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Hochschild homology and singularity categories

Yu WANG

Thèse de Doctorat de Mathématiques

Dirigée par Bernhard Keller

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À mes parents.

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Résumé

L'homologie de Hochschild a été introduite par Gerhard Hochschild en 1945. Elle est un invariant important d'une algèbre associative sur un corps k. La définition s'étend facilement des k-algèbres aux k-catégories et aux k-catégories différentielles graduées (=dg-catégories). Les catégories des singularités ont été découvertes par Buchweitz dans un contexte algébrique en 1986 et redécouvertes par Orlov dans un contexte géométrique en 2003. Dans cette thèse, nous étudions l'homologie de Hochschild des dg-catégories qui enrichissent canoniquement la catégorie dérivée et la catégorie des singularités d'une algèbre de dimension finie. L'un de nos résultats principaux affirme que l'homologie de Hochschild du dg-enrichissement de la catégorie des singularités est calculée par le complexe de Hochschild double, qui est issu de la fusion du complexe de Hochschild classique et de son k-dual. Nous rapportons aussi des applications de l'homologie de Hochschild et de l'homologie cyclique dans l'étude des structures de Calabi–Yau relatives et des complétions de Calabi–Yau relatives suivant des travaux récents de Brav–Dyckerhoff et Yeung.

Mots-clé. Homologie de Hochschild, homologie cyclique, catégorie dérivée, catégorie différentielle graduée, catégorie des singularités.

Abstract

Hochschild homology was introduced by Gerhard Hochschild in 1945. It is an important invariant of an associative algebra over a field k. The definition extends from k-algebras to k-categories and further to differential graded (=dg) categories. Singularity categories were first introduced by Buchweitz in 1986 in an algebraic setting and rediscovered by Orlov in 2003 in a geometric setting. In this thesis, we study the Hochschild homology of the canonical differential graded categories which enhance the bounded derived category and the singularity category of a finite-dimensional algebra. One of our main results states that Hochschild homology of the dg enhanced singularity category is computed by the double Hochschild complex, which is spliced together from the classical Hochschild complex and its k-dual. We also report on applications of Hochschild and cyclic homology in the study of relative Calabi–Yau structures and relative Calabi–Yau completions following recent work by Brav–Dyckerhoff and Yeung.

Keywords. Hochschild homology, cyclic homology, derived category, differential graded category, singularity category.

Résumé détaillé en français

Cette thèse contient plusieurs contributions à l'étude de l'homologie de Hochschild des enrichissements différentiels gradués de catégories dérivées et de catégories de singularités d'algèbres de dimension finie. Elle est divisée en quatre chapitres.

Le chapitre 1 sert à introduire les notions et notations principales et à résumer l'état de l'art. Soit k un corps. L'homologie et la cohomologie de Hochschild sont des invariants classiques associés à une k-algèbre associative ou, plus généralement, à une (petite) k-catégorie. On peut les interpréter comme les versions dérivées de l'espace des traces et du centre de l'algèbre. Rappelons rapidement ces constructions : soit A une k-algèbre associative unifère éventuellement non commutative. Le *centre* $Z(A)$ *de* A est formé des éléments $a \in A$ tels que $ab = ba$ pour tous $b \in A$. *L'espace des graves* $\mathrm{Tr}(A)$ est le quotient $A/[A, A]$, où $[A, A]$ désigne le sous-espace de A engendré par tous les commutateurs $[a, b] = ab - ba$. Notons que le centre est naturellement une algèbre commutative tandis que l'espace des traces est seulement un espace vectoriel. Par exemple, si $n \geq 1$ est un entier et $M_n(A)$ l'algèbre des matrices $n \times n$ à coefficients dans A, nous avons un isomorphisme

$$Z(A) \xrightarrow{\sim} Z(M_n(A))$$

qui envoie un élément $a \in A$ sur la matrice diagonale dont les coefficients diagonaux sont tous égaux à a. Nous avons aussi un isomorphisme

$$\mathrm{Tr}(M_n(A)) \xrightarrow{\sim} \mathrm{Tr}(A)$$

qui envoie la classe d'une matrice $M = (m_{ij})$ sur la classe de sa trace

$$\sum_i m_{ii},$$

voir [69]. Notons que par définition, nous avons une suite exacte courte

$$0 \longrightarrow Z(A) \longrightarrow A \longrightarrow \mathrm{Hom}_k(A, A)\,,$$

où la deuxième application est l'inclusion et la troisième envoie $a \in A$ sur le commutateur $[a, ?]$. De façon

duale, en quelque sorte, nous avons une suite exacte courte

$$0 \longleftarrow \mathrm{Tr}(A) \longleftarrow A \longleftarrow A \otimes_k A$$

où la troisième application envoie $a \otimes b$ sur les commutateur $ab - ba$ et la deuxième est la projection naturelle. Ce parallélisme suggère une certaine similarité entre le centre et l'espace des traces mais, au moins pour des algèbres non commutatives, les deux sont en fait très différents. On voit cela en examinant leurs propriétés de fonctorialité : clairement, un morphisme d'algèbres $f : A \to B$ induit une application linéaire $\mathrm{Tr}(f) : \mathrm{Tr}(A) \to \mathrm{Tr}(B)$. De l'autre côté, si $a \in A$ est central dans A, il n'y a a priori aucune raison pour que $f(a)$ soit central dans B de façon que l'association $A \mapsto Z(A)$ n'est pas un foncteur. On peut néanmoins constater une fonctorialité restreinte contravariante si on étend la construction du centre aux petites k-catégories, comme nous allons le voir au Chapitre 1. Notons $\otimes$ le produit tensoriel $\otimes_k$ sur le corps de base. Le *complexe des chaînes de Hochschild* est le complexe C_*A concentré en des degrés homologiques ≥ 0

$$A \longleftarrow A \otimes A \longleftarrow \dots \longleftarrow A^{\otimes p} \longleftarrow A^{\otimes (p+1)} \longleftarrow \dots$$

où $C_pA = A^{\otimes(p+1)}$, $p \geq 0$, et la différentielle est donnée par

$$d(a_0, \dots, a_p) = \sum_{i=0}^{p-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_p) + (-1)^p (a_p a_0, \dots, a_{p-1}) \,,$$

où l'on écrit $(a_0, \dots, a_p)$ pour $a_0 \otimes \cdots \otimes a_p$. Notons que la première différentielle envoie $a \otimes b$ sur le commutateur $ab - ba$. *L'homologie de Hochschild* $HH_*(A)$ est l'espace gradué avec $HH_p(A) = H_p(C_*A)$. Le *complexe des cochaînes de Hochschild* est le complexe C^*A concentré en degrés cohomologiques ≥ 0 donné par

$$A \longrightarrow \mathrm{Hom}_k(A, A) \longrightarrow \mathrm{Hom}_k(A \otimes A, A) \longrightarrow \dots \longrightarrow \mathrm{Hom}_k(A^{\otimes p}, A) \longrightarrow \dots$$

dont la différentielle est donnée par

$$(df)(a_0, \dots, a_p) = a_0 f(a_1, \dots, a_p) - \sum_{i=0}^{p-1} (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_p) + (-1)^p f(a_0, \dots, a_{p-1}) a_p.$$

Notons que les premières deux différentielles sont données par

$$a \mapsto [a, ?] \quad \text{and} \quad f \mapsto (a \otimes b \mapsto f(a)b - f(ab) + af(b)).$$

Nous voyons qu'en degré 0, nous retrouvons l'espace des traces $\mathrm{Tr}(A) = HH_0(A)$ respectivement le centre $Z(A) = HH^0(A)$. Nous constatons également que $HH^1(A)$ est égal à l'algèbre de Lie des dérivations extérieures de A (dont le crochet est induit par le commutateur des dérivations). Les deux structures, celle d'algèbre commutative sur $Z(A)$ et celle d'algèbre de Lie sur $HH^1(A)$ s'étendent à la cohomologie de Hochschild toute entière, comme nous allons le voir à la section 1.7. Au chapitre 1, nous allons

étendre ces constructions des k-algèbres A aux petites k-catégories $\mathcal{A}$. Par exemple, si $\mathrm{proj}\,(A)$ désigne la catégorie des modules projectifs de type fini sur une k-algèbre A, alors son homologie de Hochschild est bien définie et l'inclusion de k-catégories $A \to \mathrm{proj}\,(A)$ induit des isomorphismes

$$HH_*(A) \xrightarrow{\sim} HH_*(\mathrm{proj}\,(A))$$

ce qui donne une autre démonstration de l'invariance de $HH_*(A)$ par équivalences de Morita. A la section 1.4, nous allons rappeler la construction de la catégorie dérivée et la notion d'équivalence dérivée. Nous verrons que l'homologie et la cohomologie de Hochschild sont aussi invariants par équivalences dérivées. Ces résultats suggèrent qu'on devrait pouvoir les construire directement à partir de la catégorie dérivée. Ceci n'est pas possible en général mais nous montrons à la section 1.6 qu'on peut les obtenir à partir de l'enrichissement différentiel gradué canonique de la catégorie dérivée. L'homologie et la cohomologie de Hochschild sont munies de structures supérieures : le complexe des chaînes de Hochschild est un complexe mixte et le complexe des cochaînes de Hochschild une B_∞-algèbre, comme nous le rappelons à la section 1.7. Dans cette section, nous énonçons également les résultats d'invariance correspondants. A la section finale du Chapitre 1, nous rapportons des résultats récents sur la cohomologie de Hochschild–Tate (appelée aussi cohomologie de Hochschild singulière) suivant des travaux de Keller [58] et de Hua–Keller [46]. En particulier, nous esquissons une démonstration du fait que, sous des conditions de séparabilité, la cohomologie de Hochschild–Tate est isomorphe, en tant qu'algèbre graduée, à la cohomologie de Hochschild classique de la catégorie des singularités avec son enrichissement différentiel gradué canonique. Dans ce chapitre, nous nous inspirons largement du survol 'Hochschild (co)homology and derived categories' (prépublication 2020) de B. Keller. Nous y rajoutons des rappels de thèorèmes de localisation pour l'homologie de Hochschild.

Le Chapitre 2 est consacré à l'étude de l'homologie de Hochschild de l'enrichissement différentiel gradué (=dg-enrichissement) canonique $\mathcal{D}^b_{dg}(A)$ de la catégorie dérivée bornée et du dg-enrichissement $\mathrm{sg}_{dg}(A)$ de la catégorie des singularités

$$\mathrm{sg}(A) = \mathcal{D}^b(A)/\mathrm{per}\,(A).$$

Ici, le symbole $\mathcal{D}^b(A)$ désigne la catégorie dérivée bornée de la catégorie $\mathrm{mod}\,A$ des modules de dimension finie (sur k) sur une algèbre de dimension finie A et $\mathrm{per}\,(A)$ sa catégorie dérivée parfaite, c'est-à-dire la sous-catégorie épaisse de $\mathcal{D}^b(A)$ formée des complexes quasi-isomorphes à des complexes bornés de modules projectifs de type fini. Nous observons que nous avons une équivalence

$$\mathrm{per}\,(A^!) \xrightarrow{\sim} \mathcal{D}^b(A)$$

entre la catégorie dérivée dérivée parfaite du dual de Koszul $A^!$ de A et la catégorie dérivée bornée de A. Par le thèorème d'invariance, l'homologie de Hochschild de A est donc isomorphe à celle de $A^!$. Par un thèorème de Van den Bergh (théorème 2.1.1), l'homologie de Hochschild de $A^!$ est isomorphe à l'espace

gradué $DHH_*(A)$ dual sur k de l'homologie de Hochschild de A. Par conséquent, nous obtenons donc un isomorphisme

$$HH_*(\mathcal{D}^b_{dg}(A)) \xrightarrow{\sim} DHH_*(A).$$

Par le théorème de localisation (Thèorème 1.7.4), nous en déduisons le résultat principal, à savoir le thèorème 2.2.2 qui montre que l'homologie de Hochschild de $\mathrm{sg}_{dg}(A)$ est calculée par le *complexe de Hochschild double*, à savoir le complexe

$$\ldots \xrightarrow{b} A \otimes A \xrightarrow{b} A \xrightarrow{\tau} DA \xrightarrow{Db} D(A \otimes A) \xrightarrow{Db} \ldots ,$$

où DA est placé en degré 0, les différentielles b sont celles du complexe de Hochschild et les différentielles Db leurs duales.

Au Chapitre 3, nous donnons une nouvelle démonstration d'un théorème dû à Chen–Wang qui décrit les catégories des singularités des algèbres de dimension finie en termes d'algèbres de chemins de Leavitt différentielles graduées. Notre démonstration est basée sur

1) la dualité de Koszul–Moore élaborée dans la thèse [68] de Lefèvre-Hasegawa et décrite dans [56] et

2) la théorie de la localisation dérivée développée par Braun–Chuang–Lazarev dans [11].

De façon plus précise, soit $A = kQ/I$ une algèbre de dimension finie sur un corps k donnée comme quotient de l'algèbre des chemins kQ d'un carquois fini Q par un idéal admissible I. Notons R le quotient de A par son radical. Il s'identifie canoniquement avec la sous-algèbre de A engendré comme espace vectoriel par les idempotents associés aux sommets de Q. Alors A devient une algèbre dans la catégorie monoïdale $\mathrm{mod}\, R^e$ des R-bimodules munie du produit tensoriel sur R. Comme A est de dimension finie, nous avons une cogèbre duale $C = A^\vee = \mathrm{Hom}_{R^e}(A, R^e)$ dans cette catégorie. L'algèbre A et la cogèbre C sont augmentées sur R et nous avons des décompositions canoniques de R-bimodules $A = R \oplus J$ et $C = R \oplus J^\vee$. Si ΩC désigne la construction cobar de C sur R, nous avons une équivalence triangulée canonique $\mathcal{D}^c(C) \xrightarrow{\sim} \mathcal{D}(\Omega C)$, où $\mathcal{D}^c$ désigne la catégorie codérivée de la cogèbre C. Elle envoie R sur ΩC et induit donc une équivalence

$$\mathcal{D}^b(\mathrm{com}\, C) \xrightarrow{\sim} \mathrm{per}\,(\Omega C)\,,$$

où $\mathrm{com}\, C$ est la catégorie des C-comodules de dimension finie sur k. Par composition avec l'isomorphisme $\mathcal{D}^b(\mathrm{mod}\, A) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{com}\, C)$, nous obtenons une équivalence $\mathcal{D}^b(\mathrm{mod}\, A) \xrightarrow{\sim} \mathrm{per}\,(\Omega C)$ qui envoie R sur ΩC. L'isomorphisme $\mathcal{D}^b(\mathrm{mod}\, A) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{com}\, C)$ envoie le cogénérateur injectif $DA \xrightarrow{\sim} A^\vee$ sur le module colibre C et l'équivalence $\mathcal{D}^b(\mathrm{com}\, C) \xrightarrow{\sim} \mathrm{per}\,(\Omega C)$ envoie C sur R. Par conséquent, nous obtenons une équivalence induite

$$\mathcal{D}^b(\mathrm{mod}\, A)/\mathrm{thick}\,(DA) \xrightarrow{\sim} \mathrm{per}\,(\Omega C)/\mathrm{thick}\,(R).$$

En composant avec le foncteur de dualité

$$D : \mathcal{D}^b(\mathrm{mod}\, A^{op})^{op} \xrightarrow{\sim} \mathcal{D}^b(\mathrm{mod}\, A)$$

nous trouvons une équivalence

$$\mathrm{sg}(A^{op})^{op} \xrightarrow{\sim} \mathrm{per}\,(\Omega C)/\mathrm{thick}\,(R)\ ,$$

où $\mathrm{sg}(A^{op})$ désigne la catégorie des singularités de A^{op}. Nous appliquons au quotient à droite la théorie de la localisation dérivée de Braun–Chuang–Lazarev [11]. Ce chapitre a donné lieu à un appendice, rédigé en collaboration avec B. Keller, à la prépublication [19] de Chen–Wang.

Le chapitre final 4 reproduit des notes prises par l'auteur pour un minicours de trois séances donné par B. Keller à l'atelier de la conférence internationale sur les représentations d'algèbres (ICRA 2020) qui s'est tenue en ligne en novembre 2020. Ce minicours portait sur les structures Calabi–Yau relatives, notion d'abord esquissée par Toën dans [93] en 2014 et développée pleinement par Brav et Dyckerhoff dans [13, 12]. Une charactéristique essentielle de leur notion de structure de Calabi–Yau relative (à gauche) est une construction de recollement analogue à celle pour les cobordismes de variétés. Wai-Kit Yeung a montré comment construire de larges classes d'exemples grâce aux complétions Calabi–Yau relatives [64, 104]. Il a aussi mis en avant l'idée que ces objets devraient être considérés comme des fibrés conormaux non commutatifs. Ce point de vue a été confirmé de façon rigoureuse par Bozec–Calaque–Scherotzke [10] qui se servent du critère de Kontsevich–Rosenberg : pour vérifier qu'une version non commutative proposée est l'analogue 'correct' d'une notion donnée pour les variétés commutatives, il suffit de vérifier que sur les variétés des représentations de dimension finie, elle induit la structure commutative voulue. Le lien entre ce chapitre et le thème de la thèse consiste en l'utilisation de l'homologie de Hochschild (et l'homologie cyclique) pour déformer les structures Calabi–Yau et en le lien avec les catégories de singularités : en effet, les structures Calabi–Yau relatives servent en particulier à la construction de catégories amassées relatives associées aux carquois glacés à potentiel, voir [102, 100], et celles-ci sont étroitement liées aux catégories des singularités.

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Introduction

This thesis contains several contributions to the study of the Hochschild homology of the canonical dg enhancements of derived categories and singularity categories of finite-dimensional algebras. It consists of four chapters.

Chapter 1 serves to introduce the main notions and notations and to sum up the state of the art. Let k be a field. Hochschild homology and cohomology are classical invariants of an associative k-algebra or, more generally, a k-category. They may be viewed as derived versions of the trace space and the center of the algebra, which we recall in section 1.2. They are invariant under Morita equivalences and also under derived equivalences, as we show in section 1.5 after a reminder on derived categories in section 1.4. These invariance results suggest that Hochschild (co)homology should be defined directly using the derived category. This is not possible in general but we show in section 1.6 that they may be obtained from the canonical differential graded enhancement of the derived category. Hochschild homology and cohomology are endowed with higher structure: the Hochschild chain complex is a mixed complex and the Hochschild cochain complex is a B_∞-algebra as we recall in section 1.7, where we also state the corresponding invariance results. In the final section 1.8 of Chapter 1, we report on recent results on Tate–Hochschild cohomology after work of Keller [58] and Hua–Keller [46]. In particular, we sketch a proof of the fact that Tate–Hochschild cohomology of an algebra is isomorphic, as an algebra, to the classical Hochschild cohomology of its differential graded singularity category. In this chapter, we largely follow B. Keller's survey 'Hochschild (co)homology and derived categories' (preprint 2020). We additionally include localization results for Hochschild homology.

Chapter 2 is devoted to the study of the Hochschild homology of the canonical dg enhancement $\mathcal{D}^b_{dg}(A)$ of the bounded derived category $\mathcal{D}^b(A) = \mathcal{D}^b(\operatorname{mod} A)$ and of the canonical dg enhancement $\mathrm{sg}_{dg}(A)$ of the singularity category

$$\mathrm{sg}(A) = \mathcal{D}^b(A)/\mathrm{per}\,(A).$$

The main result is theorem 2.2.2 which shows that Hochschild homology is computed by the *double*

Hochschild complex obtained by splicing the classical Hochschild complex with its dual using a trace map from $HH_0(A)$ to its k-dual. Our main tools are Koszul duality as described in [59] and the localization theorem 1.7.4.

In Chapter 3, we give a new proof of Chen–Wang's theorem 2.4.1 describing singularity categories of finite-dimensional algebras using dg Leavitt path algebras. The proof is based on Koszul–Moore duality worked out in Lefèvre-Hasegawa's thesis [68] (and described in [56]) and on derived localisation as developped in Braun–Chuang–Lazarev's [11]. This chapter has given rise to an appendix (co-authored with Bernhard Keller) to Chen–Wang's preprint [19].

The final Chapter 4 reproduces notes taken by the author for a series of three lectures by Bernhard Keller on relative Calabi–Yau structures given at the workshop of the International Conference on Representations of Algebras which was held online in November 2020. Relative Calabi–Yau structures serve in particular to construct relative cluster categories associated with ice quivers with potential, *cf.* [102, 100], and these are closely related to singularity categories, which establishes the link to the main theme of this thesis.

Chapter 1

Hochschild (co-)homology of dg categories

1.1 Introduction

Let k be a field. Hochschild homology and cohomology are classical invariants of an associative k-algebra or, more generally, a k-category. They may be viewed as derived versions of the trace space and the center of the algebra, which we recall in section 1.2. They are invariant under Morita equivalences and also under derived equivalences, as we show in section 1.5 after a reminder on derived categories in section 1.4. These invariance results suggest that Hochschild (co)homology should be defined directly using the derived category. This is not possible in general but we show in section 1.6 that they may be obtained from the canonical differential graded enhancement of the derived category. Hochschild homology and cohomology are endowed with higher structure: the Hochschild chain complex is a mixed complex and the Hochschild cochain complex is a B_∞-algebra as we recall in section 1.7, where we also state the corresponding invariance results. In the final section 1.8, we report on recent results on Tate–Hochschild cohomology after work of Keller [58] and Hua–Keller [46]. In particular, we sketch a proof of the fact that Tate–Hochschild cohomology of an algebra is isomorphic, as an algebra, to the classical Hochschild cohomology of its differential graded singularity category.

1.2 Center and trace space for algebras and categories

For simplicity, we work over a field k. Let A be a k-algebra, *i.e.* an associative, possibly non commutative, unital k-algebra. The *center* $Z(A)$ *of* A is formed by the elements $a \in A$ such that

$$ab = ba$$

for all $b \in A$. The *trace space* $\mathrm{Tr}(A)$ is the quotient $A/[A, A]$, where $[A, A]$ denotes the subspace of A generated by all commutators $[a, b] = ab - ba$, $a, b \in A$. Notice that the center is naturally a commutative algebra whereas the trace space is just a vector space. For example, if $n \geq 1$ and $M_n(A)$ denotes the algebra of $n \times n$-matrices with coefficients in A, we have an isomorphism

$$Z(A) \xrightarrow{\sim} Z(M_n(A))$$

taking an element a to the diagonal matrix whose diagonal coefficients all equal a, and an isomorphism

$$\mathrm{Tr}(M_n(A)) \xrightarrow{\sim} \mathrm{Tr}(A)$$

taking the class of a matrix $M = (m_{ij})$ to the class of its trace

$$\sum_i m_{ii},$$

cf. [69]. Observe that by definition, we have a short exact sequence

$$0 \longrightarrow Z(A) \longrightarrow A \longrightarrow \mathrm{Hom}_k(A, A)$$

where the second map is the inclusion and the third one takes $a \in A$ to the commutator $[a, ?]$. Somehow dually, we have a short exact sequence

$$0 \longleftarrow \mathrm{Tr}(A) \longleftarrow A \longleftarrow A \otimes_k A$$

where the third map takes $a \otimes b$ to the commutator $ab - ba$ and the second map is the natural projection. This parallelism suggests some similarity between the center and the trace space but we will now argue that they are actually very different (for non commutative algebras).

Let $f : A \to B$ be an algebra morphism (it need not preserve the unit). Clearly, we have an induced map $\mathrm{Tr}(f) : \mathrm{Tr}(A) \to \mathrm{Tr}(B)$ taking the class of an element a to the class of $f(a)$. Thus, the assignment $A \mapsto \mathrm{Tr}(A)$ becomes a functor. The deeper reason for this functoriality is the following universal property: The map $\mathrm{tr} : A \to \mathrm{Tr}(A)$ taking a to the class of a is a *trace map*, *i.e.* it takes values in a vector space, is k-linear and satisfies $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in A$, and it is universal[1] among all trace maps $t : A \to V$,

[1] The Hattori-Stallings trace [41, 86] is based on an extension of this idea, cf. [38] for a beautiful application.

i.e. we have $t = \bar{t} \circ \mathrm{tr}$ for a unique linear map $\bar{t} : \mathrm{Tr}(A) \to V$:

$$\begin{tikzcd} A \arrow[d, "\mathrm{tr}"'] \arrow[dr, "\forall t"] & \\ \mathrm{Tr}(A) \arrow[r, dashed, "\exists_1 \bar{t}"'] & V \end{tikzcd}$$

On the other hand, for an element a of the center $Z(A)$, there is no reason for $f(a)$ to lie in $Z(B)$ and there is no induced map between the centers in general. Hence the assignment $A \mapsto Z(A)$ is *not a functor*. However, we will see below how, by passing from algebras to categories, we do gain some functoriality for the center.

Recall [29] that a *k-category* is a category $\mathcal{A}$ whose morphism sets are endowed with k-vector space structures such that the compositions are bilinear. We may (and will) identify a k-algebra A with the k-category with a single object whose endomorphism algebra is A. A general k-category should be viewed as a *k-algebra with several objects* [74]. Recall [29] that a *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is a set of 'vertices', Q_1 a set of 'arrows' and s and t are maps $Q_1 \to Q_0$ taking an arrow to its source respectively its target. In particular, each (small) k-category $\mathcal{A}$ has an underlying quiver and the functor taking the category to its quiver admits a left adjoint $Q \mapsto k\text{-cat}(Q)$, where the objects of $k\text{-cat}(Q)$ are the vertices of Q and the space of morphisms $x \to y$ is the vector space of formal linear combinations of paths (=formal compositions of arrows) of length ≥ 0 from x to y.

Let $\mathcal{A}$ be a small k-category. We define the *center* $Z(\mathcal{A})$ to be the algebra of endomorphisms of the identity functor $\mathbf{1}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. Thus, an element of the center is a family φX, $X \in \mathcal{A}$, of endomorphisms $\varphi X : X \to X$ of objects of $\mathcal{A}$ such that for each morphism $f : X \to Y$, the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi X"'] & Y \arrow[d, "\varphi Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. Thus, we have the short exact sequence

$$0 \longrightarrow Z(\mathcal{A}) \longrightarrow \prod_{X \in \mathcal{A}} \mathcal{A}(X,X) \longrightarrow \prod_{X,Y \in \mathcal{A}} \mathrm{Hom}_k(\mathcal{A}(X,Y), \mathcal{A}(X,Y)) \,, \tag{1.2.0.1}$$

where the third map takes a family (φX) to the family of maps taking $f : X \to Y$ to $(\varphi Y) \circ f - f \circ (\varphi X)$. 'Dually', we define the trace space $\mathrm{Tr}(\mathcal{A})$ of $\mathcal{A}$ by the sequence

$$0 \longleftarrow \mathrm{Tr}(\mathcal{A}) \longleftarrow \bigoplus_{X \in \mathcal{A}} \mathcal{A}(X,X) \longleftarrow \bigoplus_{X,Y \in \mathcal{A}} \mathcal{A}(Y,X) \otimes \mathcal{A}(X,Y) \quad , \tag{1.2.0.2}$$

where the third map takes $f \otimes g$ to $f \circ g - g \circ f$. Notice that the middle terms of the sequences are isomorphic if and only if $\mathcal{A}$ has only finitely many non zero objects.

We may ask whether we can reduce the computation of the center and the trace space of the k-category $\mathcal{A}$ to that of the center and trace space for some k-algebra A. The natural candidate is the *matrix algebra*

defined by

$$k[\mathcal{A}] = \bigoplus_{X,Y \in \mathcal{A}} \mathcal{A}(X,Y)$$

and endowed with matrix multiplication. Notice that it is associative but unital only if $\mathcal{A}$ has only finitely many non zero objects. In general, it is still locally unital, i.e. for each finite set of elements a_i, $i \in I$, we may find an idempotent e such that $ea_i = a_i = a_i e$ for all i, cf. [2] and [76].

Lemma 1.2.1. *a) We have a canonical isomorphism* $\mathrm{Tr}(k[\mathcal{A}]) \xrightarrow{\sim} \mathrm{Tr}(\mathcal{A})$.

b) We have a natural injective algebra morphism $Z(k[\mathcal{A}]) \to Z(\mathcal{A})$ *which is an isomorphism iff* $\mathcal{A}$ *has only finitely many non zero objects.*

The proof is left to the reader as an exercise. Of course, the assignment $\mathcal{A} \mapsto \mathrm{Tr}(\mathcal{A})$ is functorial with respect to k-linear functors $F : \mathcal{A} \to \mathcal{B}$. We also have some functoriality for the assignment $\mathcal{A} \mapsto Z(\mathcal{A})$. Indeed, if $\mathcal{B} \subset \mathcal{A}$ is a full subcategory, then the restriction map

$$(\varphi X)_{X \in \mathcal{A}} \mapsto (\varphi X)_{X \in \mathcal{B}}$$

is an algebra morphism $Z(\mathcal{A}) \to Z(\mathcal{B})$. We see that $\mathcal{A} \mapsto Z(\mathcal{A})$ is *contravariant with respect to fully faithful embeddings.*

Let A be a k-algebra and $\mathrm{Mod}\, A$ the k-category of all right A-modules (by choosing suitable universes, we can dispense with the set-theoretical problem that $\mathrm{Mod}\, A$ is not small). We have a fully faithful embedding $A \subset \mathrm{Mod}\, A$ taking the unique object to the free A-module of rank one.

Lemma 1.2.2. *The restriction along* $A \subset \mathrm{Mod}\, A$ *is an isomorphism* $Z(\mathrm{Mod}\, A) \xrightarrow{\sim} Z(A)$.

Proof. Let ρ denote the restriction. We define $\sigma : Z(A) \to Z(\mathrm{Mod}\, A)$ by sending an element $z \in Z(A)$ to the family $\sigma(z)$ whose component at a module M is right multiplication by z. Since z is central, the map $\sigma(z)M$ is indeed an endomorphism of M and since any $f : L \to M$ is A-linear, the family $\sigma(z)$ does lie in $Z(\mathrm{Mod}\, A)$. It is also clear that $\rho(\sigma(z)) = z$ for each $z \in Z(A)$. Thus, the map $\rho : Z(\mathrm{Mod}\, A) \to Z(A)$ is injective. Suppose that φ is in its kernel. Then $\varphi A = 0$ and in fact $\varphi F = 0$ for each free A-module. If M is an arbitrary A-module, we have an exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_1 and F_0 are free. The fact that φF_1 and φF_0 vanish then immediately implies that φM vanishes. Thus, the map ρ is also injective. $\square$

Recall that a *generator* (resp. *cogenerator*) of $\operatorname{Mod} A$ is a module G such that each module is a quotient of a coproduct of copies of G (resp. a submodule of a product of copies of G). Let $\operatorname{proj} A$ denote the category of finitely generated projective modules, $\operatorname{Proj} A$ the category of all projective modules and $\operatorname{Inj} A$ the category of all injective modules. It is easy to adapt the above proof to obtain the

Lemma 1.2.3. *Let $\mathcal{B} \subset \operatorname{Mod} A$ be a full subcategory containing a generator or a cogenerator of $\operatorname{Mod} A$. Then restriction to $\mathcal{B}$ is an isomorphism $Z(\operatorname{Mod} A) \xrightarrow{\sim} Z(\mathcal{B})$. In particular, we have isomorphisms*

$$Z(A) \xleftarrow{\sim} Z(\operatorname{proj} A) \xleftarrow{\sim} Z(\operatorname{Proj} A) \xleftarrow{\sim} Z(\operatorname{Mod} A) \xrightarrow{\sim} Z(\operatorname{Inj} A).$$

The following lemma shows that the trace space behaves entirely differently. The proof is based on the 'Eilenberg swindle'.

Lemma 1.2.4. *We have $\operatorname{Tr}(\operatorname{Mod} A) = 0$. More generally, we have $\operatorname{Tr}(\mathcal{A}) = 0$ for any k-category $\mathcal{A}$ admitting countable coproducts.*

Proof. We sketch a proof. The details may be filled in using [60]. Let $\mathcal{A}$ and $\mathcal{B}$ be k-categories and $F, G : \mathcal{A} \to \mathcal{B}$ k-linear functors. In a first step, one shows that if F and G are isomorphic functors, we have an equality $\operatorname{Tr}(F) = \operatorname{Tr}(G)$ between the induced maps $\operatorname{Tr}(\mathcal{A}) \to \operatorname{Tr}(\mathcal{B})$. Now assume that $\mathcal{B}$ has finite direct sums. Then one shows that $\operatorname{Tr}(F \oplus G) = \operatorname{Tr}(F) + \operatorname{Tr}(G)$. Now assume that $\mathcal{A}$ has countable coproducts. For $X \in \mathcal{A}$, let $X^{(\mathbb{N})}$ be the coproduct of a countable set of copies of X. Then we can choose a functorial isomorphism

$$X \oplus X^{(\mathbb{N})} \xrightarrow{\sim} X^{(\mathbb{N})}.$$

Thus, if $F : \mathcal{A} \to \mathcal{A}$ is the functor $X \mapsto X^{(\mathbb{N})}$, then we have $\mathbf{1}_{\mathcal{A}} \oplus F \xrightarrow{\sim} F$ as functors $\mathcal{A} \to \mathcal{A}$. It follows from the two previous steps that we have

$$\operatorname{Tr}(\mathbf{1}_{\mathcal{A}}) + \operatorname{Tr}(F) = \operatorname{Tr}(\mathbf{1}_{\mathcal{A}} \oplus F) = \operatorname{Tr}(F)$$

and therefore

$$\mathbf{1}_{\operatorname{Tr}(\mathcal{A})} = \operatorname{Tr}(\mathbf{1}_{\mathcal{A}}) = 0.$$

□

On the other hand, if we restrict to finite sums (and summands), this phenomenon does not occur.

Lemma 1.2.5 ([60])**.** *The inclusion $A \subset \operatorname{proj} A$ induces an isomorphism $\operatorname{Tr}(A) \xrightarrow{\sim} \operatorname{Tr}(\operatorname{proj} A)$.*

1.3 Hochschild (co)homology of algebras

Let k be a field. We write $\otimes$ for $\otimes_k$. Let A be a k-algebra. The *Hochschild chain complex of* A is the complex C_*A concentrated in homological degrees ≥ 0

$$A \longleftarrow A \otimes A \longleftarrow \dots \longleftarrow A^{\otimes p} \longleftarrow A^{\otimes(p+1)} \longleftarrow \dots$$

with $C_pA = A^{\otimes(p+1)}$, $p \geq 0$, and differential given by

$$d(a_0, \dots, a_p) = \sum_{i=0}^{p-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_p) + (-1)^p (a_p a_0, \dots, a_{p-1}) \,, \tag{1.3.0.1}$$

where we write $(a_0, \dots, a_p)$ for $a_0 \otimes \cdots \otimes a_p$. Notice that the first differential takes $a \otimes b$ to the commutator $ab - ba$. The *Hochschild homology* $HH_*(A)$ is the graded space with $HH_p(A) = H_p(C_*A)$.

The *Hochschild [44] cochain complex of* A is the complex C^*A concentrated in cohomological degrees ≥ 0

$$A \longrightarrow \mathrm{Hom}_k(A, A) \longrightarrow \mathrm{Hom}_k(A \otimes A, A) \longrightarrow \dots \longrightarrow \mathrm{Hom}_k(A^{\otimes p}, A) \longrightarrow \dots$$

whose differential is given by

$$(df)(a_0, \dots, a_p) = a_0 f(a_1, \dots, a_p) - \sum_{i=0}^{p-1} (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_p) + (-1)^p f(a_0, \dots, a_{p-1}) a_p. \tag{1.3.0.2}$$

Notice that the first two differentials are given by

$$a \mapsto [a, ?] \quad \text{and} \quad f \mapsto (a \otimes b \mapsto f(a)b - f(ab) + af(b)).$$

We see that that in degree 0 we recover the trace space $\mathrm{Tr}(A) = HH_0(A)$ respectively the center $Z(A) = HH^0(A)$. We also see that $HH^1(A)$ is equal to the Lie algebra of outer derivations of A (with the bracket induced by the commutator of derivations). Both structures, the commutative multiplication and the Lie bracket, extend to the whole of Hochschild cohomology, as we will see in section 1.7.

Let $A^e = A^{op} \otimes A$. We identify the category $\mathrm{Mod}\,(A^e)$ of right A^e-modules with the (isomorphic) category of A-A-bimodules via the rule

$$amb = m(b \otimes a).$$

In particular, we have the *identity bimodule* ${}_AA_A$ given by the algebra A considered as a bimodule over itself.

Theorem 1.3.1 (Cartan–Eilenberg [16])**.** *We have canonical isomorphisms*

$$\mathrm{Ext}^*_{A^e}(A, A) \xrightarrow{\sim} HH^*(A) \quad \textit{and} \quad \mathrm{Tor}^*_{A^e}(A, A) \xrightarrow{\sim} HH_*(A).$$

To prove the theorem, one computes the derived functors using the bar resolution of the first argument: Recall that the (augmented) bar resolution is the complex of bimodules

$$ {}_AA_A \longleftarrow A \otimes A \longleftarrow \dots \longleftarrow A^{\otimes p} \longleftarrow A^{\otimes (p+1)} \longleftarrow \dots $$

where the augmentation is the multiplication of A and the differential is given by

$$ d(a_0, \dots, a_{p+1}) = \sum_{i=0}^{p} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_{p+1}). $$

Corollary 1.3.2. *Hochschild cohomology $HH^*(A)$ carries a natural graded multiplication, the* cup product*, extending that of $Z(A) = HH^0(A)$.*

It is shown in [16] that the cup product is induced by the following associative multiplication on cochains: For $f \in C^p A$ and $g \in C^q A$, define

$$ (f \cup g)(a_1, \dots, a_p, a_{p+1}, \dots, a_{p+q}) = (-1)^{pq} f(a_1, \dots, a_p) g(a_{p+1}, \dots, a_{p+q}). $$

1.4 Reminder on derived categories

We collect basic definitions and results on derived categories. We refer to [62] for more details and references. As before, k is a field and A a k-algebra. The category $\mathcal{C}A = \mathcal{C}\mathrm{Mod}\, A$ has as objects the cochain complexes

$$ \dots \longrightarrow M^p \xrightarrow{d_M} M^{p+1} \longrightarrow \dots $$

of (right) A-modules. Notice that each such complex has an underlying $\mathbb{Z}$-graded A-module given by the sequence of the M^p, $p \in \mathbb{Z}$. It is endowed with the differential d_M, which is a homogeneous endomorphism of degree $+1$. The *morphisms* of $\mathcal{C}A$ are the morphisms $f : L \to M$ of graded A-modules which are homogeneous of degree 0 and satisfy $d_M \circ f = f \circ d_L$. The *suspension functor* $\Sigma : \mathcal{C}A \to \mathcal{C}A$ takes a complex M to ΣM with components $(\Sigma M)^p = M^{p+1}$ and differential $d_{\Sigma M} = -d_M$. Two morphisms f and $g : L \to M$ of $\mathcal{C}A$ are *homotopic* if there is a homogeneous morphism of graded A-modules $h : L \to M$ of degree -1 such that

$$ f - g = d_M \circ h + h \circ d_L. $$

The *homotopy category* $\mathcal{H}A$ has the same objects as $\mathcal{C}A$; its morphisms are homotopy classes of morphisms of $\mathcal{C}A$. A morphism $s : L \to M$ of $\mathcal{C}A$ or $\mathcal{H}A$ is a *quasi-isomorphism* if the induced morphism in homology $H^*(s) : H^*(L) \to H^*(M)$ is invertible. The *derived category* is the localization of $\mathcal{C}A$ (or $\mathcal{H}A$) at the class of all quasi-isomorphisms. Thus, it has the same objects as $\mathcal{C}A$ and its morphisms are equivalence classes of formal compositions of morphisms of $\mathcal{C}A$ (or $\mathcal{H}A$) and formal inverses of quasi-isomorphisms.

The homotopy category $\mathcal{H}A$ is canonically triangulated with suspension functor Σ. Each componentwise split short exact sequence of $\mathcal{C}A$

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

yields a canonical triangle

$$L \longrightarrow M \longrightarrow N \longrightarrow \Sigma L\ .$$

The derived category $\mathcal{D}A$ is triangulated with suspension functor Σ. Each short exact sequence of $\mathcal{C}A$ yields a canonical triangle.

We identify A-modules $M \in \operatorname{Mod} A$ with complexes concentrated in degree 0

$$\ldots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \ldots .$$

Then, for A-modules L and M, we have canonical isomorphisms

$$\operatorname{Ext}^p_A(L, M) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}A}(L, \Sigma^p M)$$

for all $p \in \mathbb{Z}$ (with the convention that Ext^p vanishes for $p < 0$). Moreover, the composition in $\mathcal{D}A$ identifies with the Yoneda product on Ext.

Theorem 1.4.1. *The projection $\mathcal{H}A \to \mathcal{D}A$ admits a fully faithful left adjoint* $\mathbf{p}$ *and a fully faithful right adjoint* $\mathbf{i}$.

Notice that the analogous statement for the category of complexes $\mathcal{C}A$ instead of the homotopy category $\mathcal{H}A$ is wrong. This is one of the main reasons for introducing $\mathcal{H}A$. The functors $\mathbf{p}$ and $\mathbf{i}$ generalize projective respectively injective resolutions. Indeed, if M is an A-module and $P \to M$ a projective resolution (*i.e.* a quasi-isomorphism where P is right bounded with projective components), then we have $P \xrightarrow{\sim} \mathbf{p}M$ in $\mathcal{H}A$. Similarly, if $M \to I$ is an injective resolution, then $\mathbf{i}M \xrightarrow{\sim} I$ in $\mathcal{H}A$.

Now let B be another algebra and X a complex of A-B-bimodules. For $M \in \mathcal{C}A$, define the complex $M \otimes_A X \in \mathcal{C}B$ by

$$(M \otimes_A X)^n = \bigoplus_{p+q=n} M^p \otimes_A X^q \quad \text{and} \quad d(m \otimes a) = (dm) \otimes a + (-1)^p m \otimes (dx).$$

For $L \in \mathcal{C}B$, define $\operatorname{Hom}_B(X, L) \in \mathcal{C}A$ as the complex whose nth component is formed by the morphisms $f : X \to L$ of graded B-modules, homogeneous of degree n and whose differential is given by

$$d(f) = d_L \circ f - (-1)^n f \circ d_X.$$

Define objects of the derived categories

$$L \overset{L}{\otimes}_A X = (\mathbf{p}L) \otimes_A X \quad \text{and} \quad \operatorname{RHom}_B(X, L) = \operatorname{Hom}_B(X, \mathbf{i}L).$$

Example 1.4.2. *For example, we have*

$$HH_*(A) = \mathrm{Tor}_*^{A^e}(A, A) = H_*(A \overset{L}{\otimes}_{A^e} A) \quad \textit{and} \quad HH^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = H^*(\mathrm{RHom}_{A^e}(A, A)).$$

The following lemma is easy to prove.

Lemma 1.4.3. *We have an adjoint pair*

$$? \overset{L}{\otimes}_A X : \mathcal{D}(A) \rightleftarrows \mathcal{D}(B) : \mathrm{RHom}_B(X, ?)$$

It is natural to ask when these adjoints are equivalences. To answer this question, let us observe that $\mathcal{D}A$ has arbitrary coproducts, which are given by coproducts of complexes. An object P of $\mathcal{D}A$ is *compact* if the functor $\mathrm{Hom}(P, ?) : \mathcal{D}A \to \mathrm{Mod}\, k$ commutes with arbitrary coproducts. It is *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projective modules. For example, the free A-module A_A is compact because

$$\mathrm{Hom}(A, M) \xrightarrow{\sim} H^0 M$$

and of course it is also perfect.

Proposition 1.4.4. *An object of $\mathcal{D}A$ is compact if and only if it is perfect.*

The *perfect derived category* $\mathrm{per}\,(A)$ is the full subcategory of $\mathcal{D}A$ formed by the perfect objects. Clearly it is a *thick subcategory*, *i.e.* a triangulated subcategory stable under taking direct summands.

Proposition 1.4.5. *The functor $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ is an equivalence if and only if*

a) X_B is perfect in $\mathcal{D}B$ and

b) X_B generates $\mathcal{D}B$ as a triangulated category with arbitrary coproducts and

c) the natural map $A \to \mathrm{Hom}_{\mathcal{D}B}(X_B, X_B)$ given by left multiplication is an isomorphism and $\mathrm{Hom}_{\mathcal{D}B}(X_B, \Sigma^p X_B) = 0$ for all $p \neq 0$.

By definition, the bimodule complex X is a *two-sided tilting complex* if these conditions hold. We have the following important class of examples:

Theorem 1.4.6 (Happel [39])**.** *If T is an A-B-bimodule, then T is a two-sided tilting complex iff T_B is a tilting module and the left action yields an isomorphism $A \xrightarrow{\sim} \mathrm{End}_B(T)$.*

For a proof of the theorem in this form, *cf.* [62].

Theorem 1.4.7 (Rickard [84, 85])**.** *There is a triangle equivalence $\mathcal{D}A \to \mathcal{D}B$ if and only if there is a two-sided tilting complex X.*

Notice that the theorem does not claim that a given triangle equivalence $F : \mathcal{D}A \xrightarrow{\sim} \mathcal{D}B$ is isomorphic to a derived functor $? \overset{L}{\otimes}_A X$ for a two-sided tilting complex X. It is open whether this always holds.

Define $\operatorname{rep}(A, B)$ to be the full subcategory of $\mathcal{D}(A^{op} \otimes B)$ formed by the bimodule complexes X such that X_B is perfect. We think of the objects of $\operatorname{rep}(A, B)$ as 'representations up to homotopy' of A in $\operatorname{per}(B)$. Notice that a bimodule complex X belongs to $\operatorname{rep}(A, B)$ if and only if the functor $? \overset{L}{\otimes}_A X$ takes $\operatorname{per}(A)$ to $\operatorname{per}(B)$. For $X \in \operatorname{rep}(A, B)$, put

$$X^\vee = \operatorname{RHom}_B(X, B)$$

and notice that this is naturally an object of $\mathcal{D}(B^{op} \otimes A)$.

Lemma 1.4.8. *We have a canonical isomorphism*

$$? \overset{L}{\otimes}_B X^\vee \xrightarrow{\sim} \operatorname{RHom}_B(X, ?).$$

Thus, the functor $? \overset{L}{\otimes}_B X^\vee$ is right adjoint to $? \overset{L}{\otimes}_A X$. The adjunction morphisms are produced by the *action morphism*

$$\alpha : A \to \operatorname{RHom}_B(X, X) \xleftarrow{\sim} X \overset{L}{\otimes}_B X^\vee \text{ in } \mathcal{D}(A^e)$$

and the *evaluation morphism*

$$\varepsilon : X^\vee \overset{L}{\otimes}_A X = \operatorname{RHom}_B(X, B) \overset{L}{\otimes}_A X \to B \text{ in } \mathcal{D}(B^e).$$

Notice that $? \overset{L}{\otimes}_A X$ is fully faithful if and only if the action morphism $A \to X \overset{L}{\otimes}_B X^\vee$ is invertible.

1.5 Invariance and localization theorems

Let A and B be k-algebras. Let $X \in \operatorname{rep}(A, B)$ and

$$X^\vee = \operatorname{RHom}_B(X, B) \in \mathcal{D}(B^{op} \otimes A).$$

Note that in general, $X^\vee$ is *not perfect over* A and so does not belong to $\operatorname{rep}(B, A)$. Recall the canonical action and evaluation morphisms

$$\alpha : A \to X \overset{L}{\otimes}_B X^\vee \text{ in } \mathcal{D}(A^e) \quad \text{and} \quad \varepsilon : X^\vee \overset{L}{\otimes}_A X \to B \text{ in } \mathcal{D}(B^e).$$

Theorem 1.5.1. *We have a canonical morphism*

$$HH_*(X) : HH_*(A) \to HH_*(B)$$

It is an isomorphism if $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ *is an equivalence.*

Sketch of proof. Recall that $HH_*(A) = H_*(A \overset{L}{\otimes}_{A^e} A)$. We have a canonical morphism $\psi(X)$ defined as the composition

$$\begin{tikzcd}
A \overset{L}{\otimes}_{A^e} A \arrow[r, "\mathbf{1}\otimes\alpha"] & A \overset{L}{\otimes}_{A^e} (X \overset{L}{\otimes}_B X^\vee) \arrow[r, equal] & A \overset{L}{\otimes}_{A^e} (X \otimes_k X^\vee) \otimes_{B^e} B \arrow[d, "\text{flip}"] \\
B \overset{L}{\otimes}_{B^e} B & B \overset{L}{\otimes}_{B^e} (X^\vee \overset{L}{\otimes}_A X) \arrow[l, "\mathbf{1}\otimes\varepsilon"'] \arrow[r, equal] & B \overset{L}{\otimes}_{B^e} (X^\vee \otimes_k X) \overset{L}{\otimes}_{A^e} A
\end{tikzcd}$$

Then $HH_*(X)$ is $H_*(\psi(X))$. One shows that $\psi(A) = \mathbf{1}$ and for $X \in \operatorname{rep}(A, B)$ and $Y \in \operatorname{rep}(B, C)$, we have $\psi(X \overset{L}{\otimes}_B Y) = \psi(Y) \circ \psi(X)$, *cf.* Theorem 2.1 in [5]. This implies the second claim. □

Now let A, B and C be k-algebras and suppose that we are given bimodules $X \in \mathcal{D}(A^{op} \otimes B)$ and $Y \in \mathcal{D}(B^{op} \otimes C)$ such that the sequence

$$0 \longrightarrow \mathcal{D}A \xrightarrow{?\overset{L}{\otimes}_A X} \mathcal{D}B \xrightarrow{?\overset{L}{\otimes}_B Y} \mathcal{D}C \longrightarrow 0$$

is an exact sequence of triangulated categories, *i.e.* the functor $? \overset{L}{\otimes}_A X$ is fully faithful and the functor $? \overset{L}{\otimes}_B Y$ induces an equivalence from the Verdier quotient of $\mathcal{D}B$ by its image onto $\mathcal{D}C$.

Theorem 1.5.2 ([60])**.** *There is a canonical long exact sequence*

$$\ldots \longrightarrow HH_n(A) \xrightarrow{HH_n(X)} HH_n(B) \xrightarrow{HH_n(Y)} HH_n(C) \longrightarrow HH_{n-1}(A) \longrightarrow \ldots$$

Now recall that a fully faithful functor $F : \mathcal{A} \to \mathcal{B}$ between k-categories yields a restriction morphism $F^* : Z(\mathcal{B}) \to Z(\mathcal{A})$. Since Hochschild cohomology may be viewed as a 'derived center', the following theorem is quite natural.

Theorem 1.5.3. *Suppose we have* $X \in \operatorname{rep}(A, B)$ *such that the functor* $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ *is fully faithful. Then we have a canonical 'restriction' morphism*

$$HH^*(X) : HH^*(B) \to HH^*(A).$$

It is an isomorphism if $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ *is an equivalence.*

Below, we will show using other methods that even if X_B is not perfect but $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ is fully faithful, we still have such a restriction morphism.

Sketch of proof. It is not hard to show, without any hypothesis on $X \in \operatorname{rep}(A, B)$, that we have an adjoint pair

$$X^\vee \overset{L}{\otimes}_A ? \overset{L}{\otimes}_A X : \mathcal{D}(A^e) \rightleftarrows \mathcal{D}(B^e) : X \overset{L}{\otimes}_B ? \overset{L}{\otimes}_B X^\vee$$

Now we construct a map from

$$HH^p(B) = \operatorname{Hom}_{\mathcal{D}B^e}(B, \Sigma^p B)$$

to $HH^p(A)$ as follows:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{D}(B^e)}(B, \Sigma^p B) \arrow[r] & \operatorname{Hom}_{\mathcal{D}(B^e)}(X^\vee \overset{L}{\otimes}_A X, \Sigma^p B) \arrow[d, "\mathrm{adj}"] \\
\operatorname{Hom}_{\mathcal{D}(A^e)}(A, \Sigma^p A) \arrow[r, "\sim"] & \operatorname{Hom}_{\mathcal{D}(A^e)}(A, \Sigma^p X \overset{L}{\otimes}_B X^\vee)
\end{tikzcd}$$

□

Each morphism $f : A \to \Sigma^p A$ of $\mathcal{D}A^e$ induces a morphism

$$f \overset{L}{\otimes}_{A^e} \mathbf{1}_A : A \overset{L}{\otimes}_{A^e} A \to \Sigma^p A \overset{L}{\otimes}_{A^e} A.$$

In this way, we obtain an action

$$\cap : HH^*(A) \otimes HH_*(A) \to HH_*(A)$$

called the cap product (not to be confused with the cup product $\cup$ on Hochschild cohomology).

Theorem 1.5.4 ([4])**.** *Suppose that $X \in \operatorname{rep}(A, B)$ is such that the functor $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ is an equivalence. Then the induced isomorphisms*

$$HH_*(X) : HH_*(A) \xrightarrow{\sim} HH_*(B) \quad \textit{and} \quad HH^*(X) : HH^*(A) \xrightarrow{\sim} HH^*(B)$$

are compatible with the cap product.

1.6 Differential graded categories

Recall from Lemma 1.2.2, that for an algebra A, the restriction along the inclusion $A \subset \operatorname{Mod} A$ is an isomorphism

$$Z(\operatorname{Mod} A) \xrightarrow{\sim} Z(A).$$

It is natural to ask what the derived version of this fact is. In the derived version, Hochschild cohomology should replace the center and the derived category should replace the module category. So we would

like to know how to recover Hochschild cohomology from the derived category $\mathcal{D}A$. Unfortunately, this seems to be an ill-posed question nobody knows how to answer. However, it is easy to recover Hochschild cohomology from the differential graded (=dg) version of $\mathcal{D}A$, namely the dg category $\mathcal{D}_{dg}A$. Our first aim in this section is to define the dg category $\mathcal{D}_{dg}A$ in an intrinsic way, via a universal property.

1.6.1 Dg categories and their derived categories

Recall that the category of complexes $\mathcal{C}k$ is *monoidal*, *i.e.* it is endowed with the bifunctor $(L, M) \mapsto L \otimes M$ given by

$$(L \otimes M)^n = \bigoplus_{p+q=n} L^p \otimes M^q, \quad d(l \otimes m) = (dl) \otimes m + (-1)^{|l|} l \otimes dm$$

enjoying a number of desirable properties. A *dg category* is a category $\mathcal{A}$ enriched in the monoidal category of complexes. Thus, the morphism spaces $\mathcal{A}(X, Y)$ are complexes and the compositions

$$\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)$$

and units $k \to \mathcal{A}(X, X)$ are morphisms of complexes.

For example, if B is an algebra, the dg category $\mathcal{C}_{dg}B$ has the same objects as $\mathcal{C}B$ and its morphism complexes are defined by

$$(\mathcal{C}_{dg}B)(L, M) = \mathrm{Hom}_B(L, M)\,,$$

cf. section 1.4. As in the case of k-categories, we identify dg categories with one object with dg algebras.

If $\mathcal{A}$ is a dg category, the *category* $H^0\mathcal{A}$ has the same objects as $\mathcal{A}$ and the morphism spaces $H^0(\mathcal{A}(X, Y))$ with the natural compositions. For example, this yields another viewpoint on the homotopy category $\mathcal{H}B$ via the equality of categories

$$H^0(\mathcal{C}_{dg}B) = \mathcal{H}B.$$

A *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor such that

$$F : \mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)$$

is a morphism of complexes for all $X, Y \in \mathcal{A}$. It is a *quasi-equivalence* if $F : \mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)$ is a quasi-isomorphism for all $X, Y \in \mathcal{A}$ and the induced functor $H^0F : H^0\mathcal{A} \to H^0\mathcal{B}$ is an equivalence. The *category* Hqe is the localization of the category dgcat of small dg categories at the class of all quasi-equivalences. For example, if $f : A \to B$ is a quasi-isomorphism between dg algebras, it may be viewed as a quasi-equivalence between dg categories with one object.

Theorem 1.6.2 (Tabuada [90]). *The category* dgcat *carries a (cofibrantly generated) Quillen model structure whose weak equivalences are the quasi-equivalences. In particular, the morphisms $\mathcal{A} \to \mathcal{B}$ in* Hqe *form a* set *for all small dg categories $\mathcal{A}$, $\mathcal{B}$.*

Thanks to the theorem, we can speak about representable functors on the category Hqe without having to 'enlarge the universe'.

Theorem 1.6.3 ([61]). *Let $\mathcal{A}$ be a dg category and $\mathcal{N} \subseteq \mathcal{A}$ a full dg subcategory. Then there is a morphism $Q : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ of* Hqe *which kills $\mathcal{N}$ (*i.e. *we have $\mathbf{1}_{QN} = 0$ in $H^0(\mathcal{A}/\mathcal{N})$ for all $N \in \mathcal{N}$) and which is universal in* Hqe *among the morphisms killing $\mathcal{N}$.*

We define the *dg quotient of $\mathcal{A}$ by $\mathcal{N}$* to be the dg category $\mathcal{A}/\mathcal{N}$ of the theorem. It is thus unique up to unique isomorphism in Hqe.

Theorem 1.6.4 (Drinfeld 2004 [25]). *The dg category $\mathcal{A}/\mathcal{N}$ is obtained from $\mathcal{A}$ by adjoining a contracting homotopy h_N for each object N on $\mathcal{N}$ (*i.e. *h_N is of degree -1 and $d(h_N) = \mathbf{1}_N$).*

For an algebra B, we define the *dg derived category of B* to be the dg quotient

$$\mathcal{D}_{dg}B = \mathcal{C}_{dg}B/\mathcal{A}c_{dg}B\,,$$

where $\mathcal{A}c_{dg}B$ is the full dg subcategory of $\mathcal{C}_{dg}B$ whose objects are the acyclic complexes.

Theorem 1.6.5 ([61, 25]). *We have a canonical equivalence $H^0(\mathcal{D}_{dg}B) \xrightarrow{\sim} \mathcal{D}B$.*

Our next aim is to define the derived category $\mathcal{D}\mathcal{A}$ (as well as its dg version $\mathcal{D}_{dg}\mathcal{A}$) of a dg category $\mathcal{A}$. Define the *opposite dg category $\mathcal{A}^{op}$* to be the dg category with the same objects, with the morphism complexes $\mathcal{A}^{op}(X,Y) = \mathcal{A}(Y,X)$ and the compositions given by

$$f \circ_{\mathcal{A}^{op}} g = (-1)^{|f||g|} g \circ f$$

for all homogeneous $f \in \mathcal{A}^{op}(Y,Z)$ and $g \in \mathcal{A}^{op}(X,Y)$. For two dg functors $F, G : \mathcal{A} \to \mathcal{B}$, define the complex $\mathrm{Hom}(F,G)$ to be the subcomplex of

$$\prod_{X \in \mathcal{A}} \mathcal{B}(FX, GX)$$

formed by the families (φX) such that

$$(Gf) \circ (\varphi X) = (-1)^{|\varphi||f|} (\varphi Y) \circ (Ff)$$

for all $X, Y \in \mathcal{A}$ and $f : X \to Y$. In this way, the category $\mathrm{Fun}_{dg}(\mathcal{A}, \mathcal{B})$ of dg functors from $\mathcal{A}$ to $\mathcal{B}$ becomes a dg category. We define the dg category of dg right $\mathcal{A}$-modules to be

$$\mathcal{C}_{dg}\mathcal{A} = \mathrm{Fun}_{dg}(\mathcal{A}^{op}, \mathcal{C}_{dg}k)$$

and the category of right $\mathcal{A}$-modules to be $Z^0\mathcal{C}_{dg}\mathcal{A}$ (same objects as $\mathcal{C}_{dg}\mathcal{A}$ and morphism spaces $Z^0(\mathcal{C}_{dg}\mathcal{A})(L, M)$). We define the *homotopy category of dg $\mathcal{A}$-modules* as

$$\mathcal{H}\mathcal{A} = H^0\mathcal{C}_{dg}\mathcal{A}.$$

For example, for each object $X \in \mathcal{A}$, we have the *representable dg module*

$$X^\wedge = \mathcal{A}(?, X) : \mathcal{A}^{op} \to \mathcal{C}_{dg}k.$$

Whence the *dg Yoneda functor*

$$\mathcal{A} \to \mathcal{C}_{dg}\mathcal{A}\ ,\ X \mapsto X^\wedge.$$

As an exercise, the reader may want to prove the dg Yoneda lemma:

Lemma 1.6.6. *For X in $\mathcal{A}$ and M in $\mathcal{C}\mathcal{A}$, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(X^\wedge, M) \xrightarrow{\sim} MX\ ,\ f \mapsto f(\mathbf{1}_X).$$

Notice that when B is an ordinary algebra and $\mathcal{A}$ the dg category whose endomorphism algebra is B (concentrated in degree 0), then $\mathcal{C}_{dg}\mathcal{A} = \mathcal{C}_{dg}B$

Let $\mathcal{A}$ be a small dg category. A morphism $s : L \to M$ of dg $\mathcal{A}$-modules is a *quasi-isomorphism* if

$$sX : LX \to MX$$

is a quasi-isomorphism for each $X \in \mathcal{A}$. We define the *derived category* $\mathcal{D}\mathcal{A}$ to be the localization of $\mathcal{C}\mathcal{A}$ (respectively $\mathcal{H}\mathcal{A}$) at the class of quasi-isomorphisms and the *dg derived category* $\mathcal{D}_{dg}\mathcal{A}$ to be the dg quotient

$$\mathcal{C}_{dg}(\mathcal{A})/\mathcal{A}c_{dg}(\mathcal{A}).$$

As in Theorem 1.4.1, the quotient functor $\mathcal{H}\mathcal{A} \to \mathcal{D}\mathcal{A}$ admits a left adjoint $\mathbf{p}$ and a right adjoint $\mathbf{i}$, *cf.* [59], and the construction of the derived functors generalizes.

Let us give an example where we have a beautiful description of the derived category of a non trivial dg category: Let A be a right noetherian algebra (concentrated in degree 0) and $\mathrm{mod}\, A$ the abelian category of finitely generated (right) A-modules. Let $\mathcal{C}^b_{dg}(\mathrm{mod}\, A) \subset \mathcal{C}_{dg}A$ be the full dg subcategory of bounded complexes over $\mathrm{mod}\, A$ and

$$\mathcal{D}^b_{dg}(\mathrm{mod}\, A) = \mathcal{C}^b_{dg}(\mathrm{mod}\, A)/\mathcal{A}c^b_{dg}(\mathrm{mod}\, A).$$

Let $\operatorname{Inj} A$ denote the category of all injective modules and $\mathcal{H}\operatorname{Inj} A$ the homotopy category of (unbounded) complexes of injective modules.

Theorem 1.6.7 (Krause [66])**.** *We have a canonical triangle equivalence*

$$\mathcal{H}\operatorname{Inj} A \xrightarrow{\sim} \mathcal{D}(\mathcal{D}^b_{dg}(\operatorname{mod} A)).$$

1.6.8 Hochschild (co)homology of dg categories

Let $\mathcal{A}$ be a small dg category. We have the following generalization of Proposition 1.4.4. Recall that a thick subcategory of a triangulated category is a full triangulated subcategory stable under taking direct factors.

Proposition 1.6.9. *An object $P \in \mathcal{D}\mathcal{A}$ is compact if and only if it is* perfect*, i.e. contained in the thick subcategory generated by the representable modules $X^\wedge$, $X \in \mathcal{A}$.*

Let $\mathcal{B}$ be another dg category. The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ is the dg category whose objects are the pairs (X, Y), $X \in \mathcal{A}$, $Y \in \mathcal{B}$, and whose morphisms are given by

$$(\mathcal{A} \otimes \mathcal{B})((X, Y), (X', Y')) = \mathcal{A}(X, X') \otimes \mathcal{B}(Y, Y').$$

We define $\operatorname{rep}(\mathcal{A}, \mathcal{B})$ as the full subcategory of $\mathcal{D}(\mathcal{B} \otimes \mathcal{A}^{op})$ formed by the dg bimodules X whose restriction to $\mathcal{B}$ is perfect.

Let $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op}$. The *identity bimodule* $I_{\mathcal{A}}$ sends (X, Y) to $\mathcal{A}(X, Y)$, $X, Y \in \mathcal{A}$. We put

$$HH^*(\mathcal{A}) = H^* \operatorname{RHom}_{\mathcal{A}^e}(I_{\mathcal{A}}, I_{\mathcal{A}}) \quad \text{and} \quad HH_*(\mathcal{A}) = H_*(I_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}^e} I_{\mathcal{A}}).$$

These may also be computed as the (co)homologies of the complexes $C_*\mathcal{A}$ and $C^*\mathcal{A}$ constructed as follows: The complex $C_*\mathcal{A}$ is the sum total complex of the bicomplex

$$\ldots \longrightarrow \bigoplus \mathcal{A}(X_1, X_0) \otimes \mathcal{A}(X_0, X_1) \longrightarrow \bigoplus \mathcal{A}(X_0, X_0)$$

whose pth column ($p \geq 0$) is the sum

$$\bigoplus \mathcal{A}(X_p, X_0) \otimes \mathcal{A}(X_{p-1}, X_p) \otimes \cdots \otimes \mathcal{A}(X_0, X_1)$$

taken over all sequences of objects X_0, X_1, $\ldots$, X_p of $\mathcal{A}$ and whose horizontal differential is given by formula (1.3.0.1) adjusted following the Koszul sign rule. The complex $C^*\mathcal{A}$ is the product total complex of the bicomplex

$$\prod \mathcal{A}(X_0, X_0) \longrightarrow \prod \operatorname{Hom}_k(\mathcal{A}(X_0, X_1), \mathcal{A}(X_0, X_1)) \longrightarrow \ldots$$

whose pth column ($p \geq 0$) is

$$\prod \mathrm{Hom}_k(\mathcal{A}(X_{p-1}, X_p) \otimes \ldots \otimes \mathcal{A}(X_0, X_1), \mathcal{A}(X_0, X_p))$$

where the product is taken over all sequences of objects $X_0, \ldots, X_p$ of $\mathcal{A}$ and whose horizontal differential is given by formula (1.3.0.2) adjusted following the Koszul sign rule.

Theorem 1.6.10 ([60, 54])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories and $X \in \mathcal{D}(\mathcal{B} \otimes \mathcal{A}^{op})$.*

*a) If $X \in \mathrm{rep}\,(\mathcal{A}, \mathcal{B})$ (*i.e. *X is right perfect), then there is a canonical induced morphism*

$$HH_*(X) : HH_*(\mathcal{A}) \to HH_*(\mathcal{B}).$$

It only depends on the class of X in $K_0(\mathrm{rep}\,(A, B))$ and is an isomorphism if the functor $? \overset{L}{\otimes}_{\mathcal{A}} X : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$ is an equivalence.

b) Suppose that the functor $? \overset{L}{\otimes}_A X : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$ is fully faithful (but X is not necessarily right perfect). Then there is a canonical restriction morphism

$$\mathrm{res}_X : HH^*(\mathcal{B}) \to HH^*(\mathcal{A}).$$

It is an isomorphism if the functor $X \overset{L}{\otimes}_B ? : \mathcal{D}(\mathcal{B}^{op}) \to \mathcal{D}(\mathcal{A}^{op})$ is also fully faithful.

We will sketch a proof of b) after Theorem 1.7.6.

Corollary 1.6.11 (Lowen–Van den Bergh [71])**.** *Let $\mathcal{A}$ be a dg category. The restriction along the Yoneda functor $\mathcal{A} \to \mathcal{D}_{dg}\mathcal{A}$ induces an isomorphism*

$$HH^*(\mathcal{D}_{dg}\mathcal{A}) \xrightarrow{\sim} HH^*(\mathcal{A}).$$

A similar result was obtained by Toën in [92]. It should be viewed as the derived version of the isomorphism

$$Z(\mathrm{Mod}\, A) \xrightarrow{\sim} Z(A)$$

of Lemma 1.2.2 for an algebra A.

1.7 Higher structure

1.7.1 Higher structure on Hochschild homology

Let A be an algebra (for simplicity). Recall the Connes–Quillen cyclic bicomplex

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & \cdots \arrow[l, "1-t"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & \cdots \arrow[l, "1-t"']
\end{tikzcd}$$

Here we put

$$t(a_1 \otimes \ldots \otimes a_n) = (-1)^{n-1} a_2 \otimes \ldots \otimes a_n \otimes a_1 \quad \text{and} \quad N = 1 + t + \cdots + t^{n-1}.$$

The bicocomplex is 2-periodic in the horizontal direction. Its even columns are copies of the Hochschild chain complex and its odd columns copies of the bimodule bar resolution of A. The homology of the sum total complex is the cyclic homology of A. Let MA denote the cone over the subcomplex formed by the first two columns. Let us write d for its differential. Let $d' : MA \to MA$ be the homogeneous map of (cohomological) degree -1 given by the projection onto the first column followed by the map N followed by the inclusion of the second column. We have

$$d^2 = 0\,,\ d'^2 = 0\,,\ dd' + d'd = 0.$$

This means that MA is a *mixed complex*, *i.e.* a dg module over the dg algebra $\Lambda = k[\varepsilon]/(\varepsilon^2)$, where ε is of degree -1 and $d = 0$. Of course, ε acts via d' in MA. We write $\mathcal{D}\mathrm{Mix}$ for the *mixed derived category*, *i.e.* the category $\mathcal{D}\Lambda$.

Notice that MA is functorial with respect to algebra morphisms which do not necessarily preserve the unit like the morphism

$$A \to M_2(A)\,,\ a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

The inclusion of the first column clearly yields an isomorphism $A \overset{L}{\otimes}_{A^e} A \xrightarrow{\sim} MA$ in $\mathcal{D}k$. Thus we have $HH_*(A) = H_*(MA)$. Moreover, the mixed complex MA contains the information on cyclic, negative cyclic and periodic cyclic homology as shown by the canonical isomorphisms (*cf.* [53])

$$HC_*(A) = H_*(MA \overset{L}{\otimes}_\Lambda k)\,,\ HN_*(A) = H_*\mathrm{RHom}_\Lambda(k, MA)\,,\ HP_*(A) = \mathrm{holim}_n(MA \overset{L}{\otimes}_\Lambda \Sigma^{-2n} k).$$

Here, the transition morphisms of the inverse system are induced by the morphism $k \to \Sigma^2 k$ of the canonical triangle

$$\Sigma k \longrightarrow \Lambda \longrightarrow k \longrightarrow \Sigma^2 k.$$

This triangle induces the ISB-sequence

$$HC_{*-1} \xrightarrow{B} HH_* \xrightarrow{I} HC_* \xrightarrow{S} HC_{*-2} \xrightarrow{B} HH_{*-1}\ .$$

Now let B be another algebra and $X \in \operatorname{rep}(A, B)$. We may and will assume that the restriction X_B is right bounded with projective components. Then we have $X^\wedge = \mathrm{RHom}_B(X, B) = \mathrm{Hom}_B(X, B)$. We have natural morphisms of differential graded algebras (not preserving the unit)

$$A \xrightarrow{\alpha} \begin{bmatrix} A & X \\ X^\wedge & B \end{bmatrix} \xleftarrow{\beta} B$$

taking a to $\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and b to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right]$.

Lemma 1.7.2 ([60])**.** *The morphism $M\beta$ is invertible in $DMix$.*

We define $MX = (M\beta)^{-1} \circ M\alpha$ in $\mathcal{D}$Mix.

Theorem 1.7.3 ([60])**.** *a) We have $M({}_A A_A) = \mathbf{1}_{MA}$ and for $X \in \operatorname{rep}(A, B)$ and $Y \in \operatorname{rep}(B, C)$, we have $M(X \overset{L}{\otimes}_B Y) = MY \circ MX$.*

b) The morphism MX only depends on the class of X in $K_0(\operatorname{rep}(A, B))$.

It is not hard to generalize the definitions and results of this subsection from algebras to dg categories. In particular, we have the following generalization of the localization theorem 1.5.2.

Theorem 1.7.4 ([60])**.** *Let*

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

be a sequence of dg categories such that the induced sequence of derived categories

$$0 \longrightarrow \mathcal{D}\mathcal{A} \xrightarrow{F^*} \mathcal{D}\mathcal{B} \xrightarrow{G^*} \mathcal{D}\mathcal{C} \longrightarrow 0$$

is exact. Then there is a canonical triangle

$$M\mathcal{A} \xrightarrow{MF} M\mathcal{B} \xrightarrow{MG} M\mathcal{C} \longrightarrow \Sigma M\mathcal{A}$$

in the mixed derived category and hence long exact sequences in Hochschild and cyclic homology.

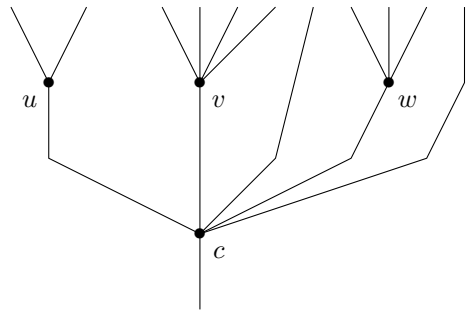


Figure 1.1: The brace operations

1.7.5 Higher structure on Hochschild cohomology

Let A be an algebra (for simplicity) and C^*A its Hochschild cochain complex, *cf.* section 1.3. If c and u, v, $\ldots$, w are Hochschild cochains, one defines [52] the *brace operation* $c\{u, v, \ldots, w\}$ by substituting the cochains u, v, $\ldots$, w for some of the arguments of c, inserting suitable signs and summing over all possibilities of doing this, cf. Figure 1.1.

The complex C^*A together with the cup product $\cup$ and the brace operations is an example of a B_∞-algebra in the sense of Getzler–Jones [32], *i.e.* a graded vector space V such that the tensor coalgebra $T^c(\Sigma V)$ (with the deconcatenation coproduct) is endowed with a dg bialgebra structure whose comultiplication is deconcatenation. Here the letter 'B' stands for 'Baues', in honour of Hans–Joachim Baues, who showed [8] that the singular cochain complex with integer coefficients of any topological space carries a natural B_∞-algebra structure. Notice that B_∞–structures are closely related to monoidal structures (*cf.* [70] and the references given there). In the case of the Hochschild cochain complex, the monoidal category is the derived category $\mathcal{D}(A^e)$ with the derived tensor product $\overset{L}{\otimes}_A$ over A; in the case of the singular cochain complex on a topological space X, it is the derived category of sheaves of abelian groups on X with the derived tensor product. The B_∞-algebra structure on C^*A contains in particular the information on the Gerstenhaber bracket, which may be recovered via

$$[c, u] = c\{u\} \mp u\{c\}.$$

By the *homotopy category of B_∞-algebras* we mean the localization of the category of B_∞-algebras with respect to all morphisms of B_∞-algebras inducing isomorphisms in homology.

Let B be another algebra and $X \in \mathcal{D}(B \otimes A^{op})$ a complex of A-B-bimodules.

Theorem 1.7.6 ([54])**.** *Suppose that the functor* $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ *is fully faithful (but* X_B *is not*

necessarily perfect). Then there is a canonical restriction morphism

$$\mathrm{res}_X : C^*B \to C^*A$$

in the homotopy category of B_∞-algebras. It is invertible if the functor $X \overset{L}{\otimes}_B ? : \mathcal{D}(B^{op}) \to \mathcal{D}(A^{op})$ is also fully faithful (for example if $? \overset{L}{\otimes}_A$ is an equivalence).

We sketch the proof: Let G be the 'glued' dg category with two objects 1 and 2 such that $G(1,1) = B$, $G(2,2) = A$, $G(1,2) = X$ and $G(2,1) = 0$. We have obvious forgetful (or 'restriction') maps

$$C^*B \xleftarrow{\mathrm{res}_B} C^*G \xrightarrow{\mathrm{res}_A} C^*A.$$

which clearly respect the B_∞-structure. It is a classical fact, *cf.* [40, 21, 73, 34, 22, 36, 9, 33], that we have a homotopy bicartesian square

$$\begin{tikzcd} C^* \arrow[r, "\mathrm{res}_A"] \arrow[d, "\mathrm{res}_B"] & C^*A \arrow[d, "\alpha"] \\ C^*B \arrow[r] & \mathrm{RHom}_{B\otimes A^{op}}(X,X) \end{tikzcd}$$

We claim that α and hence res_B is invertible in $\mathcal{D}k$. Indeed, we have a commutative square in $\mathcal{D}k$

$$\begin{tikzcd} C^*A \arrow[r, "\sim"] \arrow[d, "\alpha"] & \mathrm{RHom}_{A^e}(A,A) \arrow[d, "\beta"] \\ \mathrm{RHom}_{B\otimes A^{op}}(X,X) \arrow[r, "\sim"] & \mathrm{RHom}_{A^e}(A, \mathrm{RHom}_B(X,X)) \end{tikzcd}$$

where β is induced by the action morphism

$$A \to \mathrm{RHom}_B(X,X).$$

This is invertible by our assumption that the functor $? \overset{L}{\otimes}_A X$ is fully faithful. We put $\mathrm{res}_X = \mathrm{res}_A \circ \mathrm{res}_B^{-1}$.

Corollary 1.7.7. *For an algebra A, the isomorphism*

$$HH^*(\mathcal{D}_{dg}A) \xrightarrow{\sim} HH^*(A)$$

of Corollary 1.6.11 lifts to an isomorphism in the homotopy category of B_∞-algebras.

1.8 Tate–Hochschild cohomology

Let A be a right noetherian algebra and $\mathrm{mod}\, A$ the abelian category of finitely generated (right) A-modules. Let $\mathcal{D}^b(\mathrm{mod}\, A)$ be its bounded derived category. The perfect derived category $\mathrm{per}\,(A)$ is a thick

subcategory of $\mathcal{D}^b(\operatorname{mod} A)$. They coincide if A is of finite global dimension. We define the *singularity category* $\operatorname{sg}(A)$ *to be the quotient* $\mathcal{D}^b(\operatorname{mod} A)/\operatorname{per}(A)$. This category was first considered by Buchweitz [14] in 1986 and rediscovered by Orlov [77] in 2003 in a geometric setting. It measures the non regularity of the algebra A.

As an example, let $S = k[x_1, \ldots, x_n]$ and let $A = S/(f)$ for a non zero $f \in S$. It follows from the work of Eisenbud [27] that the singularity category $\operatorname{sg}(A)$ is triangle equivalent to the homotopy category of matrix factorizations of f. By definition, this category equals $H^0(\operatorname{mf}_{dg}(f))$, where $\operatorname{mf}_{dg}(f)$ is the *dg category of matrix factorizations of* f: its objects are the 2-periodic diagrams (not complexes!)

$$\ldots \xrightarrow{d_0} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \ldots$$

where the P_i are finitely generated projective S-modules and d^2 is the multiplication with f. For two such objects P and Q, the *morphism complex* $\operatorname{Hom}(P, Q)$ has as its nth component the space of homogeneous S-linear maps $g : P \to Q$ of degree n. The differential is given by $d(g) = d \circ g - (-1)^n g \circ d$. We leave it as an exercise for the reader to check that $d^2(g) = 0$. It is an important point that the complexes $\operatorname{Hom}(P, Q)$ are also 2-periodic, so that we may also view $\operatorname{mf}_{dg}(f)$ as a differential $\mathbb{Z}/2\mathbb{Z}$–graded category!

Now suppose that $A^e = A \otimes A^{op}$ is also (right) noetherian. We define *Tate–Hochschild cohomology of* A (sometimes also called *singular Hochschild cohomology of* A) to be the Yoneda algebra $HH^*_{sg}(A) = \operatorname{Ext}^*_{\operatorname{sg}(A^e)}(A, A)$ of the identity bimodule in the singularity category of the enveloping algebra. By definition, it is an algebra and it is not hard to check directly that it is (graded) commutative. However, the singularity category $\operatorname{sg}(A^e)$ is *not monoidal* in any natural way. We may nevertheless ask whether Tate–Hochschild cohomology carries the same rich structure as classical Hochschild cohomology. This problem was open for some time and finally solved in the thesis of Zhengfang Wang:

Theorem 1.8.1 (Zhengfang Wang). *a) $HH^*_{sg}(A)$ carries a natural (but intricate!) Gerstenhaber bracket [99].*

b) There is a natural B_∞-algebra $C^*_{sg}A$ computing $HH^*_{sg}(A)$ with its Gerstenhaber bracket [98].

Thus, we see that there is a complete structural analogy between Tate–Hochschild cohomology and classical Hochschild cohomology. It is therefore natural to ask whether Tate–Hochschild cohomology is not an instance of classical Hochschild cohomology, i.e. whether the Tate–Hochschild cohomology of A is classical Hochschild cohomology of some more complicated object associated with A in analogy with Corollary 1.6.11. Recall that a dg category $\mathcal{A}$ is *smooth* if the identity bimodule $I_{\mathcal{A}} : (X, Y) \mapsto \mathcal{A}(X, Y)$ is perfect in the derived category $\mathcal{D}(\mathcal{A}^e)$ of bimodules. Define the *dg singularity category of* A as the dg quotient

$$\operatorname{sg}_{dg}(A) = \mathcal{D}^b_{dg}(\operatorname{mod} A)/\operatorname{per}_{dg}(A).$$

Theorem 1.8.2 ([58]). *There is a canonical morphism of graded algebras*

$$HH^*_{sg}(A) \to HH^*(\mathrm{sg}_{dg}(A)).$$

It is an isomorphism if the dg category $\mathcal{D}^b_{dg}(\mathrm{mod}\, A)$ is smooth.

According to Theorem A of Elagin–Lunts–Schnürer's [28], the dg category $\mathcal{D}^b_{dg}(\mathrm{mod}\, A)$ is smooth if A is a finite-dimensional algebra over any field k such that $A/\mathrm{rad}(A)$ is separable over k (which is automatic if k is perfect). By Theorem B of [loc. cit.], it also holds if the algebra A is right noetherian and finitely generated over its center and the center is a finitely generated algebra over k.

Conjecture 1.8.3 (Keller). *The morphism of the theorem lifts to a morphism in the homotopy category of B_∞-algebras.*

Note that this morphism will be an isomorphism if the bounded dg derived category $\mathcal{D}^b_{dg}(\mathrm{mod}\, A)$ is smooth. In particular, this should hold for each finite-dimension algebra defined by a quiver with an admissible ideal of relations. The following theorem confirms the conjecture for radical square 0 algebras.

Theorem 1.8.4 (Chen–Li–Wang [18]). *The conjecture holds if $A = kQ/(Q_1)^2$, where Q is a finite quiver without sinks or sources and $(Q_1)^2$ the square of the ideal of the path algebra kQ generated by the arrows.*

To show why the conjecture is probably not easy to prove, let us sketch the construction of the isomorphism in Theorem 1.8.2. Let $\mathcal{M} = \mathcal{D}^b_{dg}(\mathrm{mod}\, A)$ and $\mathcal{S} = \mathrm{sg}_{dg}(A)$. We have natural dg functors

$$A \xrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{S}$$

whose composition vanishes in the homotopy category of dg categories. We construct the following square (commutative up to isomorphism)

$$\begin{tikzcd}
\mathcal{D}^b(A \otimes A^{op}) \arrow[r, "(\mathbf{1}\otimes i)^*"] \arrow[d] & \mathcal{D}(A \otimes \mathcal{M}^{op}) \arrow[r, "(i\otimes \mathbf{1})^!"] & \mathcal{D}(\mathcal{M} \otimes \mathcal{M}^{op}) \arrow[d, "(p\otimes p)^*"] \\
\mathrm{sg}(A \otimes A^{op}) \arrow[rr, dashed] & & \mathcal{D}(\mathcal{S} \otimes \mathcal{S}^{op})
\end{tikzcd}$$

Here, for a dg functor $f : \mathcal{A}_1 \to \mathcal{A}_2$, we denote by f^* the left adjoint and by $f^!$ the right adjoint of the restriction functor $f_* : \mathcal{D}\mathcal{A}_2 \to \mathcal{D}\mathcal{A}_1$. One checks that the dashed triangle functor exists, takes the identity bimodule A to the identity bimodule $\mathcal{S}(?, -)$ and induces an isomorphism between the Yoneda algebras of these objects. Since the functor is induced by the composition of a right derived with a left derived functor, it is hard to compute it explicitly and that is why the conjecture is not obvious.

1.9 Application: two reconstruction theorems

We apply the results of the preceding section to the reconstruction of singularities.

1.9.1 Isolated hypersurface singularities

Theorem 1.9.2 (Hua-Keller [46])**.** *Let $S = \mathbb{C}[[x_1, \ldots, x_n]]$ and suppose that $R = S/(f)$ is an isolated singularity. Then R is determined up to isomorphism by its dimension and the dg singularity category $\mathrm{sg}_{dg}(R)$.*

Notice that because of Knörrer periodicity, the dg singularity category alone does not determine R.

Sketch of proof. We consider the center

$$Z(\mathrm{sg}_{dg}(R)) = HH^0(\mathrm{sg}_{dg}(R)).$$

By Theorem A of [28], the bounded dg derived category $\mathcal{D}^b(\operatorname{mod} R)$ is smooth. By Theorem 1.8.2, the algebra $HH^0(\mathrm{sg}_{dg}(R))$ is isomorphic to $HH^0_{sg}(R)$. Now since R is a hypersurface, the dg singularity category may be described by matrix factorizations and is therefore 2-periodic. Hence its Hochschild cohomology is 2-periodic and, again by Theoreom 1.8.2, so is $HH^*_{sg}(R)$. Thus, we have an isomorphism

$$HH^0_{sg}(R) \xrightarrow{\sim} HH^{2r}_{sg}(R).$$

Now by a theorem of Buchweitz [14], for Gorenstein algebras, in sufficiently high degrees, Tate–Hochschild cohomology agrees with classical Hochschild cohomology and we get an isomorphism

$$HH^{2r}_{sg}(R) \xrightarrow{\sim} HH^{2r}(R).$$

Thus, we are reduced to the computation of Hochschild cohomology of a hypersurface. Thanks to the results of [37], we find that $HH^{2r}(R)$ and $HH^0(R)$ are isomorphic to the Tyurina algebra

$$S/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}).$$

Now the Tyurina algebra together with the dimension determine R by the Mather–Yau theorem [72], more precisely its formal series version proved in [35]. Notice that in this sketch, we have neglected the technical problems arising from the fact that R is a topological algebra. □

Notice that in the above computation, we have considered the dg singularity category $\mathrm{sg}_{dg}(R)$ as a differential $\mathbb{Z}$-graded category. If one considers it as a differential $\mathbb{Z}/2$-graded category, one obtains a different result for the center, namely the Milnor algebra

$$Z_{\mathbb{Z}/2}(\mathrm{sg}_{dg}(R)) = S/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$$

as shown by Dyckerhoff [26].

1.9.3 Compound Du Val singularities

Let R be a complete local isolated compound Du Val singularity (thus, it is 3-dimensional, normal and a generic hyperplane section through the origin is a Kleinian surface singularity). Let

$$f : Y \to X = \mathrm{Spec}(R)$$

be a small crepant resolution (thus, it is birational, an isomorphism in codimension 1, an isomorphism outside the exceptional fibre and $f^*(\omega_X) \cong \omega_Y$). Then the reduced exceptional fibre $\mathcal{F}$ of f is a tree of rational curves $\mathbb{P}^1$. The morphism f contracts this tree to a point. Associated to this situation is the *contraction algebra* Λ introduced by Donovan–Wemyss [24]. It is a finite-dimensional algebra which represents the deformations with non commutative base of the exceptional fibre of f. It is known that numerous invariants of the singularity can be computed from the algebra Λ. This has lead Donovan–Wemyss to conjecture [6] that the derived equivalence class of Λ determines R up to isomorphism. We show a weakened version: Thanks to work of Van den Bergh and de Thanhoffer de Voelcsey [97], it is known that Λ is the Jacobian algebra of a quiver Q with potential W. By definition, the potential is an element of $HH_0(\mathbb{C}Q)$. Let $\overline{W}$ be its image in $HH_0(\Lambda)$.

Theorem 1.9.4 (Hua-Keller [46])**.** *The derived equivalence class of the pair* $(\Lambda, \overline{W})$ *determines* R*.*

The proof uses, among other things, Theorem 1.9.2 and silting theory.

Let us point out the link to cluster theory: It turns out that the singularity category $\mathrm{sg}(R)$ is triangle equivalent to a generalized cluster category in the sense of Amiot [1], namely the generalized cluster category $\mathcal{C}_{Q,W}$ associated with the quiver Q with potential W. This category has therefore the same main properties as the categories appearing in the (additive) categorification of Fomin–Zelevinsky cluster algebras, *cf.* [65]. However, the quivers that appear are quite different: Whereas the quivers in Donovan–Wemyss' theory have many loops and 2-cycles, the quivers appearing in cluster theory never have loops or 2-cycles.

Chapter 2

On the Hochschild homology of singularity categories

Let k be an algebraically closed field and A a finite-dimensional k-algebra. In this chapter, we determine complexes which compute the Hochschild homology of the canonical dg enhancement $\mathcal{D}^b_{dg}(A)$ of the bounded derived category $\mathcal{D}^b(A) = \mathcal{D}^b(\operatorname{mod} A)$ and of the canonical dg enhancement $\mathrm{sg}_{dg}(A)$ of the singularity category

$$\mathrm{sg}(A) = \mathcal{D}^b(A)/\mathrm{per}\,(A).$$

Our main tools are Koszul duality as described in [59] and the localization theorem 1.7.4.

2.1 Hochschild homology and Koszul duality

Let k be an algebraically closed field. We refer to Chapter 1 for the definition and the most important properties of Hochschild homology of k-algebras and (dg) k-categories. In particular, we denote by $HH(A)$ the Hochschild complex of a k-algebra A. In the sequel, in addition, we need the computation of Hochschild homology of the Koszul dual: Let Q be a finite quiver and I an admissible ideal in kQ, *i.e.* a two-sided ideal contained in the square of the ideal generated by the arrows and such that the quotient kQ/I is finite-dimensional. Let R be the quotient of A by its radical. Thus, as an A-module, the algebra R is the direct sum of the simple A-modules. Following [59], we define the Koszul dual of A to be the dg algebra

$$A^! = \mathrm{RHom}_A(R, R).$$

Thus, if P is a projective resolution of the A-module R, then the Koszul dual is quasi-isomorphic to the dg endomorphism algebra $\mathrm{Hom}_A(P,P)$ of P. The following theorem is a special case of Corollary D.2 of Van den Bergh's [95]. We write D for the dual $\mathrm{Hom}_k(?,k)$ over the ground field.

Theorem 2.1.1 (Van den Bergh)**.** *We have a canonical isomorphism*

$$HH(A^!) \xrightarrow{\sim} DHH(A).$$

We refer to [43] for a comparison taking into account much more structure.

2.2 Hochschild homology of derived categories and singularity categories

Let Q be a finite quiver and I an admissible ideal in kQ. Let $\mathrm{mod}\, A$ be the category of k-finite-dimensional right A-modules. Denote by $\mathcal{D}^b(A) = \mathcal{D}^b(\mathrm{mod}\, A)$ the bounded derived category of A and by $\mathrm{per}\,(A)$ the perfect derived category, *i.e.* the thick subcategory generated by the free A-module of rank 1. Following Buchweitz [14] and Orlov [77], one defines the singularity category of A as the Verdier quotient

$$\mathrm{sg}(A) = \mathcal{D}^b(A)/\mathrm{per}\,(A).$$

Using the canonical dg enhancements of $\mathcal{D}^b(A)$ and $\mathrm{per}\,(A)$, *cf.* section 1.6, we obtain a canonical exact sequence of dg categories

$$0 \longrightarrow \mathrm{per}_{dg}(A) \longrightarrow \mathcal{D}^b_{dg}(A) \longrightarrow \mathrm{sg}_{dg}(A) \longrightarrow 0$$

It is not hard to see that, in the homotopy category of dg categories, it is functorial with respect to bimodules complexes $X \in \mathcal{D}(A^{op} \otimes B)$ such that X_B is perfect over B and ${}_AX$ is perfect over A. From the localization theorem 1.7.4, we deduce a triangle

$$HH(\mathrm{per}_{dg}(A) \longrightarrow HH(\mathcal{D}^b_{dg}(A)) \longrightarrow HH(\mathrm{sg}_{dg}(A)) \longrightarrow \Sigma HH(\mathrm{per}_{dg}(A)) \tag{2.2.0.1}$$

in the derived category of vector spaces.

Theorem 2.2.1. *We have a canonical isomorphism* $HH(\mathcal{D}^b_{dg}(A)) \xrightarrow{\sim} DHH(A)$.

Proof. Recall that we have defined R to be the quotient of A by its radical and the Koszul dual $A^!$ as $\mathrm{RHom}_A(R,R)$. Since the module R is a classical generator of the bounded derived category $\mathcal{D}^b(A)$, we deduce from the results of [59] that we have a triangle equivalence

$$\mathrm{RHom}_A(R,?) : \mathcal{D}^b(A) \xrightarrow{\sim} \mathrm{per}\,(A^!).$$

This lifts to a quasi-equivalence

$$\mathcal{D}^b_{dg}(A) \xrightarrow{\sim} \mathrm{per}_{dg}(A^!).$$

By Morita invariance of Hochschild homology, we have

$$HH(A^!) \xrightarrow{\sim} HH(\mathrm{per}_{dg}(A^!)).$$

By Van den Bergh's theorem 2.1.1, we have

$$HH(A!) \xrightarrow{\sim} DHH(A).$$

The claim follows if we combine these isomorphisms. □

Define a linear map $\tau : A \to DA$ by sending an element $a \in A$ to the linear form which takes $b \in A$ to the trace of the linear map

$$\lambda_a \rho_b : A \to A\,,\ x \mapsto axb.$$

Notice that since A is finite-dimensional, this is well-defined. Define the *double Hochschild complex of A* to be the complex

$$\ldots \xrightarrow{b} A \otimes A \xrightarrow{b} A \xrightarrow{\tau} DA \xrightarrow{Db} D(A \otimes A) \xrightarrow{Db} \ldots\ ,$$

where DA sits in degree 0, the differentials b are those of the Hochschild complex and the Db their duals.

Let us abbreviate $\mathcal{S} = \mathrm{sg}_{dg}(A)$.

Theorem 2.2.2. *In $\mathcal{D}k$, we have a canonical isomorphism between $HH(\mathcal{S})$ and the double Hochschild complex of A.*

Notice that this implies in particular that $HH_n(\mathcal{S})$ is finite-dimensional for all n. This is surprising since the singularity category $\mathrm{sg}(A)$ is usually not Hom-finite (except if A is Gorenstein), *cf.* for example [17].

Proof. We use the triangle

$$HH(\mathrm{per}_{dg}(A)) \longrightarrow HH(\mathcal{D}^b_{dg}(A)) \longrightarrow HH(\mathcal{S}) \longrightarrow \Sigma HH(\mathrm{per}_{dg}(A))\ .$$

We have already seen that it is isomorphic to a triangle

$$HH(A) \to HH(A^!) \to HH(\mathcal{S}) \to \Sigma HH(A)\,,$$

where the first morphism is induced by the inclusion $\mathrm{per}_{dg}(A) \to \mathcal{D}^b_{dg}(A)$. Thus, the complex $HH(\mathcal{S})$ identifies with the mapping cone over the morphism $HH(A) \to HH(A^!)$. Let us determine this morphism

explicitly. Recall that the functor HH, considered as a functor on the homotopy category of small dg categories with values in the derived category $\mathcal{D}k$, commutes with tensor products. We have the following commutative square

$$\begin{array}{ccc} \mathrm{per}_{dg}(A^{op}) \otimes \mathrm{per}_{dg}(A) & \longrightarrow & \mathrm{per}_{dg}(k) \\ \downarrow & & \| \\ \mathrm{per}_{dg}(A)^{op} \otimes \mathcal{D}^b_{dg}(A) & \longrightarrow & \mathrm{per}_{dg}(k) \end{array}$$

Here, a pair (P_1, P_2), $P_1 \in \mathrm{proj}\,(A^{op})$, $P_2 \in \mathrm{proj}\,(A)$ is taken to $P_2 \otimes_A P_1$ by the top arrow and to $(\mathrm{Hom}_A(P_1, A), P_2)$ by the left vertical arrow. It follows from Appendix D in [95] that the lower horizontal arrow induces a non degenerate pairing

$$HH(A) \otimes HH(\mathcal{D}^b_{dg}(A)) \to HH(k) = k.$$

A direct computation now shows that the morphism

$$HH(A) \to DHH(A)$$

is the composition

$$HH(A) \to HH_0(A) \to DHH_0(A) \to DHH(A)$$

where the middle morphism is induced by the map τ. □

Corollary 2.2.3. *a) For $n \geq 2$, we have canonical isomorphisms*

$$HH_n(\mathcal{S}) \xrightarrow{\sim} HH_{n-1}(A) \xrightarrow{\sim} DHH_{1-n}(\mathcal{S}).$$

Moreover, we have

$$HH_1(\mathcal{S}) \xrightarrow{\sim} \ker(HH_0(A) \xrightarrow{\tau} DHH_0(A)) \xrightarrow{\sim} DHH_0(\mathcal{S}).$$

b) If A is symmetric, then $HH_*(\mathcal{S})$ agrees with the singular Hochschild homology (=Tate–Hochschild homology) of A.

2.3 Application: Hochschild homology of dg Leavitt path algebras

Let Q be a finite quiver, for example a quiver with one vertex and a unique loop α. Let A be the associated radical square zero algebra, *i.e.* the quotient of kQ by the square of the ideal generated by the arrows. So for the one-loop quiver, we have $A = k[\varepsilon]/(\varepsilon^2)$. Let Q^* be the graded quiver obtained from the opposite quiver of Q by assigning each arrow $\alpha^* : j \to i$ corresponding to an arrow $\alpha : i \to j$ of Q the degree $+1$. For each vertex i of Q, consider the arrows $\alpha^*_s : i \to j_{i_s}$, $1 \leq s \leq t_i$ starting in Q^* at i. Let

$$\varphi_i : P_i \to \bigoplus_{s=1}^{t_i} \Sigma P_{j_{i_s}}$$

be the morphism with components α_s^*, where $P_i = e_i kQ^*$. For example, for the one-loop quiver, we just have $\varphi(1) = \alpha^* : P_1 \to \Sigma P_1$. Note that if i is a sink of Q, then

$$\bigoplus_{s=1}^{t_i} P_{j_{i_s}} = 0.$$

For each vertex $i \in Q_0$, let

$$\varphi(i)^{-1} = [\beta_{i,1}, \ldots, \beta_{i,t_i}] : \bigoplus_{s=1}^{t_i} \Sigma P_{j_{i_s}} \to P_i$$

be the formal inverse of $\varphi(i)$. The *graded Leavitt path algebra of* Q is obtained from kQ^* by adjoining all coefficients β_{ij} of all formal inverses $\varphi(i)^{-1}$, $i \in Q_0$. We endow L_Q with the grading inherited from Q^* and with $d = 0$.

Theorem 2.3.1 (Smith [78] , Chen–Yang [20])**.** *We have a triangle equivalence* $\mathrm{per}\,(L_Q) \xrightarrow{\sim} \mathrm{sg}(A)$ *taking* $e_i L_Q$ *to the simple* S_i*.*

Corollary 2.3.2. *The Hochschild homology* $HH_*(L_Q)$ *of the Leavitt path algebra is computed by the double Hochschild complex*

$$\ldots \xrightarrow{b} A \otimes A \xrightarrow{b} A \xrightarrow{\tau} DA \xrightarrow{Db} D(A \otimes A) \xrightarrow{Db} \ldots ,$$

(with DA *in degree* 0*). In particular, we have*

$$\dim HH_p(L_Q) = 0 < \infty$$

for all $p \in \mathbb{Z}$*.*

A different description of the Hochschild homology of Leavitt path algebras is due to Ara–Cortinñas [3].

2.4 Beyond radical square zero

Let Q be a finite quiver and $A = kQ/I$ the quotient of its path algebra by an admissible ideal. Let J be the radical of A and $R = kQ_0$ so that we have $A = R \oplus J$ as R-bimodules. Let $A_0 = (T_R J)/(J \otimes_R J)$ be the radical square zero algebra associated with A. Thus, we have $A_0 = R \oplus J = A$ as R-bimodules but we have $xy = 0$ in A_0 for any two elements of J. We view A_0 as a degeneration of A and A as a deformation of A_0. As pointed out by Chen–Wang [19], this suggests that the singularity category $\mathrm{sg}(A)$ is a deformation of the singularity category $\mathrm{sg}(A_0)$, which is equivalent to the perfect derived category $\mathrm{per}\,(L_{A_0})$ of the graded Leavitt path algebra L_{A_0}. Hence we can hope for the existence of a dg algebra L_A obtained from L_{A_0} by deformation such that $\mathrm{per}\,(L_A)$ is equivalent to $\mathrm{sg}(A)$. We sum up the situation in the following diagram

$$
\begin{tikzcd}
A_0 \arrow[r, squiggly, "\text{deformation}"'] & A \\
\mathrm{sg}(A_0) \arrow[r, squiggly, "\text{deformation}"'] \arrow[d, dash, "\wr"] & \mathrm{sg}(A) \arrow[d, dash, "\wr"] \\
\mathrm{per}\,(L_{A_0}) \arrow[r, squiggly, "\text{deformation?}"'] & \mathrm{per}\,(L_A) \\
L_{A_0} \arrow[r, squiggly, "\text{deformation?}"'] & L_A
\end{tikzcd}
\qquad ?
\qquad ?
$$

The following theorem confirms this hope.

Theorem 2.4.1 (Chen–Wang [19])**.** *The graded algebra L_{A_0} admits a canonical differential d_A such that for $L_A = (L_{A_0}, d_A)$, we have a triangle equivalence*

$$\mathrm{per}\,(L_A) \xrightarrow{\sim} \mathrm{sg}(A).$$

In chapter 3, we will give an alternative proof of this theorem.

Corollary 2.4.2. *The Hochschild homology of the dg Leavitt path algebra L_A is computed by the double Hochschild complex of A.*

Chapter 3

New proof of a theorem by Chen–Wang

We give a new proof of Chen–Wang's theorem 2.4.1 describing singularity categories of finite-dimensional algebras using dg Leavitt path algebras. The proof is based on Koszul–Moore duality worked out in Lefèvre-Hasegawa's thesis [68] as described in Keller's [56] and on derived localisation as developed in Braun–Chuang–Lazarev's [11].

3.1 Modules and comodules

Let k be a field and Q a finite quiver. Let A be the quotient kQ/I of the path algebra kQ by an admissible ideal I, *i.e.* an ideal contained in the square of the ideal generated by the arrows such that kQ/I is finite-dimensional. Let $R = kQ_0$ be the subalgebra of A generated by the lazy idempotents e_i, $i \in Q_0$, and let J be the Jacobson radical of A (which equals the ideal generated by the arrows). We have the decomposition $A = R \oplus J$ in the category of R-bimodules and we view A as an augmented algebra in the monoidal category of R-bimodules with the tensor product $\otimes_R$. Notice that the vector space kQ_1 whose basis is formed by the arrows of Q is naturally an R-bimodule and that the path algebra kQ identifies with the tensor algebra $T_R(kQ_1)$.

For an R-bimodule M, we define the dual bimodule by

$$M^\vee = \mathrm{Hom}_{R^e}(M, R^e).$$

For example, for $M = kQ_1$, the dual bimodule $M^\vee$ canonically identifies with kQ_1^*, where Q^* is the quiver with the same vertices as Q and whose arrows are the $\alpha^* \colon j \to i$ for each arrow $\alpha \colon i \to j$ of Q. Notice

that for an arbitrary R-bimodule M, the underlying vector space of $M^\vee$ identifies with the dual

$$DM = \mathrm{Hom}_k(M, k)$$

via the map taking an R-bilinear map $f : M \to R^e$ to the linear form $t \circ f$, where $t : R^e \to k$ takes $e_i \otimes e_j$ to $\delta_{ij} \in k$.

As an R-bimodule, the algebra A is finitely generated projective so that $C = A^\vee$ becomes a coalgebra in the category of R-bimodules. We have $C = R \oplus J^\vee$ and the induced comultiplication

$$J^\vee \to J^\vee \otimes_R J^\vee$$

is conilpotent because the Jacobson radical J of A is nilpotent. Thus, we may view C as an augmented cocomplete differential graded coalgebra (in the sense of section 2 of [56]), which is moreover concentrated in degree 0.

Since A is finitely generated projective as an R-bimodule, for each right R-module M, we have natural isomorphisms

$$\mathrm{Hom}_R(M \otimes_R A, M) \xrightarrow{\sim} M \otimes_R A^\vee \otimes_R \mathrm{Hom}_R(M, R) \xrightarrow{\sim} \mathrm{Hom}_R(M, M \otimes_R C).$$

This allows us to convert right A-modules into right C-comodules. More precisely, we have an isomorphism of categories

$$\mathrm{Mod}\, A \xrightarrow{\sim} \mathrm{Com}\, C\,,$$

where $\mathrm{Mod}\, A$ denotes the category of all right A-modules and $\mathrm{Com}\, C$ the category of all right C-comodules. Clearly, this isomorphism restricts to an isomorphism

$$\mathrm{mod}\, A \xrightarrow{\sim} \mathrm{com}\, C$$

between the categories of finite-dimensional modules respectively comodules.

3.2 Koszul–Moore duality

We refer to section 4 of [56] for all undefined terminology and for proofs or references to proofs of the claims we make. Let ΩC be the cobar construction of C over R. Thus, the underlying graded algebra of ΩC is the tensor algebra $T_R(\Sigma^{-1} J^\vee)$ on the desuspension $\Sigma^{-1} J^\vee = J^\vee[-1]$ of $J^\vee = C/R$. The differential of ΩC encodes the comultiplication $J^\vee \to J^\vee \otimes_R J^\vee$. The projection $C \to J^\vee$ composed with the inclusion $\Sigma^{-1} J^\vee \to \Omega C$ is the canonical *twisting cochain* $\tau \colon C \to \Omega C$. It is an R-bimodule morphism of degree 1 satisfying

$$d(\tau) + \tau * \tau = 0\,,$$

where $d(\tau) = d_{\Omega C} \circ \tau + \tau \circ d_C$ and $*$ is the convolution product on $\mathrm{Hom}_{R^e}(C, \Omega C)$. For a dg ΩC-module L, the twisted tensor product $L \otimes_\tau C$ is defined by twisting the differential on $L \otimes_R C$ using τ, and similarly, for a cocomplete dg C-comodule M, the twisted tensor product $M \otimes_\tau (\Omega C)$ is defined by twisting the differential on $M \otimes_R (\Omega C)$. We get a pair of adjoint functors

$$F = ? \otimes_\tau (\Omega C) \colon \mathrm{dgCom}\,(C) \leftrightarrow \mathrm{dgMod}\,(\Omega C) \colon ? \otimes_\tau C = G\,,$$

where $\mathrm{dgMod}\,(\Omega C)$ denotes the category of dg right ΩC-modules and $\mathrm{dgCom}\,(C)$ the category of cocomplete dg right C-comodules. These functors form in fact a Quillen equivalence for the standard Quillen model structure on $\mathrm{dgMod}\,(\Omega C)$ and a suitable Quillen model structure on $\mathrm{dgCom}\,(C)$, cf. [68]. Thus, they induce quasi-inverse equivalences

$$F \colon \mathcal{D}^c(C) \xrightarrow{\sim} \mathcal{D}(\Omega C) \colon G\,,$$

where $\mathcal{D}^c(C)$ is the coderived category of C and $\mathcal{D}(\Omega C)$ the derived category of ΩC. The equivalence F sends C to R and R to ΩC.

Although it is not necessary for the sequel, let us point out that $\mathcal{D}^c(C)$ is equivalent to the homotopy category $\mathcal{H}(\mathrm{Inj}\,A)$ of complexes of injective A-modules. Indeed, we know from [68] that the fibrant-cofibrant objects of $\mathrm{dgCom}\,C$ are exactly the retracts of the cofree dg comodules (which are automatically cocomplete since C is conilpotent) and that two morphisms between fibrant-cofibrant dg comodules are homotopic in the model-theoretic sense if and only if they are homotopic in the classical sense. Thus, the homotopy category $\mathcal{D}^c(C)$ of the Quillen model category $\mathrm{dgCom}\,C$ is equivalent to the usual homotopy category of all complexes of right C-comodules which are retracts of complexes of cofree comodules. It is not hard to see that this homotopy category is equivalent to the (slightly larger) homotopy category of complexes of dg comodules with injective components. Via the isomorphism $\mathrm{Com}\,C \to \mathrm{Mod}\,A$, the complexes of C-comodules with injective components correspond exactly to the complexes of A-modules with injective components so that we get the equivalence

$$\mathcal{H}(\mathrm{Inj}\,A) \xrightarrow{\sim} \mathcal{D}^c(C).$$

3.3 Description of the singularity category

Since R generates $D^b(\mathrm{com}\,C)$ and ΩC generates $\mathrm{per}\,(\Omega C)$, the equivalence $\mathcal{D}^c(C) \xrightarrow{\sim} \mathcal{D}(\Omega C)$ induces an equivalence

$$\mathcal{D}^b(\mathrm{com}\,C) \xrightarrow{\sim} \mathrm{per}\,(\Omega C).$$

By composition with the isomorphism $\mathcal{D}^b(\operatorname{mod} A) \xrightarrow{\sim} \mathcal{D}^b(\operatorname{com} C)$, we get an equivalence $\mathcal{D}^b(\operatorname{mod} A) \xrightarrow{\sim} \operatorname{per}(\Omega C)$ which sends R to ΩC. In particular, we get an induced algebra isomorphism

$$\operatorname{Ext}^*_A(R,R) \xrightarrow{\sim} \operatorname{Ext}^*_{\Omega C}(\Omega C, \Omega C) \xrightarrow{\sim} H^*(\Omega C).$$

The isomorphism $\mathcal{D}^b(\operatorname{mod} A) \xrightarrow{\sim} \mathcal{D}^b(\operatorname{com} C)$ sends the injective cogenerator $DA \xrightarrow{\sim} A^\vee$ to the cofree comodule C and the equivalence $\mathcal{D}^b(\operatorname{com} C) \xrightarrow{\sim} \operatorname{per}(\Omega C)$ sends C to R. Thus, we get an induced equivalence

$$\mathcal{D}^b(\operatorname{mod} A)/\operatorname{thick}(DA) \xrightarrow{\sim} \operatorname{per}(\Omega C)/\operatorname{thick}(R).$$

By composing with the duality functor

$$D : \mathcal{D}^b(\operatorname{mod} A^{op})^{op} \xrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} A)$$

we find an equivalence

$$\operatorname{sg}(A^{op})^{op} \xrightarrow{\sim} \operatorname{per}(\Omega C)/\operatorname{thick}(R)\,,$$

where $\operatorname{sg}(A^{op})$ denotes the singularity category of A^{op}.

3.4 Description of the singularity category as a derived localization

We put $V = \Sigma^{-1} J^\vee$ so that $\Omega C = T_R(V)$ as a graded algebra. We have an exact sequence of dg ΩC-modules

$$0 \to K \to \Omega C \to R \to 0.$$

Its underlying sequence of graded modules identifies with

$$0 \to V \otimes_R T_R(V) \to T_R(V) \to R \to 0\,,$$

where the morphism $V \otimes_R T_R(V) \to T_R(V)$ is just multiplication. Notice that the differential on $K = V \otimes_R T_R(V)$ is not $\mathbf{1}_V \otimes d_{\Omega C}$ but is induced by that of ΩC via the inclusion $K \to \Omega C$. It is not hard to see that the cone over $K \to \Omega C$ is isomorphic to $C \otimes_\tau \Omega C$. This shows in particular that this cone lies in $\operatorname{pretr}(\Omega C)$, the closure of ΩC under shifts and graded split extensions in the category of dg modules. It follows that K also lies in this category. Thus, we wish to describe the localization of $\operatorname{per}(\Omega C)$ with respect to the thick subcategory generated by the cone over the morphism

$$V \otimes_R (\Omega C) = K \to \Omega C$$

between two dg ΩC-modules in $\operatorname{pretr}(\Omega C)$. Let L be the graded algebra obtained from $T_R(V)$ by adjoining all the matrix coefficients of a formal inverse of the morphism

$$V \otimes_R T_R(V) \to T_R(V)$$

between free graded $T_R(V)$-modules. Since $V = \Sigma^{-1}J^\vee$, we see that via the choice of a basis of $J^\vee$ compatible with its isotypical decomposition as a semi-simple R^e-module, the algebra L becomes isomorphic to a graded Leavitt path algebra. We endow L with the unique differential such that the canonical morphism

$$\Omega C \to L$$

becomes a morphism of dg algebras. Clearly induction along the morphism $\Omega C \to L$ induces a triangle functor

$$\mathrm{per}\,(\Omega C) \to \mathrm{per}\,(L)$$

which annihilates the cone over $K \to \Omega C$ (indeed, the image of this morphism in $\mathrm{per}\,(L)$ is invertible and hence its cone becomes contractible) and thus induces a triangle functor

$$\mathrm{per}\,(\Omega C)/\mathrm{thick}\,(R) \to \mathrm{per}\,(L).$$

We claim that this functor is an equivalence. Indeed, this follows by combining Theorem 4.36 with (a slight generalization with a similar proof of) Corollary 4.15 in [11]. In the sections below, we sketch an alternative, alas not yet complete approach to the proof of the equivalence based on a theorem of Neeman–Ranicki [75]. By composition, we obtain an equivalence

$$\mathrm{sg}(A^{op})^{op} \xrightarrow{\sim} \mathrm{per}\,(L).$$

It is clear how to obtain a similar description of $\mathrm{sg}(A)$ itself.

3.5 Conjectural approach via Neeman–Ranicki's theorem

3.5.1 For rings concentrated in degree 0

Let R be a hereditary ring and S a set of morphisms $s : P_1 \to P_0$ between finitely generated projective (right) R-modules. Let R_S be the universal localization of R with respect to S in the sense of Cohn [23]. Thus, the ring R_S is endowed with a morphism $R \to R_S$ which is universal among the ring morphisms $R \to R'$ such that $s \otimes_R R' : P_1 \otimes_R R' \to P_0 \otimes_R R'$ is invertible for each $s \in S$. If s is a morphism between finitely generated free modules given by left multiplication by a $p \times q$-matrix M, then R_s is obtained from R by formally adjoining the entries of a matrix M' satisfying $MM' = I_p$ and $M'M = I_q$.

Let $\mathcal{R}$ be the localizing subcategory of the derived category $\mathcal{S} = \mathcal{D}R$ generated by the cones N_s over the morphisms $s \in S$. Put $\mathcal{T} = \mathcal{S}/\mathcal{R}$ so that we have an exact sequence of triangulated categories

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{T} \to 0.$$

Clearly the extension of scalars functor $? \overset{L}{\otimes}_R R_S : \mathcal{D}(R) \to \mathcal{D}(R_S)$ kills the N_s, $s \in S$, and commutes with arbitrary coproducts. Thus, it kills $\mathcal{R}$ and we have an induced canonical triangle functor

$$\mathcal{T} = \mathcal{S}/\mathcal{R} \to \mathcal{D}(R_S).$$

The following theorem is an immediate consequence of Neeman–Ranicki's work in [75].

Theorem 3.5.2 (Neeman–Ranicki)**.** *The canonical functor $\mathcal{T} \to \mathcal{D}(R_S)$ is an equivalence.*

Sketch of proof. Let $\mathcal{T} = \mathcal{D}(R)/\mathcal{N}$ and let $\pi : \mathcal{D}(R) \to \mathcal{D}(R)/\mathcal{N}$ be the quotient functor. One first shows that $\mathrm{Hom}_{\mathcal{T}}(\pi(R), \Sigma^n \pi(R))$ vanishes for $n \neq 0$. Thus, the image $\pi(R)$ is a tilting object in $\mathcal{T}$ and we have a triangle equivalence $\mathcal{D}(E) \xrightarrow{\sim} \mathcal{T}$, where $E = \mathrm{Hom}_{\mathcal{T}}(\pi(R), \pi(R))$. Now one shows that the morphism $R \to E$ given by π identifies with the universal localization $R \to R_s$. The detailed arguments are contained in [75]. $\square$

Example 3.5.3. *Let k be a field and Q a finite quiver. For each vertex i of Q which is the source of at least one arrow of Q, let s_i be the morphism*

$$P_i \to \bigoplus_{\alpha : i \to j} P_j$$

where $P_i = e_i kQ$ and the component of the map associated with α is the left multiplication by α. Let S be the (finite) set of the s_i. Clearly the hypotheses of the theorem hold so that $\mathcal{D}(R_S)$ identifies with the quotient of $\mathcal{D}(R)$ by the localizing subcategory generated by the cokernels (equivalently: cones) of the s_i.

Let us observe that the above theorem easily generalizes from rings to small categories and to small graded categories. Of course, its analogue holds for small graded k-categories, where k is a field. So let $\mathcal{P}$ be a small graded k-category whose category of graded modules (i.e. the category of k-linear graded functors with values in the category of $\mathbb{Z}$-graded vector spaces) is hereditary. Let S be a set of morphisms of $\mathcal{P}$ and $\mathcal{P}_S$ the localization of $\mathcal{P}$ at the set S in the sense of Gabriel–Zisman [30]. For example, if A is a graded k-algebra and S a set of homogeneous morphisms in the category of finitely generated graded projective right A-modules, then $\mathcal{P}_S$ is Morita-equivalent to the universal localization A_S of A at S.

Now let us denote by $\mathcal{P}_{hS}$ the localization of $\mathcal{P}$ as a dg category in the sense of Drinfeld [25]. By the main result of [25], the canonical functor

$$\mathcal{T} = \mathcal{D}(\mathcal{P})/\mathcal{N} \to \mathcal{D}(\mathcal{P}_{hS})$$

is an equivalence. As a consequence, we obtain the following variant of Neeman–Ranicki's theorem.

Theorem 3.5.4 (Neeman–Ranicki)**.** *The canonical morphism $\mathcal{P}_{hS} \to \mathcal{P}_S$ is a quasi-equivalence.*

3.5.5 The differential graded case

Let k be a commutative ring and dgcat_k the category of small dg k-categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a dg functor between small dg categories. The functor F is a *derived localization* if the induced functor $F^* : \mathcal{DA} \to \mathcal{DB}$ is a Verdier localization. Recall the two main model structures on the category of dg categories due to Tabuada [87, 89]: the Dwyer-Kan model structure of [87], whose weak equivalences are the quasi-equivalences, and the Morita model structure of [89], whose weak equivalences are the Morita functors, *i.e.* the dg functors $F : \mathcal{A} \to \mathcal{B}$ such that $F^* : \mathcal{DA} \to \mathcal{DB}$ is an equivalence. Recall that these are cofibrantly generated model structures whose sets of generating cofibrations are identical and consist of the following dg functors:

a) the inclusion $\emptyset \to k$ of the empty dg category into the one-object dg category given by the dg algebra k and

b) the dg functors $S_n : \mathcal{C}(n) \to \mathcal{P}(n)$, defined for $n \in \mathbb{Z}$, where $\mathcal{C}(n)$ and $\mathcal{P}(n)$ are the dg categories with two objects $1, 2$ respectively $3, 4$ whose only non trivial morphism complexes are $\mathcal{C}(n)(1, 2) = S^{n-1}$ respectively $\mathcal{P}(n) = D^n$, where S^{n-1} is $\Sigma^{n-1}k$ and D^n the cone over the identity of S^{n-1}. The dg functor S_n maps 1 to 3 and 2 to 4 and induces the inclusion $S^{n-1} \to D^n$ of S^{n-1} into the cone over its identity morphism.

A dg category is *finitely cellular* if it is obtained from the empty dg category by a finite number of pushouts along functors in a) or b). Equivalently, it is the path category of a graded quiver Q such that the set of arrows Q_1 admits a filtration

$$\emptyset = F_0Q_1 \subset F_1Q_1 \subset \cdots \subset F_nQ_1 = Q_1$$

such that the differential maps the graded path category of (Q_0, F_pQ_1) to that of $(Q_0, F_{p-1}Q_1)$ for each $1 \leq p \leq n$.

From now on, let us assume that k is a field and that A is a finitely cellular dg algebra given by a graded quiver Q and a differential on kQ. Let $\mathcal{P}$ be the full subcategory of the dg category of right dg A-modules whose objects are the finite direct sums of the $\Sigma^p P_i = e_iA$, $i \in Q_0$, $p \in \mathbb{Z}$. Let S be a set of (closed) morphisms of $\mathcal{P}$.

Conjecture 3.5.6. *The canonical dg functor $\mathcal{P}_{hS} \to \mathcal{P}_S$ is a quasi-equivalence.*

We have not yet proved the conjecture but believe the following strategy is promising: The given filtration on A yields a filtration on $\mathcal{P}$ indexed by $\mathbb{N}$. The localization $\mathcal{P}$ admits a filtration indexed by

$\mathbb{Z}$ such that the functor $\mathcal{P} \to \mathcal{P}_S$ becomes universal among the dg functors respecting the filtration and making the elements of S invertible. We would like to describe the associated graded category $\mathrm{gr}\,(\mathcal{P}_S)$. Each morphism s in S is given by a matrix whose entries are linear combinations of paths of Q. The filtration degree d of s is the maximum of the degrees of the paths appearing with non zero coefficients. We write $\sigma(s)$ for the image of s in the dth graded component of $\mathrm{gr}\,(\mathcal{P})$ and we write $\sigma(S)$ for the set of morphisms of $\mathrm{gr}\,(\mathcal{P})$ formed by the $\sigma(s)$, $s \in S$. It is clear that the $\sigma(s)$ become invertible in $\mathrm{gr}\,(\mathcal{P}_S)$.

Lemma 3.5.7. *The canonical morphism functor*

$$\mathrm{gr}\,(\mathcal{P})_{\sigma(S)} \to \mathrm{gr}\,(\mathcal{P}_S)$$

is invertible.

Recall that $\mathcal{P}_{hS}$ is obtained from $\mathcal{P}$ by adjoining, for each $s : P_1 \to P_2$ in S,

- a morphism $t : P_2 \to P_1$ of degree 0,
- endomorphisms h_i of P_i homogeneous of degree -1 such that $d(h_1) = ts - \mathbf{1}_{P_1}$ and $d(h_2) = st - \mathbf{1}_{P_2}$,
- a morphism $u : P_1 \to P_2$ of degree -2 such that $d(u) = h_2 s - s h_1$.

We see that $\mathcal{P}_{hS}$ admits a $\mathbb{Z}$-indexed filtration such that the canonical functor $\mathcal{P} \to \mathcal{P}_{hS}$ becomes universal among the functors respecting the filtration.

Lemma 3.5.8. *The canonical functor*

$$\mathrm{gr}\,(\mathcal{P})_{h\sigma(S)} \to \mathrm{gr}\,(\mathcal{P}_{hS})$$

is invertible.

Clearly we have a commutative square

$$\begin{array}{ccc} \mathrm{gr}(\mathcal{P})_{h\sigma(S)} & \longrightarrow & \mathrm{gr}(\mathcal{P}_{hS}) \\ \downarrow & & \downarrow \\ \mathrm{gr}(\mathcal{P})_{\sigma(S)} & \longrightarrow & \mathrm{gr}(\mathcal{P}_S) \end{array}$$

By the two preceding lemmas, the horizontal functors are invertible. By Neeman–Ranicki's theorem, the left vertical arrow is a quasi-equivalence. Thus, the canonical functor

$$\mathrm{gr}\,(\mathcal{P}_{hS}) \to \mathrm{gr}\,(\mathcal{P}_S)$$

is a quasi-equivalence. We would like to conclude that the canonical functor $\mathcal{P}_{hS} \to \mathcal{P}_S$ is a quasi-equivalence. Unfortunately, this is not clear because the filtrations are indexed by $\mathbb{Z}$ rather than $\mathbb{N}$.

Chapter 4

Cyclic homology and relative Calabi–Yau structures

These are notes taken by the author for a series of three lectures by Bernhard Keller on relative Calabi–Yau structures given at the workshop of the International Conference on Representations of Algebras which was held online in November 2020. Relative Calabi–Yau structures serve in particular to construct relative cluster categories associated with ice quivers with potential, *cf.* [102, 100], and these are closely related to singularity categories, which establishes the link to the main theme of this thesis.

4.1 Introduction

In 1957, the Italo-American geometer Eugenio Calabi conjectured [15] that each Kähler manifold whose first Chern class vanishes admits a Ricci flat metric. His conjecture was proved twenty years later by Shing-Tung Yau [103]. Algebraic varieties with trivial canonical bundle are now called Calabi–Yau varieties. The proper ones among these are characterized by the fact that their bounded derived category admits a Serre functor isomorphic to a power (the dimension) of the suspension functor. Following Kontsevich, a Hom-finite (algebraic) triangulated category with this property is now called a Calabi–Yau triangulated category. In these notes, we are concerned with a relative version of this notion that was first sketched by Toën [93] in 2014 and fully developed by Brav and Dyckerhoff in [13, 12]. One of the key features of their notion of (left) relative Calabi–Yau structure is a gluing construction analogous to that in cobordism of manifolds. Wai-Kit Yeung showed how to construct large classes of examples using relative Calabi–Yau completions in [64, 104] and advocated the idea that these should be viewed as non commutative conormal bundles. This was justified using Kontsevich–Rosenberg's criterion by Bozec–Calaque–Scherotzke in [10].

In the representation theory of quivers and finite-dimensional algebras, the motivations for studying relative Calabi–Yau structures come from at least three sources:

- applications in the study of Fukaya categories (cf. for example Brav–Dyckerhoff's [13, 12]),
- the categorification of cluster algebras with coefficients (as in the work of Geiss–Leclerc–Schröer [31], Leclerc [67], Jensen–King–Su [51], Pressland [81, 80, 79, 82] ... as well as [101]),
- its close links with higher Auslander–Reiten theory (to be illustrated below, cf. also [101, 102]).

In these notes, after an informal illustration of the main notions and a quick reminder on dg (=differential graded) algebras and their derived categories, our first aim will be to present many examples of dg algebras and morphisms endowed with Calabi–Yau structures, respectively relative Calabi–Yau structures. These will be obtained using (relative) Calabi–Yau completions and should illustrate the relevance of these for (higher) Auslander–Reiten theory. Our second aim will be to sketch the foundations of the subject for which we will need to recall the necessary material on Hochschild and cyclic homology.

4.2 Intuition and first examples

In this purely introductory section, we informally discuss the key notions and constructions to be developed in the sequel.

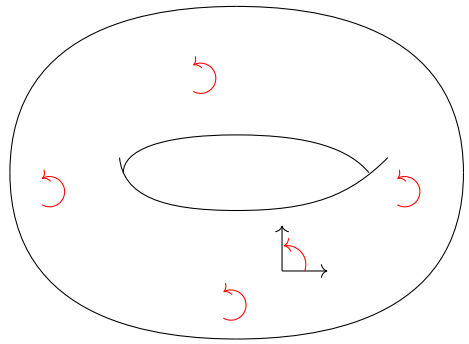
There is a close analogy (which actually goes deeper) between the notion of orientation of a (real, smooth) n-dimensional manifold M and the notion of (absolute) n-Calabi–Yau structure on a dg algebra A. This extends to a relative setting[1] where we obtain a close analogy between an n-dimensional manifold *with boundary* $\partial M \subset M$ both endowed with compatible orientations and a *relative* n-Calabi–Yau structure on a *morphism* $B \to A$ of dg algebras.

[1] Notice that we do not write *the* relative setting because the setting considered here is *not* the relative one in the usual sense of algebraic geometry.

dg algebra

+

Calabi–Yau structure
(absolute)

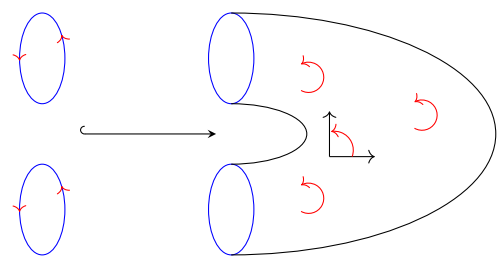


manifold

+

orientation

dg algebra morphism
$B \longrightarrow A$
+
relative Calabi–Yau structure



manifold with
boundary
+
orientation

In differential geometry, we obtain examples of oriented manifolds with boundary by taking the conormal bundle of the inclusion of a submanifold. In the homological theory of dg algebras, we obtain examples of morphisms endowed with relative Calabi–Yau structures by forming the n-Calabi–Yau completion $B \to A$ of a morphism $i_0 : B_0 \to A_0$ of smooth dg algebras over a field k. For example, let us take B_0 to be k, the algebra A_0 to be the path algebra of the quiver $1 \to 2 \to 3$ and i_0 the morphism given by the inclusion of the vertex 3 into this quiver as in Figure 4.1. If we apply the relative 2-Calabi–Yau completion to this morphism, we find the algebra morphism from the polynomial algebra $k[t]$ to the Auslander algebra of the truncated polynomial algebra $k[x]/(x^3)$ which takes the unique object to the indecomposable projective $P = k[x]/(x^3)$ and the indeterminate t to the multiplication by x.

In Figure 4.2, we consider an example of a relative 3-Calabi–Yau completion. The given morphism is the inclusion of the path algebra A of the linearly oriented A_3-quiver into its Auslander algebra. The 3-Calabi–Yau completion is a morphism from the 2-dimensional Ginzburg algebra of type A_3 to a relative Jacobian algebra, namely the one associated with the ice quiver in the lower half of the figure endowed with the potential given by the difference between the sum of the 3-cycles rotating clockwise and those rotating counter-clockwise.

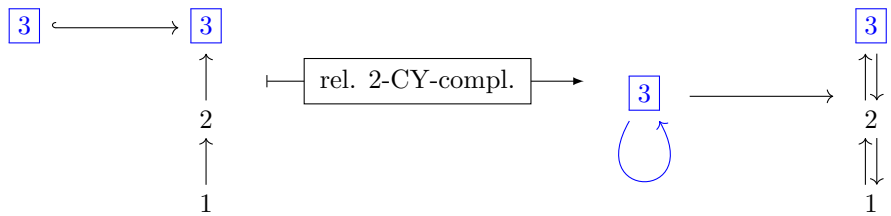


Figure 4.1: Relative 2-Calabi–Yau completion

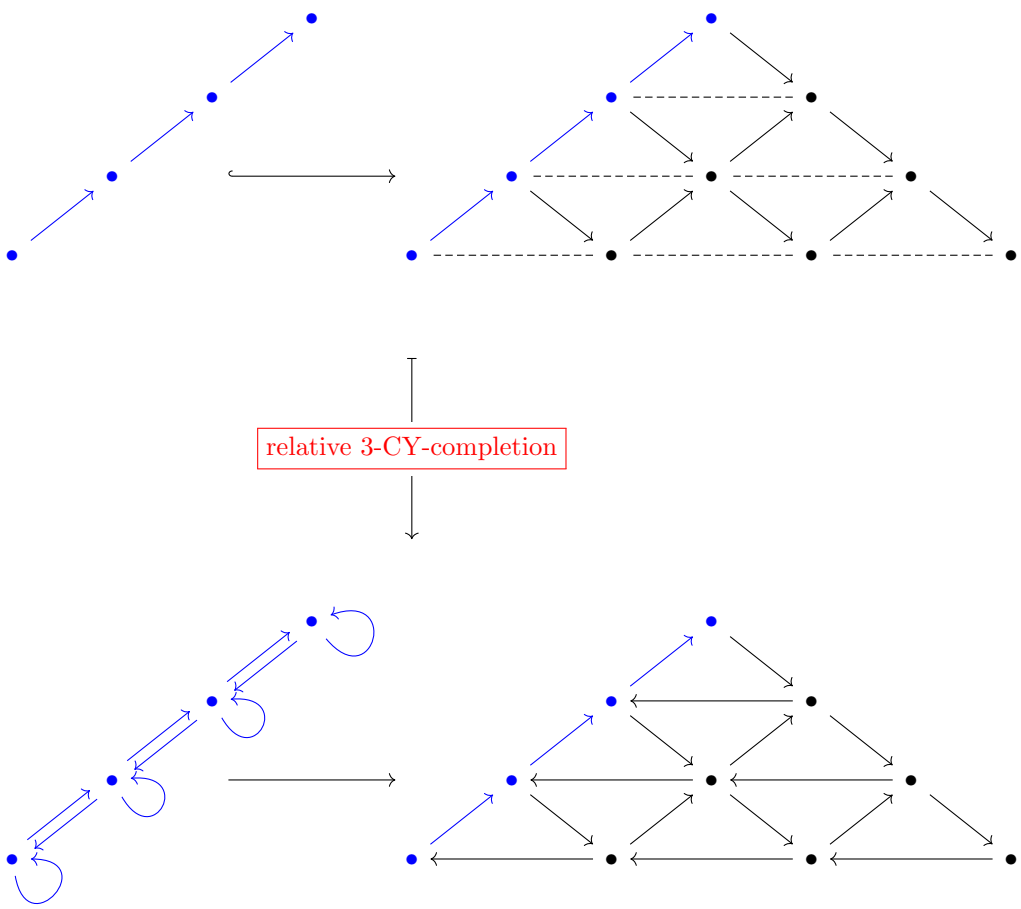


Figure 4.2: Example of a relative 3-Calabi–Yau completion

4.3 Complements on dg algebra resolutions

Let k be a field. Let Q be a *graded quiver*, *i.e.* a quiver (Q_0, Q_1, s, t) endowed with a function $|?| : Q_1 \to \mathbb{Z}$. For $n \in \mathbb{Z}$ we write Q_1^n for the set of arrows α of Q of degree $|\alpha| = n$. The associated path algebra kQ then becomes a graded algebra if we define the degree of a path $a_1 \dots a_l$ to be the sum of the degrees of the composable arrows a_i. As a typical example, let A be the path algebra of the quiver

$$1 \xrightarrow{b} 2 \xrightarrow{a} 3$$

subject to the relation $ab = 0$ (all arrows in degree 0). Let $\widetilde{A}$ be the graded path algebra of the graded quiver

$$\begin{tikzcd} 1 \arrow[r, "b"] \arrow[rr, bend right, "c"'] & 2 \arrow[r, "a"] & 3 \end{tikzcd}$$

where a and b have degree 0 and c has degree -1. We endow the graded path algebra $\widetilde{A}$ with the unique algebra differential such that $d(c) = ab$. Then the algebra morphism $\widetilde{A} \to A$ taking a to a, b to b and c to zero is compatible with the differential (that of A being zero) and is in fact a quasi-isomorphism of dg algebras. This is an example of a cofibrant dg algebra resolution.

Proposition 4.3.1. *For each quotient $A = kQ/I$ of a path algebra with finite Q_0, there is a dg algebra quasi-isomorphism*

$$\widetilde{A} \xrightarrow{\varepsilon} A$$

where $\widetilde{A} = (k\widetilde{Q}, d)$ for a non positively graded quiver $\widetilde{Q}$ with $\widetilde{Q}_0 = Q_0$, $Q_1^0 = Q_1$ such that ε induces the identity $\widetilde{Q}_0 = Q_0$ and the canonical projection $k\widetilde{Q}_1 \to kQ_1$. The morphism ε is a cofibrant dg algebra resolution.

In general, there is no 'minimal choice' for $\widetilde{A}$. However, there is if kQ/I is a monomial algebra, *cf.* [91], or if we work in the setup of pseudocompact dg algebras, *cf.* [95].

4.4 Derived categories and the inverse dualizing bimodule

Let k be a *perfect* field and A a dg algebra. Recall that $\mathcal{C}A$ denotes the category of right dg A-modules, $\mathcal{H}A$ the associated category up to homotopy and $\mathcal{D}A$ the derived category, *i.e.* the localization of $\mathcal{C}A$ or $\mathcal{H}A$ at the class of all quasi-isomorphisms. It is triangulated with suspension functor $\Sigma : \mathcal{D}A \to \mathcal{D}A$ given by the shift of dg modules and each short exact sequence of dg modules yields a canonical triangle. The derived category $\mathcal{D}A$ admits arbitrary (set-indexed) coproducts. The perfect derived category $\operatorname{per}(A)$ is

its subcategory of compact objects. It equals the thick subcategory of $\mathcal{D}A$ generated by the free module A_A of rank 1.

The *perfectly valued derived category* $\mathrm{pvd}\,(A)$ is the full subcategory of $\mathcal{D}A$ whose objects are the dg modules M whose underlying dg k-module is perfect. Equivalently, we may require that we have

$$\sum_{p\in\mathbb{Z}} \dim \mathrm{Hom}(P, \Sigma^p M) < \infty$$

for each perfect dg module P or that we have

$$\sum_{p\in\mathbb{Z}} \dim H^p(M) < \infty.$$

The dg algebra A is *proper* if it belongs to $\mathrm{pvd}\,(A)$ or, in other words, if the sum of its homologies is finite-dimensional. For example, if $A = A^0$ is concentrated in degree 0, then A is proper if and only if A^0 is finite-dimensional and in this case, we have

$$\mathrm{pvd}\,(A) = \mathcal{D}^b(\mathrm{mod}\, A^0)\,,$$

where $\mathrm{mod}\, A^0$ denotes the category of k-finite-dimensional right A^0-modules.

The dg algebra A is *connective* if $H^pA = 0$ for all $p > 0$. In this case, the canonical morphism $\tau_{\leq 0}A \to A$ of dg algebras is a quasi-isomorphism so that for most purposes, we may assume that $A^p = 0$ for $p > 0$ when A is connective. If A is connective, then $\mathcal{D}A$ has a canonical t-structure whose left (resp. right) aisle is formed by the dg modules M such that $H^pM = 0$ for $p > 0$ (resp. for $p < 0$). Its heart identifies with $\mathrm{Mod}\, H^0(A)$ via the restriction along the canonical morphism $A \to H^0(A)$ (we may suppose $A^p = 0$ for $p > 0$). We say that A is a *stalk dg algebra* or simply *stalk algebra* if $H^p(A) = 0$ for all $p \neq 0$. In this case, A is linked to the dg algebra $H^0(A)$ concentrated in degree 0 via the two dg algebra quasi-isomorphisms

$$H^0(A) \leftarrow \tau_{\leq 0}A \to A.$$

Let A be a dg algebra. Its *enveloping algebra* is

$$A^e = A \otimes A^{op}$$

so that right A^e-modules identify with A-A-bimodules via the rule

$$m.(a \otimes b) = (-1)^{|b||m|+|b||a|} bma.$$

For a dg bimodule M, its *bimodule dual* $M^\vee$ is defined by

$$M^\vee = \mathrm{RHom}_{A^e}(M, A^e).$$

We still view it as an object of $\mathcal{D}(A^e)$ using the canonical isomorphism $(A^e)^{op} = A^e$. More explicitly, the derived Hom-space is computed using the outer bimodule structure on A^e and the bimodule structure on RHom comes from the inner bimodule structure on A^e. Notice that if P is perfect in $\mathcal{D}(A^e)$, we have a canonical isomorphism

$$P \xrightarrow{\sim} (P^\vee)^\vee.$$

Indeed, it suffices to check this for $P = A_A$ and then it is clear.

The dg algebra A is (homologically) *smooth* if the *identity bimodule* ${}_AA_A$ is perfect in $\mathcal{D}(A^e)$. For example, if $A = A^0$ is concentrated in degree 0, then A is smooth if and only if ${}_AA_A$ has a bounded resolution by finitely generated projective bimodules. If $A = A^0$ is finite-dimensional, then A is smooth if and only if it is of finite global dimension (here we use the assumption that k is perfect!).

Lemma 4.4.1. *a) If A is smooth, the category* $\mathrm{pvd}\,(A)$ *is Hom-finite.*

b) If Q is a finite non positively graded quiver, then for any choice of algebra differential d on the graded path algebra kQ, the dg algebra (kQ, d) is smooth.

Suppose that A is a smooth dg algebra. Its *inverse dualizing bimodule* is

$$\Omega_A = A^\vee = \mathrm{RHom}_{A^e}(A, A^e).$$

Denote by D the duality $\mathrm{Hom}_k(?, k)$ over the ground field. The following is Lemma 4.1 of [63]

Lemma 4.4.2. *For $L \in \mathrm{pvd}\,(A)$ and $M \in \mathcal{D}A$, we have a canonical isomorphism*

$$D\mathrm{Hom}_{\mathcal{D}A}(L, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(M \overset{L}{\otimes}_A \Omega_A, L).$$

Corollary 4.4.3. *The functor $S^{-1} = ? \overset{L}{\otimes}_A \Omega_A$ induces an* inverse Serre functor *on* $\mathrm{pvd}\,(A)$, i.e. *an autoequivalence such that we have isomorphisms*

$$D\mathrm{Hom}_{\mathcal{D}A}(L, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(S^{-1}M, L)$$

which are bifunctorial in $L, M \in \mathrm{pvd}\,(A)$.

4.5 Calabi–Yau completions

Fix a perfect field k. Let A be a dg k-algebra. Fix an integer $n \in \mathbb{Z}$. By definition, a *bimodule n-Calabi–Yau structure on A* is an isomorphism

$$\Sigma^n \Omega_A \xrightarrow{\sim} A$$

in $\mathcal{D}(A^e)$. Notice that by Lemma 4.4.2, the category $\mathrm{pvd}\,(A)$ is then an n-Calabi–Yau category, *i.e.* it is Hom-finite and Σ^n is a Serre functor. We say that A is a *bimodule n-Calabi–Yau dg algebra* if it is endowed with a bimodule n-CY structure.

Exercise 4.5.1. *Suppose that A is bimodule n-Calabi–Yau and $H^0(A)$ is not semi-simple.*

a) Show that if A is connective and H^0A is finite-dimensional, then $H^pA \neq 0$ for infinitely many $p < 0$.

b) Show that if A is a stalk dg algebra, then H^0A is infinite-dimensional.

4.5.2 Absolute Calabi–Yau completions

Suppose that B is a smooth dg algebra. Let $n \in \mathbb{Z}$ be an integer and put

$$\omega = \Sigma^{n-1}\Omega_B.$$

We may and will assume that ω is cofibrant as a dg B-bimodule. Following [57], we define the *n-Calabi–Yau completion* of B to be the tensor dg algebra

$$\Pi_n B = T_B(\omega) = B \oplus \omega \oplus (\omega \otimes_B \omega) \oplus \cdots \oplus \omega^{\otimes_B p} \oplus \cdots \ .$$

It is not hard to check that up to quasi-isomorphism, $\Pi_n B$ does not depend on the choice of ω in its homotopy class.

Theorem 4.5.3 ([64, 104, 55])**.** *The dg algebra $\Pi_n B$ is smooth and carries a canonical bimodule n-Calabi–Yau structure. In particular, the category* $\mathrm{pvd}\,(\Pi_n B)$ *is n-Calabi–Yau.*

For example, consider $B = k$. Then $\Omega_B = k$ and $\Pi_n(B) = k[t]$, where t is of degree $1 - n$ and $\Pi_n B$ carries the zero differential. In particular, $\Pi_n k$ is concentrated in degree 0 iff $n = 1$. As another example, let Q be a connected non Dynkin quiver and $B = kQ$. Then the 2-Calabi–Yau completion $\Pi_2 B$ has its homology concentrated in degree 0 and is quasi-isomorphic to the preprojective algebra of Q.

Connective dg algebras form a particularly important class. It is natural to ask to which extent this class is stable under forming CY-completions. This question is not hard to answer: Suppose that B is a smooth and connective dg algebra. Define the *bimodule dimension* d of B to be the supremum of the integers p such that $H^p\Omega_B \neq 0$. Then $\Pi_n B$ is connective if and only if $n \geq d + 1$.

Let us investigate 2-CY-completions. Let B be a smooth dg algebra. For $n = 2$, the bimodule ω is $\omega = \Sigma\Omega_B$ and the functor

$$? \overset{L}{\otimes}_B \omega = \overset{L}{\otimes}_B(\Sigma\Omega_B) = \Sigma \circ (? \overset{L}{\otimes}_B \Omega_B)$$

induces the composed functor

$$\tau^{-1} = S^{-1}\Sigma$$

in the perfect derived category $\operatorname{pvd}(B)$, where S^{-1} is the inverse Serre functor. We denote this functor by τ^{-1} because if B is a finite-dimensional algebra of finite global dimension, then it is the inverse Auslander–Reiten translation of the category $\operatorname{pvd}(B) = \mathcal{D}^b(\operatorname{mod} B)$. For arbitrary smooth and proper B, the restriction of $\Pi_2(B)$ to a dg B-module is

$$(\Pi_2(B))|_B = T_B(\omega)|_B = B \oplus (B \overset{L}{\otimes}_B \omega) \oplus (B \overset{L}{\otimes}_B \omega \overset{L}{\otimes}_B \omega) \oplus \cdots = \bigoplus_{p \geq 0} \tau^{-p} B.$$

Notice that in general, it will not be perfectly valued. Now suppose that $B = kQ$ for a connected non Dynkin quiver Q. Then we know that $\tau^{-p}B$ lies in $\operatorname{mod} B$ for all $p \geq 0$ so that $\Pi_2(B)$ is a stalk dg algebra:

$$\Pi_2(B) \xrightarrow{\sim} H^0(\Pi_2 B) = T_B(H^0\omega).$$

It follows that $\Pi_2(B)$ is quasi-isomorphic to the classical preprojective algebra of Q by the description of this algebra as a tensor algebra due to Baer–Geigle–Lenzing [7].

If Q is an arbitrary finite acyclic quiver and $B = kQ$, then $\Pi_2(B)$ can be described as the 2-dimensional Ginzburg algebra of B. For example, for the quiver

$$\begin{tikzcd} 1 \arrow[r, "b"] \arrow[rr, bend right, "c"'] & 2 \arrow[r, "a"] & 3 \end{tikzcd}$$

the dg algebra $\Pi_2(B)$ is given by the graded quiver

$$Q: \quad \begin{tikzcd} 1 \arrow[loop above, "t_1"] \arrow[r, shift left, "b"] & 2 \arrow[l, shift left, "\overline{b}"] \arrow[loop above, "t_2"] \arrow[r, shift left, "a"] & 3 \arrow[l, shift left, "\overline{a}"] \arrow[loop above, "t_3"] \end{tikzcd}$$

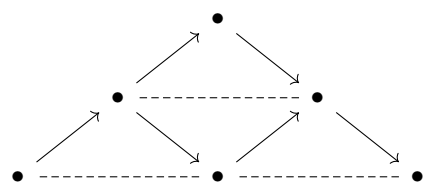
with the arrows a, b, $\overline{a}$ and $\overline{b}$ in degree 0 and the three loops t_i in degree -2. The differentials of the loops yield the preprojective relations:

$$dt_1 = -\overline{b}b\,,\ dt_2 = b\overline{b} - \overline{a}a\,,\ dt_2 = a\overline{a}.$$

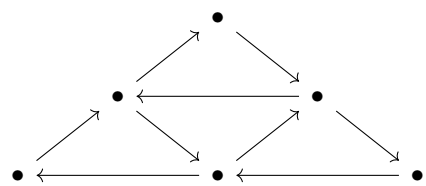
Thus, we always have an isomorphism between $H^0(\Pi_2(B))$ and the classical preprojective algebra but the homologies $H^p(\Pi_2(B))$ are non zero in infinitely many degrees $p < 0$ unless all connected components of Q are non Dynkin.

Let us now consider an example of a 3-CY-completion. Consider the Auslander algebra B of the

equioriented A_3-quiver given by



Using a cofibrant dg algebra resolution of B (*cf.* 4.3.1) it is not hard to check that $\Pi_3(B)$ is quasi-isomorphic to the 3-dimensional Ginzburg algebra of the quiver with potential (R, W), where R is the quiver of the 'relation completion' of B



and W the difference of the sum of the 3-cycles of R turning clockwise minus the unique 3-cycle turning counterclockwise. For completeness, let us recall how to construct the 3-dimensional Ginzburg algebra $\Gamma(R, W)$ associated with a quiver with potential (R, W): starting from R, construct a quiver $\widetilde{R}$ as follows:

a) for each arrow $a : i \to j$ of R, add an arrow $\overline{a} : j \to i$ of degree -1 and

b) for each vertex i of R, add a loop $t_i : i \to i$ of degree -2.

Now define the differential on the graded path algebra $k\widetilde{R}$ by

$$d(t_i) = e_i \sum_{a \in R_1} (a\overline{a} - \overline{a}a)e_i$$

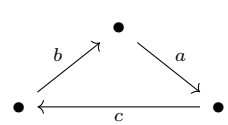
for each vertex i of $\widetilde{R}$ and

$$d\overline{a} = \partial_a W$$

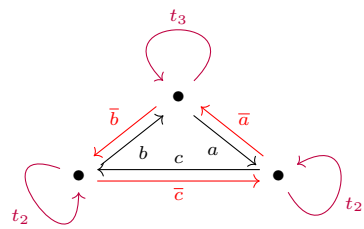
for each arrow a of R. Here $\partial_a : HH_0(kR) \to kR$ is the cyclic derivative defined on a path p by

$$\partial_a p = \sum_{p=uav} vu$$

where the sum ranges over all decompositions $p = uav$ with paths u, v of length ≥ 0. For example, starting from



with the potential $W = abc$ we obtain the quiver $\widetilde{R}$



with the differential determined by

$$d(t_1) = c\overline{c} - \overline{b}b\,,\ d(t_2) = a\overline{a} - \overline{c}c\,,\ d(t_3) = b\overline{b} - \overline{a}a$$

and

$$d(\overline{a}) = bc\,,\ d(\overline{b}) = ca\,,\ d(\overline{c}) = ab.$$

4.6 Relative Calabi–Yau completions

Let k be a perfect field and A and B smooth dg k-algebras. Let $f : B \to A$ be a dg algebra morphism. Recall that we do not require f to preserve the unit. A typical example would be the inclusion of a finite-dimensional representation-finite algebra of finite global dimension into its Auslander algebra. Let n be an integer.

Following Yeung [104], we make the following definitions:

a) the *relative inverse dualizing bimodule* is the bimodule dual of the cone over the natural morphism

$$A \overset{L}{\otimes}_B A \to A$$

considered as an object in $\mathcal{D}(A^e)$.

b) the n-dimensional *relative derived preprojective algebra* of $B \to A$ is

$$\Pi_n(A, B) = T_A(\omega)\,,$$

where ω is a cofibrant resolution of $\Sigma^{n-1}\Omega_{A,B}$.

c) the *relative n-Calabi–Yau completion* of $f : B \to A$ is the canonical morphism of dg algebras

$$\Pi_{n-1}(B) \to \Pi_n(A, B).$$

Let us explain how the canonical morphism in c) is obtained: By construction, we have a triangle in $\mathcal{D}(A^e)$

$$\Omega_{A,B} \to A^\vee \to (A \overset{L}{\otimes}_B A)^\vee \to \Sigma\Omega_{A,B}.$$

This yields morphisms

$$\Omega_B = B^\vee \to A \overset{L}{\otimes}_B B^\vee \overset{L}{\otimes}_B A \xrightarrow{\sim} (A \overset{L}{\otimes}_B A)^\vee \to \Sigma\Omega_{A,B}.$$

The canonical morphism between the tensor algebras is induced by their composition. We will define the notion of a relative (left) n-CY structure below. Assuming it we can state the following theorem.

Theorem 4.6.1 (Yeung [104], Bozec–Calaque–Scherotzke [10])**.** *The dg algebra $\Pi_n(A, B)$ is smooth and the morphism*

$$\Pi_{n-1}(B) \to \Pi_n(A, B)$$

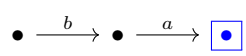
carries a canonical relative (left) n-CY structure.

Let us emphasize that all constructions and theorems generalize easily from the setting of dg algebras and morphisms to that of dg categories and dg functors and are proved in this generality in the references.

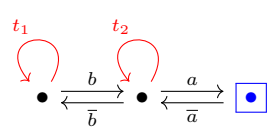
4.7 Examples of relative 2-Calabi–Yau completions

Let us recall the following example from section 4.2: Let us take B to be k, the algebra A to be the path algebra of the quiver $1 \to 2 \to 3$ and i the morphism given by the inclusion of the vertex 3 into this quiver as in Figure 4.3. If we apply the relative 2-Calabi–Yau completion to this morphism, we find a morphism of dg algebras $\Pi_1(B) \to \Pi_2(A, B)$ which, remarkably, are both stalk algebras: the dg algebra $\Pi_1(B)$ is quasi-isomorphic to the polynomial algebra $k[t]$ and the dg algebra $\Pi_2(A, B)$ to the truncated polynomial algebra $k[x]/(x^3)$. The dg functor $\Pi_1(B) \to \Pi_2(A, B)$ takes the unique object to the indecomposable projective $P = k[x]/(x^3)$ and the indeterminate t to the multiplication by x.

More generally, let us consider a finite quiver Q and a subset $F \subset Q_0$ of frozen vertices. We consider F as a subquiver with empty set of arrows. Then we have a natural algebra morphism $kF \to kQ$. We then find that $\Pi_2(kQ, kF)$ is the 2-*dimensional relative Ginzburg algebra of* (Q, F). For example, for the quiver



we find that $\Pi_2(kQ, kF)$ is given by



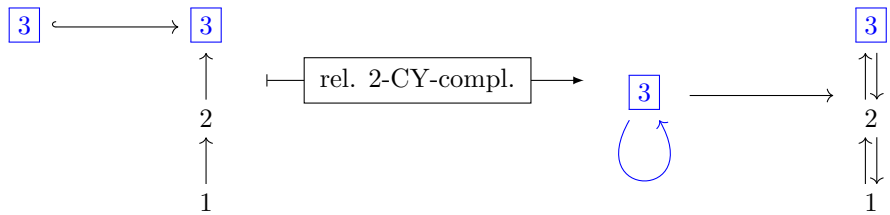
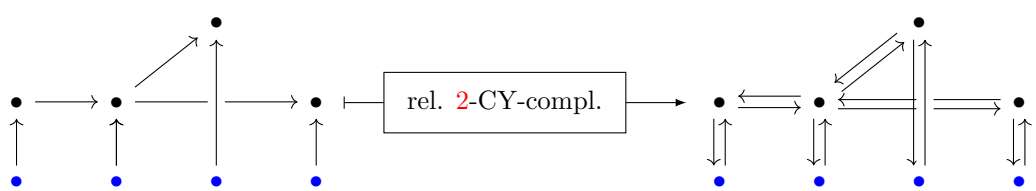


Figure 4.3: Relative 2-Calabi–Yau completion

where the arrows $a, b, \overline{a}$ and $\overline{b}$ are in degree 0, the loops t_i in degree -1 and the differential is determined by

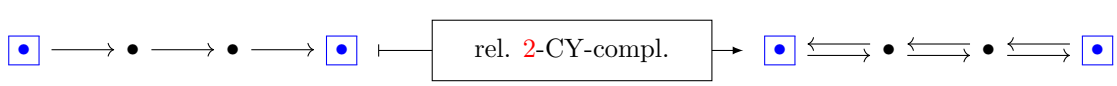
$$d(t_1) = -\overline{b}b\ ,\ d(t_2) = b\overline{b} - \overline{a}a.$$

Thus, the algebra $H^0\Pi_2(kQ, kF)$ is the *relative preprojective algebra*: it has the same quiver as the classical preprojective algebra but no relations at the frozen vertices. In our example here, we see that remarkably, $\Pi_2(kQ, kF)$ is a *stalk algebra*. Here is another example:



Here we start from a 'framed' quiver of type D_4 and obtain again a relative preprojective algebra (in particular a stalk algebra) that could be called the 'Nakajima algebra' because its (stable) representations (up to isomorphism) with a given dimension vector identify with the points of the corresponding (regular) Nakajima quiver variety.

As a final example, let us consider



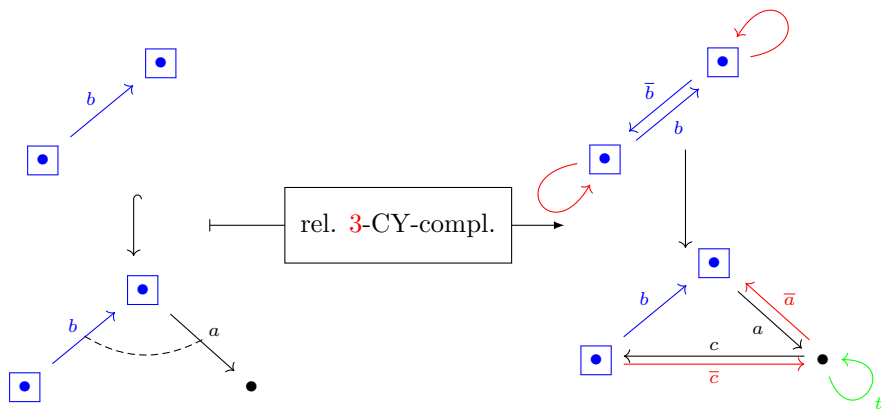
Again, the dg algebra $\Pi_2(A, B)$ is a stalk algebra. Let us consider its variant $\widehat{\Pi}_2(A, B)$ where we replace the path algebra with the completed path algebra. This is easily seen to be isomorphic to the Auslander algebra of the *Bass order*

$$B_3 = \begin{bmatrix} R & Rx^3 \\ R & R \end{bmatrix},$$

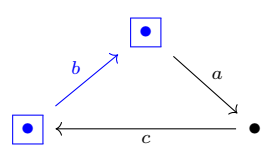
where $R = k[[x]]$ whose indecomposables are the $[R, Rx^i]$, $0 \leq i \leq 4$. For more information on Bass orders, we refer to page 72, of [47].

4.8 Examples of relative 3-Calabi–Yau completions

Consider the following example of a 3-CY-completion:



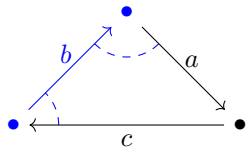
Here, the morphism $B \to A$ on the left hand side is the embedding of the path algebra of the quiver A_2 into its Auslander algebra. On the right hand side, we have the (absolute) 2-dimensional Ginzburg algebra $\Pi_2(B)$ at the top and the 3-dimensional relative preprojective algebra $\Pi_3(A, B)$ at the bottom. It is quasi-isomorphic to the relative 3-dimensional Ginzburg algebra of the ice quiver with potential (R, W) with R given by



and the potential $W = abc$. The construction of the relative 3-dimensional Ginzburg algebra is similar to that of the absolute one but

- we do not add reversed arrows $\overline{b}$ for frozen arrows b and
- we do not add loops t_i for frozen vertices i.

In the example here, the relative 3-dimensional algebra turns out to be a stalk algebra, *i.e.* quasi-isomorphic to the corresponding relative Jacobian algebra, which is given by



Note that the second example in section 4.2 is the analogous example for equioriented A_4 instead of A_2. In the next section, we will exhibit a general framework into which these examples fit.

4.9 Higher Auslander algebras and the stalk property in dimension ≥ 3

Let $n \geq 1$ be an integer and B a finite-dimensional algebra of finite global dimension over a perfect field k. Denote by S the Serre functor of the bounded derived category $\mathcal{D}^b(B) = \mathcal{D}^b(\operatorname{mod} B)$. Explicitly, we have

$$S = ? \overset{L}{\otimes}_B DB.$$

The composition of functors

$$\operatorname{mod} B \xrightarrow{\text{can}} \mathcal{D}^b(B) \xrightarrow{\Sigma^n S^{-1}} \mathcal{D}^b(B) \xrightarrow{H^0} \operatorname{mod} B$$

equals by definition τ_n^{-1}, the *higher inverse Auslander-Reiten translation* introduced by Iyama [48]. Define

$$\mathcal{M} = \operatorname{add}\{\tau_n^{-p} B \mid p \geq 0\}.$$

The motivating example is the case where $n = 1$ and B is the path algebra kQ of a Dynkin quiver Q. Then B is 1-dimensional and $\operatorname{mod}(B) = \mathcal{M}$ is 2-dimensional and in the previous section, we have seen examples where $\Pi_3(\operatorname{mod}(B), \operatorname{proj}(B))$ is concentrated in degree 0.

Put $\mathcal{B} = \operatorname{proj} B$ and $\mathcal{A} = \mathcal{M}$. As shown by Iyama in [48], if $\operatorname{mod} B$ admits an n-cluster-tilting object M, then the category $\mathcal{M}$ equals $\operatorname{add}(M)$. Let us assume that the category $\mathcal{A}$ (considered as a dg category concentrated in degree 0) is smooth (this assumption can be weakened to 'local smoothness'). We consider the question of when the relative $(n+2)$-dimensional derived preprojective algebra $\Pi_{n+2}(\mathcal{A}, \mathcal{B})$ is a stalk category, *i.e.* has its homology concentrated in degree 0. Recall that we have

$$\Pi_{n+2}(\mathcal{A}, \mathcal{B}) = T_{\mathcal{B}}(\omega)\ ,$$

where $\omega = \Sigma^{n+1}\Omega_{\mathcal{A},\mathcal{B}}$ and $\Omega_{\mathcal{A},\mathcal{B}}$ is a cofibrant replacement of the bimodule dual of the cone over

$$\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A} \to \mathcal{A}.$$

For $M \in \mathcal{M}$, let us abbreviate

$$M^\wedge = \mathrm{Hom}_B(?, M)|_{\mathcal{A}}$$

considered as a finitely generated projective module over $\mathcal{A} = \mathcal{M}$. We have the following key lemma.

Key lemma 4.9.1. *Suppose that $\mathcal{M}$ is n-rigid,* i.e. *we have*

$$\mathrm{Ext}^i_B(L, M) = 0$$

for all L, M in $\mathcal{M}$ and all $1 \leq i \leq n$. Then for $M \in \mathcal{M}$, we have a canonical isomorphism

$$M^\wedge \overset{L}{\otimes}_{\mathcal{A}} \omega \xrightarrow{\sim} (\tau_n^- M)^\wedge.$$

A proof can be found in Proposition 8.6 of [102].

Corollary 4.9.2. *If $\mathcal{M}$ is n-rigid, then the dg algebra $\Pi_{n+1}(\mathrm{proj}\, B, \mathcal{M})$ is concentrated in degree 0.*

Notice that the n-rigidity assumption holds if

- B is n-representation finite in the sense of Iyama–Oppermann [50] or
- B^{op} is n-complete in the sense of [49] or
- B is n-representation infinite in the sense of Herschend–Iyama–Oppermann [42]

4.10 Relative Calabi–Yau structures

We have announced in the preceding sections that the n-CY-completion $\Pi_n(B)$ of a smooth dg algebra carries a canonical (left) n-Calabi–Yau structure and that the relative n-CY-completion of a morphism $B \to A$ between smooth dg algebras carries a canonical (left) relative n-CY-structure. Our aim in this section is to define these structures and their right counterparts.

4.10.1 Absolute left and right Calabi–Yau-structures

Let $\mathcal{A}$ be a smooth dg category. We have canonical isomorphisms in $\mathcal{D}k$

$$\mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A} \xrightarrow{\sim} \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} (\mathcal{A}^\vee)^\vee \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{A}^e}(\mathcal{A}^\vee, \mathcal{A}).$$

We deduce a canonical morphism

$$HN_n(\mathcal{A}) \to HH_n(\mathcal{A}) \xrightarrow{\sim} H^{-n}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}\mathcal{A}^e}(\mathcal{A}^\vee, \Sigma^{-n}\mathcal{A}).$$

Definition 4.10.2 (Kontsevich)**.** *A* left n-CY-structure on $\mathcal{A}$ *is a class in* $HN_n(\mathcal{A})$ *which yields an isomorphism* $\mathcal{A}^\vee \xrightarrow{\sim} \Sigma^{-n}\mathcal{A}$.

We see that if $\mathcal{A}$ carries a left n-CY-structure, it is in particular bimodule n-Calabi–Yau. The negative cyclic homology group appears naturally when one studies the deformations of bimodule n-CY-algebras, *cf.* [96]. One can show that each n-CY-completion of a smooth dg category carries a canonical left n-CY-structure, *cf.* [55, 10].

Now let $\mathcal{A}$ be a proper dg category (*i.e.* we have $\mathcal{A}(X,Y) \in \operatorname{per}(k)$ for all $X, Y \in \mathcal{A}$). Let $D\mathcal{A}^{op}$ denote the dg bimodule

$$(X,Y) \mapsto D\mathcal{A}(Y,X).$$

We have canonical isomorphisms

$$D(\mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}) \xrightarrow{\sim} \operatorname{Hom}_k(\mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}, k) \xrightarrow{\sim} R\operatorname{Hom}_{\mathcal{A}^e}(\mathcal{A}, D\mathcal{A}^{op}).$$

We deduce morphisms

$$DHC_{-n}(\mathcal{A}) \to DHH_{-n}(\mathcal{A}) \xrightarrow{\sim} H^{-n}\operatorname{RHom}_{\mathcal{A}^e}(\mathcal{A}, D\mathcal{A}^{op}) = \operatorname{Hom}_{\mathcal{D}\mathcal{A}^e}(\mathcal{A}, \Sigma^{-n}D\mathcal{A}^{op}).$$

Definition 4.10.3 (Kontsevich)**.** *A* right n-CY-structure on $\mathcal{A}$ *is a class in* $DHC_{-n}(\mathcal{A})$ *which yields an isomorphism* $\mathcal{A} \xrightarrow{\sim} \Sigma^{-n}D\mathcal{A}^{op}$.

If $\mathcal{A}$ carries a right n-CY-structure, we have the Serre duality formula

$$D\mathcal{A}(X,Y) \xrightarrow{\sim} \Sigma^n \mathcal{A}(Y,X)$$

bifunctorially in $X, Y \in \mathcal{A}$. In particular we see that $\mathcal{A}$ is *perfectly n-Calabi–Yau*, *i.e.* the perfect derived category $\operatorname{per}(\mathcal{A})$ is n-Calabi–Yau as a triangulated category. The definition above extends to 'componentwise proper' dg categories, *i.e.* dg categories $\mathcal{A}$ such that $H^p\mathcal{A}(X,Y)$ is finite-dimensional for all $p \in \mathbb{Z}$ and all $X, Y \in \mathcal{A}$. These are important for our applications. For example, if Q is a finite acyclic quiver, then the canonical dg enhancement

$$(\mathcal{C}_Q)_{dg} = \operatorname{per}_{dg}(\Gamma_{Q,0})/\operatorname{pvd}_{dg}(\Gamma_{Q,0})$$

of the classical cluster category $\mathcal{C}_Q$ is componentwise proper but not proper. Here the subscripts dg on the right hand side denote the corresponding subcategories of the dg derived category of $\Gamma_{Q,0} = \Pi_3(kQ)$, *cf.* section 1.6.1. For 'componentwise proper' dg categories endowed with a right n-CY-structure, the perfect derived category $\operatorname{per}(\mathcal{A})$ is still a Hom-finite n-CY triangulated category.

Suppose that $\mathcal{A}$ is a dg category which is both smooth and proper. Then, as shown by Toën–Vaquié in [94], the Yoneda functor yields a quasi-equivalence

$$\mathcal{A} \xrightarrow{\sim} \mathrm{pvd}_{dg}(\mathcal{A}).$$

Now suppose that moreover $\mathcal{A}$ is augmented (in the sense of [59]) and $\mathrm{pvd}_{dg}(\mathcal{A})$ is generated, as a triangulated category, by the image of the restriction functor $\mathrm{pvd}\,(\overline{\mathcal{A}}) \to \mathrm{pvd}\,(\mathcal{A})$, where $\mathcal{A} \to \overline{\mathcal{A}}$ is the augmentation. Then we have a Morita equivalence

$$\mathcal{A}^! \xrightarrow{\sim} \mathrm{pvd}_{dg}(\mathcal{A}).$$

We deduce an isomorphism

$$HN_n(\mathcal{A}) \xrightarrow{\sim} HN_n(\mathcal{A}^!) \xrightarrow{\sim} DHC_{-n}(\mathcal{A}).$$

Proposition 4.10.4. *This isomorphism establishes a bijection between the left and the right n-Calabi–Yau structures on $\mathcal{A}$.*

Left CY-structures are inherited by dg localizations whereas right CY-structures are inherited by full dg subcategories. More precisely, we have the following proposition.

Proposition 4.10.5 ([64])**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a dg functor. Suppose that F is a localization,* i.e. *the functor $F^* : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$ is a Verdier localization. If $\mathcal{A}$ is smooth, then so is $\mathcal{B}$ and the image under the induced morphism $HN_n(\mathcal{A}) \to HN_n(\mathcal{B})$ of a left n-CY-structure on $\mathcal{A}$ is a left n-CY-structure on $\mathcal{B}$.*

As a consequence, if $\Gamma = \Gamma_{Q,0} = \Pi_3(kQ)$ for an acyclic quiver Q, then $\mathrm{per}_{dg}(\Gamma)$ carries a left 3-CY-structure and so does its localization $(\mathcal{C}_Q)_{dg}$, the dg cluster category. This seems to be contradictory with the well-known fact that the cluster category is 2-Calabi–Yau as a triangulated category. The explanation is that the 2-Calabi–Yau property of the triangulated category $\mathcal{C}_Q$ comes from a *right* 2-*Calabi–Yau structure* on $(\mathcal{C}_Q)_{dg}$ (*cf.* below). The fact that $2 \neq 3$ is not a contradiction because $(\mathcal{C}_Q)_{dg}$ is smooth (as a localization of $\mathrm{per}_{dg}(\Gamma)$) *but not proper* (only componentwise proper).

Theorem 4.10.6 (Brav–Dyckerhoff [13])**.** *Let $\mathcal{A}$ be a smooth dg category. Each left n-CY structure on $\mathcal{A}$ yields a canonical right n-CY structure on* $\mathrm{pvd}_{dg}(\mathcal{A})$.

For example, the canonical left 3-CY structure on Γ as above yields a right 3-CY structure on $\mathrm{pvd}_{dg}(\Gamma)$ which is responsible for the 3-CY property of the triangulated category $\mathrm{pvd}\,(\Gamma)$. The right 3-CY structure on $\mathrm{pvd}_{dg}(\Gamma)$ yields the right 2-CY structure on $(\mathcal{C}_Q)_{dg}$ via the connecting morphism in cyclic homology associated with the exact sequence

$$0 \to \mathrm{pvd}_{dg}(\Gamma) \to \mathrm{per}_{dg}(\Gamma) \to (\mathcal{C}_Q)_{dg} \to 0\,,$$

cf. Theorem 1.7.4.

4.11 Left and right relative Calabi–Yau structures

4.11.1 The derived category of morphisms

Let k be a perfect field and A a dg k-algebra. Let $I = k\vec{A}_2$ (the letter I stands for 'interval'). Then the objects of $\mathcal{D}(I^{op} \otimes A)$ identify with morphisms $f : M_1 \to M_2$ of dg A-modules. Each object gives rise to a triangle in $\mathcal{D}(A)$

$$M_1 \xrightarrow{f} M_2 \longrightarrow C(f) \longrightarrow \Sigma M_1$$

functorial in $(M_1 \to M_2)$ considered as an object of $\mathcal{D}(I^{op} \otimes A)$. For two objects $f : M_1 \to M_2$ and $f' : M_1' \to M_2'$, consider a diagram

$$\begin{tikzcd}
M_1 \arrow[r, "f"] \arrow[d, dotted, "a"'] & M_2 \arrow[r] \arrow[d, "b"'] & C(f) \arrow[r] \arrow[d, dotted, "c"'] & \Sigma M_1 \arrow[d, dotted, "\Sigma a"'] \\
M_1' \arrow[r, "f'"] & M_2' \arrow[r, "g'"] & C(f') \arrow[r] & \Sigma M_1'
\end{tikzcd}$$

It is well-known (and easy to check) that for a given morphism b, there are morphisms a and c making the diagram commutative if and only if we have $g'ba = 0$ and that in this case, the pair (a, b) lifts to a morphism of $\mathcal{D}(I^{op} \otimes A)$. This statement is refined by the following lemma

Lemma 4.11.2. *We have a canonical isomorphism bifunctorial in $f, f' \in \mathcal{D}(I^{op} \otimes A)$*

$$\mathrm{RHom}_{I^{op}\otimes A}(f, f') \xrightarrow{\sim} \mathrm{fib}\,(\mathrm{RHom}_A(M_2, M_2') \to \mathrm{RHom}_A(M, C(f'))\,,$$

where we write $\mathrm{fib}\,(g)$ *for* $\Sigma^{-1}C(g)$.

A proof may be found in Lemma 3.1 of [102].

4.11.3 Definition of relative Calabi–Yau structures

Let $n \in \mathbb{Z}$ be an integer and $f : B \to A$ a morphism between smooth dg algebras. Consider the diagram

$$\begin{tikzcd}
\mathrm{fib}\,(\mu) \arrow[r, "\nu"] & A \overset{L}{\otimes}_B A \arrow[r, "\mu"] & A \arrow[r] & \dots \\
\Sigma^{1-n}A^\vee \arrow[r, "\Sigma^{1-n}\mu^\vee"] \arrow[u, dotted, "\wr"] & \Sigma^{1-n}(A \overset{L}{\otimes}_B A)^\vee \arrow[r] \arrow[u, dotted, "\wr"] & \Sigma^{1-n}\mathrm{cof}\,(\mu^\vee) \arrow[r] \arrow[u, dotted, "\wr"] & \dots
\end{tikzcd} \tag{4.11.3.1}$$

From Lemma 4.11.2, via isomorphisms like

$$\mathrm{RHom}_{A^e}(A^\vee, A) \xrightarrow{\sim} A \overset{L}{\otimes}_{A^e} A\,,$$

we get the following chain of morphisms

$$\begin{aligned}\mathrm{RHom}(\mu^\vee,\nu) &\xrightarrow{\sim} \mathrm{fib}\,(A \overset{L}{\otimes}_{B^e} A \to A \overset{L}{\otimes}_{A^e} A)\\ &\leftarrow \mathrm{fib}\,(B \overset{L}{\otimes}_{B^e} B \to A \overset{L}{\otimes}_{A^e} A)\\ &\xrightarrow{\sim} \Sigma^{-1} HH(A,B)\\ &\leftarrow \Sigma^{-1} HN(A,B)\,,\end{aligned}$$

where $HH(A,B)$ denotes the relative Hochschild complex of $B \to A$.

Definition 4.11.4 (Brav–Dyckerhoff [13])**.** *A* relative left n-CY-structure on $B \to A$ *is a class in* $HH_n(A,B)$ *giving rise to an isomorphism of triangles in 4.11.3.1.*

Notice that for $B = 0$, we recover the absolute notion. If A and B are concentrated in degree 0, one easily deduces that A is bimodule internally n-Calabi–Yau (in the sense of Pressland [81]) with respect to the image e of 1_B in A.

Suppose that $B \to A$ carries a left relative n-CY structure. Put $\mathcal{A} = \mathrm{pvd}_{dg}(A)$ and $\mathcal{B} = \mathrm{pvd}_{dg}(B)$ and let $\mathcal{A} \to \mathcal{B}$ be the restriction functor.

Theorem 4.11.5 (Brav–Dyckerhoff [13])**.** *The functor* $\mathcal{A} \to \mathcal{B}$ *inherits a canonical relative right* n*-CY structure,* i.e. *there is a canonical class in* $DHC_{1-n}(\mathcal{B},\mathcal{A})$ *which yields an isomorphism of triangles*

$$\begin{tikzcd}
\Sigma^{n-1}\mathcal{A} \arrow[r, "\Sigma^{n-1}r"] \arrow[d, "\wr"] & \Sigma^{n-1}\mathcal{B}|_{\mathcal{A}^e} \arrow[r] \arrow[d, "\wr"] & \Sigma^{n-1}\mathrm{cof}\,(r) \arrow[r] \arrow[d, "\wr"] & \dots \\
\mathrm{fib}\,(Dr) \arrow[r] & D\mathcal{B}^{op}|_{\mathcal{A}^e} \arrow[r, "Dr"] & D\mathcal{A} \arrow[r] & \dots
\end{tikzcd}$$

Concretely, for objects L and M of $\mathrm{pvd}\,(A)$, the above diagram becomes

$$\begin{tikzcd}
 & \mathrm{RHom}_{\mathcal{A}}(L,\Sigma^{n-1}M) \arrow[r, "\mathrm{res}"] \arrow[d, "\wr"] & \mathrm{RHom}_{\mathcal{B}}(L,\Sigma^{n-1}M) \arrow[r] \arrow[d, "\wr"] & C(L,\Sigma^{n-1}M) \arrow[r] \arrow[d, "\wr"] & \dots \\
D\mathrm{RHom}_{\mathcal{A}}(M,\Sigma^{-1}L) \arrow[r] & DC(M,L) \arrow[r] & D\mathrm{RHom}_{\mathcal{B}}(M,L) \arrow[r, "D\mathrm{res}"] & D\mathrm{RHom}_{\mathcal{A}}(M,L) \arrow[r] & \dots
\end{tikzcd}$$

If we have $B = 0$, we recover an isomorphism

$$D\mathrm{RHom}_A(L,\Sigma^{-1}M) \xrightarrow{\sim} \mathrm{RHom}_A(L,\Sigma^{n-1}M)$$

so the theorem is a generalization of Theorem 4.10.6.

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