Mathematical study of a plasma model and turbulence in fluid mechanics

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Aux étoiles qui éclairent ma vie, mes parents Guolin et Youlan

Résumé

Cette thèse est consacrée à l'étude mathématique de deux modèles différents: le système de la magnéto-hydrodynamique prenant en compte l'effet Hall (Hall-MHD), et d'une équation scalaire de type Burgers avec terme non local et diffusion non linéaire.

Le système Hall-MHD décrit l'évolution temporelle d'un plasma tridimensionnel. Nous nous intéressons aux questions d'existence (locale dans le temps pour les grandes données ou globale dans le temps pour les petites données) et d'unicité pour le système Hall-MHD couplé à une vitesse et un champ magnétique initial dans des espaces de régularité critique, dans l'esprit du théorème de Fujita-Kato et, plus tard, de Chemin pour les équations classiques de Navier-Stokes. Nous étudions les estimations de décroissance temporelle, le critère d'explosion et les propriétés de stabilité de ces solutions. Un énoncé d'unicité fort-faible est également prouvé. Lorsque les données du système Hall-MHD sont indépendantes de la variable verticale (les flux dits de dimension $2\,1/2$), nous établissons l'existence globale d'une solution forte, en supposant seulement que le champ magnétique initial est petit. Notre stratégie s'appuie fortement sur l'utilisation de champs vectoriels auxiliaires et d'une formulation impliquant ce que l'on appelle la vitesse de l'électron. La preuve est basée sur la théorie de Littlewood-Paley et la méthode de l'énergie.

Dans la deuxième partie de cette thèse, nous nous concentrons sur l'étude d'une équation de Burgers non locale généralisée dans un domaine périodique multidimensionnel. Nous construisons des solutions classiques globales à partir de données positives et régulières, et des solutions faibles globales à partir de données positives et bornées. Nous montrons que toute solution faible est instantanément régularisée en une fonction infiniment différentiable. Nous décrivons également le comportement asymptotique en temps long de toutes les solutions. Nos méthodes s'appuient sur plusieurs avancées récentes dans la théorie de la régularité des équations intégro-différentielles paraboliques.

Mots-clé:

Magnéto-hydrodynamique, effet Hall, problème bien posé, régularité critique, asymptotique en temps long, critères d'explosion, unicité fort-faible, équation de Burgers non locale, estimations de Schauder.

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Abstract

This thesis is dedicated to the mathematical study of two different models: the Hall-magnetohydrodynamics (Hall-MHD) system and a non-local Burgers equation with non-linear diffusion.

Hall-MHD system describes the time evolution of three-dimensional plasma. We are concerned with existence (local-in-time for large data or global-in-time for small data) and uniqueness issues for Hall-MHD supplemented with initial velocity and magnetic field in critical regularity spaces, in the spirit of Fujita-Kato and, later, Chemin's theorem for the classical Navier-Stokes equations. We investigate the time-decay estimates, blow-up criterion and stability properties of such solutions. A weak-strong uniqueness statement is also proven. When the data for the Hall-MHD system are independent of the vertical variable (the so-called $2\frac{1}{2}D$ flows), we establish the global existence of a strong solution, assuming only that the initial magnetic field is small. Our strategy strongly relies on the use of auxiliary vector-fields and of an *extended formulation* involving the so-called *velocity of electron*. The proof is based on the Littlewood-Paley theory and energy method.

In the second part of this thesis, we focus on the study of a generalized non-local Burgers equation in a multi-dimensional periodic domain. We construct global classical solutions starting from smooth positive data, and global weak solutions starting from bounded positive data. We show that any weak solution is instantaneously regularized into infinitely differentiable. We also describe the long-time asymptotics of all solutions. Our methods follow several recent advances in the regularity theory of parabolic integro-differential equations.

Keywords:

Magneto-hydrodynamics, Hall effect, non-local Burgers equation, well-posedness, critical regularity, long-time asymptotics, blow-up criterion, weak-strong uniqueness, Schauder estimates.

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Chapter 1

Résumé étendu en français

Cette thèse est consacrée à l'étude mathématique d'un système de la magnéto-hydrodynamique prenant en compte l'effet Hall, et d'une équation scalaire de type Burgers avec terme non local et diffusion non linéaire.

Les résultats principaux obtenus sur ces deux types d'équations sont présentés dans les paragraphes qui suivent.

1.1 Etude mathématique d'un modèle de plasma

1.1.1 Présentation du système

La première partie de la thèse concerne le système de la magnéto-hydrodynamique incompressible, prenant en compte l'effet Hall suivant[1]:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = (\nabla \wedge \boldsymbol{B}) \wedge \boldsymbol{B} + \mu \Delta \boldsymbol{u}, \tag{1.1.1}$$

$$\operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{B} = 0, \tag{1.1.2}$$

$$\partial_t \boldsymbol{B} = \nabla \wedge \big((\boldsymbol{u} - h \nabla \wedge \boldsymbol{B}) \wedge \boldsymbol{B} \big) + \nu \Delta \boldsymbol{B}. \tag{1.1.3}$$

Les inconnues sont :

- la vitesse $\boldsymbol{u} = \boldsymbol{u}(t,x)$ (avec $t \geq 0$ et $x \in \mathbb{R}^3$) qui est un champ de vecteurs sur $\mathbb{R}^3$ dépendant du temps;
- le champ magnétique $\boldsymbol{B} = \boldsymbol{B}(t,x)$ qui est également un champ de vecteurs sur $\mathbb{R}^3$ dépendant du temps;
- la pression $P = P(t,x)$, qui est à valeurs réelles.

Les nombres réels strictement positifs μ et ν représentent la viscosité cinématique et la diffusivité magnétique, et $h > 0$ représente la magnitude de l'effet Hall.

On souhaite résoudre le problème de Cauchy pour le système ci-dessus avec conditions initiales:

$$(\boldsymbol{u}(0,x), \boldsymbol{B}(0,x)) = (\boldsymbol{u}_0(x), \boldsymbol{B}_0(x)), \quad x \in \mathbb{R}^3, \tag{1.1.4}$$

où les champs de vecteurs $\boldsymbol{u}_0$ et $\boldsymbol{B}_0$ sont à divergence nulle. Au moins formellement, la condition de divergence nulle pour B est préservée au cours de l'évolution, alors que la présence de ∇P assure $\operatorname{div} \boldsymbol{u} = 0$.

[1] Dans $\mathbb{R}^3$, nous notons $\boldsymbol{w} \wedge \boldsymbol{z} := (w_2 z_3 - w_3 z_2,\, w_3 z_1 - w_1 z_3,\, w_1 z_2 - w_2 z_1)^{\mathrm{tr}}$ et $\operatorname{div} \boldsymbol{w} := \sum_j^3 \partial_j w_j$.

Le système (1.1.1)–(1.1.3) que l'on désignera dorénavant par *système de la Hall-MHD* est utilisé dans la modélisation de fluides conducteurs tels que plasmas ou electrolytes ($\boldsymbol{u}$ représente la vitesse des ions) lorsque l'on prend en compte le fait que dans un fluide conducteur en mouvement, le champ magnétique induit des courants qui, à leur tour, polarisent le fluide et changent le champ magnétique. Ce phénomène qui est négligé dans la MHD classique (i.e. pour $h = 0$) est modélisé par le champ électrique de Hall $\boldsymbol{E}_H := h\boldsymbol{J} \wedge \boldsymbol{B}$, avec $\boldsymbol{J} := \nabla \wedge \boldsymbol{B}$ désignant le courant associé à $\boldsymbol{J}$. L'effet Hall doit être pris en compte dans la description des vents solaires, des étoiles à neutrons ou de la géo-dynamo (voir par exemple [1, 2, 3, 4, 5, 6]).

Si le système de la Hall-MHD est bien connu des physiciens, il n'a attiré l'attention des mathématiciens qu'assez récemment à la suite, notamment, de l'article d'Acheritogaray, Degond, Frouvelle et Liu [7] (dérivation formelle du système à partir de la théorie cinétique, voir le sous-secteur 2.1.2). De nombreux autres travaux mathématiques ont suivi, qui seront décrits dans l'introduction en anglais.

Notre but principal ici est d'établir que le système de la Hall-MHD est localement (ou globalement) bien posé dans un cadre fonctionnel à régularité critique similaire à celui que l'on utilise pour le système de Navier-Stokes incompressible ou le système de la MHD classique.

On notera cependant que la présence du terme de Hall tue l'invariance d'échelle du système. De ce fait, on considère la fonction $\boldsymbol{J} := \nabla \wedge \boldsymbol{B}$ comme une inconnue supplémentaire, afin d'obtenir le système de la Hall MHD augmenté suivant :

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \mu \Delta \boldsymbol{u} + \nabla P = (\nabla \wedge \boldsymbol{B}) \wedge \boldsymbol{B}, \\ \operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{B} = \operatorname{div} \boldsymbol{J} = 0, \\ \partial_t \boldsymbol{B} - \nabla \wedge ((\boldsymbol{u} - h\boldsymbol{J}) \wedge \boldsymbol{B}) - \nu \Delta \boldsymbol{B} = 0, \\ \partial_t \boldsymbol{J} - \nabla \wedge \big(\nabla \wedge ((\boldsymbol{u} - h\boldsymbol{J}) \wedge \operatorname{curl}^{-1} \boldsymbol{J})\big) - \nu \Delta \boldsymbol{J} = 0, \\ (\boldsymbol{u}(0,x), \boldsymbol{B}(0,x), \boldsymbol{J}(0,x)) = (\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{J}_0), \end{cases} \tag{1.1.5}$$

où $\operatorname{curl}^{-1} := (-\Delta)^{-1} \nabla \wedge$ désigne l'opérateur de Biot-Savart.

On notera que le système augmenté est bien invariant pour tout $\lambda > 0$ par la transformation :

$$(\boldsymbol{u}(x), \boldsymbol{B}(x), \boldsymbol{J}(x)) \rightsquigarrow (\lambda \boldsymbol{u}, \lambda \boldsymbol{B}, \lambda \boldsymbol{J})(\lambda^2 t, \lambda x).$$

Par analogie avec l'approche classique pour le système de Navier-Stokes incompressible, on cherche à résoudre (1.1.2) dans des espaces fonctionnels à normes invariantes par la transformation ci-dessus, ce qui correspond à des espaces fonctionnels critiques pour les données initiales au sens de l'invariance de la norme pour les transformations suivantes :

$$(\boldsymbol{u}_0(x), \boldsymbol{B}_0(x), \boldsymbol{J}_0(x)) \rightsquigarrow (\lambda \boldsymbol{u}_0(\lambda x), \lambda \boldsymbol{B}_0(\lambda x), \lambda \boldsymbol{J}_0(\lambda x)), \qquad \lambda > 0.$$

Dans la famille des espaces de Besov homogènes $\dot{B}^s_{p,r}$ (qui seront définis plus loin dans le texte), cette invariance correspond à la relation $s = 3/p - 1$.

1.1.2 Principaux résultats

Dans cette thèse, nous apporterons des éléments de réponses aux problèmes suivants :

- **Problème 1 :** Etablir l'existence globale à données petites dans les espaces critiques $\dot{B}^{\frac{3}{p}-1}_{p,1}$ avec $1 \leq p < \infty$;
- **Problème 2 :** Etablir l'existence locale à données grandes et des critères d'explosion dans les espaces critiques $\dot{B}^{\frac{1}{2}}_{2,1}$;
- **Problème 3 :** Démontrer l'équivalent du théorème de Fujita-Kato (i.e. caractère bien posé dans l'espace de Sobolev homogène $\dot{H}^{1/2}$);
- **Problème 4 :** Construire des solutions en dimension $2\,1/2$ c'est-à-dire ne dépendant que de deux variables.

Les résultats obtenus pour ces problèmes, et présentés ci-dessous, sont les fruits d'une collaboration avec l'un de mes directeurs de thèse (le professeur Raphäel Danchin) et ont été publiés dans [8, 9] ou, en ce qui concerne les estimations de decay, en collaboration avec Lvqiao Liu [10].

Problème 1

Dans ce paragraphe, on s'intéresse à la résolution globale du système de la Hall-MHD avec coefficients strictement positifs et données initiales telles que $(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{J}_0) \in \dot{B}^{\frac{3}{p}-1}_{p,1}$.

Forts des résultats classiques sur les équations de Navier-Stokes incompressibles (voir en particulier l'article [11] de J.-Y. Chemin), on s'attend à obtenir des solutions dans l'espace :

$$E_p(T) := \Big\{ \boldsymbol{w} \in \mathcal{C}([0,T], \dot{B}^{\frac{3}{p}-1}_{p,1}),\ \nabla^2 \boldsymbol{w} \in L^1(0,T; \dot{B}^{\frac{3}{p}-1}_{p,1}) \ \text{ et } \ \operatorname{div} \boldsymbol{w} = 0 \Big\}$$

ou dans sa version globale E_p (avec $\boldsymbol{w} \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{3}{p}-1}_{p,1})$) si les données sont petites.

En utilisant la formulation étendue et le théorème du point fixe de Picard dans l'espace E_p, on obtient le résultat suivant.

Théorème 1.1.1. *Soit $1 \leq p < \infty$. Soit $(\boldsymbol{u}_0, \boldsymbol{B}_0) \in \dot{B}^{\frac{3}{p}-1}_{p,1}$ tel que $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{B}_0 = 0$ et $\boldsymbol{J}_0 := \nabla \wedge \boldsymbol{B}_0 \in \dot{B}^{\frac{3}{p}-1}_{p,1}$. Il existe une constante $c_0 > 0$ ne dépendant que de p et de μ/ν et telle que, si*

$$\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|\boldsymbol{B}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + h\|\boldsymbol{J}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} < c_0 \mu,$$

alors le système (1.1.1)-(1.1.3) avec données initiales $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ admet une unique solution globale $(\boldsymbol{u}, \boldsymbol{B}) \in E_p$, avec $\boldsymbol{J} := \nabla \wedge \boldsymbol{B} \in E_p$. De plus, pour tout $m \geq 0$,

on a

$$\|D^m \boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|D^m \boldsymbol{B}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \leq C_0 c_0 t^{-\frac{m}{2}}, \quad \textit{pour tout } \ t \geq 0,$$

avec C_0 dépendant seulement de μ, ν, h, p et m.

Problème 2

Dans le cas où $\mu = \nu$, le système vérifié par $\boldsymbol{u}$, $\boldsymbol{B}$ et $\boldsymbol{v} := \boldsymbol{u} - h\boldsymbol{J}$ (qui, physiquement, représente la vitesse d'un élection) a une structure bien particulière, à savoir :

$$\begin{cases} \partial_t \boldsymbol{u} - \mu\Delta \boldsymbol{u} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla\Pi, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \partial_t \boldsymbol{B} - \mu\Delta\boldsymbol{B} = \nabla\wedge(\boldsymbol{u}_e\wedge\boldsymbol{B}), \\ \partial_t \boldsymbol{v} - \mu\Delta\boldsymbol{v} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - h\nabla\wedge((\nabla\wedge\boldsymbol{v})\wedge\boldsymbol{B}) \\ \qquad\qquad\qquad\qquad + \nabla\wedge(\boldsymbol{v}\wedge\boldsymbol{u}) + 2h\nabla\wedge(\boldsymbol{v}\cdot\nabla\boldsymbol{B}) - \nabla\Pi, \\ (\boldsymbol{u}(0,x),\boldsymbol{B}(0,x),\boldsymbol{v}(0,x)) = (\boldsymbol{u}_0,\boldsymbol{B}_0,\boldsymbol{v}_0) \end{cases}$$

On constate que l'unique terme quasi-linéaire vérifie

$$\big(\nabla\wedge\big((\nabla\wedge\boldsymbol{v})\wedge\boldsymbol{B}\big),\boldsymbol{v}\big)_{L^2} = \big((\nabla\wedge\boldsymbol{v})\wedge\boldsymbol{B},\nabla\wedge\boldsymbol{v}\big)_{L^2} = 0.$$

Cette propriété d'annulation demeure après localisation en fréquence, à un terme d'ordre inférieur près. En combinant localisation à l'aide de la décomposition de Littlewood-Paley et méthode d'énergie, on obtient le résultat suivant.

Théorème 1.1.2. *Supposons que $\mu = \nu$ et considérons $(\boldsymbol{u}_0,\boldsymbol{B}_0) \in \dot{B}^{\frac{1}{2}}_{2,1}$ avec $\operatorname{div}\boldsymbol{u}_0 = \operatorname{div}\boldsymbol{B}_0 = 0$ et $\boldsymbol{J}_0 := \nabla\wedge\boldsymbol{B}_0 \in \dot{B}^{\frac{1}{2}}_{2,1}$.*

Sans aucune condition de petitesse, il existe $T > 0$ tel que (1.1.1)-(1.1.3) avec donnée initiale $(\boldsymbol{u}_0,\boldsymbol{B}_0)$ admet une unique solution $(\boldsymbol{u},\boldsymbol{B}) \in E_2(T)$ avec $\boldsymbol{J} := \nabla\wedge\boldsymbol{B} \in E_2(T)$. De plus, si la solution explose au temps T^*, alors

$$\int_0^{T^*} \|(\boldsymbol{u},\boldsymbol{B},\nabla\boldsymbol{B})(t)\|^2_{L^\infty}\,dt = \infty \quad \textit{et} \quad \int_0^{T^*} \|(\boldsymbol{u},\boldsymbol{B},\nabla\boldsymbol{B})(t)\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\,dt = \infty$$

et, pour tout $\varrho \in (2,\infty)$,

$$\int_0^{T^*} \|(\boldsymbol{u},\boldsymbol{B},\nabla\boldsymbol{B})(t)\|^{\varrho}_{\dot{B}^{\frac{2}{\varrho}-1}_{\infty,\infty}}\,dt = \infty.$$

Pour les solutions construites dans le théorème 1.1.2, on démontre le résultat de stabilité suivant :

Théorème 1.1.3. *Soit $(\boldsymbol{u}_{0,i},\boldsymbol{B}_{0,i}) \in \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)$ avec $\operatorname{div}\boldsymbol{u}_{0,i} = \operatorname{div}\boldsymbol{B}_{0,i} = 0$, tel que $\boldsymbol{v}_{0,i} := \boldsymbol{u}_{0,i} - h\nabla\wedge\boldsymbol{B}_{0,i} \in \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)$, pour $i = 1,2$. On suppose de plus que $\mu = \nu$ et que le problème de Cauchy (1.1.1)-(1.1.4) avec données initiales $(\boldsymbol{u}_{0,1},\boldsymbol{B}_{0,1})$ admet*

une solution $(\boldsymbol{u}_1, \boldsymbol{B}_1)$ *sur* $[0,T]$, *telle que*

$$(\boldsymbol{u}_1, \boldsymbol{B}_1, \nabla \wedge \boldsymbol{B}_1) \in L^1(0,T; \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)).$$

Il existe une constante $c_1 > 0$ *qui ne dépend que de* μ, h *et de normes critiques de* $(\boldsymbol{u}_1, \boldsymbol{B}_1)$ *telle que, si*

$$\|(\boldsymbol{u}_{0,1} - \boldsymbol{u}_{0,2}, \boldsymbol{B}_{0,1} - \boldsymbol{B}_{0,2}, \boldsymbol{v}_{0,1} - \boldsymbol{v}_{0,2})\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \leq c_1,$$

alors $(\boldsymbol{u}_{0,2}, \boldsymbol{B}_{0,2})$ *génère une unique solution* $(\boldsymbol{u}_2, \boldsymbol{B}_2) \in E_2(T)$.

Problème 3

Toujours sous l'hypothèse $\mu = \nu$, on démontre le résultat inspiré du travail de Fujita-Kato [12] pour les équations de Navier-Stokes incompressibles, suivant :

Théorème 1.1.4. *Sous l'hypothèse* $\mu = \nu$, *soit* $(\boldsymbol{u}_0, \boldsymbol{B}_0) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ *avec* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{B}_0 = 0$ *et* $\boldsymbol{J}_0 := \nabla \wedge \boldsymbol{B}_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. *Il existe* $c_2 > 0$ *tel que, si*

$$\|\boldsymbol{u}_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \|\boldsymbol{B}_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \|\boldsymbol{u}_0 - h\boldsymbol{J}_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < c_2\mu,$$

alors il existe une unique solution globale

$$(\boldsymbol{u}, \boldsymbol{B}) \in \mathcal{C}_b(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$$

au système de la Hall-MHD, tel que $\boldsymbol{J} := \nabla \wedge \boldsymbol{B}$ *appartienne à* $L^\infty(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$. *De plus, pour tout* $t \geq t_0 \geq 0$, *on a*

$$\|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{u} - h\boldsymbol{J})(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{\mu}{2}\int_{t_0}^t \|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{u} - h\boldsymbol{J})\|^2_{\dot{H}^{\frac{3}{2}}}\, d\tau \leq \|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{u} - h\boldsymbol{J})(t_0)\|^2_{\dot{H}^{\frac{1}{2}}}.$$

En particulier, la fonction

$$t \mapsto \|\boldsymbol{u}(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{B}(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{u}(t) - h\boldsymbol{J}(t)\|^2_{\dot{H}^{\frac{1}{2}}}$$

est décroissante.

Finalement, si de plus $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ *est dans* $L^2(\mathbb{R}^3)$, *alors* $(\boldsymbol{u}, \boldsymbol{B})$ *est continue à valeurs dans* $L^2(\mathbb{R}^3)$, *vérifie l'identité énergie pour tout* $t \geq t_0 \geq 0$:

$$\|\boldsymbol{u}(t)\|^2_{L^2} + \|\boldsymbol{B}(t)\|^2_{L^2} + 2\mu\int_{t_0}^t \left(\|\nabla \boldsymbol{u}\|^2_{L^2} + \|\nabla \boldsymbol{B}\|^2_{L^2}\right) d\tau = \|\boldsymbol{u}(t_0)\|^2_{L^2} + \|\boldsymbol{B}(t_0)\|^2_{L^2},$$

et l'on a

$$\lim_{t \to +\infty} \left(\|\boldsymbol{u}(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{B}(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{J}(t)\|^2_{\dot{H}^{\frac{1}{2}}}\right) = 0.$$

On démontre finalement le résultat d'unicité fort-faible suivant :

Théorème 1.1.5. *Soit* $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ *dans* $L^2(\mathbb{R}^3)$ *avec* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{B}_0 = 0$, *et* $(\boldsymbol{u}, \boldsymbol{B})$ *une solution de Leray-Hopf du système de la Hall-MHD avec données initiales*

$(\boldsymbol{u}_0, \boldsymbol{B}_0)$. Si, de plus, $\boldsymbol{u}$ et $\nabla \wedge \boldsymbol{B}$ sont dans $L^4(0,T;\dot{H}^1(\mathbb{R}^3))$ pour un $T > 0$, alors toutes les solutions de Leray-Hopf qui correspondent à $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ coïncident avec $(\boldsymbol{u}, \boldsymbol{B})$ sur l'intervalle $[0,T]$.

Problème 4

On s'intéresse enfin à la construction de solutions ne dépendant que de deux variables pour le système de la Hall-MHD (comme suggéré par Chae et Lee dans [13]). Par analogie avec les solutions de ce type pour les équations de Navier-Stokes (décrites par exemple dans le livre [14] de Bertozzi et Majda), on parle de solution de dimension 2 1/2. Le système à résoudre devient alors :

$$\partial_t \boldsymbol{u} + \widetilde{\boldsymbol{u}} \cdot \widetilde{\nabla} \boldsymbol{u} + \nabla \Pi = \widetilde{\boldsymbol{B}} \cdot \widetilde{\nabla} \boldsymbol{B} + \mu \widetilde{\Delta} \boldsymbol{u}, \tag{1.1.6}$$

$$\widetilde{\operatorname{div}}\, \widetilde{\boldsymbol{u}} = 0, \tag{1.1.7}$$

$$\partial_t \boldsymbol{B} + \widetilde{\boldsymbol{u}} \cdot \widetilde{\nabla} \boldsymbol{B} + h\widetilde{\boldsymbol{B}} \cdot \widetilde{\nabla} \boldsymbol{j} - h\widetilde{\boldsymbol{j}} \cdot \widetilde{\nabla} \boldsymbol{B} = \nu \widetilde{\Delta} \boldsymbol{B} + \widetilde{\boldsymbol{B}} \cdot \widetilde{\nabla} \boldsymbol{u}, \tag{1.1.8}$$

$$\widetilde{\operatorname{div}}\, \widetilde{\boldsymbol{B}} = 0, \tag{1.1.9}$$

$$\big(\boldsymbol{u}(0,x), \boldsymbol{B}(0,x)\big) = (\boldsymbol{u}_0, \boldsymbol{B}_0), \tag{1.1.10}$$

où les inconnues $\boldsymbol{u}$ et $\boldsymbol{B}$ sont des fonctions de $\mathbb{R}^+ \times \mathbb{R}^2$ dans $\mathbb{R}^3$, $\widetilde{\boldsymbol{u}} := (u_1, u_2)$, $\widetilde{\boldsymbol{B}} := (B_1, B_2)$, $\widetilde{\nabla} := (\partial_1, \partial_2)$, $\nabla := (\partial_1, \partial_2, 0)$, $\widetilde{\operatorname{div}} := \widetilde{\nabla}\cdot$, $\widetilde{\Delta} := \partial_1^2 + \partial_2^2$ et $\boldsymbol{j} := \widetilde{\nabla} \wedge \boldsymbol{B} = (\partial_2 B_3, -\partial_1 B_3, \partial_1 B_2 - \partial_2 B_1)^{\mathrm{tr}}$.

Contrairement au cas de Navier-Stokes ou même de la MHD classique, les équations vérifiées par les inconnues 'horizontales' sont couplées avec l'équation sur la troisième composante via le terme $\widetilde{\boldsymbol{B}} \cdot \widetilde{\nabla} \boldsymbol{j} - \widetilde{\boldsymbol{j}} \cdot \widetilde{\nabla} \boldsymbol{B}$, ce qui laisse peut d'espoir de démontrer un résultat global à données grandes.

On parviendra néanmoins à démontrer un résultat global avec une condition de petitesse ne portant que sur le champ magnétique, en considérant l'équation de la vorticité $\boldsymbol{\omega} := \widetilde{\nabla} \wedge \boldsymbol{u}$ suivante :

$$\partial_t \boldsymbol{\omega} + \widetilde{\nabla} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{u}) = \widetilde{\nabla} \wedge (\boldsymbol{j} \wedge \boldsymbol{B}) + \mu \widetilde{\Delta} \boldsymbol{\omega}.$$

Lorsque $\mu = \nu$, on constate que le champ de vecteur $\boldsymbol{\Omega} := h\boldsymbol{\omega} + \boldsymbol{B}$ vérifie

$$\partial_t \boldsymbol{\Omega} - \widetilde{\boldsymbol{\Omega}} \cdot \widetilde{\nabla} \boldsymbol{u} + \widetilde{\boldsymbol{u}} \cdot \widetilde{\nabla} \boldsymbol{\Omega} = \mu \widetilde{\Delta} \boldsymbol{\Omega}.$$

À l'aide de cette égalité et d'une méthode d'énergie, on obtient un contrôle global de $\boldsymbol{\Omega}$ dans l'espace $L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^2))$ et, finalement, de $\boldsymbol{j}$ dans l'espace $L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^2))$, pourvu que le champ magnétique initial soit suffisamment petit dans $H^1(\mathbb{R}^2)$. Cela permet d'obtenir finalement l'énoncé suivant :

Théorème 1.1.6. *Supposons que $\mu = \nu$. Soit $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ un couple de champs de vecteurs à divergence nulle, tel que $\boldsymbol{u}_0$ et $\boldsymbol{B}_0$ soient dans $H^1(\mathbb{R}^2; \mathbb{R}^3)$. Il existe une constante c_3 dépendant seulement de la norme L^2 de $\boldsymbol{u}_0$, $\widetilde{\nabla} \boldsymbol{u}_0$, et de μ, h telle que, si*

$$\|\boldsymbol{B}_0\|_{H^1(\mathbb{R}^2)} \leq c_3,$$

alors (1.1.6)-(1.1.10) *admet une unique solution globale* $(\boldsymbol{u}, \boldsymbol{B})$ *vérifiant*

$$(\boldsymbol{u}, \boldsymbol{B}) \in \mathcal{C}_b(\mathbb{R}_+; H^1(\mathbb{R}^2)) \quad et \quad (\widetilde{\nabla}\boldsymbol{u}, \widetilde{\nabla}\boldsymbol{B}) \in L^2(\mathbb{R}_+; H^1(\mathbb{R}^2)).$$

1.2 Étude mathématique d'une équation non-locale avec une diffusion non linéaire

1.2.1 Motivations

Dans son livre [15], Lemarié-Rieusset a introduit le modèle

$$\partial_t u + u|\nabla|u - |\nabla|(u^2) = \nu\Delta u, \quad x \in \mathbb{R}^d \text{ or } \mathbb{T}^d$$

comme une variante scalaire de l'équation de Navier-Stokes classique, où $|\nabla| = (-\Delta)^{1/2}$ représente la racine carrée du Laplacien et a pour symbole $|\xi|$.

Plus récemment, les travaux de Imbert, Shvydkoy et Vigneron [16, 17] portent sur le modèle non visqueux (noter l'inversion de signe):

$$\partial_t u - u|\nabla|u + |\nabla|(u^2) = 0, \quad \text{i.e.}^2 \quad \partial_t u = [u, |\nabla|]u. \tag{1.2.1}$$

Dans l'article [16], ils construisent des solutions classiques à partir de données régulières positives et des solutions faibles à partir de données L^∞. Dans [17], ils établissent des estimées de Schauder pour des équations intégro-différentielles générales qui peuvent être appliquées à (1.2.1).

Le modèle ci-dessus présente une certaine ressemblance avec les modèles hydrodynamiques non-visqueux. Par exemple, à un facteur $1/2$ près, l'équation de Burgers peut s'écrire sous la forme d'un commutateur :

$$\partial_t u = [u, \partial_x]u.$$

Le modèle (1.2.1) peut s'interpréter comme une variante de Burgers où on remplace ∂_x par un opérateur non-local $|\nabla|$ du même ordre. L'équation d'Euler incompressible s'écrit

$$\partial_t u + u \cdot \nabla u + \nabla P = 0,$$

où P est la pression, qui est donnée par $P = \mathcal{N}(u \otimes u) + local$, où $\mathcal{N}$ est un opérateur d'intégrale singulière avec un symbole pair. On peut donc aussi faire une analogie entre les termes $u \cdot \nabla u \sim -u|\nabla|u$ et $\nabla P \sim |\nabla|(u^2)$. En vertu de ces analogies formelles avec Euler et Burgers, le modèle (1.2.1) s'appelle *l'équation de Burgers Non-locale*.

Dans ce travail, nous souhaitons étudier *l'équation de Burgers Non-locale Généralisée*:

$$\partial_t u = [F(u), |\nabla|^s]u \quad x \in \mathbb{R}^d \text{ ou } \mathbb{T}^d, \tag{1.2.2}$$

avec une condition initiale

$$u(0, x) = u_0, \tag{1.2.3}$$

[2] Le commutateur de deux opérateurs f and g est noté $[f, g] := fg - gf$.

où $|\nabla|^s = (-\Delta)^{s/2}$ représente le Laplacian fractionnaire dans le régime non-local $s \in (0,1]$.

La fonction F est donnée; on la suppose de régularité $\mathcal{C}^\infty_{\text{loc}}(\mathbb{R})$ et $F' > 0$ p.p. sur $(0,+\infty)$. Par exemple, on peut prendre $F(u) = u^n$ avec $n > 0$ mais aussi $F(u) = u - \sin u$, e^u, $\ln(1+u)$, etc. Dans ce qui suit, on fait l'hypothèse que $F(0) = 0$ qui n'est pas restrictif puisque (1.2.2) n'est pas modifiée si on remplace $F(u)$ par $F(u) - F(0)$.

Nous aimerions signaler le parallèle entre notre modèle (1.2.2) et la classe très générale des équations non-locales de la forme:

$$\partial_t u + |\nabla|^s \varphi(u) = f, \quad \text{in } (0,+\infty) \times \mathbb{R}^d. \tag{1.2.4}$$

Ces modèles apparaissent en particulier dans le contrôle par le bord de l'équation de la chaleur, comme indiqué par Athanasopoulos et Caffarelli [18] et qui renvoient au livre de Duvaut et Lions [19]. Ce type de modèle est aussi utile dans l'étude des milieux poreux.

L'existence, l'unicité et la régularité des solutions faibles[3] du cas homogène ($f \equiv 0$) ont été complètement etudiés par Vázquez *et al.* [20] (voir aussi [21, 22] pour l'équation des milieux poreux). Plus précisément, Vázquez *et al.* démontrent que

Théorème 1.2.1. *[20] Soit $f \equiv 0$ et $\varphi \in \mathcal{C}(\mathbb{R})$ une fonction croissante. Étant donné $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, il existe une unique solution faible bornée au problème (1.2.4)-(1.2.3). Si on suppose en plus que $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ et $\varphi' > 0$ sur $\mathbb{R}$, alors $u \in \mathcal{C}^\infty\big((0,+\infty) \times \mathbb{R}^d\big)$.*

Inspirés par [16, 17], nous développons une théorie du caractère bien posé et de la régularité du modèle (1.2.2) et étudions son comportement en temps long. Tous les résultats qui suivent sont établis dans le cadre des fonctions *périodiques*, à l'exception de l'existence locale qui est valable à la fois en périodique et dans l'espace entier.

1.2.2 Principaux résultats

Donnons maintenant un aperçu de nos résultats (qui ont été obtenus sous la direction du Professeur François Vigneron).

Existence locale avec critère de BKM.

Le premier résultat concerne les données $u_0 \in H^m(\Omega^d)$ positives où $\Omega^d = \mathbb{R}^d$ ou $\mathbb{T}^d$.

Théorème 1.2.2. *Soit $m > \frac{d}{2} + 1$ un entier, $s \in (0,1]$ et F une fonction satisfaisant les hypothèses précédentes. Pour toute donnée initiale strictement positive $u_0 \in H^m(\Omega^d)$, il existe un temps $T > 0$ et une unique solution locale $u \in \mathcal{C}([0,T); H^m(\Omega^d)) \cap \mathcal{C}^1([0,T); H^{m-1}(\Omega^d))$ au problème de Cauchy (1.2.2)-(1.2.3). De plus, $u(t,x) > 0$ pour tout $(t,x) \in [0,T) \times \Omega^d$ et le maximum $\max_{x \in \Omega^d} u(t,\cdot)$ décroît strictement au cours du temps.*

[3] Le lecteur peut se référer à l'Appendice A.2 pour la définition des solutions faibles pour l'équation (1.2.4).

On dispose aussi d'un critère d'explosion de type Beale-Kato-Majda :

Théorème 1.2.3. *Supposons que*

$$u \in \mathcal{C}([0,T); H^m(\Omega)) \cap \mathcal{C}^1([0,T); H^{m-1}(\Omega))$$

est une solution strictement positive de (1.2.2)*, où* $m > \frac{d}{2} + 1$*. Supposons aussi que*

$$\int_0^T \|\nabla u(t)\|_{L^\infty}\, dt < \infty.$$

Alors u s'étend au delà du temps T dans la même classe de régularité.

Regularisation instantanée et caractère globalement bien posé.

Toute solution classique positive de (1.2.2) sur un intervalle de temps $[0,T)$ satisfait aux estimations suivantes : pour tout $k \in \mathbb{N}$ et tout $0 < t_0 < T$

$$\|\partial_t^k u, D_x^k u\|_{L^\infty_{t,x}((t_0,T)\times\mathbb{T}^d)} \leq C(d,s,k,t_0,T,\min u_0, \max u_0). \tag{1.2.5}$$

Théorème 1.2.4. *Soit* $s_0 \in (0,1]$*,* $s_0 \leq s \leq 1$*. Pour toute donnée positive* $u_0 \in H^m(\mathbb{T}^d)$ *avec un entier* $m > \frac{d}{2} + 1$ *la solution du problème* (1.2.2)-(1.2.3) *construite au théorème 1.2.2 existe globalement en temps. De plus, la solution devient instantanément régulière et satisfait les estimations* (1.2.5)*.*

Pour obtenir ce résultat, nous symétrisons le membre de droite de (1.2.2) en le multipliant par $2u$ puis en utilisant l'identité[4] :

$$F(u(t,y)) - F(u(t,x)) = (u(t,y) - u(t,x)) \int_0^1 F'\big((1-\lambda)u(x) + \lambda u(y)\big)\, d\lambda.$$

On obtient ainsi l'équation d'évolution pour la densité énergétique $w = u^2$:

$$\partial_t w = \text{p.v.} \int_{\mathbb{R}^d} \big(w(y) - w(x)\big) \mathcal{K}^s(t,x,y)\, dy$$

où

$$\mathcal{K}^s(t,x,y) = \frac{c_{d,s}}{|x-y|^{d+s}} \frac{2u(x)u(y)}{u(x)+u(y)} \int_0^1 F'\big((1-\lambda)u(x) + \lambda u(y)\big)\, d\lambda.$$

Le noyau actif $\mathcal{K}^s$ est symétrique et possède des bornes elliptiques uniformes $\frac{\Lambda^{-1}}{|x-y|^{d+s}} \leq \mathcal{K}^s \leq \frac{\Lambda}{|x-y|^{d+s}}$ pour un certain $\Lambda > 0$. Cette transformation place le modèle (1.2.2) dans le champ d'application des résultats récents de Kassmann *et al.* [23, 24] et de Caffarelli, Chan & Vasseur [25] où des techniques de De Giorgi-Nash-Moser permettent d'obtenir un premier gain de régularité Höldérienne pour w, puis, par positivité, aussi pour u. Nous suivons ensuite l'idée de [17] pour obtenir une estimée de Schauder pour une équation intégro-différentielle parabolique générale, qui entraîne en particulier les bornes (1.2.5). Le Théorème 1.2.3 et la propriété de régularisation instantanée impliquent que la solution existe globalement en temps.

[4] Dans $\mathbb{R}^d$, le Laplacien fractionnel $|\nabla|^s$ possède une définition comme intégrale singulière: $|\nabla|^s f := \text{p.v.} \int_{\mathbb{R}^d} \big(f(x) - f(y)\big) K^s(x-y)\, dy := \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} \big(f(x) - f(y)\big) K^s(x-y)\, dy$.

Existence globale des solutions faibles.

Puisque les bornes (1.2.5) dépendent essentiellement uniquement de la norme L^∞ des données initiales, on peut construire des solutions régulières approchées à partir de n'importe quelle donnée $u_0 \in L^\infty(\mathbb{T}^d)$ avec $u_0 > 0$. De plus, ces solutions possèdent des bornes a-priori dans l'espace $L^\infty(\mathbb{R}_+ \times \mathbb{T}^d) \cap L^2(\mathbb{R}_+; \dot{H}^{\frac{s}{2}}(\mathbb{T}^d))$, ce qui entraîne de la compacité en appliquant le lemme d'Aubin-Lions.

Théorème 1.2.5. *Soit $s \in [s_0, 1]$ avec $s_0 \in (0, 1]$. Pour toute donnée $u_0 \in L^\infty(\mathbb{T}^d)$, $u_0 > 0$, il existe une solution faible globale de (1.2.2) dans*

$$L^\infty(\mathbb{R}_+ \times \mathbb{T}^d) \cap L^2(\mathbb{R}_+; \dot{H}^{\frac{s}{2}}(\mathbb{T}^d)) \cap \mathcal{C}(\mathbb{R}_+; L^2(\mathbb{T}^d)).$$

L'énergie est conservée, le moment $\int_{\mathbb{T}^d} u(t,x)\,dx$ est continu sur $\mathbb{R}^+$ et satisfait

$$\int_{\mathbb{T}^d} u(t',x)\,dx - \int_{\mathbb{T}^d} u(t,x)\,dx = \int_t^{t'} \int_{\mathbb{T}^d} F(u(\tau,x))|\nabla|^s u(\tau,x)\,dx\,d\tau.$$

De plus, pour tout $t > 0$, la solution u satisfait les estimées de régularisation instantanée (1.2.5) ainsi que l'équation (1.2.2) au sens classique.

Asymptotique en temps long

Lorsque $t \to \infty$, toute solution faible de l'équation (1.2.2) converge vers une constante, à savoir $\frac{1}{\sqrt{|\mathbb{T}^d|}}\|u_0\|_{L^2(\mathbb{T}^d)}$, au sens fort suivant:

Théorème 1.2.6. *Soit $u_0 \in L^\infty(\mathbb{T}^d), u_0 > 0$. Supposons que u est une solution faible associée à la donnée u_0 par le Théorème 1.2.5. Alors il existe un temps $\widetilde{T}$ qui ne dépend que de $s, d, \bar{u}(0), \underline{u}(0)$ et des valeurs maximales de F', F'' sur l'intervalle $0\,[\underline{u}(0), \bar{u}(0)]$ de sorte que pour tout $t \geq \widetilde{T}$, la norme $\|\nabla u(t,\cdot)\|_{L^\infty}$ décroît vers zéro exponentiellement.*

Enfin, nous présentons deux estimées de stabilité.

Corollaire 1.2.1. *Soient F_1, F_2 deux fonctions qui satisfont les hypothèses précédentes sur la fonction F. On considère deux données initiales positives $u_{0,1}, u_{0,2} \in H^m(\mathbb{T}^d)$. Pour $i = 1, 2$, on note u_i la solution du problème de Cauchy (1.2.2)-(1.2.3) associée à F_i et à la donnée $u_{0,i}$. On a alors l'estimée de stabilité suivante pour tout $t > 0$:*

$$\|u_1 - u_2\|_{L^2(\mathbb{T}^d)} \leq \big(\|u_{0,1} - u_{0,2}\|_{L^2(\mathbb{T}^d)} + \|F_1' - F_2'\|_{L^\infty(\mathbb{T}^d)}\big)e^{C_1 t}$$

avec une constante C_1 qui ne dépend que de $d, \|u_1\|_{H^m}, \|u_2\|_{H^m}$. On a aussi :

$$\begin{aligned}\|u_1 - u_2\|_{L^\infty(\mathbb{T}^d)} \leq\ & 2\sqrt{d}\pi\big(\|\nabla u_1(t)\|_{L^\infty(\mathbb{T}^d)} + \|\nabla u_2(t)\|_{L^\infty(\mathbb{T}^d)}\big) \\ & + \frac{1}{\sqrt{\mathbb{T}^d}}\|u_{0,1} - u_{0,2}\|_{L^2(\mathbb{T}^d)}.\end{aligned}$$

Chapter 2

Introduction

The present thesis is devoted to the mathematical study of a magneto-hydrodynamics system taking into account the Hall effect, and of a Burgers type scalar equation with non-local term and nonlinear diffusion.

The main results obtained on these two types of equations are presented in the following paragraphs.

2.1 Mathematical study of a plasma model

2.1.1 Presentation of the system

In this section, we are concerned with the following three dimensional incompressible resistive and viscous Hall-magnetohydrodynamics (Hall-MHD) system[1]:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \mu \Delta \boldsymbol{u}, \tag{2.1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2.1.2}$$

$$\partial_t \boldsymbol{B} = \nabla \times \big((\boldsymbol{u} - h \nabla \times \boldsymbol{B}) \times \boldsymbol{B} \big) + \nu \Delta \boldsymbol{B}, \tag{2.1.3}$$

$$\operatorname{div} \boldsymbol{B} = 0. \tag{2.1.4}$$

The unknowns are:

- the vector-field $\boldsymbol{u}(t,x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$, that represents the velocity of the fluid;
- the vector-field $\boldsymbol{B}(t,x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$, that represents the magnetic field interacting with the fluid;
- the scalar function $P(t,x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$, that represents the pressure.

The positive parameters μ and ν are the kinematic viscosity and the magnetic diffusivity, while the number $h > 0$ measures the magnitude of the Hall effect compared to the typical length scale of the fluid.

We wish to solve the Cauchy problem for the above system with initial conditions:

$$(\boldsymbol{u}(0,x), \boldsymbol{B}(0,x)) = (\boldsymbol{u}_0(x), \boldsymbol{B}_0(x)), \quad x \in \mathbb{R}^3, \tag{2.1.5}$$

[1] In $\mathbb{R}^3$, we note $\boldsymbol{w} \times \boldsymbol{z} := (w_2 z_3 - w_3 z_2,\, w_3 z_1 - w_1 z_3,\, w_1 z_2 - w_2 z_1)^{\mathrm{tr}}$ and $\operatorname{div} \boldsymbol{w} := \sum_j^3 \partial_j w_j$.

where we assume that $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{B}_0 = 0$. Notice that, at least formally, the divergence free condition for magnetic field is conserved through the evolution.

The above system is used to model the evolution of electrically conducting fluids such as plasmas or electrolytes (then, $\boldsymbol{u}$ represents the ion velocity), and takes into account the fact that in a moving conductive fluid, the magnetic field can induce currents which, in turn, polarize the fluid and change the magnetic field. That phenomenon which is neglected in the following classical MHD equations (corresponding to $h = 0$):

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \mu \Delta \boldsymbol{u}, \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \nu \Delta \boldsymbol{B}, \\ \operatorname{div} \boldsymbol{B} = 0, \end{cases} \tag{2.1.6}$$

is represented by the *Hall electric field* $\boldsymbol{E}_H := h\boldsymbol{J} \times \boldsymbol{B}$ where the current $\boldsymbol{J}$ is defined by $\boldsymbol{J} := \nabla \times \boldsymbol{B}$. Hall term plays an important role in magnetic reconnection, as observed in e.g. plasmas, star formation, solar flares, neutron stars or geo-dynamo. For more explanation on the physical background of the Hall-MHD system, one can refer to [1, 2, 3, 4, 5, 6].

2.1.2 Two-fluid plasma description

We first provide a derivation of Hall-MHD system without viscosity from a two-fluid isothermal Euler-Maxwell system for electrons and ions, through a set of scaling limits. To simplify the presentation, we assume that the fluid domain is the whole space $\mathbb{R}^3$ (namely boundary effects are neglected).

A plasma is a collection of fast-moving charged particles. At high temperature and velocity, ions and electrons in an isothermal plasma tend to become two separate fluids due to their different physical properties (inertia, charge). We refer to [26] for physics references in book form. In the *Eulerian description*, the following basic laws and equations from physics will enable us to determine the system of PDEs governing the dynamical evolution of such plasmas:

- the mass conservation for the ions:

$$\partial_t n^+ + \operatorname{div}(n^+ \boldsymbol{u}^+) = 0;$$

- the momentum balance for the ions ($D_t^+ := \partial_t + \boldsymbol{u}^+ \cdot \nabla$):

$$m^+ n^+ D_t^+ \boldsymbol{u}^+ = -\nabla P^+ + Z n_+ e\big(\boldsymbol{E} + \boldsymbol{u}^+ \times \boldsymbol{B}\big) + \boldsymbol{R}^+;$$

- the mass conservation for the electrons:

$$\partial_t n^- + \operatorname{div}(n^- \boldsymbol{u}^-) = 0;$$

- the momentum balance for the electrons ($D_t^- := \partial_t + \boldsymbol{u}^- \cdot \nabla$):

$$m^- n^- D_t^- \boldsymbol{u}^- = -\nabla P^- - n_- e\big(\boldsymbol{E} + \boldsymbol{u}^- \times \boldsymbol{B}\big) + \boldsymbol{R}^-;$$

- Maxwell's equations:

$$\frac{1}{c^2}\partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\mu_0 \boldsymbol{j}, \quad \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0;$$

- Gauss's law for electric field:

$$\epsilon_0 \operatorname{div} \boldsymbol{E} = \rho; \tag{2.1.7}$$

- Gauss's law for magnetism:

$$\operatorname{div} \boldsymbol{B} = 0; \tag{2.1.8}$$

- equation for the charge density:

$$\rho = e(Zn^+ - n^-);$$

- equation for the current density:

$$\boldsymbol{j} = e(Zn^+\boldsymbol{u}^+ - n^-\boldsymbol{u}^-).$$

In order to close the system we prescribe the state equations for *perfect gases*:

$$P^+ = Zn^+\theta, \quad P^- = n^-\theta$$

and the collisional moments as in [26]:

$$\boldsymbol{R}^+ = e^2\nu_e n^-(Zn^+\boldsymbol{u}^+ - n^-\boldsymbol{u}^-), \quad \boldsymbol{R}^+ = -\boldsymbol{R}^-.$$

Note that the last identity is due to the conservation of momentum when collision happens. These equations describe a plasma composed of electrons and one species of ions. e denotes the elementary positive charge, ν_e is the resistivity due to the electron-ion collisions, ϵ_0, μ_0 and c are respectively the vacuum permittivity, the vacuum permeability and the speed of light, related by the relation $\epsilon_0\mu_0 c^2 = 1$. The ions have charge Ze, density n^+, mass m^+, velocity $\boldsymbol{u}^+$, and pressure P^+, and the electrons have charge $-e$, density n^-, mass m^-, velocity $\boldsymbol{u}^-$, and pressure P^-. In addition, $\boldsymbol{E}$ and $\boldsymbol{B}$ denote the electric and magnetic field while ρ and $\boldsymbol{j}$ denote the charge and current density. The two relations (2.1.7) and (2.1.8) are propagated by the dynamic flow, provided that we assume that they are satisfied at the initial time.

Now, we follow the idea of Acheritogaray *et al.* [7] to derive the compressible Hall-MHD system and MHD system without viscosity. We introduce scaling units $n_0, u_0, E_0, B_0, L_0, t_0, \rho_0, j_0$ for respectively the densities, velocities, electric field, magnetic field, space, time, charge and current. We assume that these units are related by the following relations:

$$L_0 = u_0\, t_0, \quad u_0 = \sqrt{\frac{T}{m^+}}, \quad E_0 = u_0\, B_0, \quad \rho_0 = e\, n_0.$$

The first relation means that we observe the system at the convection time scale. The second relation states that the convection velocity is that of the ion

thermal speed. The third relation is typical of a MHD scaling and states that the main contribution to the electric field is induction due to the motion of the charged fluid. Finally, the last relation expresses the consistency between the density and charge units.

Then, six dimensionless parameters appear:

- $\varepsilon^2 = \dfrac{m^-}{m^+}$ is the electron to ion mass ratio and is very small;
- $\alpha^2 = \dfrac{eE_0L_0}{\theta}$ is the ratio of the electric energy to the thermal energy;
- $\beta = \dfrac{e^2\nu_e n_0 u_0 L_0}{\theta}$ measures the relaxation frequency of the electron and ion velocities due to collisions;
- $\gamma = \dfrac{u_0}{c}$ is the ratio of the fluid velocity to the speed of light;
- $\lambda^2 = \dfrac{\epsilon_0\theta}{e^2 n_0 L_0^2}$, where λ is the scaled Debye length and measures the closeness to quasi-neutrality;
- $\eta = \dfrac{j_0}{en_0u_0}$ is the ratio of the charge current scale to the electron or ion current scales.

The dimensionless two-fluid Euler-Maxwell system is obtained:

$$\begin{cases} \partial_t n^+ + \operatorname{div}(n^+\boldsymbol{u}^+) = 0, \\ n^+ D_t^+ \boldsymbol{u}^+ = -\nabla(Zn^+\theta) + \alpha^2 Z n^+ \big(\boldsymbol{E} + \boldsymbol{u}^+ \times \boldsymbol{B}\big) - \beta n^- (Zn^+\boldsymbol{u}^+ - n^-\boldsymbol{u}^-), \\ \partial_t n^- + \operatorname{div}(n^-\boldsymbol{u}^-) = 0, \\ \varepsilon^2 n^- D_t^- \boldsymbol{u}^- = -\nabla(n^-\theta) - \alpha^2 n^- \big(\boldsymbol{E} + \boldsymbol{u}^- \times \boldsymbol{B}\big) + \beta n^- (Zn^+\boldsymbol{u}^+ - n^-\boldsymbol{u}^-), \\ \gamma^2 \partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\dfrac{\gamma^2\eta}{\alpha^2\lambda^2}\,\boldsymbol{j}, \\ \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0, \\ \alpha^2\lambda^2 \operatorname{div} \boldsymbol{E} = \rho, \\ \operatorname{div} \boldsymbol{B} = 0, \\ \rho = Zn^+ - n^-, \\ \boldsymbol{j} = \dfrac{1}{\eta}(Zn^+\boldsymbol{u}^+ - n^-\boldsymbol{u}^-). \end{cases}$$

In all what follows, we assume that $\dfrac{\gamma^2\eta}{\alpha^2\lambda^2} = \alpha^2\eta = 1$ in order to give rise to the standard magnetostatic Ampère's law and keep the Lorentz force term in the momentum equation of ions of order 1. Then by taking the simultaneous independent limits:

- $\gamma^2 \to 0$ which amounts to neglecting the displacement current in Ampère's equation;

- $\varepsilon^2 \to 0$ which corresponds to neglecting the convection term in the electron momentum equation and gives the *generalized Ohm's law*[2]: (notice that $\alpha^4\lambda^2 \to 0$)

$$\boldsymbol{E} + \boldsymbol{u}^+ \times \boldsymbol{B} = \frac{1}{\alpha^2}\Big(\theta\nabla(\ln n^-) + \frac{\boldsymbol{j}\times\boldsymbol{B}}{n^-}\Big) + \frac{\beta}{\alpha^4}\boldsymbol{j};$$

- $\alpha^2\lambda^2 \to 0$ which uses the fact that a plasma, on the average, is almost electrically neutral. A condition commonly termed "quasi-neutrality."

The resulting system is the so-called compressible isothermal resistive Hall-MHD equations without viscosity. Denoting by $\boldsymbol{u}$ the ion velocity and n the ion density, this system is written

$$\begin{cases} \partial_t n + \operatorname{div}(n\boldsymbol{u}) = 0, \\ n\big(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\big) + \nabla(2Zn\theta) = \boldsymbol{j}\times\boldsymbol{B}, \\ \nabla\times\boldsymbol{B} = \boldsymbol{j}, \\ \partial_t\boldsymbol{B} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \dfrac{1}{\alpha^2}\nabla\times\Big(\theta\nabla(\ln n) + \dfrac{\boldsymbol{j}\times\boldsymbol{B}}{Zn}\Big) = \dfrac{\beta}{\alpha^4}\Delta\boldsymbol{B}, \\ \operatorname{div}\boldsymbol{B} = 0. \end{cases}$$

Meanwhile, we have

$$\boldsymbol{j} = Z\alpha^2 n(\boldsymbol{u}^+ - \boldsymbol{u}^-). \tag{2.1.9}$$

Note that the term $\theta\nabla(\ln n)$ at the left-hand side of magnetic equation has no contribution since the curl operator cancels it. We consider $\dfrac{\beta}{\alpha^4} = O(1)$ which corresponds to resistive MHD model and there is only one dimensionless parameter left: $\dfrac{1}{\alpha^2}$. In particular, we have

- if $\dfrac{1}{\alpha^2} \to 0$, then the electron and ion velocities become identical thanks to (2.1.9). This gives rise to compressible MHD system without viscosity.

- if $\dfrac{1}{\alpha^2 Z} = O(1)$, then the ion and electron velocities differ and Hall term appears. This gives rise to the compressible Hall-MHD system without viscosity.

Our study takes place in the context of the last regime, where both the resistive and Hall terms are important. We note that it is easy to extend this system to the viscous isothermal resistive compressible Hall-MHD system. Additionally, we assume incompressible homogeneous viscous fluid motion. In this case, the Hall-MHD system can be written according to (2.1.1)-(2.1.4). From a similar argument one can derive the classical MHD equations (2.1.6).

[2]The reader may refer to the paper [27] by Jang and Masmoudi where this generalized Ohm's law is derived from the kinetic equations.

2.1.3 Criticality of the Hall-MHD system

Despite its physical relevance, the Hall-MHD system has been considered only rather recently in mathematics, following the work by Acheritogaray, Degond, Frouvelle and Liu in [7] where the Hall-MHD system was formally derived both from a two fluids system and from a kinetic model. Then, in [28], Chae, Degond and Liu showed the global existence of weak solutions[3] in the spirit of Leray [29], as well as the local well-posedness for initial data $\boldsymbol{u}_0$ and $\boldsymbol{B}_0$ in Sobolev spaces[4] H^s with $s > 5/2$. Weak solutions have been further investigated by Dumas and Sueur in [30] both for the Maxwell-Landau-Lifshitz system and for the Hall-MHD system. Very recently, Dai [31] focused on the non-uniqueness of weak solutions in the Leray-Hopf class.

In Chae *et al.* [13, 32], Serrin type continuation criteria for smooth solutions and the global existence of strong solutions emanating from small initial data have been obtained (see also Dai [33] for more sharp criteria). In Ahmad *et al.* [34], it has been observed that the possible blow-up of smooth solutions may be controlled in terms of the velocity only. More well-posedness results of strong solutions for less regular data in Sobolev or Besov spaces, have been established in [35, 36, 37, 38], and the convergence to the MHD system with no Hall-term for $h \to 0$ has been addressed in [39].

In Chae & Schonbek [40], Weng [41, 42] and Dai & Liu [43], long time behavior of solutions have been investigated. Examples of smooth data with arbitrarily large L^∞ norms giving rise to global unique solutions have been exhibited very recently in Li & Zhu [44].

Our main goal here is to establish the well-posedness of the Hall-MHD system with initial data in *critical spaces*. In contrast with the classical MHD system (2.1.6) however, the system under consideration does not have any scaling invariance owing to the coexistence of the Hall term in (2.1.3) and of the Lorentz force in (2.1.1), and we have to explain what we mean by critical regularity.

On the one hand, for $\boldsymbol{B} \equiv \boldsymbol{0}$, the Hall-MHD system reduces to the incompressible Navier-Stokes equation:

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = \mu \Delta \boldsymbol{u} & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.1.10}$$

System (2.1.10) is invariant for all positive λ by the rescaling

$$(\boldsymbol{u}, P)(t,x) \rightsquigarrow (\lambda \boldsymbol{u}, \lambda^2 P)(\lambda^2 t, \lambda x) \quad \text{and} \quad \boldsymbol{u}_0(x) \rightsquigarrow \lambda \boldsymbol{u}_0(\lambda x), \tag{2.1.11}$$

[3] The reader may refer to the Definition A.2.1 for the definition of Leray-Hopf weak solutions for 3D Hall-MHD.

[4] For reader's convenience, definitions of Sobolev spaces (resp. Besov spaces) and its properties used in the thesis are given at the Appendix B.5 (resp. Appendix B.1).

and optimal global well-posedness results for (2.1.10) supplemented with small initial data are obtained in functional spaces endowed with norms having the scaling invariance[5] (2.1.11), for all $\lambda > 0$. One can read the book [45] by Bahouri, Chemin and Danchin for more details.

On the other hand, if the fluid velocity in (2.1.3) is $\mathbf{0}$, then we get the following *Hall equation* for $\boldsymbol{B}$:

$$\begin{cases} \partial_t \boldsymbol{B} + h\nabla \times ((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}) = \nu \Delta \boldsymbol{B}, \\ \boldsymbol{B}(0,x) = \boldsymbol{B}_0, \end{cases}$$

which is invariant by the rescaling

$$\boldsymbol{B}(t,x) \leadsto \boldsymbol{B}(\lambda^2 t, \lambda x) \quad \text{and} \quad \boldsymbol{B}_0(x) \leadsto \boldsymbol{B}_0(\lambda x).$$

Therefore, if $h > 0$ and if we neglect the Lorentz force in (2.1.1), then it is natural to work at the same level of regularity for $\boldsymbol{u}$ and $\nabla \boldsymbol{B}$, while for $h = 0$ (the classical MHD system) $\boldsymbol{u}$ and $\boldsymbol{B}$ have the same scaling invariance.

The way to reconcile the two viewpoints is to look at the current function $\boldsymbol{J} = \nabla \times \boldsymbol{B}$ as an additional unknown. Now, owing to the vector identity

$$\nabla \times (\nabla \times \boldsymbol{w}) + \Delta \boldsymbol{w} = \nabla \mathrm{div}\, \boldsymbol{w}$$

and since $\boldsymbol{B}$ is divergence free, we have $\Delta \boldsymbol{B} = -\nabla \times \boldsymbol{J}$, whence

$$\boldsymbol{B} = \mathrm{curl}^{-1} J := (-\Delta)^{-1} \nabla \times \boldsymbol{J},$$

where the -1 order homogeneous Fourier multiplier curl^{-1} (the so-called Boit-Savart operator) is defined on the Fourier side by

$$\mathcal{F}(\mathrm{curl}^{-1} J)(\xi) := \frac{i\xi \times \widehat{\boldsymbol{J}}(\xi)}{|\xi|^2}. \tag{2.1.12}$$

With that notation, one gets the following *extended Hall-MHD system*:

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \mu \Delta \boldsymbol{u} + \nabla P = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \\ \mathrm{div}\, \boldsymbol{u} = \mathrm{div}\, \boldsymbol{B} = \mathrm{div}\, \boldsymbol{J} = 0, \\ \partial_t \boldsymbol{B} - \nabla \times ((\boldsymbol{u} - h\boldsymbol{J}) \times \boldsymbol{B}) - \nu \Delta \boldsymbol{B} = 0, \\ \partial_t \boldsymbol{J} - \nabla \times \big(\nabla \times ((\boldsymbol{u} - h\boldsymbol{J}) \times \mathrm{curl}^{-1} \boldsymbol{J})\big) - \nu \Delta \boldsymbol{J} = 0, \\ (\boldsymbol{u}(0,x), \boldsymbol{B}(0,x), \boldsymbol{J}(0,x)) = (\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{J}_0). \end{cases} \tag{2.1.13}$$

In contrast with the original Hall-MHD system, the above extended system has a scaling invariance (the same as for the incompressible Navier-Stokes equations (2.1.10)). This reformulation motivates us to give the following definition of criticality on the original Cauchy problem (2.1.1)-(2.1.5):

[5] Pointing out a smallness condition for $\boldsymbol{u}_0$ leading to global existence in a space that does not obey this invariance is essentially equivalent to solving the global well-posedness issue for (2.1.10) in the large data case.

Definition 2.1.1. *A critical space for initial data* $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ *is any Banach space* $\mathcal{X} \subset \mathcal{S}'(\mathbb{R}^3)$ *whose norm is invariant for all* $\lambda > 0$ *by* $(\boldsymbol{u}_0(x), \boldsymbol{B}_0(x), \boldsymbol{J}_0(x)) \mapsto (\lambda \boldsymbol{u}_0(\lambda x), \lambda \boldsymbol{B}_0(\lambda x), \lambda \boldsymbol{J}_0(\lambda x))$.

A critical space for solutions $(\boldsymbol{u}, \boldsymbol{B})$ *is any Banach space* $X \subset \mathcal{S}'(\mathbb{R}^3)$ *whose norm is invariant for all* $\lambda > 0$ *by* $(\boldsymbol{u}(x), \boldsymbol{B}(x), \boldsymbol{J}(x)) \mapsto (\lambda \boldsymbol{u}, \lambda \boldsymbol{B}, \lambda \boldsymbol{J})(\lambda^2 t, \lambda x)$.

Studying whether the Hall-MHD system written in terms of $(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{J})$ is well-posed in the same critical functional spaces as the velocity in (2.1.10), and if similar blow-up criteria and results related to the structure of the system may be established is part of the main aim of the present thesis.

2.1.4 Main results

After the celebrated work by Fujita & kato [12] and later by Cannone [46], Chemin [11], it is by now classical that the incompressible Navier-Stokes equations are well-posed (locally for large data or globally for small data) in all homogeneous Besov spaces $\dot{B}^{\frac{3}{p}-1}_{p,r}$ with $1 \leq p < \infty$ and $1 \leq r \leq \infty$. Similar results have been obtained for the classical incompressible MHD system (2.1.6) by Miao and Yuan in [47]. According to the scaling considerations of the above paragraph, it is natural to look for similar results for the Hall-MHD system written in its extended formulation (2.1.13).

In what follows, we present the main results for the Hall-MHD system obtained in the thesis concerning the following four problems.

- **Problem 1:** Global existence and time-decay estimates for small data in the critical spaces $\dot{B}^{\frac{3}{p}-1}_{p,1}$ with general $1 \leq p < \infty$;
- **Problem 2:** Local well-posedness for large data and blow-up criterion in the critical spaces $\dot{B}^{\frac{1}{2}}_{2,1}$;
- **Problem 3:** The Fujita-Kato type theorem (i.e. the solvability in homogeneous Sobolev spaces $\dot{H}^{\frac{1}{2}}$), and properties related to the structure of the system;
- **Problem 4** The $2\frac{1}{2}D$ flows with large velocity fields.

The results obtained for **Problem 1–4** and presentation below are supervised under one of my thesis director, professor Raphäel Danchin, and have been recently published or accepted in [8, 9]. The problem concerning time-decay estimates and stability is based on a collaboration with Lvqiao Liu [10].

Problem 1

Let us first consider the case of positive general coefficients μ, ν and h, for data $(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{J}_0)$ in $\dot{B}^{\frac{3}{p}-1}_{p,1}$. Following the paper [11] by Chemin dedicated

to the incompressible Navier-Stokes equations, we introduce for $T > 0$, the space[6]

$$E_p(T) := \Big\{ \boldsymbol{w} \in \mathcal{C}([0,T], \dot{B}^{\frac{3}{p}-1}_{p,1}),\ \nabla^2 \boldsymbol{w} \in L^1(0,T; \dot{B}^{\frac{3}{p}-1}_{p,1}) \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = 0 \Big\}$$

and its global version E_p (with $\boldsymbol{w} \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{3}{p}-1}_{p,1})$) if $T = +\infty$.

Using the extended formulation and Picard's fixed point theorem in the space E_p, we obtain the following result, which works for *any positive coefficients* μ, ν *and* h.

Theorem 2.1.2. *Let* $1 \leq p < \infty$ *and* $(\boldsymbol{u}_0, \boldsymbol{B}_0) \in \dot{B}^{\frac{3}{p}-1}_{p,1}$ *with* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{B}_0 = 0$ *and* $\boldsymbol{B}_0 := \nabla \times \boldsymbol{B}_0 \in \dot{B}^{\frac{3}{p}-1}_{p,1}$. *There exists a constant* $c_0 > 0$ *depending only on* p *and* μ/ν *such that, if*

$$\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|\boldsymbol{B}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + h\|\boldsymbol{J}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} < c_0\mu, \tag{2.1.14}$$

then the Cauchy problem (2.1.1)-(2.1.5) *has a unique global solution* $(\boldsymbol{u}, \boldsymbol{B}) \in E_p$, *with* $\boldsymbol{J} := \nabla \times \boldsymbol{B} \in E_p$. *Furthermore,*

$$\|\boldsymbol{u}\|_{E_p} + \|\boldsymbol{B}\|_{E_p} + h\|\boldsymbol{J}\|_{E_p} < 2c_0\mu. \tag{2.1.15}$$

Moreover, for any integer $m \geq 1$, *we have*

$$\|D^m \boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|D^m \boldsymbol{B}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \leq C_0 c_0 t^{-\frac{m}{2}},$$

for all $t > 0$, *where*

$$\|D^m \boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} := \sup_{|\alpha|=m} \|D^\alpha \boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}},$$

and the positive constant C_0 *depends only on* μ, ν, h, p, m.

If, in addition, $\boldsymbol{u}_0 \in H^s$ *and* $\boldsymbol{B}_0 \in H^r$ *with*

$$\frac{3}{p} - 1 < s \leq r \quad \text{and} \quad \frac{3}{p} < r \leq 1 + s, \tag{2.1.16}$$

then $(\boldsymbol{u}, \boldsymbol{B}) \in \mathcal{C}_b(\mathbb{R}_+; H^s \times H^r)$, $\nabla \boldsymbol{u} \in L^2(\mathbb{R}_+; H^s)$ *and* $\nabla \boldsymbol{B} \in L^2(\mathbb{R}_+; H^r)$ *and the following energy balance is fulfilled for all* $t \geq 0$:

$$\|\boldsymbol{u}(t)\|_{L^2}^2 + \|\boldsymbol{B}(t)\|_{L^2}^2 + 2\int_0^t \left(\mu\|\nabla \boldsymbol{u}\|_{L^2}^2 + \nu\|\nabla \boldsymbol{B}\|_{L^2}^2\right) d\tau = \|\boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{B}_0\|_{L^2}^2. \tag{2.1.17}$$

Finally, in the case where only $\boldsymbol{J}_0$ *fulfills* (2.1.14), *there exists some time* $T > 0$ *such that* (2.1.1)-(2.1.5) *has a unique local-in-time solution on* $[0,T]$ *with* $(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{J})$ *in* $E_p(T)$, *and additional Sobolev regularity is preserved.*

[6] The reader may refer to the Notations in the Appendix A.1 for definitions of time-space spaces

The key to the proof of Theorem 2.1.2 is to consider the extended Hall-MHD system (2.1.13), suitably rewritten in the form of a generalized Navier-Stokes system that may be solved by implementing the classical fixed point theorem in the Banach space E_p. In order to derive an appropriate formulation of the system, we need to use some algebraic identities. The first one is that for any couple of $\mathcal{C}^1$ divergence free vector-fields z and w on $\mathbb{R}^3$, we have

$$w\cdot\nabla z = \mathrm{div}(z\otimes w), \quad \text{where} \quad \big(\mathrm{div}(z\otimes w)\big)_j := \sum_{k=1}^{3}\partial_k(z_j w_k). \tag{2.1.18}$$

Observe also that

$$(\nabla\times w)\times w = w\cdot\nabla w - \nabla\Big(\frac{|w|^2}{2}\Big)\cdot$$

Hence, setting $\Pi := P + |B|^2/2$, equation (2.1.1) recasts in

$$\partial_t u + \mathrm{div}(u\otimes u) + \nabla\Pi = \mathrm{div}(B\otimes B) + \mu\Delta u. \tag{2.1.19}$$

After projecting (2.1.19) onto the set of divergence free vector fields by means of the *Leray projector* $\mathcal{P} := \mathrm{Id} - \nabla\Delta^{-1}\mathrm{div}$, we get

$$\partial_t u - \mu\Delta u = Q_a(B,B) - Q_a(u,u),$$

where the bilinear form Q_a is defined by

$$Q_a(z,w) := \frac{1}{2}\mathcal{P}(\mathrm{div}(z\otimes w) + \mathrm{div}(w\otimes z)).$$

Next, by using the identity

$$\nabla\times(w\times z) = z\cdot\nabla w - w\cdot\nabla z, \tag{2.1.20}$$

one can rewrite Hall term as

$$\nabla\times(J\times B) = B\cdot\nabla J - J\cdot\nabla B.$$

Hence, combining with (2.1.18), equation (2.1.3) recasts in

$$\partial_t B - \nu\Delta B = Q_b(B, hJ-u),$$

where

$$Q_b(z,w) := \mathrm{div}(z\otimes w) - \mathrm{div}(w\otimes z),$$

and the equation for J may thus be written

$$\partial_t J - \nu\Delta J = \nabla\times Q_b(\mathrm{curl}^{-1}J, hJ-u).$$

Altogether, we conclude that the extended Hall-MHD system (2.1.13) recasts in

$$\begin{cases} \partial_t \boldsymbol{u} - \mu\Delta\boldsymbol{u} = Q_a(\boldsymbol{B},\boldsymbol{B}) - Q_a(\boldsymbol{u},\boldsymbol{u}), \\ \partial_t \boldsymbol{B} - \nu\Delta\boldsymbol{B} = Q_b(\boldsymbol{B}, h\boldsymbol{J} - \boldsymbol{u}), \\ \partial_t \boldsymbol{J} - \nu\Delta\boldsymbol{J} = \nabla\times Q_b(\mathrm{curl}^{-1}\boldsymbol{J}, h\boldsymbol{J} - \boldsymbol{u}), \\ \big(\boldsymbol{u}(0,x),\boldsymbol{B}(0,x),\boldsymbol{J}(0,x)\big) = (\boldsymbol{u}_0,\boldsymbol{B}_0,\boldsymbol{J}_0). \end{cases} \tag{2.1.21}$$

Set $\boldsymbol{U} := (\boldsymbol{U}_1,\boldsymbol{U}_2,\boldsymbol{U}_3)$ with $\boldsymbol{U}_1 := \boldsymbol{u}$, $\boldsymbol{U}_2 := \boldsymbol{B}$ and $\boldsymbol{U}_3 := \boldsymbol{J}$. Then, (2.1.21) may be shortened into:

$$\begin{cases} \partial_t \boldsymbol{U} - \Delta_{\mu,\nu}\boldsymbol{U} = Q(\boldsymbol{U},\boldsymbol{U}), \\ \boldsymbol{U}(0,x) = \boldsymbol{U}_0, \end{cases} \qquad \text{with} \quad \Delta_{\mu,\nu}\boldsymbol{U} := \begin{pmatrix} \mu\Delta\boldsymbol{u} \\ \nu\Delta\boldsymbol{B} \\ \nu\Delta\boldsymbol{J} \end{pmatrix} \tag{2.1.22}$$

and where $Q : \mathbb{R}^9\times\mathbb{R}^9 \to \mathbb{R}^9$ is defined by

$$Q(\boldsymbol{V},\boldsymbol{W}) := \begin{pmatrix} Q_a(\boldsymbol{V}_2,\boldsymbol{W}_2) - Q_a(\boldsymbol{V}_1,\boldsymbol{W}_1) \\ Q_b(\boldsymbol{V}_2, h\boldsymbol{W}_3 - \boldsymbol{W}_1) \\ \nabla\times Q_b(\mathrm{curl}^{-1}\boldsymbol{V}_3, h\boldsymbol{W}_3 - \boldsymbol{W}_1) \end{pmatrix}.$$

The gain of considering the above extended system rather than the initial one is that it is semi-linear, while the Hall-MHD system for $(\boldsymbol{u},\boldsymbol{B})$ is quasi-linear. The quadratic terms in the first two lines of (2.1.22) are essentially the same as for the system (2.1.10).

Owing to the Hall term in the last line however, one has to go beyond the theory of the generalized Navier-Stokes equations as presented in e.g. [45, Chap. 5] since the differentiation is *outside* instead of being inside the first variable of Q_b. This actually prevents us from considering **Problem 2** with large $\boldsymbol{J}_0$ and **Problem 3, 4**.

Problem 2

In the case[7] $\mu = \nu$, we find that there is a cancellation property that eliminates the Hall term when performing an energy estimate. This will enable us to prove the following local well-posedness result in the sense of Hadamard *for large data* in the critical Besov space $\dot{B}^{\frac{1}{2}}_{2,1}$, together with blow-up criteria involving critical norms.

Theorem 2.1.3. *Assume that $\mu = \nu$. For any initial data $(\boldsymbol{u}_0,\boldsymbol{B}_0)$ in $\dot{B}^{\frac{1}{2}}_{2,1}$ with $\operatorname{div}\boldsymbol{u}_0 = \operatorname{div}\boldsymbol{B}_0 = 0$ and $\boldsymbol{J}_0 := \nabla\times\boldsymbol{B}_0 \in \dot{B}^{\frac{1}{2}}_{2,1}$, there exists a positive time T such that the Cauchy problem (1.1.1)-(1.1.4) has a unique solution $(\boldsymbol{u},\boldsymbol{B}) \in E_2(T)$ with $\boldsymbol{J} := \nabla\times\boldsymbol{B} \in E_2(T)$. Moreover, if the maximal time of existence T^* of that solution is finite, then*

$$\int_0^{T^*} \|(\boldsymbol{u},\boldsymbol{B},\nabla\boldsymbol{B})(t)\|^2_{\dot{L}^\infty}\,dt = \infty$$

[7] That is made in most mathematical papers devoted to the Hall-MHD system even though it is not physically motivated.

$$\int_0^{T^*} \|(\boldsymbol{u},\boldsymbol{B},\nabla\boldsymbol{B})(t)\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\,dt=\infty \tag{2.1.23}$$

and, for any $\varrho\in(2,\infty)$,

$$\int_0^{T^*} \|(\boldsymbol{u},\boldsymbol{B},\nabla\boldsymbol{B})(t)\|^{\varrho}_{\dot{B}^{\frac{2}{\varrho}-1}_{\infty,\infty}}\,dt=\infty. \tag{2.1.24}$$

As an application, the following corollary states the (global) stability for possible large solutions of Hall-MHD system in the spaces $\dot{B}^{\frac{1}{2}}_{2,1}$.

Corollary 2.1.4. *Assume that* $(\boldsymbol{u}_{0,i},\boldsymbol{B}_{0,i})\in\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)$ *with* $\operatorname{div}\boldsymbol{u}_{0,i}=\operatorname{div}\boldsymbol{B}_{0,i}=0$ *such that* $\boldsymbol{v}_{0,i}:=\boldsymbol{u}_{0,i}-h\nabla\times\boldsymbol{B}_{0,i}\in\dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3)$, $i=1,2$. *Suppose in addition that for* $\mu=\nu$ *the Cauchy problem* (1.1.1)-(1.1.4) *supplemented with initial data* $(\boldsymbol{u}_{0,1},\boldsymbol{B}_{0,1})$ *admits a solution* $(\boldsymbol{u}_1,\boldsymbol{B}_1)$ *such that for some time* $T>0$,

$$(\boldsymbol{u}_1,\boldsymbol{B}_1,\nabla\times\boldsymbol{B}_1)\in L^1(0,T;\dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)).$$

There exist a positive constant c_1 *depends only on* μ, h *and some critical norms of the solution* $(\boldsymbol{u}_1,\boldsymbol{B}_1)$ *such that if*

$$\|(\boldsymbol{u}_{0,1}-\boldsymbol{u}_{0,2},\boldsymbol{B}_{0,1}-\boldsymbol{B}_{0,2},\boldsymbol{v}_{0,1}-\boldsymbol{v}_{0,2})\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\le c_1, \tag{2.1.25}$$

then $(\boldsymbol{u}_{0,2},\boldsymbol{B}_{0,2})$ *generates a unique solution* $(\boldsymbol{u}_2,\boldsymbol{B}_2)\in E_2(T)$.

In order to explain where that cancellation comes from, let us introduce the auxiliary unknown $\boldsymbol{v}:=\boldsymbol{u}-h\boldsymbol{J}$ (that may be interpreted as the *velocity of the electron*, see [1] page 5). Recall the following vector identities:

$$\nabla(\boldsymbol{w}\cdot\boldsymbol{z})=(\nabla\boldsymbol{w})^{\mathrm{tr}}\boldsymbol{z}+(\nabla\boldsymbol{z})^{\mathrm{tr}}\boldsymbol{w}\quad\text{and}\quad(\nabla\boldsymbol{w}-(\nabla\boldsymbol{w})^{\mathrm{tr}})\boldsymbol{z}=(\nabla\times\boldsymbol{w})\times\boldsymbol{v}.$$

Hence, combining with (2.1.20) yields

$$\begin{aligned}\nabla\times(\boldsymbol{w}\times\boldsymbol{z})&=\boldsymbol{z}\cdot\nabla\boldsymbol{w}-\boldsymbol{w}\cdot\nabla\boldsymbol{z}\\&=(\nabla\boldsymbol{w}-(\nabla\boldsymbol{w})^{\mathrm{tr}})\boldsymbol{z}+(\nabla\boldsymbol{z}-(\nabla\boldsymbol{z})^{\mathrm{tr}})\boldsymbol{w}-2\boldsymbol{w}\cdot\nabla\boldsymbol{z}+\nabla(\boldsymbol{w}\cdot\boldsymbol{z})\\&=(\nabla\times\boldsymbol{w})\times\boldsymbol{z}+(\nabla\times\boldsymbol{z})\times\boldsymbol{w}-2\boldsymbol{w}\cdot\nabla\boldsymbol{z}+\nabla(\boldsymbol{w}\cdot\boldsymbol{z}).\end{aligned} \tag{2.1.26}$$

Then, applying identity (2.1.26) to the term $\nabla\times(\boldsymbol{v}\times\boldsymbol{B})$, equation (1.1.3) turns into

$$\partial_t\boldsymbol{B}-\mu\Delta\boldsymbol{B}=(\nabla\times\boldsymbol{v})\times\boldsymbol{B}+\boldsymbol{J}\times\boldsymbol{u}-2\boldsymbol{v}\cdot\nabla\boldsymbol{B}+\nabla(\boldsymbol{v}\cdot\boldsymbol{B}).$$

Taking $h\cdot$curl of the above equation, and subtracting it from (1.1.1), we get

$$\begin{aligned}\partial_t\boldsymbol{v}-\mu\Delta\boldsymbol{v}=\boldsymbol{B}\cdot\nabla\boldsymbol{B}-\boldsymbol{u}\cdot\nabla\boldsymbol{u}-h\nabla\times((\nabla\times\boldsymbol{v})\times\boldsymbol{B})\\+h\nabla\times(\boldsymbol{u}\times\boldsymbol{J})+2h\nabla\times(\boldsymbol{v}\cdot\nabla\boldsymbol{B})-\nabla\Pi.\end{aligned}$$

Therefore, since $h\boldsymbol{J} = \boldsymbol{u} - \boldsymbol{v}$, in terms of unknowns $(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})$, the extended Hall-MHD system reads

$$\begin{cases} \partial_t \boldsymbol{u} - \mu\Delta\boldsymbol{u} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla\Pi, \\ \operatorname{div}\boldsymbol{u} = 0, \\ \partial_t \boldsymbol{B} - \mu\Delta\boldsymbol{B} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}), \\ \partial_t \boldsymbol{v} - \mu\Delta\boldsymbol{v} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - h\nabla\times((\nabla\times\boldsymbol{v})\times\boldsymbol{B}) \\ \qquad\qquad + \nabla\times(\boldsymbol{v}\times\boldsymbol{u}) + 2h\nabla\times(\boldsymbol{v}\cdot\nabla\boldsymbol{B}) - \nabla\Pi. \end{cases} \tag{2.1.27}$$

The only quasilinear term cancels out when performing an energy estimate, since

$$\big(\nabla\times((\nabla\times\boldsymbol{v})\times\boldsymbol{B}),\boldsymbol{v}\big)_{L^2} = \big((\nabla\times\boldsymbol{v})\times\boldsymbol{B},\nabla\times\boldsymbol{v}\big)_{L^2} = 0. \tag{2.1.28}$$

After localization of the system by means of the Littlewood-Paley spectral cut-off operators $\dot{\Delta}_j$ defined in the Appendix B.1, the above identity still holds, up to some lower order commutator term[8].

Problem 3

Still for $\mu = \nu$, we show that the Hall-MHD system is indeed globally well-posed if u_0, B_0 and v_0 are small in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, a Fujita-Kato type result for the Hall-MHD system in the spirit of the celebrated work [12]. We also get some informations on the long time behavior of the solutions, similar to that are presented for the System (2.1.10) in e.g. [45, Chap. 5].

Theorem 2.1.5. *Assume that $\mu = \nu$. Let $(\boldsymbol{u}_0, \boldsymbol{B}_0) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with $\operatorname{div}\boldsymbol{u}_0 = \operatorname{div}\boldsymbol{B}_0 = 0$, and $\boldsymbol{J}_0 := \nabla\times\boldsymbol{B}_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. There exists a constant $c_2 > 0$ such that, if*

$$\|\boldsymbol{u}_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \|\boldsymbol{B}_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + \|\boldsymbol{u}_0 - h\boldsymbol{J}_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < c_2\mu, \tag{2.1.29}$$

then there exists a unique global solution

$$(\boldsymbol{u},\boldsymbol{B}) \in \mathcal{C}_b(\mathbb{R}_+;\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+;\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$$

to the Cauchy problem (1.1.1)-(1.1.4), such that $\boldsymbol{J} := \nabla\times\boldsymbol{B} \in L^\infty(\mathbb{R}_+;\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+;\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$. Furthermore, for all $t \geq t_0 \geq 0$, one has

$$\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{u}-h\boldsymbol{J})(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{\mu}{2}\int_{t_0}^t \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{u}-h\boldsymbol{J})\|^2_{\dot{H}^{\frac{3}{2}}}\,d\tau \leq \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{u}-h\boldsymbol{J})(t_0)\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{2.1.30}$$

In particular, the function

$$t \mapsto \|\boldsymbol{u}(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{B}(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{u}(t)-h\boldsymbol{J}(t)\|^2_{\dot{H}^{\frac{1}{2}}}$$

is nonincreasing.

[8] In this thesis, the commutator of two elements, f and g, is the element $[f,g] := fg - gf$.

If, in addition, $\boldsymbol{u}_0 \in H^s$ *and* $\boldsymbol{B}_0 \in H^{s+1}$ *for some* $s \geq 0$ *with* $s \neq 1/2$, *then* $(\boldsymbol{u}, \boldsymbol{B}) \in \mathcal{C}_b(\mathbb{R}_+; H^s \times H^{s+1})$, $\nabla \boldsymbol{u} \in L^2(\mathbb{R}_+; H^s)$ *and* $\nabla \boldsymbol{B} \in L^2(\mathbb{R}_+; H^{s+1})$.

Remark 1. Note that the functional framework used in the above statement ensures that all the terms of System (2.1.1)-(2.1.4) are well defined. In particular, the most nonlinear one, namely $\nabla \times ((\nabla \times \boldsymbol{B}) \times \boldsymbol{B})$, makes sense since both $\boldsymbol{B}$ and $\nabla \times \boldsymbol{B}$ have positive Sobolev regularity.

Remark 2. A small variation on the proof yields local well-posedness if assuming only that $\|\boldsymbol{u}_0 - h\boldsymbol{J}_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$ is small. In contrast with the incompressible Navier-Stokes equations however, whether a local-in-time result may be proved without any smallness condition is an open question.

Corollary 2.1.6. *If* $(\boldsymbol{u}, \boldsymbol{B})$ *denotes the solution given by Theorem 2.1.5 and if, in addition,* $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ *is in* $L^2(\mathbb{R}^3)$, *then* $(\boldsymbol{u}, \boldsymbol{B})$ *is continuous with values in* $L^2(\mathbb{R}^3)$, *satisfies the energy balance* (2.1.17) *for all* $t \geq 0$, *and we have*

$$\lim_{t \to +\infty} \left(\|\boldsymbol{u}(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{B}(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{J}(t)\|^2_{\dot{H}^{\frac{1}{2}}} \right) = 0. \tag{2.1.31}$$

The following corollary states that global solutions, even if large and with infinite energy, enjoying a suitable integrability property (that is obviously satisfied by the solutions constructed in Theorem 2.1.5) have to decay to zero at infinity.

Corollary 2.1.7. *Assume that* $(\boldsymbol{u}_0, \boldsymbol{B}_0) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ *with* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{B}_0 = 0$ *and* $\boldsymbol{J}_0 = \nabla \times \boldsymbol{B}_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. *Suppose in addition that the Hall-MHD system with* $\mu = \nu$ *supplemented with initial data* $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ *admits a global solution* $(\boldsymbol{u}, \boldsymbol{B})$ *such that*

$$(\boldsymbol{u}, \boldsymbol{B}, \nabla \times \boldsymbol{B}) \in L^4(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3)).$$

Then, $(\boldsymbol{u}, \boldsymbol{B})$ *has the regularity properties of Theorem 2.1.5, and* (2.1.31) *is satisfied.*

Our second purpose about **Problem 3** is to prove a weak-strong uniqueness result for Hall-MHD system, namely that all Leray-Hopf weak solutions (in the tempered distribution meaning) coincide with the unique Fujita-Kato solution whenever the latter one exists. That result turns out to be less sensitive to the very structure of the system, and is valid for all values of μ, ν and h.

Theorem 2.1.8. *Consider initial data* $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ *in* $L^2(\mathbb{R}^3)$ *with* $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{B}_0 = 0$. *Let* $(\boldsymbol{u}, \boldsymbol{B})$ *be any Leray-Hopf solution of the Hall-MHD system associated with initial data* $(\boldsymbol{u}_0, \boldsymbol{B}_0)$. *Assume in addition that* $\boldsymbol{u}$ *and* $\nabla \times \boldsymbol{B}$ *are in* $L^4(0, T; \dot{H}^1(\mathbb{R}^3))$ *for some time* $T > 0$. *Then, all Leray-Hopf solutions associated with* $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ *coincide with* $(\boldsymbol{u}, \boldsymbol{B})$ *on the time interval* $[0, T]$.

In fact, for $\mu = \nu$, one can even prove well-posedness in any critical space $\dot{B}^{\frac{1}{2}}_{2,r}$ with $r \in [1, \infty]$ (see Appendix C). Then, the components of the solution will belong to the following space[9]:

$$E_{2,r}(T) := \left\{ z \in \widetilde{\mathcal{C}}_T(\dot{B}^{\frac{1}{2}}_{2,r}),\ \nabla^2 z \in \widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,r}) \quad \text{and} \quad \operatorname{div} z = 0 \right\},$$

[9] The reader may refer to Definition B.2.2 for the definition of 'tilde spaces'

where the letter T is omitted if the time interval is $\mathbb{R}_+$.

Theorem 2.1.9. *Assume that $\mu = \nu$. Consider initial data $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ in $\dot{B}^{\frac{1}{2}}_{2,r}$ with $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \boldsymbol{B}_0 = 0$ and $\boldsymbol{J}_0 := \nabla \times \boldsymbol{B}_0 \in \dot{B}^{\frac{1}{2}}_{2,r}$ for some $r \in [1, \infty]$. Then, the following results hold true:*

1. *there exists a universal positive constant c_2 such that, if*

$$\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,r}} + \|\boldsymbol{B}_0\|_{\dot{B}^{\frac{1}{2}}_{2,r}} + \|\boldsymbol{u}_0 - h\boldsymbol{J}_0\|_{\dot{B}^{\frac{1}{2}}_{2,r}} < c_2\mu, \tag{2.1.32}$$

   then the Hall-MHD system has a unique global solution $(\boldsymbol{u}, \boldsymbol{B})$ with $(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{J})$ in $E_{2,r}$.

2. *If only $\|\boldsymbol{u}_0 - h\boldsymbol{J}_0\|_{\dot{B}^{\frac{1}{2}}_{2,r}} < c_2\mu$, then there exists $T > 0$ such that the Hall-MHD system has a unique solution $(\boldsymbol{u}, \boldsymbol{B})$ on $[0, T]$, with $(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{J})$ in $E_{2,r}(T)$.*

Problem 4

As proposed by Chae and Lee in [13], we consider the $2\frac{1}{2}D$ flows for the Hall-MHD system, that is, $3D$ flows depending only on two space variables. This issue is well-known for the incompressible Navier-Stokes equations (see e.g. the book by Bertozzi and Majda [14]). In our case, the corresponding system in $\mathbb{R}^+ \times \mathbb{R}^2$ reads:

$$\partial_t \boldsymbol{u} + \widetilde{\boldsymbol{u}} \cdot \widetilde{\nabla} \boldsymbol{u} + \nabla \Pi = \widetilde{\boldsymbol{B}} \cdot \widetilde{\nabla} \boldsymbol{B} + \mu \widetilde{\Delta} \boldsymbol{u}, \tag{2.1.33}$$

$$\widetilde{\operatorname{div}}\, \widetilde{\boldsymbol{u}} = 0, \tag{2.1.34}$$

$$\partial_t \boldsymbol{B} + \widetilde{\boldsymbol{u}} \cdot \widetilde{\nabla} \boldsymbol{B} + h\widetilde{\boldsymbol{B}} \cdot \widetilde{\nabla} \boldsymbol{j} - h\widetilde{\boldsymbol{j}} \cdot \widetilde{\nabla} \boldsymbol{B} = \nu \widetilde{\Delta} \boldsymbol{B} + \widetilde{\boldsymbol{B}} \cdot \widetilde{\nabla} \boldsymbol{u}, \tag{2.1.35}$$

$$\widetilde{\operatorname{div}}\, \widetilde{\boldsymbol{B}} = 0, \tag{2.1.36}$$

$$\big(\boldsymbol{u}(0,x), \boldsymbol{B}(0,x)\big) = (\boldsymbol{u}_0, \boldsymbol{B}_0), \tag{2.1.37}$$

where the unknowns $\boldsymbol{u}$ and $\boldsymbol{B}$ are functions from $\mathbb{R}^+ \times \mathbb{R}^2$ to $\mathbb{R}^3$, $\widetilde{\boldsymbol{u}} := (u_1, u_2)$, $\widetilde{\boldsymbol{B}} := (B_1, B_2)$, $\widetilde{\nabla} := (\partial_1, \partial_2)$, $\nabla := (\partial_1, \partial_2, 0)$, $\widetilde{\operatorname{div}} := \widetilde{\nabla}\cdot$, $\widetilde{\Delta} := \partial_1^2 + \partial_2^2$ and $\boldsymbol{j} := \widetilde{\nabla} \times \boldsymbol{B} = (\partial_2 B_3, -\partial_1 B_3, \partial_1 B_2 - \partial_2 B_1)^{\mathrm{tr}}$.

A small modification of the proof of [28] allows to establish that for any initial data $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ in $L^2(\mathbb{R}^2; \mathbb{R}^3)$ with $\widetilde{\operatorname{div}}\, \widetilde{\boldsymbol{u}}_0 = \widetilde{\operatorname{div}}\, \widetilde{\boldsymbol{B}}_0 = 0$, there exists a global-in-time Leray-Hopf weak solution[10] $(\boldsymbol{u}, \boldsymbol{B})$ of (2.1.33)–(2.1.37) that satisfies:

$$\|\boldsymbol{u}(t)\|^2_{L^2(\mathbb{R}^2)} + \|\boldsymbol{B}(t)\|^2_{L^2(\mathbb{R}^2)} + 2\int_0^t \big(\mu \|\widetilde{\nabla} \boldsymbol{u}\|^2_{L^2(\mathbb{R}^2)} + \nu \|\widetilde{\nabla} \boldsymbol{B}\|^2_{L^2(\mathbb{R}^2)}\big)\, d\tau \leq \|\boldsymbol{u}_0\|^2_{L^2(\mathbb{R}^2)} + \|\boldsymbol{B}_0\|^2_{L^2(\mathbb{R}^2)}. \tag{2.1.38}$$

Whether that solution is unique and equality is true in (2.1.38) are open questions.

[10] In the $2\frac{1}{2}D$ setting, one needs to slightly adjust the Definition A.2.1.

The difficulty here is that, unlike for the $2\frac{1}{2}D$ Navier-Stokes equations or for the $2\frac{1}{2}D$ MHD flows with no Hall term, the two-dimensional system satisfied by the first two components of the flow is coupled with the equation satisfied by the third component, through the term $\widetilde{\boldsymbol{B}} \cdot \widetilde{\nabla} j - \widetilde{j} \cdot \widetilde{\nabla} \boldsymbol{B}$, thus hindering any attempt to prove the global well-posedness for any large data by means of classical arguments.

Our aim here is to take advantage of the special structure of the system so as to get a global well-posedness statement for $2\frac{1}{2}$D data such that *only the initial magnetic field is small*. Since it has been pointed out in [13] that controlling just $\boldsymbol{j}$ in the space $L^2(0,T;\mathrm{BMO}(\mathbb{R}^2))$ prevents blow-up of a smooth solution at time T and because the space $\dot{H}^1(\mathbb{R}^2)$ is continuously embedded in $\mathrm{BMO}(\mathbb{R}^2)$, it is natural to look for a control on $\boldsymbol{j}$ in the space $L^2(0,T;\dot{H}^1(\mathbb{R}^2))$ for all $T>0$.

Then, our idea is to look at the equation satisfied by the vorticity $\boldsymbol{\omega} := \widetilde{\nabla} \times \boldsymbol{u}$, namely

$$\partial_t \boldsymbol{\omega} + \widetilde{\nabla} \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \widetilde{\nabla} \times (\boldsymbol{j} \times \boldsymbol{B}) + \mu \widetilde{\Delta} \boldsymbol{\omega}.$$

In the case $\mu = \nu$, the vector-field $\boldsymbol{\Omega} := h\boldsymbol{\omega} + \boldsymbol{B}$ thus satisfies

$$\partial_t \boldsymbol{\Omega} - \widetilde{\boldsymbol{\Omega}} \cdot \widetilde{\nabla} \boldsymbol{u} + \widetilde{\boldsymbol{u}} \cdot \widetilde{\nabla} \boldsymbol{\Omega} = \mu \widetilde{\Delta} \boldsymbol{\Omega}. \tag{2.1.39}$$

From that identity and energy inequality (2.1.38), obvious energy estimates, one can get a global control of $\boldsymbol{\Omega}$ in the space $L^2(\mathbb{R}^+;\dot{H}^1(\mathbb{R}^2))$ and, finally, of $\boldsymbol{j}$ in $L^2(\mathbb{R}^+;\dot{H}^1(\mathbb{R}^2))$, provided the initial magnetic field is small enough in $H^1(\mathbb{R}^2)$. This leads to a global well-posedness statement, under the sole assumption that the initial magnetic field is small.

Theorem 2.1.10. *Assume that $\mu = \nu$. Let $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ be divergence free vector-fields with $\boldsymbol{u}_0$ in $L^2(\mathbb{R}^2;\mathbb{R}^3)$ and $\boldsymbol{B}_0$ in $H^1(\mathbb{R}^2;\mathbb{R}^3)$. There exists a constant $c_3 > 0$ such that, if*

$$\|\boldsymbol{u}_0\|_{L^2(\mathbb{R}^2)} + \|\boldsymbol{B}_0\|_{L^2(\mathbb{R}^2)} + \|\boldsymbol{u}_0 - h\widetilde{\nabla} \times \boldsymbol{B}_0\|_{L^2(\mathbb{R}^2)} < c_3 \mu,$$

then there exists a unique global solution $(\boldsymbol{u}, \boldsymbol{B})$ to the Cauchy problem (2.1.33)-(2.1.37), with $(\boldsymbol{u}, \boldsymbol{B}) \in \mathcal{C}_b(\mathbb{R}_+;L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}_+;\dot{H}^1(\mathbb{R}^2))$ and

$$\widetilde{\nabla} \boldsymbol{B} \in L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^2)) \quad \text{with} \quad \widetilde{\nabla}^2 \boldsymbol{B} \in L^2(\mathbb{R}_+;L^2(\mathbb{R}^2)).$$

If both $\boldsymbol{u}_0$ and $\boldsymbol{B}_0$ are in $H^1(\mathbb{R}^2)$, then there exists a constant c_4 depending only on the L^2 norm of $\boldsymbol{u}_0$, $\widetilde{\nabla} \boldsymbol{u}_0$, and on μ, h such that if

$$\|\boldsymbol{B}_0\|_{H^1(\mathbb{R}^2)} \leq c_4,$$

then there exists a unique global solution $(\boldsymbol{u}, \boldsymbol{B})$ to (2.1.33)-(2.1.37), with

$$(\boldsymbol{u}, \boldsymbol{B}) \in \mathcal{C}_b(\mathbb{R}_+;H^1(\mathbb{R}^2)) \quad \text{and} \quad (\widetilde{\nabla} \boldsymbol{u}, \widetilde{\nabla} \boldsymbol{B}) \in L^2(\mathbb{R}_+;H^1(\mathbb{R}^2)).$$

Remark 3. In fact, one can obtain a similar stability result for the $2\frac{1}{2}D$ flows as in the Theorem 2.1.4, by changing the estimates and spaces according to the dimension. Then we see that for any initial data $(\boldsymbol{u}_0, \boldsymbol{0}) \in \dot{B}^0_{2,1}(\mathbb{R}^2)$, it

will generate a global solution $(\boldsymbol{u}, \boldsymbol{0})$ for the system due to the theory of $2\frac{1}{2}D$ Navier-Stokes equation in [14]. Thus one can conclude that if $\|(\boldsymbol{B}_0, \widetilde{\nabla} \times \boldsymbol{B}_0)\|_{\dot{B}^0_{2,1}}$ is small enough then $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ also generates a unique global solution. This gives a global well-posedness result with large velocity and small magnetic field in the critical spaces.

2.2 Mathematical study of a non-local equation with nonlinear diffusion

The second part of this thesis is focused on the well-posedness, global regularity and long-time asymptotics of a non-local Burgers equation with non-linear diffusion.

2.2.1 Motivations

In the book [15], Lemarié-Rieusset proposed the following model

$$\partial_t u + u|\nabla| u - |\nabla|(u^2) = \nu \Delta u, \quad x \in \mathbb{R}^d \text{ or } \mathbb{T}^d$$

as a scalar case study of the Navier- Stokes equations (2.1.10), where $|\nabla| = (-\Delta)^{1/2}$ denotes the square root of the Laplacian and has the symbol $|\xi|$.

The works of Lelièvre [48, 49, 50] presented the construction of global Kato-type mild solutions for initial data in $L^3(\mathbb{R}^3)$ and of global weak Leray-Hopf type solutions for initial data in $L^2(\mathbb{R}^3)$ and so on. A local energy inequality obtained for this model was suggestive of possible uniqueness for small initial data in critical spaces, in a similar fashion to 3D Navier-Stokes equations stated in e.g. [45].

Very recently, the works of Imbert, Shvydkoy and Vigneron [16, 17] have focused on the model without viscosity (note the opposite signs),

$$\partial_t u - u|\nabla| u + |\nabla|(u^2) = 0, \quad \text{i.e.} \quad \partial_t u = [u, |\nabla|] u. \tag{2.2.1}$$

In [16], global classical solutions starting from smooth positive data, and global weak solutions starting from data in L^∞ are constructed. In [17], the authors established Schauder estimates for a general integro-differential equations which can be applied to (2.2.1). Indeed the above model bears a resemblance to some of the "even" inviscid cases. For example, dropping the $1/2$ factor, the Burgers equation can be written in the form of a commutator:

$$\partial_t u = [u, \partial_x] u.$$

Thus the model (2.2.1) can be seen as replacing ∂_x with the non-local operator $|\nabla|$ of the same order. The classical incompressible Euler equation is given by

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = 0,$$

where P is the associated pressure given by $P = \mathcal{N}(\boldsymbol{u} \otimes \boldsymbol{u}) + local$, where $\mathcal{N}$ is a singular integral operator with an even symbol. We thus can draw an analogy between terms: $\boldsymbol{u} \cdot \nabla \boldsymbol{u} \sim -u|\nabla| u$ and $\nabla P \sim |\nabla|(u^2)$. Because of

the formal analogies with Euler or Burgers equation, the model (2.2.1) was named the *Non-local Burgers equation*.

We are motivated to study the following *Generalized Non-local Burgers type equation*:

$$\partial_t u = [F(u), |\nabla|^s]u \quad x \in \mathbb{R}^d \text{ or } \mathbb{T}^d, \tag{2.2.2}$$

with some initial conditions

$$u(0,x) = u_0, \tag{2.2.3}$$

where $s \in (0,1]$ and $|\nabla|^s = (-\Delta)^{s/2}$ denotes the fractional Laplacian. The function F is given and assumed to be $\mathcal{C}^\infty_{\text{loc}}(\mathbb{R})$ smooth with $F' > 0$ a.e. on $(0,+\infty)$. Such a function can be chosen like $F(u) = u^n$ with $n > 0$ or $F(u) = u - \sin u$, e^u, $\ln(1+u)$ and so on. In the sequel, we assume that $F(0) = 0$ since (2.2.2) is invariant by replacing $F(u)$ to $F(u) - F(0)$.

We would like to mention that our model (2.2.2) bears some semblance to following general non-local equation:

$$\partial_t u + |\nabla|^s \varphi(u) = f, \quad \text{in } (0,+\infty) \times \mathbb{R}^d, \tag{2.2.4}$$

which occurs in the boundary heat control, as already mentioned by Athanasopoulos and Caffarelli [18], who refer to the model formulated in the book by Duvaut and Lions [19]. Based on some suitable assumptions on the nonlinearity φ, the existence, uniqueness and regularity properties of weak solutions[11] on the homogeneous case ($f \equiv 0$) have been fully investigated in Vázquez *et al.* [20]. See also [21, 22] for porous medium equation.

Precisely, Vázquez *et al.* [20] proved that

Theorem 2.2.1. *Let $f \equiv 0$. Let $\varphi \in \mathcal{C}(\mathbb{R})$ be nondecreasing. Given $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique bounded weak solution to the problem (2.2.4)-(2.2.3). If in additional, $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ and $\varphi' > 0$ in $\mathbb{R}$, then $u \in \mathcal{C}^\infty\big((0,+\infty) \times \mathbb{R}^d\big)$.*

Their idea of the proof is to apply the results of Athanasopoulos and Caffarelli [18], that ensure that bounded weak solutions are Hölder continuous. In order to improve the regularity they rewrite the equation as a fractional linear heat equation with a source term, which has good properties.

Let us recall that, in $\mathbb{R}^d$, the operator $|\nabla|^s$ enjoys a singular integral representation[12]:

$$|\nabla|^s f := \text{p.v.} \int_{\mathbb{R}^d} \big(f(x) - f(y)\big) K^s(x-y)\, dy$$

with the kernel $K^s(z) = \dfrac{c_{d,s}}{|z|^{d+s}}$, where $c_{d,s}$ is a constant depending on dimension d and s. See [51] for more equivalent definitions of the fractional Laplace

[11] The reader may refer to the Definition A.2.2 for the concept of weak solutions for (2.2.4).

[12] The principal-value distribution is defined by $\lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(x)} \big(f(x) - f(y)\big) K^s(x-y)\, dy$.

operator. The model (2.2.2) can thus be rewritten in an integral form:

$$\partial_t u = \text{p.v.} \int_{\mathbb{R}^d} \Big(F(u(t,y)) - F(u(t,x)) \Big) u(t,y) K^s(x-y)\, dy. \tag{2.2.5}$$

If u is periodic with period 2π in all coordinates, the representation above can alternatively be written as

$$\partial_t u = \text{p.v.} \int_{\mathbb{T}^d} \Big(F(u(t,y)) - F(u(t,x)) \Big) u(t,y) K^s_{\text{per}}(x-y)\, dy,$$

where $\mathbb{T}^d$ is the torus and $K^s_{\text{per}}(z) = \sum_{j\in\mathbb{Z}^d} \frac{c_{d,s}}{|z+2\pi j|^{d+s}}$. For periodic solutions, both representations are valid due to a sufficient decay of K^s at infinity, while the former is more amenable to an analytical study due to the explicit nature of the kernel and applicability of known results, we shall consider that (2.2.5) holds on $(0,+\infty)\times\mathbb{R}^d$ with u periodic.

Formally, the following basic structural properties of the model (2.2.2) can be readily obtained from either representation.

- Translation invariance: if $t_0 > 0$, $x_0 \in \mathbb{R}^d$ then $u(t+t_0, x+x_0)$ is another solution. In particular, the periodicity of the initial condition is preserved.
- Time reversibility: if $t_0 > 0$, then $-u(t_0 - t, x)$ is a solution too.
- Max / Min principle: if $u > 0$, then its maximum is decreasing and its minimum is increasing.
- Energy conservation: $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ is obtained by testing (2.2.5) with u.
- Higher power law: for any $p \in (2,\infty)$ the following quantity is conserved:

$$\|u(t,\cdot)\|^p_{L^p(\mathbb{T}^d)} + \frac{p}{2}\int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} (|u(\tau,y)|^{p-2} - |u(\tau,x)|^{p-2}) \\ \cdot \big(F(u(\tau,y)) - F(u(\tau,x))\big) u(\tau,x) u(\tau,y) K^s_{\text{per}}(x-y)\, dx\, dy\, d\tau.$$

Inspired by [16, 17], our goal is to develop a well-posedness theory for the model (2.2.2) and study its long-time behaviour. We blend classical techniques relevant to the Euler equation, such as energy estimates, a Beale-Kato-Majda criterion, etc [14], with recently developed tools of regularity theory for parabolic integro-differential equations [25, 52, 53, 54, 55]. Let us point out that all the results are proved in the **periodic setting**, except local existence, which holds in both the periodic and the open case. Periodicity provides extra compactness of the underlying domain, which for positive data, due to the minimum principle, warrants uniform support from below in space and time, which in turn further entails uniform ellipticity of the right-hand side of (2.2.5).

2.2.2 Main results

Let us give a brief summary of our results, where these results are obtained under the supervision of Professor François Vigneron.

Local existence with a BKM criterion

For any initial data $u_0 \in H^m(\Omega^d)$ on $\Omega^d = \mathbb{R}^d$ or $\mathbb{T}^d$, with $u_0 > 0$ pointwise and $m > \frac{d}{2} + 1$, there exists a unique local solution in $C([0,T); H^m(\Omega^d)) \cap C^1([0,T); H^{m-1}(\Omega^d))$. Even for this local existence result, the positivity of the initial data seems essential.

Theorem 2.2.2. *Let $m > \frac{d}{2} + 1$ be an integer. Given a pointwise positive initial data $u_0 \in H^m(\Omega^d)$, then there exists a time $T > 0$ such that there exists a unique local solution $u \in \mathcal{C}([0,T); H^m(\Omega^d)) \cap \mathcal{C}^1([0,T); H^{m-1}(\Omega^d))$ to the Cauchy problem (2.2.2)-(2.2.3). Moreover, $u(t,x) > 0$ for all $(t,x) \in ([0,T) \times \Omega^d$, and the maximum $\max_{x\in\Omega^d} u(t,x)$ is strictly decreasing in time.*

The proof is based on a smoothing scheme with a regularization of the kernel. In the case of $\Omega^d = \mathbb{R}^d$ the proof requires slightly more technical care to ensure the maximum principle, while being similar in the rest of the argument. However, in the case of $\Omega = \mathbb{T}^d$, we will additionally obtain a complementary statement for the minimum: $\min_{x\in\mathbb{T}^d} u(t,x)$ is a strictly increasing function of time, thus the amplitude $A(t) := \max_{x\in\mathbb{T}^d} u(t,x) - \min_{x\in\mathbb{T}^d} u(t,x)$ is shrinking. In Lemma 2.2.7 we elaborate much more on the asymptotic behaviour of the amplitude.

We also have a blow-up criterion of Beale-Kato-Majda type:

Theorem 2.2.3. *Suppose*

$$u \in \mathcal{C}([0,T); H^m(\Omega)) \cap \mathcal{C}^1([0,T); H^{m-1}(\Omega))$$

is a positive solution to (2.2.2), where $m > \frac{d}{2} + 1$. Suppose also that

$$\int_0^T \|\nabla u(t)\|_{L^\infty}\, dt < \infty. \tag{2.2.6}$$

Then u can be extended beyond the time T in the same regularity class.

Instant regularization and global well-posedness

Any positive classical solution to (2.2.2) on a time interval $[0,T)$ satisfies the following bounds: for any $k \in \mathbb{N}$, for any $0 < t_0 < T$

$$\|\partial_t^k u, D_x^k u\|_{L^\infty_{t,x}((t_0,T)\times\mathbb{T}^d)} \leq C(d,s,k,t_0,T,\min u_0, \max u_0). \tag{2.2.7}$$

Theorem 2.2.4. *Let $s_0 \in (0,1], s_0 \leq s \leq 1$. Given a pointwise positive initial data $u_0 \in H^m(\mathbb{T}^d)$ and $m > \frac{d}{2} + 1$ an integer. The solution of problem (2.2.2)-(2.2.3) obtained from Theorem 2.2.2 exists globally in time. Furthermore, the solution is regularized instantly and satisfies the bounds (2.2.7).*

To achieve this we symmetrize the right-hand side of (2.2.5) by multiplying it by $2u$. One uses also:

$$F(u(t,y)) - F(u(t,x)) = (u(t,y) - u(t,x)) \int_0^1 F'\big((1-\lambda)u(x) + \lambda u(y)\big)\, d\lambda,$$

the evolution equation for $w = u^2$ thus becomes:

$$\partial_t w = \text{p.v.} \int_{\mathbb{R}^d} (w(y) - w(x))\, \mathcal{K}^s(t,x,y)\, dy \tag{2.2.8}$$

with

$$\mathcal{K}^s(t,x,y) = \frac{c_{d,s}}{|x-y|^{d+s}} \frac{2u(x)u(y)}{u(x)+u(y)} \int_0^1 F'\big((1-\lambda)u(x) + \lambda u(y)\big)\, d\lambda. \tag{2.2.9}$$

The active kernel $\mathcal{K}^s$ is symmetric and satisfies uniform ellipticity bound $\frac{\Lambda^{-1}}{|x-y|^{d+s}} \leq \mathcal{K}^s \leq \frac{\Lambda}{|x-y|^{d+s}}$ for some $\Lambda > 0$. This puts the model within the scope of recent results of Kassmann et al. [23, 24] and of Caffarelli-Chan-Vasseur [25]. De Giorgi-Nash-Moser techniques yield initial Hölder regularity for w and hence for u by positivity. We then follow the idea of [17] to get Schauder estimates for parabolic integro-differential equations with a general fractional kernel (see also [54, 55]). Finally, we obtain the bounds (2.2.7). It readily follows from the Theorem 2.2.3 and the instant regularization property that the solution exists globally in time.

Global existence of weak solutions

Since the bounds (2.2.7) depend essentially only on the L^∞ norm of the initial condition, we can construct global smooth approximate solutions by smoothing out any initial data $u_0 \in L^\infty(\mathbb{T}^d)$, $u_0 > 0$. Moreover, these solutions have an a priori bound in the space $L^\infty(\mathbb{R}_+ \times \mathbb{T}^d) \cap L^2(\mathbb{R}_+; \dot{H}^{s/2}(\mathbb{T}^d))$, which enables us to obtain compactness by applying the well-known Aubin-Lions lemma.

Theorem 2.2.5. *Let $s \in [s_0, 1]$ with $s_0 \in (0,1]$. For any initial data $u_0 \in L^\infty(\mathbb{T}^d)$, $u_0 > 0$, there exists a global weak solution to (2.2.2) in the class*

$$L^\infty(\mathbb{R}_+ \times \mathbb{T}^d) \cap L^2(\mathbb{R}_+; \dot{H}^{s/2}(\mathbb{T}^d)) \cap \mathcal{C}(\mathbb{R}_+; L^2(\mathbb{T}^d)).$$

The energy is conserved, the momentum $\int_{\mathbb{T}^d} u(t,x)\, dx$ is continuous on $\mathbb{R}_+$ and satisfies

$$\int_{\mathbb{T}^d} u(t',x)\, dx - \int_{\mathbb{T}^d} u(t,x)\, dx = \int_t^{t'} \int_{\mathbb{T}^d} F(u(\tau,x)) |\nabla|^s u(\tau,x)\, dx\, d\tau.$$

Furthermore, for all $t > 0$, solution u satisfies the instant regularization estimates (2.2.7) and the original equation (2.2.2) is satisfied in the classical sense.

Remark 4. The continuity of the momentum at $t = 0$ prevents any concentration of the $\dot{H}^{s/2}$ norm in our weak solutions.

As a corollary, we show that some negative smooth initial data could develop singularity in finite time.

Corollary 2.2.6. *For any $t^* > 0$, there exists a negative initial condition $u_0 \in \mathcal{C}^\infty(\mathbb{T}^d)$ and there exists a classical solution to (2.2.2) om $[0, t^*)$, which is discontinuous at time t^* i.e. $u(t^*) \in L^\infty(\mathbb{T}^d)/\mathcal{C}(\mathbb{T}^d)$.*

Long-time asymptotics

As $t \to \infty$, any weak solution to (2.2.2) converges to a constant, namely $\frac{1}{\sqrt{|\mathbb{T}^d|}}\|u_0\|_{L^2(\mathbb{T}^d)}$, in the following strong sense: the amplitude of $u(t)$ and the semi-norm $\|\nabla u(t)\|_{L^\infty}$ tend to 0 exponentially fast.

Lemma 2.2.7. *Let $u_0 \in L^\infty(\mathbb{T}^d), u_0 > 0$. Suppose that u is a weak solution correspond to u_0 obtained from Theorem 2.2.5. Then the amplitude $A(t)$ decays to zero exponentially fast for all $t > 0$.*

Theorem 2.2.8. *Let $u_0 \in L^\infty(\mathbb{T}^d), u_0 > 0$. Suppose that u is a weak solution corresponding to u_0 obtained from Theorem 2.2.5. Then there exists a time $\widetilde{T}$ that depends only on $s, d, \bar{u}(0), \underline{u}(0)$ and the maximal and minimal values of F', F'' on the interval $[\underline{u}(0), \bar{u}(0)]$ such that $\|\nabla u(t, \cdot)\|_{L^\infty}$ decays to zero exponentially on $[\widetilde{T}, \infty)$.*

Finally, concerning stability, we have the following two estimates.

Corollary 2.2.9. *Let F_1, F_2 be two functions that satisfy our assumptions on the function F. Given two pointwise positive initial data $u_{0,1}, u_{0,2} \in H^m(\mathbb{T}^d)$. For $i = 1, 2$, suppose that u_i is the solution of Cauchy problem (2.2.2)-(2.2.3) with F_i and initial data $u_{0,i}$, respectively. Then we have the following stability estimates:*

$$\|u_1 - u_2\|_{L^2(\mathbb{T}^d)} \leq \left(\|u_{0,1} - u_{0,2}\|_{L^2(\mathbb{T}^d)} + \|F_1' - F_2'\|_{L^\infty(\mathbb{T}^d)}\right) e^{C_1 t}, \ \forall\, t > 0$$

with a constant C_1 that depends on $d, \|u_1\|_{H^m}, \|u_2\|_{H^m}$, and also

$$\begin{aligned}\|u_1 - u_2\|_{L^\infty(\mathbb{T}^d)} \leq & 2\sqrt{d}\pi\left(\|\nabla u_1(t)\|_{L^\infty(\mathbb{T}^d)} + \|\nabla u_2(t)\|_{L^\infty(\mathbb{T}^d)}\right) \\ & + \frac{1}{\sqrt{\mathbb{T}^d}}\|u_{0,1} - u_{0,2}\|_{L^2(\mathbb{T}^d)}.\end{aligned}$$

Chapter 3

On the well-posedness of the Hall-MHD system in critical spaces

3.1 Global existence and time-decay estimates for small data in the spaces $\dot{B}^{\frac{3}{p}}_{p,1}$ with general $1 \leq p < \infty$

The present section is dedicated to proving Th. 2.1.2. Before starting, a fundamental observation (that will be also used in the next sections) is in order: the triplet $(\boldsymbol{u}, \boldsymbol{B}, P)$ satisfies the Hall-MHD system (2.1.1)-(2.1.4) with coefficients (μ, ν, h) if and only if the *rescaled* triplet:

$$(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{P})(t,x) := \frac{h}{\mu}\Big(\boldsymbol{u}, \boldsymbol{B}, \frac{h}{\mu}P\Big)\Big(\frac{h^2}{\mu}t, hx\Big) \tag{3.1.1}$$

satisfies (2.1.1)-(2.1.3) with coefficients $(1, 1, \nu/\mu)$.

Consequently, taking advantage of the scaling invariance of the homogeneous Besov norms (see Proposition B.1.3 (vi)), it is enough to prove the statement in the case where the viscosity μ and the Hall number h are equal to 1.

For expository purpose, we shall assume in addition that the magnetic resistivity ν is equal to 1 (to achieve the general case it is only a matter of changing the heat semi-group in the definition of $\mathcal{B}$ in (3.1.3) below accordingly).

Throughout this section and the following ones, we shall repeatedly use the fact that, as a consequence of Proposition B.1.3 (vii), one has the following equivalence of norms for all $s \in \mathbb{R}$ and $(p,r) \in [1,+\infty]^2$:

$$\|\nabla \boldsymbol{B}\|_{\dot{B}^s_{p,r}} \sim \|\nabla \times \boldsymbol{B}\|_{\dot{B}^s_{p,r}} \quad \text{and} \quad \|\nabla \boldsymbol{B}\|_{\dot{H}^s} = \|\nabla \times \boldsymbol{B}\|_{\dot{H}^s}. \tag{3.1.2}$$

3.1.1 Global existence for small data

In order to establish the global existence of a solution of the Hall-MHD system in the case of small data, we shall first prove the corresponding result for the extended system (2.1.22). It relies on the following well known corollary of the fixed point theorem in complete metric spaces.

Lemma 3.1.1. *Let $(X, \|\cdot\|_X)$ be a Banach space and $\mathcal{B} : X \times X \to X$, a bilinear continuous operator with norm K. Then, for all $\boldsymbol{y} \in X$ such that $4K\|\boldsymbol{y}\|_X < 1$,*

equation

$$x = y + \mathcal{B}(x, x)$$

has a unique solution x *in the ball* $\mathcal{B}(0, \frac{1}{2K})$. *Besides,* x *satisfies* $\|x\|_X \leq 2\|y\|_X$.

We shall take for X the set of triplets of (time dependent) divergence free vector-fields with components in E_p endowed with the norm

$$\|V\|_X := \|V\|_{L^1(\dot{B}^{\frac{3}{p}+1}_{p,1})} + \|V\|_{L^\infty(\dot{B}^{\frac{3}{p}-1}_{p,1})}.$$

Let $(e^{t\Delta})_{t\geq 0}$ denote the heat semi-group defined in (B.2.2). We set $y : t \mapsto e^{t\Delta}U_0$ and define the bilinear functional $\mathcal{B}$ by the formula

$$\mathcal{B}(V, W)(t) = \int_0^t e^{(t-\tau)\Delta} Q(V, W)\, d\tau. \tag{3.1.3}$$

By virtue of Duhamel's formula (B.2.1), System (2.1.22) recasts in

$$U(t) = y(t) + \mathcal{B}(U, U)(t). \tag{3.1.4}$$

In order to apply Lemma 3.1.1, it suffices to show that y is small in X, and that $\mathcal{B}$ maps $X \times X$ to X. The former property holds true if Condition (2.1.14) is fulfilled for a small enough $c_0 > 0$, as Proposition B.2.3 ensures that y belongs to X and that

$$\|y\|_X \leq C\|U_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}.$$

In order to prove the latter property, one can use the fact that, by virtue of Identity (2.1.18), Proposition B.1.3 (i), (iii), (vii), and Inequality (B.3.2), we have

$$\begin{aligned} \|\mathrm{div}(v \otimes w)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} &\lesssim \|v \otimes w\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|v\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|w\|_{\dot{B}^{\frac{3}{p}}_{p,1}}, \end{aligned} \tag{3.1.5}$$

$$\begin{aligned} \|\mathrm{div}((\mathrm{curl}^{-1} v) \otimes w)\|_{\dot{B}^{\frac{3}{p}}_{p,1}} &= \|w \cdot \nabla(\mathrm{curl}^{-1} v)\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|\nabla \mathrm{curl}^{-1} v\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|w\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|v\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|w\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \\ &\lesssim \|v\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|w\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|v\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \|w\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}+1}_{p,1}}, \end{aligned} \tag{3.1.6}$$

and, since $\mathrm{div}(\mathrm{curl}^{-1} v) = 0$, owing to Proposition B.1.3 (viii),

$$\|\mathrm{div}(w \otimes (\mathrm{curl}^{-1} v))\|_{\dot{B}^{\frac{3}{p}}_{p,1}} = \|(\mathrm{curl}^{-1} v) \cdot \nabla w\|_{\dot{B}^{\frac{3}{p}}_{p,1}}$$

$$\lesssim \|\mathrm{curl}^{-1}\boldsymbol{v}\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|\nabla \boldsymbol{w}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}$$
$$\lesssim \|\boldsymbol{v}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|\boldsymbol{w}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}. \tag{3.1.7}$$

Hence, integrating on $\mathbb{R}_+$ and observing that the Leray projector $\mathcal{P}$ maps $\dot{B}^{\frac{3}{p}}_{p,1}$ to itself according to Proposition B.1.3 (vii), we get

$$\|Q_a(\boldsymbol{v},\boldsymbol{w})\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})} \lesssim \|\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{w}) + \mathrm{div}(\boldsymbol{w}\otimes\boldsymbol{v})\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})}$$
$$\lesssim \|\boldsymbol{v}\|_X \|\boldsymbol{w}\|_X, \tag{3.1.8}$$

$$\|Q_b(\boldsymbol{v},\boldsymbol{w})\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})} = \|\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{w}) - \mathrm{div}(\boldsymbol{w}\otimes\boldsymbol{v})\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})}$$
$$\lesssim \|\boldsymbol{v}\|_X \|\boldsymbol{w}\|_X, \tag{3.1.9}$$

$$\|\nabla\times Q_b(\mathrm{curl}^{-1}\boldsymbol{v},\boldsymbol{w})\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})} \lesssim \|Q_b(\mathrm{curl}^{-1}\boldsymbol{v},\boldsymbol{w})\|_{L^1(\dot{B}^{\frac{3}{p}}_{p,1})}$$
$$\lesssim \|\mathrm{div}((\mathrm{curl}^{-1}\boldsymbol{v})\otimes\boldsymbol{w})\|_{L^1(\dot{B}^{\frac{3}{p}}_{p,1})} + \|\mathrm{div}(\boldsymbol{w}\otimes(\mathrm{curl}^{-1}\boldsymbol{v}))\|_{L^1(\dot{B}^{\frac{3}{p}}_{p,1})}$$
$$\lesssim \|\boldsymbol{v}\|_X \|\boldsymbol{w}\|_X. \tag{3.1.10}$$

Now, by definition of $\mathcal{B}(\boldsymbol{V},\boldsymbol{W})$, we have

$$\begin{cases} \partial_t \mathcal{B}(\boldsymbol{V},\boldsymbol{W}) - \Delta\mathcal{B}(\boldsymbol{V},\boldsymbol{W}) = Q(\boldsymbol{V},\boldsymbol{W}), \\ \mathcal{B}(\boldsymbol{V},\boldsymbol{W})|_{t=0} = 0. \end{cases}$$

Hence, by Proposition B.2.3 and the definition of Q in (2.1.4), we get

$$\|\mathcal{B}(\boldsymbol{V},\boldsymbol{W})\|_X \lesssim \|Q_a(\boldsymbol{V}_2,\boldsymbol{W}_2) - Q_a(\boldsymbol{V}_1,\boldsymbol{W}_1)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})}$$
$$+\|Q_b(\boldsymbol{V}_2,\boldsymbol{W}_3-\boldsymbol{W}_1)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\nabla\times Q_b(\mathrm{curl}^{-1}\boldsymbol{V}_3,\boldsymbol{W}_3-\boldsymbol{W}_1)\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})}.$$

Remembering (3.1.8)-(3.1.10), one can conclude that $\mathcal{B}$ maps $X\times X$ to X. Hence, System (2.1.22) has a global solution (u,B,J) in X.

For completing the proof of the global existence for the original Hall-MHD system, we have to show that if $\boldsymbol{J}_0 = \nabla\times\boldsymbol{B}_0$, then $\boldsymbol{J} = \nabla\times\boldsymbol{B}$ so that $(\boldsymbol{u},\boldsymbol{B})$ is indeed a distributional solution of (2.1.1)–(2.1.5). Now, we have

$$(\partial_t - \Delta)(\nabla\times\boldsymbol{B} - \boldsymbol{J}) = \nabla\times Q_b(\mathrm{curl}^{-1}(\nabla\times\boldsymbol{B}-\boldsymbol{J}), \boldsymbol{J}-\boldsymbol{u}).$$

Remember that $\boldsymbol{J}-\boldsymbol{u}$ belongs to $L^2(\dot{B}^{\frac{3}{p}}_{p,1})$ (use interpolation for the space regularity and Hölder inequality for the time variable), that $\boldsymbol{J}$ and $\boldsymbol{u}$ are in $L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$ for all $T>0$ since they are in $L^\infty(\dot{B}^{\frac{3}{p}-1}_{p,1})$ and observe that $\nabla\times\boldsymbol{B}$ is

in $L^2(\dot{B}^{\frac{3}{p}-1}_{p,1})$. Therefore, from the definition of Q_b, the properties of continuity of operator curl $^{-1}$, and product laws, we gather that $\nabla\times Q_b(\mathrm{curl}^{-1}(\nabla\times \boldsymbol{B}-\boldsymbol{J}),\boldsymbol{J}-\boldsymbol{u})$ is in $L^1_T(\dot{B}^{\frac{3}{p}-2}_{p,1})$ for all $T>0$. Because $(\nabla\times\boldsymbol{B}-\boldsymbol{J})|_{t=0}=0$, Proposition B.2.3 thus guarantees that $\nabla\times\boldsymbol{B}-\boldsymbol{J}$ is in $\mathcal{C}([0,T];\dot{B}^{\frac{3}{p}-2}_{p,1})$ for all $T>0$. Furthermore, we have

$$\|(\nabla\times\boldsymbol{B}-\boldsymbol{J})(T)\|_{\dot{B}^{\frac{3}{p}-2}_{p,1}}+\int_0^T\|\nabla\times\boldsymbol{B}-\boldsymbol{J}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\,dt \leq C\int_0^T\|\boldsymbol{J}-\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\|\nabla\times\boldsymbol{B}-\boldsymbol{J}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\,dt.$$

The right-hand side may be handled by means of an interpolation inequality:

$$\|\boldsymbol{J}-\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\|\nabla\times\boldsymbol{B}-\boldsymbol{J}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\leq\frac{1}{2}\|\nabla\times\boldsymbol{B}-\boldsymbol{J}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}+C\|\boldsymbol{J}-\boldsymbol{u}\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}}\|\nabla\times\boldsymbol{B}-\boldsymbol{J}\|_{\dot{B}^{\frac{3}{p}-2}_{p,1}}.$$

Hence, using Gronwall's lemma ensures that $\|(\nabla\times\boldsymbol{B}-\boldsymbol{J})(t)\|_{\dot{B}^{\frac{3}{p}-2}_{p,1}}=0$, whence $\nabla\times\boldsymbol{B}-\boldsymbol{J}\equiv 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}^3$. This yields the existence part of Theorem 2.1.2 in the small data case.

3.1.2 Local existence for small current

Let us explain how the above arguments have to be modified so as to prove local existence in the case where only $\boldsymbol{J}_0$ is small. The idea is to control the existence time according to the solution $\boldsymbol{U}^L$ of the heat equation:

$$\begin{cases}\partial_t\boldsymbol{U}^L-\Delta\boldsymbol{U}^L=0,\\ \boldsymbol{U}^L|_{t=0}=U_0.\end{cases}$$

By Proposition B.2.3, we have

$$\|\boldsymbol{J}^L\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\leq C\|\boldsymbol{J}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}, \tag{3.1.11}$$

and, using also the dominated convergence theorem yields

$$\lim_{T\to 0}\|\boldsymbol{U}^L\|_{L^\varrho_T(\dot{B}^{\frac{3}{p}+\frac{2}{\varrho}-1}_{p,1})}=0,\quad\text{whenever }1\leq\varrho<\infty.$$

Clearly, $\boldsymbol{U}$ is a solution of (2.1.22) on $[0,T]\times\mathbb{R}^3$ with data $\boldsymbol{U}_0$ if and only if

$$\boldsymbol{U}:=\boldsymbol{U}^L+\widetilde{\boldsymbol{U}}$$

with, for all $t\in[0,T]$,

$$\widetilde{\boldsymbol{U}}(t):=\int_0^t e^{(t-\tau)\Delta}(Q(\widetilde{\boldsymbol{U}},\boldsymbol{U}^L)+Q(\boldsymbol{U}^L,\widetilde{\boldsymbol{U}})+Q(\widetilde{\boldsymbol{U}},\widetilde{\boldsymbol{U}})+Q(\boldsymbol{U}^L,\boldsymbol{U}^L))\,d\tau.$$

Then, proving local existence relies on the following generalization of Lemma 3.1.1.

Lemma 3.1.2. *Let $(X, \|\cdot\|_X)$ be a Banach space, $\mathcal{B} : X \times X \to X$, a bilinear continuous operator with norm K and $\mathcal{L} : X \to X$, a continuous linear operator with norm $M < 1$. Let $\boldsymbol{y} \in X$ satisfy $4K\|\boldsymbol{y}\|_X < (1-M)^2$. Then, equation*

$$\boldsymbol{x} = \boldsymbol{y} + \mathcal{L}(\boldsymbol{x}) + \mathcal{B}(\boldsymbol{x}, \boldsymbol{x})$$

has a unique solution $\boldsymbol{x}$ in the ball $B(0, \frac{1-M}{2K})$.

Take $\mathcal{B}$ as in (3.1.3), set $\boldsymbol{y} := \mathcal{B}(\boldsymbol{U}^L, \boldsymbol{U}^L)$ and define the linear map $\mathcal{L}$ by

$$\mathcal{L}(\boldsymbol{V}) := \mathcal{B}(\boldsymbol{V}, \boldsymbol{U}^L) + \mathcal{B}(\boldsymbol{U}^L, \boldsymbol{V}). \tag{3.1.12}$$

Our problem recasts in

$$\widetilde{\boldsymbol{U}} = \boldsymbol{y} + \mathcal{L}(\widetilde{\boldsymbol{U}}) + \mathcal{B}(\widetilde{\boldsymbol{U}}, \widetilde{\boldsymbol{U}}).$$

For X, we now take the space (denoted by X_T) of triplets of divergence free vector-fields with components in $E_p(T)$. Then, arguing as for getting (3.1.5), (3.1.6), integrating on $[0,T]$ we get

$$\|\mathrm{div}\,(\boldsymbol{v}\otimes\boldsymbol{w})\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\mathrm{div}\,(\mathrm{curl}^{-1}\boldsymbol{v}\otimes\boldsymbol{w})\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \lesssim \|\boldsymbol{v}\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})}\|\boldsymbol{w}\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})}.$$

Hence, using also (3.1.7) and the definition of $\mathcal{B}(\boldsymbol{V},\boldsymbol{W})$, we end up with

$$\begin{aligned}\|\mathcal{B}(\boldsymbol{V},\boldsymbol{W})\|_{X_T} \lesssim{}& \|\boldsymbol{V}\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})}\|\boldsymbol{W}\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})} \\ &+ (\|\boldsymbol{W}_1\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})} + \|\boldsymbol{W}_3\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})})\|\boldsymbol{V}_3\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}.\end{aligned} \tag{3.1.13}$$

For justifying that $\mathcal{L}$ defined in (3.1.12) is indeed a continuous linear operator on X_T *with small norm if* $T \to 0$, the troublemakers in the right-hand side of (3.1.13) are

$$\|\widetilde{\boldsymbol{u}}\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})}\|\boldsymbol{J}^L\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \quad\text{and}\quad \|\widetilde{\boldsymbol{J}}\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})}\|\boldsymbol{J}^L\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}$$

since, for large $\boldsymbol{J}_0$, the term $\|\boldsymbol{J}^L\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}$ need not to be small. One thus have to assume that $\|\boldsymbol{J}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}$ is small so as to guarantee that the norm of $\mathcal{L}$ is smaller than 1 for T small enough. Then, one can conclude thanks to Lemma 3.1.2, to the local-in-time existence statement of Theorem 2.1.2.

3.1.3 Uniqueness

To prove the uniqueness part of Theorem 2.1.2. Consider two solutions $(\boldsymbol{u}^1, \boldsymbol{B}^1)$ and $(\boldsymbol{u}^2, \boldsymbol{B}^2)$ of (2.1.1)–(2.1.4) emanating from the same data, and denote by $\boldsymbol{U}^1$ and $\boldsymbol{U}^2$ the corresponding solutions of the extended system

(2.1.22). Since one can take (with no loss of generality) for $\boldsymbol{u}^2$ the solution built previously, and as $\|\boldsymbol{J}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \leq c_0$ is assumed, we have

$$\|\boldsymbol{J}^2\|_{X_T} \leq 2c_0. \tag{3.1.14}$$

Denoting $\delta\boldsymbol{u} := \boldsymbol{u}^2 - \boldsymbol{u}^1$, we find that $\delta\boldsymbol{u}$ satisfies

$$\partial_t \delta\boldsymbol{u} - \Delta\delta\boldsymbol{u} = Q(\boldsymbol{u}^2, \delta\boldsymbol{u}) + Q(\delta\boldsymbol{u}, \boldsymbol{u}^1)$$

with $\delta\boldsymbol{u}|_{t=0}$=0, and thus

$$\delta\boldsymbol{u} = \mathcal{B}(\boldsymbol{u}^2, \delta\boldsymbol{u}) + \mathcal{B}(\delta\,\boldsymbol{u}, \boldsymbol{u}^1).$$

Arguing as in the proof of (3.1.13) yields

$$\begin{aligned}\|\mathcal{B}(\boldsymbol{u}^2, \delta\boldsymbol{u})\|_{X_T} &\lesssim \int_0^T \|\boldsymbol{u}^2\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|\delta\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\, dt + \int_0^T \|\boldsymbol{J}^2\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|\delta\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\, dt \\ &\lesssim \int_0^T \|\boldsymbol{u}^2\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|\delta\boldsymbol{u}\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|\delta\boldsymbol{u}\|^{\frac{1}{2}}_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\, dt + \int_0^T \|\boldsymbol{J}^2\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|\delta\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\, dt\end{aligned}$$

whence there exists $C > 0$ such that

$$\begin{aligned}\|\mathcal{B}(\boldsymbol{u}^2, \delta\boldsymbol{u})\|_{X_T} \leq \Big(\frac{1}{2} + C\|\boldsymbol{J}^2\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\Big)\|\delta\boldsymbol{u}\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})} \\ + C\int_0^T \|\boldsymbol{u}^2\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}} \|\delta\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\, dt.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\mathcal{B}(\delta\boldsymbol{u}, \boldsymbol{u}^1)\|_{X_T} &\leq C\bigg(\int_0^T \|\boldsymbol{u}^1\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \|\delta\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\, dt + \int_0^T \|\boldsymbol{u}^1\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \|\delta\boldsymbol{J}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\, dt\bigg) \\ &\leq \frac{1}{2}\|\delta\boldsymbol{u}\|_{L^1_T(\dot{B}^{\frac{3}{p}+1}_{p,1})} + C\int_0^T \Big(\|\boldsymbol{u}^1\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \|\boldsymbol{u}^1\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}}\Big)\|\delta\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\, dt.\end{aligned}$$

Remembering (3.1.14), one gets

$$\|\delta\boldsymbol{u}\|_{X_T} \leq C\int_0^T \Big(\|\boldsymbol{u}^1\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \|\boldsymbol{u}^1\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}} + \|\boldsymbol{u}^2\|^2_{\dot{B}^{\frac{3}{p}}_{p,1}}\Big)\|\delta\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\, dt.$$

Gronwall's lemma thus implies that $\delta\boldsymbol{u} \equiv 0$ in X_T, whence uniqueness on $[0,T]\times\mathbb{R}^3$. Of course, in the case where the data $\boldsymbol{u}_0$ are small, then $\boldsymbol{J}^2$ remains small for all $T > 0$, and one gets uniqueness on $\mathbb{R}^+ \times \mathbb{R}^3$.

3.1.4 Propagation of Sobolev regularity

Let us finally justify the propagation of Sobolev regularity in the case where, additionally, $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ is in $H^s \times H^r$ with (r,s) satisfying (2.1.16). For expository purpose, assume that the data fulfill (2.1.14) (the case where only

$\boldsymbol{J}_0$ is small being left to the reader). Our aim is to prove that the solution $(\boldsymbol{u}, \boldsymbol{B})$ we constructed above satisfies

$$(\boldsymbol{u}, \boldsymbol{B}) \in \mathcal{C}_b(\mathbb{R}_+; H^s \times H^r) \quad \text{and} \quad (\nabla \boldsymbol{u}, \nabla \boldsymbol{B}) \in L^2(\mathbb{R}_+; H^s \times H^r).$$

For the time being, let us assume that $(\boldsymbol{u}, \boldsymbol{B})$ is smooth. Then, taking the L^2 scalar product of (2.1.1) and (2.1.3) by $\boldsymbol{u}$ and $\boldsymbol{B}$, respectively, adding up the resulting identities, and using the fact that

$$(\nabla \times (\boldsymbol{J} \times \boldsymbol{B}),\ \boldsymbol{B}) = (\boldsymbol{J} \times \boldsymbol{B},\ \boldsymbol{J}) = 0,$$

one gets the following energy balance:

$$\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{B}\|_{L^2}^2) + \|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{B}\|_{L^2}^2 = 0. \tag{3.1.15}$$

Since $\|z\|_{\dot{H}^a} = \|\Lambda^a z\|_{L^2}$ and $\|z\|_{H^a} \sim \|z\|_{L^2} + \|z\|_{\dot{H}^a}$, in order to prove estimates in $H^s \times H^r$, it suffices to get a suitable control on $\|\Lambda^s \boldsymbol{u}\|_{L^2}$ and on $\|\Lambda^r \boldsymbol{B}\|_{L^2}$. To this end, apply Λ^s to (2.1.1), then take the L^2 scalar product with $\Lambda^s \boldsymbol{u}$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^s \boldsymbol{u}\|_{L^2}^2 + \|\Lambda^s \nabla \boldsymbol{u}\|_{L^2}^2 &= (\Lambda^s(\boldsymbol{B} \cdot \nabla \boldsymbol{B}), \Lambda^s \boldsymbol{u}) - (\Lambda^s(\boldsymbol{u} \cdot \nabla \boldsymbol{u}), \Lambda^s \boldsymbol{u}) \\ &=: A_1 + A_2.\end{aligned}$$

Similarly, apply Λ^r to (2.1.3) and take the L^2 scalar product with $\Lambda^r \boldsymbol{B}$ to get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^r \boldsymbol{B}\|_{L^2}^2 + \|\Lambda^r \nabla \boldsymbol{B}\|_{L^2}^2 &= (\Lambda^r(\boldsymbol{u} \times \boldsymbol{B}), \Lambda^r \boldsymbol{J}) - (\Lambda^r(\boldsymbol{J} \times B), \Lambda^r \boldsymbol{J}) \\ &=: A_3 + A_4.\end{aligned}$$

To bound A_1, A_2, A_3 and A_4, we shall use repeatedly the following classical tame estimate in homogeneous Sobolev spaces:

$$\|\Lambda^\sigma(fg)\|_{L^2} \lesssim \|f\|_{L^\infty}\|\Lambda^\sigma g\|_{L^2} + \|g\|_{L^\infty}\|\Lambda^\sigma f\|_{L^2}, \qquad \sigma \geq 0. \tag{3.1.16}$$

Using first the Cauchy-Schwarz inequality, then (3.1.16), the fact that $s \leq r \leq 1 + s$ and Young inequality, we readily get

$$\begin{aligned}
|A_1| &\leq C(\|\Lambda^s \boldsymbol{B}\|_{L^2}\|\nabla \boldsymbol{B}\|_{L^\infty} + \|\boldsymbol{B}\|_{L^\infty}\|\Lambda^s \nabla \boldsymbol{B}\|_{L^2})\|\boldsymbol{u}\|_{H^s} \\
&\leq C(\|\boldsymbol{B}\|_{H^s}^2 + \|\boldsymbol{u}\|_{H^s}^2)\|\nabla \boldsymbol{B}\|_{L^\infty} + \frac{1}{8}\|\nabla \boldsymbol{B}\|_{H^r}^2 + C\|\boldsymbol{B}\|_{L^\infty}^2\|\boldsymbol{u}\|_{H^s}^2, \\
|A_2| &\leq C\|\nabla \boldsymbol{u}\|_{L^\infty}\|\boldsymbol{u}\|_{H^s}^2, \\
|A_3| &\leq C\|\Lambda^r(\boldsymbol{u} \times \boldsymbol{B})\|_{L^2}\|\Lambda^r \boldsymbol{J}\|_{L^2} \\
&\leq C(\|\Lambda^r \boldsymbol{u}\|_{L^2}^2\|\boldsymbol{B}\|_{L^\infty}^2 + \|\Lambda^r \boldsymbol{B}\|_{L^2}^2\|\boldsymbol{u}\|_{L^\infty}^2) + \frac{1}{8}\|\nabla \boldsymbol{B}\|_{H^r}^2 \\
&\leq C\big(\|\boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{u}\|_{H^s}^2\big)\|\boldsymbol{B}\|_{L^\infty}^2 + C\|\boldsymbol{B}\|_{H^r}^2\|\boldsymbol{u}\|_{L^\infty}^2 + \frac{1}{8}\|\nabla \boldsymbol{B}\|_{H^r}^2, \\
|A_4| &\leq C\|\boldsymbol{J} \times \boldsymbol{B}\|_{H^r}\|\boldsymbol{J}\|_{H^r} \\
&\leq C(\|\boldsymbol{J}\|_{H^r}^2\|\boldsymbol{B}\|_{L^\infty} + \|\boldsymbol{J}\|_{L^\infty}\|\boldsymbol{B}\|_{H^r}\|\boldsymbol{J}\|_{H^r})
\end{aligned}$$

$$\leq C\|\boldsymbol{B}\|_{L^\infty}\|\nabla \boldsymbol{B}\|_{H^r}^2 + C\|\boldsymbol{J}\|_{L^\infty}^2\|\boldsymbol{B}\|_{H^r}^2 + \frac{1}{8}\|\boldsymbol{J}\|_{H^r}^2.$$

Putting the above estimates and (3.3.17) together, and using the fact that $\|\boldsymbol{B}\|_{L^\infty}$ is small since, according to Proposition B.1.3 and the first part of the proof, we have

$$\|\boldsymbol{B}\|_{L^\infty} \lesssim \|\boldsymbol{B}\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \lesssim \|\boldsymbol{J}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \lesssim \|(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{J}_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}},$$

one gets

$$\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{u}\|_{H^s}^2 + \|\boldsymbol{B}\|_{H^r}^2) + \|\nabla \boldsymbol{u}\|_{H^s}^2 + \|\nabla \boldsymbol{B}\|_{H^r}^2 \leq C(\|\boldsymbol{u}\|_{H^s}^2 + \|\boldsymbol{B}\|_{H^r}^2)S(t),$$

with

$$S(t) := \|\nabla \boldsymbol{u}(t)\|_{L^\infty} + \|\nabla \boldsymbol{B}(t)\|_{L^\infty} + \|\boldsymbol{u}(t)\|_{L^\infty}^2 + \|\boldsymbol{B}(t)\|_{L^\infty}^2 + \|\boldsymbol{J}(t)\|_{L^\infty}^2.$$

By Gronwall's lemma, we conclude that for all $t \geq 0$,

$$\|\boldsymbol{u}(t)\|_{H^s}^2 + \|\boldsymbol{B}(t)\|_{H^r}^2 + \int_0^t (\|\nabla \boldsymbol{u}(\tau)\|_{H^s}^2 + \|\nabla \boldsymbol{B}(\tau)\|_{H^r}^2)\, d\tau \leq (\|\boldsymbol{u}_0\|_{H^s}^2 + \|\boldsymbol{B}_0\|_{H^r}^2) \exp\left(C\int_0^t S(\tau)\, d\tau\right).$$

As $\int_0^t S(\tau)\, d\tau$ is bounded thanks to the first part of the theorem and embedding (use Proposition B.1.3 (ii)), we get a control of the Sobolev norms for all time.

Let us briefly explain how those latter computations may be made rigorous. Let us consider data $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ fulfilling (2.1.14) and such that, additionally, we have $\boldsymbol{u}_0$ in H^s and $\boldsymbol{B}_0$ in H^r with (r, s) satisfying (2.1.16). Then, there exists a sequence $(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n)$ in the Schwartz space $\mathcal{S}$ such that

$$(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n) \to (\boldsymbol{u}_0, \boldsymbol{B}_0) \text{ in } \left(\dot{B}^{\frac{3}{p}-1}_{p,1} \cap H^s\right) \times \left(\dot{B}^{\frac{3}{p}-1}_{p,1} \cap B^{\frac{3}{p}}_{p,1} \cap H^r\right).$$

The classical well-posedness theory in Sobolev spaces (see e.g. [28]) ensures that the Hall-MHD system with data $(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n)$ has a unique maximal solution $(\boldsymbol{u}^n, \boldsymbol{B}^n)$ on some interval $[0, T^n)$ belonging to all Sobolev spaces. For that solution, the previous computations hold, and one ends up for all $t < T^n$ with

$$\|\boldsymbol{u}^n(t)\|_{H^s}^2 + \|\boldsymbol{B}^n(t)\|_{H^r}^2 + \int_0^t (\|\nabla \boldsymbol{u}^n(\tau)\|_{H^s}^2 + \|\nabla \boldsymbol{B}^n(\tau)\|_{H^r}^2)\, d\tau \leq (\|\boldsymbol{u}_0^n\|_{H^s}^2 + \|\boldsymbol{B}_0^n\|_{H^r}^2) \exp\left(C\int_0^t S^n(\tau)\, d\tau\right),$$

where

$$S^n(t) := \|\nabla \boldsymbol{u}^n(t)\|_{L^\infty} + \|\nabla \boldsymbol{B}^n(t)\|_{L^\infty} + \|\boldsymbol{u}^n(t)\|_{L^\infty}^2 + \|\boldsymbol{B}^n(t)\|_{L^\infty}^2 + \|\boldsymbol{J}^n(t)\|_{L^\infty}^2.$$

Since the regularized data $(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n)$ fulfill (2.1.14) for large enough n, they generate a global solution $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n)$ in E_p which, actually, coincides with $(\boldsymbol{u}^n, \boldsymbol{B}^n)$ on $[0, T^n)$ by virtue of the uniqueness result that has been proved before. Therefore, S^n belongs to $L^1(0, T^n)$ and thus $(\boldsymbol{u}^n, \boldsymbol{B}^n)$ is in $L^\infty(0, T^n; H^s \times H^r)$. Combining with the continuation argument of e.g. [28], one can conclude that $T^n = +\infty$.

At this stage, one can assert that:

1. $(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{J}^n)_{n\in\mathbb{N}}$ is bounded in E_p;
2. $(\boldsymbol{u}^n, \boldsymbol{B}^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}(\mathbb{R}_+; H^s \times H^r)$ and also $(\nabla \boldsymbol{u}^n, \nabla \boldsymbol{B}^n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}_+; H^s \times H^r)$.

Hence, up to subsequence,

1. $(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{J}^n)$ converges weakly $*$ in E_p;
2. $(\boldsymbol{u}^n, \boldsymbol{B}^n)$ converges weakly $*$ in $L^\infty(\mathbb{R}_+; H^s \times H^r)$;
3. $(\nabla \boldsymbol{u}^n, \nabla \boldsymbol{B}^n)$ converges weakly in $L^2(\mathbb{R}_+; H^s \times H^r)$.

Clearly, a small variation of the proof of uniqueness in E_p allows to prove the continuity of the flow map. Hence, given that $(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n, \boldsymbol{J}_0^n)$ converges to $(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{J}_0)$ in $\dot{B}^{\frac{3}{p}-1}_{p,1}$, one gets $(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{J}^n) \to (\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{J})$ strongly in E_p, where $(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{J})$ stands for the solution of (2.1.22) with data $(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{J}_0)$.

Since the weak convergence results listed above imply the convergence in the sense of distributions, one can conclude that the weak limit coincides with the strong one in E_p. Hence $(\boldsymbol{u}, \boldsymbol{B})$ (resp. $(\nabla \boldsymbol{u}, \nabla \boldsymbol{B})$) is indeed in $L^\infty(\mathbb{R}_+; H^s \times H^r)$ (resp. $L^2(\mathbb{R}_+; H^s \times H^r)$). Then, looking at $(\boldsymbol{u}, \boldsymbol{B})$ as the solution of a heat equation with right-hand side in $L^2(\mathbb{R}_+; H^{s-1} \times H^{r-1})$ yields the time continuity with values in Sobolev spaces (use for instance Proposition B.2.3).

3.1.5 Time-decay estimates

In this subsection, we prove the decay estimates of the solution provided by Theorem 2.1.2. In fact, we have the following general result for small initial data $(\boldsymbol{u}_0, \boldsymbol{B}_0) \in \dot{B}^{\frac{3}{p}-1}_{p,1} \times (\dot{B}^{\frac{3}{p}-1}_{p,1} \cap \dot{B}^{\frac{3}{p}}_{p,1})$.

Theorem 3.1.3. *Let $1 \leq p \leq q < \infty$ be such that*

$$-\min\{\frac{1}{3}, \frac{1}{2p}\} \leq \frac{1}{q} - \frac{1}{p}. \tag{3.1.17}$$

Assume that $\boldsymbol{u}_0 \in \dot{B}^{\frac{3}{p}-1}_{p,1}$ and $\boldsymbol{B}_0$, $\boldsymbol{J}_0 := \nabla \times \boldsymbol{B}_0 \in \dot{B}^{\frac{3}{q}-1}_{q,1}$. There exists a positive constant c_5, which depends on μ, ν, h, p, q such that if

$$\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|\boldsymbol{B}_0\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}} + \|\boldsymbol{J}_0\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}} \leq c_5, \tag{3.1.18}$$

then the Cauchy problem (2.1.1)-(2.1.5) *admits a unique global-in-time solution*

$$(\boldsymbol{u},\boldsymbol{B})\in E_p\times E_q,\quad \textit{and}\quad \boldsymbol{J}:=\nabla\times\boldsymbol{B}\in E_q,$$

with

$$\begin{aligned}\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{3}{p}-1}_{p,1})}+\mu\|\boldsymbol{u}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{3}{p}+1}_{p,1})}\\ +\|(\boldsymbol{B},\boldsymbol{J})\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{3}{q}-1}_{q,1})}+\nu\|(\boldsymbol{B},\boldsymbol{J})\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{3}{q}+1}_{q,1})}\le Cc_5.\end{aligned}\tag{3.1.19}$$

If only $\boldsymbol{J}_0$ *fulfills* (3.1.18) *and in addition*

$$-\frac{1}{3}<\frac{1}{q}-\frac{1}{p},\tag{3.1.20}$$

there exists a time $T>0$ *such that Hall-MHD system admits a unique local-in-time solution*

$$(\boldsymbol{u},\boldsymbol{B})\in E_p(T)\times E_q(T)\quad \textit{with}\quad \boldsymbol{J}\in E_q(T).$$

Finally, if p,q *satisfy* (3.1.17) *and* (3.1.20) *then for any integer* $m\ge 1$*, we have*

$$\|D^m\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+\|D^m\boldsymbol{B}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\le C_0c_5t^{-\frac{m}{2}},\tag{3.1.21}$$

for all $t>0$*, where the positive constant* C_0 *depends only on* μ,ν,h,p,m*.*

Proof. The proof of existence statement is based on the classical energy method and we will not offer the proof here, the reader can refer to [10] for details.

Now, we use a time-weighted method to prove the time-decay estimates (3.1.21).

For any fixed $m\ge 1$, let $T\ge 0$ be the largest t such that

$$W(t):=\sup_{0\le\tau\le t}\tau^{\frac{m}{2}}\left(\|D^m\boldsymbol{u}(\tau)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+\|D^m\boldsymbol{B}(\tau)\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\right)\le C_0c_5,$$

where C_0 will be chosen later.

Decay estimates for velocity fields

Applying $\dot{\Delta}_j$ to the equation (2.1.1) and taking D^α_x (with $|\alpha|=m$) on the resulting equation leads to

$$\partial_t\dot{\Delta}_jD^\alpha_x\boldsymbol{u}-\mu\Delta\dot{\Delta}_jD^\alpha_x\boldsymbol{u}=\dot{\Delta}_jD^\alpha_x\mathcal{P}\mathrm{div}\,(\boldsymbol{B}\otimes\boldsymbol{B})-\dot{\Delta}_jD^\alpha_x\mathcal{P}\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u}).$$

Then

$$\dot{\Delta}_jD^\alpha_x\boldsymbol{u}=e^{t\mu\Delta}\dot{\Delta}_jD^\alpha_x\boldsymbol{u}_0+\int_0^t e^{(t-s)\mu\Delta}\mathcal{P}\left(\dot{\Delta}_jD^\alpha_x\mathrm{div}\,(\boldsymbol{B}\otimes\boldsymbol{B})-\dot{\Delta}_jD^\alpha_x\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u})\right)ds.$$

Lemma B.2.1 thus implies that

$$\begin{aligned}
&\|\dot{\Delta}_j D_x^\alpha \boldsymbol{u}\|_{L^p} \\
\le& C\int_0^t e^{-\bar{c}\mu 2^{2j}(t-s)}\Big(\|\dot{\Delta}_j D_x^\alpha \mathcal{P}\mathrm{div}\,(\boldsymbol{B}\otimes\boldsymbol{B})\|_{L^p} + \|\dot{\Delta}_j D_x^\alpha \mathcal{P}\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^p}\Big)ds \\
&+ Ce^{-\bar{c}\mu 2^{2j}t}\|\dot{\Delta}_j D_x^\alpha \boldsymbol{u}_0\|_{L^p} \\
\le& A_1 + A_2 + A_3 + Ce^{-\bar{c}\mu 2^{2j}t}\|\dot{\Delta}_j D_x^\alpha \boldsymbol{u}_0\|_{L^p},
\end{aligned} \tag{3.1.22}$$

where

$$\begin{aligned}
A_1 :=& C\int_0^{\frac{t}{2}} e^{-\bar{c}\mu 2^{2j}(t-s)}\Big(\|\dot{\Delta}_j D_x^\alpha \mathcal{P}\mathrm{div}\,(\boldsymbol{B}\otimes\boldsymbol{B})\|_{L^p} + \|\dot{\Delta}_j D_x^\alpha \mathcal{P}\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^p}\Big)ds, \\
A_2 :=& C\int_{\frac{t}{2}}^t e^{-\bar{c}\mu 2^{2j}(t-s)}2^j\|\dot{\Delta}_j D_x^{\alpha-1}\mathcal{P}\mathrm{div}\,(\boldsymbol{B}\otimes\boldsymbol{B})\|_{L^p}\,ds, \\
A_3 :=& C\int_{\frac{t}{2}}^t e^{-\bar{c}\mu 2^{2j}(t-s)}2^j\|\dot{\Delta}_j D_x^{\alpha-1}\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^p}\,ds.
\end{aligned}$$

Notice that there exists a constant $\widetilde{c} > 0$ such that

$$e^{-\bar{c}\mu 2^{2j}t}2^{jm} \le e^{-\widetilde{c}\mu 2^{2j}t}t^{-\frac{m}{2}}, \quad \text{for any } m \ge 1. \tag{3.1.23}$$

By employing Proposition B.1.1, a straightforward calculation shows that

$$\begin{aligned}
A_1 \le& Ct^{-\frac{m}{2}}\int_0^{\frac{t}{2}} e^{-\widetilde{c}\mu 2^{2j}(t-s)}\Big(\|\dot{\Delta}_j\mathrm{div}\,(\boldsymbol{B}\otimes\boldsymbol{B})\|_{L^p} + \|\dot{\Delta}_j\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^p}\Big)ds \\
\le& Cg_j t^{-\frac{m}{2}}2^{-\left(\frac{3}{p}-1\right)j}\Big(\|\mathrm{div}\,(\boldsymbol{B}\otimes\boldsymbol{B})\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^1(\dot{B}^{\frac{3}{p}-1}_{p,1})}\Big) \\
\le& Cg_j t^{-\frac{m}{2}}2^{-\left(\frac{3}{p}-1\right)j}\left(\|\boldsymbol{u}\|^2_{\widetilde{E}_p} + \|\boldsymbol{B}\|^2_{\widetilde{E}_q}\right) \\
\le& Cg_j t^{-\frac{m}{2}}2^{-\left(\frac{3}{p}-1\right)j}(1+\frac{1}{\mu}+\frac{1}{\nu})c_5^2,
\end{aligned} \tag{3.1.24}$$

where $\{g_j\}_{j\in\mathbb{Z}} \in \ell^1$ and $\|\{g_j\}\|_{\ell^1} \le 1$.

Thanks to product law (B.3.6), we have

$$\begin{aligned}
&\|\dot{\Delta}_j D_x^{\alpha-1}\mathcal{P}\mathrm{div}\,(\boldsymbol{B}\otimes\boldsymbol{B})\|_{L^p} \\
\le& C2^j\|\dot{\Delta}_j(D_x^{\alpha-1}\boldsymbol{B}\otimes\boldsymbol{B})\|_{L^p} \\
\le& Cg_j 2^{-\left(\frac{3}{q}-1\right)j}\|D_x^{\alpha-1}\boldsymbol{B}\otimes\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}}_{p,1}} \\
\le& Cg_j 2^{-\left(\frac{3}{q}-1\right)j}\|D_x^{\alpha-1}\boldsymbol{B}\|_{\dot{B}^{\frac{6}{q}-\frac{3}{p}}_{q,1}}\|\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}}_{q,1}} + Cg_j 2^{-\left(\frac{3}{q}-1\right)j}\|D_x^{\alpha-1}\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\|\boldsymbol{B}\|_{\dot{B}^{\frac{6}{q}-\frac{3}{p}}_{q,1}}.
\end{aligned}$$

By means of interpolation , we get

$$\begin{aligned}
&\|D_x^{\alpha-1}\boldsymbol{B}\|_{\dot{B}^{\frac{6}{q}-\frac{3}{p}}_{q,1}}\|\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}+\|D_x^{\alpha-1}\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\|\boldsymbol{B}\|_{\dot{B}^{\frac{6}{q}-\frac{3}{p}}_{q,1}}\\
\lesssim&\Big(\|D_x^{\alpha}\boldsymbol{B}\|^{1-r}_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\|\boldsymbol{B}\|^{r}_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\Big)\Big(\|D_x^{\alpha}\boldsymbol{B}\|^{\frac{1}{m}}_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\|\boldsymbol{B}\|^{1-\frac{1}{m}}_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\Big)\\
&\quad+\Big(\|D_x^{\alpha}\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\Big)\Big(\|D_x^{\alpha}\boldsymbol{B}\|^{\frac{1}{m}-r}_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\|\boldsymbol{B}\|^{1-\frac{1}{m}+r}_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\Big)\\
\lesssim&\|D_x^{\alpha}\boldsymbol{B}\|^{1+\frac{1}{m}-r}_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\|\boldsymbol{B}\|^{1-\frac{1}{m}+r}_{\dot{B}^{\frac{3}{q}-1}_{q,1}},
\end{aligned}$$

where $r:=\frac{3}{m}(\frac{1}{p}-\frac{1}{q})\leq\frac{1}{m}$.

Since $\frac{3}{p}-\frac{3}{q}<1$, Inequality (3.1.23) and Hölder's inequality imply that

$$\begin{aligned}
A_2\leq&Cg_j2^{-\left(\frac{3}{p}-1\right)j}\int_{\frac{t}{2}}^t(t-s)^{-\frac{1}{2}(1+\frac{3}{p}-\frac{3}{q})}\|D_x^{\alpha}\boldsymbol{B}\|^{1+\frac{1}{m}-r}_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\|\boldsymbol{B}\|^{1-\frac{1}{m}+r}_{\dot{B}^{\frac{3}{q}-1}_{q,1}}ds\\
\leq&Cg_j2^{-\left(\frac{3}{p}-1\right)j}(\frac{t}{2})^{-\frac{m}{2}(1+\frac{1}{m}-r)}W^{1+m-r}c_5^{1-\frac{1}{m}+r}\int_{\frac{t}{2}}^t(t-s)^{-\frac{1}{2}(1+\frac{3}{p}-\frac{3}{q})}ds\\
\leq&2^{\frac{m}{2}+1}Cg_j2^{-\left(\frac{3}{p}-1\right)j}t^{-\frac{m}{2}}W^{1+\frac{1}{m}-r}c_5^{1-\frac{1}{m}+r}.
\end{aligned}\tag{3.1.25}$$

Thanks to (B.3.2), we have

$$\begin{aligned}
\|\dot{\Delta}_jD_x^{\alpha-1}\mathcal{P}\mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^p}\leq&C2^j\|\dot{\Delta}_j(D_x^{\alpha-1}\boldsymbol{u}\otimes\boldsymbol{u})\|_{L^\infty(L^p)}\\
\leq&Cg_j2^{-\left(\frac{3}{p}-1\right)j}\|D_x^{\alpha-1}\boldsymbol{u}\otimes\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\\
\leq&Cg_j2^{-\left(\frac{3}{p}-1\right)j}\|D_x^{\alpha-1}\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\|\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}.
\end{aligned}$$

By means of interpolation,

$$\|D_x^{\alpha-1}\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\|\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\lesssim\|D_x^{\alpha}\boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\|D_x^{\alpha}\boldsymbol{u}\|^{\frac{1}{m}}_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\|\boldsymbol{u}\|^{1-\frac{1}{m}}_{\dot{B}^{\frac{3}{p}-1}_{p,1}}.$$

Thus, Inequality (3.1.23) and Hölder's inequality imply that

$$\begin{aligned}
A_3\leq&Cg_j2^{-\left(\frac{3}{p}-1\right)j}\int_{\frac{t}{2}}^t(t-s)^{-\frac{1}{2}}\|D_x^{\alpha}\boldsymbol{u}\|^{1+\frac{1}{m}}_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\|\boldsymbol{u}\|^{1-\frac{1}{m}}_{\dot{B}^{\frac{3}{p}-1}_{p,1}}ds\\
\leq&Cg_j2^{-\left(\frac{3}{p}-1\right)j}(\frac{t}{2})^{-\frac{m+1}{2}}W^{1+\frac{1}{m}}c_5^{1-\frac{1}{m}}\int_{\frac{t}{2}}^t(t-s)^{-\frac{1}{2}}ds\\
\leq&2^{\frac{m}{2}+1}Cg_j2^{-\left(\frac{3}{p}-1\right)j}t^{-\frac{m}{2}}W^{1+\frac{1}{m}}c_5^{1-\frac{1}{m}}.
\end{aligned}\tag{3.1.26}$$

Putting (3.1.24), (3.1.25) and (3.1.26) together, one has

$$t^{\frac{m}{2}}\|D^m \boldsymbol{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \lesssim c_5 + (1+\frac{1}{\mu}+\frac{1}{\nu}){c_5}^2 + W^{1+\frac{1}{m}-r}{c_5}^{1-\frac{1}{m}+r} + W^{1+\frac{1}{m}}{c_5}^{1-\frac{1}{m}}. \tag{3.1.27}$$

Decay estimates for Magnetic fields

Applying $\dot{\Delta}_j$ to equation (2.1.3) and taking D^α_x on the resulting equation leads to

$$\partial_t \dot{\Delta}_j D^\alpha_x \boldsymbol{B} - \nu\Delta\dot{\Delta}_j D^\alpha_x \boldsymbol{B} = \dot{\Delta}_j D^\alpha_x \nabla\times((\boldsymbol{u}-h\boldsymbol{J})\times\boldsymbol{B}). \tag{3.1.28}$$

Then

$$\dot{\Delta}_j D^\alpha_x \boldsymbol{B} = e^{t\nu\Delta}\Delta_j D^\alpha_x \boldsymbol{B}_0 + \int_0^t e^{(t-s)\nu\Delta}\left(\dot{\Delta}_j D^\alpha_x \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - h\dot{\Delta}_j D^\alpha_x\nabla\times(\boldsymbol{J}\times\boldsymbol{B})\right) ds.$$

Lemma B.2.1 thus implies that

$$\begin{aligned}
&\|\dot{\Delta}_j D^\alpha_x \boldsymbol{B}\|_{L^q}\\
\le& C\int_0^t e^{-\bar{c}\nu 2^{2j}(t-s)}\Big(\|\dot{\Delta}_j D^\alpha_x\nabla\times(\boldsymbol{u}\times\boldsymbol{B})\|_{L^p} - \|\dot{\Delta}_j D^\alpha_x\nabla\times(h\boldsymbol{J}\times\boldsymbol{B})\|_{L^q}\Big)ds\\
&+ Ce^{-\bar{c}\nu 2^{2j}t}\|\dot{\Delta}_j D^\alpha_x \boldsymbol{B}_0\|_{L^q}\\
\le& A_4 + A_5 + A_6 + Ce^{-\bar{c}\nu 2^{2j}t}\|\dot{\Delta}_j D^\alpha_x \boldsymbol{B}_0\|_{L^q},
\end{aligned} \tag{3.1.29}$$

where

$$\begin{aligned}
A_4 :=& C\int_0^{\frac{t}{2}} e^{-\bar{c}\nu 2^{2j}(t-s)}\Big(\|\dot{\Delta}_j D^\alpha_x\nabla\times(\boldsymbol{u}\times\boldsymbol{B})\|_{L^q} + \|\dot{\Delta}_j D^\alpha_x\nabla\times(h\boldsymbol{J}\times\boldsymbol{B})\|_{L^q}\Big)ds,\\
A_5 :=& C\int_{\frac{t}{2}}^t e^{-\bar{c}\nu 2^{2j}(t-s)}2^j\|\dot{\Delta}_j D^{\alpha-1}_x\nabla\times(\boldsymbol{u}\times\boldsymbol{B})\|_{L^q}\,ds,\\
A_6 :=& C\int_{\frac{t}{2}}^t e^{-\bar{c}\nu 2^{2j}(t-s)}2^j\|\dot{\Delta}_j D^{\alpha-1}_x\nabla\times(h\boldsymbol{J}\times\boldsymbol{B})\|_{L^q}\,ds.
\end{aligned}$$

Similar with the steps of getting estimates (3.1.24) and (3.1.26), one can easily show that

$$\begin{aligned}
A_4 \le& Ct^{-\frac{m}{2}}\int_0^{\frac{t}{2}} e^{-\tilde{c}\mu 2^{2j}(t-s)}\Big(\|\dot{\Delta}_j\nabla\times(\boldsymbol{u}\times\boldsymbol{B})\|_{L^q} + \|\dot{\Delta}_j\nabla\times(h\boldsymbol{J}\times\boldsymbol{B})\|_{L^q}\Big)ds\\
\le& Cg_j t^{-\frac{m}{2}}2^{-\left(\frac{3}{q}-1\right)j}\Big(\|\nabla\times(\boldsymbol{u}\times\boldsymbol{B})\|_{L^1(\dot{B}^{\frac{3}{q}-1}_{q,1})} + \|\nabla\times(h\boldsymbol{J}\times\boldsymbol{B}))\|_{L^1(\dot{B}^{\frac{3}{q}-1}_{q,1})}\Big)\\
\le& Cg_j t^{-\frac{m}{2}}2^{-\left(\frac{3}{q}-1\right)j}\left(\|\boldsymbol{u}\|^2_{E_p} + \|(\boldsymbol{B},\boldsymbol{J})\|^2_{E_q}\right)\\
\le& Cg_j t^{-\frac{m}{2}}2^{-\left(\frac{3}{q}-1\right)j}(1+\frac{1}{\mu}+\frac{1}{\nu})c_5^2,
\end{aligned} \tag{3.1.30}$$

and

$$\begin{aligned}
A_5 \leq & C g_j 2^{-\left(\frac{3}{q}-1\right)j} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} \|D_x^{\alpha-1}(\boldsymbol{u}\times\boldsymbol{B})\|_{\dot{B}^{\frac{3}{q}}_{q,1}} ds \\
\leq & C g_j 2^{-\left(\frac{3}{q}-1\right)j} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} \left(\|D_x^{\alpha-1}\boldsymbol{u}\times\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}}_{q,1}} + \|\boldsymbol{u}\times D_x^{\alpha-1}\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \right) ds \\
\leq & C g_j 2^{-\left(\frac{3}{q}-1\right)j} W^{1+\frac{1}{m}} c_5^{1-\frac{1}{m}} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}(1+m)} ds \\
\leq & 2^{\frac{m}{2}+1} C g_j 2^{-\left(\frac{3}{q}-1\right)j} t^{-\frac{m}{2}} W^{1+\frac{1}{m}} c_5^{1-\frac{1}{m}}.
\end{aligned} \tag{3.1.31}$$

Because $\operatorname{div}\boldsymbol{B}=0$, one can rewrite

$$\nabla\times(h\boldsymbol{J}\times\boldsymbol{B}) = h\nabla\times\left(\operatorname{div}(\boldsymbol{B}\otimes\boldsymbol{B}) - \nabla\left(\frac{|\boldsymbol{B}|^2}{2}\right)\right) = h\nabla\times\left(\operatorname{div}(\boldsymbol{B}\otimes\boldsymbol{B})\right),$$

then, Hölder's inequality yields

$$\begin{aligned}
A_6 \leq & Ch\int_{\frac{t}{2}}^t e^{-\bar{c}\nu 2^{2j}(t-s)} 2^{3}j \|\dot{\Delta}_j D_x^{\alpha-1}\boldsymbol{B}\otimes\boldsymbol{B}\|_{L^q} ds \\
\leq & Ch 2^{-(\frac{3}{q}-1)j} g_j \int_{\frac{t}{2}}^t e^{-\bar{c}\nu 2^{2j}(t-s)} 2^{2j} \|D_x^{\alpha-1}(\boldsymbol{B}\otimes\boldsymbol{B})\|_{\dot{B}^{\frac{3}{q}}_{q,1}} ds \\
\leq & Ch 2^{-(\frac{3}{q}-1)j} g_j \int_{\frac{t}{2}}^t e^{-\bar{c}\nu 2^{2j}(t-s)} 2^{2j} \|D_x^{\alpha}\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}} \|\nabla\times\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}} ds \\
\leq & Ch c_5 2^{-(\frac{3}{q}-1)j} g_j \left(\frac{t}{2}\right)^{-\frac{m}{2}} \int_{\frac{t}{2}}^t e^{-\bar{c}\nu 2^{2j}(t-s)} 2^{2j} s^{\frac{m}{2}} \|D_x^{\alpha}\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}} ds \\
\leq & 2^{\frac{m}{2}+1} C\frac{h}{\nu} 2^{-(\frac{3}{q}-1)j} g_j t^{-\frac{m}{2}} W(T) c_5,
\end{aligned} \tag{3.1.32}$$

here we use the fact that

$$\int_{\frac{t}{2}}^t e^{-\bar{c}\nu 2^{2j}(t-s)} 2^{2j} ds = \frac{2}{\bar{c}\nu}\left(1-e^{-\frac{1}{2}\bar{c}\nu t 2^{2j}}\right) \leq \frac{2}{\bar{c}\nu}.$$

Putting (3.1.30), (3.1.31) and (3.1.32) together, one has

$$t^{\frac{m}{2}} \|D^m\boldsymbol{B}\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}} \lesssim c_5 + (1+\frac{1}{\mu}+\frac{1}{\nu}){c_5}^2 + W^{1+\frac{1}{m}} {c_5}^{1-\frac{1}{m}} + \frac{h}{\nu} W c_5.$$

This combined with (3.1.27), implies that

$$W(T) \leq C c_5 + C\left(1+\frac{1}{\mu}+\frac{1}{\nu}+\frac{h}{\nu}C_0 + C_0^{1+\frac{1}{m}-r} + C_0^{1+\frac{1}{m}}\right) c_5^2.$$

Choosing c_5 small enough, one can take suitable C_0 such that $W(T) < \frac{1}{2}C_0 c_5$. By the continuous induction, we have $W(t) \leq C_0 c_5$ for all $t \geq 0$.

This completes the proof of Theorem 2.1.2. □

3.2 Local well-posedness for large data in the spaces $\dot{B}^{\frac{1}{2}}_{2,1}$, and blow-up criterion

Proving Theorem 2.1.3 is based on a priori estimates in the space $E_2(T)$ for smooth solutions $(\boldsymbol{u}, \boldsymbol{B}, v)$ of (2.1.27). Those estimates will be obtained by implementing an energy method on (2.1.27) after localization in the Fourier space. A slight modification of the method will yield uniqueness, stability and blow-up criteria.

Throughout this section and the following one, we shall take advantage of the rescaling (3.1.1), so as to reduce our study to the case $\mu = \nu = h = 1$ (remember that we have $\mu/\nu = 1$ in Theorem 2.1.3).

3.2.1 Local existence by energy method

First step: a priori estimates

Our main aim here is to prove the following result.

Proposition 3.2.1. *Consider a smooth solution $(\boldsymbol{u}, \boldsymbol{B}, P)$ to the Hall-MHD System on $[0,T]\times\mathbb{R}^3$ for some $T>0$, and denote $\boldsymbol{v} := \boldsymbol{u} - \nabla\times\boldsymbol{B}$. Let $\boldsymbol{u}^L := e^{t\Delta}\boldsymbol{u}_0$, $\boldsymbol{B}^L := e^{t\Delta}\boldsymbol{B}_0$, $\boldsymbol{v}^L := e^{t\Delta}\boldsymbol{v}_0$ and $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{v}}) := (\boldsymbol{u}-\boldsymbol{u}^L, \boldsymbol{B}-\boldsymbol{B}^L, \boldsymbol{v}-\boldsymbol{v}^L)$. Let*

$$c_1(t) := \|\boldsymbol{v}^L(t)\|_{\dot{B}^{\frac{5}{2}}_{2,1}},$$
$$c_2(t) := \|\boldsymbol{u}^L(t)\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\boldsymbol{B}^L(t)\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \big(\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\boldsymbol{v}_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\big)\|\boldsymbol{v}^L(t)\|_{\dot{B}^{\frac{5}{2}}_{2,1}}.$$

There exist three positive constants κ, C and C_1 such that if

$$\int_0^T c_2(\tau) e^{C\int_\tau^T c_1(\tau')\,d\tau'}\,d\tau < \kappa, \tag{3.2.1}$$

then we have

$$\|(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{v}})\|_{L^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} + C_1\|(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{v}})\|_{L^1_T(\dot{B}^{\frac{5}{2}}_{2,1})} \leq C\kappa \quad \text{and} \tag{3.2.2}$$
$$\|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})\|_{L^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} + C_1\|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})\|_{L^1_T(\dot{B}^{\frac{5}{2}}_{2,1})} \leq \|(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + C\kappa. \tag{3.2.3}$$

Proof. From (B.2.2), Plancherel identity and the definition of $\|\cdot\|_{\dot{B}^s_{2,1}}$, we have for some universal constant C_1,

$$\|z\|_{L^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})} + C_1\|z\|_{L^1_T(\dot{B}^{\frac{5}{2}}_{2,1})} \leq \|z_0\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \quad \text{for } z = \boldsymbol{u}^L, \boldsymbol{B}^L, \boldsymbol{v}^L. \tag{3.2.4}$$

Hence Inequality (3.2.3) follows from Inequality (3.2.2).

In order to prove (3.2.2), we use the fact that $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{v}, \Pi)$ satisfies

$$\begin{cases} \partial_t \widetilde{\boldsymbol{u}} - \Delta \widetilde{\boldsymbol{u}} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla\Pi, \\ \partial_t \widetilde{\boldsymbol{B}} - \Delta \widetilde{\boldsymbol{B}} = \nabla\times(v\times\boldsymbol{B}), \\ \partial_t \widetilde{v} - \Delta \widetilde{v} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla\times((\nabla\times\widetilde{v})\times\boldsymbol{B}) \\ \quad - \nabla\times((\nabla\times v^L)\times\boldsymbol{B}) + \nabla\times(v\times\boldsymbol{u}) + 2\nabla\times(v\cdot\nabla\boldsymbol{B}) - \nabla\Pi, \end{cases} \tag{3.2.5}$$

with null initial condition.

Apply operator $\dot{\Delta}_j$ to both sides of (3.2.5), then take the L^2 scalar product with $\dot{\Delta}_j\widetilde{\boldsymbol{u}}$, $\dot{\Delta}_j\widetilde{\boldsymbol{B}}$, $\dot{\Delta}_j\widetilde{v}$, respectively. To handle the third equation of (3.2.5), use that

$$-\nabla\times\dot{\Delta}_j((\nabla\times\widetilde{v})\times\boldsymbol{B}) = \nabla\times([\dot{\Delta}_j,\boldsymbol{B}\times](\nabla\times\widetilde{v})) + \nabla\times(\boldsymbol{B}\times\dot{\Delta}_j(\nabla\times\widetilde{v})),$$

and that the L^2 scalar product of the last term with $\dot{\Delta}_j\widetilde{v}$ is 0. Then, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\widetilde{\boldsymbol{u}}\|_{L^2}^2 + \|\nabla\dot{\Delta}_j\widetilde{\boldsymbol{u}}\|_{L^2}^2 \le (\|\dot{\Delta}_j(\boldsymbol{B}\cdot\nabla\boldsymbol{B})\|_{L^2} + \|\dot{\Delta}_j(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{L^2})\|\dot{\Delta}_j\widetilde{\boldsymbol{u}}\|_{L^2}, \\
&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\widetilde{\boldsymbol{B}}\|_{L^2}^2 + \|\nabla\dot{\Delta}_j\widetilde{\boldsymbol{B}}\|_{L^2}^2 \le \|\nabla\times\dot{\Delta}_j(v\times\boldsymbol{B})\|_{L^2}\|\dot{\Delta}_j\widetilde{\boldsymbol{B}}\|_{L^2}, \\
&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\widetilde{v}\|_{L^2}^2 + \|\nabla\dot{\Delta}_j\widetilde{v}\|_{L^2}^2 \le \big(\|\dot{\Delta}_j(\boldsymbol{B}\cdot\nabla\boldsymbol{B})\|_{L^2} + \|\dot{\Delta}_j(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{L^2}\big)\|\dot{\Delta}_j\widetilde{v}\|_{L^2} \\
&\qquad + \big(\|[\dot{\Delta}_j,\boldsymbol{B}\times](\nabla\times\widetilde{v})\|_{L^2} + \|\dot{\Delta}_j((\nabla\times v^L)\times\boldsymbol{B})\|_{L^2} + \|\dot{\Delta}_j(v\times\boldsymbol{u})\|_{L^2} \\
&\qquad\qquad + 2\|\dot{\Delta}_j(v\cdot\nabla\boldsymbol{B})\|_{L^2}\big)\|\nabla\times\dot{\Delta}_j\widetilde{v}\|_{L^2}.
\end{aligned}$$

Hence, using Bernstein inequalities, one can deduce after time integration that for some universal constants C_1 and C_2,

$$\begin{aligned}
&\|(\dot{\Delta}_j\widetilde{\boldsymbol{u}}, \dot{\Delta}_j\widetilde{\boldsymbol{B}}, \dot{\Delta}_j\widetilde{v})(t)\|_{L^2} + C_1 2^{2j}\int_0^t \|(\dot{\Delta}_j\widetilde{\boldsymbol{u}}, \dot{\Delta}_j\widetilde{\boldsymbol{B}}, \dot{\Delta}_j\widetilde{v})\|_{L^2}\,d\tau \\
&\quad\le \int_0^t \Big(\|\dot{\Delta}_j(\boldsymbol{B}\cdot\nabla\boldsymbol{B})\|_{L^2} + \|\dot{\Delta}_j(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\|_{L^2} + C_2 2^j\Big(\|[\dot{\Delta}_j,\boldsymbol{B}\times](\nabla\times\widetilde{v})\|_{L^2} \\
&\qquad\qquad + \|\dot{\Delta}_j((\nabla\times v^L)\times\boldsymbol{B})\|_{L^2} + \|\dot{\Delta}_j(v\times\boldsymbol{u})\|_{L^2} \\
&\qquad\qquad + \|\dot{\Delta}_j(v\cdot\nabla\boldsymbol{B})\|_{L^2} + \|\dot{\Delta}_j(v\times\boldsymbol{B})\|_{L^2}\Big)\Big)\,d\tau.
\end{aligned} \tag{3.2.6}$$

Multiplying both sides of (3.2.6) by $2^{\frac{j}{2}}$ and summing up over $j\in\mathbb{Z}$, we obtain that

$$\begin{aligned}
&\|(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{v})(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + C_1\int_0^t \|(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{v})\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\,d\tau \\
&\quad\le C_2\int_0^t \Big(\|\boldsymbol{B}\cdot\nabla\boldsymbol{B}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|v\times\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|v\times\boldsymbol{u}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \\
&\quad + \|v\cdot\nabla\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|(\nabla\times v^L)\times\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \sum_j 2^{\frac{3j}{2}}\|[\dot{\Delta}_j,\boldsymbol{B}\times](\nabla\times\widetilde{v})\|_{L^2}\Big)\,d\tau.
\end{aligned} \tag{3.2.7}$$

Using (B.3.2), Proposition B.1.3 (i), (ii), (iii) and Young's inequality yields

$$\begin{aligned}\|\boldsymbol{B}\cdot\nabla\boldsymbol{B}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} &\lesssim \|\boldsymbol{B}^L\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\widetilde{\boldsymbol{B}}\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}}\\ &\lesssim \|\boldsymbol{B}^L\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\widetilde{\boldsymbol{B}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\widetilde{\boldsymbol{B}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}},\end{aligned}$$

$$\begin{aligned}\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\boldsymbol{v}\times\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\boldsymbol{v}\times\boldsymbol{u}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} &\lesssim \|\boldsymbol{u}\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \big(\|\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\boldsymbol{u}\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\big)\|\boldsymbol{v}\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\\ &\lesssim \|\boldsymbol{u}^L\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\boldsymbol{B}^L\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\boldsymbol{v}^L\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\boldsymbol{v}^L\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\\ &\quad + \|\widetilde{\boldsymbol{u}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\widetilde{\boldsymbol{u}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\widetilde{\boldsymbol{B}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\widetilde{\boldsymbol{B}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\widetilde{\boldsymbol{v}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\widetilde{\boldsymbol{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}}.\end{aligned}$$

Using that $\boldsymbol{B} = \mathrm{curl}^{-1}(\boldsymbol{u}-\boldsymbol{v})$ and that $\nabla\mathrm{curl}^{-1}$ is a self-map on $\dot{B}^{\frac{3}{2}}_{2,1}$ (see Proposition B.1.3 (vii)) yields

$$\begin{aligned}\|\boldsymbol{v}\cdot\nabla\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} &\lesssim \|\boldsymbol{v}\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|\nabla\mathrm{curl}^{-1}(\boldsymbol{u}-\boldsymbol{v})\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\\ &\lesssim \|\boldsymbol{v}\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\boldsymbol{u}\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}}\\ &\lesssim \|\boldsymbol{u}^L\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\boldsymbol{v}^L\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\boldsymbol{v}^L\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\widetilde{\boldsymbol{u}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\widetilde{\boldsymbol{u}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\widetilde{\boldsymbol{v}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\widetilde{\boldsymbol{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\end{aligned}$$

and, using also (B.3.2),

$$\begin{aligned}\|(\nabla\times\boldsymbol{v}^L)\times\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} &\lesssim \|\nabla\times\boldsymbol{v}^L\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\\ &\lesssim \|\boldsymbol{v}^L\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|\mathrm{curl}^{-1}(\boldsymbol{u}-\boldsymbol{v})\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\\ &\lesssim \|\boldsymbol{v}^L\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\big(\|\boldsymbol{u}^L\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\boldsymbol{v}^L\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\widetilde{\boldsymbol{u}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\widetilde{\boldsymbol{v}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\big).\end{aligned}$$

From the estimate (B.4.1) with $s = 3/2$ and the embedding $\dot{B}^{\frac{3}{2}}_{2,1} \hookrightarrow L^\infty$, we get

$$\sum_j 2^{\frac{3j}{2}}\|[\dot{\Delta}_j, b]a\|_{L^2} \lesssim \|\nabla b\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|a\|_{\dot{B}^{\frac{1}{2}}_{2,1}}, \tag{3.2.8}$$

whence

$$\begin{aligned}\sum_j 2^{\frac{3j}{2}}\|[\dot{\Delta}_j, \boldsymbol{B}\times](\nabla\times\widetilde{\boldsymbol{v}})\|_{L^2} &\lesssim \|\boldsymbol{v}-\boldsymbol{u}\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\widetilde{\boldsymbol{v}}\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}}\\ &\lesssim \|\boldsymbol{u}^L\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\boldsymbol{v}^L\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\boldsymbol{v}^L\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\\ &\quad + \|\widetilde{\boldsymbol{u}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\widetilde{\boldsymbol{u}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \|\widetilde{\boldsymbol{v}}\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\|\widetilde{\boldsymbol{v}}\|_{\dot{B}^{\frac{5}{2}}_{2,1}}.\end{aligned}$$

Plugging the above estimates into the right-hand side of (3.2.7) and using (3.2.4), we end up with

$$X(t)+C_1\int_0^t D(\tau)\,d\tau \le C\int_0^t X(\tau)D(\tau)\,d\tau + C\int_0^t (c_1(\tau)X(\tau)+c_2(\tau))\,d\tau, \tag{3.2.9}$$

where c_1 and c_2 have been defined in the proposition,

$$\begin{aligned} X(t) &:= \|\widetilde{\boldsymbol{u}}(t)\|_{\dot{B}^{\frac12}_{2,1}} + \|\widetilde{\boldsymbol{B}}(t)\|_{\dot{B}^{\frac12}_{2,1}} + \|\widetilde{\boldsymbol{v}}(t)\|_{\dot{B}^{\frac12}_{2,1}} \\ \text{and}\quad D(t) &:= \|\widetilde{\boldsymbol{u}}(t)\|_{\dot{B}^{\frac52}_{2,1}} + \|\widetilde{\boldsymbol{B}}(t)\|_{\dot{B}^{\frac52}_{2,1}} + \|\widetilde{\boldsymbol{v}}(t)\|_{\dot{B}^{\frac52}_{2,1}}. \end{aligned}$$

Note that whenever

$$2C\sup_{\tau\in[0,t]} X(\tau) \le C_1, \tag{3.2.10}$$

Inequality (3.2.9) combined with Gronwall lemma implies that

$$X(t)+\frac{C_1}{2}\int_0^t D(\tau)\,d\tau \le C\int_0^t c_2(\tau)e^{C\int_\tau^t c_1(\tau')\,d\tau'}\,d\tau. \tag{3.2.11}$$

Now, if Condition (3.2.1) is satisfied with $\kappa := C_1/2C^2$, then the fact that the left-hand side of (3.2.9) is a continuous function on $[0,T]$ that vanishes at 0 combined with a standard bootstrap argument allows to prove that (3.2.10) and thus (3.2.1) is satisfied. Renaming the constants completes the proof of the proposition. □

Second step: constructing approximate solutions

It is based on Friedrichs' method : consider the spectral cut-off operator $\mathbb{E}_n$ defined by

$$\mathcal{F}(\mathbb{E}_n f)(\boldsymbol{\xi}) = 1_{\{n^{-1}\le|\boldsymbol{\xi}|\le n\}}(\boldsymbol{\xi})\mathcal{F}(f)(\boldsymbol{\xi}).$$

We want to solve the following truncated system:

$$\begin{cases} \partial_t \boldsymbol{u} - \Delta\boldsymbol{u} = \mathbb{E}_n\mathcal{P}(\mathbb{E}_n\boldsymbol{B}\cdot\mathbb{E}_n\nabla\boldsymbol{B} - \mathbb{E}_n\boldsymbol{u}\cdot\nabla\mathbb{E}_n\boldsymbol{u}), \\ \partial_t \boldsymbol{B} - \Delta\boldsymbol{B} = \nabla\times\mathbb{E}_n(\mathbb{E}_n(\boldsymbol{u}-\nabla\times\boldsymbol{B})\times\mathbb{E}_n\boldsymbol{B}), \end{cases} \tag{3.2.12}$$

supplemented with initial data $(\mathbb{E}_n\boldsymbol{u}_0, \mathbb{E}_n\boldsymbol{B}_0)$.

We need the following obvious lemma:

Lemma 3.2.2. *Let $s\in\mathbb{R}$ and $k\ge 0$. Let $f\in\dot{B}^s_{2,1}$. Then, for all $n\ge1$, we have*

$$\|\mathbb{E}_n f\|_{\dot{B}^{s+k}_{2,1}} \lesssim n^k\|f\|_{\dot{B}^s_{2,1}}, \tag{3.2.13}$$

$$\lim_{n\to\infty}\|\mathbb{E}_n f - f\|_{\dot{B}^s_{2,1}} = 0, \tag{3.2.14}$$

$$\|\mathbb{E}_n f - f\|_{\dot{B}^s_{2,1}} \lesssim \frac{1}{n^k}\|f\|_{\dot{B}^{s+k}_{2,1}}. \tag{3.2.15}$$

We claim that (3.2.12) is an ODE in the Banach space $L^2(\mathbb{R}^3;\mathbb{R}^3\times\mathbb{R}^3)$ for which the standard Cauchy-Lipschitz theorem applies. Indeed, the above lemma ensures that $\mathbb{E}_n$ maps L^2 to all Besov spaces, and that the right-hand side of (3.2.12) is a continuous bilinear map from $L^2(\mathbb{R}^3;\mathbb{R}^3\times\mathbb{R}^3)$ to itself. We thus deduce that (3.2.12) admits a unique maximal solution $(\boldsymbol{u}^n,\boldsymbol{B}^n)\in\mathcal{C}^1([0,T^n);L^2(\mathbb{R}^3;\mathbb{R}^3\times\mathbb{R}^3))$. Furthermore, as $\mathbb{E}_n^2=\mathbb{E}_n$, uniqueness implies $\mathbb{E}_n\boldsymbol{u}^n=\boldsymbol{u}^n$ and $\mathbb{E}_n\boldsymbol{B}^n=\boldsymbol{B}^n$, and we clearly have $\operatorname{div}\boldsymbol{u}^n=\operatorname{div}\boldsymbol{B}^n=0$. Being spectrally supported in the annulus $\{n^{-1}\leq|\boldsymbol{\xi}|\leq n\}$, one can also deduce that the solution belongs to $\mathcal{C}^1([0,T^n);\dot{B}^s_{2,1})$ for all $s\in\mathbb{R}$. Hence, setting $\boldsymbol{J}^n:=\nabla\times\boldsymbol{B}^n$ and $\boldsymbol{v}^n:=\boldsymbol{u}^n-\boldsymbol{J}^n$, we see that $\boldsymbol{u}^n$, $\boldsymbol{B}^n$ and $\boldsymbol{v}^n$ belong to the space $E_2(T)$ for all $T<T^n$ and fulfill:

$$\begin{cases}\partial_t\boldsymbol{u}^n-\Delta\boldsymbol{u}^n=\mathbb{E}_n\mathcal{P}(\boldsymbol{B}^n\cdot\nabla\boldsymbol{B}^n-\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n),\\ \partial_t\boldsymbol{B}^n-\Delta\boldsymbol{B}^n=\nabla\times\mathbb{E}_n(\boldsymbol{v}^n\times\boldsymbol{B}^n),\\ \partial_t\boldsymbol{v}^n-\Delta\boldsymbol{v}^n=\mathbb{E}_n\mathcal{P}\Big(\boldsymbol{B}^n\cdot\nabla\boldsymbol{B}^n-\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n+\nabla\times(\boldsymbol{v}^n\times\boldsymbol{u}^n)\\ \qquad\qquad-\nabla\times((\nabla\times\boldsymbol{v}^n)\times\boldsymbol{B}^n)+2\nabla\times(\boldsymbol{v}^n\cdot\nabla\boldsymbol{B}^n)\Big).\end{cases}\tag{3.2.16}$$

Third step: uniform estimates

We want to apply Proposition 3.2.1 to our approximate solution $(\boldsymbol{u}^n,\boldsymbol{B}^n,\boldsymbol{v}^n)$. The key point is that since $\mathbb{E}_n$ is an L^2 orthogonal projector, it has no effect on the energy estimates. We claim that T^n may be bounded from below by the supremum T of all the times satisfying (3.2.1), and that $(\boldsymbol{u}^n,\boldsymbol{B}^n,\boldsymbol{v}^n)_{n\geq1}$ is bounded in $E_2(T)$. To prove our claim, , we split $(\boldsymbol{u}^n,\boldsymbol{B}^n,\boldsymbol{v}^n)$ into

$$(\boldsymbol{u}^n,\boldsymbol{B}^n,\boldsymbol{v}^n)=(\boldsymbol{u}^{n,L},\boldsymbol{B}^{n,L},\boldsymbol{v}^{n,L})+(\widetilde{\boldsymbol{u}}^n,\widetilde{\boldsymbol{B}}^n,\widetilde{\boldsymbol{v}}^n),$$

where

$$\boldsymbol{u}^{n,L}:=\mathbb{E}_ne^{t\Delta}\boldsymbol{u}_0,\quad \boldsymbol{B}^{n,L}:=\mathbb{E}_ne^{t\Delta}\boldsymbol{B}_0\quad\text{and}\quad \boldsymbol{v}^{n,L}:=\mathbb{E}_ne^{t\Delta}\boldsymbol{v}_0.$$

Since $\mathbb{E}_n$ maps any Besov space $\dot{B}^s_{2,1}$ to itself with norm 1, Condition (3.2.1) may be made independent of n and thus, so does the corresponding time T. Now, as $(\widetilde{\boldsymbol{u}}^n,\widetilde{B}^n,\widetilde{\boldsymbol{v}}^n)$ is spectrally supported in $\{\boldsymbol{\xi}\in\mathbb{R}^3\,|\,n^{-1}\leq|\boldsymbol{\xi}|\leq n\}$, the estimate (3.2.2) ensures that it belongs to $L^\infty([0,T];L^2(\mathbb{R}^3))$. So, finally, the standard continuation criterion for ordinary differential equations implies that T^n is greater than any time T satisfying (3.2.1) and that we have, for all $n\geq1$,

$$\|(\widetilde{\boldsymbol{u}}^n,\widetilde{\boldsymbol{B}}^n,\widetilde{\boldsymbol{v}}^n)\|_{L^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}+C_1\|(\widetilde{\boldsymbol{u}}^n,\widetilde{\boldsymbol{B}}^n,\widetilde{\boldsymbol{v}}^n)\|_{L^1_T(\dot{B}^{\frac{5}{2}}_{2,1})}\leq C\kappa\quad\text{and}\tag{3.2.17}$$

$$\|(\boldsymbol{u}^n,\boldsymbol{B}^n,\boldsymbol{v}^n)\|_{L^\infty_T(\dot{B}^{\frac{1}{2}}_{2,1})}+C_1\|(\boldsymbol{u}^n,\boldsymbol{B}^n,\boldsymbol{v}^n)\|_{L^1_T(\dot{B}^{\frac{5}{2}}_{2,1})}\leq\|(\boldsymbol{u}_0,\boldsymbol{B}_0,\boldsymbol{v}_0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}+C\kappa.\tag{3.2.18}$$

Fourth step: existence of a solution

We claim that, up to an extraction, the sequence $(\boldsymbol{u}^n,\boldsymbol{B}^n,\boldsymbol{v}^n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^3)$ to a solution $(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})$ of (2.1.27) supplemented with data $(\boldsymbol{u}_0,\boldsymbol{B}_0,\boldsymbol{v}_0)$ having the desired regularity properties. The definition of $\mathbb{E}_n$

entails that

$$(\mathbb{E}_n \boldsymbol{u}_0, \mathbb{E}_n \boldsymbol{B}_0, \mathbb{E}_n \boldsymbol{v}_0) \to (\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0) \quad \text{in} \quad \dot{B}^{\frac{1}{2}}_{2,1},$$

and Proposition B.2.3 thus ensures that $(\boldsymbol{u}^{n,L}, \boldsymbol{B}^{n,L}, \boldsymbol{v}^{n,L}) \to (\boldsymbol{u}^L, \boldsymbol{B}^L, \boldsymbol{v}^L)$ in $E_2(T)$.

Proving the convergence of $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{\boldsymbol{v}}^n)$ will be achieved from classical compactness arguments: we shall exhibit uniform bounds in suitable spaces for $(\partial_t \boldsymbol{u}^n, \partial_t \boldsymbol{B}^n, \partial_t \boldsymbol{v}^n)_{n\in\mathbb{N}}$ so as to glean some Hölder regularity with respect to the time variable. Then, combining with compact embedding will enable us to apply Ascoli's theorem and to get the existence of a limit $(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})$ for a subsequence. Furthermore, the uniform bounds of the previous steps provide us with additional regularity and convergence properties so that we will be able to pass to the limit in (3.2.16). Let us start with a lemma.

Lemma 3.2.3. *Sequence $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{\boldsymbol{v}}^n)_{n\geq 1}$ is bounded in $\mathcal{C}^{\frac{1}{2}}([0,T]; \dot{B}^{-\frac{1}{2}}_{2,1})$.*

Proof. Observe that $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{\boldsymbol{v}}^n)$ satisfies

$$\begin{cases} \partial_t \widetilde{\boldsymbol{u}}^n = \Delta \widetilde{\boldsymbol{u}}^n + \mathbb{E}_n \mathcal{P}(\boldsymbol{B}^n \cdot \nabla \boldsymbol{B}^n - \boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n), \\ \partial_t \widetilde{\boldsymbol{B}}^n = \Delta \widetilde{\boldsymbol{B}}^n + \nabla \times \mathbb{E}_n(\boldsymbol{v}^n \times \boldsymbol{B}^n), \\ \partial_t \widetilde{\boldsymbol{v}}^n = \Delta \widetilde{\boldsymbol{v}}^n + \mathbb{E}_n \mathcal{P}\Big(\boldsymbol{B}^n \cdot \nabla \boldsymbol{B}^n - \boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n + \nabla \times (\boldsymbol{v}^n \times \boldsymbol{u}^n) \\ \qquad\qquad - \nabla \times ((\nabla \times \boldsymbol{v}^n) \times \boldsymbol{B}^n) + 2\nabla \times (\boldsymbol{v}^n \cdot \nabla \boldsymbol{B}^n)\Big). \end{cases} \tag{3.2.19}$$

According to the uniform bounds (3.2.17), (3.2.18) and to the product laws:

$$\|ab\|_{\dot{B}^{-\frac{1}{2}}_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \|b\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \quad \text{and} \quad \|ab\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \|b\|_{\dot{B}^{\frac{3}{2}}_{2,1}},$$

the right-hand side of (3.2.19) is uniformly bounded in $L^2_T(\dot{B}^{-\frac{1}{2}}_{2,1})$. Since $\widetilde{\boldsymbol{u}}^n(0) = \widetilde{\boldsymbol{B}}^n(0) = \widetilde{\boldsymbol{v}}^n(0) = 0$, applying Hölder's inequality completes the proof. □

We can now come to the proof of the existence of a solution. Let $(\phi_j)_{j\in\mathbb{N}}$ be a sequence of $\mathcal{C}^\infty_0(\mathbb{R}^3)$ cut-off functions supported in the ball $B(0, j+1)$ of $\mathbb{R}^3$ and equal to 1 in a neighborhood of $B(0,j)$. Lemma 3.2.3 tells us that $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{\boldsymbol{v}}^n)_{n\geq 1}$ is uniformly equicontinuous in the space $\mathcal{C}([0,T]; \dot{B}^{-\frac{1}{2}}_{2,1})$ and (3.2.17) ensures that it is bounded in $L^\infty([0,T]; \dot{B}^{\frac{1}{2}}_{2,1})$. Using the fact that the application $\boldsymbol{u} \mapsto \phi_j \boldsymbol{u}$ is compact from $\dot{B}^{\frac{1}{2}}_{2,1}$ into $\dot{B}^{-\frac{1}{2}}_{2,1}$, combining Ascoli's theorem and Cantor's diagonal process ensures that there exists some triplet $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{v}})$ such that for all $j \in \mathbb{N}$,

$$(\phi_j \widetilde{\boldsymbol{u}}^n, \phi_j \widetilde{\boldsymbol{B}}^n, \phi_j \widetilde{\boldsymbol{v}}^n) \to (\phi_j \widetilde{\boldsymbol{u}}, \phi_j \widetilde{\boldsymbol{B}}, \phi_j \widetilde{\boldsymbol{v}}) \quad \text{in} \quad \mathcal{C}([0,T]; \dot{B}^{-\frac{1}{2}}_{2,1}). \tag{3.2.20}$$

This obviously entails that $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{\boldsymbol{v}}^n)$ tends to $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{v}})$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3)$.

Coming back to the uniform estimates of third step and using the argument of [45, p. 443] to justify that there is no time concentration, we get that $(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{B}},\widetilde{\boldsymbol{v}})$ belongs to $L^\infty(0,T;\dot{B}^{\frac12}_{2,1})\cap L^1(0,T;\dot{B}^{\frac52}_{2,1})$ and to $\mathcal{C}^{\frac12}([0,T];\dot{B}^{-\frac12}_{2,1})$.

Let us now prove that $(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v}):=(\boldsymbol{u}^L+\widetilde{\boldsymbol{u}},\boldsymbol{B}^L+\widetilde{\boldsymbol{B}},\boldsymbol{v}^L+\widetilde{\boldsymbol{v}})$ solves (2.1.27). The only problem is to pass to the limit in the non-linear terms. By way of example, let us explain how to handle the term $\mathbb{E}_n\mathcal{P}\nabla\times((\nabla\times\boldsymbol{v}^n)\times\boldsymbol{B}^n)$ in (3.2.16) (actually, $\mathcal{P}$ may be omitted as a curl is divergence free). Let $\boldsymbol{\theta}\in\mathcal{C}^\infty_0(\mathbb{R}_+\times\mathbb{R}^3;\mathbb{R}^3)$ and $j\in\mathbb{N}$ be such that $\mathrm{Supp}\,\boldsymbol{\theta}\subset[0,j]\times B(0,j)$. We use the decomposition

$$\begin{aligned}\langle\nabla\times\mathbb{E}_n((\nabla\times\boldsymbol{v}^n)\times\boldsymbol{B}^n),\boldsymbol{\theta}\rangle-\langle\nabla\times((\nabla\times\boldsymbol{v})\times\boldsymbol{B}),\boldsymbol{\theta}\rangle\\ =\langle(\nabla\times\boldsymbol{v}^n)\times\phi_j(\boldsymbol{B}^n-\boldsymbol{B}),\nabla\times\mathbb{E}_n\boldsymbol{\theta}\rangle+\langle(\nabla\times\phi_j(\boldsymbol{v}^n-\boldsymbol{v}))\times\boldsymbol{B},\nabla\times\mathbb{E}_n\boldsymbol{\theta}\rangle\\ +\langle\mathbb{E}_n((\nabla\times\boldsymbol{v})\times\boldsymbol{B})-(\nabla\times\boldsymbol{v})\times\boldsymbol{B},\nabla\times\boldsymbol{\theta}\rangle.\end{aligned}$$

As $\nabla\times\boldsymbol{v}^n$ is uniformly bounded in $L^1_T(\dot{B}^{\frac32}_{2,1})$ and $\phi_j\boldsymbol{B}^n$ tends to $\phi_j\boldsymbol{B}$ in $L^\infty_T(\dot{B}^{\frac12}_{2,1})$, the first term tends to 0. According to the uniform estimates (3.2.18) and (3.2.20), $\nabla\times\phi_j(\boldsymbol{v}^n-\boldsymbol{v})$ tends to 0 in $L^1_T(\dot{B}^{\frac12}_{2,1})$ so that the second term tends to 0 as well. Finally, thanks to (3.2.14), the third term tends to 0.

The other non-linear terms can be treated similarly, and the continuity of $(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})$ stems from Proposition B.2.3 since the right-hand side of (2.1.27) belongs to $L^1_T(\dot{B}^{\frac12}_{2,1})$.

3.2.2 Uniqueness

Let $(\boldsymbol{u}_1,\boldsymbol{B}_1)$ and $(\boldsymbol{u}_2,\boldsymbol{B}_2)$ be two solutions of the Hall-MHD system on $[0,T]\times\mathbb{R}^3$, with the same initial data, and such that $(\boldsymbol{u}_i,\boldsymbol{B}_i,\boldsymbol{v}_i)\in E_2(T)$ for $i=1,2$. Then, the difference $(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v}):=(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{B}_1-\boldsymbol{B}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)$ is in $E_2(T)$ and satisfies

$$\begin{cases}\partial_t\delta\boldsymbol{u}-\Delta\delta\boldsymbol{u}:=\boldsymbol{R}_1,\\ \partial_t\delta\boldsymbol{B}-\Delta\delta\boldsymbol{B}:=\boldsymbol{R}_2,\\ \partial_t\delta\boldsymbol{v}-\Delta\delta\boldsymbol{v}:=\boldsymbol{R}_1+\boldsymbol{R}_3+\boldsymbol{R}_4+\boldsymbol{R}_5,\end{cases}\tag{3.2.21}$$

where

$$\begin{aligned}\boldsymbol{R}_1&:=\mathcal{P}(\boldsymbol{B}_1\cdot\nabla\delta\boldsymbol{B}+\delta\boldsymbol{B}\cdot\nabla\boldsymbol{B}_2-\boldsymbol{u}_1\cdot\nabla\delta\boldsymbol{u}-\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}_2),\\ \boldsymbol{R}_2&:=\nabla\times(\boldsymbol{v}_1\times\delta\boldsymbol{B}+\delta\boldsymbol{v}\times\boldsymbol{B}_2),\\ \boldsymbol{R}_3&:=-\nabla\times((\nabla\times\boldsymbol{v}_1)\times\delta\boldsymbol{B}+(\nabla\times\delta\boldsymbol{v})\times\boldsymbol{B}_2),\\ \boldsymbol{R}_4&:=\nabla\times(\boldsymbol{v}_1\times\delta\boldsymbol{u}+\delta\boldsymbol{v}\times\boldsymbol{u}_2),\\ \boldsymbol{R}_5&:=2\nabla\times(\boldsymbol{v}_1\cdot\nabla\delta\boldsymbol{B}+\delta\boldsymbol{v}\cdot\nabla\boldsymbol{B}_2).\end{aligned}$$

Hence, arguing as in the first step of the proof gives for all $t\in[0,T]$,

$$\|(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})(t)\|_{\dot{B}^{\frac12}_{2,1}}+\int_0^t\|(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})\|_{\dot{B}^{\frac52}_{2,1}}\,d\tau\lesssim\int_0^t\Big(\|(\boldsymbol{R}_1,\boldsymbol{R}_2,\boldsymbol{R}_4,\boldsymbol{R}_5)\|_{\dot{B}^{\frac12}_{2,1}}$$

$$+\|\nabla\times((\nabla\times v_1)\times\delta B)\|_{\dot B^{\frac12}_{2,1}}+\sum_{j\in\mathbb{Z}}2^{\frac{3j}{2}}\|[\dot\Delta_j,B_2\times](\nabla\times\delta v)\|_{L^2}\Big)d\tau. \quad (3.2.22)$$

Putting together the product laws (B.3.2) and the commutator estimate (3.2.8) yields

$$\begin{aligned}
\|R_1\|_{\dot B^{\frac12}_{2,1}} &\lesssim \|(u_1,B_1,u_2,B_2)\|_{\dot B^{\frac32}_{2,1}}\|(\delta u,\delta B)\|_{\dot B^{\frac32}_{2,1}},\\
\|R_2\|_{\dot B^{\frac12}_{2,1}} &\lesssim \|(B_2,v_1)\|_{\dot B^{\frac32}_{2,1}}\|(\delta B,\delta v)\|_{\dot B^{\frac32}_{2,1}},\\
\|R_4\|_{\dot B^{\frac12}_{2,1}} &\lesssim \|(u_2,v_1)\|_{\dot B^{\frac32}_{2,1}}\|(\delta u,\delta v)\|_{\dot B^{\frac32}_{2,1}},\\
\|R_5\|_{\dot B^{\frac12}_{2,1}} &\lesssim \|(\nabla B_2,v_1)\|_{\dot B^{\frac32}_{2,1}}\|(\nabla\delta B,\delta v)\|_{\dot B^{\frac32}_{2,1}}\\
&\lesssim \|(u_2,v_1,v_2)\|_{\dot B^{\frac32}_{2,1}}\|(\delta u,\delta v)\|_{\dot B^{\frac32}_{2,1}},\\
\|\nabla\times((\nabla\times v_1)\times\delta B)\|_{\dot B^{\frac12}_{2,1}} &\lesssim \|\nabla\times v_1\|_{\dot B^{\frac32}_{2,1}}\|\delta B\|_{\dot B^{\frac32}_{2,1}}\\
&\lesssim \|v_1\|_{\dot B^{\frac52}_{2,1}}\|(\delta u,\delta v)\|_{\dot B^{\frac12}_{2,1}}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}2^{\frac{3j}{2}}\|[\dot\Delta_j,B_2\times](\nabla\times\delta v)\|_{L^2} &\lesssim \|\nabla B_2\|_{\dot B^{\frac32}_{2,1}}\|\nabla\times\delta v\|_{\dot B^{\frac12}_{2,1}}\\
&\lesssim \|(u_2,v_2)\|_{\dot B^{\frac32}_{2,1}}\|\delta v\|_{\dot B^{\frac32}_{2,1}}.
\end{aligned}$$

Hence, by interpolation and Young's inequality, Inequality (3.2.22) becomes

$$\begin{aligned}
&\|(\delta u,\delta B,\delta v)(t)\|_{\dot B^{\frac12}_{2,1}}+\int_0^t\|(\delta u,\delta B,\delta v)(\tau)\|_{\dot B^{\frac52}_{2,1}}d\tau\\
&\le\int_0^t Z(\tau)\|(\delta u,\delta B,\delta v)(\tau)\|_{\dot B^{\frac12}_{2,1}}d\tau \qquad (3.2.23)
\end{aligned}$$

with $Z(t):=C\big(\|(u_1,u_2,B_1,B_2,v_1,v_2)\|^2_{\dot B^{\frac32}_{2,1}}+\|v_1\|_{\dot B^{\frac52}_{2,1}}\big)$.

Thus, Gronwall's lemma and our assumptions on the solutions ensure that

$$(\delta u,\delta B,\delta v)\equiv 0 \quad\text{on}\quad [0,T].$$

3.2.3 Stability

We now focus on the proof of Corollary 2.1.4. Given that initial data $(u_{0,2},B_{0,2})$, we know from the local-in-time existence part in subsection 3.2.1 that there exists a solution (u_2,B_2) on the maximal time interval $[0,T^*)$ fulfilling

$$(u_2,B_2,\nabla\times B_2)\in E_2(T_1),$$

for all $T_1<T^*$. Without lost of generality, we assume that $T^*\le T$.

Now, following the same steps as to get (3.2.23), one can obtain similar estimate for the difference of $(\boldsymbol{u}_1, \boldsymbol{B}_1, v_1)$ and $(\boldsymbol{u}_2, \boldsymbol{B}_2, v_2)$. We have

$$\begin{aligned}
&\|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \int_0^t \|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)(\tau)\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\, d\tau \\
&\leq \int_0^t \big(Z_1(\tau) + C\|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)(\tau)\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}}\big)\|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\, d\tau \\
&\qquad + \|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)(0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}
\end{aligned}$$

with $Z_1(t) := C\big(\|(\boldsymbol{u}_1, \boldsymbol{B}_1, v_1)\|^2_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|v_1\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\big)$.

Hence, by interpolation and Young's inequality, we further have

$$\begin{aligned}
&\|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \int_0^t \|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)(\tau)\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\, d\tau \\
&\leq \int_0^t \big(Z_1(\tau)\|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + C\|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)(\tau)\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)\|^2_{\dot{B}^{\frac{1}{2}}_{2,1}}\big)\, d\tau \\
&\qquad + \|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta v)(0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}}. \qquad (3.2.24)
\end{aligned}$$

Now, one needs to prove the following lemma.

Lemma 3.2.4. *Let X, D, Z_1 be three nonnegative measurable functions on $[0, T_1]$. Assume that there exists a nonnegative real constant C such that for any $t \in [0, T_1]$,*

$$X(t) + \int_0^t D(\tau)\, d\tau \leq X(0) + \int_0^t \Big(Z_1(\tau)X(\tau) + CX^2(\tau)D(\tau)\Big)\, d\tau. \qquad (3.2.25)$$

If, in addition,

$$2CX^2(0)\exp\Big(2\int_0^{T_1} Z_1(\tau)\, d\tau\Big) < \mu, \qquad (3.2.26)$$

then, for any $t \in [0, T_1]$, one has

$$X(t) + \frac{1}{2}\int_0^t D\, d\tau \leq X(0)\exp\Big(\int_0^t Z_1(\tau)\, d\tau\Big). \qquad (3.2.27)$$

Proof. Let $\widetilde{T}_1$ be the largest $t \leq T_1$ such that

$$2C \sup_{0\leq t'\leq t} X^2(t') \leq 1. \qquad (3.2.28)$$

Then, (3.2.25) implies that for all $t \in [0, \widetilde{T}_1]$, we have

$$X(t) + \frac{1}{2}\int_0^t D\, d\tau \leq X(0) + \int_0^t Z_1(\tau)X(\tau)\, d\tau. \qquad (3.2.29)$$

By Gronwall's lemma, this yields for all $t \in [0, \widetilde{T}_1]$,

$$X(t) + \frac{1}{2}\int_0^t D(\tau)\,d\tau \leq X(0)\exp\Big(\int_0^t Z_1(\tau)\,d\tau\Big)\cdot$$

Hence, it is clear that if (3.2.26) is satisfied, then (3.2.28) is satisfied with a strict inequality. A continuity argument thus ensures that we must have $\widetilde{T}_1 = T_1$ and thus (3.2.27) on $[0, T_1]$. □

Noticing our assumptions on $(\boldsymbol{u}_1, \boldsymbol{B}_1)$ ensure that $Z_1 \in L^1(0,T)$ and thus $Z_1 \in L^1(0,T^*)$.

By virtue of (2.1.25), let c_1 satisfies

$$2Cc_1^2\exp\big(2\|Z_1\|_{L^1(0,T)}\big) < 1,$$

and applying Lemma 3.2.4 to inequality (3.2.24) in time interval $[0, T_1]$, we have for any $0 < t \leq T_1 < T^*$,

$$\|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta\boldsymbol{v})(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \frac{1}{2}\int_0^t \|(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta\boldsymbol{v})(\tau)\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\,d\tau \leq c_1\exp\big(\|Z_1\|_{L^1(0,T)}\big).$$

At this stage, one can either follow the contradiction argument presented in [56] or use the forthcoming blow-up criteria (2.1.23), to show that indeed, the solution $(\delta\boldsymbol{u}_2, \delta\boldsymbol{B}_2, \delta\boldsymbol{v}_2)$ can continued beyond T^*, which finally implies that $T^* = T$.

3.2.4 Blow-up criterion

Let us assume that we are given a solution $(\boldsymbol{u}, \boldsymbol{B})$ on some *finite* time interval $[0, T^*)$ fulfilling the regularity properties listed in Theorem 2.1.3 for all $t < T^*$. Then, applying the method of the first step to (2.1.27) yields for all $t < T^*$,

$$\begin{aligned}\|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + C_1\int_0^t \|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\,d\tau &\leq \|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})(0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \\ &+ \int_0^t \Big(\|\boldsymbol{B}\cdot\nabla\boldsymbol{B}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \big(\|\boldsymbol{v}\times\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\boldsymbol{v}\times\boldsymbol{u}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \\ &+ \|\boldsymbol{v}\cdot\nabla\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \sum_j 2^{\frac{3j}{2}}\|[\dot{\Delta}_j, \boldsymbol{B}\times](\nabla\times\boldsymbol{v})\|_{L^2}\big)\Big)d\tau. \quad (3.2.30)\end{aligned}$$

Using the tame estimates (B.3.1), the fact that $\dot{B}^{\frac{3}{2}}_{2,1}$ is an algebra embedded in L^∞, interpolation inequalities and Young's inequality, we get for all $\eta > 0$,

$$\begin{aligned}\|\boldsymbol{B}\cdot\nabla\boldsymbol{B}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} &\leq C\|\boldsymbol{B}\otimes\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \\ &\leq C\|\boldsymbol{B}\|_{L^\infty}\|\boldsymbol{B}\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \\ &\leq \frac{C}{\eta}\|\boldsymbol{B}\|^2_{L^\infty}\|\boldsymbol{B}\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \eta\|\boldsymbol{B}\|_{\dot{B}^{\frac{5}{2}}_{2,1}},\end{aligned}$$

and, similarly,

$$\|u\cdot\nabla u\|_{\dot{B}^{\frac12}_{2,1}}\leq\frac{C}{\eta}\|u\|^2_{L^\infty}\|u\|_{\dot{B}^{\frac12}_{2,1}}+\eta\|u\|_{\dot{B}^{\frac52}_{2,1}}.$$

We also have

$$\begin{aligned}\|v\times B\|_{\dot{B}^{\frac32}_{2,1}}&\leq C(\|v\|_{L^\infty}\|B\|_{\dot{B}^{\frac32}_{2,1}}+\|B\|_{L^\infty}\|v\|_{\dot{B}^{\frac32}_{2,1}})\\&\leq\frac{C}{\eta}\|(B,v)\|^2_{L^\infty}\|(B,v)\|_{\dot{B}^{\frac12}_{2,1}}+\eta\|(B,v)\|_{\dot{B}^{\frac52}_{2,1}},\\\|v\times u\|_{\dot{B}^{\frac32}_{2,1}}&\leq\frac{C}{\eta}\|(u,v)\|^2_{L^\infty}\|(u,v)\|_{\dot{B}^{\frac12}_{2,1}}+\eta\|(u,v)\|_{\dot{B}^{\frac52}_{2,1}},\\\|v\cdot\nabla B\|_{\dot{B}^{\frac32}_{2,1}}&\leq\frac{C}{\eta}\|(\nabla B,v)\|^2_{L^\infty}\|(\nabla B,v)\|_{\dot{B}^{\frac12}_{2,1}}+\eta\|(\nabla B,v)\|_{\dot{B}^{\frac52}_{2,1}}.\end{aligned}$$

As, according to (B.4.1) with $s=3/2$ and to the fact that $\nabla:L^\infty\to\dot{\boldsymbol{B}}^{-1}_{\infty,\infty}$, we have

$$\sum_j 2^{\frac{3j}{2}}\|[\dot{\Delta}_j,B\times](\nabla\times v)\|_{L^2}\leq C\big(\|\nabla B\|_{L^\infty}\|v\|_{\dot{B}^{\frac32}_{2,1}}+\|v\|_{L^\infty}\|\nabla B\|_{\dot{B}^{\frac32}_{2,1}}\big),\quad(3.2.31)$$

that term may be bounded as $v\cdot\nabla B$.

Therefore, if we choose η small enough, then (3.2.30) becomes:

$$\begin{aligned}\|(u,B,v)(t)\|_{\dot{B}^{\frac12}_{2,1}}+\frac{C_1}{2}\int_0^t\|(u,B,v)\|_{\dot{B}^{\frac52}_{2,1}}d\tau&\leq\|(u,B,v)(0)\|_{\dot{B}^{\frac12}_{2,1}}\\&\quad+C\int_0^t\|(u,B,\nabla B)\|^2_{L^\infty}\|(u,B,v)\|_{\dot{B}^{\frac12}_{2,1}}d\tau\end{aligned}$$

and Gronwall's inequality implies that for all $t\in[0,T^*)$,

$$\begin{aligned}\|(u,B,v)(t)\|_{\dot{B}^{\frac12}_{2,1}}+\frac{C_1}{2}\int_0^t\|(u,B,v)\|_{\dot{B}^{\frac52}_{2,1}}d\tau&\\\leq\|(u,B,v)(0)\|_{\dot{B}^{\frac12}_{2,1}}\exp\Big(C\int_0^t\|(u,B,\nabla B)\|^2_{L^\infty}dt\Big)&.\end{aligned}$$

Now, if one assumes that

$$\int_0^{T^*}\|(u,B,\nabla B)(t)\|^2_{L^\infty}dt<\infty,$$

then the above inequality ensures that (u,B,v) belongs to $L^\infty(0,T^*;\dot{B}^{\frac12}_{2,1})$ and one may conclude by classical arguments that the solution may be continued beyond T^*.

In order to prove the second blow-up criterion, one uses the following inequalities, based on (B.3.2) and interpolation inequalities:

$$\begin{aligned}
\|\boldsymbol{B}\cdot\nabla\boldsymbol{B}\|_{\dot{B}^{\frac12}_{2,1}} &\lesssim \|\boldsymbol{B}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{B}\|_{\dot{B}^{\frac52}_{2,1}},\\
\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{\dot{B}^{\frac12}_{2,1}} &\lesssim \|\boldsymbol{u}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{u}\|_{\dot{B}^{\frac52}_{2,1}},\\
\|\boldsymbol{v}\times\boldsymbol{B}\|_{\dot{B}^{\frac32}_{2,1}} &\lesssim \|\boldsymbol{v}\|_{\dot{B}^{\frac32}_{2,1}}\|\boldsymbol{B}\|_{\dot{B}^{\frac32}_{2,1}}\\
&\lesssim \|\boldsymbol{v}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{v}\|_{\dot{B}^{\frac52}_{2,1}}+\|\boldsymbol{B}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{B}\|_{\dot{B}^{\frac52}_{2,1}},\\
\|\boldsymbol{v}\times\boldsymbol{u}\|_{\dot{B}^{\frac32}_{2,1}} &\lesssim \|\boldsymbol{v}\|_{\dot{B}^{\frac32}_{2,1}}\|\boldsymbol{u}\|_{\dot{B}^{\frac32}_{2,1}}\\
&\lesssim \|\boldsymbol{v}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{v}\|_{\dot{B}^{\frac52}_{2,1}}+\|\boldsymbol{u}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{u}\|_{\dot{B}^{\frac52}_{2,1}},\\
\|\boldsymbol{v}\cdot\nabla\boldsymbol{B}\|_{\dot{B}^{\frac32}_{2,1}} &\lesssim \|\boldsymbol{v}\|_{\dot{B}^{\frac32}_{2,1}}\|\nabla\boldsymbol{B}\|_{\dot{B}^{\frac32}_{2,1}}\\
&\lesssim \|\boldsymbol{v}\|^2_{\dot{B}^{\frac32}_{2,1}}+\|J\|^2_{\dot{B}^{\frac32}_{2,1}}\\
&\lesssim \|\boldsymbol{v}\|^2_{\dot{B}^{\frac32}_{2,1}}+\|\boldsymbol{u}\|^2_{\dot{B}^{\frac32}_{2,1}}\\
&\lesssim \|\boldsymbol{v}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{v}\|_{\dot{B}^{\frac52}_{2,1}}+\|\boldsymbol{u}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{u}\|_{\dot{B}^{\frac52}_{2,1}}
\end{aligned}$$

and by (3.2.8) and Proposition B.1.3 (iii) (vii),

$$\begin{aligned}
\sum_j 2^{\frac{3j}{2}}\|[\dot{\Delta}_j,\boldsymbol{B}\times](\nabla\times\boldsymbol{v})\|_{L^2} &\lesssim \|\nabla\boldsymbol{B}\|_{\dot{B}^{\frac32}_{2,1}}\|\boldsymbol{v}\|_{\dot{B}^{\frac32}_{2,1}}\\
&\lesssim \|\boldsymbol{v}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{v}\|_{\dot{B}^{\frac52}_{2,1}}+\|\boldsymbol{u}\|_{\dot{B}^{\frac12}_{2,1}}\|\boldsymbol{u}\|_{\dot{B}^{\frac52}_{2,1}}.
\end{aligned}$$

Plugging those estimates in (3.2.30), we find that

$$\begin{aligned}
\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(t)\|_{\dot{B}^{\frac12}_{2,1}}+C_1\int_0^t\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\dot{B}^{\frac52}_{2,1}}\,d\tau &\le \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(0)\|_{\dot{B}^{\frac12}_{2,1}}\\
&\quad+\int_0^t\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\dot{B}^{\frac12}_{2,1}}\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\dot{B}^{\frac52}_{2,1}}\,d\tau.
\end{aligned}$$

Hence, if

$$\int_0^{T^*}\|(\boldsymbol{u},\boldsymbol{B},J)\|_{\dot{B}^{\frac52}_{2,1}}\,dt<\infty,$$

then the solution may be continued beyond T^*.

For proving the last blow-up criterion, one can use that for $\varrho\in(2,\infty]$, most of the terms of (3.2.30) may be bounded by means of Inequality (B.3.3). The last commutator term may be bounded from (B.4.2) (without time integration) with $r=1$ and $s=3/2$ as follows:

$$\sum_j 2^{\frac{3j}{2}}\|[\dot{\Delta}_j,\boldsymbol{B}\times](\nabla\times\boldsymbol{v})\|_{L^2}\lesssim\|\nabla\boldsymbol{B}\|_{\dot{B}^{\frac{2}{\varrho}-1}_{\infty,\infty}}\|\boldsymbol{v}\|_{\dot{B}^{\frac52-\frac{2}{\varrho}}_{2,1}}+\|\boldsymbol{v}\|_{\dot{B}^{\frac{2}{\varrho}-1}_{\infty,\infty}}\|\nabla\boldsymbol{B}\|_{\dot{B}^{\frac52-\frac{2}{\varrho}}_{2,1}}.$$

Since, by interpolation, we have

$$\|z\|_{\dot{B}^{\frac{5}{2}-\frac{2}{\varrho}}_{2,1}} \lesssim \|z\|^{\frac{1}{\varrho}}_{\dot{B}^{\frac{1}{2}}_{2,1}} \|z\|^{\frac{1}{\varrho'}}_{\dot{B}^{\frac{5}{2}}_{2,1}} \quad \text{with} \quad \frac{1}{\varrho'} = 1 - \frac{1}{\varrho},$$

using Young inequality and reverting to (3.2.30) yields

$$\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(t)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + \int_0^t \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\dot{B}^{\frac{5}{2}}_{2,1}}\, d\tau \leq \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(0)\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + C\int_0^t \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|^{\varrho}_{\dot{B}^{\frac{2}{\varrho}-1}_{\infty,\infty}} \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\dot{B}^{\frac{1}{2}}_{2,1}}\, d\tau.$$

As before, one can conclude that if $T^* < \infty$ and (2.1.24) is fulfilled, then the solution may be continued beyond T^*.

This completes the proof of the theorem. □

3.3 The Fujita-Kato type theorem

This section is devoted to the proof of Theorem 2.1.5 and its two corollaries. Throughout this section, we assume for simplicity that $\mu = \nu = h = 1$ (the general case $\mu = \nu > 0$ and $h > 0$ may be deduced after suitable rescaling, see (3.1.1)).

3.3.1 Global existence

Proving the existence part of Theorem 2.1.5 is based on the following result:

Proposition 3.3.1. *Let $(\boldsymbol{u},\boldsymbol{B})$ be a smooth solution of the 3D Hall-MHD system with $h = \mu = \nu = 1$, on the time interval $[0,T]$. Let $\boldsymbol{v} := \boldsymbol{u} - \nabla\times\boldsymbol{B}$. There exists a universal constant C such that on $[0,T]$, we have*

$$\frac{d}{dt}(\|\boldsymbol{u}\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{B}\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{v}\|^2_{\dot{H}^{\frac{1}{2}}}) + (\|\boldsymbol{u}\|^2_{\dot{H}^{\frac{3}{2}}} + \|\boldsymbol{B}\|^2_{\dot{H}^{\frac{3}{2}}} + \|\boldsymbol{v}\|^2_{\dot{H}^{\frac{3}{2}}}) \leq C\sqrt{\|\boldsymbol{u}\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{B}\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{v}\|^2_{\dot{H}^{\frac{1}{2}}}}\,(\|\boldsymbol{u}\|^2_{\dot{H}^{\frac{3}{2}}} + \|\boldsymbol{B}\|^2_{\dot{H}^{\frac{3}{2}}} + \|\boldsymbol{v}\|^2_{\dot{H}^{\frac{3}{2}}}). \quad (3.3.1)$$

Proof. Applying the fractional derivative operator $\Lambda^{\frac{1}{2}}$ to both sides of System (2.1.27), taking the L^2 scalar product with $\Lambda^{\frac{1}{2}}\boldsymbol{u}$, $\Lambda^{\frac{1}{2}}\boldsymbol{B}$ and $\Lambda^{\frac{1}{2}}\boldsymbol{v}$, respectively, and using (2.1.18), we get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{u}\|^2_{\dot{H}^{\frac{3}{2}}} = (\Lambda^{\frac{1}{2}}(\boldsymbol{u}\otimes\boldsymbol{u}) \,|\, \nabla\Lambda^{\frac{1}{2}}\boldsymbol{u}) - (\Lambda^{\frac{1}{2}}(\boldsymbol{B}\otimes\boldsymbol{B}) \,|\, \nabla\Lambda^{\frac{1}{2}}\boldsymbol{u}) =: A_1 + A_2,$$
$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{B}\|^2_{\dot{H}^{\frac{3}{2}}} = (\Lambda^{\frac{1}{2}}(\boldsymbol{v}\times\boldsymbol{B}) \,|\, \nabla\times\Lambda^{\frac{1}{2}}\boldsymbol{B}) =: A_3,$$
$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}\|^2_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{v}\|^2_{\dot{H}^{\frac{3}{2}}} = A_4 + A_5 + \cdots + A_8,$$

where

$$
\begin{aligned}
A_4 &:= (\Lambda^{\frac{1}{2}}(\boldsymbol{u}\otimes\boldsymbol{u})\,|\,\nabla\Lambda^{\frac{1}{2}}\boldsymbol{v}),\\
A_5 &:= -(\Lambda^{\frac{1}{2}}(\boldsymbol{B}\otimes\boldsymbol{B})\,|\,\nabla\Lambda^{\frac{1}{2}}\boldsymbol{v}),\\
A_6 &:= -(\Lambda^{\frac{1}{2}}((\nabla\times\boldsymbol{v})\times\boldsymbol{B}) - (\Lambda^{\frac{1}{2}}\nabla\times\boldsymbol{v})\times\boldsymbol{B}\,|\,\Lambda^{\frac{1}{2}}\nabla\times\boldsymbol{v}),\\
A_7 &:= (\Lambda^{\frac{1}{2}}(\boldsymbol{v}\times\boldsymbol{u})\,|\,\nabla\times\Lambda^{\frac{1}{2}}\boldsymbol{v}),\\
A_8 &:= 2(\Lambda^{\frac{1}{2}}(\boldsymbol{v}\cdot\nabla\boldsymbol{B})\,|\,\nabla\times\Lambda^{\frac{1}{2}}\boldsymbol{v}).
\end{aligned}
$$

By Lemma B.5.3 and Sobolev embedding (B.5.2), we get

$$
\begin{aligned}
|A_1| &\le C\|\Lambda^{\frac{1}{2}}\boldsymbol{u}\|_{L^3}\|\boldsymbol{u}\|_{L^6}\|\nabla\Lambda^{\frac{1}{2}}\boldsymbol{u}\|_{L^2}\\
&\le C\|\boldsymbol{u}\|_{\dot H^1}^2\|\boldsymbol{u}\|_{\dot H^{\frac{3}{2}}},\\
|A_6| &\le C(\|\nabla\boldsymbol{B}\|_{L^6}\|\Lambda^{\frac{1}{2}}\boldsymbol{v}\|_{L^3} + \|\Lambda^{\frac{1}{2}}\boldsymbol{B}\|_{L^6}\|\nabla\times\boldsymbol{v}\|_{L^3})\|\boldsymbol{v}\|_{\dot H^{\frac{3}{2}}}\\
&\le C\big(\|\nabla\boldsymbol{B}\|_{\dot H^1}\|\boldsymbol{v}\|_{\dot H^1} + \|\nabla\boldsymbol{B}\|_{\dot H^{\frac{1}{2}}}\|\boldsymbol{v}\|_{\dot H^{\frac{3}{2}}}\big)\|\boldsymbol{v}\|_{\dot H^{\frac{3}{2}}},\\
|A_8| &\le C(\|\Lambda^{\frac{1}{2}}\boldsymbol{v}\|_{L^3}\|\nabla\boldsymbol{B}\|_{L^6} + \|\boldsymbol{v}\|_{L^6}\|\nabla\Lambda^{\frac{1}{2}}\boldsymbol{B}\|_{L^3})\|\boldsymbol{v}\|_{\dot H^{\frac{3}{2}}}\\
&\le C\|\nabla\boldsymbol{B}\|_{\dot H^1}\|\boldsymbol{v}\|_{\dot H^1}\|\boldsymbol{v}\|_{\dot H^{\frac{3}{2}}}.
\end{aligned}
$$

Terms A_2, A_3, A_4, A_5 and A_7 may be bounded similarly as A_1:

$$
\begin{aligned}
|A_2| &\le C\|\boldsymbol{B}\|_{\dot H^1}^2\|\boldsymbol{u}\|_{\dot H^{\frac{3}{2}}},\\
|A_3| &\le C\|\boldsymbol{v}\|_{\dot H^1}\|\boldsymbol{B}\|_{\dot H^1}\|\boldsymbol{B}\|_{\dot H^{\frac{3}{2}}},\\
|A_4| &\le C\|\boldsymbol{u}\|_{\dot H^1}^2\|\boldsymbol{v}\|_{\dot H^{\frac{3}{2}}},\\
|A_5| &\le C\|\boldsymbol{B}\|_{\dot H^1}^2\|\boldsymbol{v}\|_{\dot H^{\frac{3}{2}}},\\
|A_7| &\le C\|\boldsymbol{v}\|_{\dot H^1}\|\boldsymbol{u}\|_{\dot H^1}\|\boldsymbol{v}\|_{\dot H^{\frac{3}{2}}}.
\end{aligned}
$$

Hence, using repeatedly the fact that

$$
\|z\|_{\dot H^1} \le \sqrt{\|z\|_{\dot H^{\frac{1}{2}}}\|z\|_{\dot H^{\frac{3}{2}}}}
$$

and Young inequality and, sometimes, (3.1.2), it is easy to deduce (3.3.1) from the above inequalities. □

The following result has been used several times to establish global a priori estimates.

Lemma 3.3.2. *Let X, D, W be three nonnegative measurable functions on $[0,T]$ such that X is also differentiable. Assume that there exist two nonnegative real numbers C and α such that*

$$
\frac{d}{dt}X^2 + D^2 \le CWX^2 + CX^\alpha D^2. \tag{3.3.2}
$$

If, in addition,

$$2CX^{\alpha}(0)\exp\left(\frac{C\alpha}{2}\int_0^T W\,dt\right) < 1, \tag{3.3.3}$$

then, for any $t \in [0,T]$, *one has*

$$X^2(t) + \frac{1}{2}\int_0^t D^2\,d\tau \leq X^2(0)\exp\left(C\int_0^t W\,d\tau\right). \tag{3.3.4}$$

Proof. Let $T^\star$ be the largest $t \leq T$ such that

$$2C \sup_{0\leq t'\leq t} X^{\alpha}(t') \leq 1. \tag{3.3.5}$$

Then, (3.3.2) implies that for all $t \in [0,T^*]$, we have

$$\frac{d}{dt}X^2 + \frac{1}{2}D^2 \leq CWX^2. \tag{3.3.6}$$

By Gronwall lemma, this yields for all $t \in [0,T^\star]$,

$$X^2(t) + \frac{1}{2}\int_0^t D^2\,d\tau \leq X^2(0)\exp\left(C\int_0^t W\,d\tau\right).$$

Hence, it is clear that if (3.3.3) is satisfied, then (3.3.5) holds with a strict inequality. Then, a continuity argument ensures that we must have $T^\star = T$, and thus (3.3.4) on $[0,T]$. □

Now, combining Proposition 3.3.1 with Lemma 3.3.2 (take $\alpha = 1$, $W \equiv 0$) implies that there exists a constant $c_2 > 0$ such that if

$$\|\boldsymbol{u}_0\|_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{B}_0\|_{\dot{H}^{\frac{1}{2}}} + \|\boldsymbol{v}_0\|_{\dot{H}^{\frac{1}{2}}} < c_2, \tag{3.3.7}$$

then we have for all $t \in [0,T]$

$$\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{1}{2}\int_0^t \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|^2_{\dot{H}^{\frac{3}{2}}}\,d\tau \leq \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(0)\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{3.3.8}$$

That inequality implies that Condition (3.3.7) is satisfied for all $t_0 \in [0,T]$. Hence, the previous argument may be repeated on $[t_0,T]$, and one gets for all $0 \leq t_0 \leq t \leq T$,

$$\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{1}{2}\int_{t_0}^t \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|^2_{\dot{H}^{\frac{3}{2}}}\,d\tau \leq \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(t_0)\|^2_{\dot{H}^{\frac{1}{2}}}. \tag{3.3.9}$$

In order to prove rigorously the existence part of Theorem 2.1.5, we shall resort to the following classical procedure:

1. smooth out the initial data and get a sequence $(\boldsymbol{u}^n,\boldsymbol{B}^n)_{n\in\mathbb{N}}$ of smooth solutions to Hall-MHD system on the maximal time interval $[0,T^n)$;
2. apply (3.3.8) to $(\boldsymbol{u}^n,\boldsymbol{B}^n)_{n\in\mathbb{N}}$ so as to prove that $T^n = \infty$ and that the sequence $(\boldsymbol{u}^n,\boldsymbol{B}^n,\boldsymbol{v}^n)_{n\in\mathbb{N}}$ with $\boldsymbol{v}^n := \boldsymbol{u}^n - \nabla\times\boldsymbol{B}^n$ is bounded in $L^\infty(\mathbb{R}_+;\dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}_+;\dot{H}^{\frac{3}{2}})$;

3. use compactness to prove that $(\boldsymbol{u}^n, \boldsymbol{B}^n)_{n\in\mathbb{N}}$ converges, up to extraction, to a solution of Hall-MHD system supplemented with initial data $(\boldsymbol{u}_0, \boldsymbol{B}_0)$;
4. prove stability estimates in L^2 to get the uniqueness of the solution.

Step 1

To proceed, let us smooth out the initial data as follows:

$$\boldsymbol{u}_0^n := \mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_n}\widehat{\boldsymbol{u}_0}) \quad \text{and} \quad \boldsymbol{B}_0^n := \mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_n}\widehat{\boldsymbol{B}_0}),$$

where $\mathcal{C}_n$ stands for the annulus with small radius n^{-1} and large radius n. Clearly, $\boldsymbol{u}_0^n$ and $\boldsymbol{B}_0^n$ belong to all Sobolev spaces and, setting $\boldsymbol{v}_0^n := \boldsymbol{u}_0^n - \nabla\times\boldsymbol{B}_0^n$, we have

$$\|(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n, \boldsymbol{v}_0^n)\|_{\dot{H}^{\frac12}} \leq \|(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0)\|_{\dot{H}^{\frac12}}. \tag{3.3.10}$$

The classical well-posedness theory in Sobolev spaces (see e.g. [28]) ensures that the Hall-MHD system with data $(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n)$ has a unique maximal solution $(\boldsymbol{u}^n, \boldsymbol{B}^n)$ on $[0, T^n)$ for some $T^n > 0$, belonging to $\mathcal{C}([0,T]; H^m)\cap L^2(0,T; H^{m+1})$ for all $m\in\mathbb{N}$ and $T < T^n$. Since the solution is smooth, we have according to (3.3.9) for all $0\leq t_0\leq t < T^n$,

$$\|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)(t)\|^2_{\dot{H}^{\frac12}} + \frac12\int_{t_0}^t \|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|^2_{\dot{H}^{\frac32}}\,d\tau \leq \|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)(t_0)\|^2_{\dot{H}^{\frac12}}. \tag{3.3.11}$$

Step 2

Combining (3.1.2) and Theorem 1.48 in [45], we get

$$\int_0^{T^n} \|(\boldsymbol{u}^n, \nabla\boldsymbol{B}^n)\|^2_{BMO}\,dt \lesssim \|(\boldsymbol{u}^n, \boldsymbol{v}^n)\|^2_{L^2_{T^n}(\dot{H}^{\frac32})}.$$

Hence, the continuation criterion of [13] guarantees that $T^n = +\infty$. This means that the solution is global. Hence (3.3.11) is satisfied for all $0\leq t_0\leq t<\infty$ and, owing to (3.3.10), we have in addition

$$\|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|^2_{L^\infty(\dot{H}^{\frac12})} + \frac12\|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|^2_{L^2(\dot{H}^{\frac32})} \leq \|(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0)\|^2_{\dot{H}^{\frac12}}. \tag{3.3.12}$$

Step 3

We claim that, up to extraction, the sequence $(\boldsymbol{u}^n, \boldsymbol{B}^n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^3)$ to a solution $(\boldsymbol{u}, \boldsymbol{B})$ of (2.1.1)–(2.1.4) supplemented with data $(\boldsymbol{u}_0, \boldsymbol{B}_0)$. The definition of $(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n)$ and the fact that $(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0)$ belongs to $\dot{H}^{\frac12}$ already entail that

$$(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n, \boldsymbol{v}_0^n) \to (\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0) \quad \text{in} \quad \dot{H}^{\frac12}.$$

Proving the convergence of $(\boldsymbol{u}^n, \boldsymbol{B}^n, v^n)_{n\in\mathbb{N}}$ can be achieved from compactness arguments, after exhibiting bounds in suitable spaces for the solution sequences $(\partial_t \boldsymbol{u}^n, \partial_t \boldsymbol{B}^n, \partial_t v^n)_{n\in\mathbb{N}}$. Then, combining with compact embedding will enable us to apply Ascoli's theorem and to get the existence of a limit $(\boldsymbol{u}, \boldsymbol{B}, v)$ for a subsequence. Furthermore, the uniform bound (3.3.12) will provide us with additional regularity and convergence properties so that we will be able to pass to the limit in the Hall-MHD system.

To proceed, let us introduce $(\boldsymbol{u}_L, \boldsymbol{B}_L, v_L) := e^{t\Delta}(\boldsymbol{u}_0, \boldsymbol{B}_0, v_0)$, $(\boldsymbol{u}_L^n, \boldsymbol{B}_L^n, v_L^n) := \mathcal{F}^{-1}\big(\mathbf{1}_{\mathcal{C}_n}(\widehat{\boldsymbol{u}}_L, \widehat{\boldsymbol{B}}_L, \widehat{v}_L)\big)$ and $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{v}^n) := (\boldsymbol{u}^n - \boldsymbol{u}_L^n, \boldsymbol{B}^n - \boldsymbol{B}_L^n, v^n - v_L^n)$.

It is clear that $(\boldsymbol{u}_L^n, \boldsymbol{B}_L^n, v_L^n)$ tends to $(\boldsymbol{u}_L, \boldsymbol{B}_L, v_L)$ in $\mathcal{C}(\mathbb{R}_+; \dot{H}^{\frac12}) \cap L^2(\mathbb{R}_+; \dot{H}^{\frac32})$, which implies that $v_L = \boldsymbol{u}_L - \nabla \times \boldsymbol{B}_L$, since $v_L^n = \boldsymbol{u}_L^n - \nabla \times \boldsymbol{B}_L^n$ for all $n \in \mathbb{N}$.

Proving the convergence of $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{v}^n)_{n\in\mathbb{N}}$ relies on the following lemma:

Lemma 3.3.3. *Sequence $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{v}^n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}^{\frac14}_{loc}(\mathbb{R}_+; \dot{H}^{-1})$.*

Proof. Observe that $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{v}^n)(0) = (0,0,0)$ and that

$$\begin{cases} \partial_t \widetilde{\boldsymbol{u}}^n = \Delta \widetilde{\boldsymbol{u}}^n + \mathcal{P}\big(\operatorname{div}(\boldsymbol{B}^n \otimes \boldsymbol{B}^n) - \operatorname{div}(\boldsymbol{u}^n \otimes \boldsymbol{u}^n)\big), \\ \partial_t \widetilde{\boldsymbol{B}}^n = \Delta \widetilde{\boldsymbol{B}}^n + \nabla \times (v^n \times \boldsymbol{B}^n), \\ \partial_t \widetilde{v}^n = \Delta \widetilde{v}^n + \mathcal{P}\big(\operatorname{div}(\boldsymbol{B}^n \otimes \boldsymbol{B}^n) - \operatorname{div}(\boldsymbol{u}^n \otimes \boldsymbol{u}^n)\big) \\ \qquad\quad - \nabla \times \big((\nabla \times v^n) \times \boldsymbol{B}^n - v^n \times \boldsymbol{u}^n\big) + 2\nabla \times (v^n \cdot \nabla \boldsymbol{B}^n) \end{cases} \tag{3.3.13}$$

where $\mathcal{P} = \mathrm{Id} - \nabla\Delta^{-1}\operatorname{div}$ still stands for the Leray projector on divergence-free vector-fields.

Using the uniform bound (3.3.12) and the product law (B.5.3), we discover that the right-hand side of (3.3.13) is uniformly bounded in $L^{\frac43}_{loc}(\mathbb{R}_+; \dot{H}^{-1})$. Indeed, for example, owing to (B.5.3), the most nonlinear term $\nabla \times ((\nabla \times v^n) \times \boldsymbol{B}^n)$ is uniformly bounded in $L^{\frac43}(\mathbb{R}_+; \dot{H}^{-1})$ since $\nabla \times v^n$ and $\boldsymbol{B}^n$ are uniformly bounded in the spaces $L^2(\mathbb{R}_+; \dot{H}^{\frac12})$ and $L^4(\mathbb{R}_+; \dot{H}^1)$, respectively.

Once it is known that the sequences $(\partial_t \widetilde{\boldsymbol{u}}^n)$, $(\partial_t \widetilde{\boldsymbol{B}}^n)$ and $(\partial_t \widetilde{\boldsymbol{B}}^n)$ are bounded in the space $L^{\frac43}_{loc}(\mathbb{R}_+; \dot{H}^{-1})$, applying Hölder's inequality with respect to time completes the proof of the lemma as, obviously, $\widetilde{\boldsymbol{u}}^n(0) = \widetilde{\boldsymbol{B}}^n(0) = \widetilde{v}^n(0) = \mathbf{0}$. □

One can now turn to the proof of the existence of a solution. Let $(\phi_j)_{j\in\mathbb{N}}$ be a sequence of $\mathcal{C}^\infty_c(\mathbb{R}^3)$ cut-off functions supported in the ball $B(\mathbf{0}, j+1)$ of $\mathbb{R}^3$ and equal to 1 in a neighborhood of $B(\mathbf{0}, j)$.

Lemma 3.3.3 ensures that $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{v}^n)_{n\in\mathbb{N}}$ is uniformly equicontinuous in the space $\mathcal{C}([0,T]; \dot{H}^{-1})$ for all $T > 0$, and (3.3.12) tells us that it is bounded in $L^\infty(\mathbb{R}_+; \dot{H}^{\frac12})$. Using the fact that the application $z \mapsto \phi_j z$ is compact from $\dot{H}^{\frac12}$ into $\dot{H}^{-1}$, combining Ascoli's theorem and Cantor's diagonal process enables to conclude that there exists some triplet $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{v})$ such that for all $j \in \mathbb{N}$,

$$(\phi_j \widetilde{\boldsymbol{u}}^n, \phi_j \widetilde{\boldsymbol{B}}^n, \phi_j \widetilde{v}^n) \to (\phi_j \widetilde{\boldsymbol{u}}, \phi_j \widetilde{\boldsymbol{B}}, \phi_j \widetilde{v}) \quad \text{in} \quad \mathcal{C}(\mathbb{R}_+; \dot{H}^{-1}). \tag{3.3.14}$$

This obviously entails that $(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{v}^n)$ tends to $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{v})$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3)$, which is enough to pass to the limit in all the linear terms of (2.1.27) and to ensure that $\widetilde{v} = \widetilde{\boldsymbol{u}} - \nabla \times \widetilde{\boldsymbol{B}}$, and thus $v = \boldsymbol{u} - \nabla \times \boldsymbol{B}$.

From the estimates (3.3.12), interpolation and classical functional analysis arguments, we gather that $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{v})$ belongs to $L^\infty(0,T;\dot{H}^{\frac12}) \cap L^2(0,T;\dot{H}^{\frac32})$ and to $\mathcal{C}^{\frac14}([0,T];\dot{H}^{-1})$ for all $T > 0$, and better properties of convergence like, for instance,

$$\phi_j(\widetilde{\boldsymbol{u}}^n, \widetilde{\boldsymbol{B}}^n, \widetilde{v}^n) \to \phi_j(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{B}}, \widetilde{v}) \quad \text{in } L^2_{loc}(\mathbb{R}_+;\dot{H}^1) \quad \text{for all } j \in \mathbb{N}. \tag{3.3.15}$$

As an example, let us explain how to pass to the limit in the 'worst' term, namely $\nabla \times ((\nabla \times v^n) \times \boldsymbol{B}^n)$. Let $\boldsymbol{\theta} \in \mathcal{C}^\infty_c(\mathbb{R}_+ \times \mathbb{R}^3;\mathbb{R}^3)$ and $j \in \mathbb{N}$ be such that $\operatorname{Supp}\boldsymbol{\theta} \subset [0,j] \times B(\mathbf{0},j)$. We write

$$\begin{aligned}\langle \nabla \times ((\nabla \times v^n) \times \boldsymbol{B}^n), \boldsymbol{\theta}\rangle &- \langle \nabla \times ((\nabla \times v) \times \boldsymbol{B}), \boldsymbol{\theta}\rangle \\ &= \langle (\nabla \times v^n) \times \phi_j(\boldsymbol{B}^n - \boldsymbol{B}), \nabla \times \boldsymbol{\theta}\rangle + \langle (\nabla \times \phi_j(v^n - v)) \times \boldsymbol{B}, \nabla \times \boldsymbol{\theta}\rangle.\end{aligned}$$

Now, we have for all $T > 0$, owing to (B.5.3),

$$\|(\nabla \times v^n) \times \phi_j(\boldsymbol{B}^n - \boldsymbol{B})\|_{L^1(0,T;L^2)} \lesssim \|\nabla \times v^n\|_{L^2(0,T;\dot{H}^{\frac12})} \|\phi_j(\boldsymbol{B}^n - \boldsymbol{B})\|_{L^2(0,T;\dot{H}^1)},$$

$$\|(\nabla \times \phi_j(v^n - v)) \times \boldsymbol{B}\|_{L^{\frac43}(0,T;L^2)} \lesssim \|(\nabla \times \phi_j(v^n - v))\|_{L^2(0,T\times\mathbb{R}^3)} \|\boldsymbol{B}\|_{L^4(0,T;L^\infty)}.$$

Thanks to (3.3.12) and to (3.3.15), we see that the right-hand sides above converge to 0. Hence

$$\nabla \times ((\nabla \times v^n) \times \boldsymbol{B}^n) \to \langle \nabla \times ((\nabla \times v) \times \boldsymbol{B}) \quad \text{in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3).$$

Arguing similarly to pass to the limit in the other nonlinear terms, one may conclude that $(\boldsymbol{u}, \boldsymbol{B}, v)$ satisfies the extended formulation (2.1.27). Besides, as we know that $v = \boldsymbol{u} - \nabla \times \boldsymbol{B}$, the couple $(\boldsymbol{u}, \boldsymbol{B})$ satisfies the Hall-MHD system for some suitable pressure function P.

To prove that $(\boldsymbol{u}, \boldsymbol{B})$ is continuous in $\dot{H}^{\frac12}$, it suffices to notice that the properties of regularity of the solution ensure that $\boldsymbol{u}$ and $\boldsymbol{B}$ satisfy a heat equation with initial data in $\dot{H}^{\frac12}$ and right-hand side in $L^2(\mathbb{R}_+;\dot{H}^{-\frac12})$ (we do not know how to prove the time continuity with values in $\dot{H}^{\frac12}$ for $\nabla \boldsymbol{B}$ or, equivalently, v, though).

3.3.2 Uniqueness

Let us finally prove the uniqueness part of the theorem. Let $(\boldsymbol{u}_1, \boldsymbol{B}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{B}_2)$ be two solutions of the Hall-MHD system on $[0,T] \times \mathbb{R}^3$, supplemented with the same initial data $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ and such that, denoting $v_i = \boldsymbol{u}_i - \nabla \times \boldsymbol{B}_i$ for $i = 1,2$,

$$(\boldsymbol{u}_i, \boldsymbol{B}_i, v_i) \in L^\infty([0,T];\dot{H}^{\frac12}) \quad \text{and} \quad (\nabla \boldsymbol{u}_i, \nabla \boldsymbol{B}_i, \nabla v_i) \in L^2(0,T;\dot{H}^{\frac12}).$$

In order to prove the result, we shall estimate the difference $(\delta u, \delta B, \delta v) := (u_1 - u_2, B_1 - B_2, v_1 - v_2)$ in the space $\mathcal{C}([0,T];L^2) \cap L^2(0,T;\delta otH^1)$. In order to justify that, indeed, $(\delta u, \delta B, \delta v)$ belongs to that space, one can observe that

$$\begin{cases} \partial_t \delta u - \Delta \delta u := R_1, \\ \partial_t \delta B - \Delta \delta B := R_2, \\ \partial_t \delta v - \Delta \delta v := R_1 + R_3 + R_4 + R_5, \end{cases} \tag{3.3.16}$$

where

$$\begin{aligned} R_1 &:= \mathcal{P}(B_1 \cdot \nabla \delta B + \delta B \cdot \nabla B_2 - u_1 \cdot \nabla \delta u - \delta u \cdot \nabla u_2), \\ R_2 &:= \nabla \times (v_1 \times \delta B + \delta v \times B_2), \\ R_3 &:= -\nabla \times ((\nabla \times v_1) \times \delta B + (\nabla \times \delta v) \times B_2), \\ R_4 &:= \nabla \times (v_1 \times \delta u + \delta v \times u_2), \\ R_5 &:= 2\nabla \times (v_1 \cdot \nabla \delta B + \delta v \cdot \nabla B_2). \end{aligned}$$

Since $(\delta u, \delta B, \delta v)|_{t=0}$=0, in order to achieve our goal, it suffices to prove that R_1 to R_5 belong to the space $L^2(0,T;\dot{H}^{-1})$. Now, since $(\delta u, \delta B, \delta v) \in L^\infty(0,T;\dot{H}^{\frac{1}{2}}) \cap L^2(0,T;\dot{H}^{\frac{3}{2}})$, we have, by interpolation and Hölder inequality that $(\delta u, \delta B, \delta v) \in L^4(0,T;\dot{H}^1)$. Hence, using repeatedly (B.5.3), we get

$$\begin{aligned} \|R_1\|_{L^2(0,T;\dot{H}^{-1})} &\lesssim \|B_1 \otimes \delta B\|_{L^2(0,T;L^2)} + \|B_2 \otimes \delta B\|_{L^2(0,T;L^2)} \\ &\qquad + \|u_1 \otimes \delta u\|_{L^2(0,T;L^2)} + \|u_2 \otimes \delta u\|_{L^2(0,T;L^2)} \\ &\lesssim T^{\frac{1}{4}} \|(u_1, u_2, B_1, B_2)\|_{L^\infty(0,T;\dot{H}^{\frac{1}{2}})} \|(\delta u, \delta B)\|_{L^4(0,T;\dot{H}^1)}, \\ \|R_2\|_{L^2(0,T;\dot{H}^{-1})} &\lesssim \|v^1 \times \delta B\|_{L^2(0,T;L^2)} + \|\delta v \times B_2\|_{L^2(0,T;L^2)} \\ &\lesssim T^{\frac{1}{4}} \|(v_1, B_2)\|_{L^\infty(0,T;\dot{H}^{\frac{1}{2}})} \|(\delta v, \delta B)\|_{L^4(0,T;\dot{H}^1)}, \end{aligned}$$

$$\begin{aligned} \|R_3\|_{L^2(0,T;\dot{H}^{-1})} &\lesssim \|(\nabla \times v_1) \times \delta B\|_{L^2(0,T;L^2)} + \|(\nabla \times \delta v) \times B_2\|_{L^2(0,T;L^2)} \\ &\lesssim \|v_1\|_{L^2(0,T;\dot{H}^{\frac{3}{2}})} \|\delta B\|_{L^\infty(0,T;\dot{H}^1)} + \|\delta v\|_{L^2(0,T;\dot{H}^{\frac{3}{2}})} \|B_2\|_{L^\infty(0,T;\dot{H}^1)} \\ &\lesssim \|(v_1, \delta v)\|_{L^2(0,T;\dot{H}^{\frac{3}{2}})} \|(B_2, \delta B)\|_{L^\infty(0,T;\dot{H}^1)}. \end{aligned}$$

Note that our assumptions ensure that B_i and ∇B_i are in $L^\infty(0,T;\dot{H}^{\frac{1}{2}})$ and thus we do have, by interpolation inequality (B.5.1), B_i in $L^\infty(0,T;\dot{H}^1)$ for $i = 1, 2$. Terms R_4 and R_5 may be treated similarly.

Estimating $(\delta u, \delta B, \delta v)$ in the space $L^\infty(0,T;L^2) \cap L^2(0,T;\dot{H}^1)$ follows from a standard energy method applied on (3.3.16), Hölder's inequality Sobolev embedding and, again, (B.5.3). More precisely, we have[1]

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\delta u\|_{L^2}^2 + \|\delta u\|_{\dot{H}^1}^2 &\lesssim (\|B_1 \otimes \delta B\|_{L^2} + \|B_2 \otimes \delta B\|_{L^2} + \|u_2 \otimes \delta u\|_{L^2}) \|\nabla \delta u\|_{L^2} \\ &\lesssim (\|(B_1, B_2)\|_{\dot{H}^1} \|\delta B\|_{\dot{H}^{\frac{1}{2}}} + \|u_2\|_{\dot{H}^1} \|\delta u\|_{\dot{H}^{\frac{1}{2}}}) \|\delta u\|_{\dot{H}^1}, \end{aligned}$$

[1] Somehow, the following computations are formal. One can make them rigorous by using a regularization procedure as in Subsection **??**.

$$\frac{1}{2}\frac{d}{dt}\|\delta\boldsymbol{B}\|_{L^2}^2+\|\delta\boldsymbol{B}\|_{\dot{H}^1}^2 \lesssim (\|\boldsymbol{v}_1\times\delta\boldsymbol{B}\|_{L^2}+\|\delta\boldsymbol{v}\times\boldsymbol{B}_2\|_{L^2})\|\nabla\delta\boldsymbol{B}\|_{L^2}$$
$$\lesssim (\|\boldsymbol{v}_1\|_{\dot{H}^1}\|\delta\boldsymbol{B}\|_{\dot{H}^{\frac{1}{2}}}+\|\delta\boldsymbol{v}\|_{\dot{H}^{\frac{1}{2}}}\|\boldsymbol{B}_2\|_{\dot{H}^1})\|\delta\boldsymbol{B}\|_{\dot{H}^1}$$

and, using (3.1.2),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\delta\boldsymbol{v}\|_{L^2}^2+\|\delta\boldsymbol{v}\|_{\dot{H}^1}^2\\
&\lesssim (\|\boldsymbol{B}_1\otimes\delta\boldsymbol{B}\|_{L^2}+\|\boldsymbol{B}_2\otimes\delta\boldsymbol{B}\|_{L^2}+\|\boldsymbol{u}_1\otimes\delta\boldsymbol{u}\|_{L^2}+\|\boldsymbol{u}_2\otimes\delta\boldsymbol{u}\|_{L^2})\|\nabla\delta\boldsymbol{v}\|_{L^2}\\
&\quad+\|(\nabla\times\boldsymbol{v}_1)\times\delta\boldsymbol{B}\|_{L^2}\|\nabla\times\delta\boldsymbol{v}\|_{L^2}\\
&\quad+(\|\boldsymbol{v}_1\times\delta\boldsymbol{u}\|_{L^2}+\|\delta\boldsymbol{v}\times\boldsymbol{u}_2\|_{L^2})\|\nabla\times\delta\boldsymbol{v}\|_{L^2}\\
&\quad+(\|\boldsymbol{v}_1\cdot\nabla\delta\boldsymbol{B}\|_{L^2}+\|\delta\boldsymbol{v}\cdot\nabla\boldsymbol{B}_2\|_{L^2})\|\nabla\times\delta\boldsymbol{v}\|_{L^2}\\
&\lesssim \Big(\|\boldsymbol{B}_1\|_{\dot{H}^1}\|\delta\boldsymbol{B}\|_{\dot{H}^{\frac{1}{2}}}+\|\boldsymbol{B}_2\|_{\dot{H}^1}\|\delta\boldsymbol{B}\|_{\dot{H}^{\frac{1}{2}}}+\|\boldsymbol{u}_1\|_{\dot{H}^1}\|\delta\boldsymbol{u}\|_{\dot{H}^{\frac{1}{2}}}+\|\boldsymbol{u}_2\|_{\dot{H}^1}\|\delta\boldsymbol{u}\|_{\dot{H}^{\frac{1}{2}}}\\
&\quad+\|\nabla\times\boldsymbol{v}_1\|_{\dot{H}^{\frac{1}{2}}}(\|\delta\boldsymbol{u}\|_{L^2}+\|\delta\boldsymbol{v}\|_{L^2})+\|\boldsymbol{v}_1\|_{\dot{H}^1}\|\delta\boldsymbol{u}\|_{\dot{H}^{\frac{1}{2}}}+\|\delta\boldsymbol{v}\|_{\dot{H}^1}\|\boldsymbol{u}_2\|_{\dot{H}^{\frac{1}{2}}}\\
&\quad+\|\boldsymbol{v}_1\|_{\dot{H}^1}(\|\delta\boldsymbol{u}\|_{\dot{H}^{\frac{1}{2}}}+\|\delta\boldsymbol{v}\|_{\dot{H}^{\frac{1}{2}}})+\|\delta\boldsymbol{v}\|_{\dot{H}^{\frac{1}{2}}}\|\nabla\boldsymbol{B}_2\|_{\dot{H}^1}\Big)\|\delta\boldsymbol{v}\|_{\dot{H}^1}.
\end{aligned}$$

At this stage, interpolation and Young's inequality imply that

$$\|\boldsymbol{B}_1\|_{\dot{H}^1}\|\delta\boldsymbol{B}\|_{\dot{H}^{\frac{1}{2}}}\|\delta\boldsymbol{u}\|_{\dot{H}^1}\lesssim\|\boldsymbol{B}_1\|_{\dot{H}^1}\|\delta\boldsymbol{B}\|_{L^2}^{\frac{1}{2}}\|\delta\boldsymbol{B}\|_{\dot{H}^1}^{\frac{1}{2}}\|\delta\boldsymbol{u}\|_{\dot{H}^1}$$
$$\leq\frac{1}{10}\|(\delta\boldsymbol{B},\delta\boldsymbol{u})\|_{\dot{H}^1}^2+C\|\delta\boldsymbol{B}\|_{L^2}^2\|\boldsymbol{B}_1\|_{\dot{H}^1}^4,$$

and similar inequalities for all the terms of the right-hand sides of the above inequalities, except for the one with $\|\nabla\times\boldsymbol{v}_1\|_{\dot{H}^{\frac{1}{2}}}$ that we bound as follows:

$$\|\nabla\times\boldsymbol{v}_1\|_{\dot{H}^{\frac{1}{2}}}\|(\delta\boldsymbol{u},\delta\boldsymbol{v})\|_{L^2}\|\delta\boldsymbol{v}\|_{\dot{H}^1}\leq\frac{1}{10}\|\delta\boldsymbol{v}\|_{\dot{H}^1}^2+C\|\boldsymbol{v}_1\|_{\dot{H}^{\frac{3}{2}}}^2\|(\delta\boldsymbol{u},\delta\boldsymbol{v})\|_{L^2}^2.$$

In the end, we get for all $t\in(0,T)$,

$$\frac{1}{2}\frac{d}{dt}\|(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})\|_{L^2}^2+\|(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})\|_{\dot{H}^1}^2\lesssim V(t)\|(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})\|_{L^2}^2$$

with

$$V(t):=\|(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{B}_1,\boldsymbol{B}_2,\boldsymbol{v}_1,\boldsymbol{v}_2)(t)\|_{\dot{H}^1}^4+\|\boldsymbol{v}_1(t)\|_{\dot{H}^{\frac{3}{2}}}^2.$$

Since our assumptions ensure that V is integrable on $[0,T]$ and $(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})(0)=0$, applying Gronwall's inequality yields

$$(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})\equiv 0\ \text{ in }\ L^\infty(0,T;L^2(\mathbb{R}^3)).$$

3.3.3 Propagation of higher Sobolev regularity

Let us finally explain the propagation of regularity $H^s\times H^{s+1}$ if the initial data $(\boldsymbol{u}_0,\boldsymbol{B}_0)$ are, additionally, in $H^s\times H^{s+1}$ for some $s\geq 0$. Our aim is to prove that the solution $(\boldsymbol{u},\boldsymbol{B})$ constructed above is in $\mathcal{C}_b(\mathbb{R}_+;H^s\times H^{s+1})$, and such that $(\nabla\boldsymbol{u},\nabla\boldsymbol{B})\in L^2(\mathbb{R}_+;H^s\times H^{s+1})$.

For the time being, let us assume that $(\boldsymbol{u},\boldsymbol{B})$ is smooth and explain how to perform estimates in Sobolev spaces. First, we multiply (2.1.1) and (1.1.3) by $\boldsymbol{u}$ and $\boldsymbol{B}$, respectively, integrate and add up the resulting equations. Using the fact that

$$(\nabla\times(\boldsymbol{J}\times\boldsymbol{B}),\boldsymbol{B})=(\boldsymbol{J}\times\boldsymbol{B},\boldsymbol{J})_{L^2}=0,$$

one gets the following energy balance:

$$\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{u}\|_{L^2}^2+\|\boldsymbol{B}\|_{L^2}^2)+\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla\boldsymbol{B}\|_{L^2}^2=0. \tag{3.3.17}$$

Let Λ^s denote the fractional derivative operator defined in the Appendix B. Since

$$\|\boldsymbol{u}\|_{H^s}+\|\boldsymbol{B}\|_{H^{s+1}}\approx\|(\boldsymbol{u},\boldsymbol{B})\|_{L^2}+\|(\Lambda^s\boldsymbol{u},\Lambda^s\boldsymbol{B},\Lambda^s\boldsymbol{v})\|_{L^2},$$

in order to prove the desired Sobolev estimates, it suffices to get a suitable control on $\|\Lambda^s\boldsymbol{u}\|_{L^2}$ and on $\|\Lambda^{s+1}\boldsymbol{B}\|_{L^2}$. To this end, apply Λ^s to (1.1.1), then take the L^2 scalar product with $\Lambda^s\boldsymbol{u}$. We get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^s\boldsymbol{u}\|_{L^2}^2+\|\Lambda^s\nabla\boldsymbol{u}\|_{L^2}^2&=(\Lambda^s(\boldsymbol{B}\otimes\boldsymbol{B})\,|\,\Lambda^s\nabla\boldsymbol{u})-(\Lambda^s(\boldsymbol{u}\otimes\boldsymbol{u})\,|\,\Lambda^s\nabla\boldsymbol{u})\\&=:E_1+E_2.\end{aligned}$$

In order to control $\|\Lambda^{s+1}\boldsymbol{B}\|_{L^2}$, one has to use the cancellation property (2.1.28). Then, applying Λ^s to the second and third equation of (2.1.27) and taking the L^2 scalar product with $\Lambda^s\boldsymbol{B}$, $\Lambda^s\boldsymbol{v}$, respectively, yields:

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\boldsymbol{B}\|_{L^2}^2+\|\Lambda^s\nabla\boldsymbol{B}\|_{L^2}^2=(\Lambda^s(\boldsymbol{v}\times\boldsymbol{B})\,|\,\Lambda^s\nabla\times\boldsymbol{B})=:E_3,$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Lambda^s\boldsymbol{v}\|_{L^2}^2+\|\Lambda^s\nabla\boldsymbol{v}\|_{L^2}^2\\&=(\Lambda^s(\boldsymbol{B}\otimes\boldsymbol{B})\,|\,\Lambda^s\nabla\boldsymbol{v})-(\Lambda^s(\boldsymbol{u}\otimes\boldsymbol{u})\,|\,\Lambda^s\nabla\boldsymbol{v})\\&\quad-(\Lambda^s((\nabla\times\boldsymbol{v})\times\boldsymbol{B})-(\Lambda^s\nabla\times\boldsymbol{v})\times\boldsymbol{B}\,|\,\Lambda^s\nabla\times\boldsymbol{v})+(\Lambda^s(\boldsymbol{v}\times\boldsymbol{u})\,|\,\Lambda^s\nabla\times\boldsymbol{v})\\&\quad+2(\Lambda^s(\boldsymbol{v}\cdot\nabla\boldsymbol{B})\,|\,\Lambda^s\nabla\times\boldsymbol{v})=:E_4+E_5+\cdots+E_8.\end{aligned}$$

Sobolev embedding, Young's inequality and Lemma B.5.3, imply that

$$\begin{aligned}|E_1|&\lesssim\|\Lambda^s\boldsymbol{B}\|_{L^6}\|\boldsymbol{B}\|_{L^3}\|\Lambda^s\nabla\boldsymbol{u}\|_{L^2}\\&\lesssim\|\boldsymbol{B}\|_{\dot{H}^{\frac{1}{2}}}(\|\Lambda^s\nabla\boldsymbol{B}\|_{L^2}^2+\|\Lambda^s\nabla\boldsymbol{u}\|_{L^2}^2),\\|E_2|&\lesssim\|\boldsymbol{u}\|_{\dot{H}^{\frac{1}{2}}}\|\Lambda^s\nabla\boldsymbol{u}\|_{L^2}^2,\\|E_3|&\lesssim(\|\Lambda^s\boldsymbol{v}\|_{L^6}\|\boldsymbol{B}\|_{L^3}+\|\Lambda^s\boldsymbol{B}\|_{L^6}\|\boldsymbol{v}\|_{L^3})\|\Lambda^s\nabla\times\boldsymbol{B}\|_{L^2}\\&\lesssim\|(\boldsymbol{B},\boldsymbol{v})\|_{\dot{H}^{\frac{1}{2}}}(\|\Lambda^s\nabla\boldsymbol{B}\|_{L^2}^2+\|\Lambda^s\nabla\boldsymbol{v}\|_{L^2}^2),\\|E_4|&\lesssim\|\boldsymbol{B}\|_{\dot{H}^{\frac{1}{2}}}(\|\Lambda^s\nabla\boldsymbol{B}\|_{L^2}^2+\|\Lambda^s\nabla\boldsymbol{v}\|_{L^2}^2),\\|E_5|&\lesssim\|\boldsymbol{u}\|_{\dot{H}^{\frac{1}{2}}}(\|\Lambda^s\nabla\boldsymbol{u}\|_{L^2}^2+\|\Lambda^s\nabla\boldsymbol{v}\|_{L^2}^2),\\|E_6|&\lesssim(\|\nabla\boldsymbol{B}\|_{L^3}\|\Lambda^{s-1}\nabla\times\boldsymbol{v}\|_{L^6}+\|\Lambda^s\boldsymbol{B}\|_{L^6}\|\nabla\times\boldsymbol{v}\|_{L^3})\|\Lambda^s\nabla\boldsymbol{v}\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\leq (C\|\nabla \boldsymbol{B}\|_{\dot{H}^{\frac{1}{2}}} + \tfrac{1}{2})\|\Lambda^s \nabla \boldsymbol{v}\|_{L^2}^2 + C\|\Lambda^s \nabla \boldsymbol{B}\|_{L^2}^2 \|\boldsymbol{v}\|_{\dot{H}^{\frac{3}{2}}}^2,\\
|E_7| &\lesssim (\|\Lambda^s \boldsymbol{v}\|_{L^6}\|\boldsymbol{u}\|_{L^3} + \|\Lambda^s \boldsymbol{u}\|_{L^6}\|\boldsymbol{v}\|_{L^3})\|\Lambda^s \nabla \boldsymbol{v}\|_{L^2}\\
&\lesssim \|(\boldsymbol{u},\boldsymbol{v})\|_{\dot{H}^{\frac{1}{2}}}(\|\Lambda^s \nabla \boldsymbol{u}\|_{L^2}^2 + \|\Lambda^s \nabla \boldsymbol{v}\|_{L^2}^2),\\
|E_8| &\lesssim (\|\Lambda^s \boldsymbol{v}\|_{L^6}\|\nabla \boldsymbol{B}\|_{L^3} + \|\Lambda^s \nabla \boldsymbol{B}\|_{L^6}\|\boldsymbol{v}\|_{L^3})\|\Lambda^s \nabla \boldsymbol{v}\|_{L^2}\\
&\lesssim \|(\nabla \boldsymbol{B},\boldsymbol{v})\|_{\dot{H}^{\frac{1}{2}}}(\|\Lambda^s \nabla \boldsymbol{B}\|_{\dot{H}^1}^2 + \|\Lambda^s \nabla \boldsymbol{v}\|_{L^2}^2).
\end{aligned}
$$

Since $\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\dot{H}^{\frac{1}{2}}}$ is small, putting the above estimates and (3.3.17) together, and using (3.1.2) and $\boldsymbol{J} = \boldsymbol{u} - \boldsymbol{v}$, one gets after time integration that

$$
\begin{aligned}
&\|\boldsymbol{u}(t)\|_{\dot{H}^s}^2 + \|\boldsymbol{B}(t)\|_{\dot{H}^s}^2 + \|\boldsymbol{v}(t)\|_{\dot{H}^s}^2 + \int_0^t (\|\nabla \boldsymbol{u}\|_{\dot{H}^s}^2 + \|\nabla \boldsymbol{B}\|_{\dot{H}^s}^2 + \|\nabla \boldsymbol{v}\|_{\dot{H}^s}^2)\,d\tau\\
&\qquad \leq \|\boldsymbol{u}_0\|_{\dot{H}^s}^2 + \|\boldsymbol{B}_0\|_{\dot{H}^s}^2 + \|\boldsymbol{v}_0\|_{\dot{H}^s}^2 + C\int_0^t (\|\boldsymbol{u}\|_{\dot{H}^s}^2 + \|\boldsymbol{v}\|_{\dot{H}^s}^2)\|\boldsymbol{v}\|_{\dot{H}^{\frac{3}{2}}}^2\,d\tau.
\end{aligned}
$$

By Gronwall's inequality, we then get for all $t \geq 0$,

$$
\begin{aligned}
&\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(t)\|_{\dot{H}^s}^2 + \int_0^t \|(\nabla \boldsymbol{u}, \nabla \boldsymbol{B}, \nabla \boldsymbol{v})\|_{\dot{H}^s}^2\,d\tau\\
&\qquad\qquad \leq \|(\boldsymbol{u}_0,\boldsymbol{B}_0,\boldsymbol{v}_0)\|_{\dot{H}^s} \exp\biggl(C\int_0^t \|\boldsymbol{v}\|_{\dot{H}^{\frac{3}{2}}}^2\,d\tau\biggr).
\end{aligned}
$$

Putting together with (3.3.17) and using that $\int_0^t \|\boldsymbol{v}(\tau)\|_{\dot{H}^{\frac{3}{2}}}^2\,d\tau$ is bounded thanks to the first part of the theorem, we get a global-in-time control of the Sobolev norms.

Of course, to make the proof rigorous, one has to smooth out the data. For that, one can proceed exactly as in Subsection 3.1.4. This completes the proof of Theorem 2.1.5. □

3.3.4 Proof of Corollary 2.1.6

As the solution $(\boldsymbol{u},\boldsymbol{B})$ belongs to

$$
\mathcal{C}_b(\mathbb{R}_+;\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+;\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))
$$

and $J \in \mathcal{C}_b(\mathbb{R}_+;\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+;\dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$, the interpolation inequality between Sobolev norms (B.5.1) implies that $(\boldsymbol{u},\boldsymbol{B},\boldsymbol{J})$ belongs to $L^8_{loc}(\mathbb{R}_+;\dot{H}^{\frac{3}{4}}(\mathbb{R}^3))$, which, in view of Sobolev embedding (B.5.2), is a subspace of $L^4_{loc}(\mathbb{R}_+;L^4(\mathbb{R}^3))$. Now, the right-hand sides of the first two equations of (2.1.27) belong to $L^2_{loc}(\mathbb{R}_+;\dot{H}^{-1}(\mathbb{R}^3))$. As, furthermore, $\boldsymbol{u}_0$ and $\boldsymbol{B}_0$ are in $L^2(\mathbb{R}^3)$, one can conclude to time continuity with values in $L^2(\mathbb{R}^3)$ thanks to e.g. [45, Th. 5.10].

In order to prove that the energy balance is fulfilled, one can use the same approximation scheme as in the proof of existence (the energy balance is clearly satisfied by $(\boldsymbol{u}_n,\boldsymbol{B}_n)$) then observe that $(\boldsymbol{u}_n,\boldsymbol{B}_n)_{n\in\mathbb{N}}$ is actually a Cauchy sequence in $L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+;\dot{H}^1(\mathbb{R}^3))$, as may be checked by arguing as in the proof of uniqueness.

Let us next prove that (u, B, v) goes to $\mathbf{0}$ in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ when $t \to \infty$. Inequality (2.1.30) and interpolation guarantee that $B \in L^4(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3))$. Hence, as $v = u - \nabla \times B$, one can find some $t_0 \geq 0$ so that $v(t_0) \in L^2(\mathbb{R}^3)$. Then, performing an energy estimate on the equation satisfied by v, we get for all $t \geq t_0$,

$$\|v(t)\|_{L^2}^2 + \int_{t_0}^t \|\nabla v\|_{L^2}^2 \, d\tau \leq \|v(t_0)\|_{L^2}^2 + \int_{t_0}^t \big(\|B \otimes B - u \otimes u\|_{L^2} + \|v \times u\|_{L^2} + \|v \cdot \nabla B\|_{L^2}\big)^2 \, d\tau.$$

Using (B.5.3) repeatedly, the equivalence (3.1.2) and adding up to (2.1.17) yields

$$\|(u, B, v)(t)\|_{L^2}^2 + \int_{t_0}^t \|(u, B, v)\|_{\dot{H}^1}^2 \, d\tau \leq \|(u, B, v)(t_0)\|_{L^2}^2 + C \int_{t_0}^t \|(u, B, v)\|_{\dot{H}^1}^2 \|(u, B, v)\|_{\dot{H}^{\frac{1}{2}}}^2 \, d\tau.$$

Since $\|(u, B, v)(t)\|_{\dot{H}^{\frac{1}{2}}}$ is small for all $t \geq 0$, the last term may be absorbed by the left-hand side, and one can conclude (by interpolation) that $(u, B, v) \in L^4(t_0, \infty; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$. Therefore, for all $\sigma > 0$ one may find some $t_1 \geq t_0$ so that $\|(u, B, v)(t_1)\|_{\dot{H}^{\frac{1}{2}}} \leq \sigma$. Observing that (2.1.30) implies that $t \mapsto \|(u, B, v)(t)\|_{\dot{H}^{\frac{1}{2}}}$ is non-increasing on $[t_1, \infty)$ allows to conclude the proof of (2.1.31). □

3.3.5 Proof of Corollary 2.1.7

We shall argue as in [45] and [56], splitting the data into a small part in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and a (possibly) large part in $L^2(\mathbb{R}^3)$. More precisely, we set

$$v_0 = u_0 - \nabla \times B_0, \quad u_0 = u_{0,\ell} + u_{0,h}, \quad B_0 = B_{0,\ell} + B_{0,h}, \quad v_0 = v_{0,\ell} + v_{0,h}$$

with

$$u_{0,\ell} := \mathcal{F}^{-1}(\mathbf{1}_{B(\mathbf{0},\rho)} \widehat{u_0}), \; B_{0,\ell} := \mathcal{F}^{-1}(\mathbf{1}_{B(\mathbf{0},\rho)} \widehat{B_0}), \; v_{0,\ell} := \mathcal{F}^{-1}(\mathbf{1}_{B(\mathbf{0},\rho)} \widehat{v_0}).$$

Fix some $\eta \in (0, c_2)$ (with c_2 being the constant of (2.1.29)) and choose ρ such that

$$\|(u_{\ell,0}, B_{\ell,0}, v_{\ell,0})\|_{\dot{H}^{\frac{1}{2}}} < \frac{\eta}{2}.$$

By Theorem 2.1.5, we know that there exists a unique global solution (u_ℓ, B_ℓ) to the Hall-MHD system supplemented with data $(u_{\ell,0}, B_{\ell,0})$, that satisfies

$$\|(u_\ell, B_\ell, v_\ell)\|_{L^\infty(\dot{H}^{\frac{1}{2}})}^2 + \frac{1}{2}\|(u_\ell, B_\ell, v_\ell)\|_{L^2(\dot{H}^{\frac{3}{2}})}^2 < \frac{\eta}{2} \quad \text{with} \quad v_\ell := u_\ell - \nabla \times B_\ell. \tag{3.3.18}$$

Let $(u_h, B_h, v_h) := (u - u_\ell, B - B_\ell, v - v_\ell)$. We have $(u_h, B_h, v_h) \in \mathcal{C}(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}) \cap L^4(\mathbb{R}_+; \dot{H}^1)$ since that result holds for both (u, B, v) and (u_ℓ, B_ℓ, v_ℓ) (use interpolation as in the proof of Corollary 2.1.7). Furthermore, $(u_{0,h}, B_{h,0}, J_{h,0})$ is

in $L^2(\mathbb{R}^3)$ owing to the low-frequency cut-off and we have

$$\begin{cases} \partial_t \boldsymbol{u}_h - \Delta \boldsymbol{u}_h := \widetilde{\boldsymbol{R}}_1, \\ \partial_t \boldsymbol{B}_h - \Delta \boldsymbol{B}_h := \widetilde{\boldsymbol{R}}_2, \\ \partial_t \boldsymbol{v}_h - \Delta \boldsymbol{v}_h := \widetilde{\boldsymbol{R}}_1 + \widetilde{\boldsymbol{R}}_3 + \widetilde{\boldsymbol{R}}_4 + \widetilde{\boldsymbol{R}}_5, \\ (\boldsymbol{u}_h, \boldsymbol{B}_h, \boldsymbol{v}_h)|_{t=0} = (\boldsymbol{u}_{0,h},\ \boldsymbol{B}_{h,0},\ J_{h,0}), \end{cases} \tag{3.3.19}$$

where

$$\begin{aligned} \widetilde{\boldsymbol{R}}_1 &:= \mathcal{P}(\boldsymbol{B}\cdot\nabla \boldsymbol{B}_h + \boldsymbol{B}_h\cdot\nabla \boldsymbol{B}_\ell - \boldsymbol{u}\cdot\nabla \boldsymbol{u}_h - \boldsymbol{u}_h\cdot\nabla \boldsymbol{u}_\ell), \\ \widetilde{\boldsymbol{R}}_2 &:= \nabla\times(\boldsymbol{v}\times\boldsymbol{B}_h + \boldsymbol{v}_h\times\boldsymbol{B}_\ell), \\ \widetilde{\boldsymbol{R}}_3 &:= -\nabla\times((\nabla\times\boldsymbol{v})\times\boldsymbol{B}_h + (\nabla\times\boldsymbol{v}_h)\times\boldsymbol{B}_\ell), \\ \widetilde{\boldsymbol{R}}_4 &:= \nabla\times(\boldsymbol{v}\times\boldsymbol{u}_h + \boldsymbol{v}_h\times\boldsymbol{u}_\ell), \\ \widetilde{\boldsymbol{R}}_5 &:= 2\nabla\times(\boldsymbol{v}\cdot\nabla \boldsymbol{B}_h + \boldsymbol{v}_h\cdot\nabla \boldsymbol{B}_\ell). \end{aligned}$$

Let us bound the terms $\widetilde{\boldsymbol{R}}_1$, $\widetilde{\boldsymbol{R}}_2$, $\widetilde{\boldsymbol{R}}_4$ and $\widetilde{\boldsymbol{R}}_5$ as in the proof of the uniqueness part of Theorem 2.1.5, and estimate $\widetilde{\boldsymbol{R}}_3$ as follows:

$$\begin{aligned} \|\widetilde{\boldsymbol{R}}_3\|_{L^2_T(\dot H^{-1})} &\le \|(\nabla\times\boldsymbol{v})\times\boldsymbol{B}_h\|_{L^2_T(L^2)} + \|(\nabla\times\boldsymbol{v}_h)\times\boldsymbol{B}_\ell\|_{L^2_T(L^2)} \\ &\le \|\boldsymbol{v}\|_{L^4_T(\dot H^1)}\|\boldsymbol{B}_h\|_{L^4_T(L^\infty)} + \|\boldsymbol{v}_h\|_{L^4_T(\dot H^1)}\|\boldsymbol{B}_\ell\|_{L^4_T(L^\infty)}. \end{aligned}$$

Note that our assumptions ensure that $\boldsymbol{B}_\ell$ and $\boldsymbol{B}_h$ are in $L^4(0,T;\dot H^1\cap\dot H^2)$, thus in $L^4(0,T;L^\infty)$ owing to the Gagliardo-Nirenberg inequality (B.5.5) with $s=1$ and $s'=1$. Then one can conclude by a straightforward energy argument that

$$\begin{aligned} &\|(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{v}_h)(t)\|_{L^2}^2 + 2\int_0^t \|(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{v}_h)\|_{\dot H^1}^2\,d\tau \le \|(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{v}_h)(0)\|_{L^2}^2 \\ &\quad + C\int_0^t \widetilde V\|(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{v}_h)\|_{L^2}^2\,d\tau \quad\text{with}\quad \widetilde V(t) := \|(\boldsymbol{u},\boldsymbol{u}_\ell,\boldsymbol{B},\boldsymbol{B}_\ell,\boldsymbol{v},\boldsymbol{v}_\ell)(t)\|_{\dot H^1}^4 \end{aligned}$$

and Gronwall's lemma thus implies that

$$\begin{aligned} &\|(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{v}_h)(t)\|_{L^2}^2 + \int_0^t \|(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{v}_h)(\tau)\|_{\dot H^1}^2\,d\tau \\ &\qquad\qquad \le \|(\boldsymbol{u}_{h,0},\boldsymbol{B}_{h,0},\boldsymbol{v}_{h,0})\|_{L^2}^2 \exp\Big(C\int_0^t \widetilde V(\tau)\,d\tau\Big). \end{aligned}$$

Since $\widetilde V$ is globally integrable on $\mathbb{R}_+$ thanks to our assumptions and (3.3.18), we see by interpolation that $(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{v}_h)$ is in $L^4(\mathbb{R}_+;\dot H^{\frac12})$. This in particular implies that there exists some $t_0\ge 0$ such that $\|(\boldsymbol{u}_h,\boldsymbol{B}_h,\boldsymbol{v}_h)(t_0)\|_{\dot H^{\frac12}} < \eta/2$. Hence $\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(t_0)\|_{\dot H^{\frac12}} < \eta$ and global existence result in Theorem 2.1.5 thus ensures that $\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(t)\|_{\dot H^{\frac12}} < \eta$ for all $t\ge t_0$. This completes the proof of the corollary. □

3.3.6 Weak-strong uniqueness

This section is devoted to the proof of Theorem 2.1.8. Let us underline that the proof works for any positive coefficients μ, ν and h. Furthermore, it could be adapted to the $2\frac{1}{2}$D flows of the next section. For expository purpose however, we focus on the 3D case.

Throughout, we shall repeatedly use the following result.

Lemma 3.3.4. *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ be three divergence free vector fields in $\mathbb{R}^3$. The following inequalities hold:*

- *If, in addition, $\boldsymbol{b}$ belongs to $L^4(0,T;\dot{H}^1(\mathbb{R}^3))$, then*

$$\Big|\int_0^T (\boldsymbol{a}\cdot\nabla\boldsymbol{b}\,|\,\boldsymbol{c})\,d\tau\Big| \lesssim \|\boldsymbol{a}\|^{\frac12}_{L^\infty_T(L^2)}\|\boldsymbol{a}\|^{\frac12}_{L^2_T(\dot{H}^1)}\|\boldsymbol{b}\|_{L^4_T(\dot{H}^1)}\|\boldsymbol{c}\|_{L^2_T(\dot{H}^1)}. \tag{3.3.20}$$

- *If, in addition, $\boldsymbol{c}$ belongs to $L^4(0,T;\dot{H}^1(\mathbb{R}^3))$, then*

$$\Big|\int_0^T (\boldsymbol{a}\cdot\nabla\boldsymbol{b}\,|\,\boldsymbol{c})\,d\tau\Big| \lesssim \|\boldsymbol{a}\|^{\frac12}_{L^\infty_T(L^2)}\|\boldsymbol{a}\|^{\frac12}_{L^2_T(\dot{H}^1)}\|\boldsymbol{b}\|_{L^2_T(\dot{H}^1)}\|\boldsymbol{c}\|_{L^4_T(\dot{H}^1)}. \tag{3.3.21}$$

- *If, in addition, $\nabla\times\boldsymbol{c}$ belongs to $L^4(0,T;\dot{H}^1(\mathbb{R}^3))$, then*

$$\begin{aligned}&\Big|\int_0^T (\nabla\times((\nabla\times\boldsymbol{a})\times\boldsymbol{b})\,|\,\boldsymbol{c})\,d\tau\Big|\\ &\lesssim \|\boldsymbol{a}\|_{L^2_T(\dot{H}^1)}\|\boldsymbol{b}\|^{\frac12}_{L^\infty_T(L^2)}\|\boldsymbol{b}\|^{\frac12}_{L^2_T(\dot{H}^1)}\|\nabla\times\boldsymbol{c}\|_{L^4_T(\dot{H}^1)}.\end{aligned} \tag{3.3.22}$$

Proof. To prove the first inequality, we use the identity (2.1.18) and the duality inequality between $\dot{H}^1$ and $\dot{H}^{-1}$ so as to write

$$\Big|\int_0^T (\boldsymbol{a}\cdot\nabla\boldsymbol{b}\,|\,\boldsymbol{c})\,d\tau\Big| \le \int_0^T \|\boldsymbol{b}\otimes\boldsymbol{a}\|_{L^2}\|\boldsymbol{c}\|_{\dot{H}^1}\,d\tau.$$

Hence, thanks to Hölder and Gagliardo-Nirenberg inequalities, and Sobolev embedding (B.5.2),

$$\begin{aligned}\Big|\int_0^T (\boldsymbol{a}\cdot\nabla\boldsymbol{b}\,|\,\boldsymbol{c})\,d\tau\Big| &\le \int_0^t \|\boldsymbol{a}\|_{L^3}\|\boldsymbol{b}\|_{L^6}\|\nabla\boldsymbol{c}\|_{L^2}\,d\tau\\ &\lesssim \int_0^T \|\boldsymbol{a}\|_{\dot{H}^{\frac12}}\|\boldsymbol{b}\|_{\dot{H}^1}\|\boldsymbol{c}\|_{\dot{H}^1}\,d\tau\\ &\lesssim \|\boldsymbol{a}\|_{L^4_T(\dot{H}^{\frac12})}\|\boldsymbol{b}\|_{L^4_T(\dot{H}^1)}\|\boldsymbol{c}\|_{L^2_T(\dot{H}^1)}.\end{aligned}$$

Using the interpolation inequality (B.5.1) yields (3.3.20).

Proving (3.3.21) is similar. To get the last inequality, we take advantage of the identity

$$(\nabla\times\boldsymbol{w},\boldsymbol{z})_{L^2} = (\boldsymbol{w},\nabla\times\boldsymbol{z})_{L^2},$$

and of Hölder and Gagliardo-Nirenberg inequalities, and Sobolev embedding:

$$\Big|\int_0^T (\nabla\times((\nabla\times\boldsymbol{a})\times\boldsymbol{b})\,|\,\boldsymbol{c})\,d\tau\Big| = \Big|\int_0^T ((\nabla\times\boldsymbol{a})\times\boldsymbol{b}\,|\,\nabla\times\boldsymbol{c})\,d\tau\Big|$$

$$\begin{aligned}
&\leq \int_0^T \|\nabla\times \boldsymbol{a}\|_{L^2}\|\boldsymbol{b}\|_{L^3}\|\nabla\times\boldsymbol{c}\|_{L^6}\,d\tau\\
&\lesssim \int_0^T \|\boldsymbol{a}\|_{\dot H^1}\|\boldsymbol{b}\|_{\dot H^{\frac12}}\|\nabla\times\boldsymbol{c}\|_{\dot H^1}\,d\tau\\
&\lesssim \|\boldsymbol{a}\|_{L^2_T(\dot H^1)}\|\boldsymbol{b}\|_{L^4_T(\dot H^{\frac12})}\|\nabla\times\boldsymbol{c}\|_{L^4_T(\dot H^1)}.
\end{aligned}$$

Using the interpolation inequality (B.5.1) completes the proof. □

One can now start the proof of Theorem 2.1.8. Let us recall our situation: we are given two Leray-Hopf weak solutions $(\boldsymbol{u},\boldsymbol{B})$ and $(\bar{\boldsymbol{u}},\bar{\boldsymbol{B}})$ in $L^\infty_T(L^2)\cap L^2_T(\dot H^1)$ corresponding to the same initial data $(\boldsymbol{u}_0,\boldsymbol{B}_0)\in L^2$ with $\operatorname{div}\boldsymbol{u}_0 = \operatorname{div}\boldsymbol{B}_0 = 0$, and assume in addition that $\boldsymbol{u}$ and $\boldsymbol{J} := \nabla\times\boldsymbol{B}$ are in $L^4_T(\dot H^1)$. Note that we also have $\boldsymbol{B}$ in $L^4_T(\dot H^1)$ for finite T. Indeed, $\boldsymbol{B}$ is in $L^\infty_T(L^2)$ and thus in $L^4_T(L^2)$, and $\nabla^2\boldsymbol{B}$ is in $L^4_T(L^2)$ since J is in $L^4_T(\dot H^1)$.

We want to prove that the two solutions coincide on $[0,T]$, that is to say

$$(\overline{\delta\boldsymbol{u}},\overline{\delta\boldsymbol{B}})\equiv(\boldsymbol{0},\boldsymbol{0})\quad\text{with}\quad(\overline{\delta\boldsymbol{u}},\overline{\delta\boldsymbol{B}}) := (\boldsymbol{u}-\bar{\boldsymbol{u}},\boldsymbol{B}-\bar{\boldsymbol{B}}). \tag{3.3.23}$$

By definition of what a Leray-Hopf solution is, both $(\boldsymbol{u},\boldsymbol{B})$ and $(\bar{\boldsymbol{u}},\bar{\boldsymbol{B}})$ satisfy the energy inequality (2.1.38) on $[0,T]$, which implies that for all $t\geq T$,

$$\begin{aligned}
&\|(\overline{\delta\boldsymbol{u}},\overline{\delta\boldsymbol{B}})(t)\|_{L^2}^2 + 2\mu\int_0^t\|\nabla\overline{\delta\boldsymbol{u}}\|_{L^2}^2\,d\tau + 2\nu\int_0^t\|\overline{\delta\boldsymbol{B}}\|_{L^2}^2\,d\tau\\
&\qquad\leq 2\|(\boldsymbol{u}_0,\boldsymbol{B}_0)\|_{L^2}^2 - 2(\boldsymbol{u}(t)\,|\,\bar{\boldsymbol{u}}(t)) - 2(\boldsymbol{B}(t)\,|\,\bar{\boldsymbol{B}}(t))\\
&\qquad\qquad - 4\mu\int_0^t(\nabla\boldsymbol{u}\,|\,\nabla\bar{\boldsymbol{u}})\,d\tau - 4\nu\int_0^t(\nabla\boldsymbol{B}\,|\,\nabla\bar{\boldsymbol{B}})\,d\tau.
\end{aligned}\tag{3.3.24}$$

Then, the key to proving (3.3.23) is the following lemma, which is an adaptation to our setting of a similar result for the Navier-Stokes equations in [57].

Lemma 3.3.5. *Under the assumptions of Theorem 2.1.8, we have for all time $t\leq T$,*

$$\begin{aligned}
&(\boldsymbol{u}(t)\,|\,\bar{\boldsymbol{u}}(t)) + (\boldsymbol{B}(t)\,|\,\bar{\boldsymbol{B}}(t)) + 2\mu\int_0^t(\nabla\boldsymbol{u}\,|\,\nabla\bar{\boldsymbol{u}})\,d\tau + 2\nu\int_0^t(\nabla\boldsymbol{B}\,|\,\nabla\bar{\boldsymbol{B}})\,d\tau\\
&= \|\boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{B}_0\|_{L^2}^2 + \int_0^t\Big((\overline{\delta\boldsymbol{u}}\cdot\nabla\boldsymbol{B}\,|\,\overline{\delta\boldsymbol{B}}) + (\overline{\delta\boldsymbol{u}}\cdot\nabla\boldsymbol{u}\,|\,\overline{\delta\boldsymbol{u}}) - (\overline{\delta\boldsymbol{B}}\cdot\nabla\boldsymbol{B}\,|\,\overline{\delta\boldsymbol{u}})\\
&\qquad - (\overline{\delta\boldsymbol{B}}\cdot\nabla\boldsymbol{u}\,|\,\overline{\delta\boldsymbol{B}}) - h(\nabla\times((\nabla\times\overline{\delta\boldsymbol{B}})\times\overline{\delta\boldsymbol{B}})\,|\,\boldsymbol{B})\Big)d\tau.
\end{aligned}$$

Proof. The result is obvious if $(\boldsymbol{u},\boldsymbol{B})$ and $(\bar{\boldsymbol{u}},\bar{\boldsymbol{B}})$ are smooth and decay at infinity. In our setting where the solutions are rough, it requires some justification. Therefore, we consider two sequences $(\boldsymbol{u}_n,\boldsymbol{B}_n)_{n\in\mathbb{N}}$ and $(\bar{\boldsymbol{u}}_n,\bar{\boldsymbol{B}}_n)_{n\in\mathbb{N}}$ of smooth and divergence free vector fields, such that

$$\lim_{n\to\infty}(\boldsymbol{u}_n,\boldsymbol{B}_n,\nabla\times\boldsymbol{B}_n) = (\boldsymbol{u},\boldsymbol{B},\nabla\times\boldsymbol{B})\quad\text{in}\quad L^4_T(\dot H^1) \tag{3.3.25}$$

$$\text{and}\quad\lim_{n\to\infty}(\boldsymbol{u}_n,\boldsymbol{B}_n,\bar{\boldsymbol{u}}_n,\bar{\boldsymbol{B}}_n) = (\boldsymbol{u},\boldsymbol{B},\bar{\boldsymbol{u}},\bar{\boldsymbol{B}})\quad\text{in}\quad L^2_T(\dot H^1)\cap L^\infty_T(L^2). \tag{3.3.26}$$

Since our assumptions on $(\boldsymbol{u},\boldsymbol{B})$ also ensure that $(\partial_t\boldsymbol{u},\partial_t\boldsymbol{B})$ is in $L^2_T(\dot H^{-1})$, one can require in addition that

$$\lim_{n\to\infty}(\partial_t\boldsymbol{u}_n,\partial_t\boldsymbol{B}_n)=(\partial_t\boldsymbol{u},\partial_t\boldsymbol{B})\quad\text{in}\quad L^2_T(\dot H^{-1}). \tag{3.3.27}$$

Likewise, that $(\bar{\boldsymbol{u}},\bar{\boldsymbol{B}})$ is a Leray-Hopf solution guarantees that $(\partial_t\bar{\boldsymbol{u}},\partial_t\bar{\boldsymbol{B}})$ is in $L^{4/3}_T(\dot H^{-1}\times\dot H^{-2})$ (observe for example that $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{B}}$ are in $L^{8/3}_T(L^4)$ and thus $\boldsymbol{u}\otimes\boldsymbol{u}$ and $\boldsymbol{B}\otimes\boldsymbol{B}$ are in $L^{4/3}_T(L^2)$ and similar properties for the other nonlinear terms of the Hall-MHD system). One shall thus require also that

$$\lim_{n\to\infty}(\partial_t\bar{\boldsymbol{u}}_n,\partial_t\bar{\boldsymbol{B}}_n)=(\partial_t\bar{\boldsymbol{u}},\partial_t\bar{\boldsymbol{B}})\quad\text{in}\quad L^{4/3}_T(\dot H^{-1}\times\dot H^{-2}). \tag{3.3.28}$$

Now, taking $(\bar{\boldsymbol{u}}_n,\bar{\boldsymbol{B}}_n)$ and $(\boldsymbol{u}_n,\boldsymbol{B}_n)$ as test functions in the weak formulation of (2.1.1), (2.1.3) for $(\boldsymbol{u},\boldsymbol{B})$ and $(\bar{\boldsymbol{u}},\bar{\boldsymbol{B}})$, respectively, we get for all $t\le T$,

$$\int_0^t\Big((\partial_\tau\boldsymbol{u}\,|\,\bar{\boldsymbol{u}}_n)+\mu(\nabla\boldsymbol{u}\,|\,\nabla\bar{\boldsymbol{u}}_n)+(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,|\,\bar{\boldsymbol{u}}_n)-(\boldsymbol{B}\cdot\nabla\boldsymbol{B}\,|\,\bar{\boldsymbol{u}}_n)\Big)\,d\tau=0, \tag{3.3.29}$$

$$\int_0^t\Big((\partial_\tau\bar{\boldsymbol{u}}\,|\,\boldsymbol{u}_n)+\mu(\nabla\bar{\boldsymbol{u}}\,|\,\nabla\boldsymbol{u}_n)+(\bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}}\,|\,\boldsymbol{u}_n)-(\bar{\boldsymbol{B}}\cdot\nabla\bar{\boldsymbol{B}}\,|\,\boldsymbol{u}_n)\Big)\,d\tau=0, \tag{3.3.30}$$

$$\int_0^t\Big((\partial_\tau\boldsymbol{B}\,|\,\bar{\boldsymbol{B}}_n)+\nu(\nabla\boldsymbol{B}\,|\,\nabla\bar{\boldsymbol{B}}_n)+(\boldsymbol{u}\cdot\nabla\boldsymbol{B}\,|\,\bar{\boldsymbol{B}}_n)-(\boldsymbol{B}\cdot\nabla\boldsymbol{u}\,|\,\bar{\boldsymbol{B}}_n)\\ +h(\nabla\times(J\times\boldsymbol{B})\,|\,\bar{\boldsymbol{B}}_n)\Big)\,d\tau=0, \tag{3.3.31}$$

$$\int_0^t\Big((\partial_\tau\bar{\boldsymbol{B}}\,|\,\boldsymbol{B}_n)+\nu(\nabla\bar{\boldsymbol{B}}\,|\,\nabla\boldsymbol{B}_n)+(\bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{B}}\,|\,\boldsymbol{B}_n)-(\bar{\boldsymbol{B}}\cdot\nabla\bar{\boldsymbol{u}}\,|\,\boldsymbol{B}_n)\\ +h(\nabla\times(\bar J\times\bar{\boldsymbol{B}})\,|\,\boldsymbol{B}_n)\Big)\,d\tau=0. \tag{3.3.32}$$

Since $\nabla\bar{\boldsymbol{u}}_n$ and $\nabla\boldsymbol{u}_n$ converge to $\nabla\bar{\boldsymbol{u}}$ and $\nabla\boldsymbol{u}$, in $L^2_T(L^2)$, we deduce that

$$\lim_{n\to\infty}\left(\int_0^t(\nabla\boldsymbol{u}\,|\nabla\bar{\boldsymbol{u}}_n)\,d\tau+\int_0^t(\nabla\bar{\boldsymbol{u}}\,|\nabla\boldsymbol{u}_n)\,d\tau\right)=2\int_0^t(\nabla\boldsymbol{u}\,|\nabla\bar{\boldsymbol{u}})\,d\tau.$$

Thanks to (3.3.25) and (3.3.26), and Lemma 3.3.4, we have

$$\lim_{n\to\infty}\int_0^t(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,|\,\bar{\boldsymbol{u}}_n)\,d\tau=\int_0^t(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,|\,\bar{\boldsymbol{u}})\,d\tau,$$

$$\lim_{n\to\infty}\int_0^t(\bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}}\,|\,\boldsymbol{u}_n)\,d\tau=\int_0^t(\bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}}\,|\,\boldsymbol{u})\,d\tau,$$

and one can pass to the limit similarly in all the quadratic terms that do not contain $\boldsymbol{J}$ or $\bar{\boldsymbol{J}}$. Finally, using the following vector identity

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}=(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}, \tag{3.3.33}$$

Inequality (3.3.22) and (3.3.25), we get, since $\bar{\boldsymbol{B}}_n$ is smooth,

$$\begin{aligned}\int_0^t (\nabla\times(\boldsymbol{J}\times\boldsymbol{B})\,|\,\bar{\boldsymbol{B}}_n)(\tau)\,d\tau &= \int_0^t (\boldsymbol{J}\times\boldsymbol{B}\,|\,\nabla\times\bar{\boldsymbol{B}}_n)\,d\tau\\ &= \int_0^t (\boldsymbol{B}\times(\nabla\times\bar{\boldsymbol{B}}_n)\,|\,\boldsymbol{J})\,d\tau\\ &= -\int_0^t (\nabla\times((\nabla\times\bar{\boldsymbol{B}}_n)\times\boldsymbol{B})\,|\,\boldsymbol{B})\,d\tau.\end{aligned}$$

Hence, by (3.3.26), (3.3.22) and (3.3.33),

$$\begin{aligned}\lim_{n\to\infty}\int_0^t (\nabla\times(\boldsymbol{J}\times\boldsymbol{B})\,|\,\bar{\boldsymbol{B}}_n)\,d\tau &= -\int_0^t (\nabla\times((\nabla\times\bar{\boldsymbol{B}})\times\boldsymbol{B})\,|\,\boldsymbol{B})\,d\tau\\ &= \int_0^t (\nabla\times(\boldsymbol{J}\times\boldsymbol{B})\,|\,\bar{\boldsymbol{B}})\,d\tau.\end{aligned}$$

In order to prove that

$$\lim_{n\to\infty}\int_0^t (\nabla\times(\bar{\boldsymbol{J}}\times\bar{\boldsymbol{B}})\,|\,\boldsymbol{B}_n)\,d\tau = \int_0^t (\nabla\times(\bar{\boldsymbol{J}}\times\bar{\boldsymbol{B}})\,|\,\boldsymbol{B})\,d\tau,$$

one may use directly (3.3.22) and (3.3.25).

In order to pass to the limit in the term of (3.3.29) with a time derivative, one may use (3.3.26) and the fact that $\partial_t\boldsymbol{u}$ is in $L^2_T(\dot{H}^{-1})$. This gives

$$\lim_{n\to\infty}\int_0^t (\partial_\tau\boldsymbol{u}\,|\,\bar{\boldsymbol{u}}_n)\,d\tau = \int_0^t (\partial_\tau\boldsymbol{u}\,|\,\bar{\boldsymbol{u}})\,d\tau.$$

Next, since $\partial_t\bar{\boldsymbol{u}}$ is in $L^{4/3}_T(\dot{H}^{-1})$, (3.3.25) enables us to write that

$$\lim_{n\to\infty}\int_0^t (\partial_\tau\bar{\boldsymbol{u}}\,|\,\boldsymbol{u}_n)\,d\tau = \int_0^t (\partial_\tau\bar{\boldsymbol{u}}\,|\,\boldsymbol{u})\,d\tau.$$

In order to pass to the limit in the term of (3.3.31) with $\partial_t\boldsymbol{B}$, it suffices to use the fact that $\partial_t\boldsymbol{B}$ is in $L^2_T(\dot{H}^{-1})$ and (3.3.26). Passing to the limit in the term of (3.3.31) with $\partial_t\bar{\boldsymbol{B}}$, relies on the property that $\partial_t\bar{\boldsymbol{B}}$ is in $L^{\frac{4}{3}}_T(\dot{H}^{-2})$ and on (3.3.25).

Finally, passing to the limit in (3.3.29) and (3.3.30), and adding up the resulting equalities yields for all $t\in[0,T]$,

$$\begin{aligned}&\int_0^t \Big((\partial_\tau\boldsymbol{u}\,|\,\bar{\boldsymbol{u}})+(\partial_\tau\bar{\boldsymbol{u}}\,|\,\boldsymbol{u})\Big)d\tau + 2\mu\int_0^t(\nabla\boldsymbol{u}\,|\,\nabla\bar{\boldsymbol{u}})\,d\tau\\ &\quad+\int_0^t\Big((\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,|\,\bar{\boldsymbol{u}})-(\boldsymbol{B}\cdot\nabla\boldsymbol{B}\,|\,\bar{\boldsymbol{u}})+(\bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}}\,|\,\boldsymbol{u})-(\bar{\boldsymbol{B}}\cdot\nabla\bar{\boldsymbol{B}}\,|\,\boldsymbol{u})\Big)\,d\tau = 0.\end{aligned}\tag{3.3.34}$$

Applying the same procedure for (3.3.29) and (3.3.30), we get

$$\begin{aligned}&\int_0^t \Big((\partial_\tau\boldsymbol{B}\,|\,\bar{\boldsymbol{B}})+(\partial_\tau\bar{\boldsymbol{B}}\,|\,\boldsymbol{B})\Big)d\tau + 2\nu\int_0^t(\nabla\boldsymbol{B}\,|\,\nabla\bar{\boldsymbol{B}})\,d\tau\\ &\quad+\int_0^t\Big((\boldsymbol{u}\cdot\nabla\boldsymbol{B}\,|\,\bar{\boldsymbol{B}})-(\boldsymbol{B}\cdot\nabla\boldsymbol{u}\,|\,\bar{\boldsymbol{B}})+(\bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{B}}\,|\,\boldsymbol{B})-(\bar{\boldsymbol{B}}\cdot\nabla\bar{\boldsymbol{u}}\,|\,\boldsymbol{B})\Big)d\tau\end{aligned}$$

$$+h\int_0^t\Big((\nabla\times(\boldsymbol{J}\times\boldsymbol{B})\,|\,\bar{\boldsymbol{B}})+(\nabla\times(\bar{\boldsymbol{J}}\times\bar{\boldsymbol{B}})\,|\,\boldsymbol{B})\Big)d\tau=0. \quad (3.3.35)$$

We claim that

$$\int_0^t\big((\partial_\tau\boldsymbol{u}\,|\,\bar{\boldsymbol{u}})+(\boldsymbol{u}\,|\,\partial_\tau\bar{\boldsymbol{u}})\big)\,d\tau=(\boldsymbol{u}(t)\,|\,\bar{\boldsymbol{u}}(t))-\|\boldsymbol{u}_0\|_{L^2}^2. \quad (3.3.36)$$

Indeed, since both $\boldsymbol{u}_n$ and $\bar{\boldsymbol{u}}_n$ are smooth, we have

$$\int_0^t\big((\partial_\tau\boldsymbol{u}_n\,|\,\bar{\boldsymbol{u}}_n)+(\boldsymbol{u}_n\,|\,\partial_\tau\bar{\boldsymbol{u}}_n)\big)\,d\tau=(\boldsymbol{u}_n(t)\,|\,\bar{\boldsymbol{u}}_n(t))-(\boldsymbol{u}_n(0)\,|\bar{\boldsymbol{u}}_n(0)).$$

One can pass to the limit in the right-hand side thanks to (3.3.26). For the left-hand side, we write

$$\int_0^t\big(\partial_\tau\boldsymbol{u}_n\,|\,\bar{\boldsymbol{u}}_n\big)\,d\tau-\int_0^t\big(\partial_\tau\boldsymbol{u}\,|\,\bar{\boldsymbol{u}}\big)\,d\tau=\int_0^t\big(\partial_\tau\boldsymbol{u}\,|\,(\bar{\boldsymbol{u}}_n-\bar{\boldsymbol{u}})\big)\,d\tau+\int_0^t(\partial_\tau(\boldsymbol{u}_n-\boldsymbol{u})\,|\,\bar{\boldsymbol{u}}_n)\,d\tau.$$

$$\int_0^t\big(\boldsymbol{u}_n\,|\,\partial_\tau\bar{\boldsymbol{u}}_n\big)\,d\tau-\int_0^t\big(\boldsymbol{u}\,|\,\partial_\tau\bar{\boldsymbol{u}}\big)\,d\tau=\int_0^t((\boldsymbol{u}_n-\boldsymbol{u})\,|\,\partial_\tau\bar{\boldsymbol{u}})\,d\tau+\int_0^t(\boldsymbol{u}_n\,|\,\partial_\tau(\bar{\boldsymbol{u}}_n-\bar{\boldsymbol{u}}))\,d\tau.$$

We already proved that the first terms of the right-hand side converge to 0. For the second ones, this is due to (3.3.27),(3.3.28) and to the fact that $(\bar{\boldsymbol{u}}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ are bounded in $L^2_T(\dot{H}^1)$ and $L^4_T(\dot{H}^1)$, respectively. This proves (3.3.36).

In order to prove that

$$\int_0^t\Big((\partial_\tau\boldsymbol{B}\,|\,\bar{\boldsymbol{B}})+(\partial_\tau\bar{\boldsymbol{B}}\,|\,\boldsymbol{B})\Big)\,d\tau=(\boldsymbol{B}(t)\,|\,\bar{\boldsymbol{B}}(t))-\|\boldsymbol{B}_0\|_{L^2}^2, \quad (3.3.37)$$

we start from the fact that

$$\int_0^t\big((\partial_\tau\boldsymbol{B}_n\,|\,\bar{\boldsymbol{B}}_n)+(\boldsymbol{B}_n\,|\,\partial_\tau\bar{\boldsymbol{B}}_n)\big)\,d\tau=(\boldsymbol{B}_n(t)\,|\,\bar{\boldsymbol{B}}_n(t))-(\boldsymbol{B}_n(0)\,|\bar{\boldsymbol{B}}_n(0)).$$

Passing to the limit in the right-hand side may be done thanks to (3.3.26). For the left-hand side, we write

$$\int_0^t\big(\partial_\tau\boldsymbol{B}_n\,|\,\bar{\boldsymbol{B}}_n\big)d\tau-\int_0^t\big(\partial_\tau\boldsymbol{B}\,|\,\bar{\boldsymbol{B}}\big)d\tau=\int_0^t\big(\partial_\tau\boldsymbol{B}\,|\,(\bar{\boldsymbol{B}}_n-\bar{\boldsymbol{B}})\big)d\tau+\int_0^t(\partial_\tau(\boldsymbol{B}_n-\boldsymbol{B})\,|\,\bar{\boldsymbol{B}}_n)\,d\tau,$$

$$\int_0^t\big(\boldsymbol{B}_n\,|\,\partial_\tau\bar{\boldsymbol{B}}_n\big)d\tau-\int_0^t\big(\boldsymbol{B}\,|\,\partial_\tau\bar{\boldsymbol{B}}\big)d\tau=\int_0^t((\boldsymbol{B}_n-\boldsymbol{B})\,|\,\partial_\tau\bar{\boldsymbol{B}})d\tau+\int_0^t(\boldsymbol{B}_n\,|\,\partial_\tau(\bar{\boldsymbol{B}}_n-\bar{\boldsymbol{B}}))d\tau.$$

The convergence of the first terms of the right-hand side has already been shown before, and that of the second terms is due to the boundedness of $(\boldsymbol{B}_n)_{n\in\mathbb{N}}$ and $(\bar{\boldsymbol{B}}_n)_{n\in\mathbb{N}}$ in $L^4_T(\dot{H}^2)$ and $L^2_T(\dot{H}^1)$, respectively, and to (3.3.27),(3.3.28).

To conclude the proof of the lemma, one has to notice that

$$\int_0^t\big((\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,|\,\bar{\boldsymbol{u}})+(\bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}}\,|\,\boldsymbol{u})\big)\,d\tau=-\int_0^t(\overline{\delta\boldsymbol{u}}\cdot\nabla\boldsymbol{u}\,|\,\overline{\delta\boldsymbol{u}})\,d\tau, \quad (3.3.38)$$

$$\int_0^t\big((\boldsymbol{u}\cdot\nabla\boldsymbol{B}\,|\,\bar{\boldsymbol{B}})+(\bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{B}}\,|\,\boldsymbol{B})\big)\,d\tau=-\int_0^t(\overline{\delta\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{B}}\,|\,\overline{\delta\boldsymbol{B}})\,d\tau, \quad (3.3.39)$$

$$\int_0^t \Big((\boldsymbol{B}\cdot\nabla\boldsymbol{B}\,|\,\bar{\boldsymbol{u}}) + (\bar{\boldsymbol{B}}\cdot\nabla\bar{\boldsymbol{u}}\,|\,\boldsymbol{B}) + (\bar{\boldsymbol{B}}\cdot\nabla\bar{\boldsymbol{B}}\,|\,\boldsymbol{u}) + (\boldsymbol{B}\cdot\nabla\boldsymbol{u}\,|\,\bar{\boldsymbol{B}}) \Big)\, d\tau$$
$$= -\int_0^t \Big((\overline{\delta\boldsymbol{B}}\cdot\nabla\boldsymbol{B}\,|\,\overline{\delta\boldsymbol{u}}) + (\overline{\delta\boldsymbol{B}}\cdot\nabla\boldsymbol{u}\,|\,\overline{\delta\boldsymbol{B}}) \Big)\, d\tau, \quad (3.3.40)$$

$$\int_0^t \Big((\nabla\times(\boldsymbol{J}\times\boldsymbol{B})\,|\,\bar{\boldsymbol{B}}) + (\nabla\times(\bar{\boldsymbol{J}}\times\bar{\boldsymbol{B}})\,|\,\boldsymbol{B}) \Big) d\tau = \int_0^t (\nabla\times(\overline{\delta\boldsymbol{J}}\times\overline{\delta\boldsymbol{B}})\,|\,\boldsymbol{B})\, d\tau. \quad (3.3.41)$$

The above relations are obvious in the case of smooth vector-fields (just perform suitable integrations by parts and use the divergence-free property). Now, since the trilinear form corresponding to (3.3.20) is continuous on the spaces

$$L^4_T(L^3)\times L^2_T(\dot H^1)\times L^4_T(L^6) \quad\text{and}\quad L^4_T(L^3)\times L^4_T(\dot H^1)\times L^2_T(L^6),$$

we deduce that (3.3.38), (3.3.39) and (3.3.40) are still valid under our assumptions.

For justifying (3.3.41) under our regularity framework, one just has to use the fact that the trilinear form $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\mapsto\int_0^t(\nabla\times(\boldsymbol{a}\times\boldsymbol{b})\,|\,\boldsymbol{c})\,d\tau$ is continuous on

$$L^4_T(L^3)\times L^4_T(L^6)\times L^2_T(\dot H^1) \quad\text{and}\quad L^2_T(L^2)\times L^4_T(L^3)\times L^4_T(\dot H^2)$$

and thus on

$$L^4_T(\dot H^{\frac12})\times L^4_T(\dot H^1)\times L^2_T(\dot H^1) \quad\text{and}\quad L^2_T(L^2)\times L^4_T(\dot H^{\frac12})\times L^4_T(\dot H^2).$$

This completes the proof of Lemma 3.3.5. □

Let us finish the proof of the theorem. Reformulating the right-hand side of (3.3.24) by means of Lemma 3.3.5, we get

$$\|(\overline{\delta\boldsymbol{u}},\overline{\delta\boldsymbol{B}})(t)\|^2_{L^2} + 2\mu\int_0^t\|\nabla\overline{\delta\boldsymbol{u}}\|^2_{L^2}\,d\tau + 2\nu\int_0^t\|\nabla\overline{\delta\boldsymbol{B}}\|^2_{L^2}\,d\tau$$
$$\le 2\int_0^t\Big((\overline{\delta\boldsymbol{B}}\cdot\nabla\boldsymbol{B}\,|\,\overline{\delta\boldsymbol{u}}) + (\overline{\delta\boldsymbol{B}}\cdot\nabla\boldsymbol{u}\,|\,\overline{\delta\boldsymbol{B}}) + h(\nabla\times((\nabla\times\overline{\delta\boldsymbol{B}})\times\overline{\delta\boldsymbol{B}})\,|\,\boldsymbol{B})$$
$$- (\overline{\delta\boldsymbol{u}}\cdot\nabla\boldsymbol{B}\,|\,\overline{\delta\boldsymbol{B}}) - (\overline{\delta\boldsymbol{u}}\cdot\nabla\boldsymbol{u}\,|\,\overline{\delta\boldsymbol{u}})\Big)\,d\tau. \quad (3.3.42)$$

Arguing as in the proof of Lemma 3.3.4 and using Young's inequality, we see that

$$|(\overline{\delta\boldsymbol{B}}\cdot\nabla\boldsymbol{B}\,|\,\overline{\delta\boldsymbol{u}})| \lesssim \|\overline{\delta\boldsymbol{B}}\|^{\frac12}_{L^2}\|\overline{\delta\boldsymbol{B}}\|^{\frac12}_{\dot H^1}\|\boldsymbol{B}\|_{\dot H^1}\|\overline{\delta\boldsymbol{u}}\|_{\dot H^1}$$
$$\le \frac{C}{\mu^2\nu}\|\overline{\delta\boldsymbol{B}}\|^2_{L^2}\|\boldsymbol{B}\|^4_{\dot H^1} + \frac{\mu}{8}\|\overline{\delta\boldsymbol{u}}\|^2_{\dot H^1} + \frac{\nu}{8}\|\overline{\delta\boldsymbol{B}}\|^2_{\dot H^1},$$

$$|(\overline{\delta \boldsymbol{B}} \cdot \nabla \boldsymbol{u} \,|\, \overline{\delta \boldsymbol{B}})| \leq \frac{C}{\nu^3} \|\overline{\delta \boldsymbol{B}}\|_{L^2}^2 \|\boldsymbol{u}\|_{\dot{H}^1}^4 + \frac{\nu}{4} \|\overline{\delta \boldsymbol{B}}\|_{\dot{H}^1}^2,$$
$$|(\overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{B} \,|\, \overline{\delta \boldsymbol{B}})| \leq \frac{C}{\mu\nu^2} \|\overline{\delta \boldsymbol{u}}\|_{L^2}^2 \|\boldsymbol{B}\|_{\dot{H}^1}^4 + \frac{\mu}{8} \|\overline{\delta \boldsymbol{u}}\|_{\dot{H}^1}^2 + \frac{\nu}{8} \|\overline{\delta \boldsymbol{B}}\|_{\dot{H}^1}^2,$$
$$|(\overline{\delta \boldsymbol{u}} \cdot \nabla \boldsymbol{u} \,|\, \overline{\delta \boldsymbol{u}})| \leq \frac{C}{\mu^3} \|\overline{\delta \boldsymbol{u}}\|_{L^2}^2 \|\boldsymbol{u}\|_{\dot{H}^1}^4 + \frac{\mu}{4} \|\overline{\delta \boldsymbol{u}}\|_{\dot{H}^1}^2,$$

and

$$h|(\nabla \times ((\nabla \times \overline{\delta \boldsymbol{B}}) \times \overline{\delta \boldsymbol{B}}) \,|\, \boldsymbol{B})| \|\overline{\delta \boldsymbol{B}}\|_{L^2}^{\frac{1}{2}} \|\overline{\delta \boldsymbol{B}}\|_{\dot{H}^1}^{\frac{3}{2}} \|\nabla \times \boldsymbol{B}\|_{\dot{H}^1}$$
$$\leq \frac{Ch^4}{\nu^3} \|\overline{\delta \boldsymbol{B}}\|_{L^2}^2 \|\boldsymbol{J}\|_{\dot{H}^1}^4 + \frac{\nu}{4} \|\overline{\delta \boldsymbol{B}}\|_{\dot{H}^1}^2.$$

Thus, reverting to (3.3.42), we conclude that, for all $t \leq T$,

$$\|(\overline{\delta \boldsymbol{u}}, \overline{\delta \boldsymbol{B}})(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla \overline{\delta \boldsymbol{u}}\|_{L^2}^2 \, d\tau + \nu \int_0^t \|\nabla \overline{\delta \boldsymbol{B}}\|_{L^2}^2 \, d\tau$$
$$\leq \frac{Ch^4}{\min\{\mu, \nu\}^3} \int_0^t \|(\overline{\delta \boldsymbol{B}}, \overline{\delta \boldsymbol{u}})\|_{L^2}^2 \|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{J})\|_{\dot{H}^1}^4 \, d\tau.$$

Recall that both $\boldsymbol{u}$ and $\boldsymbol{J}$ are in $L_T^4(\dot{H}^1)$ and that this also holds true for $\boldsymbol{B}$ as pointed out at the beginning of the proof. Hence Gronwall's lemma implies that $(\overline{\delta \boldsymbol{u}}, \overline{\delta \boldsymbol{B}}) \equiv \boldsymbol{0}$ on $[0, T]$. This completes the proof of Theorem 2.1.8. □

3.4 The $2\frac{1}{2}$D flows with large velocity fields

This section is devoted to the proof of Theorem 2.1.10. It essentially relies on the following estimates and on an inequality for the vector-field $\boldsymbol{\Omega}$ defined by $\boldsymbol{\Omega} = h\boldsymbol{\omega} + \boldsymbol{B}$, that will be proved at the end of the section.

3.4.1 A priori estimates

Proposition 3.4.1. *Let $(\boldsymbol{u}, \boldsymbol{B})$ be a smooth solution of (2.1.33)–(2.1.37) with $h = \mu = \nu = 1$. Let $v := \boldsymbol{u} - j$. Then, we have*

$$\frac{1}{2} \frac{d}{dt} \|(\boldsymbol{u}, \boldsymbol{B})\|_{L^2}^2 + \|(\widetilde{\nabla} \boldsymbol{u}, \widetilde{\nabla} \boldsymbol{B})\|_{L^2}^2 = 0, \tag{3.4.1}$$

and there exists a universal constant C such that

$$\frac{d}{dt} \|\boldsymbol{v}\|_{L^2}^2 + \|\widetilde{\nabla} \boldsymbol{v}\|_{L^2}^2 \leq C \|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})\|_{L^2}^2 \|(\widetilde{\nabla} \boldsymbol{u}, \widetilde{\nabla} \boldsymbol{B}, \widetilde{\nabla} \boldsymbol{v})\|_{L^2}^2, \tag{3.4.2}$$

$$\frac{d}{dt} \|\boldsymbol{B}\|_{H^1}^2 + \|\widetilde{\nabla} \boldsymbol{B}\|_{H^1}^2 \leq CW \|\boldsymbol{B}\|_{H^1}^2 + C \|\widetilde{\nabla} \boldsymbol{B}\|_{L^2} \|\widetilde{\Delta} \boldsymbol{B}\|_{L^2}^2, \tag{3.4.3}$$

with $W := \|\boldsymbol{u}\|_{L^2}^2 \|\widetilde{\nabla} \boldsymbol{u}\|_{L^2}^2 + \|\widetilde{\nabla} \boldsymbol{u}\|_{L^2} \|\widetilde{\nabla} \boldsymbol{\omega}\|_{L^2}$ and $\boldsymbol{\omega} := \widetilde{\nabla} \times \boldsymbol{u}$.

Proof. The first identity is just the energy balance. For proving (3.4.2), we use the fact that the third equation of (2.1.27) rewrites for $2\frac{1}{2}$D flows a follows:

$$\partial_t v - \widetilde{\Delta} v = \mathcal{P}(\widetilde{\boldsymbol{B}}\cdot\widetilde{\nabla}\boldsymbol{B} - \widetilde{\boldsymbol{u}}\cdot\widetilde{\nabla}\boldsymbol{u}) - \widetilde{\nabla}\times\big((\widetilde{\nabla}\times v)\times\boldsymbol{B}\big) + \widetilde{\nabla}\times(v\times\boldsymbol{u}) + 2\widetilde{\nabla}\times(\widetilde{v}\cdot\widetilde{\nabla}\boldsymbol{B}).$$

Therefore, taking the L^2 scalar product with v, integrating by parts in some terms, and using the Cauchy-Schwarz inequality, one gets:

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \|\widetilde{\nabla} v\|_{L^2}^2 \leq \big(\|\boldsymbol{B}\otimes\boldsymbol{B} - \boldsymbol{u}\otimes\boldsymbol{u}\|_{L^2} + \|v\times\boldsymbol{u}\|_{L^2} + 2\|v\cdot\widetilde{\nabla}\boldsymbol{B}\|_{L^2}\big)\|\widetilde{\nabla} v\|_{L^2}.$$

Thanks to Hölder's inequality, Sobolev embedding (B.5.2), interpolation inequality (B.5.1) and Young's inequality, we have

$$\begin{aligned}\|\boldsymbol{B}\otimes\boldsymbol{B}\|_{L^2}\|\widetilde{\nabla} v\|_{L^2} &\leq \|\boldsymbol{B}\|_{L^4}^2\|\widetilde{\nabla} v\|_{L^2}\\ &\leq C\|\boldsymbol{B}\|_{L^2}\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}\|\widetilde{\nabla} v\|_{L^2}\\ &\leq C\|\boldsymbol{B}\|_{L^2}^2\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}^2 + \frac{1}{10}\|\widetilde{\nabla} v\|_{L^2}^2,\end{aligned}$$

a similar inequality for the term with $\boldsymbol{u}\otimes\boldsymbol{u}$,

$$\begin{aligned}\|v\times\boldsymbol{u}\|_{L^2}\|\widetilde{\nabla} v\|_{L^2} &\leq \|v\|_{L^4}\|\boldsymbol{u}\|_{L^4}\|\widetilde{\nabla} v\|_{L^2}\\ &\leq C\|\boldsymbol{u}\|_{L^2}\|v\|_{L^2}\|\widetilde{\nabla}\boldsymbol{u}\|_{L^2}\|\widetilde{\nabla} v\|_{L^2} + \frac{1}{10}\|\widetilde{\nabla} v\|_{L^2}^2\\ &\leq C\|(\boldsymbol{u},v)\|_{L^2}^2\|(\widetilde{\nabla}\boldsymbol{u},\widetilde{\nabla} v)\|_{L^2}^2 + \frac{1}{10}\|\widetilde{\nabla} v\|_{L^2}^2,\end{aligned}$$

and, using that $\boldsymbol{B} = (-\widetilde{\Delta})^{-1}\widetilde{\nabla}\times(\boldsymbol{u}-v)$ and that $\widetilde{\nabla}^2(-\widetilde{\Delta})^{-1}$ maps L^4 to L^4,

$$\begin{aligned}2\|\widetilde{v}\cdot\widetilde{\nabla}\boldsymbol{B}\|_{L^2}\|\widetilde{\nabla} v\|_{L^2} &\leq C\|v\|_{L^4}\|\boldsymbol{u}-v\|_{L^4}\|\widetilde{\nabla} v\|_{L^2}\\ &\leq C\|v\|_{L^4}(\|\boldsymbol{u}\|_{L^4} + \|v\|_{L^4})\|\widetilde{\nabla} v\|_{L^2}\\ &\leq C\|(\boldsymbol{u},v)\|_{L^2}^2\|(\widetilde{\nabla}\boldsymbol{u},\widetilde{\nabla} v)\|_{L^2}^2 + \frac{1}{10}\|\widetilde{\nabla} v\|_{L^2}^2.\end{aligned}$$

This yields (3.4.2).

For proving (3.4.3), use the following identity (valid if $\widetilde{\operatorname{div}}\,\widetilde{w} = \widetilde{\operatorname{div}}\,\widetilde{z} = 0$):

$$\widetilde{\nabla}\times(w\times z) = \widetilde{z}\cdot\widetilde{\nabla} w - \widetilde{w}\cdot\widetilde{\nabla} z,$$

to rewrite the equation for $\boldsymbol{B}$ as follows:

$$\partial_t\boldsymbol{B} - \widetilde{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) + h\widetilde{\nabla}\times(\boldsymbol{j}\times\boldsymbol{B}) = \nu\widetilde{\Delta}\boldsymbol{B}. \tag{3.4.4}$$

Taking the L^2 scalar product with $\boldsymbol{B}$ yields

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}\|_{L^2}^2 + \|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}^2 = (\boldsymbol{u}\times\boldsymbol{B}\,|\,\boldsymbol{j}).$$

To get an estimate for $\widetilde{\nabla}\boldsymbol{B}$, apply the following relations:

$$\widetilde{\nabla}\times(\boldsymbol{w}\times\boldsymbol{z}) = (\widetilde{\nabla}\times\boldsymbol{w})\times\boldsymbol{z} + (\widetilde{\nabla}\times\boldsymbol{z})\times\boldsymbol{w} - 2\widetilde{\boldsymbol{w}}\cdot\widetilde{\nabla}\boldsymbol{z} + \widetilde{\nabla}(\boldsymbol{w}\cdot\boldsymbol{z}),$$
$$\widetilde{\nabla}\times(\widetilde{\nabla}\times\boldsymbol{w}) + \widetilde{\Delta}\boldsymbol{w} = 0,$$

so as to rewrite (3.4.4) as

$$\partial_t\boldsymbol{B} + \widetilde{\boldsymbol{u}}\cdot\widetilde{\nabla}\boldsymbol{B} - \widetilde{\boldsymbol{B}}\cdot\widetilde{\nabla}\boldsymbol{u} - \widetilde{\Delta}\boldsymbol{B}\times\boldsymbol{B} - 2\widetilde{j}\cdot\widetilde{\nabla}\boldsymbol{B} + \widetilde{\nabla}(j\cdot\boldsymbol{B}) = \widetilde{\Delta}\boldsymbol{B}.$$

Taking the L^2 scalar product with $-\widetilde{\Delta}\boldsymbol{B}$ and using that $\widetilde{\operatorname{div}}\,\widetilde{\Delta}\boldsymbol{B} = 0$, we get

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}^2 + \|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}^2 = -(\widetilde{\boldsymbol{B}}\cdot\widetilde{\nabla}\boldsymbol{u}\,|\,\widetilde{\Delta}\boldsymbol{B}) + (\widetilde{\boldsymbol{u}}\cdot\widetilde{\nabla}\boldsymbol{B}\,|\,\widetilde{\Delta}\boldsymbol{B}) - 2(\widetilde{j}\cdot\widetilde{\nabla}\boldsymbol{B}\,|\,\widetilde{\Delta}\boldsymbol{B}).$$

Thanks to Hölder's inequality, (B.5.1), (B.5.4) with $p = 4$ and Young's inequality, we have

$$\begin{aligned}
|(\boldsymbol{u}\times\boldsymbol{B}\,|\,j)| &\le \|\boldsymbol{u}\times\boldsymbol{B}\|_{L^2}\|\boldsymbol{j}\|_{L^2}\\
&\le \|\boldsymbol{u}\|_{L^4}\|\boldsymbol{B}\|_{L^4}\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}\\
&\le C\|\boldsymbol{u}\|_{L^2}^{\frac12}\|\boldsymbol{u}\|_{\dot H^1}^{\frac12}\|\boldsymbol{B}\|_{L^2}^{\frac12}\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}^{\frac32}\\
&\le C\|\boldsymbol{u}\|_{L^2}^2\|\widetilde{\nabla}\boldsymbol{u}\|_{L^2}^2\|\boldsymbol{B}\|_{L^2}^2 + \frac18\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}^2\\
|(\widetilde{\boldsymbol{u}}\cdot\widetilde{\nabla}\boldsymbol{B}\,|\,\widetilde{\Delta}\boldsymbol{B})| &\le \|\boldsymbol{u}\|_{L^4}\|\widetilde{\nabla}\boldsymbol{B}\|_{L^4}\|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}\\
&\le C\|\boldsymbol{u}\|_{L^2}^{\frac12}\|\boldsymbol{u}\|_{\dot H^1}^{\frac12}\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}^{\frac12}\|\widetilde{\nabla}\boldsymbol{B}\|_{\dot H^1}^{\frac12}\|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}\\
&\le C\|\boldsymbol{u}\|_{L^2}^2\|\widetilde{\nabla}\boldsymbol{u}\|_{L^2}^2\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}^2 + \frac18\|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}^2,\\
|(\widetilde{\boldsymbol{B}}\cdot\widetilde{\nabla}\boldsymbol{u}\,|\,\widetilde{\Delta}\boldsymbol{B})| &\le \|\boldsymbol{B}\|_{L^4}\|\widetilde{\nabla}\boldsymbol{u}\|_{L^4}\|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}\\
&\le C\|\boldsymbol{B}\|_{L^2}^{\frac12}\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}^{\frac12}\|\widetilde{\nabla}\boldsymbol{u}\|_{L^2}^{\frac12}\|\widetilde{\nabla}^2\boldsymbol{u}\|_{L^2}^{\frac12}\|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}\\
&\le C(\|\boldsymbol{B}\|_{L^2}^2 + \|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}^2)\|\widetilde{\nabla}\boldsymbol{u}\|_{L^2}\|\widetilde{\nabla}^2\boldsymbol{u}\|_{L^2} + \frac18\|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}^2,\\
|(\widetilde{j}\cdot\widetilde{\nabla}\boldsymbol{B}\,|\,\widetilde{\Delta}\boldsymbol{B})| &\le \|\widetilde{j}\cdot\widetilde{\nabla}\boldsymbol{B}\|_{L^2}\|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}\\
&\le \|\boldsymbol{j}\|_{L^4}\|\widetilde{\nabla}\boldsymbol{B}\|_{L^4}\|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}\\
&\le C\|\widetilde{\nabla}\boldsymbol{B}\|_{L^2}\|\widetilde{\Delta}\boldsymbol{B}\|_{L^2}^2.
\end{aligned}$$

Summing up the above estimates and using $\|\widetilde{\nabla}^2\boldsymbol{u}\|_{L^2} \lesssim \|\widetilde{\nabla}\boldsymbol{\omega}\|_{L^2}$ yields (3.4.3). □

3.4.2 Existence and uniqueness

It is now easy to prove the first part of Theorem 2.1.10: adding up (3.4.1) and (3.4.2) yields for some universal constant C and all $t \ge 0$:

$$\begin{aligned}
&\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})(t)\|_{L^2}^2 + \int_0^t \|(\widetilde{\nabla}\boldsymbol{u},\widetilde{\nabla}\boldsymbol{B},\widetilde{\nabla}\boldsymbol{v})\|_{L^2}^2\,d\tau\\
&\qquad\le \|(\boldsymbol{u}_0,\boldsymbol{B}_0,\boldsymbol{v}_0)\|_{L^2}^2 + C\int_0^t \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{L^2}^2\|(\widetilde{\nabla}\boldsymbol{u},\widetilde{\nabla}\boldsymbol{B},\widetilde{\nabla}\boldsymbol{v})\|_{L^2}^2\,d\tau.
\end{aligned}$$

Lemma 3.3.2 (take $\alpha = 2$, $W \equiv 0$) thus implies that if $2C\|(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0)\|_{L^2}^2 < 1$, then we have for all time,

$$\|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|(\widetilde{\nabla}\boldsymbol{u}, \widetilde{\nabla}\boldsymbol{B}, \widetilde{\nabla}\boldsymbol{v})\|_{L^2}^2\, d\tau \leq \|(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0)\|_{L^2}^2. \tag{3.4.5}$$

From that stage, applying a regularization scheme similar to the one that we used for handling the 3D case allows to conclude to the first part of Theorem 2.1.10 (uniqueness being also similar).

In order to prove the second part of the statement, we observe that Inequality (3.4.3) reads

$$\frac{d}{dt}X^2 + D^2 \leq CX^2W + CXD^2 \quad \text{with } X(t) = \|\boldsymbol{B}(t)\|_{H^1},\ D^2(t) = \int_0^t \|\widetilde{\nabla}\boldsymbol{B}\|_{H^1}^2\, d\tau$$
$$\text{and} \quad W(t) = \|\boldsymbol{u}(t)\|_{L^2}^2\|\widetilde{\nabla}\boldsymbol{u}(t)\|_{L^2}^2 + \|\widetilde{\nabla}\boldsymbol{u}(t)\|_{L^2}\|\widetilde{\nabla}^2\boldsymbol{u}(t)\|_{L^2}.$$

The first term of W may be controlled thanks to (2.1.38). To handle the second one, the idea is to get a bound for $\boldsymbol{\omega}$ (the curl of $\boldsymbol{u}$) through Equation (2.1.39). More precisely, taking the scalar product of (2.1.39) with $\boldsymbol{\Omega}$ and integrating by parts, we get (remember that $\mu = 1$):

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\Omega}\|_{L^2}^2 + \|\widetilde{\nabla}\boldsymbol{\Omega}\|_{L^2}^2 + (\boldsymbol{u}\otimes\widetilde{\boldsymbol{\Omega}}\,|\,\widetilde{\nabla}\boldsymbol{\Omega}) = 0.$$

Combining Hölder and Gagliardo-Nirenberg inequalities thus yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\Omega}\|_{L^2}^2 + \|\widetilde{\nabla}\boldsymbol{\Omega}\|_{L^2}^2 &\leq \|\widetilde{\boldsymbol{\Omega}}\|_{L^4}\|\boldsymbol{u}\|_{L^4}\|\widetilde{\nabla}\boldsymbol{\Omega}\|_{L^2}\\ &\leq C\|\boldsymbol{\Omega}\|_{L^2}^{\frac{1}{2}}\|\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\widetilde{\nabla}\boldsymbol{\Omega}\|_{L^2}^{\frac{3}{2}}\\ &\leq C\|\boldsymbol{\Omega}\|_{L^2}^2\|\boldsymbol{u}\|_{L^2}^2\|\nabla\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2}\|\widetilde{\nabla}\boldsymbol{\Omega}\|_{L^2}^2.\end{aligned}$$

Taking advantage of (2.1.38) and using Gronwall's lemma, we thus get

$$\|\boldsymbol{\Omega}(t)\|_{L^2}^2 + \int_0^t \|\widetilde{\nabla}\boldsymbol{\Omega}\|_{L^2}^2\, d\tau \leq \|\boldsymbol{\Omega}_0\|_{L^2}^2 \exp\Bigl(C\|(\boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2}^4\Bigr)\cdot$$

Since $\boldsymbol{\omega} = \boldsymbol{\Omega} - \boldsymbol{B}$, using again (2.1.38) eventually yields

$$\|\boldsymbol{\omega}(t)\|_{L^2}^2 + \int_0^t \|\widetilde{\nabla}\boldsymbol{\omega}\|_{L^2}^2\, d\tau \leq \|(\boldsymbol{\omega}_0, \boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2}^2 \exp\Bigl(C\|(\boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2}^4\Bigr)\cdot \tag{3.4.6}$$

Taking advantage of the energy balance (3.4.1) and of Inequality (3.4.6), we discover that the function W of Proposition 3.4.1 may be bounded as follows for all $t \geq 0$:

$$\int_0^t W\, d\tau \leq C\Bigl(\|(\boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2}^4 + \|(\boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2}\|(\boldsymbol{\omega}_0, \boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2} \exp\Bigl(C\|(\boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2}^4\Bigr)\Bigr)\cdot$$

Hence, applying Lemma 3.3.2 to Inequality (3.4.3), one can conclude that, if

$$C\|\boldsymbol{B}_0\|_{H^1} \exp\Biggl(C\|(\boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2}\|(\boldsymbol{\omega}_0, \boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2} \exp\Bigl(C\|(\boldsymbol{u}_0, \boldsymbol{B}_0)\|_{L^2}^4\Bigr)\Biggr) < 1,$$

then we have for all $t \geq 0$,

$$\|\boldsymbol{B}(t)\|_{H^1}^2 + \int_0^t \|\widetilde{\nabla}\boldsymbol{B}\|_{H^1}^2 \, d\tau \leq 1.$$

From that latter inequality, (2.1.38) and Inequality (3.4.6), one can work out a regularization procedure similar to that of Section 3.3.1 and complete the proof of the second part of Theorem 2.1.10. □

Chapter 4

Global regularity and long-time asymptotics for a generalized non-local Burgers type equation

4.1 Global regularity for smooth positive periodic initial data

4.1.1 Local existence

We first discuss the local existence in the Theorem 2.2.2. In the case of $\Omega^d = \mathbb{R}^d$, the proof requires slightly more technical care in the maximum principle part, while being similar in the rest of the argument. We therefore present the case $\mathbb{R}^d$ only. However, in the case of $\Omega = \mathbb{T}^d$, we will additionally obtain a complementary statement for the minimum: $\min_{x\in\mathbb{T}^d} u(t,x)$ is a strictly increasing function of time, thus the amplitude is shrinking. In Section 4.3 we will elaborate much more on the asymptotic behaviour of the amplitude.

Proof of Theorem 2.2.2 The proof is based on a regularization of the kernel and by classical energy method. We will split it into several steps.

Step 1: *Regularization.* Given $\delta \in (0,1]$. Let us consider the following regularization of the kernel from Fourier side

$$\widehat{K^s_\delta}(\xi) := \int e^{-i\xi\cdot y} K^s_\delta(y)\, dy = \frac{1}{\delta} e^{-\delta|\xi|^s} \tag{4.1.1}$$

and the corresponding operator

$$|\nabla|^s_\delta f := \int_{\mathbb{R}^d} \big(f(x) - f(y)\big) K^s_\delta(x-y)\, dy = \widehat{K^s_\delta}(0) f - T^s_\delta f = \frac{1}{\delta} f - T^s_\delta f \tag{4.1.2}$$

where $T^s_\delta f = K^s_\delta \star f$ is the convolution. Note that from [58] there is an explicit formula on above regularized kernel, specially

$$K^s_1(y) = \frac{1}{(2\pi)^{\frac{d}{2}} |y|^{\frac{d}{2}-1}} \int_0^\infty e^{-t^s} t^{\frac{d}{2}} J_{\frac{d-2}{2}}(|y|t)\, dt$$

where J_α denotes the Bessel function of first kind of order α and $K^s_1(y)$ is a continuous strictly positive radial function on $\mathbb{R}^d$. Then we have by scaling

property of Fourier transform that

$$K_\delta^s(y) = \delta^{-(\frac{d}{s}+1)} K_1^s(\frac{y}{\delta^{\frac{1}{s}}}).$$

Just remark that in [16], $K_\delta^1(y)$ (that is $s = 1$) is given by

$$K_\delta^1(y) = \frac{c_{d,1}}{(\delta^2 + |y|^2)^{\frac{d+1}{2}}}$$

where the fact that the Fourier transform of Abel kernel is Poisson kernel have been used:

$$\int e^{-iy\cdot\xi} e^{-\delta|\xi|}\, d\xi = (2\pi)^d \frac{c_{d,1}}{(\delta^2 + |y|^2)^{\frac{d+1}{2}}}.$$

The regularized equation will takes the form

$$\partial_t u = [F(u), |\nabla|_\delta^s] u \tag{4.1.3}$$
$$= \int_{\mathbb{R}^d} \big(F(u(y)) - F(u(x))\big) u(y) K_\delta^s(x-y)\, dy \tag{4.1.4}$$
$$= -[G(u), T_\delta^s] u \tag{4.1.5}$$

where $G(u) := F(u) - F(0)$. Note that T_δ^s is infinitely smoothing since its symbol is exponential decreasing and $\|T_\delta^s u\|_{H^m} \le C_{d,s,\delta} \|u\|_{H^m}$.

Now, we would like to introduce a kind of composition lemma, which have been wildly used in compressible fluid dynamics with a pressure law depends on the density of the fluid, see [59] for an application to the well-posedness issue of compressible Navier-Stokes equations in the critical Besov space. In order to adapt to our setting, we state the version in Sobolev spaces, the proof and various generalization can be found in e.g. [60, 45, 61].

Lemma 4.1.1. *(Proposition 1.4.8 in [60]) Let I be an open interval of $\mathbb{R}$. Let $r > 0$ and σ be the smallest integer such that $\sigma \ge r$. Let $\mathcal{G} : I \to \mathbb{R}$ satisfy $\mathcal{G}(0) = 0$ and $\mathcal{G}' \in W^{\sigma,\infty}(I;\mathbb{R})$. Assume that $f \in H^r$ has values in $J \subset\subset I$. Then $\mathcal{G}(f) \in H^r$ and there exists a constant C_1 depending only on r, I, J, d and $\|f\|_{L^\infty}$ such that*

$$\|\mathcal{G}(f)\|_{H^r} \le C_1 \|\mathcal{G}'\|_{W^{\sigma,\infty}(I)} \|f\|_{H^r}.$$

Lemma 4.1.2. *(Corollary 1.4.9 in [60]) Let I be an open interval of $\mathbb{R}$. Let $r > d/2$ and σ be the smallest integer such that $\sigma \ge r$. Let $\mathcal{G} : I \to \mathbb{R}$ satisfy $\mathcal{G}(0) = 0$ and $\mathcal{G}'' \in W^{\sigma,\infty}(I;\mathbb{R})$. Let $f, g \in H^r$ has values in $J \subset\subset I$. Then there exists a constant C_2 depending only on r, I, J, d and $\|f\|_{L^\infty}, \|g\|_{L^\infty}$ such that*

$$\|\mathcal{G}(f) - \mathcal{G}(g)\|_{H^r} \le C_2 \|\mathcal{G}''\|_{W^{\sigma,\infty}(I)} \Big(\|f-g\|_{H^r} \sup_{\tau\in[0,1]} \|f + \tau(g-f)\|_{L^\infty} + \|f-g\|_{L^\infty} \sup_{\tau\in[0,1]} \|f + \tau(g-f)\|_{H^r} \Big).$$

With above Lemma 4.1.1, we will show the right-hand side of (4.1.5) is quadratically bounded and locally Lipschitz in any open set of H^m:

$$O^M := \{u \in H^m, \|u\|_{H^m} < M\}.$$

Indeed, by the standard quadratic estimates (note that $H^m(\mathbb{R}^d)$ is an algebra) and the facts that $G(0) = 0$, $H^m(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, one has for any $u, v \in O^M$

$$\|[G(u), T_\delta^s]u\|_{H^m} \leq C_\delta \|u\|_{H^m}^2$$

and

$$\|[G(u), T_\delta^s]u - [G(v), T_\delta^s]v\|_{H^m} \leq C_\delta \|u - v\|_{H^m}(\|u\|_{H^m} + \|v\|_{H^m})^2.$$

In particular, C_δ is a constant depends on δ. Thus, by the Picard theorem on Banach spaces (see [14] for example), for any $u(0,x) \in O^M$ there is a unique local solution $u \in \mathcal{C}^1([0,T); O^M)$ to (4.1.5). Here T depends on $\|u\|_{H^m}$ and δ. For late use, note that $\|u(t)\|_{L^2} = \|u(0,x)\|_{L^2}$ is conserved thanks to the basic feature of (4.1.3).

Step 2: *Maximum principle.* Suppose $u(0,x) \in O^M$ and in addition $u(0,x) > 0$. Let $u \in \mathcal{C}^1([0,T); H^m(\mathbb{R}^d))$ be a corresponding local solution to (4.1.5). As $H^m(\mathbb{R}^d) \hookrightarrow \mathcal{C}(\mathbb{R}^d)$ for $m > \frac{d}{2} + 1$, and $u(t,x) \to 0$ as $x \to \infty$, then for fixed time t, $u(t,x)$ has and attains its maximum $M(t) = \max_{x \in \mathbb{R}^d} u(t,x)$. We claim that $u(t,x) > 0$, for all $(t,x) \in [0,T) \times \mathbb{R}^d$, and the maximum function $M(t)$ is strictly decreasing on $[0,T)$. Let us prove the first claim first.

Let us fix $R > 0$ and show that u never vanishes on $(0,T) \times B_R(0)$. Suppose it does. Let us consider

$$t_0 := \inf\{t \in (0,T) : \exists\, x \in B_R(0), s.t.\, u(t,x) = 0\}.$$

By the boundedness of $(0,T) \times B_R(0)$ and the continuity of u, t_0 is attained. Since $u_0 > 0$, then $t_0 > 0$. We next show that $u(t_0,x) \geq 0$ for all $x \in B_R(0)$. If it does not satisfied, then there exists $x_\star$ such that $u(t_0,x_\star) < 0$. Thanks to the continuity of u there exists a constant $\delta > 0$ such that for all $(t,x) \in (t_0 - \delta, t_0) \times B_\delta(x_\star)$, we have

$$|u(t,x) - u(t_0,x_\star)| \leq \frac{1}{2}|u(t_0,x_\star)|$$

and particularly

$$\frac{3}{2}u(t_0,x_\star) \leq u(t_0 - \frac{\delta}{2}, x) \leq \frac{1}{2}u(t_0,x_\star) < 0.$$

This is a contradiction to the definition of t_0. Thus $u(t_0,x) \geq 0$ for all $x \in B_R(0)$.

Now, let $x_0 \in B_R(0)$ be such that $u(t_0,x_0) = 0$. Evaluating (4.1.4) at (t_0,x_0) we obtain

$$\partial_t u(t_0,x_0) = \int_{\mathbb{R}^d} \int_0^1 F'(\lambda u(y)) u^2(y) K_\delta^s(x_0 - y)\, \lambda\, dy > 0.$$

The right-hand side is strictly positive since the energy of solutions to (4.1.3) is conserved and $F' > 0$ a.e. on $(0, +\infty)$. This shows that for some earlier time $t < t_0$, $u(t, x_0)$ vanishes, which is a contraction. Since the argument holds for all $R > 0$, the claim follows.

Let us prove the second claim now. Suppose that $M(t)$ is not strictly decreasing on $[0, T)$. This implies that there exists a pair of times $0 < t_1 < t_2 < T$ such that $M(t_1) \leq M(t_2)$. Let us first show that there exists a $t_0 > t_1$, such that $M(t_0) \geq M(t)$ for all $t \in [t_1, t_0]$. If $M(t_1) < M(t_2)$, then by the continuity of $M(t)$ (use the fact that u is continuous), $M(t)$ attains its maximum on the interval $[t_1, t_2]$. Choose $t_0 \in [t_1, t_2]$ be the left utmost point where the maximum of $M(t)$ is attained. Then $t_0 > t_1$, and $M(t_0) \geq M(t)$ for all $t_1 \leq t \leq t_0$. If, on the contrary $M(t_1) = M(t_2)$ then either one can shrink the interval to fulfil the previous assumption or $M(t)$ is constant throughout $[t_1, t_2]$. In either case, there exists, as claimed, a $t_0 > t_1$, such that $M(t_0) \geq M(t)$ on $[t_1, t_0]$. Let us consider a point $x_0 \in \mathbb{R}^d$ such that $u(t_0, x_0) = M(t_0)$. Then

$$\partial_t u(t_0, x_0) = \int_{\mathbb{R}^d} \big(F(u(y)) - F(u(x_0))\big) u(y) K_\delta^s(x_0 - y)\, dy < 0.$$

This implies at an earlier time $t < t_0$, one must have $u(t, x_0) > u(t_0, x_0) = M(t_0)$ which in contradiction with the initial assumption.

Step 3: *uniform bounds.* Let us first state the following uniform estimates in term of δ

$$\| |\nabla|_\delta^s f \|_{H^r} \leq C_{d,s} \, \| |\nabla|^s f \|_{H^r}, \tag{4.1.6}$$

for all $r \in \mathbb{R}_+$. In fact, using the definition of nonhomogeneous Sobolev spaces (see for example [45])

$$\|f\|_{H^r} := \|(1 + |\xi|^2)^{\frac{r}{2}} \widehat{f}\|_{L^2},$$

by virtue of (4.1.2) we write

$$\begin{aligned}
\| |\nabla|_\delta^s f \|_{H^r} &= \|(1 + |\xi|^2)^{\frac{r}{2}} \mathcal{F}(|\nabla|_\delta^s f)\|_{L^2} \\
&= \|(1 + |\xi|^2)^{\frac{r}{2}} \widehat{f}\, \frac{1}{\delta}(1 - e^{-\delta|\xi|^s})\|_{L^2} \\
&\leq \|(1 + |\xi|^2)^{\frac{r}{2}} |\xi|^s \widehat{f}\|_{L^2} \sup_{\delta,\, \xi} \frac{|1 - e^{-\delta|\xi|^s}|}{\delta |\xi|^s} \\
&\leq \| |\nabla|^s f \|_{H^r}.
\end{aligned}$$

Let α be a multi-index of order $|\alpha| = m$, obviously $m \geq 2$ so that there exists $j \in \{1 \cdots d\}$ such that $\partial^\alpha = \partial^{\alpha - e_j} \partial_j$. Differentiating (4.1.3), we obtain

$$\begin{aligned}
&\partial_t \partial^\alpha u \\
&= [F(u), |\nabla|_\delta^s] \partial^\alpha u + \sum_{0 < \beta_1 \leq \alpha} \partial^{\beta_1}(F(u)) |\nabla|_\delta^s \partial^{\alpha - \beta_1} u \\
&\quad - |\nabla|_\delta^s \Big(\sum_{0 < \beta_2 < \alpha} \partial^{\beta_2}(F(u)) \partial^{\alpha - \beta_2} u + \sum_{0 < \beta_3 \leq \alpha - e_j} u\, \partial^{\beta_3}(F'(u)) \partial^{\alpha - \beta_3} u + u\, F'(u) \partial^\alpha u \Big).
\end{aligned}$$

Here, for multi-indexes α, β, we point out that $0 \le \beta \le \alpha$ means $0 \le \beta_j \le \alpha_j$ for each $j = 1, \dots, d$.

Let us take L^2 inner product of above equation with $\partial^\alpha u$. Using the property

$$\int f|\nabla|_\delta^s g = \int g|\nabla|_\delta^s f, \tag{4.1.7}$$

and thus

$$\int g \cdot [f, |\nabla|_\delta^s] g = 0.$$

We have

$$\begin{aligned}
&\frac{d}{dt}\|\partial^\alpha u\|_{L^2}^2 \\
&= \int \partial^\alpha u \sum_{0<\beta_1\le\alpha} \partial^{\beta_1}(F(u))|\nabla|_\delta^s \partial^{\alpha-\beta_1} u - \sum_{0<\beta_2<\alpha} \int \partial^\alpha u\, |\nabla|_\delta^s \Big(\partial^{\beta_2}(F(u))\partial^{\alpha-\beta_2} u\Big) \\
&\quad - \sum_{0<\beta_3\le\alpha-e_j} \int \partial^\alpha u\, |\nabla|_\delta^s \big(u\, \partial^{\beta_3}(F'(u))\partial^{\alpha-\beta_3} u\big) - \int \partial^\alpha u\, |\nabla|_\delta^s \big(uF'(u)\partial^\alpha u\big).
\end{aligned} \tag{4.1.8}$$

Only the last term is of order $m+1$. Using (4.1.7) and the definition of $|\nabla|_\delta^s$, one has

$$\begin{aligned}
&- \int \partial^\alpha u\, |\nabla|_\delta^s \big(uF'(u)\partial^\alpha u\big) \\
&= - \int \big(|\nabla|_\delta^s \partial^\alpha u\big)\, \big(uF'(u)\partial^\alpha u\big) \\
&= - \int\int u(x)F'(u(x))\, \partial^\alpha u(x) \big(\partial^\alpha u(x) - \partial^\alpha u(y)\big) K_\delta^s(x-y)\, dx\, dy.
\end{aligned}$$

Then by the positivity of u, F', a bound for the above term will follows from the elementary identity

$$-a(a-b) \le -\frac{1}{2}(a^2 - b^2).$$

More precisely, we use the embedding $H^{m-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ and estimate (4.1.6) to find that

$$\begin{aligned}
&- \int\int u(x)F'(u(x))\, \partial^\alpha u(x) \big(\partial^\alpha u(x) - \partial^\alpha u(y)\big) K_\delta^s(x-y)\, dx\, dy \\
&\le -\frac{1}{2} \int\int u(x)F'(u(x)) \big((\partial^\alpha u)^2(x) - (\partial^\alpha u)^2(y)\big) K_\delta^s(x-y)\, dx\, dy \\
&= -\frac{1}{2} \int u(x)F'(u(x))\, |\nabla|_\delta^s \big((\partial^\alpha u(x))^2\big) \\
&= -\frac{1}{2} \int (\partial^\alpha u)^2 |\nabla|_\delta^s \big(uF'(u)\big) \\
&\le \frac{1}{2} \|\partial^\alpha u\|_{L^2}^2 \, \big\||\nabla|_\delta^s \big(uF'(u)\big)\big\|_{L^\infty} \\
&\lesssim \|u\|_{H^m}^2 \, \|uF'(u)\|_{H^m} \lesssim \|u\|_{H^m}^3,
\end{aligned} \tag{4.1.9}$$

where in the last step we have apply Lemma 4.1.1 to the function $uF'(u)$ and use the maximal principle for u proved in Step 2.

The rest of the expression (4.1.8) is simpler to deal with as it does not contain any other derivatives of order $m+1$. To estimate it, we will use Lemma 4.1.1 and the Gagliardo-Nirenberg inequalities:

$$\|\partial^\gamma f\|_{L^{\frac{2r}{|\gamma|}}} \lesssim \|f\|_{L^\infty}^{1-\frac{|\gamma|}{r}} \|u\|_{H^r}^{\frac{|\gamma|}{r}}, \qquad 0 \le |\gamma| \le r, \tag{4.1.10}$$

and the following Kato-Ponce inequality (see [62]):

$$\||\nabla|^s(fg)\|_{L^2} \lesssim \||\nabla|^s f\|_{L^p}\|g\|_{L^{\bar p}} + \|f\|_{L^q}\||\nabla|^s g\|_{L^{\bar q}} \tag{4.1.11}$$

for $p, \bar q \in [2,\infty)$, $\bar p, q \in (2,\infty]$ such that $\frac{1}{2} = \frac{1}{p} + \frac{1}{\bar p} = \frac{1}{q} + \frac{1}{\bar q}$.

Remembering that $G(u) = F(u) - F(0)$ with $G(0) = 0$. So for any $0 < \beta_1 \le \alpha$, we write by Hölder's inequality

$$\begin{aligned}
&|\int \partial^\alpha u \partial^{\beta_1}(F(u))|\nabla|_\delta^s \partial^{\alpha-\beta_1} u\,| \\
&= |\int \partial^\alpha u \partial^{\beta_1}(G(u))|\nabla|_\delta^s \partial^{\alpha-\beta_1} u\,| \\
&\le \|\partial^\alpha u\|_{L^2}\|\partial^{\beta_1}(G(u))\|_{L^{\frac{2(m-1)}{|\beta_1|-1}}} \||\nabla|_\delta^s \partial^{\alpha-\beta_1} u\|_{L^{\frac{2(m-1)}{m-|\beta_1|}}} \\
&\lesssim \|u\|_{H^m}\|\nabla(G(u))\|_{L^\infty}^{1-\frac{|\beta_1|-1}{m-1}} \|\nabla(G(u))\|_{H^{m-1}}^{\frac{|\beta_1|-1}{m-1}} \||\nabla|_\delta^s u\|_{L^\infty}^{1-\frac{m-|\beta_1|}{m-1}} \||\nabla|_\delta^s u\|_{H^{m-1}}^{\frac{m-|\beta_1|}{m-1}} \\
&\lesssim \|u\|_{H^m}\|G(u)\|_{H^m}^{1-\frac{|\beta_1|-1}{m-1}} \|G(u)\|_{H^m}^{\frac{|\beta_1|-1}{m-1}} \|u\|_{H^m}^{1-\frac{m-|\beta_1|}{m-1}} \||u\|\|_{H^m}^{\frac{m-|\beta_1|}{m-1}} \lesssim \|u\|_{H^m}^3.
\end{aligned} \tag{4.1.12}$$

For the second term in the expression of (4.1.8), by the estimate (4.1.6) and Kato-Ponce inequality (4.1.11) we have for any $0 < \beta_2 < \alpha$

$$\begin{aligned}
&|\int \partial^\alpha u |\nabla|_\delta^s \Big(\partial^{\beta_2}(F(u))\partial^{\alpha-\beta_2} u\Big)| \\
&= |\int \partial^\alpha u |\nabla|_\delta^s \Big(\partial^{\beta_2}(G(u))\partial^{\alpha-\beta_2} u\Big)| \\
&\le \|\partial^\alpha u\|_{L^2}\|\partial^{\beta_2}(G(u))\partial^{\alpha-\beta_2} u\|_{\dot H^s} \\
&\lesssim \|u\|_{H^m}\|\partial^{\beta_2}|\nabla|^s(G(u))\|_{L^{\frac{2(m-1)}{|\beta_2|}}} \|\partial^{\alpha-\beta_2} u\|_{L^{\frac{2(m-1)}{m-|\beta_2|-1}}} \\
&\qquad + \|u\|_{H^m}\|\partial^{\beta_2}(G(u))\|_{L^{\frac{2(m-1)}{|\beta_2|-1}}} \|\partial^{\alpha-\beta_2}|\nabla|^s u\|_{L^{\frac{2(m-1)}{m-|\beta_2|}}}.
\end{aligned} \tag{4.1.13}$$

Using Gagliardo-Nirenberg inequalities (4.1.10) and Lemma 4.1.1 again, we get

$$\begin{aligned}
&\|\partial^{\beta_2}|\nabla|^s(G(u))\|_{L^{\frac{2(m-1)}{|\beta_2|}}} \|\partial^{\alpha-\beta_2} u\|_{L^{\frac{2(m-1)}{m-|\beta_2|-1}}} \\
&\lesssim \||\nabla|^s(G(u))\|_{L^\infty}^{1-\frac{|\beta_2|}{m-1}} \||\nabla|^s(G(u))\|_{H^{m-1}}^{\frac{|\beta_2|}{m-1}} \|\nabla u\|\|_{L^\infty}^{1-\frac{m-|\beta_2|-1}{m-1}} \|\nabla u\|\|_{H^{m-1}}^{\frac{m-|\beta_2|-1}{m-1}} \\
&\lesssim \|G(u)\|_{H^{m-1+s}}^{1-\frac{|\beta_2|}{m-1}} \|G(u)\|_{H^{m-1+s}}^{\frac{|\beta_2|}{m-1}} \|u\|\|_{H^m}^{1-\frac{m-|\beta_2|-1}{m-1}} \|u\|\|_{H^m}^{\frac{m-|\beta_2|-1}{m-1}} \\
&\lesssim \|u\|_{H^m}^2
\end{aligned}$$

and we estimate $\|\partial^{\beta_2}(G(u))\|_{L^{\frac{2(m-1)}{|\beta_2|-1}}}\|\partial^{\alpha-\beta_2}|\nabla|^s u\|_{L^{\frac{2(m-1)}{m-|\beta_2|}}}$ as same as in (4.1.12). Hence

$$|\int \partial^\alpha u|\nabla|^s_\delta\Big(\partial^{\beta_2}(F(u))\partial^{\alpha-\beta_2}u\Big)| \lesssim \|u\|^3_{H^m}. \tag{4.1.14}$$

The remaining term in the expression of (4.1.8) is new comparing to [16]. We take an advantage of the following commutator estimate developed recently due to Li [44]: for any $s\in(0,1], 1<p<\infty$,

$$\||\nabla|^s(fg)-f|\nabla|^s g\|_{L^p} \lesssim \||\nabla|^s f\|_{L^p}\|g\|_{L^\infty}. \tag{4.1.15}$$

For any $0<\beta_3\le\alpha-e_j$ we have by (4.1.15)

$$\begin{aligned}
&|\int \partial^\alpha u|\nabla|^s_\delta\Big(u\,\partial^{\beta_3}(F'(u))\partial^{\alpha-\beta_3}u\Big)|\\
&\le \|\partial^\alpha u\|_{L^2}\|u\,\partial^{\beta_3}(F'(u))\partial^{\alpha-\beta_3}u\|_{\dot H^s}\\
&\lesssim \|u\|_{H^m}\big(\||\nabla|^s u\|_{L^\infty}\|\partial^{\beta_3}(F'(u))\partial^{\alpha-\beta_3}u\|_{L^2}+\|u\|_{L^\infty}\|\partial^{\beta_3}(F'(u))\partial^{\alpha-\beta_3}u\|_{\dot H^s}\big).
\end{aligned} \tag{4.1.16}$$

The term $\|\partial^{\beta_3}(F'(u))\partial^{\alpha-\beta_3}u\|_{\dot H^s}$ can be estimated similarly as we did in (4.1.13), i.e. we have

$$\|\partial^{\beta_3}(F'(u))\partial^{\alpha-\beta_3}u\|_{\dot H^s}\lesssim\|u\|^2_{H^m}. \tag{4.1.17}$$

Notice that we can estimate $\|\partial^{\beta_3}(F'(u))\partial^{\alpha-\beta_3}u\|_{L^2}$ by simply taking $s=0$ in (4.1.17) and thus it is bounded by $\|u\|^2_{H^m}$. However, in order to prove a Beale-Kato-Majda blow-up criterion in the next (see Theorem 2.2.3), we shall prove it actually has shape bound $\|u\|_{H^m}$. Indeed, thanks to Gagliardo-Nirenberg inequalities (4.1.10) and the fact that u is essentially bounded, we have

$$\begin{aligned}
\|\partial^{\beta_3}(F'(u))\partial^{\alpha-\beta_3}u\|_{L^2} &\lesssim \|\partial^{\beta_3}\big(F'(u)-F'(0)\big)\|_{L^{\frac{2m}{|\beta_3|}}}\|\partial^{\alpha-\beta_3}u\|_{L^{\frac{2m}{m-|\beta_3|}}}\\
&\lesssim \|F'(u)\|^{1-\frac{|\beta_3|}{m}}_{L^\infty}\|F'(u)-F'(0)\|^{\frac{|\beta_3|}{m}}_{H^m}\|u\|^{1-\frac{m-|\beta_3|}{m}}_{L^\infty}\|u\|^{\frac{m-|\beta_3|}{m}}_{H^m}\\
&\lesssim \|u\|_{H^m}.
\end{aligned} \tag{4.1.18}$$

Put (4.1.9), (4.1.12) and (4.1.14), (4.1.17), (4.1.18) into (4.1.8), we thus obtain a differential inequality

$$\frac{d}{dt}\|u\|^2_{H^m}\le C\|u\|^3_{H^m}$$

with a positive constant C independent of δ or $\|u\|_{H^m}$. By continuation theorem of an autonomous ODE on a Banach Space, it shows that the solution u obtained from Step 1 can be extended to a time T which is independent of δ as well. Namely, there exists a time $T^*=(C\|u_0\|_{H^m})^{-1}$ such that we have the bound

$$\|u(t)\|_{H^m}\le\frac{\|u_0\|_{H^m}}{1-Ct\|u_0\|_{H^m}},\quad t\in[0,T^*). \tag{4.1.19}$$

Step 4: *Convergence.* We now turn to the convergence issue. For each $\delta\in(0,1]$, let u_δ be the solution to (4.1.3) from previous steps with the same

initial data $u(0,x) = u_0(x)$. Thanks to (4.1.19), $u_\delta \in \mathcal{C}([0,T);H^m)$ uniformly in terms of δ for any fixed $T < T^*$. Then we estimate the right-hand side of (4.1.3), recall that $G(u) = F(u) - F(0)$, we have

$$\begin{aligned}
&\|[F(u_\delta), |\nabla|_\delta^s]u_\delta\|_{H^{m-1}} \\
=& \|[G(u_\delta), |\nabla|_\delta^s]u_\delta\|_{H^{m-1}} \\
\lesssim& \|G(u_\delta)\|_{H^{m-1}}\||\nabla|_\delta^s u_\delta\|_{H^{m-1}} + \||\nabla|_\delta^s(u_\delta G(u_\delta))\|_{H^{m-1}} \\
\lesssim& \|u_\delta\|_{H^m}^2.
\end{aligned}$$

This shows that the family $\{\partial_t u_\delta\}$ is uniformly bounded in $\mathcal{C}([0,T);H^{m-1})$ with respect to δ.

At this moment, one can perform the classical weak compactness method, which provides a limit for a subsequence. Instead, we will show that the family $\{u_\delta\}$ is a Cauchy sequence in $\mathcal{C}([0,T];L^2)$. To prove our claim , we first need to prove an estimate on the difference of operator $|\nabla|_\delta^s$. Recall definition (4.1.2) thus for any fixed $\delta,\ \epsilon \in (0,1]$ we have

$$\begin{aligned}
\||\nabla|_\delta^s f - |\nabla|_\epsilon^s f\|_{L^2} &= \|(\frac{1}{\delta} - T_\delta^s)f - (\frac{1}{\epsilon} - T_\epsilon^s)f\|_{L^2} \\
&= \| \int_\epsilon^\delta \partial_\tau (\frac{1}{\tau}(1 - e^{-\tau|\xi|^s}))\, d\tau\, \widehat{f}\|_{L^2} \\
&= \| \int_\epsilon^\delta \frac{1}{\tau^2}\big(1 - (1+\tau|\xi|^s)e^{-\tau|\xi|^s}\big)\, d\tau\, \widehat{f}\|_{L^2} \\
&\le \| \int_\epsilon^\delta \frac{1}{\tau^2}\big(\frac{1}{2}(\tau|\xi|^s)^2\big)\, d\tau\, \widehat{f}\|_{L^2} \\
&\le \frac{1}{2}|\delta - \epsilon|\, \||\xi|^{2s}\widehat{f}\|_{L^2} \\
&\le \frac{1}{2}|\delta - \epsilon|\, \|f\|_{H^m}. \qquad (4.1.20)
\end{aligned}$$

Writing the equation for the difference of two solutions, we obtain

$$\begin{aligned}
&\partial_t(u_\delta - u_\epsilon) \\
=&\big(G(u_\delta) - G(u_\epsilon)\big)|\nabla|_\delta^s u_\delta + G(u_\epsilon)(|\nabla|_\delta^s - |\nabla|_\epsilon^s)u_\delta + G(u_\epsilon)|\nabla|_\epsilon^s(u_\delta - u_\epsilon) \\
&(|\nabla|_\delta^s - |\nabla|_\epsilon^s)(G(u_\delta)u_\delta) - |\nabla|_\epsilon^s\big((G(u_\delta) - G(u_\epsilon))u_\delta\big) - |\nabla|_\epsilon^s\big(G(u_\epsilon)(u_\delta - u_\epsilon)\big).
\end{aligned}$$

Taking L^2 inner product with $u_\delta - u_\epsilon$, we further obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_\delta - u_\epsilon\|_{L^2}^2 \\
=&\int (u_\delta - u_\epsilon)\,\big(G(u_\delta) - G(u_\epsilon)\big)|\nabla|_\delta^s u_\delta + \int (u_\delta - u_\epsilon)\, G(u_\epsilon)(|\nabla|_\delta^s - |\nabla|_\epsilon^s)u_\delta \\
&+ \int (u_\delta - u_\epsilon)\,\Big([G(u_\epsilon), |\nabla|_\epsilon^s](u_\delta - u_\epsilon)\Big) - \int (u_\delta - u_\epsilon)\,(|\nabla|_\delta^s - |\nabla|_\epsilon^s)(G(u_\delta)u_\delta) \\
&- \int (u_\delta - u_\epsilon)\,|\nabla|_\epsilon^s\Big((G(u_\delta) - G(u_\epsilon))u_\delta\Big).
\end{aligned}$$

We see that the third term cancels out, and we have for the last term, using the same trick as (4.1.9):

$$\begin{aligned}
&-\int (u_\delta - u_\epsilon)\,|\nabla|_\epsilon^s\Big((G(u_\delta)-G(u_\epsilon))u_\delta\Big)\\
&=-\int (u_\delta - u_\epsilon)\,|\nabla|_\epsilon^s\Big((u_\delta-u_\epsilon)\,u_\delta\int_0^1 F'\big((1-\lambda)u_\delta+\lambda u_\epsilon\big)\,d\lambda\Big)\\
&\le -\frac{1}{2}\int (u_\delta - u_\epsilon)^2\,|\nabla|_\epsilon^s\Big(u_\delta\int_0^1 F'\big((1-\lambda)u_\delta+\lambda u_\epsilon\big)\,d\lambda\Big)\\
&\lesssim \|u_\delta-u_\epsilon\|_{L^2}^2\,\|u_\delta\int_0^1 F'\big((1-\lambda)u_\delta+\lambda u_\epsilon\big)\,d\lambda\|_{H^m}\\
&\lesssim \|u_\delta-u_\epsilon\|_{L^2}^2, \qquad (4.1.21)
\end{aligned}$$

in view of the uniform bounds on u_δ, u_ϵ in H^m and a generalization of the composition Lemma 4.1.1 with two variables (see Chap. 5 in [61]). The rest terms can be estimated by (4.1.20):

$$\begin{aligned}
&|\int (u_\delta-u_\epsilon)\,\big(G(u_\delta)-G(u_\epsilon)\big)|\nabla|_\delta^s u_\delta| + |\int (u_\delta-u_\epsilon)\,G(u_\epsilon)(|\nabla|_\delta^s-|\nabla|_\epsilon^s)u_\delta|\\
&\quad + |\int (u_\delta-u_\epsilon)\,(|\nabla|_\delta^s-|\nabla|_\epsilon^s)(G(u_\delta)u_\delta)|\\
&\lesssim \|u_\delta-u_\epsilon\|_{L^2}\Big(\|G(u_\delta)-G(u_\epsilon)\|_{L^2}\||\nabla|_\delta^s u_\delta\|_{L^\infty}+\|G(u_\epsilon)\|_{L^\infty}\|(|\nabla|_\delta^s-|\nabla|_\epsilon^s)u_\delta\|_{L^2}\\
&\quad + \|(|\nabla|_\delta^s-|\nabla|_\epsilon^s)(G(u_\delta)u_\delta)\|_{L^2}\Big)\\
&\lesssim \|u_\delta-u_\epsilon\|_{L^2}\big(\|u_\delta-u_\epsilon\|_{L^2}+|\delta-\epsilon|\big),
\end{aligned}$$

where in order to estimate $\|G(u_\delta)-G(u_\epsilon)\|_{L^2}$ and $\|G(u_\epsilon)\|_{L^\infty}$ we have specially use the fact that $u_\delta,\ u_\epsilon \in L^\infty$. We thus get

$$\frac{1}{2}\frac{d}{dt}\|u_\delta-u_\epsilon\|_{L^2}^2 \le C\big(\|u_\delta-u_\epsilon\|_{L^2}^2+|\delta-\epsilon|\|u_\delta-u_\epsilon\|_{L^2}\big),$$

where C depends only on the initial conditions and other absolute dimensional quantities, but not on δ,ϵ. Given that the solutions start with the same initial condition, the Grönwall's lemma implies that

$$\|u_\delta(t)-u_\epsilon(t)\|_{L^2} \le C|\delta-\epsilon|(e^{Ct}-1)$$

for all $t<T$. This proves our claim.

As a consequence of the interpolation inequality

$$\|f\|_{H^{m'}} \le \|f\|_{L^2}^{1-\frac{m'}{m}}\|f\|_{H^m}^{\frac{m'}{m}}, \quad 0<m'<m$$

and the uniform bound for u_δ in $\mathcal{C}([0,T);H^m)$, so u_δ converges strongly to some u in all $\mathcal{C}([0,T);H^{m'})$, $\frac{d}{2}+1 < m' < m$. Moreover, $\partial_t u_\delta$ converges distributionally to $\partial_t u$, and in view of the uniform bound of $\partial_t u_\delta$ in H^{m-1}, it does so strongly in $H^{m'-1}$. This shows that the limit $u \in \mathcal{C}([0,T);H^{m'})\cap \mathcal{C}^1([0,T);H^{m'-1})$ solves (2.2.2) classically with initial condition u_0. Uniqueness is guaranteed by performing similar estimate like in Step 4. Note that

for the solution u that we constructed, the maximum principle proved earlier for u_δ still holds. The argument is the same, due to the positivity of the kernel.

Step 6: *Continuity of the solution.* At last, we prove that the unique solution u belongs to $\mathcal{C}([0,T],H^m)\cap\mathcal{C}^1([0,T),H^{m-1})$. By virtue of the equation it is sufficient to show that $u\in\mathcal{C}([0,T),H^m)$. For that, we first show that $u\in\mathcal{C}_w([0,T);H^m)$ which is the space of weakly continuous H^m-valued functions. In view of the uniform bounds of $\partial_t u_\delta$ in $\mathcal{C}([0,T);H^{m-1})$ and u_δ in $\mathcal{C}([0,T);H^m)$, we know that $u\in L^\infty(0,T;H^m)$ and $\partial_t u\in L^\infty(0,T;H^{m-1})$, in particular u is almost everywhere equal to a continuous function from $[0,T]$ into H^{m-1}. Finally, $H^{-(m-1)}$ is dense in H^{-m} imply that u is weakly continuous from $[0,T]$ into H^m. More precisely, let $\langle\phi,u\rangle$, $\phi\in H^m$ denote the dual paring of H^{-m}, there exist $\psi\in H^{-(m-1)}$ close to ϕ in the sense of H^m-norm and we write the decomposition

$$\langle\phi,u\rangle(t)=\langle\phi-\psi,u\rangle(t)+\langle\psi,u\rangle(t),$$

it implies the continuity of $\langle\phi,u\rangle(t)$ on $[0,T)$.

From the fact that $u\in\mathcal{C}_w([0,T);H^m)$ we have $\liminf_{t\to0+}\|u(t)\|_{H^m}\geq\|u_0\|_{H^m}$. For fixed $t\in[0,T)$, as the sequence $u_\delta(t)$ is uniformly bounded in H^m, so that it also have a subsequence weakly converge to $u(t)$ in H^m. Thus we have $\|u(t)\|_{H^m}\leq\limsup_{\delta\to0}\|u_\delta(t)\|_{H^m}$. Recall (4.1.19) we further obtain

$$\begin{aligned}\limsup_{t\to0+}\|u(t)\|_{H^m}&\leq\limsup_{t\to0+}\limsup_{\delta\to0}\|u_\delta(t)\|_{H^m}\\&\leq\limsup_{t\to0+}\frac{\|u_0\|_{H^m}}{1-Ct\|u_0\|_{H^m}}\leq\|u_0\|_{H^m}.\end{aligned}$$

In particular, $\lim_{t\to0+}\|u(t)\|_{H^m}=\|u_0\|_{H^m}$. This gives us strong right continuity at $t=0$. Because the analysis that we performed for the equation (2.2.2) is time reversible, we could likewise show strong left continuity at $t=0$. Moreover, as the equation is also translation invariant, we conclude that u is continuous on $[0,T)$.

This completes the proof of Theorem 2.2.2. □

4.1.2 A Beale-Kato-Majda type criterion

We now state the proof of the classical BKM criterion for our model. In fact we will see that $\int_0^T\||\nabla|u(t)\|_{L^\infty}dt$ is also a BKM criterion.

Proof of Theorem 2.2.3 The proof relies on an available log-Besov interpolation inequality (see Lemma B.3.5). The reader may also refer to the Appendix B for definition of Besov spaces and its properties such as interpolation inequalities, embedding.

In fact, we shall prove the following stronger criterion:

$$\int_0^T\|u(t)\|_{\dot{B}^1_{\infty,\infty}}dt<\infty. \tag{4.1.22}$$

According to the Bernstein's inequalities in Proposition B.1.1 and the fact that Δ_j is a uniformly bounded operator in terms of j in any L^p spaces ($p \in [0,\infty]$), we have $2^j\|\Delta_j u\|_{L^\infty} \lesssim \|\Delta_j \nabla u\|_{L^\infty} \lesssim \|\nabla u\|_{L^\infty}$. Hence (2.2.6) implies (4.1.22). Similarly, since the symbol of operator $|\nabla|$ is $|\xi| \sim 2^j$, we have $2^j\|\Delta_j u\|_{L^\infty} \lesssim \|\Delta_j|\nabla u|\|_{L^\infty} \lesssim \||\nabla|u\|_{L^\infty}$, thus the condition $\int_0^T \||\nabla|u(t)\|_{L^\infty}dt < \infty$ also implies (4.1.22).

Now, we prove once (4.1.22) holds, the solution will not blow-up at time T. Performing exactly the same estimates as (4.1.9), (4.1.12), (4.1.14), (4.1.17) on Step 3 of the proof of Theorem 2.2.2 but now with $|\nabla|^s$ instead of $|\nabla|_\delta^s$, we can arrive at the following a priori bound (we will specially use (4.1.18))

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u\|_{L^2}^2\\
\lesssim\ & \|u\|_{H^m}^2\Big(\||\nabla|^s(uF'(u))\|_{L^\infty} + \|\nabla G(u)\|_{L^\infty}^{1-\frac{|\beta_1|-1}{m-1}}\||\nabla|^s u\|_{L^\infty}^{1-\frac{m-|\beta_1|}{m-1}}\\
&+ \||\nabla|^s G(u)\|_{L^\infty}^{1-\frac{|\beta_2|}{m-1}}\|\nabla u\|_{L^\infty}^{1-\frac{m-|\beta_2|-1}{m-1}} + \|\nabla G(u)\|_{L^\infty}^{1-\frac{|\beta_2|-1}{m-1}}\||\nabla|^s u\|_{L^\infty}^{1-\frac{m-|\beta_2|}{m-1}}\\
&+ \||\nabla|^s u\|_{L^\infty}\|F'(u)\|_{L^\infty}^{1-\frac{|\beta_3|}{m}}\|u\|_{L^\infty}^{1-\frac{m-|\beta_3|}{m}}\\
&+ \|u\|_{L^\infty}\|\nabla F'(u)\|_{L^\infty}^{1-\frac{|\beta_3|-1}{m-1}}\||\nabla|^s u\|_{L^\infty}^{1-\frac{m-|\beta_3|-1}{m-1}}\\
&+ \|u\|_{L^\infty}\||\nabla|^s F'(u)\|_{L^\infty}^{1-\frac{|\beta_3|}{m-1}}\|\nabla u\|_{L^\infty}^{1-\frac{m-|\beta_3|-1}{m-1}}\Big).
\end{aligned}
$$

It is clear that as $u \in \mathcal{C}([0,T);H^m(\Omega)) \cap \mathcal{C}^1([0,T);H^{m-1}(\Omega))$, maximal principle holds for u, thus $\|u\|_{L^\infty} \le \|u_0\|_{L^\infty}$. Now, by Young's inequality and obvious embedding $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$ and Proposition **??** in order, we rewrite above inequality to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|_{H^m}^2\\
\lesssim\ & \|u\|_{H^m}^2\Big(\|uF'(u)\|_{\dot{B}^s_{\infty,1}} + \|G(u)\|_{\dot{B}^1_{\infty,1}} + \||u\|_{\dot{B}^s_{\infty,1}} + \|G(u)\|_{\dot{B}^s_{\infty,1}} + \|u\|_{\dot{B}^1_{\infty,1}}\\
&+ \|G(u)\|_{\dot{B}^1_{\infty,1}} + \|u\|_{\dot{B}^s_{\infty,1}} + \|u\|_{\dot{B}^s_{\infty,1}}(\|F'(u_0)\|_{L^\infty} + \|u_0\|_{L^\infty})\\
&+ \|u_0\|_{L^\infty}(\|F'(u) - F'(0)\|_{\dot{B}^1_{\infty,1}} + \|u\|_{\dot{B}^s_{\infty,1}})\\
&+ \|u_0\|_{L^\infty}(\||F'(u) - F'(0)\|_{\dot{B}^s_{\infty,1}} + \|u\|_{\dot{B}^1_{\infty,1}})\Big)\\
\lesssim\ & \|u\|_{H^m}^2\Big(\|uF'(u)\|_{\dot{B}^s_{\infty,1}} + \|G(u)\|_{\dot{B}^1_{\infty,1}} + \||u\|_{\dot{B}^s_{\infty,1}} + \|G(u)\|_{\dot{B}^s_{\infty,1}} + \|u\|_{\dot{B}^1_{\infty,1}}\\
&+ \|F'(u) - F'(0)\|_{\dot{B}^s_{\infty,1}} + \|F'(u) - F'(0)\|_{\dot{B}^1_{\infty,1}}\Big)\\
\lesssim\ & \|u\|_{H^m}^2\big(\|u\|_{\dot{B}^s_{\infty,1}} + \|u\|_{\dot{B}^1_{\infty,1}}\big),
\end{aligned}
\tag{4.1.23}
$$

where in the last step we used a composition lemma for homogeneous Besov spaces, see Theorem 2.61 in [45].

Next, one can get from interpolation inequality in Proposition B.1.3 (iii) and Young's inequality that

$$\|u\|_{\dot{B}^s_{\infty,1}} \lesssim \|u\|^{1-s}_{\dot{B}^0_{\infty,\infty}} \|u\|^s_{\dot{B}^1_{\infty,\infty}} \lesssim \|u_0\|^{1-s}_{L^\infty} \|u\|^s_{\dot{B}^1_{\infty,\infty}} \lesssim 1 + \|u\|_{\dot{B}^1_{\infty,1}} \tag{4.1.24}$$

Moreover, by taking $r = 1$, $p = \infty$, $\theta_1 = 1$, $\theta_2 = m - d/2 - 1$ in Lemma B.3.5 and from the embedding $H^m \hookrightarrow \dot{B}^{m-d/2}_{\infty,\infty}$, and the inequality $-a \log_2 a \leq 2$ on $\mathbb{R}^*_+$ we have

$$\begin{aligned}
\|u\|_{\dot{B}^1_{\infty,1}} &\lesssim \|u\|_{\dot{B}^1_{\infty,\infty}} \Big(1 + \log_2\Big(\frac{\|u\|_{\dot{B}^0_{\infty,\infty}} + \|u\|_{\dot{B}^{m-d/2}_{\infty,\infty}}}{\|u\|_{\dot{B}^1_{\infty,\infty}}}\Big)\Big) \\
&\lesssim \|u\|_{\dot{B}^1_{\infty,\infty}} \Big(1 + \log_2\big(\|u_0\|_{L^\infty} + \|u\|_{H^m}\big)\Big) - \|u\|_{\dot{B}^1_{\infty,\infty}} \log_2 \|u\|_{\dot{B}^1_{\infty,\infty}} \\
&\lesssim \|u\|_{\dot{B}^1_{\infty,\infty}} \Big(1 + \log_2\big(\|u_0\|_{L^\infty} + \|u\|_{H^m}\big)\Big) + 1.
\end{aligned} \tag{4.1.25}$$

Hence, taking (4.1.24) and (4.1.25) into (4.1.23), we obtain the following differential inequality (note that $\|u(t)\|_{H^m}$ will not vanishing)

$$\frac{d}{dt}\|u\|_{H^m} \lesssim \|u\|_{H^m} \Big(1 + \|u\|_{\dot{B}^1_{\infty,\infty}} \big(1 + \log_2\big(\|u_0\|_{L^\infty} + \|u\|_{H^m}\big)\big)\Big).$$

Define $X(t) := \ln(\|u\|_{H^m} + \|u_0\|_{L^\infty})$, we further obtain

$$\frac{d}{dt} X(t) \lesssim 1 + \|u\|_{\dot{B}^1_{\infty,\infty}} \Big(1 + X(t)\Big),$$

and thus

$$\frac{d}{dt} \ln(1 + X(t)) \lesssim 1 + \|u\|_{\dot{B}^1_{\infty,\infty}}. \tag{4.1.26}$$

One finally gets a double-exponential estimate of the form:

$$\|u(T)\|_{H^m} \lesssim \|u_0\|_{H^m} \exp\Big(\exp\Big(T + \int_0^T \|u(t)\|_{\dot{B}^1_{\infty,\infty}}\, dt\Big)\Big).$$

Theorem 2.2.3 follows immediately. □

4.1.3 Instant regularization and global regularity

In this subsection we study the question of the global existence in the periodic case through regularity theory.

Suppose $u_0(x) > 0, x \in \mathbb{T}^d$ and $u_0 \in H^m(\mathbb{T}^d)$. From local existence theory Theorem 2.2.2, there exists a unique classical positive solution $u \in \mathcal{C}([0,T); H^m(\mathbb{T}^d)) \cap \mathcal{C}^1([0,T); H^{m-1}(\mathbb{T}^d))$ on the torus $\mathbb{T}^d$. Let T^* be its maximal time of existence. We will show that $T^* = \infty$. Let us assume, on the contrary, that it is finite. Then we can quickly infer from BKM criterion Theorem 2.2.3 that

$$\int_0^{T^*} \|\nabla u(t)\|_{L^\infty(\mathbb{T}^d)}\, dt = \infty. \tag{4.1.27}$$

In what follows we apply the De Giorgi-Nash-Moser regularization result of [25] to the equation (2.2.8).

Theorem 4.1.3. *[25] Let ω be a weak solution of the evolution equations of the type*

$$\partial_t \omega = \int_{\mathbb{R}^d} (\omega(y) - \omega(x)) K(t,x,y)\, dy. \tag{4.1.28}$$

There exists $0 < \bar{s} < 2$ and $0 < \Lambda$, such that if the kernel K satisfying the properties:

$$K(t,x,y) = K(t,y,x) \quad \text{for any } x \neq y,$$
$$\frac{\Lambda^{-1}}{|x-y|^{d+\bar{s}}} \leq K(t,x,y) \leq \frac{\Lambda}{|x-y|^{d+\bar{s}}}.$$

Then for every $t_0 > 0$, $\omega \in \mathcal{C}^\alpha((t_0,\infty) \times \mathbb{R}^d)$. The constant α and the norm of ω depend only on $t_0, d, \|\omega_0\|_{L^2}$, and Λ.

Since u is a classical solution, the formal passage from the u-equation (2.2.5) to the $w = u^2$-equation (2.2.8) holds true. Moreover, in virtue of Max/Min principle, i.e. $\bar{u}(t) := \max_{x\in\mathbb{T}^d} u(t,x)$ is a strictly decreasing function of time t while $\underline{u}(t) := \min_{x\in\mathbb{T}^d} u(t,x)$ is a strictly increasing function of time t. By rewriting $\frac{2u(x)u(u)}{u(x)+u(y)} = \frac{2}{\frac{1}{u(x)}+\frac{1}{u(y)}}$ in (2.2.9), one finds that

$$\begin{aligned}&\frac{2u(x)u(u)}{u(x)+u(y)} \int_0^1 F'\big((1-\lambda)u(x) + \lambda u(y)\big)\, d\lambda \\ &\leq \bar{u}(t) \max_{a\in[\underline{u}(t),\bar{u}(t)]} F'(a) \;\leq\; \bar{u}(0) \max_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\end{aligned}$$

and

$$\begin{aligned}&\frac{2u(x)u(u)}{u(x)+u(y)} \int_0^1 F'\big((1-\lambda)u(x) + \lambda u(y)\big)\, d\lambda \\ &\geq \underline{u}(t) \min_{a\in[\underline{u}(t),\bar{u}(t)]} F'(a) \;\geq\; \underline{u}(0) \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a).\end{aligned}$$

Thus, one may define

$$\Lambda := c_{d,s} \max\Big\{\bar{u}(0) \max_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a), \quad \frac{1}{\underline{u}(0)\, \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)}\Big\}$$

and get

$$\frac{\Lambda^{-1}}{|x-y|^{d+s}} \leq \mathcal{K}^s(t,x,y) \leq \frac{\Lambda}{|x-y|^{d+s}}.$$

Obviously, the active kernel $\mathcal{K}^s$ given by (2.2.9) is symmetric with respect to (x,y). Hence above Theorem 4.1.3 applies verbatim to our periodic solutions of (2.2.8), there exists an $\alpha_0 > 0$ which depends only on $t_0, d, \Lambda, \|u_0\|_{L^\infty}$ such that for any $0 < t_0 < T^*$ we have $w \in \mathcal{C}^{\alpha_0}((t_0,T^*) \times \mathbb{T}^d)$ with the bound

$$\|w\|_{\mathcal{C}^{\alpha_0}((t_0,T^*)\times\mathbb{T}^d)} \leq C(t_0, d, \Lambda, \|u_0\|_{L^\infty}).$$

Moreover, we also have $\mathcal{C}^{\alpha_0}$ regularity for

$$m(t,x,y) := c_{d,s}\frac{2u(x)u(y)}{u(x)+u(y)}\int_0^1 F'\big((1-\lambda)u(x)+\lambda u(y)\big)\,d\lambda$$
$$= c_{d,s}\frac{2\sqrt{w(x)}\sqrt{w(y)}}{\sqrt{w(x)}+\sqrt{w(y)}}\int_0^1 F'\Big((1-\lambda)\sqrt{w(x)}+\lambda\sqrt{w(y)}\Big)\,d\lambda \tag{4.1.29}$$

on $(t_0,T^*)\times\mathbb{T}^d\times\mathbb{T}^d$, since the solution u and thus w is bounded away from zero.

Based on the gained $\mathcal{C}^{\alpha_0}$ bound in space-time, the next step is to prove Schauder estimates on equation (2.2.8) or more general type equations. Parallel to this, the Schauder estimates is obtained in [17] for a general class of linear integro-differential equations without evenness assumption on the kernel, which have been applied successfully to (2.2.8) of the case $s=1, F'\equiv 1$. Remark that the lack of evenness of the kernel makes our model out of the range of immediate applicability of recently results concerning the regularity theory of nonlinear integro-differential equations, such as Caffarelli–Silvestre [63], Lara-Dàvila [64], Mikulevicius– Pragarauskas [12] and Jin–Xiong [65, 54], Dong-Zhang [66].

It therefore needs to be addressed separately to get higher order estimates. We have the following result for a class of general equations (proved in the Appendix D).

Theorem 4.1.4. *Let $s_0\in(0,1]$, $s_0\le s\le 1$. Suppose $\omega\in\mathcal{C}^{1+\alpha,(1+\alpha)s}((-6,0]\times\mathbb{R}^d)$ is a solution of the linear integro-differential equation:*

$$\partial_t\omega = \int_{\mathbb{R}^d}\big(\omega(t,x+y)-\omega(t,x)\big)\,L(t,x,y)\,dy + \int_{\mathbb{R}^d}\big(\omega(t,x+y)-\omega(t,x)\big)\,Q(t,x,y)\,dy+\phi(t,x)$$

with $\phi(t,x)\in\mathcal{C}^{\alpha,\alpha s}((-6,0]\times\mathbb{R}^d)$. Suppose L and Q satisfy

$$L(t,x,y)=L(t,x,-y)\quad\text{for all }\ (t,x,y)\in(-6,0]\times\mathbb{R}^d\times\mathbb{R}^d, \tag{4.1.30}$$

$$\Lambda_1|y|^{-d-s}\le L(t,x,y)\le\Lambda_2|y|^{-d-s}\quad\text{for all }\ (t,x,y)\in(-6,0]\times\mathbb{R}^d\times\mathbb{R}^d, \tag{4.1.31}$$

$$|L(t_1,x_1,y)-L(t_2,x_2,y)|\le\Lambda_2(|t_1-t_2|^\alpha+|x_1-x_2|^{\alpha s})|y|^{-d-s} \tag{4.1.32}$$
$$\text{for all }\ (t_1,x_1,y),\ (t_2,x_2,y)\in(-6,0]\times\mathbb{R}^d\times\mathbb{R}^d,$$

and

$$|Q(t,x,y)|\le\Lambda_2\min\{1,|y|^{\alpha s}\}|y|^{-d-s}\quad\text{for all }\ (t,x,y)\in(-6,0]\times\mathbb{R}^d\times\mathbb{R}^d, \tag{4.1.33}$$

$$|Q(t_1,x_1,y)-Q(t_2,x_2,y)|\le\Lambda_2\min\{|t_1-t_2|^\alpha+|x_1-x_2|^{\alpha s},|y|^{\alpha s}\}|y|^{-d-s} \tag{4.1.34}$$

$$\text{for all } (t_1, x_1, y),\ (t_2, x_2, y) \in (-6, 0] \times \mathbb{R}^d \times \mathbb{R}^d,$$

respectively. Then for every $\beta < \alpha$, there exists $C > 0$ depending only on $s, d, \Lambda_1, \Lambda_2, \alpha, \beta$ such that

$$\|\omega\|_{\mathcal{C}^{1+\beta,(1+\beta)s}((-1,0]\times\mathbb{R}^d)} \le C\big(\|\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)} + \|\phi\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)}\big). \quad (4.1.35)$$

Now, we follow the idea of [17]. Assume that $w \in \mathcal{C}^{1+\alpha_1,(1+\alpha_1)s}((t_0, T^*) \times \mathbb{T}^d), 0 < \alpha_1 < 1$ is a solution to (2.2.8). we then restore the evenness by rewriting equation (2.2.8) to

$$\begin{aligned}\partial_t w = &\int_{\mathbb{R}^d} \Big(w(t,y) - w(t,x)\Big) \frac{m(t,x,x)}{|x-y|^{d+s}}\, dy \\ &+ \int_{\mathbb{R}^d} \Big(w(t,y) - w(t,x)\Big) \frac{m(t,x,y) - m(t,x,x)}{|x-y|^{d+s}}\, dy. \end{aligned} \quad (4.1.36)$$

Let $m(t,x,y)$ defined as in (4.1.29), thus $\|m\|_{\mathcal{C}^{\alpha_0}((t_0,T^*))\times\mathbb{T}^d\times\mathbb{T}^d)} \le C(d, \Lambda, \|u_0\|_{L^\infty})$. In this form it is clear that the $\mathcal{C}^{1+\alpha_1,(1+\alpha_1)s}$ regularity of w and thus C^{α_0} regularity of m are sufficient to make sense of both integrals classically. Define

$$L(t,x,y) := \frac{m(t,x,x)}{|y|^{d+s}} \quad \text{and} \quad Q(t,x,y) := \frac{m(t,x,x+y) - m(t,x,x)}{|y|^{d+s}}.$$

Then, it is elementary to check that L and Q satisfy the assumptions in Theorem 4.1.4. Therefore, we have for every $\beta_1 < \alpha_0$,

$$\|w\|_{\mathcal{C}^{1+\beta_1,(1+\beta_1)s}((t_0,T^*)\times\mathbb{T}^d)} \le C(s, d, \Lambda, \alpha_0, \beta_1).$$

Without loss of generality, we assume that $(1+\beta_1)s > 1$. In fact, if it is not satisfied, we can apply Theorem 4.1.4 again and again until for some $k_0 \in \mathbb{N}$ such that $(k_0 + \beta_{k_0})s > 1$ and

$$\|w\|_{\mathrm{Lip}_{t,x}((t_1,T^*)\times\mathbb{T}^d)} \le \|w\|_{\mathcal{C}^{1+\beta_1,(k_0+\beta_{k_0})s}((t_1,T^*)\times\mathbb{T}^d)} \le C(s, d, \Lambda, \alpha_0, \beta_{k_0}, k_0).$$

This is a contradiction to (4.1.27). Thus we conclude that $T^* = \infty$.

Next, we prove the higher regularity bounds (2.2.7) for w and hence for u. Differentiating (4.1.36) in x, we have for $v := \nabla_x w$,

$$\begin{aligned}\partial_t v = &\ \text{p.v.} \int_{\mathbb{R}^d} \big(v(t,x+y) - v(t,x)\big) \frac{m(t,x,x)}{|y|^{d+s}}\, dy \\ &+ \text{p.v.} \int_{\mathbb{R}^d} \big(v(t,x+y) - v(t,x)\big) \frac{m(t,x,x+y) - m(t,x,x)}{|y|^{d+s}}\, dy \\ &+ \text{p.v.} \int_{\mathbb{R}^d} \big(w(t,x+y) - w(t,x)\big) \frac{2\nabla_x m(t,x,x)}{|y|^{d+s}}\, dy \\ &+ \text{p.v.} \int_{\mathbb{R}^d} \big(w(t,x+y) - w(t,x)\big) \frac{2\nabla_x m(t,x,x+y) - 2\nabla_x m(t,x,x)}{|y|^{d+s}}\, dy \\ = &\ I + II + III + IV. \end{aligned}$$

Above, for simplicity, we used the property that $m(t,x,y) = m(t,y,x)$.

Notice that we have $\|m\|_{\mathcal{C}^{1+\beta_1,(1+\beta_1)s}((t_0,T^*)\times\mathbb{T}^d\times\mathbb{R}^d)} \leq C$ with $(1+\beta_1)s > 1$. And we can write by the definition of fractional derivatives that $III = 2|\nabla|^s w\, \nabla_x m(t,x,x)$. It is classic that the application $w \longmapsto |\nabla|^s w$ is bounded from $\mathcal{C}^{(1+\beta)s}$ into $\mathcal{C}^{\beta s}$. Thus we see that for $\beta_{1*} := \min\{\beta_1, \frac{(1+\beta_1)s-1}{s}\}$,

$$\begin{aligned}
&\|III\|_{\mathcal{C}^{\beta_{1*},\beta_{1*}s}((t_0,T^*)\times\mathbb{T}^d)}\\
\leq& C\|w\|_{\mathcal{C}^{\beta_1,\beta_1 s}((t_0,T^*)\times\mathbb{T}^d\times\mathbb{R}^d)}\|m\|_{\mathcal{C}^{\beta_1,(1+\beta_1)s-1}((t_0,T^*)\times\mathbb{T}^d\times\mathbb{T}^d)}\\
\leq& C(s,d,\Lambda,\alpha_0,\beta_1).
\end{aligned}$$

Meanwhile, it follows from Proposition D.1.2 that for every $\beta_2 < \beta_{1*}$ we have

$$\|IV\|_{\mathcal{C}^{\beta_2,\beta_2 s}((t_0,T^*)\times\mathbb{T}^d)} \leq C\|w\|_{\mathcal{C}^{1,s}((t_0,T^*)\times\mathbb{T}^d\times\mathbb{T}^d)} \leq C(s,d,\Lambda,\beta_{1*},\beta_2).$$

Applying Theorem 4.1.4 to the equation of v, we then obtain

$$\|\nabla_x w\|_{\mathcal{C}^{1+\beta_2,(1+\beta_2)s}((t_0,T^*)\times\mathbb{T}^d)} \leq C(s,d,\Lambda,\alpha_0,\beta_1,\beta_2).$$

In a exactly similar way, we can differentiate (4.1.36) in time t and obtain

$$\|\partial_t w\|_{\mathcal{C}^{1+\beta_2,(1+\beta_2)s}((t_0,T^*)\times\mathbb{T}^d)} \leq C(s,d,\Lambda,\alpha_0,\beta_{1*},\beta_2).$$

Then (2.2.7) follows from keeping differentiating (4.1.36) and applying Theorem 4.1.4 as above. The particular connection between w and m is crucial: a given regularity of w implies the same regularity of m.

We have completes the proof of Theorem 2.2.4. □

4.2 Global existence of weak solutions for positive bounded periodic data

In view of Theorem 2.2.4, for smooth enough and positive initial data $u_0 \in L^\infty(\mathbb{T}^d)$, the corresponding solution $u(t,x)$ is bounded and bounded below and satisfies the higher order bounds (2.2.7), where all of these bounds depend on the maximal and minimal value of u_0. Thus, for any positive time t, we shall have classical solution. However, in order to set up a weak formulation of the equation near $t = 0$ and restore the initial data, we need to have additional a priori bounds for solutions. Recall that in [16], we consider the case $s = 1, F(u) = u$ in (2.2.2), and we do have "first momentum law": by integrating (2.2.2), we have

$$\begin{aligned}
\int_{\mathbb{T}^d} u(t',x)\,dx - \int_{\mathbb{T}^d} u(t,x)\,dx &= \int_t^{t'}\int_{\mathbb{T}^d} u(\tau,x)|\nabla|u(\tau,x)\,dx\,d\tau\\
&= \|u\|^2_{L^2(t,t';\dot{H}^{1/2}(\mathbb{T}^d))}.
\end{aligned}$$

This combined nicely with energy conservation ensures that we have $u \in L^2(\mathbb{R}_+;\dot{H}^{1/2}(\mathbb{T}^d))$, regardless of the sign of u_0.

A difficulty emerged here, since above property is not obvious for the case $F(u) = u^n$, not to say for general function $F(u)$. Nevertheless, we find that

the following quantity is conserved for smooth 2π-periodic solution $u(t,x)$ to our generalized model (2.2.2),

$$\|u(t,\cdot)\|^p_{L^p(\mathbb{T}^d)} + \frac{p}{2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}(|u(\tau,y)|^{p-2} - |u(\tau,x)|^{p-2}) \cdot \big(F(u(\tau,y)) - F(u(\tau,x))\big)u(\tau,x)u(\tau,y)K^s_{\text{per}}(x-y)\,dx\,dy\,d\tau \tag{4.2.1}$$

for any $p \in (2,\infty)$. This property can be obtained by testing (2.2.2) with $|u|^{p-2}u$. Instantly, by taking $p = 3$ in above, we get

$$\frac{3}{2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|u(\tau,y)-u(\tau,x)|^2}{|y-x|^{d+s}}\mathcal{M}(\tau,x,y)\,dx\,dy\,d\tau \le \|u_0\|^3_{L^3(\mathbb{T}^d)}$$

where

$$\mathcal{M}(\tau,x,y) := u(\tau,x)u(\tau,y)\sum_{j\in\mathbb{Z}^d}\frac{c_{d,s}|x-y|^{d+s}}{|x-y+2\pi j|^{d+s}} \cdot \int_0^1 F'((1-\lambda)u(\tau,x)+\lambda u(\tau,y))\,d\lambda.$$

Hence by finite differences representation of the $\dot{H}^{s/2}(\mathbb{T}^d)$-norm, we find that for $s \in [s_0,1]$ with $s_0 \in (0,1]$

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}_+;\dot{H}^{s/2}(\mathbb{T}^d))} &= \int_0^\infty\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|u(\tau,y)-u(\tau,x)|^2}{|x-y|^{d+s}}\,dx\,dy\,d\tau \\ &\le \frac{2}{3}\frac{1}{\min_{\tau,x,y}\mathcal{M}}\|u_0\|^3_{L^3(\mathbb{T}^d)} \le C\|u_0\|_{L^\infty(\mathbb{T}^d)}.\end{aligned} \tag{4.2.2}$$

Then the Max / min principle and above a priori bound allow us to construct weak solutions from arbitrary positive data in $L^\infty(\mathbb{T}^d)$. By global weak solutions of (2.2.2), we mean that for any $\varphi \in \mathcal{C}^\infty(\mathbb{R}_+\times\mathbb{T}^d)$ the following weak formulation is satisfied for all $t > 0$,

$$\begin{aligned}&\int_{\mathbb{T}^d}u(t,x)\varphi(t,x)\,dx - \int_{\mathbb{T}^d}u_0(x)\varphi(0,x)\,dx - \int_0^t\int_{\mathbb{T}^d}u(\tau,x)\partial_t\varphi(\tau,x)\,dx\,d\tau \\ &= \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\big(F(u(\tau,y)) - F(u(\tau,x))\big)\varphi(\tau,x)u(y)\,K^s_{\text{per}}(x-y)\,dx\,dy\,d\tau.\end{aligned} \tag{4.2.3}$$

4.2.1 Proof of Theorem 2.2.5

Proof. In order to prove the existence of weak solution in the class $L^\infty(\mathbb{R}_+\times\mathbb{T}^d)\cap L^2(\mathbb{R}_+;\dot{H}^{s/2}(\mathbb{T}^d))$, one can resort to the following classical procedure:

(1) smooth out the positive and bounded initial data u_0 by taking standard mollifications $(u_0)_{\epsilon>0}$, and get a sequence of global smooth solutions $(u_\epsilon)_{\epsilon>0}$ which satisfied Max / Min principle, regularization properties and thus (4.2.1), (4.2.2);

(2) find out $(\partial_t u_\epsilon)_{\epsilon>0}$ is uniformly bounded in $L^2(\mathbb{R}_+ \times \mathbb{T}^d)$ by using commutator estimate (4.1.15), then use Aubin-Lions lemma to get strong convergence of solution sequence in $L^2(\mathbb{R}_+ \times \mathbb{T}^d)$;

(3) based on the obtained convergence results to show that $(u_\epsilon)_{\epsilon>0}$ converges, up to extraction, to a solution u of (2.2.2) in the sense of distributions.

Remark that, we have $u \geq \min_x u_0(x) > 0$. The only remaining problem is to restore the initial data and prove the announced time continuity in the theorem. We first prove the continuity of momentum. Now, for steady test function $\varphi(x)$ we rewrite (4.2.3) as

$$\begin{aligned}
&\int_{\mathbb{T}^d} u(t,x)\varphi(x)\,dx - \int_{\mathbb{T}^d} u_0(x)\varphi(x)\,dx \\
=&\frac{1}{2}\int_0^t \int_{\mathbb{T}^d\times\mathbb{T}^d} \big(F(u(\tau,y)) - F(u(\tau,x))\big)\big(u(\tau,y)-u(\tau,x)\big)\varphi(x)\,K^s_{\mathrm{per}}(x-y) \\
&+\frac{1}{2}\int_0^t \int_{\mathbb{T}^d\times\mathbb{T}^d} \big(F(u(\tau,y)) - F(u(\tau,x))\big)u(\tau,x)\big(\varphi(x)-\varphi(y)\big)\,K^s_{\mathrm{per}}(x-y).
\end{aligned} \tag{4.2.4}$$

At this moment, there are no a priori bounds that guarantee the smallness near the time $t = 0$ of the first integral on the right-hand side. However, we shall show that a possible concentration of the $\dot{H}^{s/2}$ norm near $t = 0$ is not the case. This goes back to an observation of the following lemma.

Lemma 4.2.1. *(Lemma 2.4 in [16]) Suppose that a sequence of functions $\{u_n\} \subset L^\infty(\mathbb{T}^d)$, bounded away from zero, enjoys both limits $u_n \rightharpoonup u'$ and $u_n^2 \rightharpoonup (u'')^2$ in the weak* topology of $L^\infty(\mathbb{T}^d)$. Then $u'' \geq u'$.*

Following the same line of steps (1)-(3), we know there exists a weak solution w in the class $L^\infty(\mathbb{R}_+ \times \mathbb{T}^d) \cap L^2(\mathbb{R}_+; \dot{H}^{s/2}(\mathbb{T}^d))$ (since $L^\infty \cap \dot{H}^{\frac{1}{2}}$ is an algebra) satisfies $w = u^2$-equation (2.2.8) in the weak sense, that is ($\mathcal{K}^s_{\mathrm{per}}$ represents the periodic version of $\mathcal{K}^s$, which is symmetric in terms of x, y)

$$\begin{aligned}
&\int_{\mathbb{T}^d} w(t,x)\varphi(t,x)\,dx - \int_{\mathbb{T}^d} w(0,x)\varphi(0,x)\,dx - \int_0^t\int_{\mathbb{T}^d} w(\tau,x)\partial_t\varphi(\tau,x)\,dx\,d\tau \\
=&\frac{1}{2}\int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \big(w(\tau,y)-w(\tau,x)\big)\big(\varphi(\tau,x)-\varphi(\tau,y)\big)\mathcal{K}^s_{\mathrm{per}}(\tau,x,y)\,dx\,dy\,d\tau.
\end{aligned}$$

In particular, we take φ independent of t, and find that $u^2(t) \rightharpoonup u_0^2$ weakly* in $L^\infty(\mathbb{T}^d)$ as $t \to 0$. Then we notice that for $\varphi(x) \geq 0$ the first integral of the right-hand side of (4.2.4):

$$\int_0^t \int_{\mathbb{T}^d\times\mathbb{T}^d} \big(F(u(\tau,y)) - F(u(\tau,x))\big)\big(u(\tau,y)-u(\tau,x)\big)\varphi(x)\,K^s_{\mathrm{per}}(x-y) \geq 0.$$

Meanwhile, by Hölder' inequality and composition lemma, we have for the second integral of the right-hand side of (4.2.4):

$$\begin{aligned}
&\Big|\int_0^t \int_{\mathbb{T}^d\times\mathbb{T}^d} \big(F(u(\tau,y)) - F(u(\tau,x))\big)u(\tau,x)\big(\varphi(x)-\varphi(y)\big)\,K^s_{\mathrm{per}}(x-y)\Big| \\
\leq& C\sqrt{t}\,\|F(u)\|_{L^2(0,t;\dot{H}^{s/2})}\|\varphi\|_{\dot{H}^{s/2}}\|u\|_{L^\infty}
\end{aligned}$$

$$\leq C\sqrt{t}\,\|u\|_{L^2(0,t;\dot{H}^{s/2})}\|\varphi\|_{\dot{H}^{s/2}}\|u\|_{L^\infty} \to 0 \quad \text{as} \quad t \to 0.$$

Hence, any weak* limit of a subsequence of $(u(t))_{t>0}$ would converge to a function u' satisfying

$$\int_{\mathbb{T}^d}(u' - u_0)\,\varphi(x)\,dx \geq 0 \quad \forall\ \varphi(x) \geq 0.$$

Thus $u' \geq u_0$, which combined with $u^2(t) \rightharpoonup u_0^2$ implies that $\lim_{t\to 0} u(t) = u' = u_0$ in the weak* topology of $L^\infty(\mathbb{T}^d)$, in particular, testing this weak* limit with $\varphi \equiv 1$ ensures that the momentum $\int_{\mathbb{T}^d} u(t,x)\,dx$ is continuous at $t = 0$. Look back at (4.2.4), it is clear that

$$\|u\|_{L^2(0,t;\dot{H}^{s/2}(\mathbb{T}^d))} \to 0 \quad \text{as} \quad t \to 0.$$

Let us finally point out that u is weakly continuous in $L^2(\mathbb{T}^d)$ at $t = 0$ and $\|u(t,\cdot)\|_{L^2(\mathbb{T}^d)}$ is continuous at $t = 0$. So the L^2 norm of u is also continuous at time $t = 0$ and therefore overall preserved since u is instantaneously regularized into $\mathcal{C}^\infty$. □

In view of the time reversibility property mentioned above, if u is a positive solution to (2.2.2), then $-u(t^* - t)$ is a negative solution for any $t^* > 0$. Thus starting with positive data $u_0 \in L^\infty(\mathbb{T}^d)/\mathcal{C}(\mathbb{T}^d)$ we obtain a solution u from Theorem 2.2.5 which becomes smooth instantaneously. Then $-u(t^*)$ serves as negative initial data that develop singularity at time $t = t^*$.

Corollary 4.2.2. *(Finite time singularity) For any $t^* > 0$, there exists a negative initial condition $u_0 \in \mathcal{C}^\infty(\mathbb{T}^d)$ and there exists a classical solution to (2.2.2) that develops into a discontinuous solution at time t^* i.e. $u(t^*) \in L^\infty(\mathbb{T}^d)/\mathcal{C}(\mathbb{T}^d)$.*

4.3 Long-time asymptotics and stability

As the solution is squeezed by the maximum and minimum principles, it is expected that the long-time dynamics of the model converges to a constant state consistent with the conservation of energy, namely,

$$u(t,x) \to \frac{\|u_0\|_{L^2(\mathbb{T}^d)}}{\sqrt{|\mathbb{T}^d|}} \quad \text{as} \quad t \to +\infty.$$

In this section, we first show that the amplitude of such weak solutions tends to zero exponentially fast. Then, we will exclude the non-trivial oscillations by showing that $|\nabla u|_{L^\infty}$ also tends to zero exponentially fast.

Recall notations:

$$\bar{u}(t) = \max_{x\in\mathbb{T}^d} u(t,x), \quad \underline{u}(t) = \min_{x\in\mathbb{T}^d} u(t,x)$$

and we define amplitude by

$$A(t) := \bar{u}(t) - \underline{u}(t).$$

4.3.1 Proof of Lemma 2.2.7

Proof. The proof is similar with our previous result [17], which relies on an idea from [67]. For initial data $u_0 \in L^\infty(\mathbb{T}^d), u_0 > 0$, we infer from Theorem 2.2.5 that there exists a global weak solution, which is smooth for all $t > 0$. Let us unfold such solution on $\mathbb{R}^d$. It is reasonable to say that there exist two points $\bar{x}, \underline{x} \in \mathbb{T}^d$ such that $\bar{u}(t) = u(t, \bar{x})$ and $\underline{u}(t) = u(t, \underline{x})$. The gradient $\nabla_x u$ vanish at both $\bar{x}, \underline{x}$. Now, we are going to evaluate (2.2.5) at $\bar{x}, \underline{x}$. By $F' \geq 0$ and minimal principle, we have (we dropped the reference to time for readability)

$$\begin{aligned}\frac{d}{dt}\bar{u}(t) &= \int_{\mathbb{R}^d} u(y)\big(F(u(y)) - F(u(\bar{x}))\big)K^s(\bar{x}-y)\,dy \\ &\leq \underline{u}(0)\int_{|y-\bar{x}|\geq 1, |y-\underline{x}|\geq 1}\big(F(u(y)) - F(u(\bar{x}))\big)K^s(\bar{x}-y)\,dy \\ &\leq \underline{u}(0)\int_{|y-\bar{x}|\geq 1, |y-\underline{x}|\geq 1}\big(F(u(y)) - F(u(\bar{x}))\big)\min\{K^s(\bar{x}-y), K^s(\underline{x}-y)\}\,dy,\end{aligned}$$

and similarly

$$\begin{aligned}\frac{d}{dt}\underline{u}(t) &= \int_{\mathbb{R}^d} u(y)\big(F(u(y)) - F(u(\underline{x}))\big)K^s(\underline{x}-y)\,dy \\ &\geq \underline{u}(0)\int_{|y-\bar{x}|\geq 1, |y-\underline{x}|\geq 1}\big(F(u(y)) - F(u(\bar{x}))\big)K^s(\underline{x}-y)\,dy \\ &\geq \underline{u}(0)\int_{|y-\bar{x}|\geq 1, |y-\underline{x}|\geq 1}\big(F(u(y)) - F(u(\underline{x}))\big)\min\{K^s(\bar{x}-y), K^s(\underline{x}-y)\}\,dy.\end{aligned}$$

Then mean value theorem implies that

$$\begin{aligned}&\frac{d}{dt}A(t) \\ &\leq -\underline{u}(0)\big(F(u(\bar{x})) - F(u(\underline{x}))\big)\int_{|y-\bar{x}|\geq 1, |y-\underline{x}|\geq 1}\min\{K^s(\bar{x}-y), K^s(\underline{x}-y)\}\,dy \\ &\leq -\underline{u}(0)\min_{a\in[\underline{u}(0),\bar{u}(0)]}F'(a)\,A(t)\int_{|y|\geq 1+|\bar{x}|+|\underline{x}|}\frac{c_{d,s}}{(|y|+|\bar{x}|+|\underline{x}|)^{d+s}}\,dy \\ &\leq -\eta A(t),\end{aligned}$$

where $\eta = \underline{u}(0)\min_{a\in[\underline{u}(0),\bar{u}(0)]}F'(a)\int_{|y|\geq 1+2\sqrt{d}\pi}\frac{c_{d,s}}{(|y|+2\sqrt{d}\pi)^{d+s}}\,dy$. An application of Grönwall's lemma completes the proof. □

4.3.2 Proof of Theorem 2.2.8

Proof. Let u be a weak solution obtained from Theorem 2.2.5, we unfold such solution on $\mathbb{R}^d$. Let us first rewrite (2.2.5) as

$$\partial_t u = \text{p.v.}\int_{\mathbb{R}^d}\big(F(u(t,x+z)) - F(u(t,x))\big)u(t,x+z)K^s(z)\,dz.$$

Then we consider the equation of ∇u from above and multiply by ∇u to get (the integrals being understood as principal values and we dropped the

reference to time for readability)

$$\frac{1}{2}\partial_t|\nabla u(t,x)|^2 = \nabla u(x)\int_{\mathbb{R}^d}\nabla_x\big(F(u(x+z))-F(u(x))\big)u(x+z)K^s(z)\,dz + \nabla u(x)\int_{\mathbb{R}^d}\big(F(u(x+z))-F(u(x))\big)\nabla_x u(x+z)K^s(z)\,dz. \tag{4.3.1}$$

For the first integral in (4.3.1), we write

$$\begin{aligned}&\nabla u(x)\int_{\mathbb{R}^d}\nabla_x\big(F(u(x+z))-F(u(x))\big)u(x+z)K^s(z)\,dz\\ =&\nabla u(x)\int_{\mathbb{R}^d}F'(u(x))\big(\nabla_x u(x+z)-\nabla_x u(x)\big)u(x+z)K^s(z)\,dz\\ &+\nabla u(x)\int_{\mathbb{R}^d}\big(F'(u(x+z))-F'(u(x))\big)\nabla_x u(x+z)u(x+z)K^s(z)\,dz.\end{aligned}$$

Notice that for $x\in\mathbb{T}^d$ the point of the maximum of $|\nabla u|$, one has

$$\begin{aligned}&\nabla u(x)\int_{\mathbb{R}^d}F'(u(x))\big(\nabla_x u(x+z)-\nabla_x u(x)\big)u(x+z)K^s(z)\,dz\\ =&\frac{1}{2}\int_{\mathbb{R}^d}F'(u(x))\big(|\nabla_x u(x+z)|^2-|\nabla_x u(x)|^2\big)u(x+z)K^s(z)\,dz\\ &-\frac{1}{2}\int_{\mathbb{R}^d}F'(u(x))|\nabla_x u(x+z)-\nabla_x u(x)|^2u(x+z)K^s(z)\,dz\\ \le&-\min_{a\in[\underline{u}(0),\bar{u}(0)]}F'(a)\,\underline{u}(0)\frac{1}{2}\int_{\mathbb{R}^d}|\nabla_x u(x+z)-\nabla_x u(x)|^2K^s(z)\,dz,\end{aligned}$$

hence we get from (4.3.1) that

$$\begin{aligned}&\frac{1}{2}\partial_t|\nabla u(t,x)|^2+\min_{a\in[\underline{u}(0),\bar{u}(0)]}F'(a)\,\underline{u}(0)\frac{1}{2}\int_{\mathbb{R}^d}|\nabla_x u(x+z)-\nabla_x u(x)|^2K^s(z)\,dz\\ \le&\nabla u(x)\int_{\mathbb{R}^d}\big(F(u(x+z))-F(u(x))\big)\nabla_x u(x+z)K^s(z)\,dz\\ &+\nabla u(x)\int_{\mathbb{R}^d}\big(F'(u(x+z))-F'(u(x))\big)\nabla_x u(x+z)u(x+z)K^s(z)\,dz\\ :=&\,J_1+J_2.\end{aligned}\tag{4.3.2}$$

Meanwhile, we find that by integration by parts

$$\begin{aligned}&\int_{\mathbb{R}^d}|\nabla_x u(x+z)-\nabla_x u(x)|^2K^s(z)\,dz\\ \ge&|\nabla_x u(x)|^2\int_{|z|\ge r}K^s(z)\,dz-2\nabla_x u(x)\int_{|z|\ge r}\nabla_z\big(u(x+z)-u(x)\big)K^s(z)\,dz\\ \ge&\frac{C_1}{r^s}|\nabla_x u(x)|^2+2\nabla_x u(x)\cdot\int_{|z|\ge r}\big(u(x+z)-u(x)\big)\nabla_z K^s(z)\,dz\\ &-2\nabla_x u(x)\cdot\int_{|z|=r}\nu_z(r)\big(u(x+z)-u(x)\big)K^s(z)\,d\sigma(r)\end{aligned}$$

where $\sigma(r)$ is the surface $|z|=r$ and $\nu_z(r)$ is the outward-pointing normal to $\sigma(r)$ at point z. It follows that

$$\int_{\mathbb{R}^d}|\nabla_x u(x+z)-\nabla_x u(x)|^2K^s(z)\,dz$$

$$\geq \frac{C_1}{r^s}|\nabla_x u(x)|^2 - C_2|\nabla_x u(x)|\frac{A(t)}{r^{1+s}}$$

and taking $r = \frac{2C_2A(t)}{C_1|\nabla_x u(x)|}$ gives

$$\int_{\mathbb{R}^d} |\nabla_x u(x+z) - \nabla_x u(x)|^2 K^s(z)\, dz \geq 2C_3 \frac{\|\nabla u\|_{L^\infty}^{2+s}}{A^s}. \tag{4.3.3}$$

To estimate J_1, we shall split it depending on whether $|z| > \rho_1$ or $|z| \leq \rho_1$ and write $J_1 = J_{11} + J_{12}$. We estimate J_{11} after rewriting it in the following form

$$J_{11} = \frac{1}{2}\nabla u(x) \cdot \int_{|z|>\rho_1} \int_0^1 F'\big((1-\lambda)u(x) + \lambda u(x+z)\big) \cdot \nabla_z |u(x+z) - u(x)|^2 K^s(z)\, d\lambda\, dz.$$

Define $H(\lambda, t, x, z) = F'\big((1-\lambda)u(x) + \lambda u(x+z)\big)K^s(z)$, we see that there exists $C_5 > 0$ such that

$$|H(\lambda, t, x, z)| \leq \max_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\, K^s(z) \leq \frac{C_4}{|z|^{d+s}}$$

and

$$\begin{aligned} &|\nabla_z H(\lambda, t, x, z)| \\ &\leq \max_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\, |\nabla_z K^s(z)| + \|\nabla u\|_{L^\infty} \max_{a\in[\underline{u}(0),\bar{u}(0)]} |F''(a)|\, K^s(z) \\ &\leq \frac{C_5}{|z|^{d+s+1}} + \|\nabla u\|_{L^\infty} \frac{C_5}{|z|^{d+s}}. \end{aligned}$$

Thus by integration by parts we have for J_{11},

$$\begin{aligned} |J_{11}| \leq{}& \frac{1}{2}|\nabla u(x)| \int_{|z|=\rho_1} \int_0^1 |u(x+z) - u(x)|^2 |H(\lambda, t, x, z)|\, d\lambda\, d\sigma(\rho_1) \\ &+ \frac{1}{2}|\nabla u(x)| \int_{|z|>\rho_1} \int_0^1 |u(x+z) - u(x)|^2 |\nabla_z H(\lambda, t, x, z)|\, d\lambda\, dz \\ \leq{}& C_6\big(\|\nabla u\|_{L^\infty} \frac{A^2(t)}{\rho_1^{1+s}} + \|\nabla u\|_{L^\infty}^2 \frac{A^2(t)}{\rho_1^s}\big). \end{aligned}$$

As to J_{12}, we use the first order Taylor formula for the increment of u :

$$\begin{aligned} J_{12} &= \nabla u(x) \cdot \int_{|z|\leq\rho_1} \int_0^1 F'(u(x+\lambda_1 z))z \cdot \nabla_x u(x+\lambda_1 z) \\ &\qquad \cdot \nabla_x u(x+z) K^s(z)\, d\lambda_1\, dz \\ &= \nabla u(x) \cdot \int_{|z|\leq\rho_1} \int_0^1 F'(u(x+\lambda_1 z))z \cdot \big(\nabla_x u(x+\lambda_1 z) - \nabla_x u(x)\big) \\ &\qquad \cdot \nabla_x u(x+z) K^s(z)\, d\lambda_1\, dz \\ &\quad + \nabla u(x) \cdot \int_{|z|\leq\rho_1} \int_0^1 F'(u(x+\lambda_1 z))z \cdot \nabla_x u(x) \end{aligned}$$

$$\cdot \big(\nabla_x u(x+z) - \nabla_x u(x)\big) K^s(z)\, d\lambda_1\, dz$$
$$+ |\nabla u(x)|^2 \cdot \int_{|z|\le\rho_1} \int_0^1 F'(u(x+\lambda_1 z))z \cdot \nabla_x u(x) K^s(z)\, d\lambda_1\, dz$$
$$:= J_{12}^1 + J_{12}^2 + J_{12}^3.$$

We get by Cauchy-Schwarz inequality that

$$\begin{aligned}
|J_{12}^1| \le& C_7 \|\nabla u\|_{L^\infty}^2 \int_{|z|\le\lambda_1\rho_1} \int_0^1 |\nabla_x u(x+z) - \nabla_x u(x)| \sqrt{K^s(z)} \frac{(\lambda_1)^{s-1}}{|z|^{\frac{d+s}{2}-1}}\, d\lambda_1\, dz \\
\le& C_7 \|\nabla u\|_{L^\infty}^2 \Big(\int_{|z|\le\rho_1} |\nabla_x u(x+z) - \nabla_x u(x)|^2 K^s(z)\, dz \Big)^{\frac{1}{2}} \rho_1^{1-s/2} \\
\le& \frac{1}{16} \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\, \underline{u}(0) \int_{\mathbb{R}^d} |\nabla_x u(x+z) - \nabla_x u(x)|^2 K^s(z)\, dz \\
&+ \frac{16 C_7^2}{\min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\, \underline{u}(0)} \|\nabla u\|_{L^\infty}^4 \rho_1^{2-s}.
\end{aligned}$$

The estimate for J_{12}^2 is completely analogous, since without the integral for λ'. For J_{12}^3, it is clear that

$$\begin{aligned}
J_{12}^3 \le& \max_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a) \|\nabla u\|_{L^\infty}^3 \int_{|z|\le\rho} |z| K^s(z)\, dz \\
\le& C_8 \|\nabla u\|_{L^\infty}^3 \rho_1^{1-s}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|J_{12}| \le& |J_{12}^1| + |J_{12}^2| + |J_{12}^2| \\
\le& \frac{1}{8} \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\, \underline{u}(0) \int_{\mathbb{R}^d} |\nabla_x u(x+z) - \nabla_x u(x)|^2 K^s(z)\, dz \\
&+ C_9 \big(\|\nabla u\|_{L^\infty}^4 \rho_1^{2-s} + \|\nabla u\|_{L^\infty}^3 \rho_1^{1-s} \big)
\end{aligned}$$

and

$$\begin{aligned}
|J_1| \le& |J_{11}| + |J_{12}| \\
\le& \frac{1}{8} \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\, \underline{u}(0) \int_{\mathbb{R}^d} |\nabla_x u(x+z) - \nabla_x u(x)|^2 K^s(z)\, dz + (C_6 + C_9) \\
&\cdot \big(\|\nabla u\|_{L^\infty}^4 \rho_1^{2-s} + \|\nabla u\|_{L^\infty}^3 \rho_1^{1-s} + \|\nabla u\|_{L^\infty} \frac{A^2(t)}{\rho_1^{1+s}} + \|\nabla u\|_{L^\infty}^2 \frac{A^2(t)}{\rho_1^s} \big).
\end{aligned}$$

Choosing $\rho_1 = A(t)/\|\nabla u\|_{L^\infty}$ gives

$$\begin{aligned}
|J_1| \le& \frac{1}{8} \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\, \underline{u}(0) \int_{\mathbb{R}^d} |\nabla_x u(x+z) - \nabla_x u(x)|^2 K^s(z)\, dz \\
&+ 2(C_6 + C_9) \|\nabla u\|_{L^\infty}^{2+s} (A^{1-s}(t) + A^{2-s}(t)).
\end{aligned}$$

Now, we are going to estimate J_2 by rewriting it similar with J_1:

$$\begin{aligned}
J_2 =& \frac{1}{2}\nabla u(x)\int_{\mathbb{R}^d}\big(F'(u(x+z))-F'(u(x))\big)\nabla_z|u(x+z)|^2K^s(z)\,dz\\
=& \frac{1}{2}\nabla u(x)\int_{|z|>\rho_2}\big(F'(u(x+z))-F'(u(x))\big)\nabla_z\big(|u(x+z)|^2-|u(x)|^2\big)K^s(z)\,dz\\
&+\nabla u(x)\int_{|z|\le\rho_2}\int_0^1 F''(u(x+\lambda_2 z))z\cdot\nabla_x u(x+\lambda_2 z)\\
&\qquad\qquad\cdot\nabla_x u(x+z)u(x+z)K^s(z)\,d\lambda_2\,dz\\
:=& J_{21}+J_{22}.
\end{aligned}$$

We see J_{21} is the kind of type that we studied for J_{11}. Indeed, we have

$$\begin{aligned}
|J_{21}| &\le \bar{u}(0)\|\nabla u\|_{L^\infty}A(t)\Big(\int_{|z|>\rho_2}\nabla_z\big(F'(u(x+z))K^s(z)\big)\,dz\\
&\qquad+2\max_{a\in[\underline{u}(0),\bar{u}(0)]}F'(a)\Big(\int_{|z|=\rho_2}K^s(z)\,d\sigma(\rho_2)\Big)\\
&\le C_{10}\|\nabla u\|_{L^\infty}\Big(\frac{A(t)}{\rho_2^{1+s}}+\|\nabla u\|_{L^\infty}\frac{A(t)}{\rho_2^s}\Big).
\end{aligned}$$

For J_{22}, we split it like for J_{12} :

$$\begin{aligned}
|J_{22}| &\le \bar{u}(0)\|\nabla u\|_{L^\infty}^2\|F''\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+)}\Big(\int_{|z|\le\rho_2}\int_0^1\big(\nabla_x u(x+\lambda_2 z)-\nabla_x u(x)\big)\\
&\qquad\cdot|z|K^s(z)\,d\lambda_2\,dz+\int_{|z|\le\rho_2}\int_0^1\big(\nabla_x u(x+z)-\nabla_x u(x)\big)|z|K^s(z)\,d\lambda_2\,dz\\
&\qquad+\|\nabla u\|_{L^\infty}\int_{|z|\le\rho_2}\int_0^1|z|K^s(z)\,d\lambda_2\,dz\Big)\\
&\le\frac{1}{8}\min_{a\in[\underline{u}(0),\bar{u}(0)]}F'(a)\,\underline{u}(0)\int_{\mathbb{R}^d}|\nabla_x u(x+z)-\nabla_x u(x)|^2K^s(z)\,dz\\
&\quad+C_{11}\big(\|\nabla u\|_{L^\infty}^4\rho_2^{2-s}+\|\nabla u\|_{L^\infty}^3\rho_2^{1-s}\big).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|J_2| &\le\frac{1}{8}\min_{a\in[\underline{u}(0),\bar{u}(0)]}F'(a)\,\underline{u}(0)\int_{\mathbb{R}^d}|\nabla_x u(x+z)-\nabla_x u(x)|^2K^s(z)\,dz\\
&\quad+C_{12}\Big(\|\nabla u\|_{L^\infty}\frac{A(t)}{\rho_2^{1+s}}+\|\nabla u\|_{L^\infty}^2\frac{A(t)}{\rho_2^s}+\|\nabla u\|_{L^\infty}^4\rho_2^{2-s}+\|\nabla u\|_{L^\infty}^3\rho_2^{1-s}\Big).
\end{aligned}$$

Choosing $\rho_2=\sqrt{A(t)}/\|\nabla u\|_{L^\infty}$ gives

$$\begin{aligned}
|J_2| &\le\frac{1}{8}\min_{a\in[\underline{u}(0),\bar{u}(0)]}F'(a)\,\underline{u}(0)\int_{\mathbb{R}^d}|\nabla_x u(x+z)-\nabla_x u(x)|^2K^s(z)\,dz\\
&\quad+C_{13}\|\nabla u\|_{L^\infty}^{2+s}(A^{\frac{1}{2}(1-s)}+A^{\frac{1}{2}(2-s)}).
\end{aligned}$$

Taking estimates for J_1, J_2 and (4.3.3) into (4.3.2), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\nabla u\|_{L^\infty}^2 + C_3 \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\,\underline{u}(0)\frac{\|\nabla u\|_{L^\infty}^{2+s}}{A^s(t)} \\
\le\, & C_{14}\|\nabla u\|_{L^\infty}^{2+s}\big(A^{1-s}(t) + A^{2-s}(t) + A^{\frac{1}{2}(1-s)}(t) + A^{\frac{1}{2}(2-s)}(t)\big) \\
\le\, & 4C_{14}\|\nabla u\|_{L^\infty}^{2+s}\max\{A^{2-s}(t), A^{\frac{1}{2}(1-s)}(t)\}.
\end{aligned}$$

In view of Lemma 2.2.7, there exists a time T^* such that

$$4C_{14}\max\{A^{2-s}(t), A^{\frac{1}{2}(1-s)}(t)\} \le \frac{1}{2}C_3 \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\,\underline{u}(0).$$

This implies that

$$\frac{d}{dt}\|\nabla u\|_{L^\infty}^2 + \frac{1}{2}C_3 \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\,\underline{u}(0)\frac{\|\nabla u\|_{L^\infty}^{2+s}}{A^s(t)} \le 0$$

Using the precise estimate from Lemma 2.2.7, we further get

$$\frac{d}{dt}\|\nabla u\|_{L^\infty}^2 + \frac{e^{s\eta t}}{2C_3A^s(0)} \min_{a\in[\underline{u}(0),\bar{u}(0)]} F'(a)\underline{u}(0)\|\nabla u\|_{L^\infty}^{2+s} \le 0.$$

This finally completes the proof of Theorem 2.2.8. □

Concerning stability, we also established the following two estimates.

Corollary 4.3.1. *Let F_1, F_2 be two functions satisfied our assumptions on the function F. Given two pointwise positive initial data $u_{1,0}, u_{2,0} \in H^m(\mathbb{T}^d)$. Suppose that u_i is the solution of Cauchy problem (2.2.2)-(2.2.3) with F_i and initial data $u_{i,0}$, respectively. Then we have the following stability estimate in $L^2(\mathbb{T}^d)$*

$$\|u_1 - u_2\|_{L^2} \le \big(\|u_{1,0} - u_{2,0}\|_{L^2} + \|F_1' - F_2'\|_{L^\infty}\big)e^{C_0 t},\ \forall\, t > 0$$

with a constant C_0 depends on d, $\|u_1\|_{H^m}$, $\|u_2\|_{H^m}$. And in $L^\infty(\mathbb{T}^d)$, we have

$$\|u_1 - u_2\|_{L^\infty} \le 2\sqrt{d}\pi\big(\|\nabla u_1(t)\|_{L^\infty} + \|\nabla u_2(t)\|_{L^\infty}\big) + \frac{1}{\sqrt{\mathbb{T}^d}}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{T}^d)}.$$

Remark 5. From Theorem 2.2.8 we see that when $t > \widetilde{T}$, the difference between two solutions is controlled by $\frac{1}{\sqrt{\mathbb{T}^d}}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{T}^d)}$, this is consistent with the long time asymptotics of our model.

Proof. From Theorem 2.2.4 we know that $u_1, u_2 \in \mathcal{C}(\mathbb{R}_+; H^m)$ are positive and bounded by the maximal value of its initial data. We first write the equation of the difference of two solutions:

$$\partial_t(u_1 - u_2) = [F_1(u_1) - F_2(u_2), |\nabla|^s]u_1 + [F_2(u_2), |\nabla|^s](u_1 - u_2).$$

Then we take L^2 inner product of above equation with $u_1 - u_2$. We obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_1-u_2\|_{L^2}^2\\ &=\int (u_1-u_2)(F_1(u_1)-F_2(u_2))|\nabla|^s u_1 - \int (u_1-u_2)|\nabla|^s\big(u_1(F_1(u_1)-F_2(u_2))\big).\end{aligned}$$

We write the decomposition

$$F_1(u_1)-F_2(u_2) = F_1(u_1)-F_1(u_2)+F_1(u_2)-F_2(u_2).$$

As mentioned before, we can assume $F_1(0) = 0$. By composition lemma Corollary 2.66 in [45], the first integral can be estimated as

$$\begin{aligned}&|\int (u_1-u_2)(F_1(u_1)-F_2(u_2))|\nabla|^s u_1|\\ &\le \|u_1-u_2\|_{L^2}(\|F_1(u_1)-F_1(u_2)\|_{L^2}+\|F_1(u_2)-F_2(u_2)\|_{L^2})\||\nabla|^s u_1\|_{L^\infty}\\ &\le C\|u_1-u_2\|_{L^2}(\|u_1-u_2\|_{L^2}+\|F_1-F_2\|_{L^\infty})\|u_1\|_{H^m}.\end{aligned}$$

We then split the second integral as

$$\begin{aligned}&\int (u_1-u_2)|\nabla|^s\big(u_1(F_1(u_1)-F_2(u_2))\big)\\ &=\int (u_1-u_2)|\nabla|^s\big(u_1(F_1(u_1)-F_1(u_2))\big)+\int (u_1-u_2)|\nabla|^s\big(u_1(F_1(u_2)-F_2(u_2))\big).\end{aligned}$$

One can estimate $-\int (u_1-u_2)|\nabla|^s\big(u_1(F_1(u_1)-F_1(u_2))\big)$ similarly as (4.1.21). Indeed,

$$\begin{aligned}&-\int (u_1-u_2)|\nabla|^s\big(u_1(F_1(u_1)-F_1(u_2))\big)\\ &\le C\|u_1-u_2\|_{L^2}^2\|u_1\int_0^1 F_1'\big((1-\lambda)u_1+\lambda u_2\big)\,d\lambda\|_{H^m}\\ &\le C\|u_1-u_2\|_{L^2}^2\|u_1\|_{H^m}\|u_1,u_2\|_{H^m}.\end{aligned}$$

While use the fact that $L^\infty\cap\dot{H}^s$ is an algebra and interpolation inequality, we have

$$\begin{aligned}&|\int (u_1-u_2)|\nabla|^s\big(u_1(F_1(u_2)-F_2(u_2))\big)|\\ &\le \|u_1-u_2\|_{L^2}\||\nabla|^s\big(u_1(F_1(u_2)-F_2(u_2))\big)\|_{L^2}\\ &\le C\|u_1-u_2\|_{L^2}\Big(\|u_1\|_{L^\infty}\|F_1(u_2)-F_2(u_2)\|_{\dot{H}^s}+\|u_1\|_{\dot{H}^s}\|F_1-F_2\|_{L^\infty}\Big)\\ &\le C\|u_1-u_2\|_{L^2}\|u_1\|_{H^m}\Big(\|\nabla\big(F_1(u_2)-F_2(u_2)\big)\|_{L^2}+\|F_1-F_2\|_{L^\infty}\Big)\\ &\le C\|u_1-u_2\|_{L^2}\|u_1,u_2\|_{H^m}\Big(\|F_1-F_2\|_{L^\infty}+\|F_1'-F_2'\|_{L^\infty}\Big)\\ &\le C\|u_1-u_2\|_{L^2}\|u_1,u_2\|_{H^m}^2\|F_1'-F_2'\|_{L^\infty}.\end{aligned}$$

Thus we obtain

$$\frac{d}{dt}\|u_1-u_2\|_{L^2}\le C_0\Big(\|u_1-u_2\|_{L^2}+\|F_1'-F_2'\|_{L^\infty}\Big)$$

and finish the first statement with the help of Gronwall's lemma.

Now, we find that for $i = 1, 2$

$$\underline{u}(t) \leq \frac{\|u_{i,0}\|_{L^2(\mathbb{T}^d)}}{\sqrt{\mathbb{T}^d}} = \frac{\|u_i(t)\|_{L^2(\mathbb{T}^d)}}{\sqrt{\mathbb{T}^d}} \leq \bar{u}(t),$$

thus there exists $y_i \in \mathbb{T}^d$ such that $u(t, y_i) = \frac{1}{\sqrt{\mathbb{T}^d}}\|u_{i,0}\|_{L^2(\mathbb{T}^d)}$. And so

$$\begin{aligned}
&\|u_1(t) - u_2(t)\|_{L^\infty_x} \\
&\leq \|u_1(t,x) - u_1(t,y_1)\|_{L^\infty_x} + \|u_2(t,x) - u_2(t,y_2)\|_{L^\infty_x} + \frac{1}{\sqrt{\mathbb{T}^d}}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{T}^d)} \\
&\leq 2\sqrt{d}\pi\big(\|\nabla u_1(t)\|_{L^\infty} + \|\nabla u_2(t)\|_{L^\infty}\big) + \frac{1}{\sqrt{\mathbb{T}^d}}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{T}^d)}.
\end{aligned}$$

□

Appendix A

Notations and definitions of some weak solutions

A.1 Notations

Here is for a few notations. We denote by C harmless positive constants that may change from one line to the other, and we sometimes write $A \lesssim B$ instead of $A \leq CB$. Likewise, $A \sim B$ means that $C_1 B \leq A \leq C_2 B$ with absolute constants C_1, C_2. For X a Banach space, $p \in [1, \infty]$ and $T \in (0, \infty]$, the notation $L^p(0,T;X)$ or $L^p_T(X)$ designates the set of measurable functions $f : [0,T] \to X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0,T)$, endowed with the norm $\|\cdot\|_{L^p_T(X)} := \|\|\cdot\|_X\|_{L^p(0,T)}$. For any interval I of $\mathbb{R}$, we agree that $\mathcal{C}(I;X)$ (resp. $\mathcal{C}_b(I;X)$) denotes the set of continuous (resp. continuous and bounded) functions from I to X. We keep the same notation for functions with several components.

We define the Hölder semi-norm as follows: for any $\alpha, \beta \in (0,1]$, and function f,

$$[f]_{\mathcal{C}^{\alpha,\beta}_{t,x}(I\times\mathbb{R}^d)} := \sup\Big\{ \frac{f(t,x)-f(\tau,y)}{|t-\tau|^\alpha + |x-y|^\beta}; (t,x),(\tau,y) \in I\times\mathbb{R}^d, (t,x)\neq(\tau,y)\Big\}.$$

We denote by $\mathcal{C}^{\alpha,\beta}_{t,x}(I\times\mathbb{R}^d)$ the Hölder spaces with the norm

$$\|f\|_{\mathcal{C}^{\alpha,\beta}_{t,x}(I\times\mathbb{R}^d)} := \|f\|_{L^\infty(I\times\mathbb{R}^d)} + [f]_{\mathcal{C}^{\alpha,\beta}_{t,x}(I\times\mathbb{R}^d)}.$$

For any nonnegative integers n_1 and n_2, the norm

$$\|f\|_{\mathcal{C}^{n_1+\alpha,n_2+\beta}_{t,x}(I\times\mathbb{R}^d)} := \|f\|_{L^\infty(I\times\mathbb{R}^d)} + [\partial_t^{n_2} f]_{\mathcal{C}^{\alpha,\beta}_{t,x}(I\times\mathbb{R}^d)} + [\nabla_x^{n_2} f]_{\mathcal{C}^{\alpha,\beta}_{t,x}(I\times\mathbb{R}^d)}$$

corresponding to the spaces $\mathcal{C}^{n_1+\alpha,n_2+\beta}_{t,x}(I\times\mathbb{R}^d)$. Sometimes, we omitted the subscript t,x, respectively.

A.2 Definitions of some weak solutions

In the analysis of solutions of the Hall-MHD system, we need to consider the following spaces of divergence free functions.

$$\mathcal{V} := \{\phi \in \mathcal{C}^\infty_c(\mathbb{R}^3),\ \operatorname{div}\phi = 0\},$$

and

$$H := \text{the closure of } \mathcal{V} \text{ in } L^2(\mathbb{R}^3) \quad \text{and} \quad V := \text{the closure of } \mathcal{V} \text{ in } H^1(\mathbb{R}^3).$$

Definition A.2.1. *We say that a pair $(\boldsymbol{u}, \boldsymbol{B})$ is a Leray-Hopf weak solution of the Hall-MHD problem (2.1.1)-(2.1.5), if the following conditions are satisfied.*

1. *It holds that $(\boldsymbol{u}, \boldsymbol{B}) \in \mathcal{C}_w([0,T]; H) \cap L^2(0,T;V)$;*
2. *For any $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \mathcal{V}$ and any $\psi_1, \psi_2 \in C^\infty([0,T))$, such that*

$$\begin{aligned}
&\int_0^T \Big(-(\boldsymbol{u}, \boldsymbol{\phi}_1)_{L^2} \dot{\psi}_1(t) - (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{\phi}_1)_{L^2} \psi_1(t) + \mu (\nabla \boldsymbol{u}, \nabla \boldsymbol{\phi}_{1 L^2}) \psi_1(t) \Big)\, dt \\
&\qquad = -\int_0^T (\boldsymbol{B} \otimes \boldsymbol{B}, \nabla \boldsymbol{\phi}_1)_{L^2} \psi_1(t)\, dt + (\boldsymbol{u}_0, \boldsymbol{\phi}_1)_{L^2} \psi_1(0); \\
&\int_0^T \Big(-(\boldsymbol{B}, \boldsymbol{\phi}_2)_{L^2} \dot{\psi}_2(t) - (\boldsymbol{u} \times \boldsymbol{B}, \nabla \times \boldsymbol{\phi}_2)_{L^2} \psi_2(t) + \nu (\nabla \boldsymbol{B}, \nabla \boldsymbol{\phi}_2)_{L^2} \psi_2(t) \Big)\, dt \\
&\qquad = -\int_0^T \big((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \nabla \times \boldsymbol{\phi}_2 \big)_{L^2} \psi_2(t)\, dt + (\boldsymbol{B}_0, \boldsymbol{\phi}_2)_{L^2} \psi_2(0);
\end{aligned}$$

3. *For any $t \in [0,T]$*

$$\|\boldsymbol{u}(t), \boldsymbol{B}(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla \boldsymbol{u}(\tau)\|_{L^2}^2\, d\tau + 2\nu \int_0^t \|\nabla \boldsymbol{B}(\tau)\|_{L^2}^2\, d\tau \le \|\boldsymbol{u}_0, \boldsymbol{B}_0\|_{L^2}^2.$$

If T can be taken arbitrarily large then we say $(\boldsymbol{u}, \boldsymbol{B})$ is globally exists.

Let us recall the concept of weak solution to the Cauchy problem (2.2.4)-(2.2.3).

Definition A.2.2. *We say that a scalar function u is a global weak solution of the Cauchy problem (2.2.4)-(2.2.3), if the following conditions are satisfied.*

1. *It holds that $u \in \mathcal{C}([0,\infty; L^1(\mathbb{R}^d))$ such that $\Phi(u) \in L^2((0,\infty); \dot{H}^{s/2}(\mathbb{R}^d))$;*
2. *For every $\theta \in \mathcal{C}_c^\infty((0,\infty) \times \mathbb{R}^d)$*

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^d} u \partial_t \theta\, dx\, dt - \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{s/4} \Phi(u)\, (-\Delta)^{s/4} \theta\, dx\, dt \\
+ \int_0^\infty \int_{\mathbb{R}^d} f\, \theta\, dx\, dt = 0;
\end{aligned}$$

3. *$u(\cdot, 0) = u_0$ almost everywhere.*

Appendix B

Sobolev Spaces, Besov Spaces and commutator estimates

B.1 Littlewood-Paley decomposition and Besov spaces

Here, we briefly recall the definition of the Littlewood-Paley decomposition, define Besov spaces and list some properties that have been used repeatedly in the paper. For the reader's convenience, we also prove some nonlinear and commutator estimates.

The Littlewood-Paley decomposition is a dyadic localization procedure in the frequency space for tempered distributions over $\mathbb{R}^d$. To define it, fix some nonincreasing smooth radial function χ on $\mathbb{R}^d$, supported in (say) $B(\mathbf{0}, 4/3)$ and with value 1 on $B(\mathbf{0}, 3/4)$, and set $\varphi(\boldsymbol{\xi}) := \chi(\boldsymbol{\xi}/2) - \chi(\boldsymbol{\xi})$. Then, we have

$$\forall \boldsymbol{\xi} \in \mathbb{R}^d,\ \chi(\boldsymbol{\xi}) + \sum_{j\geq 0} \varphi(2^{-j}\boldsymbol{\xi}) = 1 \quad \text{and} \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d \setminus \{\mathbf{0}\},\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\boldsymbol{\xi}) = 1.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and low-frequency cut-off operator $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\begin{aligned} \dot{\Delta}_j u &:= \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} \psi(2^j y)u(x-y)\,dy \quad \text{with} \quad \psi{:=}\mathcal{F}^{-1}\varphi, \\ \dot{S}_j u &:= \chi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} \tilde{\psi}(2^j y)u(x-y)\,dy \quad \text{with} \quad \tilde{\psi}{:=}\mathcal{F}^{-1}\chi. \end{aligned}$$

The following **Littlewood-Paley decomposition** of u:

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u$$

holds true modulo polynomials for any tempered distribution u. In order to have an equality in the sense of tempered distributions, we consider only elements of the set $\mathcal{S}'_h(\mathbb{R}^d)$ of tempered distributions u such that

$$\lim_{j\to-\infty} \|\dot{S}_j u\|_{L^\infty} = 0.$$

Next, we recall some basic facts on Littlewood-Paley theory and Besov spaces, one may check [45] for more details.

Proposition B.1.1. *Fix some $0 < r < R$. A constant C exists such that for any nonnegative integer k, any couple (p, q) in $[1, \infty]^2$ with $q \geq p \geq 1$ and any function*

u of L^p with $\operatorname{Supp}\widehat{u} \subset \{\boldsymbol{\xi} \in \mathbb{R}^d,\ |\boldsymbol{\xi}| \leq \lambda R\}$, *we have*

$$\|D^k u\|_{L^q} \leq C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p}.$$

If u satisfies $\operatorname{Supp}\widehat{u} \subset \{\boldsymbol{\xi} \in \mathbb{R}^d,\ r\lambda \leq |\boldsymbol{\xi}| \leq R\lambda\}$, *then we have*

$$C^{-k-1}\lambda^k\|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1}\lambda^k\|u\|_{L^p}.$$

Definition B.1.2. *Let s be a real number and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is the set of distributions u in $\mathcal{S}'_h$ such that*

$$\|u\|_{\dot{B}^s_{p,r}} := \|2^{js}\|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^d)}\|_{\ell^r(\mathbb{Z})} < \infty.$$

Proposition B.1.3. *The following properties hold true:*

(i) *Derivatives: for all $s \in \mathbb{R}$ and $1 \leq p,r \leq \infty$, we have*

$$\|D^k u\|_{\dot{B}^s_{p,r}} = \sup_{|\alpha|=k}\|\partial^\alpha u\|_{\dot{B}^s_{p,r}} \sim \|u\|_{\dot{B}^{s+k}_{p,r}}.$$

(ii) *Embedding: we have the following continuous embedding*

$$\dot{B}^s_{p,r} \hookrightarrow \dot{B}^{s-d(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},\tilde{r}} \quad \text{whenever } \tilde{p} \geq p \quad \text{and} \quad \tilde{r} \geq r,$$

and the space $\dot{B}^{\frac{d}{p}}_{p,1}$ is embedded in the set of bounded continuous functions.

(iii) *Real interpolation: for any $\theta \in (0,1)$ and $s < \tilde{s}$, we have*

$$\|u\|_{\dot{B}^{\theta s+(1-\theta)\tilde{s}}_{p,1}} \lesssim \|u\|^{\theta}_{\dot{B}^s_{p,\infty}}\|u\|^{1-\theta}_{\dot{B}^{\tilde{s}}_{p,\infty}}.$$

(iv) *Completeness: the space $\dot{B}^s_{p,r}$ is complete if (and only if) (s,p,r) satisfies*

$$s < \frac{d}{p}, \quad \text{or} \quad s = \frac{d}{p} \quad \text{and} \quad r = 1. \tag{B.1.1}$$

(v) *Density: the space $\mathcal{S}_0(\mathbb{R}^d)$ of Schwartz functions on $\mathbb{R}^d$ with Fourier transform supported away from the origin is dense in $\dot{B}^s_{p,r}$ whenever both p and r are finite.*

(vi) *Scaling invariance: for any $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$, there exists a constant C such that for all positive λ and $u \in \dot{B}^s_{p,r}$, we have*

$$C^{-1}\lambda^{s-\frac{d}{p}}\|u\|_{\dot{B}^s_{p,r}} \leq \|u(\lambda\cdot)\|_{\dot{B}^s_{p,r}} \leq C\lambda^{s-\frac{d}{p}}\|u\|_{\dot{B}^s_{p,r}}.$$

(vii) *Let f be a smooth function on $\mathbb{R}^d \setminus \{0\}$ which is homogeneous of degree 0. Define $f(D)$ on $\mathcal{S}(\mathbb{R}^d)$ by*

$$\mathcal{F}(f(D)u)(\boldsymbol{\xi}) := f(\boldsymbol{\xi})\mathcal{F}u(\boldsymbol{\xi}),$$

Then, for all exponents (s,p,r), we have the estimate

$$\|f(D)u\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{\dot{B}^s_{p,r}}.$$

If in addition $f(D)$ extends to a map from $\mathcal{S}'_h(\mathbb{R}^d)$ to itself and (B.1.1) is fulfilled, then $f(D)$ is continuous from $\dot{B}^s_{p,r}$ to $\dot{B}^s_{p,r}$.

(viii) *Operator* curl^{-1} *maps* $\dot{B}^{s-1}_{p,1}$ *to* $\dot{B}^s_{p,1}$ *if* $1 \leq p < \infty$ *and* $s \leq d/p$.

Proof. We only prove the last item as it is fundamental in our analysis. Owing to the definition in (2.1.12), it is obvious that curl^{-1} maps $\mathcal{S}'_h(\mathbb{R}^d)$ to itself, and homogeneity of degree -1 implies that we have for all u in $\mathcal{S}'_h(\mathbb{R}^d)$:

$$\|\mathrm{curl}^{-1}u\|_{\dot{B}^s_{p,1}} \lesssim \|u\|_{\dot{B}^{s-1}_{p,1}}.$$

As $\mathcal{S}'_h(\mathbb{R}^d)$ is dense in $\dot{B}^{s-1}_{p,1}$ and since the space $\dot{B}^s_{p,1}$ is complete (owing to $s \leq d/p$), we get the result. □

B.2 A priori estimates in Besov spaces for the heat equation

A great deal of our analysis relies on regularity estimates for the heat equation:

$$(H) \qquad \begin{cases} \partial_t u - \Delta u = f, \\ u|_{t=0} = u_0. \end{cases}$$

It is classical that for all $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in L^1_{loc}(\mathbb{R}^+;\mathcal{S}'(\mathbb{R}^d))$, equation (H) has a unique tempered distribution solution, given by the following Duhamel formula:

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}f(\tau)\,d\tau, \qquad t \geq 0. \tag{B.2.1}$$

Above, $(e^{t\Delta})_{t\geq 0}$ stands for the heat semi-group. It is defined on $\mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}(e^{t\Delta}z)(\xi) := e^{-t|\xi|^2}\widehat{z}(\xi), \tag{B.2.2}$$

and is extended to the set of tempered distributions by duality. It is classical that

Lemma B.2.1. *Let $\mathscr{C}$ be a ring of $\mathbb{R}^3$, if the support of $\hat{u}$ is included in $\lambda\mathscr{C}$. Then, there exist two positive constants $\bar{c}$ and C such that for all $1 \leq p \leq \infty$,*

$$\|e^{t\Delta}u\|_{L^p} \leq Ce^{\bar{c}\lambda^2 t}\|u\|_{L^p},$$

where $e^{t\Delta}$ denotes the heat semi-group operator.

As observed by Chemin in [11], the following spaces are suitable for describing the maximal regularity properties of the heat equation.

Definition B.2.2. *For $T>0$, $s\in\mathbb{R}$, $1\leq\rho\leq\infty$, we set*

$$\|u\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})}:=\big\|2^{js}\|\dot{\Delta}_j u\|_{L^\rho_T(L^p)}\big\|_{\ell^r(\mathbb{Z})}.$$

We define the space $\widetilde{L}^\rho_T(\dot{B}^s_{p,r})$ to be the set of tempered distribution u on $(0,T)\times\mathbb{R}^d$ such that $\lim\limits_{j\to-\infty}\|\dot{S}_j u(t)\|_{L^\infty}=0$ a.e. in $(0,T)$, and $\|u\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})}<\infty$. The space $\widetilde{L}^\rho_T(\dot{B}^s_{p,r})\cap\mathcal{C}([0,T];\dot{B}^s_{p,r})$ is denoted by $\widetilde{\mathcal{C}}_T(\dot{B}^s_{p,r})$. In the case $T=+\infty$, one denotes the corresponding space and norm by $\widetilde{L}^\rho(\dot{B}^s_{p,r})$ and $\|\cdot\|_{\widetilde{L}^\rho(\dot{B}^s_{p,r})}$, respectively.

The above spaces or norms may be compared to more classical ones according to Minkowski's inequality:

$$\|u\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})}\leq\|u\|_{L^\rho_T(\dot{B}^s_{p,r})}\ \text{ if }\ r\geq\rho\quad\text{and}\quad\|u\|_{\widetilde{L}^\rho_T(\dot{B}^s_{p,r})}\geq\|u\|_{L^\rho_T(\dot{B}^s_{p,r})}\ \text{ if }\ r\leq\rho.$$

The following fundamental result has been proved in [11].

Proposition B.2.3. *Let $T>0$, $s\in\mathbb{R}$ and $1\leq\rho,p,r\leq\infty$. Assume that $u_0\in\dot{B}^s_{p,r}$ and $f\in\widetilde{L}^\rho_T(\dot{B}^{s-2+\frac{2}{\rho}}_{p,r})$. Then, (H) has a unique solution u in $\widetilde{L}^\rho_T(\dot{B}^{s+\frac{2}{\rho}}_{p,r})\cap\widetilde{L}^\infty_T(\dot{B}^s_{p,r})$ and there exists a constant C depending only on d and such that for all $\rho_1\in[\rho,\infty]$, we have*

$$\|u\|_{\widetilde{L}^{\rho_1}_T(\dot{B}^{s+\frac{2}{\rho_1}}_{p,r})}\leq C\big(\|u_0\|_{\dot{B}^s_{p,r}}+\|f\|_{\widetilde{L}^\rho_T(\dot{B}^{s-2+\frac{2}{\rho}}_{p,r})}\big).\tag{B.2.3}$$

Furthermore, if r is finite, then u belongs to $\mathcal{C}([0,T];\dot{B}^s_{p,r})$.

B.3 Nonlinear estimates

Let us now recall a few nonlinear estimates in Besov spaces, that we used in the paper. They all may be easily proved by using the following so-called Bony decomposition (from [68]) for the (formal) product of two distributions u and v:

$$u\,v=T_u v+T_v u+R(u,v).$$

Above, T designates the paraproduct bilinear operator defined by

$$T_u v:=\sum_j\dot{S}_{j-1}u\dot{\Delta}_j v,\quad T_v u:=\sum_j\dot{S}_{j-1}v\,\dot{\Delta}_j u$$

and R stands for the remainder operator given by

$$R(u,v):=\sum_j\sum_{|j'-j|\leq1}\dot{\Delta}_j u\,\dot{\Delta}_{j'}v.$$

The following properties of the paraproduct and remainder operators are classical:

Proposition B.3.1. *For any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ and $t<0$, there exists a constant C such that*

$$\|T_u v\|_{\dot{B}^s_{p,r}}\leq C\,\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}}\quad\text{and}\quad\|T_u v\|_{\dot{B}^{s+t}_{p,r}}\leq C\,\|u\|_{\dot{B}^t_{\infty,\infty}}\|v\|_{\dot{B}^s_{p,r}}.$$

For any (s_1,p_1,r_1) *and* (s_2,p_2,r_2) *in* $\mathbb{R}\times[1,\infty]^2$ *satisfying*

$$s_1+s_2>0,\quad \frac{1}{p}:=\frac{1}{p_1}+\frac{1}{p_2}\leq 1\quad\text{and}\quad \frac{1}{r}:=\frac{1}{r_1}+\frac{1}{r_2}\leq 1,$$

there exists a constant C *such that*

$$\|R(u,v)\|_{\dot B^{s_1+s_2}_{p,r}}\leq C\,\|u\|_{\dot B^{s_1}_{p_1,r_1}}\|v\|_{\dot B^{s_2}_{p_2,r_2}}.$$

Combining the above proposition with the Bony decomposition allows to get a number of inequalities like, for instance:

- tame estimates: for any $s>0$ and $1\leq p,r\leq\infty$,

$$\|uv\|_{\dot B^s_{p,r}}\lesssim \|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}}+\|v\|_{L^\infty}\|u\|_{\dot B^s_{p,r}};\tag{B.3.1}$$

- the following product estimate:

$$\|uv\|_{\dot B^{s_1+s_2-\frac{d}{p}}_{p,1}}\lesssim \|u\|_{\dot B^{s_1}_{p,1}}\|v\|_{\dot B^{s_2}_{p,1}}\tag{B.3.2}$$

  whenever[1] $s_1,s_2\leq\frac{d}{p}$ satisfy $s_1+s_2>d\max(0,\frac{2}{p}-1)$;

- the following inequality (in the case $d=3$ and $\rho>2$) that has been used in the proof of (2.1.24):

$$\|ab\|_{\dot B^{\frac32}_{2,1}}\lesssim \|a\|_{\dot B^{\frac{2}{\rho}-1}_{\infty,\infty}}\|b\|_{\dot B^{\frac52-\frac{2}{\rho}}_{2,1}}+\|b\|_{\dot B^{\frac{2}{\rho}-1}_{\infty,\infty}}\|a\|_{\dot B^{\frac52-\frac{2}{\rho}}_{2,1}}.\tag{B.3.3}$$

Remark 6. Proposition B.3.1 and estimates like (B.3.2) or (B.3.3) may be adapted to the spaces $\widetilde L^\rho_T(\dot B^s_{p,r})$. The general principle is that the time exponent behaves according to Hölder inequality. For example, we have

$$\|T_ab\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}+\|T_ba\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}+\|R(a,b)\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,r})}\lesssim \|a\|_{\widetilde L^{\frac43}_t(\dot B^{-\frac12}_{\infty,\infty})}\|b\|_{\widetilde L^4_t(\dot B^1_{2,r})}.$$

Then, combining with embedding (in the case $d=3$) gives Inequality (C.1.2). Similarly, Inequality (C.1.3) stems from

$$\|T_ab\|_{\widetilde L^1_t(\dot B^{\frac32}_{2,r})}+\|T_ba\|_{\widetilde L^1_t(\dot B^{\frac32}_{2,r})}+\|R(a,b)\|_{\widetilde L^1_t(\dot B^{\frac32}_{2,r})}\lesssim \|a\|_{\widetilde L^4_t(\dot B^{-\frac12}_{\infty,\infty})}\|b\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}+\|b\|_{\widetilde L^4_t(\dot B^{-\frac12}_{\infty,\infty})}\|a\|_{\widetilde L^{\frac43}_t(\dot B^2_{2,r})}.$$

[1] In particular, $\dot B^{\frac{d}{p}}_{p,1}$ is an algebra for any $1\leq p<\infty$.

In order to prove Inequality (C.1.7), it suffices to use the fact that

$$\begin{aligned}\|T_a b\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} &\lesssim \|a\|_{\widetilde{L}^4_t(\dot{B}^{-\frac{1}{2}}_{\infty,\infty})}\|b\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^0_{2,r})},\\ \|T_b a\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} &\lesssim \|b\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^{-\frac{3}{2}}_{\infty,\infty})}\|a\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})},\\ \|R(a,b)\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} &\lesssim \|a\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}\|b\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^0_{2,r})}.\end{aligned}$$

Proving Inequality (C.1.8) is similar.

In fact, as an application of Bony's decomposition, the following product laws in Besov spaces will play a crucial role in the proof of Theorem 3.1.3.

Lemma B.3.2. *(see [69]) Let $q \geq p \geq 1$, and $s_1 \leq \frac{3}{p}$, $s_2 \leq \frac{3}{q}$ with*

$$s_1 + s_2 \geq 3\min\{0, \frac{1}{p} + \frac{1}{q} - 1\}.$$

Let $a \in \dot{B}^{s_1}_{p,1}(\mathbb{R}^3)$, $b \in \dot{B}^{s_2}_{q,1}(\mathbb{R}^3)$. Then $a\,b \in \dot{B}^{s_1+s_2-\frac{3}{p}}_{q,1}(\mathbb{R}^3)$, and

$$\|a\,b\|_{\dot{B}^{s_1+s_2-\frac{3}{p}}_{q,1}} \lesssim \|a\|_{\dot{B}^{s_1}_{p,1}}\|b\|_{\dot{B}^{s_2}_{q,1}}. \tag{B.3.4}$$

Lemma B.3.3. *Let $q \geq p \geq 1$ and*

$$\frac{1}{q} - \frac{1}{p} \geq -\min\{\frac{1}{3}, \frac{1}{2p}\}.$$

Assume θ satisfies

$$\frac{3}{p} - \frac{3}{q} \leq \theta \leq 1.$$

Let $a, b \in \dot{B}^{\frac{3}{q}-\theta}_{q,1}(\mathbb{R}^3) \cap \dot{B}^{\frac{3}{q}+\theta}_{q,1}(\mathbb{R}^3)$. Then $a\,b \in \dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)$, and

$$\|a\,b\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \lesssim \|a\|_{\dot{B}^{\frac{3}{q}-\theta}_{q,1}}\|b\|_{\dot{B}^{\frac{3}{q}+\theta}_{q,1}} + \|a\|_{\dot{B}^{\frac{3}{q}+\theta}_{q,1}}\|b\|_{\dot{B}^{\frac{3}{q}-\theta}_{q,1}}. \tag{B.3.5}$$

Let $a, b \in \dot{B}^{\frac{6}{q}-\frac{3}{p}}_{q,1}(\mathbb{R}^3) \cap \dot{B}^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)$. Then $a\,b \in \dot{B}^{\frac{3}{q}}_{p,1}(\mathbb{R}^3)$, and

$$\|a\,b\|_{\dot{B}^{\frac{3}{q}}_{p,1}} \lesssim \|a\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\|b\|_{\dot{B}^{\frac{6}{q}-\frac{3}{p}}_{q,1}} + \|a\|_{\dot{B}^{\frac{6}{q}-\frac{3}{p}}_{q,1}}\|b\|_{\dot{B}^{\frac{3}{q}}_{q,1}}. \tag{B.3.6}$$

Proof. The proof is standard, we follow the method of [69] and only give a proof of (B.3.5). In fact, we shall focus on the case $q > p$, since when $q = p$, it is obvious.
By Bony's decomposition, we can write

$$ab = T_a b + T_b a + R(a,b).$$

By Hölder's inequality and Proposition B.1.1, under $p < q \leq 2p$ we have

$$\begin{aligned}2^{\frac{3}{p}j}\|S_{j-1}a\dot{\Delta}_j b\|_{L^p} &\lesssim (2^{(\frac{3}{q}+\theta)j}\|\dot{\Delta}_j b\|_{L^q})\sum_{k\leq j-2}2^{(\frac{3}{p}-\frac{3}{q}-\theta)k}\|\dot{\Delta}_k a\|_{L^{\frac{pq}{q-p}}}2^{(\frac{3}{p}-\frac{3}{q}-\theta)(j-k)}\\ &\lesssim (2^{(\frac{3}{q}+\theta)j}\|\dot{\Delta}_j b\|_{L^q})\sum_{k\leq j-2}2^{(\frac{3}{q}-\theta)k}\|\dot{\Delta}_k a\|_{L^q}2^{(\frac{3}{p}-\frac{3}{q}-\theta)(j-k)}.\end{aligned}$$

Thus by Young's inequality with $\frac{3}{p}-\frac{3}{q}\leq\theta$

$$\|T_a b\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \lesssim \|a\|_{\dot{B}^{\frac{3}{q}-\theta}_{q,1}}\|b\|_{\dot{B}^{\frac{3}{q}+\theta}_{q,1}}.$$

Similarly,

$$\|T_b a\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \lesssim \|a\|_{\dot{B}^{\frac{3}{q}+\theta}_{q,1}}\|b\|_{\dot{B}^{\frac{3}{q}-\theta}_{q,1}}.$$

Thanks to Lemma B.3.1, we have

$$\begin{aligned}\|R(a,b)\|_{\dot{B}^{\frac{3}{p}}_{p,1}} &\lesssim \|a\|_{\dot{B}^{\frac{3}{p}-\frac{3}{q}-\theta}_{\frac{pq}{q-p},\infty}}\|b\|_{\dot{B}^{\frac{3}{q}+\theta}_{q,1}}\\ &\lesssim \|a\|_{\dot{B}^{\frac{3}{q}-\theta}_{q,1}}\|b\|_{\dot{B}^{\frac{3}{q}+\theta}_{q,1}}.\end{aligned}$$

□

Lemma B.3.4. *Let $1 \leq q < \infty$. For any homogeneous function σ of degree - 1 smooth outside of 0, there hold:*

Let $a \in \dot{B}^{\frac{3}{q}-1}_{q,1}$ and $b \in \dot{B}^{\frac{3}{q}+1}_{q,1}$, then

$$\|(\sigma(D)a)\cdot\nabla b\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \lesssim \|a\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\|b\|_{\dot{B}^{\frac{3}{q}+1}_{q,1}}. \tag{B.3.7}$$

Let $a \in \dot{B}^{\frac{3}{q}}_{q,1}$ and $b \in \dot{B}^{\frac{3}{q}}_{q,1}$, then

$$\|a\cdot\nabla(\sigma(D)b)\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \lesssim \|a\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\|b\|_{\dot{B}^{\frac{3}{q}}_{q,1}}. \tag{B.3.8}$$

Proof. Thanks to Lemma B.3.2, we know that when $1 \leq q < \infty$, $\dot{B}^{\frac{3}{q}}_{q,1}$ is an algebra, thus by Proposition B.1.3 (vii), we have

$$\begin{aligned}\|(\sigma(D)a)\cdot\nabla b\|_{\dot{B}^{\frac{3}{q}}_{q,1}} &\lesssim \|\sigma(D)a\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\|\nabla b\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\\ &\lesssim \|a\|_{\dot{B}^{\frac{3}{q}-1}_{q,1}}\|b\|_{\dot{B}^{\frac{3}{q}+1}_{q,1}}.\end{aligned}$$

Similarly,

$$\|a\cdot\nabla(\sigma(D)b)\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \lesssim \|a\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\|\nabla(\sigma(D)b)\|_{\dot{B}^{\frac{3}{q}}_{q,1}}$$

$$\lesssim \|a\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \|\sigma(D)\nabla b\|_{\dot{B}^{\frac{3}{q}}_{q,1}}$$
$$\lesssim \|a\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \|b\|_{\dot{B}^{\frac{3}{q}}_{q,1}}.$$

□

Lemma B.3.5. *Let $s \in \mathbb{R}$. For all $\theta_1, \theta_2 > 0$ and $1 \leq p \leq \infty$, there exists a constant C depends on θ_1, θ_2 such that*

$$\|f\|_{\dot{B}^r_{p,1}} \leq C(\theta_1,\theta_2)\|f\|_{\dot{B}^r_{p,\infty}}\Big(1+\log_2\Big(\frac{\|f\|_{\dot{B}^{r-\theta_1}_{p,\infty}}+\|f\|_{\dot{B}^{r+\theta_2}_{p,\infty}}}{\|f\|_{\dot{B}^r_{p,\infty}}}\Big)\Big). \tag{B.3.9}$$

Proof. The proof is exactly the same as the case $\theta_1 = \theta_2$, while the latter is classic and can be found in [60]. □

B.4 Commutator estimates

We end this appendix with the proof of commutator estimates that were crucial in our analysis.

Proposition B.4.1. *Let s be in $(0, d/2]$. Then, we have:*

$$\sum_{j\in\mathbb{Z}} 2^{js}\|[\dot{\Delta}_j, b]a\|_{L^2} \lesssim \|\nabla b\|_{L^\infty}\|a\|_{\dot{B}^{s-1}_{2,1}} + \|a\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla b\|_{\dot{B}^s_{2,1}}. \tag{B.4.1}$$

Furthermore, for all $r \in [1,\infty]$ and $\rho \in (2,\infty]$, we have if we set $1/\rho' := 1 - 1/\rho$,

$$\big\|2^{js}\|[\dot{\Delta}_j, b]a\|_{L^1_t(L^2)}\big\|_{\ell^r(\mathbb{Z})} \lesssim \|\nabla b\|_{\widetilde{L}^\rho_t(\dot{B}^{\frac{2}{\rho}-1}_{\infty,\infty})}\|a\|_{\widetilde{L}^{\rho'}_t(\dot{B}^{s-\frac{2}{\rho}}_{2,r})} + \|\nabla b\|_{\widetilde{L}^{\rho'}_t(\dot{B}^{s+1-\frac{2}{\rho}}_{2,r})}\|a\|_{\widetilde{L}^\rho_t(\dot{B}^{\frac{2}{\rho}-2}_{\infty,\infty})}. \tag{B.4.2}$$

Proof. Proving the two inequalities relies on the decomposition

$$[\dot{\Delta}_j, b]a = [\dot{\Delta}_j, T_b]a + \dot{\Delta}_j(T_a b + R(a,b)) - (T_{\dot{\Delta}_j a}b + R(\dot{\Delta}_j a, b)). \tag{B.4.3}$$

For getting (B.4.1), we bound the first term of (B.4.3) as follows (use [45, Ineq. (2.58)]):

$$\sum_{j\in\mathbb{Z}} 2^{js}\|[\dot{\Delta}_j, T_b]a\|_{L^2} \lesssim \|\nabla b\|_{L^\infty}\|a\|_{\dot{B}^{s-1}_{2,1}}.$$

The next two terms of (B.4.3) may be bounded by using the fact that the remainder and paraproduct operator map $\dot{B}^{-1}_{\infty,\infty} \times \dot{B}^{s+1}_{2,1}$ to $\dot{B}^s_{2,1}$. Finally, owing to the properties of localization of the Littlewood-Paley decomposition, we have

$$T_{\dot{\Delta}_j a}b + R(\dot{\Delta}_j a, b) = \sum_{j'\geq j-2} \dot{S}_{j'+2}\dot{\Delta}_j a\, \dot{\Delta}_{j'} b. \tag{B.4.4}$$

From Bernstein inequality and $\|\dot S_{j'+2}a\|_{L^\infty}\lesssim 2^{j'}\|a\|_{\dot B^{-1}_{\infty,\infty}}$, we gather

$$\begin{aligned}\sum_j 2^{js}\|T_{\dot\Delta_j a}b+R(\dot\Delta_j a,b)\|_{L^2}&\lesssim\sum_j\sum_{j'\geq j-2}2^{js}\|\dot S_{j'+2}a\|_{L^\infty}\|\dot\Delta_{j'}b\|_{L^2}\\&\lesssim\|a\|_{\dot B^{-1}_{\infty,\infty}}\sum_j\sum_{j'\geq j-2}2^{s(j-j')}\,2^{j's}\|\nabla\dot\Delta_{j'}b\|_{L^2}\\&\lesssim\|a\|_{\dot B^{-1}_{\infty,\infty}}\|\nabla b\|_{\dot B^s_{2,1}}.\end{aligned}$$

To prove (B.4.2), we observe that owing to the localization properties of the Littlewood-Paley decomposition, the first term of (B.4.3) may be decomposed into

$$[\dot\Delta_j,T_b]a=\sum_{|j'-j|\leq 4}[\dot\Delta_j,\dot S_{j'-1}b]\dot\Delta_{j'}a.$$

Now, according to [45, Lem. 2.97], we have

$$\|[\dot\Delta_j,\dot S_{j'-1}b]\dot\Delta_{j'}a\|_{L^2}\lesssim 2^{-j}\|\nabla\dot S_{j'-1}b\|_{L^\infty}\|\dot\Delta_{j'}a\|_{L^2},$$

and, since $\frac{2}{\rho}-1<0$,

$$\|\nabla S_{j'-1}b\|_{L^\rho_t(L^\infty)}\lesssim 2^{j'(1-\frac{2}{\rho})}\|\nabla b\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-1}_{\infty,\infty})}.$$

Hence, for all $(j,j')\in\mathbb{Z}^2$ such that $|j-j'|\leq 4$,

$$2^{js}\|[\dot\Delta_j,\dot S_{j'-1}b]\dot\Delta_{j'}a\|_{L^1_t(L^2)}\lesssim 2^{js}2^{-\frac{2}{\rho}j'}\|\dot\Delta_{j'}a\|_{L^{\rho'}_t(L^2)}\|\nabla b\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-1}_{\infty,\infty})}.$$

Therefore, summing up on $j'\in\{j-4,j+4\}$, then taking the $\ell^r(\mathbb{Z})$ norm,

$$\big\|2^{js}\|[\dot\Delta_j,T_b]a\|_{L^1_t(L^2)}\big\|_{\ell^r}\lesssim\|\nabla b\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-1}_{\infty,\infty})}\|a\|_{\widetilde L^{\rho'}_t(\dot B^{s-\frac{2}{\rho}}_{2,r})}.$$

The next two terms may be bounded according to Proposition B.3.1 and Remark 6:

$$\big\|2^{js}\|\dot\Delta_j T_a b\|_{L^1_t(L^2)}\|_{\ell^r}+\big\|2^{js}\|\dot\Delta_j R(a,b)\|_{L^1_t(L^2)}\|_{\ell^r}\lesssim\|a\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-2}_{\infty,\infty})}\|\nabla b\|_{\widetilde L^{\rho'}_t(\dot B^{s+1-\frac{2}{\rho}}_{2,r})}.$$

Finally, use (B.4.4) and the fact that

$$\|\dot S_{j'+2}a\|_{L^\rho_t(L^\infty)}\lesssim 2^{(2-\frac{2}{\rho})j'}\|a\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-2}_{\infty,\infty})}$$

to get

$$\begin{aligned}2^{js}\|T_{\dot\Delta_j a}b+R(\dot\Delta_j a,b)\|_{L^1_t(L^2)}&\lesssim\sum_{j'\geq j-2}2^{js}\|\dot S_{j'+2}a\|_{L^\rho_t(L^\infty)}\|\dot\Delta_{j'}b\|_{L^{\rho'}_t(L^2)}\\&\lesssim\|a\|_{\widetilde L^\rho_t(\dot B^{\frac{2}{\rho}-2}_{\infty,\infty})}\sum_{j'\geq j-2}2^{s(j-j')}\,2^{(s+2-\frac{2}{\rho})j'}\|\dot\Delta_{j'}b\|_{L^{\rho'}_t(L^2)}.\end{aligned}$$

Taking the $\ell^r(\mathbb{Z})$ norm of both sides and using a convolution inequality for series (remember that $s > 0$), we end up with

$$\big\|2^{js}\|T_{\dot\Delta_j a}b + R(\dot\Delta_j a, b)\|_{L^1_t(L^2)}\big\|_{\ell^r(\mathbb{Z})} \lesssim \|a\|_{\widetilde{L}^\rho_t(\dot B^{\frac{2}{\rho}-2}_{\infty,\infty})}\|\nabla b\|_{\widetilde{L}^{\rho'}_t(\dot B^{s+1-\frac{2}{\rho}}_{2,r})}.$$

This completes the proof of Inequality (B.4.2). □

B.5 Sobolev spaces

Let us first recall the definitions of Sobolev spaces and fractional derivation operators (more details may be found in e.g. [45, Chap. 1]).

Definition B.5.1. *Let s be in $\mathbb{R}$. The* homogeneous Sobolev space $\dot H^s(\mathbb{R}^d)$ *(also denoted by $\dot H^s$) is the set of tempered distributions u on $\mathbb{R}^d$, with Fourier transform in $L^1_{loc}(\mathbb{R}^d)$, satisfying*

$$\|u\|_{\dot H^s} := \|\Lambda^s u\|_{L^2} < \infty,$$

where Λ^s stands for the fractional derivative operator defined in terms of the Fourier transform by

$$\mathcal{F}(\Lambda^s u)(\xi) := |\xi|^s \mathcal{F}u(\xi), \qquad \xi \in \mathbb{R}^d.$$

The nonhomogeneous Sobolev space $H^s(\mathbb{R}^d)$ *(also denoted by H^s) is the set of tempered distributions u on $\mathbb{R}^d$, with Fourier transform in $L^1_{loc}(\mathbb{R}^d)$, satisfying*

$$\|u\|_{H^s} := \|\langle D\rangle^s u\|_{L^2} < \infty \quad \textit{with} \quad \mathcal{F}(\langle D\rangle^s u)(\xi) := (1+|\xi|^2)^{s/2}\mathcal{F}u(\xi).$$

We have the following proposition.

Proposition B.5.2. *Let $s_0 \le s \le s_1$. Then, $\dot H^{s_0} \cap \dot H^{s_1}$ is included in $\dot H^s$, and we have for all θ in $[0,1]$,*

$$\|u\|_{\dot H^s} \le \|u\|^{1-\theta}_{\dot H^{s_0}}\|u\|^{\theta}_{\dot H^{s_1}} \quad \text{with} \quad s = (1-\theta)s_0 + \theta s_1. \tag{B.5.1}$$

We also often used the following Sobolev embedding for $0 \le s < d/2$:

$$\dot H^s(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2s}}(\mathbb{R}^d). \tag{B.5.2}$$

Note that in dimension $d = 3$, as a consequence of Hölder's inequality and (B.5.2), we have the following product law:

$$\|uv\|_{L^2(\mathbb{R}^3)} \le \|u\|_{L^3(\mathbb{R}^3)}\|v\|_{L^6(\mathbb{R}^3)} \le C\|u\|_{\dot H^{\frac12}(\mathbb{R}^3)}\|v\|_{\dot H^1(\mathbb{R}^3)}. \tag{B.5.3}$$

We also needed the following Gagliardo-Nirenberg inequalities:

$$\|u\|_{L^p(\mathbb{R}^2)} \lesssim \|u\|^{\frac{2}{p}}_{L^2(\mathbb{R}^2)}\|\nabla u\|^{1-\frac{2}{p}}_{L^2(\mathbb{R}^2)}, \quad 2 \le p < \infty, \tag{B.5.4}$$

$$\|u\|_{L^\infty(\mathbb{R}^3)} \lesssim \|u\|^{1-\theta}_{\dot H^s(\mathbb{R}^3)}\|u\|^{\theta}_{\dot H^{s'}(\mathbb{R}^3)}, \quad s < \frac32 < s', \ \theta = \frac{\frac32 - s}{s'-s}\cdot \tag{B.5.5}$$

Finally, we used the following inequalities (see e.g. [62], Lemma 2.10):

Lemma B.5.3. *Let $s > 0$ and $1 < p,\ p_1,\ p_2,\ p_3,\ p_4 < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. There exists a constant $C > 0$ such that*

$$\|\Lambda^s(uv) - u\Lambda^s v\|_{L^p} \leq C(\|\nabla u\|_{L^{p_1}}\|\Lambda^{s-1}v\|_{L^{p_2}} + \|\Lambda^s u\|_{L^{p_3}}\|v\|_{L^{p_4}}) \tag{B.5.6}$$

and

$$\|\Lambda^s(uv)\|_{L^p} \leq C(\|\Lambda^s u\|_{L^{p_1}}\|v\|_{L^{p_2}} + \|u\|_{L^{p_3}}\|\Lambda^s v\|_{L^{p_4}}). \tag{B.5.7}$$

Appendix C

The well-posedness theory for Hall-MHD system in spaces $\dot{B}^{\frac{1}{2}}_{2,r}$ for general r

Here, we present the proof of Theorem 2.1.9. Let us first prove the a priori estimates leading to global existence.

C.1 Global existence for small data

Proposition C.1.1. *Assume that $(\boldsymbol{u},\boldsymbol{B})$ is a smooth solution of the Hall-MHD system on $[0,T]\times\mathbb{R}^3$ with $h=\mu=\nu=1$. Let $\boldsymbol{v}:=\boldsymbol{u}-\nabla\times\boldsymbol{B}$. There exists a universal constant C such that for any $r\in[1,\infty]$, we have*

$$\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{E_{2,r}(T)}\leq C\big(\|(\boldsymbol{u}_0,\boldsymbol{B}_0,\boldsymbol{v}_0)\|_{\dot{B}^{\frac{1}{2}}_{2,r}}+\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|^2_{E_{2,r}(T)}\big)\cdot \tag{C.1.1}$$

Proof. We argue as in the proof of Inequality (3.2.30), but take the $\ell^r(\mathbb{Z})$ norm instead of the $\ell^1(\mathbb{Z})$ norm. We get for all $t\in[0,T]$,

$$\begin{aligned}\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{1}{2}}_{2,r})}+\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\widetilde{L}^1_t(\dot{B}^{\frac{5}{2}}_{2,r})}&\lesssim\|(\boldsymbol{u}_0,\boldsymbol{B}_0,\boldsymbol{v}_0)\|_{\dot{B}^{\frac{1}{2}}_{2,r}}+\|\boldsymbol{B}\cdot\nabla\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})}\\&+\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})}+\|\boldsymbol{v}\cdot\nabla\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})}+\|\boldsymbol{B}\cdot\nabla\boldsymbol{v}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})}+\|\boldsymbol{v}\cdot\nabla\boldsymbol{u}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})}\\&+\|\boldsymbol{u}\cdot\nabla\boldsymbol{v}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})}+\|\boldsymbol{v}\cdot\nabla\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{3}{2}}_{2,r})}+\Big\|2^{\frac{3j}{2}}\|[\dot{\Delta}_j,\boldsymbol{B}\times](\nabla\times\boldsymbol{v})\|_{L^1_t(L^2)}\Big\|_{\ell^r(\mathbb{Z})}\cdot\end{aligned}$$

The first six nonlinear terms in the right-hand side may be bounded according to the following product law that is proved after Proposition B.3.1:

$$\|\boldsymbol{a}\,\boldsymbol{b}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})}\lesssim\|\boldsymbol{a}\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}\|\boldsymbol{b}\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^1_{2,r})}\cdot \tag{C.1.2}$$

The last but one term may be bounded as follows:

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{3}{2}}_{2,r})}\lesssim\|\boldsymbol{v}\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}\|\nabla\boldsymbol{B}\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^2_{2,r})}+\|\nabla\boldsymbol{B}\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}\|\boldsymbol{v}\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^2_{2,r})}\cdot \tag{C.1.3}$$

Finally, in light of (B.4.2) with $\boldsymbol{b} = \boldsymbol{B}$, $\boldsymbol{a} = \nabla \times \boldsymbol{v}$, $s = 3/2$ and $\rho = 4$, and embedding, one discovers that the commutator term may be bounded exactly as $\boldsymbol{v} \cdot \nabla \boldsymbol{B}$.

Putting together all the above inequalities eventually yields for all $t \geq 0$,

$$\|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})\|_{E_{2,r}(t)} \lesssim \|(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0)\|_{\dot{B}^{\frac{1}{2}}_{2,r}} + \|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^2_{2,r})} \|(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}. \tag{C.1.4}$$

Since one can prove by making use of Hölder's inequality and interpolation that

$$\|\boldsymbol{z}\|_{\widetilde{L}^{\rho}_t(\dot{B}^{\frac{1}{2}+\frac{2}{\rho}}_{2,r})} \leq \|\boldsymbol{z}\|_{E_{2,r}(t)} \quad \text{for all } \rho \in [1, +\infty],$$

Inequality (C.1.4) implies (C.1.1). □

In order to prove Theorem 2.1.9, we proceed as follows:

1. smooth out the data and get a sequence $(\boldsymbol{u}^n, \boldsymbol{B}^n)_{n\in\mathbb{N}}$ of global smooth solutions to the Hall-MHD system;
2. apply Proposition C.1.1 to $(\boldsymbol{u}^n, \boldsymbol{B}^n)_{n\in\mathbb{N}}$ and obtain uniform estimates for $(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)_{n\in\mathbb{N}}$ in the space $E_{2,r}$;
3. use compactness to prove that $(\boldsymbol{u}^n, \boldsymbol{B}^n)_{n\in\mathbb{N}}$ converges, up to extraction, to a solution of the Hall-MHD system supplemented with initial data $(\boldsymbol{u}_0, \boldsymbol{B}_0)$;
4. prove stability estimates *in a larger space* to get the uniqueness of the solution.

To proceed, let us smooth out the initial data as follows[1]:

$$\boldsymbol{u}_0^n := (\dot{S}_n - \dot{S}_{-n})\boldsymbol{u}_0 \quad \text{and} \quad \boldsymbol{B}_0^n := (\dot{S}_n - \dot{S}_{-n})\boldsymbol{B}_0.$$

Clearly, $\boldsymbol{u}_0^n$ and $\boldsymbol{B}_0^n$ belong to all Sobolev spaces, and we have for $\boldsymbol{z} = \boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v}$ and all $n \in \mathbb{N}$,

$$\forall j \in \mathbb{Z}, \quad \|\dot{\Delta}_j \boldsymbol{z}_0^n\|_{L^2} \leq \|\dot{\Delta}_j \boldsymbol{z}_0\|_{L^2} \quad \text{and} \quad \|\boldsymbol{z}_0^n\|_{\dot{B}^{\frac{1}{2}}_{2,r}} \leq \|\boldsymbol{z}_0\|_{\dot{B}^{\frac{1}{2}}_{2,r}}. \tag{C.1.5}$$

Since in particular $(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n, \boldsymbol{v}_0^n)$ is in $\dot{B}^{\frac{1}{2}}_{2,1}$, Theorem 2.1.3 guarantees that the Hall-MHD system with data $(\boldsymbol{u}_0^n, \boldsymbol{B}_0^n)$ has a unique maximal solution on $[0, T^n)$ for some $T^n > 0$, that belongs to $E_{2,1}(T)$ for all $T < T^n$. Now, take some positive real number M to be chosen later on and define

$$T_n := \sup\big\{t \in [0, T^n)\,,\ \|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|_{E_{2,r}(t)} \leq Mc\big\}.$$

We are going to show first that $T_n = T^n$, then that $T^n = +\infty$.

According to Proposition C.1.1 and to (C.1.5), we have

$$\|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|_{E_{2,r}(T_n)} \leq C\big(\|(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0)\|_{\dot{B}^{\frac{1}{2}}_{2,r}} + \|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|^2_{E_{2,r}(T_n)}\big).$$

[1] The reader may refer to the Appendix B.1 for the definition of $\dot{S}_j$

Hence, using the smallness condition on $(\boldsymbol{u}_0, \boldsymbol{B}_0, \boldsymbol{v}_0)$ and the definition of T_n,

$$\|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|_{E_{2,r}(T_n)} \leq Cc(1 + M^2 c).$$

If we take $M = 2C$, then c so that $4C^2 c < 1$, then we have

$$\|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|_{E_{2,r}(T_n)} < Mc,$$

and thus, by a classical continuity argument, $T_n = T^n$.

Now, using functional embedding and interpolation arguments, we discover that

$$\left(\int_0^{T^n} \|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|^4_{\dot{\boldsymbol{B}}^{-\frac{1}{2}}_{\infty,\infty}} \, dt\right)^{\frac{1}{4}} \lesssim \|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|_{\widetilde{L}^4_{T^n}(\dot{B}^1_{2,r})} \lesssim \|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|_{E_{2,r}(T^n)}.$$

Hence, the continuation criterion (2.1.24) guarantees that, indeed, $T^n = +\infty$. This means that the solution is global and that, furthermore,

$$\|(\boldsymbol{u}^n, \boldsymbol{B}^n, \boldsymbol{v}^n)\|_{E_{2,r}} \leq Mc \quad \text{for all} \ \ n \in \mathbb{N}. \tag{C.1.6}$$

At this stage, proving that $(\boldsymbol{u}^n, \boldsymbol{B}^n)_{n\in\mathbb{N}}$ converges (up to subsequence) to a global solution $(\boldsymbol{u}, \boldsymbol{B})$ of the Hall-MHD system with data $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ and $(\boldsymbol{u}, \boldsymbol{B}, \boldsymbol{v})$ in $E_{2,r}$ follows from the same arguments as in the previous section.

Let us now prove the uniqueness part of the theorem. Suppose that $(\boldsymbol{u}_1, \boldsymbol{B}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{B}_2)$ are two solutions of the Hall-MHD system on $[0, T] \times \mathbb{R}^3$ supplemented with the same initial data $(\boldsymbol{u}_0, \boldsymbol{B}_0)$ and such that

$$(\boldsymbol{u}_i, \boldsymbol{B}_i, \boldsymbol{v}_i) \in \widetilde{\mathcal{C}}([0, T]; \dot{B}^{\frac{1}{2}}_{2,r}) \cap \widetilde{L}^1(0, T; \dot{B}^{\frac{5}{2}}_{2,r}), \quad i = 1, 2.$$

In order to prove the uniqueness, we look at the difference $(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta\boldsymbol{v}) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{B}_1 - \boldsymbol{B}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)$ as a solution of System (3.2.21). In contrast with the previous section however, we do not know how to estimate the difference in the space $E_{2,r}(T)$ since the term $\nabla \times ((\nabla \times \boldsymbol{v}_1) \times \delta\boldsymbol{B})$ cannot be bounded in the space $\widetilde{L}^1_T(\dot{B}^{\frac{1}{2}}_{2,r})$ from the norm of $\boldsymbol{v}_1$ and $\delta\boldsymbol{B}$ in $E_{2,r}(T)$ (this is due to the fact that the norm of $E_{2,r}(T)$ fails to control $\|\cdot\|_{L^\infty(0,T\times\mathbb{R}^3)}$ by a little if $r > 1$).

For that reason, we shall accept to lose some regularity in the stability estimates and prove uniqueness in the space

$$F_{2,r}(T) := \widetilde{L}^\infty_T(\dot{B}^{-\frac{1}{2}}_{2,r}).$$

We need first to justify that $(\delta\boldsymbol{u}, \delta\boldsymbol{B}, \delta\boldsymbol{v})$ belongs to that space, though. According to Proposition B.2.3, it is enough to check that the terms $\boldsymbol{R}_1$ to $\boldsymbol{R}_5$ defined just below (3.2.21) belong to $\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,r})$. Now, from (C.1.2) and Hölder inequality, we have

$$\|\boldsymbol{R}_1\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,r})} \lesssim T^{\frac{1}{2}} \|(\boldsymbol{u}_1, \boldsymbol{B}_1, \boldsymbol{u}_2, \boldsymbol{B}_2)\|_{\widetilde{L}^4_T(\dot{B}^1_{2,r})} \|(\delta\boldsymbol{u}, \delta\boldsymbol{B})\|_{\widetilde{L}^4_T(\dot{B}^1_{2,r})},$$

$$
\begin{aligned}
\|\boldsymbol{R}_2\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,r})} &\lesssim T^{\frac{1}{2}}\|(\boldsymbol{B}_2,\boldsymbol{v}_1)\|_{\widetilde{L}^4_T(\dot{B}^1_{2,r})}\|(\delta\boldsymbol{B},\delta\boldsymbol{v})\|_{\widetilde{L}^4_T(\dot{B}^1_{2,r})},\\
\|\boldsymbol{R}_3\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,r})} &\lesssim \|(\nabla\boldsymbol{v}_1,\nabla\delta\boldsymbol{v})\|_{\widetilde{L}^{\frac{4}{3}}_T(\dot{B}^1_{2,r})}\|(\delta\boldsymbol{B},\boldsymbol{B}_2)\|_{\widetilde{L}^4_T(\dot{B}^1_{2,r})},\\
\|\boldsymbol{R}_4\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,r})} &\lesssim T^{\frac{1}{2}}\|(\boldsymbol{u}_2,\boldsymbol{v}_1)\|_{\widetilde{L}^4_T(\dot{B}^1_{2,r})}\|(\delta\boldsymbol{u},\delta\boldsymbol{v})\|_{\widetilde{L}^4_T(\dot{B}^1_{2,r})},\\
\|\boldsymbol{R}_5\|_{\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,r})} &\lesssim \|(\nabla\boldsymbol{B}_2,\nabla\delta\boldsymbol{B})\|_{\widetilde{L}^{\frac{4}{3}}_T(\dot{B}^1_{2,r})}\|(\delta\boldsymbol{v},\boldsymbol{v}_1)\|_{\widetilde{L}^4_T(\dot{B}^1_{2,r})}.
\end{aligned}
$$

Since the norm in $E_{2,r}(T)$ bounds the norm in $\widetilde{L}^4_T(\dot{B}^1_{2,r})\cap\widetilde{L}^{\frac{4}{3}}_T(\dot{B}^2_{2,r})$, one can indeed conclude that the terms $\boldsymbol{R}_1$ to $\boldsymbol{R}_5$ are in $\widetilde{L}^1_T(\dot{B}^{-\frac{1}{2}}_{2,r})$.

Next, estimating $(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})$ in $F_{2,r}(T)$ may be achieved by a slight modification of the beginning of the proof of Proposition C.1.1. We get for all $t\in[0,T]$,

$$
\begin{aligned}
\|(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})\|_{F_{2,r}(t)} \lesssim{}& \|\boldsymbol{B}_1\cdot\nabla\delta\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} + \|\delta\boldsymbol{B}\cdot\nabla\boldsymbol{B}_2\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})}\\
&+ \|\boldsymbol{u}_1\cdot\nabla\delta\boldsymbol{u}\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} + \|\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}_2\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} + \|\boldsymbol{v}_1\cdot\nabla\delta\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})}\\
&+ \|\delta\boldsymbol{B}\cdot\nabla\boldsymbol{v}_1\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} + \|\boldsymbol{B}_2\cdot\nabla\delta\boldsymbol{v}\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} + \|\delta\boldsymbol{v}\cdot\nabla\boldsymbol{B}_2\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})}\\
&+ \|\boldsymbol{v}_1\cdot\nabla\delta\boldsymbol{u}\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} + \|\delta\boldsymbol{u}\cdot\nabla\boldsymbol{v}_1\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} + \|\boldsymbol{u}_2\cdot\nabla\delta\boldsymbol{v}\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})}\\
&+ \|\delta\boldsymbol{v}\cdot\nabla\boldsymbol{u}_2\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} + \|(\nabla\times\boldsymbol{v}_1)\times\delta\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})} + \|\boldsymbol{v}_1\cdot\nabla\delta\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})}\\
&+ \|\delta\boldsymbol{v}\cdot\nabla\boldsymbol{B}_2\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})} + \Big\|2^{\frac{j}{2}}\|[\dot{\Delta}_j,\boldsymbol{B}_2\times](\nabla\times\delta\boldsymbol{v})\|_{L^1_t(L^2)}\Big\|_{\ell^r(\mathbb{Z})}.
\end{aligned}
$$

Most of the terms on the right-hand side can be bounded by means of the following inequalities that are proved in Appendix B:

$$
\|\boldsymbol{a}\,\boldsymbol{b}\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} \lesssim \|\boldsymbol{a}\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}\|\boldsymbol{b}\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^0_{2,r})}, \tag{C.1.7}
$$

$$
\|\boldsymbol{a}\,\boldsymbol{b}\|_{\widetilde{L}^1_t(\dot{B}^{-\frac{1}{2}}_{2,r})} \lesssim \|\boldsymbol{a}\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^1_{2,r})}\|\boldsymbol{b}\|_{\widetilde{L}^4_t(\dot{B}^0_{2,r})}. \tag{C.1.8}
$$

Next, owing to Inequality (C.1.2) and interpolation, we have

$$
\begin{aligned}
\|(\nabla\times\boldsymbol{v}_1)\times\delta\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})} + \|\boldsymbol{v}_1\cdot\nabla\delta\boldsymbol{B}\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})} + \|\delta\boldsymbol{v}\cdot\nabla\boldsymbol{B}_2\|_{\widetilde{L}^1_t(\dot{B}^{\frac{1}{2}}_{2,r})}&\\
\lesssim \|\delta\boldsymbol{B}\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}\|\nabla\boldsymbol{v}_1\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^1_{2,r})} + \|\nabla\delta\boldsymbol{B}\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^1_{2,r})}\|\boldsymbol{v}_1\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}&\\
+ \|\delta\boldsymbol{v}\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^1_{2,r})}\|\nabla\boldsymbol{B}_2\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}&\\
\lesssim \big(\|(\boldsymbol{u}_2,\boldsymbol{v}_2,\boldsymbol{v}_1)\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})} + \|\boldsymbol{v}_1\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^2_{2,r})}\big)\|(\delta\boldsymbol{u},\delta\boldsymbol{v})\|_{F_{2,r}(t)}.&
\end{aligned}
$$

Finally, applying (B.4.2) with $\rho = 4$, $s = 1/2$ and using embedding $\dot{B}^0_{2,r} \hookrightarrow \dot{B}^{-\frac{3}{2}}_{\infty,\infty}$ and $\dot{B}^1_{2,r} \hookrightarrow \dot{B}^{-\frac{1}{2}}_{\infty,\infty}$ yields

$$\Big\|2^{\frac{j}{2}}\|[\dot{\Delta}_j, \boldsymbol{B}_2\times](\nabla\times\delta\boldsymbol{v})\|_{L^1_t(L^2)}\Big\|_{\ell^r(\mathbb{Z})} \lesssim \|\nabla\boldsymbol{B}_2\|_{L^4_t(\dot{B}^1_{2,r})}\|\nabla\times\delta\boldsymbol{v}\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^0_{2,r})}.$$

Thus, one can conclude that

$$\|(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})\|_{F_{2,r}(t)} \leq Y(t)\|(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v})\|_{F_{2,r}(t)}$$

with $Y(t) := \sum_{i=1,2}\|(\boldsymbol{u}_i,\boldsymbol{B}_i,\boldsymbol{v}_i)\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})} + \|\boldsymbol{v}_1\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^2_{2,r})}$.

Now, Lebesgue dominated convergence theorem ensures that Y is a continuous nondecreasing function vanishing at zero. Hence $(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v}) \equiv 0$ in $\widetilde{L}^\infty_t(\dot{B}^{-\frac{1}{2}}_{2,r}) \cap \widetilde{L}^1_t(\dot{B}^{\frac{3}{2}}_{2,r})$ for small enough t. Combining with a standard connectivity argument allows to conclude that $(\delta\boldsymbol{u},\delta\boldsymbol{B},\delta\boldsymbol{v}) \equiv \boldsymbol{0}$ on $\mathbb{R}_+$. This completes the proof of the theorem in the small data case. □

C.2 Ideas on the local existence for small initial velocity of election

Let us briefly explain how the above arguments have to be modified so as to handle the case where only $\boldsymbol{v}_0$ is small. Note that no smallness condition is needed whatsoever in the proof of uniqueness. As regards the existence part, we split $\boldsymbol{u}$ and $\boldsymbol{B}$ (not $\boldsymbol{v}$) into $\boldsymbol{u} = \boldsymbol{u}^L + \widetilde{\boldsymbol{u}}$ and $B = B^L + \widetilde{B}$ and repeat the proof of Proposition C.1.1 on the system fulfilled by $(\widetilde{\boldsymbol{u}},\widetilde{B},\boldsymbol{v})$ rather than (2.1.27). Instead of (C.1.4), we get

$$\|(\widetilde{\boldsymbol{u}},\widetilde{B},\boldsymbol{v})\|_{E_{2,r}(t)} \lesssim \|\boldsymbol{v}_0\|_{\dot{B}^{\frac{1}{2}}_{2,r}} + \|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^2_{2,r})}\|(\boldsymbol{u},\boldsymbol{B},\boldsymbol{v})\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}$$

from which we deduce that

$$\begin{aligned}\|(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{B}},\boldsymbol{v})\|_{E_{2,r}(t)} \lesssim{}& \|\boldsymbol{v}_0\|_{\dot{B}^{\frac{1}{2}}_{2,r}} + \|(\boldsymbol{u}^L,\boldsymbol{B}^L)\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^2_{2,r})}\|(\boldsymbol{u}^L,\boldsymbol{B}^L)\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}\\ &+\|(\boldsymbol{u}^L,\boldsymbol{B}^L)\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^2_{2,r})\cap\widetilde{L}^4_t(\dot{B}^1_{2,r})}\|(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{B}},\boldsymbol{v})\|_{E_{2,r}(t)} + \|(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{B}},\boldsymbol{v})\|^2_{E_{2,r}(t)}.\end{aligned}$$

Since, by dominated convergence theorem, we have

$$\lim_{t\to 0}\big(\|(\boldsymbol{u}^L,\boldsymbol{B}^L)\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^2_{2,r})} + \|(\boldsymbol{u}^L,\boldsymbol{B}^L)\|_{\widetilde{L}^4_t(\dot{B}^1_{2,r})}\big) = 0,$$

it is easy to see that if $\|\boldsymbol{v}_0\|_{\dot{B}^{\frac{1}{2}}_{2,r}}$ is small enough, then one can get a control on $\|(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{B}},\boldsymbol{v})\|_{E_{2,r}(t)}$ for small enough t. From this, repeating essentially the same arguments as in the small data case, one gets a local-in-time existence statement.

Appendix D

Schauder Estimates for a class of general non-local equations

This Appendix is for the proof of Theorem 4.1.4, where the case $s = 1$ has been proved in [17]. We will use the ideas from there to prove the general case $s \in (0,1)$, and thus we need the following propositions and lemmas.

D.1 Preparation

Proposition D.1.1. *Let $\phi(t,x) \in \mathcal{C}^{\alpha,\alpha s}((-6,0]\times\mathbb{R}^d)$, and $\omega \in \mathcal{C}^{1+\alpha,(1+\alpha)s}((-6,0]\times\mathbb{R}^d)$ be a solution of the integro-differential equation (the equation presented in Theorem 4.1.4 with $Q \equiv 0$)*

$$\partial_t \omega = \int_{\mathbb{R}^d} \big(\omega(t,x+y) - \omega(t,x)\big)\, L(t,x,y)\, dy + \phi(t,x). \tag{D.1.1}$$

Suppose that L satisfy (4.1.30), (4.1.31), (4.1.32). There exists a constant $C > 0$ depending only on $s, d, \Lambda_1, \Lambda_2$ such that

$$\|\omega\|_{\mathcal{C}^{1+\alpha,(1+\alpha)s}((-2,0]\times\mathbb{R}^d)} \le C(\|u\|_{L^\infty((-5,0]\times\mathbb{R}^d)} + \|\phi\|_{\mathcal{C}^{\alpha,\alpha s}((-5,0]\times\mathbb{R}^d)}).$$

Proof. At first, we know from the Hölder estimates in [64] (see also [53]) that there exist positive γ_1 and C depending only on $s, d, \Lambda_1, \Lambda_2$ such that

$$\|\omega\|_{\mathcal{C}^{\gamma_1,\gamma_1 s}((-4,0]\times\mathbb{R}^d)} \le C(\|\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)} + \|\phi\|_{L^\infty((-5,0]\times\mathbb{R}^d)}).$$

Then we infer from Theorem 1.1 in [66] that

$$\begin{aligned}\|\omega\|_{\mathcal{C}^{1+\gamma_1,(1+\gamma_1)s}((-3,0]\times\mathbb{R}^d)} &\le C(\|\omega\|_{\mathcal{C}^{\gamma_1,\gamma_1 s}((-4,0]\times\mathbb{R}^d)} + \|\phi\|_{\mathcal{C}^{\gamma_1,\gamma_1 s}((-5,0]\times\mathbb{R}^d)}) \\ &\le C(\|\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)} + \|\phi\|_{\mathcal{C}^{\gamma_1,\gamma_1 s}((-5,0]\times\mathbb{R}^d)}).\end{aligned}$$

If $\gamma_1 = \alpha$, then we have done. If $\gamma_1 < \alpha$, we can apply above theorem one more time to finish our proof. □

Proposition D.1.2. *Let $0 < \alpha < 1, 0 < \beta < \alpha$. Suppose Q satisfy (4.1.33) and (4.1.34). Define*

$$Q_\omega(t,x) := \int_{\mathbb{R}^d} \big(\omega(t,x+y) - \omega(t,x)\big) Q(t,x,y)\, dy.$$

There exists $C > 0$ depending only on $s, d, \Lambda_2, \beta, \alpha$ such that

$$\|Q_\omega\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)} \le C\|\omega\|_{\mathcal{C}^{1,s}((-5,0]\times\mathbb{R}^d)}.$$

Proof. We firstly find by (4.1.33) that

$$\begin{aligned}
&\|Q_\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)}\\
&\le \|\omega\|_{\mathcal{C}^s_x((-5,0]\times\mathbb{R}^d)} \int_{B_1(y)} |y|^s|Q|\,dy + 2\|\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)} \int_{\mathbb{R}^d/B_1(y)} |Q|\,dy\\
&\le 2\|\omega\|_{\mathcal{C}^s_x((-5,0]\times\mathbb{R}^d)} \int_{\mathbb{R}^d} \min\{1,|y|^s\}|Q|\,dy\\
&\le C\|\omega\|_{\mathcal{C}^s_x((-5,0]\times\mathbb{R}^d)} \int_{\mathbb{R}^d} \min\{1,|y|^s\}\min\{1,|y|^{\alpha s}\}|y|^{-d-s}\,dy\\
&\le C\|\omega\|_{\mathcal{C}^s_x((-5,0]\times\mathbb{R}^d)}. \qquad \text{(D.1.2)}
\end{aligned}$$

For any $(t,x),(\tau,z) \in (-5,0]\times\mathbb{R}^d$ with $0 < |x-z| \le \frac{1}{e}$. Similar with (D.1.2), we have

$$\begin{aligned}
&|Q_\omega(t,x) - Q_\omega(t,z)|\\
&\le |\int_{\mathbb{R}^d} \big(\omega(t,x+y)-\omega(t,x)\big)\big(Q(t,x,y)-Q(t,z,y)\big)\,dy|\\
&\quad + |\int_{\mathbb{R}^d} \big(\omega(t,x+y)-\omega(t,z+y)+\omega(t,z)-\omega(t,x)\big)Q(t,z,y)\,dy|\\
&\le 2\|\omega\|_{\mathcal{C}^s_x((-5,0]\times\mathbb{R}^d)} \int_{\mathbb{R}^d} \min\{1,|y|^s\}|Q(t,x,y)-Q(t,z,y)|\,dy\\
&\quad + 2\|\omega\|_{\mathcal{C}^s_x((-5,0]times\mathbb{R}^d)} \int_{\mathbb{R}^d} \min\{|x-z|^s,|y|^s\}|Q(t,z,y)|\,dy\\
&\le C\|\omega\|_{\mathcal{C}^s_x((-5,0]\times\mathbb{R}^d)} \Big(\int_{\mathbb{R}^d} \min\{1,|y|^s\}\min\{|x-z|^{\alpha s},|y|^{\alpha s}\}|y|^{-d-s}\,dy\\
&\quad + \int_{\mathbb{R}^d} \min\{|x-z|^s,|y|^s\}\min\{1,|y|^{\alpha s}\}|y|^{-d-s}\,dy\Big)\\
&\le C\|\omega\|_{\mathcal{C}^s_x((-5,0]\times\mathbb{R}^d)}|x-z|^{\alpha s}(1+|\ln|x-z||)\\
&\le C\|\omega\|_{\mathcal{C}^s_x((-5,0]\times\mathbb{R}^d)}|x-z|^{\beta s}.
\end{aligned}$$

Moreover, for any $0 < |t-\tau| \le \frac{1}{e}$ we have

$$\begin{aligned}
&|Q_\omega(t,x) - Q_\omega(\tau,x)|\\
&\le |\int_{\mathbb{R}^d} \big(\omega(t,x+y)-\omega(t,x)\big)\big(Q(t,x,y)-Q(\tau,x,y)\big)\,dy|\\
&\quad + |\int_{\mathbb{R}^d} \big(\omega(t,x+y)-\omega(t,x)+\omega(\tau,x+y)-\omega(\tau,x)\big)Q(\tau,x,y)\,dy|\\
&\le 2\|\omega\|_{\mathcal{C}^s_x((-5,0]\times\mathbb{R}^d)} \int_{\mathbb{R}^d} \min\{1,|y|^s\}|Q(t,x,y)-Q(\tau,x,y)|\,dy\\
&\quad + \|\omega\|_{\mathcal{C}^{1,s}_{t,x}((-5,0]\times\mathbb{R}^d)} \int_{\mathbb{R}^d} \min\{|t-\tau|,|y|^s\}|Q(\tau,x,y)|\,dy\\
&\le C\|\omega\|_{\mathcal{C}^{1,s}_{t,x}((-5,0]\times\mathbb{R}^d)} \Big(\int_{\mathbb{R}^d} \min\{1,|y|^s\}\min\{|t-\tau|^{\alpha},|y|^{\alpha s}\}|y|^{-d-s}\,dy\\
&\quad + \int_{\mathbb{R}^d} \min\{|t-\tau|,|y|^s\}\min\{1,|y|^{\alpha s}\}|y|^{-d-s}\,dy\Big)\\
&\le C\|\omega\|_{\mathcal{C}^{1,s}_{t,x}((-5,0]\times\mathbb{R}^d)}|t-\tau|^{\alpha}(1+|\ln|t-\tau||)
\end{aligned}$$

$$\leq C\|\omega\|_{\mathcal{C}^{1,s}_{t,x}((-5,0]\times\mathbb{R}^d)}|t-\tau|^{\beta}.$$

Thus we conclude that for $\beta < \alpha$,

$$\|Q_\omega\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)} \leq C\|\omega\|_{\mathcal{C}^{1,s}((-5,0]\times\mathbb{R}^d)}.$$

□

We also need the following iteration lemma, which is Lemma 1.1 in [70].

Lemma D.1.3. *Let $h : [T_0, T_1] \to \mathbb{R}$ be nonnegative and bounded. Suppose that for all $0 \leq T_0 \leq t < \tau \leq T_1$ we have*

$$h(t) \leq A(\tau - t)^{-\gamma} + \frac{1}{2}h(\tau)$$

with $\gamma > 0$ and $A > 0$. Then there exists $C = C(\gamma)$ such that for all $T_0 \leq t < \tau \leq T_1$ we have

$$h(t) \leq CA(\tau - t)^{-\gamma}.$$

D.2 Proof of Theorem 4.1.4

We are ready to prove Theorem 4.1.4.

Proof. By Proposition D.1.1 and Proposition D.1.2, for $\omega \in \mathcal{C}^{1+\alpha,(1+\alpha)s}((-6,0]\times\mathbb{R}^d)$ a solution of

$$\begin{aligned}\partial_t\omega = &\int_{\mathbb{R}^d}\big(\omega(t,x+y)-\omega(t,x)\big)\,L(t,x,y)\,dy\\ &+\int_{\mathbb{R}^d}\big(\omega(t,x+y)-\omega(t,x)\big)\,Q(t,x,y)\,dy+\phi(t,x),\end{aligned}$$

we have for $\beta \in (0,\alpha)$,

$$\begin{aligned}\|\omega\|_{\mathcal{C}^{1+\beta,(1+\beta)s}((-2,0]\times\mathbb{R}^d)} &\leq C(\|\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)} + \|Q_\omega,\phi\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)})\\ &\leq C(\|\omega\|_{\mathcal{C}^{1,s}((-5,0]\times\mathbb{R}^d)} + \|\phi\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)}).\end{aligned}$$

We first prove the case $(1+\beta)s \leq 1$. Define

$$h_\omega(\gamma,\tau) := \begin{cases}[\partial_t\omega]_{\mathcal{C}^{\gamma,(1+\gamma)s}((\tau,0]\times\mathbb{R}^d)} + [\omega]_{\mathcal{C}^{\gamma,(1+\gamma)s}((\tau,0]\times\mathbb{R}^d)} & \text{if } \gamma\in(0,\beta]\\ \|[\partial_t\omega]_{\mathcal{C}^s_x(\mathbb{R}^d)}\|_{L^\infty_t((\tau,0])} + \|[\omega]_{\mathcal{C}^s_x(\mathbb{R}^d)}\|_{L^\infty_t((\tau,0])} & \text{if } \gamma = 0.\end{cases}$$

Then we just proved that

$$h(\beta,-2) \leq C(\|\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)} + \|\phi\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)} + h(0,-5)). \tag{D.2.1}$$

For every $-2 < \tau_0 < \tau \leq -1$, we let

$$\psi(t,x) := \omega(\mu t + t_*, \mu x) \quad \text{with} \quad \mu := \frac{\tau-\tau_0}{3},\ t_* := \frac{5\tau-2\tau_0}{3},$$

then $\psi(t,x)$ satisfies that

$$\partial_t\psi(t,x) = \int_{\mathbb{R}^d}\big(\psi(t,x+y)-\psi(t,x)\big)\,\widetilde{L}(t,x,y)\,dy + \int_{\mathbb{R}^d}\big(\psi(t,x+y)-\psi(t,x)\big)\,\widetilde{Q}(t,x,y)\,dy + \widetilde{\phi}(t,x) \quad \text{in}(-6,0]\times\mathbb{R}^d,$$

with

$$\widetilde{L}(t,x,y) = \mu^{d+1}L(\mu t+t_*,\mu x,\mu y),\ \widetilde{Q}(t,x,y) = \mu^{d+1}Q(\mu t+t_*,\mu x,\mu y)$$

and $\widetilde{\phi}(t,x) = \phi(\mu t+t_*,\mu x)$. As $\mu<1$, and each of $\widetilde{L},\widetilde{Q},\widetilde{\phi}$ satisfies the same assumptions on L,Q,ϕ, respectively. Therefore, (D.2.1) holds true for ψ as well. Noticing that

$$h_\psi(\beta,-2)\geq(\mu+1)\min\{\mu^\beta,\mu^{(\beta+1)s}\}h_\omega(\beta,\tau)\geq\mu^{\beta+s}h_\omega(\beta,\tau)$$

and

$$h_\psi(0,-5)\leq(\mu+1)\mu^s h_\omega(\beta,\tau_0)\leq 2\mu^s h_\omega(0,\tau_0).$$

Thus we have

$$h_\omega(\beta,\tau)\leq\frac{C}{|\tau-\tau_0|^{\beta+s}}(\|\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)}+\|\phi\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)}) + \frac{C}{|\tau-\tau_0|^\beta}h_\omega(0,\tau_0).$$

By interpolation inequality in Hölder spaces and Young's inequalitywe know that for every $\epsilon_0<1$, there exists $C>0$ independent of ϵ_0 such that

$$\begin{aligned}
h_\omega(0,\tau_0)&\leq[\partial_t\omega]_{\mathcal{C}^{\frac{\beta}{(1+\beta)s+1},s}((\tau_0,0]\times\mathbb{R}^d)}+[\omega]_{\mathcal{C}^{\frac{\beta}{(1+\beta)s+1},s}((\tau_0,0]\times\mathbb{R}^d)}\\
&\leq[\partial_t\omega]^{\frac{1+s}{(1+\beta)s+1}}_{\mathcal{C}^{\beta,(1+\beta)s}}\|\omega\|^{1-\frac{1+s}{(1+\beta)s+1}}_{L^\infty}+[\omega]^{\frac{1+s}{(1+\beta)s+1}}_{\mathcal{C}^{\beta,(1+\beta)s}}\|\omega\|^{1-\frac{1+s}{(1+\beta)s+1}}_{L^\infty}\\
&\leq\epsilon_0([\partial_t\omega]_{\mathcal{C}^{\beta,(1+\beta)s}}+[\omega]_{\mathcal{C}^{\beta,(1+\beta)s}})+C\epsilon_0^{-\frac{1+s}{\beta s}}\|\omega\|_{L^\infty}\\
&=\epsilon_0 h_\omega(\beta,\tau_0)+C\epsilon_0^{-\frac{1+s}{\beta s}}\|\omega\|_{L^\infty((\tau_0,0]\times\mathbb{R}^d)}.
\end{aligned}$$

Choosing $\epsilon_0=\frac{|\tau-\tau_0|^\beta}{2C}$, we get

$$h_\omega(\beta,\tau)\leq\frac{C}{|\tau-\tau_0|^{\beta+1+1/s}}(\|\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)}+\|\phi\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)}) + \frac{1}{2}h_\omega(\beta,\tau_0).$$

Thanks to Lemma D.1.3, we have that

$$h_\omega(\beta,-1)\leq C(\|\omega\|_{L^\infty((-1,0]\times\mathbb{R}^d)}+\|\phi\|_{\mathcal{C}^{\beta,\beta s}((-1,0]\times\mathbb{R}^d)}).$$

For the case $(1+\beta)s > 1$, we need to rewrite

$$h_\omega(\gamma,\tau) = \begin{cases} [\partial_t \omega]_{\mathcal{C}^{\gamma,(1+\gamma)s-1}((\tau,0]\times\mathbb{R}^d)} + [\nabla_x \omega]_{\mathcal{C}^{\gamma,(1+\gamma)s-1}((\tau,0]\times\mathbb{R}^d)} & \text{if } \gamma \in (0,\beta] \\ \|\partial_t \omega\|_{L^\infty((\tau,0]\times\mathbb{R}^d)} + \|\nabla_x \omega\|_{L^\infty_t((\tau,0]\times\mathbb{R}^d)} & \text{if } \gamma = 0, \end{cases}$$

as in [17] for the case $s = 1$. Then by Proposition D.1.2, we have

$$\begin{aligned} \|\omega\|_{\mathcal{C}^{1+\beta,(1+\beta)s}((-2,0]\times\mathbb{R}^d)} &\le C(\|\omega\|_{L^\infty((-5,0]\times\mathbb{R}^d)} + \|Q_\omega, \phi\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)}) \\ &\le C(\|\omega\|_{\mathcal{C}^{1,1}((-5,0]\times\mathbb{R}^d)} + \|\phi\|_{\mathcal{C}^{\beta,\beta s}((-5,0]\times\mathbb{R}^d)}). \end{aligned}$$

This shows that (D.2.1) is satisfied. We omitted another details here and complete the proof of Theorem 4.1.4. □

Bibliography

[1] P.D. Mininni. "Hall-MHD dynamos and turbulence". In: `http://www.ens-lyon.fr/PHYSIQUE/Equipe2/LesHouches11/Talks_files/Mininni_Houches2011.pdf`. 2011.

[2] S.A. Balbus and C. Terquem. "Linear analysis of the Hall effect in protostellar disks". In: *Astrophys. J.* 552 (2001), pp. 235–247.

[3] T.G. Forbes. "Magnetic reconnection in solar flares". In: *Geophys. Astrophys. Fluid Dyn.* 62 (1991), pp. 15–36.

[4] J.D. Huba. *Hall Magnetohydrodynamics - A Tutorial In: Büchner J., Scholer M., Dum C.T. (eds) Space Plasma Simulation*. Vol. 615. Springer, 2003.

[5] D.A. Shalybkov and V.A. Urpin. "The Hall effect and the decay of magnetic fields". In: *Astron. Astrophys.* 321 (1997), pp. 685–690.

[6] M. Wardle. "Star formation and the Hall effect". In: *Astrophys. Space Sci.* 292 (2004), pp. 317–323.

[7] M. Acheritogaray et al. "Kinetic formulation and global existence for the Hall-Magnetohydrodynamics system". In: *Kinet. Relat. Models* 4.1 (2011), pp. 901–918.

[8] R. Danchin and J. Tan. "On the well-posedness of the Hall-Magnetohydrodynamics system in critical Besov spaces". In: *Comm. Partial Differ. Equ.* 46.1 (2021), pp. 31–65.

[9] R. Danchin and J. Tan. "The global solvability of the Hall-magnetohydrodynamics System in critical Sobolev spaces". In: *arXiv:1912.09194* (2019).

[10] L. Liu and J. Tan. "Global well-posedness for the Hall-magnetohydrodynamics system in larger critical Besov spaces". In: *J. Differ. Equ.* 274 (2021), pp. 382–413.

[11] J.-Y. Chemin. "Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel". In: *Journal d'Analyse Mathématique* 77 (1999), pp. 25–50.

[12] H. Fujita and T. Kato. "On the Navier-Stokes initial value problem I". In: *Arch. Ration. Mech. Anal.* 16 (1964), pp. 269–315.

[13] D. Chae and J. Lee. "On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics". In: *J. Differ. Equ.* 256 (2014), pp. 3835–3858.

[14] A.J. Majda and A.L. Bertozzi. *Vorticity and Incompressible Flow*. Cambridge University Press, 2001.

[15] P.G. Lemarie-Rieusset. *Recent developments in the Navier–Stokes problem*. Chapman & Hall/CRC, 2002.

[16] C. Imbert, R. Shvydkoy, and F. Vigneron. "Global well-posedness of a non-local Burgers equation: The periodic case". In: *Ann. Fac. Sci. Toulouse, Math.* 25.4 (2016), pp. 723–758.

[17] C. Imbert, T. Jin, and R. Shvydkoy. "Schauder estimates for an integro-differential equation with applications to a nonlocal Burgers equation". In: *Ann. Fac. Sci. Toulouse, Math.* 27.4 (2018), pp. 667–677.

[18] I. Athanasopoulos and L. A. Caffarelli. "Continuity of the temperature in boundary heat control problems". In: *Adv. Math.* 224 (2010), pp. 293–315.

[19] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique*. Vol. 21. Dunod, Paris: Travaux et Recherches Math., 1972.

[20] F. Quirós A. de Pablo, A. Rodríguez, and J.L. Vázquez. "Classical solutions and higher regularity for nonlinear fractional diffusion equations". In: *J. Eur. Math. Soc.* 19 (2017), pp. 1949–1975.

[21] A. de Pablo et al. "A fractional porous medium equation". In: *Adv. Math.* 226 (2011), pp. 1378–1409.

[22] A. de Pablo et al. "A general fractional porous medium equation". In: *Comm. Pure Appl. Math.* 65 (2012), pp. 1242–1284.

[23] M.T. Barlow et al. "Non-local Dirichlet forms and symmetric jump processes". In: *Trans. Amer. Math. Soc.* 361.4 (2009), pp. 1963–1999.

[24] M. Kassmann. "A priori estimates for integro-differential operators with measurable kernels". In: *Calc. Var. Partial Differ. Equ.* 34.1 (2009), pp. 1–21.

[25] L. Caffarelli, C.H. Chan, and A. Vasseur. "Regularity theory for parabolic nonlinear integral operators". In: *J. Amer. Math. Soc.* 24.3 (2011), pp. 849–869.

[26] J.A. Bittencourt. *Fundamentals of Plasma Physics*. New York: Springer-Verlag, 2004.

[27] J. Jang and N. Masmoudi. "Derivation of Ohm's law from the Kinetic Equations". In: *SIAM J. Math. Anal.* 44.6 (2012), pp. 3649–3669.

[28] D. Chae, P. Degond, and J.-G. Liu. "Well-posedness for Hall-magnetohydrodynamics". In: *Ann. IHP, Analyse non linéaire* 31 (2014), pp. 555–565.

[29] J. Leray. "Sur le mouvement d'un liquide visqueux remplissant l'espace". In: *Acta Math.* 63 (1934), pp. 193–248.

[30] E. Dumas and F. Sueur. "On the weak solutions to the Maxwell-Landau-Lifshitz equations and to the Hall-Magneto-Hydrodynamic equations". In: *Com. Math. Phys.* 330 (2014), pp. 1179–1225.

[31] M. Dai. "Non-unique weak solutions in Leray-Hopf class for the 3D Hall-MHD system". In: *arXiv:1812.11311v4* (2021).

[32] Z. Ye. "Regularity criteria and small data global existence to the generalized viscous Hall-magnetohydrodynamics". In: *Comp. and Math. with Applications* 70 (2015), pp. 2137–2154.

[33] M. Dai. "Regularity criterion for the 3D Hall-magnetohydrodynamics". In: *J. Differ. Equ.* 261.1 (2016), pp. 573–591.

[34] B. Ahmad et al. "On regularity criteria for the 3D Hall-MHD equations in terms of the velocity". In: *Nonlinear Anal. Real World Appl.* 32.2 (2016), pp. 35–51.

[35] M. Benvenutti and L. Ferreira. "Existence and stability of global large strong solutions for the Hall-MHD system". In: *Differential Integral Equations* 29 (2016), pp. 977–1000.

[36] X. Wu, Y. Yu, and Y. Tang. "Well-posedness for the Incompressible Hall-MHD Equations in Low Regularity Spaces". In: *Mediterr. J. Math.* 15.2 (2018), Art. 48.

[37] R. Wan and Y. Zhou. "On global existence, energy decay and blow-up criteria for the Hall-MHD system". In: *J. Differ. Equ.* 259 (2015), pp. 5982–6008.

[38] M. Dai. "Local well-posedness for the Hall-MHD system in optimal Sobolev spaces". In: *J. Differ. Equ.* 289 (2021), pp. 150–181.

[39] R. Wan and Y. Zhou. "Global well-posedness for the 3D incompressible Hall-magnetohydrodynamic equations with Fujita-Kato type initial data". In: *J. Math. Fluid Mech.* 21 (2019), Art. 5, 16 pp.

[40] D. Chae and M. Schonbek. "On the temporal decay for the Hall-magnetohydrodynamic equations". In: *J. Differ. Equ.* 255 (2013), pp. 3971–3982.

[41] S. Weng. "On analyticity and temporal decay rates of solutions to the viscous resistive Hall-MHD system". In: *J. Differ. Equ.* 260.8 (2016), pp. 6504–6524.

[42] S. Weng. "Space-time decay estimates for the incompressible viscous resistive MHD and Hall-MHD equations". In: *J. Funct. Anal.* 270.6 (2016), pp. 2168–2187.

[43] M. Dai and H. Liu. "Long time behavior of solutions to the 3D Hall-magneto-hydrodynamics system with one diffusion". In: *J. Differ. Equ.* 266.11 (2019), pp. 7658–7677.

[44] J. Li, Y. Yu, and W. Zhu. "A class large solution of the 3D Hall-magnetohydrodynamic equations". In: *J. Differ. Equ.* 268.10 (2020), pp. 5811–5822.

[45] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Berlin/Heidelberg: Springer-Verlag, 2011.

[46] M. Cannone. *Ondelettes, paraproduits et Navier-Stokes*. Paris: Diderot Editeur, 1995.

[47] C. Miao and B. Yuan. "On the well-posedness of the Cauchy problem for an MHD system in Besov spaces". In: *Math. Methods Appl. Sci.* 32 (2009), pp. 53–76.

[48] F. Lelièvre. *Approximation des équations de navier-stokes préservant le changement d'échelle*. Vol. 49. 2010.

[49] F. Lelièvre. "A scaling and energy equality preserving approximation for the 3D Navier-Stokes equations in the finite energy case". In: *Nonlinear Anal.* 74.17 (2011), pp. 5902–5919.

[50] F. Lelièvre. "Un modèle scalaire analogue aux équations de Navier-Stokes". In: *C. R. Math. Acad. Sci. Paris* 349.7-8 (2011), pp. 411–416.

[51] M. Kwaśnicki. "Ten equivalent definitions of the fractional Laplace operator". In: *Fractional Calculus and Applied Analysis* 20.1 (2017), pp. 7–51.

[52] P. Constantin and V. Vicol. "Nonlinear maximum principles for dissipative linear nonlocal operators and applications". In: *Geom. Funct. Anal.* 22.5 (2012), pp. 1289–1321.

[53] M. Felsinger and M. Kassmann. "Local regularity for parabolic nonlocal operators". In: *Comm. Partial Differ. Equ.* 38.9 (2013), pp. 1539–1573.

[54] T. Jin and J. Xiong. "Schauder estimates for nonlocal fully nonlinear equations". In: *Ann. IHP, Analyse non linéaire* 33 (2016), pp. 1375–1407.

[55] R. Mikulevicius and H. Pragarauska. "On the Cauchy problem for integro-differential operators in Holder classes and the uniqueness of the martingale problem". In: *Potential Anal.* 40.4 (2014), pp. 539–563.

[56] I. Gallagher, D. Iftimie, and F. Planchon. "Asymptotics and stability for global solutions to the Navier-Stokes equations". In: *Ann. Inst. Fourier* 53.5 (2003), pp. 1387–1424.

[57] I. Gallagher and F. Planchon. "On global infinite energy solutions to the Navier-Stokes equations in two dimensions". In: *Arch. Ration. Mech. Anal.* 161 (2002), pp. 307–337.

[58] R.M. Blumenthal and R. K. Getoor. "Some theorems on stable processes". In: *Trans. Amer. Math. Soc.* 95 (1960), 263–273.

[59] R. Danchin. "Global existence in critical spaces for compressible Navier-Stokes equations". In: *Invent. math.* 141 (2000), pp. 579–614.

[60] R. Danchin. "Fourier Analysis Methods for PDE's". In: `https://perso.math.u-pem.fr/danchin.raphael/cours/courschine`. 2005.

[61] T. Runst and W. Sickel. *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, de Gruyter Series in Nonlinear Analysis and Applications*. Berlin: Walter de Gruyter & Co., 1996.

[62] C. Kenig, G. Ponce, and L. Vega. "Well-posedness of the initial value problem for the Korteweg de Vries equation". In: *J. Amer. Math. Soc.* 4 (1991), pp. 323–347.

[63] L. Caffarelli and L. Silvestre. "Regularity theory for fully nonlinear integro-differential equations." In: *Comm. Pure Appl. Math.* 62.5 (2009), pp. 597–638.

[64] H.C. Lara and G. Dàvila. "Regularity for solutions of non local parabolic equations". In: *Calc. Var. Partial Differ. Equ.* 49 (2014), pp. 139–172.

[65] T. Jin and J. Xiong. "Schauder estimates for solutions of linear parabolic integro-differential equations". In: *Discrete Contin. Dyn. Syst.* 35.12 (2015), pp. 5977–5998.

[66] H. Dong and H. Zhang. "On Schauder estimates for a class of nonlocal fully nonlinear parabolic equations". In: *Calc. Var. Partial Differ. Equ.* 58.40 (2019).

[67] C. Imbert, R. Monneau, and E. Rouy. "Homogenization of first order equations with (u/ϵ)-periodic Hamiltonians. II. Application to dislocations dynamics". In: *Comm. Partial Differ. Equ.* 33.3 (2008), pp. 479–516.

[68] J. M. Bony. "Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires". In: *Ann. Sci. École Norm. Sup.* 14.4 (1986), pp. 209–246.

[69] M. Paicu and P. Zhang. "Global solutions to the 3-D incompressible inhomogeneous Navier-Stokes system". In: *J. Funct. Anal.* 262 (2012), pp. 3556–3584.

[70] M. Giaquinta and E. Giusti. "On the regularity of the minima of variational integrals". In: *Acta Math.* 148 (1982), pp. 31–46.