the dual variables as individual problems, which highlights that only $\boldsymbol{\mu}$ dual variables are impacted by \mathcal{Q} , while all dual variables are dependent on \mathbb{P} . In the third line, we redefine the supremum problem in terms of the support function. In the fourth line, we redefine the support function as an infimum problem, and we gather the dual variables in the same the supremum problem dependent on \mathbb{P} . In the fifth line, Sion's minimax theorem is exploited to reverse the order of the supremum and infimum problems. Finally, last line indicates the reformulation in term of a support function for \mathbb{P} .

$$\begin{aligned} & \sup_{\{\frac{\boldsymbol{\mu}_s}{p_s}\} \in \mathcal{Q}_s, s \in S, \mathbf{p} \in \mathbb{P}} \boldsymbol{\alpha}^\top \mathbf{p} + \boldsymbol{\beta}^\top \boldsymbol{\mu} \\ &= \sup_{\mathbf{p} \in \mathbb{P}} \boldsymbol{\alpha}^\top \mathbf{p} + \sum_{s \in S} p_s \sup_{\{\frac{\boldsymbol{\mu}_s}{p_s}\} \in \mathcal{Q}_s} \frac{\boldsymbol{\beta}_s^\top \boldsymbol{\mu}_s}{p_s} \\ &= \sup_{\mathbf{p} \in \mathbb{P}} \boldsymbol{\alpha}^\top \mathbf{p} + \sum_{s \in S} p_s \delta^*(\boldsymbol{\beta}_s | \mathcal{Q}_s) \end{aligned} \quad (4.5)$$

$$\begin{aligned}
&= \sup_{\mathbf{p} \in \mathbb{P}} \left(\boldsymbol{\alpha}^\top \mathbf{p} + \mathbf{p} \inf_{\nu: \nu \geq \delta^*(\boldsymbol{\beta}_s | \mathcal{Q}_s)} \nu \right) \\
&= \inf_{\nu: \nu \geq \delta^*(\boldsymbol{\beta}_s | \mathcal{Q}_s)} \sup_{\mathbf{p} \in \mathbb{P}} \mathbf{p}(\boldsymbol{\alpha} + \nu) \\
&= \inf_{\nu: \nu \geq \delta^*(\boldsymbol{\beta} | \mathcal{Q})} \delta^*(\boldsymbol{\alpha} + \nu | \mathbb{P})
\end{aligned}$$

The final reformulation from equation (4.4b) is, then, $\inf_{\nu: \nu \geq \delta^*(\boldsymbol{\beta} | \mathcal{Q})} \delta^*(\boldsymbol{\alpha} + \nu | \mathbb{P})$. The final linear reformulation is defined when we replace the general support function with the specific linear support function of the ambiguity set considered.

For the remaining constraints subject to the uncertain production yield, we assume that the Slater's condition holds, as does the strong duality. Thus, we derive equivalent robust counterpart formulations for these constraints. The reformulation of the constraints subject to uncertainty as robust counterpart requires the definition of the constraints as a convex optimization problem whose variables are the random variables, and whose objective is to find the supremum of the variables over a support set [Wang and Delage, 2021]. Then, we apply the strong duality to redefine the obtained sup problem (defined in terms of the support functions of the support set) in terms of its dual inf (which is formulated in terms of the perspective of the conjugate of the epigraph g). First, we demonstrate how to obtain the counterpart reformulation for Constraint (4.4c). Then, we provide the reformulation for the remaining constraints in C.1.

If we rewrite the constraint to isolate the terms with the random variables on the right side, we obtain the following reformulation:

$$\alpha_s - \sum_{i \in N} \sum_{t \in T} H_{its}^0 \geq \left(\sum_{i \in N} \sum_{t \in T} \mathbf{H}'_{its} - \boldsymbol{\beta}'_s \right)^\top \boldsymbol{\rho} + \left(\sum_{i \in N} \sum_{t \in T} \mathbf{H}''_{its} - \boldsymbol{\beta}''_s \right)^\top \mathbf{m} \quad \forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; s \in S$$

Note that the right side term can be rewritten as a convex optimization problem based on the support function of \mathcal{W}_s . If we exploit Slater's condition and the strong duality, since the random variable $\boldsymbol{\rho}$ is bounded by $\underline{\boldsymbol{\rho}}$ and $\overline{\boldsymbol{\rho}}$ on \mathcal{W}_s , we can assume $\boldsymbol{\rho} = \frac{\overline{\boldsymbol{\rho}} + \underline{\boldsymbol{\rho}}}{2}$ and $\mathbf{v} = \mathbf{g}_s(\boldsymbol{\rho}) + 1$. Then, we define a new random variable given by $\boldsymbol{\xi}_j \in \mathcal{W}_s : \boldsymbol{\xi}_j = \boldsymbol{\rho} \forall j \in (N \times T)_{|m|}$ for any scenario $s \in S$. Finally, we obtain the following convex optimization problem:

$$\sup_{\boldsymbol{\rho}, \mathbf{m}, \{\boldsymbol{\xi}_j\}_{j \in (N \times T)_{|m|}} \in \mathbb{R}, \boldsymbol{\rho} \in [\underline{\boldsymbol{\rho}}, \overline{\boldsymbol{\rho}}], g_{js}(\boldsymbol{\xi}_j) \leq m_j, (\boldsymbol{\xi}_j) = \boldsymbol{\rho}, \forall j \in (N \times T)_{|m|}} \boldsymbol{\rho}^\top \mathbf{c}_s^1 + \mathbf{m}^\top \mathbf{c}_s^2$$

where $\mathbf{c}_s^1 = \sum_{i \in N} \sum_{t \in T} \mathbf{H}'_{its} - \boldsymbol{\beta}'_s$ and $\mathbf{c}_s^2 = \sum_{i \in N} \sum_{t \in T} \mathbf{H}''_{its} - \boldsymbol{\beta}''_s$.

As the strong duality holds, we can then reformulate the of this sup problem in terms of its inf dual problem through the Lagrangian duality as follows:

$$\begin{aligned}
 \sup_{\substack{\underline{\rho} - \rho \geq 0 \\ -(\underline{\rho} - \rho) \geq 0 \\ m_j - g_{js}(\xi_j) \geq 0 \\ \xi_j - \rho \geq 0 \\ \rho, \mathbf{m}, \xi_j \in \mathbb{R}}} \rho^\top \mathbf{c}_s^1 + \mathbf{m}^\top \mathbf{c}_s^2 &= \inf_{\substack{\bar{\rho}^\top \boldsymbol{\eta}_s^1 - \underline{\rho}^\top \boldsymbol{\eta}_s^2 + \sum_{j \in (N \times T)_{|m|}} \lambda_{js}^1 (m_j - g_{js}(\xi_j)) + \mathbf{w}_{js}^1{}^\top (\xi_j - \rho) \\ \sum_{j \in (N \times T)_{|m|}} \mathbf{w}_{js}^1 = \mathbf{c}_s^1 - \boldsymbol{\eta}_s^1 + \boldsymbol{\eta}_s^2 \\ \mathbf{c}_s^2 + \boldsymbol{\lambda}_s^1 = 0}} \\
 &\quad \boldsymbol{\eta}_s^1, \boldsymbol{\eta}_s^2, \boldsymbol{\lambda}_s^1 \geq 0 \\
 &\quad \mathbf{w}_{js}^1 \in \mathbb{R}
 \end{aligned}$$

where $\boldsymbol{\eta}^1$, $\boldsymbol{\eta}^2$, $\boldsymbol{\lambda}^1$ and \mathbf{w}^1 are the dual variables, and equations indexed by j are defined for all $j \in (N \times T)_{|m|}$.

Finally, we rewrite the convex optimization problem as linear robust constraints based on the obtained dual problem and the epigraph in the support set \mathcal{W}_s as given below. From the assumption that Slater's condition is valid, the strong duality holds. Thus, the optimal dual variables from the infimum yield an optimal supremum solution. For that, we apply the minimax theorem to inverse the order of the sup and inf problems on the second line, we rewrite the dual objective function in terms of its conjugate function on the third line, and we isolate the sup problem that is now only dependent on the \mathbf{w} and $\boldsymbol{\lambda}$ dual variables on the fourth line. Then, we replace the later conjugate function by its perspective on the fifth line given below.

$$\begin{aligned}
 \sup_{\rho, \mathbf{m}, \{\xi_j\} \in \mathbb{R}; \rho \in [\underline{\rho}, \bar{\rho}]; g_{js}(\xi_j) \leq m_j; (\xi_j) = \rho} \rho^\top \mathbf{c}_s^1 + \mathbf{m}^\top \mathbf{c}_s^2 &= \sup_{\rho, \mathbf{m}, \xi_j \in \mathbb{R}; \boldsymbol{\eta}_s^1, \boldsymbol{\eta}_s^2, \boldsymbol{\lambda}_s \geq 0; \mathbf{w}_{js} \in \mathbb{R}} \inf_{\boldsymbol{\eta}_s^1, \boldsymbol{\eta}_s^2, \boldsymbol{\lambda}_s \geq 0; \{\mathbf{w}_{js}^1\}_{j \in (N \times T)_{|m|}} \in \mathbb{R}, \rho, \mathbf{m}, \xi_j \in \mathbb{R}} \rho^\top \mathbf{c}_s^1 + \mathbf{m}^\top \mathbf{c}_s^2 + (\bar{\rho} - \rho)^\top \boldsymbol{\eta}_s^1 - (\underline{\rho} - \rho)^\top \boldsymbol{\eta}_s^2 + \sum_{j \in (N \times T)_{|m|}} \lambda_{js}^1 (m_j - g_{js}(\xi_j)) + \mathbf{w}_{js}^1{}^\top (\xi_j - \rho) \\
 &= \inf_{\boldsymbol{\eta}_s^1, \boldsymbol{\eta}_s^2, \boldsymbol{\lambda}_s \geq 0; \{\mathbf{w}_{js}^1\}_{j \in (N \times T)_{|m|}} \in \mathbb{R}} \sup_{\rho, \mathbf{m}, \xi_j \in \mathbb{R}} \rho^\top \mathbf{c}_s^1 + \mathbf{m}^\top \mathbf{c}_s^2 + (\bar{\rho} - \rho)^\top \boldsymbol{\eta}_s^1 - (\underline{\rho} - \rho)^\top \boldsymbol{\eta}_s^2 + \sum_{j \in (N \times T)_{|m|}} \lambda_{js}^1 (m_j - g_{js}(\xi_j)) + \mathbf{w}_{js}^1{}^\top (\xi_j - \rho) \\
 &= \inf_{\boldsymbol{\eta}_s^1, \boldsymbol{\eta}_s^2, \boldsymbol{\lambda}_s \geq 0; \mathbf{w}_{js}^1 \in \mathbb{R}; \sum_{j \in (N \times T)_{|m|}} \mathbf{w}_{js}^1 = \mathbf{c}_s^1 - \boldsymbol{\eta}_s^1 + \boldsymbol{\eta}_s^2; \mathbf{c}_s^2 + \boldsymbol{\lambda}_s^1 = 0} \inf_{\rho, \mathbf{m}, \xi_j \in \mathbb{R}} \bar{\rho}^\top \boldsymbol{\eta}_s^1 - \underline{\rho}^\top \boldsymbol{\eta}_s^2 + \sum_{j \in (N \times T)_{|m|}} \sup_{\xi_j} \mathbf{w}_{js}^1{}^\top (\xi_j) - \lambda_{js}^1 g_{js}(\xi_j) \\
 &= \inf_{\boldsymbol{\eta}_s^1, \boldsymbol{\eta}_s^2, \boldsymbol{\lambda}_s \geq 0; \mathbf{w}_{js}^1 \in \mathbb{R}; \sum_{j \in (N \times T)_{|m|}} \mathbf{w}_{js}^1 = \mathbf{c}_s^1 - \boldsymbol{\eta}_s^1 + \boldsymbol{\eta}_s^2; \mathbf{c}_s^2 + \boldsymbol{\lambda}_s^1 = 0} \bar{\rho}^\top \boldsymbol{\eta}_s^1 - \underline{\rho}^\top \boldsymbol{\eta}_s^2 + \sum_{j \in (N \times T)_{|m|}} \lambda_{js}^1 g_{js}^* \left(\frac{\mathbf{w}_{js}^1}{\lambda_{js}^1} \right)
 \end{aligned}$$

where $j \in (N \times T)_{|m|}$ whenever j index appears. As a result, we obtain the following reformulation for the sup convex model:

$$\sup_{\rho, \mathbf{m}, \xi_j \in \mathbb{R}; \rho \in [\underline{\rho}, \bar{\rho}]; g_{js}(\xi_j) \leq m_j; (\xi_j) = \rho \forall j \in (N \times T)_{|m|}} \rho^\top \mathbf{c}_s^1 + \mathbf{m}^\top \mathbf{c}_s^2$$

4.3. The distributionally robust multi-item multi-period lot-sizing problem under yield uncertainty

$$= \inf_{\boldsymbol{\eta}_s^1, \boldsymbol{\eta}_s^2, \boldsymbol{\lambda}_s^1 \geq 0; \mathbf{w}_{j_s}^1 \in \mathbb{R} \forall j \in (N \times T)_{|m|}; \mathbf{c}_s^2 + \boldsymbol{\lambda}_s^1 = 0; \sum_{j \in (N \times T)_{|m|}} \mathbf{w}_{j_s}^1 = \mathbf{c}_s^1 - \boldsymbol{\eta}_s^1 + \boldsymbol{\eta}_s^2} \bar{\boldsymbol{\rho}}^\top \boldsymbol{\eta}_s^1 - \underline{\boldsymbol{\rho}}^\top \boldsymbol{\eta}_s^2 + \sum_{j \in (N \times T)_{|m|}} \lambda_{j_s}^1 g_{j_s}^* \left(\frac{\mathbf{w}_{j_s}^1}{\lambda_{j_s}^1} \right)$$

where $\lambda_{j_s}^1 g_{j_s}^* \left(\frac{\mathbf{w}_{j_s}^1}{\lambda_{j_s}^1} \right)$ is the perspective function of the conjugate function of $g_{j_s}^*$ on $(\mathbf{w}_{j_s}^1, \lambda_{j_s}^1)$ for all $j \in (N \times T)_{|m|}$. Note that the superscript 1 on the dual variables $\boldsymbol{\lambda}$ and \mathbf{w} , (resp. the subscripts 1 and 2 on the coefficients \mathbf{c} and the dual variables $\boldsymbol{\eta}$) indicates the terms associated to Constraints (4.4c), while the subscript 2 (resp. 3 and 4) is associated to Constraints (4.3b), subscript 3 (resp. 5 and 6) to Constraints (4.3c), and subscript 4 (resp. 7 and 8) to Constraints (4.3d).

Repeating the aforementioned reformulation technique to obtain a linear robust counterpart to all constraints subject to the uncertain production yield, we obtain the final MILP robust reformulation of problem (4.4), which is given as follows:

$$\min \sum_{i \in N} \sum_{t \in T} (s_{it} Y_{it} + v_{it} X_{it}) + \gamma \quad (4.6a)$$

s.t.

$$\gamma \geq \delta^* (\boldsymbol{\alpha} + \mathbf{1} \nu | \mathbb{P}) \quad (4.6b)$$

$$\nu \geq \delta^* (\boldsymbol{\beta} | \mathcal{Q}) \quad (4.6c)$$

$$\alpha_s - \sum_{i \in N} \sum_{t \in T} H_{its}^0 \geq \sum_{i \in N} \sum_{t \in T} (\bar{\rho}_{it} \eta_{its}^1 - \underline{\rho}_{it} \eta_{its}^2) + \sum_{j \in (N \times T)_{|m|}} \lambda_{j_s}^1 g_{j_s}^* \left(\frac{\mathbf{w}_{j_s}^1}{\lambda_{j_s}^1} \right) \quad \forall s \in S \quad (4.6d)$$

$$\lambda_{j_s}^1 = \beta_{j_s}'' - \sum_{i \in N} \sum_{t \in T} H_{itj_s}'' \quad \forall j \in (N \times T)_{|m|}; s \in S \quad (4.6e)$$

$$\sum_{j \in (N \times T)_{|m|}} \mathbf{w}_{j_s}^1 = \sum_{i \in N} \sum_{t \in T} \left(\sum_{k \in N} \sum_{l \in T} \mathbf{H}'_{itkls} - \beta'_{its} - \eta_{its}^1 + \eta_{its}^2 \right) \quad \forall s \in S \quad (4.6f)$$

$$H_{its}^0 + h_{it} \sum_{\tau=1}^t d_{i\tau} \geq \sum_{k \in N} \sum_{l \in T} (\bar{\rho}_{kl} \eta_{itkls}^3 - \underline{\rho}_{kl} \eta_{itkls}^4) + \sum_{j \in (N \times T)_{|m|}} \lambda_{itj_s}^2 g_{j_s}^* \left(\frac{\mathbf{w}_{itj_s}^2}{\lambda_{itj_s}^2} \right) \quad \forall i \in N; t \in T; s \in S \quad (4.6g)$$

$$\lambda_{itj_s}^2 = H_{itj_s}^m \quad \forall i \in N; t \in T; j \in (N \times T)_{|m|}; s \in S \quad (4.6h)$$

$$\sum_{j \in (N \times T)_{|m|}} w_{itj_s}^2 = -H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4 \quad \forall i, k \in N; k \neq i; t, l \in T; s \in S \quad (4.6i)$$

$$\sum_{j \in (N \times T)_{|m|}} w_{itj_s}^2 = -H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4 \quad \forall i \in N; t, l \in T; l > t; s \in S \quad (4.6j)$$

$$\sum_{j \in (N \times T)_{|m|}} w_{itj_s}^2 = h_{it} X_{il} - H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4 \quad \forall i \in N; t, l \in T; l \leq t; s \in S \quad (4.6k)$$

$$H_{its}^0 + b_{it} \sum_{\tau=1}^t (\rho_{i\tau} X_{i\tau} - d_{i\tau}) \geq \sum_{k \in N} \sum_{l \in T} (\bar{\rho}_{kl} \eta_{itkls}^5 - \underline{\rho}_{kl} \eta_{itkls}^6) + \sum_{j \in (N \times T)_{|m|}} \lambda_{itjs}^2 g_{js}^* \left(\frac{\mathbf{w}_{itjs}^3}{\lambda_{itjs}^2} \right) \quad \forall i \in N; t \in T; s \in S \quad (4.6l)$$

$$\lambda_{itjs}^3 = H_{itjs}^m \quad \forall i \in N; t \in T; j \in (N \times T)_{|m|}; s \in S \quad (4.6m)$$

$$\sum_{j \in (N \times T)_{|m|}} w_{itjs}^3 = -H'_{itkls} - \eta_{itkls}^5 + \eta_{itkls}^6 \quad \forall i, k \in N; k \neq i; t, l \in T; s \in S \quad (4.6n)$$

$$\sum_{j \in (N \times T)_{|m|}} w_{itjs}^3 = -H'_{itkls} - \eta_{itkls}^5 + \eta_{itkls}^6 \quad \forall i \in N; t, l \in T; l > t; s \in S \quad (4.6o)$$

$$\sum_{j \in (N \times T)_{|m|}} w_{itjs}^3 = -b_{it} X_{il} - H'_{itkls} - \eta_{itkls}^5 + \eta_{itkls}^6 \quad \forall i \in N; t, l \in T; l \leq t; s \in S \quad (4.6p)$$

$$H_{its}^0 \geq \sum_{k \in N} \sum_{l \in T} (\bar{\rho}_{kl} \eta_{itkls}^7 - \underline{\rho}_{kl} \eta_{itkls}^8) + \sum_{j \in (N \times T)_{|m|}} \lambda_{itjs}^2 g_{js}^* \left(\frac{\mathbf{w}_{itjs}^4}{\lambda_{itjs}^2} \right) \quad \forall s \in S \quad (4.6q)$$

$$\lambda_{itjs}^4 = H_{itjs}^m \quad \forall i \in N; t \in T; j \in (N \times T)_{|m|}; s \in S \quad (4.6r)$$

$$\sum_{j \in (N \times T)_{|m|}} \mathbf{w}_{itjs}^4 = -\mathbf{H}'_{its} - \boldsymbol{\eta}_{its}^7 + \boldsymbol{\eta}_{its}^8 \quad \forall s \in S \quad (4.6s)$$

$$X_{it} \leq M_{it} Y_{it} \quad \forall i \in N; t \in T \quad (4.6t)$$

$$X_{it}, H_{its}^0, \boldsymbol{\eta}_s^1, \boldsymbol{\eta}_s^2, \boldsymbol{\eta}_{its}^3, \boldsymbol{\eta}_{its}^4, \boldsymbol{\eta}_{its}^5, \boldsymbol{\eta}_{its}^6, \boldsymbol{\eta}_{its}^7, \boldsymbol{\eta}_{its}^8, \lambda_{js}^1, \lambda_{itjs}^2 \geq 0 \quad \forall i \in N; t \in T; j \in (N \times T)_m; s \in S \quad (4.6u)$$

$$\mathbf{H}'_{its}, \mathbf{H}''_{its}, \gamma, \mathbf{w}_{js}^1, \mathbf{w}_{itjs}^2, \mathbf{w}_{itjs}^3, \mathbf{w}_{itjs}^4 \in \mathbb{R} \quad \forall i \in N; t \in T; j \in (N \times T)_m \quad (4.6v)$$

$$\boldsymbol{\alpha} \in \mathbb{R}^S, \boldsymbol{\beta}'_s \in \mathbb{R}^{S \times N \times T}; \boldsymbol{\beta}''_s \in \mathbb{R}_m^{S \times (N \times T)} \quad (4.6w)$$

$$Y_{it} \in \{0, 1\} \quad \forall i \in N; t \in T \quad (4.6x)$$

where: a) Constraints (4.4b) are reformulated as Constraints (4.6b)-(4.6c), b) Constraints (4.4c) are reformulated as Constraints (4.6d)-(4.6f), c) Constraints (4.3b) are reformulated as Constraints (4.6g)-(4.6k), d) Constraints (4.3c) are reformulated as Constraints (4.6l)-(4.6p), e) Constraints (4.3d) are reformulated as Constraints (4.6q)-(4.6s).

Reformulation based on the moment based ambiguity set

Replacing the general scenario-wise ambiguity set \mathcal{F} by the mean absolute ambiguity set \mathcal{F}_M given in section 4.3.1 on problem (4.6) we obtain the mean absolute distributionally robust formulation (MDRLSP). The support function for \mathcal{F}_M is given by $\delta^* \left((\mathbf{z}_\rho, \mathbf{z}_m) | \mathcal{Q}_k \right) = \bar{\boldsymbol{\rho}}_s^\top \mathbf{z}_\rho + \hat{\boldsymbol{\rho}}_s^\top \mathbf{z}_m$. Therefore, MDRLSP is given as follows:

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$$\min \sum_{i \in N} \sum_{t \in T} (s_{it} Y_{it} + v_{it} X_{it}) + \gamma \quad (4.7a)$$

s.t.

$$\gamma \geq \frac{1}{S} \sum_{s \in S} \left(\alpha_s + \sum_{i \in N} \sum_{t \in T} (\bar{\rho}_{its} \beta'_{its} + \hat{\rho}_{its} \beta''_{its}) \right) \quad (4.7b)$$

$$\alpha_s \geq \sum_{i \in N} \sum_{t \in T} \left(H_{its}^0 + \bar{\rho}_{it} \eta_{its}^1 - \underline{\rho}_{it} \eta_{its}^2 + \left(\sum_{k \in N} \sum_{l \in T} H'_{itkls} - \beta'_{its} - \eta_{its}^1 + \eta_{its}^2 \right) \bar{\rho}_{its} \right) \quad \forall s \in S \quad (4.7c)$$

$$w_{kls}^1 + \sum_{i \in N} \sum_{t \in T} H''_{itkls} - \beta''_{kls} \leq 0 \quad \forall k \in N; l \in T; s \in S \quad (4.7d)$$

$$\sum_{i \in N} \sum_{t \in T} H'_{itkls} - \eta_{kls}^1 + \eta_{lks}^2 \leq w_{kls}^1 \quad \forall k \in N; l \in T; s \in S \quad (4.7e)$$

$$- \sum_{i \in N} \sum_{t \in T} H'_{itkls} + \eta_{kls}^1 - \eta_{lks}^2 \leq w_{kls}^1 \quad \forall k \in N; l \in T; s \in S \quad (4.7f)$$

$$\begin{aligned} H_{its}^0 + h_{it} \sum_{l=1}^t (d_{itl}) \geq & \sum_{k=1}^{i-1} \sum_{l=1}^T \left(\bar{\rho}_{kl} \eta_{itkls}^3 - \underline{\rho}_{kl} \eta_{itkls}^4 + (-H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4) \bar{\rho}_{kls} \right) \\ & + \sum_{k=i+1}^N \sum_{l=1}^T \left(\bar{\rho}_{kl} \eta_{itkls}^3 - \underline{\rho}_{kl} \eta_{itkls}^4 + (-H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4) \bar{\rho}_{kls} \right) \\ & + \sum_{l=t+1}^T \left(\bar{\rho}_{il} \eta_{itils}^3 - \underline{\rho}_{il} \eta_{itils}^4 + (-H'_{itils} - \eta_{itils}^3 + \eta_{itils}^4) \bar{\rho}_{ils} \right) \\ & + \sum_{l=1}^t \left(\bar{\rho}_{il} \eta_{itils}^3 - \underline{\rho}_{il} \eta_{itils}^4 + (h_{it} X_{il} - H'_{itils} - \eta_{itils}^3 + \eta_{itils}^4) \bar{\rho}_{ils} \right) \end{aligned} \quad \forall i \in N; t \in T; s \in S \quad (4.7g)$$

$$w_{itkls}^2 - H''_{itkls} \leq 0 \quad \forall i, k \in N; t, l \in T; s \in S \quad (4.7h)$$

$$-H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4 \leq w_{itkls}^2 \quad \forall i, k \in N; k \neq i; t, l \in T; s \in S \quad (4.7i)$$

$$-H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4 \leq w_{itkls}^2 \quad \forall i, k \in N; k = i; t, l \in T; l > t; s \in S \quad (4.7j)$$

$$h_{it} X_{il} - H'_{itils} - \eta_{itils}^3 + \eta_{itils}^4 \leq w_{itils}^2 \quad \forall i \in N; t, l \in T; l \leq t; s \in S \quad (4.7k)$$

$$H'_{itkls} + \eta_{itkls}^3 - \eta_{itkls}^4 \leq w_{itkls}^2 \quad \forall i, k \in N; k \neq i; t, l \in T; s \in S \quad (4.7l)$$

$$H'_{itkls} + \eta_{itkls}^3 - \eta_{itkls}^4 \leq w_{itkls}^2 \quad \forall i, k \in N; k = i; t, l \in T; l > t; s \in S \quad (4.7m)$$

$$-h_{it} X_{il} + H'_{itils} + \eta_{itils}^3 - \eta_{itils}^4 \leq w_{itils}^2 \quad \forall i \in N; t, l \in T; l \leq t; s \in S \quad (4.7n)$$

$$\begin{aligned}
 H_{its}^0 - b_{it} \sum_{l=1}^t (d_{itl}) &\geq \sum_{k=1}^{i-1} \sum_{l=1}^T (\bar{\rho}_{kl} \eta_{itkls}^5 - \underline{\rho}_{kl} \eta_{itkls}^6 + (-H'_{itkls} - \eta_{itkls}^5 + \eta_{itkls}^6) \bar{\rho}_{kls}) \\
 &+ \sum_{k=i+1}^N \sum_{l=1}^T (\bar{\rho}_{kl} \eta_{itkls}^5 - \underline{\rho}_{kl} \eta_{itkls}^6 + (-H'_{itkls} - \eta_{itkls}^5 + \eta_{itkls}^6) \bar{\rho}_{kls}) \\
 &+ \sum_{l=t+1}^T (\bar{\rho}_{il} \eta_{itils}^5 - \underline{\rho}_{il} \eta_{itils}^6 + (-H'_{itils} - \eta_{itils}^5 + \eta_{itils}^6) \bar{\rho}_{ils}) \\
 &+ \sum_{l=1}^t (\bar{\rho}_{il} \eta_{itils}^5 - \underline{\rho}_{il} \eta_{itils}^6 + (-b_{it} X_{il} - H'_{itils} - \eta_{itils}^5 + \eta_{itils}^6) \bar{\rho}_{ils})
 \end{aligned} \tag{4.7o} \quad \forall i \in N; t \in T; s \in S$$

$$\begin{aligned}
 w_{itkls}^3 - H''_{itkls} &\leq 0 & \forall i, k \in N; t, l \in T; s \in S & \tag{4.7p} \\
 -H'_{itkls} - \eta_{itkls}^5 + \eta_{itkls}^6 &\leq w_{itkls}^3 & \forall i, k \in N; k \neq i; t, l \in T; s \in S & \tag{4.7q} \\
 -H'_{itkls} - \eta_{itkls}^5 + \eta_{itkls}^6 &\leq w_{itkls}^3 & \forall i, k \in N; k = i; t, l \in T; l > t; s \in S & \tag{4.7r} \\
 -b_{it} X_{il} - H'_{itils} - \eta_{itils}^5 + \eta_{itils}^6 &\leq w_{itils}^3 & \forall i \in N; t, l \in T; l \leq t; s \in S & \tag{4.7s} \\
 H'_{itkls} + \eta_{itkls}^5 - \eta_{itkls}^6 &\leq w_{itkls}^3 & \forall i, k \in N; k \neq i; t, l \in T; s \in S & \tag{4.7t} \\
 H'_{itkls} + \eta_{itkls}^5 - \eta_{itkls}^6 &\leq w_{itkls}^3 & \forall i, k \in N; k = i; t, l \in T; l > t; s \in S & \tag{4.7u} \\
 b_{it} X_{il} + H'_{itils} + \eta_{itils}^5 - \eta_{itils}^6 &\leq w_{itils}^3 & \forall i \in N; t, l \in T; l \leq t; s \in S & \tag{4.7v}
 \end{aligned}$$

$$\begin{aligned}
 H_{its}^0 &\geq \sum_{k=1}^{i-1} \sum_{l=1}^T (\bar{\rho}_{kl} \eta_{itkls}^7 - \underline{\rho}_{kl} \eta_{itkls}^8 + (-H'_{itkls} - \eta_{itkls}^7 + \eta_{itkls}^8) \bar{\rho}_{kls}) \\
 & \quad \forall i \in N; t \in T; s \in S
 \end{aligned} \tag{4.7w}$$

$$\begin{aligned}
 w_{itkls}^4 - H''_{itkls} &\leq 0 & \forall i, k \in N; t, l \in T; s \in S & \tag{4.7x} \\
 -H'_{itkls} - \eta_{itkls}^7 + \eta_{itkls}^8 &\leq w_{itkls}^4 & \forall i, k \in N; t, l \in T; s \in S & \tag{4.7y} \\
 H'_{itkls} + \eta_{itkls}^7 - \eta_{itkls}^8 &\leq w_{itkls}^4 & \forall i, k \in N; t, l \in T; s \in S & \tag{4.7z}
 \end{aligned}$$

$$\begin{aligned}
 X_{it} &\leq M_{it} Y_{it} & \forall i \in N; t \in T & \tag{4.7} \\
 X_{it}, \boldsymbol{\eta}_s^1, \boldsymbol{\eta}_s^2, \boldsymbol{\eta}_{its}^3, \boldsymbol{\eta}_{its}^4, \boldsymbol{\eta}_{its}^5, \boldsymbol{\eta}_{its}^6, \boldsymbol{\eta}_{its}^7, \boldsymbol{\eta}_{its}^8 &\geq 0 & \forall i \in N; t \in T; s \in S & \tag{4.7} \\
 H_{its}^0, \mathbf{H}'_{its}, \mathbf{H}''_{its}, \mathbf{w}_s^1, \mathbf{w}_{its}^2, \mathbf{w}_{its}^3, \mathbf{w}_{its}^4 &\in \mathbb{R}, \gamma, \boldsymbol{\beta} \in \mathbb{R}^{(N \times T)}, \boldsymbol{\alpha} \in \mathbb{R}^S & \forall i \in N; t \in T; s \in S & \tag{4.7} \\
 Y_{it} &\in \{0, 1\} & \forall i \in N; t \in T & \tag{4.7}
 \end{aligned}$$

Reformulation for the Wasserstein ambiguity set

If the Wasserstein ambiguity set \mathcal{F}_W given in section 4.3.1 replaces the general scenario-wise ambiguity set \mathcal{F} on problem (4.6), we obtain the Wasserstein distributionally robust formulation (WDRISP). We assume that the support function for \mathcal{F}_W is given by

$$\delta^*((\mathbf{z}_\rho, \mathbf{z}_m) | \mathcal{Q}_k) = \begin{cases} \theta z_m, & \text{if } \mathbf{z}_\rho = \mathbf{0} \\ \infty, & \text{otherwise} \end{cases}$$

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As a result, the WDRLSP is given as follows:

$$\min_{Y, X} \sum_{i \in N} \sum_{t \in T} (s_{it} Y_{it} + v_{it} X_{it}) + \gamma \quad (4.8a)$$

s.t.

$$\gamma \geq \frac{1}{S} \sum_{s \in S} \alpha_s + \theta \beta \quad (4.8b)$$

$$\alpha_s \geq \sum_{i \in N} \sum_{t \in T} (H_{its}^0 + \bar{\rho}_{it} \eta_{its}^1 - \underline{\rho}_{it} \eta_{its}^2 + (\sum_{k \in N} \sum_{l \in T} H'_{itkls} - \beta'_{its} - \eta_{its}^1 + \eta_{its}^2) \hat{\rho}_{its}) \quad \forall s \in S \quad (4.8c)$$

$$w_s^1 + \sum_{i \in N} \sum_{t \in T} H''_{its} - \beta \leq 0 \quad \forall s \in S \quad (4.8d)$$

$$\sum_{i \in N} \sum_{t \in T} H'_{itkls} - \eta_{kls}^1 + \eta_{lks}^2 \leq w_s^1 \quad \forall k \in N, l \in T, s \in S \quad (4.8e)$$

$$- \sum_{i \in N} \sum_{t \in T} H'_{itkls} + \eta_{kls}^1 - \eta_{lks}^2 \leq w_s^1 \quad \forall k \in N, l \in T, s \in S \quad (4.8f)$$

$$\begin{aligned} H_{its}^0 + h_{it} \sum_{l=1}^t (d_{it}) &\geq \sum_{k=1}^{i-1} \sum_{l=1}^T (\bar{\rho}_{kl} \eta_{itkls}^3 - \underline{\rho}_{kl} \eta_{itkls}^4 + (-H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4) \hat{\rho}_{kls}) \\ &+ \sum_{k=i+1}^N \sum_{l=1}^T (\bar{\rho}_{kl} \eta_{itkls}^3 - \underline{\rho}_{kl} \eta_{itkls}^4 + (-H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4) \hat{\rho}_{kls}) \\ &+ \sum_{l=t+1}^T (\bar{\rho}_{il} \eta_{itils}^3 - \underline{\rho}_{il} \eta_{itils}^4 + (-H'_{itils} - \eta_{itils}^3 + \eta_{itils}^4) \hat{\rho}_{ils}) \\ &+ \sum_{l=1}^t (\bar{\rho}_{il} \eta_{itils}^3 - \underline{\rho}_{il} \eta_{itils}^4 + (h_{it} X_{il} - H'_{itils} - \eta_{itils}^3 + \eta_{itils}^4) \hat{\rho}_{ils}) \end{aligned} \quad \forall i \in N; t \in T; s \in S \quad (4.8g)$$

$$w_{its}^2 - H''_{its} \leq 0 \quad \forall i \in N; t \in T; s \in S \quad (4.8h)$$

$$-H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4 \leq w_{its}^2 \quad \forall i, k \in N; k \neq i; t, l \in T; s \in S \quad (4.8i)$$

$$-H'_{itkls} - \eta_{itkls}^3 + \eta_{itkls}^4 \leq w_{its}^2 \quad \forall i, k \in N; k = i; t, l \in T; l > t; s \in S \quad (4.8j)$$

$$h_{it} X_{il} - H'_{itils} - \eta_{itils}^3 + \eta_{itils}^4 \leq w_{its}^2 \quad \forall i \in N; t, l \in T; l \leq t; s \in S \quad (4.8k)$$

$$H'_{itkls} + \eta_{itkls}^3 - \eta_{itkls}^4 \leq w_{its}^2 \quad \forall i, k \in N; k \neq i; t, l \in T; s \in S \quad (4.8l)$$

$$H'_{itkls} + \eta_{itkls}^3 - \eta_{itkls}^4 \leq w_{its}^2 \quad \forall i, k \in N; k = i; t, l \in T; l > t; s \in S \quad (4.8m)$$

$$-h_{it} X_{il} + H'_{itils} + \eta_{itils}^3 - \eta_{itils}^4 \leq w_{its}^2 \quad \forall i \in N; t, l \in T; l \leq t; s \in S \quad (4.8n)$$

4.4 Results and discussion

The goals of the computational experiments are: (i) demonstrate the performance of distributionally robust models for the LSPs with uncertain production yield compared to robust or stochastic production plans; (ii) present an in-depth investigation into distributionally robust plans for LSPs with uncertain production yield in terms of cost distribution and production planning quality, as well as computational efficiency. (iii) present a case study that demonstrates the performance of the DRO in real cases

For this, we consider the following models in the experiments:

1. *RO*, the static robust model presented in C.2
2. *SP_N*, the two stage stochastic model presented in C.3, whose uncertainty is drawn from a normal distribution
3. *SP_U*, the two stage stochastic model presented in C.3, whose uncertainty is drawn from a uniform distribution
4. *MDRO*, our mean absolute distributionally robust model given in (4.7)
5. *WDRO*, our Wasserstein distributionally robust model given in (4.8)

4.4.1 Instance generation and simulation approach

In a similar manner to Metzker et al. [2023a], the experiments are performed with instances randomly generated following the standard approach in the literature for LSP problems. Each parameter was defined with an uniform distribution, such that the production cost, the holding stock cost, the demand, the nominal value, and the maximum deviation of the uncertain production yield supports correspond to the following intervals: $v_{it} \in U(30, 50)$, $h_{it} \in U(1, 10)$, $d_{it} \in U(450, 780)$, $\bar{\rho}'_{it} \in U(0.7, 0.9)$, and $\hat{\rho}'_{it} \in U(0.01, 0.1)$, respectively. The setup costs for each item i are computed with the time between order formula: $s_{it} = \frac{\bar{D}_{it} \cdot TBO^2 \cdot h_t}{2}$, where \bar{D}_{it} represents the average demand for item i in periods up to t . We also set the inventory and the backorder levels at the beginning of the horizon to zero.

For the optimization step, we have drawn 100 scenarios to represent historical data from a uniform distribution with support $[\bar{\rho}'_{it} - \hat{\rho}'_{it}; \bar{\rho}'_{it} + \hat{\rho}'_{it}]$ for each item i in period t . For each scenario $s \in S$, we define a average production yield $\bar{\rho}_s$ and an average standard deviation of the production yield $\hat{\rho}_s$. Then, we gather these values over all scenarios to compute the average production yield $\bar{\rho}_{it}$, standard deviation of the production yield $\hat{\rho}_{it}$,

and finally the upper and the lower bounds on the production yield given by $\bar{\rho}_{it}$ and $\underline{\rho}_{it}$, respectively, for all item $i \in N$ and all period $t \in T$ that should be represented on the ambiguity set, uncertainty set or probability distribution.

We assume that the ambiguity sets build up from the generated historical data, and so does the uncertainty set and the parameters to define the distributions for the stochastic models. We assume that a completely robust production plan can be computed with the most conservative uncertainty set from the robust optimization, the box uncertainty set. This set is equivalent to the budgeted uncertainty set proposed by Bertsimas and Sim [2004] when the budget is set to $\Gamma_t = t$ for each period t and the production yield is mapped in terms of the nominal value and maximum deviation of the production yield calculated from the average over all the scenarios. For the stochastic programs, we consider two possible estimations for the probability distribution of the uncertain yield. While SP_U uses 100 scenarios drawn from a uniform distribution based on the maximum and minimum values of the production yield from the historical data, SP_N is drawn from a normal distribution based on the average and standard deviation of the production yield obtained from the historical data. The algorithms were implemented in Python 3.6, the MILP and the RSOME models are solved with CPLEX version 12.10. The experiments were run on Intel(R) Gold 6148/2.4GHz processors with 92G of RAM. All the models for all the instances were solved until optimality.

First, we investigate the quality of the distributionally robust models compared to the other optimization methods. We evaluate the models performance through a Monte Carlo simulation with $|\Omega| = 1000$ scenarios, where each scenario ω gives a possible production yield rate ρ_{it}^ω for each item i in each period t based on a uniform distribution with support $[\hat{\rho}'_{it} - \hat{\rho}'_{it}; \hat{\rho}'_{it} + \hat{\rho}'_{it}]$. The expected value of perfect information $EVPI$ gives a lower bound on the costs, since $EVPI$ is the cost from a deterministic formulation of the problem where the true realization of the production yield is ρ^ω . We evaluate each model by solving the deterministic model for each scenario ω with the setups and lot size decisions fixed to the values obtained from the optimization step. Next, we investigate the performance of the same models in a case study presented in Section 4.4.3

4.4.2 Performance of the models

This section reports the performance and quality of production plans for LSP under production yield uncertainty. Figure 4.1 and Figure 4.2 report the inventory management levels of the production plans obtained for each method throughout the simulation. The

characteristics of plans *DRO* compared to plans calculated with other optimization methods (namely *SP_N*, *SP_U* and *RO*). The graphs on the left of the figure represent the cases where models are optimized based on historical data drawn from a uniform distribution and production plans are also simulated following a uniform distribution. On the right side we present the results of the simulation of the planes following a normal distribution.

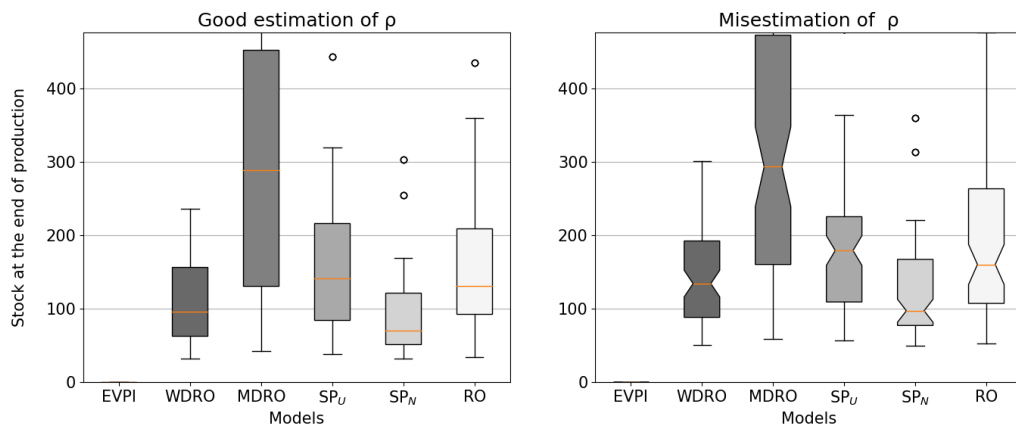


Figure 4.1 – Stock level at the end of the production horizon

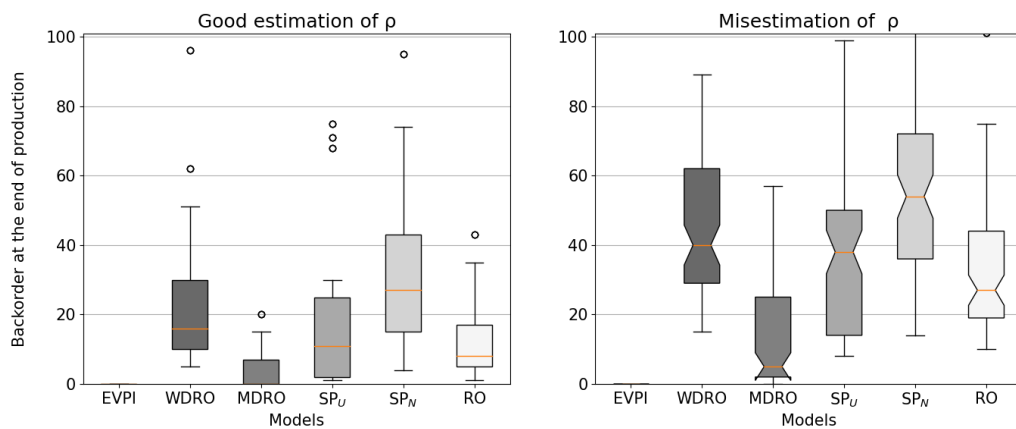


Figure 4.2 – Stockout at the end of the production horizon

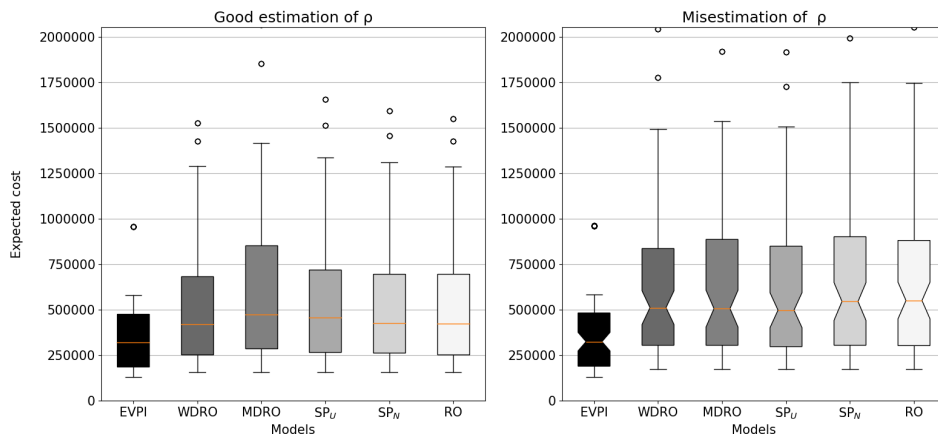


Figure 4.3 – Average expected cost of the production plans

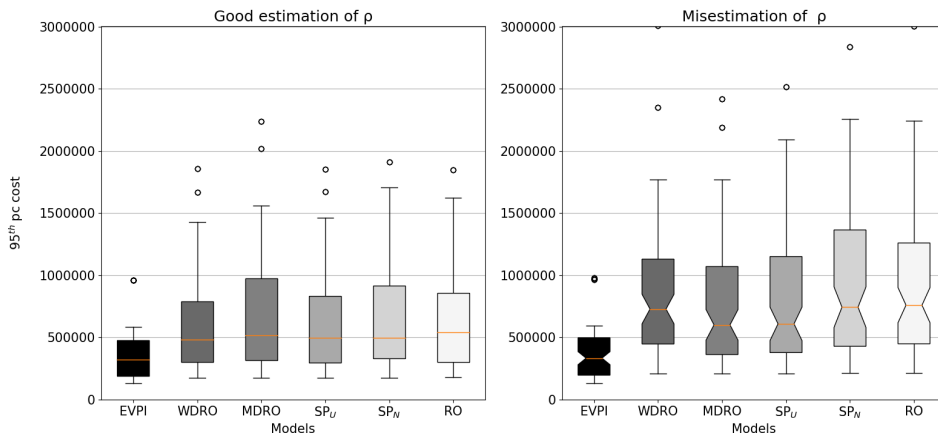


Figure 4.4 – Average 95th percentile cost of the production plans

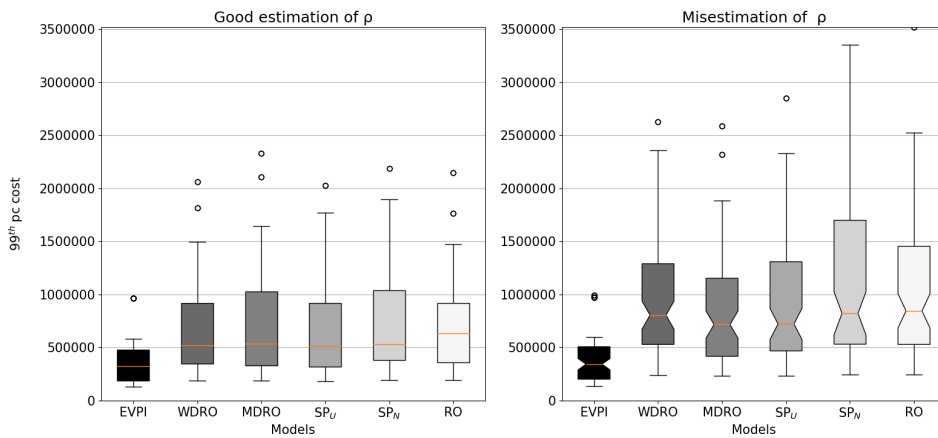


Figure 4.5 – Average 99th percentile cost of the production plans

Figure 4.1 and Figure 4.2 show that, as *MDRO* is the most conservative *DRO* model, it keeps the largest number of items in stock, which indicates a protection strategy against uncertainties in production, avoiding stockouts. As a result, *MDRO* outperforms stochastic programs that are much more susceptible to backorders due to a reduced amount of items held in stock. Compared to *RO*, *MDRO* achieves better tradeoffs in the inventory management activity. It uses all the knowledge on the available data to define an optimal inventory level (much higher than that defined by *RO*) but which results in a very low backlog level. As *WDRO* reduces the conservatism of the solution with θ , it leads to a lower inventory level, which implies a small increase in stockouts. However, like *MDRO*, it achieves a good tradeoff in inventory management, which leads to lower average costs despite a higher occurrence of stockouts. *WDRO* outperforms the *RO* plans, with very low inventory and backorder levels. In addition, few items are kept in stocks, or suffer from stockouts, and the variability in the inventory and backorder levels is also reduced, even if its backorder is higher than the one for the robust model in a case of misestimation. Compared to the stochastic programs, *WDRO* is not only propose a lean production, but it is also less impacted by misestimation and variabilities. However, the strategy adopted by the *DRO* models achieves good cost savings as we present later (see Figure 4.3, Figure 4.4 and Figure 4.5).

It is expected that *SP* models take more risks of stock outs to reduce the inventory and production costs, while *RO* plans produce a medium amount of goods and maintain a low stock level that fully satisfy demands with occasional backorders [Metzker et al., 2023a]. Figure Figure 4.1 and Figure 4.2 show that when compared with the other optimization methods, *DRO* models have larger lot sizes, which implies larger stock of goods and backorder levels as sporadic and as low as possible. Note that the inventory and backorder levels at the end of the production horizon confirms the production strategies of each model. *DRO* models and *RO* models have the lowest backorder levels and the largest stock of goods, even when the behaviour of the uncertain parameter is misestimated. In fact, *DRO* models propose a strategy between robust and stochastic methods, which leads to a medium stock level and makes the occurrence of backorders more frequent, yielding a sufficiently robust and risk-sensitive production plan. In addition, *DRO* models are less impacted by the misestimation of the uncertain parameter, since their results suffer lower variation between the corrected estimation and the misestimation of the uncertain yield.

Figure 4.3, Figure 4.4 and Figure 4.5 report the impact of the realization of the production yield on the expected costs, the 95th and 99th percentile costs, respectively, when

the value of the production yield is correctly estimated (the charts in the left) and misestimated (the charts in the right) on the simulation. The *DRO* models propose the best costs, with the lowest average costs. Since *MDRO* is more conservative than *WDRO*, *WDRO* present the best expected costs based on average scenarios (even for the case of misestimation of the production yield), while *MDRO* yields the lowest costs in the more pessimistic scenarios (here represented by the 95th and 99th percentile costs and also the cases of misestimation of the uncertainty. Compared to the other methodologies, *DRO* present the best production plans, with the lowest average costs and highest cost savings. In addition, *DRO* presents the lowest variation on the costs based on the realization of the production yield, based on the highest and lowest costs they yield on the simulation and on the low variability presented in the right side of the figures.

Since *DRO* models use historical data-driven ambiguity sets, it provides a more robust and cost-effective production plan. Contrarily to *SP*, *DRO* models are not sensitive to the risk of misestimation of the probability, since they provide a plan that performs better for contexts not considered within the stochastic program formulation. Therefore, *DRO* solutions are still robust within a misestimated context, and they lead to lower costs in both cases, when the uncertain yield is correctly estimated, and for the situation of misestimation. Therefore, the experimental results indicate that *DRO* plans have a better cost cutting strategy compared with robust or stochastic program solutions, while they are more immunized from errors in predictions.

4.4.3 Case study

In this case study, we have the historical production yield rate data for 2 products in four different scenarios, namely: machine down, machine with low battery, ambient temperature and low temperature. From the available data, we obtain the average production yield $\bar{\rho}_{its}$, the standard deviation of the production yield $\hat{\rho}_{its}$, and finally the upper and lower output yield limits given by $\bar{\bar{\rho}}_{its}$ and $\underline{\rho}_{its}$, respectively, for all elements $i \in N$, in period t and scenario s . These data become the samples of our ambiguity set for *DRO*, our uncertainty set for *RO* and the probability distribution for the *SP* models. Following the simulation process presented in the Section 4.4.1, we compare the performance of the models. To facilitate the analysis, we assume the same model for the representation of uncertainty. In this case, we assume an optimization on a space close to a uniform distribution of the sample. Therefore, *DRO* is assumed to be the model *MDRO* and *SP* is represented by *SP_U*. Therefore, *RO*, *SP* and *DRO* are going to be compared when they

are all optimized in a space bounded by the minimum and maximum measure of production yields from the available data. We solve each model to optimality and simulate 1000 scenarios to analyse the performance of the obtained production plans.

Table 4.1 reports the characteristics of the simulated production plans. Column $\|X\|$ gives the quantity of items produced, while Column $\|I\|$ (resp. $\|B\|$) gives the cumulative stock (resp. backorder) over the production planning, Column $\|I_T\|$ (resp. Column $\|B_T\|$) gives the amount of items kept on stock (resp. stockouts) at the end of the production horizon, and, finally, Column $\|Y\|$ gives the frequency of setup over the entire production horizon. *DRO* produces more than *RO* and *SP* (with a difference of about 160 units), reducing the stockout at the end of the production horizon in 25 units, with a maintenance of larger level of stocks over the production horizon. Therefore, it is clear that *DRO* adopts a strategy to maintain a sufficient amount of inventory to avoid cost overruns due to stockouts and to be able to satisfy all demands regardless of dysfunctions in production performance.

Table 4.1 – Characteristic of the production plans

Model	$\ X\ $	$\ I\ $	$\ B\ $	$\ I_T\ $	$\ B_T\ $	$\% \ Y\ $
EVPI	3,479	1,498	242	-	146	58.5%
DRO	3,649	1,847	164	145	163	62.0%
SP	3,489	1,538	196	25	188	62.0%
RO	3,486	1,538	196	23	189	62.0%

Table 4.2 reports the average costs from the simulation of the performance of the production plans. Table 4.2 reports the overall costs, computational time, standard deviation and coefficient variation for 20 instances with 2 items and 12 periods. We compare the methods based on the average computational time (column *Time*, in seconds) from the optimization, and the expected value (column *Exp.Cost*) of each solution approach evaluated in the simulation, along with the 75th, 95th and 99th percentile cost (p.c.), where the 99th percentile cost gives the approximate behaviour of the models for an adverse context. We also indicate the coefficient of variation *CV* of the costs, which gives the percentage of variability of the costs in the simulation. Thus, *CV* gives the ratio of the standard deviation to the mean, where a high *CV* indicates costs widely dispersed from the average expected cost.

Table 4.2 shows that the *DRO* models outperform the stochastic programs and the robust models for all criteria but the computational time. Although *RO* models are faster

Table 4.2 – Performance of the models in a real application

Model	Exp. Cost	95 th p.c.	99 th p.c.c	Time	CV
EVPI	1,460,799	1,483,854	1,496,477		2.83%
DRO	1,495,142	1,527,551	1,546,791	0.08	3.00%
SP	1,496,609	1,546,409	1,577,002	0.02	3.47%
RO	1,496,421	1,545,033	1,575,301	0.01	3.46%

to be computed, *DRO* leads to greater cost savings (since the expected costs and even the 99th percentile costs with *DRO* are lower than the costs with *RO*) and higher robustness (since its coefficient of variation is the lowest) in an average situation. In fact, the *DRO* expected cost represents a cost saving of 1.2K compared to *RO*, and about 1.5K compared to *SP*. *DRO* also has a reduction of 17.5K and 18.9K (resp. 28.5K and 30.2K) on the 95th percentile costs (resp. 99th percentile costs) compared to *RO* and *SP*, respectively. In addition, *DRO* coefficient of variation is reduced by 13% and 14% when compared with *RO* and *SP*, respectively. Regarding the computational time, Table 4.2 all the models are easily computed. Therefore, scalability is not a strategical nor operational issue.

4.5 Conclusion

In this chapter, we have introduced the distributionally robust formulation for multi-item multi period lot-sizing under yield uncertainty. We show that with a scenario-wise ambiguity set the problem can be easily solved with the help of a python library, and it can also be reformulated as a MILP robust model. Our experimental results show that although the distributionally robust models suffer from scalability issues for large instances and large ambiguity sets, distributionally robust LSP models propose sufficiently robust production plans that integrate a good cost cutting strategy and lower risk sensitivity to the realization of the production yield. Other major advantages of the event-wise distributionally robust optimization over the other considered approaches are that the construction of the ambiguity set is data driven, free from strong assumptions about uncertain parameter patterns. Further investigation is still needed to improve the quality of the distributionally robust formulations, and to reduce their sensitivity to greater disturbance on the uncertain production yield for adverse scenarios. A decomposition approach is also envisaged to improve the quality of the distributionally robust production plans and obtain better bounds with less computational effort. In addition, an extension of this distributionally robust model with a clustering-based ambiguity set is envisaged to bring the proposed model closer to real-world applications.

CONCLUSION AND PERSPECTIVES

This thesis considers the robust and adaptive robust single-item multi-period lot-sizing problems with backorders and uncertain production yield, and it intends to provide a data-driven robust approach to handle the multi-item version of the problem via the distributionally robust optimization. Since the adaptive robust model is difficult to solve due to its non-fixed recourse that relates the uncertain production yield to the adaptive lot size decisions, we propose an approximation and a column and constraint generation approach to compute adaptive robust plans with commercial solvers. The growing interest about data-driven techniques to define lot-sizing decisions has motivated us to study a distributionally robust model for the multi-item lot-sizing problem under yield uncertainty. We show that with an event-wise ambiguity set the problem can be easily solved with the help of a python library, and the it can also be reformulated as a MILP robust model. Furthermore, we provide different methodological strategies to easily compute sufficiently robust lot-sizing decisions, and thus help manufacturers to enhance the system to better account for shop floor volatility.

To conclude this manuscript, in this section first we highlight the managerial contributions, then we summarise the methodological contributions, and finally, we review the conclusions from our numerical experiments in order to indicate some perspectives of research.

5.1 Managerial contributions

Figure 5.1 summarises the managerial insights gained from the work carried out. It is evident that the conservatism of the solution can be reduced by controlling the decision maker aversion to risk. It is also really recommended to integrate more up to date information to the model in order to improve the quality of the production plan, and even make the final model closer to real cases. In fact, the large amount of data that can nowadays be collected from the shop floor allows inferring more accurate information on the LSP parameters and their variability. It allows us to avoid the natural nervousness arising from the decision making process based on distributive information and from misestimating of the production context with the help of a robust perspective of the optimization process.

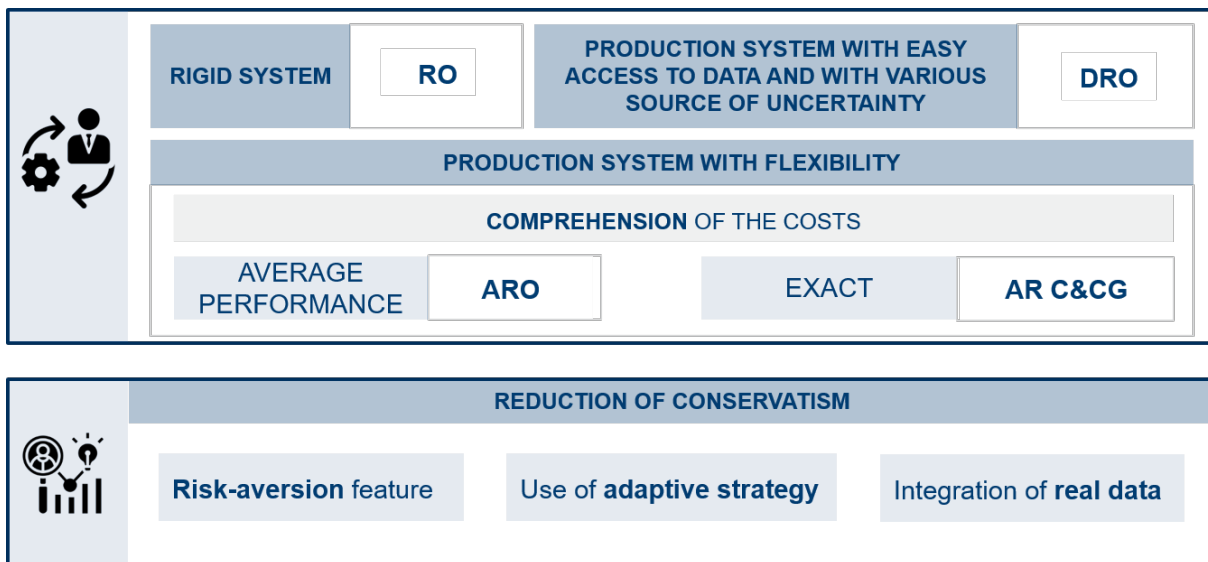


Figure 5.1 – Managerial insights from the contributions

Figure 5.1 also highlights some guidance to identify the best strategy and modeling to handle different production contexts. For instance, when facing rigid production systems where changes are not easily implementable, one should apply the RO methods. However, when facing adaptive production systems for which new accurate information is available, one could decide between the ARO and the C&CG approach. The first is more suitable for cases where risks are acceptable and only a good average performance should be guaranteed. If a complete and exact comprehension of the impact of the uncertainty of the costs is needed, then the later and optimal adaptive robust approach should be used. Finally, in a production context with various sources of uncertainty for which the

estimation of the uncertain parameter can be complex or hard to perform, then it would be better to use D R methods. Here, any information, accurate or not, new or historical, can be integrated in the system to improve the quality of the obtained production plan, yet still robust and mitigate nervousness. With respect to the reduction of the conservatism, we learned how the manufacturer risk aversion could be integrated into the robust modeling to respond to this strategic need.

The adaptive approximation should be adopted in adaptive production systems when an average optimal performance in terms of quantities and costs is enough for reassuring the decision maker about the quality and the robustness of the production plan. Since the problem is relaxed and not handled in its entire complexity, a lower computational effort is required to compute a solution. However, there is an increase of the risk sensitivity of the model to changes in the production. On the other hand, the optimal solution obtained by the decomposition approach is recommended for systems where an exact comprehension of the impact of the yield on the costs is needed. It allows us to compute optimal adaptive robust plans for the problems in its real complexity (even if the complexity from the quadratic uncertainty is indirectly handled). This leads to more conservative solutions, where the risks taken by the decision maker are fully controlled, at the expense of higher computational effort which compensates the robustness of an adaptive plan fully immunized from adverse events. Hence, the adaptive strategy helps us to adapt the system to react to the availability of new and accurate data, while a data-driven decision process has made possible through the distributionally robust method.

5.2 Methodological contributions

Figure 5.2 summarizes the methodological contributions proposed in this manuscript. First, it was proposed a state of the art on stochastic LSPs and the recent trends to handle with it. The Figure shows that, as analytical contribution, we proposed properties to define robust and adaptive robust optimal lot size decisions, which can guide the decision maker on its decision process. It was also derived an optimal robust policy and a myopic adaptive robust policy that disregards the impact of the uncertain yield on future costs of the system. In a technical-computational perspective, a dynamic program for the static strategy, a decomposition method for the static-dynamic strategy and a rolling horizon simulation of the dynamic strategy were also provided to easily compute optimal plans. In a modeling perspective, the manuscript presents the first models on the lot-sizing literature that addresses the production yield uncertainty in a robust perspective. It also provides MILP formulations to easily compute robust, adaptive robust and distributionally robust plans.

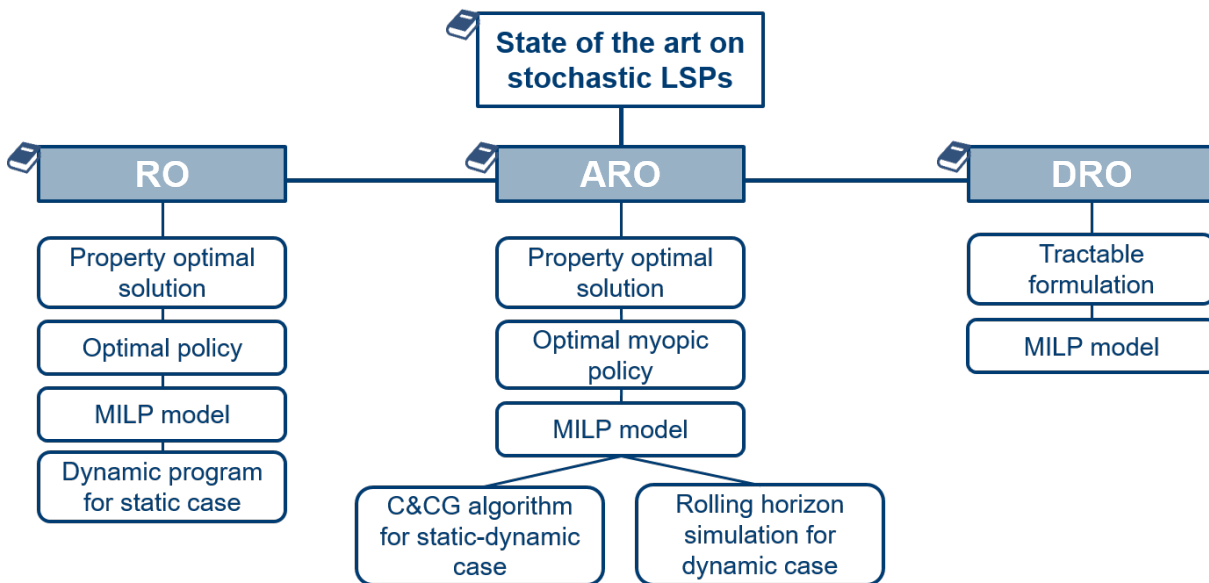


Figure 5.2 – Methodological insights from the contributions

5.3 Final discussion and perspectives

Our results show that with a proper budget of uncertainty, the robust model mitigates uncertainties with a balance between production quantities, setup costs, and inventory management costs. In addition, the robust optimization method requires less computational effort than stochastic programming, and its solutions are less conservative, yet more robust, compared to the classical approaches to dealing with uncertainties on LSPs (represented here by the nominal problem and the deterministic problem with safety stock). The adaptive approach offers a production plan that is both robust and easily changeable in an adaptive decision context, while the production system remains immunized from uncertain production yield. Finally, our experiments indicate that although the distributionally robust models suffer from scalability issues for large instances and large ambiguity sets, the distributionally robust production plans provide cost savings and lower risk sensitivity to the realization of the production yield.

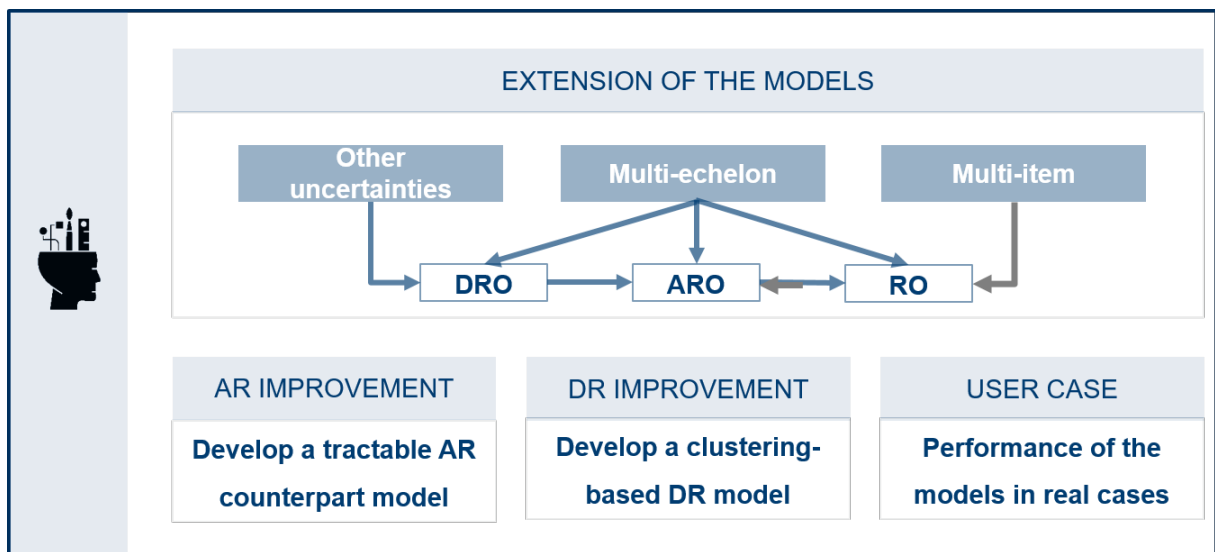


Figure 5.3 – Perspective of research

Figure 5.3 gives some of the research perspectives. The immediate extension of the current work is to design a decomposition approach to solve large DR problems easily. It is also intended to handle real cases with it to measure the quality of the method when real errors or misestimating are taken into account. In addition, we can cite some perspectives of research. The extension of the robust models for the multi-item and multi-echelon variants would bring the formulation closer to real applications, as well as the integration of different uncertainties in the same model, for example combining the uncertain demand

and uncertain lead time with the production yield rate. It would also be interesting to develop a tractable formulation do directly compute an A R production plan. For the distributionally robust model, it is intended to develop a clustering based ambiguity set. As a result, we could work with different features and measurements describing the uncertainty in a same ambiguity set. And finally, case studies would be great tools to better analyse the benefits and impact of robust decisions on the production process under yield uncertainty.

Figure 5.3 also shows that further investigation is still needed to extend the robust perspective on lot-sizing problems under yield uncertainty to the multi-item and multi-echelon version of problems. Other extensions of the problem, such as the integration of different uncertainties are also envisaged to better represent real case applications. Additional studies are also needed to improve the quality of the distributionally robust formulations and to reduce their risk sensitiveness to greater disturbance on the uncertain production yield considering adverse scenarios. A decomposition approach is also envisaged to improve the quality of the distributionally robust production plans and obtain better bounds with less computational effort. In addition, an extension of this distributionally robust model a clustering-based ambiguity set is envisaged to bring the proposed model closer to real-world applications.

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Appendices

ROBUST LOT-SIZING PROBLEMS UNDER YIELD UNCERTAINTY

A.1 Proof of Proposition 2.3.1

Proof. In the robust single-period inventory management problem under box uncertainty set, the worst inventory (resp. backorder) cost corresponds to the largest (resp. lowest) production yield, and this situation corresponds to $Z = 1$ (resp. $Z = -1$). For this special case, we denote by $\hat{I}(X)$ the worst inventory cost and by $\hat{B}(X)$ the worst backorder cost. These costs depend only on the lot size X , and they are given as follows:

$$\hat{I}(X) = \max \{h [X(\bar{\rho} + \hat{\rho}) - d] , 0\} \quad \hat{B}(X) = \max \{b [d - X(\bar{\rho} - \hat{\rho})] , 0\}$$

The total cost $\text{TC} = \hat{I}(X) + \hat{B}(X)$ is piecewise linear convex in the lot size X , and it reaches its minimum when $\hat{I}(X) = \hat{B}(X)$. \square

A.2 Proof of Proposition 2.3.2

Proof. Since the optimal policy achieves its minimum cost when the worst inventory cost is equal to the worst backorder cost, from Proposition 2.3.1, we can directly derive the optimal policy as the production quantity X for which $\hat{I}(X) = \hat{B}(X)$. Thus, with a little algebra, we can isolate X in the equality $h[X(\bar{\rho} + \hat{\rho}) - d] = b[d - X(\bar{\rho} - \hat{\rho})]$, that results in our optimal robust policy $X = \frac{d}{\bar{\rho} + \hat{\rho} \left(\frac{h-b}{h+b} \right)}$. \square

A.3 Proof of Proposition 2.3.3

Proof. As in the proof for the single-period problem, we define the inventory and backorder costs in terms of the production quantities $X_{[t]}$ and disturbance from the mean $Z_{[t]}$ up to period t as follows:

$$I_t(X_{[t]}, Z_{[t]}) = \max \left\{ h_t \sum_{\tau=1}^t [X_\tau(\bar{\rho} + \hat{\rho}Z_\tau^t) - d_\tau] ; 0 \right\}$$

$$B_t(X_{[t]}, Z_{[t]}) = \max \left\{ b_t \sum_{\tau=1}^t [d_\tau - X_\tau(\bar{\rho} + \hat{\rho}Z_\tau^t)] ; 0 \right\}$$

where $X_{[t]}$ (resp. $Z_{[t]}$) is the vector of X_τ (resp. Z_τ^q) $\forall \tau \leq t$; $q \leq \tau$. We assume that the values of Z_τ^t are chosen independently for each period t . Therefore, the worst case perspective in a period t sets Z_τ^t to 1 (for inventory) or -1 (for backorder) for all $\tau \leq t$. Let us define the cumulative demand as $\bar{D}_t = \sum_{\tau=1}^t d_\tau$ and the cumulative production quantity up to period t as $\bar{X}_t = \sum_{\tau=1}^t X_\tau$. The worst inventory cost ($\hat{I}_t(\bar{X}_t)$) and the worst backorder cost ($\hat{B}_t(\bar{X}_t)$) in period t can be written as follows:

$$\hat{I}_t(\bar{X}_t) = \max \left\{ h_t [\bar{X}_t(\bar{\rho} + \hat{\rho}) - \bar{D}_t] ; 0 \right\}$$

$$\hat{B}_t(\bar{X}_t) = \max \left\{ b_t [\bar{D}_t - \bar{X}_t(\bar{\rho} - \hat{\rho})] ; 0 \right\}$$

The total cost TC_t for period t is given by $TC_t = v_t X_t + \hat{I}_t(\bar{X}_t) + \hat{B}_t(\bar{X}_t)$. TC_t is a piecewise linear convex function in \bar{X}_t . From Proposition 2.3.1, TC_t reaches its minimum when $\hat{I}_t(\bar{X}_t) = \hat{B}_t(\bar{X}_t)$. When $v_t \geq 0$, the function defining the total cost at period t only has an upward shift equivalent to the total production costs in period t where $\hat{I}_t(\bar{X}_t) = \hat{B}_t(\bar{X}_t)$. The period where the minimum is reached does not change. Therefore, the condition $v_t \leq (\bar{\rho} - \hat{\rho})b_t$ indicates that it is profitable to produce while the production cost is lower than the backorder cost for the worst realization of the production yield, that is, for the lowest production yield rate. Note that if $v_t > (\bar{\rho} - \hat{\rho})b_t$, the production cost becomes larger than the backorder cost, and TC_t is a strictly increasing function. Thus, TC_t reaches its minimum when $\hat{I}_t(\bar{X}_t) = \hat{B}_t(\bar{X}_t)$, if condition $v_t \leq (\bar{\rho} - \hat{\rho})b_t$ is respected.

Since the cumulative demand increases with t , we seek for the cumulative production quantities that optimize the total cost for each period and lead to a minimization of the sum maximum cost over the horizon up to t . With a little algebra, from the equality $\hat{I}_t(\bar{X}_t) = \hat{B}_t(\bar{X}_t)$, we define the optimal robust cumulative policy for each period t as

follows:

$$\bar{X}_t = \frac{\bar{D}_t}{\bar{\rho} + \hat{\rho} \left(\frac{h_t - b_t}{h_t + b_t} \right)} = \frac{\sum_{\tau=1}^t d_\tau}{\bar{\rho} + \hat{\rho} \left(\frac{h_t - b_t}{h_t + b_t} \right)} \quad (\text{A.1})$$

Since the production in period t is equivalent to the difference between the cumulative production up to t given by \bar{X}_t and the cumulative production up to period $t - 1$ given by \bar{X}_{t-1} , with a little algebra we define the optimal robust policy for the multi-period inventory management problem under box uncertainty set and yield uncertainty, and with non-stationary inventory and backorder costs as follows:

$$X_t = \bar{X}_t - \bar{X}_{t-1} = \frac{\bar{D}_t}{\bar{\rho} + \hat{\rho} \left(\frac{h_t - b_t}{h_t + b_t} \right)} - \frac{\bar{D}_{t-1}}{\bar{\rho} + \hat{\rho} \left(\frac{h_{t-1} - b_{t-1}}{h_{t-1} + b_{t-1}} \right)} = \frac{\bar{\rho} d_t + \hat{\rho} \left[\left(\frac{h_{t-1} - b_{t-1}}{h_{t-1} + b_{t-1}} \right) \bar{D}_t - \left(\frac{h_t - b_t}{h_t + b_t} \right) \bar{D}_{t-1} \right]}{\left(\bar{\rho} + \hat{\rho} \left(\frac{h_t - b_t}{h_t + b_t} \right) \right) \left(\bar{\rho} + \hat{\rho} \left(\frac{h_{t-1} - b_{t-1}}{h_{t-1} + b_{t-1}} \right) \right)}$$

□

A.4 Proof of Proposition 2.3.6

Proof. Since our dynamic program relies on Proposition 2.3.4 and Proposition 2.3.5, it addresses the uncapacitated LSPs under a box uncertainty set, with uncertain yield, with non-stationary costs and demands, and with stationary nominal value and maximum deviation of the production yield. Clearly, the execution time is dominated by the search for successive regeneration intervals and their respective setup periods for each t in function $F(t)$. The search for the best production setup β and regeneration interval $[m, n]$ that gives the minimal $MCI(m, \beta, n)$ cost takes at most $\mathcal{O}(T^2)$. One should also consider the function $MCR(r, T)$ that minimizes the costs from the last regeneration period to the end of the planning horizon in $\mathcal{O}(T)$ run time. However, $\mathcal{O}(T)$ is dominated by $\mathcal{O}(T^2)$. Since dominated complexities can be ignored, then we only consider $\mathcal{O}(T^2)$. With the forward approach, we can access previously computed values in $\mathcal{O}(1)$. Thus, the recursive function $F(t)$ takes only $\mathcal{O}(1)$ to access the values for $F(\tau)$, $\tau < t$. Running the dynamic program calls T times the recursive function $F(t)$. Consequently, the dynamic program takes at most $\mathcal{O}(T^3)$ run time to propose a solution over the entire planning horizon. \square

A.5 The newsboy-based LSP under yield uncertainty

Khouja [1999] present an excellent review on the single period newsboy problem. An introduction to the problem under yield uncertainty is presented by Noori and Keller [1986]. Some considerations about the lot sizing as a single-stage, single-item, multi-period newsboy problem with deterministic demand random yields was proposed in Yano [1989]. The authors present the dynamic program for the newsboy-based formulation of the LSP with uncertain yield. Inspired by the newsboy-based LSP formulation proposed by Yano [1989], for each period $t \in T$, we can implement a newsboy-based problem (*NB*) to deal with a lot-sizing problem with uncertain yield with a lot-for-lot policy that is given as follows:

$$NB : \min E(Z_t) = v_t Q_t + b_t \int_0^{\beta_t} (d_t - \rho_t Q_t) f(\rho) d\rho + h_t \int_{\beta_t}^1 (\rho_t X_t - d_t) f(\rho) d\rho$$

where $\beta_t = \frac{d_t}{Q_t}$ is the expected production yield rate, and Q_t the production quantity in period t .

The conditions for minimization optimality are $Q_t^* \Leftrightarrow \frac{\partial E(Z_t)}{\partial Q_t} = 0$; $\frac{\partial^2 E(Z_t)}{\partial Q_t^2} \geq 0$. We have $\frac{\partial E(Z_t)}{\partial Q_t} = v_t + h_t \bar{\rho}_t + (b_t + h_t) E(\beta)$ as a first condition for optimality. Therefore, production happens if and only if the first condition of optimality is met. Thus, we may produce when:

$$\frac{\partial E(Z_t^*)}{\partial Q_t^*} = v_t + h_t \bar{\rho}_t E\left(\frac{d_t}{d_t} = 1\right) + (b_t + h_t) E(\beta_t^*) = 0 \quad \therefore \quad E(\beta_t^*) = \frac{v_t + h_t \bar{\rho}_t}{b_t + h_t}$$

where $\beta_t^* = \frac{d_t}{X_t^*}$. The second condition for optimality is $\frac{\partial^2 E(Z_t)}{\partial Q_t^2} = \frac{(b_t + h_t) d_t^2}{Q_t^3}$. The optimality conditions say that given the expected production yield we may produce at least $Q_t = d_t$ otherwise, we may backorder the demands. Then, if any production is made, we have $\frac{\partial^2 E(Z_t)}{\partial Q_t^2} \geq 0$.

For the capacitated LSP, we check if the quantity to be produced respects the resource availability. Any quantity not supported by available capacity must be backordered. We repeat this process over the production planning horizon to compute the optimal newsboy-based LSP solution.

A.6 The deterministic LSP with safety stock

Inspired by the work of Absi and Kedad-Sidhoum [2009], we consider DET_{SS} , the deterministic LSP with production yield ρ_t and safety stock SS_t , which is given as follows:

$$\begin{aligned} \min \quad & \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t + b_t B_t \\ \text{s.t.} \quad & \\ & I_t - B_t = I_{t-1} - B_{t-1} + \rho_t X_t - d_t & \forall t \in T \\ & X_t \leq M_t Y_t & \forall t \in T \\ & I_t \geq SS_t & \forall t \in T \\ & X_t, I_t, B_t \geq 0 & \forall t \in T \\ & Y_t \in \{0, 1\} & \forall t \in T \end{aligned}$$

We compute the safety stock with the newsboy-based solution, such that $SS_t = Q_t - d_t$, where Q_t is the production quantity defined with NB . This leads to a situation where backorders and inventory can occur in the same period, so that safety stock that is not respected is penalized through backorder costs.

A.7 The two-stage stochastic programming LSP with uncertain production yield

SP handles uncertainty through a mathematical program whose objective is to minimize the expected cost [Birge and Louveaux, 2011]. The uncertain parameter is described by a probability distribution and some statistical indicators (e.g., mean and standard deviation) that are usually gathered by processing and analysis of data from historical data and other available data about the decision system. The stochastic programs are a natural benchmark to compare the production plans proposed by other methodologies and verify their performance and quality.

To evaluate the performance of our robust model, we propose a scenario-based stochastic program to represent the lot-sizing problem under yield uncertainty, based on the case study presented by Kazemi Zanjani et al. [2010]. While the authors present a multi-stage formulation which deals with a dynamic decision problem, our work addresses a static decision strategy. A two stage approach corresponds to a static decision in which the plan for the entire production horizon is defined before the realization of the uncertainty. Since our work investigates the static version of the LSP with uncertain yield, a two stage model is a more appropriate benchmark to measure the quality of the static robust model. We consider a set Ω of possible yield scenarios, where each scenario ω has a probability p_ω of realization. ρ_t^ω is the realization of the uncertain yield for the period t of in scenario ω . The two-stage stochastic program for the LSP with uncertain yield is given as follows:

$$\begin{aligned}
 & \min \sum_{\omega \in \Omega} p_\omega \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t^\omega + b_t B_t^\omega \\
 & \text{s.t. :} \\
 & I_t^\omega - B_t^\omega = I_{t-1}^\omega - B_{t-1}^\omega + \rho_t^\omega X_t - d_t & \forall t \in T; \omega \in \Omega \\
 & X_t \leq M_t \cdot Y_t & \forall t \in T \\
 & X_t \geq 0 & \forall t \in T \\
 & I_t^\omega, B_t^\omega \geq 0 & \forall t \in T; \omega \in \Omega \\
 & Y_t \in \{0, 1\} & \forall t \in T
 \end{aligned}$$

Although SP is largely applied within optimization under uncertainties, this approach often suffers from scalability issues, being computationally prohibitive, and requiring advanced techniques to generate possible scenarios. For this, as many scenarios as possible are generated in order to reflect the uncertainty distribution, even though the number of

scenarios may be limited to restrict the computational efforts. For our numerical experiments, we used this stochastic problem with 500 scenarios, where the yield realizations are randomly drawn from a uniform distribution with support $[\bar{\rho}_t - \hat{\rho}_t; \bar{\rho}_t + \hat{\rho}_t]$ for each period t .

ADAPTIVE ROBUST LOT-SIZING PROBLEMS UNDER YIELD UNCERTAINTY

B.1 Adaptive robust reformulations per constraints and dualization

In this section, we present the dualization proposed for each constraint subject to uncertainty in period t . This dualization consists of the second step of the reformulation per constraint and dualization approach applied to the adaptive lot-sizing problem under yield uncertainty. a) **Dual formulation for Constraints (3.7b)**

$$\begin{array}{ll}
\max \sum_{\tau=1}^{t-1} \hat{\rho}_\tau (v_t X_t^\tau + H_t^\tau) Z_t^\tau & \min \Gamma_t \gamma_t + \sum_{\tau=1}^{t-1} (\alpha_t^\tau + \delta_t^\tau) \\
\sum_{\tau=1}^{t-1} R_\tau \leq \Gamma_t & \gamma_t - \alpha_t^{\prime\tau} - \delta_t^{\prime\tau} \geq 0 \quad \forall \tau < t \\
Z_t^\tau \leq 1 & \alpha_t^\tau - \delta_t^\tau + \alpha_t^{\prime\tau} - \delta_t^{\prime\tau} = \hat{\rho}_\tau (v_t X_t^\tau + H_t^\tau) \quad \forall \tau < t \\
-R_t^\tau \leq 1 & \gamma_t \geq 0 \quad \forall \tau < t \\
-R_\tau + Z_t^\tau \leq 0 & \alpha_t^\tau, \alpha_t^{\prime\tau}, \delta_t^\tau, \delta_t^{\prime\tau} \geq 0 \quad \forall \tau < t \\
-R_\tau - Z_t^\tau \leq 0 & \\
R_\tau \geq 0 & \\
Z_t^\tau \in \mathbb{R} &
\end{array} \xrightarrow{\text{dualized}}$$

b) Dual formulation for Constraints (3.8)

Replacing the uncertain yield and the adjustable variables by their respective parameterizations, and developing a little algebra in Constraints (3.7c) leads to:

$$\begin{aligned}
H_t^0 + \sum_{\tau=1}^{t-1} H_t^\tau \rho_\tau &\geq h_t \sum_{\tau=1}^t \left[\rho_\tau (X_\tau^0 + \sum_{r=1}^{\tau-1} X_\tau^r \rho_r) - d_\tau \right] & \forall t \in T \\
H_t^0 + \sum_{\tau=1}^{t-1} H_t^\tau (\bar{\rho}_\tau + \hat{\rho}_\tau Z_t^\tau) &\geq h_t \sum_{\tau=1}^t \left[(\bar{\rho}_\tau + \hat{\rho}_\tau Z_t^\tau) [X_\tau^0 + \sum_{r=1}^{\tau-1} X_\tau^r (\bar{\rho}_r + \hat{\rho}_r Z_r^\tau)] - d_\tau \right] & \forall t \in T \\
H_t^0 + \sum_{\tau=1}^{t-1} \bar{\rho}_\tau H_t^\tau + \sum_{\tau=1}^{t-1} \hat{\rho}_\tau H_t^\tau Z_t^\tau &\geq h_t \left[\sum_{\tau=1}^t \left(\bar{\rho}_\tau X_\tau^0 - d_\tau + \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \bar{\rho}_r X_\tau^r \right) \right] + h_t \left[\sum_{\tau=1}^t \left(\hat{\rho}_\tau X_\tau^0 + \sum_{r=1}^{\tau-1} \hat{\rho}_\tau \bar{\rho}_r X_\tau^r \right) Z_t^\tau \right] & \forall t \in T \\
&+ h_t \left[\sum_{\tau=1}^t \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \hat{\rho}_r X_\tau^r Z_t^\tau \right] + h_t \left[\sum_{\tau=1}^t \sum_{r=1}^{\tau-1} \hat{\rho}_\tau \hat{\rho}_r Z_t^\tau X_\tau^r Z_t^\tau \right] \\
H_t^0 + \sum_{\tau=1}^{t-1} \bar{\rho}_\tau H_t^\tau &\geq h_t \left[\sum_{\tau=1}^t \left(\bar{\rho}_\tau X_\tau^0 - d_\tau + \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \bar{\rho}_r X_\tau^r \right) \right] + \left[\sum_{\tau=1}^{t-1} \left(h_t \hat{\rho}_\tau X_\tau^0 - \hat{\rho}_\tau H_t^\tau + \sum_{r=1}^{\tau-1} h_t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r \right) Z_t^\tau \right] & \forall t \in T \\
&+ h_t (\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau) Z_t^t + h_t \left[\sum_{\tau=1}^t \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \hat{\rho}_r X_\tau^r Z_t^\tau \right] + h_t \left[\sum_{\tau=1}^t \sum_{r=1}^{\tau-1} \hat{\rho}_\tau \hat{\rho}_r Z_t^\tau X_\tau^r Z_t^\tau \right]
\end{aligned}$$

We concentrate our analysis on the case that ignores the quadratic term, thus we consider the following reformulation of Constraints (3.8):

$$H_t^0 + \sum_{\tau=1}^{t-1} \bar{\rho}_\tau H_t^\tau \geq h_t \left[\sum_{\tau=1}^t \left(\bar{\rho}_\tau X_\tau^0 - d_\tau + \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \bar{\rho}_r X_\tau^r \right) \right] + \left[\sum_{\tau=1}^{t-1} \left(h_t \hat{\rho}_\tau X_\tau^0 - \hat{\rho}_\tau H_t^\tau + \sum_{r=1}^{\tau-1} h_t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r \right) Z_t^\tau \right] \quad \forall t \in T$$

$$+ h_t (\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau) Z_t^t + h_t \left[\sum_{\tau=1}^t \sum_{r=1}^{\tau-1} \bar{\rho}_\tau \hat{\rho}_r X_\tau^r Z_t^\tau \right]$$

The dualization of the terms that take into account the disturbance Z in constraints is given as follows:

$$\max \sum_{\tau=1}^{t-1} \left(\hat{\rho}_\tau (h_t X_\tau^0 - H_t^\tau) + h_t (\sum_{r=1}^{\tau-1} \hat{\rho}_\tau \bar{\rho}_r X_\tau^r + \sum_{r=\tau+1}^t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r) \right) Z_t^\tau + h_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau \right) Z_t^t$$

$$\begin{aligned} \sum_{\tau=1}^t R_\tau &\leq \Gamma_t && \forall \tau \leq t \\ Z_t^\tau &\leq 1 && \forall \tau \leq t \\ -Z_\tau &\leq 1 && \forall \tau \leq t \\ -R_\tau + Z_t^\tau &\leq 0 && \forall \tau \leq t \\ -R_\tau - Z_t^\tau &\leq 0 && \forall \tau \leq t \\ Z_t^\tau &\in \mathbb{R} && \forall \tau \leq t \\ R_\tau &\geq 0 && \forall \tau \leq t \end{aligned}$$

dualized

$$\begin{aligned} \min \Gamma_t \lambda_t + \sum_{\tau=1}^t (\mu_t^\tau + \varepsilon_t^\tau) \\ \mu_t^\tau - \varepsilon_t^\tau + \mu_t^{\prime\tau} - \varepsilon_t^{\prime\tau} = \hat{\rho}_\tau (h_t X_\tau^0 - H_t^\tau) + h_t (\sum_{r=1}^{\tau-1} \hat{\rho}_\tau \bar{\rho}_r X_\tau^r + \sum_{r=\tau+1}^t \hat{\rho}_\tau \bar{\rho}_r X_\tau^r) & \quad \forall \tau < t \\ \lambda_t - \mu_t^{\prime\tau} - \varepsilon_t^{\prime\tau} \geq 0 & \quad \forall \tau < t \\ \mu_t^t - \varepsilon_t^t + \mu_t^{\prime t} - \varepsilon_t^{\prime t} = h_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau \right) \\ \lambda_t - \mu_t^{\prime t} - \varepsilon_t^{\prime t} \geq 0 \\ \lambda_t \geq 0 \\ \mu_t^\tau, \mu_t^{\prime\tau}, \varepsilon_t^\tau, \varepsilon_t^{\prime\tau} \geq 0 & \quad \forall \tau < t \end{aligned}$$

c) Dual formulation for Constraints (3.9)

Similarly to the formulation presented for Constraints (3.8), we have

$$\max \sum_{\tau=1}^{t-1} \left(-\hat{\rho}_\tau (b_t X_\tau^0 + H_t^\tau) - b_t (\sum_{r=1}^{\tau-1} \hat{\rho}_\tau \bar{\rho}_r X_r^\tau + \sum_{r=\tau+1}^t \hat{\rho}_\tau \bar{\rho}_r X_r^\tau) \right) Z_t^\tau - b_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau \right) Z_t^t$$

$$\begin{aligned} \sum_{\tau=1}^t R_\tau &\leq \Gamma_t && \\ Z_t^\tau &\leq 1 && \forall \tau \leq t \\ -Z_t^\tau &\leq 1 && \forall \tau \leq t \\ -R_\tau + Z_t^\tau &\leq 0 && \forall \tau \leq t \\ -R_\tau - Z_t^\tau &\leq 0 && \forall \tau \leq t \\ Z_t^\tau &\in \mathbb{R} && \forall \tau \leq t \\ R_\tau &\geq 0 && \forall \tau \leq t \end{aligned}$$

dualized

$$\begin{aligned} \min \Gamma_t \psi_t + \sum_{\tau=1}^t (\varpi_t^\tau + \xi_t^\tau) \\ \varpi_t^\tau - \xi_t^\tau + \varpi_t'^\tau - \xi_t'^\tau = -\hat{\rho}_\tau (b_t X_\tau^0 + H_t^\tau) - b_t (\sum_{r=1}^{\tau-1} \hat{\rho}_\tau \bar{\rho}_r X_r^\tau + \sum_{r=\tau+1}^t \hat{\rho}_\tau \bar{\rho}_r X_r^\tau) & \forall \tau < t \\ \psi_t - \varpi_t'^\tau - \xi_t'^\tau \geq 0 & \forall \tau < t \\ \varpi_t^t - \xi_t^t + \varpi_t'^t - \xi_t'^t = -b_t \left(\hat{\rho}_t X_t^0 + \sum_{\tau=1}^{t-1} \hat{\rho}_t \bar{\rho}_\tau X_t^\tau \right) \\ \psi_t - \varpi_t'^t - \xi_t'^t \geq 0 \\ \psi_t \geq 0 \\ \varpi_t^\tau, \varpi_t'^\tau, \xi_t^\tau, \xi_t'^\tau \geq 0 & \forall \tau < t \end{aligned}$$

d) Dual formulation for Constraints (3.7e)

$$\begin{array}{ll} \max \sum_{\tau=1}^{t-1} \hat{\rho}_\tau X_t^\tau Z_t^\tau & \min \Gamma_t \eta_t + \sum_{\tau=1}^{t-1} (\beta_t^\tau + \theta_t^\tau) \\ \sum_{\tau=1}^{t-1} R_\tau \leq \Gamma_t & \eta_t - \beta_t'^\tau - \theta_t'^\tau \geq 0 \quad \forall \tau < t \\ Z_t^\tau \leq 1 & \forall \tau < t \xrightarrow{\text{dualized}} \beta_t^\tau - \theta_t^\tau + \beta_t'^\tau - \theta_t'^\tau = \hat{\rho}_\tau X_t^\tau \quad \forall \tau < t \\ -Z_t^\tau \leq 1 & \forall \tau < t & \eta_t \geq 0 \\ -R_\tau + Z_t^\tau \leq 0 & \forall \tau < t & \beta_t^\tau, \beta_t'^\tau, \theta_t^\tau, \theta_t'^\tau \geq 0 \quad \forall \tau < t \\ -R_\tau - Z_t^\tau \leq 0 & \forall \tau < t \\ R_\tau \geq 0 & \forall \tau < t \\ Z_t^\tau \in \mathbb{R} & \forall \tau < t \end{array}$$

e) Dual formulation for Constraints (3.7f)

$$\begin{array}{ll}
\min \sum_{\tau=1}^{t-1} \hat{\rho}_\tau X_t^\tau Z_t^\tau & \max -\Gamma_t \phi_t - \sum_{\tau=1}^{t-1} (\pi_t^\tau + \chi_t^\tau) \\
-\sum_{\tau=1}^{t-1} R_\tau \geq -\Gamma_t & \phi_t - \pi_t^{\prime\tau} - \chi_t^{\prime\tau} \geq 0 \quad \forall \tau < t \\
-Z_t^\tau \geq -1 \quad \forall \tau < t & \xrightarrow{\text{dualized}} -\pi_t^\tau + \chi_t^\tau - \pi_t^{\prime\tau} + \chi_t^{\prime\tau} = \hat{\rho}_\tau X_t^\tau \quad \forall \tau < t \\
Z_t^\tau \geq -1 \quad \forall \tau < t & \phi_t \geq 0 \\
R_\tau - Z_t^\tau \geq 0 \quad \forall \tau < t & \pi_t^\tau, \pi_t^{\prime\tau}, \chi_t^\tau, \chi_t^{\prime\tau} \geq 0 \quad \forall \tau < t \\
R_\tau + Z_t^\tau \geq 0 \quad \forall \tau < t & \\
R_\tau \geq 0 \quad \forall \tau < t & \\
Z_t^\tau \in \mathbb{R} \quad \forall \tau < t &
\end{array}$$

f) **Dual formulation for Constraints (3.7g)**

$$\begin{array}{ll}
\min \sum_{\tau=1}^{t-1} \hat{\rho}_\tau H_t^\tau Z_t^\tau & \max -\Gamma_t \epsilon_t - \sum_{\tau=1}^{t-1} (\sigma_t^\tau + \kappa_t^\tau) \\
-\sum_{\tau=1}^{t-1} R_\tau \geq -\Gamma_t & \epsilon_t - \sigma_t^{\prime\tau} - \kappa_t^{\prime\tau} \geq 0 \quad \forall \tau < t \\
-Z_t^\tau \geq -1 \quad \forall \tau < t & \xrightarrow{\text{dualized}} -\sigma_t^\tau + \kappa_t^\tau - \sigma_t^{\prime\tau} + \kappa_t^{\prime\tau} = \hat{\rho}_\tau H_t^\tau \quad \forall \tau < t \\
Z_t^\tau \geq -1 \quad \forall \tau < t & \epsilon_t \geq 0 \\
R_\tau - Z_t^\tau \geq 0 \quad \forall \tau < t & \sigma_t^\tau, \sigma_t^{\prime\tau}, \kappa_t^\tau, \kappa_t^{\prime\tau} \geq 0 \quad \forall \tau < t \\
R_\tau + Z_t^\tau \geq 0 \quad \forall \tau < t & \\
R_\tau \geq 0 \quad \forall \tau < t & \\
Z_t^\tau \in \mathbb{R} \quad \forall \tau < t &
\end{array}$$

B.2 The robust LSPs with yield uncertainty model

We remind readers of the robust LSP with uncertain production yield proposed by Metzker et al. [2023a], that is given as follows:

$$\min \sum_{t \in T} s_t Y_t + v_t X_t + H_t \quad (\text{B.1})$$

s.t. :

$$H_t \geq h_t \max_{\tilde{\rho} \in \mathcal{U}_t} \left[\sum_{\tau=1}^t (\tilde{\rho}_\tau X_\tau - d_\tau) \right] \quad \forall t \in T \quad (\text{B.2})$$

$$H_t \geq -b_t \max_{\tilde{\rho} \in \mathcal{U}_t} \left[\sum_{\tau=1}^t (\tilde{\rho}_\tau X_\tau - d_\tau) \right] \quad \forall t \in T \quad (\text{B.3})$$

$$X_t \leq M_t Y_t \quad \forall t \in T$$

$$X_t, H_t \geq 0 \quad \forall t \in T$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

where $\tilde{\rho}_t$ is the uncertain yield in the budgeted uncertainty set \mathcal{U}_t . For each period t , the budgeted uncertainty set is given by $\mathcal{U}_t = \{-\mathbf{1} \leq \mathbf{Z}^t \leq \mathbf{1} : \sum_{\tau=1}^t |Z_\tau^t| \leq \Gamma_t\}$.

B.3 Performance of the ARO_{Γ} linear approximations

In this section, we investigate their performance of the linear approximations for the capacitated version of the problem. We recall that ARO_{Γ}^0 refers to the linear approximation that omit the quadratic terms, whereas the ARO_{Γ}^1 model replaces the quadratic terms by 1 such a way the inventory management costs are as high as possible.

Table B.1 – Result comparisons of the capacitated ARO_{Γ}^0 and ARO_{Γ}^1 models

Γ	ARO_{Γ}^0				ARO_{Γ}^1				$C\&CG_{\Gamma}$			
	Exp. Cost	99 th p.c.	Time	CV	Exp. Cost	99 th p.c.	Time	CV	Exp. Cost	99 th p.c.	Time	CV
0.1T	596404	676540	2.5	5.63%	650394	737784	0.9	10.42%	605300	683398	53	5.55%
0.2T	608438	678528	2.0	4.63%	657433	733167	1.7	9.10%	627972	707360	61	5.15%
0.3T	608750	665814	2.0	4.11%	638314	764943	1.9	6.98%	592309	656176	82	4.47%
0.4T	554899	609310	1.5	4.32%	622313	702454	2.2	5.38%	629111	691160	122	4.16%
0.5T	507478	559888	1.3	4.58%	597079	680908	2.2	5.53%	608804	668834	180	4.33%
0.6T	608388	660711	1.4	4.12%	617024	695733	2.1	5.2%	629265	683915	184	4.01%
0.7T	608362	661090	1.2	4.15%	615577	693255	2.0	5.18%	639830	698554	263	4.29%
0.8T	618258	672314	1.0	4.26%	615051	690595	1.9	5.15%	631537	690523	141	4.34%
0.9T	608481	661027	1.0	4.15%	618232	692787	1.7	5.11%	644250	703460	120	4.4%
T	611931	663306	0.9	4.16%	619104	693349	1.5	5.12%	637219	696036	161	4.62%

Table B.1 indicates that ARO_{Γ}^0 gives the best simulated results. ARO_{Γ}^0 has the lowest average costs, about 593139 over all budgets of uncertainty compared to the average expected cost of 625052 for ARO_{Γ}^1 . Furthermore, ARO_{Γ}^0 has the lowest variation in costs with a coefficient of variation of about 4.4%, while ARO_{Γ}^1 has a CV of about 6.45% for all Γ . However, both models converge to the same optimal adaptive solution with the help of the decomposition method $C\&CG_{\Gamma}$, with average expected cost of 624560 over all Γ and a CV of about 4.5%.

B.4 Average cost and solution quality of the proposed models

This section indicates the quality of the production plans based on the uncapacitated and capacitated models. To investigate the performance of the adaptive models, we analyse the average and worst case performance of the models in their uncapacitated and capacitated versions. The results in Table B.2 (for the capacitated models) and Table B.3 (for the uncapacitated models) provide insights for decision-makers on the approach to deal with the uncertain production yield within a static-dynamic strategy. Table B.2 and Table B.3 report the results for RO_Γ , ARO_Γ , and $C\&CG_\Gamma$ over all the instances. All models are solved to optimality, and we evaluate their costs depending on Γ , where the case with $\Gamma = T$ is the most conservative solution that corresponds to the box uncertainty set. These tables give the following outcomes from the simulation step: the 95th percentile costs (95th p.c. column), the 99th percentile costs (99th p.c. column), and the coefficient of variation (*CV* column).

From the optimization step, Table B.2 reports the computational time (Time column), in seconds, for each model for each Γ , and the average number of iterations performed by $C\&CG_\Gamma$ to obtain an optimal solution (Iter column). These tables also show the relative suboptimality between an optimization method and the optimal adaptive model. We assume GAP_{ARO} (resp. GAP_{RO}) as the relative suboptimality between $C\&CG_\Gamma$ and ARO_Γ (resp. RO_Γ). The relative suboptimality indicates the difference between the objective function value $\mathbf{Obj}(C\&CG_\Gamma)$ computed by $C\&CG_\Gamma$, and the objective function value $\mathbf{Obj}(ARO_\Gamma)$ (resp. $\mathbf{Obj}(RO_\Gamma)$) computed by ARO_Γ (resp. RO_Γ), and it is given by $GAP_{ARO} = 100 \times \frac{\mathbf{Obj}(ARO_\Gamma) - \mathbf{Obj}(C\&CG_\Gamma)}{\mathbf{Obj}(C\&CG_\Gamma)}$ (resp. $GAP_{RO} = 100 \times \frac{\mathbf{Obj}(RO_\Gamma) - \mathbf{Obj}(C\&CG_\Gamma)}{\mathbf{Obj}(C\&CG_\Gamma)}$). Note that this relative suboptimality compares the objective value computed by the linear approximation with regard to the objective value computed by the decomposition method for the setup and lot size decision fixed as the values obtained from the approximation. Therefore, we compute the actual cost difference from the linear approximation and an adaptive solution that considers the robust adaptive model in its full complexity (with more real quadratic terms).

Table B.2 reports the experimental results of the uncapacitated models. ARO_Γ and $C\&CG_\Gamma$ outperform RO_Γ in terms of average costs in the simulation since the 95th percentile costs of ARO_Γ (resp. $C\&CG_\Gamma$) are about 16.2% (resp. 13.9%) lower than the 95th percentile costs obtained with RO_Γ . In addition, the 99th percentile costs of ARO_Γ (resp.

$C\&CG_\Gamma$) are 19.3% (resp. 16.7%) lower than the one computed with the static solution. Not only do the adaptive solutions give better production plans in terms of cost savings, but they also provide more stable production plans, since their coefficients of variation are the lowest. The CV of ARO_Γ and $C\&CG_\Gamma$ are 49.6% lower than the CV obtained from the static solutions.

Table B.2 – Result of the uncapacitated RO , ARO and $C\&CG$ models

Γ	RO_Γ					ARO_Γ					$C\&CG_\Gamma$				
	95 th p.c.	99 th p.c.	Time	CV	GAP_{RO}	95 th p.c.	99 th p.c.	Time	CV	GAP_{ARO}	95 th p.c.	99 th p.c.	Time	CV	Iter
0.1T	578602	671295	0.05	10.47%	3.4%	551283	580476	2.43	6.0%	-0.56%	516302	552256	92	6.67%	8
0.2T	552325	630267	0.06	8.05%	4.75%	373057	388081	0.03	4.98%	-2.57%	500963	520730	112	4.49%	7
0.3T	561483	587864	0.07	6.1%	8.85%	483144	494676	1.60	3.79%	1.07%	494352	513679	115	3.93%	8
0.4T	571708	601045	0.08	6.59%	7.9%	530701	537867	2.09	3.29%	2.4%	501564	511691	101	3.69%	7
0.5T	581408	608250	0.06	6.54%	11.18%	533864	541922	1.69	3.44%	-5.06%	493438	503536	92	3.08%	7
0.6T	589393	611821	0.05	6.73%	11.02%	481766	489727	1.49	3.26%	-15.47%	499023	509251	585	2.78%	23
0.7T	599702	621196	0.05	6.49%	11.27%	428062	435820	1.03	3.52%	3.64%	504789	512210	239	2.54%	14
0.8T	602630	624073	0.04	7.63%	10.2%	520735	529562	1.63	3.15%	-2.95%	512857	523926	274	3.42%	16
0.9T	603192	628096	0.03	7.83%	11.81%	519642	527989	1.28	3.04%	-2.4%	502548	511333	213	3.55%	15
T	603268	628155	0.02	7.84%	10.09%	477693	485666	1.17	3.02%	-1.17%	508807	515872	311	3.31%	20

Regarding the relative suboptimality, Table B.2 shows that ARO_Γ has the lowest objective value since it underestimates the optimal adaptive robust costs. As the static models are overly-conservative, and not adapted to the adaptive strategy (with a solution that is optimal only for a static context), the GAP_{RO} results from the optimal adaptive solution are very high (about 9%). ARO_Γ underestimates the inventory management costs due to linearization that mitigates complexity of the quadratic uncertainty, which is demonstrated by the negative GAP_{ARO} on the GAP_{ARO} Column. As a result, its objective value is lower than the optimal adaptive cost.

The adaptive plans take more time to be computed than the static plans, where ARO_Γ (resp. $C\&CG_\Gamma$) takes about 1.5 second (resp. 3.5 minutes) more than RO_Γ to compute a solution. However, the adaptive plans are still robust and less conservative in terms of relative suboptimality, more stable in terms of the coefficient of variation and less costly with regard to the average costs. As a result, the adaptive plans propose a sufficiently robust solution that hedges against uncertainties better, and that yields greater cost savings.

$C\&CG_\Gamma$ leads to more stable simulated costs since its CV of is on average 7.8% lower than the CV of ARO_Γ , yet ARO_Γ is robust to a sufficiently large number of realizations of the yield. Since the quadratic term in $C\&CG_\Gamma$ represents the impact of the yield uncertainty on the adaptive part of the production quantity, $C\&CG_\Gamma$ accounts better for the extreme cases where the disturbance of the yield is large for several periods. The average computation time of approximately 1 second required by ARO_Γ is much less than

the time needed to calculate the optimal adaptive solution by $C\&CG_\Gamma$ (around 4 minutes on average). The greater computational effort required by $C\&CG_\Gamma$ is explained by the average amount of 15 iterations to find an optimal solution, while our approximation obtains satisfactory results much faster by solving a MILP model. Therefore, our results show that ignoring the quadratic term in ARO_Γ leads to less conservative solutions, while they remain sufficiently robust. Furthermore, ARO_Γ is easier to solve since it only requires solving a MILP, while $C\&CG_\Gamma$ requires a lot of iterations and computational effort to compute the sub-problems and the upper and lower bounds on the costs.

Table B.3 – Result of the capacitated RO , ARO and $C\&CG$ models

Γ	RO_Γ					ARO_Γ					$C\&CG_\Gamma$				
	95 th p.c.	99 th p.c.	Time	CV	GAP_{RO}	95 th p.c.	99 th p.c.	Time	CV	GAP_{ARO}	95 th p.c.	99 th p.c.	Time	CV	Iter
0.1T	732895	828207	0.04	9.51%	34.61%	649975	676540	2.45	5.63%	0.78%	657246	683398	53	5.55%	6
0.2T	712162	782660	0.04	7.24%	37.19%	653014	678528	2.04	4.63%	-4.68%	677701	707360	61	5.15%	7
0.3T	726915	771313	0.05	6.55%	40.87%	647254	665814	2.03	4.11%	-2.3%	634023	656176	82	4.47%	8
0.4T	733529	763458	0.06	6.08%	40.54%	593370	609310	1.46	4.32%	-3.2%	670847	691160	122	4.16%	10
0.5T	739163	766138	0.06	5.8%	44.06%	544922	559888	1.30	4.58%	-11.48%	650971	668834	180	4.33%	14
0.6T	746244	770341	0.05	5.58%	43.66%	646152	660711	1.41	4.12%	-1.47%	668822	683915	184	4.01%	13
0.7T	753257	776072	0.04	5.74%	42.45%	646323	661090	1.22	4.15%	-1.64%	682712	698554	263	4.29%	15
0.8T	751346	774898	0.03	6.23%	41.02%	657486	672314	0.98	4.26%	0.72%	674699	690523	141	4.34%	12
0.9T	750097	773725	0.02	6.24%	43.0%	646404	661027	1.02	4.15%	-2.47%	687532	703460	120	4.4%	12
T	750544	774042	0.02	6.24%	40.86%	649474	663306	0.91	4.16%	-3.3%	681110	696036	161	4.62%	14

Table B.3 reports the experimental results for the capacitated models. As in the uncapacitated case, the adaptive approximation and the optimal adaptive robust plans outperform the static model. However, the average costs increase when compared to the uncapacitated case. The limited capacity leads to more frequent backorder of the demands yielding higher costs. As a result, the production plans become highly dependent on the realization of the uncertain yield. Nevertheless, ARO_Γ provides less conservative robust adaptive plans. When compared to $C\&CG_\Gamma$ (resp. RO_Γ) plan, ARO_Γ reduces the average 95th percentile costs by approximately 5.6% (resp. 9.6%), and it also drops the average 99th percentile costs by 5.7% (resp. 11.6%). Furthermore, the adaptive models propose sufficiently robust solutions.

The relative suboptimality between the optimal static and optimal adaptive models (GAP_{RO}) is about 41.8%. The difference between the optimal value from the adaptive approximation and the optimal adaptive value (GAP_{ARO}) is around -2.9%. Therefore, $C\&CG_\Gamma$ gives the optimal objective value within the adaptive strategy in its full complexity of quadratic terms. Since ARO_Γ underestimates the costs, it leads to suboptimal but sufficiently good solutions.

Table B.3 also shows that ARO_Γ provides a reasonably stable production plan as its coefficient of variation is slightly higher (resp. 9.4% lower) than the CV obtained with the

$C\&CG_\Gamma$ (resp. RO_Γ) solution. ARO_Γ proposes good adaptive plans yet robust. Thus, even if the system has limited availability of production resources, the adaptive approximation achieves a good trade-off on the robustness of the production plan to mitigate uncertainties and a less conservative cost-saving strategy.

B.5 Remarks on the optimality of the myopic adaptive robust policy

This section compares the robust models with the adaptive myopic robust policy presented in Section 3.3.2 in a rolling horizon simulation. A rolling horizon technique allows us to handle LSPs under a dynamic strategy. We develop a rolling horizon framework similar to Venkataraman [1996] and Meistering and Stadtler [2017], where the models are solved in period t with the horizon $[t; T]$ to compute the lot sizes for the next $\|F\|$ periods, where F represents the frozen horizon. We freeze the setup computed in period 1 for the frozen horizon, and we update lot size and inventory/backorder decisions with the rolling horizon technique. The models are solved every $|F|$ periods, for which we fix the lot size for the corresponding periods. Then we perform the production for the frozen horizon, so that the lot size for the period $p + 1$ can be updated as soon as the quality inspection for the period p ends. Finally, after the production of the frozen horizon is complete, we repeat the process until the whole production horizon is complete.

Table B.4 – Performance of the models for the special case presented in Section 3.3.2

Model	Exp. Cost	95th p.c.	99th p.c.	Worst Cost
$RO_{\Gamma=T}$	798	2,028	2,204	2,249
$ARO_{\Gamma=T}$	798	2,028	2,204	2,249
$C\&CG_{\Gamma=T}$	798	2,028	2,204	2,249
<i>Policy</i>	798	2,028	2,204	2,249

We consider the optimal myopic policy (*Policy*) and $RO_{\Gamma=T}$, $ARO_{\Gamma=T}$ and $C\&CG_{\Gamma=T}$ for the special case of the problem with zero setup and unit production costs, with constant nominal value and the maximum deviation of the uncertain production yield, and with a box uncertainty set. Table B.4 reports the results for the rolling horizon simulation with $F = 1$. Since the optimal robust model generated with the help of $C\&CG$ reduces to the adaptive approximation, and both adaptive models are equivalent to the robust model when $F = 1$, the results confirm that our policy is valid.

DISTRIBUTIONALLY ROBUST LOT-SIZING PROBLEMS UNDER YIELD UNCERTAINTY

C.1 Counterpart reformulation for DRLSP constraints

In this section, we present the reformulation of the remaining constraints subject to uncertainties from problem (4.4) in a robust fashion. For that, we assume that Slater's condition and the strong duality hold on each constraint, which help us to define the constraint in a robust counterpart form.

C.1.1 Reformulation for Constraints (4.3b)

If Slater's condition holds on Constraints (4.3b), then Constraints (4.3b) is equivalent to

$$H_{its}^0 + h_{it}d_{it} \geq \sup_{\boldsymbol{\rho}, \mathbf{m}, \{\boldsymbol{\xi}_j\}_{j \in (N \times T)_{|m|}}, \underline{\boldsymbol{\rho}} \in [\underline{\boldsymbol{\rho}}, \bar{\boldsymbol{\rho}}], g_{js}(\boldsymbol{\xi}_j) \leq m_j, (\boldsymbol{\xi}_j) = \boldsymbol{\rho}, \forall j \in (N \times T)_{|m|}} \boldsymbol{\rho}^\top \mathbf{c}_{its}^3 + \mathbf{m}^\top \mathbf{c}_{its}^4$$

$\forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; i, k \in N; t, l \in T; j \in (N \times T)_{|m|}; s \in S$, where \mathbf{c}_{its}^3 is given in (C.1) and $c_{itjs}^4 = -H_{itjs}''$.

$$c_{itkls}^3 = \begin{cases} h_{it}X_{il} - H'_{itils}, & \text{if } i = k, l \leq t \\ -H'_{itils}, & \text{otherwise} \end{cases} \quad (\text{C.1})$$

If we reformulate the supremum in terms of the dual infimum for a scenario $s \in S$, we obtain:

$$\begin{aligned} & \sup_{\boldsymbol{\rho}, \mathbf{m}, \{\boldsymbol{\xi}_j\}_{j \in (N \times T)_{|m|}}, \underline{\boldsymbol{\rho}} \in [\underline{\boldsymbol{\rho}}, \bar{\boldsymbol{\rho}}], g_{js}(\boldsymbol{\xi}_j) \leq m_j, (\boldsymbol{\xi}_j) = \boldsymbol{\rho}, \forall j \in (N \times T)_{|m|}} \boldsymbol{\rho}^\top \mathbf{c}_{its}^3 + \mathbf{m}^\top \mathbf{c}_{its}^4 \\ = & \inf_{\boldsymbol{\eta}_{its}^3, \boldsymbol{\eta}_{its}^4, \lambda_{its}^2 \geq 0, \{\mathbf{w}_{js}^2\}_{j \in (N \times T)_{|m|}}, \mathbf{c}_{its}^4 + \lambda_{its}^2 = 0, \sum_{j \in (N \times T)_{|m|}} \mathbf{w}_{js}^2 = \mathbf{c}_{its}^3 - \boldsymbol{\eta}_{its}^3 + \boldsymbol{\eta}_{its}^4} \bar{\boldsymbol{\rho}}^\top \boldsymbol{\eta}_{its}^3 - \underline{\boldsymbol{\rho}}^\top \boldsymbol{\eta}_{its}^4 + \sum_{j \in (N \times T)_{|m|}} \lambda_{js}^2 g_{js}^* \left(\frac{\mathbf{w}_{js}^2}{\lambda_{js}^2} \right) \end{aligned}$$

where $\boldsymbol{\xi}_j = \boldsymbol{\rho}$, $\mathbf{m} = \mathbf{g}_s(\boldsymbol{\rho}) + 1$, and $\lambda_{js}^2 g_{js}^* \left(\frac{\mathbf{w}_{js}^2}{\lambda_{js}^2} \right)$ is the perspective function of the conjugate function of g_{js}^* on $(\mathbf{w}_{js}^2, \lambda_{js}^2)$ for $j \in (N \times T)_{|m|}$. As a consequence we can replace Constraints (4.3b) by Constraints (4.6g)-(4.6k) in our final reformulation.

C.1.2 Reformulation for Constraints (4.3c)

If Slater's condition holds on Constraints (4.3c), then Constraints (4.3c) is equivalent to

$$H_{its}^0 - b_{it}d_{it} \geq \sup_{\boldsymbol{\rho}, \mathbf{m}, \{\boldsymbol{\xi}_j\}_{j \in (N \times T)_{|m|}}, \underline{\boldsymbol{\rho}} \in [\underline{\boldsymbol{\rho}}, \bar{\boldsymbol{\rho}}], g_{js}(\boldsymbol{\xi}_j) \leq m_j, (\boldsymbol{\xi}_j) = \boldsymbol{\rho}, \forall j \in (N \times T)_{|m|}} \boldsymbol{\rho}^\top \mathbf{c}_{its}^5 + \mathbf{m}^\top \mathbf{c}_{its}^6$$

$\forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; i, k \in N; t, l \in T; j \in (N \times T)_{|m|}; s \in S$, where \mathbf{c}_{its}^5 is given in (C.2) and $\mathbf{c}_{itjs}^6 = -H''_{itjs}$.

$$\mathbf{c}_{itkls}^5 = \begin{cases} -b_{it}X_{il} - H'_{itils}, & \text{if } i = k, l \leq t \\ -H'_{itils}, & \text{otherwise} \end{cases} \quad (\text{C.2})$$

If we reformulate the supremum in terms of the dual infimum for a scenario $s \in S$, we obtain:

$$\begin{aligned} & \sup_{\boldsymbol{\rho}, \mathbf{m}, \{\boldsymbol{\xi}_j\}_{j \in (N \times T)_{|m|}}, \boldsymbol{\rho} \in [\underline{\boldsymbol{\rho}}, \bar{\boldsymbol{\rho}}], g_{js}(\boldsymbol{\xi}_j) \leq m_j, (\boldsymbol{\xi}_j) = \boldsymbol{\rho}, \forall j \in (N \times T)_{|m|}} \boldsymbol{\rho}^\top \mathbf{c}_{its}^5 + \mathbf{m}^\top \mathbf{c}_{its}^6 \\ = & \inf_{\boldsymbol{\eta}_{its}^5, \boldsymbol{\eta}_{its}^6, \boldsymbol{\lambda}_{its}^3 \geq 0, \{\mathbf{w}_{js}^3\}_{j \in (N \times T)_{|m|}}, \mathbf{c}_{its}^6 + \boldsymbol{\lambda}_{its}^3 = 0, \sum_{j \in (N \times T)_{|m|}} \mathbf{w}_{js}^3 = \mathbf{c}_{its}^5 - \boldsymbol{\eta}_{its}^5 + \boldsymbol{\eta}_{its}^6} \bar{\boldsymbol{\rho}}^\top \boldsymbol{\eta}_{its}^5 - \underline{\boldsymbol{\rho}}^\top \boldsymbol{\eta}_{its}^6 + \sum_{j \in (N \times T)_{|m|}} \lambda_{js}^3 g_{js}^* \left(\frac{\mathbf{w}_{js}^3}{\lambda_{js}^3} \right) \end{aligned}$$

where $\boldsymbol{\xi}_j = \boldsymbol{\rho}$, $\mathbf{m} = \mathbf{g}_s(\boldsymbol{\rho}) + 1$, and $\lambda_{js}^3 g_{js}^* \left(\frac{\mathbf{w}_{js}^3}{\lambda_{js}^3} \right)$ is the perspective function of the conjugate function of g_{js}^* on $(\mathbf{w}_{js}^3, \lambda_{js}^3)$ for $j \in (N \times T)_{|m|}$. As a consequence we can reformulate Constraints (4.3c) as Constraints (4.6l)-(4.6p) in our final reformulation.

C.1.3 Reformulation for Constraints (4.3d)

If Slater's condition holds on Constraints (4.3d), then Constraints (4.3d) is equivalent to

$$H_{its}^0 \geq \sup_{\boldsymbol{\rho}, \mathbf{m}, \{\boldsymbol{\xi}_j\}_{j \in (N \times T)_{|m|}}, \boldsymbol{\rho} \in [\underline{\boldsymbol{\rho}}, \bar{\boldsymbol{\rho}}], g_{js}(\boldsymbol{\xi}_j) \leq m_j, (\boldsymbol{\xi}_j) = \boldsymbol{\rho}, \forall j \in (N \times T)_{|m|}} \boldsymbol{\rho}^\top \mathbf{c}_{its}^7 + \mathbf{m}^\top \mathbf{c}_{its}^8$$

$\forall (\boldsymbol{\rho}, \mathbf{m}) \in \mathcal{W}_s; i, k \in N; t, l \in T; j \in (N \times T)_{|m|}; s \in S$, where $\mathbf{c}_{itkls}^7 = H'_{itkls}$ and $\mathbf{c}_{itjs}^8 = -H''_{itjs}$.

If we reformulate the supremum in terms of the dual infimum for a scenario $s \in S$, we obtain:

$$\begin{aligned} & \sup_{\boldsymbol{\rho}, \mathbf{m}, \{\boldsymbol{\xi}_j\}_{j \in (N \times T)_{|m|}}, \boldsymbol{\rho} \in [\underline{\boldsymbol{\rho}}, \bar{\boldsymbol{\rho}}], g_{js}(\boldsymbol{\xi}_j) \leq m_j, (\boldsymbol{\xi}_j) = \boldsymbol{\rho}, \forall j \in (N \times T)_{|m|}} \boldsymbol{\rho}^\top \mathbf{c}_{its}^7 + \mathbf{m}^\top \mathbf{c}_{its}^8 \\ = & \inf_{\boldsymbol{\eta}_{its}^7, \boldsymbol{\eta}_{its}^8, \boldsymbol{\lambda}_{its}^4 \geq 0, \{\mathbf{w}_{js}^4\}_{j \in (N \times T)_{|m|}}, \mathbf{c}_{its}^8 + \boldsymbol{\lambda}_{its}^4 = 0, \sum_{j \in (N \times T)_{|m|}} \mathbf{w}_{js}^4 = \mathbf{c}_{its}^7 - \boldsymbol{\eta}_{its}^7 + \boldsymbol{\eta}_{its}^8} \bar{\boldsymbol{\rho}}^\top \boldsymbol{\eta}_{its}^7 - \underline{\boldsymbol{\rho}}^\top \boldsymbol{\eta}_{its}^8 + \sum_{j \in (N \times T)_{|m|}} \lambda_{js}^4 g_{js}^* \left(\frac{\mathbf{w}_{js}^4}{\lambda_{js}^4} \right) \end{aligned}$$

where $\boldsymbol{\xi}_j = \boldsymbol{\rho}$, $\mathbf{m} = \mathbf{g}_s(\boldsymbol{\rho}) + 1$, and $\lambda_{js}^4 g_{js}^* \left(\frac{\mathbf{w}_{js}^4}{\lambda_{js}^4} \right)$ is the perspective function of the conjugate function of g_{js}^* on $(\mathbf{w}_{js}^4, \lambda_{js}^4)$ for $j \in (N \times T)_{|m|}$. As a consequence we can reformulate Constraints (4.3d) as Constraints (4.6q)-(4.6s) in our final reformulation.

C.2 The robust LSPs with yield uncertainty model

Based on the robust single item LSP with uncertain production yield proposed by Metzker et al. [2023a], the multi-item lot-sizing problem under yield uncertainty is given as follows:

$$\begin{aligned}
 & \min \sum_{i \in N} \sum_{t \in T} s_{it} Y_t + v_{it} X_{it} + H_{it} \\
 & \text{s.t. :} \\
 & H_{it} \geq h_{it} \max_{\rho \in \mathcal{U}_{it}} \left[\sum_{\tau=1}^t (\tilde{\rho}_{i\tau} X_{i\tau} - d_{i\tau}) \right] \quad \forall t \in T; i \in N \\
 & H_{it} \geq -b_{it} \max_{\tilde{i}\rho \in \mathcal{U}_{it}} \left[\sum_{\tau=1}^t (\tilde{i}\rho_{\tau} X_{i\tau} - d_{i\tau}) \right] \quad \forall t \in T; i \in N \\
 & X_{it} \leq M_{it} Y_{it} \quad \forall t \in T; i \in N \\
 & X_{it}, H_{it} \geq 0 \quad \forall t \in T; i \in N \\
 & Y_{it} \in \{0, 1\} \quad \forall t \in T; i \in N
 \end{aligned}$$

where $\tilde{\rho}$ mapped by a affine rule that bounds its realization to a range centered on its nominal value $\bar{\rho}$ and spread by its maximum deviation $\hat{\rho}$. The uncertain production yield is represented into the budgeted uncertainty set \mathcal{U}_{it} where a budget Γ controls the size of the uncertainty set according to the decision maker's sensitivity to risk. Thus, for each item i in each period t , the uncertain production yield $\tilde{\rho}_{it}$ belongs to \mathcal{U}_{it} that is given by $\mathcal{U}_{it} = \{-\mathbf{1} \leq \mathbf{Z}_i^t \leq \mathbf{1} : \sum_{\tau=1}^t |Z^{t,i\tau}| \leq \Gamma_t\}$.

C.3 The stochastic programming LSP with yield uncertainty model

Similar to the model proposed in the appendix from Metzker et al. [2023a], we propose a scenario-based stochastic program to represent the multi-item lot-sizing problem under yield uncertainty. To represent the static strategy, we rely on the two-stage formulation, where only the inventory and backorder levels react to the different scenarios.

We consider a set Ω of possible yield scenarios, where each scenario ω has a probability p_ω of realization. ρ_{it}^ω is the realization of the uncertain yield for item i in period t and scenario ω . The two-stage stochastic program for the LSP with uncertain yield is given as follows:

$$\begin{aligned}
& \min \sum_{\omega \in \Omega} p_\omega \sum_{t \in T} \sum_{i \in N} s_{it} Y_{it} + v_{it} X_{it} + h_{it} I_{it}^\omega + b_{it} B_{it}^\omega \\
& \text{s.t. :} \\
& I_{it}^\omega - B_{it}^\omega = I_{it-1}^\omega - B_{it-1}^\omega + \rho_{it}^\omega X_{it} - d_{it} & \forall t \in T; i \in N; s \in \Omega \\
& X_{it} \leq M_{it} Y_{it} & \forall t \in T; i \in N \\
& X_{it}, I_{it}^\omega, B_{it}^\omega \geq 0 & \forall t \in T; i \in N; s \in \Omega \\
& Y_{it} \in \{0, 1\} & \forall t \in T; i \in N
\end{aligned}$$

To better represent the realization of the uncertain yield, we need to generate as many scenarios as possible. However, to avoid the drawbacks from scalability issues, such as prohibitive computational time, we generate the scenarios with a Monte Carlo approach. Thus, the yield rate is randomly drawn from a uniform distribution with support $[\bar{\rho}_{it} - \hat{\rho}_{it}; \bar{\rho}_{it} + \hat{\rho}_{it}]$ over 200 scenarios.

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Titre : L'optimisation robuste pour des problèmes de lotissement dans un contexte de rendement incertain

Mot clés : problème du lotissement, optimisation robuste, optimisation robuste adaptative, programmation stochastique sous la perspective robuste, optimisation combinatoire

Résumé :

Les fabricants doivent gérer leurs capacités de production et leur performance pour satisfaire la demande des clients avec des produits de qualité. De plus, ils sont constamment mis au défi d'optimiser l'utilisation de leurs ressources et leur performance de production dans un contexte de marché dynamique et volatil, et ce, de manière rentable. Pour atteindre cet objectif commercial, un défi tactique à relever parmi d'autres, les fabricants doivent minutieusement prendre des décisions de dimensionnement de lots de production afin de déterminer la configuration et les quantités de production répondant aux demandes avec des produits de qualité, tout en minimisant les coûts globaux de production et de gestion de stocks. Nous proposons différents mo-

dèles et stratégies méthodologiques pour calculer des plans de production optimaux robustes à conservation moindre. Nous fournissons également une analyse de risque pour aider les décideurs à adapter leurs plans décisionnels à leurs objectifs stratégiques et opérationnels. Les résultats expérimentaux montrent que les plans de production robustes ont une meilleure stratégie de réduction des coûts par rapport aux autres approches de résolution. Les expériences numériques montrent la robustesse et l'efficacité du modèle robuste à travers une analyse des cas moyens et les plus défavorables. Les expériences démontrent également les performances et la valeur des solutions adaptatives robustes et la faible sensibilité aux erreurs de prédiction des modèles distributionnellement robustes.

Title: Robust optimization for lot-sizing problems under production yield uncertainty

Keywords: lot-sizing problems, robust optimization, adaptive robust optimization, distributionally robust optimization, combinatorial optimization

Abstract:

Manufacturers must efficiently manage their production capacities and their performances to satisfy customer demands with quality goods. They also have been constantly challenged to optimize resource usage and production performance in a dynamic and volatile market context in a cost effective manner. To achieve this business objective, we consider lot-sizing problems under yield uncertainty via a robust, adaptive robust and distributionally robust optimization methodologies to determine the production setups and quantities that meet demands with quality goods, while minimizing the overall production and inventory management costs. We propose

different models and methodological strategies to compute optimal robust and less conservative production plans. We also provide a risk analysis to help decision-makers to tailor their decision plans to their strategic and operational goals. The experimental results show that robust-wise production plans have a better cost cutting strategy compared with other resolution approaches. The computational experiments show the robustness and effectiveness of the robust model through an average and worst case analysis. The experiments also demonstrate the performances and the value of the adaptive robust solutions and the low sensitiveness to errors in predictions of the distributionally robust models.