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Céline Bonandrini

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Céline Bonandrini. On the numerical Grothendieck group of very general Küchle fourfolds of type c_5 . Algebraic Geometry [math.AG]. Université Paris-Saclay, 2022. English. NNT : 2022UPASM037 . tel-04022272

HAL Id: tel-04022272

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Submitted on 9 Mar 2023

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On the numerical Grothendieck group of very general K uchle fourfolds of type c5

Sur le groupe de Grothendieck num rique des vari t s de
K uchle de dimension 4 et de type c5 tr s g n rales

Th se de doctorat de l'Universit  Paris-Saclay

 cole Doctorale de Math matiques Hadamard (EDMH) n  574
Sp cialit  de doctorat : Math matiques fondamentales
Graduate School : Math matiques.
R f rent : Facult  des sciences d'Orsay

Th se pr par e dans l'unit  de recherche **Laboratoire de
Math matiques d'Orsay (Universit  Paris-Saclay, CNRS)**
sous la direction d'**Emanuele MACR **, Professeur des
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Titre: Sur le groupe de Grothendieck numérique des variétés de Kuchle de dimension 4 et de type c5 très générales

Mots clés: Mathématiques, Géométrie algébrique, variétés de Kuchle de dimension 4 et de type c5, caractéristique d'Euler, catégories dérivées

Résumé: Nous essayons de trouver une base du groupe de Grothendieck numérique G d'une variété de Kuchle de dimension 4 et de type c5 très générale X_4 . Pour ce faire, nous nous intéressons d'abord à la géométrie de telles variétés, et nous essayons d'en déduire une famille F d'éléments de G qui pourrait être une base. Ensuite nous essayons de calculer la matrice dont les coefficients sont les caractéristiques d'Euler entre les éléments de F . Cette matrice devrait nous permettre de mieux appréhender

les catégories dérivées bornées des variétés considérées.

Dans cette thèse nous ne parvenons pas à trouver une base de G mais nous proposons deux familles F_1 et F_2 qui pourraient être des bases du groupe de Grothendieck numérique d'un éclatement de X_4 , et à partir desquelles il pourrait être possible de trouver une base de G . Nous calculons les matrices de caractéristiques d'Euler associées à ces deux familles, à un nombre d'intersection près dans un cas et entièrement dans l'autre.

Title: On the numerical Grothendieck group of very general Kuchle fourfolds of type c5

Keywords: Mathematics, Algebraic geometry, Kuchle fourfolds of type c5, Euler characteristic, derived categories

Abstract: We try to find a basis of the numerical Grothendieck group G of very general Kuchle fourfolds of type c5, denoted X_4 . To do so we first study the geometry of such varieties, and try to deduce from this a family F of elements in G which could be a basis. Then we try to compute the matrix whose coefficients are Euler characteristics between elements in F . This matrix is expected to provide intuition on

the bounded derived category of very general Kuchle fourfolds of type c5.

In this thesis we don't manage to find a basis of G but we give two families F_1 and F_2 which could be basis of the numerical Grothendieck group of a blow-up of X_4 , and from which it may be possible to deduce a basis of G . We compute the matrices of Euler characteristics associated to these families, up to an intersection number in one case and entirely in the other case.

Acknowledgements

I would like to thank my advisor, Emanuele Macrì, for his help, all his advice, his encouragements to explore the multiple sides of algebraic geometry and for numerous stimulating discussions.

I would also like to thank Benjamin Sung for very helpful discussions and comments. His ideas allowed me to see things from a different perspective and move on, while I was afraid to be in a dead end.

I am very thankful to Enrico Fatighenti, who helped me greatly to understand how to apply the Borel-Bott-Weil Theorem and how to use Macaulay2.

Finally I would like to thank all the people who contributed to make these three years interesting and pleasant. Among them, Jean, who helped me to find the Weyl formula, Pietro, who kindly listened to my unfruitful tries to understand some geometrical contractions, Carlo, Luigi, Alessio, Jieao, Amadou, Anne, Vladimiro, Guillaume, Louise, Dorian, Xiaozong, Corinne, Cyril, Nicolas, Brice, Dominique, Calla, Pierre-Louis, Lucien...

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Introduction

0.1 Introduction en français

Dans cette thèse, on s'intéresse aux variétés de K uchle de type c5 et de dimension 4 sur \mathbb{C} . Pour X_4 une telle vari et e tr es g en erale, l'objectif est de calculer la matrice $M = (\chi(e_i, e_j))$ o u χ d esigne la caract eristique d'Euler et les e_i forment une base du groupe de Grothendieck num erique. Une fois cette matrice calcul ee, trouver une autre base dans laquelle M a une forme sp ecifique, que nous d ecrirons un peu plus tard, donnerait une indication sur une possible d ecomposition semi-orthogonale de la cat egorie d eriv ee born ee des faisceaux coh erents sur X_4 . En effet certaines similitudes avec les cubiques de dimension 4 dans \mathbb{P}^5 et certaines vari et es de Gushel-Mukai de dimension 4 sugg erent qu'une telle d ecomposition pourrait exister.

Bref r esum e

Dans son article [22] sur les cat egories Calabi-Yau et Calabi-Yau fractionnelles, A. Kuznetsov donne un Th eor eme qui permet dans de multiples cas de trouver des d ecompositions semi-orthogonales de cat egories d eriv ees born ees de vari et es projectives lisses.

Plus pr ecis ement, quand un foncteur $\phi : D^b(X) \rightarrow D^b(M)$ entre les cat egories d eriv ees born ees de deux vari et es projectives lisses satisfait certaines conditions, il est possible de construire une d ecomposition semi-orthogonale de $D^b(X)$  a partir d'une d ecomposition de Lefschetz rectangulaire de $D^b(M)$ (l a encore sous certaines conditions techniques que nous n'allons pas d etailler ici). Cette d ecomposition semi-orthogonale de $D^b(X)$ consiste principalement en sous-cat egories obtenues  a partir des sous-cat egories de la d ecomposition de Lefschetz de $D^b(M)$, ainsi que leur orthogonal  a droite, une sous-cat egorie souvent appel ee "composante de Kuznetsov". De plus, ce m eme Th eor eme 3.5 de [22] nous permet de calculer le foncteur de Serre de la composante de Kuznetsov.

Certaines des composantes de Kuznetsov obtenues gr ace  a ce Th eor eme sont des cat egories K3 non-commutatives, c'est- a-dire des cat egories qui "ressemblent"  a des cat egories d eriv ees born ees de surfaces K3. En fait, tous les exemples non triviaux de cat egories K3 non-commutatives qui sont connus jusqu' a pr esent nous viennent de composantes de Kuznetsov apparaissant dans les cat egories d eriv ees born ees de certaines vari et es projectives. De plus, ces cat egories K3 non-commutatives semblent li ees  a la g eom etrie des vari et es qui leur sont associ ees.

En effet, on peut trouver de telles cat egories K3 non-commutatives dans $D^b(C)$, o u C est une conique de dimension 4 dans \mathbb{P}^5 , sous la forme de composantes de Kuznetsov, et il a  et e conjectur e par A. Kuznetsov (voir la Conjecture 1.1 de [18]) qu'une cubique de dimension 4 est rationnelle si et seulement si sa composante de Kuznetsov est  equivalente  a la cat egorie d eriv ee born ee d'une surface K3. Cette condition n'est pas satisfaite pour les cubiques de dimension 4 tr es g en erales : le groupe de Grothendieck num erique de leur composante de Kuznetsov ne pourrait pas contenir la classe d'un faisceau gratte-ciel, car il ne contient aucun  el ement non nul v tel que $\chi(v, v) = 0$ (voir par exemple la Remarque 3.19 de [24] pour plus de d etails). Cette conjecture implique donc que les cubiques de dimension 4 tr es g en erales ne sont pas rationnelles.

Les vari et es de Gushel-Mukai de dimension 4 ont aussi une composante de Kuznetsov qui est une cat egorie

K3 non-commutative, et il a été conjecturé par A. Kuznetsov et A. Perry dans [23] (Conjecture 3.12) que cette composante est équivalente à la catégorie dérivée bornée d'une surface K3 quand la variété de Gushel-Mukai associée est rationnelle.

Une autre caractéristique intéressante des composantes de Kuznetsov qui sont des catégories K3 non-commutatives est qu'à chacune des trois familles d'exemples (non triviaux) connues jusqu'à présent, on peut associer une famille de variétés hyperkähleriennes : à une cubique de dimension 4, C , correspond la famille des droites dans C , à une variété de Gushel-Mukai de dimension 4 correspond une double sextique EPW, et à une variété de Debarre-Voisin, dont la dernière famille d'exemples est constituée, on peut associer une variété hyperkählienne de dimension 4 dans $\text{Gr}(6, 10)$. Tous ces exemples sont listés dans la partie 3.6.1 de [6].

Il serait donc intéressant de trouver une nouvelle famille non triviale de catégories K3 non-commutatives qui soient des composantes de Kuznetsov dans les catégories dérivées bornées de variétés projectives lisses : il serait alors possible d'étudier les liens entre ces catégories et la géométrie de leurs variétés associées. De plus, un espace de modules des objets contenus dans une telle composante de Kuznetsov comporterait une forme symplectique (voir la partie 4.4 de [22] pour plus de détails), on pourrait donc également espérer trouver une famille de variétés hyperkähleriennes associées.

Parmi toutes les variétés de Fano de dimension 4 classées par O. Küchle dans [15], trois types ont un diamant de Hodge qui suggère que leur catégorie dérivée bornée pourrait contenir une catégorie K3 non-commutative. Deux de ces trois types ont été étudiés dans [19] et ne donnent pas de nouvelles familles non triviales de catégories K3 non-commutatives. Cependant pour le troisième type, c'est-à-dire le type c5, la question reste ouverte.

Dans [20], Kuznetsov a étudié la géométrie des variétés de Küchle de dimension 4 et de type c5, notées X_4 , et de variétés de dimension 5 associées, appelées variétés de Küchle de dimension 5 et de type c5. Cela lui a permis de calculer le motif de Chow de variétés de Küchle de dimension 5 et de type c5 générales. Cependant, aucune décomposition de Lefschetz rectangulaire de ces variétés n'a encore été trouvée, ce qui serait nécessaire pour appliquer le Théorème 3.5 de [22] mentionné précédemment. Or, nous aurions besoin de ce Théorème pour prouver l'existence d'une composante de Kuznetsov dans $D^b(X_4)$.

Même s'il est assez compliqué de prouver directement l'existence d'une composante de Kuznetsov dans $D^b(X_4)$, si une telle composante \mathcal{A} existe, il pourrait être possible de trouver une base du groupe de Grothendieck numérique $K_{\text{num}}(\mathcal{A})$, par analogie avec le cas des cubiques de dimension 4. En effet, si cette analogie est correcte, dans une base bien choisie \mathcal{B} de $K_{\text{num}}(X_4)$ et pour X_4 très général, la matrice dont les coefficients sont les caractéristiques d'Euler entre les éléments de \mathcal{B} devrait avoir cette forme :

$$M = \begin{pmatrix} A & * & \dots & * \\ 0 & 1 & * & * \\ \vdots & 0 & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

où A est la matrice dont les coefficients sont les caractéristiques d'Euler entre les éléments d'une base de $K_{\text{num}}(\mathcal{A})$.

Par conséquent, s'il était possible de trouver une base de $K_{\text{num}}(X_4)$ dont la matrice de caractéristiques d'Euler a la forme ci-dessus, alors les éléments de cette base correspondant au bloc A seraient de "bons candidats" pour former une base de $K_{\text{num}}(\mathcal{A})$. Cela pourrait alors donner une indication sur ce à quoi une composante de Kuznetsov dans $D^b(X_4)$ pourrait ressembler.

C'est pourquoi le premier objectif de cette thèse était de trouver une base de $K_{\text{num}}(X_4)$ (en tant que réseau sur \mathbb{Z}), calculer la matrice M de caractéristiques d'Euler entre les éléments de cette base, et finalement chercher une matrice inversible R à coefficients dans \mathbb{Z} telle que $RM R^{-1}$ a la forme mentionnée ci-dessus.

Cependant, cela s'est avéré assez difficile. En effet, le diamant de Hodge de X_4 suggère qu'il devrait y avoir dans $K_{\text{num}}(X_4)$ quatre classes associées à des surfaces dans X_4 . En utilisant des résultats de [20] sur la

géométrie des variétés de K uchle de dimension 4 et 5 de type c5 nous sommes parvenus   trouver des surfaces dans X_4 , mais malheureusement nous ne sommes pas capables de calculer certaines des caract ristiques d'Euler impliquant les faisceaux structurels de ces surfaces.

Malgr  tout, nous savons d'apr s [20] qu'il existe un  clatement \tilde{X}_4 de X_4 , et il pourrait  tre possible de retrouver $D^b(X_4)$   partir de $D^b(\tilde{X}_4)$ en utilisant des foncteurs de mutation   droite. Le second objectif de cette th se a donc  t  de trouver une base de $K_{\text{num}}(\tilde{X}_4)$ et de calculer la matrice \tilde{M} de caract ristiques d'Euler entre les  l ments de cette base.

Nous avons trouv  deux familles libres de 14  l ments dans $K_{\text{num}}(\tilde{X}_4)$, mais n'avons pas pu prouver que ces familles sont des bases. Pour l'une d'entre elles nous avons calcul  la matrice associ e \tilde{M}   un nombre (correspondant   une intersection entre sous-vari t s de \tilde{X}_4) pr s, et pour la seconde nous avons calcul  la matrice de caract ristiques d'Euler associ e \tilde{M}_2 enti rement. Cependant ces deux matrices ne peuvent pas avoir le m me d terminant, ce qui sugg re que nos deux familles ne g n rent pas les m mes sous-r seaux de $K_{\text{num}}(\tilde{X}_4)$. Il serait donc int ressant de savoir si l'une de ces familles correspond   un sous-r seau primitif de $K_{\text{num}}(\tilde{X}_4)$ ou non.

D finition des vari t s de K uchle de dimension 4 et 5 de type c5

Pour commencer, notons par G la Grassmannienne $\text{Gr}(3, \mathbb{C}^7)$ et \mathcal{U}_3 le sous-fibr  tautologique de G . Une vari t  de K uchle de dimension 4 et de type c5 est le lieu des z ros d'une section globale g n rale de $\mathcal{U}_3^\perp(1) \oplus \mathcal{U}_3(1) \oplus \mathcal{O}_G(1)$, o  \mathcal{U}_3^\perp est le quotient de l'injection $\mathcal{U}_3 \hookrightarrow \mathbb{C}^7 \otimes \mathcal{O}_G$. Pour une section globale assez g n rale, cette construction donne une vari t  de Fano lisse et de dimension 4. L'appellation "type c5" vient de la classification, donn e par O. K uchle dans [15], des vari t s de Fano de dimension 4 et d'index 1 qui peuvent  tre obtenues en consid rant le lieu des z ros d'une section globale g n rale d'une somme directe de fibr s vectoriels irr ductibles, homog nes et globalement g n r s, sur une Grassmannienne. De plus, chaque vari t  de K uchle de dimension 4 et de type c5 est une section hyperplane d'une vari t  de dimension 5, que nous noterons X_5 et que nous appellerons vari t  de K uchle de type c5 et de dimension 5. En effet, si X_4 est le lieu des z ros de la section globale $\lambda \oplus \mu \oplus \nu$ de $\mathcal{U}_3^\perp(1) \oplus \mathcal{U}_3(1) \oplus \mathcal{O}_G(1)$ alors nous pouvons d finir X_5 comme le lieu des z ros de $\lambda \oplus \mu$, laquelle est une section globale de $\mathcal{U}_3^\perp(1) \oplus \mathcal{U}_3(1)$. Dans ce cas, puisque ν est une section globale de $\mathcal{O}_G(1)$, son lieu des z ros H_ν est un hyperplan et $X_4 = X_5 \cap H_\nu$. Dans [20], A. Kuznetsov d crit la g om trie des ces vari t s de K uchle de dimension 5, et en d duit certaines propri t s g om triques des vari t s de K uchle de dimension 4 associ es. Nous nous r f rerons souvent   cet article [20] et conserverons la majeure partie de ses notations.

Composantes de Kuznetsov et cat gories K3 non-commutatives

Il est attendu que la cat gorie d riv e born e des faisceaux coh rents sur X_4 , not e $D^b(X_4)$, admet une d composition semi-orthogonale int ressante, par analogie avec d'autres exemples et   cause du diamant de Hodge de X_4 .

Commen ons par le cas des cubiques de dimension 4 dans \mathbb{P}^5 : ces vari t s sont des sections de vari t s de dimension 5, notons-les Y_5 , qui admettent une d composition de Lefschetz rectangulaire, c'est- -dire une d composition semi-orthogonale de leur cat gorie d riv e born e des faisceaux coh rents comme suit :

$$D^b(Y_5) = \langle \mathcal{B}, \mathcal{B} \otimes \mathcal{L}, \dots, \mathcal{B} \otimes \mathcal{L}^{\otimes m} \rangle$$

o  $m \in \mathbb{N}$, \mathcal{B} est une sous-cat gorie triangul e de $D^b(Y_5)$, \mathcal{L} est un fibr  en droite de Y_5 et une d composition semi-orthogonale $\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ est une suite de sous-cat gories pleines et admissibles telles que $\text{Hom}^\bullet(\mathcal{A}_j, \mathcal{A}_i) = 0$ pour tout $j > i$, et telles que la plus petite sous-cat gorie triangul e pleine de \mathcal{D} contenant tous les \mathcal{A}_i est  quivalente   \mathcal{D} (voir la D finition 1.59 de [14]).

Le Th or me 3.5 de l'article de A. Kuznetsov [22] donne alors une d composition semi-orthogonale de $D^b(Y_4)$:

$$D^b(Y_4) = \langle \mathcal{A}, \mathcal{B}', \mathcal{B}' \otimes \mathcal{L}', \dots, \mathcal{B}' \otimes \mathcal{L}'^{\otimes k} \rangle$$

où $k \in \mathbb{N}$, \mathcal{A} est appelée la composante de Kuznetsov de Y_4 et le reste de la décomposition est induit par la décomposition de Lefschetz de $D^b(Y_5)$. De plus, lorsque Y_4 est une cubique de dimension 4, sa composante de Kuznetsov \mathcal{A}_{Y_4} est une catégorie K3 non-commutative : son foncteur de Serre est [2] et son homologie de Hochschild est la même que celle de la catégorie dérivée bornée d'une surface K3. En tant que sous-catégorie triangulée de $D^b(Y_4)$, \mathcal{A}_{Y_4} satisfait donc la Définition 2.31 de [24], donnée par E. Macrì et P. Stellari, et est donc une catégorie K3 non-commutative, autrement dit une catégorie qui "ressemble" à la catégorie dérivée bornée d'une surface K3.

Cependant il existe des catégories K3 non-commutatives qui ne peuvent pas être équivalentes à la catégorie dérivée bornée d'une surface K3 (voir par exemple la Proposition 4.8 de [18]), alors que les catégories dérivées bornées de surfaces K3 sont toutes des catégories K3 non-commutatives. Les catégories K3 non-commutatives peuvent donc être considérées comme une "généralisation" de $D^b(S)$ où S est une surface K3.

Revenant aux variétés de Kùchle de dimension 4 et 5, on peut voir en regardant le diamant de Hodge de X_4 :

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 0 & 0 \\
 & & & & & 0 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 1 & 24 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 1 & 0 & & \\
 & & & & & 0 & 0 & & & \\
 & & & & & 1 & & & &
 \end{array}$$

donné par exemple dans l'introduction de [20], qu'il pourrait "contenir" le diamant de Hodge d'une surface K3. Les variétés de Kùchle de dimension 4 et de type c5 ne sont pas les seules parmi les variétés de Fano de dimension 4 classées par Kùchle dont le diamant de Hodge suggère que leur catégorie dérivée bornée pourrait contenir une catégorie K3 non-commutative, mais les deux autres "candidats" à la contenance de telles sous-catégories (les variétés de Kùchle de dimension 4 et de types d3 et c7) sont des exemples déjà connus tandis que les variétés de Kùchle de type c5 pourraient donner une nouvelle famille de catégories K3 non-commutatives.

Pourquoi s'intéresser aux catégories K3 non-commutatives : lien conjecturé avec la rationalité et variétés hyperkählériennes

Une telle famille de catégories K3 non-commutatives serait intéressante pour principalement deux raisons. Tout d'abord, dans le cas d'une cubique C de dimension 4 il a été conjecturé par A. Kuznetsov que C est rationnelle si et seulement si sa composante de Kuznetsov \mathcal{A}_C est équivalente à $D^b(S)$ où S est une surface K3.

Il a aussi été conjecturé par A. Kuznetsov et A. Perry que la composante de Kuznetsov d'une variété GM de Gushel-Mukai et de dimension 4, qui est aussi une catégorie K3 non-commutative, est équivalente à la catégorie dérivée bornée d'une surface K3 lorsque GM est rationnelle.

Ces catégories K3 non-commutatives semblent donc liées à la géométrie des leurs variétés associées et il pourrait être intéressant d'en trouver de nouveaux exemples (apparaissant comme composante de Kuznetsov dans la catégorie dérivée bornée d'une variété), tout particulièrement parce qu'à l'exception des catégories dérivées bornées de surfaces K3, seulement trois familles d'exemples sont connues jusqu'à présent : les cubiques mentionnées ci-dessus, les variétés de Gushel-Mukai de dimension 4 et les variétés de Debarre-Voisin.

De plus, comme détaillé par A. Kuznetsov dans la partie 4.4 de [22], à chaque composante de Kuznetsov qui est aussi une catégorie K3 non-commutative il pourrait être possible d'associer une variété hyperkählérienne. C'est pourquoi trouver une famille de variétés avec une composante K3 non-commutative pourrait aussi donner une famille de variétés hyperkählériennes associée.

À chacune des familles non triviales de catégories K3 non-commutatives connues jusqu'à présent correspond une famille de variétés hyperkählériennes associée : ces dernières sont listées dans la partie 3.6.1 de [6], et nous allons en parler plus en détails un peu plus tard.

Principal obstacle à la preuve de l'existence d'une composante de Kuznetsov dans $D^b(X_4)$

Malheureusement, dans le cas des variétés de K uchle X_4 de dimension 4 et de type c5 il est assez difficile de prouver qu'une telle composante de Kuznetsov existe : en effet dans le cas des trois familles d'exemples (non-triviales) qui sont connues jusqu'ici, pour prouver l'existence de la composante de Kuznetsov et certaines des propri et es requises pour ˆetre une cat egorie K3 non-commutative, il faut utiliser le Th eor eme 3.5 de [22]. Mais dans le cas de X_4 l'utilisation de ce Th eor eme n ecessiterait de conna tre une d ecomposition de Lefschetz rectangulaire de $D^b(X_5)$ et une telle d ecomposition n'est pas connue   ce jour.

Cependant, il est attendu qu'une d ecomposition de Lefschetz rectangulaire existe bien dans ce cas. Pour pouvoir utiliser une telle d ecomposition (avec le Th eor eme 3.5 de [22]) dans le but de trouver une d ecomposition semi-orthogonale d'une vari et  de K uchle de type c5 et de dimension 4, il faudrait que la d ecomposition soit de la forme d ecrite dans l'exemple 3.1 de [22]. Puisque notre vari et  de K uchle X_4 est le lieu des z eros d'une section globale de $\mathcal{O}_{X_5}(1)$ dans la vari et  de dimension 5 associ ee X_5 , il faudrait alors que le fibr e en droites \mathcal{L} de la d ecomposition de Lefschetz rectangulaire soit $\mathcal{O}_{X_5}(1)$. D'apr es le Lemme 2.8 de [22] et comme nous savons d'apr es [20] que $\omega_{X_5} = \mathcal{O}_{X_5}(-2)$, il n'y a qu'une seule possibilit e pour une telle d ecomposition de Lefschetz rectangulaire : $\langle \mathcal{B}_0, \mathcal{B}_0(1) \rangle$. Dans un tel cas, la preuve de la Proposition 3.4 et le Th eor eme 3.5 de [22] impliquent que le foncteur de Serre de la composante de Kuznetsov \mathcal{A}_{X_4} est $[-4 + 6] = [2]$. Combin e avec le diamant de Hodge mentionn e ci-dessus, cela implique que si une vari et  de K uchle de dimension 5 et de type c5 admet une d ecomposition de Lefschetz rectangulaire comme d ecrite pr ec edemment, alors tous les X_4 associ es ont une composante de Kuznetsov qui est tr es probablement une cat egorie K3 non-commutative.

En cherchant une d ecomposition de Lefschetz rectangulaire de X_5 , A. Kuznetsov a  tudi e en d etails la g eom etrie de vari et s de K uchle de type c5 et de dimension 5 g en erales et prouv e dans le Th eor eme 5.3 de [20] que le motif de Chow d'un X_5 g en eral est de type Lefschetz. Il en a d eduit le diamant de Hodge et les groupes de Chow de X_5 , affirmant que des g en erateurs explicites de ces derniers peuvent ˆetre d eduits de sa description g eom etrique de X_5 .

Comment avoir une intuition au sujet d'une possible composante de Kuznetsov sans conna tre de d ecomposition de Lefschetz rectangulaire de $D^b(X_5)$

Cependant, nous allons nous int eresser ici aux X_4 plutˆot qu'aux X_5 . En effet, nous savons que si une d ecomposition semi-orthogonale de la forme $D^b(X_4) = \langle \mathcal{A}_{X_4}, \mathcal{B}, \mathcal{B} \otimes \mathcal{L}, \dots, \mathcal{B} \otimes \mathcal{L}^{\otimes k} \rangle$ existe, alors dans une base du groupe de Grothendieck num erique $K_{\text{num}}(X_4)$ dont les  el ements $(e_i)_i$ appartiennent dans l'ordre   $\mathcal{A}_{X_4}, \mathcal{B}, \dots, \mathcal{B} \otimes \mathcal{L}^{\otimes k}$, la matrice $M = (\chi(e_i, e_j))_{i,j}$ devrait ˆetre de la forme :

$$M = \begin{pmatrix} A & * & \dots & * \\ 0 & B_0 & * & * \\ \vdots & 0 & \ddots & * \\ 0 & \dots & 0 & B_k \end{pmatrix}$$

o 

$$A = (\chi(e_i, e_j))_{e_i, e_j \in K_{\text{num}}(\mathcal{A}_{X_4})}, B_0 = (\chi(e_i, e_j))_{e_i, e_j \in K_{\text{num}}(\mathcal{B})}, \dots, B_k = (\chi(e_i, e_j))_{e_i, e_j \in K_{\text{num}}(\mathcal{B} \otimes \mathcal{L}^{\otimes k})}.$$

De plus, dans le cas des cubiques de dimension 4 il se trouve que la sous-cat egorie triangul ee \mathcal{B} est en fait g en er e par un objet exceptionnel (un  el ement E de $D^b(C)$ tel que $\text{Hom}(E, E[m]) = \mathbb{C}\delta_{0,m}$). Dans le cas des cubiques tr es g en erales on sait  galement que la matrice A est de taille 2×2 , donc si les vari et s de K uchle de

dimension 4 et de type c5 ont une décomposition semi-orthogonale similaire à celle des cubiques de dimension 4, pour des variétés très générales on peut même espérer que la matrice M soit :

$$M = \begin{pmatrix} m_1 & m_2 & * & \dots & * \\ m_3 & m_4 & * & \dots & * \\ 0 & 0 & 1 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Par conséquent, trouver une base de $K_{\text{num}}(X_4)$ dans laquelle M a cette forme nous donnerait une indication sur ce qu'une décomposition semi-orthogonale de $D^b(X_4)$ pourrait être : les deux premiers éléments de la base pourraient correspondre à des générateurs de la composante de Kuznetsov et les autres éléments à des objets exceptionnels générant les sous-catégories de la forme $\mathcal{B} \otimes \mathcal{L}^{\otimes j}$ de la décomposition.

Lien entre les variétés hyperkählériennes associées et $K_{\text{num}}(X_4)$: les exemples déjà connus

On peut également espérer que la sous-matrice de M associée à la composante de Kuznetsov de X_4 soit en fait :

$$\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}.$$

En effet, à chacune des familles de catégories K3 non-commutatives déjà connues correspond une famille de variétés hyperkählériennes polarisées, et cette correspondance nous permet de formuler une telle hypothèse.

Avant de détailler ceci plus avant, rappelons nous que si $\Lambda := (H^2(X, \mathbb{Z}), q_X)$ est le réseau associé à une variété hyperkählérienne X de type $K3^{[2]}$, muni de sa forme de Beauville (voir par exemple le paragraphe 3.5 de [6]), alors Λ est l'orthogonal d'un vecteur primitif v de carré 2 dans $\tilde{\Lambda} := U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$. De plus, d'après la partie 3.7.1 de [6] nous savons que $\text{Pic}(X) = v^\perp \cap \tilde{\Lambda}_{\text{alg}}$. Pour X très général on peut s'attendre à ce que $\text{Pic}(X)$, qui contient la polarisation h , soit de rang 1. En particulier nous devons alors avoir $h.v = 0$. Le sous-réseau de $\tilde{\Lambda}$ généré par v et h a alors pour matrice d'intersection :

$$\begin{pmatrix} 2 & 0 \\ 0 & h^2 \end{pmatrix}.$$

Comme détaillé sous la Remarque 3.6 dans [6], le plongement de ce sous-réseau dans $\tilde{\Lambda}$ est primitif si et seulement si la divisibilité de h est 1, c'est ce qu'on appelle le "split case". Dans le cas contraire, le "non split case", quand la divisibilité de h est 2, combiné avec la condition $h.v = 0$ cela donne que nous pouvons écrire $h = v + 2\lambda$ où λ est tel que $\lambda.v = -1$ et $\lambda^2 = 2d'$ avec $d' \in \mathbb{N}$. La matrice d'intersection du réseau généré par v et λ est alors :

$$\begin{pmatrix} 2 & -1 \\ -1 & 2d' \end{pmatrix}.$$

Finalement, nous avons $h^2 = (v + 2\lambda)^2 = 8d' - 2$.

D'après la partie 2.1 de [13] et (7) dans [6], nous avons qu'à une cubique lisse de dimension 4 correspond une variété hyperkählérienne X comme ci-dessus (plus précisément la variété des droites dans C), avec $d' = 1$ dans le "non split case". Cette correspondance est aussi mentionnée dans la partie 3.6.1 de [6] : le cas des cubiques de dimension 4 correspond à $q(h) = 6$ et $\gamma = 2$, alors que les variétés de Debarre-Voisin et leurs variétés hyperkählériennes associées (le lieu des zéros d'un certain fibré vectoriel sur $\text{Gr}(6, 10)$) correspondent à $q(h) = 22$ et $\gamma = 2$, c'est-à-dire $d' = 3$ dans le "non split case". Pour finir, les variétés de Gushel-Mukai correspondent (voir [5]) au cas où $q(h) = 2$ et $\gamma = 1$, autrement dit $h^2 = 2$ dans le "split case" : leurs variétés hyperkählériennes associées sont appelées sextiques doubles EPW.

Cependant, nous ne connaissons pas d'exemples du "non split case" quand $d' = 2$ (qui correspond à $q(h) = 14$ et $\gamma = 2$ avec les notations de [6]). Puisque les exemples pour $d' = 1$ et $d' = 3$ sont reliés à des

familles de variétés de Fano de dimension 4 dont les catégories dérivées bornées admettent une composante de Kuznetsov K3 non-commutative, on peut espérer que si les catégories dérivées bornées de variétés de K uchle de dimension 4 et de type c5 contiennent effectivement une nouvelle famille de telles cat egories K3, alors il y a une famille de vari et es hyperk ahl eriennes associ ees qui pourrait donner un exemple du "non split case" quand $d' = 2$.

Si c' etait le cas, la matrice d'intersection de $K_{\text{num}}(\mathcal{A}_{X_4})$, pour X_4 tr es g en eral, devrait avoir la forme mentionn ee plus haut,  a supposer que ce r eseau soit, comme dans les cas d ej a connus, isomorphe au r eseau g en er e par v et λ de la vari et e hyperk ahl erienne associ ee (voir la partie 3.4 de [24] pour plus de d etails).

Il est aussi attendu (comme pr ec edemment, d'apr es ce qu'on peut observer dans le cas des cubiques, vari et es de Gushel-Mukai et vari et es de Debarre-Voisin) qu'une telle vari et e hyperk ahl erienne pourrait param etrer certains objets g eom etriques, ou  tre isomorphe   un espace de modules des conditions de stabilit e sur \mathcal{A}_{X_4} . Finalement, dans un tel cas les vari et es de K uchle de dimension 4 et de type c5 tr es g en erales correspondraient   une famille dans l'espace des modules ${}^4\mathcal{M}_{14}^{(2)}$ des vari et es hyperk ahl eriennes polaris ees de dimension 4 et de type $K3^{[2]}$ telles que $h^2 = 14$ et $\text{div}(h) = 2$, lequel est conjectur e unirationnel, l a encore par analogie avec les exemples d ej a connus.

Premier objectif

Le premier objectif de cette th ese  tait donc le suivant :

- trouver une base (e_1, \dots, e_n) de $K_{\text{num}}(X_4)$ sur \mathbb{Z} ;
- calculer la matrice $M = (\chi(e_i, e_j))_{1 \leq i, j \leq n}$ dont les coefficients sont les caract eristiques d'Euler entre les  l ements de notre base;
- trouver une matrice $R \in GL(n, \mathbb{Z})$ telle que $RM R^{-1}$ ait la forme mentionn ee pr ec edemment.

Comment trouver une famille dans $K_{\text{num}}(X_4)$ qui pourrait  tre une base

D'apr es le diamant de Hodge de X_4 , si $D^b(X_4)$ admet une d ecomposition semi-orthogonale comme d ecrite ci-dessus, et si la composante de Kuznetsov correspond bien   un sous-espace de dimension 2 de $K_{\text{num}}(X_4)$, alors pour X_4 tr es g en eral $K_{\text{num}}(X_4)$ devrait  tre de dimension 8. De plus, dans ce cas, pour X_4 tr es g en eral, dans une base de $K_{\text{num}}(X_4)$, cinq  l ements devraient venir de $H^2(X_4)$ (en supposant que l'on puisse g en erer la classe du faisceau structural d'une droite et d'un faisceau gratte-ciel   partir de ces cinq  l ements), et trois de $H^4(X_4)$.

C'est pourquoi, pour commencer, nous cherchons une famille libre de huit  l ements dans $K_{\text{num}}(X_4)$ telle que cinq  l ements viennent de faisceaux en droites et les trois autres sont associ es   des faisceaux structurels de surfaces dans X_4 .

Comme  l ements associ es   des faisceaux en droites nous pouvons simplement prendre

$$\{\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(1), \mathcal{O}_{X_4}(2), \mathcal{O}_{X_4}(3), \mathcal{O}_{X_4}(4)\}$$

(nous verrons plus tard qu'en fait cela ne donnerait qu'un sous-r eseau de $K_{\text{num}}(X_4)$, et nous remplacerons donc les deux derniers faisceaux en droites par le faisceau structural d'une droite et un faisceau gratte-ciel); les coefficients de M correspondants peuvent alors  tre calcul es gr ace au Th eor eme de Borel-Bott-Weil.

  propos des trois derniers  l ements cependant, les choses se compliquent. Nous pouvons utiliser la description de la g eom etrie de vari et es g en erales de K uchle de dimensions 4 et 5 et de type c5 donn ee dans [20] pour trouver des surfaces dont les faisceaux structurels donneront les trois derniers  l ements de la base. En

où

$$(e_1, \dots, e_5) = (\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(1), \mathcal{O}_{X_4}(2), \mathcal{O}_l, \mathcal{O}_p)$$

avec \mathcal{O}_l le faisceau structurel d'une droite dans Σ_2 et \mathcal{O}_p un faisceau gratte-ciel. Cette matrice correspond à un sous-réseau de $K_{\text{num}}(X_4)$, mais malheureusement il n'est pas évident que ce sous-réseau soit primitif.

En ajoutant un sixième élément $e_6 := \mathcal{O}_{\Sigma_2}$ à la famille $\{e_1, \dots, e_5\}$ on trouve la matrice :

$$M_2 = (\chi(e_i, e_j))_{1 \leq i, j \leq 6} = \begin{pmatrix} 1 & 20 & 124 & 1 & 1 & 1 \\ 1 & 1 & 20 & 0 & 1 & 1 \\ 20 & 1 & 1 & -1 & 1 & 7 \\ 0 & -1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 7 & 19 & 0 & 0 & 2 \end{pmatrix}.$$

Cependant comme détaillé plus haut, il nous manque encore au moins 2 éléments pour pouvoir espérer que notre famille soit une base de $K_{\text{num}}(X_4)$.

Dans la partie 2.1.2, nous considérons une famille libre \mathcal{F}_1 de 14 éléments dans $K_{\text{num}}(\tilde{X}_4)$, qui semble être une "bonne candidate" pour être une base. Nous ne pouvons calculer la matrice de caractéristiques d'Euler associée qu'à une intersection b près, ce qui donne :

$$\tilde{M} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 & 1 & 2 & 3 & -1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 & 1 & 2 & 3 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 1 & 2 & 3 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 6 & -1 & -7 & -19 & -5 & 12 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & -3 & -12 & -6 & 12+b & 19 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -6 & -6 & 12+2b & 28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 20 & 124 & 0 & -12 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 20 & 0 & 0 & 37 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 1 & 1 & 12 & 156 & 127 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 4 & 1 & 7 & 19 & 1 & -36 & 56 & 0 & 0 \\ 0 & 0 & 0 & 36 & 36+b & 36+2b & -36 & 0 & 180 & -156 & 48 & -24 & 0 & 0 \\ 3 & 3 & 3 & -42 & -33 & -24 & 79 & 79 & 205 & 76 & -24 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Dans la partie 2.2, nous considérons une autre famille libre \mathcal{F}_2 de 14 éléments dans $K_{\text{num}}(\tilde{X}_4)$, qui pourrait être une base si $K_{\text{num}}(X_4)$ a pour dimension 8, et la matrice des caractéristiques d'Euler entre les éléments de cette famille est :

$$\tilde{M}_2 = \begin{pmatrix} 1 & 20 & 124 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & -12 & 1 & 1 & 1 \\ 1 & 1 & 20 & 1 & 0 & 10 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 \\ 20 & 1 & 1 & 7 & 0 & 43 & 11 & 11 & 11 & 7 & 156 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & -84 & 0 & 1 & 0 \\ -1 & -7 & -19 & -1 & 1 & -2 & -2 & -2 & -2 & -2 & 12 & 0 & 1 & 0 \\ 0 & 24 & 72 & 10 & 10 & -1 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 \\ 0 & 6 & 20 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 6 & 20 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 6 & 20 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & -4 & 0 & 0 & 0 \\ 1 & 7 & 19 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ -36 & 0 & 180 & -60 & 36 & -12 & -4 & -4 & -4 & 0 & 48 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Il nous manque une intersection (le b mentionné ci-dessus) entre les classes de deux surfaces de \tilde{X}_4 pour pouvoir utiliser des foncteurs de mutation à droite et trouver une famille de $\pi^* K_{\text{num}}(X_4)$ induite par \mathcal{F}_2 . Malgré tout, nous pouvons exprimer les images de ces foncteurs de mutation qui nous intéressent en fonction de cette intersection b . La famille et la matrice associées sont détaillées dans la partie 2.3.2, en fonction de b . Cette nouvelle approche est basée sur les idées de Benjamin Sung.

Ce que l'on pourrait faire ensuite

Avant de pouvoir obtenir une indication sur une possible décomposition semi-orthogonale de $D^b(X_4)$, il y a donc au moins quatre étapes manquantes :

- Il faudrait vérifier que les matrices que nous avons obtenues (ou au moins certaines d'entre elles) ne sont pas associées à des sous-réseaux stricts de $K_{\text{num}}(\tilde{X}_4)$ ou $K_{\text{num}}(X_4)$.
- Il faudrait également calculer l'intersection b qui nous manque. Cela pourrait aussi nous permettre de trouver une matrice associée à $\pi^*(K_{\text{num}}(X_4))$ et donc M , comme nous allons le détailler dans la partie 2.3.2.
- Ensuite il faudrait chercher une matrice inversible $R \in GL(8, \mathbb{N})$ telle que $RM R^{-1}$ ait la forme décrite précédemment.
- Enfin, prouver que $K_{\text{num}}(X_4)$ est bien de dimension 8 pour X_4 très général nous assurerait que nous ne "manquons" aucune partie d'une possible décomposition de $D^b(X_4)$ en considérant M .

Organisation de la thèse

Dans la partie 1 nous rappelons plusieurs résultats et notations de [20], que nous allons utiliser par la suite. Dans la partie 2.1 nous trouvons une première famille \mathcal{F}_1 de 14 éléments dans $K_{\text{num}}(\tilde{X}_4)$ et calculons la matrice de caractéristiques d'Euler associée à une intersection entre surfaces près. Dans la partie 2.2 nous donnons une autre famille \mathcal{F}_2 de 14 éléments dans $K_{\text{num}}(\tilde{X}_4)$ et la matrice associée \tilde{M}_2 . Dans la partie 2.3.1 nous donnons deux matrices de caractéristiques d'Euler entre respectivement 5 et 6 éléments de $K_{\text{num}}(X_4)$. Pour pouvoir obtenir une base de $K_{\text{num}}(X_4)$, il faudrait plus de 6 éléments, comme nous l'avons détaillé précédemment. Cependant il serait intéressant de savoir si les sous-réseaux générés par nos 5 et 6 éléments sont primitifs dans $K_{\text{num}}(X_4)$. Nous nous intéressons brièvement à cette question sans parvenir à une conclusion. Dans la partie 2.3.2, nous expliquons comment trouver une famille dans $K_{\text{num}}(X_4)$ et la matrice associée en utilisant une famille de $K_{\text{num}}(\tilde{X}_4)$, sa matrice associée et des foncteurs de mutation à droite. Nous appliquons cette méthode à la famille \mathcal{F}_1 et calculons une matrice, qui dépend de l'intersection b et qui devrait correspondre à une famille dans $\pi^* K_{\text{num}}(X_4)$. Dans la partie 3 nous donnons une suite de petits résultats sur la géométrie de \tilde{X}_4 , dont certains nous seront utiles pour calculer des caractéristiques d'Euler. Dans la partie 4 nous détaillons les calculs de caractéristiques d'Euler entre éléments de \mathcal{F}_1 et dans la partie 5 nous faisons de même avec les éléments de \mathcal{F}_2 . Enfin, dans l'annexe 6.1 nous détaillons brièvement certains calculs utilisés dans la partie 2.3, dans l'annexe 6.2 nous donnons une autre manière (pas nécessairement plus rapide mais que nous avons utilisée pour vérifier certains calculs) de calculer certaines des caractéristiques d'Euler mentionnées dans la partie 5, et dans les annexes 6.3 et 6.4 nous donnons les codes que nous avons utilisés, respectivement dans Macaulay2 et Scilab pour calculer certaines des dimensions mentionnées dans la partie 4.2.

0.2 Introduction in English

In this thesis, we are interested in Kuchle fourfolds of type c5 over \mathbb{C} . For X_4 a very general such fourfold, the aim was to compute the matrix $M = (\chi(e_i, e_j))$ where χ is the Euler characteristic and the e_i give a basis of the numerical Grothendieck group. Once this matrix computed, finding another basis in which M has a specific form that will be described below would give a hint on a possible semi-orthogonal decomposition of the bounded derived category of coherent sheaves on X_4 . Indeed some similarities with cubic fourfolds in \mathbb{P}^5 and some Gushel-Mukai fourfolds suggest that there could be such a decomposition.

Short summary

In his article on Calabi-Yau and fractional Calabi-Yau categories [22], A. Kuznetsov gives a Theorem which allows in multiple cases to find semi-orthogonal decompositions of the bounded derived categories of some smooth projective varieties.

More precisely, when a functor $\phi : D^b(X) \rightarrow D^b(M)$ between the bounded derived categories of two smooth projective varieties satisfies some conditions, one can induce a semi-orthogonal decomposition of

$D^b(X)$ if one knows a rectangular Lefschetz decomposition of $D^b(M)$ (again under some technical conditions which we are not going to detail here). This semi-orthogonal decomposition of $D^b(X)$ consists mainly in subcategories induced by those of the Lefschetz decomposition of $D^b(M)$ and their right orthogonal, a subcategory often referred to as the "Kuznetsov component". Moreover, this same Theorem 3.5 of [22] allows to compute the Serre functor of the Kuznetsov component.

Some of the Kuznetsov components obtained thanks to this Theorem turn out to be non-commutative K3-categories, that is to say categories which "look like" the bounded derived category of a K3 surface. In fact, all the non trivial examples of non-commutative K3-categories known so far arise as Kuznetsov components in the bounded derived categories of some projective varieties. Furthermore, these non-commutative K3-categories are believed to be related to the geometry of their associated variety.

Indeed, such non-commutative K3-categories appear as Kuznetsov components of cubic fourfolds in \mathbb{P}^5 , and it has been conjectured by A. Kuznetsov (see Conjecture 1.1 of [18]) that a cubic fourfold is rational if and only if its Kuznetsov component is equivalent to the bounded derived category of a K3 surface. This condition is not satisfied for very general cubic fourfolds : the numerical Grothendieck group of their Kuznetsov component couldn't contain the class of skyscraper sheaf, since it has no non zero element v such that $\chi(v, v) = 0$ (see for instance Remark 3.19 of [24] for more details). Therefore, this conjecture implies that very general cubic fourfolds are not rational.

Gushel-Mukai fourfolds also have a Kuznetsov component which is a non-commutative K3-category, and it has been conjectured by A. Kuznetsov and A. Perry in [23] (Conjecture 3.12) that it is equivalent to the bounded derived category of a K3 surface when the associated Gushel-Mukai fourfold is rational.

Another interesting feature of non-commutative K3-categories appearing as Kuznetsov components is that in the three families of (non trivial) examples known so far, one can find a corresponding family of Hyper-Kähler varieties : to a cubic fourfold C corresponds the family of lines in C , to a Gushel-Mukai fourfold is associated a double EPW sextic, and to a Debarre-Voisin variety, the last family of examples, one can associate a Hyper-Kähler fourfold in $\text{Gr}(6, 10)$. All these examples are listed in section 3.6.1 of [6].

Therefore, it would be interesting to find a new non trivial family of non-commutative K3-categories, appearing as Kuznetsov components in the bounded derived categories of some smooth projective varieties : one could then study the relation between these categories and the geometry of their associated varieties. Moreover, a moduli space of objects in such a Kuznetsov component would carry a symplectic form (see section 4.4 of [22] for more details), hence one could also hope to find a corresponding family of Hyper-Kähler manifolds.

Among all the Fano fourfolds classified by O. Küchle in [15], three types have a Hodge diamond suggesting that their bounded derived category could contain a non-commutative K3-category. Out of these three types, two have been studied in [19] and do not provide new non trivial families of non-commutative K3-categories. Yet for the third type, namely the type c5, the question is still open.

In [20], Kuznetsov studied the geometry of Küchle fourfolds of type c5, denoted X_4 , and of some associated fivefolds, called Küchle fivefolds of type c5. This allowed him to compute the Chow motive of general Küchle fivefolds of type c5. However, a rectangular Lefschetz decomposition of such fivefolds has not been found yet, and is necessary to apply the above-mentioned Theorem 3.5 of [22], which would be needed to prove the existence of Kuznetsov components in $D^b(X_4)$.

Although it is quite complicated to prove directly the existence of a Kuznetsov component in $D^b(X_4)$, if such a component \mathcal{A} exists, using some analogy with the cubic fourfolds case we mentioned earlier, it may be possible to find a basis of the numerical Grothendieck group $K_{\text{num}}(\mathcal{A})$. Indeed, if the analogy with cubic fourfolds is correct, in an appropriate basis \mathcal{B} of $K_{\text{num}}(X_4)$ for very general X_4 , the matrix of Euler characteristics between

elements of \mathcal{B} should have this form :

$$M = \begin{pmatrix} A & * & \dots & * \\ 0 & 1 & * & * \\ \vdots & 0 & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

where A is a matrix of Euler characteristics between elements of a basis of $K_{\text{num}}(\mathcal{A})$.

If one could thus find a basis of $K_{\text{num}}(X_4)$ whose associated matrix of Euler characteristics has the form above, then the elements of this basis corresponding to the block A would be "good candidates" for being a basis of $K_{\text{num}}(\mathcal{A})$. This could therefore give an indication about what a Kuznetsov component in $D^b(X_4)$ could be.

This is why the first goal of this thesis was to find a basis of $K_{\text{num}}(X_4)$ (as a lattice over \mathbb{Z}), compute the matrix M of Euler characteristics between elements of this basis, and then look for an invertible matrix R with coefficients in \mathbb{Z} such that $RM R^{-1}$ has the form mentioned above.

However, it turns out to be quite difficult. Indeed, the Hodge diamond of X_4 suggests that there should be in $K_{\text{num}}(X_4)$ four classes associated to surfaces in X_4 . Using some results about the geometry of K uchle fivefolds and fourfolds of type c5 given in [20], we manage to find some surfaces in X_4 , but unfortunately we are unable to compute Euler characteristics involving some of them.

Yet, we know from [20] the existence of a blow-up \tilde{X}_4 of X_4 , and it may be possible to recover $D^b(X_4)$ from $D^b(\tilde{X}_4)$ using right mutation functors. Hence, the second goal of this thesis was to find a basis of $K_{\text{num}}(\tilde{X}_4)$ and compute the matrix \tilde{M} of Euler characteristics between elements of this basis.

Although we didn't prove there are basis, we found two free families of 14 elements in $K_{\text{num}}(\tilde{X}_4)$. For one of them we computed the associated matrix \tilde{M} up to an intersection number and for the second family we did compute the matrix of Euler characteristics \tilde{M}_2 entirely. Yet these two matrices can't have the same determinant, which suggests that our two families don't generate the same sublattices of $K_{\text{num}}(\tilde{X}_4)$. It would thus be interesting to know whether any of these families correspond to a primitive sublattice of $K_{\text{num}}(\tilde{X}_4)$ or not.

Definition of K uchle fourfolds and fivefolds of type c5

To start with, let us denote by G the Grassmannian $\text{Gr}(3, \mathbb{C}^7)$ and by \mathcal{U}_3 the tautological sub-bundle on G . A K uchle fourfold of type c5 is the zero locus of a general global section of $\mathcal{U}_3^\perp(1) \oplus \mathcal{U}_3(1) \oplus \mathcal{O}_G(1)$ where \mathcal{U}_3^\perp is the quotient of the injection $\mathcal{U}_3 \hookrightarrow \mathbb{C}^7 \otimes \mathcal{O}_G$. For a general enough global section this gives a smooth Fano fourfold. The name "type c5" comes from the classification, given by O. K uchle in [15], of Fano fourfolds of index 1 that can be obtained as the zero locus of a general global section of a direct sum of globally generated irreducible homogeneous vector bundles over a Grassmannian. Moreover, each K uchle fourfold of type c5 is a hyperplane section of a fivefold X_5 which we will call K uchle fivefold of type c5. Indeed if X_4 is the zero locus of the global section $\lambda \oplus \mu \oplus \nu$ of $\mathcal{U}_3^\perp(1) \oplus \mathcal{U}_3(1) \oplus \mathcal{O}_G(1)$ then we can define X_5 to be the zero locus of $\lambda \oplus \mu$, global section of $\mathcal{U}_3^\perp(1) \oplus \mathcal{U}_3(1)$. Then since ν is a global section of $\mathcal{O}_G(1)$ its zero locus H_ν is a hyperplane and $X_4 = X_5 \cap H_\nu$. In [20], A. Kuznetsov describes the geometry of such K uchle fivefolds, which allows him to deduce some geometrical properties of the associated K uchle fourfolds. We will often refer to this article [20] and mostly keep its notations.

Kuznetsov components and non-commutative K3-categories

It is expected that the bounded derived category of coherent sheaves on X_4 , denoted by $D^b(X_4)$, admits an interesting semi-orthogonal decomposition, by analogy with some other examples and because of the Hodge diamond of X_4 .

Let us first consider the case of cubic fourfolds in \mathbb{P}^5 : these varieties are sections of fivefolds Y_5 with a rectangular Lefschetz decomposition, that is to say a semi-orthogonal decomposition of their bounded derived category of coherent sheaves as follows :

$$D^b(Y_5) = \langle \mathcal{B}, \mathcal{B} \otimes \mathcal{L}, \dots, \mathcal{B} \otimes \mathcal{L}^{\otimes m} \rangle$$

where $m \in \mathbb{N}$, \mathcal{B} is a triangulated subcategory of $D^b(Y_5)$, \mathcal{L} is a line bundle on Y_5 and a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is a sequence of full admissible triangulated subcategories such that $\text{Hom}^\bullet(\mathcal{A}_j, \mathcal{A}_i) = 0$ for any $j > i$ and such that the smallest full triangulated subcategory of \mathcal{D} containing all the \mathcal{A}_i is equivalent to \mathcal{D} (see Definition 1.59 of [14]).

Applying Theorem 3.5 of A. Kuznetsov's [22] gives then a semi-orthogonal decomposition of $D^b(Y_4)$:

$$D^b(Y_4) = \langle \mathcal{A}, \mathcal{B}', \mathcal{B}' \otimes \mathcal{L}', \dots, \mathcal{B}' \otimes \mathcal{L}'^{\otimes k} \rangle$$

where $k \in \mathbb{N}$, \mathcal{A} is called the Kuznetsov component of Y_4 and the rest of the decomposition is induced by the Lefschetz decomposition of $D^b(Y_5)$. Moreover when Y_4 is a cubic fourfold, its Kuznetsov component \mathcal{A}_{Y_4} is a non-commutative K3-category : its Serre functor is the shift by 2 and its Hochschild homology is the same as the one of the bounded derived category of a K3 surface. So as a triangulated subcategory of $D^b(Y_4)$, \mathcal{A}_{Y_4} satisfies the Definition 2.31 of [24] given by E. Macrì and P. Stellari and is thus a non-commutative K3 category, that is to say a category that "looks like" the bounded derived category of a K3 surface.

However there exists non-commutative K3 categories that cannot be equivalent to the bounded derived category of a K3 surface (see for instance Proposition 4.8 of [18]), while bounded derived categories of K3 surfaces are all non-commutative K3 categories. Thus, non-commutative K3 categories can be considered as a "generalisation" of $D^b(S)$ for S a K3 surface.

Coming back to our K3 fourfolds and fivefolds, one can see by looking at X_4 's Hodge diamond :

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 0 \\ 0 & 1 & 24 & 1 & 0 & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 1 & 0 & & \\ & 0 & 0 & & & \\ & & & & & 1 \end{array}$$

given for instance in the introduction of [20], that it could "contain" the Hodge diamond of a K3 surface. K3 fourfolds of type c5 are not the only ones among the Fano fourfolds classified by K3 whose Hodge diamond suggests that their bounded derived category could have a non-commutative K3 subcategory but the two other "candidates" for having such subcategories (namely K3 fourfolds of type d3 and c7) are already known examples while the fourfolds of type c5 could give a new family of non-commutative K3 categories.

Why are we interested in non-commutative K3-categories : conjectured link to rationality and Hyper-Kähler varieties

Such families of non-commutative K3-categories are interesting for mainly two reasons. First of all, in the case of a cubic fourfold C it has been conjectured by A. Kuznetsov that C is rational if and only if its Kuznetsov component \mathcal{A}_C is equivalent to $D^b(S)$ where S is a K3 surface.

It has also been conjectured by A. Kuznetsov and A. Perry that the Kuznetsov component of a Gushel-Mukai fourfold GM , which is also a non-commutative K3-category, is equivalent to the bounded derived category of a K3 surface when GM is rational.

These non-commutative K3 categories seem thus to be linked to the geometry of their associated varieties and it could be interesting to find new examples of them (appearing as Kuznetsov component in the bounded derived category of a variety), especially since except for the bounded derived categories of K3 surfaces, there are only three families of known examples so far : the cubic fourfolds mentioned above, Gushel-Mukai fourfolds and the Debarre-Voisin varieties.

Moreover, as detailed by A. Kuznetsov in section 4.4 of [22], to each Kuznetsov component which is also a non-commutative K3 category it may be possible to associate a Hyper-Kähler variety, hence finding a family of varieties with a non-commutative K3-Kuznetsov component could also give an associated family of Hyper-Kähler varieties.

To each of the non-trivial families of non-commutative K3-categories known so far corresponds an associated family of Hyper-Kähler varieties : these are listed in section 3.6.1 of [6], and we are going to talk about them more in detail later.

Main difficulty in proving the existence of a Kuznetsov component in $D^b(X_4)$

Unfortunately, in the case of K3 fourfolds X_4 of type c5 it is quite difficult to prove that such a Kuznetsov component exists : indeed in the three families of (non-trivial) examples which are known so far, to prove the existence of the Kuznetsov component and some of the properties required to be a non-commutative K3 category, one has to use Theorem 3.5 of [22]. But using this Theorem in the case of X_4 would require to know a rectangular Lefschetz decomposition of $D^b(X_5)$ and such a decomposition is still to be found.

Yet, it is expected that such a rectangular Lefschetz decomposition exists. In order to be able to use this type of decomposition (together with Theorem 3.5 in [22]) to induce a semi-orthogonal decomposition of a K3 fourfold of type c5 though, we would need it to be of the form described in Example 3.1 of [22]. Since our K3 fourfold X_4 is the zero locus of a global section of $\mathcal{O}_{X_5}(1)$ in the associated fivefold X_5 , we would then need the line bundle \mathcal{L} of the rectangular Lefschetz decomposition to be $\mathcal{O}_{X_5}(1)$. By Lemma 2.8 of [22] and since we know from [20] that $\omega_{X_5} = \mathcal{O}_{X_5}(-2)$, there is then only one possibility for such a rectangular Lefschetz decomposition :

$$\langle \mathcal{B}_0, \mathcal{B}_0(1) \rangle .$$

In this case, the proof of Proposition 3.4 and Theorem 3.5 of [22] yield that the Serre functor of the Kuznetsov component \mathcal{A}_{X_4} is the shift $[-4 + 6] = [2]$. Together with the above-mentioned Hodge diamond, it implies that if a K3 fivefold of type c5 admits a rectangular Lefschetz decomposition as just described, then all the associated X_4 have a Kuznetsov component which is very likely to be a non-commutative K3 category.

While looking for a rectangular Lefschetz decomposition of X_5 , A. Kuznetsov studied closely the geometry of general K3 fivefolds of type c5 and proved in Theorem 5.3 of [20] that the Chow motive of a general X_5 is of Lefschetz type. He deduced from this result the Hodge diamond and the Chow groups of X_5 , claiming that explicit generators of the latter can be deduced from his geometrical description of X_5 .

How to find a hint at what the Kuznetsov component could be without knowing a rectangular Lefschetz decomposition of $D^b(X_5)$

However, we are going here to look at X_4 rather than X_5 . Indeed, we know that if a semi-orthogonal decomposition of the form $D^b(X_4) = \langle \mathcal{A}_{X_4}, \mathcal{B}, \mathcal{B} \otimes \mathcal{L}, \dots, \mathcal{B} \otimes \mathcal{L}^{\otimes k} \rangle$ exists, then in a basis of the numerical Grothendieck group $K_{\text{num}}(X_4)$ whose elements $(e_i)_i$ belong in order to $\mathcal{A}_{X_4}, \mathcal{B}, \dots, \mathcal{B} \otimes \mathcal{L}^{\otimes k}$, the matrix $M = (\chi(e_i, e_j))_{i,j}$ should be of the form :

$$M = \begin{pmatrix} A & * & \dots & * \\ 0 & B_0 & * & * \\ \vdots & 0 & \ddots & * \\ 0 & \dots & 0 & B_k \end{pmatrix}$$

where

$$A = (\chi(e_i, e_j))_{e_i, e_j \in K_{\text{num}}(\mathcal{A}_{X_4})}, B_0 = (\chi(e_i, e_j))_{e_i, e_j \in K_{\text{num}}(\mathcal{B})}, \dots, B_k = (\chi(e_i, e_j))_{e_i, e_j \in K_{\text{num}}(\mathcal{B} \otimes \mathcal{L}^{\otimes k})}.$$

Moreover, in the case of the cubic fourfolds it turns out that the triangulated subcategory \mathcal{B} is actually generated by an exceptional object (an element E in $D^b(C)$ such that $\text{Hom}(E, E[m]) = \mathbb{C}\delta_{0,m}$). Furthermore, in the very general case the matrix A is then of size 2 by 2 so if the Küchle fourfolds of type c5 have a semi-orthogonal decomposition similar to the one of the cubic fourfolds, one can even expect the matrix M to be :

$$M = \begin{pmatrix} m_1 & m_2 & * & \dots & * \\ m_3 & m_4 & * & \dots & * \\ 0 & 0 & 1 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Thus, finding a basis of $K_{\text{num}}(X_4)$ in which M has this form would give us a hint about what a semi-orthogonal decomposition of $D^b(X_4)$ could be : the first two elements of the basis could correspond to generators of the Kuznetsov component and the others to exceptional objects generating the $\mathcal{B} \otimes \mathcal{L}^{\otimes j}$ parts of the decomposition.

Link between the associated Hyper-Kähler and $K_{\text{num}}(\mathcal{A}_{X_4})$: the already known examples

Furthermore, we hope that the part of M associated to the Kuznetsov component of X_4 is actually :

$$\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}.$$

Indeed to each of the already known families of non-commutative $K3$ categories corresponds a family of polarized Hyper-Kähler manifolds, in a way that allows us to formulate such a hope.

Before detailing this, let us recall that if $\Lambda := (H^2(X, \mathbb{Z}), q_X)$ is the lattice associated to a $K3^{[2]}$ -type Hyper-Kähler manifold X , endowed with its Beauville form (see for instance part 3.5 of [6]), then Λ is the orthogonal of a primitive vector v of square 2 in $\tilde{\Lambda} := U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$. Moreover, from 3.7.1 of [6] we know that $\text{Pic}(X) = v^\perp \cap \tilde{\Lambda}_{\text{alg}}$. For very general X one can expect $\text{Pic}(X)$, which contains the polarization h , to be rank 1. In particular we then must have $h.v = 0$. The sublattice of $\tilde{\Lambda}$ generated by v and h has then intersection matrix :

$$\begin{pmatrix} 2 & 0 \\ 0 & h^2 \end{pmatrix}.$$

As detailed below Remark 3.6 in [6], the embedding of this sublattice in $\tilde{\Lambda}$ is primitive if and only if the divisibility of h is 1, this is the so-called "split case". In the "non split case", when the divisibility is 2, together with the condition $h.v = 0$ it implies that we can write $h = v + 2\lambda$ where λ is such that $\lambda.v = -1$ and $\lambda^2 = 2d'$ for some $d' \in \mathbb{N}$. The intersection matrix of the lattice generated by v and λ is then :

$$\begin{pmatrix} 2 & -1 \\ -1 & 2d' \end{pmatrix}.$$

Finally we have $h^2 = (v + 2\lambda)^2 = 8d' - 2$.

By part 2.1 in [13] and (7) in [6], we have that to a smooth cubic fourfold C corresponds a HK manifold X as above (namely the variety of lines in C), in the non-split case and with $d' = 1$. This is also mentioned in section 3.6.1 of [6] : the case of cubic fourfolds corresponds to $q(h) = 6$ and $\gamma = 2$, while Debarre-Voisin varieties and their associated HK manifolds (namely the zero locus of some vector bundle on $\text{Gr}(6, 10)$) correspond to $q(h) = 22$ and $\gamma = 2$, that is to say $d' = 3$ in the non-split case. Finally Gushel-Mukai varieties are in correspondence (see [5]) with the case $q(h) = 2$ and $\gamma = 1$, in other words $h^2 = 2$ in the split case : their associated HK

varieties are called double EPW sextic.

However, we don't know examples of the non-split case when $d' = 2$ (which corresponds to $q(h) = 14$ and $\gamma = 2$ with [6]'s notations). Since the examples for $d' = 1$ and 3 both arise in relation with families of Fano fourfolds whose bounded derived category admits a non-commutative K3-Kuznetsov component, one could hope that if the bounded derived categories of Kuchle fourfolds of type c5 indeed contain a new family of such K3-categories, then there is a family of associated HK manifolds which will provide an example of the non-split case with $d' = 2$.

If it was the case, the intersection matrix of $K_{\text{num}}(\mathcal{A}_{X_4})$, for X_4 very general, should be of the above-mentioned form, provided that it is, similarly to the already known cases, isomorphic to the lattice generated by v and λ of the associated HK manifold (see part 3.4 of [24] for more details).

It is also expected (as before, from what happens in the case of the cubic fourfolds, Gushel-Mukai fourfolds and Debarre-Voisin varieties) that this HK manifold could parameterize some geometrical objects or be isomorphic to a moduli space of stability conditions on \mathcal{A}_{X_4} . Finally, in such a case very general Kuchle fourfolds of type c5 would give rise to a family in the moduli space ${}^4\mathcal{M}_{14}^{(2)}$ of polarized HK fourfolds of $K3^{[2]}$ -type such that $h^2 = 14$ and $\text{div}(h) = 2$, which is believed to be unirational, again by analogy with the already known cases.

First goal

The first goal of this thesis was therefore the following :

- find a basis (e_1, \dots, e_n) of $K_{\text{num}}(X_4)$ over \mathbb{Z} ;
- compute the matrix $M = (\chi(e_i, e_j))_{1 \leq i, j \leq n}$ whose coefficients are Euler characteristics between elements of our basis;
- find a matrix $R \in GL(n, \mathbb{Z})$ such that RMR^{-1} has the form mentioned earlier.

How to find a family in $K_{\text{num}}(X_4)$ which could be a basis

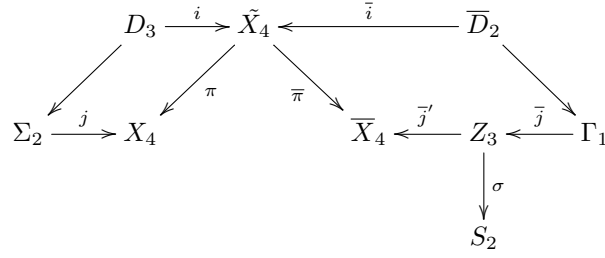
According to the Hodge diamond of X_4 , if $D^b(X_4)$ admits a semi-orthogonal decomposition as described above and if the Kuznetsov component induces a 2-dimensional subspace of $K_{\text{num}}(X_4)$, then for X_4 very general $K_{\text{num}}(X_4)$ should be 8-dimensional. Moreover, in this case, for X_4 very general we should have that in a basis of $K_{\text{num}}(X_4)$, five elements should come from $H^2(X_4)$ (assuming we can recover the structure sheaf of a line and a skyscraper sheaf from these five elements), and three from $H^4(X_4)$.

Therefore, to start with, we look for a free family of eight elements in $K_{\text{num}}(X_4)$ such that five elements come from line bundles and the three others are associated to structure sheaves of surfaces in X_4 .

As elements associated to line bundles we can simply take $\{\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(1), \mathcal{O}_{X_4}(2), \mathcal{O}_{X_4}(3), \mathcal{O}_{X_4}(4)\}$ (we will actually see later that this would only give a sublattice of $K_{\text{num}}(X_4)$, and will thus replace the last two line bundles by the structure sheaf of a line and a skyscraper sheaf); the corresponding coefficients of M can then be computed using Borel-Bott-Weil Theorem.

Concerning the last three elements however, things become more complicated. We can use the description of the geometry of general Kuchle fourfolds and fivefolds of type c5 given in [20] to find surfaces whose structure sheaves will give the last three elements in our basis. In particular, as proved in [20], for general X_4 there

is a diagram :



where the indices give the dimension of the associated varieties, and π and $\bar{\pi}$ are blow-ups of the fourfolds X_4 and \bar{X}_4 , respectively in Σ_2 and Z_3 with respectively exceptional divisor D_3 and exceptional locus \bar{D}_2 . The maps i, \bar{i}, j, \bar{j}' and \bar{j} are embeddings, Z_3 is a scroll over S_2 and \bar{D}_2 is a ruled surface over the curve Γ_1 .

There are three surfaces which appear in this diagram : Σ_2, \bar{D}_2 and S_2 . However neither S_2 nor \bar{D}_2 are actual surfaces in \tilde{X}_4 ; we can use them to induce such surfaces but in both cases it requires to project surfaces in \tilde{X}_4 to X_4 using the contraction π . Unfortunately, it is quite complicated to understand whether the images of these surfaces by π have "bad" contractions or not, which makes it very hard to compute Euler characteristics involving them. This is why we didn't manage to complete our first goal, and adopted a new strategy, with the help of Benjamin Sung.

We can look at $K_{\text{num}}(\tilde{X}_4)$ instead

On the other hand, looking at a matrix of Euler characteristics between elements in a basis of $K_{\text{num}}(\tilde{X}_4)$ instead of $K_{\text{num}}(X_4)$ turns out to be easier, since in this case we don't have problems with contractions. It may also be possible to recover the matrix M related to X_4 from the one, let us denote it \tilde{M} , related to \tilde{X}_4 , using mutation functors and the semi-orthogonal decomposition of $D^b(\tilde{X}_4)$ given by Theorem 3.5 of [2]. Indeed this Theorem allows to find a semi-orthogonal decomposition of bounded derived categories of blow-ups. All this is detailed in subsection 2.3.2.

Moreover, as we will see, the determinants of these two matrices M and \tilde{M} are equal, hence if there is indeed a basis in which the matrix M has the above-mentioned form, then we can expect this determinant to be $m_1 m_4 - m_2 m_3$, which we hope thus to be 7 as mentioned earlier.

Second goal

Our second goal became then to find a basis of $K_{\text{num}}(\tilde{X}_4)$ over \mathbb{Z} and then compute the associated matrix of Euler characteristics.

Main results

In section 2.3.1 we compute the following matrix :

$$M_1 = (\chi(e_i, e_j))_{1 \leq i, j \leq 5} = \begin{pmatrix} 1 & 20 & 124 & 1 & 1 \\ 1 & 1 & 20 & 0 & 1 \\ 20 & 1 & 1 & -1 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

where

$$(e_1, \dots, e_5) = (\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(1), \mathcal{O}_{X_4}(2), \mathcal{O}_l, \mathcal{O}_p)$$

with \mathcal{O}_l the structure sheaf of a line in Σ_2 and \mathcal{O}_p a skyscraper sheaf. It corresponds to a sublattice of $K_{\text{num}}(X_4)$, but unfortunately it is not clear whether this sublattice is primitive or not.

Adding a sixth element $e_6 := \mathcal{O}_{\Sigma_2}$ to the family $\{e_1, \dots, e_5\}$ gives then the matrix :

$$M_2 = (\chi(e_i, e_j))_{1 \leq i, j \leq 6} = \begin{pmatrix} 1 & 20 & 124 & 1 & 1 & 1 \\ 1 & 1 & 20 & 0 & 1 & 1 \\ 20 & 1 & 1 & -1 & 1 & 7 \\ 0 & -1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 7 & 19 & 0 & 0 & 2 \end{pmatrix}.$$

Yet as discussed above, we are still missing at least two elements to be able to hope that we are considering a basis of $K_{\text{num}}(X_4)$.

In section 2.1.2, we consider a free family \mathcal{F}_1 of 14 elements in $K_{\text{num}}(\tilde{X}_4)$, which seems to be a "good candidate" for being a basis. We can compute the associated matrix of Euler characteristics only up to an intersection number b , which gives :

$$\tilde{M} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 & 1 & 2 & 3 & -1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 & 1 & 2 & 3 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 1 & 2 & 3 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 6 & -1 & -7 & -19 & -5 & 12 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & -3 & -12 & -6 & 12+b & 19 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -6 & -6 & 12+2b & 28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 20 & 124 & 0 & -12 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 20 & 0 & 0 & 37 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 1 & 1 & 12 & 156 & 127 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 4 & 1 & 7 & 19 & 1 & -36 & 56 & 0 & 0 \\ 0 & 0 & 0 & 36 & 36+b & 36+2b & -36 & 0 & 180 & -156 & 48 & -24 & 0 & 0 \\ 3 & 3 & 3 & -42 & -33 & -24 & 79 & 79 & 205 & 76 & -24 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In section 2.2, we consider another free family \mathcal{F}_2 of 14 elements in $K_{\text{num}}(\tilde{X}_4)$ which may be a basis if $K_{\text{num}}(X_4)$ is 8-dimensional, and the matrix of Euler characteristics between elements of this family is :

$$\tilde{M}_2 = \begin{pmatrix} 1 & 20 & 124 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & -12 & 1 & 1 & 1 \\ 1 & 1 & 20 & 1 & 0 & 10 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 \\ 20 & 1 & 1 & 7 & 0 & 43 & 11 & 11 & 11 & 7 & 156 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & -84 & 0 & 1 & 0 \\ -1 & -7 & -19 & -1 & 1 & -2 & -2 & -2 & -2 & -2 & 12 & 0 & 1 & 0 \\ 0 & 24 & 72 & 10 & 10 & -1 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 \\ 0 & 6 & 20 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 6 & 20 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 6 & 20 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & -4 & 0 & 0 & 0 \\ 1 & 7 & 19 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ -36 & 0 & 180 & -60 & 36 & -12 & -4 & -4 & -4 & 0 & 48 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We are missing one intersection number (namely b) between two surfaces in \tilde{X}_4 to be able to use right mutation functors and find a family in $\pi^*K_{\text{num}}(X_4)$ induced by \mathcal{F}_2 . Yet we can express the images of these mutation functors we are interested in, in terms of this intersection number b . The associated family and matrix are detailed in section 2.3.2, still in terms of b . This new approach is based on the ideas of Benjamin Sung.

What could be done next

Before being able to get an intuition about a possible semi-orthogonal decomposition of $D^b(X_4)$ there are thus at least four steps missing :

- One should check that the matrices we obtained (or at least some of them) do not correspond to strict sublattices of $K_{\text{num}}(\tilde{X}_4)$ or $K_{\text{num}}(X_4)$.

- One should compute the intersection number b we are missing. This could also allow one to find a matrix associated to $\pi^*(K_{\text{num}}(X_4))$ and thus M , as we will discuss in section 2.3.2.
- Then one should look for an invertible matrix $R \in GL(8, \mathbb{Z})$ such that RMR^{-1} has the above-mentioned form.
- Finally, proving that $K_{\text{num}}(X_4)$ is 8-dimensional for X_4 very general would insure us that we would not be missing any part of a possible decomposition of $D^b(X_4)$ by studying M .

Organization of the thesis

In section 1 we recall various results and notations from [20] which we are going to use all along. In section 2.1 we find a first family \mathcal{F}_1 of 14 elements in $K_{\text{num}}(\tilde{X}_4)$ and compute the associated matrix of Euler characteristics up to an intersection number. In section 2.2 we give another family \mathcal{F}_2 of 14 elements in $K_{\text{num}}(\tilde{X}_4)$ and the associated matrix \tilde{M}_2 . In section 2.3.1 we give two matrices of Euler characteristics between respectively 5 and 6 elements in $K_{\text{num}}(X_4)$. In order to get a basis of $K_{\text{num}}(X_4)$, one would need more than 6 elements, as we detailed it previously, yet it would already be interesting to know whether the sublattices generated by our 5 and 6 elements are primitive in $K_{\text{num}}(X_4)$. We briefly discuss this without reaching a conclusion. In section 2.3.2, we explain how we hope to find a family in $K_{\text{num}}(X_4)$ and the associated matrix using a family in $K_{\text{num}}(\tilde{X}_4)$, its matrix and right mutation functors. We apply this method to the family \mathcal{F}_1 and compute a matrix, depending on the intersection number b , which should correspond to a family in $\pi^*K_{\text{num}}(X_4)$. In section 3 we give a collection of small results about the geometry of \tilde{X}_4 , some of which will be used to compute some Euler characteristics. In section 4 we detail the computations of Euler characteristics between elements of \mathcal{F}_1 and in section 5 we do the same with elements of \mathcal{F}_2 . Finally in Appendix 6.1 we briefly detail some calculations used in section 2.3, in Appendix 6.2 we give another way (not necessarily faster but which we used to check some calculations) to compute some of the Euler characteristics appearing in section 5, while in Appendix 6.3 and 6.4 we give the codes we used in respectively Macaulay2 and Scilab to compute some dimensions mentioned in section 4.2.

Chapter 1

Some results on the geometry of K uchle fourfolds and fivefolds of type c_5

In this section we are going to recall some results and notations from [20] which we are going to use in the next sections.

In his article [20], A. Kuznetsov describes the geometry of general K uchle fivefolds of type c_5 , in particular he shows that under some general assumptions that we are going to mention later, such a K uchle fivefold X_5 contains a threefold F_3 , such that the blow-up of X_5 in F_3 is also the blow-up of another fivefold in a smooth scroll over a del Pezzo surface of degree 6. This will be mentioned in Theorem 1.2.1 below. Moreover, since each K uchle fourfold of type c_5 X_4 is a hyperplane section of a K uchle fivefold of type c_5 , the results about the geometry of X_5 induce similar statements for X_4 , which are gathered here in Theorem 1.2.3. We will need Theorem 1.2.3 to find elements in $K_{\text{num}}(X_4)$ and $K_{\text{num}}(\tilde{X}_4)$ associated to structure sheaves of surfaces (namely Σ_2 , \overline{D}_2 and a blow-up of S_2). Since these surfaces actually come from the geometry of X_5 , we will also need a lot of the results about X_5 in [20] to compute the associated Euler characteristics. We will then keep the same notations as in [20] in most cases.

1.1 Notations and assumptions

First of all, here is a list of some notations and conventions we are going to keep from [20].

- If \mathcal{V} is a vector bundle over a scheme S then $p : \mathbb{P}_S(\mathcal{V}) \rightarrow S$ is the associated projective bundle and $\mathcal{O}(1)$ (also denoted sometimes $\mathcal{O}(V_{\mathbb{P}_S(\mathcal{V})})$) will be the ample generator of the relative Picard group such that $p_*\mathcal{O}(1) \cong \mathcal{V}^\vee$.
- Unless otherwise mentioned, all functors will be considered to be derived so for instance f_* will actually mean R^*f_* .
- W will be a 7-dimensional vector space over \mathbb{C} and we will denote by U_k a k -dimensional vector subspace of W .
- We will denote by \mathcal{U}_k the tautological sub-bundle of $\text{Gr}(k, W)$ and define $\mathcal{U}_k^\perp := (W/\mathcal{U}_k)^\vee$ where (W/\mathcal{U}_k) fits in the following short exact sequence : $0 \rightarrow \mathcal{U}_k \rightarrow W \otimes \mathcal{O}_{\text{Gr}(k, W)} \rightarrow W/\mathcal{U}_k \rightarrow 0$.
- Let ϕ be a p -form on a vector space V and $U \subset V$ a vector subspace of dimension k . Then we will say that U is ϕ -isotropic if $k \geq p$ and $\bigwedge^k U \bullet \phi = 0$, where $\bullet : \bigwedge^r V \otimes \bigwedge^s V^\vee \rightarrow \bigwedge^{r-s} V$ is the convolution induced by the pairing $V \otimes V^\vee \rightarrow \mathbb{C}$ for $r \geq s$. If $k \leq p$ and $\phi \bullet \bigwedge^k U = 0$ we will say that U is annihilated by ϕ .

By definition, a K uchle fivefold of type c5 is the zero locus of a global section of $\mathcal{U}_3^\perp(1) \oplus \mathcal{U}_3(1)$ in $\text{Gr}(3, 7)$, and a K uchle fourfold of type c5 is the zero locus of a global section of $\mathcal{U}_3^\perp(1) \oplus \mathcal{U}_3(1) \oplus \mathcal{O}_{\text{Gr}(3,7)}(1)$ in $\text{Gr}(3, 7)$. We will always denote by λ, μ and ν , global sections of respectively $\mathcal{U}_3^\perp(1), \mathcal{U}_3(1)$ and $\mathcal{O}(1)$ on $\text{Gr}(3, W)$. Then λ is a 4-form on W , μ is a 2-form on W and ν is a 3-form on W (see the introduction of [20]). We will also denote by $X_5^{\mu,\lambda}$ and $X_4^{\mu,\lambda,\nu}$ the associated fivefold and fourfold. However when there is no confusion about which global sections we are considering, we will simply use X_5 and X_4 . Moreover, we are going to make the same assumptions as in [20] in order to insure good properties of X_5 and X_4 (for instance smoothness). These assumptions are mainly taken from section 3 of [20].

- We assume that the same assumptions on λ, μ and ν hold as in the article. In particular this implies, using again the same notations :

1. $\text{rank}(\mu) = 6$ and $W = \mathbb{C}w_0 \oplus \overline{W}$ where $\mu(w_0, \cdot) = 0$ on W and μ is non degenerate on \overline{W} .
2. We can write $\lambda = w_0^\vee \wedge \overline{\lambda} + \lambda'$ where λ' is a 4-form on \overline{W} and $\overline{\lambda}$ is a 3-form on \overline{W} .
3. We can write $\overline{W} = A_1 \oplus A_2$ with $A_1 = \langle x_1, x_2, x_3 \rangle, A_2 = \langle x_4, x_5, x_6 \rangle$ and $\overline{\lambda} = x_{123} + x_{456}, \lambda' = x_{1256} + x_{1346} + x_{2345}$. Moreover, λ' induces a non degenerate pairing between $\bigwedge^2 A_1$ and $\bigwedge^2 A_2$.
4. In the basis mentioned just above, we have $\mu^2 = \mu \wedge \mu = M_1 x_{1456} + M_2 x_{2456} + M_3 x_{3456} + M_4 x_{1234} + M_5 x_{1235} + M_6 x_{1236} + K_1 x_{2345} + K_2 x_{1346} + K_3 x_{1256}$ and

$$\mu = \begin{pmatrix} 0 & M_4 K_3 & -M_5 K_2 & M_1 M_4 & M_1 M_5 & K_2 K_3 + M_1 M_6 \\ -M_4 K_3 & 0 & M_6 K_1 & M_2 M_4 & K_1 K_3 + M_2 M_5 & M_2 M_6 \\ M_5 K_2 & -M_6 K_1 & 0 & K_1 K_2 + M_3 M_4 & M_3 M_5 & M_3 M_6 \\ -M_1 M_4 & -M_2 M_4 & -K_1 K_2 - M_3 M_4 & 0 & M_1 K_1 & -M_2 K_2 \\ -M_1 M_5 & -K_1 K_3 - M_2 M_5 & -M_3 M_5 & -M_1 K_1 & 0 & M_3 K_3 \\ -K_2 K_3 - M_1 M_6 & -M_2 M_6 & -M_3 M_6 & M_2 K_2 & -M_3 K_3 & 0 \end{pmatrix}$$

Moreover the K_i are pairwise distinct, all the M_i and K_i are different from 0 and we assume :

$$M_1 M_6 K_1 + M_2 M_5 K_2 + M_3 M_4 K_3 + K_1 K_2 K_3 \neq 0. \quad (1.1)$$

5. We assume that μ induces a non degenerate pairing between A_1 and A_2 , and that the A_i are not Lagrangian for μ .
 6. We will denote by U_{k,A_i} a k -dimensional vector subspace of A_i . Moreover if $U_{2,A_1} \oplus U_{1,A_2}$ is μ isotropic then λ' does not annihilate it. Similarly, if $U_{1,A_1} \oplus U_{2,A_2}$ is μ isotropic then λ' does not annihilate it.
- We will denote by \overline{U}_k a k dimensional subspace of \overline{W} and by \overline{U}_k the tautological subbundle of $\text{Gr}(k, \overline{W})$.
 - Unless otherwise mentioned, for any vector space U, U^\perp will denote the μ -orthogonal of U .

Finally here are some more notations and results that we take from section 4 of [20]. We will use them for the statement of the main results from [20].

- $\text{SGr}_\mu(3, \overline{W})$ is the subset of $\text{Gr}(3, \overline{W})$ parametrizing all the $\overline{U}_3 \subset \overline{W}$ which are μ isotropic.
- Under the above mentioned hypothesis, X_5 is a smooth Fano fivefold which parametrizes the $U_3 \subset W$ annihilated by λ and μ isotropic.
- $\overline{X}_5 = \text{SGr}_\mu(3, \overline{W})_{\overline{\lambda}}$ parametrizes all the $\overline{U}_3 \subset \overline{W}$ which are μ and $\overline{\lambda}$ isotropic.
- $\text{SF1}_\mu(3, 4; W) \subset \text{SF1}(3, 4; W)$ parametrizes all the $U_3 \subset U_4 \subset W$ such that U_3 and U_4 are μ isotropic.

1.2 Main results

Here are the main results of [20] that we are going to use all along the next sections. Some of these results are gathered here in one Theorem or Proposition, but it is indicated next to each of these in parenthesis from which parts of [20] they are taken.

Theorem 1.2.1 (Theorem 4.2, section 4.2, section 4.3, Proposition 4.13, section 4.6, Proposition 4.6, section 4.4). *If X_5 is a Küchel fivefold as defined in the previous subsection, we have the following diagram :*

$$\begin{array}{ccccc}
 & E_4 & \xrightarrow{i} & \tilde{X}_5 & \xleftarrow{\bar{i}} & \bar{E}_4 & & \\
 & \searrow^{\mathbb{P}^1} & & \swarrow^{\pi} & & \searrow^{\mathbb{P}^1} & & \\
 F_3 & \xrightarrow{i} & X_5 & & \bar{X}_5 & \xleftarrow{\bar{i}} & Z_3 & \\
 & & & & & & \downarrow^{\sigma} & \\
 & & & & & & S_2 &
 \end{array} \tag{1.2}$$

Where

1. i and \bar{i} are embeddings.
2. The maps having the label \mathbb{P}^1 have fibers isomorphic to \mathbb{P}^1 .
3. π is the blowup of X_5 in F_3 , E_4 is the associated exceptional divisor.
4. $\bar{\pi}$ is the blowup of \bar{X}_5 in Z_3 , \bar{E}_4 is the associated exceptional divisor.
5. $F_3 \cong \text{Fl}(1, 2; A_1) \cong \text{Fl}(1, 2; A_2)$ and F_3 parametrizes all the $U_3 \in X_5$ such that $U_3 \cong \mathbb{C}w_0 \oplus \bar{U}_2$.
6. $\tilde{X}_5 \subset \text{SFl}_\mu(3, 4; W)$ parametrizes the $U_3 \subset U_4$ such that both U_3 and U_4 are μ and λ isotropic.
7. π is induced by the map $\text{SFl}_\mu(3, 4; W) \rightarrow \text{SGr}_\mu(3, W)$ which is defined by $U_3 \subset U_4 \mapsto U_3$ so $\pi(U_3, U_4) = U_3$.
8. $\bar{\pi}$ is induced by the map $\text{SFl}_\mu(3, 4; W) \rightarrow \text{SGr}_\mu(3, \bar{W})$ which is defined by $U_3 \subset \bar{U}_3 \oplus \mathbb{C}w_0 \mapsto \bar{U}_3$ so $\bar{\pi}(U_3, U_4) = U_4 / \langle w_0 \rangle$. This is well defined since we assumed that μ has rank 6 on W , thus a 4-dimensional sub-vector space U_4 is μ -isotropic only if it contains w_0 .
9. $Z_3 = \text{SGr}_\mu(3, \bar{W}) \times_{\text{Gr}(4, W)} \text{Gr}_{\text{Gr}_\lambda(5, W)}(4, \mathcal{U}_5)$ and Z_3 parametrizes all the $\bar{U}_3 \oplus \mathbb{C}w_0 \subset \bar{U}_4 \oplus \mathbb{C}w_0$ such that \bar{U}_4 is $\bar{\lambda}$, λ' and $\mu \wedge \mu$ isotropic and \bar{U}_3 is μ isotropic.
10. $\sigma(\bar{U}_3, \bar{U}_4) = \bar{U}_4$. S_2 parametrizes the $\bar{U}_4 = U_{2, A_1} \oplus U_{2, A_2}$ such that \bar{U}_4 is λ' -isotropic and $\bar{U}_4^\perp \subset \bar{U}_4$ is 2-dimensional.
The fiber of σ for a given \bar{U}_4 is then $\{ \langle \bar{U}_4^\perp, v \rangle \subset \bar{U}_4, v \in \bar{U}_4 / \bar{U}_4^\perp \} \cong \mathbb{P}(\bar{U}_4 / \bar{U}_4^\perp) \cong \mathbb{P}^1$.
11. S_2 is a del Pezzo surface of degree 6. It is cut out by two divisors of bidegree $(1, 1)$ in $\mathbb{P}(\wedge^2 A_1) \times \mathbb{P}(\wedge^2 A_2)$. If \bar{h}_i is the hyperplane class on $\wedge^2 A_i$ then $\omega_S = \mathcal{O}(-\bar{h}_1 - \bar{h}_2)$.

The following Proposition is a technical result that we will use later.

Proposition 1.2.2 (Section 4.4, Lemma 4.8, Corollary 4.9).

1. $Z_3 = \mathbb{P}_S(\mathcal{V}_S)$ where \mathcal{V}_S is a rank 2 vector bundle on S_2 defined by the short exact sequence

$$0 \rightarrow \mathcal{O}(-\bar{h}_1) \oplus \mathcal{O}(-\bar{h}_2) \rightarrow \mathcal{U}_{2, A_1} \oplus \mathcal{U}_{2, A_2} \rightarrow \mathcal{V}_S \rightarrow 0. \tag{1.3}$$

Moreover \mathcal{V}_S is selfdual (since it is of rank 2 and $c_1(\mathcal{V}_S) = 0$).

2. There are smooth rational curves $\bar{C}_1, \bar{C}_2 \subset S_2$ such that $\bar{C}_i \in |\bar{h}_i|$ and the following conditions are equivalent :

- $p = U_{2,A_1} \oplus U_{2,A_2} \in S_2$ is in \bar{C}_i .
- U_{2,A_i} is μ isotropic.
- $\mu_i \in U_{2,A_i}$ where μ_i is a generator of $A_i^\perp \cap A_i$ (which is one dimensional since we assumed that A_i is not μ -Lagrangian).
- $U_{2,A_i} \cap (U_{2,A_1} \oplus U_{2,A_2})^\perp$ is 1-dimensional.

3. There exists smooth rational curves $C_1, C_2 \subset Z_3$ such that $C_i \cong \bar{C}_i$ (and $\sigma(C_i) = \bar{C}_i$) and $\bar{U}_3 \subset U_{2,A_1} \oplus U_{2,A_2}$ is in C_i if and only if $U_{2,A_i} \subset \bar{U}_3$ with $i \in \{1, 2\}$. Moreover, $C_1 \cap C_2 = \emptyset$.

Finally, here are the main results about the geometry of X_4 that we will use :

Theorem 1.2.3 (Theorem 5.1). *If $X_4^{\mu,\lambda,\nu}$ is as defined in the previous subsection, with ν defined up to $\lambda(w, \cdot)$ and $\mu \wedge v$ for any $w \in W$ and $v \in W^\vee$, then we have the following diagram :*

$$\begin{array}{ccccccc}
 & & D_3 & \xrightarrow{i} & \tilde{X}_4 & \xleftarrow{\bar{i}} & \bar{D}_2 & & \\
 & \swarrow \mathbb{P}^1 & & & & & & \searrow \mathbb{P}^1 & \\
 \Sigma_2 & \longrightarrow & X_4 & & \tilde{X}_4 & & \bar{X}_4 & \longleftarrow & Z_3 & \longleftarrow & \Gamma_1 \\
 & & & \searrow \pi & & \searrow \bar{\pi} & & & & & & \\
 & & & & & & & & & & & \downarrow \sigma \mathbb{P}^1 \\
 & & & & & & & & & & & S_2
 \end{array} \tag{1.4}$$

Where :

1. The maps i, \bar{i}, π and $\bar{\pi}$ are induced by the corresponding maps in (1.2).
2. The maps having the label \mathbb{P}^1 have fibers isomorphic to \mathbb{P}^1 .
3. π is the blowup of X_4 in Σ_2 and D_3 is the associated exceptional divisor.
4. $\bar{\pi}$ is the blowup of \bar{X}_4 in Z_3 and \bar{D}_2 is the associated exceptional locus.
5. \bar{X}_4 has singularities along the curve Γ_1 .
6. $X_4 = X_5 \cap H_\nu$ where H_ν is the hyperplane which is the zero locus of the global section ν of $\mathcal{O}_{\text{Gr}(3,W)}(1)$. Thus X_4 parametrizes the $U_3 \subset W$ which are μ and ν isotropic, and annihilated by λ .
7. $\Sigma_2 = H_\nu \cap F_3$ is a del Pezzo surface of degree 6. It is cut out by two divisors of bidegree $(1, 1)$ in $\mathbb{P}(A_1) \times \mathbb{P}(A_2)$. If h_i is the hyperplane class on A_i then $\omega_{\Sigma_2} = \mathcal{O}(-h_1 - h_2)$. Since $\Sigma_2 \subset F_3$ we have that $U_3 \in \Sigma_2$ only if $U_3 = \mathbb{C}w_0 \oplus \bar{U}_2$.
8. $D_3 = \Sigma_2 \times_{F_3} E_4$ is the preimage of Σ_2 by π .
9. $\tilde{X}_4 = \tilde{X}_5 \times_{X_5} X_4$ is the preimage of X_4 by π .
10. \bar{X}_4 is cut in \bar{X}_5 by the pull back of ν to \tilde{X}_4 : it is a quadratic section of \bar{X}_5 containing Z_3 .

1.3 Divisors and projective bundles

Here we are going to recall some results and notations about some divisors and projective bundles that appear in [20] and are also going to be useful for us.

Proposition 1.3.1 (section 4.7, Lemma 4.14, section 4.4). *We have $E_4 \cong \mathbb{P}_F(\mathcal{V}_F)$, $\bar{E}_4 \cong \mathbb{P}_Z(\mathcal{V}_Z)$ and $Z \cong \mathbb{P}_S(\mathcal{V}_S)$ with \mathcal{V}_S defined by (1.3), \mathcal{V}_F is the monad of*

$$0 \rightarrow \mathcal{O}(-h_1) \oplus \mathcal{O}(-h_2) \rightarrow \bar{W} \otimes \mathcal{O} \rightarrow \mathcal{O}(h_1) \oplus \mathcal{O}(h_2) \rightarrow 0 \quad (1.5)$$

on F_3 and there is a short exact sequence

$$0 \rightarrow \mathcal{V}_Z \rightarrow \bigwedge^3 \bar{U}_3 \oplus \bigwedge^2 \bar{U}_3 \rightarrow \mathcal{O}(-\bar{h}_1) \oplus \mathcal{O}(-\bar{h}_2) \rightarrow 0 \quad (1.6)$$

on Z_3 .

Moreover, we have

$$0 \rightarrow \mathcal{O}(-\bar{h}_1) \oplus \mathcal{O}(-\bar{h}_2) \rightarrow \bar{U}_3 \rightarrow \mathcal{O}(-V_Z) \rightarrow 0 \quad (1.7)$$

on Z_3 .

Corollary 1.3.2 (Theorem 5.1). *As a result, $D_3 \cong \mathbb{P}_{\Sigma_2}(\mathcal{V}_F|_{\Sigma_2})$ and $\bar{D}_2 \cong \mathbb{P}_\Gamma(\mathcal{V}_Z|_\Gamma)$.*

Definition 1.3.3 (Section 4.7). *We are going to use the following notations :*

- h_i is the hyperplane class on $\mathbb{P}(A_i)$.
- \bar{h}_i is the hyperplane class on $\mathbb{P}(\bigwedge^2 A_i)$.
- V_E is the hyperplane class on $E_4 \cong \mathbb{P}_F(\mathcal{V}_F)$ (thus the pushforward of $\mathcal{O}_{E_4}(V_E)$ on F_3 is \mathcal{V}_F^\vee).
- $V_{\bar{E}}$ is the hyperplane class on $\bar{E}_4 \cong \mathbb{P}_Z(\mathcal{V}_Z)$ (the pushforward of $\mathcal{O}_{\bar{E}_4}(V_{\bar{E}})$ on Z_3 is \mathcal{V}_Z^\vee).
- V_Z is the hyperplane class on $Z_3 \cong \mathbb{P}_S(\mathcal{V}_S)$ (the pushforward of $\mathcal{O}_{Z_3}(V_Z)$ on S_2 is \mathcal{V}_S^\vee).
- h is the hyperplane class on $\text{Gr}(3, W)$.
- \bar{h} is the hyperplane class on \bar{X}_5 .
- e is the class of E_4 and \bar{e} is the class of \bar{E}_4 .

Most of the times pull backs or restrictions of these hyperplane classes to various varieties in (1.2) and (1.4) will keep the same notation.

Lemma 1.3.4 (Lemma 4.15, Remark 4.7). *In $\text{Pic}(\bar{X}_5)$ the following holds*

$$\bar{h} = h - e, \quad h = 2\bar{h} - \bar{e}, \quad \bar{e} = h - 2e, \quad e = \bar{h} - \bar{e}. \quad (1.8)$$

In $\text{Pic}(F_3)$ we have

$$h = h_1 + h_2. \quad (1.9)$$

In $\text{Pic}(E_4)$ we have

$$V_E = -e. \quad (1.10)$$

In $\text{Pic}(Z_3)$ and $\text{CH}^3(Z_3)$ we have

$$\bar{h} = V_Z + \bar{h}_1 + \bar{h}_2, \quad \bar{h}_1^3 = \bar{h}_2^3 = \bar{h}_1 \bar{h}_2^2 = \bar{h}_1^2 \bar{h}_2 = 0. \quad (1.11)$$

In $\text{Pic}(\bar{E}_4)$ we have

$$h = V_{\bar{E}}. \quad (1.12)$$

In $\text{Pic}(S_2)$ and $\text{CH}^2(S_2)$ we have

$$\bar{h}_1 \bar{h}_2 = 2, \quad \bar{h}_1^2 = \bar{h}_2^2 = 1. \quad (1.13)$$

In $\text{Pic}(\Sigma_2)$ and $\text{CH}^2(\Sigma_2)$ we have

$$h_1 h_2 = 2, \quad h_1^2 = h_2^2 = 1. \quad (1.14)$$

In $\text{CH}^2(Z_3)$ the curve Γ_1 defined in (1.4) has class :

$$[\Gamma_1] = 4V_Z(\bar{h}_1 + \bar{h}_2). \quad (1.15)$$

Lemma 1.3.5 (Lemma 4.16). In $\text{CH}^2(E_4)$, $\text{CH}^2(\bar{E}_4)$ and $\text{CH}^2(Z_3)$ respectively, the following hold :

$$V_E^2 = -h_1 h_2, \quad (1.16)$$

$$V_{\bar{E}}^2 = 3V_Z V_{\bar{E}} + 2V_{\bar{E}}(\bar{h}_1 + \bar{h}_2) - 4V_Z(\bar{h}_1 + \bar{h}_2), \quad (1.17)$$

$$V_Z^2 = -\bar{h}_1 \bar{h}_2. \quad (1.18)$$

1.4 Embeddings of blow-ups

In this subsection we are going to recall a result from A. Kuznetsov, namely Lemma 2.1 of [20], which we are going to use to find embedded surfaces in a smooth variety X later. In [20] the Lemma is stated for S an irreducible Cohen-Macaulay scheme but since we are only going to use it for smooth varieties, we are going to consider only such varieties here. The proof follows very closely the one from [20].

Lemma 1.4.1. *Let \mathcal{E} and \mathcal{F} be vector bundles of rank respectively $r + 1$ and r on a smooth variety S . Let $\phi : \mathcal{E} \rightarrow \mathcal{F}$ be a morphism of vector bundles and let $D_k(\phi)$ denote the k th degeneracy locus of ϕ . Let $p : \mathbb{P}_S(\mathcal{E}) \rightarrow S$ be the projective bundle and $\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ be such that $p_*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)) \cong \mathcal{E}^\vee$. If $\text{codim}(D_k(\phi)) \geq k + 1$ for all $k \geq 1$ then there is an isomorphism between the blow-up of S in $D_1(\phi)$ and the zero locus of the global section of $p^*(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ induced by ϕ in $\mathbb{P}_S(\mathcal{E})$. The line bundle corresponding to the exceptional divisor of this blow-up is then isomorphic to $p^*\det(\mathcal{E}^\vee) \otimes p^*\det(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(-1)$ restricted to the zero locus of the section induced by ϕ on $\mathbb{P}_S(\mathcal{E})$. Finally if $D_2(\phi) = \emptyset$ then the exceptional divisor in $\mathbb{P}_S(\mathcal{E})$ is isomorphic to the projectivization of the rank 2 vector bundle $\text{Ker}(\phi|_{D_1(\phi)})$.*

Proof. First of all, the global section of $p^*(\mathcal{F})(1)$ induced by ϕ comes from the isomorphisms : $H^*(\mathcal{E}, \mathcal{F}) \cong H^*(\mathcal{O}_S, \mathcal{F} \otimes \mathcal{E}^\vee) \cong H^*(\mathcal{O}_S, \mathcal{F} \otimes p_*(\mathcal{O}(1))) \cong H^*(\mathcal{O}_S, p_*(p^*(\mathcal{F}) \otimes \mathcal{O}(1))) \cong H^*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}, p^*(\mathcal{F})(1))$. Let us consider now $\phi^\vee : \mathcal{F}^\vee \rightarrow \mathcal{E}^\vee$. It is injective outside of $D_1(\phi)$ hence by assumption outside of a codimension 2 locus. Thus it is an injective morphism, since \mathcal{F}^\vee is torsion free. Let \mathcal{C} be its cokernel. Let us show that \mathcal{C} is torsion free. We have a short exact sequence $0 \rightarrow \mathcal{F}^\vee \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{C} \rightarrow 0$ that induces (by dualizing) the long exact sequence $0 \rightarrow \mathcal{C}^\vee \rightarrow \mathcal{E} \xrightarrow{\phi} \mathcal{F} \rightarrow \mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_S) \rightarrow 0$, since \mathcal{E}^\vee and \mathcal{F}^\vee are both locally free so $\mathcal{E}xt^i(\mathcal{E}^\vee, \mathcal{O}_S) = 0 = \mathcal{E}xt^i(\mathcal{F}^\vee, \mathcal{O}_S)$ for all $i \neq 0$. Moreover since ϕ is surjective outside of $D_1(\phi)$ we have that $\text{supp}(\mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_S)) = D_1(\phi)$ and is thus codimension at least 2. Therefore (see for instance [14] page 78 below the definition of $\mathcal{F}^{\bullet\vee}$), $\mathcal{E}xt^q(\mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_S), \mathcal{O}_S) = 0$ for all $q \leq 1$. So splitting the previous long exact sequence in short exact sequences and dualizing gives the exact sequences :

$$0 \rightarrow \mathcal{E}xt^0(Q, \mathcal{O}_S) \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{C}^{\vee\vee} \rightarrow \mathcal{E}xt^1(Q, \mathcal{O}_S) \rightarrow 0 \quad (1.19)$$

since \mathcal{E} is locally free, and

$$0 \rightarrow \mathcal{E}xt^0(\mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_S), \mathcal{O}_S) = 0 \rightarrow \mathcal{F}^\vee \rightarrow \mathcal{E}xt^0(Q, \mathcal{O}_S) \rightarrow \mathcal{E}xt^1(\mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_S), \mathcal{O}_S) = 0 \rightarrow \quad (1.20)$$

$$0 \rightarrow \mathcal{E}xt^1(Q, \mathcal{O}_S) \rightarrow \mathcal{E}xt^2(\mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_S), \mathcal{O}_S) \rightarrow 0 \quad (1.21)$$

where Q is the cokernel of the map $\mathcal{C}^\vee \rightarrow \mathcal{E}$, and since \mathcal{F} is locally free. Then (1.20) gives $\mathcal{F}^\vee \cong \mathcal{E}xt^0(Q, \mathcal{O}_S)$ and (1.21) gives $\mathcal{E}xt^1(Q, \mathcal{O}_S) \cong \mathcal{E}xt^2(\mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_S), \mathcal{O}_S)$, and plugging these two isomorphisms in (1.19) gives the short exact sequence :

$$0 \rightarrow \mathcal{F}^\vee \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{C}^{\vee\vee} \rightarrow \mathcal{E}xt^2(\mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_S), \mathcal{O}_S) \rightarrow 0.$$

So $\mathcal{C} \hookrightarrow \mathcal{C}^{\vee\vee}$ hence \mathcal{C} is torsion free since $\mathcal{C}^{\vee\vee}$ is torsion free.

Let us consider now $\mathcal{E}^\vee \cong \mathcal{E}^\vee \otimes \bigwedge^r(\mathcal{E}^\vee \otimes \mathcal{E}) \cong \det(\mathcal{E}^\vee) \otimes \bigwedge^r \mathcal{E} \xrightarrow{\bigwedge^r \phi} \det(\mathcal{E}^\vee) \otimes \bigwedge^r \mathcal{F} \cong \det(\mathcal{E}^\vee) \otimes \det(\mathcal{F})$. The composition $(\bigwedge^r \phi) \circ \phi^\vee$ is identically zero on \mathcal{F}^\vee . Indeed let $x \in S$ be a point, then $\phi_x^\vee : \mathcal{F}_x^\vee \rightarrow \mathcal{E}_x^\vee$ sends a morphism $f : \mathcal{F}_x \rightarrow \mathcal{O}_{S,x}$ to $f \circ \phi : \mathcal{E}_x \rightarrow \mathcal{O}_{S,x}$. In other words, ϕ_x^\vee sends $\text{Ker}(f) \in \text{Gr}(r-1, \mathcal{F}_x)$ to $\text{Ker}(f \circ \phi) \supset \text{Ker}(\phi) \neq \emptyset$ in $\text{Gr}(r, \mathcal{E}_x)$. Thus $(\bigwedge^r \phi)(\bigwedge^r \text{Ker}(f \circ \phi)) = 0$ and $(\bigwedge^r \phi) \circ \phi^\vee = 0$. As a result there is an induced morphism from \mathcal{C} to $\det(\mathcal{E}^\vee) \otimes \det(\mathcal{F})$: for each $x \in S$ there is an isomorphism $\mathcal{C}_x \cong \mathcal{E}_x^\vee / \phi(\mathcal{F}_x^\vee)$ which we can compose with $\bigwedge^r \phi_x$ to get a morphism $\mathcal{C}_x \rightarrow (\det(\mathcal{E}^\vee) \otimes \det(\mathcal{F}))_x$. Moreover, outside of $D_1(\phi)$, ϕ is surjective so $\bigwedge^r \phi$ is surjective on the line bundle $\det(\mathcal{E}^\vee) \otimes \det(\mathcal{F})$, and \mathcal{C} is of rank one and torsion free so the induced map $\mathcal{C} \rightarrow \det(\mathcal{E}^\vee) \otimes \det(\mathcal{F})$ is an isomorphism outside of $D_1(\phi)$. Again since by hypothesis $D_1(\phi)$ has codimension at least 2, and since \mathcal{C} is torsion free, this map is then injective on all of S . Finally, $\bigwedge^r \phi_x = 0$ if and only if $\dim(\text{im}(\phi_x)) < r$ that is to say if and only if $x \in D_1(\phi)$, and outside of $D_1(\phi)$ we have seen that since ϕ is surjective $\bigwedge^r \phi$ is surjective as well. So we have an exact sequence :

$$0 \rightarrow \mathcal{F}^\vee \xrightarrow{\phi^\vee} \mathcal{E}^\vee \xrightarrow{\bigwedge^r \phi} \det(\mathcal{E}^\vee) \otimes \det(\mathcal{F}) \rightarrow (\det(\mathcal{E}^\vee) \otimes \det(\mathcal{F}))|_{D_1(\phi)} \rightarrow 0. \quad (1.22)$$

Let us denote by \tilde{S} the zero locus of the section of $p^*\mathcal{F} \otimes \mathcal{O}(1)$ induced by ϕ on $\mathbb{P}_S(\mathcal{E})$. Since $p^*\mathcal{F}(1)$ is a vector bundle of rank r and $\mathbb{P}_S(\mathcal{E})$ is a smooth variety of dimension $\dim(S)+r$, then any component of \tilde{S} has dimension at least $\dim(S)$. Moreover, outside of $D_1(\phi)$, \mathcal{C}_x^\vee has dimension 1 and on $D_k(\phi) - D_{k+1}(\phi)$, \mathcal{C}_x^\vee has dimension $k+1$, so by definition of $\mathbb{P}_S(\mathcal{E})$, the fibers of \tilde{S} over S are projective of dimension k over $D_k(\phi) - D_{k+1}(\phi)$ and \tilde{S} is in bijection with S outside of $D_1(\phi)$. Therefore $p|_{\tilde{S}}$ gives an isomorphism between $p^{-1}(S - D_1(\phi)) \cap \tilde{S}$ and $S - D_1(\phi)$, and for each $k \geq 1$, $p^{-1}(D_k(\phi)) \cap \tilde{S}$ is of dimension $\dim(D_k(\phi)) + k < \dim(S)$ since by hypothesis $D_k(\phi)$ has codimension at least $k+1$ in S . So \tilde{S} is irreducible (otherwise it would contradict the irreducibility of S), it has the same dimension as S , and $p|_{\tilde{S}} : \tilde{S} \rightarrow S$ is birational.

Let us now compute $p_*\mathcal{O}_{\tilde{S}}(1)$. Since $\dim(\tilde{S}) = \dim(S) = \dim(\mathbb{P}_S(\mathcal{E})) - \text{rank}(p^*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1))$, by Exercise 17.20 of 17.6 of [7], the Koszul resolution gives a resolution of $\mathcal{O}_{\tilde{S}}$ as follows :

$$0 \rightarrow \bigwedge^r(p^*\mathcal{F}^\vee) \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(-r) \rightarrow \dots \rightarrow \bigwedge^2(p^*\mathcal{F}^\vee) \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(-2) \rightarrow p^*\mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(-1) \xrightarrow{\phi^\vee} \mathcal{O}_{\mathbb{P}_S(\mathcal{E})} \rightarrow \mathcal{O}_{\tilde{S}} \rightarrow 0.$$

Twisting this resolution by $\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$ and taking its pushforward by p gives then the exact sequence :

$$\dots \rightarrow \star \rightarrow \mathcal{F}^\vee \xrightarrow{\phi^\vee} \mathcal{E}^\vee \rightarrow p_*(\mathcal{O}_{\tilde{S}}(1)) \rightarrow 0$$

since $p_*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1) = \mathcal{E}^\vee$ and since $p_*(p^*\mathcal{F}^\vee) = \mathcal{F}^\vee$. Indeed, $R^*p_*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}) = \mathcal{O}_S : R^0p_*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}) = p_*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}) = \mathcal{O}_S$ (see for instance Proposition 7.11 of II.7 of [12]) and $R^k p_*(\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}) = 0$ for $k \neq 0$ (see for instance Exercise 8.4 a) of III.8 of [12]). Thus $p_*(p^*\mathcal{F}^\vee) = \mathcal{F}^\vee \otimes p_*\mathcal{O}_{\mathbb{P}_S(\mathcal{E})} = \mathcal{F}^\vee$. Finally we have already seen that $\mathcal{F}^\vee \xrightarrow{\phi^\vee} \mathcal{E}^\vee$ is injective, so we have a short exact sequence :

$$0 \rightarrow \mathcal{F}^\vee \xrightarrow{\phi^\vee} \mathcal{E}^\vee \rightarrow p_*(\mathcal{O}_{\tilde{S}}(1)) \rightarrow 0.$$

But this short exact sequence combined with (1.22) gives us : $p_*\mathcal{O}_{\tilde{S}}(1) = \mathcal{I}_{D_1(\phi)} \otimes \det(\mathcal{E}^\vee) \otimes \det(\mathcal{F})$. Thus since $\det(\mathcal{E}^\vee)$ and $\det(\mathcal{F})$ are line bundles, $p : \tilde{S} \rightarrow S$ is the blow-up of S in $D_1(\phi)$.

Finally, we have then that $p_*(p^*\det(\mathcal{E}) \otimes p^*\det(\mathcal{F}^\vee) \otimes \mathcal{O}_{\tilde{S}}(1)) = \mathcal{I}_{D_1(\phi)}$ so the line bundle corresponding to the exceptional divisor of this blow-up is $p^*\det(\mathcal{E}^\vee) \otimes p^*\det(\mathcal{F}) \otimes \mathcal{O}_{\tilde{S}}(-1)$. Indeed if \mathcal{L} is the line bundle associated to the exceptional divisor, then by definition, $p^*\mathcal{I}_{D_1(\phi)} = \mathcal{L}^\vee$, which implies $p_*p^*\mathcal{I}_{D_1(\phi)} = \mathcal{I}_{D_1(\phi)} \otimes p_*\mathcal{O}_{\tilde{S}} = \mathcal{I}_{D_1(\phi)} = p_*\mathcal{L}^\vee$.

If $D_2(\phi) = \emptyset$, then as we have seen before \tilde{S} has projective fibers of dimension 1 over $D_1(\phi)$ and is isomorphic to S outside of $D_1(\phi)$, so the exceptional divisor is given by the projectivization of \mathcal{C}^\vee over $D_1(\phi)$, that is to say by the projectivization of $\text{Ker}(\phi|_{D_1(\phi)})$. This is a rank two vector bundle because $D_2(\phi) = \emptyset$. \square

Chapter 2

Two free families in $K_{\text{num}}(\tilde{X}_4)$, a free family in $K_{\text{num}}(X_4)$, and the associated matrices of Euler characteristics

In this section, we describe the partial results we found regarding the study of $(K_{\text{num}}(X_4), \chi)$ and $(K_{\text{num}}(\tilde{X}_4), \chi)$. In particular, we give all the Euler characteristics which we have computed between elements of $K_{\text{num}}(X_4)$ or $K_{\text{num}}(\tilde{X}_4)$ (the calculations are detailed mainly in section 4 and 5).

As explained in the Introduction, our first goal was to find a basis of $K_{\text{num}}(X_4)$ over \mathbb{Z} and then compute the matrix whose coefficients are the Euler characteristics between elements of this basis. Yet this turned out to be quite difficult, so we tried instead to find a basis of $K_{\text{num}}(\tilde{X}_4)$ and compute the associated matrix of Euler characteristics.

In subsection 2.1, we find a family \mathcal{F}_1 of 14 elements in $K_{\text{num}}(\tilde{X}_4)$ which will turn out to be free. We then give the associated matrix up to an intersection number b in Proposition 2.1.5. In subsection 2.2, we find a second family \mathcal{F}_2 of 14 elements, whose associated matrix is given in Proposition 2.2.1.

Although we didn't manage to find a basis of $K_{\text{num}}(X_4)$ and compute the corresponding matrix of Euler characteristics, we did find a free family of 6 elements in $K_{\text{num}}(X_4)$, and we give the associated matrix in Proposition 2.3.1 in subsection 2.3.1.

It would be interesting to know if this family generates a primitive sublattice of $K_{\text{num}}(X_4)$. Indeed, in such a case it may be possible to complete our family of 6 elements into a basis of $K_{\text{num}}(X_4)$ over \mathbb{Z} . On the reverse, if the sublattice is not primitive, one could only complete our family into a basis over \mathbb{Q} . This could be an indication that this family is not the most appropriate one to start studying $(K_{\text{num}}(X_4), \chi)$ as a lattice over \mathbb{Z} .

We discuss this issue a little, still in Proposition 2.3.1, and we ask the same question about a subfamily generated by 5 of our 6 initial elements. Yet we fail to give an answer to this primitivity question in both cases.

Finally, in subsection 2.3.2 we explain why we hope to be able to recover a copy of the lattice $(K_{\text{num}}(X_4), \chi)$ in $(K_{\text{num}}(\tilde{X}_4), \chi)$, using mutation functors. Unfortunately, to do so we need to apply some mutation functors to a sheaf whose image will then depend on the intersection number b mentioned previously. We thus give a matrix which should correspond to a family of $K_{\text{num}}(X_4)$ (yet we don't know if this family would correspond to a primitive sublattice or not), but only up to this number b .

2.1 A first family in $K_{\text{num}}(\tilde{X}_4)$ induced by elements in $K_{\text{num}}(X_4)$ and $K_{\text{num}}(\Sigma_2)$

In this subsection, we consider a first family in $K_{\text{num}}(\tilde{X}_4)$ which is a "good candidate" for being a basis, and we give in Proposition 2.1.5 the corresponding matrix of Euler characteristics, up to an intersection number b .

2.1.1 A semi-orthogonal decomposition of $D^b(\tilde{X}_4)$

In what follows we use a Theorem from A. Bondal and D. Orlov to find a semi-orthogonal decomposition of $D^b(\tilde{X}_4)$. We will then use this decomposition to build a family in $K_{\text{num}}(\tilde{X}_4)$ which will turn out to be free, and which may be a basis.

Before going further, let us give more precise names to some of the maps appearing in (1.4). Let us denote:

$$\begin{array}{ccc} D_3 & \xrightarrow{j} & \tilde{X}_4 \\ \downarrow p & & \downarrow \pi \\ \Sigma_2 & \xrightarrow{i} & X_4 \\ \downarrow \phi & & \\ \mathbb{P}^2 & & \end{array}$$

where as later in subsection 4.3.1 and Definition 4.3.3, ϕ is the blow-up of \mathbb{P}^2 in three points with exceptional divisors E_1, E_2 and E_3 in Σ_2 .

Lemma 2.1.1. *There is a semi-orthogonal decomposition :*

$$\begin{aligned} D^b(\Sigma_2) &= \langle \mathcal{O}_{E_1}(E_1), \mathcal{O}_{E_2}(E_2), \mathcal{O}_{E_3}(E_3), \phi^* D^b(\mathbb{P}^2) \rangle \\ &= \langle \mathcal{O}_{E_1}(E_1), \mathcal{O}_{E_2}(E_2), \mathcal{O}_{E_3}(E_3), \mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(h_1), \mathcal{O}_{\Sigma_2}(2h_1) \rangle . \end{aligned}$$

Proof. This comes from the fact that if $T \xrightarrow{f} \mathbb{P}^2$ is the blow-up of \mathbb{P}^2 in a point p with exceptional divisor E , then by Theorem 3.5 of [2] there is a semi-orthogonal decomposition $D^b(T) = \langle \mathcal{O}_E(E), f^* D^b(\mathbb{P}^2) \rangle = \langle \mathcal{O}_E(E), f^* \mathcal{O}_{\mathbb{P}^2}, f^* \mathcal{O}_{\mathbb{P}^2}(1), f^* \mathcal{O}_{\mathbb{P}^2}(2) \rangle$ where for the last equality one can use for instance Corollary 8.29 in [14]. One can then conclude using the fact that Σ_2 is the blow-up of $\mathbb{P}(A_1)$ in three points together with Definition 1.3.3. Indeed Remark 4.7 of [20] still applies when S_2 is replaced by Σ_2 and the $\bigwedge^2 A_i$ by A_i for $i \in \{1, 2\}$, because both S_2 and Σ_2 are degree 6 del Pezzo surfaces cut out by two divisors of bidegree (1, 1) in $\mathbb{P}^2 \times \mathbb{P}^2$. \square

Using again Theorem 3.5 from [2] one can then find :

Lemma 2.1.2. *There is a semi-orthogonal decomposition $D^b(\tilde{X}_4) = \langle j_* \mathcal{O}_{E_1}(E_1 - V_E), j_* \mathcal{O}_{E_2}(E_2 - V_E), j_* \mathcal{O}_{E_3}(E_3 - V_E), j_* \mathcal{O}_{D_3}(-V_E), j_* \mathcal{O}_{D_3}(h_1 - V_E), j_* \mathcal{O}_{D_3}(2h_1 - V_E), \pi^* D^b(X_4) \rangle$.*

Corollary 2.1.3. *For any $F \in \langle j_* \mathcal{O}_{E_1}(E_1 - V_E), j_* \mathcal{O}_{E_2}(E_2 - V_E), j_* \mathcal{O}_{E_3}(E_3 - V_E), j_* \mathcal{O}_{D_3}(-V_E), j_* \mathcal{O}_{D_3}(h_1 - V_E), j_* \mathcal{O}_{D_3}(2h_1 - V_E) \rangle$ and any $0 \leq i \leq 4$, one has $H^*(\pi^* \mathcal{O}_{X_4}(ih), F) = 0$.*

2.1.2 The family \mathcal{F}_1 and the associated matrix

We can now try to build a family which could be a basis of $K_{\text{num}}(\tilde{X}_4)$ (over \mathbb{Q}). Indeed, if $K_{\text{num}}(X_4)$ is of dimension 8 as we hope for (by analogy with the cubic fourfolds' case mentioned in the Introduction), then by Lemma 2.1.2, $K_{\text{num}}(\tilde{X}_4)$ should be of dimension 14. We thus look for a family of 14 elements in $K_{\text{num}}(\tilde{X}_4)$. If the determinant of the associated matrix of Euler characteristics is non zero, it will imply that this family is free.

By Lemma 2.1.2, if we know a basis \mathcal{B} of $K_{\text{num}}(X_4)$ then we can easily find a basis of $K_{\text{num}}(\tilde{X}_4)$, using $\pi^*\mathcal{B}$. As mentioned in the introduction, a first naive guess for such a basis could be

$$\{\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(h), \mathcal{O}_{X_4}(2h), \mathcal{O}_{X_4}(3h), \mathcal{O}_{X_4}(4h)\}$$

together with the structure sheaves of three surfaces. However, we know from [15] that $h^4 = 66$, and we want to be able to recover a skyscraper sheaf (without multiplicity) from our family, hence we will actually replace $\mathcal{O}_{X_4}(4h)$ by a skyscraper sheaf \mathcal{O}_p , and for the same reason we will replace $\mathcal{O}_{X_4}(3h)$ by the structure sheaf of a line with class $E_1 \cdot V_E \cdot e$, which we will call \mathcal{O}_{l_h} . Indeed, it corresponds to a curve with class E_1 in Σ_2 , which is a \mathbb{P}^1 by definition, and moreover $E_1 \cdot h = 1$ in Σ_2 since by (1.9), $h|_{\Sigma_2} = h_1 + h_2 = 3h_1 - E_1 - E_2 - E_3$ (see Remark 4.1.4 for more details). We won't need to replace $\mathcal{O}_{X_4}(2h)$ because $66 = 2 \cdot 3 \cdot 11$ doesn't contain any square, hence we should be able to recover the structure sheaf of a surface induced by h without multiplicity.

About structure sheaves of surfaces, as suggested in the Introduction and detailed below in section 3, we can consider structure sheaves of Σ_2 , \overline{D}_2 and a blow-up of S_2 , namely \tilde{S} (see (1.4) and Corollary 3.2.7). As explained in the Introduction, the last two surfaces are problematic if we want to study $K_{\text{num}}(X_4)$ because in such a case we need to consider pushforwards of their structure sheaves and we can't control how badly π may contract them to X_4 . This is why we are considering $K_{\text{num}}(\tilde{X}_4)$ now : both \overline{D}_2 and \tilde{S} are smooth surfaces in \tilde{X}_4 , and we can thus take their structure sheaves as elements in our family. Then, $\mathcal{O}_{\overline{D}_2}$ and $\mathcal{O}_{\tilde{S}}$ have no particular reason to be pull-backs of sheaves on X_4 , yet using right mutation functors (see Definition 2.4 in [21]) and Lemma 2.1.2 we may induce a family in $\pi^*K_{\text{num}}(X_4)$ whose matrix of Euler characteristics can be deduced from the one associated to $K_{\text{num}}(\tilde{X}_4)$. This will be detailed in subsection 2.3.2.

Moreover, by Lemma 2.1.2 and Corollary 2.1.3, if $\{e_1, \dots, e_n\}$ is a basis of $K_{\text{num}}(X_4)$, then the matrix $\tilde{M} := (\chi(y_i, y_j))_{i,j}$ with

$$\{y_1, \dots, y_{n+6}\} = \{j_*\mathcal{O}_{E_1}(E_1 - V_E), j_*\mathcal{O}_{E_2}(E_2 - V_E), j_*\mathcal{O}_{E_3}(E_3 - V_E), j_*\mathcal{O}_{D_3}(-V_E), j_*\mathcal{O}_{D_3}(h_1 - V_E), j_*\mathcal{O}_{D_3}(2h_1 - V_E), \pi^*e_1, \dots, \pi^*e_n\},$$

is actually $\tilde{M} = \begin{pmatrix} N & * \\ 0 & M \end{pmatrix}$ where $N = (\chi(y_i, y_j))_{1 \leq i, j \leq 6}$ is easy to compute as we will see later, and $M = (\chi(\pi^*e_i, \pi^*e_j))_{1 \leq i, j \leq n}$ is the matrix we are looking for.

This comes from the fact that $\pi_*\mathcal{O}_{\tilde{X}_4} = \mathcal{O}_{X_4}$; the projection formula then yields

$$\chi(\pi^*e_i, \pi^*e_j) = \chi(e_i, \pi_*\pi^*e_j) = \chi(e_i, e_j)$$

for any i and j . In particular, it is possible to recover the determinant of M from those of \tilde{M} and N . A small issue with $K_{\text{num}}(\tilde{X}_4)$ could come from the fact that $R^*\pi^*\mathcal{O}_{\Sigma_2}$ has nonzero R^1 , but since we are already considering $\mathcal{O}_{\overline{D}_2}$ and $\mathcal{O}_{\tilde{S}}$ which are not in $\pi^*D^b(X_4)$, for convenience we are going to replace the derived pull-back $\pi^*\mathcal{O}_{\Sigma_2}$ by $\mathcal{O}_{D_3}(V_E)$. This is possible because this sheaf doesn't belong to $\langle j_*\mathcal{O}_{E_1}(E_1 - V_E), j_*\mathcal{O}_{E_2}(E_2 - V_E), j_*\mathcal{O}_{E_3}(E_3 - V_E), j_*\mathcal{O}_{D_3}(-V_E), j_*\mathcal{O}_{D_3}(h_1 - V_E), j_*\mathcal{O}_{D_3}(2h_1 - V_E) \rangle$.

So finally, our goal is to compute Euler characteristics between elements of the following family :

$$\mathcal{F}_1 = \{j_*\mathcal{O}_{E_1}(E_1 - V_E), j_*\mathcal{O}_{E_2}(E_2 - V_E), j_*\mathcal{O}_{E_3}(E_3 - V_E), j_*\mathcal{O}_{D_3}(-V_E), j_*\mathcal{O}_{D_3}(h_1 - V_E), j_*\mathcal{O}_{D_3}(2h_1 - V_E), \mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{l_h}, \mathcal{O}_p, \mathcal{O}_{D_3}(V_E), \mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\tilde{S}}\}.$$

Such computations are detailed in section 4.

Remark 2.1.4. *Some Euler characteristics mentioned below involve $\mathcal{O}_{\tilde{X}_4}(3h)$ and $\mathcal{O}_{\tilde{X}_4}(4h)$, which are not going to be directly useful for us since we are interested in \mathcal{O}_p and \mathcal{O}_{l_h} instead. However when such Euler characteristics have been computed, I will still mention them.*

Moreover, if \mathcal{F} is any of the twists of \mathcal{O}_{D_3} in \mathcal{F}_1 , \mathcal{O}_h the structure sheaf of a hyperplane section of class h and \mathcal{O}_{h^2} is defined as follows :

$$0 \rightarrow \mathcal{O}_h(-h) \rightarrow \mathcal{O}_h \rightarrow \mathcal{O}_{h^2} \rightarrow 0$$

then we cannot recover neither $\chi(\mathcal{O}_h, \mathcal{F})$ nor $\chi(\mathcal{O}_{h^2}, \mathcal{F})$ using Euler characteristics between elements of \mathcal{F}_1 . Instead, those Euler characteristics allow to recover $\chi(\mathcal{O}_{h^2}(2h), \mathcal{F})$ and $\chi(\mathcal{O}_h(h), \mathcal{F})$.

So computing Euler characteristics between elements of \mathcal{F}_1 or of $(\mathcal{F}_1 \cup \{\mathcal{O}_h, \mathcal{O}_{h^2}\}) - \{\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{X}_4}(2h)\}$ is not the same. To fix this, one could replace $\mathcal{O}_{\tilde{X}_4}(h)$ and $\mathcal{O}_{\tilde{X}_4}(2h)$ by $\mathcal{O}_{\tilde{X}_4}(-h)$ and $\mathcal{O}_{\tilde{X}_4}(-2h)$ in \mathcal{F}_1 .

Unfortunately, we are not able to compute all the Euler characteristics between elements of \mathcal{F}_1 : some of them depend on intersection numbers among which one, as mentioned in section 4.7.4, is hard to compute. However we get the following result :

Proposition 2.1.5. *Let us define $b := j_* h_1 \cdot [\overline{D}_2]$ where $[\overline{D}_2]$ is the class of \overline{D}_2 in \tilde{X}_4 .*

If

$$\{e_1, \dots, e_{14}\} = \{j_* \mathcal{O}_{E_1}(E_1 - V_E), j_* \mathcal{O}_{E_2}(E_2 - V_E), j_* \mathcal{O}_{E_3}(E_3 - V_E), j_* \mathcal{O}_{D_3}(-V_E), j_* \mathcal{O}_{D_3}(h_1 - V_E), j_* \mathcal{O}_{D_3}(2h_1 - V_E), \mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{D_3}(V_E), \mathcal{O}_{\overline{D}_2} \mathcal{O}_{\tilde{S}}, \mathcal{O}_{l_h}, \mathcal{O}_P\},$$

then :

$$\tilde{M} = (\chi(e_i, e_j))_{1 \leq i, j \leq 14} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 & 1 & 2 & 3 & -1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 & 1 & 2 & 3 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 1 & 2 & 3 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 6 & -1 & -7 & -19 & -5 & 12 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & -3 & -12 & -6 & 12+b & 19 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -6 & -6 & 12+2b & 28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 20 & 124 & 0 & -12 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 20 & 0 & 0 & 37 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 1 & 1 & 12 & 156 & 127 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 4 & 1 & 7 & 19 & 1 & -36 & 56 & 0 & 0 \\ 0 & 0 & 0 & 36 & 36+b & 36+2b & -36 & 0 & 180 & -156 & 48 & -24 & 0 & 0 \\ 3 & 3 & 3 & -42 & -33 & -24 & 79 & 79 & 205 & 76 & -24 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. The computations are detailed in section 4. The main two Theorems we are using are Hirzebruch-Riemann-Roch and Borel-Bott-Weil (section 4.2). \square

Remark 2.1.6. *Even though we cannot compute the determinant of \tilde{M} , we know it can't be 0. Indeed, b has to be an integer, and the roots of*

$$\det(\tilde{M}) = -72(-4556496 + 102168b + 3869b^2)$$

are not integers (approximately -49.97328 and 23.56646).

Therefore, \mathcal{F}_1 is free.

2.2 A second family in $K_{\text{num}}(\tilde{X}_4)$

In this section, we consider another family of $K_{\text{num}}(\tilde{X}_4)$ which could be a basis, assuming the dimension of $K_{\text{num}}(\tilde{X}_4)$ is 8. The choice of this new family, as well as \mathcal{F}_1 defined in section 2.1.2, is based on suggestions from Benjamin Sung.

As we saw in subsection 4.7.4 and Proposition 2.1.5, to compute the matrix \tilde{M} defined in section 2.1.2, we would need to understand some intersections between \overline{D}_2 and surfaces in D_3 , which turns out to be quite difficult. This is why we want to consider another family in $K_{\text{num}}(\tilde{X}_4)$, whose associated matrix of Euler characteristics is easier to compute. This would yield a new matrix whose determinant should be the same as the one of \tilde{M} , assuming our family doesn't generate only a strict sublattice of $K_{\text{num}}(\tilde{X}_4)$. Thus, it could give us a hint on the number b from Lemma 4.7.18, and the determinant of M .

As we saw in the introduction and subsection 2.1.2, we expect that for X_4 very general, $K_{\text{num}}(X_4)$ will be generated by (or at least will contain) $\{\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{l_h}, \mathcal{O}_p\}$ and the structure sheaves of three surfaces. Since \tilde{X}_4 is the blow-up of X_4 in Σ_2 , its numerical Grothendieck group should consist in the pull-backs of the above-mentioned elements, the class of D_3 for which we will consider \mathcal{O}_{D_3} , the class of a fiber of D_3 over Σ_2 which we will represent by \mathcal{O}_{l_D} where l_D is a line of class eh_1^2 (because by (1.14), $h_1^2 = 1$ in Σ_2), and surfaces corresponding to the pull-backs of divisors in Σ_2 .

We have seen in Remark 4.1.4 that $\text{Pic}(\Sigma_2)$ is generated by four elements, so together with the three surfaces coming from X_4 , we need seven structure sheaves of surfaces to complete our family.

A first surface can be given by a section of D_3 by its relative hyperplane class : we will then consider $\mathcal{O}_{D_3}(-V_E)$ since together with \mathcal{O}_{D_3} it allows us to recover the structure sheaf of such a surface, and we have already computed some Euler characteristics involving it in section 4.

For the last six surfaces, one can notice the following : we will see in the proof of Proposition 4.5.3 that $\text{Pic}(Z_3) = \langle V_Z, \bar{h}_1, \bar{E}_1, \bar{E}_2, \bar{E}_3 \rangle$ and by Lemma 3.2.6, there exists in \tilde{X}_4 a threefold, namely \tilde{Z} , which is the blow-up of Z_3 in Γ_1 , with exceptional divisor $\bar{\pi}^{-1}(\Gamma_1) = \bar{D}_2$. Thus $\text{Pic}(\tilde{Z})$ has rank 6 and each divisor in it corresponds to a surface in \tilde{X}_4 . We will therefore complete our family with the structure sheaves of six surfaces in \tilde{Z} : as we will see we can compute all the intersections between divisors there, so we won't have the same problem as in subsection 4.7.4.

So let us consider the following family in $K_{\text{num}}(\tilde{X}_4)$:

$$\mathcal{F}_2 = \{\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{D_3}, \mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\bar{h}_1}, \mathcal{O}_{\bar{E}_1}, \mathcal{O}_{\bar{E}_2}, \mathcal{O}_{\bar{E}_3}, \mathcal{O}_{V_Z}, \mathcal{O}_{\bar{D}_2}, \mathcal{O}_{l_h}, \mathcal{O}_{l_D}, \mathcal{O}_p\}$$

where \mathcal{O}_K for $K \in \text{Pic}(\tilde{Z})$ is the structure sheaf of a surface in \tilde{X}_4 induced by a surface in \tilde{Z} with class K , l_h is the line in \tilde{X}_4 mentioned in subsection 2.1.2, l_D is a line with class $e \cdot h_1^2$ and \mathcal{O}_p is the structure sheaf of a point.

This time, we can compute the following (calculations are detailed in section 5) :

Proposition 2.2.1. *Let us denote $\mathcal{F}_2 = \{e_1, \dots, e_{14}\}$. Then :*

$$\tilde{M}_2 = (\chi(e_i, e_j))_{1 \leq i, j \leq 14} = \begin{pmatrix} 1 & 20 & 124 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & -12 & 1 & 1 & 1 \\ 1 & 1 & 20 & 1 & 0 & 10 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 \\ 20 & 1 & 1 & 7 & 0 & 43 & 11 & 11 & 11 & 7 & 156 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & -84 & 0 & 1 & 0 \\ -1 & -7 & -19 & -1 & 1 & -2 & -2 & -2 & -2 & -2 & 12 & 0 & 1 & 0 \\ 0 & 24 & 72 & 10 & 10 & -1 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 \\ 0 & 6 & 20 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 6 & 20 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 6 & 20 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & -4 & 0 & 0 & 0 \\ 1 & 7 & 19 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ -36 & 0 & 180 & -60 & 36 & -12 & -4 & -4 & -4 & 0 & 48 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its determinant is $-9510912 = -2^{13} \cdot 3^3 \cdot 43$.

Remark 2.2.2. *One can compute the determinant (as a polynomial in b) of the matrix \tilde{M} defined in Proposition 2.1.5 : it is $\det(\tilde{M}) = -72(-4556496 + 102168b + 3869b^2)$.*

One can then look for the possible values of b such that $\det(\tilde{M}) = \det(\tilde{M}_2) = -9510912 = -72 \cdot 132096$: this could give a hint on what b could be. Yet the roots of the polynomial

$$-4688592 + 102168b + 3869b^2$$

aren't integers (approximately -50.43 and 24.03), while $b \in \mathbb{Z}$. This could come from the fact that \mathcal{F}_1 and \mathcal{F}_2 don't generate the same sublattices of $K_{\text{num}}(\tilde{X}_4)$.

However, if the part of \tilde{M} corresponding to elements in $K_{\text{num}}(X_4)$ can be written, in a good basis, as we hoped for in the introduction, the matrix \tilde{M}^t in this good basis completed with the first six elements of \mathcal{F}_1 should be :

$$\begin{pmatrix} N^t & 0 \\ * & M^t \end{pmatrix} \quad \text{with} \quad M^t = \begin{pmatrix} m_1 & m_3 & 0 & \dots & 0 \\ m_2 & m_4 & 0 & \dots & 0 \\ * & * & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & 1 \end{pmatrix}$$

and \tilde{N} the matrix given in Lemma 4.1.5. As a result, \tilde{M}^t should then have an eigenvector of eigenvalue 1. It turns out that \tilde{M}^t has two eigenvectors with eigenvalue 1, namely $-[\mathcal{O}_{\tilde{E}_1}] + [\mathcal{O}_{\tilde{E}_2}]$ and $-[\mathcal{O}_{\tilde{E}_1}] + [\mathcal{O}_{\tilde{E}_3}]$.

2.3 Some matrices of Euler characteristics between elements of $K_{\text{num}}(X_4)$

In this subsection, we focus more on $K_{\text{num}}(X_4)$. Indeed in subsection 2.3.1, we give two matrices associated to two families (one being a subfamily of the other) of respectively 5 and 6 elements in $K_{\text{num}}(X_4)$. These families are free but should be too small to yield a basis of $K_{\text{num}}(X_4)$. Moreover, we don't know whether they correspond to primitive sublattices of $K_{\text{num}}(X_4)$ or not. In Proposition 2.3.1, we try to find some conditions that must be satisfied if the associated sublattices are not primitive, yet it is not obvious that this conditions are not (or are) verified in general.

Finally in subsection 2.3.2, we explain how we can recover a sublattice of $\pi^*K_{\text{num}}(X_4)$, and the associated matrix of Euler characteristics, using right mutation functors and the results we obtained in subsection 2.1.2. However, the result we obtain depends on the same intersection number b which was already mentioned in Proposition 2.1.5.

2.3.1 Two first "too small" families with five and six elements

As mentioned in the introduction, eventually we are interested in finding a free family of $K_{\text{num}}(X_4)$ and computing the associated matrix of Euler characteristics. For the exact same reasons as the ones detailed in the beginning of section 2.1.2 (and the same problem as mentioned in Remark 2.1.4), one can start building such a family with $\{\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(h), \mathcal{O}_{X_4}(2h), \mathcal{O}_l, \mathcal{O}_p\}$ where \mathcal{O}_l is the structure sheaf of a line l of class $E_1 \cdot [\Sigma_2]$ in X_4 and as before \mathcal{O}_p is a skyscraper sheaf. The associated matrix of Euler characteristics can then easily be obtained from calculations we already did to compute \tilde{M}_1 and \tilde{M}_2 , or very similar computations in the case of \mathcal{O}_l .

Furthermore, as we have seen in section 2.1.2, the structure sheaf of Σ_2 seems to be a natural candidate to start completing this family into a basis. In the case of X_4 the associated Euler characteristics are quite easy to compute with Hirzebruch-Riemann-Roch (contrarily to the case of \tilde{X}_4 where we would have needed to understand the pull-back $\pi^*\mathcal{O}_{\Sigma_2}$), and it even turns out that we already know them from calculations we made for Proposition 2.1.5 (detailed in section 4.3.1).

Let us denote by $\{u_1, \dots, u_6\} = \{\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(h), \mathcal{O}_{X_4}(2h), \mathcal{O}_l, \mathcal{O}_p, \mathcal{O}_{\Sigma_2}\}$. Then we have the following :

Proposition 2.3.1. *The following two matrices are :*

$$P_1 = (\chi(u_i, u_j))_{1 \leq i, j \leq 5} = \begin{pmatrix} 1 & 20 & 124 & 1 & 1 \\ 1 & 1 & 20 & 0 & 1 \\ 20 & 1 & 1 & -1 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = (\chi(u_i, u_j))_{1 \leq i, j \leq 6} = \begin{pmatrix} 1 & 20 & 124 & 1 & 1 & 1 \\ 1 & 1 & 20 & 0 & 1 & 1 \\ 20 & 1 & 1 & -1 & 1 & 7 \\ 0 & -1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 7 & 19 & 0 & 0 & 2 \end{pmatrix}.$$

Let us denote by Λ_1 and Λ_2 the sublattices of $K_{\text{num}}(X_4)$ generated by respectively $G_1 := \{u_1, \dots, u_5\}$ and $G_2 := \{u_1, \dots, u_6\}$.

Then if Λ_1 is not primitive in $K_{\text{num}}(X_4)$, there exists a primitive element $v \in K_{\text{num}}(X_4)$ such that nv is primitive in Λ_1 with $n \in \{2, 3, 6\}$ and $\chi(u_1 - 2u_2 + u_3 - 6u_4 - 8u_5 - 11u_6, v) = 0$.

If Λ_2 is not primitive in $K_{\text{num}}(X_4)$, it contains a primitive element with divisibility $m \in \mathbb{N}^*$ such that $m \neq 1$ and $m|96$ in $K_{\text{num}}(X_4)$.

Proof. The calculation of the coefficients of P_1 and P_2 is detailed in Appendix 6.1.

Let us assume now that the embedding of the sublattice $\Lambda_1 \hookrightarrow K_{\text{num}}(X_4)$ is not primitive. Then, there exists a

nonzero primitive element $v \in K_{\text{num}}(X_4)$ and $1 \neq n \in \mathbb{N}^*$ such that $nv \in \Lambda_1$ can be written as $nv = \sum_{i=1}^5 a_i u_i$ with $a_i \in \mathbb{Z}$ for all $1 \leq i \leq 5$ and $\gcd(a_1, \dots, a_5) = 1$. Moreover, at least one of the a_i has to be non zero, so let us assume that $a_j \neq 0$ for some $1 \leq j \leq 5$. Then the family $\langle w_1, \dots, w_5 \rangle = \langle nv, \{u_i, i \neq j\} \rangle$ generates Λ_1 and one can find a matrix $R \in GL(5, \mathbb{Z})$ such that $RP_1R^{-1} = (\chi(w_i, w_j))_{1 \leq i, j \leq 5} =: S$. In particular, one must then have $\det(P_1) = \det(S)$.

Furthermore, since for any $u \in \Lambda_1$, one must have $\chi(nv, u) = n\chi(v, u)$ and $\chi(u, nv) = n\chi(u, v)$, the line and column of S corresponding to nv are multiples of n . As a result, $\det(S)$, and thus $\det(P_1)$, is divisible by n . Since $\det(P_1) = 66$, we must have $n \in \{2, 3, 11, 6, 22, 33, 66\}$. Using P_1 , one also finds that the following equations must be satisfied :

$$\begin{cases} n\chi(u_1, v) = a_1 + 20a_2 + 124a_3 + a_4 + a_5 \\ n\chi(u_2, v) = a_1 + a_2 + 20a_3 + a_5 \\ n\chi(u_3, v) = 20a_1 + a_2 + a_3 - a_4 + a_5 \\ n\chi(u_4, v) = -a_2 - 2a_3 \\ n\chi(u_5, v) = a_1 + a_2 + a_3 \end{cases} \Leftrightarrow \begin{cases} a_4 = n\chi(u_1, v) - a_1 - 20a_2 - 124a_3 - a_5 \\ = -66a_3 + n\chi(u_1 - u_2 + 19u_4, v) \\ a_5 = n\chi(u_2, v) - a_1 - a_2 - 20a_3 = -19a_3 + n\chi(u_2 - u_5, v) \\ 66a_3 = n\chi(u_1 - 2u_2 + u_3 - 19u_5, v) \\ a_2 = -n\chi(u_4, v) - 2a_3 \\ a_1 = n\chi(u_5, v) - a_2 - a_3 = a_3 + n\chi(u_5 + u_4, v) \end{cases} \quad (2.1)$$

In particular, the condition $\gcd(a_1, \dots, a_5) = 1$ implies then that $0 \neq a_3$ is not divisible by n .

On the other hand, $nv \in \Lambda_1 \subset \Lambda_2$, hence with the same reasoning as before one finds that n must divide $\det(P_2) = 96 = 3 \cdot 2^5$. Hence, $n \in \{2, 3, 11, 6, 22, 33, 66\} \cap \{m \in \mathbb{N}^*, m \neq 1 \text{ and } \frac{96}{m} \in \mathbb{N}\} = \{2, 3, 6\}$. Finally, P_2 gives us one more equation for the a_i :

$$n\chi(u_6, v) = a_1 + 7a_2 + 19a_3 \Leftrightarrow 6a_3 = n\chi(6u_4 - u_5 + u_6, v). \quad (2.2)$$

Together with the third equation on the right of the systems (2.1) above, it yields :

$$0 = \chi(u_1 - 2u_2 + u_3 - 66u_4 - 8u_5 - 11u_6, v).$$

Let us assume now that $\Lambda_2 \hookrightarrow K_{\text{num}}(X_4)$ is not primitive, without any assumption on Λ_1 . The same reasoning as in the previous case then yield the existence of a primitive vector $r \in K_{\text{num}}(X_4)$ such that $mr \in \Lambda_2$ for some $1 \neq m \in \mathbb{N}^*$. Since $\det(P_2) = 96$ we must have $m \in \{2^a \cdot 3^b, (a, b) \neq (0, 0), 0 \leq a \leq 5, 0 \leq b \leq 1\}$, and there exists some integers b_i for $1 \leq i \leq 6$ such that $\gcd(b_1, \dots, b_6) = 1$ and $r = \sum_{i=1}^6 b_i u_i$. With P_2 one can then find the following system of equations :

$$\begin{cases} m\chi(u_1, r) = b_1 + 20b_2 + 124b_3 + b_4 + b_5 + b_6 \\ m\chi(u_2, r) = b_1 + b_2 + 20b_3 + b_5 + b_6 \\ m\chi(u_3, r) = 20b_1 + b_2 + b_3 - b_4 + b_5 + 7b_6 \\ m\chi(u_4, r) = -b_2 - 2b_3 \\ m\chi(u_5, r) = b_1 + b_2 + b_3 \\ m\chi(u_6, r) = b_1 + 7b_2 + 19b_3 + 2b_6 \end{cases} \Leftrightarrow \begin{cases} b_4 = m\chi(u_1, r) - b_1 - 20b_2 - 124b_3 - b_5 - b_6 \\ = -66b_3 + m\chi(u_1 - u_2 + 19u_4, r) \\ b_5 = m\chi(u_2, r) - b_1 - b_2 - 20b_3 - b_6 \\ = -16b_3 + m\chi(u_2 - 3u_4 - \frac{u_5}{2} - \frac{u_6}{2}, r) \\ 48b_3 = m\chi(u_1 - 2u_2 + u_3 - 18u_4 - 16u_5 - 3u_6, r) \\ b_2 = -m\chi(u_4, r) - 2b_3 \\ b_1 = m\chi(u_5, r) - b_2 - b_3 = b_3 + m\chi(u_5 + u_4, r) \\ 2b_6 = m\chi(u_6, r) - b_1 - 7b_2 - 19b_3 \\ = -6b_3 + m\chi(6u_4 - u_5 + u_6, r) \end{cases} \quad (2.3)$$

Taking $b_6 = 0$ reduces to the previous case. \square

Remark 2.3.2. The determinant $\det(P_1) = 66$ may seem quite big, but this comes from the fact that $h^4 = 66$ (see [15]). Indeed, let us denote by \mathcal{O}_h the structure sheaf of a hyperplane surface in X_4 with class h , and \mathcal{O}_{h^2} the quotient :

$$0 \rightarrow \mathcal{O}_h(-h) \rightarrow \mathcal{O}_h \rightarrow \mathcal{O}_{h^2} \rightarrow 0. \quad (2.4)$$

Then the matrix of Euler characteristics associated to the family $\{\mathcal{O}_{X_4}, \mathcal{O}_h, \mathcal{O}_{h^2}, \mathcal{O}_l, \mathcal{O}_p\}$ is :

$$R := \begin{pmatrix} 1 & \chi(\mathcal{O}_{X_4}, \mathcal{O}_h) & \chi(\mathcal{O}_{X_4}, \mathcal{O}_{h^2}) & 1 & 1 \\ \chi(\mathcal{O}_h, \mathcal{O}_{X_4}) & \chi(\mathcal{O}_h, \mathcal{O}_h) & \chi(\mathcal{O}_h, \mathcal{O}_{h^2}) & \chi(\mathcal{O}_h, \mathcal{O}_l) & 0 \\ \chi(\mathcal{O}_{h^2}, \mathcal{O}_{X_4}) & \chi(\mathcal{O}_{h^2}, \mathcal{O}_h) & \chi(\mathcal{O}_{h^2}, \mathcal{O}_{h^2}) & 0 & 0 \\ 0 & \chi(\mathcal{O}_l, \mathcal{O}_h) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

using P_1 together with the same arguments as in the proof of Lemma 4.8.1 and Lemma 4.8.2 to find the coefficients which are 0 or 1.

Its determinant, which we expect to be the same as $\det(P_1)$ (because in this specific case, replacing \mathcal{O}_h by $\mathcal{O}_h(h)$ and \mathcal{O}_{h^2} by $\mathcal{O}_{h^2}(2h)$ doesn't change the determinant of R , since twisting by a line bundle leaves both $c_1(\mathcal{O}_h)$ and $ch_2(\mathcal{O}_{h^2})$ unchanged) is thus $\chi(\mathcal{O}_l, \mathcal{O}_h)\chi(\mathcal{O}_h, \mathcal{O}_l)\chi(\mathcal{O}_{h^2}, \mathcal{O}_{h^2})$.

Using the short exact sequence

$$0 \rightarrow \mathcal{O}_{X_4}(-h) \rightarrow \mathcal{O}_{X_4} \rightarrow \mathcal{O}_h \rightarrow 0$$

one can find that

$$\chi(\mathcal{O}_l, \mathcal{O}_h) = \chi(\mathcal{O}_l, \mathcal{O}_{X_4}) - \chi(\mathcal{O}_l, \mathcal{O}_{X_4}(-h))$$

and

$$\chi(\mathcal{O}_h, \mathcal{O}_l) = \chi(\mathcal{O}_{X_4}, \mathcal{O}_l) - \chi(\mathcal{O}_{X_4}(-h), \mathcal{O}_l).$$

By Serre duality (see Theorem 4.2.14) and Lemma 4.2.13, we also have $\chi(\mathcal{O}_l, \mathcal{O}_{X_4}(-h)) = \chi(\mathcal{O}_{X_4}, \mathcal{O}_l)$ hence

$$\chi(\mathcal{O}_l, \mathcal{O}_h) = 0 - 1 = -1.$$

Using the same argument as in the proof of Lemma 6.1.1, one can find that

$$\begin{aligned} \chi(\mathcal{O}_{X_4}(-h), \mathcal{O}_l) &= \int_{\Sigma_2} (1, h, 3) \cdot (0, E_1, \frac{1}{2}) \cdot (1, \frac{h}{2}, 1) \\ &= \int_{\Sigma_2} (0, E_1, 1 + \frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = 1 + \frac{1}{2} + \frac{1}{2} = 2. \end{aligned}$$

As a result, $\chi(\mathcal{O}_h, \mathcal{O}_l) = 1 - 2 = -1$.

Finally, using the same argument as in the proof of Lemma 4.6.1 as well as (2.4), one can find that $\chi(\mathcal{O}_{h^2}, \mathcal{O}_{h^2}) = h^4 = 66$.

Altogether, this yield $\det(R) = 66$.

2.3.2 A family of eight elements obtained with mutation functors, but still in $K_{\text{num}}(\tilde{X}_4)$

In this subsection, we try to find a family of eight elements in $\pi^* K_{\text{num}}(X_4)$, following a method suggested by Benjamin Sung.

Before going further, let us recall the following definition (see for instance Definition 1.57 and Proposition 2.56 of [14]) :

Definition 2.3.3. Let E be an object in a \mathbb{C} -linear triangulated category \mathcal{D} . Then E is an exceptional object if and only if $\dim(\text{Ext}^i(E, E)) = \delta_{i,0}$.

With what we will see in the proof of Lemma 4.1.5 and Proposition 11.16 in [14], one can find that

$$\{e_1, \dots, e_6\} := \{j_* \mathcal{O}_{E_1}(E_1 - V_E), j_* \mathcal{O}_{E_2}(E_2 - V_E), j_* \mathcal{O}_{E_3}(E_3 - V_E), j_* \mathcal{O}_{D_3}(-V_E), \\ j_* \mathcal{O}_{D_3}(h_1 - V_E), j_* \mathcal{O}_{D_3}(2h_1 - V_E)\}$$

is a sequence of exceptional objects.

Moreover, if we denote by \mathcal{D} the subcategory $\langle e_1, \dots, e_6 \rangle \subset D^b(\tilde{X}_4)$, Lemma 2.1.2 yields $\pi^* D^b(X_4) \subset {}^\perp \mathcal{D}$ and

$$K_{\text{num}}(\tilde{X}_4) = \langle K_{\text{num}}(\mathcal{D}), \pi^* K_{\text{num}}(X_4) \rangle .$$

By the statement below Lemma 2.4 in [17] there is another semi-orthogonal decomposition $D^b(\tilde{X}_4) = \langle \mathcal{D}, {}^\perp \mathcal{D} \rangle$, hence

$$K_{\text{num}}(\tilde{X}_4) = \langle K_{\text{num}}(\mathcal{D}), K_{\text{num}}({}^\perp \mathcal{D}) \rangle .$$

If $K_{\text{num}}(\tilde{X}_4)$ is free this should thus imply $\pi^* K_{\text{num}}(X_4) \cong K_{\text{num}}({}^\perp \mathcal{D})$.

This is why in this subsection we are going to look for a free family of 8 elements in $K_{\text{num}}({}^\perp \mathcal{D})$: this should actually yield a family in $\pi^* K_{\text{num}}(X_4)$. Furthermore, since $\pi_* \mathcal{O}_{\tilde{X}_4} = \mathcal{O}_{X_4}$, the projection formula implies that for any sheaves \mathcal{E} and \mathcal{F} in $D^b(X_4)$, one has $H^*(\pi^* \mathcal{E}, \pi^* \mathcal{F}) = H^*(\mathcal{E}, \pi_* \pi^* \mathcal{F}) = H^*(\mathcal{E}, \mathcal{F})$, therefore if we could compute the matrix of Euler characteristics associated to such a family in $\pi^* K_{\text{num}}(X_4)$, it would directly give us the matrix associated to the corresponding family in $K_{\text{num}}(X_4)$.

Using Definition 2.4 in [21], one can use a composition of right mutation functors, namely $\mathcal{R} := \mathbb{R}_{e_6} \circ \dots \circ \mathbb{R}_{e_1}$, to project elements of $K_{\text{num}}(\tilde{X}_4)$ to ${}^\perp \mathcal{D}$. Eventually, this will give a free family of 8 elements in $K_{\text{num}}({}^\perp \mathcal{D})$; if $K_{\text{num}}(\tilde{X}_4)$ is free and this family doesn't generate a strict sublattice, this could give a basis of $K_{\text{num}}(X_4)$, whose matrix of Euler characteristics is easily calculable using Propositions 2.1.5 and 2.2.1.

With Proposition 2.1.5 and Definition 2.4 in [21], one finds :

$$\begin{aligned} \mathcal{R}(\mathcal{O}_{\tilde{S}}) &= \mathcal{O}_{\tilde{S}} - 3e_1 - 3e_2 - 3e_3 + 33e_4 - 129e_5 + 204e_6 \\ \mathcal{R}(\mathcal{O}_{D_3}(V_E)) &= \mathcal{O}_{D_3}(V_E) + e_4 - 4e_5 + 2e_6 \\ \mathcal{R}(\mathcal{O}_{\tilde{D}_2}) &= \mathcal{O}_{\tilde{D}_2} - 36e_4 + (72 - b)e_5 + (-36 + b)e_6. \end{aligned}$$

Moreover, if $\{y_1, \dots, y_8\} := \{\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{h^*}, \mathcal{O}_p, \mathcal{R}(\mathcal{O}_{\tilde{S}}), \mathcal{R}(\mathcal{O}_{D_3}(V_E)), \mathcal{R}(\mathcal{O}_{\tilde{D}_2})\}$, then :

$$M' = (\chi(y_i, y_j))_{1 \leq i, j \leq 8} = \begin{pmatrix} 1 & 20 & 124 & 1 & 1 & 1 & 0 & -12 \\ 1 & 1 & 20 & 0 & 1 & 37 & 0 & 0 \\ 20 & 1 & 1 & -1 & 1 & 127 & 12 & 156 \\ 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 37 & 217 & -125 & 0 & 0 & 10488 & -314 & -2616 + 333b \\ 0 & 12 & 36 & 0 & 0 & 46 & 8 & -48 \\ 0 & 36 + 3b & 216 + 6b & 0 & 0 & -24 + 9b & -192 & b^2 + 48 \end{pmatrix} .$$

Its determinant is $2(-6188803776 - 70302816b + 1781868b^2) = 2^3 \cdot 3(-515733648 - 5858568b + 148489b^2)$.

Remark 2.3.4. One could also use the family \mathcal{F}_2 and the associated matrix \tilde{M}_2 from subsection 2.2, and project elements from \mathcal{F}_2 on ${}^\perp \mathcal{D}$ with \mathcal{R} in order to find a free family of 8 elements. However, computing $\mathcal{R}(\mathcal{F})$ for a sheaf \mathcal{F} requires to know all the $\chi(\mathcal{F}, e_i)$ for $1 \leq i \leq 6$.

If we want to consider structure sheaves of surfaces in \tilde{Z} among \mathcal{F}_2 this should then cause similar problems as when we tried to compute $\mathcal{R}(\mathcal{O}_{\tilde{D}_2})$: we would need to understand the intersections between some surfaces in D_3 and \tilde{Z} , which is not so easy in general.

Unfortunately, if we want to project 8 elements from \mathcal{F}_2 , we need to consider at least one structure sheaf of a surface in \tilde{Z} : indeed $\mathcal{O}_{D_3}(-V_E)$ is just e_4 , so we cannot consider it (because $\mathcal{R}(e_4) = 0$), and then out of the 13 other elements, 6 are structure sheaves of surfaces in \tilde{Z} , so if we want 8 elements we need to take at least one such structure sheaf.

Chapter 3

Subvarieties of X_4 , \tilde{X}_4 and \tilde{X}_5

In this section we gather some small results about the geometry of X_4 and \tilde{X}_4 . Some of these results will be useful for later, the last part of the section will concern an attempt to describe set theoretically some of what could be contracted by the map π (see (1.2) and (1.4)) in some surfaces in \tilde{X}_4 .

3.1 A surface induced by S_2 in Z_3

As one can see in (1.2) and (1.4), S_2 is a surface which appears to be related both to \tilde{X}_4 and the associated \tilde{X}_5 , but is not embedded in any of them. We would like to use it to induce a surface in \tilde{X}_4 whose structure sheaf could be used to build a basis of $K_{\text{num}}(\tilde{X}_4)$. In this subsection, we thus try to find a surface in Z_3 , "induced" by S_2 (as it will turn out to be quite difficult to find a copy of S_2 in Z_3 , we will instead consider a blow-up of S_2) and which we can use later to find a surface in \tilde{X}_4 related to S_2 .

We will first try to find a section of the map $\sigma : Z_3 \rightarrow S_2$ (see (1.2)) which could help us to find a copy of S_2 in Z_3 and then use Lemma 1.4.1 to find a blow-up of S_2 embedded in Z_3 .

Proposition 3.1.1. *It is equivalent to be given a section of the map $\sigma : Z_3 \rightarrow S_2$ or a short exact sequence $0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{V}_S \rightarrow \mathcal{L}_2 \rightarrow 0$ where the \mathcal{L}_i are line bundles on S_2 and \mathcal{V}_S is as defined in 1. of Lemma 1.2.2.*

Proof. First of all, from 1. of Lemma 1.2.2 we know that $Z_3 = \mathbb{P}_S(\mathcal{V}_S)$ and that $\mathcal{V}_S^\vee \cong \mathcal{V}_S$. So from Exercise 7.8 of II.7 in [12], we get that there is a bijection between sections of σ and line bundles \mathcal{L}_2 on S_2 such that there is a surjective morphism $\mathcal{V}_S \rightarrow \mathcal{L}_2 \rightarrow 0$. Let us assume that we are given such a surjective morphism. Since \mathcal{V}_S is a rank 2 vector bundle (still from 1. of Lemma 1.2.2), the kernel K of this morphism has to be of rank 1. Moreover \mathcal{V}_S is locally free since it is a vector bundle, so K is a line bundle that we can call \mathcal{L}_1 . Thus we have a short exact sequence $0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{V}_S \rightarrow \mathcal{L}_2 \rightarrow 0$. Conversely, such a sequence obviously gives a surjective morphism from \mathcal{V}_S to a line bundle on S_2 . \square

Lemma 3.1.2. *If there is a short exact sequence $0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{V}_S \xrightarrow{f} \mathcal{L}_2 \rightarrow 0$ where the \mathcal{L}_i are line bundles on S_2 , then f induces a global section of $\sigma^* \mathcal{L}_2 \otimes \mathcal{O}_{Z_3}(V_Z)$ on $Z_3 = \mathbb{P}_S(\mathcal{V}_S)$ with V_Z the relative hyperplane class of Z_3 as defined in Definition 1.3.3. The zero locus of this global section is then a surface isomorphic to S_2 and with class $V_Z + [\sigma^* \mathcal{L}_2]$ in Z_3 .*

Proof. Let us assume that we have a short exact sequence $0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{V}_S \xrightarrow{f} \mathcal{L}_2 \rightarrow 0$ in S_2 . Then f induces a global section of $\sigma^* \mathcal{L}_2 \otimes \mathcal{O}_{Z_3}(V_Z)$ on Z_3 since $\text{Hom}(\mathcal{V}_S, \mathcal{L}_2) = \text{Hom}(\mathcal{O}_{S_2}, \sigma_*(\mathcal{O}_{Z_3}(V_Z) \otimes \sigma^* \mathcal{L}_2)) \cong \text{Hom}(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(V_Z) \otimes \sigma^* \mathcal{L}_2)$. Let us denote by \tilde{S} the zero locus of this section induced by f . Then since $\mathcal{O}_{Z_3}(V_Z) \otimes \sigma^* \mathcal{L}_2$ is a line bundle, \tilde{S} is a divisor in Z_3 so it is 2-dimensional. Moreover over each point p of S_2 the kernel of f is $\mathcal{L}_{1,p}$ which is one dimensional, so there is exactly one point over p in $\tilde{S} : \mathbb{P}(\mathcal{L}_{1,p})$. Thus $\sigma|_{\tilde{S}} : \tilde{S} \rightarrow S_2$ is an isomorphism. By definition of \tilde{S} , its class in Z_3 is $V_Z + [\sigma^* \mathcal{L}_2]$. \square

Lemma 3.1.3. *On S_2 , \mathcal{V}_S has no non-zero global sections.*

Proof. As mentioned in 11. of Theorem 1.2.1, $S_2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ is cut out by two $(1,1)$ divisors. So we have a resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-2, -2) \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, -1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} \rightarrow \mathcal{O}_{S_2} \rightarrow 0. \quad (3.1)$$

Moreover if we denote by Ω_i the cotangent bundle on A_i we have a short exact sequence

$$0 \rightarrow \Omega_i \rightarrow \mathcal{O}_{\mathbb{P}(\Lambda^2 A_i)}(-1)^{\oplus 3} \rightarrow \mathcal{O}_{\mathbb{P}(\Lambda^2 A_i)} \rightarrow 0. \quad (3.2)$$

On the other hand, by the proof of Proposition 4.6 in [20] we have short exact sequences on S_2 :

$$0 \rightarrow \mathcal{U}_{2,A_1} \rightarrow A_1 \otimes \mathcal{O}_{S_2} \rightarrow \mathcal{O}_{S_2}(\bar{h}_1) \rightarrow 0 \quad (3.3)$$

$$0 \rightarrow \mathcal{U}_{2,A_2} \rightarrow A_2 \otimes \mathcal{O}_{S_2} \rightarrow \mathcal{O}_{S_2}(\bar{h}_2) \rightarrow 0 \quad (3.4)$$

so on S_2 , $\Omega_i \cong \mathcal{U}_{2,A_i}(-\bar{h}_i)$. Using now (1.3) and (3.1) we can compute $H^*(\mathcal{V}_S)$ as follows : (1.3) gives us the short exact sequence $0 \rightarrow \mathcal{O}(-\bar{h}_1) \oplus \mathcal{O}(-\bar{h}_2) \rightarrow \Omega_1(\bar{h}_1) \oplus \Omega_2(\bar{h}_2) \rightarrow \mathcal{V}_S \rightarrow 0$ and (3.1) allows us to compute the cohomology of each term of the sequence.

- $\mathcal{O}(-\bar{h}_1)$ is the pullback of $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, 0)$ by definition of \bar{h}_1 in Definition 1.3.3. So we get a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-3, -2) \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-2, -1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, 0) \rightarrow \mathcal{O}_{S_2}(-\bar{h}_1) \rightarrow 0.$$

Finally, using the Künneth formula : $H^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(a, b)) \cong \bigoplus_i H^i(\mathcal{O}_{\mathbb{P}^2}(a)) \otimes H^{*-i}(\mathcal{O}_{\mathbb{P}^2}(b))$ we get that

$$H^*(\mathcal{O}(-\bar{h}_1)) = 0. \text{ Similarly } H^*(\mathcal{O}(-\bar{h}_2)) = 0.$$

- We use the same reasoning as before : we have a short exact sequence

$$0 \rightarrow \Omega_1(-1, -2) \rightarrow \Omega_1(0, -1)^{\oplus 2} \rightarrow \Omega_1(1, 0) \rightarrow \Omega_1(\bar{h}_1) \rightarrow 0$$

where $\Omega_1(a, b) \cong \Omega_1(a) \boxtimes \mathcal{O}(b)$. To compute $H^*(\Omega_i(a))$ we use (3.2) which yields :

$$\begin{aligned} H^2(\Omega_i(-2)) &\cong \mathbb{C}, & H^j(\Omega_i(-2)) &= 0 & \text{for } j \neq 2 \\ H^j(\Omega_i(-1)) &= 0 & & & \forall j \\ H^1(\Omega_i) &\cong \mathbb{C}, & H^j(\Omega_i) &= 0 & \text{for } j \neq 1 \\ H^j(\Omega_i(1)) &= 0 & & & \forall j \\ H^0(\Omega_i(2)) &\cong \mathbb{C}^3, & H^j(\Omega_i(2)) &= 0 & \text{for } j \neq 0. \end{aligned}$$

Finally, Künneth formula gives us : $H^*(\Omega_1(-1, -2)) = H^*(\Omega_1(0, -1)) = H^*(\Omega_1(1, 0)) = 0$ so $H^*(\Omega_1(\bar{h}_1)) = 0$. Similarly, $H^*(\Omega_2(\bar{h}_2)) = 0$.

Therefore, $H^*(\mathcal{V}_S) = 0$ and so \mathcal{V}_S has no non-zero global sections on S_2 . \square

However there is an injection $0 \rightarrow \mathcal{O}_{S_2} \rightarrow \mathcal{V}_S(\bar{h}_1 + \bar{h}_2)$ which if it gave rise to a copy of S_2 would be very convenient because this surface would have class $V_Z + \bar{h}_1 + \bar{h}_2 = \bar{h}$ by Lemma 3.1.2 and (1.11). Here \bar{h} , \bar{h}_1 and \bar{h}_2 are defined as in Definition 1.3.3. In particular $\omega_S = \mathcal{O}_{S_2}(-\bar{h}_1 - \bar{h}_2)$ (by 11. of Theorem 1.2.1).

Lemma 3.1.4. *On S_2 , $\mathcal{V}_S(\bar{h}_1 + \bar{h}_2)$ has nonzero global sections.*

Proof. We use the same method as in the proof of Lemma 3.1.3, except that now everything is twisted by $(\bar{h}_1 + \bar{h}_2)$. We have then a short exact sequence :

$$0 \rightarrow \mathcal{O}(\bar{h}_2) \oplus \mathcal{O}(\bar{h}_1) \rightarrow \Omega_1(2\bar{h}_1 + \bar{h}_2) \oplus \Omega_2(\bar{h}_1 + 2\bar{h}_2) \rightarrow \mathcal{V}_S(\bar{h}_1 + \bar{h}_2) \rightarrow 0. \quad (3.5)$$

- There is a resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, -2) \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, -1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 0) \rightarrow \mathcal{O}_{S_2}(\bar{h}_1) \rightarrow 0.$$

Using Künneth formula again, one can find :

$$\begin{aligned} H^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, -2)) &= 0 \\ H^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, -1)^{\oplus 2}) &= 0 \\ H^0(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 0)) &= \mathbb{C}^3, \quad H^j(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 0)) = 0 \quad \forall j \neq 0. \end{aligned}$$

So $\dim(H^j(\mathcal{O}_{S_2}(\bar{h}_1))) = 3\delta_{0j}$. Similarly, $\dim(H^j(\mathcal{O}_{S_2}(\bar{h}_2))) = 3\delta_{0j}$.

- We use now

$$0 \rightarrow \Omega_1(0, -1) \rightarrow \Omega_1(1, 0)^{\oplus 2} \rightarrow \Omega_1(2, 1) \rightarrow \Omega_1(2\bar{h}_1 + \bar{h}_2) \rightarrow 0.$$

With the same method as in the previous Lemma we find :

$$\begin{aligned} H^*(\Omega_1(0, -1)) &= 0 \\ H^*(\Omega_1(1, 0)^{\oplus 2}) &= 0 \\ H^0(\Omega_1(2, 1)) &= H^0(\Omega_1(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathbb{C}^9 \quad H^j(\Omega_1(2, 1)) = 0 \quad \forall j \neq 0. \end{aligned}$$

So $\dim(H^j(\Omega_1(2\bar{h}_1 + \bar{h}_2))) = 9\delta_{j0}$. Similarly $\dim(H^j(\Omega_2(\bar{h}_1 + 2\bar{h}_2))) = 9\delta_{j0}$.

As a result, $\dim(H^0(\mathcal{V}_S(\bar{h}_1 + \bar{h}_2))) = 2 \cdot 9 - 2 \cdot 3 = 18 - 6 = 12$. □

Yet, we have the following problem:

Lemma 3.1.5. *If we have a short exact sequence $0 \rightarrow \mathcal{O}_{S_2} \rightarrow \mathcal{V}_S(\bar{h}_1 + \bar{h}_2) \rightarrow Q \rightarrow 0$ then Q cannot be a line bundle.*

Proof. First of all since $\text{rank}(\mathcal{V}_S) = 2$ we must have $\text{rank}(Q) = 1$. We also know from (1.3) that there is a short exact sequence

$$0 \rightarrow \mathcal{O}_{S_2}(-\bar{h}_1) \oplus \mathcal{O}_{S_2}(-\bar{h}_2) \rightarrow \mathcal{U}_{2,A_1} \oplus \mathcal{U}_{2,A_2} \rightarrow \mathcal{V}_S \rightarrow 0.$$

Moreover, the short exact sequences (3.3) and (3.4) with (1.13) imply that $\text{ch}(\mathcal{U}_{2,A_1}) = (3, 0, 0) - (1, \bar{h}_1, \frac{1}{2}) = (2, -\bar{h}_1, -\frac{1}{2})$ and $\text{ch}(\mathcal{U}_{2,A_2}) = (2, -\bar{h}_2, -\frac{1}{2})$. So $\text{ch}(\mathcal{V}_S) = (4, -\bar{h}_1 - \bar{h}_2, -1) - (2, -\bar{h}_1 - \bar{h}_2, 1) = (2, 0, -2)$. Thus $\text{ch}(Q) = (2, 0, -2) \cdot (1, \bar{h}_1 + \bar{h}_2, 3) - (1, 0, 0) = (1, 2\bar{h}_1 + 2\bar{h}_2, 4)$. However $(2\bar{h}_1 + 2\bar{h}_2)^2 = 4 \cdot 6 = 24$ and $\frac{2 \cdot 4}{2} = 4 \neq 4$. As a result Q cannot be a line bundle. □

So a non-zero map $\mathcal{O}_{S_2}(-\bar{h}_1 - \bar{h}_2) \rightarrow \mathcal{V}_S$ will not induce a surface isomorphic to S_2 in Z_3 . However if we could prove that there is a short exact sequence $0 \rightarrow \mathcal{O}_{S_2} \rightarrow \mathcal{V}_S(\bar{h}_1 + \bar{h}_2) \rightarrow \mathcal{L} \otimes \mathcal{I}_D \rightarrow 0$ with \mathcal{L} a line bundle and \mathcal{I}_D the ideal sheaf of some points D , then using Lemma 1.4.1 we could still get a copy of a blow-up of S_2 in D inside Z_3 , with class \bar{h} . This is what we are going to do now.

Before going further, let us recall some geometrical properties of S_2 that we are going to use. First of all, by 10. of Theorem 1.2.1, the points of S_2 are exactly the $\bar{U}_4 \subset \bar{W}$ of the form $\bar{U}_4 = U_{2,A_1} \oplus U_{2,A_2}$ such that \bar{U}_4 is λ' -isotropic and $\bar{U}_4^\perp \subset \bar{U}_4$ is 2-dimensional. Moreover, since we assumed that the A_i are not μ -Lagrangian, where μ is the 2-form associated to the global section of $\mathcal{U}_3(1)$ we are considering, we have $\dim(A_i^\perp \cap A_i) = 1$. Let μ_i denote a generator of $A_i^\perp \cap A_i$ for $i \in \{1, 2\}$. Then (see Lemma 1.2.2) there are two smooth rational curves \bar{C}_1 and \bar{C}_2 in S_2 such that the following are equivalent :

- $p = U_{2,A_1} \oplus U_{2,A_2} \in S_2$ is in \bar{C}_i .
- U_{2,A_i} is μ -isotropic.

- $\mu_i \in U_{2,A_i}$.
- $U_{2,A_i} \cap (U_{2,A_1} \oplus U_{2,A_2})^\perp$ is 1-dimensional.

To find what \mathcal{I}_D could complete the short exact sequence above, we need first to look more in details at S_2 and the curves \overline{C}_1 and \overline{C}_2 .

Lemma 3.1.6. *If $w, z \in A_2$ are nonzero vectors such that $\text{Ker}(\mu(w, \cdot)) \cap A_1 = \text{Ker}(\mu(z, \cdot)) \cap A_1$ then w and z are colinear.*

Similarly, if $u, v \in A_1$ are nonzero vectors such that $\text{Ker}(\mu(u, \cdot)) \cap A_2 = \text{Ker}(\mu(v, \cdot)) \cap A_2$ then u and v are colinear.

Proof. Let us assume that $w, z \in A_2$ are nonzero vectors such that $\text{Ker}(\mu(w, \cdot)) \cap A_1 = \text{Ker}(\mu(z, \cdot)) \cap A_1$. Then let (u, v, s) be a basis of A_1 such that $\text{Ker}(\mu(w, \cdot)) \cap A_1 = \text{Ker}(\mu(z, \cdot)) \cap A_1 = \langle u, v \rangle$. Then since μ induces a non degenerate pairing between the A_i we have that $\mu(w, s) = a \neq 0$ and $\mu(z, s) = b \neq 0$. Thus $\mu(bw - az, A_1) = 0$ and so, again since the pairing induced by μ between A_1 and A_2 is non degenerate, we must have $bw - az = 0$. The same reasoning can be used to show the second assertion. \square

Before going further, let us define some "properties": avoiding them will make it possible for us to find an ideal \mathcal{I}_D fitting in the short exact sequence we are looking for. It will also be useful later.

Definition 3.1.7. *Let us define :*

(P1) $\mu_1 \in \overline{U}_4^\perp$ for one of the two points in $\overline{C}_1 \cap \overline{C}_2$.

(P2) $\mu_2 \in \overline{U}_4^\perp$ for one of the two points in $\overline{C}_1 \cap \overline{C}_2$.

(A1) $A_1 / \langle \mu_1 \rangle \oplus A_2 / \langle \mu_2 \rangle \in S_2$ (by $A_i / \langle \mu_i \rangle$ we actually mean the kernel of μ_i^* , so that for instance a point in S_2 such that $U_{2,A_1} = \langle u + a\mu_1, v \rangle$ and $U_{2,A_2} = \langle w + b\mu_2, v \rangle$ would not satisfy (A1)).

(B1) In $\overline{C}_1 - \overline{C}_2$, there is more than one point such that $\overline{U}_S / A_2 = \langle \mu_1 \rangle$.

(B2) In $\overline{C}_2 - \overline{C}_1$, there is more than one point such that $\overline{U}_S / A_1 = \langle \mu_2 \rangle$.

Lemma 3.1.8. *The following are equivalent :*

- (P1) holds.
- (P2) holds.
- μ_1 and μ_2 are μ -orthogonal.
- $M_1M_6K_1 + M_2M_5K_2 + M_3M_4K_3 = 0$ using the notations of 4. of section 1.1.

Proof. • Let us first see that (P1) \Leftrightarrow (P2).

Let us assume that (P1) holds. Then there is a point $p_1 \in \overline{C}_1 \cap \overline{C}_2$ such that $p = \overline{U}_4 = \langle \mu_1, u \rangle \oplus \langle \mu_2, w \rangle$ with $u \in A_1, w \in A_2$ and $\overline{U}_4^\perp = \langle \mu_1, aw + b\mu_2 \rangle$. In particular, μ_1 is μ -orthogonal to μ_2 . Moreover, since μ induces a non degenerate pairing between A_1 and A_2 , $K := \text{Ker}(\mu(\mu_2, \cdot)) \cap A_1$ is 2-dimensional. Thus $K = \langle \mu_1, v \rangle$ with some $v \in A_1$ and $\langle \mu_2, K \rangle$ is a Lagrangian for μ , by definition of K and μ_1 (by definition, $\mu(\mu_1, A_1) = 0$). By 6. of section 1.1, $\lambda'(\mu_1, \mu_2, v, \cdot)$ is non degenerate so there exists, up to multiplication by a nonzero scalar, a unique $z \in A_2 / \langle \mu_2 \rangle$ such that $\lambda'(\mu_1, \mu_2, v, z) = 0$. Then μ_2 is μ -orthogonal to μ_1, v and z and since z is not colinear to μ_2 , we must have $K' := \text{Ker}(\mu(z, \cdot)) \cap A_1 \neq K$, by Lemma 3.1.6. But then $K' \cap K$ must be a line in A_1 , so $(\langle \mu_1, \mu_2, v, z \rangle)^\perp$ contains $\langle \mu_2, K \cap K' \rangle$. Finally, $(\langle \mu_1, \mu_2, v, z \rangle)^\perp = \langle \mu_2, K \cap K' \rangle$ because $K \neq K'$ and because $\mu(v, A_1) \neq 0$ but $\mu(v, \mu_1) = \mu(v, v) = 0$ and $\mu(z, A_2) \neq 0$ but $\mu(z, \mu_2) = \mu(z, z) = 0$. So $\langle \mu_1, \mu_2, v, z \rangle$ is λ' -isotropic and it contains its 2-dimensional μ -orthogonal : by 10. of Theorem 1.2.1 it means that $p_2 := \langle \mu_1, \mu_2, v, z \rangle \in S_2$. Since it contains both μ_1 and μ_2 it has to be in $\overline{C}_1 \cap \overline{C}_2$.

Moreover $p_1 \neq p_2$. Indeed, if $p_2 = p_1$ then there exists another point $V_4 := \langle \mu_1, r \rangle \oplus \langle \mu_2, s \rangle \in \overline{C}_1 \cap \overline{C}_2$. Therefore, there exists $(\alpha, \beta) \in \mathbb{C}^2$ such that $\alpha\mu_1 + \beta r$ is μ -orthogonal to V_4 . In particular, $\alpha\mu_1 + \beta r$ is μ -orthogonal to μ_2 and since μ_2 is μ -orthogonal to μ_1 , it must then also be μ -orthogonal to r . But then $K = \langle \mu_1, r \rangle$ and again by 6. of section 1.1 this implies that $\langle \mu_2, s \rangle = \langle \mu_2, z \rangle$ and so $V_4 = p_2 = p_1$,

which is a contradiction since $\overline{C}_1 \cap \overline{C}_2$ contains two points. So $\{p_1, p_2\} = \overline{C}_1 \cap \overline{C}_2$ and if p_1 satisfies (P1) it implies that p_2 satisfies (P2).

By the same reasoning with μ_i and A_i exchanged, (P2) implies (P1).

- Let us see now that (P1) $\Leftrightarrow \mu(\mu_1, \mu_2) = 0$.
We have just seen that (P1) implies that $\mu(\mu_1, \mu_2) = 0$, so we only have to show that $\mu(\mu_1, \mu_2) = 0$ implies (P1). Let us assume thus that μ_1 is μ -orthogonal to μ_2 . Using the same reasoning and notations as before we can then construct a point in $\overline{C}_1 \cap \overline{C}_2$ satisfying (P1). Indeed as before $\langle K, \mu_2 \rangle$ is a Lagrangian for μ if $K = \text{Ker}(\mu(\mu_2, \cdot)) \cap A_1$ contains μ_1 thus by 6. of section 1.1 there exists a unique (up to a scalar) $z \in A_2 / \langle \mu_2 \rangle$ such that $\lambda'(\mu_2, \mu_1, v, z) = 0$. Exactly as before, this gives a point in $\overline{C}_1 \cap \overline{C}_2$ satisfying (P2). Then by what we did before the other point in $\overline{C}_1 \cap \overline{C}_2$ satisfies (P1).
- Finally, in the proof of Lemma 4.8 in Kuznetsov's article, we have that $\mu_1 = M_6 K_1 x_1 + M_5 K_2 x_2 + M_4 K_3 x_3$, using the notations of 4. of section 1.1. Similarly, one can compute that $\mu_2 = M_3 K_3 x_4 + M_2 K_2 x_5 + M_1 K_1 x_6$. A direct computation then shows that $\mu(\mu_1, \mu_2) = (M_1 M_6 K_1 + M_2 M_5 K_2 + M_3 M_4 K_3)(M_1 M_6 K_1 + M_2 M_5 K_2 + M_3 M_4 K_3 + K_1 K_2 K_3)$. By 4. of section 1.1, $M_1 M_6 K_1 + M_2 M_5 K_2 + M_3 M_4 K_3 + K_1 K_2 K_3 \neq 0$, so $\mu(\mu_1, \mu_2) = 0$ if and only if $M_1 M_6 K_1 + M_2 M_5 K_2 + M_3 M_4 K_3 = 0$. □

In the following, we are going to assume that (P1) and (P2) don't hold, so from now on we are going to assume

$$M_1 M_6 K_1 + M_2 M_5 K_2 + M_3 M_4 K_3 \neq 0. \quad (3.6)$$

Definition 3.1.9. We will call $K_1 := \text{Ker}(\mu(\mu_1, \cdot)) \cap A_2$ and $K_2 := \text{Ker}(\mu(\mu_2, \cdot)) \cap A_1$.

Lemma 3.1.10. We have seen in the proof of Lemma 3.1.8 that K_1 and K_2 are both 2-dimensional. To each line in K_i corresponds a unique point in \overline{C}_i so $\overline{C}_i \cong \mathbb{P}(K_i)$.

Moreover, to each line l in $A_i / \langle \mu_i \rangle$ corresponds a point p in \overline{C}_i such that $l \subset U_{2, A_i}$ so $\overline{C}_i \cong \mathbb{P}(A_i / \langle \mu_i \rangle)$.

Proof. Let v be a nonzero vector in K_2 . By hypothesis, $K := \text{Ker}(\mu(v, \cdot)) \cap A_2$ contains μ_2 . Thus, $\langle v, K \rangle$ is Lagrangian for μ and by 6. of section 1.1 we have, up to a scalar, a unique $u \in A_1 / \langle v \rangle$ such that $\lambda'(u, v, K) = 0$. We would like to show that $U := \langle u, v, K \rangle$ is a point in S_2 . For this we need to check that $U^\perp \subset U$ (indeed by construction $\lambda'(U) = 0$, and since $U = U_{2, A_1} \oplus U_{2, A_2}$, it is $\bar{\lambda}$ -isotropic). Let us distinguish two cases :

- $\mu_1 \notin U$. Then $\mu(u, v) \neq 0$ since the projections of u and v onto $A_1 / \langle \mu_1 \rangle$ must be non colinear, and μ is non degenerate on A_1 . Let us denote by $K' := \text{Ker}(\mu(u, \cdot)) \cap A_2$. Then since u and v are non colinear by Lemma 3.1.6 we must have $K \neq K'$. Since they are both 2-dimensional (again because of the non degenerate pairing) in A_1 , $K \cap K'$ must be a line. Let w be a nonzero vector in this line. Then $w \in U^\perp$ since $\mu_2 \in K$. Let z be such that $K = \langle w, z \rangle$. By construction, v is μ -orthogonal to w and z but $\mu(u, z) \neq 0$. Then $z + \frac{\mu(u, z)}{\mu(v, u)} v \in U^\perp$. Finally, $U^\perp = \langle w, z + \frac{\mu(u, z)}{\mu(v, u)} v \rangle$ because $\mu_2 \in K$ and μ is non degenerate on A_2 so there should be a vector in K which is not μ -orthogonal to A_2 / K , and because we have assumed that μ_1 is not μ -orthogonal to μ_2 .
- $\mu_1 \in U$. Then $v \in U^\perp$. For the same reason as in the previous case there should be, up to a scalar, exactly one vector w in K such that w is μ -orthogonal to u and v . Since $\mu_2 \in K$, $w \in U^\perp$. Finally, $U^\perp = \langle v, w \rangle$ because μ is non degenerate on the A_i and in this case $\mu_1 \in U_{2, A_1}$ and $\mu_2 \in U_{2, A_2}$ so the two vectors generating \overline{W} / U cannot be in U^\perp . In this case $U \in \overline{C}_1 \cap \overline{C}_2$.

So $U \in S_2$ and since $\mu_2 \in K$, $U \in \overline{C}_2$. Therefore, to each line in K_2 corresponds a point in \overline{C}_2 , and so $\mathbb{P}(K_2) \subset \overline{C}_2$. But by Proposition 1.2.2, \overline{C}_2 is rational so the inclusion is in fact an equality and each point in \overline{C}_2 can be obtained by a similar construction as what we just did.

Moreover, since μ induces a non degenerate pairing between the A_i , if v and v' are two non colinear vectors in K_2 , then the associated K and K' are distinct. But they both contain μ_2 , so we can write $K = \langle \mu_2, w \rangle$,

$K' = \langle \mu_2, z \rangle$ with w and z non colinear in $A_2 / \langle \mu_2 \rangle$. Thus two different points in \overline{C}_2 cannot have the same associated $U_{2,A_2} / \langle \mu_2 \rangle$ and therefore to each point in \overline{C}_2 corresponds a unique line in $A_2 / \langle \mu_2 \rangle$. The same reasoning exchanging the μ_i and A_i finishes the proof. \square

Lemma 3.1.11. *Let $p = \overline{U}_4 = U_{2,A_1} \oplus U_{2,A_2}$ be a point in \overline{C}_i . Then by 2. of Lemma 1.2.2, $U_{2,A_i} \cap \overline{U}_4^\perp$ is 1-dimensional. Let u_p be a generator of this intersection. Then there is a unique point q in \overline{C}_i such that $\langle u_q \rangle = \langle \mu_i \rangle$ (in other words u_q and μ_i are colinear). It is not in $\overline{C}_1 \cap \overline{C}_2$.*

Proof. Let $p = \overline{U}_4 = U_{2,A_1} \oplus U_{2,A_2}$ be a point in \overline{C}_1 . Then by 2. of Lemma 1.2.2 we can write $U_{2,A_1} = \langle \mu_1, u \rangle$, thus we can write $u_p = \alpha\mu_1 + \beta u$ for some $\alpha, \beta \in \mathbb{C}$ such that $(\alpha, \beta) \neq (0, 0)$. Let us also write $U_{2,A_2} = \langle y_1, y_2 \rangle$. Then u_p and μ_1 are colinear if and only if $\beta = 0$.

Let us assume that $\beta = 0$. As a result, $\mu_1 \in \overline{U}_4^\perp$ that is to say μ_1 is μ -orthogonal to y_1 and y_2 . Since μ is a non-degenerate pairing between A_1 and A_2 then there is only one possibility for \overline{U}_{2,A_2} , namely K_1 . Moreover, since \overline{U}_4 is a point in S_2 we have that \overline{U}_4 is λ' isotropic by 10. of Theorem 1.2.1, and the linear form $\lambda'(\mu_1, y_1, y_2, \cdot)$ is zero on A_2 by 3. of section 1.1. Thus there are two possibilities :

- There exists one $v \in A_1/\mathbb{C}\mu_1$ such that $\lambda'(\mu_1, y_1, y_2, v) \neq 0$ and u must be a nonzero vector in $\text{Ker}(\lambda'(\mu_1, y_1, y_2, \cdot)) \cap A_1 / \langle \mu_1 \rangle$, which is one dimensional. Moreover in such a case, by a reasoning similar to the one in the proof of Lemma 3.1.10, $U = \langle \mu_1, u, K_1 \rangle$ is a point in \overline{C}_1 with $\mu_1 \in U^\perp$. So in this case there is exactly one point in \overline{C}_1 such that $\mu_1 \in \overline{U}_4^\perp$, and by Lemma 3.1.8, since we assumed (3.6), this point is actually in $\overline{C}_1 - \overline{C}_2$.
- $\lambda'(\mu_1, y_1, y_2, \cdot)$ is identically zero. Then by the same reasoning as in the proof of Lemma 3.1.10, each $u \in \mathbb{P}(A_1 / \langle \mu_1 \rangle)$ gives a point $\langle \mu_1, u \rangle \oplus K_1 \in \overline{C}_1$. So $\mathbb{P}(A_1/\mathbb{C}\mu_1) \subset \overline{C}_1$, but since \overline{C}_1 is rational by Proposition 1.2.2 this yields $\overline{C}_1 \cong \mathbb{P}(A_1/\mathbb{C}\mu_1)$. But then, for each $\overline{U}_4 \in \overline{C}_1$, we have that $\mu_2 \notin U_{2,A_2} = K_1$ by Lemma 3.1.8 and (3.6). Hence by Proposition 1.2.2, $\overline{C}_1 \cap \overline{C}_2 = \emptyset$. Since by (1.13), $\overline{h}_1 \overline{h}_2 = 2$, and since $\overline{C}_i \in |\overline{h}_i|$ by 2. of Lemma 1.2.2, this is a contradiction.

So only the first possibility can happen and there is exactly one point $p \in \overline{C}_1$ such that u_p and μ_1 are colinear. Since we assumed (3.6) by Lemma 3.1.8 this point has to be in $\overline{C}_1 - \overline{C}_2$. The exact same reasoning switching the A_i and the μ_i completes the proof. \square

Proposition 3.1.12. *If (3.6) holds, and either (A1) is not satisfied, or one of (B1), (B2) is not satisfied, then there exists a global section of $\mathcal{V}_S(\overline{h}_1 + \overline{h}_2)$ such that in the associated short exact sequence $0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{V}_S(\overline{h}_1 + \overline{h}_2) \rightarrow Q \rightarrow 0$, $Q \cong \det(\mathcal{V}_S(\overline{h}_1 + \overline{h}_2)) \otimes \mathcal{I}_K$ where $\text{supp}(K) = \{p_1, p_2, p_3, p_4\}$. Moreover, $\{p_1, p_2\} = \overline{C}_1 \cap \overline{C}_2$ and $p_3 \in \overline{C}_1 - \overline{C}_2$, $p_4 \in \overline{C}_2 - \overline{C}_1$.*

Proof. By (1.3) we have a short exact sequence

$$0 \longrightarrow \mathcal{O}(-\overline{h}_1) \oplus \mathcal{O}(-\overline{h}_2) \longrightarrow \mathcal{U}_{2,A_1} \oplus \mathcal{U}_{2,A_2} \longrightarrow \mathcal{V}_S \longrightarrow 0.$$

Moreover by Lemma 3.1.4, $\mathcal{V}_S(\overline{h}_1 + \overline{h}_2)$ has global sections which are induced by the ones of $\mathcal{U}_{2,A_1}(\overline{h}_1 + \overline{h}_2) \oplus \mathcal{U}_{2,A_2}(\overline{h}_1 + \overline{h}_2)$. So we need to find a "nice" global section, with which Q as the expected form.

To start with, since there are isomorphisms $\mathcal{U}_{2,A_i} \cong \mathcal{U}_{2,A_i}^\vee(-\overline{h}_i)$ we get the short exact sequence :

$$0 \longrightarrow \mathcal{O}(\overline{h}_2) \oplus \mathcal{O}(\overline{h}_1) \longrightarrow \mathcal{U}_{2,A_1}^\vee(\overline{h}_2) \oplus \mathcal{U}_{2,A_2}^\vee(\overline{h}_1) \longrightarrow \mathcal{V}_S(\overline{h}_1 + \overline{h}_2) \longrightarrow 0. \quad (3.7)$$

So a section $s : \mathcal{O}_S \rightarrow \mathcal{V}_S(\overline{h}_1 + \overline{h}_2)$ is induced by a section of $\mathcal{U}_{2,A_1}^\vee(\overline{h}_2) \oplus \mathcal{U}_{2,A_2}^\vee(\overline{h}_1)$. Moreover, a section of \mathcal{U}_{2,A_i}^\vee corresponds to a 1-form on A_i (which is 0 on the other A_j) and a section of $\mathcal{O}(\overline{h}_i)$ corresponds to a 2-form on A_i (see the introduction of [20] and page 218 of [10], with Lemma 4.2.6). So let us choose ϕ_i some 1-forms over A_i and ψ_i some 2-forms over A_i . Then the associated section of $\mathcal{V}_S(\overline{h}_1 + \overline{h}_2)$ corresponds to the restriction of $\phi_1 \cdot \psi_2 + \phi_2 \cdot \psi_1$ to $\overline{U}_S := \overline{U}_4 / \overline{U}_4^\perp$ at any point \overline{U}_4 of S_2 , since by the proof of Proposition 4.6 in [20], \mathcal{V}_S is the rank 2 vector bundle parametrizing the $\overline{U}_4 / \overline{U}_4^\perp$ and in (1.3), $\mathcal{U}_{2,A_1} \oplus \mathcal{U}_{2,A_2}$ parametrizes all the $\overline{U}_4 \in S_2$ (and

$\mathcal{O}(-\bar{h}_1) \oplus \mathcal{O}(-\bar{h}_2)$ the associated \bar{U}_4^\perp .

We would like to show that except for the four points p_1, p_2, p_3 and p_4 as described in the proposition, the fiber of Q is a one dimensional vector space, and that for these points the fiber is 0. Then we should have $Q \cong \mathcal{L} \otimes \mathcal{I}_{p_1, p_2, p_3, p_4}$ and \mathcal{L} has to be the determinant of Q , which is also the determinant of $\mathcal{V}_S(\bar{h}_1 + \bar{h}_2)$ since $\det(\mathcal{O}) = \mathcal{O}$.

To understand what the fibers of Q can be, we use the two following examples :

- On $\text{Gr}(2, A_i)$, there is a short exact sequence $0 \rightarrow \mathcal{O}_{\text{Gr}(2, A_i)} \rightarrow \mathcal{U}_{2, A_i}^\vee \rightarrow R \rightarrow 0$. Here the fiber of R at a point U_2 is $U_2/\text{Ker}(\phi)$ where ϕ is the 1-form corresponding to the global section $\mathcal{O}_{\text{Gr}(2, A_i)} \rightarrow \mathcal{U}_{2, A_i}^\vee$ and $R \cong \mathcal{O}(\bar{h}_i) \otimes \mathcal{I}_{\text{Ker}(\phi)}$.
- On S_2 we have the short exact sequence $0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}(\bar{h}_i) \rightarrow \mathcal{O}_{\bar{C}_i}(\bar{h}_i) \rightarrow 0$ where the global section $\mathcal{O}_S \rightarrow \mathcal{O}(\bar{h}_i)$ corresponds to a 2-form ψ and $\psi(\wedge^2 \bar{U}_{2, A_i}) = 0$ exactly on \bar{C}_i , if we choose the kernel vector of ψ to be μ_i , by Proposition 1.2.2. For a different choice of ψ we would get another curve $\bar{C}'_i \in |\bar{h}_i|$.

Noticing that since \bar{U}_S is 2-dimensional at any point of S_2 , $\psi_i(\wedge^2 \bar{U}_S) \in \mathbb{C}$, we can then define the fiber of Q at a point \bar{U}_4 as $\bar{U}_4/\text{Ker}(\chi')$ where $\chi' : v \in \bar{U}_S \mapsto \phi_1(v)\psi_2(w_1, w_2) + \phi_2(v)\psi_1(w_1, w_2)$ where $\bar{U}_S = \langle w_1, w_2 \rangle$.

Let us now look at what the fibers of Q at p can be, depending on the position of p with respect to the curves \bar{C}_1 and \bar{C}_2 .

1. Let us first assume that we look at a point \bar{U}_4 which is neither on \bar{C}_1 nor on \bar{C}_2 . Then by Proposition 1.2.2 we can assume $\bar{U}_S = \langle x_1 + y_1, x_2 + y_2 \rangle$ with $x_i \in A_1$ and $y_i \in A_2$ non colinear. Here we will make two assumptions :
(H1) We assume that for at least one $i \in \{1, 2\}$, $\text{Ker}(\phi_i) \neq \bar{U}_{2, A_i}$ for any \bar{U}_S .
(H2) We assume that the kernel vectors v_i of the ψ_i (namely generators of $\text{Ker}(\psi_i(\cdot, A_i))$) satisfy $v_i \notin \bar{U}_{2, A_i}$.
Actually we don't need (H1) and (H2) to hold for exactly all the points in S_2 and we will see later how to make them hold when we need it.

By the assumption (H1) we have that one at least of $\text{Ker}(\phi_1) \cap \langle x_1, x_2 \rangle$ and $\text{Ker}(\phi_2) \cap \langle y_1, y_2 \rangle$ is one dimensional. Thus the fiber of Q is 0 only if \bar{U}_S is ψ_1 or ψ_2 isotropic. On the other hand, since y_1 and y_2 are non colinear, $U_{2, A_2} = \langle y_1, y_2 \rangle$ and up to a nonzero scalar, $\psi_2(\wedge^2 A_2) = \psi_2(y_1, y_2) \neq 0$ by (H2). Similarly, $\psi_1(\wedge^2 A_1) \neq 0$. Therefore χ' is a nonzero one form, and its kernel (and thus the fiber of Q at \bar{U}_4) is a one dimensional vector space.

2. If $\bar{U}_4 \in \bar{C}_1 \cap \bar{C}_2$ then $\bar{U}_S = \langle x_1, y_2 \rangle$ and $\bar{U}_4^\perp = \langle x_2, y_1 \rangle$ with $x_i \in A_1$ and $y_i \in A_2$. In this case the fibers of Q are 0 because both ψ_i vanish on \bar{U}_S . Moreover $\bar{C}_1 \cap \bar{C}_2 = \{p_1, p_2\}$ since by 2. of Lemma 1.2.2, $\bar{C}_i \in |\bar{h}_i|$ and by (1.13) the intersection of the \bar{h}_i is 2.
3. We assume now that \bar{U}_4 is on \bar{C}_1 but not on \bar{C}_2 (the case for $\bar{U}_4 \in \bar{C}_2 - \bar{C}_1$ is similar). Then $\bar{U}_S = \langle x_1 + y_1, y_2 \rangle$ with $x_1 \in A_1$ and $y_i \in A_i$ non colinear. In this case $\phi_1 \cdot \psi_2 + \phi_2 \cdot \psi_1$ is just $\phi_1 \cdot \psi_2$ since $\psi_1(\wedge^2 \bar{U}_S) = 0$. By the same reasoning as before and (H2) we must have $\psi_2(\wedge^2 \bar{U}_S) \neq 0$. Therefore, if $\phi_1(x_1) \neq 0$ the fiber of Q is a one dimensional vector space (namely $\bar{U}_S/(\bar{U}_S \cap \text{Ker}(\phi_1))$), otherwise it is 0. We would like that $\phi_1(x_1) = 0$ only at exactly one point in $\bar{C}_1 - \bar{C}_2$ and $\phi_2(y_1) = 0$ at exactly one point in $\bar{C}_2 - \bar{C}_1$. These would correspond to p_3 and p_4 .
By Proposition 1.2.2, \bar{U}_4 is in \bar{C}_1 if and only if $\mu_1 \in \bar{U}_{2, A_1}$, therefore we can write $x_1 = \alpha\mu_1 + \beta u$ where $\bar{U}_{2, A_1} = \langle \mu_1, u \rangle$ and $(\alpha, \beta) \in \mathbb{C}^2 - \{(0, 0)\}$. Since $U_{2, A_1} = \langle \mu_1, u \rangle$, we have that $\alpha = 0$ if and only if $\mu_1 \in \bar{U}_4^\perp$. By Lemma 3.1.11, it happens for exactly one point $p_3 \in \bar{C}_1 - \bar{C}_2$. So we have two possibilities :

- (a) (A1) doesn't hold. Then we can take $\phi_i = \mu_i^*$. Indeed by what we have seen they will vanish at exactly one point on $\bar{C}_1 - \bar{C}_2$, respectively $\bar{C}_2 - \bar{C}_1$. We still need to check that (H1) holds. But since (A1) doesn't hold, $\text{Ker}(\phi_1) \oplus \text{Ker}(\phi_2) = A_1/\langle \mu_1 \rangle \oplus A_2/\langle \mu_2 \rangle \notin S$, so (H1) holds when we need it (namely outside of $\bar{C}_1 \cup \bar{C}_2$). Indeed on $\bar{C}_1 \cap \bar{C}_2$ the ϕ_i don't matter, and on $\bar{C}_2 - \bar{C}_1$ since ψ_2 vanishes

again ϕ_1 doesn't matter. Finally we have seen that on $\overline{C}_1 - \overline{C}_2$, $\phi_1 = \mu_1^*$ behaves as we want so we actually don't need (H1) on \overline{C}_1 . For ϕ_2 it is similar.

- (b) (A1) holds but at least one of (B1), (B2) doesn't hold. Let us assume that (B1) doesn't hold. We cannot take $\phi_i = \mu_i^*$ as in the previous case, because (A1) would contradict (H1) in a case when we need it to hold (namely outside of the \overline{C}_i). However, since (B1) doesn't hold there is at most one point p in \overline{C}_1 such that $\overline{U}_S/A_2 = \langle \mu_1 \rangle$. If such a p exists, let $u \in A_1 / \langle \mu_1 \rangle$ be such that U_{2,A_1} associated to this point is $\langle u, \mu_1 \rangle$, otherwise let $u \in A_1 / \langle \mu_1 \rangle$ be any nonzero vector. Finally, let $v \in A_1$ be such that $\langle u, v \rangle = A_1 / \langle \mu_1 \rangle$. Then we can take $\phi_1 = v^*$. Indeed by Lemma 3.1.10, to each $(a, b) \neq (0, 0)$ corresponds a point in \overline{C}_1 such that $U_{2,A_1} = \langle \mu_1, au + bv \rangle$. Moreover since (B1) doesn't hold there is only one case in which v^* vanishes on \overline{U}_S in \overline{C}_1 , namely when we are looking at p if it exists, or when we look at the point associated to $(1, 0)$ otherwise.

We now need to check that this is compatible with (H1). Since $\phi_1 = v^*$, $\mu_1 \in \text{Ker}(\phi_1)$, so if $U_{2,A_1} = \text{Ker}(\phi_1)$ then we are in \overline{C}_1 , but we have already seen that in \overline{C}_1 , ϕ_1 vanishes exactly at one point outside of \overline{C}_2 , which is what we wanted, since what happens on $\overline{C}_1 \cap \overline{C}_2$ only depends on the ψ_i . So (H1) holds when we need it and we can now choose $\phi_2 = \mu_2^*$.

To satisfy (H2) we can choose the kernel vectors of the ψ_i to be the μ_i since if $\mu_i \in \overline{U}_{2,A_i}$ then we are necessarily in C_i and $\psi_i(\wedge^2 \overline{U}_S) = 0$ anyway.

Then we have that the fibers of Q are 0 in $\overline{C}_1 \cap \overline{C}_2 = \{p_1, p_2\}$ and one point of each $\overline{C}_i - (\overline{C}_1 \cap \overline{C}_2)$, let us call them p_3 and p_4 . So $Q \cong \mathcal{L} \otimes \mathcal{I}_K$ where \mathcal{L} is the determinant of Q and $\text{supp}(K) = \{p_1, p_2, p_3, p_4\}$. \square

Remark 3.1.13. In Proposition 3.1.12, $K \neq \{p_1, p_2, p_3, p_4\}$. Indeed we know from the proof of Lemma 3.1.5 that $\text{ch}(Q) = \text{ch}(\det(\mathcal{V}_S(\overline{h}_1 + \overline{h}_2)) \cdot (1, 0, -8))$. One possibility would be that all the p_i in the support of K have multiplicity 2.

Corollary 3.1.14. If we keep the same notation as in Proposition 3.1.12 there is a surface in Z_3 with class \overline{h} which is isomorphic to the blow-up of S_2 in K . We are going to call this surface \hat{S} .

Proof. We have seen in the proof of Lemma 3.1.5 that if there is an injection $0 \rightarrow \mathcal{O}_{S_2} \rightarrow \mathcal{V}_S(\overline{h}_1 + \overline{h}_2)$ then its quotient Q has determinant $\det(\mathcal{V}_S(\overline{h}_1 + \overline{h}_2)) = \mathcal{O}_{S_2}(2\overline{h}_1 + 2\overline{h}_2)$. Hence, by Proposition 3.1.12, there is a map $\phi : \mathcal{V}_S \rightarrow \mathcal{O}_{S_2}(\overline{h}_1 + \overline{h}_2)$ which is surjective outside of K . We have also seen that \mathcal{V}_S is rank 2, and $Z_3 = \mathbb{P}_S(\mathcal{V}_S)$ by 1. of Proposition 1.2.2. We can thus apply Lemma 1.4.1 to ϕ , with $D_1(\phi) = K$ and $D_2(\phi) = \emptyset$. As result, the zero locus of the global section of $\mathcal{O}_{Z_3}(\overline{h}_1 + \overline{h}_2 + V_Z)$ induced by ϕ is isomorphic to the blow-up of S_2 in K . By construction this blow-up is embedded in Z_3 and has class $\overline{h}_1 + \overline{h}_2 + V_Z = \overline{h}$ (the last equality comes from (1.11)). \square

Lemma 3.1.15. To avoid (A1) we can assume

$$M_3M_4K_3^2 + M_1M_6K_1^2 + M_2M_5K_2^2 \neq 0. \quad (3.8)$$

Proof. We have seen in Lemma 3.6 that we can choose $\mu_1 = M_6K_1x_1 + M_5K_2x_2 + M_4K_3x_3$ and $\mu_2 = M_3K_3x_4 + M_2K_2x_5 + M_1K_1x_6$. Thus we can choose basis of $A_i / \langle \mu_i \rangle : A_1 / \langle \mu_1 \rangle = \langle M_4K_3x_1 - M_6K_1x_3, M_5K_2x_1 - M_6K_1x_2 \rangle$ and $A_2 / \langle \mu_2 \rangle = \langle M_1K_1x_4 - M_3K_3x_6, M_2K_2x_4 - M_3K_3x_5 \rangle$. But using 3. of section 1.1 one gets

$$\begin{aligned} & \lambda'(M_4K_3x_1 - M_6K_1x_3, M_5K_2x_1 - M_6K_1x_2, M_1K_1x_4 - M_3K_3x_6, M_2K_2x_4 - M_3K_3x_5) \\ &= M_4K_3(-M_6K_1)M_3K_3(-M_3K_3) + (-M_5K_2)(-M_6K_1)M_2K_2M_3K_3 + (-M_6K_1)M_6K_1M_1K_1(-M_3K_3) \\ &= (M_6K_1M_3K_3)(M_4K_3M_3K_3 + M_5K_2M_2K_2 + M_6K_1M_1K_1) \end{aligned}$$

Since we have assumed in 4. of section 1.1 that all the M_i and K_i are non zero, $M_6K_1M_3K_3 \neq 0$.

So $\lambda'(A_1 / \langle \mu_1 \rangle, A_2 / \langle \mu_2 \rangle) = 0$ if and only if $M_3M_4K_3^2 + M_1M_6K_1^2 + M_2M_5K_2^2 = 0$. Hence if (3.8) holds then $A_1 / \langle \mu_1 \rangle \oplus A_2 / \langle \mu_2 \rangle$ cannot be a point in S_2 . \square

3.2 Blow-up of Z_3

We would like now to use this new surface \hat{S} in Z_3 defined in Corollary 3.1.14 to induce a surface in \tilde{X}_4 . For this we need a section of $\bar{\pi}$ or, as we did with S_2 , a blow-up of Z_3 embedded in \tilde{X}_4 . But as shown in diagram (1.2), we have that $\bar{\pi}^{-1}(Z_3) = \bar{E}_4$ in \tilde{X}_5 with $\bar{E}_4 = \mathbb{P}_Z(\mathcal{V}_Z)$ where the vector bundle \mathcal{V}_Z is defined by (1.6) in Proposition 1.3.1. So we are going to proceed as in the previous subsection, and use Lemma 1.4.1 :

Proposition 3.2.1. *It is equivalent to be given a section of $\bar{\pi} : \bar{E}_4 \rightarrow Z_3$ or a short exact sequence $0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{V}_Z^\vee \rightarrow \mathcal{L}_2 \rightarrow 0$ where the \mathcal{L}_i are line bundles on Z_3 and \mathcal{V}_Z is defined by (1.6).*

Proof. By Proposition 1.3.1 we have that $\bar{E}_4 = \mathbb{P}_Z(\mathcal{V}_Z)$. Moreover (1.6) yields that \mathcal{V}_Z is the kernel of a map between a rank $1 + 3 = 4$ vector bundle and a rank 2 vector bundle, so $\text{rank}(\mathcal{V}_Z) = 2$. We can then use the exact same reasoning as in the proof of Proposition 3.1.1, except for one thing we need to be more careful at : in the proof of Proposition 3.1.1 we use Exercise 7.8 of II.7 in [12] but in Proposition 7.11 of II.7 in [12] we see that R. Hartshorne has taken a different definition of $\mathcal{O}(1)$ and $\mathbb{P}_X(\mathcal{E})$ where \mathcal{E} is a vector bundle on a variety X . Indeed in the definition of Hartshorne, if $p : \mathbb{P}_X(\mathcal{E}) \rightarrow X$ then $p_*\mathcal{O}(1) = \mathcal{E}$ while in section 1.1 we define $\mathcal{O}(1)$ to be such that $p_*\mathcal{O}(1) = \mathcal{E}^\vee$. So here we need to consider \mathcal{V}_Z^\vee instead of \mathcal{V}_Z . In the case of Proposition 3.1.1 this was not a problem since by 1. of Lemma 1.2.2, $\mathcal{V}_S \cong \mathcal{V}_S^\vee$. \square

Lemma 3.2.2. *The dimension of $H^0(\mathcal{V}_Z^\vee)$ is at least 6 so \mathcal{V}_Z^\vee has non-zero global sections.*

Proof. By (1.6) we have a short exact sequence :

$$0 \rightarrow \mathcal{O}_{Z_3}(\bar{h}_1) \oplus \mathcal{O}_{Z_3}(\bar{h}_2) \rightarrow \bigwedge^3 \bar{\mathcal{U}}_3^\vee \oplus \bigwedge^2 \bar{\mathcal{U}}_3^\vee \rightarrow \mathcal{V}_Z^\vee \rightarrow 0. \quad (3.9)$$

We have also seen in the proof of Lemma 3.1.4 that for $i \in \{1, 2\}$, the dimension of $H^j(\mathcal{O}_{S_2}(\bar{h}_i))$ is $3\delta_{0,j}$. Moreover, $H^*(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h}_i)) = H^*(\mathcal{O}_{S_2}, \sigma_*\sigma^*\mathcal{O}_{S_2}(\bar{h}_i)) = H^*(\mathcal{O}_{S_2}, \mathcal{O}_{S_2}(\bar{h}_i))$ by the projection formula and since $\sigma_*\mathcal{O}_{Z_3} = \mathcal{O}_{S_2}$ (as we will see in Corollary 4.5.5). Thus $\dim(H^*(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h}_i))) = 3\delta_{0,*}$. Furthermore, since by (1.11) we have $\bar{h} = V_Z + \bar{h}_1 + \bar{h}_2$, the short exact sequence $0 \rightarrow \mathcal{O}_{Z_3}(-V_Z) \rightarrow \mathcal{O}_{Z_3} \rightarrow \mathcal{O}_{S_2} \rightarrow 0$ induces a short exact sequence :

$$0 \rightarrow \mathcal{O}_{Z_3}(\bar{h}_1 + \bar{h}_2) \rightarrow \mathcal{O}_{Z_3}(\bar{h}) \rightarrow \mathcal{O}_{S_2}(\bar{h}) \rightarrow 0. \quad (3.10)$$

Moreover twisting the resolution (3.1) by $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(\bar{h}_1 + \bar{h}_2)$ one gets a long exact sequence :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, -1) \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, 0)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 1) \rightarrow \mathcal{O}_{S_2}(\bar{h}_1 + \bar{h}_2) \rightarrow 0. \quad (3.11)$$

And similarly to what we did in the proof of Lemma 3.1.3, the Künneth formula yields :

- $H^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, -1)) = 0$.
- $H^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}) = \mathbb{C}\delta_{*,0}$ so $\dim(H^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}^{\oplus 2})) = 2\delta_{*,0}$.
- $H^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 1)) = 3^2\delta_{*,0} = 9\delta_{*,0}$.

Splitting now (3.11) into short exact sequences gives the two following exact sequences :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, -1) \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, 0)^{\oplus 2} \rightarrow K \rightarrow 0 \quad (3.12)$$

$$0 \rightarrow K \rightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 1) \rightarrow \mathcal{O}_{S_2}(\bar{h}_1 + \bar{h}_2) \rightarrow 0. \quad (3.13)$$

Since $H^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-1, -1)) = 0$ the long exact sequence in cohomology induced by (3.12) gives $H^*(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, 0)^{\oplus 2}) = H^*(K)$. Therefore the long exact sequence in cohomology induced by (3.13) gives an exact sequence :

$$0 \rightarrow \mathbb{C}^2 \rightarrow \mathbb{C}^9 \rightarrow H^0(\mathcal{O}_{S_2}(\bar{h}_1 + \bar{h}_2)) \rightarrow 0$$

and as a result $\dim(H^*(\mathcal{O}_{S_2}(\bar{h}_1 + \bar{h}_2))) = 7\delta_{*,0}$.

Now the long exact sequence in cohomology induced by (3.10) gives the short exact sequence :

$$0 \rightarrow \mathbb{C}^7 \rightarrow H^0(\mathcal{O}_{Z_3}(\bar{h})) \rightarrow H^0(\mathcal{O}_{S_2}(\bar{h})) \rightarrow 0 = H^1(\mathcal{O}_{Z_3}(\bar{h}_1 + \bar{h}_2))$$

since $H^*(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h}_1 + \bar{h}_2)) = H^*(\mathcal{O}_{S_2}, \mathcal{O}_{S_2}(\bar{h}_1 + \bar{h}_2))$. Moreover,

$$\begin{aligned} H^*(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})) &= H^*(\mathcal{O}_{S_2}, \sigma_*(\mathcal{O}_{Z_3}(V_Z) \otimes \sigma^*\mathcal{O}_{S_2}(\bar{h}_1 + \bar{h}_2))) \\ &= H^*(\mathcal{O}_{S_2}, \mathcal{V}_S^\vee(\bar{h}_1 + \bar{h}_2)) \\ &= H^*(\mathcal{O}_{S_2}, \mathcal{V}_S(\bar{h}_1 + \bar{h}_2)) \end{aligned}$$

using the definition of V_Z , the projection formula and 1. of Proposition 1.2.2. By Lemma 3.1.4, we then know that $\dim(H^0(\mathcal{O}_{Z_3}(\bar{h}))) = 12$. Finally the long exact sequence in cohomology given by (3.9) gives a short exact sequence :

$$0 \rightarrow H^0(\mathcal{O}_{Z_3}(\bar{h}_1)) \oplus H^0(\mathcal{O}_{Z_3}(\bar{h}_2)) \rightarrow H^0(\mathcal{O}_{Z_3}(\bar{h})) \oplus H^0(\bigwedge^2 \bar{\mathcal{U}}_3^\vee) \rightarrow H^0(\mathcal{V}_Z^\vee) \rightarrow 0$$

since we have seen that $H^1(\mathcal{O}_{Z_3}(\bar{h}_1)) = H^1(\mathcal{O}_{Z_3}(\bar{h}_2)) = 0$. Hence

$$\dim(H^0(\mathcal{V}_Z^\vee)) \geq \dim(H^0(\mathcal{O}_{Z_3}(\bar{h}))) - \dim(H^0(\mathcal{O}_{Z_3}(\bar{h}_1))) - \dim(H^0(\mathcal{O}_{Z_3}(\bar{h}_2))) = 12 - 3 - 3 = 6.$$

□

Lemma 3.2.3. *The Chern character of \mathcal{V}_Z^\vee is*

$$\text{ch}(\mathcal{V}_Z^\vee) = (2, 3\bar{h} - \bar{h}_1 - \bar{h}_2, \frac{3}{2}\bar{h}_1\bar{h}_2 + 2V_Z(\bar{h}_1 + \bar{h}_2), 3).$$

Proof. By (1.6) we have

$$0 \rightarrow \mathcal{V}_Z \rightarrow \bigwedge^3 \bar{\mathcal{U}}_3 \oplus \bigwedge^2 \bar{\mathcal{U}}_3 \rightarrow \mathcal{O}(-\bar{h}_1) \oplus \mathcal{O}(-\bar{h}_2) \rightarrow 0$$

on Z_3 . We already know that $\bigwedge^3 \bar{\mathcal{U}}_3^\vee \cong \mathcal{O}(\bar{h})$ (see Lemma 4.2.9 below) but we need to compute $\text{ch}(\bigwedge^2 \bar{\mathcal{U}}_3^\vee)$. By (1.7) we have

$$0 \rightarrow \mathcal{O}(-\bar{h}_1) \oplus \mathcal{O}(-\bar{h}_2) \rightarrow \bar{\mathcal{U}}_3 \rightarrow \mathcal{O}(-V_Z) \rightarrow 0$$

on Z_3 . Thus, using (1.11),

$$\text{ch}(\bar{\mathcal{U}}_3^\vee) = (1, V_Z, \frac{-\bar{h}_1^2 - \bar{h}_2^2}{2}, -\frac{1}{3}) + (1, \bar{h}_1, \frac{\bar{h}_1^2}{2}, 0) + (1, \bar{h}_2, \frac{\bar{h}_2^2}{2}, 0) = (3, \bar{h}, 0, -\frac{1}{3}).$$

By definition (see for instance page 430 of [12]),

$$\begin{aligned} c_t(\bar{\mathcal{U}}_3^\vee) &= (1 + a_1 t)(1 + a_2 t)(1 + a_3 t) \\ &= 1 + (a_1 + a_2 + a_3)t + (a_1 a_2 + a_1 a_3 + a_2 a_3)t^2 + (a_1 a_2 a_3)t^3. \end{aligned}$$

Thus :

$$\begin{aligned} c_1(\bar{\mathcal{U}}_3^\vee) &= a_1 + a_2 + a_3 = \bar{h} \\ c_2(\bar{\mathcal{U}}_3^\vee) &= a_1 a_2 + a_1 a_3 + a_2 a_3 = \frac{c_1(\bar{\mathcal{U}}_3^\vee)^2}{2} - \text{ch}_2(\bar{\mathcal{U}}_3^\vee) = \frac{\bar{h}^2}{2} - 0 = \frac{\bar{h}^2}{2} \\ c_3(\bar{\mathcal{U}}_3^\vee) &= 2(\text{ch}_3(\bar{\mathcal{U}}_3^\vee) - \frac{c_1(\bar{\mathcal{U}}_3^\vee)^3}{6} + \frac{c_1(\bar{\mathcal{U}}_3^\vee)c_2(\bar{\mathcal{U}}_3^\vee)}{2}) = 2(-\frac{1}{3} - \frac{\bar{h}^3}{6} + \frac{\bar{h}^3}{4}) = -\frac{2}{3} + \frac{\bar{h}^3}{6}. \end{aligned}$$

Furthermore, by (1.11), (1.18) and (1.13), one gets :

$$\begin{aligned}\bar{h}^2 &= (V_Z + \bar{h}_1 + \bar{h}_2)^2 = V_Z^2 + \bar{h}_1^2 + \bar{h}_2^2 + 2V_Z(\bar{h}_1 + \bar{h}_2) + 2\bar{h}_1\bar{h}_2 \\ &= -\bar{h}_1\bar{h}_2 + \bar{h}_1^2 + \bar{h}_2^2 + 2V_Z(\bar{h}_1 + \bar{h}_2) + 2\bar{h}_1\bar{h}_2 \\ &= 2V_Z(\bar{h}_1 + \bar{h}_2) + 2\bar{h}_1\bar{h}_2.\end{aligned}$$

As a result, $c_2(\bar{U}_3^\vee) = \frac{\bar{h}^2}{2} = V_Z(\bar{h}_1 + \bar{h}_2) + \bar{h}_1\bar{h}_2$. Then

$$\begin{aligned}\bar{h}^3 &= (V_Z + \bar{h}_1 + \bar{h}_2)(2V_Z(\bar{h}_1 + \bar{h}_2) + 2\bar{h}_1\bar{h}_2) \\ &= 2V_Z(\bar{h}_1 + \bar{h}_2)^2 + 2V_Z\bar{h}_1\bar{h}_2 \\ &= 2.6 + 2.2 = 12 + 4 = 16.\end{aligned}$$

Hence, $c_3(\bar{U}_3^\vee) = -\frac{2}{3} + \frac{\bar{h}^3}{6} = -\frac{2}{3} + \frac{8}{3} = \frac{6}{3} = 2$.

Now by the formula for $c_t(\bigwedge^p \bar{U}_3^\vee)$ page 430 of [12] we get

$$\begin{aligned}c_t(\bigwedge^2 \bar{U}_3^\vee) &= (1 + (a_1 + a_2)t)(1 + (a_1 + a_3)t)(1 + (a_2 + a_3)t) \\ &= 1 + (2a_1 + 2a_2 + 2a_3)t + (a_1^2 + a_2^2 + a_3^2 + 3a_1a_2 + 3a_1a_3 + 3a_2a_3)t^2 + \\ &\quad + (a_1^2a_2 + a_1^2a_3 + a_2^2a_1 + a_2^2a_3 + a_3^2a_1 + a_3^2a_2 + 2a_1a_2a_3)t^3 \\ &= 1 + 2c_1(\bar{U}_3^\vee)t + (c_1(\bar{U}_3^\vee)^2 + c_2(\bar{U}_3^\vee))t^2 + (c_1(\bar{U}_3^\vee)c_2(\bar{U}_3^\vee) - c_3(\bar{U}_3^\vee))t^3.\end{aligned}$$

So

$$\begin{aligned}c_1(\bigwedge^2 \bar{U}_3^\vee) &= 2\bar{h} \\ c_2(\bigwedge^2 \bar{U}_3^\vee) &= \bar{h}^2 + \frac{\bar{h}^2}{2} = \frac{3}{2}\bar{h}^2 \\ c_3(\bigwedge^2 \bar{U}_3^\vee) &= \frac{\bar{h}^3}{2} - 2 = 8 - 2 = 6 \\ \text{ch}_0(\bigwedge^2 \bar{U}_3^\vee) &= \text{rk}(\bigwedge^2 \bar{U}_3^\vee) = 3 \\ \text{ch}_1(\bigwedge^2 \bar{U}_3^\vee) &= c_1(\bigwedge^2 \bar{U}_3^\vee) = 2\bar{h} \\ \text{ch}_2(\bigwedge^2 \bar{U}_3^\vee) &= \frac{c_1(\bigwedge^2 \bar{U}_3^\vee)^2}{2} - c_2(\bigwedge^2 \bar{U}_3^\vee) = 2\bar{h}^2 - \frac{3}{2}\bar{h}^2 = \frac{\bar{h}^2}{2} = \bar{h}_1\bar{h}_2 + \bar{h}_1V_Z + \bar{h}_2V_Z \\ \text{ch}_3(\bigwedge^2 \bar{U}_3^\vee) &= \frac{c_1(\bigwedge^2 \bar{U}_3^\vee)^3}{6} - \frac{c_1(\bigwedge^2 \bar{U}_3^\vee)c_2(\bigwedge^2 \bar{U}_3^\vee)}{2} + \frac{c_3(\bigwedge^2 \bar{U}_3^\vee)}{2} = \frac{4}{3}\bar{h}^3 - \frac{3}{2}\bar{h}^3 + 3 = \frac{64}{3} - 24 + 3 = \frac{64 - 63}{3} = \frac{1}{3}.\end{aligned}$$

Here we used (1.13) and (1.11) again. So finally $\text{ch}(\bigwedge^2 \bar{U}_3^\vee) = (3, 2\bar{h}, \bar{h}_1\bar{h}_2 + \bar{h}_1V_Z + \bar{h}_2V_Z, \frac{1}{3})$.

Thus, using (1.6) and (1.11) we get

$$\begin{aligned}\text{ch}(\mathcal{V}_Z^\vee) &= \text{ch}(\mathcal{O}(\bar{h})) + \text{ch}(\bigwedge^2 \bar{U}_3^\vee) - \text{ch}(\mathcal{O}(\bar{h}_1)) - \text{ch}(\mathcal{O}(\bar{h}_2)) \\ &= (1, \bar{h}, \frac{\bar{h}^2}{2}, \frac{\bar{h}^3}{6}) + (3, 2\bar{h}, \bar{h}_1\bar{h}_2 + \bar{h}_1V_Z + \bar{h}_2V_Z, \frac{1}{3}) - (1, \bar{h}_1, \frac{\bar{h}_1^2}{2}, 0) - (1, \bar{h}_2, \frac{\bar{h}_2^2}{2}, 0).\end{aligned}$$

So finally, since $\bar{h}_1^2 + \bar{h}_2^2 = \bar{h}_1\bar{h}_2$,

$$\begin{aligned} \text{ch}(\mathcal{V}_Z^\vee) &= (1, \bar{h}, \bar{h}_1\bar{h}_2 + V_Z(\bar{h}_1 + \bar{h}_2), \frac{8}{3}) + (3, 2\bar{h}, \bar{h}_1\bar{h}_2 + \bar{h}_1V_Z + \bar{h}_2V_Z, \frac{1}{3}) - (1, \bar{h}_1, \frac{\bar{h}_1^2}{2}, 0) - (1, \bar{h}_2, \frac{\bar{h}_2^2}{2}, 0) \\ &= (2, 3\bar{h} - \bar{h}_1 - \bar{h}_2, \frac{3}{2}\bar{h}_1\bar{h}_2 + 2V_Z(\bar{h}_1 + \bar{h}_2), 3). \end{aligned}$$

□

Remark 3.2.4. To compute $\text{ch}(\bigwedge^2 \bar{\mathcal{U}}_3^\vee)$ we could have used the following : since $\bar{\mathcal{U}}_3^\vee$ is of rank 2 there is an isomorphism $\bigwedge^2 \bar{\mathcal{U}}_3^\vee \cong \bar{\mathcal{U}}_3 \otimes \det(\bar{\mathcal{U}}_3^\vee) = \bar{\mathcal{U}}_3 \otimes \mathcal{O}_{Z_3}(\bar{h})$. Hence

$$\begin{aligned} \text{ch}(\bigwedge^2 \bar{\mathcal{U}}_3^\vee) &= (3, -\bar{h}, 0, \frac{1}{3}) \cdot (1, \bar{h}, \bar{h}_1\bar{h}_2 + V_Z(\bar{h}_1 + \bar{h}_2), \frac{8}{3}) \\ &= (3, 2\bar{h}, \bar{h}_1\bar{h}_2 + V_Z(\bar{h}_1 + \bar{h}_2), \frac{1}{3}). \end{aligned}$$

Corollary 3.2.5. If there is a short exact sequence $0 \rightarrow \mathcal{O}_{Z_3} \rightarrow \mathcal{V}_Z^\vee \rightarrow Q \rightarrow 0$ then $\text{ch}(Q) = (1, 3V_Z + 2(\bar{h}_1 + \bar{h}_2), \frac{3}{2}\bar{h}_1\bar{h}_2 + 2V_Z(\bar{h}_1 + \bar{h}_2), 3) = \text{ch}(\det(\mathcal{V}_Z^\vee)) \cdot (1, 0, -4V_Z(\bar{h}_1 + \bar{h}_2), 24)$.

Proof. If we have a short sequence like the one given just above then $\text{ch}(Q) = \text{ch}(\mathcal{V}_Z^\vee) - (1, 0, 0, 0)$ so we just have to use Lemma 3.2.3 to compute $\text{ch}(Q)$. Moreover still from Lemma 3.2.3 we get that $\det(\mathcal{V}_Z^\vee) = \mathcal{O}_{Z_3}(3V_Z + 2(\bar{h}_1 + \bar{h}_2))$ whose Chern character is : $(1, 3V_Z + 2(\bar{h}_1 + \bar{h}_2), \frac{3}{2}\bar{h}_1\bar{h}_2 + 6V_Z(\bar{h}_1 + \bar{h}_2), 27)$. Finally $(1, 3V_Z + 2(\bar{h}_1 + \bar{h}_2), \frac{3}{2}\bar{h}_1\bar{h}_2 + 6V_Z(\bar{h}_1 + \bar{h}_2), 27) \cdot (1, 0, y, x) = (1, 3V_Z + 2(\bar{h}_1 + \bar{h}_2), \frac{3}{2}\bar{h}_1\bar{h}_2 + 6V_Z(\bar{h}_1 + \bar{h}_2) + y, 27 + x + y(3V_Z + 2(\bar{h}_1 + \bar{h}_2)))$. So to get the Chern character of Q we need to take $y = -4V_Z(\bar{h}_1 + \bar{h}_2)$ and $x = 3 - 27 + 8.6 = 24$. □

As before this shows that a global section of \mathcal{V}_Z^\vee cannot induce a copy of Z_3 in \bar{E}_4 , yet since $4V_Z(\bar{h}_1 + \bar{h}_2)$ is the class of the curve Γ_1 in Z_3 (from (1.15)), it may be possible to find a blow-up of Z_3 in Γ_1 inside \bar{E}_4 . Indeed :

Proposition 3.2.6. There is a threefold in \bar{E}_4 with class $[V_{\bar{E}}]$ and which is isomorphic to the blow-up of Z_3 in the curve Γ_1 (defined in 5. of Theorem 1.2.3). We are going to call this threefold \tilde{Z} .

Proof. As defined in section 1.1, let ν be a global section of $\mathcal{O}_G(1)$ such that X_4 is the zero locus of the sections μ , λ and ν . Then by definition of h in Definition 1.3.3, $\nu \in |h|$ and in the proof of Theorem 5.1 in [20], one finds that $h|_{\bar{E}_4}$ corresponds to a global section of \mathcal{V}_Z^\vee which for general ν vanishes on the curve Γ_1 (where Γ_1 is the curve defined in Theorem 1.2.3 for $X_4^{\lambda, \mu, \nu}$). This global section induces a morphism : $\mathcal{V}_Z \xrightarrow{\nu^\vee} \mathcal{O}_{Z_3}$. Since $\text{rank}(\mathcal{V}_Z) = 2$ and $\text{rank}(\mathcal{O}_{Z_3}) = 1$, with the notations of Lemma 1.4.1 we have that $D_k(\nu^\vee) = \emptyset$ for $k \geq 2$. Moreover, since the section of \mathcal{V}_Z^\vee induced by ν vanishes exactly on Γ_1 we have $D_1(\nu^\vee) = \Gamma_1$. Finally, $\bar{E}_4 = \mathbb{P}_Z(\mathcal{V}_Z)$ from Proposition 1.3.1 so we can apply Lemma 1.4.1 to $\mathcal{V}_Z \xrightarrow{\nu^\vee} \mathcal{O}_{Z_3}$ which yields that the zero locus of the global section of $\mathcal{O}_{\bar{E}_4}(V_{\bar{E}})$ induced by ν^\vee is isomorphic to the blow-up of Z_3 in the curve Γ_1 . □

Corollary 3.2.7. There is a blow-up of \hat{S} in the points of $\Gamma_1 \cap \hat{S}$ inside \bar{E}_4 , and thus inside \tilde{X}_5 . We are going to call this blow-up \tilde{S} . Its class in $\text{CH}^2(\bar{E}_4)$ is $hV_{\bar{E}}$.

Proof. First of all, $4V_Z(\bar{h}_1 + \bar{h}_2)\bar{h} = 4V_Z^2(\bar{h}_1 + \bar{h}_2) + 4V_Z(\bar{h}_1 + \bar{h}_2)^2 = 4.6 = 24$. So in general \hat{S} and Γ_1 intersect in 24 points, in particular their intersection is not empty. By Proposition 3.2.6 there is in \bar{E}_4 a blow-up of Z_3 in Γ_1 , namely \tilde{Z} . Let us denote by $p : \tilde{Z} \rightarrow Z_3$ this blow-up. Then we can take $\tilde{S} = \hat{S} \times_{Z_3} \tilde{Z} \cong p^{-1}(\hat{S})$. By definition of p it will be the blow-up of \hat{S} in $\Gamma_1 \cap \hat{S}$, and its class can be computed using the classes of \tilde{Z} and \hat{S} . □

By (1.12) in $\text{Pic}(\bar{E}_4)$ we have $V_{\bar{E}} = h$, thus we have found a surface in \tilde{X}_5 with class $h\bar{h}\bar{e}$. Since $\tilde{X}_4 = \tilde{X}_5 \cdot \tilde{H}_\nu$, where \tilde{H}_ν is the pull back of the hyperplane H_ν satisfying $X_4 = X_5 \cap H_\nu$ (see the proof of Theorem 5.1 in [20]), and since we have used ν to build \tilde{Z} (more precisely to induce the global section whose zero locus is \tilde{Z}), this should then induce an embedding $\tilde{Z} \hookrightarrow \tilde{X}_4$. As $\tilde{S} \hookrightarrow \tilde{Z}$ by construction, we have then found a surface in \tilde{X}_4 , which is isomorphic to the blow-up of S_2 in some points.

3.3 Intersections with E_4

We would like to understand what is possibly contracted by π in both \tilde{S} and \overline{D}_2 . In order to do so, we first try to compute the intersection between \overline{E}_4 and E_4 . Indeed since E_4 is the exceptional divisor of π (see Theorem 1.2.1), π is an isomorphism outside of E_4 , and $(\sigma \circ \overline{\pi})^{-1}(S_2) = \overline{E}_4$.

Definition 3.3.1. Let p and z be points in S_2 and Z_3 respectively. We are going to denote by $f_Z(p)$ and $f_{\overline{E}}(z)$ the fibers respectively over p in Z_3 and over z in \overline{E}_4 . By Theorem 1.2.1 each of these fibers is a \mathbb{P}^1 . Similarly, if x is a point in F_3 and y a point in Σ_2 , we will denote respectively by $f_E(x)$ and $f_D(y)$ their fibers in respectively E_4 and D_3 .

Before going further we need a quick "technical Lemma" :

Lemma 3.3.2. Let $\overline{U}_2 \subset \overline{W}$. Then \overline{U}_2 is $\overline{\lambda}$ -isotropic if and only if we can write $\overline{U}_2 = \langle u_1, u_2 \rangle$ where $u_i \in A_i$. Here $\overline{\lambda} = x_{123} + x_{456}$ and $A_1 = \langle x_1, x_2, x_3 \rangle$ and $A_2 = \langle x_4, x_5, x_6 \rangle$ as defined in 3. of section 1.1.

Proof. Let us assume that $\overline{U}_2 = \langle u, v \rangle$ with $u = \sum_{i=1}^6 \alpha_i x_i$ and $v = \sum_{i=1}^6 \beta_i x_i$. Since $\overline{\lambda} = x_{123} + x_{456}$ we have that \overline{U}_2 is $\overline{\lambda}$ -isotropic if and only if :

$$0 = x_1^*(\alpha_2\beta_3 - \alpha_3\beta_2) - x_2^*(\alpha_1\beta_3 - \alpha_3\beta_1) + x_3^*(\alpha_1\beta_2 - \alpha_2\beta_1) + x_4^*(\alpha_5\beta_6 - \alpha_6\beta_5) - x_5^*(\alpha_4\beta_6 - \alpha_6\beta_4) + x_6^*(\alpha_4\beta_5 - \alpha_5\beta_4)$$

that is to say if and only if the following equations are all satisfied :

$$\alpha_2\beta_3 - \alpha_3\beta_2 = 0 \quad (3.14)$$

$$\alpha_1\beta_3 - \alpha_3\beta_1 = 0 \quad (3.15)$$

$$\alpha_1\beta_2 - \alpha_2\beta_1 = 0 \quad (3.16)$$

$$\alpha_5\beta_6 - \alpha_6\beta_5 = 0 \quad (3.17)$$

$$\alpha_4\beta_6 - \alpha_6\beta_4 = 0 \quad (3.18)$$

$$\alpha_4\beta_5 - \alpha_5\beta_4 = 0. \quad (3.19)$$

Let us assume first that these equations are all satisfied. Here we have four possibilities :

1. $\alpha_1 \neq 0$. Since multiplying u by a non-zero scalar is not going to change \overline{U}_2 we can then assume that $\alpha_1 = 1$. Then (3.16) and (3.15) give respectively $\beta_2 = \alpha_2\beta_1$ and $\beta_3 = \alpha_3\beta_1$. Since $\langle u, v \rangle = \langle u, v - \beta_1.u \rangle$ we can then take $u_2 = \beta_4x_4 + \beta_5x_5 + \beta_6x_6$.
2. $\alpha_1 = 0$ and $\alpha_2 \neq 0$. As in the previous case we can then assume that $\alpha_2 = 1$. Then (3.16) gives $\beta_1 = 0$ and (3.14) gives $\beta_3 = \alpha_3\beta_2$. Since $\langle u, v \rangle = \langle u, v - \beta_2.u \rangle$ we can then take $u_2 = \beta_4x_4 + \beta_5x_5 + \beta_6x_6$.
3. $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 \neq 0$. As before we can assume $\alpha_3 = 1$ and thus (3.14) and (3.15) give $\beta_1 = \beta_2 = 0$. Since $\langle u, v \rangle = \langle u, v - \beta_3.u \rangle$ we can then take $u_2 = \beta_4x_4 + \beta_5x_5 + \beta_6x_6$.
4. $\alpha_1 = \alpha_2 = \alpha_3 = 0$. We can then take $u_2 = u = \alpha_4x_4 + \alpha_5x_5 + \alpha_6x_6$.

Similarly if β_4, β_5 or β_6 is non-zero then using (3.17), (3.18) and (3.19) we can show that $\overline{U}_2 = \langle u_1, v \rangle$ with $u_1 = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3$. In such a case we cannot have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ otherwise it would contradict $\dim(\overline{U}_2) = 2$, hence we are in case 1., 2. or 3. above and we can take $u_1 = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3$ and $u_2 = \beta_4x_4 + \beta_5x_5 + \beta_6x_6$.

If $\beta_4 = \beta_5 = \beta_6 = 0$ then we cannot be in one of the first three cases above since it would contradict again $\dim(\overline{U}_2) = 2$. Thus we are in case 4. and we can take $u_1 = v$ and $u_2 = u$.

Conversely, if $\alpha_4 = \alpha_5 = \alpha_6 = \beta_1 = \beta_2 = \beta_3 = 0$ or $\alpha_1 = \alpha_2 = \alpha_3 = \beta_4 = \beta_5 = \beta_6 = 0$ it is easy to check that the six equations (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19) are satisfied. \square

Lemma 3.3.3. *The intersection between fibers over $\overline{C}_1 \cup \overline{C}_2$ in \overline{E}_4 and E_4 is*

- two \mathbb{P}^1 intersecting at one point, over a point p such that $p \notin \overline{C}_1 \cap \overline{C}_2$;
- three \mathbb{P}^1 such that one of them intersects the two others at exactly one point each, these two intersection points being distinct, over a point $p \in \overline{C}_1 \cap \overline{C}_2$.

Let us denote by C'_1 and C'_2 the fibers in Z_3 over \overline{C}_1 and \overline{C}_2 . Then the intersection between the fibers over $C'_1 \cup C'_2$ in \overline{E}_4 and E_4 can be described as follows :

- Let $p \in \overline{C}_i - \overline{C}_1 \cap \overline{C}_2$. Then there is exactly one point $z' \in f_Z(p)$ such that $f_{\overline{E}}(z') \subset E_4$. For all the other points $z \in f_Z(p)$, we have that $f_{\overline{E}}(z) \cap E_4$ is exactly one point. Moreover if $p = U_{2,A_1} \oplus U_{2,A_2}$, then z' is the only point $\overline{U}_3 \subset \overline{U}_4$ in $f_Z(p)$ such that $U_{2,A_i} \subset \overline{U}_3$.
- Let $p \in \overline{C}_1 \cap \overline{C}_2$. Then there are exactly two points z_1 and z_2 in $f_Z(p)$ such that $f_{\overline{E}}(z_i) \subset E_4$. For all the other points $z \in f_Z(p)$, we have that $f_{\overline{E}}(z) \cap E_4$ is exactly one point. Moreover, if $p = \overline{U}_4 = U_{2,A_1} \oplus U_{2,A_2}$, then z_1 and z_2 are respectively the two points $\overline{U}_{3,i} \subset \overline{U}_{4,i}$ in $f_Z(p)$ such that $U_{2,A_i} \subset \overline{U}_{3,i}$. For all the other points $z \in f_Z(p)$, the intersection of $f_{\overline{E}}(z)$ with E_4 is $\{\overline{U}_4^\perp \oplus \mathbb{C}w_0\}$.

Proof. 1. Assume that $p \in \overline{C}_1 - \overline{C}_2$ is such that $p = \overline{U}_4 = \overline{U}_{2,A_1} \oplus \overline{U}_{2,A_2}$ and $\overline{U}_{2,A_i} = \langle u_i, v_i \rangle$. Then by hypothesis on p we can assume $\overline{U}_4^\perp = \langle u_1, u_2 + \epsilon v_1 \rangle$ with $\epsilon \neq 0$. Moreover $f_Z(p)$ is composed by all the $\overline{U}_4^\perp \subset \overline{U}_3 \subset \overline{U}_4$ (by 10. of Theorem 1.2.1) so each point in $f_Z(p)$ is of the form :

$$\overline{U}_3 = \langle u_1, u_2 + \epsilon v_1, Av_1 + Bv_2 \rangle \quad (3.20)$$

with $A, B \in \mathbb{C}$ not vanishing at the same time. On the other hand, the points in E_4 are all of the form $\overline{U}_2 \oplus \mathbb{C}w_0 \subset \overline{U}_3 \oplus \mathbb{C}w_0$ (see 5. and 6. of Theorem 1.2.1) so the points in $E_4 \cap \pi^{-1}(\hat{S})$ over p correspond to the $\overline{U}_2 \subset \overline{U}_3$ which are $\bar{\lambda} = x_{123} + x_{456}$ -isotropic. Indeed, they are all automatically μ and λ' isotropic : \overline{U}_3 contains \overline{U}_4^\perp and another vector of \overline{U}_4 so it is μ -Lagrangian, and λ' vanishes on w_0 so it annihilates any $\overline{U}_2 \oplus \mathbb{C}w_0$. We have two cases :

$$U_3 = \langle w_0, u_2 + \epsilon v_1, Av_1 + Bv_2 \rangle \quad (3.21)$$

and

$$U_3 = \langle w_0, u_1 + \Lambda(u_2 + \epsilon v_1) + \Gamma(Av_1 + Bv_2), \phi(u_2 + \epsilon v_1) + \psi(Av_1 + Bv_2) \rangle. \quad (3.22)$$

We need to check when the associated $\overline{U}_2 = U_3 / \langle w_0 \rangle$ is $\bar{\lambda}$ -isotropic. We will assume : $u_1 = \sum_{i=1}^3 \alpha_i x_i$,

$$v_1 = \sum_{i=1}^3 \beta_i x_i, u_2 = \sum_{i=4}^6 \gamma_i x_i \text{ and } v_2 = \sum_{i=4}^6 \delta_i x_i.$$

• **Case (3.21):**

By Lemma 3.3.2, \overline{U}_2 is $\bar{\lambda}$ -isotropic if and only if it can be written as $\langle w_1, w_2 \rangle$ with $w_i \in A_i$. If $B \neq 0$ then since $\langle u_2, v_2 \rangle$ is 2-dimensional and $\dim(\overline{U}_2) = 2$ we cannot write $\overline{U}_2 = \langle w_1, w_2 \rangle$ with $w_1 \in A_1$. If $B = 0$ we must have $A \neq 0$ and we can assume $A = 1$ since multiplying a generator by a non-zero scalar doesn't change a vector subspace. In this case

$$\overline{U}_2 = \langle u_2, v_1 \rangle$$

is $\bar{\lambda}$ -isotropic.

• **Case (3.22):**

Here

$$\overline{U}_2 = \langle u_1 + \Lambda u_2 + (\Lambda \epsilon + \Gamma A)v_1 + \Gamma Bv_2, \phi u_2 + (\epsilon \phi + \psi A)v_1 + \psi Bv_2 \rangle.$$

Let us consider two cases, using again Lemma 3.3.2 :

- $\phi \neq 0$, hence we can assume $\phi = 1$. Then $\bar{U}_2 = \langle u_1 + (\Lambda\epsilon + \Gamma A - \Lambda\epsilon - \Lambda\psi A)v_1 + (\Gamma B - \psi\Lambda B)v_2, u_2 + (\epsilon + \psi A)v_1 + \psi Bv_2 \rangle = \langle u_1 + A(\Gamma - \Lambda\psi)v_1 + B(\Gamma - \psi\Lambda)v_2, u_2 + (\epsilon + \psi A)v_1 + \psi Bv_2 \rangle$. As before and since $\dim(\langle u_1, v_1 \rangle) = \dim(\langle u_2, v_2 \rangle) = 2$ we must have : $B(\Gamma - \Lambda\psi) = 0$ and $\epsilon + \psi A = 0$.

If $B \neq 0$ it gives : $\Gamma = \Lambda\psi$ and $A\psi = -\epsilon$. Since $\epsilon \neq 0$ this forces $A \neq 0$ and $\psi \neq 0$, therefore we can assume $A = 1$ which yields : $\psi = -\epsilon$. Hence we find :

$$\bar{U}_2 = \langle u_1, u_2 - \epsilon Bv_2 \rangle .$$

If $B = 0$ this yields $A\psi = -\epsilon$ so as just before we can assume $A = 1$ and $\psi = -\epsilon$. Therefore we get : $\bar{U}_2 = \langle u_1 + (\Gamma + \Lambda\epsilon)v_1, u_2 \rangle$. Since here there are no conditions on Γ and Λ this actually gives :

$$\bar{U}_2 = \langle u_1 + kv_1, u_2 \rangle$$

for $k \in \mathbb{C}$.

- $\phi = 0$. Then since ϕ and ψ cannot be zero at the same time, we can assume $\psi = 1$. Thus $\bar{U}_2 = \langle u_1 + \Lambda u_2 + (\Lambda\epsilon + \Gamma A)v_1 + \Gamma Bv_2, Av_1 + Bv_2 \rangle = \langle u_1 + \Lambda u_2 + \Lambda\epsilon v_1, Av_1 + Bv_2 \rangle$. For the same reason as before we thus want $A = 0$ and $\Lambda = 0$. As a result we can assume $B = 1$ and we get :

$$\bar{U}_2 = \langle u_1, v_2 \rangle .$$

Combining what we found in the two cases (3.21) and (3.22) finally gives :

- (a) if $B \neq 0$ there is $\bar{U}_2 = \langle u_1, Au_2 - \epsilon Bv_2 \rangle$ that is to say one point.
- (b) if $B = 0$ there is $\bar{U}_2 = \langle \alpha u_1 + \beta v_1, u_2 \rangle$ for $(\alpha, \beta) \in \mathbb{C}^2 - \{(0, 0)\}$, that is to say a \mathbb{P}^1 .

Let us denote now $z' = \langle u_1, u_2 + \epsilon v_1, v_1 \rangle = \langle u_1, u_2, v_1 \rangle \subset \bar{U}_4 \in f_Z(p)$. It corresponds to $B = 0$ and it is the only point in $f_Z(p)$ such that $U_{2,A_1} \subset \bar{U}_3$ by the definition of $f_Z(p)$ given in (3.20). Moreover we have just seen that $f_{\bar{E}}(z') \cap E_4$ contains a \mathbb{P}^1 and since $f_{\bar{E}}(z')$ is a \mathbb{P}^1 (see (1.2)) this yields : $f_{\bar{E}}(z') \subset E_4$.

Finally we have found : to each $\bar{U}_3 \neq \langle u_1, u_2, v_1 \rangle$ in $f_Z(p)$ corresponds a unique point in E_4 , namely $\bar{U}_2 = \langle u_1, Au_2 - \epsilon Bv_2 \rangle$. To $\bar{U}_3 = \langle u_1, u_2, v_1 \rangle$ (that is to say $B = 0$) corresponds a \mathbb{P}^1 in E_4 , namely $\{\langle u_1 + \alpha v_1, u_2 \rangle, \alpha \in \mathbb{C}\} \cup \{\langle u_2, v_1 \rangle\}$. Since $f_Z(p)$ is also a \mathbb{P}^1 by Theorem 1.2.1, this gives two \mathbb{P}^1 intersecting at $\langle u_1, u_2 \rangle \subset \langle u_1, u_2, v_1 \rangle$: one of the \mathbb{P}^1 is isomorphic to $f_Z(p)$ and the other is $f_{\bar{E}}(z')$.

2. Let us assume now that $p \in \bar{C}_1 \cap \bar{C}_2$. For $p \in \bar{C}_2 - \bar{C}_1$ it is symmetric to the previous case. We can assume now $\bar{U}_4^\perp = \langle u_1, u_2 \rangle$ and $\bar{U}_S = \langle v_1, v_2 \rangle$ with $u_1, v_1 \in A_1$ and $u_2, v_2 \in A_2$. Then the points in $f_Z(p)$ are given by

$$\bar{U}_3 = \langle u_1, u_2, av_1 + bv_2 \rangle .$$

Again we distinguish two types of possible \bar{U}_2 .

$$\bar{U}_2 = \langle u_2, av_1 + bv_2 \rangle \tag{3.23}$$

and

$$\bar{U}_2 = \langle u_1 + \Gamma u_2 + \Lambda av_1 + \Lambda bv_2, \phi u_2 + \psi av_1 + \psi bv_2 \rangle . \tag{3.24}$$

We need now to find which ones are $\bar{\lambda}$ isotropic and for this we are going to use Lemma 3.3.2 as before.

• **Case (3.23):**

Here \bar{U}_2 can be written as $\langle w_1, w_2 \rangle$ with $w_i \in A_i$ if and only if $b = 0$ since u_2 and v_2 are not colinear. This gives :

$$\bar{U}_2 = \langle u_2, v_1 \rangle .$$

• **Case (3.24):**

Let us distinguish between two cases again :

- $\phi \neq 0$ so we can assume $\phi = 1$. Then $\bar{U}_2 = \langle u_1 + \Gamma u_2 + \Lambda a v_1 + \Lambda b v_2, u_2 + \psi a v_1 + \psi b v_2 \rangle = \langle u_1 + a(\Lambda - \psi\Gamma)v_1 + b(\Lambda - \psi\Gamma)v_2, u_2 + \psi a v_1 + \psi b v_2 \rangle$. So we need $b(\Lambda - \psi\Gamma) = 0$ and $\psi a = 0$.

If $b = 0$ we can assume $a = 1$ and thus we want $\psi = 0$. This yields :

$$\bar{U}_2 = \langle u_1 + \Lambda v_1, u_2 \rangle .$$

If $b \neq 0$ we can assume $b = 1$. Let us assume now that $a = 0$. Then we get $\Lambda = \psi\Gamma$ which gives :

$$\bar{U}_2 = \langle u_1, u_2 + \psi v_2 \rangle .$$

Here there are no conditions on ψ .

If $b \neq 0$ and $a \neq 0$ we get $\psi = 0$ and $\Lambda = \psi\Gamma = 0$. Thus :

$$\bar{U}_2 = \langle u_1, u_2 \rangle .$$

- $\phi = 0$ hence we can assume $\psi = 1$. Then $\bar{U}_2 = \langle u_1 + \Gamma u_2 + \Lambda a v_1 + \Lambda b v_2, a v_1 + b v_2 \rangle = \langle u_1 + \Gamma u_2, a v_1 + b v_2 \rangle$. Therefore we need $\Gamma = 0$ and $a = 0$ which gives :

$$\bar{U}_2 = \langle u_1, v_2 \rangle .$$

Finally we have found :

- (a) if $b = 0$ then $\bar{U}_3 = \langle u_1, u_2, v_1 \rangle$ and the possible \bar{U}_2 are

$$\{\langle u_2, v_1 \rangle\} \cup \{\langle u_1 + \Lambda v_1, u_2 \rangle, \Lambda \in \mathbb{C} - \{0\}\} \cup \{\langle u_1, u_2 \rangle\}.$$

- (b) if $a = 0$ then $\bar{U}_3 = \langle u_1, u_2, v_2 \rangle$ and the possible \bar{U}_2 are $\{\langle u_1, u_2 + \psi v_2 \rangle, \psi \in \mathbb{C}\} \cup \{\langle u_1, v_2 \rangle\}$.

- (c) if $a \neq 0$ and $b \neq 0$ then the only possible \bar{U}_2 is $\langle u_1, u_2 \rangle$.

Let us denote now by z_1 and z_2 the points in $f_Z(p)$ with $\bar{U}_3 = \langle u_1, u_2, v_1 \rangle$ respectively $\langle u_1, u_2, v_2 \rangle$. Then we have seen that for both $i = 1$ and $i = 2$, $f_{\bar{E}}(z_i) \cap E_4$ contains a \mathbb{P}^1 . Since each $f_{\bar{E}}(z_i)$ is itself a \mathbb{P}^1 this implies that $f_{\bar{E}}(z_i) \subset E_4$. For the other points z in $f_Z(p)$ we have $f_{\bar{E}}(z) \cap E_4 = \{\langle u_1, u_2, w_0 \rangle\}$. Since $\{\langle u_1, u_2, w_0 \rangle\}$ is also in each $f_{\bar{E}}(z_i) \cap E_4$ (respectively when $\Lambda = 0$ and $\psi = 0$) and $f_Z(p)$ is also a \mathbb{P}^1 , we have found three \mathbb{P}^1 (namely $f_{\bar{E}}(z_i)$) and one isomorphic to $f_Z(p)$ intersecting in two points : each $f_{\bar{E}}(z_i)$ intersects the third \mathbb{P}^1 respectively in $\langle u_1, u_2 \rangle \subset \langle u_1, u_2, v_1 \rangle$ and $\langle u_1, u_2 \rangle \subset \langle u_1, u_2, v_2 \rangle$. Finally for $i \in \{1, 2\}$, z_i is the only point in $f_Z(p)$ such that the associated \bar{U}_3 contains U_{2, A_i} . \square

Lemma 3.3.4. *Let $p \in S_2 - (\bar{C}_1 \cup \bar{C}_2)$. Then for each point $z \in f_Z(p)$ we have that $f_{\bar{E}}(z) \cap E_4$ contains exactly one point.*

Proof. Let p be a point in $S_2 - (\bar{C}_1 \cup \bar{C}_2)$. Let us assume that $p = \bar{U}_4 = U_{2, A_1} \oplus U_{2, A_2}$ with $U_{2, A_i} = \langle u_i, v_i \rangle$. Then by hypothesis up to a good choice of the u_i and v_i we can assume that $\bar{U}_4 = \langle u_1 + u_2, v_1 + v_2 \rangle$. Then we have :

$$f_Z(p) = \{\langle u_1 + u_2, v_1 + v_2, Au_1 + Bv_1 \rangle \mid A, B \in \mathbb{C}, (A, B) \neq (0, 0)\}. \quad (3.25)$$

Let $z \in f_Z(p)$ be such a \bar{U}_3 . By Lemma 3.3.2, the points in $f_{\bar{E}}(z) \cap E_4$ correspond to the $\bar{U}_2 \subset \langle u_1 + u_2, v_1 + v_2, Au_1 + Bv_1 \rangle$ which can be written as $\langle w_1, w_2 \rangle$ with $w_i \in A_i$. Since the u_i and v_i are not colinear $\langle u_1 + u_2, v_1 + v_2 \rangle$ doesn't satisfy this property. So if such a \bar{U}_2 exists it has to be of the form :

$$\langle Au_1 + Bv_1 + \phi(u_1 + u_2) + \psi(v_1 + v_2), \zeta(u_1 + u_2) + \xi(v_1 + v_2) \rangle .$$

That is to say :

$$\langle (A + \phi)u_1 + (B + \psi)v_1 + \phi u_2 + \psi v_2, \zeta(u_1 + u_2) + \xi(v_1 + v_2) \rangle .$$

There are two cases :

- $\zeta = 0$. Then we can assume $\xi = 1$ so

$$\begin{aligned}\bar{U}_2 &= \langle (A + \phi)u_1 + (B + \psi)v_1 + \phi u_2 + \psi v_2, v_1 + v_2 \rangle \\ &= \langle (A + \phi)u_1 + \phi u_2 + (\psi - B - \psi)v_2, v_1 + v_2 \rangle \\ &= \langle (A + \phi)u_1 + \phi u_2 - Bv_2, v_1 + v_2 \rangle.\end{aligned}$$

Since u_1 and v_1 are not colinear, we want $\phi = -A$ which gives $\bar{U}_2 = \langle -Au_2 - Bv_2, v_1 + v_2 \rangle$. The only possibility here is $A = 0$ which gives :

$$\bar{U}_2 = \langle v_2, v_1 \rangle.$$

- $\zeta \neq 0$ thus we can assume $\zeta = 1$. So

$$\begin{aligned}\bar{U}_2 &= \langle (A + \phi)u_1 + (B + \psi)v_1 + \phi u_2 + \psi v_2, u_1 + u_2 + \xi(v_1 + v_2) \rangle \\ &= \langle (B + \psi - A\xi - \phi\xi)v_1 + (\phi - A - \phi)u_2 + (\psi - A\xi - \phi\xi)v_2, u_1 + u_2 + \xi v_1 + \xi v_2 \rangle.\end{aligned}$$

To start with, since u_1 and v_1 are not colinear we need $B + \psi = \xi(A + \phi)$. This implies

$$\bar{U}_2 = \langle -Au_2 - Bv_2, u_1 + u_2 + \xi v_1 + \xi v_2 \rangle.$$

If $B = 0$, we can assume $A = 1$ and thus we have $\bar{U}_2 = \langle u_2, u_1 + \xi v_1 + \xi v_2 \rangle$ so we want $\xi = 0$ and we get :

$$\bar{U}_2 = \langle u_2, u_1 \rangle.$$

If $B \neq 0$ and $A \neq 0$ we can assume $A = 1$. Thus $\bar{U}_2 = \langle u_2 + Bv_2, u_1 + \xi v_1 + (\xi - B)v_2 \rangle$. Hence we need $B = \xi$ and we get :

$$\bar{U}_2 = \langle u_2 + Bv_2, u_1 + Bv_1 \rangle.$$

If $A = 0$ we have $\bar{U}_2 = \langle v_2, u_1 + u_2 + \xi v_1 \rangle$ which cannot be written as $\langle w_1, w_2 \rangle$ with $w_i \in A_i$.

Finally, we have found that for each $z = \langle u_1 + u_2, v_1 + v_2, Au_1 + Bv_1 \rangle \in f_Z(p)$ we have

$$f_{\bar{E}}(z) \cap E_4 = \{ \langle w_0, Au_1 + Bv_1, Au_2 + Bv_2 \rangle \}.$$

□

Combining Lemma 3.3.3 and Lemma 3.3.4 we have found :

Corollary 3.3.5. *Let $z = \bar{U}_3 \subset \bar{U}_4$ be a point in Z_3 . Then :*

1. *if $\bar{U}_4 \cap A_i \subset \bar{U}_3$ for $i = 1$ or 2 then $\mathbb{P}^1 \cong f_{\bar{E}}(z) \subset E_4$ (and $\sigma(p) \in \bar{C}_i$);*
2. *otherwise, $f_{\bar{E}}(z) \cap E_4$ contains one point.*

Remark 3.3.6. *From 3. of Lemma 1.2.2 we have that there exists two disjoint curves $C_i \subset Z_3$ such that $\sigma(C_i) = \bar{C}_i$ and $\bar{U}_3 \subset \bar{U}_4 \in C_i$ if and only if $\bar{U}_4 \cap A_i \subset \bar{U}_3$. Thus Corollary 3.3.5 together with $\bar{E}_4 = \mathbb{P}_Z(\mathcal{V}_Z)$ seem to suggest that $E_4 \cap \bar{E}_4$ is isomorphic to the blow-up of Z_3 in C_1 and C_2 .*

Before going further, let us introduce some notations and give names to the various \mathbb{P}^1 described in Lemma 3.3.3.

Definition 3.3.7. *Let $\bar{C}_1 \cap \bar{C}_2 = \{p_1, p_2\}$.*

For each p_i let us denote $\bar{U}_4^\perp = \langle u_1^i, u_2^i \rangle$ and $\bar{U}_S := \bar{U}_4 / \bar{U}_4^\perp = \langle v_1^i, v_2^i \rangle$ with $u_j^i, v_j^i \in A_j$.

Moreover, let p and q be points in $\bar{C}_1 - \bar{C}_2$, respectively $\bar{C}_2 - \bar{C}_1$. Then we will use the following notations :

- *for p , $\bar{U}_4^\perp = \langle u_1^p, u_2^p + \epsilon_p v_1^p \rangle$ with $\epsilon_p \neq 0$, and $\bar{U}_4 = \langle u_1^p, u_2^p, v_1^p, v_2^p \rangle$. Here $v_j^p, u_j^p \in A_j$;*

- for q , $\bar{U}_4^\perp = \langle u_2^q, u_1^q + \epsilon_q v_2^q \rangle$ with $\epsilon_q \neq 0$, and $\bar{U}_4 = \langle u_1^q, u_2^q, v_1^q, v_2^q \rangle$. Here $v_j^q, u_j^q \in A_j$.

When the point to which the u_i and v_i are associated is obvious, we will sometimes drop the index associated to this point.

Definition 3.3.8. Let us denote the \mathbb{P}^1 in $E_4 \cap \bar{E}_4$ over points in S_2 as follows :

- Over p_i , $A'_i := \{ \langle au_1 + bv_1, u_2 \rangle \subset \langle u_1, u_2, v_1 \rangle, (a, b) \in \mathbb{C}^2 - \{(0, 0)\} \}$.
- Over p_i , $B_i := \{ \langle u_1, \phi u_2 + \psi v_2 \rangle \subset \langle u_1, u_2, v_2 \rangle, (\psi, \phi) \in \mathbb{C}^2 - \{(0, 0)\} \}$.
- Over p_i , $C'_i := \{ \langle u_1, u_2 \rangle \subset \langle u_1, u_2, av_1 + bv_2 \rangle, (a, b) \in \mathbb{C}^2 - \{(0, 0)\} \}$.
- Over $p \in \bar{C}_1 - \bar{C}_2$, $E_p := \{ \langle u_1, Au_2 - \epsilon Bv_2 \rangle \subset \langle u_1, u_2 + \epsilon v_1, Av_1 + Bv_2 \rangle, (A, B) \in \mathbb{C}^2 - \{(0, 0)\} \}$.
- Over $p \in \bar{C}_1 - \bar{C}_2$, $F_p := \{ \langle \beta u_1 + \alpha v_1, u_2 \rangle \subset \langle u_1, u_2, v_1 \rangle, (\alpha, \beta) \in \mathbb{C}^2 - \{(0, 0)\} \}$.
- Over $p \in \bar{C}_2 - \bar{C}_1$, $G_p := \{ \langle u_2, Au_1 - \epsilon Bv_1 \rangle \subset \langle u_2, u_1 + \epsilon v_2, Av_2 + Bv_1 \rangle, (A, B) \in \mathbb{C}^2 - \{(0, 0)\} \}$.
- Over $p \in \bar{C}_2 - \bar{C}_1$, $H_p := \{ \langle \beta u_2 + \alpha v_2, u_1 \rangle \subset \langle u_2, u_1, v_2 \rangle, (\alpha, \beta) \in \mathbb{C}^2 - \{(0, 0)\} \}$.

3.4 Some contractions by π

As mentioned before, we want to understand what can be contracted by π in \tilde{S} and \bar{D}_2 . Indeed it is necessary to understand such contractions to be able to compute Euler characteristics between $\pi_* \mathcal{O}_{\bar{D}}$, $\pi_* \mathcal{O}_{\tilde{S}}$ and other sheaves in $D^b(X_4)$. Thus, if we want to use the two pushforwards $\pi_* \mathcal{O}_{\tilde{S}}$ and $\pi_* \mathcal{O}_{\bar{D}}$ in order to build a basis of $K_{\text{num}}(X_4)$ and then compute all the associated Euler characteristics, we need to study what can possibly be contracted by π in \tilde{S} and \bar{D}_2 .

In the following subsection we give a partial answer to this question. We first try to understand what can be contracted by π in $E_4 \cap \bar{E}_4 \subset \tilde{X}_5$ in subsection 3.4.1, and then try to understand what remains of these contractions in $E_4 \cap \bar{E}_4 \cap \tilde{X}_4 = D_3 \cap \bar{E}_4$ in subsection 3.4.2.

3.4.1 Some contractions by π in $E_4 \cap \bar{E}_4$

Here we try to understand what is contracted by π in $E_4 \cap \bar{E}_4$. We are only able to give a partial description of these contractions : we can use some geometrical descriptions given in [20] to study contractions in the preimages of two rational curves in S_2 (namely \bar{C}_1 and \bar{C}_2).

We start with some technical Lemmas, which we will use in the proof of Lemmas 3.4.6 and 3.4.7, where we list all the contractions we have found. We will use the notations of Definitions 3.3.7 and 3.3.8 all along this subsection.

Lemma 3.4.1. With the notations of Definition 3.3.7, we must have :

1. u_1^1 and u_1^2 are not colinear;
2. u_2^1 and u_2^2 are not colinear;
3. $u_1^2 \notin \langle u_1^1, v_1^1 \rangle$ and $u_1^1 \notin \langle u_1^2, v_1^2 \rangle$;
4. $u_2^2 \notin \langle u_2^1, v_2^1 \rangle$ and $u_2^1 \notin \langle u_2^2, v_2^2 \rangle$.

Proof. 1. Let us assume that u_1^1 and u_1^2 are colinear. Then up to rescaling we can assume that $u_1^1 = u_1^2$. Then since μ induces a non degenerate pairing between the A_i , and by definition of the u_j one gets : $\langle u_2^1, v_2^1 \rangle = \text{Ker}(\mu(u_1^1, \cdot)) \cap A_2 = \text{Ker}(\mu(u_1^2, \cdot)) \cap A_2 = \langle u_2^2, v_2^2 \rangle$. Moreover, since the p_i are in $\bar{C}_1 \cap \bar{C}_2$, we must have that $\mu_2 \in \langle u_2^1, v_2^1 \rangle = \langle u_2^2, v_2^2 \rangle$, hence $\langle u_2^1, v_2^1, u_1^1 \rangle = \langle u_2^2, v_2^2, u_1^2 \rangle$ is a μ -Lagrangian. Then by 6. of section 1.1, we must have that v_1^1 and v_1^2 are colinear. But then $\langle u_1^1, u_2^1, v_1^1, v_2^1 \rangle = \langle u_1^2, u_2^2, v_1^2, v_2^2 \rangle$ so $p_1 = p_2$, which is a contradiction.

2. We use the same reasoning as just before : let us assume that u_2^1 and u_2^2 are colinear. Then we can assume that $u_2^1 = u_2^2$. Then since μ induces a non degenerate pairing between the A_i , and by definition of the u_j one gets : $\langle u_1^1, v_1^1 \rangle = \text{Ker}(\mu(u_2^1, \cdot)) \cap A_1 = \text{Ker}(\mu(u_2^2, \cdot)) \cap A_1 = \langle u_1^2, v_1^2 \rangle$. Moreover, since the p_i are in $\overline{C}_1 \cap \overline{C}_2$, we must have that $\mu_1 \in \langle u_1^1, v_1^1 \rangle = \langle u_1^2, v_1^2 \rangle$, hence $\langle u_1^1, v_1^1, u_2^1 \rangle = \langle u_1^2, v_1^2, u_2^2 \rangle$ is a μ -Lagrangian. Then by 6. of section 1.1, we must have that v_2^1 and v_2^2 are colinear. But then $p_1 = p_2$, which is a contradiction.
3. Let us assume that $u_1^2 = au_1^1 + bv_1^1$. We have just seen that u_2^1 and u_2^2 are not colinear, so that $\langle u_1^1, v_1^1 \rangle = \text{Ker}(\mu(u_2^1, \cdot)) \cap A_1 \neq \text{Ker}(\mu(u_2^2, \cdot)) \cap A_1 = \langle u_1^2, v_1^2 \rangle$. So $A_1 = \langle u_1^1, v_1^1, u_1^2, v_1^2 \rangle$. On the other hand, $\mu(u_1^2, v_1^2) = 0$, $\mu(u_2^2, u_1^2) = 0$, $\mu(u_1^2, v_1^1) = \mu(au_1^1 + bv_1^1, v_1^1) = 0$ and $\mu(u_2^2, u_1^1) = \mu(au_1^1 + bv_1^1, u_1^1) = 0$ since $\mu(u_1^1, v_1^1) = 0$. Therefore, u_1^2 is μ -orthogonal to A_1 , so it is μ_1 . But then by Lemma 3.1.8, this contradicts (3.6).
With the same reasoning, if $u_1^1 \in \langle u_1^2, v_1^2 \rangle$ then $u_1^1 = \mu_1$, which contradicts (3.6).
4. The same reasoning as for 3. gives us that if $u_2^2 \in \langle u_2^1, v_2^1 \rangle$, then $u_2^2 = \mu_2$ and that if $u_2^1 \in \langle u_2^2, v_2^2 \rangle$, then $u_2^1 = \mu_2$. So none of them is possible by (3.6). □

Lemma 3.4.2. *Let $p \in \overline{C}_1 - \overline{C}_2$. Then $\overline{U}_S \cong \langle u_2^p, v_2^p \rangle$ and u_2^p is μ -orthogonal to $\langle u_1^p, v_1^p \rangle$.
Let $q \in \overline{C}_2 - \overline{C}_1$. Then $\overline{U}_S \cong \langle u_1^q, v_1^q \rangle$ and u_1^q is μ -orthogonal to $\langle u_2^q, v_2^q \rangle$.*

Proof. By Definition 3.3.7, we have that $\overline{U}_4^1 = \langle u_1^p, u_2^p + \epsilon_p v_1^p \rangle$. So we can write $\overline{U}_S = \langle v_1^p + au_2^p + bv_2^p, cu_2^p + dv_2^p \rangle$. But since in $\overline{U}_4/\overline{U}_4^1$, $v_1^p = -\frac{1}{\epsilon_p}u_2^p$ we can also write $\overline{U}_S = \langle (a - \frac{1}{\epsilon_p})u_2^p + bv_2^p, cu_2^p + dv_2^p \rangle = \langle u_2^p, v_2^p \rangle$. (Since by hypothesis $p \notin \overline{C}_2$, $\mu(u_2^p, v_2^p) \neq 0$ so this is not a contradiction.) Moreover, by hypothesis on u_1^p , we must have $\mu(u_1^p, u_2^p) = 0$ and $0 = \mu(u_2^p + \epsilon_p v_1^p, v_1^p) = \mu(u_2^p, v_1^p)$ since $\mu(v_1^p, v_1^p) = 0$. So u_2^p is μ -orthogonal to u_1^p and v_1^p .

The same reasoning, "switching" A_1 and A_2 , gives us the result for q . □

Lemma 3.4.3. *Let $p \in \overline{C}_1 - \overline{C}_2$. Then u_1^j and u_1^p are not colinear, and u_2^j and u_2^p are not colinear.
Let $q \in \overline{C}_2 - \overline{C}_1$. Then u_1^j and u_1^q are not colinear, and u_2^j and u_2^q are not colinear.*

Proof. 1. Let us assume that u_1^j and u_1^p are colinear : we can then assume $u_1^j = u_1^p$. Since $p \in \overline{C}_1$ we have then $\langle u_2^j, v_2^j \rangle = \text{Ker}(\mu(u_1^j, \cdot)) \cap A_2 = \text{Ker}(\mu(u_1^p, \cdot)) \cap A_2 = \langle u_2^p, v_2^p \rangle$. But $p_j \in \overline{C}_2$ so $\mu_2 \in \langle u_2^j, v_2^j \rangle = \langle u_2^p, v_2^p \rangle$. By Proposition 1.2.2 this implies $p \in \overline{C}_2$, which is a contradiction.

2. Let us assume that u_2^j and u_2^p are colinear : we can then assume $u_2^j = u_2^p$. By Lemma 3.4.2 we have then $\langle u_1^j, v_1^j \rangle = \text{Ker}(\mu(u_2^j, \cdot)) \cap A_1 = \text{Ker}(\mu(u_2^p, \cdot)) \cap A_1 = \langle u_1^p, v_1^p \rangle$. But $p_j \in \overline{C}_1$ so $\mu_1 \in \langle u_1^j, v_1^j \rangle = \langle u_1^p, v_1^p \rangle$. So $\langle u_1^j, v_1^j, u_2^j \rangle = \langle u_1^p, v_1^p, u_2^p \rangle$ is μ -Lagrangian. By 6. of section 1.1 we then get that v_2^j and v_2^p are colinear. This implies $p = p_j$, which is a contradiction.

3. The same reasoning gives the statement about q . □

Lemma 3.4.4. *Let $p \in \overline{C}_1 - \overline{C}_2$. Then $u_1^j \notin \langle u_1^p, v_1^p \rangle$.
Let $q \in \overline{C}_2 - \overline{C}_1$. Then $u_2^j \notin \langle u_2^p, v_2^p \rangle$.*

Proof. 1. Let us first assume that $u_1^j \in \langle u_1^p, v_1^p \rangle$. Then we can write $u_1^j = au_1^p + bv_1^p$. We have seen in Lemma 3.4.3 that u_2^j and u_2^p are not colinear, so by Lemma 3.1.6, $\langle u_1^j, v_1^j \rangle \neq \langle u_1^p, v_1^p \rangle$ and $A_1 = \langle u_1^j, v_1^j, u_1^p, v_1^p \rangle$. But $\mu(u_1^j, u_1^j) = \mu(u_1^j, v_1^j) = 0$, $\mu(u_1^j, u_1^p) = \mu(au_1^p + bv_1^p, u_1^p) = 0$ and $\mu(u_1^j, v_1^p) = \mu(au_1^p + bv_1^p, v_1^p) = 0$ since $p \in \overline{C}_1$ and so $\mu_1 \in \langle u_1^p, v_1^p \rangle$ which implies $\mu(u_1^p, v_1^p) = 0$. Hence u_1^j is μ -orthogonal to A_1 and thus $u_1^j = \mu_1$, which contradicts (3.6) by Lemma 3.1.8.

2. For q we can use the same reasoning, "switching" A_1 and A_2 . □

Lemma 3.4.5. *With the notations of Definition 3.3.7, for any two p, q in $\overline{C}_1 \cup \overline{C}_2 - \overline{C}_1 \cap \overline{C}_2$, u_1^p and u_1^q are not colinear. Similarly, u_2^p and u_2^q are not colinear.*

Proof. Let us first assume that u_1^p and u_1^q are colinear. Then up to rescaling we can assume they are equal. By Lemma 3.4.2 we then have $\langle u_2^p, v_2^p \rangle = \text{Ker}(\mu(u_1^p, \cdot)) \cap A_2 = \text{Ker}(\mu(u_1^q, \cdot)) \cap A_2 = \langle u_2^q, v_2^q \rangle$. Then we have two cases :

- One of p and q is in \overline{C}_2 . Let us assume it is p . Then by Proposition 1.2.2, $\mu_2 \in \langle u_2^p, v_2^p \rangle = \langle u_2^q, v_2^q \rangle$, so $q \in \overline{C}_2$. But then $\langle u_2^p, v_2^p, u_1^p \rangle = \langle u_2^q, v_2^q, u_1^q \rangle$ is μ -Lagrangian, so by 6. of section 1.1, this implies that $v_1^p = v_1^q$. So $p = q$ and this contradicts our hypothesis.
- Both p and q are in $\overline{C}_1 - \overline{C}_2$. Then $\mu(u_2^p, v_2^p) \neq 0$ so we cannot use the same argument as above. But by the same argument as in the previous case we must have that u_2^p and u_2^q are not colinear. So by Lemma 3.1.6, $\langle u_1^p, v_1^p \rangle \neq \langle u_1^q, v_1^q \rangle$ and $A_1 = \langle u_1^p, v_1^p, u_1^q, v_1^q \rangle$. On the other hand, $\mu(u_1^p, u_1^p) = \mu(u_1^q, v_1^p) = 0$, $\mu(u_1^p, u_1^q) = \mu(u_1^q, u_1^q) = 0$ and $\mu(u_1^p, v_1^q) = \mu(u_1^q, v_1^q) = 0$ since both p and q are in \overline{C}_1 and thus $\mu_1 \in \langle u_1^p, v_1^p \rangle$ and $\mu_1 \in \langle u_1^q, v_1^q \rangle$. But then $u_1^p = u_1^q$ is μ -orthogonal to A_1 , so $u_1^p = u_1^q = \mu_1$. But we have seen in the proof of Proposition 3.1.12 that there is a unique point p' in $\overline{C}_1 - \overline{C}_2$ such that $u_1^{p'} = \mu_1$, so $p = q$ which is a contradiction.

So u_1^p and u_1^q must be non colinear.

With a similar reasoning one can find that u_2^p and u_2^q are non colinear. □

We can now compute some obvious contractions of π , and list some contractions that cannot occur, using the \mathbb{P}^1 listed in Definition 3.3.8 :

Lemma 3.4.6. *With the notations of Definition 3.3.8 we have :*

1. Both C'_1 and C'_2 are fibers in E_4 . They are contracted by π to the points $\langle u_1^1, u_2^1, w_0 \rangle$ and $\langle u_1^2, u_2^2, w_0 \rangle$ respectively.
2. For $i \in \{1, 2\}$, π doesn't contract anything in A'_i or B_i . For any fixed $p \in \overline{C}_i - (\overline{C}_1 \cap \overline{C}_2)$, π doesn't contract anything in E_p, F_p, G_p or H_p .

Proof. From 7. of Theorem 1.2.1 we have that two points $U_3^1 \subset U_4^1$ and $U_3^2 \subset U_4^2$ in \tilde{X}_5 have same image by π and only if $U_3^1 = U_3^2$. Hence, by definition of C'_1 and C'_2 they must be contracted to respectively $\langle u_1^1, u_2^1, w_0 \rangle$ and $\langle u_1^2, u_2^2, w_0 \rangle$. Moreover by (1.2) we know that the fibers in E_4 are all \mathbb{P}^1 and C'_1 and C'_2 are also \mathbb{P}^1 so $f_E(\langle u_1^1, u_2^1, w_0 \rangle) = C'_1$ and $f_E(\langle u_1^2, u_2^2, w_0 \rangle) = C'_2$.

Using again the definition of π in 7. of Theorem 1.2.1, one finds that π doesn't contract anything in A'_i, B_i, F_p or H_p : indeed by Lemma 3.3.3 these are isomorphic to fibers in \overline{E}_4 . But by 8. of Theorem 1.2.1, $\overline{\pi}(U_3 \subset U_4) = U_4 / \langle w_0 \rangle$ so in a fiber in \overline{E}_4 all points have same U_4 and thus different U_3 . So π should be an isomorphism from $f_{\overline{E}}(z)$ to $\pi(f_{\overline{E}}(z))$ for any $z \in Z_3$.

Finally, $\langle u_1, Au_2 - \epsilon Bv_2 \rangle = \langle u_1, \tilde{A}u_2 - \epsilon \tilde{B}v_2 \rangle$ if and only if $\tilde{A}u_2 - \epsilon \tilde{B}v_2$ and $Au_2 - \epsilon Bv_2$ are colinear, if and only if $\langle u_1, u_2 + \epsilon v_1, Av_1 + Bv_2 \rangle = \langle u_1, u_2 + \epsilon v_1, \tilde{A}v_1 + \tilde{B}v_2 \rangle$ so π doesn't contract anything in any E_p . Similarly, π doesn't contract anything in any G_p . □

In the following Lemma, we try to complete the above-mentioned Lemma 3.4.6 and list all the contractions which can (or cannot) occur between points in the \mathbb{P}^1 listed in Definition 3.3.8. In order to do so, we study case by case what can be contracted between two of these \mathbb{P}^1 . There are a lot of cases to consider but most of the times we can easily understand whether there are contractions or not using the technical Lemmas from the beginning of the subsection. However, as detailed in the proof, there are some cases in which we don't manage to conclude.

Lemma 3.4.7. *With the notations of Definition 3.3.8, C'_1 and C'_2 are contracted to two distinct points by π .*

If p and q are respectively the unique point in \overline{C}_1 such that $u_1^p = \mu_1$ and the unique point in \overline{C}_2 such that $u_2^q = \mu_2$ then E_p and G_q are contracted to a \mathbb{P}^1 in the unions of respectively all F_r and all H_s .

There is also a pairing between points in $\overline{C}_1 - \overline{C}_2$ and $\overline{C}_2 - \overline{C}_1$ such that if r and s are paired then $\pi(F_r) \cap \pi(H_s)$ contains exactly one point. Moreover, $\pi(A'_1) \cap \pi(B_1)$ and $\pi(A'_2) \cap \pi(B_2)$ also contain exactly one point each.

Finally, there might be some $r \in \overline{C}_1 - \overline{C}_2$ and $s \in \overline{C}_2 - \overline{C}_1$ such that $\pi(E_r) \cap \pi(G_s) \neq \emptyset$. Such an intersection would then contain exactly one point.

Proof. We will use the notations of Definition 3.3.7 and Definition 3.3.8, so we will denote by p_1 and p_2 the points in $\overline{C}_1 \cap \overline{C}_2$ and by p, q or p' points in one of the \overline{C}_i but not the other.

We want now to compute what can be contracted by π between two distinct \mathbb{P}^1 among those listed in Definition 3.3.8. Some of these \mathbb{P}^1 already intersect in E_4 : we have seen in the proof of Lemma 3.3.3 that $A'_i \cap C'_i = \{ \langle u_1, u_2 \rangle \subset \langle u_1, u_2, v_1 \rangle \}$, $B_i \cap C'_i = \{ \langle u_1, u_2 \rangle \subset \langle u_1, u_2, v_2 \rangle \}$, $E_p \cap F_p = \{ \langle u_1, u_2 \rangle \subset \langle u_1, u_2, v_1 \rangle \}$ and $G_p \cap H_p = \{ \langle u_1, u_2 \rangle \subset \langle u_1, u_2, v_2 \rangle \}$.

By Theorem 1.2.1 we have that $\pi(\overline{U}_2 \oplus \mathbb{C}w_0 \subset \overline{U}_3 \oplus \mathbb{C}w_0) = \overline{U}_2 \oplus \mathbb{C}w_0$, so to find which points may be contracted by π we need to find which points in the above mentioned \mathbb{P}^1 have the same \overline{U}_2 . We are going to proceed case by case.

- For $i = 1$ and $i = 2$, one has $\pi(A'_i) \cap \pi(B_i) = \{ \langle w_0, u_1^i, u_2^i \rangle \}$: both A'_i and B_i intersect C'_i in one point and the whole C'_i is contracted by π as we saw in Lemma 3.4.6.
- For $i = 1$ and $i = 2$, we have $\pi(A'_i) \cap \pi(C'_i) = \pi(A'_i \cap C'_i)$ and $\pi(B_i) \cap \pi(C'_i) = \pi(B_i \cap C'_i)$.
- Can $\pi(C'_1) = \pi(C'_2)$?
If it is the case then $\langle u_1^1, u_2^1 \rangle = \langle u_1^2, u_2^2 \rangle$. But since $u_j^i \in A_i$ this implies, up to rescaling, $u_1^1 = u_1^2$ and $u_2^1 = u_2^2$. This contradicts Lemma 3.4.1. So $\pi(C'_1) \neq \pi(C'_2)$.
- $\pi(A'_i) \cap \pi(C'_j)$
If this intersection is non empty, then $\langle au_1^i + bv_1^i, u_2^i \rangle = \langle u_1^j, u_2^j \rangle$. As before then we can assume $u_2^i = u_2^j$ so this contradicts Lemma 3.4.1. Thus $\pi(A'_i) \cap \pi(C'_j) = \emptyset$.
- $\pi(B_i) \cap \pi(C'_j)$
If the intersection is non empty, then $\langle u_1^i, \phi u_2^i + \psi v_2^i \rangle = \langle u_1^j, u_2^j \rangle$ so we can assume $u_1^i = u_1^j$, which contradicts Lemma 3.4.1. Hence $\pi(B_i) \cap \pi(C'_j) = \emptyset$.
- $\pi(B_1) \cap \pi(B_2)$
If the intersection is non empty, $\langle u_1^i, \phi_i u_2^i + \psi_i v_2^i \rangle = \langle u_1^j, \phi_j u_2^j + \psi_j v_2^j \rangle$ therefore we can assume $u_1^i = u_1^j$ which contradicts Lemma 3.4.1. Therefore, $\pi(B_1) \cap \pi(B_2) = \emptyset$.
- $\pi(A'_i) \cap \pi(B_j)$
If the intersection is non empty, $\langle au_1^i + bv_1^i, u_2^i \rangle = \langle u_1^j, \phi u_2^j + \psi v_2^j \rangle$. In particular, $u_1^i \in \langle u_1^j, v_1^j \rangle$, which contradicts Lemma 3.4.1. As a result, $\pi(A'_i) \cap \pi(B_j) = \emptyset$.
- $\pi(A'_1) \cap \pi(A'_2)$
If $\langle au_1^i + bv_1^i, u_2^i \rangle = \langle \alpha u_1^j + \beta v_1^j, u_2^j \rangle$ then we can assume $u_2^i = u_2^j$, which contradicts Lemma 3.4.1. Hence $\pi(A'_1) \cap \pi(A'_2) = \emptyset$.
- $\pi(C_j) \cap \pi(E_p)$
If the intersection is non empty, $\langle u_1^j, u_2^j \rangle = \langle u_1^p, Au_2^p - \epsilon Bv_2^p \rangle$. In particular we can assume $u_1^j = u_1^p$, which contradicts Lemma 3.4.3. Thus $\pi(C_j) \cap \pi(E_p) = \emptyset$.
- $\pi(B_j) \cap \pi(E_p)$
If $\langle u_1^j, \phi u_2^j + \psi v_2^j \rangle = \langle u_1^p, Au_2^p - \epsilon Bv_2^p \rangle$, then we can assume $u_1^j = u_1^p$ which contradicts Lemma 3.4.3. Hence $\pi(B_j) \cap \pi(E_p) = \emptyset$.

• $\pi(A'_j) \cap \pi(E_p)$

If $\langle au_1^j + bv_1^j, u_2^j \rangle = \langle u_1^p, Au_2^p - \epsilon Bv_2^p \rangle$ then, up to rescaling, we can assume $au_1^j + bv_1^j = u_1^p$ and $u_2^j = Au_2^p - \epsilon Bv_2^p$. Then by Lemma 3.4.4, $p \in \overline{C}_1 - \overline{C}_2$.

If u_2^j and u_1^p are not colinear, by Lemma 3.1.6, $\langle u_1^j, v_1^j \rangle \neq \langle u_1^p, v_1^p \rangle$ and thus $A_1 = \langle u_1^j, v_1^j, u_1^p, v_1^p \rangle$. On the other hand, $\mu(u_1^p, u_1^p) = \mu(u_1^p, v_1^p) = 0$, $\mu(u_1^p, u_1^j) = \mu(au_1^j + bv_1^j, u_1^j) = 0$ and $\mu(u_1^p, v_1^j) = \mu(au_1^j + bv_1^j, v_1^j) = 0$. Hence u_1^p is μ -orthogonal to A_1 and it is therefore equal to μ_1 . So $u_1^p = au_1^j + bv_1^j = \mu_1$ and p is the unique point in \overline{C}_1 such that $u_1^p = \mu_1$.

Moreover since $u_1^p = \mu_1$ we must have $\langle u_2^j, v_2^j \rangle = K_1 = \text{Ker}(\mu(\mu_1, \cdot)) \cap A_2$. On the other hand, since u_2^j is μ -orthogonal both to u_1^j and v_1^j it must be orthogonal to $au_1^j + bv_1^j = \mu_1$. So we must have $u_2^j \in \langle u_2^p, v_2^p \rangle$. So the intersection is non empty for the unique choice of A and B such that $u_2^j = Au_2^p - \epsilon Bv_2^p$ and the unique choice of a and b such that $au_1^j + bv_1^j = \mu_1$.

If u_2^j and u_2^p are colinear, it contradicts Lemma 3.4.4.

Finally :

1. If p is the unique point in \overline{C}_1 such that $u_1^p = \mu_1$ then $\pi(A'_1) \cap \pi(E_p) = \{ \langle \mu_1, u_2^1 \rangle \}$ and $\pi(A'_2) \cap \pi(E_p) = \{ \langle \mu_1, u_2^2 \rangle \}$. These are two distinct points since $u_2^1 \neq u_2^2$ by Lemma 3.4.1.
2. Otherwise, $\pi(A'_1) \cap \pi(E_p) = \emptyset$.

• $\pi(C_j) \cap \pi(F_p)$

If $\langle u_1^j, u_2^j \rangle = \langle \beta u_1^p + \alpha v_1^p, u_2^p \rangle$ then we can assume $u_2^j = u_2^p$ but this contradicts Lemma 3.4.3. So $\pi(C_j) \cap \pi(F_p) = \emptyset$.

• $\pi(B_j) \cap \pi(F_p)$

If $\langle u_1^j, \phi u_2^j + \psi v_2^j \rangle = \langle \beta u_1^p + \alpha v_1^p, u_2^p \rangle$ then up to rescaling we can assume that $u_1^j = \beta u_1^p + \alpha v_1^p$ and $u_2^j = \phi u_2^p + \psi v_2^p$. Then by Lemma 3.4.4, $p \in \overline{C}_2 - \overline{C}_1$ and by Lemma 3.4.3, $\alpha \neq 0$. This contradicts our hypothesis : p should be in \overline{C}_1 . Thus $\pi(B_j) \cap \pi(F_p) = \emptyset$.

• $\pi(A'_j) \cap \pi(F_p)$

If $\langle au_1^j + bv_1^j, u_2^j \rangle = \langle \beta u_1^p + \alpha v_1^p, u_2^p \rangle$ then we can assume $u_2^j = u_2^p$ which contradicts Lemma 3.4.3. Thus $\pi(A'_j) \cap \pi(F_p) = \emptyset$.

• $\pi(C'_j) \cap \pi(G_p)$

If $\langle u_1^j, u_2^j \rangle = \langle u_2^p, Au_1^p - \epsilon Bv_1^p \rangle$ then we can assume $u_2^j = u_2^p$, which contradicts Lemma 3.4.3. So $\pi(C'_j) \cap \pi(G_p) = \emptyset$.

• $\pi(B_j) \cap \pi(G_p)$

Let us consider two cases :

- $\langle u_1^j, u_2^j \rangle = \langle u_2^p, Au_1^p - \epsilon Bv_1^p \rangle$. We have just seen it is impossible by Lemma 3.4.3.
- $\langle u_1^j, \phi u_2^j + v_2^j \rangle = \langle u_2^p, Au_1^p - \epsilon Bv_1^p \rangle$. Then we can assume that $u_1^j = v_1^p + \alpha u_1^p$ (since if $u_1^j = u_1^p$ it contradicts Lemma 3.4.3) and $u_2^j = \phi u_2^p + v_2^p$. Since here $p \in \overline{C}_2 - \overline{C}_1$, it doesn't contradict Lemma 3.4.4. Since u_1^j and u_1^p are not colinear, by Lemma 3.1.6, $\langle u_2^j, v_2^j \rangle = \text{Ker}(\mu(u_1^j, \cdot)) \cap A_2 \neq \text{Ker}(\mu(u_1^p, \cdot)) \cap A_2 = \langle u_2^p, v_2^p \rangle$. So $A_2 = \langle u_2^j, v_2^j, u_2^p, v_2^p \rangle$. On the other hand, since $p \in \overline{C}_2$, $\mu(u_2^p, u_2^p) = \mu(u_2^p, v_2^p) = 0$, $\mu(u_2^p, u_2^j) = \mu(\phi u_2^p + v_2^p, u_2^p) = 0$ and $\mu(u_2^p, v_2^j) = \mu(\phi u_2^p + v_2^p, v_2^p) = 0$. So u_2^p is μ -orthogonal to A_2 thus we can assume it is μ_2 . Then p is the unique point in \overline{C}_2 such that $u_2^p = \mu_2$. Since $\mu_2 = u_2^p \in \langle u_2^j, v_2^j \rangle$ and by Lemma 3.4.3 we can then write $u_2^p = \phi u_2^j + v_2^j$ for some ϕ . Moreover, $\langle u_1^p, v_1^p \rangle = \text{Ker}(\mu(\mu_2, \cdot)) \cap A_1$ and u_1^j is μ -orthogonal to μ_2 so by Lemma 3.4.3 again we can write $u_1^j = \alpha u_1^p + v_1^p$.

Thus :

1. If p is the unique point of \overline{C}_2 such that $u_2^p = \mu_2$, then $\pi(B_1) \cap \pi(G_p) = \{ \langle \mu_2, u_1^1 \rangle \}$ and $\pi(B_2) \cap \pi(G_p) = \{ \langle \mu_2, u_1^2 \rangle \}$.
 2. Otherwise, $\pi(B_j) \cap \pi(G_p) = \emptyset$.
- $\pi(A'_j) \cap \pi(G_p)$
If $\langle au_1^j + bv_1^j, u_2^j \rangle = \langle u_2^p, Au_1^p - \epsilon Bv_1^p \rangle$ then we can assume $u_2^j = u_2^p$, which contradicts Lemma 3.4.3. Hence $\pi(A'_j) \cap \pi(G_p) = \emptyset$.
 - $\pi(C_j) \cap \pi(H_p)$
If $\langle u_1^j, u_2^j \rangle = \langle \beta u_2^p + \alpha v_2^p, u_1^p \rangle$ then we can assume $u_1^j = u_1^p$, which contradicts Lemma 3.4.3. Hence $\pi(C_j) \cap \pi(H_p) = \emptyset$.
 - $\pi(B_j) \cap \pi(H_p)$
If $\langle u_1^j, \phi u_2^j + \psi v_2^j \rangle = \langle \beta u_2^p + \alpha v_2^p, u_1^p \rangle$ then we can assume $u_1^j = u_1^p$, which contradicts Lemma 3.4.3. Therefore $\pi(B_j) \cap \pi(H_p) = \emptyset$.
 - $\pi(A'_j) \cap \pi(H_p)$
If $\langle au_1^j + bv_1^j, u_2^j \rangle = \langle \beta u_2^p + \alpha v_2^p, u_1^p \rangle$ then we can assume $u_2^j = \beta u_2^p + \alpha v_2^p$. Since $p \in \overline{C}_2 - \overline{C}_1$, this contradicts Lemma 3.4.4. Thus $\pi(A'_j) \cap \pi(H_p) = \emptyset$.
 - $\pi(E_p) \cap \pi(E_q)$
If it is non empty, then $\langle u_1^p, A_p u_2^p - \epsilon_p B_p v_2^p \rangle = \langle u_1^q, A_q u_2^q - \epsilon_q B_q v_2^q \rangle$. In particular, we can assume $u_1^p = u_1^q$ which contradicts Lemma 3.4.5. Thus $\pi(E_p) \cap \pi(E_q) = \emptyset$.
 - $\pi(E_p) \cap \pi(F_q)$
If $\langle u_1^p, Au_2^p - \epsilon Bv_2^p \rangle = \langle \beta u_1^q + \alpha v_1^q, u_2^q \rangle$ then we can assume $u_1^p = \beta u_1^q + \alpha v_1^q$ with $\alpha \neq 0$ and $u_2^q = \gamma u_2^p + v_2^p$ by Lemma 3.4.5. Moreover, we have $p \in \overline{C}_1$ and $q \in \overline{C}_1$ by hypothesis.
By Lemma 3.4.5, u_2^p and u_2^q cannot be colinear so by Lemma 3.1.6, $\langle u_1^p, v_1^p \rangle \neq \langle u_1^q, v_1^q \rangle$ and $A_1 = \langle u_1^p, v_1^p, u_1^q, v_1^q \rangle$. But since both p and q are in \overline{C}_1 , $\mu(u_1^p, u_1^p) = \mu(u_1^p, v_1^p) = 0$, $\mu(u_1^p, u_1^q) = \mu(\beta u_1^q + \alpha v_1^q, u_1^q) = 0$ and $\mu(u_1^p, v_1^q) = \mu(\beta u_1^q + \alpha v_1^q, v_1^q) = 0$. So u_1^p is μ -orthogonal to A_1 , thus we can assume it is μ_1 . Then p is the unique point in \overline{C}_1 such that $u_1^p = \mu_1$.
Let q be any other point in \overline{C}_1 . Then $\mu_1 \in \langle u_1^q, v_1^q \rangle$ so $\mu_1 = u_1^q = au_1^q + bv_1^q$ with $b \neq 0$ by Lemma 3.4.5. Finally, u_2^q is μ -orthogonal to μ_1 (because $\mu_1 \in \langle u_1^q, v_1^q \rangle$) and since $\langle u_2^p, v_2^p \rangle = \text{Ker}(\mu(\mu_1, \cdot)) \cap A_2$, we must have $u_2^q = cu_2^p + dv_2^p$, again with $d \neq 0$ by Lemma 3.4.5. So $\pi(E_p) \cap \pi(F_q) = \{ \langle \mu_1, u_2^q \rangle \}$.
The only case when it is not possible is when such a q is actually in $\overline{C}_1 \cap \overline{C}_2$, but then this has been considered in the case $\pi(A'_j) \cap \pi(E_p)$. Finally :
1. If p is the unique point in \overline{C}_1 such that $u_1^p = \mu_1$, and q any other point in $\overline{C}_1 - \overline{C}_2$: $\pi(E_p) \cap \pi(F_q) = \{ \langle \mu_1, u_2^q \rangle \}$.
 2. Otherwise, $\pi(E_p) \cap \pi(F_q) = \emptyset$.
- $\pi(F_p) \cap \pi(F_q)$
If $\langle \beta_p u_1^p + \alpha_p v_1^p, u_2^p \rangle = \langle \beta_q u_1^q + \alpha_q v_1^q, u_2^q \rangle$, then we can assume that $u_2^p = u_2^q$ and it contradicts Lemma 3.4.5. Thus, $\pi(F_p) \cap \pi(F_q) = \emptyset$.
 - $\pi(G_p) \cap \pi(G_q)$
If $\langle u_2^p, A_p u_1^p - \epsilon_p B_p v_1^p \rangle = \langle u_2^q, A_q u_1^q - \epsilon_q B_q v_1^q \rangle$ then we can assume $u_2^p = u_2^q$, which contradicts Lemma 3.4.5. Thus $\pi(G_p) \cap \pi(G_q) = \emptyset$.
 - $\pi(G_p) \cap \pi(H_q)$
If $\langle u_2^p, Au_1^p - \epsilon Bv_1^p \rangle = \langle \beta u_2^q + \alpha v_2^q, u_1^q \rangle$ then we can assume $u_2^p = \beta u_2^q + \alpha v_2^q$ with $\alpha \neq 0$ by Lemma 3.4.5 and $u_1^p = v_1^q + \gamma u_1^q$. Moreover both p and q are in $\overline{C}_2 - \overline{C}_1$. Since u_1^p and u_1^q are not colinear, by Lemma

3.1.6 we have $\langle u_2^p, v_2^p \rangle = \text{Ker}(\mu(u_1^p, \cdot)) \cap A_2 \neq \text{Ker}(\mu(u_1^q, \cdot)) \cap A_2 = \langle u_2^q, v_2^q \rangle$. So $A_2 = \langle u_2^p, v_2^p, u_2^q, v_2^q \rangle$. Moreover, $\mu(u_2^p, u_2^p) = \mu(u_2^p, v_2^p) = 0$, $\mu(u_2^p, u_2^q) = \mu(\beta u_2^q + \alpha v_2^q, u_2^q) = 0$ and $\mu(u_2^p, v_2^q) = \mu(\beta u_2^q + \alpha v_2^q, v_2^q) = 0$. So u_2^p is μ -orthogonal to A_2 , thus we can assume it is μ_2 . Then p is the unique point in \overline{C}_2 such that $u_2^p = \mu_2$.

Then $\langle u_1^p, v_1^p \rangle = \text{Ker}(\mu(\mu_2, \cdot)) \cap A_1$ and since u_1^q is μ -orthogonal to μ_2 because $q \in \overline{C}_2$, we must have $u_1^q \in \langle u_1^p, v_1^p \rangle$. By Lemma 3.4.5 we can then write $u_1^q = v_1^p + \gamma u_1^p$ for some γ . Moreover, for any $q \in \overline{C}_2$, $\mu_2 = u_2^p \in \langle u_2^q, v_2^q \rangle$ so by Lemma 3.4.5 we can write $u_2^p = v_2^q + \alpha u_2^q$. If such a q is in $\overline{C}_1 \cap \overline{C}_2$, it has been considered already in $\pi(B_j) \cap \pi(G_p)$. Finally,

1. If p is the unique point in \overline{C}_2 such that $u_2^p = \mu_2$, then for any other $q \in \overline{C}_2 - \overline{C}_1$ we have $\pi(G_p) \cap \pi(H_q) = \{ \langle \mu_2, u_1^q \rangle \}$.

2. Otherwise, $\pi(G_p) \cap \pi(H_q) = \emptyset$.

• $\pi(H_p) \cap \pi(H_q)$

If $\langle u_1^p, \beta_p u_2^p + \alpha_p v_2^p \rangle = \langle u_1^q, \beta_q u_2^q + \alpha_q v_2^q \rangle$, then we can assume $u_1^p = u_1^q$, which contradicts Lemma 3.4.5. Thus $\pi(H_p) \cap \pi(H_q) = \emptyset$.

• $\pi(E_p) \cap \pi(G_q)$

If this intersection is not empty then $\langle u_1^p, A_p u_2^p - \epsilon_p B_p v_2^p \rangle = \langle u_1^q, A_q u_1^q - \epsilon_q B_q v_1^q \rangle$. Then we can assume $u_1^p = v_1^q + \alpha u_1^q$ and $u_2^p = v_2^p + \beta u_2^p$. Moreover, $p \in \overline{C}_1 - \overline{C}_2$ and $q \in \overline{C}_2 - \overline{C}_1$. Then, since $\mu_1 \in \langle u_1^p, v_1^p \rangle$ but $\mu_1 \notin \langle u_1^q, v_1^q \rangle$ and since $\mu_2 \in \langle u_2^q, v_2^q \rangle$ but $\mu_2 \notin \langle u_2^p, v_2^p \rangle$, we can assume $v_1^p = \mu_1$ and $v_2^p = \mu_2$. So we have :

$$p = \langle v_1^q + \alpha u_1^q, \mu_1 \rangle \oplus \langle u_2^p, v_2^p \rangle \quad \text{with } \overline{U}_4^\perp = \langle v_1^q + \alpha u_1^q, u_2^p \rangle$$

$$q = \langle u_1^q, v_1^q \rangle \oplus \langle \beta u_2^p + v_2^p, \mu_2 \rangle \quad \text{with } \overline{U}_4^\perp = \langle u_1^q, v_2^p + \beta u_2^p \rangle.$$

If $u_1^p = v_1^q + \alpha u_1^q$ then u_2^q is μ -orthogonal to u_1^p since it is μ -orthogonal to $\langle u_1^q, v_1^q \rangle$. On the other hand, $\text{Ker}(\mu(u_1^p, \cdot)) \cap A_2 = \langle u_2^p, v_2^p \rangle$ so by Lemma 3.4.5 we should have $u_2^q = v_2^p + \beta u_2^p$.

Similarly if $u_2^p = v_2^p + \beta u_2^p$ one can find that $u_1^p = v_1^q + \alpha u_1^q$.

So if there exists $p \in \overline{C}_1 - \overline{C}_2$ and $q \in \overline{C}_2 - \overline{C}_1$ such that $u_1^p = v_1^q + \alpha u_1^q$ or $u_2^p = v_2^p + \beta u_2^p$ then $\pi(E_p) \cap \pi(G_q) = \{ \langle u_1^p, u_2^q \rangle \}$. Otherwise $\pi(E_p) \cap \pi(G_q) = \emptyset$.

Since $q \notin \overline{C}_1$ for any $\alpha \in \mathbb{R}$ by Lemma 3.1.10 there is a unique $p \in \overline{C}_1$ such that $v_1^q + \alpha u_1^q \in \langle u_1^p, v_1^p \rangle$. In this case $\langle u_1^p, v_1^p \rangle = \langle v_1^q + \alpha u_1^q, \mu_1 \rangle$ but the problem is that there is no reason for $v_1^q + \alpha u_1^q$ to be u_1^p that is to say to be μ -orthogonal to $\langle u_2^p, v_2^p \rangle$. Indeed we choose v_2^p as a nonzero vector in the line $\text{Ker}(\lambda'(v_1^q + \alpha u_1^q, \mu_1, u_2^p, \cdot)) \cap A_2 / \langle u_2^p \rangle$ and in general it has no reason to be μ -orthogonal to $v_1^q + \alpha u_1^q$. Similarly there is no specific reason for $u_2^p = v_2^p + \beta u_2^p$ to happen, but it is not impossible.

• $\pi(E_p) \cap \pi(H_q)$

If $\langle u_1^p, A u_2^p - \epsilon B v_2^p \rangle = \langle u_1^q, \beta u_2^q + \alpha v_2^q \rangle$ then we can assume $u_1^p = u_1^q$, which contradicts Lemma 3.4.5. Hence, $\pi(E_p) \cap \pi(H_q) = \emptyset$.

• $\pi(F_p) \cap \pi(G_q)$

If $\langle u_2^p, \beta u_1^p + \alpha v_1^p \rangle = \langle u_2^q, A u_1^q - \epsilon B v_1^q \rangle$ then we can assume $u_2^p = u_2^q$, which contradicts Lemma 3.4.5. Thus $\pi(F_p) \cap \pi(G_q) = \emptyset$.

• $\pi(F_p) \cap \pi(H_q)$

If $\langle u_2^p, \beta_p u_1^p + \alpha_p v_1^p \rangle = \langle \beta_q u_2^q + \alpha_q v_2^q, u_1^q \rangle$ then we can assume $u_2^p = \beta_q u_2^q + \alpha_q v_2^q$ and $u_1^p = \beta_p u_1^p + \alpha_p v_1^p$. Moreover, $p \in \overline{C}_1 - \overline{C}_2$ and $q \in \overline{C}_2 - \overline{C}_1$.

Let p be any point in $\overline{C}_1 - \overline{C}_2$. Then u_1^p is not μ -orthogonal to μ_2 since $p \notin \overline{C}_2$, so $\langle u_1^p, v_1^p \rangle \neq \text{Ker}(\mu(\mu_2, \cdot)) \cap A_1$. Since these are both two dimensional in A_1 which is 3-dimensional, they must intersect in a line. Let w be a nonzero vector in this line. Then by Lemma 3.1.10 there exists a unique $q \in \overline{C}_2$ such that $u_1^q = w$. If this point q is also in \overline{C}_1 , then $u_1^q = u_1^i = m u_1^p + n v_1^p$. Moreover we have assumed $u_1^i \neq \mu_1$, hence :

$\langle u_1^i, \mu_1 \rangle = \langle u_1^p, v_1^p \rangle = A_1 \cap p_i = A_1 \cap p$. This contradicts Lemma 3.1.10. So let us assume now that $q \in \overline{C_2} - \overline{C_1}$. By Lemma 3.4.5 we can assume $u_1^q = v_1^p + au_1^p$. Moreover, since u_2^p is μ -orthogonal to $\langle u_1^p, v_1^p \rangle$, it is in particular μ -orthogonal to $v_1^p + au_1^p = u_1^q$. So $u_2^p \in \text{Ker}(\mu(u_1^q, \cdot)) \cap A_2 = \langle u_2^q, v_2^q \rangle$. By Lemma 3.4.5 we can then write $u_2^p = v_2^q + bu_2^q$. Therefore, to each $p \in \overline{C_1} - \overline{C_2}$ corresponds a unique $q \in \overline{C_2} - \overline{C_1}$, such that $\pi(F_p) \cap \pi(H_q) = \{\langle u_2^p, u_1^q \rangle\}$. Similarly, to each $q \in \overline{C_2} - \overline{C_1}$ corresponds a unique $p \in \overline{C_1} - \overline{C_2}$, such that $\pi(F_p) \cap \pi(H_q) = \{\langle u_2^p, u_1^q \rangle\}$.

So finally, let p be the point in $\overline{C_1} - \overline{C_2}$ such that $u_1^p = \mu_1$. Then

- $\pi(A'_1) \cap \pi(E_p) = \{\langle \mu_1, u_2^1 \rangle\}$;
- $\pi(A'_2) \cap \pi(E_p) = \{\langle \mu_1, u_2^2 \rangle\}$;
- for any other $q \in \overline{C_1} - \overline{C_2}$, $\pi(F_q) \cap \pi(E_p) = \{\langle \mu_1, u_2^q \rangle\}$.

By Lemma 3.4.5, each of these points of intersection are different. So the A'_i seem to be an "extension" of the F_q on $\overline{C_1}$. Moreover we have already seen that $\pi(E_p) \cap \pi(F_p) = \{\langle u_1^p, u_2^p \rangle\} = \{\langle \mu_1, u_2^p \rangle\}$. Then for any $q \in \overline{C_1}$, $\pi(F_q) \cap \pi(E_p) = \{\langle \mu_1, u_2^q \rangle\}$. Similarly, let q be the point in $\overline{C_2} - \overline{C_1}$ such that $u_2^q = \mu_2$. Then

- $\pi(B_1) \cap \pi(G_q) = \{\langle \mu_2, u_1^1 \rangle\}$;
- $\pi(B_2) \cap \pi(G_q) = \{\langle \mu_2, u_1^2 \rangle\}$;
- for any other $p \in \overline{C_2} - \overline{C_1}$, $\pi(G_q) \cap \pi(H_p) = \{\langle \mu_2, u_1^p \rangle\}$.

By Lemma 3.4.5, each of these points of intersection are different. So the B_i seem to be an "extension" of the H_p on $\overline{C_2}$. Moreover we have already seen that $\pi(G_q) \cap \pi(H_q) = \{\langle u_1^q, u_2^q \rangle\} = \{\langle \mu_2, u_1^q \rangle\}$. Then for any $p \in \overline{C_2}$, $\pi(G_q) \cap \pi(H_p) = \{\langle \mu_2, u_1^p \rangle\}$.

More generally, there is a pairing between points in $p \in \overline{C_1} - \overline{C_2}$ and points $q \in \overline{C_2} - \overline{C_1}$. For each such pair, $\pi(F_p) \cap \pi(H_q) = \{\langle u_2^p, u_1^q \rangle\}$. The "extension" of this phenomenon is given by : for each $i \in \{1, 2\}$, $\pi(A'_i) \cap \pi(B_i) = \{\langle w_0, u_1^i, u_2^i \rangle\}$.

Finally for some points $p \in \overline{C_1} - \overline{C_2}$ and $q \in \overline{C_2} - \overline{C_1}$ there can be $\pi(E_p) \cap \pi(G_q) = \{\langle u_1^p, u_2^q \rangle\}$.

So both C'_1 and C'_2 are contracted to distinct points by π . If p is the unique point of $\overline{C_1}$ with $u_1^p = \mu_1$, then E_p is contracted to another \mathbb{P}^1 in $\{F_q, q \in \overline{C_1}\} \cong \mathbb{P}^1 \times \mathbb{P}^1$. If q is the unique point of $\overline{C_2}$ with $u_2^q = \mu_2$, then G_q is contracted to another \mathbb{P}^1 in $\{H_p, p \in \overline{C_2}\} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Finally to each $p \in \overline{C_1} - \overline{C_2}$ corresponds a unique $p' \in \overline{C_2} - \overline{C_1}$ such that $\pi(F_p) \cap \pi(H_{p'})$ contains one point. And there may exist some other pairs of points $p \in \overline{C_1} - \overline{C_2}$ and $q \in \overline{C_2} - \overline{C_1}$ such that $\pi(E_p) \cap \pi(G_q)$ contains one point. \square

About other points in $E_4 \cap \overline{E_4}$ I don't see any reason for them not to be sometimes contracted by π either on one another or on some points in the \mathbb{P}^1 listed in Definition 3.3.8. I don't see either any reason for such contractions to have a specific pattern in general.

3.4.2 Contractions in \tilde{X}_4

We would like now to try to understand $D_3 \cap \overline{E_4}$ and more precisely $\tilde{S} \cap D_3$ and $\overline{D_2} \cap D_3$, and what π contracts in those intersections.

To do so, we can use what we saw in the previous subsection 3.4.1, together with the fact that $\tilde{X}_4 = \tilde{X}_5 \cap \pi^{-1}H_\nu$ where H_ν is a hyperplane associated to a hyperplane section ν of $\mathcal{O}_{X_5}(1)$ (this has been described more precisely in Theorem 1.2.3). As a result, $\overline{E_4} \cap D_3$ consists in all the points of $\overline{E_4} \cap E_4$ on which the section ν vanishes. Moreover, as mentioned in section 1.1, to each hyperplane section ν we can associate a three-form on $W \cong \mathbb{C}^7$, which we are also going to call ν .

We can therefore look at the points in $E_4 \cap \overline{E_4}$ which we had found to be contracted in subsection 3.4.1, and try to understand when these points also lie in $\pi^{-1}H_\nu$ using the 3-form ν and the geometric descriptions of fibers

in both \overline{E}_4 and Z_3 , which we already used in subsection 3.4.1.

Since we had only found partial results about contractions in $E_4 \cap \overline{E}_4$, we also find only partial results here.

Definition 3.4.8. Let us denote $\nu = \overline{\nu} \wedge w_0^\vee + \nu'$ where $\overline{\nu}$ is a 2-form on \overline{W} and ν' is a 3-form on \overline{W} .

Lemma 3.4.9. Let $p \in S_2 - (\overline{C}_1 \cup \overline{C}_2)$. Let us assume that $\overline{\nu}$ induces a non-degenerate pairing between A_1 and A_2 (using the notations introduced in subsection 1.1). Then in general there are two distinct points z_1 and z_2 in $f_Z(p)$ such that $f_{\overline{E}}(z_i) \cap D_3$ contains one point for $i = 1$ and $i = 2$. For all the other points $z \in f_Z(p)$, $f_{\overline{E}}(z) \cap D_3 = \emptyset$. It is also possible that $f_{\overline{E}}(z) \cap D_3$ contains one point for exactly one $z \in f_Z(p)$ or for all $z \in f_Z(p)$.

Proof. Let $p \in S_2 - (\overline{C}_1 \cup \overline{C}_2)$. As we have seen in Lemma 3.3.4, $f_Z(p) = \{ \langle u_1 + u_2, v_1 + v_2, Au_1 + Bv_1 \rangle, (A, B) \in \mathbb{C}^2 - \{(0, 0)\} \}$ and for such a point z , $f_{\overline{E}}(z) \cap E_4 = \{ \langle w_0, Au_1 + Bv_1, Au_2 + Bv_2 \rangle \}$. Moreover, by 7. and 8. of Theorem 1.2.3, D_3 consists in all the $U_3 \subset U_4$ in E_4 such that U_3 is ν -isotropic. Since the points in E_4 are all of the form $\overline{U}_2 \oplus \langle w_0 \rangle \subset \overline{U}_3 \oplus \langle w_0 \rangle$, a point in E_4 is ν -isotropic if and only if $\overline{\nu}(\overline{U}_2) = 0$. Finally, $\overline{\nu}(Au_1 + Bv_1, Au_2 + Bv_2) = A^2\overline{\nu}(u_1, u_2) + AB(\overline{\nu}(u_1, v_2) + \overline{\nu}(v_1, u_2)) + B^2\overline{\nu}(v_1, v_2)$. If $\overline{\nu}(u_1, u_2) = \overline{\nu}(v_1, u_2) = \overline{\nu}(u_1, v_2) = \overline{\nu}(v_1, v_2) = 0$ and $\overline{\nu}$ induces a non-degenerate pairing between the A_i then we have $\text{Ker}(\overline{\nu}(u_1, \cdot)) \cap A_2 = \langle u_2, v_2 \rangle = \text{Ker}(\overline{\nu}(v_1, \cdot))$ so by the same reasoning as in Lemma 3.1.6, u_1 and v_1 are colinear which is a contradiction. So under our hypothesis, $\overline{\nu}(Au_1 + Bv_1, Au_2 + Bv_2)$ is not identically zero. Let us consider several cases now :

- $A = 0$, then we can assume $B = 1$. In this case, $\overline{\nu}(Au_1 + Bv_1, Au_2 + Bv_2) = \overline{\nu}(v_1, v_2)$.
- $A \neq 0$ so we can assume $A = 1$. Then

$$\overline{\nu}(Au_1 + Bv_1, Au_2 + Bv_2) = \overline{\nu}(u_1, u_2) + B(\overline{\nu}(u_1, v_2) + \overline{\nu}(v_1, u_2)) + B^2\overline{\nu}(v_1, v_2). \quad (3.26)$$

Again there are several possibilities :

- If $\overline{\nu}(v_1, v_2) \neq 0$ then (3.26)= 0 is an equation of degree 2 in B so there are either one or two solutions.
- If $\overline{\nu}(v_1, v_2) = 0$ and $\overline{\nu}(u_1, v_2) + \overline{\nu}(v_1, u_2) \neq 0$ then (3.26)= 0 is an equation of degree 1 in B so there is one solution.
- If $\overline{\nu}(v_1, v_2) = 0$ and $\overline{\nu}(u_1, v_2) + \overline{\nu}(v_1, u_2) = 0$ then (3.26)= 0 is equivalent to $\overline{\nu}(u_1, u_2) = 0$.

So finally :

1. If $\overline{\nu}(v_1, v_2) \neq 0$ there is one or two points in $f_Z(p)$ whose $f_{\overline{E}}$ intersects D_3 in one point.
2. If $\overline{\nu}(v_1, v_2) = 0$ and $\overline{\nu}(u_1, v_2) + \overline{\nu}(v_1, u_2) \neq 0$ there are two points in $f_Z(p)$ whose $f_{\overline{E}}$ intersects D_3 in one point, namely $A = 0$ and $A = 1$, B is the unique solution of (3.26)= 0.
3. If $\overline{\nu}(v_1, v_2) = 0$, $\overline{\nu}(u_1, v_2) + \overline{\nu}(v_1, u_2) = 0$ and $\overline{\nu}(u_1, u_2) \neq 0$ then there is one point in $f_Z(p)$ whose $f_{\overline{E}}$ intersects D_3 in one point, namely $A = 0$.
4. If $\overline{\nu}(v_1, v_2) = 0$, $\overline{\nu}(u_1, v_2) + \overline{\nu}(v_1, u_2) = 0$ and $\overline{\nu}(u_1, u_2) = 0$ then for all $z \in f_Z(p)$, we have $f_{\overline{E}}(z) \cap D_3 = f_{\overline{E}}(z) \cap E_4 = \{1\text{pt}\}$.

□

Lemma 3.4.10. Let P be one of the \mathbb{P}^1 listed in Definition 3.3.8, then if $P \neq C'_i$, in general $P \cap D_3$ is one point, otherwise $P \subset D_3$. If $P = C'_i$ then in general $P \cap D_3 = \emptyset$, otherwise $P \subset D_3$.

Proof. Let $p \in \overline{C}_1 \cup \overline{C}_2$. As we saw in Lemma 3.4.9, the points in some $f_{\overline{E}}$ over $f_Z(p)$ which are in D_3 are those who are in E_4 and such that the associated \overline{U}_2 is $\overline{\nu}$ -isotropic. By Lemma 3.3.3 it is thus the points in one of the \mathbb{P}^1 listed in Definition 3.3.8 whose \overline{U}_2 is $\overline{\nu}$ -isotropic. Let us proceed case by case :

- For A'_i . We look at the $\langle au_1^i + bv_1^i, u_2^i \rangle$ which are $\bar{\nu}$ -isotropic, that is to say such that :

$$\bar{\nu}(au_1^i + bv_1^i, u_2^i) = a\bar{\nu}(u_1^i, u_2^i) + b\bar{\nu}(v_1^i, u_2^i) = 0. \quad (3.27)$$

If one at least of $\bar{\nu}(u_1^i, u_2^i)$ and $\bar{\nu}(v_1^i, u_2^i)$ is non-zero (3.27) has one solution up to multiplication by a scalar, otherwise it is satisfied for any a and b .

- For B_i . With the same reasoning as for the A'_i , if one at least of $\bar{\nu}(u_1^i, u_2^i)$ and $\bar{\nu}(u_1^i, v_2^i)$ is non-zero then $B_i \cap D_3$ contains exactly one point, otherwise $B_i \subset D_3$.
- For C'_i . Either $\bar{\nu}(u_1^i, u_2^i) \neq 0$ and $C'_i \cap D_3 = \emptyset$ or $\bar{\nu}(u_1^i, u_2^i) = 0$ and $C'_i \subset D_3$.
- For E_p . We consider the equation :

$$\bar{\nu}(u_1^p, au_2^p + bv_2^p) = a\bar{\nu}(u_1^p, u_2^p) + b\bar{\nu}(u_1^p, v_2^p) = 0 \quad (3.28)$$

where $a = A$ and $b = -\epsilon B$ with the notations of Definition 3.3.8. Similarly to the cases of A'_i and B_i , either one at least of $\bar{\nu}(u_1^p, u_2^p)$ and $\bar{\nu}(u_1^p, v_2^p)$ is non zero and $E_p \cap D_3$ is exactly one point, or $\bar{\nu}(u_1^p, u_2^p) = \bar{\nu}(u_1^p, v_2^p) = 0$ and $E_p \subset D_3$.

- Similarly, either one at least of $\bar{\nu}(u_1^p, u_2^p)$ and $\bar{\nu}(v_1^p, u_2^p)$ is non zero and $F_p \cap D_3$ is exactly one point, or $\bar{\nu}(u_1^p, u_2^p) = \bar{\nu}(v_1^p, u_2^p) = 0$ and $F_p \subset D_3$.
- With the same reasoning, either one at least of $\bar{\nu}(u_1^p, u_2^p)$ and $\bar{\nu}(v_1^p, v_2^p)$ is non zero and $G_p \cap D_3$ is exactly one point, or $\bar{\nu}(u_1^p, u_2^p) = \bar{\nu}(v_1^p, v_2^p) = 0$ and $G_p \subset D_3$.
- As before, either one at least of $\bar{\nu}(u_1^p, u_2^p)$ and $\bar{\nu}(u_1^p, v_2^p)$ is non zero and $H_p \cap D_3$ is exactly one point, or $\bar{\nu}(u_1^p, u_2^p) = \bar{\nu}(u_1^p, v_2^p) = 0$ and $H_p \subset D_3$.

□

Remark 3.4.11. Let p and q be the points in $\bar{C}_1 - \bar{C}_2$, respectively $\bar{C}_2 - \bar{C}_1$, such that μ_1 , respectively μ_2 , is in the associated \bar{U}_4^\perp . Then we have seen in Lemma 3.4.7 that each point in E_p is contracted by π to some point in a $F_{p'}$, except for the point in $E_p \cap F_p$ and that similarly each point in $G_q - (G_q \cap H_q)$ is contracted to another point in some $H_{q'}$. Obviously if a point is in D_3 any other point in its fiber in E_4 is also in D_3 , hence by Lemma 3.4.10 either $E_p \subset D_3$ and π is contracting a \mathbb{P}^1 onto a another \mathbb{P}^1 in $D_3 \cap \tilde{S}$, or $E_p \cap D_3$ is one point and unless it is precisely $E_p \cap F_p$, π will contract this point to another point in some $F_{p'}$ in $D_3 \cap \tilde{S}$. For G_q it is similar. So in general π should contract at least a point in $\tilde{S} \cap D_3$.

Before going further, let us remember from Theorem 1.2.3 that \bar{D}_2 is a ruled surface over the curve Γ_1 : indeed $\bar{D}_2 = \mathbb{P}_\Gamma(\mathcal{V}_Z|_\Gamma)$. So \bar{D}_2 consists in all the fibers $\{U_3 \subset \bar{U}_3 \oplus \langle w_0 \rangle\}$ in \bar{E}_4 which are also in \tilde{X}_4 , that is to say all the fibers in \bar{E}_4 such that all their associated U_3 are ν -isotropic.

Lemma 3.4.12. Let $p \in \Gamma_1$. If $p \in C_1 \cup C_2$, there are two possibilities :

1. $f_{\bar{E}}(p) \subset D_3$;
2. $f_{\bar{E}}(p) \cap D_3$ contains one point.

If $p \notin C_1 \cup C_2$ then either $f_{\bar{E}}(p) \cap D_3$ is one point or it is empty.

Proof. To start with, since D_3 is a hyperplane section of E_4 , $D_3 \subset E_4$ and to compute $f_{\bar{E}}(p) \cap D_3$ it is enough to compute $f_{\bar{E}}(p) \cap E_4 \cap D_3$.

Let us first consider the case $p \in C_1 \cup C_2$. From Lemma 3.3.3 and the definition of the C_i given in 3. of Proposition 1.2.2, we know that $f_{\bar{E}}(p) \subset E_4$ and is either A'_i , B_i , F_p or H_p with the notations of Definition 3.3.8. In other words, $f_{\bar{E}}(p)$ is either of the form $\{\langle au_1 + bv_1, u_2 \rangle, (a, b) \in \mathbb{C}^2 - \{(0, 0)\}\}$ or $\{\langle au_2 + bv_2, u_1 \rangle, (a, b) \in \mathbb{C}^2 - \{(0, 0)\}\}$ with $u_i \in A_i$ and $v_i \in A_i$. Let us consider the first case : then as in the proof of Lemma

3.4.9, a point $U_3 = \bar{U}_2 \oplus \langle w_0 \rangle \subset \bar{U}_3 \oplus \langle w_0 \rangle \in f_{\bar{E}}(p)$ is in D_3 if and only if $\bar{v}(\bar{U}_2) = 0$. In our case, if and only if :

$$a\bar{v}(u_1, u_2) + b\bar{v}(v_1, u_2) = 0. \quad (3.29)$$

Therefore, there are two possibilities :

1. At least one of $\bar{v}(u_1, u_2)$ and $\bar{v}(v_1, u_2)$ is not zero. Then (3.29) has one solution up to multiplication by a non zero scalar so $f_{\bar{E}}(p) \cap D_3$ is one point.
2. $\bar{v}(u_1, u_2) = 0$ and $\bar{v}(v_1, u_2) = 0$. Then (3.29) is always satisfied and $f_{\bar{E}}(p) \subset D_3$.

For the second case it is very similar : either at least one of $\bar{v}(u_1, u_2)$ and $\bar{v}(u_1, v_2)$ is not zero and $f_{\bar{E}}(p) \cap D_3$ is one point, or $\bar{v}(u_1, u_2) = 0 = \bar{v}(u_1, v_2) = 0$ and $f_{\bar{E}}(p) \subset D_3$.

Let us assume now that $p \notin C_1 \cup C_2$. Then by Lemma 3.3.3 and Lemma 3.3.4 we have that $f_{\bar{E}}(p) \cap E_4$ is exactly one point : if its associated \bar{U}_2 is \bar{v} -isotropic then $f_{\bar{E}}(p) \cap D_3$ is one point, otherwise it is empty. \square

Remark 3.4.13. *From the proof of Theorem 5.1 in [20], we know that $\sigma(\Gamma_1)$ is a curve of class $4(\bar{h}_1 + \bar{h}_2)$. Hence in S_2 and for $i \in \{1, 2\}$, one has $\sigma(\Gamma_1) \cdot \bar{C}_i = 4\bar{h}_i(\bar{h}_1 + \bar{h}_2) = 12$. Since $\sigma(C_i) = \bar{C}_i$ there are at most 24 points in $\Gamma_1 \cap (C_1 \cup C_2)$.*

3.5 \bar{D}_2 and Γ_1

In this subsection we are taking a closer look at the curve Γ_1 (see (1.4)), which will be useful to work with the surface \bar{D}_2 later.

The genus of Γ_1 which we find in Lemma 3.5.1 and the fact that $\sigma(\Gamma_1)$ is not smooth, as we will see in Lemma 3.5.2, contradict a statement in Theorem 5.1 of [20]. These two results, as well as advices about the proof of the two Lemmas, were kindly explained to me by A. Kuznetsov.

Lemma 3.5.1. *The curve Γ_1 is smooth in Z_3 , with genus 13. The Chern character of its ideal sheaf is : $\text{ch}(\mathcal{I}_{\Gamma_1}) = (1, 0, -4V_Z(\bar{h}_1 + \bar{h}_2), 24)$.*

Proof. By the proof of Lemma 2.1 in [20] and the proof of Proposition 3.2.6, there is a short exact sequence

$$0 \rightarrow \mathcal{O}_{Z_3} \xrightarrow{\nu} \mathcal{V}_Z^\vee \rightarrow \det(\mathcal{V}_Z^\vee) \otimes \mathcal{I}_{\Gamma_1} \rightarrow 0$$

where \mathcal{I}_{Γ_1} is the ideal sheaf of Γ_1 . This is exactly the short exact sequence mentioned in Corollary 3.2.5, hence $\text{ch}(\mathcal{I}_{\Gamma_1}) = (1, 0, -4V_Z(\bar{h}_1 + \bar{h}_2), 24)$.

Let us now show that Γ_1 is smooth in Z_3 . For this, we can show that \mathcal{V}_Z^\vee is globally generated and use Bertini's theorem.

By (1.6) there is a short exact sequence

$$0 \rightarrow \mathcal{O}_{Z_3}(\bar{h}_1) \oplus \mathcal{O}_{Z_3}(\bar{h}_2) \rightarrow \bigwedge^3 \bar{U}_3^\vee \oplus \bigwedge^2 \bar{U}_3^\vee \rightarrow \mathcal{V}_Z^\vee \rightarrow 0$$

hence by Lemma 2.11 in [3], if $\bigwedge^3 \bar{U}_3^\vee$ and $\bigwedge^2 \bar{U}_3^\vee$ are globally generated then so will be \mathcal{V}_Z^\vee .

- First, since \bar{U}_3 is the tautological bundle of $\text{Gr}(3, \bar{W}) = \text{Gr}(3, 6)$, we have that $\bigwedge^3 \bar{U}_3^\vee = \mathcal{O}_{\text{Gr}(3,6)}(\bar{h})$ (see for instance part 2.1 in [20] or Lemma 4.2.9). Then we have that the set of global sections of $\bigwedge^3 \bar{U}_3^\vee$ corresponds to the space of 3-forms on \bar{W} . Let now \bar{U}_3 be a point in $\text{Gr}(3, 6)$. By definition, $(\bigwedge^3 \bar{U}_3^\vee)_{\bar{U}_3} \cong \bigwedge^3 \bar{U}_3 \cong \mathbb{C}$ is generated by a 3-form. As result, taking a basis of the space of 3-forms on \bar{W} gives a finite number of global sections of $\bigwedge^3 \bar{U}_3^\vee$ which generate $(\bigwedge^3 \bar{U}_3^\vee)_{\bar{U}_3}$ for any $\bar{U}_3 \in \text{Gr}(3, 6)$. So $\bigwedge^3 \bar{U}_3^\vee$ is globally generated on $\text{Gr}(3, 6)$.

- Since $\text{Gr}(3, 6) = \text{Gr}(6 - 3, 6)$, by (2.4.1) in [15] we have a natural identification $\Lambda^2 \bar{U}_3^\vee \cong \bar{U}_3(1)$, therefore the space of global sections of $\Lambda^2 \bar{U}_3^\vee$ corresponds to the space of 2-forms on \bar{W} (as in the case of the global sections of $\mathcal{U}_3(1)$ on $\text{Gr}(3, 7)$). Let now \bar{U}_3 be a point in $\text{Gr}(3, 6)$. Then $(\Lambda^2 \bar{U}_3^\vee)_{\bar{U}_3} = \Lambda^2 \bar{U}_3 \cong \mathbb{C}^3$ is generated by three 2-forms, thus taking a basis of the space of 2-forms on \bar{W} gives a finite number of global sections of $\Lambda^2 \bar{U}_3^\vee$ which generate $(\Lambda^2 \bar{U}_3^\vee)_{\bar{U}_3}$ for any $\bar{U}_3 \in \text{Gr}(3, 6)$. Therefore $\Lambda^2 \bar{U}_3^\vee$ is globally generated on $\text{Gr}(3, 6)$.

Finally, by construction we have $Z_3 \subset \bar{X}_5 \subset \text{Gr}(3, 6)$ (see Theorem 4.2 in [20]), and a global section of a sheaf on $\text{Gr}(3, 6)$ thus induces a global section of the restriction of this sheaf to Z_3 . As a result, both $\Lambda^3 \bar{U}_3^\vee$ and $\Lambda^2 \bar{U}_3^\vee$ are globally generated on Z_3 .

Finally, one can conclude with Bertini's Theorem, using for instance Proposition A in [26]. For a general X_4 , the associated Γ_1 is thus smooth.

To compute the genus of Γ_1 we can then use Hirzebruch-Riemann-Roch (see Theorem 4.1.1). Indeed, let us denote by ι the inclusion of Γ_1 in Z_3 . We then have :

$$\begin{aligned} \chi(\mathcal{O}_{Z_3}, \iota_* \mathcal{O}_{\Gamma_1}) &= \chi(\iota^* \mathcal{O}_{Z_3}, \mathcal{O}_{\Gamma_1}) = 1 - g(\Gamma_1) \\ &= \int_{Z_3} (0, 0, 4V_Z(\bar{h}_1 + \bar{h}_2), -24) \cdot (1, V_Z + \frac{1}{2}(\bar{h}_1 + \bar{h}_2), x_2, 1) = -24 + 12 = -12 \end{aligned}$$

where we used Lemma 4.5.3, which we will see later, and $\text{ch}(\mathcal{O}_{\Gamma_1}) = (1, 0, 0, 0) - \text{ch}(\mathcal{I}_{\Gamma_1})$. This yields $g(\Gamma_1) = 1 + 12 = 13$. \square

Lemma 3.5.2. *Using the notations of (1.4), $\sigma(\Gamma_1)$ isn't smooth in S_2 . Its geometric genus is 13 and its arithmetic genus is 37.*

Proof. First of all, by (1.15), $G := \sigma(\Gamma_1)$ has class $4(\bar{h}_1 + \bar{h}_2)$ in S_2 . The genus formula then yields (by I.15 of [1]):

$$g_a(G) = 1 + \frac{1}{2}(16(\bar{h}_1 + \bar{h}_2)^2 - 4(\bar{h}_1 + \bar{h}_2)^2) = 1 + \frac{12(\bar{h}_1 + \bar{h}_2)^2}{2} = 1 + 6.6 = 1 + 36 = 37$$

since we know from 11. of Theorem 1.2.1 that $K_S = -\bar{h}_1 - \bar{h}_2$, and using (1.13).

On the other hand, if G was smooth by Proposition 1.1 in IV.1. of [12] we would have that its geometric and arithmetic genus coincide. Moreover in such a case by Theorem 8.19 in II.8. of [12] we would have $g_g(G) = g_g(\Gamma_1) = 13 \neq 37$ as we saw in Lemma 3.5.1. \square

Chapter 4

Computation of Euler characteristics between elements of \mathcal{F}_1

In this section, we try to compute all the Euler characteristics between elements in the family

$$\mathcal{F}_1 = \{j_*\mathcal{O}_{E_1}(E_1 - V_E), j_*\mathcal{O}_{E_2}(E_2 - V_E), j_*\mathcal{O}_{E_3}(E_3 - V_E), j_*\mathcal{O}_{D_3}(-V_E), j_*\mathcal{O}_{D_3}(h_1 - V_E), j_*\mathcal{O}_{D_3}(2h_1 - V_E), \mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{l_h}, \mathcal{O}_p, \mathcal{O}_{D_3}(V_E), \mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\tilde{S}}\},$$

defined in subsection 2.1.2. As there are 14 elements in this family, this requires a lot of computations, for which the two main Theorems we use are Hirzebruch-Riemann-Roch (either the general version or some more specific one for curves and surfaces) and Borel-Bott-Weil. This section is thus split into several subsections : each subsection is devoted to the computation of some of the Euler characteristics we are interested in, for which we use either the same Theorem or the same geometric characteristics of some of the varieties mentioned in section 3.

More precisely : in subsection 4.1 we compute Euler characteristics between the first six elements of \mathcal{F}_1 , which are all twists of pull-backs of sheaves from $D^b(\Sigma_2)$. Then in subsection 4.2 we compute some Euler characteristics between line bundles of the form $\pi^*\mathcal{O}_{X_4}(ih)$ with $i \in \mathbb{N}$, using the Theorem of Borel-Bott-Weil. In subsections 4.3, 4.4 and 4.5 we compute Euler characteristics involving respectively $\mathcal{O}_{D_3}(V_E)$, $\mathcal{O}_{\overline{D}_2}$ and $\mathcal{O}_{\tilde{S}}$. In subsection 4.6, we compute Euler characteristics between $\mathcal{O}_{\overline{D}_2}$ and $\mathcal{O}_{\tilde{S}}$, and in subsection 4.7, we compute, up to an intersection number b , Euler characteristics between the first six elements and the last eight elements of \mathcal{F}_1 , except for \mathcal{O}_{l_h} and \mathcal{O}_p . Finally in subsection 4.8 we compute Euler characteristics involving \mathcal{O}_{l_h} and \mathcal{O}_p .

All the results of this section are gathered in Proposition 2.1.5.

4.1 Euler characteristics between sheaves coming from $D^b(\Sigma_2)$

In this subsection, we compute Euler characteristics between the first six elements of \mathcal{F}_1 .

Before going further, let us recall the theorem of Hirzebruch-Riemann-Roch (see for instance Corollary 5.27 of [14] and the fourth formula page 84 of [14]).

Theorem 4.1.1 (Hirzebruch-Riemann-Roch). *Let X be a smooth projective variety and $E, F \in D^b(X)$. Then :*

$$\chi(F, E) = \chi(\mathcal{O}_X, E \otimes F^\vee) = \int_X \text{ch}(E) \cdot \text{ch}(F^\vee) \cdot \text{td}(X)$$

where $\text{td}(X)$ is the Todd class of X , that is to say the Todd class of its tangent sheaf.

We will need to apply this Theorem of Hirzebruch-Riemann-Roch to compute Euler characteristics in Σ_2 ; this is why we want now to compute $\text{td}(\Sigma_2)$. But for this we need first to know the canonical bundle of Σ_2 .

Lemma 4.1.2. *With the same notations as in Definition 1.3.3, the canonical bundle of Σ_2 is $\omega_\Sigma = \mathcal{O}_{\Sigma_2}(i^*(-h))$ which we are going to denote by abuse of notation as $\mathcal{O}_{\Sigma_2}(-h)$.*

Proof. First of all, we know from 7. of Theorem 1.2.3 that Σ is cut out in $\mathbb{P}(A_1) \times \mathbb{P}(A_2) \cong \mathbb{P}^2 \times \mathbb{P}^2$ by two divisors of bidegree $(1, 1)$. Moreover in the proof of Proposition 4.6 in [20], A. Kuznetsov computes the anticanonical class of S_2 using the adjunction formula and the fact that S_2 is cut out in some $\mathbb{P}^2 \times \mathbb{P}^2$ by two divisors of bidegree $(1, 1)$ (as recalled in 11. of Theorem 1.2.1). Using the same reasoning with $\bigwedge^2 A_i$ replaced by A_i and \bar{h}_i replaced by h_i one gets that $\omega_\Sigma = \mathcal{O}_{\Sigma_2}(-h_1 - h_2)$. Finally in $\text{Pic}(F)$ one has $h = h_1 + h_2$ by (1.9) and this equality still holds when restricted to Σ_2 so $\omega_\Sigma = \mathcal{O}_{\Sigma_2}(-h)$. \square

Lemma 4.1.3. *The Todd class of Σ_2 is $\text{td}(\Sigma_2) = (1, \frac{h}{2}, 1)$.*

Proof. We have seen in Lemma 4.1.2 that $\omega_\Sigma = \mathcal{O}_{\Sigma_2}(-h)$, hence $c_1(\mathcal{T}_{\Sigma_2}) = h$ since $\omega_\Sigma = \bigwedge^2 \mathcal{T}_{\Sigma_2}^\vee$. One can then conclude that $\text{td}(\Sigma_2) = (1, \frac{h}{2}, x)$ with the formula page 432 of [12]. Finally, by Hirzebruch-Riemann-Roch :

$$\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}) = \int_{\Sigma_2} (1, 0, 0) \cdot (1, \frac{h}{2}, x) = x$$

thus $x = 1$ because as we will see in the proof of Lemma 4.3.5, we have $\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}) = 1$. \square

Remark 4.1.4. *By [20], we know that S_2 and Σ_2 are two del Pezzo surfaces of degree 6 cut by two divisors of bidegree $(1, 1)$ in respectively $\mathbb{P}(\bigwedge^2 A_1) \times \mathbb{P}(\bigwedge^2 A_2)$ and $\mathbb{P}(A_1) \times \mathbb{P}(A_2)$. The same reasoning as in Remark 4.7 of [20] applies thus to Σ_2 , which can then be seen as the blow-up of $\mathbb{P}(A_1)$ in three points, with exceptional divisors E_1, E_2 and E_3 . With the exact same reasoning as later in Remark 4.5.2 and Definition 1.3.3, we then get that $\text{Pic}(\Sigma_2) = \langle h_1, E_1, E_2, E_3 \rangle$. Moreover, by d) of Proposition 4.8 in V.4 of [12], we have $\omega_\Sigma = \mathcal{O}_{\Sigma_2}(-3h_1 + E_1 + E_2 + E_3)$. Together with (1.9) and Lemma 4.1.2, this yields $h_2 = 2h_1 - E_1 - E_2 - E_3$. In particular, for $1 \leq i, j \leq 3$, one has :*

$$h_1 \cdot E_i = 0 \quad E_i \cdot E_j = -\delta_{i,j} \quad h_2 \cdot E_i = 1. \quad (4.1)$$

Lemma 4.1.5. *Let*

$$(x_1, \dots, x_6) = (j_* \mathcal{O}_{E_1}(E_1 - V_E), j_* \mathcal{O}_{E_2}(E_2 - V_E), j_* \mathcal{O}_{E_3}(E_3 - V_E), j_* \mathcal{O}_{D_3}(-V_E), j_* \mathcal{O}_{D_3}(h_1 - V_E), j_* \mathcal{O}_{D_3}(2h_1 - V_E)).$$

Then the matrix $N = (\chi(x_i, x_j))_{1 \leq i, j \leq 6}$ is
$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. Let $(y_1, \dots, y_6) = (\mathcal{O}_{E_1}(E_1), \mathcal{O}_{E_2}(E_2), \mathcal{O}_{E_3}(E_3), \mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(h_1), \mathcal{O}_{\Sigma_2}(2h_1))$. By Proposition 3.2 in [2], we have that for any $1 \leq i, j \leq 6$, $\chi(x_i, x_j) = \chi(y_i, y_j)$. We can use Hirzebruch-Riemann-Roch to compute the $\chi(y_i, y_j)$ in Σ_2 .

Let $1 \leq k \leq 3$. There is a short exact sequence

$$0 \rightarrow \mathcal{O}_{\Sigma_2} \rightarrow \mathcal{O}_{\Sigma_2}(E_k) \rightarrow \mathcal{O}_{E_k}(E_k) \rightarrow 0,$$

so $\text{ch}(\mathcal{O}_{E_k}(E_k)) = (1, E_k, -\frac{1}{2}) - (1, 0, 0) = (0, E_k, -\frac{1}{2})$. The other y_i are line bundles so we know their Chern characters, and we can now compute :

- Let $1 \leq m, n \leq 3$. Then $\text{ch}(\mathcal{O}_{E_m}(-E_m)^\vee) \cdot \text{ch}(\mathcal{O}_{E_n}(-E_n)) = (0, 0, -E_m \cdot E_n) = (0, 0, \delta_{m,n})$. Since $\text{td}(\Sigma_2) = (1, \frac{h}{2}, 1)$ by Lemma 4.1.3, we have $\chi(\mathcal{O}_{E_m}(-E_m), \mathcal{O}_{E_n}(-E_n)) = \delta_{m,n}$.

- Let $1 \leq m \leq 3$, then

$$\begin{aligned}\chi(\mathcal{O}_{E_m}(E_m), \mathcal{O}_{\Sigma_2}) &= \int_{\Sigma_2} (0, -E_k, -\frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = -\frac{1}{2} - \frac{1}{2} = -1 \\ \chi(\mathcal{O}_{E_m}(E_m), \mathcal{O}_{\Sigma_2}(h_1)) &= \int_{\Sigma_2} (0, -E_k, -\frac{1}{2}) \cdot (1, h_1, \frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = \int_{\Sigma_2} (0, -E_k, -\frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = -1 \\ \chi(\mathcal{O}_{E_m}(E_m), \mathcal{O}_{\Sigma_2}(2h_1)) &= \int_{\Sigma_2} (0, -E_k, -\frac{1}{2}) \cdot (1, 2h_1, 2) \cdot (1, \frac{h}{2}, 1) = \int_{\Sigma_2} (0, -E_k, -\frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = -1 \\ \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{E_m}(E_m)) &= \int_{\Sigma_2} (0, E_k, -\frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = -\frac{1}{2} + \frac{1}{2} = 0 \\ \chi(\mathcal{O}_{\Sigma_2}(h_1), \mathcal{O}_{E_m}(E_m)) &= \int_{\Sigma_2} (1, -h_1, \frac{1}{2}) \cdot (0, E_k, -\frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = \int_{\Sigma_2} (0, E_k, -\frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = 0 \\ \chi(\mathcal{O}_{\Sigma_2}(2h_1), \mathcal{O}_{E_m}(E_m)) &= \int_{\Sigma_2} (1, -2h_1, 2) \cdot (0, E_k, -\frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = \int_{\Sigma_2} (0, E_k, -\frac{1}{2}) \cdot (1, \frac{h}{2}, 1) = 0\end{aligned}$$

where the last three equalities can also be deduced from Lemma 2.1.1.

- Finally, if $i \in \mathbb{Z}$, then $H^*(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(ih_1)) = H^*(\phi^* \mathcal{O}_{\mathbb{P}^2}, \phi^* \mathcal{O}_{\mathbb{P}^2}(i)) = H^*(\mathcal{O}_{\mathbb{P}^2}, \phi_* \phi^* \mathcal{O}_{\mathbb{P}^2}(i)) = H^*(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(i))$. Therefore :

$$\begin{aligned}\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}) &= 1 \\ \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(h_1)) &= 3 \\ \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(2h_1)) &= 6 \\ \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-h_1)) &= 0 \\ \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-2h_1)) &= 0.\end{aligned}$$

□

Remark 4.1.6. In particular one can see that $\det(N) = 1$, hence $\det(\tilde{M}) = \det(M)$ with the notations of section 2.1.2.

4.2 Euler characteristics related to the $\mathcal{O}_{\tilde{X}_4}(ih)$

In this subsection we want to compute all the $\chi(e_i, e_j)$ with $e_i, e_j \in \{\pi^* \mathcal{O}_{X_4}(kh), 0 \leq k \leq 4\} = \{\mathcal{O}_{\tilde{X}_4}(kh), 0 \leq k \leq 4\}$.

Lemma 4.2.1. Let $0 \leq i, j \leq 4$, then $\chi(\pi^* \mathcal{O}_{X_4}(ih), \pi^* \mathcal{O}_{X_4}(jh)) = \chi(\mathcal{O}_{X_4}(ih), \mathcal{O}_{X_4}(jh))$.

Proof. For any $0 \leq i, j \leq 4$, there are isomorphisms $H^*(\pi^* \mathcal{O}_{X_4}(ih), \pi^* \mathcal{O}_{X_4}(jh)) \cong H^*(\mathcal{O}_{X_4}(ih), \pi_* \pi^* \mathcal{O}_{X_4}(jh))$. Moreover we have $\pi_* \mathcal{O}_{\tilde{X}_4} = \mathcal{O}_{X_4}$ (see for instance Example 11.14 ii) in [14]), hence the projection formula yields $\pi_* \pi^* \mathcal{O}_{X_4}(jh) = \mathcal{O}_{X_4}(jh)$. □

As a result we are rather going to compute all the $\chi(\mathcal{O}_{X_4}(ih), \mathcal{O}_{X_4}(jh))$ for $0 \leq i, j \leq 4$.

4.2.1 A general version of Borel-Bott-Weil Theorem

We will use the Borel-Bott-Weil Theorem all along this subsection. One can find a general version of this Theorem in Küchle's [15], along with explanations about how to apply it in the case of Grassmannians. To actually compute the coefficients of M , we will use a more specific version of the Theorem stated in part 2.6 of [16], but we will also use some of the information given in [15], so in this subsection we are going to reproduce mainly definitions, facts and the version of the Theorem from Küchle's [15].

Theorem 4.2.2 (Borel-Bott-Weil). *Let P be a parabolic subgroup of a semi-simple complex group G and E an irreducible P -module with highest weight λ . Let $\mathcal{E} = G \times_P E$ denote the vector bundle over G/P which is the quotient of $G \times E$ by the action of P on $G \times E$ defined by $p.(g, e) = (g.p^{-1}, p.e)$. Let δ be the sum of the fundamental weights, μ the unique dominant weight in the Weyl group orbit of $\lambda + \delta$ and $\text{ind}(\lambda + \delta)$ the number of positive roots $\alpha \in G$ such that $(\lambda + \delta, \alpha) < 0$. Then :*

1. *if $\lambda + \delta$ is singular then $H^q(G/P, \mathcal{E}) = 0$ for all $q \in \mathbb{Z}$;*
2. *if $\lambda + \delta$ is not singular then $H^q(G/P, \mathcal{E}) = (\mu - \delta)^{|G \delta_{q, \text{ind}(\lambda + \delta)}|}$.*

Here $(\mu - \delta)^{|G|}$ denotes the G -module given by the space of algebraic morphisms $s : G \rightarrow R$ such that $s(gp^{-1}) = p.s(g)$ for all $g \in G$ and $p \in P$, with R the one dimensional representation of the Borel subgroup of G associated to $\mu - \delta$.

Moreover, according to [15] part 2.2, if \mathcal{E} is a vector bundle on G/P which is homogeneous with respect to G , then \mathcal{E} can be written as $G \times_P E$ as in Theorem 4.2.2. Then \mathcal{E} is called irreducible if E is an irreducible P -module.

In our case we want to apply this theorem for $G/P = \text{Gr}(3, 7)$, and we will be interested only in the dimension of the $H^q(G/P, \mathcal{E})$. So following part 2.4 and 2.5 of [15], we can reduce to a specific case of the theorem, as follows.

Let $G = \text{SL}(7, \mathbb{C})$ and $P = \left\{ \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}, A \in \text{GL}(3, \mathbb{C}), B \in \text{GL}(4, \mathbb{C}), \det(A)\det(B) = 1 \right\}$. Then $\text{Gr}(3, 7) = G/P$.

We can then choose the positive roots of G to be $\{e_i - e_j, 1 \leq i < j \leq 7\}$ where $\{e_1, \dots, e_7\}$ is an orthonormal basis of \mathbb{R}^7 . In this case the simple roots are $\{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, e_5 - e_6, e_6 - e_7\}$ and a weight

$\alpha = \sum_{i=1}^7 \alpha_i e_i = (\alpha_1, \dots, \alpha_7)$ is dominant if and only if $\alpha_i \geq \alpha_j$ for all $j > i$. Moreover the sum of the fundamental

weights is $\delta = (6, 5, 4, 3, 2, 1, 0)$. Finally here $(\alpha, \beta) = \sum_{i=1}^7 \alpha_i \beta_i$ and the Weyl group of G is just the permutations

of the e_i so the unique dominant weight in the orbit of a weight α is the weight $(\alpha_{i_1}, \dots, \alpha_{i_7})$ such that $\alpha_{i_j} \geq \alpha_{i_k}$ for all $k > j$. Then a weight λ is non singular if and only if $\lambda_i \neq \lambda_j$ for all $i \neq j$. Thus the BBW Theorem induces the following :

Corollary 4.2.3. *Let \mathcal{E} be a vector bundle on $\text{Gr}(3, 7)$ of the form $G \times_P E$ as in Theorem 4.2.2. Let λ be the associated highest weight and $\delta = (6, 5, 4, 3, 2, 1, 0)$. Then :*

1. *if there exists $i \neq j$ such that $\lambda_i = \lambda_j$ then $H^q(\text{Gr}(3, 7), \mathcal{E}) = 0$ for all $q \in \mathbb{Z}$;*
2. *otherwise let μ be the unique weight with the same coefficients as λ but such that $\mu_i > \mu_j$ if $j > i$. Let x be the number of positive roots $\alpha = (\alpha_1, \dots, \alpha_7)$ such that $\sum_{i=1}^7 \alpha_i (\lambda + \delta)_i < 0$. Then $\dim(H^q(\text{Gr}(3, 7), \mathcal{E})) = \dim(\mu - \delta)^{|\text{Sl}_7 \delta_{q,x}|}$.*

To compute $\dim(\mu - \delta)^{|\text{Sl}_7|}$ (here it is the dimension as a Sl_7 -representation) in Corollary 4.2.3 we will use the Weyl formula that can be found in ([27], formula 3.49), which Jean Douçot pointed out to me :

Proposition 4.2.4 (Weyl formula). *Let Λ be a representation of highest weight λ . Then*

$$\dim(\Lambda) = \prod_{\alpha > 0} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)} \quad (4.2)$$

where δ is the sum of the fundamental weights.

We are also going to use the following properties (see [15] and for instance [25] §10.5, Proposition 3 page 345):

Proposition 4.2.5. *Let E and F be vector bundles associated to representations of P as in Theorem 4.2.2 and of highest weights $\alpha = (\alpha_1, \dots, \alpha_7)$ and $\beta = (\beta_1, \dots, \beta_7)$ respectively. Then E^\vee and $E \otimes F$ are also associated to representations of P (not necessarily irreducible) and their highest weights are, respectively, $(-\alpha_3, -\alpha_2, -\alpha_1, -\alpha_7, -\alpha_6, -\alpha_5, -\alpha_4)$ and $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_7 + \beta_7)$. Moreover, any $\bigwedge^n E$ is also associated to a representation of P .*

Let us go back to the definition of X_4 now. By the introduction of [20], it is the zero locus of a global section of $\mathcal{U}_3^\perp(1) \oplus \mathcal{U}_3(1) \oplus \mathcal{O}(1)$ in $\text{Gr}(3, 7)$. Here \mathcal{U}_3 is the tautological subbundle of $\text{Gr}(3, 7)$ and \mathcal{U}_3^\perp is defined as follows : there is a short exact sequence $0 \rightarrow \mathcal{U}_3 \rightarrow \mathcal{O}_{\text{Gr}(3,7)} \otimes \mathbb{C}^7 \rightarrow Q \rightarrow 0$ and we define $\mathcal{U}_3^\perp = Q^\vee$. Using part 2.5 of [15] and part 2.6 of [16] one can find :

Lemma 4.2.6. *The vector bundles $\mathcal{U}_3^\vee, \mathcal{U}_3^\perp$ and $\mathcal{O}_{\text{Gr}(3,7)}(1)$ are all of the form described in Theorem 4.2.2, and their associated highest weights are respectively : $(1, 0, 0, 0, 0, 0, 0)$, $(0, 0, 0, 1, 0, 0, 0)$ and $(1, 1, 1, 0, 0, 0, 0)$.*

Remark 4.2.7. *The highest weight associated to $\mathcal{O}_{\text{Gr}(3,7)} \cong \mathcal{O}_{\text{Gr}(3,7)}^{\otimes k} \cong \mathcal{O}_{\text{Gr}(3,7)} \otimes \mathcal{O}_{\text{Gr}(3,7)}^\vee$ is $(1, 1, 1, 1, 1, 1, 1) = (k, k, k, k, k, k, k) = (0, 0, 0, 0, 0, 0, 0)$ for any $k \in \mathbb{Z}$. In particular since tensoring with $\mathcal{O}_{\text{Gr}(3,7)}$ doesn't change a vector bundle, for any highest weight $\alpha = (\alpha_1, \dots, \alpha_7)$ we can write $\alpha = (\alpha_1 - k, \dots, \alpha_7 - k)$ for any $k \in \mathbb{Z}$. Then by convention we usually choose the coordinates $(\alpha_1, \dots, \alpha_7)$ such that one at least of the α_i is 0.*

Corollary 4.2.8. *The highest weights associated to $\mathcal{U}_3, (\mathcal{U}_3^\perp)^\vee$ and $\mathcal{U}_3(1)$ and $\mathcal{U}_3^\perp(1)$ are respectively $(0, 0, -1, 0, 0, 0, 0) = (1, 1, 0, 1, 1, 1, 1)$, $(0, 0, 0, 0, 0, 0, -1) = (1, 1, 1, 1, 1, 1, 0)$, $(1, 1, 0, 0, 0, 0, 0)$ and $(1, 1, 1, 1, 0, 0, 0)$.*

Finally we are going to use (see for instance [10] §15 just before Example 15.14, page 221 and [16] part 2.6):

Lemma 4.2.9. *The following are also associated to irreducible representations as in Theorem 4.2.2: $\bigwedge^2 \mathcal{U}_3^\vee, \bigwedge^3 \mathcal{U}_3^\vee \cong \mathcal{O}_{\text{Gr}(3,7)}(1), \bigwedge^2 \mathcal{U}_3^\perp, \bigwedge^3 \mathcal{U}_3^\perp$ and $\bigwedge^4 \mathcal{U}_3^\perp \cong \mathcal{O}_{\text{Gr}(3,7)}(-1)$. The associated highest weights are respectively :*

$$(1, 1, 0; 0, 0, 0, 0, 0), (1, 1, 1; 0, 0, 0, 0, 0), (0, 0, 0; 1, 1, 0, 0), (0, 0, 0; 1, 1, 1, 0) \text{ and } (0, 0, 0; 1, 1, 1, 1).$$

4.2.2 Application of BBW to Grassmannians

The BBW Theorem stated in section 4.2.1 requires the vector bundle \mathcal{E} to be irreducible. However in our case we will need to apply the Theorem to tensor products of irreducible representations : these do not remain irreducible in general. Since it is not necessarily easy to decompose such a tensor product into the sum of its irreducible factors, we are instead going to use the much easier approach of Kuznetsov given in [16].

In part 2.6 of [16], Kuznetsov uses a slightly different version of the BBW Theorem, which, combined with Proposition 2.12 and Corollary 2.13 in [16], yields the following Theorem. We are stating the result here only for $\text{Gr}(3, 7)$ but in [16] everything is done in the more general case of $\text{Gr}(k, n)$. Moreover here we are mainly going to keep the notations of K\"uchle's article [15], which we used in section 4.2.1.

Let \mathcal{E} be a tensor product of $(\bigwedge^i \mathcal{U}_3^\vee)^{\otimes k_i}, (\bigwedge^i \mathcal{U}_3)^{\otimes l_i}, (\bigwedge^i \mathcal{U}_3^\perp)^{\otimes m_i}$ and $(\bigwedge^i (\mathcal{U}_3^\perp)^\vee)^{\otimes n_i}$ with $i \in \{1, 2, 3, 4\}$ and the k_i, l_i, m_i and n_i in \mathbb{N} . Then by Proposition 4.2.5, Lemma 4.2.6 and Lemma 4.2.9 it is associated to a representation of P as in Theorem 4.2.2, whose highest weight is computable by the Proposition and Lemmas above-mentioned.

Theorem 4.2.10. *Let \mathcal{E} be a vector bundle on $\text{Gr}(3, 7)$ as just above. Let λ be the associated highest weight and $\delta = (6, 5, 4, 3, 2, 1, 0)$. Then :*

1. *if there are some $i \neq j$ such that $\lambda_i + \delta_i = \lambda_j + \delta_j$ then $H^*(\text{Gr}(3, 7), \mathcal{E}) = 0$;*
2. *if $\lambda_i + \delta_i \neq \lambda_j + \delta_j$ for any $i \neq j$ then $\dim(H^q(\text{Gr}(3, 7), \mathcal{E})) = \dim((\sigma(\delta + \lambda) - \delta)^{|G|} \delta_{q, l(\sigma)})$ with σ the unique permutation in S_7 such that $\sigma(\delta + \lambda)$ satisfies $\sigma(\delta + \lambda)_i > \sigma(\delta + \lambda)_j$ whenever $i < j$ and $l(\sigma)$ the length of σ .*

This is what we are going to use, along with (4.2) to compute the Euler characteristics related to the $\mathcal{O}_{X_4}(i)$.

4.2.3 Computation of the Euler characteristics

To compute the Euler characteristics associated to the $\mathcal{O}_{X_4}(i)$ we are finally going to use the Koszul complex, that is to say :

$$0 \rightarrow \bigwedge^8 \mathcal{V} \rightarrow \bigwedge^7 \mathcal{V} \rightarrow \dots \rightarrow \mathcal{V} \xrightarrow{s^\vee} \mathcal{O}_{\text{Gr}(3,7)} \rightarrow \mathcal{O}_{X_4} \rightarrow 0 \quad (4.3)$$

where $\mathcal{V} = (\mathcal{U}_3(1) \oplus \mathcal{U}_3^\perp(1) \oplus \mathcal{O}_{\text{Gr}(3,7)}(1))^\vee$ is a rank 8 vector bundle and the arrow $\mathcal{V} \xrightarrow{s^\vee} \mathcal{O}_{\text{Gr}(3,7)}$ is given by the dual of the section s of \mathcal{V}^\vee whose zero locus defines X_4 .

Thanks to a method that E. Fatighenti kindly taught me, we can now compute the $\chi(\mathcal{O}_{X_4}(i), \mathcal{O}_{X_4}(j))$. Since the method for computing these doesn't depend on i and j , we are going to detail only one of these computations here, in Proposition 4.2.11. Other Euler characteristics of the same form can easily (although with possibly a lot of calculations) be computed with the same method.

The idea consists in the following : first, $\chi(\mathcal{O}_{X_4}(i), \mathcal{O}_{X_4}(j)) = \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(j-i))$, so we can reduce to $\chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(k)) = \chi(\mathcal{O}_{X_4}(k))$ for some $k \in \mathbb{Z}$. Then, one can split an appropriate twist of (4.3) into short exact sequences. In each of these short exact sequences, at least one (twist of an) alternate product of \mathcal{V} , $\bigwedge^i \mathcal{V}(k)$, will appear, which we can decompose into direct sums of sheaves of the form $(\bigwedge^i \mathcal{U}_3^\vee)^{\otimes k} \otimes (\bigwedge^j \mathcal{U}_3)^{\otimes l} \otimes (\bigwedge^p \mathcal{U}_3^\perp)^{\otimes m} \otimes (\bigwedge^q (\mathcal{U}_3^\perp)^\vee)^{\otimes n} \otimes \mathcal{O}_{\text{Gr}(3,7)}(r)$. Then, one can use Theorem 4.2.10 together with (4.2) to compute the Euler characteristic of each summand of the decomposition of the $\bigwedge^i \mathcal{V}(k)$ we are considering. Finally, using the fact that for any short exact sequence $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0$ we have $\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{Q})$, one can compute successively the Euler characteristics of all the sheaves appearing in the short exact sequences coming from (4.3). Eventually, this will allow us to compute the Euler characteristic $\chi(\mathcal{O}_{X_4}(k))$ of the twist of \mathcal{O}_{X_4} we are interested in.

One can also use Macaulay2 and Scilab to decompose more easily the $\bigwedge^i \mathcal{V}(k)$ and to do some computations faster : this is detailed in Appendix 6.3 and Appendix 6.4.

In the following proof, we even manage to compute $H^*(\mathcal{O}_{X_4}(4))$, using long exact sequences in cohomology. Yet, this is not always possible : sometimes (when the long exact sequences in cohomology have too many non trivial sheaves) only the Euler characteristic is calculable.

Proposition 4.2.11. *With the same notations as above, $\chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(4)) = 1181$.*

Proof. First of all, if $i : X_4 \rightarrow \text{Gr}(3,7)$ is the injection of X_4 in the Grassmannian, we have $H^*(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(4)) = H^*(i^*(\mathcal{O}_{\text{Gr}(3,7)}(-4)), \mathcal{O}_{X_4}) = H^*(\mathcal{O}_{\text{Gr}(3,7)}(-4), i_*\mathcal{O}_{X_4}) = H^*(\mathcal{O}_{\text{Gr}(3,7)}, i_*(\mathcal{O}_{X_4}(4)))$. By abuse of notation we are going to note $i_*\mathcal{O}_{X_4}$ by \mathcal{O}_{X_4} ; it is the last sheaf on the right in (4.3). Then (4.3) gives us a resolution of $\mathcal{O}_{X_4}(4)$:

$$0 \rightarrow \bigwedge^8 \mathcal{V}(4) \xrightarrow{f_7} \bigwedge^7 \mathcal{V}(4) \xrightarrow{f_6} \dots \xrightarrow{f_1} \mathcal{V}(4) \xrightarrow{s^\vee} \mathcal{O}_{\text{Gr}(3,7)}(4) \rightarrow \mathcal{O}_{X_4}(4) \rightarrow 0. \quad (4.4)$$

We are going to use this resolution to compute $\chi(\mathcal{O}_{\text{Gr}(3,7)}, i_*(\mathcal{O}_{X_4}(4))) = \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(4))$ (the equality holds by what we just did), but to do this computation we need to split (4.4) into short exact sequences, as follows :

- we first look at the short exact sequence $0 \rightarrow \bigwedge^8 \mathcal{V}(4) \xrightarrow{f_7} \bigwedge^7 \mathcal{V}(4) \rightarrow K_7 \rightarrow 0$ where K_7 is the cokernel of f_7 . By definition, $\mathcal{V} = (\mathcal{U}_3(1) \oplus \mathcal{U}_3^\perp(1) \oplus \mathcal{O}_{\text{Gr}(3,7)}(1))^\vee = \mathcal{U}_3^\vee(-1) \oplus (\mathcal{U}_3^\perp)^\vee(-1) \oplus \mathcal{O}_{\text{Gr}(3,7)}(-1)$.

Thus, since \mathcal{U}_3^\vee is a rank 3 vector bundle and $(\mathcal{U}_3^\perp)^\vee$ a rank 4 vector bundle this yields :

$$\begin{aligned}
\bigwedge^8 \mathcal{V} &= \bigwedge^3 \mathcal{U}_3^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-3) \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-4) \otimes \mathcal{O}_{\text{Gr}(3,7)}(-1) \\
&= \bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-8) \\
\bigwedge^7 \mathcal{V} &= \bigwedge^3 \mathcal{U}_3^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-3) \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-4) \oplus \\
&\quad \bigwedge^3 \mathcal{U}_3^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-3) \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-3) \otimes \mathcal{O}_{\text{Gr}(3,7)}(-1) \oplus \\
&\quad \bigwedge^2 \mathcal{U}_3^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-2) \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-4) \otimes \mathcal{O}_{\text{Gr}(3,7)}(-1) \\
&= \bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-7) \oplus \bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-7) \oplus \\
&\quad \bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-7)
\end{aligned}$$

Using Lemma 4.2.9 and Proposition 4.2.5 one can find that the highest weight associated to $\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee$ is

$$(1, 1, 1; 0, 0, 0, 0) + (0, 0, 0; -1, -1, -1, -1) = (1, 1, 1; 0, 0, 0, 0) + (1, 1, 1; 0, 0, 0, 0) = (2, 2, 2; 0, 0, 0, 0).$$

Moreover the highest weight associated to $\mathcal{O}_{\text{Gr}(3,7)}(-1)$ is $(-1, -1, -1; 0, 0, 0, 0) = (0, 0, 0; 1, 1, 1, 1)$ so again by Proposition 4.2.5, the highest weight associated to $\mathcal{O}_{\text{Gr}(3,7)}(-8)$ is $(0, 0, 0; 8, 8, 8, 8)$.

So finally the highest weight associated to $\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}(-8)$ is

$$(2, 2, 2; 0, 0, 0, 0) + (0, 0, 0; 8, 8, 8, 8) = (2, 2, 2; 8, 8, 8, 8) = (0, 0, 0; 6, 6, 6, 6).$$

Therefore the highest weight associated to $(\bigwedge^8 \mathcal{V})(4)$ is

$$(0, 0, 0; 6, 6, 6, 6) + (4, 4, 4; 0, 0, 0, 0) = (0, 0, 0; 2, 2, 2, 2).$$

We can now use Theorem 4.2.10 to compute $H^*((\bigwedge^8 \mathcal{V})(4))$, since $\mathcal{O}_{\text{Gr}(3,7)}(1) = \bigwedge^3 \mathcal{U}_3^\vee$ and thus $\mathcal{O}_{\text{Gr}(3,7)}(-1) = \bigwedge^3 \mathcal{U}_3$. So we compute

$$\delta + (0, 0, 0; 2, 2, 2, 2) = (6, 5, 4, 3, 2, 1, 0) + (0, 0, 0; 2, 2, 2, 2) = (6, 5, 4, 5, 4, 3, 2).$$

Here the second and the fourth coefficients are the same so it is part 1. of Theorem 4.2.10 that applies and $H^*((\bigwedge^8 \mathcal{V})(4)) = 0$.

We want now to compute $H^*((\bigwedge^7 \mathcal{V})(4))$ with a similar reasoning. By what we have computed above,

$$\begin{aligned}
H^*((\bigwedge^7 \mathcal{V})(4)) &= H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee (-7+4)) \oplus H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee (-7+4)) \\
&\quad \oplus H^*(\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee (-7+4)).
\end{aligned}$$

So here we will have to apply Theorem 4.2.10 three times. First of all the highest weight associated to $\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee (-3)$ is $(1, 1, 1; 0, 0, 0, 0) + (1, 1, 1; 0, 0, 0, 0) + (0, 0, 0; 3, 3, 3, 3) = (0, 0, 0; 1, 1, 1, 1)$.

But $\delta + (0, 0, 0; 1, 1, 1, 1) = (6, 5, 4, 4, 3, 2, 1)$. The third and fourth coefficients are the same so $H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee (-3)) = 0$. The highest weight associated to $\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee (-3)$ is

$$(1, 1, 1; 0, 0, 0, 0) + (1, 1, 1; 1, 0, 0, 0) + (0, 0, 0; 3, 3, 3, 3) = (2, 2, 2, 4, 3, 3, 3) = (0, 0, 0; 2, 1, 1, 1).$$

But $\delta + (0, 0, 0; 2, 1, 1, 1) = (6, 5, 4, 5, 3, 2, 1)$. The second and fourth coefficients are the same so $H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee(-3)) = 0$. The highest weight associated to $\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee(-3)$ is

$$(1, 1, 0; 0, 0, 0, 0) + (1, 1, 1; 0, 0, 0, 0) + (0, 0, 0; 3, 3, 3, 3) = (2, 2, 1; 3, 3, 3, 3) = (1, 1, 0; 2, 2, 2, 2).$$

But $\delta + (1, 1, 0; 2, 2, 2, 2) = (7, 6, 4; 5, 4, 3, 2)$. The third and fifth coefficients are the same so $H^*(\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee(-3)) = 0$. So finally $H^*((\bigwedge^7 \mathcal{V})(4)) = 0$ and thus $H^*(K_7) = 0$.

- Since f_7 is injective we have that $K_7 \cong (\bigwedge^7 \mathcal{V})(4)/\text{im}(f_7)$ and since (4.4) is a long exact sequence, $\text{im}(f_7) = \ker(f_6)$ so we have another short exact sequence :

$$0 \rightarrow K_7 \xrightarrow{f_6} (\bigwedge^6 \mathcal{V})(4) \rightarrow K_6 \rightarrow 0. \quad (4.5)$$

We already know $H^*(K_7)$ so we need to compute $H^*((\bigwedge^6 \mathcal{V})(4))$ with Theorem 4.2.10. We first decompose $(\bigwedge^6 \mathcal{V})(4)$ as follows :

$$\begin{aligned} (\bigwedge^6 \mathcal{V})(4) &= \bigwedge^6 ((\mathcal{U}_3^\vee \oplus (\mathcal{U}_3^\perp)^\vee \oplus \mathcal{O}_{\text{Gr}(3,7)}) \otimes \mathcal{O}_{\text{Gr}(3,7)}(-1))(4) \\ &= \mathcal{O}_{\text{Gr}(3,7)}(-6+4) \otimes (\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee \oplus \bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)} \oplus \bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \oplus \\ &\quad \bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)} \oplus \bigwedge^1 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \otimes \mathcal{O}_{\text{Gr}(3,7)}) \end{aligned}$$

Since it can be long to compute such a decomposition a lot of times, and also to avoid making mistakes while doing it, one can use Macaulay2 as detailed in annex 6.3. We want now to apply Theorem 4.2.10 to each summand of the decomposition above.

- The highest weight of $\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee(-2)$ is $(1, 1, 1; 0, 0, 0, 0) + (1, 1, 1; 1, 0, 0, 0) + (0, 0, 0; 2, 2, 2, 2) = (2, 2, 2; 3, 2, 2, 2) = (0, 0, 0; 1, 0, 0, 0)$. In $\delta + (0, 0, 0; 1, 0, 0, 0) = (6, 5, 4; 4, 2, 1, 0)$ the third and fourth coefficients are the same so $H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee(-2)) = 0$.
- The highest weight of $\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee(-2)$ is $(1, 1, 1; 0, 0, 0, 0) + (1, 1, 1; 1, 1, 0, 0) + (0, 0, 0; 2, 2, 2, 2) = (2, 2, 2; 3, 3, 2, 2) = (0, 0, 0; 1, 1, 0, 0)$. In $\delta + (0, 0, 0; 1, 1, 0, 0) = (6, 5, 4; 4, 3, 1, 0)$ the third and fourth coefficients are the same so $H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee(-2)) = 0$.
- The highest weight of $\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee(-2)$ is $(1, 1, 0; 0, 0, 0, 0) + (1, 1, 1; 0, 0, 0, 0) + (0, 0, 0; 2, 2, 2, 2) = (2, 2, 1; 2, 2, 2, 2) = (1, 1, 0; 1, 1, 1, 1)$. In $\delta + (1, 1, 0; 1, 1, 1, 1) = (7, 6, 4; 4, 3, 2, 1)$ the third and fourth coefficients are the same so $H^*(\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee(-2)) = 0$.
- The highest weight of $\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee(-2)$ is $(1, 1, 0; 0, 0, 0, 0) + (1, 1, 1; 1, 0, 0, 0) + (0, 0, 0; 2, 2, 2, 2) = (2, 2, 1; 3, 2, 2, 2) = (1, 1, 0; 2, 1, 1, 1)$. In $\delta + (1, 1, 0; 2, 1, 1, 1) = (7, 6, 4; 5, 3, 2, 1)$ all the coefficients are different so we are in part 2. of Theorem 4.2.10. Here $\lambda = (7, 6, 4, 5, 3, 2, 1)$ is not dominant because $\lambda_3 = 4 < \lambda_4 = 5$ but $\sigma_{34}(\lambda) = (7, 6, 5, 4, 3, 2, 1)$ is dominant. Moreover $l(\sigma_{34}) = 1$ (this is also the number of positive roots α such that $(\lambda, \alpha) < 0$: indeed we have seen in section 1.1 that the positive roots are all the $e_i - e_j$ with $i < j$, and $(\lambda, e_i - e_j) = \lambda_i - \lambda_j$. It is negative only for $i = 3$ and $j = 4$). So $\dim(H^*(\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee(-2))) = \dim((\lambda - \delta)|^{\text{Sl}_7} \delta_{*1})$. Finally $\lambda - \delta = (1, 1, 1, 1, 1, 1, 1) = (0, 0, 0, 0, 0, 0, 0)$. Hence (4.2) gives $\dim((\lambda - \delta)|^{\text{Sl}_7}) = \prod_{\alpha > 0} \frac{(\delta + \alpha, \alpha)}{(\delta + \alpha, \alpha)} = 1$ and $H^*(\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee(-2)) = \mathbb{C} \cdot \delta_{*1}$.
- The highest weight of $\mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee(-2)$ is $(1, 0, 0; 0, 0, 0, 0) + (1, 1, 1; 0, 0, 0, 0) + (0, 0, 0; 2, 2, 2, 2) = (2, 1, 1; 2, 2, 2, 2) = (1, 0, 0; 1, 1, 1, 1)$. In $\delta + (1, 0, 0; 1, 1, 1, 1) = (7, 5, 4; 4, 3, 2, 1)$ the third and fourth coefficients are the same so $H^*(\mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee(-2)) = 0$.

Finally $H^*(\bigwedge^6 \mathcal{V})(4) = \mathbb{C} \cdot \delta_{*1}$. Therefore, since $H^*(K_7) = 0$ the long exact sequence in cohomology induced by (4.5) gives $H^*(K_6) = \mathbb{C} \cdot \delta_{*1}$.

- We are now going to proceed as in the previous step : there is a short exact sequence

$$0 \rightarrow K_6 \xrightarrow{f_5} \bigwedge^5 \mathcal{V}(4) \rightarrow K_5 \rightarrow 0. \quad (4.6)$$

Using Macaulay2 as explained in annex 6.3, or by hand as in the previous step, one finds

$$\begin{aligned} \bigwedge^5 \mathcal{V}(4) = & \mathcal{O}_{\text{Gr}(3,7)}(-5+4) \otimes \left(\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee \oplus \bigwedge^3 \mathcal{U}_3^\vee \otimes (\mathcal{U}_3^\perp)^\vee \oplus \bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee \oplus \right. \\ & \left. \oplus \bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee \oplus \mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \oplus \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee \oplus \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \right) \end{aligned}$$

Now for each summand we apply Theorem 4.2.10. To go faster, one can use scilab as described in annex 6.4 to check if $\delta + \lambda$ has two identical coefficients : this is what we are going to do here, this is why from now on the details about this calculation will be skipped.

- The highest weight of $\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee(-1)$ is $(1, 1, 1; 0, 0, 0, 0) + (1, 1, 1; 1, 1, 0, 0) + (0, 0, 0; 1, 1, 1, 1) = (2, 2, 2; 2, 2, 1, 1) = (1, 1, 1; 1, 1, 0, 0)$. Using scilab one finds that $\delta + (1, 1, 1; 1, 1, 0, 0)$ has pairwise distinct coefficients, moreover $\delta + (1, 1, 1; 1, 1, 0, 0) = (7, 6, 5; 4, 3, 1, 0)$ is dominant so we will have $\dim(H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee(-1))) = \dim(((7, 6, 5; 4, 3, 1, 0) - \delta)^{\text{Sl}_7} \delta_{*0}) = \dim((1, 1, 1; 1, 1, 0, 0)^{\text{Sl}_7} \delta_{*0})$. Finally we use (4.2) to compute $\dim((1, 1, 1; 1, 1, 0, 0)^{\text{Sl}_7})$. Let us denote $\lambda = (1, 1, 1; 1, 1, 0, 0)$, then :

$$\begin{aligned} \dim(\lambda)^{\text{Sl}_7} &= \prod_{\alpha > 0} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)} = \prod_{j=2}^7 \frac{(\lambda + \delta, e_1 - e_j)}{(\delta, e_1 - e_j)} \times \prod_{j=3}^7 \frac{(\lambda + \delta, e_2 - e_j)}{(\delta, e_2 - e_j)} \times \\ & \prod_{j=4}^7 \frac{(\lambda + \delta, e_3 - e_j)}{(\delta, e_3 - e_j)} \times \prod_{j=5}^7 \frac{(\lambda + \delta, e_4 - e_j)}{(\delta, e_4 - e_j)} \times \prod_{j=6}^7 \frac{(\lambda + \delta, e_5 - e_j)}{(\delta, e_5 - e_j)} \times \frac{(\lambda + \delta, e_6 - e_7)}{(\delta, e_6 - e_7)} \\ &= \frac{(7-6)(7-5)(7-4)(7-3)(7-1)(7-0)(6-5)(6-4)(6-3)(6-1)6}{(6-5)(6-4)(6-3)(6-2)(6-1)(6-0)(5-4)(5-3)(5-2)(5-1)5} \times \\ & \frac{(5-4)(5-3)(5-1)5(4-3)(4-1)4(3-1)31}{(4-3)(4-2)(4-1)4(3-2)(3-1)3(2-1)21} \\ &= \frac{6}{5} \times \frac{7}{6} \times \frac{5}{4} \times \frac{6}{5} \times \frac{4}{3} \times \frac{5}{4} \times \frac{3}{2} \times \frac{4}{3} \times \frac{2}{1} \times \frac{3}{2} = \frac{7 \times 6}{2} = 21 \end{aligned}$$

To do these long computations one can also use scilab as explained in annex 6.4, so from now on we will skip details of such calculations. Finally $H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee(-1)) = \mathbb{C}^{21} \cdot \delta_{*0}$.

- The highest weight of $\bigwedge^3 \mathcal{U}_3^\vee \otimes (\mathcal{U}_3^\perp)^\vee(-1)$ is $(1, 1, 1; 0, 0, 0, 0) + (1, 1, 1; 1, 1, 1, 0) + (0, 0, 0; 1, 1, 1, 1) = (2, 2, 2; 2, 2, 2, 1) = (1, 1, 1; 1, 1, 1, 0)$. $\delta + (1, 1, 1; 1, 1, 1, 0)$ is dominant and so using scilab one finds $H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes (\mathcal{U}_3^\perp)^\vee(-1)) = \mathbb{C}^7 \cdot \delta_{*0}$.
- The highest weight of $\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee(-1)$ is $(1, 1, 0; 0, 0, 0, 0) + (1, 1, 1; 1, 0, 0, 0) + (0, 0, 0; 1, 1, 1, 1) = (2, 2, 1; 2, 1, 1, 1) = (1, 1, 0; 1, 0, 0, 0)$. Using scilab one finds that $\delta + (1, 1, 0; 1, 0, 0, 0)$ has two identical coefficients so $H^*(\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee(-1)) = 0$.
- The highest weight of $\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee(-1)$ is $(1, 1, 0; 0, 0, 0, 0) + (1, 1, 1; 1, 1, 0, 0) + (0, 0, 0; 1, 1, 1, 1) = (2, 2, 1; 2, 2, 1, 1) = (1, 1, 0; 1, 1, 0, 0)$. Using scilab one finds that $\delta + (1, 1, 0; 1, 1, 0, 0)$ has two identical coefficients so $H^*(\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee(-1)) = 0$.
- The highest weight of $\mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee(-1)$ is $(1, 0, 0; 0, 0, 0, 0) + (1, 1, 1; 0, 0, 0, 0) + (0, 0, 0; 1, 1, 1, 1) = (2, 1, 1; 1, 1, 1, 1) = (1, 0, 0; 0, 0, 0, 0)$. $\delta + (1, 0, 0; 0, 0, 0, 0)$ is dominant and so using scilab one finds $H^*(\mathcal{U}_3^\vee \otimes \bigwedge^4 (\mathcal{U}_3^\perp)^\vee(-1)) = \mathbb{C}^7 \cdot \delta_{*0}$.

- The highest weight of $\mathcal{U}_3^\vee \otimes \bigwedge^3(\mathcal{U}_3^\perp)^\vee(-1)$ is $(1, 0, 0; 0, 0, 0, 0) + (1, 1, 1; 1, 0, 0, 0) + (0, 0, 0; 1, 1, 1, 1) = (2, 1, 1; 2, 1, 1, 1) = (1, 0, 0; 1, 0, 0, 0)$. Using scilab one finds that $\delta + (1, 0, 0; 1, 0, 0, 0)$ has two identical coefficients so $H^*(\mathcal{U}_3^\vee \otimes \bigwedge^3(\mathcal{U}_3^\perp)^\vee(-1)) = 0$.
- The highest weight of $\bigwedge^4(\mathcal{U}_3^\perp)^\vee(-1)$ is $(1, 1, 1; 0, 0, 0, 0) + (0, 0, 0; 1, 1, 1, 1) = (1, 1, 1; 1, 1, 1, 1) = (0, 0, 0; 0, 0, 0, 0)$. $\delta + (0, 0, 0; 0, 0, 0, 0)$ is dominant and so using scilab one finds $H^*(\bigwedge^4(\mathcal{U}_3^\perp)^\vee(-1)) = \mathbb{C} \cdot \delta_{*0}$.

So finally $\dim(H^*((\bigwedge^5 \mathcal{V})(4))) = (21 + 7 + 7 + 1) \cdot \delta_{*0} = 36 \cdot \delta_{*0}$. Since we have also seen at the previous step that $\dim(H^*(K_6)) = \delta_{*1}$, the long exact sequence in cohomology induced by (4.6) gives :

$$0 \rightarrow H^0(K_6) \cong 0 \rightarrow H^0((\bigwedge^5 \mathcal{V})(4)) \cong \mathbb{C}^{36} \rightarrow H^0(K_5) \rightarrow H^1(K_6) \cong \mathbb{C} \rightarrow H^1((\bigwedge^5 \mathcal{V})(4)) \cong 0 \rightarrow H^1(K_5) \rightarrow H^2(K_6) \cong 0 \rightarrow H^2((\bigwedge^5 \mathcal{V})(4)) \cong 0 \rightarrow H^2(K_5) \rightarrow 0$$

This yields $\dim(H^*(K_5)) = 37 \cdot \delta_{*0}$.

- We now look at the short exact sequence :

$$0 \rightarrow K_5 \xrightarrow{f_4} (\bigwedge^4 \mathcal{V})(4) \rightarrow K_4 \rightarrow 0 \quad (4.7)$$

As before we want to compute $H^*((\bigwedge^4 \mathcal{V})(4))$, and we start by decomposing

$$\begin{aligned} (\bigwedge^4 \mathcal{V})(4) = & \mathcal{O}_{\text{Gr}(3,7)}(-4+4) \otimes (\bigwedge^3 \mathcal{U}_3^\vee \otimes (\mathcal{U}_3^\perp)^\vee \oplus \bigwedge^3 \mathcal{U}_3^\vee \oplus \bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee \oplus \bigwedge^2 \mathcal{U}_3^\vee \otimes (\mathcal{U}_3^\perp)^\vee \oplus \\ & \oplus \mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee \oplus \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee \oplus \bigwedge^4 (\mathcal{U}_3^\perp)^\vee \oplus \bigwedge^3 (\mathcal{U}_3^\perp)^\vee) \end{aligned}$$

Then we proceed as before :

- The highest weight of $\bigwedge^3 \mathcal{U}_3^\vee \otimes (\mathcal{U}_3^\perp)^\vee$ is $(1, 1, 1; 0, 0, 0, 0) + (1, 1, 1; 1, 1, 1, 0) = (2, 2, 2; 1, 1, 1, 0)$. Since $\delta + (2, 2, 2; 1, 1, 1, 0)$ is dominant (4.2) and Theorem 4.2.10 give $\dim(H^*(\bigwedge^3 \mathcal{U}_3^\vee \otimes (\mathcal{U}_3^\perp)^\vee)) = 224 \cdot \delta_{*0}$.
- The highest weight of $\bigwedge^3 \mathcal{U}_3^\vee$ is $(1, 1, 1; 0, 0, 0, 0)$. Since $\delta + (1, 1, 1; 0, 0, 0, 0)$ is dominant (4.2) and Theorem 4.2.10 give $\dim(H^*(\bigwedge^3 \mathcal{U}_3^\vee)) = 35 \cdot \delta_{*0}$.
- The highest weight of $\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee$ is $(1, 1, 0; 0, 0, 0, 0) + (1, 1, 1; 1, 1, 0, 0) = (2, 2, 1; 1, 1, 0, 0)$. Since $\delta + (2, 2, 1; 1, 1, 0, 0)$ is dominant (4.2) and Theorem 4.2.10 give $\dim(H^*(\bigwedge^2 \mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee)) = 392 \cdot \delta_{*0}$.
- The highest weight of $\bigwedge^2 \mathcal{U}_3^\vee \otimes (\mathcal{U}_3^\perp)^\vee$ is $(1, 1, 0; 0, 0, 0, 0) + (1, 1, 1; 1, 1, 1, 0) = (2, 2, 1; 1, 1, 1, 0)$. Since $\delta + (2, 2, 1; 1, 1, 1, 0)$ is dominant (4.2) and Theorem 4.2.10 give $\dim(H^*(\bigwedge^2 \mathcal{U}_3^\vee \otimes (\mathcal{U}_3^\perp)^\vee)) = 140 \cdot \delta_{*0}$.
- The highest weight of $\mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee$ is $(1, 0, 0; 0, 0, 0, 0) + (1, 1, 1; 1, 0, 0, 0) = (2, 1, 1; 1, 0, 0, 0)$. Since $\delta + (2, 1, 1; 1, 0, 0, 0)$ is dominant (4.2) and Theorem 4.2.10 give $\dim(H^*(\mathcal{U}_3^\vee \otimes \bigwedge^3 (\mathcal{U}_3^\perp)^\vee)) = 224 \cdot \delta_{*0}$.
- The highest weight of $\mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee$ is $(1, 0, 0; 0, 0, 0, 0) + (1, 1, 1; 1, 1, 0, 0) = (2, 1, 1; 1, 1, 0, 0)$. Since $\delta + (2, 1, 1; 1, 1, 0, 0)$ is dominant (4.2) and Theorem 4.2.10 give $\dim(H^*(\mathcal{U}_3^\vee \otimes \bigwedge^2 (\mathcal{U}_3^\perp)^\vee)) = 140 \cdot \delta_{*0}$.
- The highest weight of $\bigwedge^4 (\mathcal{U}_3^\perp)^\vee$ is $(1, 1, 1; 0, 0, 0, 0)$. Since $\delta + (1, 1, 1; 0, 0, 0, 0)$ is dominant (4.2) and Theorem 4.2.10 give $\dim(H^*(\bigwedge^4 (\mathcal{U}_3^\perp)^\vee)) = 35 \cdot \delta_{*0}$.
- The highest weight of $\bigwedge^3 (\mathcal{U}_3^\perp)^\vee$ is $(1, 1, 1; 1, 0, 0, 0)$. Since $\delta + (1, 1, 1; 1, 0, 0, 0)$ is dominant (4.2) and Theorem 4.2.10 give $\dim(H^*(\bigwedge^3 (\mathcal{U}_3^\perp)^\vee)) = 35 \cdot \delta_{*0}$.

Finally $\dim(H^*(\bigwedge^4 \mathcal{V})(4))) = 1225 \cdot \delta_{*0}$, and the long exact sequence in cohomology induced by (4.7) gives that $\dim(H^*(K_4)) = (1225 - 37) \cdot \delta_{0*} = 1188 \cdot \delta_{0*}$.

- We look now at the short exact sequence

$$0 \rightarrow K_4 \xrightarrow{f_3} \bigwedge^3 \mathcal{V}(4) \rightarrow K_3 \rightarrow 0 \quad (4.8)$$

Using Macaulay2 one can find that the highest weights associated to the decomposition of $\bigwedge^3 \mathcal{V}(4)$ are: $(2, 2, 2; 0, 0, 0, 0)$, $(2, 2, 1; 0, 0, 0, 0)$, $(3, 3, 2; 1, 1, 1, 0)$, $(3, 2, 2; 1, 1, 1, 0)$, $(3, 2, 2; 1, 1, 0, 0)$, $(2, 2, 2; 1, 1, 0, 0)$ and $(2, 2, 2; 1, 0, 0, 0)$. For each of these weight, their sum with δ is non singular and dominant so using (4.2) and Theorem 4.2.10 one finds that they contribute to $\dim(H^0(\bigwedge^3 \mathcal{V}(4)))$ respectively by 490, 490, 3024, 1323, 3402, 588 and 784. Therefore, $\dim(H^*(\bigwedge^3 \mathcal{V}(4))) = 10101 \cdot \delta_{0*}$ and thus $\dim(H^*(K_3)) = (10101 - 1188) \cdot \delta_{0*} = 8913 \cdot \delta_{0*}$.

- Then we can look at the short exact sequence :

$$0 \rightarrow K_3 \xrightarrow{f_2} \bigwedge^2 \mathcal{V}(4) \rightarrow K_2 \rightarrow 0. \quad (4.9)$$

The highest weights associated to $\bigwedge^2 \mathcal{V}(4)$ are : $(3, 3, 2; 0, 0, 0, 0)$, $(3, 2, 2; 0, 0, 0, 0)$, $(4, 3, 3; 1, 1, 1, 0)$, $(3, 3, 3; 1, 1, 1, 0)$, $(3, 3, 3; 1, 1, 0, 0)$. Their sums with δ are all dominant so they contribute to $\dim(H^0(\bigwedge^2 \mathcal{V}(4)))$ by respectively 5292, 2646, 15680, 2940 and 7056. So $\dim(H^*(\bigwedge^2 \mathcal{V}(4))) = 33614 \cdot \delta_{0*}$ and thus $\dim(H^*(K_2)) = (33614 - 8913) \cdot \delta_{0*} = 24701 \cdot \delta_{0*}$.

- We look then at the short exact sequence :

$$0 \rightarrow K_2 \xrightarrow{f_1} \mathcal{V}(4) \rightarrow K_1 \rightarrow 0. \quad (4.10)$$

The highest weights associated to $\mathcal{V}(4)$ are $(3, 3, 3; 0, 0, 0, 0)$, $(4, 3, 3; 0, 0, 0, 0)$ and $(4, 4, 4; 1, 1, 1, 0)$. Their sums with δ are all dominant and non singular, and they contribute to $\dim(H^0(\mathcal{V}(4)))$ respectively by 4116, 20580 and 23520. As a result $\dim(H^*(\mathcal{V}(4))) = 48216 \cdot \delta_{*0}$ and so $\dim(H^*(K_1)) = (48216 - 24701) \cdot \delta_{0*} = 23515 \cdot \delta_{*0}$.

- Finally, there is a short exact sequence :

$$0 \rightarrow K_1 \rightarrow \mathcal{O}_{\text{Gr}(3,7)}(4) \rightarrow \mathcal{O}_{X_4}(4) \rightarrow 0. \quad (4.11)$$

And $\mathcal{O}_{\text{Gr}(3,7)}(4) = \bigwedge^3 \mathcal{U}_3^{\vee} \otimes^4$ has highest weight $(4, 4, 4; 0, 0, 0, 0)$. Its sum with δ is dominant and (4.2) and Theorem 4.2.10 give $\dim(H^*(\mathcal{O}_{\text{Gr}(3,7)}(4))) = 24696 \cdot \delta_{0*}$. So using the long exact sequence in cohomology induced by (4.11) one finds $\dim(H^*(\mathcal{O}_{X_4}(4))) = (24696 - 23515) \cdot \delta_{0*} = 1181 \cdot \delta_{0*}$.

Thus $\chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(4)) = \chi(\mathcal{O}_{\text{Gr}(3,7)}, \mathcal{O}_{X_4}(4)) = \sum_{i \in \mathbb{Z}} (-1)^i \dim(H^i(\mathcal{O}_{\text{Gr}(3,7)}, \mathcal{O}_{X_4}(4))) = 1181$. □

With a similar reasoning, but with (4.3) tensored by $\mathcal{O}_{\text{Gr}(3,7)}$, $\mathcal{O}_{\text{Gr}(3,7)}(1)$, $\mathcal{O}_{\text{Gr}(3,7)}(2)$ or $\mathcal{O}_{\text{Gr}(3,7)}(3)$ instead of $\mathcal{O}_{\text{Gr}(3,7)}(4)$, one can compute the following :

Proposition 4.2.12. *The following Euler characteristics are :*

$$\chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}) = 1, \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(1)) = 20, \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(2)) = 124 \text{ and } \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(3)) = 445.$$

To be able to compute the rest of the Euler characteristics related to the $\mathcal{O}_{X_4}(i)$ we can use Serre duality, which we are going to recall just below, but for this we need to know the canonical bundle ω_{X_4} of X_4 .

Proposition 4.2.13. *The canonical bundle of X_4 is $\omega_{X_4} = \mathcal{O}_{X_4}(-h)$.*

Proof. By ([20], proof of Lemma 4.1), for any X_5 , $\omega_{X_5} = \mathcal{O}_{X_5}(-2h)$. Since $X_{4,\lambda,\mu,\nu}$ is a hyperplane section of $X_{5,\lambda,\mu}$, we can use the adjunction formula which yields :

$$\begin{aligned}\omega_{X_4} &= j^*(\omega_{X_5} \otimes \mathcal{O}_{X_5}(h)) \\ &= j^*(\mathcal{O}_{X_5}(-2h) \otimes \mathcal{O}_{X_5}(h)) = j^*(\mathcal{O}_{X_5}(-h)) = \mathcal{O}_{X_4}(-h)\end{aligned}$$

where j is the injection $j : X_4 \rightarrow X_5$. □

Let us recall now Serre duality (see for instance Theorem 3.12 of [14]) :

Theorem 4.2.14 (Serre duality). *Let X be a smooth projective variety over a field. Then for any $E, F \in D^b(X)$, there are isomorphisms $H^i(E, F) \cong (H^{\dim(X)-i}(F, E \otimes \omega_X))^*$ for all $i \in \mathbb{Z}$. Here ω_X is the canonical bundle of X .*

We can now finish to compute the Euler characteristics related to the $\mathcal{O}_{X_4}(i)$:

Proposition 4.2.15. *Let \tilde{M}' be the matrix whose coefficients are $(m_{ij} = \chi(\mathcal{O}_{X_4}((i-1)h), \mathcal{O}_{X_4}((j-1)h)))_{1 \leq i, j \leq 5}$*

$$\text{then } \tilde{M}' = \begin{pmatrix} 1 & 20 & 124 & 445 & 1181 \\ 1 & 1 & 20 & 124 & 445 \\ 20 & 1 & 1 & 20 & 124 \\ 124 & 20 & 1 & 1 & 20 \\ 445 & 124 & 20 & 1 & 1 \end{pmatrix}.$$

Proof. From Proposition 4.2.11 and Proposition 4.2.12, we know already the first line of \tilde{M}' . Using then

$$H^*(\mathcal{O}_{X_4}(i), \mathcal{O}_{X_4}(j)) = H^*(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(j-i)) \text{ for any } i, j \in \mathbb{Z} \text{ we get } \tilde{M}' = \begin{pmatrix} 1 & 20 & 124 & 445 & 1181 \\ * & 1 & 20 & 124 & 445 \\ * & * & 1 & 20 & 124 \\ * & * & * & 1 & 20 \\ * & * & * & * & 1 \end{pmatrix}.$$

Using again $\chi(\mathcal{O}_{X_4}(i), \mathcal{O}_{X_4}(j)) = \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(j-i))$ for any $i, j \in \mathbb{Z}$ we also get that the missing coefficients are all equal to one of the following : $\{\chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(j)), -4 \leq j \leq -1\}$. But by the hypothesis we are keeping from [20], X_4 is smooth so Serre duality gives : $H^*(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(j)) \cong H^*(\mathcal{O}_{X_4}(j), \omega_{X_4})^\vee = H^*(\mathcal{O}_{X_4}, \omega_{X_4}(-j))^\vee$. By Proposition 4.2.13 we know that $\omega_{X_4} = \mathcal{O}_{X_4}(-1)$, so finally :

$$\begin{aligned}\chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(-1)) &= \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}) \\ \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(-2)) &= \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(1)) \\ \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(-3)) &= \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(2)) \\ \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(-4)) &= \chi(\mathcal{O}_{X_4}, \mathcal{O}_{X_4}(3)).\end{aligned}$$

This allows us to complete \tilde{M}' as above. □

4.3 Euler characteristics involving $\mathcal{O}_{D_3}(V_E)$

In this subsection, we want to compute the Euler characteristics related to $\mathcal{O}_{D_3}(V_E)$.

Let us recall the following notations from section 2.1.1 :

$$\begin{array}{ccc} D_3 & \xrightarrow{j} & \tilde{X}_4 \\ \downarrow p & & \downarrow \pi \\ \Sigma_2 & \xrightarrow{i} & X_4 \\ \downarrow \phi & & \\ \mathbb{P}^2 & & \end{array}$$

Lemma 4.3.1. *The following Euler characteristics are :*

$$(\chi(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{D_3}(V_E)), \dots, \chi(\pi^*\mathcal{O}_{X_4}(4h), \mathcal{O}_{D_3}(V_E))) = (0, 0, 12, 36, 72).$$

Proof. Let $0 \leq i \leq 4$, let us first show that $\chi_{\bar{X}_4}(\pi^*\mathcal{O}_{X_4}(ih), \mathcal{O}_{D_3}(V_E)) = \chi_{\Sigma_2}(\mathcal{O}_{\Sigma_2}, \mathcal{V}_F|_{\Sigma_2}^\vee(-ih))$.
Indeed,

$$\begin{aligned} H^*(\pi^*\mathcal{O}_{X_4}(ih), j_*\mathcal{O}_{D_3}(V_E)) &= H^*(\mathcal{O}_{X_4}(ih), \pi_*j_*\mathcal{O}_{D_3}(V_E)) \\ &= H^*(\mathcal{O}_{X_4}(ih), i_*p_*\mathcal{O}_{D_3}(V_E)) \\ &= H^*(i^*\mathcal{O}_{X_4}, p_*\mathcal{O}_{D_3}(V_E - ih)) \\ &= H^*(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{D_3}(V_E - ih)). \end{aligned}$$

Finally, since $D_3 = \mathbb{P}_{\Sigma_2}(\mathcal{V}_F|_{\Sigma_2})$ the projection formula yields :

$$p_*\mathcal{O}_{D_3}(V_E - ih) = p_*\mathcal{O}_{D_3}(V_E) \otimes \mathcal{O}_{\Sigma_2}(-ih) = \mathcal{V}_F|_{\Sigma_2}^\vee(-ih).$$

We know from Lemma 4.14 of [20] that \mathcal{V}_S and \mathcal{V}_F are respectively the monads of

$$0 \rightarrow \mathcal{O}_{S_2}(-\bar{h}_1) \oplus \mathcal{O}_{S_2}(-\bar{h}_2) \rightarrow \left(\bigwedge^2 A_1 \oplus \bigwedge^2 A_2 \right) \otimes \mathcal{O}_{S_2} \rightarrow \mathcal{O}_{S_2}(\bar{h}_1) \oplus \mathcal{O}_{S_2}(\bar{h}_2) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_{S_2}(-h_1) \oplus \mathcal{O}_{S_2}(-h_2) \rightarrow (A_1 \oplus A_2) \otimes \mathcal{O}_{S_2} \rightarrow \mathcal{O}_{S_2}(h_1) \oplus \mathcal{O}_{S_2}(h_2) \rightarrow 0.$$

Moreover, $A_1 \cong A_2 \cong \bigwedge^2 A_1 \cong \bigwedge^2 A_2 \cong \mathbb{C}^3$ and we know from [20] that Σ_2 and S_2 are both degree 6 del Pezzo surfaces, cut out by two divisors of bidegree $(1, 1)$ in $\mathbb{P}(A_1) \oplus \mathbb{P}(A_2)$, respectively $\mathbb{P}(\bigwedge^2 A_1) \oplus \mathbb{P}(\bigwedge^2 A_2)$, with canonical sheaf $\mathcal{O}_{\Sigma_2}(-h_1 - h_2) = \mathcal{O}_{\Sigma_2}(-h)$, respectively $\mathcal{O}_{S_2}(-\bar{h}_1 - \bar{h}_2)$.

Hence, $\text{ch}(\mathcal{V}_F|_{\Sigma_2}) = (2, 0, -2) = \text{ch}(\mathcal{V}_S)$ and

$$\chi(\mathcal{O}_{\Sigma_2}, \mathcal{V}_F|_{\Sigma_2}^\vee(-ih)) = \chi(\mathcal{O}_{S_2}, \mathcal{V}_S^\vee(-i(\bar{h}_1 + \bar{h}_2))) = \chi(\mathcal{O}_{S_2}, \mathcal{V}_S(-i(\bar{h}_1 + \bar{h}_2)))$$

since $\mathcal{V}_S \cong \mathcal{V}_S^\vee$ by Proposition 1.2.2. Furthermore :

- We saw in Lemma 3.1.3 that $\chi(\mathcal{O}_{S_2}, \mathcal{V}_S) = 0$.
- By 11. of Theorem 1.2.1 and Serre duality, we have $H^*(\mathcal{O}_{S_2}, \mathcal{V}_S(-(\bar{h}_1 + \bar{h}_2))) \cong H^{2-*}(\mathcal{V}_S, \mathcal{O}_{S_2}) = H^{2-*}(\mathcal{O}_{S_2}, \mathcal{V}_S^\vee) = 0$ again by Lemma 3.1.3 and $\mathcal{V}_S \cong \mathcal{V}_S^\vee$. So $\chi(\mathcal{O}_{S_2}, \mathcal{V}_S(-(\bar{h}_1 + \bar{h}_2))) = 0$.
- Using again Serre duality and $\mathcal{V}_S \cong \mathcal{V}_S^\vee$ one finds $H^*(\mathcal{O}_{S_2}, \mathcal{V}_S(-2(\bar{h}_1 + \bar{h}_2))) \cong H^*(\mathcal{O}_{S_2}, \mathcal{V}_S(\bar{h}_1 + \bar{h}_2))$. Hence by Lemma 3.1.4, $\chi(\mathcal{O}_{S_2}, \mathcal{V}_S(-2(\bar{h}_1 + \bar{h}_2))) = 12$.
- To compute $\chi(\mathcal{O}_{S_2}, \mathcal{V}_S(-3(\bar{h}_1 + \bar{h}_2)))$ we will use the same method as in the proof of Lemma 3.1.4. We first look at the short exact sequence induced by (1.3) :

$$0 \rightarrow \mathcal{O}_{S_2}(-4\bar{h}_1 - 3\bar{h}_2) \oplus \mathcal{O}_{S_2}(-3\bar{h}_1 - 4\bar{h}_2) \rightarrow \Omega_1(-2\bar{h}_1 - 3\bar{h}_2) \oplus \Omega_2(-3\bar{h}_1 - 2\bar{h}_2) \rightarrow \mathcal{V}_S(-3\bar{h}_1 - 3\bar{h}_2) \rightarrow 0. \quad (4.12)$$

Moreover, (3.1) induces the following resolution in $\mathbb{P}^2 \times \mathbb{P}^2$:

$$0 \rightarrow \mathcal{O}(-6, -5) \rightarrow \mathcal{O}(-5, -4)^{\oplus 2} \rightarrow \mathcal{O}(-4, -3) \rightarrow \mathcal{O}_{S_2}(-4\bar{h}_1 - 3\bar{h}_2) \rightarrow 0. \quad (4.13)$$

Using Künneth formula as in the proof of Lemma 3.1.3, one finds : $\dim(H^*(\mathcal{O}(-6, -5))) = 60 \cdot \delta_{*,4}$, $\dim(H^*(\mathcal{O}(-5, -4))^{\oplus 2}) = 36 \cdot \delta_{*,4}$ and $\dim(H^*(\mathcal{O}(-4, -3))) = 3 \cdot \delta_{*,4}$. Splitting (4.13) into short exact sequences and computing the Euler characteristics of each term finally yields :

$$\chi(\mathcal{O}_{S_2}(-4\bar{h}_1 - 3\bar{h}_2)) = 3 - (36 - 60) = 27.$$

A very similar calculation shows $\chi(\mathcal{O}_{S_2}(-3\bar{h}_1 - 4\bar{h}_2)) = 27$.

The resolution (3.1) also induces the following :

$$0 \rightarrow \Omega_1(-4, -5) \rightarrow \Omega_1(-3, -4)^{\oplus 2} \rightarrow \Omega_1(-2, -3) \rightarrow \Omega_1(-2\bar{h}_1 - 3\bar{h}_2) \rightarrow 0 \quad (4.14)$$

and using again Künneth formula and (3.2) one finds :

$$\dim(H^*(\Omega_1(-4, -5))) = 90.\delta_{*,4}, \dim(H^*(\Omega_1(-3, -4)^{\oplus 2})) = 48.\delta_{*,4} \text{ and } \dim(H^*(\Omega_1(-2, -3))) = 3.\delta_{*,4}.$$

Splitting (4.14) into short exact sequences and computing Euler characteristics then yields

$$\chi(\Omega_1(-2\bar{h}_1 - 3\bar{h}_2)) = 3 - (48 - 90) = 45.$$

A very similar reasoning gives $\chi(\Omega_2(-3\bar{h}_1 - 2\bar{h}_2)) = 45$.

So finally, using (4.12) one finds $\chi(\mathcal{V}_S(-3(\bar{h}_1 + \bar{h}_2))) = 90 - 54 = 36$.

- We will compute $\chi(\mathcal{O}_{S_2}, \mathcal{V}_S(-4(\bar{h}_1 + \bar{h}_2)))$ the same way as $\chi(\mathcal{O}_{S_2}, \mathcal{V}_S(-3(\bar{h}_1 + \bar{h}_2)))$. There is a short exact sequence :

$$0 \rightarrow \mathcal{O}_{S_2}(-5\bar{h}_1 - 4\bar{h}_2) \oplus \mathcal{O}_{S_2}(-4\bar{h}_1 - 5\bar{h}_2) \rightarrow \Omega_1(-3\bar{h}_1 - 4\bar{h}_2) \oplus \Omega_2(-4\bar{h}_1 - 3\bar{h}_2) \rightarrow \mathcal{V}_S(-4(\bar{h}_1 + \bar{h}_2)) \rightarrow 0 \quad (4.15)$$

and a resolution

$$0 \rightarrow \mathcal{O}(-7, -6) \rightarrow \mathcal{O}(-6, -5)^{\oplus 2} \rightarrow \mathcal{O}(-5, -4) \rightarrow \mathcal{O}_{S_2}(-5\bar{h}_1 - 4\bar{h}_2) \rightarrow 0. \quad (4.16)$$

By Künneth formula (and what did before), $\dim(H^*(\mathcal{O}(-7, -6))) = 150.\delta_{*,4}$, $\dim(H^*(\mathcal{O}(-6, -5)^{\oplus 2})) = 120.\delta_{*,4}$ and $\dim(H^*(\mathcal{O}(-5, -4))) = 18.\delta_{*,4}$. As a result, $\chi(\mathcal{O}_{S_2}(-5\bar{h}_1 - 4\bar{h}_2)) = 18 - (120 - 150) = 48$ and similarly, $\chi(\mathcal{O}_{S_2}(-4\bar{h}_1 - 5\bar{h}_2)) = 18 - (120 - 150) = 48$. We can then look at the resolution :

$$0 \rightarrow \Omega_1(-5, -6) \rightarrow \Omega_1(-4, -5)^{\oplus 2} \rightarrow \Omega_1(-3, -4) \rightarrow \Omega_1(-3\bar{h}_1 - 4\bar{h}_2) \rightarrow 0 \quad (4.17)$$

which yields $\dim(H^*(\Omega_1(-5, -6))) = 240.\delta_{*,4}$, $\dim(H^*(\Omega_1(-4, -5)^{\oplus 2})) = 180.\delta_{*,4}$ and $\dim(H^*(\Omega_1(-3, -4))) = 24.\delta_{*,4}$.

Therefore $\chi(\Omega_1(-3\bar{h}_1 - 4\bar{h}_2)) = 24 - (180 - 240) = 84$ and $\chi(\Omega_2(-4\bar{h}_1 - 3\bar{h}_2)) = 84$.

Finally, $\chi(\mathcal{O}_{S_2}, \mathcal{V}_S(-4(\bar{h}_1 + \bar{h}_2))) = 168 - 96 = 72$.

□

Remark 4.3.2. In Lemma 4.3.1, one could also have used Hirzebruch-Riemann-Roch together with Lemma 4.1.3 to compute the $\chi(\mathcal{O}_{\Sigma_2}, \mathcal{V}_F|_{\Sigma_2}^{\vee}(-ih))$.

Before we can compute the other Euler characteristics involving $\mathcal{O}_{D_3}(V_E)$, we need to look at Σ_2 . This is what we will do in the following subsection :

4.3.1 The surface Σ_2

Along this subsection we are going to use the following notations :

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{i} & X_4 \\ \downarrow \phi & & \\ \mathbb{P}^2 & & \end{array} \quad (4.18)$$

where Σ_2 and X_4 are as in Theorem 1.2.3, i is the associated embedding and ϕ is a blow-up of \mathbb{P}^2 in three points. Indeed we know from 7. of Theorem 1.2.3 that Σ_2 is a smooth degree 6 del Pezzo surface, hence it is the blow-up of \mathbb{P}^2 in three points which we are going to call p_1 , p_2 and p_3 (see for instance Theorem 4.6 and Remark 4.7.1 in V, 4 of [12]).

Definition 4.3.3. Let us denote by H the hyperplane class in \mathbb{P}^2 . We will also write H for its pull-back to Σ_2 , by abuse of notation.

If ϕ is the blow-up of \mathbb{P}^2 in the three points p_1, p_2 and p_3 we are going to denote by E_i the exceptional divisor associated to p_i for $1 \leq i \leq 3$.

Lemma 4.3.4. The Chern characters of the derived pushforward ϕ_* of the following line bundles are :

- $\text{ch}(R^*\phi_*\mathcal{O}_{\Sigma_2}(E_1 + E_2 + E_3)) = (1, 0, 0)$;
- $\text{ch}(R^*\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2 + 2E_3)) = (1, 0, -3)$;
- $\text{ch}(R^*\phi_*\mathcal{O}_{\Sigma_2}(3E_1 + 3E_2 + 3E_3)) = (1, 0, -9)$;
- $\text{ch}(R^*\phi_*\mathcal{O}_{\Sigma_2}(4E_1 + 4E_2 + 4E_3)) = (1, 0, -18)$;
- $\text{ch}(R^*\phi_*\mathcal{O}_{\Sigma_2}(5E_1 + 5E_2 + 5E_3)) = (1, 0, -30)$.

Proof. First of all let us consider $f : T \rightarrow \mathbb{P}^2$ the blow-up in a point p with exceptional divisor E . We will first compute the Chern characters of $R^*f_*\mathcal{O}_T(\alpha E)$ for $1 \leq \alpha \leq 5$ and then we will deduce from them the Chern characters of the sheaves listed above.

- To start with, there is a short exact sequence on $T : 0 \rightarrow \mathcal{O}_T(-E) \rightarrow \mathcal{O}_T \rightarrow \mathcal{O}_E \rightarrow 0$. Since E is a \mathbb{P}^1 in T and $E^2 = -1$, tensoring by $\mathcal{O}_T(E)$ gives a short exact sequence :

$$0 \rightarrow \mathcal{O}_T \rightarrow \mathcal{O}_T(E) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow 0 \quad (4.19)$$

which induces a long exact sequence :

$$0 \rightarrow R^0f_*\mathcal{O}_T \rightarrow R^0f_*\mathcal{O}_T(E) \rightarrow R^0f_*\mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow R^1f_*\mathcal{O}_T \rightarrow R^1f_*\mathcal{O}_T(E) \rightarrow R^1f_*\mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \quad (4.20)$$

$$\rightarrow R^2f_*\mathcal{O}_T \rightarrow R^2f_*\mathcal{O}_T(E) \rightarrow R^2f_*\mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow 0. \quad (4.21)$$

On one hand, $R^*f_*\mathcal{O}_T = f_*\mathcal{O}_T = \mathcal{O}_{\mathbb{P}^2}$ (see for instance Proposition 3.4 of V.3. in [12]), on the other hand since $f(E) = \{p\}$, $f|_E$ has image supported in one point, thus $R^*f_*\mathcal{O}_E(E) = R^*f_*\mathcal{O}_{\mathbb{P}^1}(-1) = \text{Hom}^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes k(p)$ where $k(p)$ is the skyscraper sheaf supported in p (see for instance the beginning of page 73 of [14]). Thus the long exact sequence (4.20) gives actually :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow R^0f_*\mathcal{O}_T(E) \rightarrow 0 \rightarrow 0 \rightarrow R^1f_*\mathcal{O}_T(E) \rightarrow 0 \rightarrow 0 \rightarrow R^2f_*\mathcal{O}_T(E) \rightarrow 0 \quad (4.22)$$

since $\text{Hom}^*(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. So finally $R^*f_*\mathcal{O}_T(E) = \mathcal{O}_{\mathbb{P}^2}$ and thus $\text{ch}(R^*f_*\mathcal{O}_T(E)) = (1, 0, 0)$.

- Tensoring now (4.19) by $\mathcal{O}_T(E)$ we get the short exact sequence $0 \rightarrow \mathcal{O}_T(E) \rightarrow \mathcal{O}_T(2E) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow 0$ since $2E^2 = -2$. This gives a long exact sequence similar to (4.20). Since we have just computed $R^*f_*\mathcal{O}_T(E)$ and since by Serre duality $\text{Hom}^*(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong \text{Hom}^{1-*}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}\delta_{1-*,0}$, this yields :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow R^0f_*\mathcal{O}_T(2E) \rightarrow 0 \rightarrow 0 \rightarrow R^1f_*\mathcal{O}_T(2E) \rightarrow \mathbb{C} \otimes k(p) \rightarrow 0 \rightarrow R^2f_*\mathcal{O}_T(2E) \rightarrow 0.$$

So $R^0f_*\mathcal{O}_T(2E) = \mathcal{O}_{\mathbb{P}^2}$ and $R^1f_*\mathcal{O}_T(2E) = k(p)$. Therefore $\text{ch}(R^*f_*\mathcal{O}_T(2E)) = (1, 0, 0) - (0, 0, 1) = (1, 0, -1)$.

- Proceeding the same way gives $\text{ch}(R^*f_*\mathcal{O}_T(3E)) = (1, 0, -3)$, $\text{ch}(R^*f_*\mathcal{O}_T(4E)) = (1, 0, -6)$ and $\text{ch}(R^*f_*\mathcal{O}_T(5E)) = (1, 0, -10)$.

Let us go back to Σ_2 now. By what we have just seen (replacing E by E_1 and p by p_1), $R^*\phi_*\mathcal{O}_{\Sigma_2}(E_1) = \mathcal{O}_{\mathbb{P}^2}$ and the equivalent of (4.19) with E_2 , tensored by $\mathcal{O}_{\Sigma_2}(E_1)$, gives the short exact sequence : $0 \rightarrow \mathcal{O}_{\Sigma_2}(E_1) \rightarrow \mathcal{O}_{\Sigma_2}(E_1 + E_2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow 0$ since $E_1.E_2 = 0$. Since $R^*\phi_*\mathcal{O}_{\Sigma_2} = R^*\phi_*\mathcal{O}_{\Sigma_2}(E_1) = \mathcal{O}_{\mathbb{P}^2}$, by the same reasoning as above we get that $R^*\phi_*\mathcal{O}_{\Sigma_2}(E_1 + E_2) = \mathcal{O}_{\mathbb{P}^2}$. Exactly the same way, the equivalent of (4.19) with E_3 tensored by $\mathcal{O}_{\Sigma_2}(E_1 + E_2)$ gives a short exact sequence : $0 \rightarrow \mathcal{O}_{\Sigma_2}(E_1 + E_2) \rightarrow \mathcal{O}_{\Sigma_2}(E_1 + E_2 + E_3) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow 0$

since $E_1.E_3 = E_2.E_3 = 0$. So $R^*\phi_*\mathcal{O}_{\Sigma_2}(E_1 + E_2 + E_3) = \mathcal{O}_{\mathbb{P}^2}$ and $\text{ch}(R^*\phi_*\mathcal{O}_{\Sigma_2}(E_1 + E_2 + E_3)) = (1, 0, 0)$. Similarly, one gets a short exact sequence $0 \rightarrow \mathcal{O}_{\Sigma_2}(E_1 + E_2) \rightarrow \mathcal{O}_{\Sigma_2}(2E_1 + E_2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow 0$ by tensoring (4.19) for E_1 by $\mathcal{O}_{\Sigma_2}(E_1 + E_2)$. We have seen that $R^*\phi_*\mathcal{O}_{\Sigma_2}(E_1 + E_2) = \mathcal{O}_{\mathbb{P}^2}$ so using again the same type of reasoning as before, one gets that $R^0\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + E_2) = \mathcal{O}_{\mathbb{P}^2}$ and $R^1\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + E_2) = \mathbb{C} \otimes k(p_1)$. Thus $\text{ch}(R^*\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + E_2)) = (1, 0, -1)$. Tensoring now (4.19) for E_2 by $\mathcal{O}_{\Sigma_2}(2E_1 + E_2)$ one gets a short exact sequence $0 \rightarrow \mathcal{O}_{\Sigma_2}(2E_1 + E_2) \rightarrow \mathcal{O}_{\Sigma_2}(2E_1 + 2E_2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow 0$. This yields the long exact sequence :

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow R^0\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2) \rightarrow 0 \rightarrow k(p_1) \rightarrow R^1\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2) \rightarrow \mathbb{C} \otimes k(p_2) \rightarrow \\ 0 \rightarrow R^2\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2) \rightarrow 0. \end{aligned}$$

So $R^0\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2) = \mathcal{O}_{\mathbb{P}^2}$ and $\text{ch}(R^1\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2)) = (0, 0, 2)$ thus $\text{ch}(R^*\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2)) = (1, 0, -2)$.

Finally, from the short exact sequence $0 \rightarrow \mathcal{O}_{\Sigma_2}(2E_1 + 2E_2) \rightarrow \mathcal{O}_{\Sigma_2}(2E_1 + 2E_2 + E_3) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow 0$ one gets $R^0\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2 + E_3) = \mathcal{O}_{\mathbb{P}^2}$ and $\text{ch}(R^1\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2 + E_3)) = (0, 0, 2)$. Then from the short exact sequence $0 \rightarrow \mathcal{O}_{\Sigma_2}(2E_1 + 2E_2 + E_3) \rightarrow \mathcal{O}_{\Sigma_2}(2E_1 + 2E_2 + 2E_3) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow 0$ one gets $R^0\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2 + 2E_3) = \mathcal{O}_{\mathbb{P}^2}$ and $\text{ch}(R^1\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2 + 2E_3)) = (0, 0, 2) + (0, 0, 1) = (0, 0, 3)$. Therefore $\text{ch}(R^*\phi_*\mathcal{O}_{\Sigma_2}(2E_1 + 2E_2 + 2E_3)) = (1, 0, -3)$.

The other Chern characters mentioned above can be computed using exactly the same method. \square

Lemma 4.3.5. *The following holds : $(\chi(\mathcal{O}_{X_4}, i_*\mathcal{O}_{\Sigma_2}), \dots, \chi(\mathcal{O}_{X_4}(5h), i_*\mathcal{O}_{\Sigma_2})) = (1 \ 1 \ 7 \ 19 \ 37 \ 61)$.*

Proof. First of all, since i_* is right adjoint to i^* and ϕ_* is right adjoint to ϕ^* (see for instance iii) of page 83 in [14]),

$$H^*(\mathcal{O}_{X_4}, i_*\mathcal{O}_{\Sigma_2}) \cong H^*(i^*\mathcal{O}_{X_4}, \mathcal{O}_{\Sigma_2}) = H^*(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}) = H^*(\phi^*\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\Sigma_2}) \cong H^*(\mathcal{O}_{\mathbb{P}^2}, \phi_*\mathcal{O}_{\Sigma_2}).$$

Moreover, $\phi_*\mathcal{O}_{\Sigma_2} \cong \mathcal{O}_{\mathbb{P}^2}$ (see for instance Proposition 3.4 of V.3. in [12]) so finally

$$\chi(\mathcal{O}_{X_4}, i_*\mathcal{O}_{\Sigma_2}) = \chi(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}) = 1.$$

Secondly, with a similar reasoning one gets $H^*(\mathcal{O}_{X_4}(h), i_*\mathcal{O}_{\Sigma_2}) \cong H^*(i^*\mathcal{O}_{X_4}(h), \mathcal{O}_{\Sigma_2}) = H^*(\mathcal{O}_{\Sigma_2}(h), \mathcal{O}_{\Sigma_2}) = H^*(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-h))$. By Lemma 4.1.2, $\omega_{\Sigma} = \mathcal{O}_{\Sigma_2}(-h)$ and since Σ_2 is smooth we can use Serre duality, which yields $H^*(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-h)) \cong H^{2-*}(\mathcal{O}_{\Sigma_2}(-h), \mathcal{O}_{\Sigma_2}(-h))^\vee = H^{2-*}(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2})^\vee$. Thus, since for any $q \in \mathbb{Z}$, $(-1)^q = (-1)^{2-q}$ this implies $\chi(\mathcal{O}_{X_4}(h), i_*\mathcal{O}_{\Sigma_2}) = \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}) = 1$ by the computation we did just above.

For the rest of the coefficients we are going to use Hirzebruch-Riemann-Roch's Theorem.

Let us first start as before : $H^*(\mathcal{O}_{X_4}(2h), i_*\mathcal{O}_{\Sigma_2}) \cong H^*(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-2h)) = H^*(\mathcal{O}_{\Sigma_2}, \omega_{\Sigma}^{\otimes 2})$. Moreover, if we denote by E_i the exceptional divisor in Σ_2 associated to p_i by the blow-up ϕ as in Definition 4.3.3, then using for instance Notation 4.7.3 and Proposition 4.8, in V,4 of [12], one gets that $\omega_{\Sigma} = \mathcal{O}_{\Sigma_2}(-3H + E_1 + E_2 + E_3)$ where H is as in Definition 4.3.3. Thus

$$\begin{aligned} H^*(\mathcal{O}_{\Sigma_2}, \omega_{\Sigma}^{\otimes 2}) &= H^*(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-6H + 2E_1 + 2E_2 + 2E_3)) \\ &= H^*(\phi^*\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\Sigma_2}(-6H + 2E_1 + 2E_2 + 2E_3)) \\ &\cong H^*(\mathcal{O}_{\mathbb{P}^2}, \phi_*\mathcal{O}_{\Sigma_2}(-6H + 2E_1 + 2E_2 + 2E_3)). \end{aligned}$$

By the projection formula one gets that $\phi_*\mathcal{O}_{\Sigma_2}(-6H + 2E_1 + 2E_2 + 2E_3) = \mathcal{O}_{\mathbb{P}^2}(-6H) \otimes_{\mathcal{O}_{\mathbb{P}^2}} (\phi_*(2E_1 + 2E_2 + 2E_3))$. But since here ϕ_* is $R^*\phi_*$ (as mentioned in the beginning of section 1.1) by Lemma 4.3.4, $\text{ch}(\mathcal{O}_{\mathbb{P}^2}(\phi_*(2E_1 + 2E_2 + 2E_3))) = (1, 0, -3)$.

Moreover, we know that the Chern character of $\mathcal{O}_{\mathbb{P}^2}(-6H)$ is $(1, -6H, \frac{(-6H)^2}{2}) = (1, -6H, 18)$ since $H^2 = 1$. Therefore

$$\text{ch}(\mathcal{O}_{\mathbb{P}^2}(-6H) \otimes_{\mathcal{O}_{\mathbb{P}^2}} (\phi_*(2E_1 + 2E_2 + 2E_3))) = (1, 0, -3)(1, -6H, 18) = (1, -6H, 15).$$

The Hirzebruch-Riemann-Roch Theorem then gives

$$\chi(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(-6H) \otimes_{\mathcal{O}_{\mathbb{P}^2}} (\phi_*(2E_1 + 2E_2 + 2E_3))) = \int_{\mathbb{P}^2} (1, -6H, 15). \text{td}(\mathcal{T})$$

where $\text{td}(\mathcal{T})$ is the Todd class of the tangent sheaf \mathcal{T} of \mathbb{P}^2 . Since \mathcal{T} is the dual of Ω we have a short exact sequence : $0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \rightarrow \mathcal{T} \rightarrow 0$. Thus $c_1(\mathcal{T}) = 3H$ and $c_2(\mathcal{T}) = 3$. Hence, using the formula for the Todd class page 432 in Appendix A of [12], one finds that the Todd class of \mathcal{T} is :

$$\left(1, \frac{c_1(\mathcal{T})}{2}, \frac{c_1(\mathcal{T})^2 + c_2(\mathcal{T})}{12}\right) = \left(1, \frac{3}{2}H, 1\right).$$

So finally, $\int_{\mathbb{P}^2}(1, -6H, 15).\text{td}(\mathcal{T}) = \int_{\mathbb{P}^2}(1, -\frac{9}{2}H, 15 + 1 - 9) = \int_{\mathbb{P}^2}(1, -\frac{9}{2}H, 7) = 7$.
For the remaining Euler characteristics, one can proceed similarly. \square

Remark 4.3.6. *The Euler characteristics in Lemma 4.3.5 can also be computed using Hirzebruch-Riemann-Roch and Lemma 4.1.3.*

4.3.2 Back to $\mathcal{O}_{D_3}(V_E)$

We can now compute the following :

Lemma 4.3.7. *The following Euler characteristics are : $(\chi(\mathcal{O}_{D_3}(V_E), \pi^*\mathcal{O}_{X_4}), \dots, \chi(\mathcal{O}_{D_3}(V_E), \pi^*\mathcal{O}_{X_4}(4h))) = (1, 7, 19, 37, 61)$.*

Proof. Let $0 \leq i \leq 4$. Since \tilde{X}_4 is the blow-up of X_4 in the codimension 2 subvariety Σ_2 , the canonical sheaf of \tilde{X}_4 is $\omega_{\tilde{X}_4} = \pi^*\omega_X(e) = \mathcal{O}_{\tilde{X}_4}(-h + e)$ by Lemma 4.2.13 and the fact that D_3 has class e in \tilde{X}_4 (indeed it is the intersection of \tilde{X}_4 and E_4). By Serre duality and (1.10) this implies :

$$\begin{aligned}\chi(\mathcal{O}_{D_3}(V_E), \pi^*\mathcal{O}_{X_4}(ih)) &= \chi(\pi^*\mathcal{O}_{X_4}(ih), \mathcal{O}_{D_3}(V_E + e - h)) \\ &= \chi(\pi^*\mathcal{O}_{X_4}(ih), \mathcal{O}_{D_3}(-h)) \\ &= \chi(\pi^*\mathcal{O}_{X_4}((i+1)h), \mathcal{O}_{D_3}).\end{aligned}$$

Finally,

$$\chi(\pi^*\mathcal{O}_{X_4}((i+1)h), \mathcal{O}_{D_3}) = \chi(\mathcal{O}_{X_4}((i+1)h), \pi_*\mathcal{O}_{D_3}) = \chi(\mathcal{O}_{X_4}((i+1)h), \mathcal{O}_{\Sigma_2})$$

so one can conclude with Lemma 4.3.5. \square

Lemma 4.3.8. *One has $\chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{D_3}(V_E)) = 1$.*

Proof. Since D_3 is a divisor in \tilde{X}_4 there is a short exact sequence : $0 \rightarrow \mathcal{O}_{\tilde{X}_4}(-e) \rightarrow \mathcal{O}_{\tilde{X}_4} \rightarrow \mathcal{O}_{D_3} \rightarrow 0$. Indeed E_4 has class e in $\text{Pic}(\tilde{X}_5)$ and $D_3 = \Sigma_2 \times_{F_3} E_4$ so the class of D_3 is e in $\text{Pic}(\tilde{X}_4)$. As a result,

$$\chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{D_3}(V_E)) = \chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}) - \chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{D_3}).$$

- Let us compute $\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3})$. One can use that

$$\begin{aligned}H^*(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}) &= H^*(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{D_3}) \\ &= H^*(\mathcal{O}_{X_4}, \pi_*\mathcal{O}_{D_3}) \\ &= H^*(\mathcal{O}_{X_4}, \mathcal{O}_{\Sigma_2})\end{aligned}$$

and Lemma 4.3.5 to find that $\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}) = 1$.

- Let us compute now $\chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{D_3})$. For this, one can notice that

$$\begin{aligned}H^*(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{D_3}) &= H^*(\mathcal{O}_{\tilde{X}_4}(-e), j_*\mathcal{O}_{D_3}) \\ &= H^*(j^*\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{D_3}) \\ &= H^*(\mathcal{O}_{D_3}(-e), \mathcal{O}_{D_3}).\end{aligned}$$

Moreover by (1.10) we have $-e = V_E$ in $\text{Pic}(E_4)$, hence since $D_3 \subset E_4$ we have

$$\begin{aligned} H^*(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(e)) &= H^*(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(-V_E)) \\ &= H^*(p^*\mathcal{O}_{\Sigma_2}, \mathcal{O}_{D_3}(-V_E)) \\ &= H^*(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{D_3}(-V_E)). \end{aligned}$$

Since $p_*\mathcal{O}_{D_3}(-V_E) = 0$ (see for instance Exercise 8.4 of III.8. in [12]), we get $\chi(\mathcal{O}_{\bar{X}_4}(-e), \mathcal{O}_{D_3}) = 0$.

Finally, $\chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{D_3}(V_E)) = 1 - 0 = 1$. □

4.4 The ruled surface \bar{D}_2

To start with, let us remember from Theorem 1.2.3 that \bar{D}_2 is a ruled surface over the curve Γ_1 : indeed $\bar{D}_2 = \mathbb{P}_\Gamma(\mathcal{V}_Z|_\Gamma)$. We can then use results from part V.2. of [12] to compute the Euler characteristics associated to $\mathcal{O}_{\bar{D}_2}$.

Lemma 4.4.1. *Up to numerical equivalence, the Picard group of \bar{D}_2 is generated by two classes δ and f such that : $\delta^2 = 48$, $f^2 = 0$ and $\delta.f = 1$.*

Moreover the canonical sheaves of \bar{D}_2 and Γ_1 are respectively $\omega_{\bar{D}_2} = \mathcal{O}_{\bar{D}_2}(-2\delta + 72f)$ and $\omega_\Gamma = \mathcal{O}_{\Gamma_1}(24)$.

Proof. We have just seen that $\bar{D}_2 = \mathbb{P}_\Gamma(\mathcal{V}_Z|_{\Gamma_1})$, therefore \bar{D}_2 is a ruled surface over Γ_1 (see for instance the first definition and note of V.2. in [12]). By Proposition 2.3 of V.2. in [12] we thus have : $\text{Pic}(\bar{D}_2) \cong \mathbb{Z} \oplus \pi^*(\text{Pic}(\Gamma_1))$ and $\text{Num}(\bar{D}_2) \cong \mathbb{Z}^2$ where $\text{Num}(\bar{D}_2) = \text{Pic}(\bar{D}_2)/\{D, \forall E \in \text{Pic}(\bar{D}_2), D.E = 0\}$. Furthermore we can find two generators f and δ such that $f^2 = 0$ and $\delta.f = 1$. By Proposition 2.9 of V.2. in [12] we also get $\delta^2 = \deg(\mathcal{E})$ where $\bar{D}_2 = \mathbb{P}_{\Gamma_1}(\mathcal{E})$. However we have to be careful here : as mentioned in the proof of Proposition 3.2.1, in [12] R. Hartshorne took a different definition of $\mathbb{P}(\mathcal{E})$ than A. Kuznetsov in [20], the latter being the definition we use. So for us, \mathcal{E} will be $\mathcal{V}_Z^\vee|_{\Gamma_1}$ and not $\mathcal{V}_Z|_{\Gamma_1}$. We also have by Lemma 3.2.3 and (1.11) that $\det(\mathcal{V}_Z^\vee) = \mathcal{O}_{Z_3}(3V_Z + 2(\bar{h}_1 + \bar{h}_2))$, and by (1.15), the class of Γ_1 in Z_3 is $4V_Z.(\bar{h}_1 + \bar{h}_2)$. Finally, in $\text{CH}(Z_3)$,

$$4V_Z.(\bar{h}_1 + \bar{h}_2).(3V_Z + 2(\bar{h}_1 + \bar{h}_2)) = 8V_Z.(\bar{h}_1 + \bar{h}_2)^2 = 8.6 = 48.$$

As a result, $\det(\mathcal{V}_Z^\vee|_{\Gamma_1}) = \mathcal{O}_{\Gamma_1}(48)$ and thus $\delta^2 = 48$.

We also know from Lemma 3.5.1 that Γ_1 has genus 13. Hence (see for instance Example 1.3.3 of IV.1. of [12]), $\omega_\Gamma = \mathcal{O}_{\Gamma_1}(2g(\Gamma_1) - 2) = \mathcal{O}_{\Gamma_1}(24)$.

Finally, by Lemma 2.10 of V.2. in [12],

$$\omega_{\bar{D}_2} = \mathcal{O}_{\bar{D}_2}(-2\delta + (\omega_\Gamma + \delta^2)f) = \mathcal{O}_{\bar{D}_2}(-2\delta + (24 + 48)f) = \mathcal{O}_{\bar{D}_2}(-2\delta + 72f).$$

□

Before going further, let us recall the Riemann-Roch theorem for surfaces, taken here from V.1. of [12] :

Theorem 4.4.2 (Riemann-Roch). *Let X be a non singular projective surface over an algebraically closed field k . Let D be a divisor on X and K the canonical class of X . Then :*

$$\chi(\mathcal{O}_X, \mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}D.(D - K). \quad (4.23)$$

Definition 4.4.3. *For simplicity let us denote $\bar{\pi}|_{\bar{D}_2} : \bar{D}_2 \rightarrow \Gamma_1$ by ϕ .*

Lemma 4.4.4. *The following holds : $\chi(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{\bar{D}_2}) = \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}) = -12$.*

Proof. To start with :

$$\begin{aligned} H^*(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{\overline{D}_2}) &= H^*(\mathcal{O}_{\overline{X}_4}, \overline{i}_*\mathcal{O}_{\overline{D}_2}) \\ &= H^*(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}) \\ &= H^*(\phi^*\mathcal{O}_{\Gamma_1}, \mathcal{O}_{\overline{D}_2}) \\ &= H^*(\mathcal{O}_{\Gamma_1}, \phi_*\mathcal{O}_{\overline{D}_2}). \end{aligned}$$

In particular, $\chi(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{\overline{D}_2}) = \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}) = \chi(\mathcal{O}_{\Gamma_1}, \phi_*\mathcal{O}_{\overline{D}_2})$. Moreover $\pi_*\mathcal{O}_{\overline{D}_2} = \mathcal{O}_{\Gamma_1}$: indeed by Lemma 2.1 of V.2. in [12], we have that $R^0\pi_*\mathcal{O}_{\overline{D}_2} = \pi_*\mathcal{O}_{\overline{D}_2} = \mathcal{O}_{\Gamma_1}$ and since $0.f = 0 \geq 0$, by Lemma 2.4 of V.2. in [12], $R^1\pi_*\mathcal{O}_{\overline{D}_2} = 0$. As a result, $H^*(\mathcal{O}_{\Gamma_1}, \phi_*\mathcal{O}_{\overline{D}_2}) = H^*(\mathcal{O}_{\Gamma_1}, \mathcal{O}_{\Gamma_1})$.

Finally by Riemann-Roch for curves (see for instance Theorem 1.3 of IV.1. in [12]) :

$$\chi(\mathcal{O}_{\Gamma_1}, \mathcal{O}_{\Gamma_1}(n)) = n + 1 - g(\Gamma_1).$$

Thus, $\chi(\mathcal{O}_{\Gamma_1}, \mathcal{O}_{\Gamma_1}) = 1 - g(\Gamma_1) = 1 - 13 = -12$. □

Lemma 4.4.5. *We have $h|_{\overline{D}_2} = V_{\overline{E}}|_{\overline{D}_2} = \delta + 48f$ with the notations of Lemma 4.4.1.*

Proof. We know from (1.12) that in $\text{Pic}(\overline{E}_4)$, $h = V_{\overline{E}}$ thus since $\overline{D}_2 \subset \overline{E}_4$ we have $h|_{\overline{D}_2} = V_{\overline{E}}|_{\overline{D}_2}$. Moreover we know by definition of $V_{\overline{E}}$ that $\overline{\pi}_*\mathcal{O}_{\overline{E}_4}(V_{\overline{E}}) = \mathcal{V}_Z^\vee$, hence $\phi_*\mathcal{O}_{\overline{D}_2}(V_{\overline{E}}) = \mathcal{V}_Z^\vee|_{\Gamma_1}$. On the other hand, in Lemma 4.4.1 we defined δ as C_0 in section V.2. of [12], according to which $\phi_*\mathcal{O}_{\overline{D}_2}(\delta) = \mathcal{V}_Z|_{\Gamma_1}$ (see Notation 2.8.1. in V.2. and Proposition 7.11 in II.7. of [12]). We thus have :

$$\phi_*\mathcal{O}_{\overline{D}_2}(V_{\overline{E}}) = \mathcal{V}_Z|_{\Gamma_1} \otimes \det(\mathcal{V}_Z^\vee|_{\Gamma_1}) = \phi_*\mathcal{O}_{\overline{D}_2}(\delta) \otimes \det(\mathcal{V}_Z^\vee|_{\Gamma_1}).$$

Indeed from Corollary 2.7. in V.2. of [12] we know that there exists two invertible sheaves \mathcal{L}_1 and \mathcal{L}_2 and a short exact sequence $0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{V}_Z|_{\Gamma_1} \rightarrow \mathcal{L}_2 \rightarrow 0$, inducing the following short exact sequence in $D^b(\Gamma_1)$:

$$0 \rightarrow \mathcal{L}_2^\vee \rightarrow \mathcal{V}_Z^\vee|_{\Gamma_1} \rightarrow \mathcal{L}_1^\vee \rightarrow 0.$$

Since $\det(\mathcal{V}_Z^\vee|_{\Gamma_1}) = \mathcal{L}_1^\vee \otimes \mathcal{L}_2^\vee$, tensoring the first short exact sequence with it gives another short exact sequence:

$$0 \rightarrow \mathcal{L}_2^\vee \rightarrow \det(\mathcal{V}_Z^\vee|_{\Gamma_1}) \otimes \mathcal{V}_Z|_{\Gamma_1} \rightarrow \mathcal{L}_1^\vee \rightarrow 0$$

which yields $\mathcal{V}_Z^\vee|_{\Gamma_1} = \mathcal{V}_Z|_{\Gamma_1} \otimes \det(\mathcal{V}_Z^\vee|_{\Gamma_1})$.

Finally, we have seen in the proof of Lemma 4.4.1 that $\det(\mathcal{V}_Z^\vee|_{\Gamma_1}) = \mathcal{O}_{\Gamma_1}(48)$, therefore,

$$\phi_*\mathcal{O}_{\overline{D}_2}(V_{\overline{E}}) = \phi_*\mathcal{O}_{\overline{D}_2}(\delta) \otimes \mathcal{O}_{\Gamma_1}(48) = \phi_*\mathcal{O}_{\overline{D}_2}(\delta + 48f)$$

(see for instance Proposition 2.3 in V.2. of [12]). Since we have seen in Lemma 4.4.1 that $\text{Pic}(\overline{D}_2) = \mathbb{Z}\delta \oplus \mathbb{Z}f$, it implies that $h|_{\overline{D}_2} = V_{\overline{E}}|_{\overline{D}_2} = \delta + 48f$. Indeed, let us assume that there exists α and β in \mathbb{Z} such that $\phi_*\mathcal{O}_{\overline{D}_2}(\alpha\delta + \beta f) = \mathcal{V}_Z|_{\Gamma_1}(48)$. Since $\mathcal{V}_Z|_{\Gamma_1}$ is not self-dual, by the Remark below Proposition 9.3 in [8] (taking in account the fact that [8] and [12] have different definitions of the projectivization of a vector bundle), we must have $\alpha = 1$. Then β has to be 48 because $\phi_*\mathcal{O}_{\overline{D}_2}(\delta + \beta f) = \phi_*(\mathcal{O}_{\overline{D}_2}(\delta) \otimes \phi^*\mathcal{O}_{\Gamma_1}(\beta)) = \mathcal{V}_Z|_{\Gamma_1}(\beta)$. □

Proposition 4.4.6. *One has $(\chi(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{\overline{D}_2}), \dots, \chi(\pi^*\mathcal{O}_{X_4}(4h), \mathcal{O}_{\overline{D}_2})) = (-12, 0, 156, 456, 900)$.*

Proof. We have already seen in Lemma 4.4.4 that $\chi(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{\overline{D}_2}) = -12$. Let us now compute $\chi(\pi^*\mathcal{O}_{X_4}(ih), \mathcal{O}_{\overline{D}_2})$ for $1 \leq i \leq 4$.

Let us start with $\chi(\pi^*\mathcal{O}_{X_4}(h), \mathcal{O}_{\overline{D}_2})$: as in the proof of Lemma 4.4.4 we have

$$\begin{aligned} H^*(\pi^*\mathcal{O}_{X_4}(h), \mathcal{O}_{\overline{D}_2}) &= H^*(\mathcal{O}_{\overline{X}_4}(h), \overline{i}_*\mathcal{O}_{\overline{D}_2}) \\ &= H^*(\mathcal{O}_{\overline{D}_2}(h), \mathcal{O}_{\overline{D}_2}) \\ &= H^*(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-h)). \end{aligned}$$

Moreover by Lemma 4.4.5 we have $\mathcal{O}_{\bar{D}_2}(-h) = \mathcal{O}_{\bar{D}_2}(-\delta - 48f)$.

So $H^*(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}(-h)) = H^*(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}(-\delta - 48f))$.

Using (4.23) and Lemma 4.4.4, one finally finds :

$$\chi(\pi^* \mathcal{O}_{X_4}(h), \mathcal{O}_{\bar{D}_2}) = \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}(-\delta - 48f)) = -12 + \frac{1}{2}((\delta + 48f)^2 + (\delta + 48f)K_{\bar{D}_2}).$$

By Lemma 4.4.1, $\delta^2 = 48$, $\delta \cdot f = 1$, $f^2 = 0$ and $K_{\bar{D}_2} = -2\delta + 72f$ therefore :

$$\chi(\mathcal{O}_{X_4}(h), \pi_* \mathcal{O}_{\bar{D}_2}) = -12 + \frac{1}{2}(48 + 2.48 - 2.48 + 72 - 2.48) = -12 + \frac{24}{2} = 0.$$

We will proceed similarly for $\chi(\pi^* \mathcal{O}_{X_4}(ih), \mathcal{O}_{\bar{D}_2})$ with $2 \leq i \leq 4$:

- $\chi(\pi^* \mathcal{O}_{X_4}(2h), \mathcal{O}_{\bar{D}_2}) = \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}(-2(\delta + 48f))) = -12 + \frac{1}{2}(4.144 - 120.2) = -12 + 168 = 156.$
- $\chi(\pi^* \mathcal{O}_{X_4}(3h), \mathcal{O}_{\bar{D}_2}) = \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}(-3(\delta + 48f))) = -12 + \frac{1}{2}(9.144 - 120.3) = -12 + 468 = 456.$
- $\chi(\pi^* \mathcal{O}_{X_4}(4h), \mathcal{O}_{\bar{D}_2}) = \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}(-4(\delta + 48f))) = -12 + \frac{1}{2}(16.144 - 120.4) = -12 + 912 = 900.$

□

Before going further, we need to know the canonical bundle of \tilde{X}_4 , in order to apply Serre duality later.

Lemma 4.4.7. *The canonical bundle of \tilde{X}_4 is $\omega_{\tilde{X}_4} = \mathcal{O}_{\tilde{X}_4}(-h + e)$.*

Proof. We have seen in Lemma 4.2.13 that $\omega_X = \mathcal{O}_{X_4}(-h)$ and we know from Theorem 1.2.3 that \tilde{X}_4 is the blow-up of X_4 in Σ_2 , which is codimension 2 in X_4 . Finally, the class of D_3 , which is the exceptional divisor of this blow-up, is $e|_{\tilde{X}_4}$, which we are still going to denote by e : indeed D is the intersection between \tilde{X}_4 and E_4 . Hence $\omega_{\tilde{X}_4} = \pi^* \mathcal{O}_{X_4}(-h) \otimes \mathcal{O}_{\tilde{X}_4}(e) = \mathcal{O}_{\tilde{X}_4}(-h + e)$. □

We are now able to compute :

Proposition 4.4.8. *The following Euler characteristics are :*

$$(\chi(\mathcal{O}_{\bar{D}_2}, \pi^* \mathcal{O}_{X_4}), \dots, \chi(\mathcal{O}_{\bar{D}_2}, \pi^* \mathcal{O}_{X_4}(4h))) = (-36, 0, 180, 504, 972).$$

Proof. Let $0 \leq i \leq 4$. Then by Serre duality and Lemma 4.4.7, we have :

$$H^*(\mathcal{O}_{\bar{D}_2}, \pi^* \mathcal{O}_{X_4}(ih)) \cong H^{4-*}(\pi^* \mathcal{O}_{X_4}(ih), \mathcal{O}_{\bar{D}_2}(-h + e)).$$

Since 4 is even we have $\chi(\mathcal{O}_{\bar{D}_2}, \pi^* \mathcal{O}_{X_4}(ih)) = \chi(\pi^* \mathcal{O}_{X_4}(ih), \mathcal{O}_{\bar{D}_2}(-h + e)) = \chi(\pi^* \mathcal{O}_{X_4}(ih), \mathcal{O}_{\bar{D}_2}(-\bar{h}))$ where we used (1.8) for the last equality. Moreover, using (1.11) and (1.15) one finds that

$$\bar{h}|_{\bar{D}_2} = 4V_Z(\bar{h}_1 + \bar{h}_2) \cdot (V_Z + \bar{h}_1 + \bar{h}_2) = 4V_Z(\bar{h}_1 + \bar{h}_2)^2.$$

Since $(\bar{h}_1 + \bar{h}_2)^2 = 6$ in $\text{CH}(S_2)$ we thus have $\bar{h}|_{\bar{D}_2} = 24f$. We can now use the same reasoning as in the proof

of Proposition 4.4.6 to find :

$$\begin{aligned}
\chi(\mathcal{O}_{\overline{D}_2}, \pi^* \mathcal{O}_{X_4}) &= \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-\overline{h})) = \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-24f)) \\
&= -12 + \frac{1}{2}((24f)^2 + 24f(-2\delta + 72f)) = -12 + \frac{1}{2}(-48) = -36 \\
\chi(\mathcal{O}_{\overline{D}_2}, \pi^* \mathcal{O}_{X_4}(h)) &= \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-\overline{h} - h)) = \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-\delta - 72f)) \\
&= -12 + \frac{1}{2}((\delta + 72f)^2 + (\delta + 72f)(-2\delta + 72f)) \\
&= -12 + \frac{1}{2}(48 + 2.72 - 2.48 + 72 - 2.72) = 0 \\
\chi(\mathcal{O}_{\overline{D}_2}, \pi^* \mathcal{O}_{X_4}(2h)) &= \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-\overline{h} - 2h)) = \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-2\delta - 120f)) \\
&= -12 + \frac{1}{2}((2\delta + 120f)^2 + (2\delta + 120f)(-2\delta + 72f)) \\
&= -12 + \frac{1}{2}(4.48 + 4.120 - 4.48 + 2.72 - 2.120) = 180 \\
\chi(\mathcal{O}_{\overline{D}_2}, \pi^* \mathcal{O}_{X_4}(3h)) &= \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-\overline{h} - 3h)) = \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-3\delta - 168f)) \\
&= -12 + \frac{1}{2}((3\delta + 168f)^2 + (3\delta + 168f)(-2\delta + 72f)) \\
&= -12 + \frac{1}{2}(9.48 + 6.168 - 6.48 + 3.72 - 2.168) = 504 \\
\chi(\mathcal{O}_{\overline{D}_2}, \pi^* \mathcal{O}_{X_4}(4h)) &= \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-\overline{h} - 4h)) = \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-4\delta - 216f)) \\
&= -12 + \frac{1}{2}((4\delta + 216f)^2 + (4\delta + 216f)(-2\delta + 72f)) \\
&= -12 + \frac{1}{2}(16.48 + 8.216 - 8.48 + 4.72 - 2.216) = 972.
\end{aligned}$$

□

Lemma 4.4.9. One has $\chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{\overline{D}_2}) = -36$ and $\chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{D_3}(V_E)) = -156$.

Proof. One has the short exact sequence :

$$0 \rightarrow \mathcal{O}_{\tilde{X}_4}(-2e) \rightarrow \mathcal{O}_{\tilde{X}_4}(-e) \rightarrow \mathcal{O}_{D_3}(-e) \rightarrow 0 \quad (4.24)$$

and can use (1.10) to find $\mathcal{O}_{D_3}(V_E) = \mathcal{O}_{D_3}(-e)$.

- By (4.24), one has

$$\begin{aligned}
\chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{\overline{D}_2}) &= \chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{\overline{D}_2}) - \chi(\mathcal{O}_{\tilde{X}_4}(-2e), \mathcal{O}_{\overline{D}_2}) \\
&= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(h - \overline{h})) - \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(2h - 2\overline{h}))
\end{aligned}$$

with (1.8). By Lemma 4.4.5, we know that $\mathcal{O}_{\overline{D}_2}(h) = \mathcal{O}_{\overline{D}_2}(\delta + 48f)$, and we saw in the proof of Lemma 4.4.8 that $\mathcal{O}_{\overline{D}_2}(\overline{h}) = \mathcal{O}_{\overline{D}_2}(24f)$. As a consequence, (4.23) (with the same reasoning as in the proof of Lemma 4.4.6), Lemma 4.4.4 and Lemma 4.4.1 yield :

$$\begin{aligned}
\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(h - \overline{h})) &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(\delta + 24f)) = -12 + \frac{1}{2}((\delta + 24f)^2 - (\delta + 24f)(-2\delta + 72f)) \\
&= -12 + \frac{1}{2}(48 + 48 + 2.48 - 72 + 48) = 72 \\
\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(2h - 2\overline{h})) &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(2\delta + 48f)) = -12 + \frac{1}{2}(4(\delta + 24f)^2 - 2(\delta + 24f)(-2\delta + 72f)) \\
&= -12 + \frac{1}{2}(384 - 144) = 108.
\end{aligned}$$

Therefore, $\chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{\overline{D}_2}) = 72 - 108 = -36$.

- As we saw in the proof of Lemma 4.3.7, $\omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}_4}(-h + e)$, hence by Serre duality

$$\begin{aligned}\chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{D_3}(V_E)) &= \chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{\overline{D}_2}(-h + e)) \\ &= \chi(\mathcal{O}_{D_3}, \mathcal{O}_{\overline{D}_2}(-h + 2e)) \\ &= \chi(\mathcal{O}_{D_3}, \mathcal{O}_{\overline{D}_2}(h - 2\bar{h})).\end{aligned}$$

Using (4.24) twisted by $\mathcal{O}_{\tilde{X}_4}(e)$ gives then

$$\begin{aligned}\chi(\mathcal{O}_{D_3}, \mathcal{O}_{\overline{D}_2}(h - 2\bar{h})) &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(h - 2\bar{h})) - \chi(\mathcal{O}_{D\tilde{X}_4}(-e), \mathcal{O}_{\overline{D}_2}(h - 2\bar{h})) \\ &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(h - 2\bar{h})) - \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(2h - 3\bar{h})).\end{aligned}$$

Riemann Roch then gives :

$$\begin{aligned}\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(h - 2\bar{h})) &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(\delta)) = -12 + \frac{1}{2}(\delta^2 - \delta(-2\delta + 72f)) \\ &= -12 + \frac{1}{2}(48 + 2.48 - 72) = 24. \\ \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(2h - 3\bar{h})) &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(2\delta + 24f)) = -12 + \frac{1}{2}((2\delta + 24f)^2 - (2\delta + 24f)(-2\delta + 72f)) \\ &= -12 + \frac{1}{2}(4.48 + 4.24 + 4.48 - 2.72 + 48) = 180.\end{aligned}$$

Hence, $\chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{D_3}(V_E)) = 24 - 180 = -156$.

□

4.5 The surface \tilde{S}

We would like now to compute the Euler characteristics associated to \tilde{S} . But before doing so we need some technical results.

Definition 4.5.1. By Remark 4.7 in [20] we know that S_2 is the blow-up of $\mathbb{P}(\wedge^2 A_1)$ in three points. Let us denote by \tilde{E}_1, \tilde{E}_2 and \tilde{E}_3 the associated exceptional divisors.

Remark 4.5.2. Then $\text{Pic}(S_2) = \langle \bar{h}_1, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3 \rangle$ (see for instance Proposition 4.8 in V.4. of [12]). On one hand, $\omega_S = \mathcal{O}_{S_2}(-3\bar{h}_1 + \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3)$ by d) of Proposition 4.8 and Notation 4.7.3 of V.4. in [12]. On the other hand, by 11. of Theorem 1.2.1, $\omega_S = \mathcal{O}_{S_2}(-\bar{h}_1 - \bar{h}_2)$. Hence :

$$\bar{h}_2 = 2\bar{h}_1 - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3. \quad (4.25)$$

Finally still by Proposition 4.8 of V.4. in [12], we get : $\bar{h}_1 \cdot \tilde{E}_i = 0$ and $\tilde{E}_i \cdot \tilde{E}_j = -\delta_{i,j}$ for $1 \leq i, j \leq 3$.

Proposition 4.5.3. The canonical sheaf of Z_3 is $\omega_Z = \mathcal{O}_{Z_3}(-2V_Z - 3\bar{h}_1 + \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3) = \mathcal{O}_{Z_3}(-2V_Z - \bar{h}_1 - \bar{h}_2)$ with the notations of Definition 4.5.1 and where we keep the same notation for a divisor on S_2 and its pull-back to Z_3 by σ .

Proof. Since $Z_3 = \mathbb{P}_S(\mathcal{V}_S)$, we have $\text{Pic}(Z_3) = \mathbb{Z}.V_Z \oplus \text{Pic}(S_2)$, thus using Definition 4.5.1 and Remark 4.5.2 we can write : $K_Z = aV_Z + b\bar{h}_1 + c\tilde{E}_1 + d\tilde{E}_2 + e\tilde{E}_3$ where K_Z is the canonical class of Z_3 . We will now use a very similar reasoning to the proof of Lemma 2.10 in V.2. of [12], but before going further, since we are going to use here \bar{h}_1 and \tilde{E}_1 both as a divisors in Z_3 and S_2 let us fix a notation : \bar{h}_1^S and \tilde{E}_1^S will denote the divisors in S_2 and \bar{h}_1^Z and \tilde{E}_1^Z their pull-backs to Z_3 .

- A divisor D in $|\bar{h}_1^Z|$ is the set of all $f_Z(p)$ with $p \in C$, where $C \in |\bar{h}_1^S|$. Hence it is a ruled surface above C (see for instance the first definition and note of V.2. in [12]). Let us fix such a surface in Z_3 . Then the adjunction formula gives :

$$K_D = (\bar{h}_1^Z + K_Z)|_D = (\bar{h}_1^Z)^2 + \bar{h}_1^Z \cdot K_Z. \quad (4.26)$$

On the other hand, we can also use Lemma 2.10 of V.2. in [12] to compute K_D . This yields :

$$K_D = -2C_0 + (K_C + \det(\mathcal{V}_S|_C))f \quad (4.27)$$

where C_0 is the relative hyperplane class (using the definition of [12] so here $\sigma_*\mathcal{O}(C_0) = \mathcal{V}_S|_C$) and f is a fiber. Here we also used the fact that since $Z_3 = \mathbb{P}_S(\mathcal{V}_S)$, we have $D = \mathbb{P}_C(\mathcal{V}_S|_C)$. Furthermore by definition of V_Z we know that $\sigma_*\mathcal{O}_{Z_3}(V_Z) = \mathcal{V}_S^\vee$ so using the fact that $\det(\mathcal{V}_S) = \mathcal{O}_{S_2}$ (by 1. of Proposition 1.2.2) and the same reasoning as in the proof of Lemma 4.4.5 we get : $C_0 = \bar{h}_1 \cdot V_Z$ and $f = (\bar{h}_1^Z)^2$. Moreover, the adjunction formula in S_2 gives : $K_C = (\bar{h}_1^S + K_S) \cdot \bar{h}_1^S = -\bar{h}_2 \cdot \bar{h}_1^S = -2$ by 11. of Theorem 1.2.1 and (1.13). We also just saw that from 1. of Proposition 1.2.2, $\det(\mathcal{V}_S) = \mathcal{O}_{S_2}$. Hence (4.27) actually gives :

$$K_D = -2\bar{h}_1^Z \cdot V_Z - 2(\bar{h}_1^Z)^2. \quad (4.28)$$

Together (4.26) and (4.28) yield : $\bar{h}_1^Z \cdot K_Z = a\bar{h}_1^Z \cdot V_Z + b(\bar{h}_1^Z)^2 = -2\bar{h}_1^Z \cdot V_Z - 3(\bar{h}_1^Z)^2$. Therefore, $a = -2$ and $b = -3$.

- We will now use the same reasoning as just before but this time with D' a divisor in $|\tilde{E}_1^Z|$ which is thus a ruled surface above a curve $C' \in |\tilde{E}_1^S|$. The adjunction formula yields :

$$K_{D'} = (\tilde{E}_1^Z + K_Z)|_{D'} = (\tilde{E}_1^Z)^2 + \tilde{E}_1^Z \cdot K_Z. \quad (4.29)$$

On the other hand using again Lemma 2.10 of V.2. in [12] gives :

$$K_{D'} = -2\tilde{E}_1^Z \cdot V_Z - (K_{C'} + 0)(\tilde{E}_1^Z)^2 = -2\tilde{E}_1^Z \cdot V_Z + 2(\tilde{E}_1^Z)^2 \quad (4.30)$$

since by the adjunction formula :

$$K_{C'} = (\tilde{E}_1^S - \bar{h}_1 - \bar{h}_2) \cdot \tilde{E}_1^S = -1 - \bar{h}_2 \cdot \tilde{E}_1^S$$

and

$$-\bar{h}_2 \cdot \tilde{E}_1^S = (-2\bar{h}_1^S + \tilde{E}_1^S + \tilde{E}_2 + \tilde{E}_3) \cdot \tilde{E}_1^S = -1$$

by (4.25). We also used the fact that here the fiber f should be $-(\tilde{E}_1^Z)^2$, because in S_2 we have $(\tilde{E}_1^S)^2 = -1$. To conclude we combine (4.29) and (4.30) and we get :

$$K_Z \cdot \tilde{E}_1^Z = a\tilde{E}_1^Z \cdot V_Z + c(\tilde{E}_1^Z)^2 = -2\tilde{E}_1^Z \cdot V_Z + (\tilde{E}_1^Z)^2.$$

As a result, $c = 1$.

- We can then proceed exactly the same way with \tilde{E}_2 and \tilde{E}_3 which gives : $d = e = c = 1$. □

Lemma 4.5.4. *Let \mathcal{E} and \mathcal{F} be vector bundles of rank respectively $r+1$ and r on a smooth variety X and let $\phi : \mathcal{E} \rightarrow \mathcal{F}$ be a morphism of vector bundles. By Lemma 1.4.1 we have seen that ϕ induces a global section of $p^*\mathcal{F}(1)$ whose zero locus in $\mathbb{P}_X(\mathcal{E})$ is isomorphic to the blow-up of X in $D_1(\phi)$, where p is the map $p : \mathbb{P}_X(\mathcal{E}) \rightarrow X$. Let us call $\tilde{X} \hookrightarrow \mathbb{P}_X(\mathcal{E})$ this zero locus. Then $p_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$.*

Proof. In the proof of Lemma 1.4.1 we have seen that there is a long exact sequence :

$$0 \rightarrow \bigwedge^r (p^* \mathcal{F}^\vee) \otimes \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-r) \rightarrow \dots \rightarrow \bigwedge^2 (p^* \mathcal{F}^\vee) \otimes \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-2) \rightarrow p^* \mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-1) \xrightarrow{\phi^\vee} \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow 0$$

and that $p_* \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} = \mathcal{O}_X$. Taking the push-forward of the above resolution thus yields the exact sequence :

$$\dots \rightarrow \mathcal{F}^\vee \otimes p_*(\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-1)) \rightarrow \mathcal{O}_X \rightarrow p_* \mathcal{O}_{\tilde{X}} \rightarrow 0.$$

But $R^j p_*(\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(-1)) = 0$ for all $0 \leq j \leq r$ (see for instance Exercise 8.4 of III.8 in [12]). Therefore $Rp_* \mathcal{O}_{\tilde{X}}$ which we just denote by $p_* \mathcal{O}_{\tilde{X}}$ is isomorphic to \mathcal{O}_X . \square

Corollary 4.5.5. *We have the following : $\pi_* \mathcal{O}_{\tilde{Z}} = \mathcal{O}_{Z_3}$, $\bar{\pi}_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_{\tilde{S}}$ and $\sigma_* \mathcal{O}_{\tilde{S}} = \mathcal{O}_{S_2}$.*

Moreover, since we have seen in the proof of Lemma 1.4.1 that $p_ \mathcal{O}_{\mathbb{P}_X(\mathcal{E})} = \mathcal{O}_X$, we also have : $\pi_* \mathcal{O}_{D_3} = \mathcal{O}_{\Sigma_2}$, $\bar{\pi}_* \mathcal{O}_{\tilde{E}_4} = \mathcal{O}_{Z_3}$ and $\sigma_* \mathcal{O}_{Z_3} = \mathcal{O}_{S_2}$.*

Lemma 4.5.6. *Let \mathcal{T}_Z be the tangent sheaf of Z_3 . Then the Todd class of Z_3 is*

$$\text{td}(Z_3) = \text{td}(\mathcal{T}_Z) = (1, V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3), x_2, 1) = (1, V_Z + \frac{1}{2}(\bar{h}_1 + \bar{h}_2), x_2, 1)$$

where $\bar{h}.x_2 = \frac{10}{3}$ and $\bar{h}_1.x_2 = \bar{h}_2.x_2 = \frac{3}{2}$.

Proof. Let us write $\text{td}(Z_3) = (x_0, x_1, x_2, x_3)$. Then using the formula for the Todd class given page 432 of [12], one gets that $x_0 = 1$ and $x_1 = \frac{1}{2}c_1(\mathcal{T}_Z)$. Moreover, $\omega_Z = \bigwedge^3 \mathcal{T}_Z^\vee$ hence $x_1 = \frac{-K_Z}{2} = V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3) = V_Z + \frac{1}{2}(\bar{h}_1 + \bar{h}_2)$.

Moreover, we know by Hirzebruch-Riemann-Roch that $\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}) = \int_{Z_3} (1, 0, 0, 0).(x_0, x_1, x_2, x_3) = x_3$. But $H^*(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}) = H^*(\sigma^* \mathcal{O}_{S_2}, \mathcal{O}_{Z_3}) = H^*(\mathcal{O}_{S_2}, \sigma_* \mathcal{O}_{Z_3}) = H^*(\mathcal{O}_{S_2}, \mathcal{O}_{S_2})$ by Corollary 4.5.5 and we have seen in the proof of Lemma 4.3.5 that $\chi(\mathcal{O}_{S_2}, \mathcal{O}_{S_2}) = \chi(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}) = 1$, thus $x_3 = 1$.

Since by definition of V_Z and the projection formula

$$H^*(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})) = H^*(\mathcal{O}_{S_2}, \sigma_*(\sigma^* \mathcal{O}_{S_2}(\bar{h}_1 + \bar{h}_2) \otimes \mathcal{O}_{Z_3}(V_Z))) = H^*(\mathcal{O}_{S_2}, \mathcal{V}_S^\vee(\bar{h}_1 + \bar{h}_2)),$$

the fact that $\mathcal{V}_S \cong \mathcal{V}_S^\vee$ together with Lemma 3.1.4 yield $\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})) = 12$.

On the other hand, the Chern character of the line bundle $\mathcal{O}_{Z_3}(\bar{h})$ is $(1, \bar{h}, \frac{\bar{h}^2}{2}, \frac{\bar{h}^3}{6})$. By (1.11), $\bar{h} = V_Z + \bar{h}_1 + \bar{h}_2$, hence by (1.18),

$$\bar{h}^2 = -\bar{h}_1 \bar{h}_2 + 2V_Z(\bar{h}_1 + \bar{h}_2) + (\bar{h}_1 + \bar{h}_2)^2 = 2V_Z(\bar{h}_1 + \bar{h}_2) + 2\bar{h}_1 \bar{h}_2.$$

Thus $\frac{\bar{h}^2}{2} = V_Z(\bar{h}_1 + \bar{h}_2) + \bar{h}_1 \bar{h}_2$. Moreover,

$$\bar{h}^3 = (V_Z + \bar{h}_1 + \bar{h}_2).(2V_Z(\bar{h}_1 + \bar{h}_2) + 2\bar{h}_1 \bar{h}_2) = 2V_Z(\bar{h}_1 + \bar{h}_2)^2 + 2V_Z \bar{h}_1 \bar{h}_2 = 2.6 + 4 = 16,$$

therefore $\frac{\bar{h}^3}{6} = \frac{8}{3}$. By Hirzebruch-Riemann-Roch we thus have :

$$\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})) = 12 = \int_{Z_3} (1, \bar{h}, \bar{h}_1 \bar{h}_2 + V_Z(\bar{h}_1 + \bar{h}_2), \frac{8}{3}).(1, V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3), x_2, 1) \quad (4.31)$$

$$= 1 + \bar{h}.x_2 + 2 + \frac{9}{2} - \frac{3}{2} + \frac{8}{3} = 6 + \frac{8}{3} + \bar{h}.x_2 \quad (4.32)$$

$$\Rightarrow \bar{h}.x_2 = 6 - \frac{8}{3} = \frac{10}{3} \quad (4.33)$$

Let now $i \in \{1, 2\}$. Then

$$H^*(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h}_i)) = H^*(\sigma^* \mathcal{O}_{S_2}, \sigma^* \mathcal{O}_{S_2}(\bar{h}_i)) = H^*(\mathcal{O}_{S_2}, \sigma_* \sigma^* \mathcal{O}_{S_2}(\bar{h}_i)) = H^*(\mathcal{O}_{S_2}, \mathcal{O}_{S_2}(\bar{h}_i))$$

by the projection formula, since $\sigma_*\mathcal{O}_{Z_3} = \mathcal{O}_{S_2}$. We have seen in the proof of Lemma 3.1.4 that $\dim(H^*(\mathcal{O}_{S_2}, \mathcal{O}_{S_2}(\bar{h}_i))) = 3\delta_{*,0}$, therefore $\chi(\mathcal{O}_{Z_3}(\bar{h}_i)) = 3$. Hirzebruch-Riemann-Roch then gives :

$$\begin{aligned}\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h}_1)) &= 3 = \int_{Z_3} (1, \bar{h}_1, \frac{\bar{h}_1^2}{2}, 0) \cdot (1, V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3), x_2, 1) \\ &= 1 + \bar{h}_1 \cdot x_2 + \frac{1}{2} = \frac{3}{2} + \bar{h}_1 \cdot x_2 \\ \Rightarrow \bar{h}_1 \cdot x_2 &= 3 - \frac{3}{2} = \frac{3}{2} \\ \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h}_2)) &= 3 = \int_{Z_3} (1, \bar{h}_2, \frac{\bar{h}_2^2}{2}, 0) \cdot (1, V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3), x_2, 1) \\ &= 1 + \bar{h}_2 \cdot x_2 + \frac{1}{2} = \frac{3}{2} + \bar{h}_2 \cdot x_2 \\ \Rightarrow \bar{h}_2 \cdot x_2 &= 3 - \frac{3}{2} = \frac{3}{2}\end{aligned}$$

because $\bar{h}_2^2 = (2\bar{h}_1 - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3)^2 = 4\bar{h}_1^2 + \tilde{E}_1^2 + \tilde{E}_2^2 + \tilde{E}_3^2$ and for $1 \leq j \leq 3$, $\tilde{E}_j^3 = 0$ in S_2 hence $\tilde{E}_j^3 = 0$ in Z_3 . As a result, for both $i = 1$ and $i = 2$ we have : $\bar{h}_i \cdot x_2 = \frac{3}{2}$. \square

We can now compute Euler characteristics involving $\mathcal{O}_{\tilde{S}}$. The main idea is to relate them to some Euler characteristics between sheaves in $D^b(Z_3)$, which we can compute using Hirzebruch-Riemann-Roch together with the Lemmas from the beginning of the subsection.

Lemma 4.5.7. *The following holds : $\chi(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{\tilde{S}}) = 1$.*

Proof. Since $\tilde{X}_4 = \tilde{X}_5 \times_{X_5} X_4$, there is a map $\tilde{i} : \tilde{X}_4 \rightarrow \tilde{X}_5$. This map is an embedding since the map $X_4 \rightarrow X_5$ is a closed immersion (see for instance Lemma 26.17.6 in The Stacks project). In fact, we even have that $\tilde{X}_4 = \tilde{X}_5 \cap \tilde{H}_\nu$ where \tilde{H}_ν is the pull back of the hyperplane H_ν such that $X_4 = X_5 \cap H_\nu$, by the proof of Theorem 5.1 in [20]. We can then use the following :

$$H^*(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{\tilde{S}}) = H^*(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{S}}) = H^*(\tilde{i}^*\mathcal{O}_{\tilde{X}_5}, \mathcal{O}_{\tilde{S}}) = H^*(\mathcal{O}_{\tilde{X}_5}, \tilde{i}_*\mathcal{O}_{\tilde{S}}) = H^*(\mathcal{O}_{\tilde{X}_5}, \mathcal{O}_{\tilde{S}}).$$

Moreover,

$$\begin{aligned}H^*(\mathcal{O}_{\tilde{X}_5}, \mathcal{O}_{\tilde{S}}) &= H^*(\mathcal{O}_{\tilde{X}_5}, \tilde{i}_*\mathcal{O}_{\tilde{S}}) \\ &= H^*(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\tilde{S}}) \\ &= H^*(\mathcal{O}_{Z_3}, \bar{\pi}_*\mathcal{O}_{\tilde{S}}) \\ &= H^*(\mathcal{O}_{Z_3}, \mathcal{O}_{\tilde{S}}) \\ &= H^*(\mathcal{O}_{S_2}, \sigma_*\mathcal{O}_{\tilde{S}}) \\ &= H^*(\mathcal{O}_{S_2}, \mathcal{O}_{S_2})\end{aligned}$$

by Corollary 4.5.5. Finally, if we denote now by $f : S_2 \rightarrow \mathbb{P}^2$ the blow-up of \mathbb{P}^2 in three points mentioned in Definition 4.5.1, we have $H^*(\mathcal{O}_{S_2}, \mathcal{O}_{S_2}) = H^*(f^*\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{S_2}) = H^*(\mathcal{O}_{\mathbb{P}^2}, f_*\mathcal{O}_{S_2}) = H^*(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2})$ (see for instance Proposition 3.4 of V.3. in [12] for the last equality). As a result, $\chi(\pi^*\mathcal{O}_{X_4}, \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}) = 1$. \square

Lemma 4.5.8. *The canonical sheaf of \bar{E}_4 is $\omega_{\bar{E}_4} = \mathcal{O}_{\bar{E}_4}(\bar{h} - 2V_{\bar{E}})$.*

Proof. Since \bar{E}_4 is the exceptional divisor of $\bar{\pi}$ in \tilde{X}_5 , the adjunction formula gives :

$$\omega_{\bar{E}} = \tilde{i}^*(\omega_{\tilde{X}_5} + \mathcal{O}_{\tilde{X}_5}(\bar{e}))$$

where $\omega_{\tilde{X}_5}$ is the canonical sheaf of \tilde{X}_5 . We know from the proof of Lemma 4.1 in [20] that the canonical sheaf of X_5 is $\omega_{X_5} = \mathcal{O}_{X_5}(-2h)$. Therefore, since \tilde{X}_5 is the blow-up of X_5 with exceptional divisor E_4 , we get that $\omega_{\tilde{X}_5} = \mathcal{O}_{\tilde{X}_5}(-2h + e)$. Finally :

$$\begin{aligned}\omega_{\bar{E}} &= \tilde{i}^* \mathcal{O}_{\tilde{X}_5}(\bar{e} + e - 2h) \\ &= \tilde{i}^* \mathcal{O}_{\tilde{X}_5}(2\bar{h} - h + h - \bar{h} - 2h) \quad \text{by (1.8)} \\ &= \mathcal{O}_{\bar{E}_4}(\bar{h} - 2V_{\bar{E}}) \quad \text{by (1.12)}.\end{aligned}$$

□

Lemma 4.5.9. *Let $i \in \mathbb{Z}$. We have :*

$$\begin{aligned}\chi(\pi^* \mathcal{O}_{X_4}(ih), \mathcal{O}_{\tilde{S}}) &= \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}})) - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}})) - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}} - \bar{h})) + \\ &\quad \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}} - \bar{h})) \\ &= \chi_{Z_3}(\mathcal{O}_{Z_3}(\bar{h}) \otimes \pi_* \mathcal{O}_{\bar{E}_4}((i-2)V_{\bar{E}})) - \chi_{Z_3}(\mathcal{O}_{Z_3}(\bar{h}) \otimes \pi_* \mathcal{O}_{\bar{E}_4}((i-1)V_{\bar{E}})) - \\ &\quad \chi_{Z_3}(\mathcal{O}_{Z_3}(2\bar{h}) \otimes \pi_* \mathcal{O}_{\bar{E}_4}((i-2)V_{\bar{E}})) + \chi_{Z_3}(\mathcal{O}_{Z_3}(2\bar{h}) \otimes \pi_* \mathcal{O}_{\bar{E}_4}((i-1)V_{\bar{E}})).\end{aligned}$$

Proof. To start with, let us give names to some of the embeddings we have seen before :

$$\begin{array}{ccccccc} \tilde{X}_5 & \xleftarrow{\tilde{i}} & \bar{E}_4 & \xrightarrow{\pi} & Z_3 & \xrightarrow{\sigma} & S_2 \\ \uparrow \tilde{i} & & \uparrow f & & & & \\ \tilde{X}_4 & \xleftarrow{j} & \tilde{Z} & \xleftarrow{g} & \tilde{S} & & \end{array}$$

where we saw the existence of j , f and g in Lemma 3.2.6, Corollary 3.2.7 and the remark just after, and the existence of \tilde{i} in the proof of Lemma 4.5.7. Let now $i \in \mathbb{Z}$. One can notice

$$\begin{aligned}H^*(\pi^* \mathcal{O}_{X_4}(ih), \mathcal{O}_{\tilde{S}}) &= H^*(\mathcal{O}_{\tilde{X}_4}(ih), \mathcal{O}_{\tilde{S}}) \\ &= H^*(\tilde{i}^* \mathcal{O}_{\tilde{X}_5}(ih), \mathcal{O}_{\tilde{S}}) \\ &= H^*(\mathcal{O}_{\tilde{X}_5}(ih), \tilde{i}_* \mathcal{O}_{\tilde{S}}) \\ &= H^*(\mathcal{O}_{\tilde{X}_5}(ih), \mathcal{O}_{\tilde{S}}) \\ &= H^*(\mathcal{O}_{\tilde{X}_5}(ih), \tilde{i}_* f_* g_* \mathcal{O}_{\tilde{S}}) \\ &= H^*(\mathcal{O}_{\bar{E}_4}(ih), f_* g_* \mathcal{O}_{\tilde{S}}).\end{aligned}$$

By (1.12), $h = V_{\bar{E}}$ in $\text{Pic}(\bar{E}_4)$, therefore $H^*(\mathcal{O}_{\bar{E}_4}(ih), \mathcal{O}_{\tilde{S}}) = H^*(\mathcal{O}_{\bar{E}_4}, f_* g_* \mathcal{O}_{\tilde{S}}(-iV_{\bar{E}})) = H^*(\mathcal{O}_{\tilde{Z}}, g_* \mathcal{O}_{\tilde{S}}(-iV_{\bar{E}}))$. So $\chi_{\tilde{X}_4}(\pi^* \mathcal{O}_{X_4}(ih), \mathcal{O}_{\tilde{S}}) = \chi_{\tilde{Z}}(\mathcal{O}_{\tilde{Z}}, g_* \mathcal{O}_{\tilde{S}}(-iV_{\bar{E}}))$. To compute the latter, we can use the fact that the class of \tilde{Z} in \bar{E}_4 is $V_{\bar{E}}$ by Proposition 3.2.6, while the class of \tilde{S} in \bar{E}_4 is $\bar{h} \cdot V_{\bar{E}}$ by Corollary 3.2.7. This yields the following short exact sequence in \tilde{Z} :

$$0 \rightarrow \mathcal{O}_{\tilde{Z}}(-\bar{h}) \rightarrow \mathcal{O}_{\tilde{Z}} \rightarrow g_* \mathcal{O}_{\tilde{S}} \rightarrow 0.$$

Twisting it by $\mathcal{O}_{\tilde{Z}}(-iV_{\bar{E}})$ induces the short exact sequence :

$$0 \rightarrow \mathcal{O}_{\tilde{Z}}(-\bar{h} - iV_{\bar{E}}) \rightarrow \mathcal{O}_{\tilde{Z}}(-iV_{\bar{E}}) \rightarrow g_* \mathcal{O}_{\tilde{S}}(-iV_{\bar{E}}) \rightarrow 0.$$

In particular this implies : $\chi(\mathcal{O}_{\tilde{Z}}, g_* \mathcal{O}_{\tilde{S}}(-iV_{\bar{E}})) = \chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(-iV_{\bar{E}})) - \chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(-iV_{\bar{E}} - \bar{h}))$.

- First, let us consider $\chi(\mathcal{O}_{\bar{Z}}, \mathcal{O}_{\bar{Z}}(-iV_{\bar{E}}))$. To start with, we can notice :

$$H^*(\mathcal{O}_{\bar{Z}}, \mathcal{O}_{\bar{Z}}(-iV_{\bar{E}})) = H^*(f^*\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{Z}}(-iV_{\bar{E}})) = H^*(\mathcal{O}_{\bar{E}_4}, f_*\mathcal{O}_{\bar{Z}}(-iV_{\bar{E}})).$$

We can now use the short exact sequence in \bar{E}_4 (given by Lemma 3.2.6) :

$$0 \rightarrow \mathcal{O}_{\bar{E}_4}(-V_{\bar{E}}) \rightarrow \mathcal{O}_{\bar{E}_4} \rightarrow f_*\mathcal{O}_{\bar{Z}} \rightarrow 0.$$

Twisting it by $\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}})$ yields the short exact sequence :

$$0 \rightarrow \mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}}) \rightarrow \mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}}) \rightarrow f_*\mathcal{O}_{\bar{Z}}(-iV_{\bar{E}}) \rightarrow 0.$$

As a result, $\chi_{\bar{Z}}(\mathcal{O}_{\bar{Z}}, \mathcal{O}_{\bar{Z}}(-iV_{\bar{E}})) = \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}})) - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}}))$.

- To compute $\chi(\mathcal{O}_{\bar{Z}}, \mathcal{O}_{\bar{Z}}(-iV_{\bar{E}} - \bar{h}))$, we are going to proceed similarly :

$$H^*(\mathcal{O}_{\bar{Z}}, \mathcal{O}_{\bar{Z}}(-iV_{\bar{E}} - \bar{h})) = H^*(\mathcal{O}_{\bar{E}_4}, f_*\mathcal{O}_{\bar{Z}}(-iV_{\bar{E}} - \bar{h}))$$

and the short exact sequence in \bar{E}_4 :

$$0 \rightarrow \mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}} - \bar{h}) \rightarrow \mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}} - \bar{h}) \rightarrow f_*\mathcal{O}_{\bar{Z}}(-iV_{\bar{E}} - \bar{h}) \rightarrow 0$$

yields $\chi_{\bar{Z}}(\mathcal{O}_{\bar{Z}}, \mathcal{O}_{\bar{Z}}(-iV_{\bar{E}} - \bar{h})) = \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}} - \bar{h})) - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}} - \bar{h}))$.

Finally,

$$\begin{aligned} \chi(\mathcal{O}_{\bar{Z}}, g_*\mathcal{O}_{\bar{S}}(-iV_{\bar{E}})) &= \chi(\mathcal{O}_{\bar{Z}}, \mathcal{O}_{\bar{Z}}(-iV_{\bar{E}})) - \chi(\mathcal{O}_{\bar{Z}}, \mathcal{O}_{\bar{Z}}(-iV_{\bar{E}} - \bar{h})) \\ &= \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}})) - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}})) - \\ &\quad (\chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}} - \bar{h})) - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}} - \bar{h}))) \\ &= \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}})) - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}})) - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}} - \bar{h})) + \\ &\quad \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}} - \bar{h})). \end{aligned}$$

Let us now consider $\chi(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}}))$ with $i \in \mathbb{Z}$. Here, Serre duality and Lemma 4.5.8 yield :

$$H^*(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}})) \cong H^{*-4}(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}}), \mathcal{O}_{\bar{E}_4}(\bar{h} - 2V_{\bar{E}})) = H^{*-4}(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(\bar{h} + (i-2)V_{\bar{E}})).$$

Since 4 is even we thus have $\chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}})) = \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(\bar{h} + (i-2)V_{\bar{E}}))$.

Furthermore,

$$\begin{aligned} H^*(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(\bar{h} + (i-2)V_{\bar{E}})) &= H^*(\pi^*\mathcal{O}_{Z_3}, \mathcal{O}_{\bar{E}_4}(\bar{h} + (i-2)V_{\bar{E}})) \\ &= H^*(\mathcal{O}_{Z_3}, \pi_*\mathcal{O}_{\bar{E}_4}(\bar{h} + (i-2)V_{\bar{E}})) \\ &= H^*(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h}) \otimes \pi_*\mathcal{O}_{\bar{E}_4}((i-2)V_{\bar{E}})). \end{aligned}$$

Similarly, for any $i \in \mathbb{Z}$, by Serre duality we have

$$\chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}} - \bar{h})) = \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}((i-2)V_{\bar{E}} + 2\bar{h})) = \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(2\bar{h}) \otimes \pi_*\mathcal{O}_{\bar{E}_4}((i-2)V_{\bar{E}})).$$

□

Corollary 4.5.10. *One has :*

- $\chi(\mathcal{O}_{\bar{X}_4}(h), \mathcal{O}_{\bar{S}}) = -\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(2\bar{h}))$;
- $\chi(\mathcal{O}_{\bar{X}_4}(2h), \mathcal{O}_{\bar{S}}) = \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_{\bar{Z}}^\vee(\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(2\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_{\bar{Z}}^\vee(2\bar{h}))$;

- $\chi(\mathcal{O}_{\tilde{X}_4}(3h), \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(\bar{h})) - \chi(\mathcal{O}_{Z_3}, S^2\mathcal{V}_Z^\vee(\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(2\bar{h})) + \chi(\mathcal{O}_{Z_3}, S^2\mathcal{V}_Z^\vee(2\bar{h}));$
- $\chi(\mathcal{O}_{\tilde{X}_4}(4h), \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{Z_3}, S^2\mathcal{V}_Z^\vee(\bar{h})) - \chi(\mathcal{O}_{Z_3}, S^3\mathcal{V}_Z^\vee(\bar{h})) - \chi(\mathcal{O}_{Z_3}, S^2\mathcal{V}_Z^\vee(2\bar{h})) + \chi(\mathcal{O}_{Z_3}, S^3\mathcal{V}_Z^\vee(2\bar{h})).$

Proof. Applying the Remark below Proposition 9.3 in [8] to the formula given in Lemma 4.5.9 for $1 \leq i \leq 4$ yield the above-mentioned formulas. \square

We need now to know the Chern characters of $S^2\mathcal{V}_Z^\vee$ and $S^3\mathcal{V}_Z^\vee$, in order to compute with Hirzebruch-Riemann-Roch the Euler characteristics listed in Corollary 4.5.10.

Lemma 4.5.11. *The Chern characters of the following symmetric powers of \mathcal{V}_Z^\vee are :*

$$\text{ch}(S^2\mathcal{V}_Z^\vee) = (3, 9\bar{h} - 3\bar{h}_1 - 3\bar{h}_2, 14V_Z(\bar{h}_1 + \bar{h}_2) + \frac{15}{2}\bar{h}_1\bar{h}_2, 51) \quad (4.34)$$

$$\text{ch}(S^3\mathcal{V}_Z^\vee) = (4, 18\bar{h} - 6\bar{h}_1 - 6\bar{h}_2, 44V_Z(\bar{h}_1 + \bar{h}_2) + 21\bar{h}_1\bar{h}_2, 468) \quad (4.35)$$

$$\text{ch}(S^4\mathcal{V}_Z^\vee) = (5, 30\bar{h} - 10\bar{h}_1 - 10\bar{h}_2, 92V_Z(\bar{h}_1 + \bar{h}_2) + 45\bar{h}_1\bar{h}_2, 2100). \quad (4.36)$$

Proof. Let us denote the Chern polynomial of \mathcal{V}_Z^\vee by $c_t(\mathcal{V}_Z^\vee) = (1 + a_1t)(1 + a_2t) = 1 + (a_1 + a_2)t + a_1a_2t^2$. Then (see for instance page 429 and 430 of [12]), $c_1(\mathcal{V}_Z^\vee) = a_1 + a_2$ and $c_2(\mathcal{V}_Z^\vee) = a_1a_2$. We also know from Lemma 3.2.3 that $\text{ch}(\mathcal{V}_Z^\vee) = (2, 3\bar{h} - \bar{h}_1 - \bar{h}_2, \frac{3}{2}\bar{h}_1\bar{h}_2 + 2V_Z(\bar{h}_1 + \bar{h}_2), 3)$, thus $c_1(\mathcal{V}_Z^\vee) = 3\bar{h} - \bar{h}_1 - \bar{h}_2$ and

$$\begin{aligned} c_2(\mathcal{V}_Z^\vee) &= \frac{c_1(\mathcal{V}_Z^\vee)^2}{2} - \text{ch}_2(\mathcal{V}_Z^\vee) \\ &= \frac{9\bar{h}^2 + (\bar{h}_1 + \bar{h}_2)^2 - 6\bar{h}(\bar{h}_1 + \bar{h}_2)}{2} - (\frac{3}{2}\bar{h}_1\bar{h}_2 + 2V_Z(\bar{h}_1 + \bar{h}_2)) \\ &= \frac{18\bar{h}_1\bar{h}_2 + 18V_Z(\bar{h}_1 + \bar{h}_2) + 3\bar{h}_1\bar{h}_2 - 6V_Z(\bar{h}_1 + \bar{h}_2) - 18\bar{h}_1\bar{h}_2}{2} - (\frac{3}{2}\bar{h}_1\bar{h}_2 + 2V_Z(\bar{h}_1 + \bar{h}_2)) \\ &= 4V_Z(\bar{h}_1 + \bar{h}_2). \end{aligned}$$

Moreover by Example 3.2.6. page 57 of [9], the Chern roots of $S^2(\mathcal{V}_Z^\vee)$ are : $2a_1, 2a_2$ and $a_1 + a_2$. Hence its Chern polynomial is :

$$\begin{aligned} c_t(S^2\mathcal{V}_Z^\vee) &= (1 + 2a_1t)(1 + 2a_2t)(1 + (a_1 + a_2)t) \\ &= 1 + (2a_1 + 2a_2 + a_1 + a_2)t + (4a_1a_2 + 2a_1^2 + 2a_1a_2 + 2a_1a_2 + 2a_2^2)t^2 + 4a_1a_2(a_1 + a_2)t^3 \\ &= 1 + (3a_1 + 3a_2)t + (2a_1^2 + 2a_2^2 + 8a_1a_2)t^2 + (4a_1^2a_2 + 4a_1a_2^2)t^3 \\ &= 1 + 3c_1(\mathcal{V}_Z^\vee)t + (2c_1(\mathcal{V}_Z^\vee)^2 + 4c_2(\mathcal{V}_Z^\vee))t^2 + 4c_1(\mathcal{V}_Z^\vee)c_2(\mathcal{V}_Z^\vee)t^3 \end{aligned}$$

which implies that

$$\begin{aligned} c_1(S^2\mathcal{V}_Z^\vee) &= 3c_1(\mathcal{V}_Z^\vee) = 9\bar{h} - 3\bar{h}_1 - 3\bar{h}_2 \\ c_2(S^2\mathcal{V}_Z^\vee) &= 2c_1(\mathcal{V}_Z^\vee)^2 + 4c_2(\mathcal{V}_Z^\vee) = 24V_Z(\bar{h}_1 + \bar{h}_2) + 6\bar{h}_1\bar{h}_2 + 16V_Z(\bar{h}_1 + \bar{h}_2) = 40V_Z(\bar{h}_1 + \bar{h}_2) + 6\bar{h}_1\bar{h}_2 \\ c_3(S^2\mathcal{V}_Z^\vee) &= 4c_1(\mathcal{V}_Z^\vee)c_2(\mathcal{V}_Z^\vee) = 16V_Z(\bar{h}_1 + \bar{h}_2)(3V_Z + 2(\bar{h}_1 + \bar{h}_2)) = 32.6 = 192 \\ \text{ch}_0(S^2\mathcal{V}_Z^\vee) &= \text{rk}(S^2\mathcal{V}_Z^\vee) = \text{deg}(c_t(S^2\mathcal{V}_Z^\vee)) = 3 \\ \text{ch}_1(S^2\mathcal{V}_Z^\vee) &= c_1(S^2\mathcal{V}_Z^\vee) = 9\bar{h} - 3\bar{h}_1 - 3\bar{h}_2 \\ \text{ch}_2(S^2\mathcal{V}_Z^\vee) &= \frac{c_1(S^2\mathcal{V}_Z^\vee)^2}{2} - c_2(S^2\mathcal{V}_Z^\vee) = 9(6V_Z(\bar{h}_1 + \bar{h}_2) + \frac{3}{2}\bar{h}_1\bar{h}_2) - 40V_Z(\bar{h}_1 + \bar{h}_2) - 6\bar{h}_1\bar{h}_2 \\ &= 14V_Z(\bar{h}_1 + \bar{h}_2) + \frac{15}{2}\bar{h}_1\bar{h}_2 \\ \text{ch}_3(S^2\mathcal{V}_Z^\vee) &= \frac{c_1(S^2\mathcal{V}_Z^\vee)^3}{6} - \frac{c_1(S^2\mathcal{V}_Z^\vee)c_2(S^2\mathcal{V}_Z^\vee)}{2} + \frac{c_3(S^2\mathcal{V}_Z^\vee)}{2} = \frac{27}{6}((3V_Z + 2(\bar{h}_1 + \bar{h}_2))(12V_Z(\bar{h}_1 + \bar{h}_2) + 3\bar{h}_1\bar{h}_2)) + \\ &\quad - \frac{54.2 + 40.6.6}{2} + 96 = \frac{9}{2}(9.2 + 24.6) - 54 - 720 + 96 = 81 + 648 - 774 + 96 = 51. \end{aligned}$$

For $S^3\mathcal{V}_Z^\vee$ and $S^4\mathcal{V}_Z^\vee$, one can proceed similarly. \square

One can now use Hirzebruch-Riemann-Roch, with Lemma 4.5.6, Lemma 3.2.3, Lemma 4.5.11 and Corollary 4.5.10 to compute the following.

Lemma 4.5.12. *One has $(\chi(\pi^*\mathcal{O}_{X_4}(h), \mathcal{O}_{\tilde{S}}), \dots, \chi(\pi^*\mathcal{O}_{X_4}(4h), \mathcal{O}_{\tilde{S}})) = (37, 127, 271, 469)$.*

Lemma 4.5.13. *The following holds :*

- $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4}) = -\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(2\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(3\bar{h}))$;
- $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4}(h)) = -\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(2\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(3\bar{h}))$;
- $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4}(2h)) = \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(2\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(2\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(3\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(3\bar{h}))$;
- $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4}(3h)) = \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(2\bar{h})) - \chi(\mathcal{O}_{Z_3}, S^2\mathcal{V}_Z^\vee(2\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(3\bar{h})) + \chi(\mathcal{O}_{Z_3}, S^2\mathcal{V}_Z^\vee(3\bar{h}))$;
- $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4}(4h)) = \chi(\mathcal{O}_{Z_3}, S^2\mathcal{V}_Z^\vee(2\bar{h})) - \chi(\mathcal{O}_{Z_3}, S^3\mathcal{V}_Z^\vee(2\bar{h})) - \chi(\mathcal{O}_{Z_3}, S^2\mathcal{V}_Z^\vee(3\bar{h})) + \chi(\mathcal{O}_{Z_3}, S^3\mathcal{V}_Z^\vee(3\bar{h}))$.

Proof. Let $i \in \mathbb{Z}$. By Serre duality, Lemma 4.4.7 and (1.8) one has

$$\chi(\mathcal{O}_{\tilde{S}}, \pi^*\mathcal{O}_{X_4}(ih)) = \chi(\pi^*\mathcal{O}_{X_4}(ih), \mathcal{O}_{\tilde{S}}(-h+e)) = \chi(\pi^*\mathcal{O}_{X_4}(ih), \mathcal{O}_{\tilde{S}}(-\bar{h})).$$

Using the same reasoning as in Lemma 4.5.9 with $\mathcal{O}_{\tilde{S}}$ replaced by $\mathcal{O}_{\tilde{S}}(-\bar{h})$, one can find :

$$\begin{aligned} \chi(\pi^*\mathcal{O}_{X_4}(ih), \mathcal{O}_{\tilde{S}}(-\bar{h})) &= \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}} - \bar{h})) - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}} - \bar{h})) \\ &\quad - \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}} - 2\bar{h})) + \chi_{\bar{E}_4}(\mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}} - 2\bar{h})) \\ &= \chi_{Z_3}(\mathcal{O}_{Z_3}(2\bar{h}) \otimes \bar{\pi}_*\mathcal{O}_{\bar{E}_4}((i-2)V_{\bar{E}})) - \chi_{Z_3}(\mathcal{O}_{Z_3}(2\bar{h}) \otimes \bar{\pi}_*\mathcal{O}_{\bar{E}_4}((i-1)V_{\bar{E}})) \\ &\quad - \chi_{Z_3}(\mathcal{O}_{Z_3}(3\bar{h}) \otimes \bar{\pi}_*\mathcal{O}_{\bar{E}_4}((i-2)V_{\bar{E}})) + \chi_{Z_3}(\mathcal{O}_{Z_3}(3\bar{h}) \otimes \bar{\pi}_*\mathcal{O}_{\bar{E}_4}((i-1)V_{\bar{E}})). \end{aligned}$$

As in Corollary 4.5.10 one can conclude using the Remark below Proposition 9.3 in [8]. \square

One can now use Hirzebruch-Riemann-Roch to compute the following.

Lemma 4.5.14. *One has $(\chi(\mathcal{O}_{\tilde{S}}, \pi^*\mathcal{O}_{X_4}), \dots, \chi(\mathcal{O}_{\tilde{S}}, \pi^*\mathcal{O}_{X_4}(4h))) = (79, 79, 205, 385, 619)$.*

Remark 4.5.15. *One could also have computed the Euler characteristics mentioned in Lemma 4.5.12 and Lemma 4.5.14 with Hirzebruch-Riemann-Roch in \tilde{Z} . Indeed, if we denote by j the inclusion of \tilde{Z} in \tilde{X}_4 , we have, for $i \in \mathbb{Z}$: $\chi(\mathcal{O}_{\tilde{X}_4}(ih), j_*\mathcal{O}_{\tilde{S}}) = \chi(j^*\mathcal{O}_{\tilde{X}_4}(ih), \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{Z}}(ih), \mathcal{O}_{\tilde{S}})$ since $\tilde{S} \subset \tilde{Z}$ by construction. We know from Corollary 3.2.7 that the class of \tilde{S} in \tilde{Z} is \bar{h} , hence we get $\chi(\mathcal{O}_{\tilde{X}_4}(ih), j_*\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{Z}}(ih), \mathcal{O}_{\tilde{Z}}) - \chi(\mathcal{O}_{\tilde{Z}}(ih), \mathcal{O}_{\tilde{Z}}(-\bar{h}))$. One can then conclude with Hirzebruch-Riemann-Roch, Lemma 5.0.7 and the fact that $h|_{\tilde{Z}} = 3V_Z + 2(\bar{h}_1 + \bar{h}_2) - \bar{D}$, as we will see in Lemma 4.6.2.*

For the Euler characteristics computed in Lemma 4.5.14, one can use the same reasoning together with Serre duality and Lemma 4.4.7.

Lemma 4.5.16. *The following Euler characteristics are : $\chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{\tilde{S}}) = 56$ and $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{D_3}(V_E)) = 76$.*

Proof. We can use again (4.24) to find

$$\begin{aligned} \chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{\tilde{S}}) &= \chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{\tilde{S}}) - \chi(\mathcal{O}_{\tilde{X}_4}(-2e), \mathcal{O}_{\tilde{S}}) \\ &= \chi(\mathcal{O}_{\tilde{X}_4}(\bar{h}-h), \mathcal{O}_{\tilde{S}}) - \chi(\mathcal{O}_{\tilde{X}_4}(2\bar{h}-2h), \mathcal{O}_{\tilde{S}}), \end{aligned}$$

using (1.8) for the last equality.

1. Using the same reasoning as in the proof of Lemma 4.5.9, one finds

$$\begin{aligned}\chi(\pi^* \mathcal{O}_{X_4}(\bar{h} - h), \mathcal{O}_{\bar{S}}) &= \chi_{Z_3}(\mathcal{O}_{Z_3}(2\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-3V_{\bar{E}})) - \chi_{Z_3}(\mathcal{O}_{Z_3}(2\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-2V_{\bar{E}})) \\ &\quad - \chi_{Z_3}(\mathcal{O}_{Z_3}(3\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-3V_{\bar{E}})) + \chi_{Z_3}(\mathcal{O}_{Z_3}(3\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-2V_{\bar{E}})).\end{aligned}$$

The Remark below Proposition 9.3 in [8] implies then

$$\chi(\pi^* \mathcal{O}_{X_4}(\bar{h} - h), \mathcal{O}_{\bar{S}}) = -\chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z(2\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(2\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z(3\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(3\bar{h})).$$

2. Using again the same reasoning as in the proof of Lemma 4.5.9 and the Remark below Proposition 9.3 in [8], we can get :

$$\begin{aligned}\chi(\pi^* \mathcal{O}_{X_4}(2\bar{h} - 2h), \mathcal{O}_{\bar{S}}) &= \chi_{Z_3}(\mathcal{O}_{Z_3}(3\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-4V_{\bar{E}})) - \chi_{Z_3}(\mathcal{O}_{Z_3}(3\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-3V_{\bar{E}})) \\ &\quad - \chi_{Z_3}(\mathcal{O}_{Z_3}(4\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-4V_{\bar{E}})) + \chi_{Z_3}(\mathcal{O}_{Z_3}(4\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-3V_{\bar{E}})) \\ &= -\chi(\mathcal{O}_{Z_3}, S^2 \mathcal{V}_Z(3\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z(3\bar{h})) + \chi(\mathcal{O}_{Z_3}, S^2 \mathcal{V}_Z(4\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z(4\bar{h})).\end{aligned}$$

One can then conclude with Hirzebruch-Riemann-Roch, Lemma 3.2.3, Lemma 4.5.11 and Lemma 4.5.6.

By Serre duality and Lemma 4.4.7, $\chi(\mathcal{O}_{\bar{S}}, \mathcal{O}_{D_3}(V_E)) = \chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{\bar{S}}(-h + e)) = \chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{\bar{S}}(-\bar{h}))$. Hence with (4.24), we get

$$\begin{aligned}\chi(\mathcal{O}_{\bar{S}}, \mathcal{O}_{D_3}(V_E)) &= \chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{\bar{S}}(-\bar{h})) - \chi(\mathcal{O}_{\tilde{X}_4}(-2e), \mathcal{O}_{\bar{S}}(-\bar{h})) \\ &= \chi(\mathcal{O}_{\tilde{X}_4}(2\bar{h} - h), \mathcal{O}_{\bar{S}}) - \chi(\mathcal{O}_{\tilde{X}_4}(3\bar{h} - 2h), \mathcal{O}_{\bar{S}}).\end{aligned}$$

We can then proceed as in the previous case to find :

1. First,

$$\begin{aligned}\chi(\mathcal{O}_{\tilde{X}_4}(2\bar{h} - h), \mathcal{O}_{\bar{S}}) &= \chi_{Z_3}(\mathcal{O}_{Z_3}(3\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-3V_{\bar{E}})) - \chi_{Z_3}(\mathcal{O}_{Z_3}(3\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-2V_{\bar{E}})) \\ &\quad - \chi_{Z_3}(\mathcal{O}_{Z_3}(4\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-3V_{\bar{E}})) + \chi_{Z_3}(\mathcal{O}_{Z_3}(4\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-2V_{\bar{E}})) \\ &= -\chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z(3\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(3\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z(4\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(4\bar{h}));\end{aligned}$$

2. Secondly,

$$\begin{aligned}\chi(\mathcal{O}_{\tilde{X}_4}(3\bar{h} - 2h), \mathcal{O}_{\bar{S}}) &= \chi_{Z_3}(\mathcal{O}_{Z_3}(4\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-4V_{\bar{E}})) - \chi_{Z_3}(\mathcal{O}_{Z_3}(4\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-3V_{\bar{E}})) \\ &\quad - \chi_{Z_3}(\mathcal{O}_{Z_3}(5\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-4V_{\bar{E}})) + \chi_{Z_3}(\mathcal{O}_{Z_3}(5\bar{h}) \otimes \bar{\pi}_* \mathcal{O}_{\bar{E}_4}(-3V_{\bar{E}})) \\ &= -\chi(\mathcal{O}_{Z_3}, S^2 \mathcal{V}_Z(4\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z(4\bar{h})) + \chi(\mathcal{O}_{Z_3}, S^2 \mathcal{V}_Z(5\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z(5\bar{h})).\end{aligned}$$

As before, one can then conclude with Hirzebruch-Riemann-Roch. □

4.6 Euler characteristics between structure sheaves of surfaces

We would like now to compute the four Euler characteristics between $\mathcal{O}_{\bar{D}_2}$ and $\mathcal{O}_{\bar{S}}$.

Lemma 4.6.1. *Let \mathcal{O}_S and \mathcal{O}_T be the structure sheaves of two surfaces S and T in \tilde{X}_4 , with possibly $S = T$. Then $\chi(\mathcal{O}_S, \mathcal{O}_T) = \chi(\mathcal{O}_T, \mathcal{O}_S) = \text{ch}_2(\mathcal{O}_T) \cdot \text{ch}_2(\mathcal{O}_S)$.*

Proof. Since \mathcal{O}_S and \mathcal{O}_T are structure sheaves of surfaces in \tilde{X}_4 , one has $\text{ch}_0(\mathcal{O}_S) = \text{ch}_0(\mathcal{O}_T) = \text{ch}_1(\mathcal{O}_S) = \text{ch}_1(\mathcal{O}_T) = 0$. Moreover $\text{ch}_2(\mathcal{O}_S) = \text{ch}_2(\mathcal{O}_S^\vee)$ and $\text{ch}_2(\mathcal{O}_T) = \text{ch}_2(\mathcal{O}_T^\vee)$ (see for instance pages 430 and 432 of [12]), hence Hirzebruch-Riemann-Roch yields :

$$\begin{aligned}\chi(\mathcal{O}_S, \mathcal{O}_T) &= \int_{\tilde{X}_4} \text{ch}(\mathcal{O}_S^\vee) \text{ch}(\mathcal{O}_T) \text{td}(\tilde{X}_4) = \int_{\tilde{X}_4} (0, 0, 0, 0, \text{ch}_2(\mathcal{O}_T) \cdot \text{ch}_2(\mathcal{O}_S)) \text{td}(\tilde{X}_4) \\ &= \int_{\tilde{X}_4} \text{ch}(\mathcal{O}_S) \text{ch}(\mathcal{O}_T^\vee) \text{td}(\tilde{X}_4) = \chi(\mathcal{O}_T, \mathcal{O}_S).\end{aligned}$$

□

Lemma 4.6.2. *One has $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}) = 48$, $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\bar{D}_2}) = -24$ and $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{S}}) = -4$.*

Proof. Along this proof, let us denote by a, b, c and d the following embeddings : $a : \bar{D}_2 \hookrightarrow \tilde{X}_4$, $b : \tilde{S} \hookrightarrow \tilde{Z}$, $c : \bar{D}_2 \hookrightarrow \tilde{Z}$ and $d : \tilde{Z} \hookrightarrow \tilde{X}_4$.

1. First, $\chi(a_* \mathcal{O}_{\bar{D}_2}, a_* \mathcal{O}_{\bar{D}_2}) = \chi(a^* a_* \mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2})$ and by Theorem 13.7 in [8], $\text{ch}(a^* a_* \mathcal{O}_{\bar{D}_2}) = (0, 0, c_2(\mathcal{N}_{\bar{D}_2/\tilde{X}_4}))$. So by Hirzebruch-Riemann-Roch, $\chi(a_* \mathcal{O}_{\bar{D}_2}, a_* \mathcal{O}_{\bar{D}_2}) = c_2(\mathcal{N}_{\bar{D}_2/\tilde{X}_4})$.

Moreover, by Proposition 3.4 in [11], since $\bar{D}_2 \subset \tilde{Z} \subset \tilde{X}_4$ and all these inclusions are smooth, there is a short exact sequence in \bar{D}_2 :

$$0 \rightarrow \mathcal{N}_{\bar{D}_2/\tilde{Z}} \rightarrow \mathcal{N}_{\bar{D}_2/\tilde{X}_4} \rightarrow \mathcal{N}_{\tilde{Z}/\tilde{X}_4}|_{\bar{D}_2} \rightarrow 0. \quad (4.37)$$

So knowing $\mathcal{N}_{\bar{D}_2/\tilde{Z}}$ and $\mathcal{N}_{\tilde{Z}/\tilde{X}_4}|_{\bar{D}_2}$ would allow us to compute $\chi(a_* \mathcal{O}_{\bar{D}_2}, a_* \mathcal{O}_{\bar{D}_2})$. Moreover, using the equalities just above Proposition 13.1 in [8] and the fact that \tilde{Z} is a divisor in \tilde{X}_4 and \bar{D}_2 a divisor in \tilde{Z} (because \tilde{Z} is the blow up of Z_3 in Γ_1 and thus by construction \bar{D}_2 is its exceptional divisor), one finds :

- $\tilde{N}_{\tilde{Z}/\tilde{X}_4} = \mathcal{O}_{\tilde{Z}}(\bar{e})$. Indeed, from Proposition 3.2.6 and (1.12) we know that \tilde{Z} has class $\bar{e}h$ in \tilde{X}_5 , hence since \tilde{X}_4 is cut out in \tilde{X}_5 by h we have that the class of \tilde{Z} in \tilde{X}_4 is $\bar{e}|_{\tilde{X}_4}$ which we are still going to denote by \bar{e} . With (1.8), this yields : $\tilde{N}_{\tilde{Z}/\tilde{X}_4} = \mathcal{O}_{\tilde{Z}}(2\bar{h} - h)$.
- With the notations of Lemma 4.4.1, $\mathcal{N}_{\bar{D}_2/\tilde{Z}} = \mathcal{O}_{\bar{D}_2}(-\delta)$. Indeed, by the above mentioned formula in [8], $\mathcal{N}_{\bar{D}_2/\tilde{Z}} = \mathcal{O}_{\bar{D}_2}([\bar{D}_2])|_{\bar{D}_2}$ where by $[\bar{D}_2]$ we mean the class of \bar{D}_2 in \tilde{Z} .

So we need to compute $[\bar{D}_2]^2$ in \tilde{Z} . For this one can notice that on one hand by Lemma 4.4.5, $h|_{\bar{D}_2} = \delta + 48f$. On the other hand, by (1.12) and (1.17) one finds that in E_4 the following holds :

$$h^2 = V_{\bar{E}}^2 = 3V_Z V_{\bar{E}} + 2V_{\bar{E}}(\bar{h}_1 + \bar{h}_2) - 4V_Z(\bar{h}_1 + \bar{h}_2).$$

Since \tilde{Z} has class $h\bar{e} = V_{\bar{E}}\bar{e}$ in \bar{E}_4 , with (1.15) and the formula below Definition 1.7 in [4] this implies

$$h|_{\tilde{Z}} = 3V_Z + 2(\bar{h}_1 + \bar{h}_2) - [\bar{D}_2]$$

(indeed by (1.15) we have that the class of \bar{D}_2 in \bar{E}_4 is $-V_{\bar{E}}^2 + 3V_Z V_{\bar{E}} + 2(\bar{h}_1 + \bar{h}_2)V_{\bar{E}} = V_{\bar{E}}(-V_{\bar{E}} + 3V_Z + 2(\bar{h}_1 + \bar{h}_2))$ and thus $[\bar{D}_2] = -V_{\bar{E}} + 3V_Z + 2(\bar{h}_1 + \bar{h}_2) = -h + 3V_Z + 2(\bar{h}_1 + \bar{h}_2)$).

But one should have $h|_{\bar{D}_2} = (h|_{\tilde{Z}})|_{\bar{D}_2}$ so finally :

$$\delta + 48f = 3V_Z \cdot [\bar{D}_2] + 2(\bar{h}_1 + \bar{h}_2) \cdot [\bar{D}_2] - [\bar{D}_2]^2.$$

By (1.13) and (1.18), $V_Z \cdot 4V_Z(\bar{h}_1 + \bar{h}_2) = 0$ in Z_3 , so $V_Z \cdot [\bar{D}_2] = 0$, and $(\bar{h}_1 + \bar{h}_2) \cdot 4V_Z(\bar{h}_1 + \bar{h}_2) = 24$ in Z_3 so $(\bar{h}_1 + \bar{h}_2) \cdot [\bar{D}_2] = 24f$. As a result, $\delta + 48f = 48f - [\bar{D}_2]^2$, that is to say $[\bar{D}_2]^2 = -\delta$.

Finally, we saw in the proof of Lemma 4.4.8 that $\bar{h}|_{\overline{D}_2} = 24f$, which together with $h|_{\overline{D}_2} = \delta + 48f$ and what we did above implies $\mathcal{N}_{\tilde{Z}/\tilde{X}_4}|_{\overline{D}_2} = \mathcal{O}_{\overline{D}_2}(48f - \delta - 48f) = \mathcal{O}_{\overline{D}_2}(-\delta)$. Then (4.37) gives :

$$0 \rightarrow \mathcal{O}_{\overline{D}_2}(-\delta) \rightarrow \mathcal{N}_{\overline{D}_2/\tilde{X}_4} \rightarrow \mathcal{O}_{\overline{D}_2}(-\delta) \rightarrow 0$$

and so $\text{ch}(\mathcal{N}_{\overline{D}_2/\tilde{X}_4}) = (1, -\delta, 24) + (1, -\delta, 24) = (2, -2\delta, 48)$. Therefore,

$$c_2(\mathcal{N}_{\overline{D}_2/\tilde{X}_4}) = \frac{(2\delta)^2}{2} - 48 = 2.48 - 48 = 48.$$

2. One has $\chi(d_*c_*\mathcal{O}_{\overline{D}_2}, d_*b_*\mathcal{O}_{\tilde{S}}) = \chi(d^*d_*c_*\mathcal{O}_{\overline{D}_2}, b_*\mathcal{O}_{\tilde{S}})$ and by Theorem 13.7 in [8] we have $d^*d_*c_*[\overline{D}_2] = [\overline{D}_2].c_1(\mathcal{N}_{\tilde{Z}/\tilde{X}_4}) = [\overline{D}_2].(2\bar{h} - h)$. Finally Hirzebruch-Riemann-Roch and the fact that the class of \tilde{S} in \tilde{Z} is \bar{h} give :

$$\begin{aligned} \chi(d^*d_*c_*\mathcal{O}_{\overline{D}_2}, b_*\mathcal{O}_{\tilde{S}}) &= \int_{\tilde{Z}} (0, 0, [\overline{D}_2].(2\bar{h} - h), *) \cdot (0, \bar{h}, *, *) \cdot \text{td}(\tilde{Z}) = \int_{\tilde{Z}} (0, 0, 0, [\overline{D}_2].(2\bar{h} - h)\bar{h}) \cdot \text{td}(\tilde{Z}) \\ &= [\overline{D}_2].(2\bar{h} - h)\bar{h} = 2\bar{h}|_{\overline{D}_2}^2 - h|_{\overline{D}_2}\bar{h}|_{\overline{D}_2} = 2(24f)^2 - 24f(\delta + 48f) = -24. \end{aligned}$$

Lemma 4.6.1 then yields $\chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\overline{D}_2})$.

3. By Lemma 4.6.1, $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{S}}) = (\bar{e}\bar{h})^2 = \bar{e}\bar{h}^2\bar{e} = \bar{h}|_{\tilde{Z}}^2\bar{e}|_{\tilde{Z}}$ using the formula below Definition 1.7 in [4] for the last equality. Furthermore, since \tilde{Z} is cut out in \overline{E}_4 by h , we can use [4] again to get : $\bar{h}|_{\tilde{Z}}^2\bar{e}|_{\tilde{Z}} = h|_{\overline{E}_4}\bar{h}|_{\overline{E}_4}^2\bar{e}|_{\overline{E}_4}$. So finally we need to compute $h\bar{h}^2\bar{e}$ in \overline{E}_4 , which by (1.8), (1.11), (1.13) and (1.12) gives :

$$\begin{aligned} h\bar{h}^2\bar{e} &= V_{\overline{E}}(2\bar{h} - h)\bar{h}^2 = (V_Z^2 + (\bar{h}_1 + \bar{h}_2)^2 + 2V_Z(\bar{h}_1 + \bar{h}_2))(2\bar{h}V_{\overline{E}} - V_{\overline{E}}^2) \\ &= (2\bar{h}_1\bar{h}_2 + 2V_Z(\bar{h}_1 + \bar{h}_2))(2(V_Z + \bar{h}_1 + \bar{h}_2)V_{\overline{E}} - 3V_ZV_{\overline{E}} - 2V_{\overline{E}}(\bar{h}_1 + \bar{h}_2) + 4V_Z(\bar{h}_1 + \bar{h}_2)) \\ &= 8 + 24 - 12 - 24 = -4. \end{aligned}$$

□

4.7 Euler characteristics between sheaves coming from $D^b(\Sigma_2)$ and...

In this subsection, we try to compute Euler characteristics between the first six elements of \mathcal{F}_1 , namely the twists of pull-backs of sheaves on Σ_2 , and the other sheaves of $\mathcal{F}_1 - \{\mathcal{O}_{l_h}, \mathcal{O}_p\}$. Again, as there are a lot of Euler characteristics to compute, we divide the subsection into smaller subsections, whose titles indicate which elements of \mathcal{F}_1 are concerned.

In most cases, the methods are not different from those used before, yet we don't manage to compute all the Euler characteristics between the first six elements of \mathcal{F}_1 and $\mathcal{O}_{\overline{D}_2}$: some depend on an intersection number which we don't know how to compute.

Before going further, let us compute the Todd class of D_3 .

Lemma 4.7.1. *The canonical bundle of D_3 is $\omega_{D_3} = \mathcal{O}_{D_3}(-2V_E - 3h_1 + E_1 + E_2 + E_3)$, with the notations of section 2.1.1. By Remark 4.1.4 this implies $\omega_{D_3} = \mathcal{O}_{D_3}(-2V_E - h_1 - h_2)$.*

Proof. One can use the same reasoning as in the proof of Proposition 4.5.3, but with $V_E, h_1, D_3, \Sigma_2, \mathcal{V}_F|_{\Sigma_2}$ instead of $V_Z, \bar{h}_1, Z_3, S_2, \mathcal{V}_S$. □

Lemma 4.7.2. *The Todd class of D_3 is $\text{td}(D_3) = (1, V_E + \frac{3}{2}h_1 - \frac{1}{2}(E_1 + E_2 + E_3), x_2, 1) = (1, V_E + \frac{1}{2}(h_1 + h_2), x_2, 1)$ where $h_1.x_2 = h_2.x_2 = \frac{3}{2}, E_1.x_2 = E_2.x_2 = E_3.x_2 = \frac{1}{2}$ and $V_E.x_2 = \frac{1}{3}$.*

Proof. Using the same reasoning as in the proof Lemma 4.5.6 one can find that $\text{td}(D_3) = (1, V_E + \frac{3}{2}h_1 - \frac{1}{2}(E_1 + E_2 + E_3), x_2, 1) = (1, V_E + \frac{1}{2}(h_1 + h_2), x_2, 1)$. We can then use Hirzebruch-Riemann-Roch and some Euler characteristics we have already computed to find some x_2 . D' where D' is a divisor in D_3 .

First, $H^*(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(h_1)) = H^*(p^*\phi^*\mathcal{O}_{\mathbb{P}^2}, p^*\phi^*\mathcal{O}_{\mathbb{P}^2}(1)) = H^*(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))$ using the projection formula, $p_*\mathcal{O}_{D_3} = \mathcal{O}_{\Sigma_2}$ (see Corollary 4.5.5) and $\phi_*\mathcal{O}_{\Sigma_2} = \mathcal{O}_{\mathbb{P}^2}$ (see for instance the proof of Proposition 3.1 in [2]). As a result, $\chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(h_1)) = 3$. On the other hand by Hirzebruch-Riemann-Roch :

$$\begin{aligned}\chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(h_1)) &= 3 = \int_{D_3} (1, h_1, \frac{h_1^2}{2}, 0) \cdot (1, V_E + \frac{3}{2}h_1 - \frac{1}{2}(E_1 + E_2 + E_3), x_2, 1) \\ &= 1 + h_1 \cdot x_2 + \frac{1}{2} = \frac{3}{2} + h_1 \cdot x_2\end{aligned}$$

thus $h_1 \cdot x_2 = 3 - \frac{3}{2} = \frac{3}{2}$.

Similarly, $H^*(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(E_1)) = H^*(p^*\phi^*\mathcal{O}_{\mathbb{P}^2}, p^*\mathcal{O}_{\Sigma_2}(E_1)) = H^*(\mathcal{O}_{\mathbb{P}^2}, \phi_*\mathcal{O}_{\Sigma_2}(E_1))$. We saw in the proof of Lemma 4.3.4 that $\phi_*\mathcal{O}_{\Sigma_2}(E_1) = \mathcal{O}_{\mathbb{P}^2}$, hence $\chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(E_1)) = 1$. Thus :

$$\begin{aligned}\chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(E_1)) &= 1 = \int_{D_3} (1, E_1, \frac{E_1^2}{2}, 0) \cdot (1, V_E + \frac{3}{2}h_1 - \frac{1}{2}(E_1 + E_2 + E_3), x_2, 1) \\ &= 1 + E_1 \cdot x_2 - \frac{1}{2} = \frac{1}{2} + E_1 \cdot x_2 \\ \Rightarrow E_1 \cdot x_2 &= 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

The same reasoning then yields $E_2 \cdot x_2 = E_3 \cdot x_2 = \frac{1}{2}$.

Finally, $H^*(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(V_E)) = H^*(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{D_3}(V_E)) = H^*(\mathcal{O}_{\Sigma_2}, \mathcal{V}_F|_{\Sigma}) \cong H^*(\mathcal{O}_{S_2}, \mathcal{V}_S^{\vee}) = 0$ using the proof of Lemma 4.3.1 and Lemma 3.1.3 for the last two equalities. As a result, with (1.16) one gets :

$$\begin{aligned}\chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(V_E)) &= 0 = \int_{D_3} (1, V_E, -\frac{h_1 h_2}{2}, -\frac{1}{3}) \cdot (1, V_E + \frac{3}{2}h_1 - \frac{1}{2}(E_1 + E_2 + E_3), x_2, 1) \\ &= 1 + V_E \cdot x_2 - 1 - \frac{1}{3} = -\frac{1}{3} + V_E \cdot x_2 \\ \Rightarrow V_E \cdot x_2 &= \frac{1}{3}.\end{aligned}$$

To conclude by Remark 4.1.4 one has $h_2 = 2h_1 - E_1 - E_2 - E_3$, hence $h_2 \cdot x_2 = 3 - \frac{3}{2} = \frac{3}{2}$. □

Remark 4.7.3. We have found that $h_i \cdot x_2 = \bar{h}_i \cdot x_2$ and $(V_Z + \bar{h}_1 + \bar{h}_2) \cdot x_2 = \frac{10}{3} = (V_E + h_1 + h_2) \cdot x_2$. These numbers should only depend on the fact that S_2 and Σ_2 are degree 6 del Pezzo surfaces and on \mathcal{V}_S and $\mathcal{V}_F|_{\Sigma_2}$ so it is not surprising.

We can now start to compute Euler characteristics between the sheaves considered in section 4.1 and the other sheaves of the family $\mathcal{F}_1 - \{\mathcal{O}_{l_h}, \mathcal{O}_p\}$.

4.7.1 ... $\mathcal{O}_{\tilde{X}_4}(ih)$

By Corollary 2.1.3 we already know that $\chi(\mathcal{O}_{\tilde{X}_4}(ih), F) = 0$ for any $0 \leq i \leq 4$ and F one of the first six elements of \mathcal{F}_1 . Let us compute the other Euler characteristics $\chi(F, \mathcal{O}_{\tilde{X}_4}(ih))$.

We are going to use the fact that $\mathcal{O}_{\tilde{X}_4}(ih) = \pi^*\mathcal{O}_{X_4}(ih)$ for any $i \in \mathbb{Z}$.

Proposition 4.7.4. *The following Euler characteristics are :*

$$(\chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \pi^*\mathcal{O}_{X_4}), \dots, \chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \pi^*\mathcal{O}_{X_4}(4h))) = (1, 2, 3, 4, 5).$$

Proof. Let $0 \leq i \leq 4$, we want to compute all the $\chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \pi^*\mathcal{O}_{X_4}(ih))$. We can start by noticing that by Serre duality,

$$\begin{aligned}\chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \pi^*\mathcal{O}_{X_4}(ih)) &= \chi(\pi^*\mathcal{O}_{X_4}(ih), j_*\mathcal{O}_{E_1}(E_1 - V_E - h + e)) \\ &= \chi(\pi^*\mathcal{O}_{X_4}((i+1)h), j_*\mathcal{O}_{E_1}(E_1 - V_E + e)).\end{aligned}$$

Moreover, taking the derived pushforward by j and twisting by $\mathcal{O}_{D_3}(-V_E + e)$ the following short exact sequence $0 \rightarrow \mathcal{O}_{D_3} \rightarrow \mathcal{O}_{D_3}(E_1) \rightarrow \mathcal{O}_{E_1}(E_1) \rightarrow 0$ yields :

$$\begin{aligned}\chi(\pi^*\mathcal{O}_{X_4}((i+1)h), j_*\mathcal{O}_{E_1}(E_1 - V_E + e)) &= \chi(\pi^*\mathcal{O}_{X_4}((i+1)h), j_*\mathcal{O}_{D_3}(E_1 - V_E + e)) \\ &\quad - \chi(\pi^*\mathcal{O}_{X_4}((i+1)h), j_*\mathcal{O}_{D_3}(-V_E + e)).\end{aligned}$$

Since $\pi \circ j = i \circ p$ (with the notations of section 2.1.1), we have

$$\begin{aligned}\chi(\pi^*\mathcal{O}_{X_4}((i+1)h), j_*\mathcal{O}_{D_3}(E_1 - V_E + e)) &= \chi(\mathcal{O}_{X_4}((i+1)h), \pi_*j_*\mathcal{O}_{D_3}(E_1 - V_E + e)) \\ &= \chi(\mathcal{O}_{X_4}((i+1)h), i_*p_*\mathcal{O}_{D_3}(E_1 - V_E + e)) \\ &= \chi(i^*\mathcal{O}_{X_4}((i+1)h), p_*\mathcal{O}_{D_3}(E_1 - V_E + e)) \\ &= \chi(\mathcal{O}_{\Sigma_2}((i+1)h), p_*\mathcal{O}_{D_3}(E_1 - V_E + e))\end{aligned}$$

and similarly, $\chi(\pi^*\mathcal{O}_{X_4}((i+1)h), j_*\mathcal{O}_{D_3}(-V_E + e)) = \chi(\mathcal{O}_{\Sigma_2}((i+1)h), p_*\mathcal{O}_{D_3}(-V_E + e))$.

Finally by (1.10), we have $\mathcal{O}_{D_3}(e) = \mathcal{O}_{D_3}(-V_E)$ therefore

$$\chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \pi^*\mathcal{O}_{X_4}(ih)) = \chi(\mathcal{O}_{\Sigma_2}((i+1)h), p_*\mathcal{O}_{D_3}(E_1 - 2V_E)) - \chi(\mathcal{O}_{\Sigma_2}((i+1)h), p_*\mathcal{O}_{D_3}(-2V_E)).$$

By the projection formula, $\chi(\mathcal{O}_{\Sigma_2}((i+1)h), p_*\mathcal{O}_{D_3}(E_1 - 2V_E)) = \chi(\mathcal{O}_{\Sigma_2}((i+1)h), \mathcal{O}_{\Sigma_2}(E_1) \otimes p_*\mathcal{O}_{D_3}(-2V_E))$. By the Remark below Proposition 9.3 in [8] we have $R^*p_*\mathcal{O}_{D_3}(-2V_E) = \mathcal{O}_{\Sigma_2}\delta_{*,1}$ so

$$\chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \pi^*\mathcal{O}_{X_4}(ih)) = -\chi(\mathcal{O}_{\Sigma_2}((i+1)h), \mathcal{O}_{\Sigma_2}(E_1)) + \chi(\mathcal{O}_{\Sigma_2}((i+1)h), \mathcal{O}_{\Sigma_2}).$$

One can conclude with Hirzebruch-Riemann-Roch, Lemma 4.1.3 and (1.14). □

Corollary 4.7.5. *The same result holds with E_1 replaced by E_2 or E_3 .*

Lemma 4.7.6. *The following Euler characteristics are :*

$$\begin{pmatrix} \chi(j_*\mathcal{O}_{D_3}(-V_E), \pi^*\mathcal{O}_{X_4}) & \dots & \chi(j_*\mathcal{O}_{D_3}(-V_E), \pi^*\mathcal{O}_{X_4}(4h)) \\ \chi(j_*\mathcal{O}_{D_3}(h_1 - V_E), \pi^*\mathcal{O}_{X_4}) & \dots & \chi(j_*\mathcal{O}_{D_3}(h_1 - V_E), \pi^*\mathcal{O}_{X_4}(4h)) \\ \chi(j_*\mathcal{O}_{D_3}(2h_1 - V_E), \pi^*\mathcal{O}_{X_4}) & \dots & \chi(j_*\mathcal{O}_{D_3}(2h_1 - V_E), \pi^*\mathcal{O}_{X_4}(4h)) \end{pmatrix} = \begin{pmatrix} -1 & -7 & -19 & -37 & -61 \\ 0 & -3 & -12 & -27 & -48 \\ 0 & 0 & -6 & -18 & -36 \end{pmatrix}.$$

Proof. Let $0 \leq k \leq 2$ and $0 \leq i \leq 4$. We want to compute $\chi(j_*\mathcal{O}_{D_3}(kh_1 - V_E), \pi^*\mathcal{O}_{X_4}(ih))$. First, we can use the same reasoning as in the beginning of the proof of Proposition 4.7.4 to find

$$\begin{aligned}\chi(j_*\mathcal{O}_{D_3}(kh_1 - V_E), \pi^*\mathcal{O}_{X_4}(ih)) &= \chi(\pi^*\mathcal{O}_{X_4}(ih), j_*\mathcal{O}_{D_3}(kh_1 - V_E - h + e)) \\ &= \chi(\pi^*\mathcal{O}_{X_4}((i+1)h), j_*\mathcal{O}_{D_3}(kh_1 - 2V_E)) \\ &= \chi(\mathcal{O}_{X_4}((i+1)h), \pi_*j_*\mathcal{O}_{D_3}(kh_1 - 2V_E)) \\ &= \chi(\mathcal{O}_{X_4}((i+1)h), i_*p_*\mathcal{O}_{D_3}(kh_1 - 2V_E)) \\ &= \chi(i^*\mathcal{O}_{X_4}((i+1)h), p_*\mathcal{O}_{D_3}(kh_1 - 2V_E)) \\ &= \chi(\mathcal{O}_{\Sigma_2}((i+1)h), \mathcal{O}_{\Sigma_2}(kh_1) \otimes p_*\mathcal{O}_{D_3}(-2V_E)) \\ &= -\chi(\mathcal{O}_{\Sigma_2}((i+1)h), \mathcal{O}_{\Sigma_2}(kh_1)) \\ &= -\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(kh_1 - (i+1)h)).\end{aligned}$$

We can compute such Euler characteristics using Hirzebruch-Riemann-Roch. □

4.7.2 ... $\mathcal{O}_{D_3}(V_E)$

Let us start with a small technical lemma.

Lemma 4.7.7. *One has $\text{ch}(S^2\mathcal{V}_F|_{\Sigma_2}^\vee) = \text{ch}(S^2\mathcal{V}_F|_{\Sigma_2}) = (3, 0, -8)$.*

Proof. One can use the same reasoning as in the proofs of Lemma 3.1.5 and Lemma 4.3.1 to find that $\text{ch}(\mathcal{V}_F|_{\Sigma_2}^\vee) = \text{ch}(\mathcal{V}_F|_{\Sigma_2}) = (2, 0, -2)$. This yields $c_1(\mathcal{V}_F|_{\Sigma_2}^\vee) = 0$ and $c_2(\mathcal{V}_F|_{\Sigma_2}^\vee) = 2$.

Using the same reasoning as in the proof of Lemma 4.5.11, one then finds :

$$\begin{aligned} c_1(S^2\mathcal{V}_F|_{\Sigma_2}^\vee) &= 3c_1(\mathcal{V}_F|_{\Sigma_2}^\vee) = 3 \cdot 0 = 0 \\ c_2(S^2\mathcal{V}_F|_{\Sigma_2}^\vee) &= 2c_1(\mathcal{V}_F|_{\Sigma_2}^\vee)^2 + 4c_2(\mathcal{V}_F|_{\Sigma_2}^\vee) = 4 \cdot 2 = 8 \\ \text{ch}_0(S^2\mathcal{V}_F|_{\Sigma_2}^\vee) &= 3 \\ \text{ch}_1(S^2\mathcal{V}_F|_{\Sigma_2}^\vee) &= 0 \\ \text{ch}_2(S^2\mathcal{V}_F|_{\Sigma_2}^\vee) &= \frac{0^2}{2} - 8 = -8. \end{aligned}$$

Since $c_1(S^2\mathcal{V}_F|_{\Sigma_2}^\vee) = 0$ we also have $\text{ch}(S^2\mathcal{V}_F|_{\Sigma_2}) = \text{ch}(S^2\mathcal{V}_F|_{\Sigma_2}^\vee)$. □

Lemma 4.7.8. *Let*

$$j_*F \in \{j_*\mathcal{O}_{E_1}(E_1 - V_E), j_*\mathcal{O}_{E_2}(E_2 - V_E), j_*\mathcal{O}_{E_3}(E_3 - V_E), j_*\mathcal{O}_{D_3}(-V_E), j_*\mathcal{O}_{D_3}(h_1 - V_E), j_*\mathcal{O}_{D_3}(2h_1 - V_E)\}.$$

Then :

1. $\chi(j_*F, \mathcal{O}_{D_3}(V_E)) = \chi(\mathcal{O}_{\Sigma_2}(h), p_*F(-2V_E)) - \chi(\mathcal{O}_{\Sigma_2}(h), p_*F(-3V_E));$
2. $\chi(\mathcal{O}_{D_3}(V_E), j_*F) = \chi(\mathcal{O}_{\Sigma_2}, p_*F(-V_E)) - \chi(\mathcal{O}_{\Sigma_2}, p_*F(-2V_E)).$

Proof. To start with, let $j_*F \in \{j_*\mathcal{O}_{E_1}(E_1 - V_E), j_*\mathcal{O}_{E_2}(E_2 - V_E), j_*\mathcal{O}_{E_3}(E_3 - V_E), j_*\mathcal{O}_{D_3}(-V_E), j_*\mathcal{O}_{D_3}(h_1 - V_E), j_*\mathcal{O}_{D_3}(2h_1 - V_E)\}$. Then since there is a short exact sequence :

$$0 \rightarrow \mathcal{O}_{\tilde{X}_4}(-2e) \rightarrow \mathcal{O}_{\tilde{X}_4}(-e) \rightarrow \mathcal{O}_{D_3}(V_E) \rightarrow 0$$

in \tilde{X}_4 , we have :

- $\chi(j_*F, \mathcal{O}_{D_3}(V_E)) = \chi(j_*F, \mathcal{O}_{\tilde{X}_4}(-e)) - \chi(j_*F, \mathcal{O}_{\tilde{X}_4}(-2e)) = \chi(j_*F(e), \pi^*\mathcal{O}_{X_4}) - \chi(j_*F(2e), \pi^*\mathcal{O}_{X_4});$
- $\chi(\mathcal{O}_{D_3}(V_E), j_*F) = \chi(\mathcal{O}_{\tilde{X}_4}(-e), j_*F) - \chi(\mathcal{O}_{\tilde{X}_4}(-2e), j_*F) = \chi(\pi^*\mathcal{O}_{X_4}, j_*F(e)) - \chi(\pi^*\mathcal{O}_{X_4}, j_*F(2e)).$

We can now use the same method as in the proof of Lemma 4.7.6.

- By Serre duality, (1.10) and since $\pi \circ j = i \circ p$, one has

$$\begin{aligned} \chi(j_*F(e), \pi^*\mathcal{O}_{X_4}) &= \chi(\pi^*\mathcal{O}_{X_4}, j_*F(e - h + e)) = \chi(\pi^*\mathcal{O}_{X_4}(h), j_*F(2e)) = \chi(\mathcal{O}_{X_4}(h), \pi_*j_*F(2e)) \\ &= \chi(\mathcal{O}_{X_4}(h), i_*p_*F(2e)) = \chi(i^*\mathcal{O}_{X_4}(h), p_*F(2e)) = \chi(\mathcal{O}_{\Sigma_2}(h), p_*F(-2V_E)). \end{aligned}$$

Furthermore, $\chi(j_*F(2e), \pi^*\mathcal{O}_{X_4}) = \chi(\mathcal{O}_{\Sigma_2}(h), p_*F(-3V_E)).$

- For the second case we can use the same reasoning, we just don't need to use Serre duality here. This implies : $\chi(\pi^*\mathcal{O}_{X_4}, j_*F(e)) = \chi(\mathcal{O}_{\Sigma_2}, p_*F(-V_E))$ and $\chi(\pi^*\mathcal{O}_{X_4}, j_*F(2e)) = \chi(\mathcal{O}_{\Sigma_2}, p_*F(-2V_E)).$

□

Lemma 4.7.9. *One has : $\chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \mathcal{O}_{D_3}(V_E)) = -1$ and $\chi(\mathcal{O}_{D_3}(V_E), j_*\mathcal{O}_{E_1}(E_1 - V_E)) = 0$.*

Proof. • By Lemma 4.7.8, we have :

$$\chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \mathcal{O}_{D_3}(V_E)) = \chi(\mathcal{O}_{\Sigma_2}(h), p_*\mathcal{O}_{E_1}(-3V_E + E_1)) - \chi(\mathcal{O}_{\Sigma_2}(h), p_*\mathcal{O}_{E_1}(-4V_E + E_1)).$$

1. Using the short exact sequence $0 \rightarrow \mathcal{O}_{D_3} \rightarrow \mathcal{O}_{D_3}(E_1) \rightarrow \mathcal{O}_{E_1}(E_1) \rightarrow 0$ in D_3 , one finds :

$$\chi(\mathcal{O}_{\Sigma_2}(h), p_*\mathcal{O}_{E_1}(-3V_E + E_1)) = \chi(\mathcal{O}_{\Sigma_2}(h), p_*\mathcal{O}_{D_3}(-3V_E + E_1)) - \chi(\mathcal{O}_{\Sigma_2}(h), p_*\mathcal{O}_{D_3}(-3V_E)).$$

By the remark below Proposition 9.3 in [8], we know that $R^*p_*\mathcal{O}_{D_3}(-3V_E) = \mathcal{V}_F|_{\Sigma_2}\delta_{*,1}$. Together with the projection formula, this gives :

$$\chi(\mathcal{O}_{\Sigma_2}(h), p_*\mathcal{O}_{E_1}(-3V_E + E_1)) = -\chi(\mathcal{O}_{\Sigma_2}(h), \mathcal{V}_F|_{\Sigma_2}(E_1)) + \chi(\mathcal{O}_{\Sigma_2}(h), \mathcal{V}_F|_{\Sigma_2}).$$

We saw in the proof of Lemma 4.7.7 that $\text{ch}(\mathcal{V}_F|_{\Sigma_2}) = (2, 0, -2)$, therefore we can compute the above-mentioned Euler characteristics with Hirzebruch-Riemann-Roch.

2. With the same reasoning as in 1., except that $R^*p_*\mathcal{O}_{D_3}(-4V_E) = S^2\mathcal{V}_F|_{\Sigma_2}\delta_{*,1}$, one gets

$$\chi(\mathcal{O}_{\Sigma_2}(h), p_*\mathcal{O}_{E_1}(-4V_E + E_1)) = -\chi(\mathcal{O}_{\Sigma_2}(h), S^2\mathcal{V}_F|_{\Sigma_2}(E_1)) + \chi(\mathcal{O}_{\Sigma_2}(h), S^2\mathcal{V}_F|_{\Sigma_2}).$$

Again these Euler characteristics can be computed with Hirzebruch-Riemann-Roch and Lemma 4.7.7.

• By Lemma 4.7.8, $\chi(\mathcal{O}_{D_3}(V_E), j_*\mathcal{O}_{E_1}(E_1 - V_E)) = \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{E_1}(E_1 - 2V_E)) - \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{E_1}(E_1 - 3V_E))$. With the same reasoning as above, one can find :

1.

$$\begin{aligned} \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{E_1}(E_1 - 2V_E)) &= \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{D_3}(E_1 - 2V_E)) - \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{D_3}(-2V_E)) \\ &= -\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(E_1)) + \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}); \end{aligned}$$

2.

$$\begin{aligned} \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{E_1}(E_1 - 3V_E)) &= \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{D_3}(E_1 - 3V_E)) - \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{D_3}(-3V_E)) \\ &= -\chi(\mathcal{O}_{\Sigma_2}, \mathcal{V}_F|_{\Sigma_2}(E_1)) + \chi(\mathcal{O}_{\Sigma_2}, \mathcal{V}_F|_{\Sigma_2}). \end{aligned}$$

All these Euler characteristics can again be computed with Hirzebruch-Riemann-Roch. □

Corollary 4.7.10. *The same results hold when E_1 is replaced by E_2 or E_3 .*

Lemma 4.7.11. *One has: $\chi(j_*\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{D_3}(V_E)) = -5$, $\chi(j_*\mathcal{O}_{D_3}(h_1 - V_E), \mathcal{O}_{D_3}(V_E)) = -6$ and $\chi(j_*\mathcal{O}_{D_3}(2h_1 - V_E), \mathcal{O}_{D_3}(V_E)) = -6$.*

Proof. Let $0 \leq i \leq 2$. We want to compute $\chi(j_*\mathcal{O}_{D_3}(ih_1 - V_E), \mathcal{O}_{D_3}(V_E))$. By Lemma 4.7.8, we have

$$\chi(j_*\mathcal{O}_{D_3}(ih_1 - V_E), \mathcal{O}_{D_3}(V_E)) = \chi(\mathcal{O}_{\Sigma_2}(h), p_*\mathcal{O}_{D_3}(ih_1 - 3V_E)) - \chi(\mathcal{O}_{\Sigma_2}(h), p_*\mathcal{O}_{D_3}(ih_1 - 4V_E)).$$

Using the projection formula and the Remark below Proposition 9.3 in [8], we can then say

$$\chi(j_*\mathcal{O}_{D_3}(ih_1 - V_E), \mathcal{O}_{D_3}(V_E)) = -\chi(\mathcal{O}_{\Sigma_2}(h), \mathcal{V}_F|_{\Sigma_2}(ih_1)) + \chi(\mathcal{O}_{\Sigma_2}(h), S^2\mathcal{V}_F|_{\Sigma_2}(ih_1)).$$

Finally we can use Lemma 4.7.7 and Hirzebruch-Riemann-Roch. □

Lemma 4.7.12. *The following holds: $\chi(\mathcal{O}_{D_3}(V_E), j_*\mathcal{O}_{D_3}(-V_E)) = -1$, $\chi(\mathcal{O}_{D_3}(V_E), j_*\mathcal{O}_{D_3}(h_1 - V_E)) = 1$ and $\chi(\mathcal{O}_{D_3}(V_E), j_*\mathcal{O}_{D_3}(2h_1 - V_E)) = 4$.*

Proof. Let $0 \leq i \leq 2$. We want to compute $\chi(\mathcal{O}_{D_3}(V_E), j_*\mathcal{O}_{D_3}(ih_1 - V_E))$. By Lemma 4.7.8, we know

$$\chi(\mathcal{O}_{D_3}(V_E), j_*\mathcal{O}_{D_3}(ih_1 - V_E)) = \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{D_3}(ih_1 - 2V_E)) - \chi(\mathcal{O}_{\Sigma_2}, p_*\mathcal{O}_{D_3}(ih_1 - 3V_E)).$$

Again by the projection formula and the Remark below Proposition 9.3 in [8], this is equal to: $-\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(ih_1)) + \chi(\mathcal{O}_{\Sigma_2}, \mathcal{V}_F|_{\Sigma_2}(ih_1))$. □

4.7.3 ... $\mathcal{O}_{\tilde{S}}$

Lemma 4.7.13. *One has $\chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{S}}, j_*\mathcal{O}_{E_1}(E_1 - V_E)) = 3$.*

Proof. First, the short exact sequence $0 \rightarrow \mathcal{O}_{D_3}(-V_E) \rightarrow \mathcal{O}_{D_3}(E_1 - V_E) \rightarrow \mathcal{O}_{E_1}(E_1 - V_E) \rightarrow 0$ in D_3 gives :

$$\begin{aligned} \text{ch}(\mathcal{O}_{E_1}(E_1 - V_E)) &= \text{ch}(\mathcal{O}_{D_3}(E_1 - V_E)) - \text{ch}(\mathcal{O}_{D_3}(-V_E)) \\ &= (1, E_1 - V_E, \frac{E_1^2 - h_1 h_2 - 2E_1 V_E}{2}, \frac{5}{6}) - (1, -V_E, \frac{-h_1 h_2}{2}, \frac{2}{6}) \\ &= (0, E_1, \frac{E_1^2 - 2E_1 V_E}{2}, \frac{7}{6}). \end{aligned}$$

By Theorem 5.26 in [14], we thus have

$$\begin{aligned} \text{ch}(j_*\mathcal{O}_{E_1}(E_1 - V_E)).\text{td}(\tilde{X}_4) &= j_*((0, E_1, \frac{E_1^2 - 2E_1 V_E}{2}, \frac{7}{6}).(1, V_E + \frac{3}{2}h_1 - \frac{1}{2}(E_1 + E_2 + E_3), x_2, 1)) \\ &= j_*(0, E_1, 0, \frac{7}{6} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2}) \\ &= j_*(0, E_1, 0, \frac{2}{3}) = (0, 0, j_*E_1, *, *) \end{aligned}$$

where for the last equality we used the definition of pushforward on cohomology page 126 of [14]. Moreover, we saw in the proof of Corollary 4.6.2 that $\text{ch}(\mathcal{O}_{\tilde{S}}) = (0, 0, \bar{e}\bar{h}, *, *)$, hence by Hirzebruch-Riemann-Roch and (1.8) :

$$\begin{aligned} \chi(j_*\mathcal{O}_{E_1}(E_1 - V_E), \mathcal{O}_{\tilde{S}}) &= \int_{\tilde{X}_4} (0, 0, j_*E_1, *, *) \cdot (0, 0, \bar{e}\bar{h}, *, *) = j_*E_1 \cdot \bar{e} \cdot \bar{h} \\ &= j_*E_1 \cdot (h - 2e) \cdot (h - e) = j_*E_1 \cdot (h^2 - 3e \cdot h + 2e^2). \end{aligned}$$

Before going further, let us notice that a divisor D' in $|E_1|$ corresponds to a ruled surface over a curve in $C \in |E_1|$ in Σ_2 : namely, $D' = \mathbb{P}_C(\mathcal{V}_F|_C)$. Then the Picard group of D' is generated by two elements f_E and δ_E where f_E corresponds to a fiber over a point in C and δ_E is the relative hyperplane class (see V.2. of [12]). Furthermore, we know that $f_E \cdot \delta_E = 1$, $f_E^2 = 0$ and $\delta_E^2 = \det(\mathcal{V}_F|_C) = 0$ because $\mathcal{V}_F|_C = (\mathcal{V}_F|_{\Sigma_2})|_C$ and $\det(\mathcal{V}_F|_{\Sigma_2}) = \mathcal{O}_{\Sigma_2}$ as we saw in the proof of Lemma 4.7.7. From now on let us abuse of notations and still denote by E_1 some closed subscheme of D_3 with class in $|E_1|$. Then, since $h \cdot E_1 = h_2 \cdot E_1 = 1$ in Σ_2 we have that $h|_{E_1} = f_E$ and the same reasoning as in the proof of Lemma 4.4.5 together with $\det(\mathcal{V}_F|_{\Sigma_2}) = \mathcal{O}_{\Sigma_2}$ yield $V_E|_{E_1} = \delta_E$. We can now finish the computation :

$$j_*E_1 \cdot (h^2 - 3e \cdot h + 2e^2) = h|_{E_1}^2 + 3V_E|_{E_1} \cdot h|_{E_1} + 2V_E|_{E_1}^2 = f_E^2 + 3\delta_E \cdot f_E + 2\delta_E^2 = 3$$

using (1.10) and $E_1 \subset D_3 \subset E_4$ to find $j_*E_1 \cdot e = -V_E \cdot j_*E_1$.

We can do the exact same computation for $\chi(\mathcal{O}_{\tilde{S}}, j_*\mathcal{O}_{E_1}(-E_1 - V_E))$ using the same argument as in the proof of Lemma 4.6.1. \square

Corollary 4.7.14. *One has $\chi(j_*\mathcal{O}_{E_2}(E_2 - V_E), \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{S}}, j_*\mathcal{O}_{E_2}(E_2 - V_E)) = \chi(j_*\mathcal{O}_{E_3}(E_3 - V_E), \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{S}}, j_*\mathcal{O}_{E_3}(E_3 - V_E)) = 3$.*

Lemma 4.7.15. *One has $\chi(j_*\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{S}}) = 10$ and $\chi(\mathcal{O}_{\tilde{S}}, j_*\mathcal{O}_{D_3}(-V_E)) = -42$.*

Proof. We are going to proceed as in the proof of Lemma 4.5.16. First, using (4.24) twisted by $\mathcal{O}_{\tilde{X}_4}(2e)$ and (1.10) one can find

$$\chi(\mathcal{O}_{\tilde{S}}, j_*\mathcal{O}_{D_3}(-V_E)) = \chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4}(e)) - \chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4})$$

and

$$\chi(j_*\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{X}_4}(e), \mathcal{O}_{\tilde{S}}) - \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{S}}).$$

- Let us consider $\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(-V_E)) = \chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4}(e)) - \chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4})$.
 1. We saw in Lemma 4.5.14 that $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4}) = 79$.
 2. By Serre duality, $\chi(\mathcal{O}_{\tilde{S}}, \mathcal{O}_{\tilde{X}_4}(e)) = \chi(\mathcal{O}_{\tilde{X}_4}(e), \mathcal{O}_{\tilde{S}}(-h + e)) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{S}}(-h)) = 37$ with Lemma 4.5.12.

As a result, $\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(-V_E)) = 37 - 79 = -42$.

- Let us look at $\chi(j_* \mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{X}_4}(e), \mathcal{O}_{\tilde{S}}) - \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{S}})$.
 1. We saw in Lemma 4.5.7 that $\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{S}}) = 1$.
 2. By (1.8) we have $\chi(\mathcal{O}_{\tilde{X}_4}(e), \mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{S}}(\bar{h} - h)) = \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{S}}(\bar{h}))$. With the same reasoning as in the proof of Lemma 4.5.9 one finds

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{S}}(\bar{h})) &= \chi_{Z_3}(\pi_* \mathcal{O}_{\bar{E}_4}(-V_{\bar{E}})) - \chi_{Z_3}(\pi_* \mathcal{O}_{\bar{E}_4}) \\ &\quad - \chi_{Z_3}(\mathcal{O}_{Z_3}(\bar{h}) \otimes \pi_* \mathcal{O}_{\bar{E}_4}(-V_{\bar{E}})) + \chi_{Z_3}(\mathcal{O}_{Z_3}(\bar{h}) \otimes \pi_* \mathcal{O}_{\bar{E}_4}) \\ &= -\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})). \end{aligned}$$

By (1.11), the projection formula and Lemma 3.1.4, $\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})) = \chi(\mathcal{O}_{S_2}, \mathcal{V}_S^\vee(\bar{h}_1 + \bar{h}_2)) = 12$.
As a result, $\chi(\mathcal{O}_{\tilde{X}_4}(e), \mathcal{O}_{\tilde{S}}) = -1 + 12 = 11$.

This yields $\chi(j_* \mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{S}}) = 11 - 1 = 10$.

□

Lemma 4.7.16. *One has $\chi(j_* \mathcal{O}_{D_3}(h_1 - V_E), \mathcal{O}_{\tilde{S}}) = 19$, $\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(h_1 - V_E)) = -33$, $\chi(j_* \mathcal{O}_{D_3}(2h_1 - V_E), \mathcal{O}_{\tilde{S}}) = 28$ and $\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(2h_1 - V_E)) = -24$.*

Proof. • There is a short exact sequence in D_3 :

$$0 \rightarrow \mathcal{O}_{D_3}(-V_E) \rightarrow \mathcal{O}_{D_3}(h_1 - V_E) \rightarrow \mathcal{O}_{h_1}(h_1 - V_E) \rightarrow 0$$

where by \mathcal{O}_{h_1} we mean the structure sheaf of a divisor in $|h_1|$. This implies

$$\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(h_1 - V_E)) = \chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(-V_E)) + \chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{h_1}(h_1 - V_E))$$

and

$$\chi(j_* \mathcal{O}_{D_3}(h_1 - V_E), \mathcal{O}_{\tilde{S}}) = \chi(j_* \mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{S}}) + \chi(j_* \mathcal{O}_{h_1}(h_1 - V_E), \mathcal{O}_{\tilde{S}}).$$

We have seen in Lemma 4.7.15 that $\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(-V_E)) = -42$ and $\chi(j_* \mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{S}}) = 10$. Moreover, we can use the same method as in the proof of Lemma 4.7.13, writing h_1 instead of a divisor in $|h_1|$ by abuse of notation, to find : $\text{Pic}(h_1) = \mathbb{Z}f_h \oplus \mathbb{Z}\delta_h$ with $f_h^2 = 0$, $f_h \cdot \delta_h = 1$ and $\delta_h^2 = 0$. Then, using again the same method as in the proof of Lemma 4.7.13,

$$\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{h_1}(h_1 - V_E)) = \chi(j_* \mathcal{O}_{h_1}(h_1 - V_E), \mathcal{O}_{\tilde{S}}) = j_* h_1(h^2 - 3eh + 2e^2) = 9f_h^2 + 3\delta_h \cdot 3f_h + 2\delta_h^2 = 9$$

since $h \cdot h_1 = 3$ in Σ_2 and thus $h|_{h_1} = 3f_h$. Therefore, $\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(h_1 - V_E)) = -42 + 9 = -33$ and $\chi(j_* \mathcal{O}_{D_3}(h_1 - V_E), \mathcal{O}_{\tilde{S}}) = 10 + 9 = 19$.

- With the same method as above,

$$\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(2h_1 - V_E)) = \chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(h_1 - V_E)) + \chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{h_1}(2h_1 - V_E))$$

and

$$\chi(j_* \mathcal{O}_{D_3}(2h_1 - V_E), \mathcal{O}_{\tilde{S}}) = \chi(j_* \mathcal{O}_{D_3}(h_1 - V_E), \mathcal{O}_{\tilde{S}}) + \chi(j_* \mathcal{O}_{h_1}(2h_1 - V_E), \mathcal{O}_{\tilde{S}}).$$

Then $\chi(\mathcal{O}_{\tilde{S}}, j_* \mathcal{O}_{D_3}(2h_1 - V_E)) = -33 + 9 = -24$ and $\chi(j_* \mathcal{O}_{D_3}(2h_1 - V_E), \mathcal{O}_{\tilde{S}}) = 19 + 9 = 28$.

□

4.7.4 ... $\mathcal{O}_{\overline{D}_2}$

Lemma 4.7.17. *One has $\chi(j_*\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\overline{D}_2}) = 12$ and $\chi(\mathcal{O}_{\overline{D}_2}, j_*\mathcal{O}_{D_3}(-V_E)) = 36$.*

Proof. We can proceed as in the proof of Lemma 4.7.15 : (4.24) twisted by $\mathcal{O}_{\tilde{X}_4}(2e)$ and (1.10) give

$$\chi(\mathcal{O}_{\overline{D}_2}, j_*\mathcal{O}_{D_3}(-V_E)) = \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\tilde{X}_4}(e)) - \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\tilde{X}_4})$$

and

$$\chi(j_*\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\overline{D}_2}) = \chi(\mathcal{O}_{\tilde{X}_4}(e), \mathcal{O}_{\overline{D}_2}) - \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}).$$

- Let us look at $\chi(\mathcal{O}_{\overline{D}_2}, j_*\mathcal{O}_{D_3}(-V_E)) = \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\tilde{X}_4}(e)) - \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\tilde{X}_4})$.
We saw in Lemma 4.4.8 that $\chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\tilde{X}_4}) = -36$.
By Serre duality, $\chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\tilde{X}_4}(e)) = \chi(\mathcal{O}_{\tilde{X}_4}(e), \mathcal{O}_{\overline{D}_2}(-h+e)) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(-h))$ and we saw in Lemma 4.4.6 that $\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(-h)) = 0$.
Therefore, $\chi(\mathcal{O}_{\overline{D}_2}, j_*\mathcal{O}_{D_3}(-V_E)) = 36$.
- Let us look now at $\chi(j_*\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\overline{D}_2}) = \chi(\mathcal{O}_{\tilde{X}_4}(e), \mathcal{O}_{\overline{D}_2}) - \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2})$.
We saw in Lemma 4.4.6 that $\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}) = -12$.
By (1.8), $\chi(\mathcal{O}_{\tilde{X}_4}(e), \mathcal{O}_{\overline{D}_2}) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(\bar{h}-h))$. Finally, using the same reasoning as in the proof of Lemma 4.4.9, one finds :

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\overline{D}_2}(\bar{h}-h)) &= \chi(\mathcal{O}_{\overline{D}_2}, \mathcal{O}_{\overline{D}_2}(-\delta-24f)) = -12 + \frac{1}{2}((\delta+24f)^2 + (\delta+24f)(-2\delta+72f)) \\ &= -12 + \frac{1}{2}(48+48-2.48+72-48) = -12+12=0. \end{aligned}$$

Finally, $\chi(j_*\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\overline{D}_2}) = 12$.

□

Lemma 4.7.18. *Let us define $a_i := j_*E_i \cdot [\overline{D}_2]$ and $b := j_*h_1 \cdot [\overline{D}_2]$ where $[\overline{D}_2]$ is the class of \overline{D}_2 in \tilde{X}_4 . Then :*

- For $1 \leq i \leq 3$, $\chi(j_*\mathcal{O}_{E_i}(E_i - V_E), \mathcal{O}_{\overline{D}_2}) = \chi(\mathcal{O}_{\overline{D}_2}, j_*\mathcal{O}_{E_i}(E_i - V_E)) = x_i$.
- $\chi(j_*\mathcal{O}_{D_3}(h_1 - V_E), \mathcal{O}_{\overline{D}_2}) = 12 + b$ and $\chi(\mathcal{O}_{\overline{D}_2}, j_*\mathcal{O}_{D_3}(h_1 - V_E)) = 36 + b$.
- $\chi(j_*\mathcal{O}_{D_3}(2h_1 - V_E), \mathcal{O}_{\overline{D}_2}) = 12 + 2b$ and $\chi(\mathcal{O}_{\overline{D}_2}, j_*\mathcal{O}_{D_3}(2h_1 - V_E)) = 36 + 2b$.

Proof. For the first formula one can use the same reasoning as in the proof of Lemma 4.7.13, and for the last four formulas one can proceed as in the proof of Lemma 4.7.16, using the results of Lemma 4.7.17. □

To conclude, we would thus need to know a_1, a_2, a_3 and b . However this is not so easy, since we cannot write the class of \overline{D}_2 as a product of two divisors in \tilde{X}_4 , as it was the case with \tilde{S} .

About the a_i , up to assuming that some equations are not satisfied we can find the following Lemma.

The main idea is that $j_*E_i \cdot [\tilde{Z}]$ should be a linear combination of \mathbb{P}^1 in \tilde{Z} , whose intersection with \overline{D} (as a divisor in \tilde{Z}) will give exactly a_i . But the \mathbb{P}^1 of \tilde{Z} either come from \mathbb{P}^1 in Z_3 , in which case their intersection with \overline{D} is 0, or are fibers in \tilde{Z} over Γ_1 , in which case the intersection with \overline{D} is non zero. Finally, for very general X_4 , one can use the geometry and some results from section 3 to show that the intersection $j_*E_i \cdot [\tilde{Z}]$ should have no component in the fibers of \tilde{Z} , hence when intersected with \overline{D} it should give 0.

Lemma 4.7.19. *For X_4 very general, we can assume that $\mu(\mu_2, p_i) \neq 0$ for $1 \leq i \leq 3$. In such a case, $a_1 = a_2 = a_3 = 0$.*

Proof. • Each point in Σ_2 corresponds to a threefold $\bar{U}_2 \oplus \mathbb{C}.w_0$ such that \bar{U}_2 is $\bar{\lambda}$ -isotropic (see 7. of Theorem 1.2.3). In the basis such that $\bar{\lambda}$ can be written as $x_{123} + x_{456}$ it implies that $\bar{U}_2 = \langle u_1, u_2 \rangle$ where $u_i \in A_i$ (see 3. of section 1.1). Furthermore any such $\bar{U}_2 \in \Sigma_2$ must satisfy : $\mu(u_1, u_2) = 0 = \bar{\nu}(u_1, u_2)$ with the notations of Definition 3.4.8.

- By analogy between Σ_2 and S_2 (as detailed in Remark 4.1.4), Remark 4.7 in [20] implies that there is a map $\phi : \Sigma_2 \rightarrow \mathbb{P}(A_1)$ which is the blow up of $\mathbb{P}(A_1)$ in three points. Let p be one of these three points. Then the associated exceptional divisor in Σ_2 is $E_p = \{\bar{U}_2 = \langle p, u_2 \rangle \in \Sigma_2\}$. Since μ induces a non-degenerate pairing between A_1 and A_2 we must then have : $E_p = \langle p, \text{Ker}(\mu(p, \cdot)) \cap A_2 \rangle$. Moreover for each $\langle p, u_2 \rangle \in E_p$ we also must have $\bar{\nu}(p, u_2) = 0$, so $\text{Ker}(\mu(p, \cdot)) \cap A_2 \subset \text{Ker}(\bar{\nu}(p, \cdot)) \cap A_2$. Conversely if a point $p \in A_1$ satisfies $\text{Ker}(\mu(p, \cdot)) \cap A_2 \subset \text{Ker}(\bar{\nu}(p, \cdot)) \cap A_2$ then any $\langle p, u_2 \rangle$ for $u_2 \in \text{Ker}(\mu(p, \cdot)) \cap A_2$ is μ and $\bar{\nu}$ isotropic (and λ and ν' isotropic by construction), hence p (the associated line in $\mathbb{P}(A_1)$) must be one of the three points in which ϕ is blowing up $\mathbb{P}(A_1)$.
- Let us fix $p \in A_1$ one of the points generating a line in which $\mathbb{P}(A_1)$ is blown up to give Σ_2 . Let $\bar{U}_4 = U_{2,A_1} \oplus U_{2,A_2} = \langle u_1, u_2, v_1, v_2 \rangle$ be a point of S_2 . If there is an element $x \in \bar{E}_4$ such that $\sigma \circ \bar{\pi}(x) = \bar{U}_4$ and $\pi(x) \in E_p$ then we must have $p \in U_{2,A_1}$. Using the same argument as in the proof of Lemma 4.8 in [20], one can find that the set of $\bar{U}_4 \in S_2$ such that $p \in U_{2,A_1}$ is a curve in $|\bar{h}_1|$. Let us denote this curve by C_p .
- Let $\bar{U}_4 \in C_p$. Then $f_Z(\bar{U}_4) = \{\langle \bar{U}_4^\perp, v \rangle, v \in \bar{U}_4/\bar{U}_4^\perp\}$. There are two possibilities :
 1. $p \in \bar{U}_4^\perp$. Then any point in $f_Z(\bar{U}_4)$ contains p .
 2. $p \notin \bar{U}_4^\perp$. Then there is exactly one point in $f_Z(\bar{U}_4)$ which contains p , namely $\langle \bar{U}_4^\perp, p \rangle$.

Let now $q \in f_Z(\bar{U}_4)$ and assume that $p \in q$, we have $f_{\bar{E}}(q) = \{U_3 \subset q \oplus \mathbb{C}.w_0, (U_3 \subset q \oplus \mathbb{C}.w_0) \in \tilde{X}_5\}$. If there is a point $v \in q$ such that $v \in \text{Ker}(\mu(p, \cdot)) \cap A_2$ then as we saw above $\langle p, v, w_0 \rangle$ is a point in Σ_2 , thus $\langle p, v, w_0 \rangle \subset \langle q, w_0 \rangle$ is a point in \tilde{X}_5 whose image by π is in E_p . As a result there is a one to one correspondence between such $v \in q \cap \text{Ker}(\mu(p, \cdot)) \cap A_2$ (up to multiplication by a non zero scalar) and points $x \in f_{\bar{E}}(q)$ such that $\pi(x) \in E_p$.

Let us now consider the following possibilities for $\bar{U}_4 \in C_p$:

- If $p \in \bar{U}_4^\perp$ then $\bar{U}_4 \in \bar{C}_1$. Moreover since μ induces a non-degenerate pairing between A_1 and A_2 we have that $\bar{U}_4 \cap A_2 = \text{Ker}(\mu(p, \cdot)) \cap A_2$ and as we did in Definition 3.3.7 we can write $\bar{U}_4^\perp = \langle p, v_1 + u_2 \rangle$ with $U_{2,A_1} = \langle p, v_1 \rangle$ and $u_2 \in A_2$. Hence $f_Z(\bar{U}_4) = \{\langle p, v_1 + u_2, \alpha u_2 + \beta v_2 \rangle\}$ with $U_{2,A_2} = \langle u_2, v_2 \rangle$. For each point $\langle p, v_1 + u_2, \alpha u_2 + \beta v_2 \rangle =: q$ in $f_Z(\bar{U}_4)$ there is exactly one $x \in f_{\bar{E}}(q)$ such that $\pi(x) \in E_p$, namely $\langle p, \alpha u_2 + \beta v_2, w_0 \rangle \subset q$. So there is a copy of E_p isomorphic to $f_Z(\bar{U}_4)$ in \bar{E}_4 . In fact this copy lies in \tilde{Z} because $\pi^{-1}(E_p) \subset D_3 \subset \tilde{X}_4$ and $\tilde{Z} = \bar{E}_4 \cap \tilde{X}_4$. In this case though, since $\bar{U}_4 \in \bar{C}_1$, we must have either $p = \mu_1$ or $\lambda'(p, \mu_1, \text{Ker}(\mu(p, \cdot)) \cap A_2) = 0$.
- If $p \notin \bar{U}_4^\perp$ but $\bar{U}_4 \in \bar{C}_1$ then we can write $\bar{U}_4 = \langle p, u_1, u_2, v_2 \rangle$ with $u_1 \in \bar{U}_4^\perp \cap A_1$, u_2 and v_2 in A_2 and $\mu(p, u_2) = 0 \neq \mu(p, v_2)$. We can also write $\bar{U}_4^\perp = \langle u_1, \alpha p + u_2 \rangle$, hence there is a unique point q in $f_Z(\bar{U}_4)$ containing p , namely $\langle u_1, \alpha p + u_2, u_2 \rangle$. There is exactly one point x in $f_{\bar{E}}(q)$ such that $\pi(x) \in E_p$, namely $\langle p, u_2 \rangle$.
- If $p \notin \bar{U}_4^\perp$ and $\bar{U}_4 \in \bar{C}_2$ we can write $\bar{U}_4 = \langle p, w, u_2, v_2 \rangle$ with $w \in A_1$ and $u_2, v_2 \in A_2$ such that $\bar{U}_4^\perp = \langle u_2, \alpha p + \beta w + v_2 \rangle$. There are two possibilities :
 1. $\beta = 0$ (and thus $\alpha \neq 0$). Then $f_Z(\bar{U}_4) = \{\langle u_2, \alpha p + v_2, \gamma p + \eta w \rangle\}$ and $\langle u_2, v_2, p \rangle$ is the only point in $f_Z(\bar{U}_4)$ containing p . Moreover, in this case we know from Lemma 3.4.2 that $\mu(u_2, p) = \mu(v_2, p) = 0$. This implies that

$$f_{\bar{E}}(\langle u_2, v_2, p \rangle) = \{\langle \alpha' u_2 + \beta' v_2, p, w_0 \rangle \subset \langle u_2, v_2, p, w_0 \rangle, (\alpha', \beta') \in \mathbb{C}^2\} \cong E_p.$$

Indeed $E_p = \{ \langle p, \alpha' u_2 + \beta' v_2 \rangle, (\alpha', \beta') \in \mathbb{C}^2 \}$, hence

$$\{ \langle \alpha' u_2 + \beta' v_2, p, w_0 \rangle \subset \langle u_2, v_2, p, w_0 \rangle, (\alpha', \beta') \in \mathbb{C}^2 \} \subset f_{\bar{E}}(\langle u_2, v_2, p \rangle),$$

and since the fibers in \bar{E}_4 are \mathbb{P}^1 , this inclusion is in fact an equality. As in the case $p \in \bar{U}_4$ we also have that $f_{\bar{E}}(\langle u_2, v_2, p \rangle)$ is in fact in \tilde{Z} , that is to say a fiber in \bar{D}_2 .

This can happen only if $\mu_2 \in \text{Ker}(\mu(p, \cdot))$, by definition of \bar{C}_2 . By the reasoning in the proof of Lemma 3.1.10, if $\mu_2 \in \text{Ker}(\mu(p, \cdot))$ then there is a unique $\bar{U}_4 \in \bar{C}_2$ such that \bar{U}_4 is of the form $\langle p, \text{Ker}(\mu(p, \cdot)) \cap A_2, w \rangle$.

2. $\beta \neq 0$. Then $\mu(p, v_2) \neq 0$ and the only point in $f_Z(\bar{U}_4)$ containing p is $\langle u_2, \beta w + v_2, p \rangle$. Furthermore $\langle u_2, p, w_0 \rangle \subset \langle u_2, \beta w + v_2, p \rangle$ is the only point in $f_{\bar{E}}(\langle u_2, \beta w + v_2, p \rangle)$ whose image by π is in E_p .
- If $p \notin \bar{U}_4^\perp$ and $U_4 \notin (\bar{C}_1 \cup \bar{C}_2)$. Then we can write $\bar{U}_4 = \langle p, w, u_2, v_2 \rangle$ with $w \in A_1$ and $u_2, v_2 \in A_2$ such that $\bar{U}_4^\perp = \langle u_2 + p + aw, v_2 + bp + w \rangle$. Then $\mu(p, \alpha u_2 + \beta v_2) = -\alpha a \mu(w, u_2) - \beta \mu(p, w)$.

If $a = 0$ the only line in $\langle u_2, v_2 \rangle$ being orthogonal to p is generated by u_2 . Indeed we must have $\mu(p, w) \neq 0$ since $\mu_1 \notin \langle p, w \rangle$.

If $a \neq 0$ then $\mu(u_2, w) \neq 0$ otherwise $\mu(u_2 + p + aw, u_2) = 0$ would imply $\mu(u_2, p) = 0$ and then $\mu(v_2 + bp + w, u_2) = 0$ would give $\mu(u_2, v_2) = 0$, that is to say $u_2 \in \bar{U}_4^\perp$.

Thus in this case $-\alpha a \mu(w, u_2) - \beta \mu(p, w) = 0$ if and only if $-\alpha = \frac{\beta b \mu(p, w)}{a \mu(w, u_2)}$, therefore the only line in $\langle u_2, v_2 \rangle$ orthogonal to p is spanned by $-\frac{\mu(p, w)}{a \mu(w, u_2)} u_2 + v_2$. Moreover

$$f_Z(\bar{U}_4) = \{ \langle u_2 + p + aw, v_2 + bp + w, \alpha p + \beta w \rangle \}$$

hence the only point in $f_Z(\bar{U}_4)$ containing p is $q := \langle u_2 + aw, v_2 + w, p \rangle = \langle u_2 - av_2, v_2 + w, p \rangle$. We saw in the proof of Lemma 3.3.4 that there is then only one point in $E_4 \cap f_{\bar{E}}(\langle u_2 - av_2, v_2 + w, p \rangle)$: it is $\langle u_2 - av_2, p, w_0 \rangle \subset \langle u_2 - av_2, v_2 + w, p, w_0 \rangle$. This point is mapped to E_p by π if and only if $u_2 - av_2 = -\frac{\mu(p, w)}{a \mu(w, u_2)} u_2 + v_2$ that is to say if and only if $a \neq 0$ and

$$-\frac{1}{a} = -\frac{\mu(p, w)}{a \mu(w, u_2)} \Leftrightarrow \mu(w, u_2) = \mu(p, w) \Leftrightarrow \mu(w, u_2 + p) = 0.$$

The last equality holds because $u_2 + p + aw \in \bar{U}_4^\perp$ and is thus orthogonal to w .

Finally if $a = 0$ then the only point in $f_Z(\bar{U}_4)$ containing p is $\langle u_2, p, v_2 + bp + w \rangle$ and there is exactly one $x \in f_{\bar{E}}(\langle u_2, p, v_2 + bp + w \rangle)$ such that $\pi(x) \in E_p$, namely $\langle u_2, p \rangle \subset \langle u_2, p, v_2 + bp + w \rangle$.

As before, if $x \in f_{\bar{E}}(\langle u_2, p, v_2 + bp + w \rangle)$ is such that $\pi(x) \in E_p$, then $x \in \tilde{Z}$.

• So finally :

1. If $p = \mu_1$ or $\lambda'(p, \mu_1, \text{Ker}(\mu(p, \cdot)) \cap A_2) = 0$, then there is a line in \bar{E}_4 which is mapped isomorphically by $\bar{\pi}$ to a fiber in Z_3 and by π to E_p . Moreover as we saw above this line should also be in \tilde{Z} , with class \bar{h}_1^2 (since $\bar{h}_1^2 = 1$ in S_2).
2. If $\mu(p, \mu_2) = 0$ then there is a fiber in \tilde{Z} which is mapped isomorphically to E_p by π . It has class $f\bar{D}$ with the notations of Lemma 4.4.1.
3. In any case there is a line in \tilde{Z} which is mapped isomorphically by $\sigma \circ \bar{\pi}$ to a line in $|\bar{h}_1|$ and by π to E_p . Its class in \tilde{Z} is $\bar{h}_1 V_Z$.

• On the other hand, we have $E_p \cdot \tilde{Z} = E_p \cdot \bar{e} = E_p(h - 2e) = h|_{E_p} - 2e|_{E_p} = f_p + 2\delta_p$ with the same reasoning as in the proof of Lemma 4.7.13. Hence $E_p|_{\tilde{Z}}$ can be written as a linear combination of \mathbb{P}^1 , and the intersection with this curve and \bar{D} in \tilde{Z} would give $E_p \cdot \bar{D}$.

Moreover f_p is a fiber of π and thus by definition of π and $\bar{\pi}$ it must be mapped isomorphically to $\bar{\pi}(f_p)$.

- The \mathbb{P}^1 in \tilde{Z} are :
 - the \mathbb{P}^1 in $S_2 : \bar{h}_1 V_Z, \tilde{E}_i V_Z$ for $1 \leq i \leq 3$.
 - a fiber in $Z_3 : \bar{h}_1^2$.
 - a fiber in $\bar{D}_2 : f\bar{D}$.
- By Lemma 5.0.6 we have : $V_Z \bar{D} = 0$ and $\bar{h}_1^2 \bar{D} = 12^2 f^2 = 0$. As a result, if we have $E_p|_{\tilde{Z}} = \alpha \bar{h}_1 V_Z + \sum_{i=1}^3 \beta_i \tilde{E}_i V_Z + \gamma \bar{h}_1^2 + \eta f \bar{D}$ then $E_p \bar{D} = \eta f \bar{D}^2 = -\eta f \delta = -\eta$. We saw above that $\eta \neq 0$ if and only if $\mu(p, \mu_2) = 0$, hence if $\mu(p, \mu_2) \neq 0$ we have $E_p \bar{D} = 0$. Finally, the analogous of Remark 4.7 in [20] applied to Σ_2 tells us that we can compute the three points p_1, p_2 and p_3 in which $\mathbb{P}(A_1)$ is blown-up to give Σ_2 , in terms of the basis of \bar{W} mentioned in 3. of section 1.1. Since we know μ_2 from the proof of Lemma 3.1.8, the three equations $\mu(p_i, \mu_2) = 0$ for $1 \leq i \leq 3$ can be written in terms of the x_i, M_i and K_i defined in 3. and 4. of section 1.1. For X_4 very general we can thus assume that these three equations are not satisfied, which implies then that $E_i \bar{D} = 0$ for any $i \in \{1, 2, 3\}$. □

Computing b is more difficult : as for the a_i , the intersection $j_* h_1 \cdot [\tilde{Z}]$ should be a linear combination of \mathbb{P}^1 , but this time there is no reason for this linear combination to have no component in the fibers of \tilde{Z} . Yet one can make the following remarks :

- Since by definition h_1 is the pull back of the hyperplane class in $\mathbb{P}(A_1)$ we have that a line in $|h_1|$ in Σ_2 is of the form : $\{ \langle u, v, w_0 \rangle, u \in H, v \in A_2 \}$ where $H \in |h_1|$ in $\mathbb{P}(A_1)$. Let us fix one of these H and write $H = \langle u, w \rangle$.
- As in Lemma 4.7.19, we have $h_1 \cdot \tilde{Z} = h_1 \cdot (h - 2e) = 3f_h + 2\delta_h$ and thus $h_1|_{\tilde{Z}}$ is a combination of \mathbb{P}^1 . Moreover, still as in Lemma 4.7.19, $h_1 \cdot \bar{D} \neq 0$ only if $h_1|_{\tilde{Z}}$ can be written as $\eta f \bar{D} + P$ with $\eta \neq 0$ and P the pull-back of a combination of \mathbb{P}^1 in Z_3 .
- By definition of $\bar{\pi}$, a fiber in \bar{D}_2 is mapped isomorphically onto its image by π , hence if there is such a fiber F in $h_1|_{\tilde{Z}}$ we must have that $F = f_{\bar{E}}(\langle H, z \rangle)$ with $z \in A_2$ and $\text{Ker}(\mu(z, \cdot)) \cap A_1 = H$.

4.8 The line l_h and a skyscraper sheaf

In this subsection we will consider the structure sheaf of a line l_h with class $E_1 \cdot V_E \cdot e$ as explained in section 2.1.1, and a skyscraper sheaf \mathcal{O}_p . We will compute Euler characteristics between these two sheaves and the other elements in \mathcal{F}_1 , using Hirzebruch-Riemann-Roch.

Lemma 4.8.1. *One has $(\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{l_h}), \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{l_h}), \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{l_h})) = (1, 0, -1)$ and $(\chi(\mathcal{O}_{l_h}, \mathcal{O}_{\tilde{X}_4}), \chi(\mathcal{O}_{l_h}, \mathcal{O}_{\tilde{X}_4}(h)), \chi(\mathcal{O}_{l_h}, \mathcal{O}_{\tilde{X}_4}(2h))) = (0, -1, -2)$. For \mathcal{F} any other of the sheaves in the family \mathcal{F}_1 mentioned in section 2.1.1, one has $\chi(\mathcal{F}, \mathcal{O}_{l_h}) = \chi(\mathcal{O}_{l_h}, \mathcal{F}) = 0$.*

Proof. Let l_h be a line in \tilde{X}_4 with class $E_1 \cdot V_E \cdot e$. Then its Chern character is $(0, 0, 0, E_1 \cdot V_E \cdot e, x)$. Moreover by Lemma 4.4.7 and the definition of the Todd class (see page 432 of [12]) we have that $\text{td}(\tilde{X}_4) = (1, \frac{h-e}{2}, *, *, *)$. Hence by Hirzebruch-Riemann-Roch for any sheaf \mathcal{F} whose support has dimension at most 2 one has $\chi(\mathcal{F}, \mathcal{O}_{l_h}) = \chi(\mathcal{O}_{l_h}, \mathcal{F}) = 0$.

Moreover, since l_h is a \mathbb{P}^1 (because E_1 is a \mathbb{P}^1 in Σ_2), its genus is $g(l_h) = 0$, hence if we denote by f the embedding $f : l_h \hookrightarrow \tilde{X}_4$ we get $\chi(\mathcal{O}_{\tilde{X}_4}, f_* \mathcal{O}_{l_h}) = \chi(f^* \mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{l_h}) = \chi(\mathcal{O}_{l_h}, \mathcal{O}_{l_h}) = 1 - g(l_h) = 1$. By Hirzebruch-Riemann-Roch, we then get :

$$\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{l_h}) = 1 = \int_{\tilde{X}_4} (0, 0, 0, E_1 \cdot V_E \cdot e, x) \cdot (1, \frac{h-e}{2}, *, *, *) = \frac{(h-e)E_1 \cdot V_E \cdot e}{2} + x.$$

Finally, since e is the class of D_3 in \tilde{X}_4 we have that $(h-e)E_1.V_E.e$ is equal to $(h-e)E_1.V_E = (h_1+h_2+V_E).E_1.V_E$ in D_3 . With the same reasoning as in the proof of Lemma 4.7.13 we then get that $(h_1+h_2+V_E).E_1.V_E = 1$ and thus $x = \frac{1}{2}$. We can now use Hirzebruch-Riemann-Roch to compute the above-mentioned Euler characteristics. Similarly, one can use the short exact sequence $0 \rightarrow \mathcal{O}_{\tilde{X}_4}(-2e) \rightarrow \mathcal{O}_{\tilde{X}_4}(-e) \rightarrow \mathcal{O}_{D_3}(V_E) \rightarrow 0$ to find that $\text{ch}(\mathcal{O}_{D_3}(V_E)) = (0, e, *, *, *)$. Hence by Hirzebruch-Riemann-Roch and (1.9):

$$\chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{l_h}) = -e.V_E.E_1.e = -e(E_1.(-V_E^2)) = 0.$$

With a similar reasoning to the one in the proof of Lemma 4.6.1 one can then find that

$$\chi(\mathcal{O}_{l_h}, \mathcal{O}_{D_3}(V_E)) = \chi(\mathcal{O}_{D_3}(V_E), \mathcal{O}_{l_h}) = -e.V_E.E_1.e = 0.$$

The exact same method then yields $\chi(\mathcal{O}_{l_h}, \mathcal{F}) = \chi(\mathcal{F}, \mathcal{O}_{l_h}) = 0$ for any $\mathcal{F} \in \{\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{D_3}(h_1-V_E), \mathcal{O}_{D_3}(2h_1-V_E)\}$. \square

Lemma 4.8.2. *For $0 \leq i \leq 2$ we have $\chi(\mathcal{O}_p, \mathcal{O}_{\tilde{X}_4}(ih)) = \chi(\mathcal{O}_{\tilde{X}_4}(ih), \mathcal{O}_p) = 1$. For any other sheaf \mathcal{F} in the family \mathcal{F}_1 mentioned in section 2.1.1, one has $\chi(\mathcal{F}, \mathcal{O}_p) = \chi(\mathcal{O}_p, \mathcal{F}) = 0$.*

Proof. Since $\text{ch}(\mathcal{O}_p) = (0, 0, 0, 0, 1)$ by Hirzebruch-Riemann-Roch for any sheaf \mathcal{F} one has $\chi(\mathcal{F}, \mathcal{O}_p) = \chi(\mathcal{O}_p, \mathcal{F}) = \text{rk}(\mathcal{F})$. \square

Chapter 5

Computations of Euler characteristics between elements of \mathcal{F}_2

In this section, we compute Euler characteristics between elements of the family

$$\mathcal{F}_2 = \{\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{D_3}, \mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{h}_1}, \mathcal{O}_{\tilde{E}_1}, \mathcal{O}_{\tilde{E}_2}, \mathcal{O}_{\tilde{E}_3}, \mathcal{O}_{V_Z}, \mathcal{O}_{\tilde{D}_2}, \mathcal{O}_{l_h}, \mathcal{O}_{l_D}, \mathcal{O}_p\},$$

defined in subsection 2.2. Some of these Euler characteristics have already been computed in section 4, for those who haven't, we use similar reasoning as in section 4.

All these results are gathered in Proposition 2.2.1.

We already know from Proposition 4.2.15 and Lemma 4.2.1 that

$$\begin{pmatrix} \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{X}_4}) & \cdots & \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{X}_4}(2h)) \\ \vdots & \vdots & \vdots \\ \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{\tilde{X}_4}) & \cdots & \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{\tilde{X}_4}(2h)) \end{pmatrix} = \begin{pmatrix} 1 & 20 & 124 \\ 1 & 1 & 20 \\ 20 & 1 & 1 \end{pmatrix}.$$

Lemma 5.0.1. *Let $i \in \mathbb{Z}$ then $\chi(\mathcal{O}_{\tilde{X}_4}(ih), j_*\mathcal{O}_{D_3}) = \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-ih))$ and $\chi(j_*\mathcal{O}_{D_3}, \mathcal{O}_{\tilde{X}_4}(ih)) = 0$.*

Proof. With the notations of section 2.1.1, one has

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{X}_4}(ih), j_*\mathcal{O}_{D_3}) &= \chi(j^*\mathcal{O}_{\tilde{X}_4}(ih), \mathcal{O}_{D_3}) \\ &= \chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(-ih)) \\ &= \chi(p^*\mathcal{O}_{\Sigma_2}, p^*\mathcal{O}_{\Sigma_2}(-ih)) \\ &= \chi(\mathcal{O}_{\Sigma_2}, p_*p^*\mathcal{O}_{\Sigma_2}(-ih)) \\ &= \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-ih)) \end{aligned}$$

since $p_*\mathcal{O}_{D_3} = \mathcal{O}_{\Sigma_2}$ (by Corollary 4.5.5).

By Serre duality and Lemma 4.4.7, one also has

$$\begin{aligned} \chi(j_*\mathcal{O}_{D_3}, \mathcal{O}_{\tilde{X}_4}(ih)) &= \chi(\mathcal{O}_{\tilde{X}_4}(ih), j_*\mathcal{O}_{D_3}(-h+e)) \\ &= \chi(\mathcal{O}_{\tilde{X}_4}, j_*\mathcal{O}_{D_3}(-(i+1)h+e)) \\ &= \chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(-(i+1)h+e)) \\ &= \chi(p^*\mathcal{O}_{\Sigma_2}, \mathcal{O}_{D_3}(-(i+1)h+e)) \\ &= \chi(\mathcal{O}_{\Sigma_2}, p_*(p^*\mathcal{O}_{\Sigma_2}(-(i+1)h) \otimes \mathcal{O}_{D_3}(-V_E))) \\ &= \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-(i+1)h) \otimes p_*\mathcal{O}_{D_3}(-V_E)) \end{aligned}$$

where we used (1.10) and the projection formula for the last equalities. By the Remark page 328 of [8], we have $p_*\mathcal{O}_{D_3}(-V_E) = 0$, hence $\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-(i+1)h) \otimes p_*\mathcal{O}_{D_3}(-V_E)) = 0$. \square

Corollary 5.0.2. *One has $(\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}), \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{D_3}), \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{D_3})) = (1, 1, 7)$ and $(\chi(\mathcal{O}_{D_3}, \mathcal{O}_{\tilde{X}_4}), \chi(\mathcal{O}_{D_3}, \mathcal{O}_{\tilde{X}_4}(h)), \chi(\mathcal{O}_{D_3}, \mathcal{O}_{\tilde{X}_4}(2h))) = (0, 0, 0)$.*

Proof. We can use Lemma 4.1.3 and Hirzebruch-Riemann-Roch to compute the $\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}(-ih))$ for $0 \leq i \leq 2$. \square

Remark 5.0.3. *The non zero Euler characteristics above had already been computed in Lemma 4.3.5.*

As we saw in Lemma 4.3.8, we have $\chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}) = \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{D_3}(-V_E)) = 1$. Moreover by Lemma 4.7.6, we have $(\chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{X}_4}), \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{X}_4}(h)), \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{X}_4}(2h))) = (-1, -7, -19)$ and by Corollary 2.1.3, we get $(\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}(-V_E)), \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{D_3}(-V_E)), \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{D_3}(-V_E))) = (0, 0, 0)$.

Lemma 5.0.4. *One has $\chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(-V_E)) = 1$ and $\chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{D_3}) = -1$.*

Proof. We can use the short exact sequence $0 \rightarrow \mathcal{O}_{\tilde{X}_4}(-e) \rightarrow \mathcal{O}_{\tilde{X}_4} \rightarrow \mathcal{O}_{D_3} \rightarrow 0$ in \tilde{X}_4 to find :

- $\chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(-V_E)) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}(-V_E)) - \chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{D_3}(-V_E))$.
We have just seen that $\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}(-V_E)) = 0$ and by (1.10) we have

$$\chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{D_3}(-V_E)) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}(-2V_E)).$$

By the Remark below Proposition 9.3 in [8], we can find

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{X}_4}, j_*\mathcal{O}_{D_3}(-2V_E)) &= \chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(-2V_E)) \\ &= \chi(p^*\mathcal{O}_{\Sigma_2}, \mathcal{O}_{D_3}(-2V_E)) \\ &= -\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}) = -1. \end{aligned}$$

So finally $\chi(\mathcal{O}_{D_3}, \mathcal{O}_{D_3}(-V_E)) = 0 - (-1) = 1$.

- $\chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{D_3}) = \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{X}_4}) - \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{X}_4}(-e))$.
We have seen above that $\chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{X}_4}) = -1$ and by Serre duality and (1.10) we find

$$\begin{aligned} \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{X}_4}(-e)) &= \chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{D_3}(-V_E - h + e)) \\ &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}(-V_E - h + 2e)) \\ &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}(-3V_E - h)). \end{aligned}$$

Again using the Remark below Proposition 9.3 in [8], one finds

$$\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{D_3}(-3V_E - h)) = \chi(p^*\mathcal{O}_{\Sigma_2}, \mathcal{O}_{D_3}(-3V_E)) \otimes p^*\mathcal{O}_{\Sigma_2}(-h) = -\chi(\mathcal{O}_{\Sigma_2}, \mathcal{V}_F|_{\Sigma_2}(-h)).$$

With Hirzebruch-Riemann-Roch, Lemma 4.1.3 and Lemma 4.7.7 one gets :

$$\chi(\mathcal{O}_{\Sigma_2}, \mathcal{V}_F|_{\Sigma_2}(-h)) = \int_{\Sigma_2} (2, 0, -2) \cdot (1, -h, 3) \cdot (1, \frac{h}{2}, 1) = \int_{\Sigma_2} (2, -2h, 4) \cdot (1, \frac{h}{2}, 1) = 6 - 6 = 0.$$

Therefore, $\chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{D_3}) = -1$. \square

Before going further, we need the following technical result and notation :

Definition 5.0.5. *Let us denote by \bar{D} the class of \bar{D}_2 in \tilde{Z} . If D' is a divisor in Z_3 , let us still denote by D' its pull-back to \tilde{Z} .*

Lemma 5.0.6. *The Picard group of \tilde{Z} is generated by $V_Z, \bar{h}_1, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3$ and \bar{D} . Moreover, for $1 \leq i \leq 3$ we have the following equalities :*

$$\begin{aligned} \bar{D}|_{\bar{D}_2} &= -\delta & V_Z|_{\bar{D}_2} &= 0 & \bar{h}_1|_{\bar{D}_2} &= \bar{h}_2|_{\bar{D}_2} = 12f & \tilde{E}_i|_{\bar{D}_2} &= 4f \\ \bar{D}^3 &= 48 & \bar{D}^2\bar{h}_1 &= \bar{D}^2\bar{h}_2 = -12 & \bar{D}^2\tilde{E}_i &= -4. \end{aligned}$$

Proof. The first claim follows from the fact that $\text{Pic}(Z_3) = \langle V_Z, \bar{h}_1, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3 \rangle$ (see the proof of Proposition 4.5.3) together with Definition 5.0.5 and the fact that \tilde{Z} is the blow-up of Z_3 in Γ_1 (see Lemma 3.2.6). The exceptional divisor of this blow-up is then $\bar{\pi}^{-1}(\Gamma_1) = \bar{D}_2$.

We have already seen the first equality in the proof of Lemma 4.6.2, and in Z_3 the following holds :

$$V_Z.[\Gamma_1] = 4V_Z^2(\bar{h}_1 + \bar{h}_2) = 0 \quad \bar{h}_1.[\Gamma_1] = 4V_Z\bar{h}_1(\bar{h}_1 + \bar{h}_2) = 3.4 = 12 \quad \tilde{E}_i.[\Gamma_1] = 4V_Z\tilde{E}_i(\bar{h}_1 + \bar{h}_2) = 4$$

which implies the rest of the first line of equalities. Finally the second line can be deduced from the first line and Lemma 4.4.1. \square

Lemma 5.0.7. *The Todd class of \tilde{Z} is*

$$\text{td}(\tilde{Z}) = (1, V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3) - \frac{1}{2}\bar{D}, y_2, 1) = (1, V_Z + \frac{1}{2}(\bar{h}_1 + \bar{h}_2) - \frac{1}{2}\bar{D}, y_2, 1)$$

where $\bar{h}_1.y_2 = \frac{3}{2}$, $\tilde{E}_i.y_2 = \frac{1}{2}$, $V_Z.y_2 = \frac{1}{3}$ and $\bar{D}.y_2 = -38$.

Proof. By Lemma 4.5.3, we have

$$K_Z = -2V_Z - 3\bar{h}_1 + \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 = -2V_Z - \bar{h}_1 - \bar{h}_2.$$

Thus, since \tilde{Z} is the blow up of Z_3 in Γ_1 , we get

$$K_{\tilde{Z}} = -2V_Z - 3\bar{h}_1 + \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \bar{D} = -2V_Z - \bar{h}_1 - \bar{h}_2 + \bar{D}.$$

Using now the formula for the Todd class page 432 of [12] and the fact that $c_1(\mathcal{T}_{\tilde{Z}}) = -K_{\tilde{Z}}$, one finds

$$\text{td}(\tilde{Z}) = (1, V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3) - \frac{1}{2}\bar{D}, y_2, x_3) = (1, V_Z + \frac{1}{2}(\bar{h}_1 + \bar{h}_2) - \frac{1}{2}\bar{D}, y_2, x_3).$$

Moreover by Hirzebruch-Riemann-Roch $\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}) = \int_{\tilde{Z}}(1, 0, 0, 0).\text{td}(\tilde{Z}) = x_3$ and by Corollary 4.5.5 we have $\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}) = \chi(\bar{\pi}^*\mathcal{O}_{Z_3}, \mathcal{O}_{\tilde{Z}}) = \chi(\mathcal{O}_{Z_3}, \bar{\pi}_*\mathcal{O}_{\tilde{Z}}) = \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}) = 1$ as we saw in the proof of Lemma 4.5.6. So $x_3 = 1$.

Let now $K \in \{\bar{h}_1, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, V_Z\}$. On one hand, we have $\text{ch}(\mathcal{O}_{\tilde{Z}}(K)) = \text{ch}(\mathcal{O}_{Z_3}(K)) = (1, K, \frac{K^2}{2}, \frac{K^3}{6})$ so $\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(K)) = \int_{\tilde{Z}}(1, K, \frac{K^2}{2}, \frac{K^3}{6}).\text{td}(\tilde{Z})$. Moreover, $\text{td}(\tilde{Z}) - \text{td}(Z_3) = (0, -\frac{1}{2}\bar{D}, y_2 - x_2, 0)$ and from Lemma 5.0.6 we know that $\bar{h}_1|_{\bar{D}_2} = 12f$, $\tilde{E}_i|_{\bar{D}_2} = 4f$ and $V_Z|_{\bar{D}_2} = 0$. Thus :

$$\begin{aligned} \bar{h}_1^2\bar{D} &= (\bar{h}_1|_{\bar{D}_2})^2 = 12^2f^2 = 0 \\ \tilde{E}_i^2\bar{D} &= (\tilde{E}_i|_{\bar{D}_2})^2 = 16f^2 = 0 \\ V_Z^2\bar{D} &= 0. \end{aligned}$$

As a result,

$$\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(K)) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(K)) = \int_{\tilde{Z}}(1, K, \frac{K^2}{2}, \frac{K^3}{6}).\text{td}(\tilde{Z}) - \int_{Z_3}(1, K, \frac{K^2}{2}, \frac{K^3}{6}).\text{td}(Z_3) = K.y_2 - K.x_2.$$

On the other hand, we have

$$\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(K)) = \chi(\bar{\pi}^*\mathcal{O}_{Z_3}, \bar{\pi}^*\mathcal{O}_{Z_3}(K)) = \chi(\mathcal{O}_{Z_3}, \bar{\pi}_*\bar{\pi}^*\mathcal{O}_{Z_3}(K)) = \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(K))$$

by the projection formula and Corollary 4.5.5. Hence, $K.y_2 - Kx_2 = 0$ and so from Lemma 4.5.6, Lemma 4.7.2 and Remark 4.7.3 we get : $\bar{h}_1.y_2 = \frac{3}{2}$, $\bar{E}_i.y_2 = \frac{1}{2}$ and $V_Z.y_2 = \frac{1}{3}$.

Finally, we saw in the proof of Lemma 4.6.2 that $h|_{\tilde{Z}} = 3V_Z + 2(\bar{h}_1 + \bar{h}_2) - \bar{D}$ and that $\bar{D}^2 = -\delta$ with the notations of Lemma 4.4.1. In particular this implies

$$\text{ch}(\mathcal{O}_{\tilde{Z}}(h)) = (1, 3V_Z + 2(\bar{h}_1 + \bar{h}_2) - \bar{D}, \frac{3}{2}\bar{h}_1\bar{h}_2 + 6V_Z(\bar{h}_1 + \bar{h}_2) + \frac{1}{2}\bar{D}^2 - 2\bar{D}(\bar{h}_1 + \bar{h}_2), -5).$$

Hirzebruch-Riemann-Roch then yields :

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(h)) &= \int_{\tilde{Z}} (1, 3V_Z + 2(\bar{h}_1 + \bar{h}_2) - \bar{D}, \frac{3}{2}\bar{h}_1\bar{h}_2 + 6V_Z(\bar{h}_1 + \bar{h}_2) + \frac{1}{2}\bar{D}^2 - 2\bar{D}(\bar{h}_1 + \bar{h}_2), -5) \\ &\quad \cdot (1, V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\bar{E}_1 + \bar{E}_2 + \bar{E}_3) - \frac{1}{2}\bar{D}, y_2, 1) \\ &= 1 + 1 + 6 - \bar{D}.y_2 + 3 + 27 - 9 - 9 + 3 - 12 - 24 - 5 = -18 - \bar{D}.y_2. \end{aligned}$$

On the other hand, let us denote by g the embedding $g : \tilde{Z} \hookrightarrow \bar{E}_4$. Then

$$\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(h)) = \chi(g^*\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\tilde{Z}}(h)) = \chi(\mathcal{O}_{\bar{E}_4}(-h), g_*\mathcal{O}_{\tilde{Z}}).$$

Using the short exact sequence $0 \rightarrow \mathcal{O}_{\bar{E}_4}(-V_{\bar{E}}) \rightarrow \mathcal{O}_{\bar{E}_4} \rightarrow g_*\mathcal{O}_{\tilde{Z}} \rightarrow 0$ in \bar{E}_4 (which comes from the construction of \tilde{Z} in Proposition 3.2.6) one finds :

$$\chi(\mathcal{O}_{\bar{E}_4}(-h), g_*\mathcal{O}_{\tilde{Z}}) = \chi(\mathcal{O}_{\bar{E}_4}(-h), \mathcal{O}_{\bar{E}_4}) - \chi(\mathcal{O}_{\bar{E}_4}(-h), \mathcal{O}_{\bar{E}_4}(-V_{\bar{E}})).$$

By (1.12) we then get $\chi(\mathcal{O}_{\bar{E}_4}(-h), g_*\mathcal{O}_{\tilde{Z}}) = \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(V_{\bar{E}})) - \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}) = \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3})$. We saw in the proof of Lemma 4.5.6 that $\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}) = 1$ and Hirzebruch-Riemann-Roch together with Lemma 3.2.3 yields :

$$\begin{aligned} \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee) &= \int_{Z_3} (2, 3\bar{h} - \bar{h}_1 - \bar{h}_2, \frac{3}{2}\bar{h}_1\bar{h}_2 + 2V_Z(\bar{h}_1 + \bar{h}_2), 3) \cdot (1, V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\bar{E}_1 + \bar{E}_2 + \bar{E}_3), x_2, 1) \\ &= 2 + 10 - 3 + 3 + 9 - 3 + 3 = 21. \end{aligned}$$

So $\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(h)) = 21 - 1 = 20$, and thus $20 = -18 - \bar{D}.y_2$, that is to say $\bar{D}.y_2 = -38$. \square

Remark 5.0.8. One could also have computed $\bar{D}.y_2$ in Lemma 5.0.7 the following way.

We saw in Lemma 4.4.4 that $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}) = -12$, and if we denote by f the embedding $f : \bar{D}_2 \hookrightarrow \tilde{Z}$ we have $\chi(\mathcal{O}_{\tilde{Z}}, f_*\mathcal{O}_{\bar{D}_2}) = \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}) = -12$. On the other hand there is a short exact sequence in \tilde{Z} :

$$0 \rightarrow \mathcal{O}_{\tilde{Z}}(-\bar{D}) \rightarrow \mathcal{O}_{\tilde{Z}} \rightarrow f_*\mathcal{O}_{\bar{D}_2} \rightarrow 0$$

therefore $\chi(\mathcal{O}_{\tilde{Z}}, f_*\mathcal{O}_{\bar{D}_2}) = \chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}) - \chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(-\bar{D}))$. Hence, $\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(-\bar{D})) = 1 - (-12) = 13$. Moreover, $\text{ch}(\mathcal{O}_{\tilde{Z}}(-\bar{D})) = (1, -\bar{D}, \frac{\bar{D}^2}{2}, -8)$ since $\bar{D}^3 = (\bar{D}|_{\bar{D}_2})^2 = (-\delta)^2 = 48$. So by Hirzebruch-Riemann-Roch :

$$\begin{aligned} 13 &= \int_{\tilde{Z}} (1, -\bar{D}, \frac{\bar{D}^2}{2}, -8) \cdot (1, V_Z + \frac{3}{2}\bar{h}_1 - \frac{1}{2}(\bar{E}_1 + \bar{E}_2 + \bar{E}_3) - \frac{1}{2}\bar{D}, y_2, 1) \\ &= -8 - 9 + 3 - 12 - \bar{D}.y_2 + 1 = -25 - \bar{D}.y_2 \end{aligned}$$

and thus $\bar{D}.y_2 = -25 - 13 = -38$.

We can now use Hirzebruch-Riemann-Roch to compute some of the Euler characteristics we are interested in.

Lemma 5.0.9. One has
$$\begin{pmatrix} \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\bar{h}_1}) & \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\tilde{E}_i}) & \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{V_Z}) \\ \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\bar{h}_1}) & \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\tilde{E}_i}) & \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{V_Z}) \\ \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{\bar{h}_1}) & \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{\tilde{E}_i}) & \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{V_Z}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 10 & 2 & 1 \\ 43 & 11 & 7 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} \chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{\tilde{X}_4}) & \chi(\mathcal{O}_{\tilde{E}_i}, \mathcal{O}_{\tilde{X}_4}) & \chi(\mathcal{O}_{V_Z}, \mathcal{O}_{\tilde{X}_4}) \\ \chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{\tilde{X}_4}(h)) & \chi(\mathcal{O}_{\tilde{E}_i}, \mathcal{O}_{\tilde{X}_4}(h)) & \chi(\mathcal{O}_{V_Z}, \mathcal{O}_{\tilde{X}_4}(h)) \\ \chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{\tilde{X}_4}(2h)) & \chi(\mathcal{O}_{\tilde{E}_i}, \mathcal{O}_{\tilde{X}_4}(2h)) & \chi(\mathcal{O}_{V_Z}, \mathcal{O}_{\tilde{X}_4}(2h)) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 24 & 6 & 7 \\ 72 & 20 & 19 \end{pmatrix}.$$

Proof. Let $K \in \{\bar{h}_1, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, V_Z\}$. Then for any $i \in \mathbb{Z}$ one has $\chi(\mathcal{O}_{\tilde{X}_4}(ih), \mathcal{O}_K) = \chi(\mathcal{O}_{\tilde{Z}}(ih), \mathcal{O}_K)$, and by Serre duality, $\chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}(ih)) = \chi(\mathcal{O}_{\tilde{X}_4}(ih), \mathcal{O}_K(-h+e)) = \chi(\mathcal{O}_{\tilde{Z}}(ih), \mathcal{O}_K(-\bar{h}))$. Moreover there is a short exact sequence $0 \rightarrow \mathcal{O}_{\tilde{Z}}(-K) \rightarrow \mathcal{O}_{\tilde{Z}} \rightarrow \mathcal{O}_K \rightarrow 0$ in \tilde{Z} , which yields, for $1 \leq i \leq 3$:

$$\begin{aligned} \text{ch}(\mathcal{O}_{\bar{h}_1}) &= (0, \bar{h}_1, -\frac{\bar{h}_1^2}{2}, 0) \\ \text{ch}(\mathcal{O}_{\tilde{E}_i}) &= (0, \tilde{E}_i, -\frac{\tilde{E}_i^2}{2}, 0) \\ \text{ch}(\mathcal{O}_{V_Z}) &= (0, V_Z, \frac{\bar{h}_1 \bar{h}_2}{2}, -\frac{1}{3}). \end{aligned}$$

Together with the fact that $h|_{\tilde{Z}} = 3V_Z + 2(\bar{h}_1 + \bar{h}_2) - \bar{D}$ as we saw in the proof of Lemma 4.6.2, and Lemma 5.0.7, Hirzebruch-Riemann-Roch then allows us to compute the above-mentioned Euler characteristics. \square

Lemma 5.0.10. One has

$$(\chi(\mathcal{O}_{D_3}, \mathcal{O}_{\bar{h}_1}), \chi(\mathcal{O}_{D_3}, \mathcal{O}_{\tilde{E}_i}), \chi(\mathcal{O}_{D_3}, \mathcal{O}_{V_Z})) = (1, -1, 0)$$

and

$$(\chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{D_3}), \chi(\mathcal{O}_{\tilde{E}_i}, \mathcal{O}_{D_3}), \chi(\mathcal{O}_{V_Z}, \mathcal{O}_{D_3})) = (10, 2, 2).$$

Proof. Let $K \in \{\bar{h}_1, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, V_Z\}$. We are going to use the following short exact sequence in \tilde{X}_4 :

$$0 \rightarrow \mathcal{O}_{\tilde{X}_4}(-e) \rightarrow \mathcal{O}_{\tilde{X}_4} \rightarrow \mathcal{O}_{D_3} \rightarrow 0. \quad (5.1)$$

Indeed with (1.8), it implies

$$\chi(\mathcal{O}_K, \mathcal{O}_{D_3}) = \chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}) - \chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}(-e)) = \chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}) - \chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}(\bar{h}-h))$$

and

$$\chi(\mathcal{O}_{D_3}, \mathcal{O}_K) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_K) - \chi(\mathcal{O}_{\tilde{X}_4}(\bar{h}-h), \mathcal{O}_K).$$

Finally, by Serre duality and Lemma 4.4.7 we have :

$$\chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}(-e)) = \chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_K(-h+e)) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_K(-h+2h-2\bar{h})) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_K(h-2\bar{h})).$$

One can then conclude with Hirzebruch-Riemann-Roch and the proof of Lemma 5.0.9. \square

Lemma 5.0.11. One has $(\chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\bar{h}_1}), \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\tilde{E}_i}), \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{V_Z})) = (-2, -2, -2)$ and $(\chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{D_3}(-V_E)), \chi(\mathcal{O}_{\tilde{E}_i}, \mathcal{O}_{D_3}(-V_E)), \chi(\mathcal{O}_{V_Z}, \mathcal{O}_{D_3}(-V_E))) = (10, 2, 0)$.

Proof. By (1.10), twisting (5.1) by $\mathcal{O}_{\tilde{X}_4}(e)$ gives a short exact sequence :

$$0 \rightarrow \mathcal{O}_{\tilde{X}_4} \rightarrow \mathcal{O}_{\tilde{X}_4}(e) \rightarrow \mathcal{O}_{D_3}(-V_E) \rightarrow 0.$$

With (1.8), this yields

$$\chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_K) = \chi(\mathcal{O}_{\tilde{X}_4}(h-\bar{h}), \mathcal{O}_K) - \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_K)$$

and

$$\chi(\mathcal{O}_K, \mathcal{O}_{D_3}(-V_E)) = \chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}(h - \bar{h})) - \chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}).$$

Moreover by Serre duality,

$$\chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}(h - \bar{h})) = \chi(\mathcal{O}_{\tilde{X}_4}(h - \bar{h}), \mathcal{O}_K(-h + e)) = \chi(\mathcal{O}_{\tilde{X}_4}(h - \bar{h}), \mathcal{O}_K(-\bar{h})) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_K(-h)).$$

One can then conclude with the same reasoning as in the proof of Lemma 5.0.9. \square

Lemma 5.0.12. *If $\{x_1, \dots, x_5\} = \{\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{\tilde{E}_1}, \mathcal{O}_{\tilde{E}_2}, \mathcal{O}_{\tilde{E}_3}, \mathcal{O}_{V_Z}\}$ then $(\chi(x_i, x_j))_{1 \leq i, j \leq 5} =$*

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Proof. Using the same reasoning as in 2. of the proof of Lemma 4.6.2 one gets :

$$\begin{aligned} \chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{\bar{h}_1}) &= \bar{h}_1^2(2\bar{h} - h) = \bar{h}_1^2(-V_Z + \bar{D}) = -1 \\ \chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{\tilde{E}_i}) &= \chi(\mathcal{O}_{\tilde{E}_i}, \mathcal{O}_{\bar{h}_1}) = \bar{h}_1 \tilde{E}_i(-V_Z + \bar{D}) = 0 \\ \chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{V_Z}) &= \chi(\mathcal{O}_{V_Z}, \mathcal{O}_{\bar{h}_1}) = \bar{h}_1 V_Z(-V_Z + \bar{D}) = 0 \\ \chi(\mathcal{O}_{\tilde{E}_i}, \mathcal{O}_{\tilde{E}_j}) &= \tilde{E}_j \tilde{E}_i(-V_Z + \bar{D}) = \delta_{i,j} \\ \chi(\mathcal{O}_{V_Z}, \mathcal{O}_{\tilde{E}_i}) &= \chi(\mathcal{O}_{\tilde{E}_i}, \mathcal{O}_{V_Z}) = V_Z \tilde{E}_i(-V_Z + \bar{D}) = 0 \\ \chi(\mathcal{O}_{V_Z}, \mathcal{O}_{V_Z}) &= V_Z^2(-V_Z + \bar{D}) = 2. \end{aligned}$$

\square

We saw in Lemma 4.4.6 that $(\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\bar{D}_2}), \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{\bar{D}_2}), \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{\bar{D}_2})) = (-12, 0, 156)$ and in Lemma 4.4.8 that $(\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\tilde{X}_4}), \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\tilde{X}_4}(h)), \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\tilde{X}_4}(2h))) = (-36, 0, 180)$. Furthermore :

Lemma 5.0.13. *One has $\chi(\mathcal{O}_{D_3}, \mathcal{O}_{\bar{D}_2}) = -84$ and $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{D_3}) = -60$.*

Proof. We can use (5.1) again together with (1.8) and similar reasoning as in the proof of Lemma 4.4.6 to find :

1.

$$\begin{aligned} \chi(\mathcal{O}_{D_3}, \mathcal{O}_{\bar{D}_2}) &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\bar{D}_2}) - \chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{\bar{D}_2}) \\ &= \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}) - \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}(h - \bar{h})) \\ &= -12 - (-12 + \frac{1}{2}((\delta + 24f)^2 - (\delta + 24f)(-2\delta + 72f))) \\ &= -12 - (-12 + \frac{1}{2}(48 + 48 + 2.48 - 72 + 48)) = -12 - (-12 + 84) = -84. \end{aligned}$$

2. $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{D_3}) = \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\tilde{X}_4}) - \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\tilde{X}_4}(-e))$.

By Serre duality

$$\begin{aligned} \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\tilde{X}_4}(-e)) &= \chi(\mathcal{O}_{\tilde{X}_4}(-e), \mathcal{O}_{\bar{D}_2}(-h + e)) \\ &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\bar{D}_2}(-h + 2e)) \\ &= \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{\bar{D}_2}(h - 2\bar{h})) \\ &= \chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}(\delta)) \\ &= -12 + \frac{1}{2}(\delta^2 - \delta(-2\delta + 72f)) \\ &= -12 + \frac{1}{2}(48 + 2.48 - 72) = -12 + 36 = 24. \end{aligned}$$

So finally, $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{D_3}) = -36 - 24 = -60$.

□

We also know from Lemma 4.7.17 that $\chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{\bar{D}_2}) = 12$ and $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{D_3}(-V_E)) = 36$. Finally :

Lemma 5.0.14. *One has $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}) = 48$, $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{h}_1}) = \chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{\bar{D}_2}) = -12$, $\chi(\mathcal{O}_{\bar{D}_2}, \tilde{E}_i) = \chi(\tilde{E}_i, \mathcal{O}_{\bar{D}_2}) = -4$ and $\chi(\mathcal{O}_{\bar{D}_2}, V_Z) = \chi(V_Z \mathcal{O}_{\bar{D}_2}) = 0$.*

Proof. We already saw in Lemma 4.6.2 that $\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{D}_2}) = 48$. Moreover, using the same reasoning as in 2. of the proof of Lemma 4.6.2 (and Lemma 5.0.12), one finds :

$$\begin{aligned}\chi(\mathcal{O}_{\bar{D}_2}, \mathcal{O}_{\bar{h}_1}) &= \chi(\mathcal{O}_{\bar{h}_1}, \mathcal{O}_{\bar{D}_2}) = \bar{h}_1 \bar{D}(-V_Z + \bar{D}) = -\delta.12f = -12 \\ \chi(\mathcal{O}_{\bar{D}_2}, \tilde{E}_i) &= \chi(\tilde{E}_i, \mathcal{O}_{\bar{D}_2}) = \tilde{E}_i \bar{D}(-V_Z + \bar{D}) = -\delta.4f = -4 \\ \chi(\mathcal{O}_{\bar{D}_2}, V_Z) &= \chi(V_Z \mathcal{O}_{\bar{D}_2}) = V_Z \bar{D}(-V_Z + \bar{D}) = 0\end{aligned}$$

□

Since $\text{ch}(\mathcal{O}_{D_3})$ and $\text{ch}(\mathcal{O}_{D_3}(-V_E))$ are both of the form $(0, e, *, *, *)$, Lemma 4.8.1 remains unchanged for our new family of sheaves \mathcal{F}_2 .

Lemma 5.0.15. *The following holds :*

- $(\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{l_D}), \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_{l_D}), \chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_{l_D})) = (1, 1, 1)$;
- $(\chi(\mathcal{O}_{l_D}, \mathcal{O}_{\tilde{X}_4}), \chi(\mathcal{O}_{l_D}, \mathcal{O}_{\tilde{X}_4}(h)), \chi(\mathcal{O}_{l_D}, \mathcal{O}_{\tilde{X}_4}(2h))) = (0, 0, 0)$;
- $\chi(\mathcal{O}_{D_3}, \mathcal{O}_{l_D}) = \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{l_D}) = \chi(\mathcal{O}_{l_D}, \mathcal{O}_{D_3}) = \chi(\mathcal{O}_{l_D}, \mathcal{O}_{D_3}(-V_E)) = 1$;
- *for any other sheaf \mathcal{F} in our family \mathcal{F}_2 , $\chi(\mathcal{O}_{l_D}, \mathcal{F}) = \chi(\mathcal{F}, \mathcal{O}_{l_D}) = 0$.*

Proof. Since l_D is a line of class eh_1^2 , it is a fiber in D_3 and thus a \mathbb{P}^1 . We can then use the same reasoning as in the proof of Lemma 4.8.1 together with (1.10) and (1.9) to find :

$$\begin{aligned}\chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_{l_D}) &= 1 = \int_{\tilde{X}_4} (0, 0, 0, eh_1^2, x) \cdot (1, \frac{h-e}{2}, *, *, *) = x + \frac{eh_1^2(h-e)}{2} = x + \frac{e(h_1^2(h_1+h_2) - h_1^2(-V_E))}{2} \\ &= x + \frac{1}{2}.\end{aligned}$$

Thus $x = 1 - \frac{1}{2} = \frac{1}{2}$. One can then compute the first two equalities above with Hirzebruch-Riemann-Roch. Furthermore, using again the same method as in the proof of Lemma 4.8.1 and the remark just above this Lemma, one finds :

$$\chi(\mathcal{O}_{D_3}, \mathcal{O}_{l_D}) = \chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_{l_D}) = \chi(\mathcal{O}_{l_D}, \mathcal{O}_{D_3}) = \chi(\mathcal{O}_{l_D}, \mathcal{O}_{D_3}(-V_E)) = -e^2 h_1^2 = eV_E h_1^2 = 1.$$

The last equality comes again from the same reasoning as in the proof of Lemma 4.8.1. □

Using finally the proof of Lemma 4.8.2, one can find the matrix given in Proposition 2.2.1.

Chapter 6

Appendix

6.1 Computation of the matrix P_2 from section 2.3.1

In this subsection we give more details about how we obtain the matrix P_2 (and thus P_1) in Proposition 2.3.1. We mainly use calculations already done in section 4 :

- The submatrix $((P_2)_{ij})_{1 \leq i, j \leq 3}$ can be deduced from Proposition 4.2.15, together with Lemma 4.2.1.
- The fifth line and column of P_2 can easily be found using the same reasoning as in the proof of Lemma 4.8.2.
- The $(P_2)_{i6}$ for $1 \leq i \leq 3$ have been computed in Lemma 4.3.5, while the $(P_2)_{6i}$ for $1 \leq i \leq 3$ can be deduced from this same Lemma and Serre duality. Indeed for any $j \in \mathbb{Z}$, Serre duality yields $\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{X_4}(jh)) = \chi(\mathcal{O}_{X_4}((j+1)h), \mathcal{O}_{\Sigma_2})$.
- Using the same reasoning as in the proof of Lemma 4.8.1, one can find that since $\dim(\text{supp}(\mathcal{O}_{\Sigma_2})) = 2$, the coefficients $(P_2)_{46} = \chi(\mathcal{O}_l, \mathcal{O}_{\Sigma_2})$ and $(P_2)_{64} = \chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_l)$ are both equal to 0. Similarly, $\dim(\text{supp}(\mathcal{O}_l)) = 1$ implies $\chi(\mathcal{O}_l, \mathcal{O}_l) = 0$.

So far we thus now the following coefficients of P_2 :

$$\begin{pmatrix} 1 & 20 & 124 & * & 1 & 1 \\ 1 & 1 & 20 & * & 1 & 1 \\ 20 & 1 & 1 & * & 1 & 7 \\ * & * & * & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 7 & 19 & 0 & 0 & * \end{pmatrix}.$$

Let us compute the remaining coefficients.

Lemma 6.1.1. *The following holds :*

- $(\chi(\mathcal{O}_{X_4}, \mathcal{O}_l), \chi(\mathcal{O}_{X_4}(h), \mathcal{O}_l), \chi(\mathcal{O}_{X_4}(2h), \mathcal{O}_l)) = (1, 0, -1);$
- $(\chi(\mathcal{O}_l, \mathcal{O}_{X_4}), \chi(\mathcal{O}_l, \mathcal{O}_{X_4}(h)), \chi(\mathcal{O}_l, \mathcal{O}_{X_4}(2h))) = (0, -1, -2);$
- $\chi(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}) = 2.$

Proof. • Let us keep the notations of section 4.3 and denote by i the inclusion $\Sigma_2 \hookrightarrow X_4$. Then, since by definition l is a line in Σ_2 with class E_1 , what we denote by abuse of notation by \mathcal{O}_l is actually $i_*\mathcal{O}_l$. As a result, for any $j \in \mathbb{Z}$ one has $\chi(\mathcal{O}_{X_4}(jh), i_*\mathcal{O}_l) = \chi(i^*\mathcal{O}_{X_4}(jh), \mathcal{O}_l) = \chi(\mathcal{O}_{\Sigma_2}(jh), \mathcal{O}_l)$. Moreover, we know the Chern character, in Σ_2 , of \mathcal{O}_l : indeed the short exact sequence

$$0 \rightarrow \mathcal{O}_{\Sigma_2}(-E_1) \rightarrow \mathcal{O}_{\Sigma_2} \rightarrow \mathcal{O}_{E_1} \rightarrow 0$$

together with $\text{ch}(\mathcal{O}_{\Sigma_2}(-E_1)) = (1, -E_1, -\frac{1}{2})$ yields $\text{ch}(\mathcal{O}_l) = (0, E_1, \frac{1}{2})$. One can then compute the first three Euler characteristics mentioned above with Hirzebruch-Riemann-Roch and Lemma 4.1.2.

- Let $j \in \mathbb{Z}$. By Serre duality and Lemma 4.2.13, one has $\chi(\mathcal{O}_l, \mathcal{O}_{X_4}(jh)) = \chi(\mathcal{O}_{X_4}((j+1)h), \mathcal{O}_l)$. The other three Euler characteristics involving \mathcal{O}_l can then be computed with Hirzebruch-Riemann-Roch.
- To compute $\chi(i_*\mathcal{O}_{\Sigma_2}, i_*\mathcal{O}_{\Sigma_2})$, we are going to use the same reasoning as in 1. of the proof of Lemma 4.6.2.

By Theorem 13.7 in [8], we have that $\text{ch}(i^*i_*\mathcal{O}_{\Sigma_2}) = (0, 0, c_2(\mathcal{N}_{\Sigma_2/X_4}))$. Therefore, by Hirzebruch-Riemann-Roch we have $\chi(i_*\mathcal{O}_{\Sigma_2}, i_*\mathcal{O}_{\Sigma_2}) = \chi(i^*i_*\mathcal{O}_{\Sigma_2}, \mathcal{O}_{\Sigma_2}) = c_2(\mathcal{N}_{\Sigma_2/X_4})$.

By pages 251-252 of [14] we know that $D_3 = \mathbb{P}_{\Sigma_2}(\mathcal{N}_{\Sigma_2/X_4})$. On the other hand, we know from [20] that $D_3 = \mathbb{P}_{\Sigma_2}(\mathcal{V}_F|_{\Sigma_2})$ (here we don't need to worry about which definitions are chosen for the projectivization of a vector bundle in [14] and [20], because $\mathcal{V}_F|_{\Sigma_2}$ is self-dual : indeed it is a rank 2 vector bundle whose first Chern class is 0 as we saw in the proof of Lemma 4.7.7). So the normal bundle $\mathcal{N}_{\Sigma_2/X_4}$ is $\mathcal{V}_F|_{\Sigma_2}$ up to a twist by a line bundle. Since twisting by a line bundle doesn't change the c_2 , we thus have $c_2(\mathcal{N}_{\Sigma_2/X_4}) = c_2(\mathcal{V}_F|_{\Sigma_2})$. Finally, we saw in the proof of Lemma 4.7.7 that $c_2(\mathcal{V}_F|_{\Sigma_2}) = 2$. □

Remark 6.1.2. To compute the first six Euler characteristics mentioned in Lemma 6.1.1, one could also have used the same reasoning as in the proof of Lemma 4.8.1 : indeed since l is a \mathbb{P}^1 with class $E_1 \cdot [\Sigma_2]$ in X_4 , we must have

$$\begin{aligned} \chi(\mathcal{O}_{X_4}, \mathcal{O}_l) &= 1 = \int_{X_4} (0, 0, 0, E_1 \cdot [\Sigma_2], x) \cdot (1, \frac{h}{2}, *, *, *) \\ &= x + E_1 \cdot \frac{h}{2} \cdot [\Sigma_2] = x + E_1|_{\Sigma_2} \cdot \frac{h|_{\Sigma_2}}{2} = x + \frac{1}{2} \end{aligned}$$

where we used Lemma 4.2.13 as well as the formula for the Todd class page 432 of [12] to find that $\text{td}(X_4) = (1, \frac{h}{2}, *, *, *)$.

As a result, $x = \frac{1}{2}$ and we can use Hirzebruch-Riemann-Roch to compute the Euler characteristics we are interested in. For instance, it gives :

$$\begin{aligned} \chi(\mathcal{O}_l, \mathcal{O}_{X_4}(2h)) &= \int_{X_4} (0, 0, 0, -E_1 \cdot [\Sigma_2], \frac{1}{2}) \cdot (1, 2h, *, *, *) \cdot (1, \frac{h}{2}, *, *, *) \\ &= \int_{X_4} (0, 0, 0, -E_1 \cdot [\Sigma_2], \frac{1}{2} - 2h \cdot E_1 \cdot [\Sigma_2]) \cdot (1, \frac{h}{2}, *, *, *) \\ &= \frac{1}{2} - 2h \cdot E_1 \cdot [\Sigma_2] - \frac{E_1 \cdot h \cdot [\Sigma_2]}{2} = \frac{1}{2} - 2h|_{\Sigma_2} \cdot E_1|_{\Sigma_2} - \frac{E_1|_{\Sigma_2} \cdot h|_{\Sigma_2}}{2} = \frac{1}{2} - 2 - \frac{1}{2} = -2. \end{aligned}$$

The other calculations are similar.

6.2 Another way to compute some Euler characteristics between elements in \mathcal{F}_2

In this subsection we just give some other way to compute some of the Euler characteristics mentioned in section 5. This involves in most cases more calculations than simply using Hirzebruch-Riemann-Roch in \tilde{Z} , but it can be useful to check results obtained in the previous section.

Lemma 6.2.1. *Let $K \in \{\bar{h}_1, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, V_Z\}$. Then :*

- $\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_K) = \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(-K));$
- $\chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_K) = -\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(K + \bar{h}));$
- $\chi(\mathcal{O}_{\tilde{X}_4}(2h), \mathcal{O}_K) = \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(K + \bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(K + \bar{h})).$

Proof. Let $i \in \mathbb{Z}$ and $K \in \{\bar{h}_1, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, V_Z\}$. Then with the notations of Lemma 4.5.9, we have

$$\chi(\mathcal{O}_{\tilde{X}_4}(ih), j_*\mathcal{O}_K) = \chi(j^*\mathcal{O}_{\tilde{X}_4}(ih), \mathcal{O}_K) = \chi(\mathcal{O}_{\tilde{Z}}(ih), \mathcal{O}_K).$$

Furthermore, in \tilde{Z} there is a short exact sequence $0 \rightarrow \mathcal{O}_{\tilde{Z}}(-K) \rightarrow \mathcal{O}_{\tilde{Z}} \rightarrow \mathcal{O}_K \rightarrow 0$ which induces $\chi(\mathcal{O}_{\tilde{Z}}(ih), \mathcal{O}_K) = \chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(-ih)) - \chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(-ih - K))$.

If $i = 0$, since by Corollary 4.5.5, $\pi_*\mathcal{O}_{\tilde{Z}} = \mathcal{O}_{Z_3}$, it implies the first formula.

Moreover,

$$\chi(\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}}(-ih - K)) = \chi(f^*\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\tilde{Z}}(-ih - K)) = \chi(\mathcal{O}_{\bar{E}_4}, f_*\mathcal{O}_{\tilde{Z}}(-ih - K)).$$

The short exact sequence $0 \rightarrow \mathcal{O}_{\bar{E}_4}(-V_{\bar{E}}) \rightarrow \mathcal{O}_{\bar{E}_4} \rightarrow f_*\mathcal{O}_{\tilde{Z}} \rightarrow 0$ then yields

$$\begin{aligned} \chi(\mathcal{O}_{\bar{E}_4}, f_*\mathcal{O}_{\tilde{Z}}(-ih - K)) &= \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-ih - K)) - \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-ih - K - V_{\bar{E}})) \\ &= \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}} - K)) - \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}} - K)) \end{aligned}$$

by (1.12). Similarly, $\chi(\mathcal{O}_{\bar{E}_4}, f_*\mathcal{O}_{\tilde{Z}}(-ih)) = \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-iV_{\bar{E}})) - \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(-(i+1)V_{\bar{E}}))$.

Serre duality, (1.12) and Lemma 4.5.8 then yield

$$\chi(\mathcal{O}_{\bar{E}_4}, f_*\mathcal{O}_{\tilde{Z}}(-ih - K)) = \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}((i-2)V_{\bar{E}} + K + \bar{h})) - \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}((i-1)V_{\bar{E}} + K + \bar{h}))$$

and

$$\chi(\mathcal{O}_{\bar{E}_4}, f_*\mathcal{O}_{\tilde{Z}}(-ih)) = \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(\bar{h} + (i-2)V_{\bar{E}})) - \chi(\mathcal{O}_{\bar{E}_4}, \mathcal{O}_{\bar{E}_4}(\bar{h} + (i-1)V_{\bar{E}})).$$

By definition, $\mathcal{O}_{\bar{E}_4}(-K) = \pi^*\mathcal{O}_{Z_3}(-K)$, thus the projection formula together with a) of Exercice 8.4 in III.8. of [12] conclude the proof. \square

With very a similar reasoning and Serre duality, one can also find the following formulas :

Lemma 6.2.2. *Let $K \in \{\bar{h}_1, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, V_Z\}$. Then :*

- $\chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}) = \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(-\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(-\bar{h} - K));$
- $\chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}(h)) = -\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(2\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(K + 2\bar{h}));$
- $\chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}(2h)) = \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(2\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(2\bar{h})) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(K + 2\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(K + 2\bar{h}));$
- $\chi(\mathcal{O}_{D_3}, \mathcal{O}_K) = \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_K) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(-\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(-\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(-\bar{h} - K)) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(-\bar{h} - K));$
- $\chi(\mathcal{O}_K, \mathcal{O}_{D_3}) = \chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}) - \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(-2\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(-2\bar{h})) + \chi(\mathcal{O}_{Z_3}, \mathcal{V}_Z^\vee(-2\bar{h} - K)) - \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(-2\bar{h} - K));$
- $\chi(\mathcal{O}_{D_3}(-V_E), \mathcal{O}_K) = -\chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}) + \chi(\mathcal{O}_{Z_3}, \mathcal{O}_{Z_3}(K)) - \chi(\mathcal{O}_{\tilde{X}_4}, \mathcal{O}_K);$
- $\chi(\mathcal{O}_K, \mathcal{O}_{D_3}(-V_E)) = \chi(\mathcal{O}_{\tilde{X}_4}(h), \mathcal{O}_K) - \chi(\mathcal{O}_K, \mathcal{O}_{\tilde{X}_4}).$

6.3 Macaulay2

Thanks to E. Fatighenti, I could use Macaulay2 to compute the highest weights of the decomposition of an external power of $\mathcal{V} = \mathcal{U}_3^\vee(-1) \oplus (\mathcal{U}_3^\perp)^\vee(-1) \oplus \mathcal{O}_{\text{Gr}(3,7)}(-1)$. We are going to reproduce here as an example a way to get the highest weights associated to $\bigwedge^7 \mathcal{V}$. The parts in blue will be the answers given by Macaulay2 :

```
i1: loadPackage "SchurRings"
o1=SchurRings
o1: Package
i2: loadPackage "Schubert2"
o2=Schubert2
o2: Package
i3: R=schurRing(QQ,q,4)
o3: R
o3: SchurRing
i4: Q=schurRing(R,r,3)
o4=Q
o4: SchurRing
i5: F=exteriorPower(4,q_1)+r_1*exteriorPower(4,q_1)+exteriorPower(3,q_1)
o5=q_{1,1,1,1}r_1 + (q_{1,1,1,1} + q_{1,1,1})r_0
o5: Q
i6 : exteriorPower(7,F)
o6=(q_{7,6,6,6} + q_{6,6,6,6})r_{1,1,1} + q_{6,6,6,6}r_{1,1}
o6: Q
```

The commands of i3 and i4 define q_1 and r_1 that will correspond respectively to the highest weights $(0, 0, 0; 1, 0, 0, 0)$ and $(1, 0, 0; 0, 0, 0, 0)$. Then "exteriorPower(n, q_1)" corresponds to \bigwedge^n of the irreducible representation with highest weight $(0, 0, 0; 1, 0, 0, 0)$. Finally r_0 , $r_{1,1}$ and $r_{1,1,1}$ correspond to the irreducible representations with highest weight respectively $(0, 0, 0; 0, 0, 0, 0)$, $(1, 1, 0; 0, 0, 0, 0)$ and $(1, 1, 1; 0, 0, 0, 0)$. For q it is similar. The line i5 is just defining \mathcal{V} as F , using the highest weight of each summand of \mathcal{V} , and the line i6 is just asking for $\bigwedge^7 \mathcal{V}$. The answer o6 reads : "The highest weights of the decomposition of $\bigwedge^7 \mathcal{V}$ are $(1, 1, 1; 7, 6, 6, 6)$, $(1, 1, 1; 6, 6, 6, 6)$ and $(1, 1, 0; 6, 6, 6, 6)$ ". For any $\bigwedge^n \mathcal{V}$ one can proceed similarly, just replacing 7 by n in i6, and for $(\bigwedge^n \mathcal{V})(m)$ one can either compute the highest weights associated to $\bigwedge^n \mathcal{V}$ and add $(m, m, m; 0, 0, 0, 0)$ if $m > 0$, $(0, 0, 0; m, m, m, m)$ if $m < 0$, or replace i6 by "exteriorPower(n, F)*exteriorPower(3, r_1)" if $m > 0$ and by "exteriorPower(n, F)*exteriorPower(4, q_1)" if $m < 0$.

6.4 Scilab

One can also use scilab to compute the dimension of the irreducible representation of Sl_7 associated to a highest weight λ , using (4.2). Here is the function one can define in scilab :

```
function d=W(L)
delta=[7,6,5,4,3,2,1];
i=1;
j=2;
d=1;
while i<7
while j<8
d=d*(L(1,i)+delta(1,i)-L(1,j)-delta(1,j))/(delta(1,i)-delta(1,j));
j=j+1;
end
i=i+1;
```

```

j=i+1;
end
endfunction

```

The input is a vector of 7 columns and 1 line, whose coefficients are those of the highest weight λ , and the output is the dimension of the only nonzero $H^q(\mathcal{E})$ if it exists, or 0 otherwise, where \mathcal{E} is the vector bundle associated to λ , when $\lambda + \delta$ is dominant (including cases when $\lambda + \delta$ is singular as we will see). If $\mu := \lambda + \delta$ is not dominant then there are some $i < j$ such that $\mu_i < \mu_j$ and then the associated coefficient in the Weyl formula (4.2) will be $\frac{\mu_i - \mu_j}{\delta_i - \delta_j}$. Here the denominator will always be positive but then the numerator is negative,

so if μ is not dominant the result may be non-positive. Yet in our case the Weyl formula gives $\frac{\prod_{1 \leq i < j \leq 7} \mu_i - \mu_j}{\prod_{1 \leq i < j \leq 7} \delta_i - \delta_j}$

thus even if μ is not dominant the Weyl formula should give the right result up to a sign. Finally, if μ is singular then there exists $i \neq j$ such that $\mu_i = \mu_j$. But then the associated coefficient is, up to a sign, $\frac{\mu_i - \mu_j}{\delta_i - \delta_j} = 0$. Since (4.2) is a product, in this case the result will be 0 but since μ is singular by Theorem 4.2.10 $H^*(\mathcal{E}) = 0$. So this function gives $\dim(H^*(\mathcal{E})) = \bigoplus_{i \in \mathbb{Z}} \dim(H^i(\mathcal{E}))$, up to a sign, since by Theorem 4.2.10 again there is at most one $H^q(\mathcal{E}) \neq 0$. The sign will be correct whenever μ is dominant.

Remark 6.4.1. *What has just been said above may suggest that one doesn't really need to check $\lambda + \delta$, since the function W gives either $\dim(H^*(\mathcal{E}))$ or $-\dim(H^*(\mathcal{E}))$, and $\dim(H^*(\mathcal{E}))$ has to be non negative thus it should be enough to take the absolute value of the result given by W . However, if $\lambda + \delta$ is not singular, one still needs to determine for which $\sigma \in S_7$, $\sigma(\delta + \lambda)$ is dominant, in order to find for which q one has $H^q(\mathcal{E}) \neq 0$. Alternatively one can also compute the number of positive roots α such that $(\lambda + \delta, \alpha) < 0$, but in any case one needs to compute $\lambda + \delta$. So a possible way to proceed, for a given highest weight λ , would be :*

1. Compute $W(\lambda)$. If it is 0 then $\lambda + \delta$ is singular and by Theorem 4.2.10, $H^*(\mathcal{E}) = 0$.
2. If $W(\lambda) \neq 0$ then write down $\lambda + \delta$ and use one of the two methods given above (corresponding to Theorem 4.2.10 and Theorem 4.2.2 respectively) to find out for which $q \in \mathbb{Z}$ one has $H^q(\mathcal{E}) \neq 0$. Then $\dim(H^i(\mathcal{E})) = |W(\lambda)| \cdot \delta_{qi}$.

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Titre: Sur le groupe de Grothendieck numérique des variétés de K uchle de dimension 4 et de type c5 tr s g n rales

Mots cl s: Math matiques, G om trie alg brique, vari t s de K uchle de dimension 4 et de type c5, caract ristique d'Euler, cat gories d riv es

R sum : Nous essayons de trouver une base du groupe de Grothendieck num rique G d'une vari t  de K uchle de dimension 4 et de type c5 tr s g n rale X_4 . Pour ce faire, nous nous int ressons d'abord   la g om trie de telles vari t s, et nous essayons d'en d duire une famille F d' l ments de G qui pourrait  tre une base. Ensuite nous essayons de calculer la matrice dont les coefficients sont les caract ristiques d'Euler entre les  l ments de F . Cette matrice devrait nous permettre de mieux appr hender

les cat gories d riv es born es des vari t s consid r es.

Dans cette th se nous ne parvenons pas   trouver une base de G mais nous proposons deux familles F_1 et F_2 qui pourraient  tre des bases du groupe de Grothendieck num rique d'un  clatement de X_4 , et   partir desquelles il pourrait  tre possible de trouver une base de G . Nous calculons les matrices de caract ristiques d'Euler associ es   ces deux familles,   un nombre d'intersection pr s dans un cas et enti rement dans l'autre.

Title: On the numerical Grothendieck group of very general K uchle fourfolds of type c5

Keywords: Mathematics, Algebraic geometry, K uchle fourfolds of type c5, Euler characteristic, derived categories

Abstract: We try to find a basis of the numerical Grothendieck group G of very general K uchle fourfolds of type c5, denoted X_4 . To do so we first study the geometry of such varieties, and try to deduce from this a family F of elements in G which could be a basis. Then we try to compute the matrix whose coefficients are Euler characteristics between elements in F . This matrix is expected to provide intuition on

the bounded derived category of very general K uchle fourfolds of type c5.

In this thesis we don't manage to find a basis of G but we give two families F_1 and F_2 which could be basis of the numerical Grothendieck group of a blow-up of X_4 , and from which it may be possible to deduce a basis of G . We compute the matrices of Euler characteristics associated to these families, up to an intersection number in one case and entirely in the other case.