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**SORBONNE UNIVERSITÉ**  
**LJLL**

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Unité de recherche **Laboratoire Jacques-Louis Lions**

Thèse présentée par **Jules PERTINAND**

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Discipline **Mathématiques appliquées**

# Effective properties of random media : expansions in dilute regimes

Thèse dirigée par Antoine GLORIA

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**EFFECTIVE PROPERTIES OF RANDOM MEDIA : EXPANSIONS IN DILUTE REGIMES****Résumé**

Cette thèse porte sur le développement en régimes dilués des propriétés effectives de milieux aléatoires et justifie rigoureusement plusieurs formules communément utilisées en physique. Dans ce manuscrit, ces propriétés effectives sont obtenues par la théorie de l'homogénéisation stochastique. On montre que le type de dilution a un impact sur les ordres élevés de ces développements et on justifie les formules de Clausius-Mossotti et de Batchelor issues respectivement de l'électrostatique et de la mécanique des fluides.

Les différents modèles de dilution étudiés (dilatation, effacement aléatoire ou dilution générale) ainsi que quelques outils généraux du domaine (développements en cluster, intensités à plusieurs particules) sont présentés dans un premier chapitre introductif. Le deuxième chapitre est consacré à l'étude d'un problème de conductivité effective avec dilution par dilatation. Grâce à une nouvelle méthode par point fixe, on montre que cette conductivité effective dépend de manière analytique du paramètre de dilatation. Le troisième et dernier chapitre étudie la notion de vitesse effective de sédimentation de particules dans un fluide visqueux. En utilisant une approche par cluster, on en fournit un développement au premier ordre avec contrôle quantitatif de l'erreur.

**Mots clés :** EDP, homogénéisation, milieu aléatoire, homogénéisation stochastique, processus de points, Clausius-Mossotti, sédimentation, Batchelor, développement dilué

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**Abstract**

This thesis focuses on the expansion of some effective properties of random media in dilute regimes and the rigorous justification of several formulas commonly used in physics. These effective properties are obtained from the theory of stochastic homogenization. We show that the type of dilution plays an important role on the high orders of these developments and we justify the Clausius-Mossotti and Batchelor formulas originating respectively from electrostatic and fluid mechanics.

The different dilution models under consideration (dilation, random deletion, or general dilution) as well as some general tools of the field (cluster developments, multi-particle intensities) are presented in the first chapter. The second chapter is devoted to the study of an effective conductivity problem with dilution by dilation. Using a new fixed-point approach, we show that this effective conductivity depends analytically on the dilation parameter. The third and final chapter study the notion of effective sedimentation speed for particles in a viscous fluid. Using the cluster expansion, we provide a first order expansion of this effective speed with a quantitative control of the error.

**Keywords:** PDE, homogenization, stochastic homogenization, points processes, Clausius-Mossotti, Batchelor, sedimentation, dilute expansion

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# Table des matières

<b>Résumé</b>	<b>ii</b>
<b>Remerciements</b>	<b>iii</b>
<b>Table des matières</b>	<b>vii</b>
<b>1 General introduction</b>	<b>1</b>
1.1 Effective properties of heterogeneous media . . . . .	2
1.1.1 Description and motivations . . . . .	3
1.1.2 Dilute regime and physics formulas . . . . .	5
1.2 Stochastic homogenization theory : the effective conductivity case	7
1.2.1 Qualitative theory . . . . .	7
1.2.2 Approximations . . . . .	11
1.2.3 Quantification of the approximations . . . . .	14
1.2.4 Back to dilute regimes . . . . .	16
1.3 Notion of dilution . . . . .	17
1.3.1 One parameter models . . . . .	17
1.3.2 Cluster expansion formalism . . . . .	19
1.3.3 Intrinsic dilution : multi-point intensities . . . . .	21
1.3.4 Toy model . . . . .	25
1.4 Long-range interactions and renormalization . . . . .	29
1.4.1 Multi-point correlation functions . . . . .	29
1.4.2 Linear models . . . . .	30
1.4.3 Implicit renormalization . . . . .	35
1.5 The Clausius-Mossotti formula for geometric dilation models . . .	38
1.5.1 Setting . . . . .	38
1.5.2 Previous results . . . . .	39
1.5.3 Contribution . . . . .	41
1.5.4 Perspectives . . . . .	42
1.6 Toward Batchelor's formula . . . . .	43
1.6.1 Setting and previous works . . . . .	43



1.6.2	Contribution . . . . .	45
1.6.3	Perspectives . . . . .	46
1.A	Proof of the optimization Lemma 1.4 . . . . .	47
<b>2</b>	<b>A fixed-point approach to Clausius-Mossotti formulas</b>	<b>50</b>
2.1	Introduction . . . . .	51
2.1.1	Setting and main result . . . . .	51
2.1.2	General strategy . . . . .	54
2.1.3	Previous works . . . . .	59
2.1.4	Extensions . . . . .	61
2.2	Preliminaries . . . . .	63
2.2.1	Massive approximation . . . . .	63
2.2.2	Some theory of Point Processes . . . . .	73
2.3	Operator . . . . .	75
2.3.1	The key operator . . . . .	76
2.3.2	Improved bound . . . . .	80
2.4	Proof of Theorem 2.1 : analyticity of the homogenized coefficient	90
2.4.1	Main propositions . . . . .	91
2.4.2	Proof of Proposition 2.4 . . . . .	92
2.4.3	Proof of Proposition 2.3 : analyticity of the fixed-point operator . . . . .	96
<b>3</b>	<b>Mean sedimentation speed of a random suspension : First order dilute expansion</b>	<b>105</b>
3.1	Introduction . . . . .	106
3.1.1	Model and averaged settling speed . . . . .	106
3.1.2	Main result . . . . .	109
3.1.3	General strategy . . . . .	111
3.1.4	Relation to other works . . . . .	115
3.1.5	Extensions and future works . . . . .	116
3.2	Preliminaries . . . . .	117
3.2.1	Notations . . . . .	117
3.2.2	Point processes . . . . .	118
3.2.3	Finite volume approximation . . . . .	124
3.2.4	Trace estimates . . . . .	135
3.3	1 <sup>st</sup> order expansion : proof of Theorem 3.1 . . . . .	136
3.3.1	Strategy . . . . .	136
3.3.2	Control of the remainder : proof of Proposition 3.2 . . . . .	138
3.3.3	Proof of the renormalization lemmas : Lemmas 3.10 to 3.12	145
3.4	2 <sup>nd</sup> order expansion : strategy, issues and how to overcome them .	160
3.4.1	Expected results . . . . .	160

3.4.2 Difficulties . . . . . 162

**Bibliographie** **164**

# Chapter 1

## General introduction

### Outline of the current chapter

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<b>1.1</b>	<b>Effective properties of heterogeneous media</b>	<b>2</b>
1.1.1	Description and motivations . . . . .	3
1.1.2	Dilute regime and physics formulas . . . . .	5
<b>1.2</b>	<b>Stochastic homogenization theory : the effective conductivity case</b>	<b>7</b>
1.2.1	Qualitative theory . . . . .	7
1.2.2	Approximations . . . . .	11
1.2.3	Quantification of the approximations . . . . .	14
1.2.4	Back to dilute regimes . . . . .	16
<b>1.3</b>	<b>Notion of dilution</b>	<b>17</b>
1.3.1	One parameter models . . . . .	17
1.3.2	Cluster expansion formalism . . . . .	19
1.3.3	Intrinsic dilution : multi-point intensities . . . . .	21
1.3.4	Toy model . . . . .	25
<b>1.4</b>	<b>Long-range interactions and renormalization</b>	<b>29</b>
1.4.1	Multi-point correlation functions . . . . .	29
1.4.2	Linear models . . . . .	30
1.4.3	Implicit renormalization . . . . .	35
<b>1.5</b>	<b>The Clausius-Mossotti formula for geometric dilation models</b>	<b>38</b>
1.5.1	Setting . . . . .	38
1.5.2	Previous results . . . . .	39
1.5.3	Contribution . . . . .	41
1.5.4	Perspectives . . . . .	42

<b>1.6</b>	<b>Toward Batchelor's formula</b>	<b>43</b>
1.6.1	Setting and previous works . . . . .	43
1.6.2	Contribution . . . . .	45
1.6.3	Perspectives . . . . .	46
<b>1.A</b>	<b>Proof of the optimization Lemma 1.4</b>	<b>47</b>

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The main object of this thesis is the rigorous proof of several celebrated formulas in physics regarding the dilute expansion of some effective properties of random heterogeneous media. We will focus on two examples stemming from electrostatic (with the so-called Clausius-Mossotti formula for the effective conductivity of random media) and from fluid mechanics (with Batchelor's formula for the effective sedimentation speed of particles in a viscous fluid).

After reading the above, one could wonder :

1. What is a random media and how can we define its effective properties ?
2. What is a dilute regime ?
3. What are the Clausius-Mossotti's and Batchelor's formulas and why do they require a rigorous mathematical treatment ?

This introduction aims at providing some answers.

The first question fits into the framework of stochastic homogenization. We will recall the fundamentals of this theory as our approaches in the rest of this manuscript rely on it.

The second question seems naive at first glance : dilution means low concentration (a.k.a. volume fraction). In the case of random media, while the concentration suffice to coarsely describe dilution, we shall see that correlations should also be taken into account if we aim for a finer description. We shall illustrate this phenomenon on a toy model.

Regarding the third question, we will first illustrate some difficulties through the explicit study of comparable linear models. Then, we will rephrase these formulas in modern mathematical language while giving an account of the literature regarding their proof in dilute regimes. Finally, we will present our main contribution to their rigorous justification.

## 1.1 Effective properties of heterogeneous media

In this section, we give an informal definition of the effective properties of random media as well as some motivations for their use. Then, we informally describe the physics formulas under study.

### 1.1.1 Description and motivations

In this manuscript, we say that a system exhibits an effective behavior if it can be replaced by a simpler one while retaining most of its physical features. In material science, this corresponds to replacing a heterogeneous material by an equivalent homogeneous one. This phenomenon typically appears in multi-scales media : if the scale of variation of the properties of the medium (called microscale) is much smaller than the scale of variation of the external forcing (called macroscale), one expects some sort of averaging of the small scales (also called microstructure). For instance, an aircraft wing (which is several meters long) is made of panels of composite materials, a polymer matrix reinforced by a fine weaving of carbon or glass wires (with a radius of a few micrometers). To compute the deformation of the wing, it would be extremely costly to take into account these small scales variations : an approach via Finite Elements Method (FEM) would require meshing the wing at a scale finer than the micrometer ! To reduce the computational costs, engineers treat composite panels as if they were made of an effective metal (lighter and stronger than a natural one) characterized by a few constant coefficients, allowing for a way coarser mesh.

When the heterogeneities of the medium are disordered, it is natural, from a modelling perspective, to adopt a statistical description. Such media are said to be random. This assumption is particularly suited for natural materials (such as wood and bone which are natural composites but one can also think of a well stirred suspension of particles in a fluid) as there is a lot of uncertainty on how they were obtained. Even in the case of manufactured composites (such as carbon or glass fiber) which are designed using deterministic patterns, one should still take into account uncertainty in the fabrication process (which includes heating as well as chemical reactions) creating random deviations from the original patterns.

The properties of homogeneous materials are encapsulated in their (linear or nonlinear) constitutive relations. They are usually parametrized by some constants. For instance, one can think of Hooke's law in elasticity or Ohm's law in electrostatic. The linear version of the latter relates the electrical flux  $J \in \mathbb{R}^d$  to the gradient of the potential  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  through the (constant) conductivity matrix  $a \in \mathbb{R}^{d \times d}$  via  $J = a \nabla u$ . Assuming that the medium occupies a domain  $\Omega \subset \mathbb{R}^d$ , conservation of this flux then leads to the following conductivity equation

$$-\nabla \cdot a \nabla u = f \quad \text{in } \Omega \tag{1.1}$$

where  $f : \Omega \rightarrow \mathbb{R}$  is a source term (and we do not specify boundary conditions) and  $\nabla \cdot a \nabla u := \sum_{1 \leq i, j \leq d} \partial_i (a_{ij} \partial_j u) =: \partial_i (a_{ij} \partial_j u)$  (using Einstein convention). In heterogeneous materials, these parameters vary spatially so that the conductivity  $a : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  is now a spatial matrix field. For multi-scales materials, the

conductivity field  $a$  will even oscillate at very small scales. This problem admits an effective behavior if one can find a constant conductivity coefficient  $\bar{a} \in \mathbb{R}^{d \times d}$  (independent of the source term  $f$ ) such that  $u \simeq \bar{u}$  where  $\bar{u}$  is the solution of the effective conductivity equation

$$-\nabla \cdot \bar{a} \nabla \bar{u} = f \quad \text{in } \Omega \quad (1.2)$$

with the same source term.

From this conductivity example, one can see that effective properties can be more complex than a local average and can present a nonlinear nature. Indeed, if one finely mixes a conductor with an insulator (forming a dense network of small insulator pockets surrounded by conductor), the geometry of the microstructure plays an equally important role as the proportion of each phase : when the insulator percolates (i.e. when the insulating network is actually connected), the resulting effective medium is an insulator, however good the conductor may be.

Understanding the derivation of effective parameters then serves three purposes. First, if one zooms in enough, every material is in fact heterogeneous (at least at the atomic scale) and the effective theory justifies why we can still consider some materials to be homogeneous. Second, the nonlinear nature of the averaging allows the design of synthetic material with uncanny properties. This was the case of composites (which are lighter and more resistant than metals) in the 1960s. Current researches include, among others, metamaterials which exhibit a negative effective refractive index and can be used to create so-called invisibility cloak as well as active fluids which can exhibit super fluidity, i.e. an effective viscosity close to zero thanks to the addition of active particles (note that these effects are way beyond the scope of this manuscript). Finally, as already pointed out, the constant coefficient effective models are much easier to simulate, especially for random materials. Indeed, in that case, one needs to suppress statistical fluctuations using some sort of Monte Carlo method which itself requires one FEM simulation (already expensive because of the small scales) per sample !

In this manuscript, we will exclusively work with two-phases heterogeneous media constituted of a countable set of inclusions (or particles), parametrized by the (countable) point set  $\mathcal{P}$  of their centers, scattered across a background medium. Note that particles in a viscous fluid also fit into this framework. The randomness of the medium will be fully encapsulated in a suitable distribution on the point set  $\mathcal{P}$ .

For models stemming from materials science, effective properties are mainly derived using the mathematical theory of homogenization (which holds both for deterministic and random media). This approach requires a clear separation between the micro and macro scales, see details Section 1.2 below. In the sequel, we will always place ourselves in this framework. Note that, effective properties can

also be derived through any process which averages the fine scales, for instance mean field approaches, in suitable regimes.

### 1.1.2 Dilute regime and physics formulas

While the theory of composites was established in the 1960s and the theory of homogenization (which can be seen as its mathematical counterpart) in the 1970s, the first results regarding the effective properties for materials trace back to the 19<sup>th</sup> century. At that time, physicists focused on inclusion models in dilute regimes. There, dilution is characterized (at first order) by the smallness of the *volume fraction* (a.k.a. concentration)  $c$  of the inclusion phase. In a bounded domain  $\Omega$ , we have  $c := \frac{|Z|}{|\Omega|}$  where  $|Z|$  is the total volume of inclusions. In [Mos36] and [Cla79], Mossotti and Clausius independently argued that the effective conductivity of a medium constituted of a background material of constant isotropic conductivity  $\alpha \text{Id}$  perturbed by scarce spherical inclusions of (constant isotropic) conductivity  $\beta \text{Id}$  is given by the following expansion for  $c \ll 1$

$$\bar{a} = \alpha \text{Id} + \frac{\alpha d(\beta - \alpha)}{\beta + \alpha(d - 1)} \text{Id} c + o(c) \quad (1.3)$$

in any dimension  $d \geq 1$ .

Analogous formulas for similar models (effective dielectric constants and effective refractive indexes) were obtained by Maxwell [Max98], Garnett [Gar04], Lorenz [Lor80], Lorentz [Lor09], see [Mar00, Section 1.1.3 p.7] for an account of the historical context. Therefore, such expansions are called in the literature under various combinations of those names but we will always refer to them as the Clausius-Mossotti formula in the sequel. In fact, we will even call under this name the slightly more general formula

$$\bar{a} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)}c + o(c) \quad (1.4)$$

with given  $\mathcal{A}^{(0)}, \mathcal{A}^{(1)} \in \mathbb{R}^{d \times d}$ . This formulation allows us to cover potential anisotropy as well as the case of so-called stiff inclusions, formally obtained sending  $\beta \rightarrow \infty$ .

The key idea to obtain the expansion (1.3) (and its various analogues) is to consider the inclusions as isolated and therefore neglect the interactions between them. This is justified in the dilute regime as particles are quite distant from one another (at least in some average sense).

Expansions in the form of (1.4) are very robust to the distribution of the inclusion phase as long as it remains scarce. It also holds for the effective stiffness tensor in linear elasticity as it shares the same structure as the conductivity prob-

lem (1.2) seeing now  $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$  as a displacement field and  $a$  as a fourth order stiffness tensor relating the stress  $\sigma$  to the strain  $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$  (a.k.a. symmetric gradient) via Hooke's law  $\sigma = a : D(u)$ .

In [Ein05] (see also [Str05]), Einstein extended this formula to the effective viscosity  $\bar{\nu}$  of a suspension of scarce, spherical, rigid particles in a Stokes fluid of viscosity  $\nu > 0$  in the following way

$$\bar{\nu} = \nu(1 + \frac{5}{2}c) + o(c) \quad (1.5)$$

in dimension  $d = 3$ . Formally this problem corresponds to the elasticity counterpart of (1.1) adding the incompressibility constraint on the fluid and making the inclusions rigid (i.e. sending their stiffness  $\beta \rightarrow \infty$ ). This formula is the prototype (and starting point) for the study of complex rheology i.e. the study of the effective constitutive laws for complex fluids. Depending on the distribution of particles, the effective viscosity can be anisotropic and hence represented as a symmetric matrix  $\bar{B} \in \mathbb{R}_{\text{sym}}^{d \times d}$ .

In the expansion (1.5), the particles are assumed to be buoyant, i.e. having the same density as the fluid. If the particles are denser, they will start to sink in the fluid, a phenomenon called sedimentation. In its fall, each particle interacts with all the others through the fluid (hence in a very nonlocal way) which can lead to quite complex trajectories, notably when particles are not spherical, see [Gua06; GM12b]. While nowadays physicists are more interested in dense regimes, the historical main contribution is due to Batchelor who gave in [Bat72] a dilute expansion of  $\bar{V}$ , the mean sedimentation speed of the particles. He showed that, at first order, particles are falling at the speed  $\bar{V}^{(1)}$  (explicitly given for spherical particles) corresponding to a single isolated particle sinking in the fluid as well as a first correction of the following form

$$\bar{V} = \bar{V}^{(1)}(1 - 6.55c) + o(c) \quad (1.6)$$

in dimension  $d = 3$ . Note that this formula is coherent with (1.5) as adding particles increases the effective viscosity of the fluid and thus decreases the sedimentation speed.

Even if both (1.5) and (1.6) are first order dilute expansions, in the framework of Clausius-Mossotti formulas, (1.6) should be seen as a second order expansion. Indeed, in the absence of particles,  $\bar{\nu} = \nu > 0$  whereas  $\bar{V} = 0$ . While there is a wide physical consensus regarding the first order correction in (1.4), the next orders are more complex, particularly for random media, as they should involve the correlation structure. These differences appear for (1.6) as Batchelor predicted a correction  $\propto c$  for a Poisson-like random distribution of particles whereas, in



[Has59], Hasimoto predicted a correction  $\propto c^{\frac{1}{3}}$  (for  $d = 3$ ) for periodically arranged particles. Clarifying the scaling of the higher orders correction is thus one of the motivations of this thesis.

## 1.2 Stochastic homogenization theory : the effective conductivity case

We recall some elements of the stochastic homogenization theory, as it will be our approach to derive effective properties in the sequel. We illustrate it on the conductivity problem (1.1) (a.k.a. linear elliptic equation in divergence form) for inclusion models. Indeed, as previously mentioned, the conductivity problem can be seen as a prototype for more complex models (linear or nonlinear elasticity or waves for instance) and was therefore, historically, one of the first to be treated.

We start by presenting the qualitative theory as well as its assumptions. It yields a formula for the effective conductivity, called homogenized coefficient in this context, which is, in general, not computable. We thus present some (rather theoretical) approximation methods (used in Chapters 2 and 3) before reviewing how to quantify their convergence. Finally, we come back to the dilute regime and why it remains relevant today.

### 1.2.1 Qualitative theory

The main assumption of homogenization theory is to have a clear separation between a microscale and a macroscale (as we are aiming for formulas ; therefore, we voluntarily put aside compactness results such as  $H$ -convergence [MT97]). We thus start by introducing the two-phases medium under study and its rescaling.

We set a constant background conductivity matrix  $\alpha \in \mathbb{R}^{d \times d}$  that we change into  $\beta \in \mathbb{R}^{d \times d}$  in the spherical inclusions  $\cup_{x \in \mathcal{P}} B(x)$  scattered along the (countable) point set  $\mathcal{P} \subset \mathbb{R}^d$  (where  $B(x)$  is the unit ball centered at point  $x$ ). The associated conductivity field  $a(\mathcal{P}, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  then writes

$$a(\mathcal{P}, \cdot) = \alpha + (\beta - \alpha) \mathbf{1}_{\cup_{x \in \mathcal{P}} B(x)}(\cdot) \quad (1.7)$$

where we assume that  $\alpha$  and  $\beta$  are bounded strongly elliptic matrices (so that  $a$  is too), i.e. there exists  $\lambda > 0$  s.t.  $\forall x \in \mathbb{R}^d, \forall \xi \in \mathbb{R}^d, \lambda |\xi|^2 \leq \xi \cdot a(x) \xi$  and  $|a(x) \xi| \lesssim |\xi|$ . The conductivity  $a$  typically varies at scale  $\sim 1$ .

To define the microstructure, we consider, for  $0 < \varepsilon \ll 1$ , the rescaled conductivity field  $a_\varepsilon : x \mapsto a(\mathcal{P}, \frac{x}{\varepsilon})$  so that

$$a_\varepsilon(\cdot) = \alpha + (\beta - \alpha) \mathbf{1}_{\cup_{x \in \mathcal{P}} B_\varepsilon(\varepsilon x)}(\cdot) . \quad (1.8)$$

By construction,  $a_\varepsilon$  varies at scale  $\varepsilon \ll 1$  but we have conserved the same volume fraction of inclusions.

We then consider the associated family  $(u_\varepsilon)_{0 < \varepsilon \ll 1}$  of Lax-Milgram solutions in  $H_0^1(Q)$  of the rescaled conductivity problem (analogous to (1.1))

$$-\nabla \cdot a_\varepsilon \nabla u_\varepsilon = f \quad \text{in } Q \quad (1.9)$$

where  $Q := [-\frac{1}{2}, \frac{1}{2}]^d$  is the unit cube and we have chosen the forcing term  $f \in H^{-1}(Q)$ . Note that we have (implicitly) chosen here Dirichlet Boundary Conditions (BC) (i.e. that  $u_\varepsilon$  vanishes on  $\partial Q$ ) for convenience, the sequel holds for any BC independent of  $\varepsilon$ . An energy estimate yields  $\|u_\varepsilon\|_{H_0^1(Q)} \lesssim 1$  (independently of  $\varepsilon$ ), therefore  $u_\varepsilon$  admits a weak limit (up to extraction) in  $H_0^1(Q)$ . Homogenization theory identifies this (unique) limit (as the solution of a PDE) but requires additional assumptions on the coefficient field  $a$  to do so. Note that instead of decreasing the microscale  $\varepsilon \ll 1$ , one could equivalently set  $\varepsilon = 1$  and increase the macroscale replacing  $Q$  by  $Q_R := [-\frac{R}{2}, \frac{R}{2}]^d$  for  $R \gg 1$ . In that case, one also needs to rescale the forcing  $f(R \cdot)$  to keep it macroscopic.

In the context of stochastic homogenization, these additional assumptions concern the law of  $a$  i.e. the law on  $\mathcal{P}$  here (which fully encompass the randomness by (1.7)). We define the space  $\Omega$  of locally finite point sets  $\mathcal{P}$ , designated from now on as random point processes, via

$$\Omega := \{\mathcal{P} \in (\mathbb{R}^d)^\mathbb{N} \mid \forall B \subset \mathbb{R}^d \text{ Borel set, } \#(\mathcal{P} \cap B) < \infty\}. \quad (1.10)$$

$\mathcal{P} \in \Omega$  is now a random variable. Before defining its law, we first need a notion of measurability. Following [Chi+13, Section 4.1.1 p.108], we equip  $\Omega$  with the  $\sigma$ -algebra

$$\mathcal{F} := \sigma(\mathcal{P} \mapsto \#(\mathcal{P} \cap B), B \subset \mathbb{R}^d \text{ bounded Borel set})$$

(which is countably generated since we can limit ourselves to rational rectangles). Equivalently in [DV08],  $\mathcal{P}$  is represented by the locally finite measure  $\mu_{\mathcal{P}} := \sum_{x \in \mathcal{P}} \delta_x$  which acts on the space of compactly supported continuous functions via  $f \mapsto \sum_{x \in \mathcal{P}} f(x)$ . Using this representation,  $\Omega$  can be identified with a subspace of the space of Radon measures for which we have a notion of Borel sets from the topology of weak convergence. We then equip  $(\Omega, \mathcal{F})$  with a probability measure  $\mathbb{P}$  that we assume :

- *stationary* (i.e. without origin) : for all  $z \in \mathbb{R}^d$ ,  $\tau_z \mathcal{P} \sim_{\text{law}} \mathcal{P}$  where the shift  $\tau_z$  is defined by  $\tau_z \mathcal{P} := (x + z)_{x \in \mathcal{P}}$ ,
- *ergodic* (i.e. decorrelation at large scales) : for any  $F \in \mathcal{F}$ ,  $(F = F + z \text{ for all } z \in \mathbb{R}^d) \implies \mathbb{P}[F] \in \{0, 1\}$ .

By Birkhoff theorem, ergodicity implies that  $\mathbb{E}[f(\mathcal{P})] = \lim_{R \rightarrow \infty} \int_{B_R} f(\mathcal{P} + z) dz$

almost surely for any  $f \in L^1(\Omega, \mathbb{P})$  (where  $B_R := B_R(0)$  is the ball of radius  $R$  implicitly centered at 0). This means that we can recover expectations from spatial averages over large domains.

**Remark 1.1** (i) Although not required for homogenization to hold, it is useful to add a hardcore assumption on  $\mathcal{P}$ , i.e. to assume that its minimal distance  $\ell(\mathcal{P}) := \inf_{\substack{x, y \in \mathcal{P} \\ x \neq y}} |x - y|$  is strictly positive. If  $\ell(\mathcal{P}) > 2$ , the inclusions are disjoint so that (1.7) rewrites

$$a(\mathcal{P}, \cdot) = \alpha + (\beta - \alpha) \sum_{x \in \mathcal{P}} \mathbb{1}_{B(x)}(\cdot) . \quad (1.11)$$

(ii) In this formula, the random sum  $\sum_{x \in \mathcal{P}}$  arises. This sum is the natural object for random point processes. Even if the points of  $\mathcal{P}$  all play a symmetric role in (1.10), it is sometimes useful to label them so that  $\mathcal{P} = (x_n)_{n \in \mathbb{N}}$  (which is legitimate as one can construct a measurable enumeration).  $\square$

The qualitative theory of stochastic homogenization was established in the late 1970s independently in the pioneering works of Papanicolaou and Varadhan [PV81] and Kozlov [Koz80], based on the compensated compactness by Murat and Tartar [Mur78; MT97]. Under the stationary and ergodic assumptions, it states that the sequence  $u_\varepsilon$  weakly converges in  $H^1(Q)$  almost surely to the unique (Lax-Milgram) solution  $\bar{u} \in H_0^1(Q)$  of

$$-\nabla \cdot \bar{a} \nabla \bar{u} = f \quad \text{in } Q \quad (1.12)$$

(with Dirichlet BC) where  $\bar{a}$  is a constant deterministic strongly elliptic matrix. In other words, at large scales, the random heterogeneous material (1.7) behaves as an effective homogenized material with constant conductivity  $\bar{a}$  called homogenized coefficient. This coefficient only depends on the law of  $\mathcal{P}$  and in particular neither on  $f$ ,  $Q$  nor on the boundary condition. As a by product, the theory even yields a formula : given  $e \in \mathbb{R}^d$  (typically in the canonical basis  $(e_i)_{1 \leq i \leq d}$  of  $\mathbb{R}^d$ ),

$$\bar{a}e := \mathbb{E} [a(\mathcal{P})(\nabla \phi_e(\mathcal{P}) + e)] \quad (1.13)$$

where  $\phi_e(\mathcal{P})$  is the so-called corrector (in the direction  $e$ ) defined as the suitable (see below) solution of the whole space equation

$$-\nabla \cdot a(\mathcal{P})(\nabla \phi_e(\mathcal{P}) + e) = 0 \quad \text{in } \mathbb{R}^d. \quad (1.14)$$

By stationarity, the expectation in (1.13) can be taken at any point  $z \in \mathbb{R}^d$  and here implicitly at 0.

**Remark 1.2** The other key idea in homogenization (however less related to the effective properties of materials), is that the corrector allows for a better approximation of  $u_\varepsilon$ . Indeed,  $u_\varepsilon$  do not converge strongly in  $H^1$  to  $\bar{u}$  as  $u_\varepsilon$  exhibits the spacial oscillations of  $a_\varepsilon$ . But these oscillations have been averaged out in  $\bar{a}$  hence in  $\bar{u}$ . However, replacing  $\bar{u}$  by its so-called two-scale expansion,  $u_\varepsilon^{(1)} := \bar{u} + \varepsilon \phi_{e_i} \partial_i \bar{u}(\frac{\cdot}{\varepsilon})$ , we have  $u_\varepsilon - u_\varepsilon^{(1)} \rightarrow 0$  in  $H^1$  as  $\varepsilon \rightarrow 0$  : the corrector has reconstructed the oscillations of  $u_\varepsilon$ . Note that, without the corrector, we still have the strong convergence  $u_\varepsilon \rightarrow \bar{u}$  in  $L^2$  by Rellich's theorem.  $\square$

Let us now discuss the equation (1.14). It can be rewritten  $-\nabla \cdot a \nabla \phi_e = \nabla \cdot a e$  so that formally,

$$\nabla \phi_e = \nabla (-\nabla \cdot a \nabla)^{-1} \nabla \cdot a e . \quad (1.15)$$

Even if (1.9) is a linear equation, this formula highlights that the corrector and therefore the homogenized coefficient (via (1.13)) depend nonlinearly and nonlocally on the point process  $\mathcal{P}$ . The difficulty here is that the source term  $\nabla \cdot a e$  does not vanish at infinity, which implies that the solution  $\phi_e$  itself (whatever the meaning we give) does not decay at infinity either. This prevents us from using standard energy methods (such as Lax-Milgram).

If  $a$  is periodic, i.e. here if  $\mathcal{P}$  is a periodic point set, this difficulty is avoided as the problem can be folded to one bounded periodic cell (typically the unit cube if  $\mathcal{P} = \mathbb{Z}^d$ ). Lax-Milgram (in  $H_{\text{per}}^1$ ) then yields a unique periodic solution of (1.14). Note that the function is unique up to an additive constant but its gradient is unique. Moreover, in this periodic framework, stationarity should be adapted accordingly using discrete shifts and expectations should be replaced by averages over the periodicity cell.

The random case is more subtle : existence and uniqueness rely on the stationarity (and ergodicity) assumptions. Following [JKO94, Chapter 7] and [PV81, Section 2 p.840], one can construct a unique solution in the following class :

- $\phi_e \in H_{\text{loc}}^1(\mathbb{R}^d)$  almost surely,
- $\nabla \phi_e$  is a stationary field, i.e. the distribution of  $\nabla \phi_e(\cdot + z)$  do not depend on  $z \in \mathbb{Z}^d$  (actually, we even require that shifts in the physical space and in the probability space are intertwined by the so-called contravariance property  $\nabla \phi_e(\mathcal{P} + z, \cdot) = \nabla \phi_e(\mathcal{P}, \cdot - z)$  for all  $z \in \mathbb{R}^d$ ),
- $\mathbb{E}[\nabla \phi_e] = 0$ ,
- $\mathbb{E}[|\nabla \phi_e|^2] \lesssim |e|^2$ ,
- $\int_{B(0)} \phi_e = 0$  a.s (anchoring condition).

The idea is to directly consider (1.14) in the probability space which, by stationarity, somehow corresponds to the periodic torus up to the fundamental difference that the probability space is not finite dimensional. More precisely, stationarity

(in fact contravariance) allows the transfer of a notion of gradient from  $\mathbb{R}^d$  to  $\Omega$  : the stochastic counterpart of  $\partial_i$  (for  $i = 1, \dots, d$ ) is denoted by  $D_i$  and is defined by

$$D_i f(\mathcal{P}) = \lim_{h \rightarrow 0} \frac{f(\mathcal{P} + h e_i) - f(\mathcal{P})}{h}. \quad (1.16)$$

These are the infinitesimal generators of the translations (in the directions  $(e_i)_{1 \leq i \leq d}$ ) and we denote by  $\mathcal{H}(\Omega) \subset L^2(\Omega)$  the domain of  $D := (D_1, \dots, D_d)$ . For  $f, g \in \mathcal{H}(\Omega)$ , we have the following *integration by parts* formula

$$\mathbb{E} [f D_i g] = -\mathbb{E} [D_i f g] \quad (1.17)$$

so that (1.14) can be rewritten

$$\mathbb{E} [D\psi \cdot a(\mathcal{P}) D\phi_e] = -\mathbb{E} [D\psi \cdot a e] \quad \forall \psi \in \mathcal{H}(\Omega) \quad (1.18)$$

and solved by Lax-Milgram in  $L_{\text{pot}}^2(\Omega) := \overline{\{D\psi, \psi \in \mathcal{H}(\Omega)\}}^{L^2(\Omega)}$ , see the detailed lectures notes [Feh20, Section 3.3 p.25].

Note that here the gradient  $\nabla \phi_e$  is constructed first and the corrector  $\phi_e$  is only recovered as a by-product, hence the need for an anchoring condition and the absence of uniform bounds on  $\phi_e$ .

From an applied perspective, the formula (1.13) of  $\bar{a}$  is not fully satisfying : except in a few specific settings, it remains abstract. These specific settings include the unidimensional case for which  $\bar{a}$  is simply given by the harmonic average  $\mathbb{E} [a^{-1}]^{-1}$  of  $a$ , in line with the formula for equivalent series resistors (reminding that conductivity is the inverse of resistivity). We recover the physical intuition that in 1D, if the inclusions are insulators, conduction is broken and the medium effectively behaves as an insulator. In 2D, we can also obtain the geometric average from Dykhne's formula in the case of random i.i.d. checkerboard taking with probability  $\frac{1}{2}$  either the conductivity  $\alpha \text{Id}$  or  $\beta \text{Id}$  so that  $\bar{a} = \sqrt{\alpha\beta} \text{Id}$ , see [JKO94, Section 7.3 p.235]. Another explicit case (in any dimension) is given by layered materials, also called laminates. For these materials, the conductivity  $a$  only varies in one direction rendering them effectively unidimensional. The lamination procedure can be iterated in several directions while retaining an explicit formula. This defines the class of sequential laminates, a rich parametrized family of composite materials widely used in practice, see for instance [All02, Section 2.2.1 p.102].

## 1.2.2 Approximations

Formula (1.13) is (in general) not explicit and one has to resort to numerical simulations to approximate the homogenized coefficient  $\bar{a}$ . The difficulties arise from the corrector equation (1.14) which is both random and posed on the whole

space (in the random case). We focus here on the latter : the elliptic nature of the equation yields a nonlocal dependence on the randomness (see (1.15)) and the inhomogeneity of the conductivity prevents us from using potential theory (such as Single or Double Layers potentials). We thus need an approximation procedure to restrict this problem to a finite volume better suited for numerical simulation. This can be done either by changing the equation or by changing the domain.

The first strategy of approximation is the addition of a so-called massive parameter. For all  $T > 0$ , fixing  $\mathcal{P} \in \Omega$ , we consider  $\phi_T(\mathcal{P})$  (we drop the subscript  $e$  in the sequel for convenience) solution of

$$\frac{1}{T}\phi_T - \nabla \cdot a(\mathcal{P})(\nabla\phi_T(\mathcal{P}) + e) = 0 \quad \text{in } \mathbb{R}^d . \quad (1.19)$$

$\phi_T$  is uniquely defined in the space  $H_{\text{loc}}^1 := \{v \in H_{\text{loc}}^1(\mathbb{R}^d) \mid \sup_{z \in \mathbb{R}^d} \int_{B(z)} v^2 + |\nabla v|^2 < \infty\}$  by fully deterministic arguments. Indeed, the massive term  $\frac{1}{T}$  exponentially screens the medium at distance  $\sqrt{T}$  so that the elliptic operator  $\frac{1}{T} - \nabla \cdot a \nabla$  can be inverted in the whole space, see Section 2.2.1. For instance, for  $d = 3$ , the Green function of the massive Laplacian  $\frac{1}{T} - \Delta$  is explicitly given by  $G_T := e^{-\frac{|\cdot|}{\sqrt{T}}} \cdot |\cdot|^{d-2} = e^{-\frac{|\cdot|}{\sqrt{T}}} G$  (where  $G$  is the standard Green function for the Laplacian). This yields the following approximation of the homogenized coefficient

$$\bar{a}_T := \mathbb{E} [a(\mathcal{P})(\nabla\phi_T + e)] . \quad (1.20)$$

Seen as a random field,  $\nabla\phi_T \xrightarrow[T \rightarrow \infty]{L^2(\Omega)} \nabla\phi$  so that  $\bar{a}_T \rightarrow \bar{a}$  as  $T \rightarrow \infty$ .

The main feature of this approach is that it preserves stationarity. Indeed, by uniqueness,  $\phi_T$  is even contravariant. Here,  $\phi_T$  itself has finite moments (for finite  $T$ ) a priori, unlike  $\phi_e$ . The equation can even be rewritten in the probability space : for all  $\psi \in \mathcal{H}(\Omega)$ ,

$$\mathbb{E} \left[ \frac{1}{T}\phi_T\psi \right] + \mathbb{E} [D\psi \cdot a(D\phi_T + e)] = 0 .$$

The main drawback here is that we have modified the differential relations. For instance, the exponential screening is broken if we add an incompressibility constraint (i.e. divergence free fields).

While the screening drastically reduces the influence of the medium at infinity, the equation (1.19) is still posed in the whole space. The other approach then restricts the equation to a finite box, following the naive intuition. For  $R > 0$ , consider  $\phi_R(\mathcal{P})$  solution of

$$-\nabla \cdot a(\mathcal{P})(\nabla\phi_R(\mathcal{P}) + e) = 0 \quad \text{in } Q_R \quad (1.21)$$

which is deterministically uniquely defined in

$$H_{\text{per}}^1(Q_R) := \{v \in H_{\text{loc}}^1(\mathbb{R}^d) \mid v \text{ is } Q_R\text{-periodic, } \int_{Q_R} v = 0\}$$

choosing periodic boundary. The other natural choice are Dirichlet boundary conditions so that the solution belongs to  $H_0^1(Q_R)$ . We define the corresponding approximation of the homogenized coefficient by

$$a_R(\mathcal{P}) := \int_{Q_R} a(\mathcal{P})(\nabla\phi_R + e) . \quad (1.22)$$

Note that,  $a_R$  is a random variable contrary to  $\bar{a}$  and  $\bar{a}_T$ . However, by ergodicity, we have  $a_R \rightarrow \bar{a}$  almost surely as  $R \rightarrow \infty$ . We could also consider  $\mathbb{E}[a_R(\mathcal{P})]$  so that the random error for finite  $R$  can be decomposed in bias and variance writing  $a_R - \bar{a} = (\mathbb{E}[a_R] - \bar{a}) + (a_R - \mathbb{E}[a_R])$ .

This approach preserves the structure of the equation (for instance, incompressibility) but breaks stationarity as it fixes (implicitly) an origin at the center of  $Q_R$ . This creates boundary layer effects : some inclusions may intersect  $\partial Q_R$  and thus would be cut. Stationarity can be recovered if the law of  $\mathcal{P}$  admits a periodic approximation i.e. we can find a periodic  $\mathcal{P}_R \subset Q_R$  so that  $\mathcal{P}_R \rightarrow \mathcal{P}$  (in law) as  $R \rightarrow \infty$ . In that case, the inclusions intersecting  $\partial Q_R$  should then be periodically extended to the torus  $Q_R$ . This periodic approximation in law, although restrictive, is verified for a wide range of models in practice (notably when they admit an algorithmic construction).

These approximations provide an alternative construction of a solution to the corrector equation (1.14). Details on this approach can be found in [Neu17, Proposition 2.15 p.21]. It relies on three main steps :

1. Obtain a uniform bound on the gradient (for instance  $\mathbb{E}[|\nabla\phi_T|^2] \lesssim 1$ ) by energy estimations. In general, one does not get a uniform a priori estimate on the solution itself (for instance, a priori,  $\mathbb{E}[|\phi_T|^2] \lesssim T$  which diverges as  $T \rightarrow \infty$ ).
2. Extract a (weak) limit  $\Psi$  of this sequence of gradient by compactness. This limit is necessarily a gradient i.e.  $\Psi = \nabla\phi$  as the space of potential fields (a.k.a. gradients) is closed for weak convergence by Weyl's theorem. Passing to the limit in the weak formulation of the approximated problem (for instance (1.19)),  $\phi$  solves (1.14) weakly.
3. Show uniqueness (in the above class) of solution of (1.14). This step is a

consequence of ergodicity in the form of the sublinearity of the corrector

$$\lim_{R \rightarrow \infty} \frac{1}{R^2} \int_{B_R} \phi^2 = 0, \quad (1.23)$$

see [Neu17, Lemma 2.18 p.24 & Corollary 2.26 p.31].

In practice, these two approximation strategies can be combined tuning the massive parameter to dim the influence of boundaries, see [GH16]. The massive term can also be used for an iterative coarse-graining, see [Arm+21].

### 1.2.3 Quantification of the approximations

Quantifying the convergence of numerical approximations of the homogenized coefficient  $\bar{a}$  (here,  $\bar{a}_T$  and  $a_R$ ) is a first goal of quantitative stochastic homogenization, which has been a booming field in the last fifteen years. We will focus on the convergence of  $a_R$  since, heuristically,  $\bar{a}_T \sim \mathbb{E}[a_R]$  for  $R = \sqrt{T}$  by screening. As  $a_R$  converges to  $\bar{a}$  by the Birkhoff ergodic theorem, one needs to quantify ergodicity to obtain a rate. The main difficulty lies (as before) in the nonlinear and nonlocal dependence of  $\bar{a}$  on the law of  $\mathcal{P}$ , see (1.15). Surprisingly, the optimal quantitative results state that  $a_R(\mathcal{P})$  essentially behaves as if its dependence on  $\mathcal{P}$  were local : the fluctuation of  $a_R = \int_{Q_R} a(\nabla\phi_R + e)$  exhibits a similar behavior to the one of the empirical average  $\int_{Q_R} ae$ .

In the case of linear averages, it is well-known that the Law of Large Numbers (a form of ergodic theorem) is quantified by the Central Limit Theorem (CLT). This yields, for instance,

$$e \cdot \int_{Q_R} ae \simeq \mathbb{E}[e \cdot ae] + R^{-\frac{d}{2}} \mathcal{N}(0, \text{Var}[e \cdot ae])$$

where  $\mathcal{N}$  denotes a normal random variable, if  $\mathcal{P}$  is weakly correlated enough so that the CLT scaling holds for fluctuations. The rate of convergence can still be estimated for more correlated fields, for instance in the  $\alpha$ -mixing case.

**Definition 1.1** We say that  $\mathcal{P}$  is algebraically  $\alpha$ -mixing with rate  $\beta > 0$  if for all  $U, V \subset \mathbb{R}^d$  and for any  $A \in \sigma(\mathcal{P} \cap U)$  and  $B \in \sigma(\mathcal{P} \cap V)$

$$|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]| \lesssim (1 + \text{dist}(U, V))^{-\beta}. \quad \square$$

The first quantitative (yet suboptimal) approximation results are due to Yurinskii [Yur86] for  $d > 2$  in the post-processed form [EMZ05, Theorem 1.3 p.126], see also Bourgeat and Piatnitski [BP04, Theorem 4 p.163]. In the 2010s, two quantitative approaches emerged.



The first approach is due to Gloria and Otto (as well as Neukamm, Fischer, Duerinckx, Nolen among others) and rely on a quantification of ergodicity through functional inequalities (involving a derivative with respect to randomness in the spirit of Malliavin calculus). These inequalities imply a strong decorrelation of the random field  $a$ , typically Poisson-like point processes but they can be relaxed to cover more correlated cases, see [DG20a; DG20b]. Treating first the case of discrete elliptic equations in divergence form with i.i.d. conductances (the prototypical example of stochastic homogenization) [GO11; GO12; GNO15], these authors obtained in [GN16] a quantitative CLT of  $a_R$  around  $\bar{a}$  with a rate of convergence in Wasserstein distance toward normality and an explicit covariance structure : there exists  $\sigma_e > 0$  s.t.

$$\text{dist} \left( R^{\frac{d}{2}} \frac{e \cdot (a_R - \bar{a})e}{\sigma_e}, \mathcal{N}(0, 1) \right) \lesssim R^{-\frac{d}{2}} \ln^d(R)$$

(note that the expected optimal rate should be  $R^{-\frac{d}{2}} \ln(R)$ ). See [DFG22; DO20] in the continuum setting. See also [Clo+22] when the law of  $\mathcal{P}$  admits a periodic approximation.

The second approach is due Armstrong and Smart (as well as Kuusi and Mourrat among others) and relies on a quantification of ergodicity through a variational strategy. Similar results hold, see [AS16] and [AKM19; AK22] and this approach remains valid for strongly correlated fields. In particular, [AM16, Theorem 5.1 p.294] (reduced to the conductivity case, as this theorem is written in the non-linear setting) provides the best known results in the  $\alpha$ -mixing case : if  $\mathcal{P}$  is algebraically  $\alpha$ -mixing with rate  $\beta$  (see Definition 1.1), there exists  $0 < \gamma \ll \beta$  s.t. for all  $R > 0$ ,

$$|\bar{a} - \mathbb{E}[a_R(\mathcal{P})]| \lesssim R^{-\gamma} . \quad (1.24)$$

Since its assumptions are quite weak, this estimate is a useful tool. Indeed, a suitable version of (1.24) is used in [DG20c, Theorem 5 p.70] to obtain Einstein's formula (1.5) by the so-called implicit renormalization method, see Section 1.4.3 below.

**Remark 1.3 (Functional inequalities and  $\alpha$ -mixing)** The notion of  $\alpha$ -mixing (used here in a form with explicit decay rate) was introduced to handle *linear* functions of the randomness (such as empirical averages) whereas the functional inequalities used in the approach by Gloria and Otto (abbreviated in FI in the sequel) were introduced to handle *nonlinear* functions of the randomness (such as  $\bar{a}(\mathcal{P})$ ). The latter, although more general, does not fully encompass the former.

Indeed, for reasonably correlated fields, FI imply  $\alpha$ -mixing. For instance, every Gaussian field satisfies FI (it was actually the prototypical example) but a Gaussian

field is only  $\alpha$ -mixing when its covariance is integrable, see [DG20a, Proposition 1.4].

Assuming  $\alpha$ -mixing only does not yield any notion of FI. However, most models encountered in stochastic homogenization satisfy some FI.  $\square$

### 1.2.4 Back to dilute regimes

While the approximations (1.22) and (1.20) are better suited for numerical approximations, their computation remains expensive as one can hope (at best) for CLT fluctuations. Therefore, in the random setting, Clausius-Mossotti formulas (1.4) keep some relevance.

First, it provides a *first guess* of the true homogenized matrix  $\bar{a}$ . Therefore, it can be used in variance reduction methods as in [AL11; AL12]. These papers sparked a renewal of interest for Clausius-Mossotti formulas in the stochastic homogenization community [Mou15; DG16].

Second, (1.4) and its higher order generalizations already provide some information on the original medium, notably its volume fraction. In the context of homogenization of inclusion models (1.7), the volume fraction of the inclusion phases is linked to the intensity  $\lambda(\mathcal{P})$  of the point process (i.e. the average number of points per unit of volume). For a stationary ergodic point process  $\mathcal{P}$ , it is defined as

$$\lambda(\mathcal{P}) := \mathbb{E} [\#(\mathcal{P} \cap Q)] . \quad (1.25)$$

As  $\mathcal{P}$  is ergodic, it can be recovered by taking averages over large domains via

$$\lambda(\mathcal{P}) = \lim_{R \rightarrow \infty} \frac{\#(\mathcal{P} \cap Q_R)}{|Q_R|} . \quad (1.26)$$

The volume fraction is then defined similarly via

$$c(\mathcal{P}) := \lim_{R \rightarrow \infty} \frac{|\cup_{x \in \mathcal{P}} B(x) \cap Q_R|}{|Q_R|} . \quad (1.27)$$

If the inclusions are disjoint, as in (1.11),  $c(\mathcal{P}) = |B|\lambda(\mathcal{P})$ . The Clausius-Mossotti formula (1.4) can thus be seen as an expansion of  $\bar{a}$  in  $\lambda$ . In this manuscript, we aim at high order expansions in the form

$$\bar{a} = \sum_{i=0}^n \mathcal{A}^{(i)} \lambda^{\alpha_i} + o(\lambda^{\alpha_n}) \quad (1.28)$$

for  $n \geq 1$  with coefficients  $(\mathcal{A}^{(i)})_{0 \leq i \leq n} \in (\mathbb{R}^{d \times d})^n$  and where the exponents  $(\alpha_i)_{0 \leq i \leq n} \in (0, \infty)^n$  form an increasing sequence (for instance  $\alpha_i = i$ , see the

discussion regarding Batchelor formula (1.6)). From a numerical perspective, it is of course better if the coefficients  $(\mathcal{A}^{(i)})_{i \geq 0}$  are given by explicit formulas or easily computable. But, even if these coefficients remain abstract, the mere knowledge of the exponents  $(\alpha_i)_{i \geq 0}$  already provides useful information. Indeed, as the simulations are expensive, engineers often prefer to extrapolate from previous computations. The expansion (1.28) then allows for a more accurate extrapolation.

These formulas can also be used to (partially) reconstruct the microstructure of the medium from external measurements, which constitutes an inverse problem. Using the first order formula (1.4), it is possible to measure the volume fraction, *inter alia*, see for instance [AK07, Chapter 8].

## 1.3 Notion of dilution

In this section, we present different modes of dilution starting from one-parameter models to a more intrinsic notion of dilution. To do so, we introduce two important tools for dilute expansions : the cluster expansion and multi-point intensities. The former, in the spirit of Taylor expansions, provides a decomposition of many-body interactions in finite subsets contributions whereas the latter generalizes the notion of intensity providing an intrinsic measure of dilution which combines nicely with the cluster formalism. Finally, we will illustrate these dilution methods on a toy model. In particular, we will see why the volume fraction  $c$  (see (1.27)) or equivalently the intensity of the point process  $\lambda(\mathcal{P})$  (see (1.25)) is not sufficient to characterize dilution at higher orders.

### 1.3.1 One parameter models

As already mentioned, for inclusions models such as (1.7) and following the physical intuition, we usually characterize dilute regimes by the assumption  $\lambda(\mathcal{P}) \ll 1$ . However, by definition (see (1.25)), the intensity is only a suitable notion of mean for point processes. Therefore, it does not describe its correlation structure which should be of crucial importance for high order expansions.

We will see that the so-called multi-point intensities provides a description of the (local) correlations. Before their introduction, the idea in the literature to bypass this difficulty was to rely on one parameter models : fix an underlying point process  $\mathcal{P}$  (assumed stationary and ergodic so that homogenization holds) and modify it so that the parameter characterizes the dilution level. The two canonical models are geometric dilation and Bernoulli random deletion. The first one is essentially a rescaling whereas the second one can be informally described as tossing independent coins on each inclusion.

In the geometric dilation case, given  $\mathcal{P} \in \Omega$  a stationary ergodic point process, choose a dilation parameter  $L \geq 1$  and consider the dilated process

$$L\mathcal{P} := (Lx)_{x \in \mathcal{P}} \in \Omega. \quad (1.29)$$

Note that geometric dilation preserves stationarity and ergodicity. This rescaling provides indeed a dilution : by (1.26),

$$\lambda(L\mathcal{P}) = L^{-d}\lambda(\mathcal{P}) \quad (1.30)$$

so that  $\lambda(L\mathcal{P}) \ll 1$  when  $L \gg 1$ . However, the correlation structure of  $L\mathcal{P}$  is essentially similar to the one of the original process  $\mathcal{P}$ .

In the Bernoulli random deletion case, given  $\mathcal{P} \in \Omega$  (still a stationary ergodic point process), choose a deletion parameter  $0 \leq p \leq 1$ . Define a family of i.i.d. Bernoulli random variables s.t. for every  $x \in \mathcal{P}$ ,  $b_x \sim p\delta_1 + (1-p)\delta_0$  independent of  $\mathcal{P}$ . We then consider the deleted process

$$\mathcal{P}^{(p)} := \{x \in \mathcal{P} \mid b_x = 1\}. \quad (1.31)$$

As the family  $(b_x)_{x \in \mathcal{P}}$  is i.i.d., Bernoulli random deletion preserves stationarity and ergodicity. This rescaling also provides a dilution : we claim that

$$\lambda(\mathcal{P}^{(p)}) = p\lambda(\mathcal{P}) \quad (1.32)$$

so that  $\lambda(\mathcal{P}^{(p)}) \ll 1$  when  $p \ll 1$ . Indeed, using the definition (1.25) of the intensity,

$$\begin{aligned} \lambda(\mathcal{P}^{(p)}) &= \mathbb{E} [\#(\mathcal{P}^{(p)} \cap Q)] = \mathbb{E} \left[ \sum_{x \in \mathcal{P}^{(p)}} \mathbf{1}_Q(x) \right] = \mathbb{E} \left[ \sum_{x \in \mathcal{P}} b_x \mathbf{1}_Q(x) \right] \\ &= \mathbb{E}_{\mathcal{P}} \left[ \sum_{x \in \mathcal{P}} \mathbb{E}_b[b_x] \mathbf{1}_Q(x) \right] = p \mathbb{E} \left[ \sum_{x \in \mathcal{P}} \mathbf{1}_Q(x) \right] = p\lambda(\mathcal{P}) \end{aligned}$$

as the expectation can be decomposed as  $\mathbb{E} = \mathbb{E}_{\mathcal{P}}\mathbb{E}_b$  by independence between the Bernoulli variables and  $\mathcal{P}$  and since  $\mathbb{E}[b_x] = p$ . This dilution procedure modifies the correlation structure. Indeed, the Bernoulli random variables provide additional independence. For  $p \ll 1$ ,  $\mathcal{P}^{(p)}$  should essentially look like a Poisson process regardless of the underlying process  $\mathcal{P}$ .

The main difference between these dilution procedures is how they modify the minimal distance  $\ell(\mathcal{P})$  between points : random deletion leaves it unchanged, hence potentially arbitrarily small, whereas geometric dilation increases it by a (multiplicative) factor  $L$ , rendering it potentially very large for  $L \gg 1$ .

On the effective conductivity problem, we can therefore consider  $\bar{a}^{(p)}$  the homogenized coefficient associated to  $a(\mathcal{P}^{(p)})$  (see (1.7)) and  $\bar{a}^L$  associated to  $a(L\mathcal{P})$ . Their dilute expansion is thus equivalent to a Taylor expansion of the maps  $p \mapsto \bar{a}^{(p)}$  at  $p = 0$  and  $L \mapsto \bar{a}^L$  at  $L \rightarrow \infty$  instead of the higher order formula (1.28).

### 1.3.2 Cluster expansion formalism

We recall that the key idea to establish Clausius-Mossotti, Einstein's or Batchelor's formulas was to consider the particles as isolated, neglecting the many particles interactions thanks to diluteness, see Section 1.1.2. To obtain better approximations, it is fruitful to take into account finite subset contributions (pairs, triples, ...). This can efficiently be done using the so-called cluster expansion. This approach is well-known in statistical mechanics (see [Tor02, Chapter 19]) and has already been successfully used in stochastic homogenization, see for instance [DG16]. In this paper, the authors study the random deletion case for which the cluster expansion appears naturally :  $j$ -inclusions interactions appear exactly when  $j$  Bernoulli variables take the value one. See also [DG20c, Section 1.3.1].

Given a discrete countable set  $E = \{x_1, \dots, x_{|E|}\} \subset \mathbb{R}^d$ , cluster expansion formalism applies to functions

$$\Phi = \Phi^\# : F \subset E \mapsto \Phi^F$$

from the power set of  $E$  (i.e. the set of subsets of  $E$ ) to a given vector space  $V$ . A typical example is the corrector from the conductivity problem  $\phi : \mathcal{P} \mapsto \phi(\mathcal{P})$ , defined by its PDE (1.14), seen as a function of the point set  $\mathcal{P}$ . Note that, to be fully rigorous, one should rather consider one of its approximations. For instance the massive corrector  $\phi_T$  defined by the PDE (1.19) is well-posed for any given deterministic (finite or) infinite subset of particles, unlike (1.14) which requires a probabilistic input for infinite sets.

For all  $x \in E$ , we introduce the difference operator  $\delta^{\{x\}}$  acting on this space defined by

$$\delta^{\{x\}}\Phi^F := \delta^{\{x\}}\Phi^{F \cup \{x\}} := \Phi^{F \cup \{x\}} - \Phi^F \quad \forall F \subset E.$$

The difference operator  $\delta^{\{x\}}$  provides a notion of discrete derivative (or sensitivity) of  $\Phi$  with respect to adding a point at  $x$ .

For  $F \subset E$ , we recursively define higher-order difference operators via

$$\delta^F := \prod_{x \in F} \delta^{\{x\}}$$

with the natural convention  $\delta^\emptyset = \text{Id}$  and noting that for all  $x \neq y$ ,  $\delta^{\{x\}}\delta^{\{y\}} =$

$\delta\{y\}\delta\{x}$  and  $(\delta\{x\})^2 = -\delta\{x\}$ . One can check that for any  $F, H \subset E$ ,

$$\delta^F \Phi^H := \delta^F \Phi^{H \cup \#} = \sum_{G \subset F} (-1)^{|F \setminus G|} \Phi^{G \cup H} .$$

These difference operators are the building blocks to construct the so-called *cluster expansions*. For  $E$  finite, the following identity holds (using some combinatorics) :

$$\Phi^E = \Phi^\emptyset + \sum_{x \in E} \delta\{x\} \Phi^\# + \frac{1}{2!} \sum_{\substack{\neq \\ x_1, x_2 \in E}} \delta\{x_1, x_2\} \Phi^\# + \dots + \frac{1}{|E|!} \sum_{\substack{\neq \\ x_1, \dots, x_{|E|} \in E}} \delta\{x_1, \dots, x_{|E|}\} \Phi^\# \quad (1.33)$$

where we have set  $\sum_{\substack{\neq \\ x_1, \dots, x_j \in E}} := \sum_{\substack{x_1, \dots, x_j \in E \\ x_k \neq x_l \text{ for } k \neq l}}$ . This can be rewritten in the more compact form

$$\Phi^E = \sum_{F \subset E} \delta^F \Phi^\# = \sum_{i=0}^{|E|} \sum_{\substack{F \subset E \\ |F|=i}} \delta^F \Phi^\# .$$

**Example 1.1** Taking  $E = \{a, b\}$ , we have

$$\begin{aligned} \delta\{a, b\} \Phi &= \delta\{a\} \delta\{b\} \Phi = \delta\{a\} (\Phi^{\{b\}} - \Phi^\emptyset) \\ &= \Phi^{\{a, b\}} - \Phi^{\{a\}} - \Phi^{\{b\}} + \Phi^\emptyset \end{aligned}$$

so that

$$\Phi^{\{a, b\}} = \Phi^\emptyset + \delta\{a\} \Phi^\# + \delta\{b\} \Phi^\# + \delta\{a, b\} \Phi^\# .$$

This corresponds to (1.33) writing the last term in the symmetric form  $\delta\{a, b\} \Phi = \frac{1}{2}(\delta\{a, b\} \Phi + \delta\{b, a\} \Phi)$ .  $\square$

**Example 1.2** If the function  $\Phi$  is linear i.e.  $\Phi^F := \sum_{x \in F} \varphi(x)$  for all  $F \subset E$ , formula (1.33) does not provide any additional information. Indeed, for all  $x \in E$ ,

$$\delta\{x\} \Phi^\# = \varphi(x)$$

and for all  $y \neq x$

$$\delta\{x, y\} \Phi^\# = 0 .$$

Therefore, cluster expansions are most relevant when  $\Phi$  is nonlinear.  $\square$

For all  $n$ , we also introduce  $\mathcal{C}^{(n)}$ , the truncation of this series to order  $n$ , given

by

$$\mathcal{C}^{(n)}\Phi^\#(E) := \sum_{i=0}^n \sum_{\substack{F \subset E \\ |F|=i}} \delta^F \Phi^\# \quad (1.34)$$

as well as  $\overline{\Phi}^{(n)}$ , the  $n$ th cluster coefficient, via

$$\overline{\Phi}^{(n)}(E) := \sum_{\substack{F \subset E \\ |F|=n}} \delta^F \Phi^\# = \frac{1}{n!} \sum_{x_1, \dots, x_n \in E} \delta^{\{x_1, \dots, x_n\}} \Phi^\# . \quad (1.35)$$

When  $|E| = \infty$ , as it is the case for random point processes  $\mathcal{P} \in \Omega$  (seen as countable sets), the series in (1.33) does not converge in general. However, we expect that its finite truncations, referred to in the sequel as cluster expansions, provide a good ansatz.

### 1.3.3 Intrinsic dilution : multi-point intensities

While the cluster expansion appears naturally in the Bernoulli random deletion case, its coefficients are robust and can be applied for a wide variety of models. Their estimation requires the introduction of the so-called multi-point intensities, a refined measure of diluteness introduced in [DG20c, Section 1.3.2] that we now recall.

**Definition 1.2** For  $\mathcal{P} \in \Omega$ , we define its minimal distance

$$\ell := \ell(\mathcal{P}) := \inf_{\substack{x, y \in \mathcal{P} \\ x \neq y}} |x - y| , \quad (1.36)$$

which is almost surely a deterministic characteristic length of  $\mathcal{P}$  (by ergodicity as  $\ell$  is shift-invariant).

If  $\ell(\mathcal{P}) > 0$ , we say that the process is hardcore.  $\square$

In the sequel, we always consider hard-core processes.

**Definition 1.3** For  $\mathcal{P} \in \Omega$  hard-core and  $j \geq 1$ , we define its  $j$ -points intensity by

$$f\lambda_j(\mathcal{P}) := \sup_{z_1, \dots, z_j \in \mathbb{R}^d} \mathbb{E} \left[ \sum_{\substack{\neq \\ x_1, \dots, x_j \in \mathcal{P}}} \ell^{-jd} \prod_{k=1}^j \mathbf{1}_{Q_\ell(z_k)}(x_k) \right] . \quad (1.37)$$

As it proves useful for estimation in the sequel, we also define the aggregated quantity

$$\overline{\lambda}_j(\mathcal{P}) := \max_{\sum_i j_i = j} \prod_i \lambda_{j_i}(\mathcal{P}) .$$

(to which we will often improperly refer as  $j$ -point intensity as then coincide under mixing assumptions, see (1.41) below).  $\square$

Let us now motivate this definition. We have seen that a natural object for the point process  $\mathcal{P}$  is the random variable  $\#\mathcal{P} \cap B$ , the number of points in a given set  $B \subset \mathbb{R}^d$ . We first note that  $\lambda_1(\mathcal{P})$  is nothing but the intensity. Indeed, by stationarity

$$\lambda(\mathcal{P}) = \mathbb{E} [R^{-d} \#(\mathcal{P} \cap Q_R(z))] = \mathbb{E} \left[ \sum_{x \in \mathcal{P}} R^{-d} \mathbf{1}_{Q_R(z)}(x) \right]$$

for any  $R > 0$  and  $z \in \mathbb{R}^d$ . Then, we note that the RHS of (1.37) can be rewritten

$$\mathbb{E} \left[ \sum_{x_1, \dots, x_j \in \mathcal{P}}^{\neq} \prod_{i=1}^j (\ell^{-d} \mathbf{1}_{Q_\ell(z_i)}(x_i)) \right] = \mathbb{E} \left[ \prod_{i=1}^j (\ell^{-d} \#(\mathcal{P} \cap Q_\ell(z_i))) \right]$$

for  $(z_i)_{1 \leq i \leq j}$  s.t. the cubes are disjoint. Note that, by definition (1.36) of  $\ell$ , each cube  $Q_\ell(z)$  contains at most one point of  $\mathcal{P}$ . Hence,  $\lambda_j(\mathcal{P})$  corresponds to the maximum expected number of  $j$ -tuples of points in  $\mathcal{P}$  that lie in the  $\ell$ -neighborhood of an element of  $(\mathbb{R}^d)^j$  (properly normalized by  $\ell$ ): a local notion of correlation.

The multi-point intensities can be defined alternatively using the so-called multi-point densities of the process.

**Definition 1.4** Let  $\mathcal{P} \in \Omega$  be a stationary random point process and  $j \geq 1$ . We define its  $j$ -point density  $f_j$  (also known as  $j$ th-order reduced/factorial moment measure, see [Chi+13, Section 4.3.2 p.121]) as the non-negative function defined by the relation

$$\int_{(\mathbb{R}^d)^j} \chi f_j = \mathbb{E} \left[ \sum_{x_1, \dots, x_j \in \mathcal{P}}^{\neq} \chi(x_1, \dots, x_j) \right], \quad \forall \chi \in C_c^\infty((\mathbb{R}^d)^j). \quad (1.38)$$

Note that  $f_j$  is symmetric as the points are indistinguishable. Intuitively,  $f_j dx_1 \dots dx_j$  is the probability to get  $j$ -points at positions  $(x_1, \dots, x_j)$ .  $\square$

We then recover the  $j$ -point intensity  $\lambda_j(\mathcal{P})$  as the (locally averaged) maximum of  $f_j$

$$\lambda_j(\mathcal{P}) := \sup_{z_1, \dots, z_j \in \mathbb{R}^d} \int_{Q_\ell(z_1) \times \dots \times Q_\ell(z_j)} f_j. \quad (1.39)$$

This definition naturally extends to  $\ell = 0$  setting  $\lambda_j(\mathcal{P}) := \|f_j\|_{L^\infty((\mathbb{R}^d)^j)}$  in that case.



Borrowing from [DG20c, Lemma 1.1 p.8], we recall some useful properties of the multi-point intensities.

**Lemma 1.1** *Let  $\mathcal{P} \in \Omega$  be a stationary ergodic point process.*

— *For all  $j \geq 1$ , we have*

$$\lambda_{j+1}(\mathcal{P}) \leq \ell^{-d} \lambda_j(\mathcal{P}). \quad (1.40)$$

*We also have  $\lambda(\mathcal{P}) \leq \ell^{-d}$ .*

— *If  $\mathcal{P}$  is strongly mixing (see Definition 1.5 below),*

$$\lambda_j(\mathcal{P}) = \bar{\lambda}_j(\mathcal{P}) \quad (1.41)$$

*which implies in particular that  $\lambda_j(\mathcal{P}) \geq \lambda(\mathcal{P})^j$ .  $\square$*

**Definition 1.5** We say that a point process  $\mathcal{P}$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  is strongly mixing if, for all  $U, V \subset \mathbb{R}^d$  and for any  $A \in \sigma(\mathcal{P} \cap U)$  and  $B \in \sigma(\mathcal{P} \cap V)$ ,

$$\mathbb{P}[A \cap B] - \mathbb{P}[A] \mathbb{P}[B] \rightarrow 0 \text{ as } \text{dist}(U, V) \rightarrow \infty. \quad \square$$

Note that the algebraic  $\alpha$ -mixing from Definition 1.1 is a quantification of this notion and that strong mixing implies ergodicity.

For  $\mathcal{P}$  hard-core and mixing, combining (1.40) and (1.41), we have  $\lambda(\mathcal{P})^j \lesssim \lambda_j(\mathcal{P}) \lesssim \lambda(\mathcal{P})$ . One can actually construct point processes for which  $\lambda_j(\mathcal{P})$  spans this entire range. Indeed, following [DG20c, Section 5.1 p.121], for any  $\beta \in [0, 1]$ , there exist point processes s.t.  $\lambda_2(\mathcal{P}) \sim \lambda(\mathcal{P})^{1+\beta}$  and this construction can be generalized to  $k \geq 2$ . The cases  $\beta \in \{0, 1\}$  are simpler : we provide some simple examples to illustrate them.

We start by the canonical example of random point process : the Poisson process. It is to point processes what Gaussian fields are to continuous random fields. A lot of more refined models are based on it, see for instance [Chi+13, Chapter 5].

**Example 1.3 (Poisson)** The Poisson process on  $\mathbb{R}^d$  (resp.  $V$ , for bounded  $V \subset \mathbb{R}^d$ ) is fully parametrized by its intensity  $\lambda > 0$ . To be a Poisson process,  $\mathcal{P}$  should satisfy two properties :

- For any bounded Borel set  $B \subset \mathbb{R}^d$  (resp.  $\subset V$ ),  $\#(\mathcal{P} \cap B)$  follows a Poisson law of parameter  $\lambda|B|$  (which is its mean) i.e. for all  $k \geq 0$ ,  $\mathbb{P}[\#(\mathcal{P} \cap B) = k] = \frac{(\lambda|B|)^k}{k!} e^{-\lambda|B|}$ .
- For any  $k \geq 2$ , the random variables  $(\#(\mathcal{P} \cap B_j))_{1 \leq j \leq k}$  are independent if the bounded Borel sets  $(B_j)_{1 \leq j \leq k}$  are disjoint.

See [Chi+13, Section 2.3] for more details.

The Poisson process is stationary by construction. Thanks to independence it is ergodic and for all  $j \geq 1$ , we have  $f_j \equiv \lambda^j$ , see [Chi+13, (2.36) p.47], so that

$$\lambda_j(\mathcal{P}) = \lambda^j . \quad (1.42)$$

In finite volume, it is very easy to simulate. For instance, it can be constructed sequentially : first draw  $N$ , the total number of points, according to Poisson law then add  $N$  points drawn independently and uniformly in space.

However, it lacks an important feature for inclusion models : it is not hard-core ( $\ell = 0$  a.s.).  $\square$

We then define what we call, in the sequel, a hard-core Poisson process.

**Example 1.4 (Hard-core Poisson)** In the sequel, we regroup under the common name hard-core Poisson processes a variety of models constructed from the Poisson process which are still stationary and ergodic with the additional hard-core feature. The previous definition can not hold anymore as we have to require at least some local correlation to obtain a hard-core process. However, one can retain strong decorrelations at large distance. For the models that we consider, we have

$$\lambda_j(\mathcal{P}) \sim \lambda(\mathcal{P})^j \quad \forall j \geq 1$$

reminiscent of the Poisson case.

Some examples are the so-called Matérn's hard-core point processes type I, II, III [Mat86] or Penrose graphical construction of the random parking measure [Pen01]. Most of these models are obtained by an algorithmic modification of an underlying Poisson process. It is usually some sort of thinning (which means erasing points from the original process, as in the Bernoulli random deletion case) with some refinement to keep a high intensity (proportional to the volume fraction here) as naive approaches delete too many points. Those examples are stationary and ergodic. In fact, their correlation decays exponentially (which implies that they are algebraically  $\alpha$ -mixing for an arbitrarily high rate, see Definition 1.1). This can be efficiently captured through so-called multi-scales functional inequalities developed by Duerinckx and Gloria, see [DG20b, Section 3.3 and 3.4].

As these processes are derived from the Poisson process by an algorithmic procedure, they can be numerically simulated (in finite volume).  $\square$

We conclude with a simple example to illustrate the behavior of the multi-point intensities.

**Example 1.5 (Doubled process)** Given  $\mathcal{P}$  stationary ergodic and hard-core, choose  $e \in \mathbb{R}^d$  s.t.  $|e| \leq \frac{1}{2}\ell(\mathcal{P})$ . Then, the point process  $\mathcal{P}_e := \mathcal{P} \cup (\mathcal{P} + e)$  consists

of pairs of points  $(x, x + e)_{x \in \mathcal{P}}$  and thus satisfies  $\lambda_2(\mathcal{P}_e) \simeq \ell^{-d} \lambda(\mathcal{P}_e)$ . Hence,  $\lambda_2(\mathcal{P}_e) \simeq \lambda(\mathcal{P}_e)$ .  $\square$

### 1.3.4 Toy model

We have at hand three modes of dilution : the one parameter models of geometric dilation and random deletion as well as the more intrinsic way using multi-point intensities. We wish to illustrate their behavior on a toy model. In particular, we want to identify the scalings of their respective dilute expansions and we want to see how the cluster expansion and the multi-point intensities combine. We shall start by the random deletion case before presenting the general dilute case following the historical order [DG16; DG20c]. We finish with the geometric dilation model as it does not entirely fit in the cluster expansion framework.

The toy-model should retain the key features of the homogenized conductivity :  $\bar{a}$  is an averaged quantity and its dependence on the point process  $\mathcal{P}$  is both nonlinear and nonlocal (we will also say long-range in the sequel) because of the corrector (1.14), see also (1.15). We borrow this toy-model from [DG20c, Section 1.3.3] : we define the function  $\Phi^\# : F \subset \mathcal{P} \mapsto \Phi^F \in \mathbb{R}$  by

$$\Phi^\# = \mathbb{E} [\Psi^\#] \tag{1.43}$$

where, for a countable  $F \subset \mathbb{R}^d$ ,  $\Psi^\#$  is given by

$$\Psi^F := g \left( \sum_{x \in F} h(x) \right) \in \mathbb{R} \tag{1.44}$$

for some  $h : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that

- (a)  $h$  is *short-range* (a.k.a. local) in the sense that  $\int_{\mathbb{R}^d} \sup_{B_l(z)} |h| dz \leq 1$  for some  $l \geq 1$ ,
- (b)  $g$  is *smooth*, in the sense that  $g \in C_b^\infty(\mathbb{R})$  (the space of infinitely differentiable bounded functions)

and recalling the convention  $\sum_{x \in \emptyset} = 0$ .

This model keeps the average nature as well as the nonlinearity (in a very explicit form) of  $\bar{a}$  but localizes the dependence on the point process. Although cluster coefficients  $\left( \bar{\Phi}^{(k)} \right)_{k \geq 0}$  (see (1.35)) are defined by infinite series, these series are always summable in this setting thanks to the short-range assumption.

The simple form (1.44) of  $\Psi$  yields explicit expressions for the cluster coefficients and for the cluster expansion remainder.

**Lemma 1.2 ([DG20c, Lemma 1.2 p.10])** *Let  $\mathcal{P} \in \Omega$  be a random point process with  $\ell(\mathcal{P}) \leq l$  and let  $\Psi^\#$  be a set function of the form (1.44) satisfying both the short-range and smoothness assumptions (a) and (b). Then, for all  $k \geq 1$ , we have explicit formulas for the difference operators*

$$\delta^{\{x_1, \dots, x_k\}} \Psi^\# = \int_0^{h(x_1)} \cdots \int_0^{h(x_k)} g^{(k)}(t_1 + \dots + t_k) dt_1 \dots dt_k \quad \forall x_1, \dots, x_k \in \mathcal{P} \quad (1.45)$$

as well as for the cluster expansion remainder at order  $k$  (see (1.34))

$$\Psi^{\mathcal{P}} - \mathcal{C}^{(k)} \Psi^\# = \sum_{n_1 < \dots < n_{k+1}} \int_0^{h(x_{n_1})} \cdots \int_0^{h(x_{n_{k+1}})} g^{(k+1)} \left( t_1 + \dots + t_{k+1} + \sum_{n < n_1} h(x_n) \right) dt_1 \dots dt_{k+1} . \quad (1.46)$$

Note that, we have chosen in (1.46) to label the points of  $\mathcal{P}$  (see Remark 1.1 (ii)) so that  $\frac{1}{(k+1)!} \sum_{x_1, \dots, x_{k+1} \in \mathcal{P}}$  can be rewritten  $\sum_{\substack{n_1 < \dots < n_{k+1} \\ n_1, \dots, n_{k+1} \in \mathbb{N}}}$ .  $\square$

In the spirit of Taylor formulas, (1.45) and (1.46) are obtained by induction (both on the order  $k$  and the cardinality of  $\mathcal{P}$  for the remainder).

Thanks to this lemma, we are now able to measure the size of the cluster coefficients  $\left( \overline{\Phi}^{(k)} \right)_{k \geq 0}$  and that of the cluster expansion remainder  $\Phi^{\mathcal{P}} - \mathcal{C}^{(k)} \Phi^\#(\mathcal{P})$ .

In the Bernoulli random deletion case, considering  $\mathcal{P}^{(p)}$  (see (1.31)), Lemma 1.2 yields for all  $k \geq 1$  and for all  $0 \leq p \leq 1$

$$|\overline{\Phi}^{(k)}(\mathcal{P}^{(p)})| \lesssim_{k,g,h} p^k \quad (1.47)$$

and

$$|\Phi^{\mathcal{P}^{(p)}} - \mathcal{C}^{(k)} \Phi^\#(\mathcal{P}^{(p)})| \lesssim_{k,g,h} p^{k+1} . \quad (1.48)$$

For the cluster coefficients, one can be even more precise :

$$\overline{\Phi}^{(k)}(\mathcal{P}^{(p)}) = p^k \overline{\Phi}^{(k)}(\mathcal{P}) . \quad (1.49)$$

Indeed, from the definition (1.35), we get

$$\begin{aligned}\bar{\Phi}^{(k)}(\mathcal{P}^{(p)}) &= \mathbb{E} \left[ \bar{\Psi}^{(k)}(\mathcal{P}^{(p)}) \right] = \mathbb{E} \left[ \frac{1}{k!} \sum_{x_1, \dots, x_k \in \mathcal{P}^{(p)}}^{\neq} \delta^{\{x_1, \dots, x_k\}} \Psi^\# \right] \\ &= \mathbb{E} \left[ \frac{1}{k!} \sum_{x_1, \dots, x_k \in \mathcal{P}}^{\neq} \prod_{i=1}^k b_{x_i} \delta^{\{x_1, \dots, x_k\}} \Psi^\# \right] \\ &= p^k \mathbb{E} \left[ \frac{1}{k!} \sum_{x_1, \dots, x_k \in \mathcal{P}}^{\neq} \delta^{\{x_1, \dots, x_k\}} \Psi^\# \right]\end{aligned}$$

proceeding as in (1.32).

The remainder estimate (1.48) is obtained noting that (1.46) yields

$$|\Psi^{\mathcal{P}^{(p)}} - \mathcal{C}^{(k)} \Psi^\#| \leq \|g^{(k+1)}\|_{L^\infty(\mathbb{R})} \frac{1}{(k+1)!} \sum_{x_1, \dots, x_{k+1} \in \mathcal{P}^{(p)}}^{\neq} \prod_{i=1}^{k+1} |h(x_i)| \quad (1.50)$$

and proceeding similarly.

**Remark 1.4** Here, we could actually track the dependence on  $k$  in the constants appearing in (1.47) and (1.48). This would yield the analyticity of the map  $p \mapsto \Phi^{\mathcal{P}^{(p)}}$  at  $p = 0$  under the additional assumption that  $g$  itself is analytic (so that we control the growth of  $k \mapsto \|g^{(k)}\|_{L^\infty(\mathbb{R})}$ ).  $\square$

Formula (1.49) illustrates how nicely Bernoulli deletion and the cluster extension blend. With similar arguments, we would obtain a similar link for the multi-point densities and intensities. For all  $k \geq 1$ ,

$$f_k(\mathcal{P}^{(p)}) = p^k f_k(\mathcal{P}) \quad (1.51)$$

so that

$$\lambda_k(\mathcal{P}^{(p)}) = p^k \lambda_k(\mathcal{P}) . \quad (1.52)$$

From this observation, we realise that the estimates (1.47) and (1.48) are controlled in fact in terms of multi-point intensities.

For a general (stationary ergodic) point process  $\mathcal{P} \in \Omega$ , the intrinsic dilute setting is characterized by the smallness not only of the intensity  $\lambda(\mathcal{P})$  but of the family  $(\lambda_j(\mathcal{P}))_{j \geq 1}$  of multi-point intensities. For the toy model, Lemma 1.2 yields, for all  $k \geq 1$ ,

$$|\bar{\Phi}^{(k)}(\mathcal{P})| \lesssim_{k,g,h} \lambda_k(\mathcal{P}) \quad (1.53)$$

and

$$|\Phi^{\mathcal{P}} - \mathcal{C}^{(k)}\Phi^{\#}| \lesssim_{k,g,h} \lambda_{k+1}(\mathcal{P}) . \quad (1.54)$$

We recall that in general  $\lambda_k(\mathcal{P}) \not\sim \lambda(\mathcal{P})^k$  which contradicts the naive bound one could have expected for  $\bar{\Phi}^{(k)}(\mathcal{P})$ .

Indeed, from (1.45), we have

$$\begin{aligned} |\bar{\Phi}^{(k)}(\mathcal{P})| &\leq \frac{1}{k!} \mathbb{E} \left[ \sum_{x_1, \dots, x_k \in \mathcal{P}}^{\neq} \|g^{(k)}\|_{L^\infty(\mathbb{R})} \prod_{i=1}^k |h(x_i)| \right] \\ &\leq \frac{\|g^{(k)}\|_{L^\infty(\mathbb{R})}}{k!} \int_{(\mathbb{R}^d)^k} |h|^{\otimes k} f_k \end{aligned}$$

so that estimate (1.53) follows by definition (1.39) of the multi-point intensities and our short-range assumption on  $h$ . Estimate (1.54) is obtained similarly.

Finally, in the geometric dilation case, considering  $L\mathcal{P}$  (see (1.29)), we have for all  $k \geq 1$  and all  $L \geq 1$ ,

$$|\bar{\Phi}^{(k)}(L\mathcal{P})| \lesssim_{k,g,h} L^{-kd} \quad (1.55)$$

and

$$|\Phi^{\mathcal{P}} - \mathcal{C}^{(k)}\Phi^{\#}| \lesssim_{k,g,h} L^{-(k+1)d} . \quad (1.56)$$

These estimates are corollaries of (1.53) and (1.54) noting that for all  $j \geq 1$

$$f_j(L\mathcal{P}) = L^{-dj} f_j(\mathcal{P}) \left( \frac{\cdot}{L} \right) \quad (1.57)$$

so that, by definition (1.39) of the multi-point intensities,

$$\lambda_j(L\mathcal{P}) = L^{-dj} \lambda_j(\mathcal{P}) . \quad (1.58)$$

The claim (1.57) on the  $j$  point intensity is obtained by a change of variables. Indeed, for a test function  $\chi$ ,

$$\begin{aligned} \int_{(\mathbb{R}^d)^j} \chi f_j(L\mathcal{P}) &= \mathbb{E} \left[ \sum_{x_1, \dots, x_j \in L\mathcal{P}}^{\neq} \chi(x_1, \dots, x_j) \right] \\ &= \mathbb{E} \left[ \sum_{x_1, \dots, x_j \in \mathcal{P}}^{\neq} \chi(Lx_1, \dots, Lx_j) \right] \\ &= \int_{(\mathbb{R}^d)^j} \chi(L\cdot) f_j(\mathcal{P}) . \end{aligned}$$

While the cluster expansion approach still provides good estimates in the geometric dilation case, it is not fully satisfactory if one aims for an expansion in the dilation parameter  $L$  (in the dilute regime  $L \gg 1$ ). This is the main motivation for the alternative approach that we introduce in Chapter 2. Indeed, as opposed to the Bernoulli deletion case, the cluster coefficients  $\bar{\Phi}^{(k)}(L\mathcal{P})$  still depends on  $L$ . Combining their definition (1.35) with the expression (1.45) of the difference operators and the definition (1.38) of the  $j$ -point density, for all  $k \geq 1$ , we obtain

$$\begin{aligned} \bar{\Phi}^{(k)}(L\mathcal{P}) &= \frac{1}{k!} \mathbb{E} \left[ \sum_{x_1, \dots, x_k \in \mathcal{P}}^{\neq} \int_0^{h(Lx_1)} \cdots \int_0^{h(Lx_k)} g^{(k)}(t_1 + \dots + t_k) dt_1 \dots dt_k \right] \\ &= \frac{1}{k!} \int_{(\mathbb{R}^d)^k} \int_0^{h(Lx_1)} \cdots \int_0^{h(Lx_k)} g^{(k)}(t_1 + \dots + t_k) dt_1 \dots dt_k \\ &\quad f_k(x_1, \dots, x_k) dx_1 \dots dx_k . \end{aligned}$$

A dependence on  $L$  still remains in the integration domain  $\prod_{i=1}^k [0, h(Lx_i)]$ . To expand further, one would need additional assumptions on  $h$  and  $g$  for instance taking  $h = \langle x \rangle^{-(d+\gamma)}$ , short-range for any  $\gamma > 0$  (where we have used the Japanese bracket notation  $\langle x \rangle^2 := 1 + |x|^2$ ). In any case, one should look for an expansion of the map  $L \mapsto \Phi^{L\mathcal{P}}$  as  $L \gg 1$  instead of  $L^{-d} \mapsto \Phi^{L\mathcal{P}}$  (as one could have naively hoped from  $\lambda(L\mathcal{P}) \sim L^{-d}$ ).

We expect similar estimates for the effective conductivity  $\bar{a}$  (1.4), the effective viscosity  $\bar{\nu}$  (1.5) or the mean sedimentation speed  $\bar{V}$  (1.6).

## 1.4 Long-range interactions and renormalization

In this section, we illustrate how the decay of correlations allows us to handle the nonlocal (we also say long-range) dependence of the effective quantities on the point process  $\mathcal{P}$  (see (1.15) in Section 1.2). We start by introducing a suitable notion of multi-points correlations for point processes. Thanks to this notion, we then study two long-range models based on linear PDEs using a so-called renormalization approach. Finally, we see that these computations can be bypassed when a convergence rate for the finite volume approximations of effective quantities is available.

### 1.4.1 Multi-point correlation functions

Following [DG20c, Section 4.3.1 p.74] (see also Mayer's expansion in the literature or joint cumulants), we define a notion of multi-point correlations for point processes.

**Definition 1.6** Let  $\mathcal{P} \in \Omega$  be a stationary point process. For all  $j \geq 1$ , we introduce its  $j$ -point correlation function  $h_j$  from the following decomposition of its  $j$ -point density  $f_j$  (see (1.38)),

$$f_j(x_1, \dots, x_j) = \sum_{\pi} \prod_{H \in \pi} h_{|H|}(x_H) \quad (1.59)$$

where  $\pi$  runs over all the partitions of the index set  $\{1, \dots, j\}$ ,  $H$  runs over all cells of the partition  $\pi$  and we have set  $x_H := (x_{i_1}, \dots, x_{i_{|H|}})$  for  $H = \{i_1, \dots, i_{|H|}\}$ . For  $j = 1, 2, 3$ , this reads

$$f_1(x) = h_1(x) = \lambda(\mathcal{P}) , \quad (1.60)$$

$$f_2(x, y) = h_2(x, y) + \lambda(\mathcal{P})^2 , \quad (1.61)$$

$$f_3(x, y, z) = h_3(x, y, z) + \lambda(\mathcal{P})(h_2(x, y) + h_2(y, z) + h_2(x, z)) + \lambda(\mathcal{P})^3 . \quad (1.62)$$

Inverting recursively these relations,  $h_j$  is a polynomial in  $(f_i)_{i \leq j}$  and, like  $f_j$ ,  $h_j$  is a symmetric function. In particular, this yields the following bound

$$\sup_{z_1, \dots, z_j \in \mathbb{R}^d} \int_{Q_\ell(z_1) \times \dots \times Q_\ell(z_j)} |h_j| \lesssim_j \bar{\lambda}_j(\mathcal{P}) \quad (1.63)$$

(see Definition 1.3). □

Algebraic  $\alpha$ -mixing (Definition 1.1) implies the decay of correlation functions in the following form.

**Lemma 1.3 ([DG20c, Lemma 4.2 p.74])** *Assume that  $\mathcal{P}$  is algebraically  $\alpha$ -mixing with rate  $\gamma > 0$ . Then, the correlation functions satisfy for all  $j \geq 1$  and for all  $z_1, \dots, z_j \in \mathbb{R}^d$*

$$\int_{B(z_1) \times \dots \times B(z_j)} |h_j| \lesssim_j \min_{k \neq l} \langle z_k - z_l \rangle^{-\gamma} . \quad (1.64)$$
□

## 1.4.2 Linear models

Our goal here is to illustrate how the decay of correlations compensates the long-range nature of the interactions. For that purpose, we focus on the well-posedness of two linear long-range random PDEs and forget momentarily about dilute expansions. For  $\mathcal{P} \in \Omega$  stationary ergodic point process, we set  $\mathcal{I} := \cup_{x \in \mathcal{P}} B(x)$  and for  $e \in \mathbb{R}^d$ , we consider  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $v : \mathbb{R}^d \rightarrow \mathbb{R}$  suitable solutions of the whole space equations

$$-\Delta u(\mathcal{P}) = \nabla \cdot \mathbf{1}_{\mathcal{I}} e \quad \text{in } \mathbb{R}^d \quad (1.65)$$



and

$$-\Delta v(\mathcal{P}) = \mathbb{1}_{\mathcal{I}} - \mathbb{E}[\mathbb{1}_{\mathcal{I}}] \quad \text{in } \mathbb{R}^d. \quad (1.66)$$

The well-posedness of these equations shares similar difficulties with the equation of the corrector (1.14) but here, both  $u$  and  $v$  depend linearly on the point process  $\mathcal{P}$ . To construct solutions, we make use of the approximation strategy described in Section 1.2.2. In fact, we focus on the first step : obtaining uniform bounds.

**Remark 1.5** To motivate further these equations, note that (1.65) can be obtained by linearizing the equation (1.14) of the corrector setting  $a = \text{Id} + \delta \mathbb{1}_{\mathcal{I}} \text{Id}$  and Taylor expanding in the parameter  $\delta \ll 1$ .

Regarding equation (1.66), it can be seen as the (scalar) linear analogue of the sedimentation problem, see [Glo21, Section 1.3] as well as Definition 3.6 and Remark 3.9 in Chapter 3. Besides, it is natural to (try to) replace a source term in divergence form by a zero average one. Indeed, in a bounded domain  $\Omega$ , if  $q$  has zero average i.e.  $\int_{\Omega} q = 0$ , Bogovskii's theorem implies that there exists  $u$  s.t.  $\nabla \cdot u = q$ , see for instance [BF13, Theorem IV.3.1 p. 245]. This result does not hold anymore in our probabilistic framework : we shall see that the well-posedness of (1.66) requires a stronger mixing than the one of (1.65). See also [DG22b, Section 2.7] for a more detailed comparison between these models.  $\square$

In the sequel, we assume that  $\ell(\mathcal{P}) \geq 4$ , so that the inclusions are well separated and  $\mathbb{1}_{\mathcal{I}} = \sum_{x \in \mathcal{P}} \mathbb{1}_{B(x)}$ .

We start with equation (1.65) and consider, for  $T > 1$ , its massive approximation

$$\frac{1}{T} u_T(\mathcal{P}) - \Delta u_T(\mathcal{P}) = \nabla \cdot \mathbb{1}_{\mathcal{I}} e \quad \text{in } \mathbb{R}^d. \quad (1.67)$$

In the probability space (see Section 1.2.1), this equation can be rewritten in the weak form

$$\mathbb{E} \left[ \frac{1}{T} u_T \chi \right] + \mathbb{E} [Du_T \cdot D\chi] = -\mathbb{E} [D\chi \cdot \mathbb{1}_{\mathcal{I}} e]$$

for all  $\chi \in \mathcal{H}(\Omega)$ . The energy estimate then yields

$$\mathbb{E} [|Du_T|^2] \leq \mathbb{E} [|\mathbb{1}_{\mathcal{I}} e|^2] \lesssim 1. \quad (1.68)$$

In the above, we have essentially shown the boundedness of the Helmholtz projection  $f \mapsto \nabla(-\Delta)^{-1} \nabla \cdot f$  in  $L^2(\Omega)$  by energy estimate. This boundedness can also be obtained by different approaches, for instance Calderòn-Zygmund theory. We rather present what we call a renormalization approach assuming quantitative ergodicity (in the form of algebraic  $\alpha$ -mixing with rate  $\gamma > 0$ ). This is obviously much less general and pedestrian but instructive : in particular, it points out where the decay of correlations can be used. For that purpose, we recall  $G_T$ , the Green

function of the massive Laplace operator, as well its decay, see Definition 2.4 in Chapter 2 for more details.

**Definition 1.7** For  $T > 0$ , the massive Green function  $G_T$  is defined as the unique decaying (weak) solution of

$$\frac{1}{T}G_T - \Delta G_T = \delta_0 \quad \text{in } \mathbb{R}^d.$$

For  $d \geq 1$  and for all  $k \geq 1$ , we have the exponentially decaying point-wise bounds

$$|\nabla^k G_T(\cdot)| \lesssim_k |\cdot|^{-(d-2+k)} e^{-\frac{|\cdot|}{\sqrt{T}}}. \quad (1.69)$$

□

For convenience, and in order to avoid any regularity issue, we mollify the equation (1.65) replacing  $\mathbb{1}_B e$  by  $\mathcal{J} := \mathbb{1}_B e * \rho$  (where  $\rho$  is the standard mollifier). Therefore,  $\mathcal{J} \in C_c^\infty(\mathbb{R}^d)$ , is supported in  $B_2$ ,  $\mathcal{J}$  is an even function and  $\int_{\mathbb{R}^d} \mathcal{J} = e|B|$ . Still calling  $u_T$  the corresponding mollified solution, we have

$$\nabla u_T = \nabla G_T * \nabla \cdot \sum_{x \in \mathcal{P}} \mathcal{J}(\cdot - x) = \nabla^2 G_T * \sum_{x \in \mathcal{P}} \mathcal{J}(\cdot - x). \quad (1.70)$$

By stationarity,  $\mathbb{E}[|Du_T|^2] = \mathbb{E}[|\nabla u_T(0)|^2]$ . Before computing this  $L^2(\Omega)$ -norm, let us see what happens in  $L^1(\Omega)$ . By the triangle inequality,

$$\mathbb{E}[|\nabla u_T(0)|] \leq \mathbb{E}\left[\sum_{x \in \mathcal{P}} |\nabla^2 G_T(x - \cdot) * \mathcal{J}(0)|\right]$$

Setting  $F_T(x) := \nabla^2 G_T * \mathcal{J}(\cdot - x)(0) = \nabla^2 G_T * \mathcal{J}(x)$  (by parity) for all  $x \in \mathbb{R}^d$ , we have

$$|F_T(x)| \lesssim \langle x \rangle^{-d} e^{-\frac{|x|}{\sqrt{T}}} \quad (1.71)$$

from (1.69). Thus, by definition of the one-point density (1.38), we obtain

$$\mathbb{E}[|\nabla u_T(0)|] \leq \int_{\mathbb{R}^d} |F_T(x)| \lambda(\mathcal{P}) dx \lesssim \lambda(\mathcal{P}) \ln(T)$$

as

$$\int_{\mathbb{R}^d} |F_T| \lesssim \ln(T). \quad (1.72)$$

Indeed, using the bound (1.71) and polar coordinates, we have

$$\int_{\mathbb{R}^d} |F_T| \lesssim \int_{\mathbb{R}^d} \langle x \rangle^{-d} e^{-\frac{|x|}{\sqrt{T}}} \lambda(\mathcal{P}) dx \lesssim \int_0^\infty r^{d-1} \langle r \rangle^{-d} e^{-\frac{r}{\sqrt{T}}} \lambda(\mathcal{P}) dr \lesssim 1 + \int_{\sqrt{T}^{-1}}^\infty e^{-r} \frac{dr}{r}.$$

where we have split the integral between  $[0, 1]$  and  $[1, \infty]$  and changed variables in the second part. This last integral converges at infinity thanks to the exponential decay but logarithmically blows up as  $\sqrt{T}^{-1} \rightarrow 0$  because of  $\frac{dx}{r}$  yielding (1.72).

We have obtained  $\mathbb{E} [|\nabla u_T(0)|] \lesssim \ln(T)$  whereas combining the energy estimate (1.68) and Cauchy-Schwarz' inequality, we have

$$\mathbb{E} [|\nabla u_T(0)|] \leq \mathbb{E} [|\nabla u_T(0)|^2]^{\frac{1}{2}} \lesssim 1 .$$

This is due to taking absolute values, therefore preventing the use of stochastic cancellations such as  $\mathbb{E} [Du_T] = 0$  (by construction). We can recover this property in the physical space here :

$$\mathbb{E} [\nabla u_T(0)] = \mathbb{E} \left[ \sum_{x \in \mathcal{P}} F_T(x) \right] = \int_{\mathbb{R}^d} F_T(x) \lambda(\mathcal{P}) dx = 0$$

as

$$\int_{\mathbb{R}^d} F_T(x) dx = \int_{\mathbb{R}^d} \nabla(\nabla G_T * \mathcal{J})(x) dx = 0 \quad (1.73)$$

since the integral of a gradient vanishes.

For the norm in  $L^2(\Omega)$ , plugging in the representation formula (1.70), we get

$$\begin{aligned} \mathbb{E} [|\nabla u_T(0)|^2] &= \mathbb{E} \left[ \sum_{x, y \in \mathcal{P}} F_T(x) F_T(y) \right] = \mathbb{E} \left[ \sum_{x \in \mathcal{P}} |F_T(x)|^2 \right] + \mathbb{E} \left[ \sum_{x, y \in \mathcal{P}}^{\neq} F_T(x) F_T(y) \right] \\ &= \int_{\mathbb{R}^d} |F_T(x)|^2 \lambda(\mathcal{P}) dx + \iint_{(\mathbb{R}^d)^2} F_T(x) F_T(y) f_2(x, y) dx dy \quad (1.74) \end{aligned}$$

from the definition (1.38) of the two-point density. The first integral on the RHS is bounded uniformly in  $T$  from the decay (1.71) of  $F_T$  in the form  $|F_T(x)|^2 \lesssim \langle x \rangle^{-2d}$ . The second integral needs to be renormalized i.e. requires a rewriting to be absolutely convergent independently of  $T$ . Indeed, we have

$$\int_{(\mathbb{R}^d)^2} |F_T(x)| |F_T(y)| f_2(x, y) dx dy \lesssim \lambda_2(\mathcal{P}) \left( \int_{\mathbb{R}^d} |F_T| \right)^2 \lesssim \lambda_2(\mathcal{P}) \ln(T)^2$$

using (1.72). However, decomposing the two-point density into  $f_2 = h_2 + \lambda^2(\mathcal{P})$  (see (1.61)), we have

$$\iint_{(\mathbb{R}^d)^2} F_T(x) F_T(y) f_2(x, y) dx dy = \iint_{(\mathbb{R}^d)^2} F_T(x) F_T(y) h_2(x, y) dx dy$$

by the cancellation property (1.73) of  $F_T$ . Assuming that  $\mathcal{P}$  is algebraically  $\alpha$ -mixing with rate  $\gamma > 0$  (see Definition 1.1), we combine the decay (1.71) of  $F_T$  with the decay (1.64) of  $h_2$  so that

$$\iint_{(\mathbb{R}^d)^2} F_T(x)F_T(y)h_2(x,y) dx dy \lesssim \iint_{(\mathbb{R}^d)^2} \langle x \rangle^{-d} \langle y \rangle^{-d} \langle x-y \rangle^{-\gamma} \lesssim 1$$

for any  $\gamma > 0$ . Indeed, informally  $\langle \cdot \rangle^{-d} * \langle \cdot \rangle^{-\gamma} \lesssim \langle \cdot \rangle^{-(\gamma-\varepsilon)}$  for arbitrarily small  $\varepsilon > 0$ , see Lemma 3.14 from Chapter 3.

Here, the renormalization approach is suboptimal compared to working directly in the probability space : we have recovered the energy estimate (1.68) at the expense of assuming an algebraic  $\alpha$ -mixing rate  $\gamma > 0$  for  $\mathcal{P}$ . However, this rate can be chosen arbitrarily small, in line with the fact that establishing (1.68) only requires qualitative ergodicity.

The renormalization approach demonstrates its relevance on  $v$  solution of (1.66). For  $T \geq 1$ , we consider  $v_T$ , the massive approximation of  $v$ , solution of

$$\frac{1}{T}v_T - \Delta v_T = \mathbb{1}_{\mathcal{I}} - \mathbb{E}[\mathbb{1}_{\mathcal{I}}] \quad \text{in } \mathbb{R}^d. \quad (1.75)$$

Note that, as  $\ell(\mathcal{P}) > 4$ ,  $\mathbb{E}[\mathbb{1}_{\mathcal{I}}] = c(\mathcal{P}) = \lambda(\mathcal{P})|B|$ . Here, the approach in the probability space is not as fruitful : in the weak form, this equation rewrites

$$\mathbb{E} \left[ \frac{1}{T}v_T \chi \right] + \mathbb{E} [Dv_T \cdot D\chi] = \mathbb{E} [(\mathbb{1}_{\mathcal{I}} - \mathbb{E}[\mathbb{1}_{\mathcal{I}}])\chi]$$

for all  $\chi \in \mathcal{H}(\Omega)$  so that the energy estimates only yields

$$\frac{1}{T}\mathbb{E} [v_T^2] + \mathbb{E} [|Dv_T|^2] \lesssim T \text{Var} [\mathbb{1}_{\mathcal{I}}] \lesssim T.$$

using Young's inequality. Indeed, there is no analogue of Poincaré's inequality in the probability space : for  $\chi \in \mathcal{H}(\Omega)$ ,  $\mathbb{E} [|\chi - \mathbb{E}[\chi]|^2] \not\lesssim \mathbb{E} [|D\chi|^2]$  generically !

The diverging bound on  $\mathbb{E} [|\nabla v_T(0)|^2]$  from (1.4.2) can actually be improved. Representing the solution of (1.75) with the Green function  $G_T$ , we have

$$\nabla v_T = \nabla G_T * (\mathbb{1}_{\mathcal{I}} - \lambda(\mathcal{P})|B|) = \sum_{x \in \mathcal{P}} \int_{B(x)} \nabla G_T(\cdot - y) dy. \quad (1.76)$$

As previously, we set  $H_T(x) := \int_{B(x)} \nabla G_T$  for all  $x \in \mathbb{R}^d$ . From the decay (1.69), we have  $|H_T(x)| \lesssim \langle x \rangle^{-(d-1)} e^{\frac{|x|}{\sqrt{T}}}$ . Note that  $G_T$  and its gradient  $\nabla G_T$  (hence also  $H_T$ ) are integrable near the origin (and at infinity thanks to the exponential decay),

unlike the second gradient  $\nabla^2 G_T$ . Like  $F_T$  (see (1.73)),  $H_T$  has zero average

$$\int_{\mathbb{R}^d} H_T(x) dx = \int_{\mathbb{R}^d} \nabla G_T = 0 \quad (1.77)$$

justifying that we can remove the constant in (1.76). With these notations and proceeding as in (1.74) for  $u_T$ , we have

$$\mathbb{E} [|\nabla v_T(0)|^2] = \int_{\mathbb{R}^d} |H_T(x)|^2 \lambda(\mathcal{P}) dx + \iint_{(\mathbb{R}^d)^2} H_T(x) H_T(y) f_2(x, y) dx dy .$$

As above, the first term of the RHS is bounded uniformly in  $T$  but only for  $d \geq 3$  this time since  $|H_T(x)|^2 \lesssim \langle x \rangle^{-2(d-1)}$ . The second integral also needs to be renormalized. We proceed as before, replacing  $f_2$  by  $h_2$  from the cancellation property (1.77) of  $H_T$  and assuming that  $\mathcal{P}$  is algebraically  $\alpha$ -mixing with rate  $\gamma > 0$  so that

$$\begin{aligned} \iint_{(\mathbb{R}^d)^2} H_T(x) H_T(y) f_2(x, y) dx dy &= \iint_{(\mathbb{R}^d)^2} H_T(x) H_T(y) h_2(x, y) dx dy \\ &\lesssim \iint_{(\mathbb{R}^d)^2} \langle x \rangle^{-(d-1)} \langle y \rangle^{-(d-1)} \langle x - y \rangle^{-\gamma} dx dy . \end{aligned}$$

This time, we need to assume  $\gamma > 2$  to obtain a finite integral as (informally)  $\langle \cdot \rangle^{-(d-1)} * \langle \cdot \rangle^{-\gamma} \lesssim \langle \cdot \rangle^{-(\gamma-1-\varepsilon)}$ , for arbitrarily small  $\varepsilon > 0$ , see Lemma 3.14 from Chapter 3.

We have obtained that for  $\mathcal{P}$  algebraically  $\alpha$  mixing with rate  $\gamma > 2$ ,

$$\mathbb{E} [|\nabla v_T(0)|^2] \lesssim 1 .$$

From our renormalizations, we could even estimate the size of each integral in terms of multi-point intensities, see (3.86) in Chapter 3 (performed there using the finite volume approximation instead of the massive one).

Here, our renormalizations only amount to noting that second moments are in fact variances as the mean vanishes i.e.  $\mathbb{E} [|\nabla u_T|^2] = \text{Var} [\nabla u_T]$  as  $\mathbb{E} [\nabla u_T] = 0$  (and likewise for  $v_T$ ). We have chosen to present them through cancellation and decomposition of the multi-point densities in multi-point correlations as this approach is more suited to handle higher moments as in Chapter 3, see Section 3.3.3.

### 1.4.3 Implicit renormalization

We have seen that the long-range interactions can be handled by a renormalization approach assuming that the point process  $\mathcal{P}$  is  $\alpha$ -mixing. We have also seen

that if  $\mathcal{P}$  is  $\alpha$ -mixing, the variational approach of Armstrong and Smart (further developed with Kuusi and Mourrat) yields a quantitative rate of convergence of the approximated effective quantities. Following [DG20c, Theorem 5 p.70], one can obtain dilute expansions for one-parameters models from this quantitative convergence, bypassing the above renormalizations.

We shall illustrate this strategy on the toy-model from Section 1.3.4. Let  $\mathcal{P}$  be a stationary ergodic point process. For  $T > 1$ , we consider  $\Phi_T^{\mathcal{P}}$  of the form (1.43) associated to  $h_T$  defined for all  $x \in \mathbb{R}^d$  by  $h_T(x) := F_T(x) = \nabla^2 G_T * \mathcal{J}(x)$  (see (1.70)). From (1.72),  $h_T$  is short-range. Its limit  $h$ , defined by  $h(x) := \lim_{T \rightarrow \infty} h_T(x) = \nabla^2 G * \mathcal{J}(x) =: F(x)$ , is long-range :  $|h(x)| \sim \langle x \rangle^{-d}$  so that  $\int_{\mathbb{R}^d} |h| = \infty$ . However, we assume that

- $\Phi^{\mathcal{P}} = \mathbb{E} [g(\sum_{x \in \mathcal{P}} h(x))]$  is well-defined thanks to stochastic cancellations,
- there exists  $\gamma > 0$  s.t.

$$\sup_{p \in [0,1]} |\Phi_T^{\mathcal{P}(p)} - \Phi^{\mathcal{P}(p)}| \lesssim T^{-\gamma}$$

recalling that  $\mathcal{P}^{(p)}$  is the Bernoulli random deletion of  $\mathcal{P}$  (see (1.31)).

These two last hypotheses would be satisfied for  $\Phi_T^{\mathcal{P}} = \bar{a}_T(\mathcal{P})$ . The first one is a consequence of qualitative homogenization theory. The second one holds if we assume  $\mathcal{P}$  to be algebraically  $\alpha$ -mixing which yields the quantitative convergence (1.24) (adapted to the massive approximation).

Under these assumptions, we claim that the map  $p \mapsto \Phi^{\mathcal{P}(p)}$  can be Taylor expanded at any order at  $p = 0$  and that the cluster coefficients  $(\Phi^{(k)})_{k \geq 0}$  are well-defined quantities so that, for all  $n \geq 0$  and  $0 < p \ll 1$ ,

$$\Phi^{\mathcal{P}(p)} = \sum_{k=0}^n p^k \Phi^{(k)}(\mathcal{P}) + o(p^{n+1}) . \quad (1.78)$$

This claim is a corollary of the following lemma which can be seen as an interpolation between a quantitative uniform convergence and a high order Taylor expansion with (logarithmically) diverging remainder.

**Lemma 1.4** *Given a Banach space  $V$ , let the function  $f : [0, 1] \rightarrow V$  and the sequence  $(f_T)_{T > 0}$  with  $f_T : [0, 1] \rightarrow V$  for all  $T > 0$  be such that*

- $f_T$  converges to  $f$  with a quantitative rate :  
there exists  $\beta > 0$  such that  $\|f_T - f\|_{L^\infty([0,1],V)} \lesssim T^{-\beta}$ .
- for some  $n \geq 0$  and for all  $T > 0$ ,  $f_T$  admits a Taylor expansion up to order  $n$  with a controlled remainder :

there exists  $(b_T^{(i)})_{0 \leq i \leq n} \in V^n$  and  $R_T^{(n+1)} : [0, 1] \rightarrow V$  such that for all  $x \in [0, 1]$ ,

$$f_T(x) = \sum_{i=0}^n b_T^{(i)} x^i + R_T^{(n+1)}(x) \quad (1.79)$$

and  $|R_T^{(n+1)}(x)| \lesssim_n \ln(T)^{n+1} x^{n+1}$ .

Then,  $f$  admits a Taylor expansion at the same order  $n$ . More precisely, there exists  $(b^{(i)})_{0 \leq i \leq n} \in V^n$  such that

$$|b_T^{(i)} - b^{(i)}| \lesssim_i T^{-\beta 2^{-i}} \quad (1.80)$$

and for all  $x \in [0, 1]$ ,

$$f(x) = \sum_{i=0}^n b^{(i)} x^i + R^{(n+1)}(x) \quad (1.81)$$

with  $|R^{(n+1)}(x)| \lesssim_n (1 + |\ln(x)|)^{n+1} x^{n+1}$ .  $\square$

Indeed, from Section 1.3.4, for all  $n \geq 0$ , we have

$$\Phi_T(\mathcal{P}^{(p)}) = \sum_{k=0}^n p^k \Phi_T^{(k)}(\mathcal{P}) + R_T^{(n+1)}(p)$$

where for all  $k \geq 0$ ,

$$|\Phi_T(\mathcal{P})| \lesssim_k \ln^k(T)$$

and for all  $n \geq 0$  and for all  $0 \leq p \leq 1$ ,

$$|R_T^{(n+1)}(p)| \lesssim \ln^{n+1}(T) p^{n+1} \quad (1.82)$$

using Lemma 1.2 and tracking the dependence on  $\int_{\mathbb{R}^d} |h_T| \lesssim \ln(T)$ . We can thus apply Lemma 1.4 and obtain (1.78).

The proof of Lemma 1.4 relies on the fact that the approximate coefficients  $b_T^{(i)}$  are independent of  $x$  in the Taylor expansion (1.79). Hence, we have the freedom to optimize either in  $x$  or in  $T$ , see Section 1.A for details. We have voluntarily stated Lemma 1.4 in a general form to underline its versatility, in particular in the context of Clausius-Mossotti formulas. The standard approach to get a Taylor expansion would be to get a bound on the remainder which is uniform in the approximating parameter. Yet, such a uniform bound is not always accessible. The power of the method is that, even if the bound (slowly) explodes here, we can still obtain the desired Taylor expansion. This was our first strategy for the results presented in Chapter 2 before finding the approach presented in Section 2.3.2 (note that Lemma 1.4 can not yield analyticity).

Note also that in [DG20c, Theorem 5 p.70], the authors refine this analysis keeping track of the dependencies on the multi-point intensities. Indeed, they are aiming at estimates in the general dilute setting and use the Bernoulli random deletion as a form of approximation of  $\mathcal{P}$  to obtain them.

## 1.5 The Clausius-Mossotti formula for geometric dilation models

This section introduces the first main result of this manuscript : an expansion of the effective conductivity in the specific setting of geometric dilation. Its proof is the object of Chapter 2. After introducing the setting, we provide an overview of the previous related results. We then present our main contribution as well as some perspectives.

### 1.5.1 Setting

In the framework of stochastic homogenization, we consider the geometric dilation setting. More precisely, given a dilation parameter  $L \geq 1$ , we consider a stationary ergodic point process  $\mathcal{P} \in \Omega$  that we suppose hard-core with  $\ell(\mathcal{P}) > 2$  (see Remark 1.1) to which we associate the random conductivity

$$a(L\mathcal{P}, \cdot) := \text{Id} + \sum_{x \in \mathcal{P}} (\beta - \text{Id}) \mathbf{1}_{B(x)}(\cdot) . \quad (1.83)$$

Homogenization theory then yields the homogenized coefficient  $\bar{a}^L$ . We want to establish an expansion of  $L \mapsto \bar{a}^L$  in the dilute regime, fully characterized here by  $L \gg 1$ , or equivalently of  $L^{-1} \mapsto \bar{a}^L$ .

Compared to Section 1.2.1, we have assumed that the background conductivity is symmetric so that it can be taken equal to  $\text{Id}$  (without loss of generality by a suitable change of variables). We have also added a hard-core assumption on  $\mathcal{P}$  which is natural for the geometric dilation setting as we want the minimal distance  $\ell(L\mathcal{P}) = L\ell(\mathcal{P})$  to increase.

This is the most natural setting if we assume  $\mathcal{P}$  to be periodic : increasing  $L$  increases the period. In that case, note that it is equivalent to vary the size of the cell or the size of the inclusion i.e. considering the periodic conductivity

$$a(\cdot) = \text{Id} + (\beta - \text{Id}) \mathbf{1}_{B_1(0)} \quad \text{in } Q_L$$



(identifying the box  $Q_L$  with the periodic torus  $\mathbb{R}^d/L\mathbb{Z}^d$ ) or

$$a(\cdot) = \text{Id} + (\beta - \text{Id})\mathbf{1}_{B_{L^{-1}}(0)} \quad \text{in } Q .$$

This second formulation, which corresponds to varying the radius of a spherical inclusion in a fixed periodic cell, is the most common one in the periodic homogenization literature.

### 1.5.2 Previous results

The Clausius-Mossotti formula (1.4) was first justified in the periodic case : [JKO94, Section 1.7 p.45, see (1.79)] gives an expansion of  $\bar{a}^L$  in  $L^{-d}$  up to order  $2d$  with a quantitative control of the remainder  $O(L^{-(2d+2)})$ . In the constant isotropic case with spherical inclusions, this expansion recovers the explicit formula (1.3) and Berdichevskii proposes in [Ber83] a general algorithm to obtain arbitrarily high accuracy based on explicit calculation using spherical harmonics.

The key idea in [JKO94] is to compare the corrector  $\phi$  to the so-called *single inclusion solution*  $\phi^\circ$ . This idea, which can be seen as a basic form of cluster expansion, was the main inspiration for the approach developed in Chapter 2. We define the shorthand notation  $a^\circ := \text{Id} + (\beta - \text{Id})\mathbf{1}_B$  so that, given a direction  $e \in \mathbb{R}^d$ ,  $\phi_e^\circ$  is defined as the unique decaying solution of the single particle problem

$$-\nabla \cdot a^\circ(\nabla\phi_e^\circ + e) = 0 \quad \text{in } \mathbb{R}^d . \quad (1.84)$$

Existence and uniqueness are elementary (and fully deterministic). In the constant radial isotropic case i.e.  $a^\circ = \text{Id}(1 + (\beta - 1)\mathbf{1}_B)$  (with a slight abuse of notation),  $\nabla\phi_e^\circ$  is explicitly given in dimension  $d$  by

$$\nabla\phi_e^\circ = \begin{cases} Ce & |x| < 1 \\ \frac{C}{|x|^d} \left( e - d \frac{e \cdot x \otimes x}{|x|^2} \right) & |x| > 1 \end{cases} \quad (1.85)$$

where  $C := \frac{1-\beta}{\beta+1 \times (d-1)}$ .

The first justification of the first order Clausius-Mossotti formula in the random setting is due Almgren in dimension  $d = 3$ , whose results in [Alm13; Alm14] combined with elementary homogenization theory, precisely yield (1.3). The proof is based on (scalar) potential theory and crucially relies on the fact that  $d = 3$ , that  $a$  is everywhere a multiple of the identity and that the inclusions are spherical and disjoint.

The Bernoulli random deletion case was then considered by Anantharaman and Le Bris [AL11], see also [AL12; Ana10] for an underlying periodic  $\mathcal{P}$  and by Mourrat [Mou15] for a discrete scalar equation. In both cases, they obtained a

first order expansion in the Bernoulli parameter  $p$ . Extending these works, [DG16] have proved the real analyticity of the map  $p \mapsto \bar{a}^{(p)}$  at  $p = 0$  (which implies the analyticity on the full interval  $[0, 1]$  by a change of reference medium, observation due to [Mou15]) for general underlying random point processes.

**Theorem 1.1 ([DG16, Theorem 2.1 p.307])** *Let  $\mathcal{P} \in \Omega$  be random stationary ergodic hardcore point process. Then, there exists a family of matrices  $(\bar{a}^{(j)})_{j \geq 0}$  such that for all  $0 < p \ll 1$ , the homogenized coefficient  $\bar{a}^{(p)}$  associated to  $a(\mathcal{P}^{(p)})$  (see (1.7) and (1.31)) is given by the summable series*

$$\bar{a}^{(p)} = \sum_{j=0}^{\infty} p^j \bar{a}^{(j)} .$$

*In particular,  $\bar{a}^{(j)}$  is given by the  $j$ th term of the cluster expansion of  $\bar{a}$ .  $\square$*

The proof of this result relies on the cluster expansion together with a family of new discrete  $\ell^1 - \ell^2$  estimates (some refined energy estimates), which yields the existence of a universal constant  $C > 0$  s.t. for all  $j \geq 0$ ,  $\bar{a}^{(j)} \leq C^j$  (i.e. analyticity). In particular, the proof does not make use of elliptic regularity (unlike our approach in Chapter 2). The  $\ell^1 - \ell^2$  estimates are based on PDE analysis, and essentially avoid taking the modulus of the mixed second gradient of the Green function of the operator  $-\nabla \cdot a \nabla$ , which is not absolutely integrable on the whole space (i.e. long-range).

In the random case, the expansion of  $L^{-1} \mapsto \bar{a}^L$  has not yet been treated in the literature. However, in the past years, Einstein's formula (1.5), which is closely related to the Clausius-Mossotti formula, has received a lot of attention, see [HM12; HW20; GH20; NS20; GH21; GM22; DG20c], and some of these works provide some partial answers.

In [Gér22, Theorem 3], Gérard-Varet considered the Clausius-Mossotti formula with stiff inclusions (the scalar version of Einstein's formula). He proposed an alternative approach, independent of homogenization, where he quantifies the distance between the solution of a problem with small inclusions, and the solution of an effective problem (with conductivity given by the Clausius-Mossotti formula (1.4)) in terms of the volume fraction  $c$  of inclusions. Using this approach, [GH20; GM22] obtained Einstein's formula (1.5) at second order in the volume fraction in a setting that covers geometric dilation.

In parallel, Gloria and Duerinckx approached Einstein's formula through a stochastic homogenization perspective [DG21a; DG21b; Due22; DG20c]. Instead of directly looking for an effective dilute model, they decomposed the approach, first defining an effective model (with a notion of effective viscosity valid even in a non-dilute setting) then expanding the effective viscosity in a dilute regime

(or more precisely under various notions of dilution) using the cluster expansion formalism. Inter alia, these authors studied in [DG20c, Theorem 11 p.123] the validity of the cluster expansion of the effective viscosity  $\overline{B}(L\mathcal{P})$  under geometric dilation. Combining this time elliptic regularity in a crucial way with the  $\ell^1 - \ell^2$  estimates, they obtained the summability of the cluster coefficients  $\left(\overline{B}^{(j)}(L\mathcal{P})\right)_{j \geq 0}$  in the form

$$\overline{B}(L\mathcal{P}) = \sum_{j=0}^{\infty} \overline{B}^{(j)}(L\mathcal{P}) \quad (1.86)$$

with the existence of a constant  $C$  s.t. for all  $j \geq 0$ ,  $\overline{B}^{(j)}(L\mathcal{P}) \leq (CL^{-d})^j$ . However, similarly to the toy model (see Section 1.3.4), this does not provide a summable series in  $L^{-1}$ . The cluster coefficients depend themselves on  $L$  and would require an additional multi-pole expansion (see [DG20c, Remark 5.1 p.123]).

In [Gér21], Gérard-Varet used a similar cluster expansion approach (with similar drawbacks regarding the expansion in  $L^{-1}$ ) at second order. However, he does not rely on  $\ell^1 - \ell^2$  estimates to tackle the long-range interactions but rather appeals to Calderón-Zygmund theory (which also avoids taking the modulus of the mixed second gradient of the Green function).

### 1.5.3 Contribution

Our main result in Chapter 2 is the full asymptotic expansion of the homogenized coefficient  $\overline{a}^L$  in the geometric dilation setting for the dilute regime  $L \gg 1$ .

**Theorem 1.2** *Let  $\mathcal{P} \in \Omega$  be a hard-core stationary ergodic random point process. For  $L \geq 1$ , let  $\overline{a}^L$  be the homogenized matrix associated to  $a(L\mathcal{P}, \cdot)$  (given by (1.83)). Then, the map  $L^{-1} \mapsto \overline{a}^L$  is real analytic at 0.*

*In particular, there exists a family of matrices  $(\mathcal{A}^{(i)})_{i \geq 0} \in (\mathbb{R}^{d \times d})^{\mathbb{N}}$  such that  $\overline{a}^L$  is given by the summable series*

$$\overline{a}^L = \sum_{i=0}^{\infty} \mathcal{A}^{(i)} L^{-i} . \quad (1.87)$$

□

Analyticity is not obtained through abstract arguments. Our result is constructive and we have a formula for  $\overline{a}^L$  with an explicit dependence on  $L$ . Our approach relies on an operator  $\mathcal{K}(L)$  which relates the corrector  $\nabla\phi(L\mathcal{P})$  to the single inclusion solution  $\nabla\phi^\circ$  (see (1.84)) through a fixed-point formulation (which can be explicitly solved using Neumann series). The well-posedness of this operator is the main difficulty of our approach. It uses elliptic regularity (in the form of Green function estimates), stationarity, ergodicity and Palm theory. In particular, our approach is quite different from the arguments used in [DG16; DG20c].

Using this operator  $\mathcal{K}(L)$ , we obtain, for  $e \in \mathbb{R}^d$ ,

$$\bar{a}^L e = e + L^{-d} \int_{B_1} (\beta - \text{Id}) \mathbb{E}^\circ \left[ (\text{Id} - L^{-d} \mathcal{B}^\circ \mathcal{K}(L))^{-1} (\nabla \phi_e^\circ + e) \right] \lambda(\mathcal{P}) \quad (1.88)$$

where  $\lambda(\mathcal{P})$  is the intensity of the point process,  $\mathcal{B}^\circ$  some deterministic operator (related to the solution operator of  $a^\circ$ ) and  $\mathbb{E}^\circ$  is the expectation under the so-called Palm measure (a suitable notion of conditioning for stationary point processes). Here,  $\mathcal{K}(L)$  fully encompasses the dependence on the randomness and on the dilation parameter.

In (1.83), for convenience, we have chosen the inclusions spherical with constant conductivity  $\beta$  but this can be relaxed to any bounded shape and non-constant value (as long as the conductivity  $a$  stays strongly elliptic and bounded). However, we require the background conductivity to be constant and symmetric : this is our main assumption.

Our approach also includes the periodic case. As (1.88) yields computable formulas for the  $(\mathcal{A}^{(i)})_{i \geq 0}$ , we recover (and extend) the result from [JKO94].

The expansion (1.87) is a series in  $L^{-1}$ , not in  $\lambda(L\mathcal{P}) \sim L^{-d}$  nor in the higher-order intensities  $(\lambda_k(\mathcal{P}))_{k \geq 2}$ . This  $L^{-1}$  scaling is due to a multi-pole expansion effect (in the form of a Taylor expansion of the standard Green function for the Laplace operator  $-\Delta$ ).

Theorem 1.2 should be compared to Theorem 1.1 (for Bernoulli random deletion) as, in both cases, we obtain analyticity in the dilution parameter.

### 1.5.4 Perspectives

**Question 1.1** Can our fixed-point approach extend to the effective viscosity setting ?

We believe that the answer is yes : our approach mostly relies on  $L^2$  theory and elliptic regularity and should therefore be robust. This would provide an alternative proof of Einstein's formula (1.5) under dilution by geometric dilation. In this setting, we would also like to clarify the links between the cluster expansion (1.86) (see [DG20c]) and the expansion obtained by our fixed-point approach.

**Question 1.2** How much can the assumptions on the background be relaxed ?

Our approach requires a constant symmetric background. We believe that the symmetry assumption is only technical. Regarding the constant assumption, recent results [BGO20; Clo+22] provide tools to cover (to some extent) general stationarity ergodic conductivity. We refer to Section 2.1.4 for more details in both cases.

**Question 1.3** Is our approach numerically relevant ?

This last question is more speculative. We have already seen that our result is already of numerical interest as it clarifies the expected scaling of the expansion. However, it is unclear to the author if the coefficients  $(\mathcal{A}^{(i)})_{i \geq 1}$  are, in general, easier to compute numerically than  $\bar{a}^L$ . One could also wonder if the fixed-point operator yields an interesting numerical structure. In further work, we would like to investigate these numerical aspects.

## 1.6 Toward Batchelor's formula

This section introduces the second main result of this manuscript : a first order expansion of the effective sedimentation speed in the intrinsic dilute regime. Its proof is the object of Chapter 3. These results originate from an ongoing work with Mitia Duerinckx and Antoine Gloria. We start by introducing the setting along with the relevant literature. We then present our main contribution. It should be seen as a prototype to tackle Batchelor's formula (1.6).

### 1.6.1 Setting and previous works

Neglecting inertia, sedimentation can be mathematically modelled by rigid particles sinking in a Stokes flow : given the centers  $(x)_{x \in \mathcal{P}}$  of the particles, their speed  $(V_x)_{x \in \mathcal{P}}$  are then obtained solving a Stokes system. Most previous works focused on a dynamical approach : in [JO04], Jabin and Otto identified the non-interacting regime and in [Höf18; HS21; Mec19] Höfer, Mecherbet and Schubert studied the mean-field limit.

Here, following the physical contributions by Smoluchowski [Smo27], Burgers [Bur41; Bur42] and Batchelor [Bat72], we place ourselves in an equilibrium perspective. We assume that particle positions are distributed according to a stationary ergodic point process  $\mathcal{P}$  and we analyze the statistics of the corresponding velocities  $(V_x(\mathcal{P}))_{x \in \mathcal{P}}$ . In particular, we would like to define  $\bar{V}$ , the mean settling speed. If we had  $\#\mathcal{P} < \infty$ , it would be defined as the expectation of the arithmetic mean i.e.  $\mathbb{E} \left[ \frac{1}{\#\mathcal{P}} \sum_{x \in \mathcal{P}} V_x \right]$ . This definition does not make sense in our setting, first because  $\#\mathcal{P} = \infty$  but mainly because of the long-range nature of the hydrodynamical interactions : the  $(V_x)_{x \in \mathcal{P}}$  are not summable (a priori).

Let us now give the system of equations under consideration. We represent the particles by the set  $\mathcal{I} := \cup_{x \in \mathcal{P}} B(x)$  where  $\mathcal{P}$  is a random hardcore stationary point process (with  $\ell(\mathcal{P}) > 2 + \delta$  for a given  $\delta > 0$ , so that the particles are disjoint). Setting the direction of gravity  $e \in \mathbb{R}^d$ , the velocity of the fluid  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and

its associated pressure  $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}$  solve the following equation

$$\left\{ \begin{array}{ll} -\nabla \cdot \sigma(\phi, \Pi) = -\alpha e & \text{in } \mathbb{R}^d \setminus \mathcal{I} \\ \nabla \cdot \phi = 0 & \text{in } \mathbb{R}^d \setminus \mathcal{I} \\ D(\phi) = 0 & \text{in } \mathcal{I} \\ \Pi = 0 & \text{in } \mathcal{I} \\ e|B| + \int_{\partial B(x)} \sigma(\phi, \Pi) \nu = 0 & \forall x \in \mathcal{P} \\ \int_{\partial B(x)} \Theta \nu \cdot \sigma(\phi, \Pi) \nu = 0 & \forall \Theta \in \mathbb{R}_{\text{skew}}^{d \times d}, \forall x \in \mathcal{P} \end{array} \right. \quad (1.89)$$

where  $\sigma(u, p) := 2D(u) - p \text{Id} = (\nabla u + \nabla u^T) - p \text{Id}$  is the Newtonian stress tensor. The boundary conditions on  $\partial \mathcal{I}$  correspond to the static equilibrium of the (rigid) particles under the action of the fluid and the effective gravity (difference between weight and buoyancy). In  $\mathcal{I}$ , the velocity of the fluid  $\phi$  coincides with the rigid motion of the particles (using the standard no-slip boundary condition). For a particle  $x \in \mathcal{P}$ , the speed  $V_x$  is therefore given by  $V_x := \int_{B(x)} \phi(\mathcal{P})$ .

The *back-flow* term  $\alpha$  is given by  $\alpha := \frac{c(\mathcal{P})}{1-c(\mathcal{P})}$  recalling that, here, the volume fraction  $c(\mathcal{P})$  is related to the intensity  $\lambda(\mathcal{P})$  of the point process via  $c(\mathcal{P}) = |B|\lambda(\mathcal{P})$ , see (1.27). The back-flow is an averaged multiparticle effect. As it is constant, it could formally be absorbed in the pressure but we keep this formulation as  $\alpha$  will turn out to be crucial for solvability. We refer to Chapter 3 for more details on the back-flow as well as this physical content of (1.89).

The velocity  $\phi$  is reminiscent of the corrector (1.14) in stochastic homogenization. Borrowing ideas and tools from this field, Gloria [Glo21] (on a scalar version of this problem) and Gloria and Duerinckx [DG22b] succeeded in defining the mean settling speed and in quantifying its variance, the latter being the main focus of their work. In particular, they established the well-posedness of (1.89).

**Lemma 1.5 ([DG22b, extracted from Theorem 1])** *Let  $e \in \mathbb{R}^d$  s.t.  $|e| = 1$ ,  $d \geq 3$  and  $\mathcal{P}$  be a stationary ergodic point process which is algebraically  $\alpha$ -mixing with rate  $\gamma > 2$  (see Definition 1.1).*

*There exists a unique solution  $(\phi, \Pi) \in L^2(\Omega, (H_{loc}^1(\mathbb{R}^d))^d) \times L^2(\Omega, L_{loc}^2(\mathbb{R}^d))$  of (1.89) s.t.*

- *the gradient field  $\nabla \phi$  and the pressure  $\Pi$  are stationary,*
- *they have finite second moments  $\mathbb{E}[|\nabla \phi|^2 + \Pi^2] \lesssim 1$ ,*
- *$\phi$  satisfies the anchoring condition  $\int_B \phi = 0$  a.s.*

*One can then define the mean settling speed  $\bar{V}$  (in the direction  $e$ ) by*

$$\bar{V} := \alpha^{-1} \mathbb{E}[|\nabla \phi|^2] \quad (1.90)$$

*and this definition coincides with the intuitive notion of arithmetic mean.  $\square$*

Compared to the homogenized conductivity  $\bar{a}$  or the effective viscosity  $\bar{B}$ , which exists using qualitative ergodicity only, the definition of  $\bar{V}$  requires (weak) quantitative assumptions (conveniently expressed by  $\alpha$ -mixing). Note that Duerinckx and Gloria construct  $\phi$  by approximation and thus require the additional assumption that the law of  $\mathcal{P}$  can be periodically approximated (this localization procedure is indeed convenient to deal with the incompressibility constraint).

In this setting, our goal is to provide an expansion of  $\bar{V}$  in the general dilute regime and to recover Batchelor's formula (1.6). To the knowledge of the author, no mathematical result is known on this specific topic. However, the literature is much more extensive regarding the dilute expansion of the effective viscosity  $\bar{B}$  (i.e. Einstein's formula (1.5)), see Section 1.5.2. In particular, Duerinckx and Gloria in [DG20c, Section 1.4.3] justified Einstein's formula up to any order in the general dilute regime. Their approach relies on the cluster expansion. For all  $k \geq 1$ , these authors established that

$$|\bar{B}^{(k)}(\mathcal{P})| \lesssim \lambda_k |\log \lambda(\mathcal{P})|^{k-1} \quad (1.91)$$

and

$$|\bar{B} - \sum_{j=0}^k \bar{B}^{(j)}| \lesssim \lambda_{k+1} |\log \lambda(\mathcal{P})|^k. \quad (1.92)$$

Compared to the estimates (1.53) and (1.54) for the toy model (see Section 1.3.4), a logarithmic correction is needed. It is the manifestation of the long-range hydrodynamic interactions in this problem. To tackle them, these author used various strategies (energy method,  $\ell^1 - \ell^2$  estimates, implicit or explicit renormalization), see [DG20c].

### 1.6.2 Contribution

Our main result here is a quantitative first-order dilute expansion of the mean sedimentation speed  $\bar{V}$  in the general dilute setting (involving the multi-point intensities  $(\lambda_j(\mathcal{P}))_{j \geq 1}$ ). Before stating it, we introduce the single particle solution  $\phi^\circ$ , analogue in this context of the single inclusion solution (1.84). For  $e \in \mathbb{R}^d$ ,  $\phi^\circ$  and its associated pressure  $\Pi^\circ$  are the unique (deterministic) decaying solution of

$$\left\{ \begin{array}{ll} -\nabla \cdot \sigma(\phi^\circ, \Pi^\circ) = 0 & \text{in } \mathbb{R}^d \setminus B \\ \nabla \cdot \phi^\circ = 0 & \text{in } \mathbb{R}^d \setminus B \\ D(\phi^\circ) = 0 & \text{in } B \\ \Pi^\circ = 0 & \text{in } B \\ e|B| + \int_{\partial B} \sigma(\phi^\circ, \Pi^\circ) \nu = 0 \\ \int_{\partial B} \Theta \nu \cdot \sigma(\phi^\circ, \Pi^\circ) \nu = 0 \quad \forall \Theta \in \mathbb{R}_{\text{skew}}^{d \times d} \end{array} \right. \quad (1.93)$$

Note that this equation is only well-posed for  $d \geq 3$ . Having  $\phi^\circ$ , we may define  $\bar{V}^{(1)} := \int_B \phi^\circ \cdot e$  which corresponds to the sedimentation speed of a single isolated particle (in the direction  $e$ ).

**Theorem 1.3** *Let  $e \in \mathbb{R}^d$  s.t.  $|e| = 1$ ,  $d \geq 3$  and  $\mathcal{P}$  be a stationary ergodic point process which is algebraically  $\alpha$ -mixing with rate  $\gamma > 2$ . Let  $(\lambda_i)_{1 \leq i \leq 3}$  be its multi-point intensities (see Definition 1.3). The mean settling speed  $\bar{V}$  given by Lemma 1.5 admits the following quantitative approximation*

$$|\bar{V} - \bar{V}^{(1)}| \lesssim \lambda + \frac{\lambda_2^{1-\frac{2}{\gamma}}}{\lambda} + \frac{\lambda_3^{1-\frac{2}{\gamma}}}{\lambda} . \quad (1.94)$$

□

Theorem 1.3 shows that, to leading order, particles fall as if they were isolated.

Our strategy of proof relies on the cluster expansion and explicit renormalization. It can be seen as an adaptation of [DG20c, Theorem 6 p.72] (where  $\bar{B}$  is treated similarly) on the effective sedimentation problem. This adaptation is not straightforward though : the hydrodynamical interaction are stronger in the sedimentation case (see Section 1.4.2 and Remark 1.5) and one has to deal with the back-flow term.

The expansion (1.94) is not sufficient to show Batchelor's formula (1.6) as it only provides a (quantitative) upper bound on the 2<sup>nd</sup> order correction of  $\bar{V}$ . In particular, it does not settle the physical debate between Batchelor's and Hasimoto's predictions. However, as we believe our bound to be sharp, we guess that the difference lies in the mode of dilution. Particularizing (1.94) to the hard-core Poisson case (see Example 1.4), it reduces to

$$|\bar{V} - \bar{V}^{(1)}| \lesssim \lambda(\mathcal{P}) . \quad (1.95)$$

This is in line with Batchelor's prediction (1.6) from [Bat72], who considered a Poisson-like model. Hasimoto's periodically arranged particles fits into the geometric dilation setting. Its correction in  $c^{\frac{1}{3}}$  (for  $d = 3$ ) should then be understood as a correction in  $L^{-1}$ , the dilation parameter. We have seen that such a scaling appears further expanding the cluster coefficients in that setting.

### 1.6.3 Perspectives

**Question 1.4** Can we extend Theorem 1.3 to cover Batchelor's formula ?

Our result should be seen as a prototype : we believe that the strategy presented here can be extended to any order. This extension is however technically much more involved : in the spirit of [DG20c, Section 4.4 p.80], one should decompose



further the cluster expansion in elementary contributions to unravel additional cancellations, see Section 3.4 for details. It is the object of an ongoing work with Duerinckx and Gloria.

**Question 1.5** Is it really necessary to assume that the law of the point process can be periodically approximated ?

Theorem 1.3 is built on Lemma 1.5 and therefore requires this additional assumption on the point process. We believe it to be purely technical and hope to relax it combining the massive approximation with the standard approach for incompressibility by penalization of the pressure, see [BF13, Section 5.4]

**Question 1.6** Can the fixed-point approach from Theorem 1.2 be applied here ?

From formal computations (in the spirit of Section 2.1.2), we would be tempted to give a positive answer. However, the necessity of quantitative ergodicity assumptions in the definition of  $\bar{V}$  foreshadows bigger difficulties in the rigorous treatment of the fixed-point operator. Applying this method here would be a good test of its robustness. It would yield an asymptotic expansion of the map  $L^{-1} \mapsto \bar{V}(L\mathcal{P})$  as  $L \rightarrow \infty$ . In particular, this would settle the periodic case.

## 1.A Proof of the optimization Lemma 1.4

Lemma 1.4 is a reformulation of an argument used by Duerinckx and Gloria in [DG20c, Proof of Theorem 5 p. 73]. The proof relies on the fact that, in the Taylor expansion (1.79), the approximate coefficients  $b_T^{(i)}$  are independent of  $x$ . Hence, we have the freedom to optimize either in  $x$  or in  $T$ .

*Step 1* : For  $i \geq 0$ , define  $b^{(i)}$  by

$$b^{(i)} = b_1^{(i)} + \sum_{k=0}^{\infty} (b_{2^{k+1}}^{(i)} - b_{2^k}^{(i)}).$$

This quantity is well defined and satisfies for all  $T > 1$ ,

$$|b^{(i)} - b_T^{(i)}| \lesssim_i T^{-\beta 2^{-(i+1)}} \quad (1.96)$$

Both of these claims are implied summing the following bound : for all  $T > 1$ ,

$$|b_{2T}^{(i)} - b_T^{(i)}| \lesssim_i T^{-\beta 2^{-i}} \ln(T)^i. \quad (1.97)$$

We prove it by induction.

For  $i = 0$ , this is a direct consequence of the rate of convergence of  $f_T$  for  $x = 0$ .

Assume (1.97) for  $i > 0$ . Subtracting the expansion of  $f_{2T}$  and  $f_T$  for  $n = i$ , one has for all  $x \in [0, 1]$ ,

$$x^i |b_{2T}^{(i)} - b_T^{(i)}| \leq |(f_{2T} - f_T)(x)| + \sum_{k=0}^{i-1} x^k |b_{2T}^{(k)} - b_T^{(k)}| + |R_{2T}^{(i+1)}(x)| + |R_T^{(i+1)}(x)|$$

Using  $|(f_{2T} - f_T)(x)| \leq \|f_{2T} - f + f - f_T\|_\infty \lesssim T^{-\beta}$ , (1.97) for all  $0 \leq k \leq i$  and dividing by  $x^i$ , one gets

$$|b_{2T}^{(i)} - b_T^{(i)}| \lesssim_i x^{-i} T^{-\beta} + \sum_{k=0}^{i-1} x^{k-i} T^{-\beta 2^{-k}} \ln(T)^k + x \ln(T)^{i+1}$$

for all  $x \in [0, 1]$ .

To minimize the RHS, we chose

$$x = T^{-\beta 2^{-i}} \ln(T)^{-1}. \quad (1.98)$$

Although not optimal, this choice ensures that all the terms of the RHS have roughly the same size. Indeed, with this  $x$  we get

$$|b_{2T}^{(i)} - b_T^{(i)}| \lesssim_i \ln(T)^i \sum_{k=0}^{i-1} T^{-\beta(2^{-k} - (i-k)2^{-i})} + T^{-\beta 2^{-i}} \ln(T)^i \lesssim T^{-\beta 2^{-i}} \ln(T)^i$$

since  $2^{-k} - (i-k)2^{-i} \geq 2^{-i}$  (it is equivalent to  $2^{i-k} \geq (i-k) + 1$  which is true for  $i-k > 0$ ).

*Step 2 :* Using Step 1, we can define  $\sum_{i=0}^n b^{(i)} x^i$ , suitable candidate for the expansion of  $f$ . We write their difference, for all  $x \in [0, 1]$  as

$$f(x) - \sum_{i=0}^n b^{(i)} x^i = f(x) - f_T(x) - \sum_{i=0}^n (b^{(i)} - b_T^{(i)}) x^i + R_T^{(n+1)}(x).$$

Using (1.96), the uniform rate for  $f_T$  and the bound on  $R_T^{(n+1)}(x)$ , one has

$$\begin{aligned} |f(x) - \sum_{i=0}^n b^{(i)} x^i| &\lesssim_n T^{-\beta} + \sum_{i=0}^n T^{-\beta 2^{-i}} x^i + \ln(T)^{n+1} x^{n+1} \\ &\lesssim_n T^{-\beta 2^{-n}} + \ln(T)^{n+1} x^{n+1} \end{aligned}$$

for all  $T \geq 1$ .

We chose

$$T = x^{-(n+1)2^n \beta^{-1}} \tag{1.99}$$

to nearly optimally balance the two terms of the RHS and get

$$|f(x) - \sum_{i=0}^n b^{(i)} x^i| \lesssim_n |\ln(x)|^{n+1} x^{n+1}$$

which is the desired estimate.

**Remark 1.6** Even if we have been sharper and kept track of the growth in  $n$  of the constant in  $\lesssim_n$ , this type of technique does not yield analyticity (which would require this bound to grow like  $C^n$  for some  $C > 0$ ). To obtain such a result, one would need bounds on the  $b_T^{(i)}$  which are uniform in  $T$ .  $\square$

# Chapter 2

## A fixed-point approach to Clausius-Mossotti formulas

We consider the homogenized coefficient  $\bar{a}^L$  associated to a random two-phase media diluted by dilation (with dilation parameter  $L$ ). We show that  $\bar{a}^L$  depends analytically on  $L$  in the regime  $L \gg 1$ . This result gives a rigorous justification of the Clausius-Mossotti formula (and its extensions to any order). Our approach relies on a new fixed-point formulation for the homogenization corrector involving the so-called single inclusion problem. It holds without the need of any quantitative homogenization theory. The result should be compared to [DG16] where analyticity of the homogenized coefficient was obtained in the case of dilution by random Bernoulli deletion.

### Outline of the current chapter

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<b>2.1 Introduction</b>	<b>51</b>
2.1.1 Setting and main result . . . . .	51
2.1.2 General strategy . . . . .	54
2.1.3 Previous works . . . . .	59
2.1.4 Extensions . . . . .	61
<b>2.2 Preliminaries</b>	<b>63</b>
2.2.1 Massive approximation . . . . .	63
2.2.2 Some theory of Point Processes . . . . .	73
<b>2.3 Operator</b>	<b>75</b>
2.3.1 The key operator . . . . .	76
2.3.2 Improved bound . . . . .	80
<b>2.4 Proof of Theorem 2.1 : analyticity of the homogenized coefficient</b>	<b>90</b>

2.4.1 Main propositions . . . . .	91
2.4.2 Proof of Proposition 2.4 . . . . .	92
2.4.3 Proof of Proposition 2.3 : analyticity of the fixed-point operator . . . . .	96

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## 2.1 Introduction

### 2.1.1 Setting and main result

#### Setting

In this chapter, we study the effective behavior of random two-phase media in a dilute regime obtained via dilation.

We start by constructing a family of random two-phase media in the following way. First, we set a constant homogeneous reference medium (or background phase) associated to the constant conductivity matrix  $\alpha \in \mathbb{R}^{d \times d}$ . Then, we change the conductivity of this reference medium in so-called inclusions that we scatter along a hard-core random point process  $\mathcal{P} = (x)_{x \in \mathcal{P}}$ . Given a measurable set  $J \subset B_1$  and a (variable) conductivity matrix  $\beta : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ , the conductivity matrix of our two phase medium can be written

$$a(\mathcal{P}, \cdot) = \alpha \mathbb{1}_{\mathbb{R}^d \setminus \cup_{x \in \mathcal{P}} (J+x)} + \beta(\cdot) \mathbb{1}_{\cup_{x \in \mathcal{P}} (J+x)}. \quad (2.1)$$

Both  $\alpha$  and  $\beta$  are assumed uniformly elliptic and bounded (so that  $a$  is too), i.e. there exists  $\lambda > 0$  s.t. for and  $\forall x \in \mathbb{R}^d, \forall \xi \in \mathbb{R}^d, \lambda |\xi|^2 \leq \xi \cdot a(x) \xi$  and  $|a(x) \xi| \lesssim |\xi|$ .

For the clarity of exposition, in the sequel we assume that the inclusions are disjoint (see Remark 2.2), that  $\alpha = \text{Id}$  and set  $C^\circ(\cdot) := (\beta(\cdot) - \alpha) \mathbb{1}_J$  so that the conductivity matrix of our two phase medium rewrites

$$a(\mathcal{P}, \cdot) := \text{Id} + \sum_{x \in \mathcal{P}} C^\circ(\cdot - x) \quad (2.2)$$

with  $C^\circ : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  bounded and supported in  $B_1$ . Typically, one should think of  $C^\circ$  as  $\mathbb{1}_{B_1} \text{Id}$ .

**Remark 2.1** Passing from formula (2.1) to (2.2), we have implicitly assumed that the background conductivity  $\alpha$  is symmetric.

Indeed, in this case,  $\alpha$  can be diagonalized in an orthonormal basis and a change of variables (in the conductivity problem associated to the operator  $-\nabla \cdot \alpha \nabla$ ) brings us back to  $\alpha = \text{Id}$  by change of variable. This is the most common physical case, (by Onsager reciprocal relations, the conductivity matrix has to be symmetric if the medium is time reversible. It holds for instance for isotropic medium).

However, if  $\alpha$  is not symmetric the definition (2.1) and (2.2) are not equivalent anymore (the change of variable can not be performed as  $\alpha$  can not be diagonalized in an orthonormal basis). Although less common, non-symmetric background conductivities also exist in the physical literature. They appear, for instance, in presence of an external magnetic field (by the so-called Hall effect).  $\square$

To draw our random point processes, we define the space

$$\Omega := \{\mathcal{P} \in (\mathbb{R}^d)^{\mathbb{N}}, \forall x, y \in \mathcal{P}, x \neq y, |x - y| \geq 3\} \quad (2.3)$$

of hard-core point processes  $\mathcal{P}$ . Note that we have chosen a minimal distance of 3 so that the inclusions are well separated (see Remark 2.2). Following [Chi+13, Section 4.1.1 p.108], we equip  $\Omega$  with the  $\sigma$ -algebra

$$\mathcal{F} := \sigma(\mathcal{P} \mapsto \#(\mathcal{P} \cap B), B \subset \mathbb{R}^d \text{ bounded Borel set})$$

(which is countably generated since we can limit ourselves to rational rectangles) and a probability measure  $\mathbb{P}$  that we assume :

- *stationary* (i.e. without origin) : for all  $z \in \mathbb{R}^d$ ,  $\tau_z \mathcal{P} \sim_{\text{law}} \mathcal{P}$  where the shift  $\tau_z$  are defined by  $\tau_z \mathcal{P} := (x + z)_{x \in \mathcal{P}}$ ,
- *ergodic* (i.e. decorrelated at large scale) : for any  $F \in \mathcal{F}$ , ( $F = F + z$  for all  $z \in \mathbb{R}^d$ )  $\implies \mathbb{P}[F] \in \{0, 1\}$ . By Birkhoff theorem, this property implies that  $\mathbb{E}[f(\mathcal{P})] = \lim_{R \rightarrow \infty} \int_{B_R} f(\mathcal{P} + z) dz$  for any  $f \in L^1(\Omega, \mathbb{P})$  (i.e. we can recover expectations from spatial averages over large domain).

Under these assumptions, the standard theory of stochastic homogenization ([JKO94], [PV81]) yields that the heterogeneous random medium has an effective homogeneous behavior at large scales characterized by its effective conductivity (a.k.a. homogenized coefficient)  $\bar{a}$ .

The standard notion of a dilute medium is to assume the volume fraction (a.k.a. concentration)  $c$  of the inclusion phase to be small. As we have assumed the inclusions to be disjoint, the volume fraction is proportional to the intensity  $\theta$  of the point process  $\mathcal{P}$  (average amount of point per unit of volume, see Definition 2.5). In the dilute regime  $\theta \ll 1$ , Clausius-Mossotti formulas then give an approximation of  $\bar{a}$  in the form a power series in  $\theta$ .

In this work, we consider a specific dilution procedure : through dilation. For a dilation parameter  $L \geq 1$ , we define the space

$$L\Omega := \{L\mathcal{P} \mid \mathcal{P} \in \Omega\} \quad (2.4)$$

of dilated random point processes. Note that  $L\Omega \subset \Omega$  for all  $L \geq 1$ . If  $\mathcal{P}$  had intensity  $\theta \sim 1$ , its dilation  $L\mathcal{P}$  has intensity  $L^{-d}\theta \ll 1$  for  $L \gg 1$ .

**Remark 2.2** Our main assumption on the point process  $\mathcal{P}$  is to be hard-core, i.e.  $\inf_{\substack{x,y \in \mathcal{P} \\ x \neq y}} |x - y| > 0$ .

After dilatation, this minimal distance is multiplied by a factor  $L$ . As we are interested in the regime  $L \gg 1$ , we can actually make it as big as needed taking  $L \geq L_0$  with  $L_0$  suitably chosen. Hence, in the definition (2.3) of  $\Omega$ , the choice of a minimal distance of 3 is not essential.  $\square$

We define a dilated random medium through its conductivity  $a(L\mathcal{P}, \cdot)$  for  $L \geq 1$ ,  $\mathcal{P} \in \Omega$ . As stationarity and ergodicity are preserved by dilation, we can still define the dilated homogenized matrix  $\bar{a}^L$  associated to  $a(L\mathcal{P}, \cdot)$ .

### Main result

The goal of this work is to show that  $\bar{a}^L$  admits an asymptotic expansion in the dilute regime  $L \gg 1$ . Our main result reads as follows.

**Theorem 2.1** *Let  $\mathcal{P} \in \Omega$  be a hard-core stationary ergodic random point process (see (2.3)). For  $L \geq 1$ , let  $\bar{a}^L$  be the homogenized matrix associated to  $a(L\mathcal{P}, \cdot)$  (given by (2.2)). Then, the map  $L^{-1} \mapsto \bar{a}^L$  is analytic at 0.*

*In particular, there exists a family of matrices  $(\mathcal{A}^{(i)})_{i \geq 0} \in (\mathbb{R}^{d \times d})^{\mathbb{N}}$  such that  $\bar{a}^L$  is given by the summable series*

$$\bar{a}^L = \sum_{i=0}^{\infty} \mathcal{A}^{(i)} L^{-i} . \quad (2.5)$$

Before commenting on the assumption, we point out that analyticity is not obtained through abstract arguments. Our result is constructive : (2.118) gives a formula for  $\bar{a}^L$  with an explicit dependence on  $L$  involving a fixed-point operator (to be defined later, see Section 2.3). It yields computable formulas for the  $(\mathcal{A}^{(i)})_{i \geq 0}$ , in particular we have that  $\mathcal{A}^{(0)} = \text{Id}$  the background conductivity, that  $\mathcal{A}^{(i)} = 0$  for  $0 < i < d$ , that  $\mathcal{A}^{(d)}$  is the standard Clausius-Mossotti correction, that  $\mathcal{A}^{(i)} = 0$  for  $d < i < 2d$  and that generically higher order terms do not vanish.

Our stochastic assumptions (stationarity and ergodicity) are essentially minimal for homogenization to hold. In particular, we do not require any form of quantification of ergodicity. Our result also covers the periodic case seeing a periodic lattice as a deterministic hard-core process (replacing the expectation by an average over the periodic cell).

Note that we do not make any assumption regarding the smoothness of the shape of the inclusions  $\text{supp}(C^\circ)$ . Even if stated using the scalar notations, our proof of Theorem 2.1 holds in the case of uniformly elliptic systems and of linear elasticity (provided that the elasticity tensor is uniformly strongly elliptic, as standard in homogenization).

Our proof uses the symmetry of the constant background conductivity (that we have implicitly assumed using formula (2.2), see Remark 2.1). However, we believe that our strategy can be adapted to the non-symmetric case, see Section 2.1.4.

### 2.1.2 General strategy

We start by studying how the homogenized coefficient  $\bar{a}^L$  behaves at first order under dilation before refining this analysis to higher orders.

#### Dilation and first-order expansion

By standard stochastic homogenization theory [JKO94; PV81], the homogenized coefficient is given for  $e \in \mathbb{R}^d$  by the formula

$$\bar{a}^L e = \mathbb{E} [a(L\mathcal{P}, \cdot)(\nabla\varphi(L\mathcal{P}, \cdot) + e)](0) \quad (2.6)$$

where  $\varphi(L\mathcal{P}, \cdot) \in H_{\text{loc}}^1(\mathbb{R}^d)$  is the unique weak solution (up to additive constant) of

$$-\nabla \cdot a(L\mathcal{P}, \cdot)(\nabla\varphi(L\mathcal{P}, \cdot) + e) = 0 \quad \text{in } \mathbb{R}^d \quad (2.7)$$

such that  $\nabla\varphi(L\mathcal{P})$  is stationary,  $\mathbb{E}[\nabla\varphi(L\mathcal{P})] = 0$  and  $\mathbb{E}[|\nabla\varphi(L\mathcal{P})|^2] \lesssim 1$  (see Remark 2.11 for details).

By stationarity, one could evaluate (2.6) at any point  $z \in \mathbb{R}^d$  instead of 0. The function  $\varphi(L\mathcal{P}, \cdot)$  (defined up to a constant) which depends both on the point process and on the space variable is called the corrector because it corrects the affine function  $e \cdot x$  into the  $a(L\mathcal{P})$ -harmonic function  $x \mapsto e \cdot x + \varphi(L\mathcal{P}, x)$ . Better than stationarity, shifts on the point process and the physical space are intertwined by the so-called contravariance property : for all  $z \in \mathbb{R}^d$ ,

$$\nabla\varphi(L\mathcal{P}, \cdot + z) = \nabla\varphi(L\mathcal{P} - z, \cdot) . \quad (2.8)$$

Rewriting (2.7) as  $-\nabla \cdot a(L\mathcal{P})\nabla\varphi(L\mathcal{P}) = \nabla \cdot a(L\mathcal{P})e$ , one can formally write the corrector as  $\nabla\varphi(L\mathcal{P}) = \nabla(-\nabla \cdot a(L\mathcal{P})\nabla)^{-1}\nabla \cdot a(L\mathcal{P})e$ . Because of the inverse operator  $(-\nabla \cdot a(L\mathcal{P})\nabla)^{-1}$ ,  $\nabla\varphi(L\mathcal{P})$  (and consequently  $\bar{a}^L$  from (2.6)) depends non-locally and non-linearly on the point process  $\mathcal{P}$  and it is a priori unclear how to extract its scaling in  $L$ .

When dilating the point process  $\mathcal{P}$  by a factor  $L$ , the volume fraction of inclusions is reduced by a factor  $L^{-d}$ . Since  $\bar{a}^L$  is a (weighted) average of  $a(L\mathcal{P}, \cdot)$ , we expect that  $\bar{a}^L \xrightarrow{L \rightarrow \infty} \text{Id}$ , the (constant) conductivity of the background. We can



even make explicit  $\bar{a}^L - \text{Id}$  via

$$\bar{a}^L e = e + \mathbb{E} \left[ \sum_{x \in \mathcal{P}} C^\circ(\cdot - Lx)(\nabla\varphi(L\mathcal{P}, \cdot) + e) \right] (0) \quad (2.9)$$

which directly follows from the definitions (2.6) of  $\bar{a}^L$  and (2.2) of  $a(L\mathcal{P})$  after expanding the product and using that  $\mathbb{E}[\nabla\varphi] = 0$ . Based on this formula, we will show that the remainder term is  $O(L^{-d})$ .

As one can see on (2.9), we only need to describe the corrector around each inclusion since  $C^\circ$  is supported in  $B_1$ . By the hard-core hypothesis, for  $x \in \mathcal{P}$ ,  $a(L\mathcal{P}, \cdot) = \text{Id} + C^\circ(\cdot - Lx)$  on  $B_L(x)$ . Hence, for  $L \gg 1$ , we can almost treat each inclusion as if it were isolated. Similarly to the corrector associated with  $a(L\mathcal{P})$ , we define the (deterministic) single inclusion problem and its solution  $\varphi^\circ$  associated with the conductivity

$$a^\circ := \text{Id} + C^\circ \quad (2.10)$$

of a medium with one single inclusion at the origin via

$$-\nabla \cdot a^\circ(\nabla\varphi^\circ + e) = 0 \quad \text{in } \mathbb{R}^d \quad . \quad (2.11)$$

The existence and uniqueness (up to constant) for (2.11) are elementary, see Remark 2.4 for details. In the sequel, we will consider solely the gradient  $\nabla\varphi^\circ$  which we will (abusively) call the single inclusion solution.

We claim that  $\nabla\varphi(L\mathcal{P}, \cdot) \simeq \nabla\varphi^\circ(\cdot - Lx)$  on  $B_1(x)$  when  $L \gg 1$ . Let us argue how this gives a first order description of  $\bar{a}^L$ . Replacing  $\nabla\varphi(L\mathcal{P}, \cdot)$  by  $\nabla\varphi^\circ$  in (2.9) yields

$$\begin{aligned} \bar{a}^L e &\simeq e + \mathbb{E} \left[ \sum_{x \in \mathcal{P}} [C^\circ(\nabla\varphi^\circ + e)](0 - Lx) \right] \\ &= e + L^{-d} \int_{\mathbb{R}^d} [C^\circ(\nabla\varphi^\circ + e)](x)\theta \, dx \, , \end{aligned}$$

where  $\theta$  is the intensity of the point process  $\mathcal{P}$  (see Definition 2.5) and we have used the Campbell formula (see (2.56) below) to pass from the first line to the second. This is the so-called Clausius-Mossotti formula (2.21) (as  $\varphi^\circ$  is explicit in spherical coordinate for constant background with spherical constant inclusion). Here, we aim at a complete description of  $\bar{a}^L$  and therefore we now need to generalize this formula to arbitrarily high orders.

### Fixed-point approach for the corrector

We want to quantify the vague statement  $\nabla\varphi(L\mathcal{P}, \cdot)|_{B_1(x)} \simeq_{L \gg 1} \nabla\varphi^\circ(\cdot - Lx)|_{B_1(x)}$  which amounts to understanding :

*How well can one approximate  $\nabla\varphi(L\mathcal{P}, \cdot)$  using  $\nabla\varphi^\circ$ , the single inclusion solution, only ?*

The fixed-point approach that we (informally) present here answers this question in a way that allows to make explicit the dependence in  $L$  and reach accuracy to arbitrarily high order. This constitutes one of the main contributions of this chapter.

From now on, we assume that  $0 \in \mathcal{P}$  and we will characterize  $\nabla\varphi(L\mathcal{P})$  on  $B_1$  as the solution of a fixed-point problem involving  $\nabla\varphi^\circ$ . This could be seen as taking  $x = 0$  in the previous approximation and will be justified by conditioning the point process using Palm theory, see Section 2.2.2.

The first step is to derive an equation for  $\varphi(L\mathcal{P}) - \varphi^\circ$ . To this effect, we rewrite (2.7), the equation of the corrector, using that  $a(L\mathcal{P}) = a^\circ + \sum_{x \in \mathcal{P} \setminus \{0\}} C^\circ(\cdot - Lx)$  by the definition (2.10) of  $a^\circ$ .

$$-\nabla \cdot a^\circ(\nabla\varphi(L\mathcal{P}) + e) = \nabla \cdot \sum_{x \in \mathcal{P} \setminus \{0\}} C^\circ(\cdot - Lx)(\nabla\varphi(L\mathcal{P}) + e)$$

Subtracting the equation (2.11) for  $\varphi^\circ$ , this yields

$$-\nabla \cdot a^\circ(\nabla\varphi(L\mathcal{P}) - \nabla\varphi^\circ) = \nabla \cdot \sum_{x \in \mathcal{P} \setminus \{0\}} C^\circ(\cdot - Lx)(\nabla\varphi(L\mathcal{P}) + e) .$$

Since we have taken out the origin from the sum in the RHS, by the support condition on  $C^\circ$ ,  $\varphi(L\mathcal{P}) - \varphi^\circ$  solves an elliptic equation with source terms located at distance  $L$  from  $B_1$ . This should yield smallness in appropriate norms.

To go quantitative, we need to rely on precise asymptotics of the Green function associated with  $a^\circ$ . Our claim is that they can be deduced from the asymptotics of the standard Green function  $G$  associated with  $-\Delta$  by a simple argument. More precisely, defining  $\psi$  as the (formal) solution of

$$-\Delta\psi = \nabla \cdot \sum_{x \in \mathcal{P} \setminus \{0\}} C^\circ(\cdot - Lx)(\nabla\varphi(L\mathcal{P}) + e) , \quad (2.12)$$

we have

$$\nabla(\varphi(L\mathcal{P}) - \varphi^\circ) = \mathcal{B}^\circ \nabla\psi \quad (2.13)$$

with  $\mathcal{B}^\circ \nabla \psi := \nabla \psi + \nabla \rho$  where  $\nabla \rho$  is associated to  $\nabla \psi$  via

$$-\nabla \cdot a^\circ \nabla \rho = \nabla \cdot C^\circ \nabla \psi .$$

Note that  $\mathcal{B}^\circ$  is a (deterministic), bounded, local, linear operator on  $L^2(\mathbb{R}^d)$  (cf Lemmas 2.3 and 2.4 for details and Remark 2.5 regarding the notion of locality).

Using the Green representation formula, we have

$$\begin{aligned} \nabla \psi &= \nabla^2 G * \sum_{x \in \mathcal{P} \setminus \{0\}} C^\circ(\cdot - Lx)(\nabla \varphi + e)(L\mathcal{P}, \cdot) \\ &= \sum_{x \in \mathcal{P} \setminus \{0\}} \nabla^2 G(\cdot - Lx) * C^\circ(\nabla \varphi + e)(L\mathcal{P}, \cdot + Lx) . \end{aligned}$$

By the contravariance property (2.8) of the corrector, we have  $(\nabla \varphi + e)(L\mathcal{P}, \cdot + Lx) = (\nabla \varphi + e)(L(\mathcal{P} - x), \cdot)$ . Combining this with the homogeneity of the Green function in the form of  $\nabla^2 G(\cdot - Lx) = L^{-d} \nabla^2 G(\frac{\cdot}{L} - x)$ , we obtain

$$\nabla \psi = L^{-d} \sum_{x \in \mathcal{P} \setminus \{0\}} \nabla^2 G(\frac{\cdot}{L} - x) * C^\circ(\nabla \varphi + e)(L(\mathcal{P} - x), \cdot) . \quad (2.14)$$

We then define the (formal) operator  $K^L$  acting on functions  $f : (L\mathcal{P}, x) \mapsto f(L\mathcal{P}, x)$  depending both on  $L$ -dilated point process and on the space variable as

$$K^L f(L\mathcal{P}, \cdot) = \sum_{x \in \mathcal{P} \setminus \{0\}} \nabla^2 G(\frac{\cdot}{L} - x) * C^\circ f(L(\mathcal{P} - x), \cdot) \quad (2.15)$$

so that (2.14) takes the form  $\nabla \psi = L^{-d} [K^L(\nabla \varphi + e)](L\mathcal{P}, \cdot)$ . Combining this with (2.13), we obtain the identity

$$(\nabla \varphi + e)(L\mathcal{P}, \cdot) = \nabla \varphi^\circ + e + L^{-d} \mathcal{B}^\circ K^L(\nabla \varphi + e)(L\mathcal{P}, \cdot) \quad (2.16)$$

which we interpret as a fixed-point equation.

If  $K^L$  were bounded, we could then write for  $L$  large enough

$$\nabla \varphi(L\mathcal{P}, \cdot) + e = \left[ (\text{Id} - L^{-d} \mathcal{B}^\circ K^L)^{-1} (\nabla \varphi^\circ + e) \right](L\mathcal{P}, \cdot) . \quad (2.17)$$

Since  $(\nabla \varphi^\circ + e)$  is deterministic, the dependence of  $\nabla \varphi$  on the point process is completely encompassed in the operator  $K^L$ . Last, Taylor expanding the sum

$$\sum_{x \in \mathcal{P} \setminus \{0\}} \nabla^2 G(x - \frac{\cdot}{L}) = \sum_{x \in \mathcal{P} \setminus \{0\}} \nabla^2 G(x) + \sum_{x \in \mathcal{P} \setminus \{0\}} \nabla^3 G(x) \cdot \frac{\cdot}{L} + \dots \quad (2.18)$$

we can express  $\nabla\varphi$  as a power series in  $L^{-1}$  from which the desired the expansion for  $\bar{a}^L$  follows.

### Toward a rigorous proof

Our main issue in the previous computation is the definition of the operator  $K^L$  : even after defining proper spaces, our formula does not make sense a priori. The limiting object here is the (formal) sum  $\sum_{x \in \mathcal{P} \setminus \{0\}} \nabla^2 G(x)$  which is not well-defined for a given  $\mathcal{P}$ . Indeed,  $|\nabla^2 G(z)| \sim |z|^{-d}$  and this decay at infinity is (borderline) not summable. For  $k \geq 3$  however, the decay of  $\nabla^k G$  is strong enough to define all the other terms in the expansion (2.18).

To circumvent this issue and work with finite quantities, we introduce a massive approximation adding  $\frac{1}{T}$  to the operator  $-\nabla \cdot a \nabla$  in order to screen the medium at scale  $> \sqrt{T}$  (see Lemma 2.1). We could also have chosen other approximation strategies, for instance restricting ourselves to a finite part of the medium by periodization on a big torus (as in [DG22b]).

Our goal is to obtain estimates which are uniform with respect to this screening parameter so that we can pass to the limit  $T \rightarrow \infty$  and recover our fixed point formulation which yields analyticity of the homogenized coefficient. The screening allows us to properly define an approximate version of the operator  $K_T^L$  but, using only deterministic arguments, we only obtain  $\| \| K_T^L \| \| \lesssim \ln(TL^{-2})$  (see (2.65)) which diverges as  $T \rightarrow \infty$ .

This is where stationarity and ergodicity come into play : thanks to these assumptions, we can rewrite our operator  $K^L$  (2.15) as the difference of two PDE solution. Indeed, still assuming  $0 \in \mathcal{P}$  and reverting our previous computation, we obtain the identity  $L^{-d} K^L f = \nabla u - \nabla u^\circ$  for  $f(L\mathcal{P}, \cdot)$  (see Lemma 2.13) where  $u^\circ$  solves

$$-\Delta u^\circ = \nabla \cdot C^\circ f(L\mathcal{P}) \quad (2.19)$$

(which is well posed as the source term is localized) and where  $u$  solves

$$-\Delta u = \nabla \cdot \sum_{x \in \mathcal{P}} C^\circ(\cdot - Lx) f(L(\mathcal{P} - x)) , \quad (2.20)$$

(which is now well posed by stationarity and ergodicity of the source term similarly to the corrector equation, see Remark 2.11). Combining the energy estimates on (2.20) and (2.19), we obtain  $L^{-d} \| \| K^L \| \| \lesssim 1$ .

It remains to extract the smallness. This is done through elliptic regularity (in the form of mean value property and Green function decay) using a variational strategy introduced by Duerinckx and Gloria in [DG20c] (this is the only place where we use the symmetry of the background conductivity). We note that, as  $\mathcal{P}$  is hard-core, (2.19) and (2.20) have the same source terms in  $B_L$ . Our claim is that to

compare them, it is enough to compare the Dirichlet and Neumann problems with the same source term on  $B_L$  (see Lemma 2.15). Indeed, as the PDEs (2.19) and (2.20) admit a variational formulation, comparing their solutions (or minimizers) amounts to comparing their energies for which we know that the Dirichlet and Neumann boundary condition are extremal. We then conclude estimating the difference between the Dirichlet (resp. Neumann) problem and the full space problem (2.19) for which we have an explicit decay (see Lemma 2.16). This finally yields the uniform bound  $\| \| K_T^L \| \| \lesssim 1$ , see Proposition 2.1.

### 2.1.3 Previous works

In the late 19<sup>th</sup> century, way before the mathematical theory of homogenization was established, the physics community was already studying two-phase media and looking for expansions of their properties in a dilute regime. For various problems (dielectric constant, refractive index, conductivity) Clausius [Cla79], Mossotti [Mos36], Maxwell [Max98], Garnett [Gar04], Lorenz [Lor80], Lorentz [Lor09] (see [Mar00, Section 1.1.3 p.7] for an historical account) proved, in modern language and in the terminology of our problem, that given a constant background  $\text{Id}$  perturbed by spherical inclusions of constant conductivity  $\beta \text{Id}$ ,  $\beta > 0$ , the effective conductivity can be expanded at first order

$$\bar{a}^L = \text{Id} + c\mathcal{A}^{(d)} + o(c) \quad (2.21)$$

where we recall that  $c$  is the volume fraction of the inclusion phase and where, in this case,  $\mathcal{A}^{(d)} = \frac{d(\beta-1)}{\beta+(d-1)} \text{Id}$  (explicit thanks to spherical shape of the inclusions). In the literature, such expansions are called under various combinations of those names but we will always refer to them as Clausius-Mossotti formula in the sequel.

The development of the theory of homogenization in the 1970s allowed for a rigorous treatment of these results in the periodic case. In [JKO94, Section 1.7 p.45, see (1.79)] Jikov, Kozlov and Oleinik give an expansion similar to (2.5) up to order  $2d$  with a quantitative control of the remainder  $O(L^{-(2d+2)})$ . This result, which was the main inspiration for this work, relies on a far field expansion of the corrector using the explicit Green function as well as a periodization procedure (where the summability issue is treated through a suitable renormalization). Our work confirms that the scaling of the remainder is indeed sharp ( $\mathcal{A}^{(2d+1)}$ , the term of order  $2d + 1$ , vanishes by the symmetries of the cubic lattice). In [Ber83], Berdichevskii proposes a general algorithm to obtain arbitrarily high accuracy in the constant isotropic case with spherical inclusions based on explicit calculation (using spherical harmonics).

Before turning to the more recent contributions, let us point out why such approximations remain relevant today. Indeed, if homogenization theory yields a

formula for the homogenized coefficient, it is only abstract (except for a few examples : 1D, spherical symmetry, . . .) and one has to resort to numerical simulation to approximate it. These simulations could be quite costly, in particular in the stochastic case, hence the need for simpler approximations. In our approach, the matrices  $(\mathcal{A}^{(i)})_{i \geq 0}$  are not necessarily easier to compute than the original homogenized coefficient  $\bar{a}^L$  but our main point is that we know the correct exponents in the power series expansion (which were unclear at high orders, see discussion below between dilation and deletion). This could be used for scarce data interpolation in practice. This approximation can also be used for variance reduction purposes (see [AL11; AL12]). We also mention application to inverse problems, see [AK04].

The first rigorous proof of the first order formula (2.21) to hold in the random case is due to Almog in [Alm13; Alm14].

To go to higher orders, the idea in the literature was to rely on one parameter models : the parameter characterizes the dilution level and proving (2.21) now amounts to obtaining an expansion of the associated homogenized matrix. A first order expansion was obtained in [AL11; AL12; Mou15] for random Bernoulli deletion models : for  $p \in [0, 1]$  and for each  $x \in \mathcal{P}$ , define a family of i.i.d random Bernoulli variables  $b_x \stackrel{i.i.d}{\sim} p\delta_1 + (1 - p)\delta_0$ , independent of  $\mathcal{P}$  and set  $\mathcal{P}^{(p)} := \{x \in \mathcal{P} \mid b_x = 1\}$ . Then, define  $\bar{a}^{(p)}$  as the homogenized coefficient associated to  $a(\mathcal{P}^{(p)}, \cdot)$  (which is still stationary and ergodic thanks to the independence of the  $(b_x)_{x \in \mathcal{P}}$ ). Extending these works, [DG16] (we refer to its introduction for a detailed comparison between those works) have obtained the analyticity of the map  $p \mapsto \bar{a}^{(p)}$  near zero (which implies the analyticity on the full interval  $[0, 1]$  by a change of reference medium, observation due to [Mou15]). Their method is constructive and rely on the so-called cluster expansion as well as  $\ell^1 - \ell^2$  estimates (some refined energy estimates) to control the terms in the expansion. Like us, they do not need any quantitative ergodicity assumption and assume some non-clustering assumption on the inclusions (which are more general than disjointedness but still not cover the Poisson case, see Section 2.1.4). But, contrary to our result, they cover the case of any stationary ergodic (uniformly elliptic) background conductivity. This Bernoulli deletion is indeed a dilution : for  $\mathcal{P}$  with intensity  $\theta \sim 1$ ,  $\mathcal{P}^{(p)}$  has intensity  $p\theta \ll 1$  when  $p \ll 1$ . Among one parameter models, the deletion one is at the other end of the spectrum compared to our dilation one. In particular, it let the minimal distance between points unchanged (hence potentially arbitrarily small) when our dilation increase it. This difference reflects in the expansion. Reformulated in terms of the volume fraction  $c$ , the results do not scale similarly : in the deletion case,  $\bar{a}^{(p)}$  can be written as a power series in  $p \propto c$  whereas in the dilation case,  $\bar{a}^L$  is a power series in  $L^{-1} \propto c^{\frac{1}{d}}$  (as  $c \propto L^{-d}$ ). This  $L^{-1}$  scaling in our result is due to a multi-pole expansion effect (in the form of a Taylor expansion of the standard Laplace Green function). Nonethe-

less, we have the same conclusion in both cases : we recover analyticity for one parameter dilution models constructed on a (stationary ergodic) reference point process.

Another instantiation of formula (2.21) is the so-called Einstein formula ([Ein05]) which provides a dilute expansion of the effective viscosity of mixture fluid + buoyant rigid particles. It is described by a similar model considering that  $a$  is the linear elasticity tensor and formally sending the stiffness of the inclusion to infinity (making them rigid). In the past years, the rigorous proof of this relation has received a lot of attention from the mathematical fluid community, see [HM12; HW20; GH20; NS20; GH21; GM22; DG20c]. In parallel, Duerinckx and Gloria, building up on their Bernoulli deletion results, noticed that the cluster expansion coefficients appearing in their Bernoulli deletion expansion still yields a good approximation (at finite order) of the effective viscosity in the dilute regime. They used several approaches to justify this dilute expansion (energy method,  $\ell^1 - \ell^2$  estimates, implicit or explicit renormalization) which are summarized in the memoir [DG20c]. The size of the coefficients and of the remainder term are well measured using the so-called multi-point intensities which reflect the local correlation structure of the point process and which define an intrinsic notion of dilution. In general, this expansion requires some mild quantification of ergodicity (in the spirit of  $\alpha$ -mixing) and only yields a finite order approximation but in the one parameter cases (deletion or dilation), these authors recovered a full expansion (see [DG20c, Theorem 11 p.123]). However, in the dilation case, their coefficients obtained by cluster expansions still depend on the dilation parameter  $L$  and would require an additional multi-pole expansion (see [DG20c, Remark 5.1 p.123]). Our approach, despite being way less general, is tailored for the dilation case and therefore yields a more straightforward expansion in this case.

In a different setting, we also mention the recent work [Clo+22] on the periodization error for the approximation of (stochastic) homogenized coefficients. In this work, they consider an approximation  $\bar{a}_R$  of the homogenized coefficient  $\bar{a}$  obtained by periodization on a big box  $Q_R$  (through a suitable periodization in law) and give a first order (quantitative) expansion in terms of the size of the box  $\bar{a}_R \simeq \bar{a} + R^{-d} \mathcal{A}$ . Formal sums of Green functions over the periodic lattice, reminiscent of our fixed-point operator (2.15), appears in their work and their treatment was in inspiration for this work.

### 2.1.4 Extensions

Our approach still holds in the case of polydispersed inclusions : instead of assuming that each inclusion is the translation of the same deterministic shape, we can give ourselves a set of i.i.d. random shapes (still without any regularity

assumption). We will essentially see the average shape.

Our hard-core assumption (see Remark 2.2 for a discussion on the value of the minimal distance) can not be relaxed. If the definition (2.1) of the conductivity which involves a union of set still makes sense in the intersecting case (contrary to the definition (2.2) for which we have summability issue if an unbounded number of inclusions intersect) and if the standard theory of homogenization still yields a homogenized matrix  $\bar{a}^L$  (see [JKO94, Section 9]), our estimates (see Proposition 2.1) on the fixed point operator requires that  $B_L \subset V_0(L\mathcal{P}^\circ)$  i.e. that our particles are well separated. This is not surprising : our goal is to describe the corrector using the single inclusion solution only. If points are too close (for instance in the extreme case where points always appear in pair), the single inclusion solution is not a good approximation anymore. Nonetheless, using the tools developed by Duerinckx and Gloria in [DG20c, Theorem 1 p.18] (see also [DG21a]), one could maybe keep our strategy under some non-clustering assumptions.

As our approach mainly relies on energy estimates as well as a good description of the Green function  $G$  for the Laplacian, one can hope that the approach will extend to the Stokes case (using the Stokeslet  $\mathcal{G}$  instead of  $G$ ) both for the problem of effective viscosity [DG20c] and mean settling speed [DG22b]. As mentioned, one can recover this problem formally sending the conductivity (or stiffness in that case) of the inclusion to infinity. Although, this strategy does not work straightforwardly : all of our estimates implicitly depend on the bounds on the total conductivity matrix  $a$ .

Our other main assumption is the constant background conductivity.

First, we expect to be able to relax the symmetry assumption (see Remark 2.1). Indeed, this property is only used in Lemma 2.15, a variational step in the proof of Proposition 2.1, the improved bound on the fixed point operator. Recalling the strategy described in Section 2.1.2, the variable argument can be replaced in favor of a PDE argument (which should hold without symmetry) comparing directly  $u$  (solution of (2.20)) and the Dirichlet approximation of  $u^\circ$  (solution of (2.19)). This is left for future work.

Second, we can hope to cover the case of any stationary ergodic and uniformly elliptic (non-constant) background conductivity. Indeed, our approach only requires a good far field description of the background Helmholtz projector (in the form of Green function estimates in the constant case) which can be provided by the recent works [BGO20] (for the Helmholtz projector) and [Clo+22, Proposition 4 p.24] (directly at the level of the Green function). However, these approach will only yield finite order approximation : they require higher order correctors which only exist up to order  $\lceil \frac{d}{2} \rceil$  in the stochastic case (but up to any order in the periodic one), see [DGL19; Due21] for some leads on how to (partially) overcome this problem (and [AKS23] for the periodic case). Note that in the non-constant



background case, we will need to subtract the associated background corrector  $\varphi^\circ$  everywhere (it was null in the constant background case) : for instance we need to replace  $\varphi^\circ$  by  $\delta^\circ\varphi := \varphi^\circ - \varphi^\circ$ .

As we need our random point process to be hard-core, our result does not cover the Poisson case. This process was covered by Duerinckx in his PhD [Due17, Theorem 5.7.1 p.271] (see also [Giu+21] with similar tools) who proved in that case that  $p \mapsto \bar{a}^{(p)}$  is  $C^\infty$  at on  $[0, 1]$  but not analytic (from this proof, one can actually obtain a Gevrey regularity of order 2, see [DG22a]). In particular the expansion obtained via cluster is still valid up to any order. This also covers the dilation case. Indeed, the Poisson process is fully characterized by its intensity  $\theta$ . As already mentioned its intensity becomes  $p\theta$  under deletion or  $L^{-d}\theta$  under dilation : in this case, for  $p = L^{-d}$ ,  $\mathcal{P}^{(p)} =_{law} L\mathcal{P}$ . The analyticity of  $\theta \mapsto \bar{a}^\theta$  remains an open question.

## 2.2 Preliminaries

In line with the informal approach (see Section 2.1.2), it is enough to consider point processes with a point at the origin. We introduce the space

$$\Omega^\circ := \Omega \cap \{\mathcal{P}^\circ \in (\mathbb{R}^d)^\mathbb{N}, 0 \in \mathcal{P}^\circ\},$$

as well as its rescaled versions  $L\Omega^\circ$  for all  $L \geq 1$  (see (2.3) for  $\Omega$ ).

In this section, we start by providing some properties of the massive approximation, this part is fully deterministic. Then we give some elements of the theory of random point processes which are needed for our approach. In particular, we will see that  $\Omega^\circ$  may be equipped with a probability measure, the Palm measure.

### 2.2.1 Massive approximation

#### Massive corrector

We start by introducing appropriate functional spaces.

**Definition 2.1** We start with the space of uniformly locally square-integrable function

$$L^2_{\text{uloc}}(\mathbb{R}^d) := \{g \in L^2_{\text{loc}}(\mathbb{R}^d) \mid \sup_{z \in \mathbb{R}^d} \int_{B_1(z)} g^2 < \infty\} \quad (2.22)$$

which we equip with the norm  $\|\cdot\|_{L^2_{\text{uloc}}}$  defined for  $g \in L^2_{\text{loc}}(\mathbb{R}^d)$  by

$$\|g\|_{L^2_{\text{uloc}}}^2 := \sup_{z \in \mathbb{R}^d} \int_{B_1(z)} g^2.$$

Likewise, we define

$$H_{\text{uloc}}^1(\mathbb{R}^d) := \{u \in H_{\text{loc}}^1(\mathbb{R}^d) \mid u, \nabla u \in (L_{\text{uloc}}^2(\mathbb{R}^d))^d\}, \quad (2.23)$$

equipped with the norm  $\|\cdot\|_{H_{\text{uloc}}^1} := \|\cdot\|_{L_{\text{uloc}}^2} + \|\nabla \cdot\|_{L_{\text{uloc}}^2}$ .  $\square$

In particular,  $L^\infty(\mathbb{R}^d) \subset L_{\text{uloc}}^2(\mathbb{R}^d)$ .

**Remark 2.3** Note that for a vector-valued function  $g \in (L^2(\mathbb{R}^d))^d$ , we will abusively write its norm  $\|g\|_{L^2(\mathbb{R}^d)}$  omitting the exponent to lighten the notation.  $\square$

The following lemma quantifies the screening introduced by the massive term.

**Lemma 2.1** *Let  $T > 0$ ,  $A : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  be a bounded uniformly elliptic tensor field and  $f \in (L_{\text{uloc}}^2(\mathbb{R}^d))^d$ . Then, there exists a unique weak solution  $\psi_T$  in  $H_{\text{uloc}}^1(\mathbb{R}^d)$  of*

$$\frac{1}{T}\psi_T - \nabla \cdot A \nabla \psi_T = \nabla \cdot f. \quad (2.24)$$

Moreover, there exists  $c > 0$  (depending only on the bounds on  $A$ ) such that, defining  $\eta_{z,T} := e^{\frac{-c|\cdot-z|}{\sqrt{T}}}$ , we have the following weighted energy estimate for all  $z \in \mathbb{R}^d$

$$\frac{1}{T} \int_{\mathbb{R}^d} \eta_{z,T}^2 \psi_T^2 + \int_{\mathbb{R}^d} \eta_{z,T}^2 |\nabla \psi_T|^2 \lesssim \int_{\mathbb{R}^d} \eta_{z,T}^2 f^2, \quad (2.25)$$

where the constant in  $\lesssim$  is uniform in  $T$  and we can bound the RHS in the following way

$$\int_{\mathbb{R}^d} \eta_{z,T}^2 f^2 \lesssim (1 + \sqrt{T}^d) \|f\|_{L_{\text{uloc}}^2}^2. \quad (2.26)$$

$\square$

**PROOF** Existence is obtained by compactness working on  $B_R$  instead of  $\mathbb{R}^d$ , uniqueness is a consequence of (2.25) which is itself a variation of Caccioppoli's inequality. We refer to [GO17, Appendix A.1 p.3530] for a detailed proof although written there for a specific RHS  $f \in L^\infty(\mathbb{R}^d)$  (but the adaptation to the general case is straightforward).

We only justify (2.26). Fix  $T > 0$ . Since  $L_{\text{uloc}}^2(\mathbb{R}^d)$  is invariant by translation, w.l.o.g we only treat the case  $z = 0$  and use the short-hand notation  $\eta_T(|\cdot|) := \eta_{0,T}$ . We decompose the RHS of (2.26) over the  $B_{k+1} \setminus B_k$  for  $k \geq 0$  and use that  $\eta_T$  is a radially decreasing function to write

$$\int_{\mathbb{R}^d} \eta_{z,T}^2 f^2 \leq \sum_{k=0}^{\infty} \eta_T(k) \int_{B_{k+1} \setminus B_k} f^2. \quad (2.27)$$

Covering  $B_{k+1} \setminus B_k$  with balls of radius unity for all  $k > 0$ , we have

$$\int_{B_{k+1} \setminus B_k} f^2 \lesssim |B_{k+1} \setminus B_k| \|f\|_{L^2_{\text{uloc}}(\mathbb{R}^d)}^2 \lesssim k^{d-1} \|f\|_{L^2_{\text{uloc}}(\mathbb{R}^d)}^2 .$$

Plugging this bound in (2.27), we have

$$\int_{\mathbb{R}^d} \eta_{z,T}^2 f^2 \leq \left( 1 + \sum_{k=1}^{\infty} k^{d-1} \eta_T(k) \right) \|f\|_{L^2_{\text{uloc}}}^2 .$$

This concludes the proof of (2.26) noting that

$$\sum_{k=1}^{\infty} k^{d-1} \eta_T(k) \lesssim \int_0^{\infty} r^{d-1} e^{-crT^{-\frac{1}{2}}} dr \lesssim \sqrt{T}^d$$

comparing the sum to an integral then taking out the scaling by change of variables.  $\blacksquare$

Lemma 2.1 allows us to properly define the massive approximation of the corrector  $\varphi$  (see (2.7)).

**Lemma 2.2** *Let  $T > 0$ ,  $L \geq 1$ ,  $L\mathcal{P} \in L\Omega$  and  $e \in \mathbb{R}^d$ . Then, we can define the massive corrector  $\varphi_T(L\mathcal{P}, \cdot)$  associated with  $a(L\mathcal{P}, \cdot)$  as the unique solution in  $H^1_{\text{uloc}}(\mathbb{R}^d)$  of*

$$\frac{1}{T} \varphi_T - \nabla \cdot a(L\mathcal{P}, \cdot) (\nabla \varphi_T + e) = 0 \quad \text{in } \mathbb{R}^d . \quad (2.28)$$

*It satisfies the following contravariance property. For all  $z \in \mathbb{R}^d$ ,*

$$\nabla \varphi_T(\cdot + Lz, L\mathcal{P}) = \nabla \varphi_T(\cdot, L\mathcal{P} - Lz) . \quad (2.29)$$

*Moreover, for  $T \geq 1$ ,*

$$\sup_{\mathcal{P} \in \Omega} \|\nabla \varphi_T(L\mathcal{P}, \cdot)\|_{L^2(B_1)} \lesssim \sqrt{T}^{\frac{d}{2}} + 1 . \quad (2.30)$$

$\square$

**PROOF** Fix  $\mathcal{P} \in \Omega$ . Lemma 2.1 yields existence and uniqueness of  $\varphi_T(L\mathcal{P})$ . Indeed, (2.28) can be rewritten as

$$\frac{1}{T} \varphi_T - \nabla \cdot a(L\mathcal{P}) \nabla \varphi_T = \nabla \cdot \left[ \sum_{x \in \mathcal{P}} C^o(\cdot - Lx) e \right] \quad (2.31)$$

using that  $\nabla \cdot [a(L\mathcal{P}, \cdot)e] = \nabla \cdot [\sum_{x \in \mathcal{P}} C^\circ(\cdot - Lx)e]$  and since

$$\left\| \sum_{x \in \mathcal{P}} C^\circ(\cdot - Lx)e \right\|_{L^2_{\text{uloc}}} = \|C^\circ\|_{L^2(\mathbb{R}^d)} \lesssim 1. \quad (2.32)$$

This last bound is a consequence of the equality  $\left| \sum_{x \in \mathcal{P}} C^\circ(\cdot - Lx)e \right|^2 = \sum_{x \in \mathcal{P}} |C^\circ(\cdot - Lx)e|^2$  noting that the support of  $C^\circ(\cdot - Lx)$  for  $x \in \mathcal{P}$  is included in  $B_1(Lx)$  and that these balls are disjoint by the hard-core assumption on  $\mathcal{P}$ .

The contravariance property follows from uniqueness.

For (2.30), the weighted energy estimates (2.25) and (2.26) applied to (2.31) with  $z = 0$  combined with the estimate of the norm (2.32) yields

$$\int_{\mathbb{R}^d} \eta_T^2 |\nabla \varphi(L\mathcal{P})|^2 \lesssim (1 + \sqrt{T}^d) \|C^\circ\|_{L^2(\mathbb{R}^d)}^2.$$

This yields (2.30) remarking that  $\inf_{B_1} \eta_T^2 \gtrsim 1$  for  $T \geq 1$ . ■

### Single inclusion medium

We recall the short-hand notation  $a^\circ = \text{Id} + C^\circ$  (see (2.10)) for the medium with a single inclusion at the origin. Since we work with equations in divergence form, it is convenient to introduce the space

$$L^2_{\text{pot}}(\mathbb{R}^d) := \{ \nabla u \in L^2(\mathbb{R}^d)^d / u \in L^2_{\text{loc}}(\mathbb{R}^d) \}.$$

We now define the Helmholtz projector associated with  $a^\circ$  which encompasses the properties of the single inclusion medium.

**Definition 2.2** Let  $\mathcal{H}^\circ : \begin{cases} (L^2(\mathbb{R}^d))^d & \rightarrow (L^2(\mathbb{R}^d))^d \\ f & \mapsto \nabla \psi^\circ \end{cases}$  be the Helmholtz projector associated with  $a^\circ$ , where  $\nabla \psi^\circ$  is the unique Lax-Milgram solution in  $L^2_{\text{pot}}(\mathbb{R}^d)$  of

$$-\nabla \cdot a^\circ \nabla \psi^\circ = \nabla \cdot f.$$

It satisfies the bound

$$\| \mathcal{H}^\circ \|_{(L^2(\mathbb{R}^d))^d \circlearrowleft} \lesssim 1. \quad (2.33)$$

□

This allows us to properly define the single inclusion solution. Let  $e \in \mathbb{R}^d$  and define  $\nabla \varphi^\circ \in L^2_{\text{pot}}(\mathbb{R}^d)$  as  $\nabla \varphi^\circ := \mathcal{H}^\circ[C^\circ e]$ .

### Remark 2.4

- Noting that  $\nabla \cdot a^\circ e = \nabla \cdot C^\circ e$ ,  $\nabla \varphi^\circ$  indeed solves the single inclusion problem (2.11) which is similar to the corrector equation (2.7).
- This problem is deterministic and can be explicitly solved for (constant) spherical (or ellipsoidal) inclusions [JKO94, (1.92) p.48] which is used to give the explicit Clausius-Mossotti formula (2.21).  $\square$

Similarly, for  $e \in \mathbb{R}^d$ , we define the massive single inclusion solution  $\varphi_T^\circ$  using Lemma 2.1 which solves

$$\frac{1}{T} \varphi_T^\circ - \nabla \cdot a^\circ (\nabla \varphi_T^\circ + e) = 0 \quad \text{in } \mathbb{R}^d. \quad (2.34)$$

Note that since the source term  $C^\circ e$  has compact support, the weighted energy estimate (2.25) upgrades to the plain energy estimate

$$\frac{1}{T} \int_{\mathbb{R}^d} \varphi_T^\circ{}^2 + \int_{\mathbb{R}^d} |\nabla \varphi_T^\circ|^2 \lesssim 1. \quad (2.35)$$

In Definition 2.2, the Helmholtz projector is defined on  $L^2(\mathbb{R}^d)$ . Using Lemma 2.1, we define its screened version acting on the larger space  $L^2_{\text{uloc}}(\mathbb{R}^d)$  (see Definition 2.1).

**Definition 2.3** Let  $T > 0$  and  $A : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  be as in Lemma 2.1. We define  $\mathcal{H}_T^A$  the (massive) Helmholtz projector associated with  $A$  by

$$\mathcal{H}_T^A : \begin{cases} (L^2_{\text{uloc}}(\mathbb{R}^d))^d & \rightarrow (L^2_{\text{uloc}}(\mathbb{R}^d))^d \\ f & \mapsto \nabla \psi_T \end{cases}$$

where  $\psi_T$  is the unique solution of (2.24) given by Lemma 2.1.

In what follows, we shall use the shorthand notation  $\mathcal{H}_T^\circ := \mathcal{H}_T^{a^\circ}$  and  $\mathcal{H}_T := \mathcal{H}_T^{\text{Id}}$ .  $\square$

In Section 2.1.2 on the informal approach, we have claimed that we could reduce the study of the single inclusion medium to the one of the background (without inclusions). Indeed, we can rewrite  $\mathcal{H}_T^\circ$  as a local (see Remark 2.5 below), bounded, linear transformation of  $\mathcal{H}_T$ .

**Lemma 2.3** Let  $T > 0$ . We have

$$\mathcal{H}_T^\circ = \mathcal{B}_T^\circ \mathcal{H}_T \quad (2.36)$$

where  $\mathcal{B}_T^\circ : \begin{cases} (L^2_{\text{uloc}}(\mathbb{R}^d))^d & \rightarrow (L^2_{\text{uloc}}(\mathbb{R}^d))^d \\ f & \mapsto f + \mathcal{H}_T^\circ(C^\circ f) \end{cases}$  is a bounded linear operator.

Moreover, the family of operators  $\{\mathcal{B}_T^\circ\}_{T \geq 1}$  is bounded uniformly with respect to  $T$  :

$$\|\mathcal{B}_T^\circ\|_{L^2_{\text{uloc}}(\mathbb{R}^d) \circ} \lesssim 1 . \quad (2.37)$$

□

PROOF Fix  $f \in (L^2_{\text{uloc}}(\mathbb{R}^d))^d$ . By Definition 2.3,  $\mathcal{H}_T^\circ f = \nabla \psi^\circ$  where  $\psi^\circ$  is the unique weak solution of

$$\frac{1}{T} \psi^\circ - \nabla \cdot a^\circ \nabla \psi^\circ = \nabla \cdot f . \quad (2.38)$$

Likewise,  $\mathcal{H}_T f = \nabla \psi$  where  $\psi$  is the unique weak solution of

$$\frac{1}{T} \psi - \nabla \cdot \text{Id} \nabla \psi = \nabla \cdot f . \quad (2.39)$$

Since  $a^\circ = \text{Id} + C^\circ$ , (2.39) can be rewritten  $\frac{1}{T} \psi - \nabla \cdot a^\circ \nabla \psi = \nabla \cdot f - \nabla \cdot C^\circ \nabla \psi$ . Subtracting it from (2.38), we get

$$\left( \frac{1}{T} - \nabla \cdot a^\circ \nabla \right) (\psi^\circ - \psi) = \nabla \cdot C^\circ \nabla \psi$$

which yields (2.36) once reformulated in terms of Helmholtz projectors.

Regarding the uniform bound on  $\mathcal{B}_T^\circ = \text{Id} + \mathcal{H}_T^\circ(C^\circ \cdot)$ , by the triangle inequality, it is sufficient to bound  $\mathcal{H}_T^\circ(C^\circ \cdot)$ . By Definition 2.3,  $\mathcal{H}_T^\circ(C^\circ \cdot) = \nabla \rho_T$  where  $\rho_T$  is the unique solution of

$$\frac{1}{T} \rho_T - \nabla \cdot a^\circ \nabla \rho_T = \nabla \cdot C^\circ f .$$

Since  $C^\circ$  is supported in  $B_1$  and  $C^\circ \in L^\infty(\mathbb{R}^d)$ , by the plain energy estimate (taking only the gradient part), one has

$$\|\mathcal{H}_T^\circ(C^\circ \cdot)\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(B_1)}$$

uniformly in  $T$ . This concludes the proof since the  $L^2(\mathbb{R}^d)$ -norm controls the  $L^2_{\text{uloc}}(\mathbb{R}^d)$ -norm. ■

**Remark 2.5** The operator  $\mathcal{B}_T^\circ$  is local in the sense that  $(\mathcal{B}_T^\circ f)|_{B_1}$  only depends on  $f|_{B_1}$  (where  $B_1$  is the support of  $C^\circ$ ). □

**Remark 2.6** The locality of  $\mathcal{B}_T^\circ$  (see Remark 2.5) allows us to extend it to  $L^2(B_1)$  : using the same formula (which still make sense for functions in  $L^2(B_1)$ ), we define

the operator

$$\widetilde{\mathcal{B}}_T^\circ : \begin{cases} (L^2(B_1))^d & \rightarrow (L^2(B_1))^d \\ f & \mapsto f + \mathcal{H}_T^\circ(C^\circ f) \end{cases}$$

to the effect that for  $f \in L^2(B_1)$ ,  $\widetilde{\mathcal{B}}_T^\circ f = \mathcal{B}_T^\circ(\mathbb{1}_{B_1}f)|_{B_1}$ .

In the sequel, we identify these two operators and keep the same notation  $\mathcal{B}_T^\circ$ . With this identification, the uniform bound (2.37) on the family  $\{\mathcal{B}_T^\circ\}_{T \geq 1}$  also yields

$$\|\|\mathcal{B}_T^\circ\|\|_{L^2(B_1)^\circ} \lesssim 1. \quad (2.40)$$

□

A similar computation yields an analogous relation between  $\mathcal{H}^\circ$  and  $\mathcal{H} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ , the Helmholtz projector associated with Id, the constant background (defined as  $\mathcal{H}^\circ$  in Definition 2.2 with  $a^\circ$  replaced by Id).

**Lemma 2.4** *We have*

$$\mathcal{H}^\circ = \mathcal{B}^\circ \mathcal{H}, \quad (2.41)$$

where  $\mathcal{B}^\circ : \begin{cases} (L^2(\mathbb{R}^d))^d & \rightarrow (L^2(\mathbb{R}^d))^d \\ f & \mapsto f + \mathcal{H}^\circ(C^\circ f) \end{cases}$  is a linear operator satisfying

$$\|\|\mathcal{B}^\circ\|\|_{L^2(\mathbb{R}^d)^\circ} \lesssim 1. \quad (2.42)$$

Proceeding as for  $\mathcal{B}_T^\circ$ , the operator  $\mathcal{B}^\circ$  can be extended to  $L^2(B_1)$  such that

$$\|\|\mathcal{B}^\circ\|\|_{L^2(B_1)^\circ} \lesssim 1. \quad (2.43)$$

□

**Remark 2.7** With these operators, for  $e \in \mathbb{R}^d$ , we have :

- $\nabla \varphi^\circ + e = \mathcal{B}^\circ e$ ,
- $\nabla \varphi_T^\circ + e = \mathcal{B}_T^\circ e$ .

□

Working with  $\mathcal{H}$  instead of  $\mathcal{H}^\circ$  allows to precisely describe the far-field behavior. Indeed,  $\mathcal{H}$  can be represented as the convolution operator  $\mathcal{H} = \nabla^2 G * \cdot$  using the standard Green function  $G$  associated with  $-\Delta$ , the solution of

$$-\Delta G = \delta_0.$$

Similarly,  $\mathcal{H}_T$  can be represented using the massive Green function  $G_T$ . For the convenience of the reader, we recall some well-known properties of  $G_T$  which are similar to those of  $G$  with an additional exponential decay (at scale  $\sqrt{T}$ ).

**Definition 2.4** Let  $T > 0$ . Defining the massive Green function  $G_T$  as the unique distributional solution in  $W^{1,1}(\mathbb{R}^d)$  of

$$\frac{1}{T}G_T - \Delta G_T = \delta_0, \quad (2.44)$$

one can represent  $\mathcal{H}_T$  as the convolution operator

$$\mathcal{H}_T = \nabla^2 G_T * \cdot. \quad (2.45)$$

Moreover, we have the following properties of the massive Green function :

- $G_T$  is a  $C^\infty$  function on  $\mathbb{R}^d \setminus \{0\}$ .
- Since  $\frac{1}{T} - \Delta$  is invariant by rotation,  $G_T$  is a radial function.
- By scaling,

$$G_T = \frac{1}{\sqrt{T}^{d-2}} G_1 \left( \frac{\cdot}{\sqrt{T}} \right). \quad (2.46)$$

In particular, this yields the following homogeneity property. For  $\mu > 0$ ,

$$G_T(\mu \cdot) = \mu^{-(d-2)} G_{\frac{T}{\mu^2}}. \quad (2.47)$$

- There exists  $c > 0$  s.t. for all  $x \in \mathbb{R}^d \setminus \{0\}$  and for all  $k \geq 1$ ,

$$|\nabla^k G_T(x)| \lesssim_k \frac{e^{-c\frac{|x|}{\sqrt{T}}}}{|x|^{d-2+k}}. \quad (2.48)$$

□

A proof of these results in the scalar case can be found in [GO17, Definition 2.4 p.3501 and Appendix A.3 p.3532] (note also that  $G_T$  is the modified Bessel function of the second kind, see [Olv+20, Eq. 10.25.3]). These results extend to the case of system as, here,  $G_T^{\text{system}} = \text{diag}(G_T^{\text{scalar}})$ .

## Convergence

As expected, when the massive parameter  $T \rightarrow \infty$ , we recover the limit objects.

**Remark 2.8**  $\varphi_T$  is indeed an approximation of  $\varphi$  : for almost all  $\mathcal{P} \in \Omega$ ,  $\nabla \varphi_T(L\mathcal{P}) \xrightarrow{T \rightarrow \infty} \nabla \varphi(L\mathcal{P})$  in  $L^2_{\text{loc}}(\mathbb{R}^d)$  since  $\lim_{T \rightarrow \infty} \mathbb{E} [|\nabla \varphi_T - \nabla \varphi|^2] = 0$  (see [GH16, Theorem 1]). This convergence requires some stochastic averaging (thanks to ergodicity). This is in line with the deterministic bound (2.30) which diverges as  $T \rightarrow \infty$ .



**Lemma 2.5** *For the Helmholtz projectors  $\mathcal{H}_T^\circ$  and  $\mathcal{H}^\circ$  acting in  $(L^2(\mathbb{R}^d))^d$  (see Definition 2.3 and Definition 2.2), we have the pointwise convergence  $\mathcal{H}_T^\circ \xrightarrow{T \rightarrow \infty} \mathcal{H}^\circ$ .*

*Consequently, for the operators  $\mathcal{B}_T^\circ$  and  $\mathcal{B}^\circ$  acting in  $(L^2(B_1))^d$ , we have the pointwise convergence  $\mathcal{B}_T^\circ \xrightarrow{T \rightarrow \infty} \mathcal{B}^\circ$ .  $\square$*

The proof of Lemma 2.5 relies on arguments similar to [GH16, Theorem 1].

PROOF By definition,  $\mathcal{B}_T^\circ = \text{Id} + \mathcal{H}_T^\circ(C^\circ \cdot)$ . Thus, we only need to prove the pointwise convergence of  $\mathcal{H}_T^\circ$ .

Fix  $f \in (L^2(\mathbb{R}^d))^d$ . The convergence  $\mathcal{H}_T^\circ f \rightarrow \mathcal{H}^\circ f$  rewrites  $\nabla \psi_T \rightarrow \nabla \psi$  where  $\psi_T$  is the unique Lax-Milgram solution in  $H^1(\mathbb{R}^d)$  of

$$\frac{1}{T} \psi_T - \nabla \cdot a^\circ \nabla \psi_T = \nabla \cdot f \quad (2.49)$$

and  $\nabla \psi$  is the unique Lax-Milgram solution in  $L^2_{\text{pot}}(\mathbb{R}^d)$  of

$$-\nabla \cdot a^\circ \nabla \psi = \nabla \cdot f .$$

Note that we do not have to appeal to Lemma 2.1 to define  $\psi_T$  since  $f \in (L^2(\mathbb{R}^d))^d$ . The standard energy estimate for  $\psi_T$  yields

$$\frac{1}{T} \int_{\mathbb{R}^d} \psi_T^2 + \int_{\mathbb{R}^d} |\nabla \psi_T|^2 \lesssim \int_{\mathbb{R}^d} f^2 . \quad (2.50)$$

We claim first that

$$\nabla \psi_T \rightharpoonup \nabla \psi \quad \text{in } (L^2(\mathbb{R}^d))^d \quad (2.51)$$

and then that

$$\int_{\mathbb{R}^d} \nabla \psi_T \cdot a^\circ \nabla \psi_T \xrightarrow{T \rightarrow \infty} \int_{\mathbb{R}^d} \nabla \psi \cdot a^\circ \nabla \psi . \quad (2.52)$$

Since (2.52) is the conservation of the norm  $\|g\|_{a^\circ}^2 := \int_{\mathbb{R}^d} g \cdot a^\circ g$  on  $(L^2(\mathbb{R}^d))^d$  (which is equivalent to the canonical one), combined with the weak convergence (2.51), it implies the strong convergence  $\nabla \psi_T \rightarrow \nabla \psi$  in  $(L^2(\mathbb{R}^d))^d$ .

For (2.51), we first note that  $\nabla \psi_T$  is uniformly bounded in  $(L^2(\mathbb{R}^d))^d$  using (2.50). Hence, there exists  $\nabla \tilde{\psi} \in L^2_{\text{pot}}(\mathbb{R}^d)$  s.t.  $\nabla \psi_T \rightharpoonup \nabla \tilde{\psi}$  (up to extraction). The limit is indeed a gradient since, by the Helmholtz decomposition in  $\mathbb{R}^d$ ,  $L^2_{\text{pot}}(\mathbb{R}^d) = \{v \in (L^2(\mathbb{R}^d))^d \mid \text{curl } v = 0 \text{ (weakly)}\}$  which shows that this space is closed for the weak topology.

Then, we pass to the limit  $T \rightarrow \infty$  in the weak formulation of (2.49) and we obtain for all smooth compactly supported  $\chi$

$$\int_{\mathbb{R}^d} \nabla \chi \cdot a^\circ \nabla \tilde{\psi} = - \int_{\mathbb{R}^d} f \cdot \nabla \chi$$

noting that  $|\frac{1}{T} \int_{\mathbb{R}^d} \psi_T \chi| \lesssim T^{-\frac{1}{2}} \|\chi\|_{L^2(\mathbb{R}^d)}$  (from Cauchy-Schwarz inequality and the bound (2.49) in the form  $\frac{1}{\sqrt{T}} \|\psi_T\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$ ). This means that  $\nabla \tilde{\psi}$  solves the same problem as  $\nabla \psi$  and, by uniqueness, that  $\nabla \tilde{\psi} = \nabla \psi$ .

For (2.52), we notice that the quadratic form  $g \mapsto \int_{\mathbb{R}^d} g \cdot a^\circ g$  is strongly continuous and convex hence weakly l.s.c. Combined with the weak convergence (2.51), this gives

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla \psi \cdot a^\circ \nabla \psi &\leq \liminf_{T \rightarrow \infty} \int_{\mathbb{R}^d} \nabla \psi_T \cdot a^\circ \nabla \psi_T \\ &\leq \limsup_{T \rightarrow \infty} \int_{\mathbb{R}^d} \nabla \psi_T \cdot a^\circ \nabla \psi_T \\ &\leq \limsup_{T \rightarrow \infty} \int_{\mathbb{R}^d} \nabla \psi_T \cdot a^\circ \nabla \psi_T + \frac{1}{T} \int_{\mathbb{R}^d} \psi_T^2. \end{aligned}$$

This concludes the proof noting that

$$\int_{\mathbb{R}^d} \nabla \psi_T \cdot a^\circ \nabla \psi_T + \frac{1}{T} \int_{\mathbb{R}^d} \psi_T^2 = - \int_{\mathbb{R}^d} f \cdot \nabla \psi_T \xrightarrow{T \rightarrow \infty} - \int_{\mathbb{R}^d} f \cdot \nabla \psi = \int_{\mathbb{R}^d} \nabla \psi \cdot a^\circ \nabla \psi.$$

Here, we have used again the weak convergence (2.51) and have tested  $\psi_T$  and  $\nabla \psi$  is their respective equations to establish the two identities above.  $\blacksquare$

This proof holds for any field  $A$  bounded and uniformly elliptic and then also yields the pointwise convergence

$$\mathcal{H}_T \xrightarrow{T \rightarrow \infty} \mathcal{H} \tag{2.53}$$

(one could also obtain this convergence directly in Fourier by the convergence of the symbols). By standard duality arguments (and elliptic regularity), this last convergences upgrades to the Green functions in the following form.

**Lemma 2.6** *Let  $\omega \subset \mathbb{R}^d$  be a compact set such that  $\text{dist}(\omega, 0) > 0$ . Then, for all  $k \geq 2$ ,*

$$\|\nabla^k G_T - \nabla^k G\|_{L^\infty(\omega)} \xrightarrow{T \rightarrow \infty} 0. \quad \square$$

### 2.2.2 Some theory of Point Processes

We start by defining the intensity of the point process  $\mathcal{P}$ , so far informally defined as the average number of point per unit of volume.

**Definition 2.5** Let  $\mathcal{P} \in (\Omega, \mathcal{F}, \mathbb{P})$  be a random point process. We define its intensity by

$$\theta := \mathbb{E} [\#(\mathcal{P} \cap Q)] . \quad (2.54)$$

As  $\mathbb{P}$  is ergodic, we can also recover  $\theta$  by taking averages over large domain (see [DV08, Corolary 12.2.V p.201]) and

$$\theta = \lim_{R \rightarrow \infty} \frac{\#(\mathcal{P} \cap Q_R)}{|Q_R|} . \quad (2.55)$$

□

$\Omega^\circ$ , the space of point processes with a point at the origin, can be equipped with a probability measure  $\mathbb{P}^\circ$  obtained from the original probability measure  $\mathbb{P}$  on  $\Omega$  (using the same  $\sigma$ -algebra  $\mathcal{F}$  as  $\Omega^\circ \subset \Omega$ ).

Informally,  $\mathcal{P}^\circ \in \Omega^\circ$  can be thought of as a point process  $\mathcal{P} \in \Omega$  conditioned to have a point at the origin. In the theory of random point processes,  $\mathbb{P}^\circ$  is referred to as the Palm measure (for stationary processes). See [Chi+13, Section 4.4] for a first approach to the theory or [DV08, Section 13] for completeness.

Rigorously,  $\mathcal{P}$  and  $\mathcal{P}^\circ$  are linked through the Campbell-Mecke theorem (see [DV08, Theorem 13.2.III p.288]).

**Theorem 2.2 (Campbell-Mecke)** *Let  $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a non-negative measurable function. For a stationary random point process  $\mathcal{P}$  with intensity  $\theta$ , we have*

$$\mathbb{E} \left[ \sum_{x \in \mathcal{P}} f(x, \mathcal{P} - x) \right] = \int_{\mathbb{R}^d} \mathbb{E}^\circ [f(x, \mathcal{P}^\circ)] \theta \, dx . \quad \square$$

For a deterministic non-negative measurable function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$ , it reduces to the Campbell formula

$$\mathbb{E} \left[ \sum_{x \in \mathcal{P}} g(x) \right] = \int_{\mathbb{R}^d} g(x) \theta \, dx . \quad (2.56)$$

This formula is consistent with the definition of the intensity (2.54) choosing  $g = \mathbb{1}_Q$ .

This theorem can be seen as a definition for  $\mathbb{E}^\circ$ . For any  $\varepsilon > 0$ , taking  $f(\mathcal{P}, x) = h(\mathcal{P}) \mathbb{1}_{B_\varepsilon}(x)$ , (2.2) yields the following lemma.

**Lemma 2.7** *Let  $h : \Omega \rightarrow \mathbb{R}$  be a non-negative measurable function and  $\mathcal{P}$  be as in Theorem 2.2. Then, for all  $\varepsilon > 0$ , we have*

$$\mathbb{E}^\circ [h(\mathcal{P}^\circ)] = \frac{1}{\theta |B_\varepsilon|} \mathbb{E} \left[ \sum_{x \in \mathcal{P} \cap B_\varepsilon} h(\mathcal{P} - x) \right]. \quad (2.57)$$

□

As our point processes are hard-core, we have a deterministic upper bound on the local number of points.

**Lemma 2.8** *Let  $R \geq 0$  and define the annulus  $C_R := B_{R+1} \setminus B_R$  (noting that  $B_0 = \emptyset$ ). For any  $\mathcal{P} \in \Omega$ , we have*

$$\#(\mathcal{P} \cap C_R) \lesssim \max\{1, R^{d-1}\}. \quad (2.58)$$

□

PROOF Fix  $\mathcal{P} \in \Omega$ . By the hardcore hypothesis, there is at most one point  $x \in \mathcal{P}$  in a cube of size 1. Using the partition  $\{Q_1(z)\}_{z \in \mathbb{Z}^d}$ , we can cover  $C_R$  with a finite number  $N$  of such cubes with  $N \lesssim \max\{\frac{|C_R|}{|Q_1|}, 1\}$ . This implies that there is at most  $N$  points in  $\mathcal{P} \cap C_R$  and yields (2.58). ■

Note that combining (2.58) with the definition (2.54) of the intensity yields the upper bound  $\theta \lesssim 1$ .

For our PDE approach on the operator, analogously to the notion of periodization, we provide a way to render stationary functions associated to a point process  $\mathcal{P} \in \Omega$ . We start by recalling the standard notion of Voronoi tessellation (see [Chi+13, Section 9.2 p.346] for instance).

**Definition 2.6** Let  $\mathcal{P} \in \Omega$ . For a point  $x \in \mathcal{P}$ , we define  $V_x(\mathcal{P})$  the associated Voronoi cell by

$$V_x(\mathcal{P}) := \{y \in \mathbb{R}^d, |y - x| \leq |y - x'| \ \forall x' \in \mathcal{P} \text{ s.t. } x' \neq x\}. \quad (2.59)$$

The sets  $(V_x(\mathcal{P}))_{x \in \mathcal{P}}$  form a partition of  $\mathbb{R}^d$  adapted to the point process with exactly one point per cell.

Moreover, the associated indicator functions satisfy the contravariance property and in particular

$$\mathbf{1}_{V_x(\mathcal{P})}(\cdot) = \mathbf{1}_{V_0(\mathcal{P}-x)}(\cdot - x).$$

For any  $L \geq 1$  and  $\mathcal{P} \in \Omega$ , from the hard-core property, we have for all  $x \in \mathcal{P}$

$$B_{3L}(x) \subset V_x(L\mathcal{P}).$$

□

Note that if  $\mathcal{P}$  is a periodic lattice, the Voronoi cells are nothing but translation of the periodic cell.

Thanks to the Voronoi tessellation, we define our stationarization procedure.

**Lemma 2.9** *For  $f : L\Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  measurable and non-negative, we define  $\bar{f} : L\Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  its associated stationary function by*

$$\bar{f}(L\mathcal{P}, \cdot) := \sum_{x \in \mathcal{P}} \mathbf{1}_{V_0(L(\mathcal{P}-x))}(\cdot - Lx) f(L(\mathcal{P} - x), \cdot - Lx) \quad (2.60)$$

By construction  $\bar{f}$  is contravariant (see (2.29)) and if  $f$  is already contravariant itself, we have  $\bar{f} = f$ .

Moreover, we have the following equality

$$\mathbb{E} [\bar{f}(L\mathcal{P}, 0)] = L^{-d} \theta \mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} f(L\mathcal{P}^\circ) \right]. \quad (2.61)$$

□

(2.61) is a direct corollary of Theorem 2.2 (and Fubini's theorem) plugging in the definition (2.60).

## 2.3 Operator

We now turn to the precise definition of the operator for the fixed-point approach (formally given by (2.15)). We start by introducing an appropriate functional setting to handle the dilation of the point process.

**Definition 2.7** Let  $L \geq 1$ . We define the functional space

$$E^L := L^2(L\Omega^\circ, (L^2(B_1))^d) \quad (2.62)$$

that we equip with the norm  $\|\cdot\|_{E^L}$  defined for  $f \in E^L$  as

$$\|f\|_{E^L}^2 := \mathbb{E}^\circ \left[ \|f(L\mathcal{P}^\circ, \cdot)\|_{L^2(B_1)}^2 \right]. \quad (2.63)$$

The space  $(L^2(B_1))^d$  continuously embeds in  $E^L$  via the following identification : for  $g \in (L^2(B_1))^d$ , we define its extension  $\tilde{g}(L\mathcal{P}, \cdot) \in E^L$  by setting  $\tilde{g}(L\mathcal{P}, \cdot) := g$  for all  $L\mathcal{P} \in L\Omega^\circ$ . □

In the sequel, we do not distinguish between  $g \in (L^2(B_1))^d$  and its extension  $\tilde{g} \in E^L$ . We also use the short-hand notation  $f(L\mathcal{P})$  instead of  $f(L\mathcal{P}, \cdot)$  for  $f \in E^L$ .

**Remark 2.9** We have chosen  $(L^2(B_1))^d$  because of the support condition on  $C^\circ$  and because  $L^2$  is the energy space.

Note that for  $L \geq 1$ ,  $E^L \subset E^1$  as  $L\Omega^\circ \subset \Omega^\circ$ . By definition of the norm,  $f \in E^L$  implies  $\mathbb{P}^\circ$ -a.s,  $f(L\mathcal{P}^\circ) \in L^2(B_1)$ .  $\square$

For  $L \geq 1$ , we introduce the operator norm  $\| \cdot \|_L$  associated with  $\| \cdot \|_{E^L}$  defined for  $M : E^L \rightarrow E^L$  as

$$\| \| M \| \|_L := \sup_{\substack{f \in E^L \\ \|f\|_{E^L} \leq 1}} \|Mf\|_{E^L} .$$

Operators on  $(L^2(B_1))^d$  can also be seen as operators acting on  $E^L$  using, as above, the same extension technique ; this procedure preserves their norm. In particular, we may consider  $\mathcal{B}^\circ$  and  $\mathcal{B}_T^\circ$  as operators on  $E^L$  which are bounded uniformly in  $L$  (trivially since they do not involve point processes).

### 2.3.1 The key operator

We start with the definition of the massive approximation of the operator  $K^L$  informally introduced in (2.15).

**Lemma 2.10** *Let  $T > 0$  and  $L \geq 1$ . The linear operator  $K_T^L : E^L \rightarrow E^L$  given for all  $f \in E^L$  and  $L\mathcal{P}^\circ \in L\Omega^\circ$  by*

$$K_T^L f(L\mathcal{P}^\circ) := \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^2 G_{\frac{T}{L^2}}(\frac{\cdot}{L} - x) * [C^\circ f(L(\mathcal{P}^\circ - x))] \quad (2.64)$$

*is well-defined and satisfies the bound*

$$\| \| K_T^L \| \|_L \lesssim \ln \left( 2 + \frac{T}{L^2} \right) . \quad (2.65)$$

$\square$

We note that re-centering  $\mathcal{P}^\circ$  at  $x \in \mathcal{P}^\circ$  still yields a hard-core point process that contains the origin so that  $\mathcal{P}^\circ - x \in \Omega^\circ$  and we may evaluate  $f(L(\mathcal{P}^\circ - x))$ . Because of the structure of the operator  $K_T^L$  and in order to estimate its norm, we need the following lemma.

**Lemma 2.11** *Let  $\omega : \mathbb{R}^d \rightarrow \mathbb{R}_+$  be a non negative weight. For all measurable*

$g : \Omega^\circ \rightarrow \mathbb{R}_+$  and for all  $0 < \varepsilon < 3$ , we have

$$\begin{aligned} \mathbb{E}^\circ \left[ \left| \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \omega(x) g(\mathcal{P}^\circ - x) \right|^2 \right] \\ \lesssim \left( \sup_{\mathcal{P}^\circ \in \Omega^\circ} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \omega(x) \right) \left( \int_{\mathbb{R}^d \setminus B_{3-\varepsilon}} \|\omega\|_{L^\infty(B_\varepsilon(x))} dx \right) \mathbb{E}^\circ [g(\mathcal{P}^\circ)^2] . \end{aligned} \quad (2.66) \quad \square$$

PROOF Fix  $\varepsilon > 0$  and a measurable non negative  $g$ . We claim that

$$\left| \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \omega(x) g(\mathcal{P}^\circ - x) \right|^2 \leq \left( \sup_{\mathcal{P}^\circ \in \Omega^\circ} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \omega(x) \right) \left( \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \omega(x) g(\mathcal{P}^\circ - x)^2 \right) \quad (2.67)$$

and

$$\mathbb{E}^\circ \left[ \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \omega(x) g(\mathcal{P}^\circ - x)^2 \right] \lesssim \left( \int_{\mathbb{R}^d \setminus B_{3-\varepsilon}} \|\omega\|_{L^\infty(B_\varepsilon(x))} dx \right) \mathbb{E}^\circ [g(\mathcal{P}^\circ)^2] \quad (2.68)$$

from which (2.66) follows.

(2.67) is a direct consequence of Cauchy-Schwarz' inequality for the scalar product  $(g_1, g_2) \mapsto \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \omega(x) g_1(x) g_2(x)$ .

For (2.68), we go back to the definition of  $\mathbb{E}^\circ$  and apply Lemma 2.7 to  $\mathcal{P} \mapsto \sum_{x \in \mathcal{P}} \mathbb{1}_{x \neq 0} \omega(x) g(\mathcal{P} - x)^2$  to the effect that

$$\mathbb{E}^\circ \left[ \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \omega(x) g(\mathcal{P}^\circ - x)^2 \right] \lesssim \mathbb{E} \left[ \sum_{y \in \mathcal{P} \cap B_\varepsilon} \sum_{z \in \mathcal{P} - y} \mathbb{1}_{z \neq 0} \omega(z) g(\mathcal{P} - y - z)^2 \right] . \quad (2.69)$$

Using Fubini theorem and the change of variables  $z = x - y$ , we rewrite the RHS

$$\sum_{y \in \mathcal{P} \cap B_\varepsilon} \sum_{z \in \mathcal{P} - y} \mathbb{1}_{z \neq 0} \omega(z) g(\mathcal{P} - y - z)^2 = \sum_{x \in \mathcal{P}} \sum_{y \in \mathcal{P} \cap B_\varepsilon} \mathbb{1}_{x \neq y} \omega(x - y) g(\mathcal{P} - x)^2 . \quad (2.70)$$

Then, we note that for all  $x \in \mathcal{P}$ ,

$$\sum_{y \in \mathcal{P} \cap B_\varepsilon} \mathbb{1}_{x \neq y} \omega(x - y) \lesssim \mathbb{1}_{|x| \geq 3-\varepsilon} \|\omega\|_{L^\infty(B_\varepsilon(x))} . \quad (2.71)$$

Indeed, by Lemma 2.8  $\#(\mathcal{P} \cap B_\varepsilon) \lesssim 1$  and for all  $y \in \mathcal{P} \cap B_\varepsilon$ ,  $\omega(x - y) \leq$

$\|\omega\|_{L^\infty(B_\varepsilon(x))}$ . Moreover, since  $\mathcal{P}$  is hard-core, the condition  $x \neq y$  turns into  $|x - y| > 3$  and implies that  $|x| > 3 - \varepsilon$  (since  $|y| \leq \varepsilon$ ). Plugging (2.70) and (2.71) in (2.69), we have

$$\mathbb{E}^\circ \left[ \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \omega(x) g(\mathcal{P}^\circ - x)^2 \right] \lesssim \mathbb{E} \left[ \sum_{x \in \mathcal{P}} \mathbb{1}_{|x| \geq 3 - \varepsilon} \|\omega\|_{L^\infty(B_\varepsilon(x))} g(\mathcal{P} - x)^2 \right]. \quad (2.72)$$

Applying Theorem 2.2 to the RHS then yields (2.68) and concludes the proof.  $\blacksquare$

PROOF (LEMMA 2.10) Fix  $T > 0$ ,  $L \geq 1$ ,  $\mathcal{P}^\circ \in \Omega^\circ$  and  $f \in E^L$ .

In (2.64), we have taken out the origin from the sum avoiding the singularity of the massive Green function. Its exponential decay at scale  $\sqrt{T}$  yields summability and the bound (2.65).

Indeed, fix  $z \in B_1$ . Using the triangle inequality on the convolution and since  $C^\circ$  is supported in  $B_1$ , we have

$$|K_T^L f(L\mathcal{P}^\circ, z)| \leq \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \int_{B_1} |\nabla^2 G_{\frac{T}{L^2}}(x - \frac{z - y}{L})| |C^\circ f|(L(\mathcal{P}^\circ - x), y) dy. \quad (2.73)$$

Since  $|\frac{y - z}{L}| \leq 2$  for all  $y, z \in B_1$ , we obtain

$$\|K_T^L f(L\mathcal{P}^\circ)\|_{L^\infty(B_1)} \leq \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \|\nabla^2 G_{\frac{T}{L^2}}\|_{L^\infty(B_2(x))} \int_{B_1} |C^\circ f|(L(\mathcal{P}^\circ - x), y) dy.$$

We estimate the RHS using Lemma 2.11 with  $\omega = \|\nabla^2 G_{\frac{T}{L^2}}\|_{L^\infty(B_2(\cdot))}$  and  $g(\mathcal{P}^\circ) := \int_{B_1} |C^\circ f|(L\mathcal{P}^\circ)$  for all  $\mathcal{P}^\circ \in \Omega^\circ$  for a small  $\varepsilon > 0$  such that  $3 - \varepsilon > 2 + \varepsilon$  (for instance  $\varepsilon = \frac{1}{4}$ ). Noting that  $g(\mathcal{P}^\circ)^2 \leq \int_{B_1} f(L\mathcal{P}^\circ)^2$  by Cauchy-Schwarz' inequality, (2.65) then follows after proving that

$$\sup_{\mathcal{P}^\circ \in \Omega^\circ} \left( \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \|\nabla^2 G_{\frac{T}{L^2}}\|_{L^\infty(B_2(x))} \right) \lesssim \ln \left( 2 + \frac{T}{L^2} \right) \quad (2.74)$$

and that

$$\int_{\mathbb{R}^d \setminus B_{3 - \varepsilon}} \|\nabla^2 G_{\frac{T}{L^2}}\|_{L^\infty(B_{2 + \varepsilon}(x))} dx \lesssim \ln \left( 2 + \frac{T}{L^2} \right). \quad (2.75)$$

For that purpose, we claim that there exists  $c > 0$  such that for all  $|x| > 3 - \varepsilon$ ,

$$\|\nabla^2 G_{\frac{T}{L^2}}\|_{L^\infty(B_{2 + \varepsilon}(x))} \lesssim |x|^{-d} e^{-c|x| \frac{L}{\sqrt{T}}}. \quad (2.76)$$

Indeed, using the bound (2.48) on the massive Green function, there exists  $c > 0$



such that  $|\nabla^2 G_{\frac{T}{L^2}}| \lesssim |\cdot|^{-d} e^{-c|\cdot|/\sqrt{T}}$ . For  $y \in B_{2+\varepsilon}(x)$ ,  $|y| = |x - (x - y)| \geq |x| - (2 + \varepsilon) \geq (1 - \frac{2+\varepsilon}{3-\varepsilon})|x|$ . This yields (2.76) (up to modification of  $c$ ) remarking that  $|\cdot|^{-d} e^{-c|\cdot|/\sqrt{T}}$  is a radially decreasing function.

For (2.75), we integrate (2.76) over  $\mathbb{R}^d \setminus B_{3-\varepsilon}$  and, using polar coordinates, we have

$$\int_{\mathbb{R}^d \setminus B_{3-\varepsilon}} |x|^{-d} e^{-c|x|/\sqrt{T}} dx \lesssim \int_1^\infty e^{-cr/\sqrt{T}} \frac{dr}{r}. \quad (2.77)$$

By change of variables,  $\int_1^\infty e^{-cr/\sqrt{T}} \frac{dr}{r} = \int_{\frac{L}{\sqrt{T}}}^\infty e^{-cr} \frac{dr}{r}$ . This last integral is bounded in the regime  $L > \sqrt{T}$  thanks to the exponential decay and logarithmically explodes in the regime  $\frac{L}{\sqrt{T}} \rightarrow 0$  because of  $\frac{dr}{r}$  which yields (2.75).

For (2.74), by the hard-core condition,  $x \in \mathcal{P}^\circ \setminus \{0\}$  implies that  $|x| \geq 3$  and we can use the radial bound on the Green function (2.76) to the effect that

$$\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \|\nabla^2 G_{\frac{T}{L^2}}\|_{L^\infty(B_2(x))} \lesssim \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-d} e^{-c|x|/\sqrt{T}}.$$

We then decompose the sum over the annuli  $C_k = B_{k+1} \setminus B_k$  for  $k > 0$  (note that  $k = 0$  is left out because we have taken out the origin) and we use again the radial monotony of  $|\cdot|^{-d} e^{-c|\cdot|/\sqrt{T}}$  to write

$$\begin{aligned} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-d} e^{-c|x|/\sqrt{T}} &= \sum_{k=1}^\infty \sum_{x \in \mathcal{P}^\circ \setminus \{0\} \cap C_k} |x|^{-d} e^{-c|x|/\sqrt{T}} \\ &\leq \sum_{k=1}^\infty \#(\mathcal{P}^\circ \cap C_k) k^{-d} e^{-ck/\sqrt{T}}. \end{aligned}$$

Lemma 2.8 gives  $\#(\mathcal{P}^\circ \cap C_k) \lesssim k^{d-1}$ . We compare the sum over  $k \geq 1$  with an integral so that

$$\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-d} e^{-c|x|/\sqrt{T}} \lesssim \int_1^\infty e^{-cr/\sqrt{T}} \frac{dr}{r}. \quad \blacksquare$$

As for (2.77), we obtain a logarithmically diverging bound which concludes the proof of (2.74).

**Remark 2.10**  $K_T^L$  is a linear operator on  $E^L$  but depends (a priori) nonlinearly on the point process. Even if summing the massive Green function over the point process is a linear operation, the evaluation  $f(L(\mathcal{P} - x))$  for a given  $f \in E^L$  (for instance  $f = \nabla \varphi_T$ ) can introduce a nonlinearity. It is not the case when  $f$  itself is linear (for instance, for deterministic  $f \in (L^2(B_1))^d$ ).  $\square$

The dependence of the operator  $K_T^L$  in  $L$  is twofold : it is partly explicit by its formula (2.64) but it is also partly implicit due to the space  $E^L$  (which depends itself on  $L$ ). In order to distinguish both dependencies, we introduce a new operator acting on the original point process  $\mathcal{P}^\circ$  (without dilation).

**Lemma 2.12** *Let  $\tau > 0$  and  $L \geq 1$ . The linear operator  $\mathcal{K}_\tau(L) : E^1 \rightarrow E^1$  given for all  $f \in E^1$  and  $\mathcal{P}^\circ \in \Omega^\circ$  by*

$$\mathcal{K}_\tau(L)f(\mathcal{P}^\circ) := \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^2 G_\tau(x - \frac{\cdot}{L}) * [C^\circ f(\mathcal{P}^\circ - x)] \quad (2.78)$$

is well-defined and satisfies the bound

$$\|\|\|\mathcal{K}_\tau(L)\|\|\|_1 \lesssim \ln(2 + \tau) . \quad (2.79)$$

□

The proof is a direct rephrasing of the one of Lemma 2.10.

By construction, for a deterministic function  $g \in (L^2(B_1))^d$ , the operators  $K_T^L$  and  $\mathcal{K}_{\frac{T}{L^2}}$  are related via

$$K_T^L g(L\mathcal{P}^\circ) = \mathcal{K}_{\frac{T}{L^2}}(L)g(\mathcal{P}^\circ) , \quad (2.80)$$

where the equality holds in  $(L^2(B_1))^d$  for all  $\mathcal{P}^\circ \in \Omega^\circ$  and  $L \geq 1$ .

## 2.3.2 Improved bound

### Main propositions

The naive logarithmically diverging bound (2.65) on the operator  $K_T^L$  can be avoided.

**Proposition 2.1** *Let  $L \geq 1$  and  $T > 0$ . The operator  $K_T^L$  (defined in Lemma 2.10) satisfies*

$$\|\|\|K_T^L\|\|\|_L \lesssim 1 . \quad (2.81)$$

□

Thanks to this uniform bound, we can pass to the limit  $T \rightarrow \infty$  and give a rigorous definition of the operator  $K^L$  (see (2.15) in the informal approach).

**Proposition 2.2** *Let  $L \geq 1$ . There exists a linear operator  $K^L : E^L \rightarrow E^L$  such that  $K_T^L \xrightarrow{T \rightarrow \infty} K^L$  pointwise in  $E^L$ . Moreover, it satisfies the bound*

$$\|\|\|K^L\|\|\|_L \lesssim 1 . \quad (2.82)$$

□

### Proof of Proposition 2.1 : improved bound

Fix  $T > 0$ ,  $L \geq 1$  and  $f \in E^L$  s.t.  $\|f\|_{E^L} \leq 1$ . The first step in the proof of the improved bound (2.81) is to notice that the operator  $K_T^L$  originally defined through the sum (2.64) can be rewritten as the difference of the solutions of two PDEs.

**Lemma 2.13** *Let  $T > 0$  and  $L \geq 1$ . For all  $f \in E^L$ , we have*

$$L^{-d}K_T^L f = \nabla u_T - \nabla u_T^\circ \quad (2.83)$$

where,  $\mathbb{P}^\circ$ -a.s, we define  $u_T(L\mathcal{P}^\circ) \in H_{\text{uloc}}^1(\mathbb{R}^d)$  as the unique solution (given by Lemma 2.1) of

$$\frac{1}{T}u_T(L\mathcal{P}^\circ) - \Delta u_T(L\mathcal{P}^\circ) = \nabla \cdot \overline{C^\circ f}(L\mathcal{P}^\circ) \quad (2.84)$$

and  $u_T^\circ(L\mathcal{P}^\circ) \in H^1(\mathbb{R}^d)$  as the unique Lax-Milgram solution of

$$\frac{1}{T}u_T^\circ(L\mathcal{P}^\circ) - \Delta u_T^\circ(L\mathcal{P}^\circ) = \nabla \cdot [C^\circ f(L\mathcal{P}^\circ)] . \quad (2.85)$$

□

Proving (2.81) then amounts to establishing

$$\mathbb{E}^\circ \left[ \int_{B_1} |\nabla u_T(L\mathcal{P}^\circ) - \nabla u_T^\circ(L\mathcal{P}^\circ)|^2 \right] \lesssim L^{-2d} \quad (2.86)$$

where  $\nabla u_T$  and  $\nabla u_T^\circ$  are given by Lemma 2.13.

We make a first factor  $L^{-d}$  appear using the (massive) mean-value property. Subtracting,  $\mathbb{P}^\circ$ -a.s, the equation for  $u_T(L\mathcal{P}^\circ)$  and  $u_T^\circ(L\mathcal{P}^\circ)$ , we have  $(\frac{1}{T} - \Delta)(u_T(L\mathcal{P}^\circ) - u_T^\circ(L\mathcal{P}^\circ)) = 0$  in  $B_L$  by the hardcore property.

**Lemma 2.14** *There exists  $C(d) > 0$  such that for all  $0 < \rho < R$ ,  $T > 0$  and all massive harmonic  $u \in H^1(B_R)$  (i.e. weakly solving  $\frac{1}{T}u - \Delta u = 0$ ) in  $B_R$ , we have*

$$\int_{B_\rho} u^2 \leq C \int_{B_R} u^2 . \quad (2.87)$$

□

Lemma 2.14 implies that  $\mathbb{P}^\circ$ -a.s,

$$\int_{B_1} |\nabla u_T(L\mathcal{P}^\circ) - \nabla u_T^\circ(L\mathcal{P}^\circ)|^2 \lesssim L^{-d} \int_{B_L} |\nabla u_T(L\mathcal{P}^\circ) - \nabla u_T^\circ(L\mathcal{P}^\circ)|^2 .$$

Taking the expectation of this inequality and since  $B_L \subset V_0(L\mathcal{P}^\circ)$  (by the hardcore

property), we obtain

$$\mathbb{E}^\circ \left[ \int_{B_1} |\nabla u_T(L\mathcal{P}^\circ) - \nabla u_T^\circ(L\mathcal{P}^\circ)|^2 \right] \lesssim L^{-d} \mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} |\nabla u_T(L\mathcal{P}^\circ) - \nabla u_T^\circ(L\mathcal{P}^\circ)|^2 \right].$$

To conclude the proof of (2.86), it remains to show that

$$\mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} |\nabla u_T(L\mathcal{P}^\circ) - \nabla u_T^\circ(L\mathcal{P}^\circ)|^2 \right] \lesssim L^{-d}. \quad (2.88)$$

For that purpose, we use a strategy introduced by Duerinckx and Gloria, see [DG20c, Theorem 1 p.18]. We start by introducing the Dirichlet and Neumann approximations of  $u_T^\circ$  on  $V_0$ .

**Definition 2.8** Let  $T > 0$ ,  $L \geq 1$  and  $f \in E^L$ .  $\mathbb{P}^\circ$ -a.s, we define  $\nabla u_T^{\circ,D}(L\mathcal{P}^\circ) \in H_0^1(V_0(L\mathcal{P}^\circ))$  as the unique solution of the Dirichlet problem

$$\begin{cases} \frac{1}{T} u_T^{\circ,D}(L\mathcal{P}^\circ) - \Delta u_T^{\circ,D}(L\mathcal{P}^\circ) &= \nabla \cdot [C^\circ f(L\mathcal{P}^\circ)] & \text{in } V_0(L\mathcal{P}^\circ) \\ u_T^{\circ,D}(L\mathcal{P}^\circ) &= 0 & \text{on } \partial V_0(L\mathcal{P}^\circ) \end{cases}. \quad (2.89)$$

□

Similarly, we define  $\nabla u_T^{\circ,N}(L\mathcal{P}^\circ) \in H^1(V_0(L\mathcal{P}^\circ))$   $\mathbb{P}^\circ$ -a.s as the unique solution of the corresponding Neumann problem.

It is well known that (2.89) is equivalent to an energy minimization problem on  $V_0$  and that, in terms of boundary conditions, the Dirichlet and Neumann problems are extremal. Using that  $u_T$  also admits a variational formulation, this allows to control the difference between  $u_T$  and  $u_T^{\circ,D}$  by the one between  $u_T^{\circ,D}$  and  $u_T^{\circ,N}$ .

**Lemma 2.15** Let  $T > 0$ ,  $L \geq 1$ ,  $f \in E^L$  and, associated to  $f$ ,  $\nabla u_T$  (see (2.84)),  $\nabla u_T^{\circ,D}$  and  $\nabla u_T^{\circ,N}$  (see Definition 2.8). We have

$$\begin{aligned} \mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} \frac{1}{T} (u_T - u_T^{\circ,D})^2 + |\nabla(u_T - u_T^{\circ,D})|^2 \right] \\ \leq \mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} \frac{1}{T} (u_T^{\circ,N} - u_T^{\circ,D})^2 + |\nabla(u_T^{\circ,N} - u_T^{\circ,D})|^2 \right]. \end{aligned} \quad (2.90)$$

□

The last step is to control the difference between these Dirichlet and Neumann approximations of  $u_T^\circ$ . To do so, we quantitatively compare  $u_T^{\circ,D}$  and  $u_T^{\circ,N}$  to  $u_T^\circ$ . This becomes a fully deterministic question : we are comparing the solution to a Laplace equation on  $\mathbb{R}^d$  with a compactly supported source term to its approximations on the representative volume  $V_0$  (the border of which is at distance  $\sim L$  from the source term).

**Lemma 2.16** *Let  $T > 0$ ,  $L \geq 1$ ,  $f \in E^L$ ,  $\nabla u_T^{\circ,N}$  and  $\nabla u_T^{\circ,D}$  (see Definition 2.8).  $\mathbb{P}^\circ$ -a.s, we have*

$$\int_{V_0(L\mathcal{P}^\circ)} \frac{1}{T} (u_T^{\circ,D} - u_T^\circ)^2(L\mathcal{P}^\circ) + |\nabla(u_T^{\circ,D} - u_T^\circ)|^2(L\mathcal{P}^\circ) \lesssim L^{-d} \int_{B_1} f(L\mathcal{P}^\circ)^2 \quad (2.91)$$

and

$$\int_{V_0(L\mathcal{P}^\circ)} \frac{1}{T} (u_T^{\circ,N} - u_T^\circ)^2(L\mathcal{P}^\circ) + |\nabla(u_T^{\circ,N} - u_T^\circ)|^2(L\mathcal{P}^\circ) \lesssim L^{-d} \int_{B_1} f(L\mathcal{P}^\circ)^2. \quad (2.92)$$

□

With these results at hand, we conclude the proof of (2.88). Combining (2.90) with the expectation  $\mathbb{E}^\circ$  of (2.91) and (2.92) (using the triangle inequality), we get

$$\mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} |\nabla u_T(L\mathcal{P}^\circ) - \nabla u_T^{\circ,D}(L\mathcal{P}^\circ)|^2 \right] \lesssim L^{-d}. \quad (2.93)$$

Lemma 2.16 directly gives

$$\mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} |\nabla u_T^{\circ,D}(L\mathcal{P}^\circ) - \nabla u_T^\circ(L\mathcal{P}^\circ)|^2 \right] \lesssim L^{-d}. \quad (2.94)$$

Combined with (2.93), this yields (2.88) by the triangle inequality and concludes the proof of Proposition 2.1.

We now turn to the proofs of the previous lemmas, starting with the PDE rewriting of the operator  $K_T^L$ .

PROOF (LEMMA 2.13) Fix  $T > 0$ ,  $L \geq 1$  and  $f \in E^L$ . By definition of  $K_T^L$  (see Lemma 2.10), we have

$$L^{-d} K_T^L f(L\mathcal{P}^\circ) = L^{-d} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^2 G_{\frac{T}{L^2}}(\frac{\cdot}{L} - x) * [C^\circ f(L(\mathcal{P}^\circ - x))].$$

Using the scaling property (2.46),  $L^{-d} \nabla^2 G_{\frac{T}{L^2}}(\frac{\cdot}{L} - x) = \nabla^2 G_T(\cdot - Lx)$ , we write

$$\begin{aligned} L^{-d} K_T^L f(L\mathcal{P}^\circ) &= \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^2 G_T * [C^\circ f(L(\mathcal{P}^\circ - x))](\cdot - Lx) \\ &= \sum_{x \in \mathcal{P}^\circ} \nabla^2 G_T * [C^\circ f(L(\mathcal{P}^\circ - x))](\cdot - Lx) \\ &\quad - \nabla^2 G_T * [C^\circ f(L(\mathcal{P}^\circ))]. \end{aligned}$$

This concludes the proof recognizing  $u_T$  and  $u_T^\circ$  from the Definition 2.4 of  $G_T$ . ■

The (massive) mean value result of Lemma 2.14 is standard. We refer for instance to [GM12a, (5.13) Prop 5.8 p.80] (though written without the massive term, their approach is robust to its addition).

Before moving to the proof Lemma 2.15, we first argue how we can equivalently define  $u_T$  in the spirit of the corrector.

**Remark 2.11** Following [JKO94, Chapter 7] and [PV81, Section 2 p.840], we define the space  $H^1(L\Omega)$  that we identify to the subspace of functions of  $L^2(L\Omega, H_{\text{loc}}^1(\mathbb{R}^d))$  which satisfy the contravariance property (2.29). The stationarity of  $\mathcal{P}$  allows to transfer a notion of gradient from  $H_{\text{loc}}^1(\mathbb{R}^d)$  to  $H^1(L\Omega)$  and to equip this space with the norm  $\mathbb{E}[v^2 + |\nabla v|^2]^{\frac{1}{2}}$  for  $v \in H^1(L\Omega)$ .

Consequently, we can define  $u_T$  as the unique Lax-Milgram solution in  $H^1(L\Omega)$  of the problem

$$\mathbb{E} \left[ \frac{1}{T} u_T v + \nabla u_T \cdot \nabla v \right] = -\mathbb{E} [\overline{C^\circ f} v]$$

for all  $v \in H^1(L\Omega)$ . See for instance [GH16, Section 2.1] for details on the functional setting.

This formulation yields the following bound

$$\mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} \frac{1}{T} u_T (L\mathcal{P}^\circ)^2 + |\nabla u_T(L\mathcal{P}^\circ)|^2 \right] \lesssim \|f\|_{E^L}^2. \quad (2.95)$$

□

PROOF By the Lax-Milgram theorem, we have the following energy estimate

$$\mathbb{E} \left[ \frac{1}{T} u_T^2 + |\nabla u_T|^2 \right] \leq \mathbb{E} [|\overline{C^\circ f}|^2]$$

which turns into (2.95) using Lemma 2.9 noting that  $|\overline{C^\circ f}|^2 = \overline{|C^\circ f|^2}$  (by construction of the extension since the Voronoi cells form a partition) and that  $C^\circ$  is supported in  $B_1$ . ■

PROOF (LEMMA 2.15) Fix  $f \in E^L$ .  $\mathbb{P}^\circ$ -a.s,  $f(L\mathcal{P}^\circ) \in L^2(B_1)$  and for  $v \in H^1(V_0(L\mathcal{P}^\circ))$ , we define the energy  $E_T^\circ(v) := \int_{V_0(L\mathcal{P}^\circ)} \frac{1}{2} (\frac{1}{T} v^2 + |\nabla v|^2) + C^\circ f(L\mathcal{P}^\circ) \cdot \nabla v$ .

We claim that

$$\mathbb{E}^\circ \left[ E_T^\circ(u_T^{\circ,N}) \right] \leq \mathbb{E}^\circ [E_T^\circ(u_T)] \leq \mathbb{E}^\circ \left[ E_T^\circ(u_T^{\circ,D}) \right], \quad (2.96)$$

$$E_T^\circ(u_T^{\circ,D}) - E_T^\circ(u_T^{\circ,N}) = \frac{1}{2} \int_{V_0(L\mathcal{P}^\circ)} |\nabla u_T^{\circ,D} - \nabla u_T^{\circ,N}|^2 + \frac{1}{T} (u_T^{\circ,D} - u_T^{\circ,N})^2, \quad (2.97)$$

and

$$\mathbb{E}^\circ \left[ E_T^\circ(u_T^{\circ,D}) - E_T^\circ(u_T) \right] = \frac{1}{2} \mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} |\nabla u_T - \nabla u_T^{\circ,D}|^2 + \frac{1}{T} (u_T - u_T^{\circ,D})^2 \right] \quad (2.98)$$

from which we get (2.90). Indeed, (2.96) implies that

$$\mathbb{E}^\circ \left[ E_T^\circ(u_T^{\circ,D}) - E_T^\circ(u_T) \right] \leq \mathbb{E}^\circ \left[ E_T^\circ(u_T^{\circ,D}) - E_T^\circ(u_T^{\circ,N}) \right].$$

The identities (2.97) and (2.98) then convert these differences of energies into norms differences.

It is well-known that  $u_T^{\circ,N}(L\mathcal{P}^\circ)$  minimizes  $E_T^\circ$  over  $H^1(V_0(L\mathcal{P}^\circ))$ . Since  $\mathbb{P}^\circ$ -a.s.,  $u_T(L\mathcal{P}^\circ) \in H^1(V_0(L\mathcal{P}^\circ))$  (using the bound (2.95)), we conclude that  $E_T^\circ(u_T^{\circ,N}(L\mathcal{P}^\circ)) \leq E_T^\circ(u_T(L\mathcal{P}^\circ))$  which proves the left inequality in (2.96).

The optimality of  $u_T^{\circ,N}(L\mathcal{P}^\circ)$  also justifies (2.97). Indeed, for  $u, v \in H^1(V_0(L\mathcal{P}^\circ))$ , we define  $b(u, v) = \int_{V_0(L\mathcal{P}^\circ)} \frac{1}{T} uv + \nabla u \cdot \nabla v$  and  $l(v) = \int_{V_0(L\mathcal{P}^\circ)} C^\circ f(L\mathcal{P}^\circ) \cdot \nabla v$ , so that  $E_T^\circ(v) = \frac{1}{2} b(v, v) + l(v)$ . We compute  $E_T^\circ(v) - E_T^\circ(u_T^{\circ,N})$ :

$$E_T^\circ(v) - E_T^\circ(u_T^{\circ,N}) = \frac{1}{2} b(v + u_T^{\circ,N}, v - u_T^{\circ,N}) + l(v - u_T^{\circ,N}).$$

Noting that  $b(u_T^{\circ,N}, v - u_T^{\circ,N}) = -l(v - u_T^{\circ,N})$  (testing  $v - u_T^{\circ,N}$  in the weak formulation of the equation defining  $u_T^{\circ,N}$ ), we obtain

$$E_T^\circ(v) - E_T^\circ(u_T^{\circ,N}) = \frac{1}{2} b(v - u_T^{\circ,N}, v - u_T^{\circ,N}).$$

This yields (2.97) taking  $v = u_T^{\circ,D}(L\mathcal{P}^\circ) \in H_0^1(L\mathcal{P}^\circ)$ .

Using the Lax-Milgram definition of  $u_T$  from Remark 2.11,  $u_T$  minimizes  $E_T(v) := \mathbb{E} \left[ \frac{1}{2} \left( \frac{1}{T} v^2 + |\nabla v|^2 \right) + \overline{C^\circ f} \cdot \nabla v \right]$  over  $H^1(L\Omega)$ . We claim that  $\overline{u_T^{\circ,D}} \in H^1(L\Omega)$ . Hence,  $E_T(u_T) \leq E_T(\overline{u_T^{\circ,D}})$  which yields the right inequality in (2.96) (using Lemma 2.9 to pass from  $\mathbb{E}$  to  $\mathbb{E}^\circ$ ). This also proves (2.98) proceeding as for (2.97) with  $E_T$  instead of  $E_T^\circ$  and using again Lemma 2.9.

To show that  $\overline{u_T^{\circ,D}}$  is admissible, the key point is to notice that  $\nabla \overline{u_T^{\circ,D}} = \overline{\nabla u_T^{\circ,D}}$  since  $u_T^{\circ,D}(L\mathcal{P}^\circ) \in H_0^1(L\mathcal{P}^\circ)$  which allows to glue the functions together (without jump) in the definition of  $\overline{u_T^{\circ,D}}$ . We also check that it has a finite  $H^1(L\Omega)$ -norm. By the energy estimate for  $u_T^{\circ,D}(L\mathcal{P}^\circ)$ , we have

$$\int_{V_0(L\mathcal{P}^\circ)} \frac{1}{T} u_T^{\circ,D}(L\mathcal{P}^\circ)^2 + |\nabla u_T^{\circ,D}(L\mathcal{P}^\circ)|^2 \lesssim \int_{B_1} f(L\mathcal{P}^\circ)^2.$$

Taking the expectation  $\mathbb{E}^\circ$  and using Lemma 2.9 then yields

$$\mathbb{E} \left[ \frac{1}{T} \overline{u_T^{\circ,D^2}} + |\nabla \overline{u_T^{\circ,D}}|^2 \right] \lesssim_L \|f\|_{E^L}^2 < \infty$$

which concludes the proof.  $\blacksquare$

PROOF (LEMMA 2.16) Let  $f \in E^L$ .  $\mathbb{P}^\circ$ -a.s,  $f(L\mathcal{P}^\circ) \in L^2(B_1)$  and we fix such a  $L\mathcal{P}^\circ$ . For the rest of the proof and to lighten the notation, we omit the dependence upon  $L\mathcal{P}^\circ$  of  $V_0$  and of the functions  $u_T^\circ$ ,  $u_T^{\circ,N}$  and  $u_T^{\circ,D}$  and  $f$ .

We start with the Neumann problem and the proof of (2.92). Introducing the short-hand notation  $\delta^N u := u_T^\circ - u_T^{\circ,N}$ ,  $\delta^N u$  solves

$$\begin{cases} \frac{1}{T} \delta^N u - \Delta \delta^N u &= 0 & \text{in } V_0 \\ \partial_n \delta^N u &= \partial_n u_T^\circ & \text{on } \partial V_0 \end{cases} . \quad (2.99)$$

Note that  $\partial_n u_T^\circ$  is well-defined since  $u_T^\circ$  is massive-harmonic outside  $B_1$  thus smooth on  $\partial V_0$ . Testing (2.99) with  $\delta^N u$ , we have

$$\int_{V_0} \frac{1}{T} (\delta^N u)^2 + |\nabla \delta^N u|^2 = \int_{\partial V_0} u_T^\circ \partial_n u_T^\circ - \int_{\partial V_0} u_T^{\circ,N} \partial_n u_T^\circ . \quad (2.100)$$

We claim that

$$\int_{\partial V_0} u_T^\circ \partial_n u_T^\circ \leq 0 \quad (2.101)$$

and

$$\left| \int_{\partial V_0} u_T^{\circ,N} \partial_n u_T^\circ \right| \lesssim L^{-\frac{d}{2}} \|f\|_{L^2(B_1)} \|\nabla \delta^N u\|_{L^2(V_0)} . \quad (2.102)$$

Combining (2.100) with (2.101) and (2.102), we obtain (2.92) absorbing  $\|\nabla \delta^N u\|_{L^2(V_0)}$  into the LHS by Young's inequality.

For (2.101), we test the equation of  $u_T^\circ$  (2.85) with  $u_T^\circ$  itself on  $V_0$  and integrate by parts to the effect that

$$\int_{\partial V_0} u_T^\circ \partial_n u_T^\circ = \int_{V_0} \frac{1}{T} u_T^{\circ 2} + |\nabla u_T^\circ|^2 + \int_{B_1} C^\circ f \cdot \nabla u_T^\circ .$$

Integrating this time on  $\mathbb{R}^d$ , it yields

$$\int_{V_0} C^\circ f \cdot \nabla u_T^\circ = \int_{B_1} C^\circ f \cdot \nabla u_T^\circ = - \int_{\mathbb{R}^d} \frac{1}{T} u_T^{\circ 2} + |\nabla u_T^\circ|^2 .$$



Combining both, we obtain

$$\int_{\partial V_0} u_T^\circ \partial_n u_T^\circ = - \int_{\mathbb{R}^d \setminus V_0} \frac{1}{T} u_T^\circ{}^2 + |\nabla u_T^\circ|^2 \leq 0 .$$

For (2.102), we claim that

$$\int_{\partial V_0} u_T^{\circ, N} \partial_n u_T^\circ = - \int_{V_0} C^\circ f \cdot \nabla \delta^N u . \quad (2.103)$$

Using then the support of  $C^\circ$  and Cauchy-Schwarz' inequality, (2.103) implies that

$$\left| \int_{\partial V_0} u_T^{\circ, N} \partial_n u_T^\circ \right| \leq \|f\|_{L^2(B_1)} \|\nabla \delta^N u\|_{L^2(B_1)} .$$

We obtain (2.101) noting that  $\|\nabla \delta^N u\|_{L^2(B_1)} \lesssim L^{-\frac{d}{2}} \|\nabla \delta^N u\|_{L^2(V_0)}$ . Indeed,  $\delta^N u$  is massive-harmonic in  $V_0$  and we apply Lemma 2.14 between  $B_1$  and  $B_L \subset V_0$ .

To prove (2.103), we first test the equation (2.85) for  $u_T^\circ$  with  $u_T^{\circ, N}$  on  $V_0$  and integrate by parts so that

$$\int_{\partial V_0} u_T^{\circ, N} \partial_n u_T^\circ = \int_{V_0} \frac{1}{T} u_T^{\circ, N} u_T^\circ + \nabla u_T^{\circ, N} \cdot \nabla u_T^\circ + \int_{V_0} C^\circ f \cdot \nabla u_T^{\circ, N} .$$

Then, we test the equation for  $u_T^{\circ, N}$  with  $u_T^\circ$  and obtain

$$\int_{V_0} \frac{1}{T} u_T^{\circ, N} u_T^\circ + \nabla u_T^{\circ, N} \cdot \nabla u_T^\circ = - \int_{V_0} C^\circ f \cdot \nabla u_T^\circ .$$

Combining both yields (2.103).

For (2.91), the proof is similar considering  $\delta^D u := u_T^\circ - u_T^{\circ, D}$  which solves

$$\begin{cases} \frac{1}{T} \delta^D u - \Delta \delta^D u = 0 & \text{in } V_0(L\mathcal{P}^\circ) \\ \delta^D u = u_T^\circ & \text{on } \partial V_0(L\mathcal{P}^\circ) \end{cases} . \quad (2.104)$$

Testing (2.104) with  $\delta^D u$ , we have

$$\int_{V_0} \frac{1}{T} (\delta^D u)^2 + |\nabla \delta^D u|^2 = \int_{\partial V_0} u_T^\circ \partial_n u_T^\circ - \int_{\partial V_0} u_T^\circ \partial_n u_T^{\circ, D} .$$

Note that  $\partial_n u_T^{\circ, D}$  is well-defined. Indeed,  $u_T^{\circ, D}$  is massive-harmonic outside  $B_1$ , zero on  $\partial V_0$  and  $V_0$  is a Lipschitz domain thus  $u_T^{\circ, D}$  is smooth up to the boundary.

Proceeding as for (2.92), we only need to prove that

$$\int_{\partial V_0} u_T^\circ \partial_n u_T^{\circ, D} = \int_{V_0} C^\circ f \cdot \nabla \delta^D u \quad \blacksquare$$

from which we conclude as before. Like (2.103), this identity is obtained subtracting the equation (2.85) of  $u_T^\circ$  tested with  $u_T^{\circ, D}$  and the equation (2.89) of  $u_T^{\circ, D}$  tested with  $u_T^\circ$  (each time on  $V_0$  and integrating by part).

### Proof of Proposition 2.2 : limit operator

We first note that the bound (2.82) on  $K^L$  is inherited from the pointwise convergence  $K_T^L \rightarrow K^L$  and the bound (2.81) on  $K_T^L$  (uniform in  $T$ ) using the Banach-Steinhaus theorem (a.k.a. the uniform boundedness principle).

Then, following Lemma 2.13, we use the PDE rewriting of the operator  $K_T^L$  to rigorously define the operator  $K^L$ .

**Definition 2.9** Let  $L \geq 1$ . We define the operator  $K^L : E^L \rightarrow E^L$  for  $f \in E^L$  by

$$L^{-d} K^L f = \nabla u - \nabla u^\circ \quad (2.105)$$

where  $\mathbb{P}^\circ$ -a.s, we define  $\nabla u^\circ(L\mathcal{P}^\circ)$  as the unique Lax-Milgram solution in  $L_{\text{pot}}^2(\mathbb{R}^d)$  of

$$-\Delta u^\circ(L\mathcal{P}^\circ) = \nabla \cdot [C^\circ f(L\mathcal{P}^\circ)] \quad \text{in } \mathbb{R}^d \quad (2.106)$$

and  $\nabla u$  is defined similarly to the corrector as the unique weak solution in  $H_{\text{loc}}^1(\mathbb{R}^d)$  (up to additive constant) of

$$-\Delta u = \nabla \cdot \overline{C^\circ f}(L\mathcal{P}) \quad \text{in } \mathbb{R}^d \quad (2.107)$$

such that  $\nabla u$  is stationary,  $\mathbb{E}[\nabla u(L\mathcal{P})] = 0$  and  $\mathbb{E}[|\nabla u(L\mathcal{P})|^2] < \infty$ .  $\square$

**Remark 2.12** With these definitions,  $\nabla u \in E^L$  and  $\nabla u^\circ \in E^L$  so that the operator  $K^L$  is well-defined.

Indeed, for  $\nabla u$ , as in Remark 2.11, we can define a space  $L_{\text{pot}}^2(L\Omega)$  and (with a slight abuse of notation)  $\nabla u$  can be defined as the unique Lax-Milgram solution in this space of the problem

$$\mathbb{E}[\nabla u \cdot \nabla v] = -\mathbb{E}[\overline{C^\circ f} \cdot \nabla v]$$

for all  $\nabla v \in L_{\text{pot}}^2(L\Omega)$ .

Similarly to the bound (2.95), we have

$$\mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} |\nabla u(L\mathcal{P}^\circ)|^2 \right] \lesssim \|f\|_{E^L}^2$$

which justifies that  $\nabla u \in E^L$ .

For  $\nabla u^\circ$ , we just need to take the expectation  $\mathbb{E}^\circ$  of the energy estimate obtained from (2.106) for fixed  $L\mathcal{P}^\circ \in L\Omega^\circ$ .  $\square$

Lastly, given  $f \in E^L$ , the pointwise convergence of  $K_T^L f \rightarrow K^L f$  is a direct consequence of the convergences  $\nabla u_T \rightarrow \nabla u$  and  $\nabla u_T^\circ \rightarrow \nabla u^\circ$  in  $E^L$ .

**Lemma 2.17** *Let  $L \geq 1$ ,  $f \in E^L$  and  $T > 0$ . For  $u_T^\circ$  and  $\nabla u_T^\circ$  associated to  $f$  (see Definition 2.9), we have*

$$\|\nabla u_T^\circ - \nabla u^\circ\|_{E^L} \xrightarrow{T \rightarrow \infty} 0. \quad (2.108)$$

For  $u_T$  and  $\nabla u$  associated to  $f$  (see (2.84) and (2.107)), we have

$$\|\nabla u_T - \nabla u\|_{E^L} \xrightarrow{T \rightarrow \infty} 0. \quad (2.109)$$

$\square$

PROOF Fix  $L \geq 1$  and  $f \in E^L$ .

For (2.108), we appeal to the dominated convergence theorem. Noticing that  $\mathbb{P}^\circ$ -a.s,  $\nabla u_T^\circ(L\mathcal{P}^\circ) = \mathcal{H}_T[C^\circ f(L\mathcal{P}^\circ)]$  and  $\nabla u^\circ(L\mathcal{P}^\circ) = \mathcal{H}[C^\circ f(L\mathcal{P}^\circ)]$ , the pointwise convergence  $\mathcal{H}_T \rightarrow \mathcal{H}$  (see Lemma 2.5 and (2.53)) implies that  $\mathbb{P}^\circ$ -a.s,  $\nabla u_T^\circ(L\mathcal{P}^\circ) \rightarrow \nabla u^\circ(L\mathcal{P}^\circ)$  in  $(L^2(\mathbb{R}^d))^d$  as  $T \rightarrow \infty$ . The energy estimates on  $u_T^\circ(L\mathcal{P}^\circ)$  and  $\nabla u^\circ(L\mathcal{P}^\circ)$  provides the  $\mathbb{P}^\circ$ -a.s bounds

$$\int_{\mathbb{R}^d} |\nabla u_T^\circ(L\mathcal{P}^\circ)|^2 + |\nabla u^\circ(L\mathcal{P}^\circ)|^2 \lesssim \int_{B_1} |f(L\mathcal{P}^\circ)|^2.$$

Since  $f \in E^L$ , this allows to conclude that  $\mathbb{E}^\circ \left[ \int_{\mathbb{R}^d} |\nabla u_T^\circ(L\mathcal{P}^\circ) - \nabla u^\circ(L\mathcal{P}^\circ)|^2 \right] \rightarrow 0$ .

For (2.109), we claim that

$$\mathbb{E} \left[ |\nabla u_T - \nabla u|^2 \right] \xrightarrow{T \rightarrow \infty} 0. \quad (2.110)$$

Using Lemma 2.9 and since  $\nabla u_T$  and  $\nabla u$  are contravariant by construction, this convergence upgrades to  $\mathbb{E}^\circ \left[ \int_{V_0(L\mathcal{P}^\circ)} |\nabla u_T - \nabla u|^2(L\mathcal{P}^\circ) \right] \rightarrow 0$  which implies (2.109).

Using the spaces introduced in Remark 2.11 and Remark 2.12, the proof of (2.110) is similar to the one of Lemma 2.5 (see [GH16, Theorem 1]).  $\blacksquare$

## 2.4 Proof of Theorem 2.1 : analyticity of the homogenized coefficient

Proceeding as in the informal approach and using the theory of point processes, we can rewrite  $\bar{a}^L$ , the homogenized coefficient associated with  $a(L\mathcal{P}, \cdot)$  (see (2.2)) in the following form (see (2.9)).

**Lemma 2.18** *Let  $e \in \mathbb{R}^d$  and  $L \geq 1$ . We can rewrite  $\bar{a}^L$  in the following form :*

$$\bar{a}^L e = e + L^{-d} \int_{B_1} C^\circ(x) \mathbb{E}^\circ[\nabla\varphi(L\mathcal{P}^\circ, x) + e] \theta \, dx, \quad (2.111)$$

where  $\theta$  is the intensity of the point process  $\mathcal{P}$  (see Definition 2.5).  $\square$

This lemma allows us to take out a first  $L^{-d}$  factor in the expansion of  $\bar{a}^L$  and rigorously shows that it is enough to describe the corrector on  $B_1$  for a point process with an inclusion at the origin.

**PROOF** Let  $L \geq 1$  and  $e \in \mathbb{R}^d$ . Similarly to the informal approach, we start from (2.9) that we rewrite

$$\bar{a}_r^L e = e + \mathbb{E} \left[ \sum_{x \in \mathcal{P}} C^\circ(-Lx) (\nabla\varphi_T(L(\mathcal{P} - x), -Lx) + e) \right]$$

using that  $\nabla\varphi(L\mathcal{P}, \cdot) = \nabla\varphi(L(\mathcal{P} - x), \cdot - Lx)$  for all  $x \in \mathcal{P}$  by contravariance of the corrector (see (2.8)).

By the hard-core assumption and the support condition of  $C^\circ$ , there is only one  $x \in \mathcal{P}$  such that  $C^\circ(-Lx) \neq 0$ . By stationarity, we recenter the process around this point and average over its typical position. Theorem 2.2 makes this statement precise and yields

$$\bar{a}^L e = e + \int_{\mathbb{R}^d} C^\circ(-Lx) \mathbb{E}^\circ [(\nabla\varphi(L\mathcal{P}^\circ, -Lx) + e)] \theta \, dx.$$

A change of variables in the space variable then gives (2.111).

We conclude the proof checking that  $\mathbb{E}^\circ \left[ \int_{B_1} |\nabla\varphi(L\mathcal{P}^\circ)|^2 \right] \lesssim 1$  so that the RHS of (2.111) is well-defined. Using again that  $a(L\mathcal{P}) = \text{Id} + \overline{C^\circ e}$ , the equation (2.7) of the corrector can be rewritten

$$-\nabla \cdot a(L\mathcal{P}) \nabla\varphi(L\mathcal{P}) = \nabla \cdot \overline{C^\circ e}.$$

The energy estimate in the probability space then writes

$\mathbb{E} [|\nabla\varphi(LP)|^2] \lesssim \mathbb{E} [|\overline{C^\circ e}|^2]$ . We upgrade it to  $\mathbb{E}^\circ \left[ \int_{V_0} |\nabla\varphi(LP^\circ)|^2 \right] \lesssim \|C^\circ\|_{L^2(\mathbb{R}^d)}^2$  by Lemma 2.9 noting that  $|\overline{C^\circ e}|^2 = |\overline{C^\circ e}|^2$  by the hard-core assumption (proceeding as in (2.32)).  $\blacksquare$

### 2.4.1 Main propositions

**Proposition 2.3** *The family of operators  $(\mathcal{K}_\tau(L))_{\tau>0, L\geq 1}$  satisfies the uniform bound*

$$\sup_{\tau>0, L\geq 1} \|\mathcal{K}_\tau(L)\|_1 \lesssim 1. \quad (2.112)$$

Moreover, for all  $L \geq 1$ , the sequence  $(\mathcal{K}_\tau(L))_{\tau>0}$  of operator  $E^1 \rightarrow E^1$  converges pointwise as  $\tau \rightarrow \infty$  and we define the operator  $\mathcal{K}(L) := \lim_{\tau \rightarrow \infty} \mathcal{K}_\tau(L)$ .

Defined in this way, the map  $L^{-1} \mapsto \mathcal{K}(L)$  is analytic at 0 : there exist a family  $(\mathcal{K}^{(i)})_{i\geq 0}$  of bounded operators  $E^1 \rightarrow E^1$  and a constant  $C > 0$  such that, for all  $L^{-1} < C^{-1}$ ,

$$\mathcal{K}(L) = \sum_{i=0}^{\infty} \mathcal{K}^{(i)} L^{-i} \quad (2.113)$$

and for all  $i \geq 0$

$$\|\mathcal{K}^{(i)}\|_1 \leq C^i. \quad (2.114)$$

In addition, for all  $i > 0$ ,  $\mathcal{K}^{(i)}$  is given for  $f \in E^1$  by

$$\mathcal{K}^{(i)} f := (\mathcal{P}^\circ, z) \mapsto \frac{1}{i!} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^{2+i} G_\tau(x) \cdot \int_{B_1} (y-z)^{\otimes i} C^\circ(y) f(\mathcal{P}^\circ - x, y) dy, \quad (2.115)$$

(which are well-defined formulas) and  $\mathcal{K}^{(0)}$  is given for  $f \in E^1$  by

$$\mathcal{K}^{(0)} f := \lim_{\tau \rightarrow \infty} \left[ (\mathcal{P}^\circ, z) \mapsto \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^2 G_\tau(x) \cdot \int_{B_1} C^\circ(y) f(\mathcal{P}^\circ - x, y) dy \right] \quad (2.116)$$

where the limit is taken in  $E^1$  and exists.  $\square$

We are now in position to give the fixed-point formulation of the corrector using the single inclusion solution.

**Proposition 2.4** *Let  $L \gg 1$ . Then, using the operator  $\mathcal{K}(L)$  defined in Proposition 2.2, we have  $\mathbb{P}^\circ$ -a.s*

$$\nabla\varphi(LP^\circ) + e = (\text{Id} - L^{-d} \mathcal{B}^\circ \mathcal{K}(L))^{-1} (\nabla\varphi^\circ + e)(\mathcal{P}^\circ). \quad (2.117)$$

$\square$

Even if the corrector is defined on the whole space, (2.117) is an equality between function in  $L^2(\Omega^\circ, (L^2(B_1))^d)$  and therefore only holds on  $B_1$ .

Since  $\mathcal{B}^\circ$  and  $\nabla\varphi^\circ + e$  are deterministic, the dependence of the corrector on the point process and on the dilation parameter  $L$  is fully encompassed in the operator  $\mathcal{K}(L)$ .

Combining these two propositions, we directly obtain Theorem 2.1. Indeed, for  $e \in \mathbb{R}^d$  and  $L \geq 1$ , plugging the fixed-point formulation of the corrector (2.117) in the preliminary expansion (2.111) of  $\bar{a}^L$ , we have

$$\bar{a}^L e = e + L^{-d} \int C^\circ(x) \mathbb{E}^\circ \left[ (\text{Id} - L^{-d} \mathcal{B}^\circ \mathcal{K}(L))^{-1} (\nabla\varphi^\circ + e)(\mathcal{P}^\circ, x) \right] \theta dx . \quad (2.118)$$

The analyticity of the map  $L^{-1} \mapsto \bar{a}^L$  then results from the one of  $L^{-1} \mapsto \mathcal{K}(L)$  given by Proposition 2.3 noting that the inverse is also analytic.

## 2.4.2 Proof of Proposition 2.4

### Structure of proof

Recalling the informal computation of Section 2.1.2 and the need for a massive term, we start by establishing the massive counterpart of Proposition 2.4.

**Lemma 2.19** *Let  $T > 0$ ,  $L \gg 1$ ,  $e \in \mathbb{R}^d$  and the associated massive corrector  $\nabla\varphi_T$  (see Lemma 2.2). Using the operator  $\mathcal{K}_{\frac{T}{L^2}}(L)$  (defined in Lemma 2.12),  $\mathbb{P}^\circ$ -a.s, we have*

$$\nabla\varphi_T(L\mathcal{P}^\circ) + e = (\text{Id} - L^{-d} \mathcal{B}_T^\circ \mathcal{K}_{\frac{T}{L^2}}(L))^{-1} (\nabla\varphi_T^\circ + e)(\mathcal{P}^\circ) . \quad (2.119)$$

□

Fix  $L \gg 1$  and  $e \in \mathbb{R}^d$ . We have

$$\nabla\varphi_T \xrightarrow[T \rightarrow \infty]{E^L} \nabla\varphi \quad (2.120)$$

and

$$(\text{Id} - L^{-d} \mathcal{B}_T^\circ \mathcal{K}_{\frac{T}{L^2}}(L))^{-1} (\nabla\varphi_T^\circ + e) \xrightarrow[T \rightarrow \infty]{E^1} (\text{Id} - L^{-d} \mathcal{B}^\circ \mathcal{K}(L))^{-1} (\nabla\varphi^\circ + e) \quad (2.121)$$

which yield (2.117) taking the limit  $T \rightarrow \infty$  in (2.119).

(2.120) is a consequence of the convergence  $\mathbb{E}[|\nabla\varphi_T - \nabla\varphi|^2] \rightarrow 0$  (see Remark 2.8) which we upgrade to  $E^L$  proceeding as for (2.109).

For (2.121), as  $T \rightarrow \infty$ , we have that  $(\text{Id} - L^{-d} \mathcal{B}_T^\circ \mathcal{K}_{\frac{T}{L^2}}(L))^{-1} \rightarrow (\text{Id} - L^{-d} \mathcal{B}^\circ \mathcal{K}(L))^{-1}$  pointwise in  $E^1$  combining that  $\mathcal{B}_T^\circ \rightarrow \mathcal{B}^\circ$  pointwise in

$L^2(B_1)$  (hence in  $E^1$ ) by Lemma 2.5 and that  $\mathcal{K}_{\frac{T}{L^2}}(L) \rightarrow \mathcal{K}(L)$  pointwise in  $E^1$  by Proposition 2.3. Besides  $\nabla\varphi_T^\circ + e \rightarrow \nabla\varphi^\circ + e$  since  $\nabla\varphi_T^\circ + e = \mathcal{B}_T^\circ e$ . Although only pointwise, the convergence of the operators implies the (pointwise) convergence of the inverse thanks to the uniform bounds (2.40) and (2.112) recalling the following standard result.

**Lemma 2.20** *Let  $H$  be a Hilbert space and  $(L_T)_{T>0}$  be a sequence of bounded operator  $H \rightarrow H$  converging pointwise to an operator  $T : H \rightarrow H$  and let  $(x_T)_{T>0}$  be a sequence in  $H$  converging to  $x \in H$ . If there exists  $\rho > 0$  s.t. for all  $T > 0$ ,  $\|L_T\| \leq \rho$ , then  $L_T x_T \xrightarrow{T \rightarrow \infty} Lx$ .*

*Moreover, if  $\rho < 1$ ,  $(\text{Id} - L_T)^{-1} \xrightarrow{T \rightarrow \infty} (\text{Id} - L)^{-1}$  pointwise.  $\square$*

PROOF Fix  $T > 0$ .

$$\|L_T x_T - Lx\| \leq \|L_T(x_T - x)\| + \|L_T x - Lx\| \leq \rho \|x_T - x\| + \|L_T x - Lx\| \xrightarrow{T \rightarrow \infty} 0$$

The condition  $\rho < 1$  guarantees that the inverses exists and that  $\|(\text{Id} - L_T)^{-1}\| \leq (1 - \rho)^{-1}$ . The pointwise convergence is then a consequence of the previous result and the resolvent identity  $(\text{Id} - L_T)^{-1} - (\text{Id} - L)^{-1} = (\text{Id} - L_T)^{-1}(L - L_T)(\text{Id} - L)^{-1}$ .  $\blacksquare$

### Proof of Lemma 2.19

The proof is essentially a rigorous rewriting of the informal fixed-point approach using the well-defined operator  $K_T^L$ . In the last step, we post-process the operator to eliminate the artificial dependencies on  $L$ .

*Step 1 : Equation for  $\varphi_T(L\mathcal{P}^\circ) - \varphi_T^\circ$*

Fix  $T > 0$ ,  $L \geq 1$  and  $\mathcal{P}^\circ \in \Omega^\circ$ . The difference between the massive corrector and the massive single inclusion solution  $\varphi_T(L\mathcal{P}^\circ) - \varphi_T^\circ$  satisfies the following equation in  $\mathbb{R}^d$

$$\left(\frac{1}{T} - \nabla \cdot a^\circ \nabla\right)(\varphi_T(L\mathcal{P}^\circ) - \varphi_T^\circ) = \nabla \cdot \left[ \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} C^\circ(\cdot - Lx)(\nabla\varphi_T(L\mathcal{P}^\circ) + e) \right]. \quad (2.122)$$

Indeed, since  $a(L\mathcal{P}^\circ) = a^\circ + \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} C^\circ(\cdot - Lx)$ , the (2.28) for  $\varphi_T(L\mathcal{P}^\circ)$  can be rewritten as

$$\frac{1}{T}\varphi_T(L\mathcal{P}^\circ) - \nabla \cdot a^\circ(\nabla\varphi_T(L\mathcal{P}^\circ) + e) = \nabla \cdot \left[ \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} C^\circ(\cdot - Lx)(\nabla\varphi_T(L\mathcal{P}^\circ) + e) \right].$$

This equation is well-posed by Lemma 2.1 noticing that  $\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} C^\circ(\cdot - Lx)(\nabla\varphi_T(L\mathcal{P}^\circ) + e) \in L^2_{\text{uloc}}(\mathbb{R}^d)$  since  $\nabla\varphi_T(L\mathcal{P}^\circ) \in L^2_{\text{uloc}}(\mathbb{R}^d)$ . We obtain (2.122) by subtracting the (2.34) defining  $\varphi_T^\circ$  from the previous equation.

*Step 2 : Operator reformulation*

Using the operator  $\mathcal{B}_T^\circ$  and  $K_T^L$  introduced in Lemmas 2.3 and 2.10, we claim that (2.122) can be rewritten

$$(\text{Id} - L^{-d}\mathcal{B}_T^\circ K_T^L)(\nabla\varphi_T + e)(L\mathcal{P}^\circ) = \nabla\varphi_T^\circ + e. \quad (2.123)$$

We start by solving (2.122) using the massive single inclusion Helmholtz projector (see Definition 2.3) and get

$$\nabla(\varphi_T(L\mathcal{P}^\circ) - \varphi_T^\circ) = \mathcal{H}_T^\circ \left[ \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} C^\circ(\cdot - Lx)(\nabla\varphi_T(L\mathcal{P}^\circ) + e) \right].$$

Then, we combine the rewriting  $\mathcal{H}_T^\circ = \mathcal{B}_T^\circ \mathcal{H}_T$  from Lemma 2.3 with the Green representation of  $\mathcal{H}_T$  from Definition 2.4 to obtain

$$\nabla(\varphi_T(L\mathcal{P}^\circ) - \varphi_T^\circ) = \mathcal{B}_T^\circ \left[ \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^2 G_T * C^\circ(\cdot - Lx)(\nabla\varphi_T(L\mathcal{P}^\circ) + e) \right].$$

By the contravariance property (2.29) of the massive corrector,  $(\nabla\varphi_T + e)(L\mathcal{P}^\circ, \cdot) = (\nabla\varphi_T + e)(L(\mathcal{P}^\circ - x), \cdot - Lx)$ , so that

$$\nabla(\varphi_T(L\mathcal{P}^\circ) - \varphi_T^\circ) = \mathcal{B}_T^\circ \left[ \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^2 G_T(\cdot - Lx) * C^\circ(\nabla\varphi_T + e)(L(\mathcal{P}^\circ - x)) \right].$$

By the homogeneity property (2.47) of  $G_T$ ,  $\nabla^2 G_T(\cdot - Lx) = L^{-d}\nabla^2 G_{\frac{T}{L^2}}(\frac{\cdot}{L} - x)$  and we get

$$\nabla(\varphi_T(L\mathcal{P}^\circ) - \varphi_T^\circ) = L^{-d}\mathcal{B}_T^\circ \left[ \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^2 G_{\frac{T}{L^2}}\left(\frac{\cdot}{L} - x\right) * C^\circ(\nabla\varphi_T + e)(L(\mathcal{P}^\circ - x)) \right]. \quad (2.124)$$

Recognizing the operator  $K_T^L$  of Lemma 2.10, we rewrite this identity as

$$\nabla(\varphi_T(L\mathcal{P}^\circ) - \varphi_T^\circ) = L^{-d}\mathcal{B}_T^\circ K_T^L(\nabla\varphi_T + e)(L\mathcal{P}^\circ)$$



which yields (2.123).

*Step 3 : Banach fixed-point theorem*

We start by noticing that  $\nabla\varphi_T + e \in E^L$  using the crude bound (2.30) and the triangle inequality.

Choosing  $L \gg 1$ , we claim that  $L^{-d}\mathcal{B}_T^\circ K_T^L$  is a contraction in the Banach space  $E^L$ . This allows us to invert the operator in (2.123) and yields  $\mathbb{P}^\circ$ -a.s

$$\nabla\varphi_T(L\mathcal{P}^\circ) = (\text{Id} - L^{-d}\mathcal{B}_T^\circ K_T^L)^{-1}(\nabla\varphi_T^\circ + e)(L\mathcal{P}^\circ) . \quad (2.125)$$

We now prove that we indeed have a contraction, i.e.  $\|L^{-d}\mathcal{B}_T^\circ K_T^L\|_L < 1$ . Using our extension from  $(L^2(B_1))^d$  to  $E^L$ , the uniform bound (2.40) on  $\mathcal{B}_T^\circ$  can (trivially) be rewritten in the form

$$\sup_{L \geq 1} \|\mathcal{B}_T^\circ\|_L \lesssim 1 .$$

We combine it with the improved bound (2.81) on  $K_T^L$  from (2.1) to the effect that

$$\|L^{-d}\mathcal{B}_T^\circ K_T^L\|_L \lesssim L^{-d} < 1$$

by the assumption  $L \gg 1$ .

*Step 4 : Post-processing*

We claim that we can replace the operator  $K_T^L$  by the operator  $\mathcal{K}_{\frac{T}{L^2}}(L)$  (see Lemma 2.12) in (2.125) which yields (2.117). The key here is that  $K_T^L$  acts on  $\nabla\varphi_T^\circ + e$  which is a deterministic function.

Indeed, fix  $L \geq 1$ ,  $g \in (L^2(B_1))^d$  and  $\mathcal{P}^\circ \in \Omega^\circ$ . We claim that

$$\left[ (\text{Id} - L^{-d}\mathcal{B}_T^\circ K_T^L)^{-1} g \right] (L\mathcal{P}^\circ) = \left[ \left( \text{Id} - L^{-d}\mathcal{B}_T^\circ \mathcal{K}_{\frac{T}{L^2}}(L) \right)^{-1} g \right] (\mathcal{P}^\circ) \quad (2.126)$$

which yields (2.117) taking  $g = \nabla\varphi_T^\circ + e \in (L^2(B_1))^d$ .

First,  $\left( \text{Id} - L^{-d}\mathcal{B}_T^\circ \mathcal{K}_{\frac{T}{L^2}}(L) \right)^{-1}$  is well-defined in  $E^1$  thanks to the uniform bound (2.112) proceeding as in Step 3.

Then, similarly to the (2.80) relating  $K_T^L$  to  $\mathcal{K}_{\frac{T}{L^2}}$ , we notice that the operators are constructed in such a way that for all  $k > 0$ ,

$$(K_T^L)^k g(L\mathcal{P}^\circ) = \left( \mathcal{K}_{\frac{T}{L^2}}(L) \right)^k g(\mathcal{P}^\circ) .$$

We even have

$$(\mathcal{B}_T^\circ K_T^L)^k g(L\mathcal{P}^\circ) = \left( \mathcal{B}_T^\circ \mathcal{K}_{\frac{T}{L^2}}(L) \right)^k g(\mathcal{P}^\circ) \quad (2.127)$$

since the operator  $\mathcal{B}_T^\circ$  does not act on the point processes. Note that in this equality,  $\mathcal{B}_T^\circ$  is seen as an operator acting on  $E^L$  in the LHS and acting on  $E^1$  in the RHS.

Finally, (2.127) implies (2.126) using Neumann series (respectively in  $E^L$  and  $E^1$ ) in the form of

$$(\text{Id} - L^{-d} \mathcal{B}_T^\circ K_T^L)^{-1} = \sum_{k=0}^{\infty} (L^{-d} \mathcal{B}_T^\circ K_T^L)^k .$$

As in Step 3, these expansions are justified by the assumption  $L \gg 1$  which guarantees smallness of the operators in the appropriate norms.

### 2.4.3 Proof of Proposition 2.3 : analyticity of the fixed-point operator

#### Structure of proof

Informally, the expansion (2.113) of  $\mathcal{K}(L)$  is obtained by first Taylor expanding the massive Green function in  $L^{-1}$  in the definition (2.78) of  $\mathcal{K}_\tau(L)$  and then passing to the limit  $\tau \rightarrow \infty$ . The key point here is the analyticity statement which is captured in the bounds (2.114) controlling the growth of  $i \mapsto \mathcal{K}^{(i)}$ .

**Remark 2.13** To keep track of this growth, in the sequel (when it is necessary), we drop the notation  $A \lesssim B$  in favor of  $A \leq C \times B$ . In the proofs, the constant  $C$  is generic and may change from one line to another but remains independent of the parameters of interest (notably  $\tau$ ,  $L$  and its exponent, specific dependence will be mentioned).  $\square$

We start by establishing a Taylor expansion of the map  $L^{-1} \mapsto \mathcal{K}_\tau(L)$  up to any order with controlled growth of the coefficients and the remainder term.

**Lemma 2.21** *Let  $\tau > 0$ . For all  $n \geq 0$  and for all  $L \geq 1$ , we have*

$$\mathcal{K}_\tau(L) = \sum_{i=0}^n \mathcal{K}_\tau^{(i)} L^{-i} + R_\tau^{n+1}(L) L^{-(n+1)} \quad (2.128)$$

where, for all  $f \in E^1$ , the family  $(\mathcal{K}_\tau^{(i)})_{i \geq 0}$  of bounded operators  $E^1 \rightarrow E^1$  are given by

$$\mathcal{K}_\tau^{(i)} f(\mathcal{P}^\circ) = \frac{1}{i!} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^{2+i} G_\tau(x) \cdot \int_{B_1} (y - \cdot)^{\otimes i} C^\circ(y) f(\mathcal{P}^\circ - x, y) dy , \quad (2.129)$$

and the remainder operator  $R_\tau^{n+1}(L) : E^1 \rightarrow E^1$  is given by

$$R_\tau^{n+1}(L)f(\mathcal{P}^\circ) = \frac{1}{n!} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \int_{B_1} \int_0^1 \nabla^{3+n} G_\tau \left( x + t \frac{y - \cdot}{L} \right) \cdot (y - \cdot)^{\otimes n+1} (1-t)^n C^\circ(y) f(\mathcal{P}^\circ - x, y) dt dy . \quad (2.130)$$

In addition, there exists a constant  $C > 0$  (independent of  $\tau$  and  $L$ ) such that for all  $i \geq 1$

$$\| \| \mathcal{K}_\tau^{(i)} \| \|_1 \leq C^i \quad (2.131)$$

and for all  $n \geq 0$  and  $L \geq 1$ ,

$$\| \| R_\tau^{n+1}(L) \| \|_1 \leq C^{n+1} . \quad (2.132)$$

□

Fix  $n \geq 0$ ,  $L \geq 1$ . We want to pass to the limit  $\tau \rightarrow \infty$  in the expansion (2.128). For the RHS, we start with the higher order terms.

**Lemma 2.22** *For all  $i \geq 1$ , the operator  $\mathcal{K}_\tau^{(i)} : E^1 \rightarrow E^1$  (defined in (2.129)) converges in the operator norm  $\| \| \cdot \| \|_1$  toward the operator  $\mathcal{K}^{(i)} : E^1 \rightarrow E^1$  (see (2.115)).*

*For all  $n \geq 0$  and for all  $L \geq 1$ , the operator  $R_\tau^{n+1}(L) : E^1 \rightarrow E^1$  (defined in (2.130)) converges in the operator norm  $\| \| \cdot \| \|_1$  toward the operator  $R^{n+1}(L) : E^1 \rightarrow E^1$  defined for  $f \in E^1$  by*

$$R^{n+1}(L)f(\mathcal{P}^\circ) = \frac{1}{n!} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \int_{B_1} \int_0^1 \nabla^{3+n} G \left( x + t \frac{y - \cdot}{L} \right) \cdot (y - \cdot)^{\otimes n+1} (1-t)^n C^\circ(y) f(\mathcal{P}^\circ - x, y) dt dy . \quad (2.133)$$

□

The 0<sup>th</sup>-order terms needs a special treatment. A priori, combining the bound (2.79) on  $\mathcal{K}_\tau(L)$  with the bounds (2.131) and (2.132) from Lemma 2.21, we only have  $\| \| \mathcal{K}_\tau^{(0)} \| \| \lesssim \ln(2 + \tau)$ . This logarithmic divergence can be avoided using the improved bound on  $K_\tau^L$  from Proposition 2.1.

**Lemma 2.23** *The operator  $\mathcal{K}_\tau^{(0)} : E^1 \rightarrow E^1$  satisfies the bound*

$$\| \| \mathcal{K}_\tau^{(0)} \| \| \lesssim 1 . \quad (2.134)$$

*Moreover, it converges pointwise as  $\tau \rightarrow \infty$  and we define the linear operator  $\mathcal{K}^{(0)} : E^1 \rightarrow E^1$  as its limit.*

□

Note that  $\mathcal{K}_\tau^{(0)}$  only converges pointwise contrary to the  $(\mathcal{K}_\tau^{(i)})_{i \geq 0}$  which converge in norm. In both cases, the bounds (2.114) are inherited from the bounds (2.131) and (2.134) (using Banach-Steinhaus theorem in the pointwise convergence case).

For the convergence of the operator  $\mathcal{K}_\tau$  itself, we use Lemma 2.21 with  $n = 0$  to write for all  $\tau > 0$

$$\mathcal{K}_\tau(L) = \mathcal{K}_\tau^{(0)} + R_\tau^1(L) .$$

This formula implies the uniform bound (2.112) combining the bounds (2.134) and (2.132) on  $\mathcal{K}_\tau^{(0)}$  and  $R_\tau^1(L)$ . It also yields the pointwise convergence  $\mathcal{K}_\tau(L) \xrightarrow{\tau \rightarrow \infty} \mathcal{K}(L) := \mathcal{K}^{(0)} + R^1(L)$  combining Lemma 2.22 with Lemma 2.23.

Sending  $\tau \rightarrow \infty$  in (2.128), we have obtained that

$$\mathcal{K}(L) = \sum_{i=0}^n \mathcal{K}^{(i)} L^{-i} + R^{n+1}(L) L^{-(n+1)} . \quad (2.135)$$

Using Lemma 2.22,  $R^{n+1}$  inherits the bound (2.132) and there exists  $C > 0$  s.t.  $\|R^{n+1}(L)\| \lesssim C^{n+1}$ . We then chose  $L$  s.t.  $L^{-1}C < 1$  and send  $n \rightarrow \infty$  in (2.135) so that  $\|R^{n+1}(L)\| L^{-(n+1)} \rightarrow 0$ . This yields the series expansion (2.113) and concludes the proof of Proposition 2.3.

We now turn to the proofs of the auxiliary results Lemmas 2.22 and 2.23. The proof Lemma 2.21 is postponed as it requires another technical ingredient.

PROOF (LEMMA 2.22) We only provide the details for the convergence of the  $(\mathcal{K}_\tau^{(i)})_{i \geq 1}$ . The proof of the convergence of the remainder operators  $(R_\tau^{n+1}(L))_{n \geq 0}$  is similar.

Fix  $i > 0$ . We claim that

$$\|\mathcal{K}_\tau^{(i)} - \mathcal{K}^{(i)}\|_1 \lesssim_i \int_{\mathbb{R}^d \setminus B_2} \|\nabla^{2+i}(G_\tau - G)\|_{B_1(x)} dx . \quad (2.136)$$

Indeed, we fix  $f \in E^1$  and  $\mathcal{P}^\circ \in \Omega^\circ$ . From the definitions (2.129) of  $\mathcal{K}_\tau^{(i)}$  and (2.115) of  $\mathcal{K}^{(i)}$  and since  $|(z - y)^{\otimes i}| \lesssim_i 1$  for all  $z, y \in B_1$ , we have for all  $\mathcal{P}^\circ \in \Omega^\circ$

$$\|(\mathcal{K}_\tau^{(i)} - \mathcal{K}^{(i)})f(\mathcal{P}^\circ)\|_{L^\infty(B_1)} \lesssim_i \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |\nabla^{2+i}(G_\tau - G)(x)| \int_{B_1} |C^\circ f(\mathcal{P}^\circ - x)| . \quad (2.137)$$

(2.136) then follows from applying Lemma 2.11 to the RHS with  $\omega = |\nabla^{2+i}(G_\tau - G)(\cdot)|$ ,  $g(\mathcal{P}^\circ) = \int_{B_1} |C^\circ f(\mathcal{P}^\circ - x)|$  (so that  $\mathbb{E}^\circ [g^2] \lesssim \|f\|_{E^1}^2$ ) and  $\varepsilon = 1$  and from

noticing that

$$\sup_{\mathcal{P}^\circ \in \Omega^\circ} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |\nabla^{2+i}(G_\tau - G)(x)| \lesssim_i 1. \quad (2.138)$$

For (2.138), we note that for all  $x \neq 0$

$$|\nabla^{2+i}G_\tau(x)| + |\nabla^{2+i}G(x)| \lesssim_i |x|^{-(d+i)} \quad (2.139)$$

from the bound (2.48) on the massive Green function and its counterpart for the standard Green function. Summing (2.139), we obtain for all  $\mathcal{P}^\circ \in \Omega^\circ$

$$\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |\nabla^{2+i}(G_\tau - G)(x)| \lesssim_i \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-(d+i)}$$

which concludes the proof of (2.138) since, for all  $\mathcal{P}^\circ \in \Omega^\circ$ , we have

$$\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-(d+i)} \lesssim_i 1.$$

To show this inequality, we proceed as we did for (2.74). We decompose the sum over the annuli  $C_k = B_{k+1} \setminus B_k$  and use that  $|\cdot|^{-(d+i)}$  is a radially decaying function to write

$$\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-(d+i)} \lesssim \sum_{k=1}^{\infty} \#(\mathcal{P}^\circ \cap C_k) k^{-(d+i)}. \quad (2.140)$$

Recalling that  $\#(\mathcal{P}^\circ \cap C_k) \lesssim k^{d-1}$  by Lemma 2.8, we get

$$\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-(d+i)} \lesssim \sum_{k=1}^{\infty} k^{-(1+i)} \lesssim_i 1$$

since  $i > 0$ .

We conclude the proof of this lemma applying Lebesgue's dominated convergence theorem to the RHS of (2.136). To do so, we check that we have, for all  $x \in \mathbb{R}^d \setminus B_2$ , the convergence  $\|\nabla^{2+i}(G_\tau - G)\|_{B_1(x)} \xrightarrow{\tau \rightarrow \infty} 0$  by Lemma 2.6 and the integrable bound  $\|\nabla^{2+i}(G_\tau - G)\|_{B_1(x)} \lesssim_i |x|^{-(d+i)}$  which is an upgrade of the pointwise bound (2.139) proceeding as we did for (2.76).  $\blacksquare$

PROOF (LEMMA 2.23) On the one hand, the Taylor expansion (2.128) implies that for all  $L \geq 1$   $\mathcal{K}_\tau^{(0)} = K_\tau(L) - R_\tau^1(L)$ . On the other hand, we note that  $\mathcal{K}_\tau(1) = K_\tau^1$ . Combining both, we obtain

$$\mathcal{K}_\tau^{(0)} = K_\tau^1 - R_\tau^1(1). \quad (2.141)$$

This identity implies the bound (2.134) combining the improved bound (2.81) on  $K_\tau^1$  and the bound (2.132) on the remainder.

It also implies the pointwise convergence of  $\mathcal{K}_\tau^{(0)}$  as  $\tau \rightarrow \infty$  combining the pointwise convergence of  $K_\tau^1$  from Proposition 2.2 with the convergence in operator norm of  $R^1$  from Lemma 2.22.  $\blacksquare$

### Growth of the derivatives

In Lemma 2.21, to establish the bounds (2.131) and (2.132) on the  $(\mathcal{K}_\tau^{(i)})_{i \geq 1}$  and  $(R_\tau^{n+1}(L))_{n \geq 0}$ , it is necessary to control the growth of function of the form  $i \mapsto \sup_{\mathcal{P}^\circ \in \Omega^\circ} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |\nabla^{2+i} G_\tau|(x)$ .

If the well known bound of Lemma 2.24 on  $\nabla^{2+i} G_\tau$  ensures summability, it does not keep track of the growth of the constants in  $i$ . We make it explicit in the following lemma which controls the growth of the derivatives of the massive Green function while ensuring summability (far from the origin) uniformly in the massive parameter.

**Lemma 2.24** *Let  $T > 0$  and  $G_T$  be the massive Green function defined in Definition 2.4. Let  $\lambda > 0$ . There exists  $C(d, \lambda) > 0$  (depending only on the dimension and the margin parameter  $\lambda$ ) such that for all  $i \geq 1$ , for all  $r > 0$  and for all  $z \in \mathbb{R}^d$  s.t.  $|z| \geq (1 + \lambda)r$ , we have*

$$\|\nabla^{2+i} G_T\|_{L^\infty(B_r(z))} \leq C \left( \frac{C(i-1)}{r} \right)^{i-1} |z|^{-(d+1)}. \quad (2.142)$$

For  $i \geq 2$ , the RHS of (2.142) reduces to  $\left( \frac{C(i-1)}{r} \right)^{i-1} |z|^{-(d+1)}$  (up to increasing  $C$ ).

Note that the sole purpose of the margin parameter  $\lambda$  is to quantify the condition  $|z| > r$  (which guaranties that  $0 \notin B_r(z)$ ).  $\square$

The proof relies on the following iterated version of the Caccioppoli inequality where we have kept track of the growth of the constant.

**Lemma 2.25** *There exists  $C(d) > 0$  such that for all  $0 < \rho < R$ ,  $T > 0$ ,  $u \in H^1(B_R)$  massive harmonic (i.e. weakly solving  $\frac{1}{T}u - \Delta u = 0$ ) in  $B_R$  and  $n > 0$ , we have*

$$\|\nabla^n u\|_{L^2(B_\rho)} \leq \left( \frac{Cn}{R - \rho} \right)^n \|u\|_{L^2(B_R)}. \quad (2.143)$$

$\square$

PROOF Fix  $\rho, R, T$  and  $u$  as above. There exists  $C(d) > 0$  (independent of  $T$ ) such that

$$\|\nabla u\|_{L^2(B_\rho)} \leq \frac{C}{R-\rho} \|u\|_{L^2(B_R)}. \quad (2.144)$$

Indeed, take  $\eta$  a smooth cut-off from  $B_\rho$  to  $B_R$  such that  $|\nabla \eta| \lesssim \frac{1}{R-\rho}$ . Testing  $\eta^2 u$  in the equation for  $u$  and using Young inequality, we get

$$\frac{1}{T} \int u^2 \eta^2 + \int |\nabla u|^2 \eta^2 \lesssim \int |\nabla \eta|^2 u^2$$

which yields (2.144).

We obtain (2.143) by iterating (2.144) : for  $n > 0$ , define  $r_0 = \rho < r_1 < \dots < r_n = R$ . For each  $k \in \llbracket 0, n \rrbracket$ , we recursively apply (2.144) to  $\nabla^{n-k} u$  (which still verifies  $(\frac{1}{T} - \Delta) \nabla^{n-k} u = 0$  in  $B_R$ ) between  $r_k$  and  $r_{k+1}$  and obtain

$$\|\nabla^n u\|_{L^2(B_\rho)} \leq C^n \prod_{k=0}^{n-1} (r_{k+1} - r_k)^{-1} \|u\|_{L^2(B_R)}.$$

Maximizing  $\prod_{k=0}^{n-1} (r_{k+1} - r_k)$  under the constraint  $\sum_{k=0}^{n-1} (r_{k+1} - r_k) = R - \rho$  yields the optimal choice  $r_{k+1} - r_k = \frac{R-\rho}{n}$  for all  $k$ . Injecting this in the previous inequality yields (2.143) and concludes the proof.  $\blacksquare$

PROOF (LEMMA 2.24) Fix  $r$  and  $z$  as above.

For  $i = 1$ , we claim that

$$\|\nabla^3 G_T\|_{L^\infty(B_{(1+\frac{\lambda}{3})r}(z))} \leq C |z|^{-(d+1)} \quad (2.145)$$

which is a strengthening of (2.142). To prove (2.145), we use that  $|\nabla^3 G_T| \leq C |\cdot|^{-(d+1)}$  uniformly in  $T$  from the bound (2.48) on the massive Green function. Then, we remark that for  $x \in B_{(1+\frac{\lambda}{3})r}(z)$  and  $|z| \geq (1+\lambda)r$ ,  $|x| = |z - (z-x)| \geq |z| - (1+\frac{\lambda}{3})r \geq \frac{2\lambda}{3(1+\lambda)}|z|$ . Since  $|\cdot|^{-(d+1)}$  is a radially decaying function, this yields (2.145).

Fix now  $i \geq 2$ . The Sobolev embedding allows us to use  $L^2$ -based norms controlling the  $L^\infty$ -norm by the  $H^{k_0}$ -norm for some  $k_0 > \lceil \frac{d}{2} \rceil$ . By scaling (considering  $G_T(r(\cdot - z))$  on  $B_1$ ) we can make explicit the dependence of the Sobolev constant on  $r$  and write

$$\|\nabla^{2+i} G_T\|_{L^\infty(B_r(z))} \leq C r^{-\frac{d}{2}} \sum_{n=0}^{k_0} r^n \|\nabla^{2+i+n} G_T\|_{L^2(B_r(z))}. \quad (2.146)$$

To estimate this norm, we remark that  $r$  and  $z$  are chosen such that  $\nabla^{2+i} G_T$

is massive harmonic in  $B_{(1+\frac{\lambda}{2}r)}(z)$ . Applying Lemma 2.25 between  $r$  and  $(1+\frac{\lambda}{6})r$  for  $n = 1, \dots, k_0$  gives

$$\begin{aligned} \|\nabla^{2+i+n}G_T\|_{L^2(B_r(z))} &\leq \left(\frac{Cn}{r}\right)^n \|\nabla^{2+i}G_T\|_{L^2(B_{(1+\frac{\lambda}{6})r}(z))} \\ &\leq Cr^{-n} \|\nabla^{2+i}G_T\|_{L^2(B_{(1+\frac{\lambda}{6})r}(z))} . \end{aligned}$$

Plugging these bounds in (2.146) yields

$$\|\nabla^{2+i}G_T\|_{L^\infty(B_r(z))} \leq Cr^{-\frac{d}{2}} \|\nabla^{2+i}G_T\|_{L^2(B_{(1+\frac{\lambda}{6})r}(z))} .$$

Using again Lemma 2.25 on  $\nabla^3G_T$  between  $(1+\frac{\lambda}{6})r$  and  $(1+\frac{\lambda}{3})r$  gives

$$\|\nabla^{2+i}G_T\|_{L^2(B_{(1+\frac{\lambda}{6})r}(z))} \leq \left(\frac{C(i-1)}{r}\right)^{i-1} \|\nabla^3G_T\|_{L^2(B_{(1+\frac{\lambda}{3})r}(z))} .$$

We conclude the proof of (2.142) noting that

$$r^{-\frac{d}{2}} \|\nabla^3G_T\|_{L^2(B_{(1+\frac{\lambda}{3})r}(z))} \leq C \|\nabla^3G_T\|_{L^\infty(B_{(1+\frac{\lambda}{3})r}(z))} \leq C|z|^{-(d+1)}$$

by the triangle inequality and the estimate (2.145) on  $\nabla^3G_T$ . ■

With Lemma 2.24 at hand, we now have all the ingredients to prove Lemma 2.21.

PROOF (LEMMA 2.21) Recalling the definition (2.78) of  $\mathcal{K}_\tau(L)$  and using that  $C^\circ$  is supported in  $B_1$ , for all  $\mathcal{P}^\circ \in \Omega^\circ$ ,  $L \geq 1$  and  $f \in E^1$ , we have

$$\mathcal{K}_\tau(L)f(\mathcal{P}^\circ) = \int_{B_1} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \nabla^2 G_\tau \left( x - \frac{\cdot - y}{L} \right) C^\circ(y) f(\mathcal{P}^\circ - x, y) dy . \quad (2.147)$$

From Definition 2.4,  $G_\tau$  is  $C^\infty$  outside the origin and for all  $n \geq 0$ , the expansion (2.128) ensue from Taylor formula (with integral remainder).

The proof of the bounds (2.131) and (2.132) follows the same structure as the proof of the bound (2.65) from Lemma 2.10 where  $\frac{\tau}{L^2}$  played the role of  $\tau$  here. These bounds also justify that the operators are well-defined. We provide the details for the bound (2.132) on the remainder operator only. The bounds (2.131) on  $\left(\mathcal{K}_\tau^{(i)}\right)_{i \geq 0}$  follow similarly (with  $2+i$ ,  $i \geq 1$  instead of  $3+n$ ,  $n \geq 0$ ).

Fix  $n \geq 0$ ,  $L \geq 1$ ,  $\tau > 0$ ,  $\mathcal{P}^\circ \in \Omega^\circ$  and  $f \in E^1$ .

From the definition (2.130) of  $R_\tau^{n+1}(L)f$ , we apply the triangular inequality then note that  $t \frac{|z-y|}{L} \leq 2$  for all  $z, y \in B_1$  and  $t \in [0, 1]$  and finally use that



$\int_0^1 (1-t)^n dt = \frac{1}{n+1}$  to the effect that

$$\|R_\tau^{n+1}(L)f(\mathcal{P}^\circ)\|_{L^\infty(B_1)} \leq \frac{C^{(n+1)}}{(n+1)!} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \|\nabla^{3+n} G_\tau\|_{L^\infty(B_2(x))} \int_{B_1} |C^\circ f(\mathcal{P}^\circ - x)|.$$

Then, we claim that

$$\sup_{\mathcal{P}^\circ \in \Omega^\circ} \left( \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \|\nabla^{3+n} G_\tau\|_{L^\infty(B_2(x))} \right) \leq (C(n+1))^{n+1} \quad (2.148)$$

and that

$$\int_{\mathbb{R}^d \setminus B_{3-\varepsilon}} \|\nabla^{3+n} G_\tau\|_{L^\infty(B_{2+\varepsilon}(x))} dx \leq (C(n+1))^{n+1}. \quad (2.149)$$

Using Lemma 2.11 with  $\omega = \|\nabla^{3+n} G_\tau\|_{L^\infty(B_2(\cdot))}$ ,  $g(\mathcal{P}^\circ) := \int_{B_1} |C^\circ f(L\mathcal{P}^\circ)|$  (so that  $\mathbb{E}^\circ [g^2] \lesssim \|f\|_{E^1}^2$ ) and  $\varepsilon > 0$  s.t.  $3 - \varepsilon > 2 + \varepsilon$  ( $\varepsilon = \frac{1}{4}$  for instance), (2.148) and (2.149) implies that  $\|R_\tau^{n+1} f\|_{E^1} \leq \frac{C^{(n+1)}}{(n+1)!} \|f\|_{E^1}$ . This yields (2.132) using Stirling formula in the form  $\frac{n^n}{n!} \leq C^n$ .

Lemma 2.24 with  $r = 2 + \varepsilon$  and  $\lambda > 0$  s.t.  $(1 + \lambda)(2 + \varepsilon) = 3 - \varepsilon$ , implies that for all  $|x| > 3 - \varepsilon$ ,

$$\|\nabla^{3+n} G_\tau\|_{L^\infty(B_{2+\varepsilon}(x))} \leq (C(n+1))^{n+1} |x|^{-(d+1)}. \quad (2.150)$$

For (2.148), we sum (2.150) for  $x \in \mathcal{P}^\circ \setminus \{0\}$  (implying  $|x| \geq 3$  by the hard-core property) and obtain

$$\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} \|\nabla^{3+n} G_\tau(x)\|_{L^\infty(B_2(x))} \leq (C(n+1))^{n+1} \sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-(d+1)}.$$

As we did for (2.74), we decompose this last sum over the annuli  $C_k = B_{k+1} \setminus B_k$  and use that  $|\cdot|^{-(d+1)}$  is a radially decaying function to write

$$\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-(d+1)} \lesssim \sum_{k=1}^{\infty} \#(\mathcal{P}^\circ \cap C_k) k^{-(d+1)}. \quad (2.151)$$

Recalling that  $\#(\mathcal{P}^\circ \cap C_k) \lesssim k^{d-1}$  by Lemma 2.8, we get

$$\sum_{x \in \mathcal{P}^\circ \setminus \{0\}} |x|^{-(d+1)} \lesssim \sum_{k=1}^{\infty} k^{-2} \lesssim 1$$

which concludes the proof of (2.148).

(2.149) directly follows from (2.150) integrating over  $\mathbb{R}^d \setminus B_{3-\varepsilon}$  since  $\int_{\mathbb{R}^d \setminus B_1} |x|^{-(d+1)} \lesssim 1$ . ■

# Chapter 3

## Mean sedimentation speed of a random suspension : First order dilute expansion

The results in this chapter come from an ongoing work with Mitia DUERINCKX and Antoine GLORIA. In [DG22b], these author showed that one can define a notion of mean sedimentation speed  $\bar{V}$  for particles in a viscous fluid, in the spirit of stochastic homogenization theory. Using the cluster expansion, we provide a first order dilute expansion of this sedimentation speed  $\bar{V} \simeq \bar{V}^{(1)}$  (where  $\bar{V}^{(1)}$  corresponds to the settling speed of a single isolated particle) with a quantitative control of the error in terms of so-called multi-points intensities, an intrinsic notion of dilution introduced in [DG20c]. This first order expansion is not sufficient to justify the (second order) Batchelor formula  $\bar{V} \simeq \bar{V}^{(1)}(1 - 6.55\varphi)$  (where  $\varphi$  is the volume fraction of particles) but we believe that our strategy can be extended to higher orders.

### Outline of the current chapter

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<b>3.1 Introduction</b>	<b>106</b>
3.1.1 Model and averaged settling speed . . . . .	106
3.1.2 Main result . . . . .	109
3.1.3 General strategy . . . . .	111
3.1.4 Relation to other works . . . . .	115
3.1.5 Extensions and future works . . . . .	116
<b>3.2 Preliminaries</b>	<b>117</b>
3.2.1 Notations . . . . .	117
3.2.2 Point processes . . . . .	118

3.2.3	Finite volume approximation . . . . .	124
3.2.4	Trace estimates . . . . .	135
<b>3.3</b>	<b>1<sup>st</sup> order expansion : proof of Theorem 3.1</b>	<b>136</b>
3.3.1	Strategy . . . . .	136
3.3.2	Control of the remainder : proof of Proposition 3.2 .	138
3.3.3	Proof of the renormalization lemmas : Lemmas 3.10 to 3.12 . . . . .	145
<b>3.4</b>	<b>2<sup>nd</sup> order expansion : strategy, issues and how to over- come them</b>	<b>160</b>
3.4.1	Expected results . . . . .	160
3.4.2	Difficulties . . . . .	162

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## 3.1 Introduction

### 3.1.1 Model and averaged settling speed

Consider a large number of identical spherical rigid particles in a tank filled with a viscous fluid at rest. If the particles are denser than the fluid, they will start to sink under their effective weight (gravity force minus buoyancy). We refer to this phenomenon as sedimentation. In this paper, we are interested in the bulk behavior of sedimentation. We hence take an infinite tank filled with a homogeneous, viscous and incompressible fluid. Assuming that the particles are small enough, we can neglect their inertia and the motion of the fluid can be modeled by the Stokes equation. Moreover, we postulate that the particles reach a stationary state so that we can statistically describe their positions. Following the physical intuition, this statistic should be without origin (we will say stationary in the sequel) and there should be disorder in the form of large scale decorrelation (which rigorously corresponds to ergodicity).

Mathematically, we then represent the particles by the set  $\mathcal{I} := \cup_{x \in \mathcal{P}} B(x)$  where  $\mathcal{P}$  is a random point process. For that purpose, we define the space

$$\Omega := \{\mathcal{P} \in (\mathbb{R}^d)^{\mathbb{N}}, \forall x, y \in \mathcal{P}, x \neq y, |x - y| \geq 3\} \quad (3.1)$$

of hard-core point processes  $\mathcal{P}$  that we equip, following [Chi+13, Section 4.1.1 p.108], with the  $\sigma$ -algebra

$$\mathcal{F} := \sigma(\mathcal{P} \mapsto \#(\mathcal{P} \cap B), B \subset \mathbb{R}^d \text{ bounded Borel set})$$

(which is countably generated since we can limit ourselves to rational rectangles) and a probability measure  $\mathbb{P}$  that we assume

- *stationary* : for all  $z \in \mathbb{R}^d$ ,  $\tau_z \mathcal{P} \sim_{\text{law}} \mathcal{P}$  where the shift  $\tau_z$  are defined by  $\tau_z \mathcal{P} := (x + z)_{x \in \mathcal{P}}$ ,
- *ergodic* : for any  $F \in \mathcal{F}$ ,  $(F = F + z \text{ for all } z \in \mathbb{R}^d) \implies \mathbb{P}[F] \in \{0, 1\}$ . By Birkhoff theorem, this property implies that  $\mathbb{E}[f(\mathcal{P})] = \lim_{R \rightarrow \infty} \int_{B_R} f(\mathcal{P} + z) dz$  for any  $f \in L^1(\Omega, \mathbb{P})$  (i.e. we can recover expectations from spatial averages over large domain).

Note that we have chosen a minimal distance of 3 between the points so that there is a minimal distance of 1 between the particles (but any positive number would do).

We assume that  $\mathcal{P}$  has intensity  $\lambda$  (i.e. average amount of points per unit of volume) so that the volume fraction of particles writes  $\varphi := \lambda|B|$ . Setting the direction of gravity  $e \in \mathbb{R}^d$ , the velocity of the fluid  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and its associated pressure  $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}$  solve the following equation

$$\left\{ \begin{array}{ll} -\nabla \cdot \sigma(\phi, \Pi) = -\alpha e & \text{in } \mathbb{R}^d \setminus \mathcal{I} \\ \nabla \cdot \phi = 0 & \text{in } \mathbb{R}^d \setminus \mathcal{I} \\ D(\phi) = 0 & \text{in } \mathcal{I} \\ \Pi = 0 & \text{in } \mathcal{I} \\ e|B| + \int_{\partial B(x)} \sigma(\phi, \Pi) \nu = 0 & \forall x \in \mathcal{P} \\ \int_{\partial B(x)} \Theta \nu \cdot \sigma(\phi, \Pi) \nu = 0 & \forall \Theta \in \mathbb{R}_{\text{skew}}^{d \times d}, \forall x \in \mathcal{P} \end{array} \right. \quad (3.2)$$

where, setting the symmetric gradient  $D(u) := \frac{1}{2}(\nabla u + \nabla u^T)$ ,  $\sigma(u, p) := 2D(u) - p \text{Id}$  is the Newtonian stress tensor and we have set the *back-flow* term  $\alpha := \frac{\varphi}{1-\varphi}$ . We have extended the pressure  $\Pi$  by zero inside the particles. For the velocity  $\phi$ , we assume that it coincides with the rigid motion of the particles on their boundaries (from the standard no-slip boundary condition). Hence, for all  $x \in \mathcal{P}$ , we impose  $D(\phi) = 0$  in  $B(x)$  or equivalently that there exist  $V_x \in \mathbb{R}^d$ ,  $\Theta_x \in \mathbb{R}_{\text{skew}}^{d \times d}$  s.t.  $\phi = V_x + \Theta_x(\cdot - x)$  in  $B(x)$ . The boundary conditions on each particle correspond to the equilibrium of forces (effective weight and stress exerted by the fluid) and moments (written in a form which is valid in any dimension ; in the case  $d = 3$ , for any  $\Theta \in \mathbb{R}_{\text{skew}}^{3 \times 3}$ , there exists  $\omega \in \mathbb{R}^3$  s.t.  $\Theta y = \omega \wedge y$ ,  $\forall y \in \mathbb{R}^3$  and we recover the standard definition of the torque exerted by the fluid).

The back-flow term  $\alpha$ , which could be surprising at first glance, can be seen as an averaged multiparticle effect. As it is constant, it could formally be absorbed in the pressure but it is crucial for solvability (see Remark 3.7 for a motivation when considering a finite volume).

**Remark 3.1** Our model (3.2) could also be interpreted as the snapshot of a dynamical one (or its time discretization).

Here, the data of the positions  $x$  of the particles' centers fully determines the velocity field  $\phi$  and in particular the instantaneous speeds of the particle  $V_x$ .

Given an initial particles' configuration, we thus could define a (deterministic) dynamic setting  $\dot{x} = V_x$  for each particle  $x$ . A natural choice for  $\mathbb{P}$  would then be the invariant (or equilibrium) measure for this dynamic should it exist or some suitable transient statistic.  $\square$

In our statistical setting, one could naturally want to define a notion of averaged settling speed  $\bar{V}$ . This has been a challenge in the physics literature of the previous century ([Smo27; Bur41; Bur42]). Indeed, a single particle in a Stokes fluid generates a flow which decays as  $O(r^{2-d})$  at distance  $r$ . This long-range effect is not integrable over the whole space and therefore, any naive summation of the particles' contributions would diverge. In [Bat72], Batchelor first found how to deal with this divergence through a suitable renormalization (using the back-flow and random cancellations) allowing him to define a mean settling speed. These ideas (and much more) have been mathematically formalized by Duerinckx and Gloria in [DG22b, Theorem 1] from which we extract the following result.

**Proposition 3.1 ([DG22b])** *Let  $e \in \mathbb{R}^d$  s.t.  $|e| = 1$ ,  $d \geq 3$  and  $\mathcal{P}$  be a stationary ergodic point process which is  $\gamma$ -algebraically mixing to order 2 (see Definition 3.4 below) with  $\gamma > 2$  and which could be periodically approximated to order 2 (see Definition 3.5 below).*

*There exists a unique solution  $(\phi, \Pi) \in L^2(\Omega, (H_{loc}^1(\mathbb{R}^d))^d) \times L^2(\Omega, L_{loc}^2(\mathbb{R}^d))$  of (3.2) s.t. the gradient field  $\nabla\phi$  and the pressure  $\Pi$  are stationary, they have vanishing expectation  $\mathbb{E}[(\nabla\phi, \Pi)] = 0$ , they have finite second moments  $\mathbb{E}[|\nabla\phi|^2 + \Pi^2] \lesssim 1$  and  $\phi$  satisfies the anchoring condition  $\int_B \phi = 0$  a.s.*

*Noting  $\lambda$  the intensity of  $\mathcal{P}$  and recalling that the back-flow term is given by*

$$\alpha := \frac{\lambda|B|}{1 - \lambda|B|}, \quad (3.3)$$

*one can then define the mean settling speed  $\bar{V}$  (in the direction  $e$ ) by*

$$\bar{V} := \alpha^{-1} \mathbb{E}[|\nabla\phi|^2] \quad (3.4)$$

*and this definition coincides with the intuitive notion of arithmetic mean (see Remark 3.10 below).  $\square$*

Note that here,  $\bar{V}$  is in fact the projection on the direction of the gravity  $e$  of the mean settling speed, thus a scalar. This fully determines the sedimentation speed as it needs to be aligned with the gravity. In the sequel, we will omit this slight distinction.

The assumptions on  $\mathcal{P}$  essentially amount to a slight quantification of the ergodicity and to a good finite volume approximation of  $\mathcal{P}$  which keeps the same

properties, in particular stationarity. The quantification is a weakened version of  $\alpha$ -mixing (with explicit algebraic rate of decay  $\gamma$ ), see Section 3.2.2 for details. The decay of the two-point correlation only is needed to define  $\bar{V}$ , hence the condition at order 2.

The decorrelation at large scale (in the form of the stationarity and quantitative mixing) is key for the well-posedness of (3.2). Indeed, Stokes equation is elliptic and the boundary conditions essentially impose a source-term at each  $x \in \mathcal{P}$ , hence non decaying : a standard deterministic approach through energy estimate (via Lax-Milgram for instance) would thus fail, see Remark 3.16 below.

**Remark 3.2** For  $d \geq 3$ , [DG22b, Theorem 1] states that the two-point correlation should be integrable i.e.  $\gamma > d$  but the condition  $\gamma > 2$  is actually sufficient inspecting the proof (see the key estimate (3.86)).

The main focus of these authors was the notion of hyperuniformity (see [Tor18]) a type of long range order on the disordered point set  $\mathcal{P}$ . Assuming in addition that  $\mathcal{P}$  is hyperuniform (see [DG22b, Appendix A]), the results of Proposition 3.1 can be extended to  $d = 2$ . This is very surprising and only possible thanks to the hyperuniformity assumption. Indeed, this threshold at  $d = 3$  is linked to the well-known Stokes paradox in physics : the Stokes flow around an obstacle  $d = 2$  is not uniquely defined and we have the same issue for the flow generated by a single sedimenting particle.

We will always assume  $d \geq 3$  in the sequel to avoid this issue.  $\square$

Note that  $\bar{V}$  can also be defined as the mean of  $V_0$ , the speed of the typical particle, under the Palm distribution (well-defined since  $\mathcal{P}$  is stationary), see Section 2.2.2 in Chapter 2.

### 3.1.2 Main result

In this chapter, we rigorously establish a 1<sup>st</sup> order expansion of the mean sedimentation speed  $\bar{V}$  in a dilute regime. Our long term goal (not reached in this thesis) would be to provide a rigorous higher-order dilute expansion of  $\bar{V}$  and in particular to settle the correct scaling for the 2<sup>nd</sup> order term.

A priori, a dilute regime is characterized by the smallness of the volume fraction of particles i.e.  $\varphi \ll 1$ . In this regime, the particles are scarce and should essentially behave as if they were isolated. Hence, one expects that  $\bar{V} \sim \bar{V}^{(1)}$  where  $\bar{V}^{(1)}$  is the settling speed of a single isolated particle in the whole space (solving a single particle problem analogous to (3.2)). The physical debate lies in the next order in this approximation. In [Bat72], Batchelor derived (for  $d = 3$ ) the expansion

$$\bar{V} \simeq \bar{V}^{(1)}(1 - 6.55\varphi) \tag{3.5}$$

with a correction  $\propto \varphi$  for a Poisson-like model. This was in contradiction with [Has59] who found a correction  $\propto \varphi^{\frac{1}{3}}$  (again for  $d = 3$ ) for a model of periodically arranged particles.

In [DG20c], Duerinckx and Gloria have established a rigorous expansion for the effective viscosity of a related Stokes problem. In particular, they have highlighted that, at higher-order, the volume fraction  $\varphi$  is not sufficient to characterize dilution : one need to appeal to the so-called multi-point intensities  $(\lambda_i)_{i \geq 1}$  of the point process. They reflect, inter alia, the local correlation structure of the process, see Definition 3.2 for details.

Using tools and ideas developed in [DG20c], we shall establish the following quantitative 1<sup>st</sup> order expansion of  $\bar{V}$ .

**Theorem 3.1** *Let  $e \in \mathbb{R}^d$  s.t.  $|e| = 1$ ,  $d \geq 3$  and  $\mathcal{P}$  be a stationary ergodic point process which is  $\gamma$ -algebraically mixing to order 4 (see Definition 3.4 below) with  $\gamma > 2$  and that can be periodically approximated to order 4 (see Definition 3.5 below). Let  $(\lambda_i)_{1 \leq i \leq 3}$  be its associated multi-point intensities (see Definition 3.2 below). The mean settling speed  $\bar{V}$  given by (3.4) (see Proposition 3.1) admits the following quantitative approximation*

$$|\bar{V} - \bar{V}^{(1)}| \lesssim \lambda + \frac{\lambda_2^{1-\frac{2}{\gamma}}}{\lambda} + \frac{\lambda_3^{1-\frac{2}{\gamma}}}{\lambda} \quad (3.6)$$

where  $\bar{V}^{(1)} := \int_B \phi^\circ \cdot e$  (see Lemma 3.4 below) corresponds to the settling speed of a single isolated particle.  $\square$

Before commenting on the assumptions, let us just particularize the result of Theorem 3.1 to the hard-core Poisson case. It satisfies our mixing and approximation assumptions up to any order, we have  $\gamma = \infty$  (formally) and  $\lambda_i \sim \lambda^i$  for all  $i \geq 1$  so that the RHS of (3.6) reduces to  $\lesssim \lambda$  (which is coherent with Batchelor's prediction (3.5)).

As we are aiming at a dilute expansion, it is surprising that there is no explicit assumption on the smallness of the multi-point intensities  $(\lambda_{1 \leq i \leq 3})$ . It is actually already implicitly encapsulated in the hard-core assumption (3.1) on  $\Omega$  which implies  $\lambda_i < 1$ , for  $i = 1, 2, 3$  (see (3.16)). Of course, (3.6) only yields a good approximation if  $\lambda_i \ll 1$ , for  $i = 1, 2, 3$  and one can find models so that these quantities scales independently of one another (see Remark 3.3).

As our result build up on the definition of  $\bar{V}$ , we require at least the same mixing ( $\gamma > 2$ ) and approximation assumptions as in Proposition 3.1. We also limit ourselves to the case  $d \geq 3$ , see Remark 3.2. We expect the approximation assumption to be purely technical and hope to relax it once we will have designed a good finite volume approximation procedure for any mixing point process (in



particular, it should not break stationarity). Our additional assumption on the mixing (order 4 instead of 2) is necessary for the control of the remainder. As  $\bar{V}$  depends non-linearly on  $\mathcal{P}$ , it is actually unexpected that a truncated mixing assumption suffices : this is specific to our problem thanks to the rigidity constraint on the particles, as observed in [Gér21, Proposition 3.16] (see also [DG20c, Theorem 3 p.38]).

The expansion (3.6) only provides a (quantitative) upper bound on the 2<sup>nd</sup> order correction of  $\bar{V}$  and thus does not settle the physical debate. Nonetheless, we see this result as a prototype : we believe that the strategy presented here can be extended to any order, see Section 3.4. This is left for future work.

### 3.1.3 General strategy

#### Cluster expansion formalism

Because of the dilute regime, we expect the mean settling speed  $\mathcal{P} \mapsto \bar{V}(\mathcal{P})$  (that we now see as a function of the particles' positions  $\mathcal{P}$ ) to behave like  $\bar{V}^{(1)}$  the settling speed of a single isolated particle. Doing so, we have neglected all the multi-particles interactions. It could then be fruitful to decompose  $\mathcal{P} \mapsto \bar{V}(\mathcal{P})$  in finite subset contributions (pairs, triples, ...) in order to account for these interactions thus obtaining a better approximation. This can efficiently be done using the so-called cluster expansion which is well-known in statistical mechanics (see [Tor02, Chapter 19]) and has already been successfully used in stochastic homogenization (see [DG16]). We recall this formalism in our context.

Given a discrete finite set  $E = \{x_1, \dots, x_{|E|}\} \subset \mathbb{R}^d$ , we consider functions acting on the power set of  $E$

$$\Phi = \Phi^\# : F \subset E \mapsto \Phi^F .$$

For all  $x \in E$ , we introduce the difference operator  $\delta^{\{x\}}$  acting on this space defined by

$$\delta^{\{x\}}\Phi^F := \delta^{\{x\}}\Phi^{F \cup \{x\}} := \Phi^{F \cup \{x\}} - \Phi^F \quad \forall F \subset E .$$

$\delta^{\{x\}}$  provides a notion of discrete derivative (or sensitivity) of  $\Phi$  at  $x$ .

For  $F \subset E$ , we recursively define higher-order difference operator

$$\delta^F := \prod_{x \in F} \delta^{\{x\}}$$

with the natural convention  $\delta^\emptyset = \text{Id}$ .

With these definitions, one can check the following identity using some combi-

natorics

$$\Phi^E = \Phi^\emptyset + \sum_{x \in E} \delta^{\{x\}} \Phi^\# + \frac{1}{2!} \sum_{\substack{\neq \\ x_1, x_2 \in E}} \delta^{\{x_1, x_2\}} \Phi^\# + \dots + \frac{1}{|E|!} \sum_{\substack{\neq \\ x_1, \dots, x_{|E|} \in E}} \delta^{\{x_1, \dots, x_{|E|}\}} \Phi^\# \quad (3.7)$$

where we have set  $\sum_{x_1, \dots, x_j}^\neq := \sum_{x_k \neq x_l \text{ for } k \neq l}^{x_1, \dots, x_j}$ . (3.7) can be rewritten in the more compact form

$$\Phi^E = \sum_{F \subset E} \delta^F \Phi^\# = \sum_{i=0}^{|E|} \sum_{\substack{F \subset E \\ |F|=i}} \delta^F \Phi^\# .$$

For all  $n$ , we also introduce  $\mathcal{C}^{(n)}$  the truncation of this series to order  $n$  given by

$$\mathcal{C}^{(n)} \Phi^\# := \sum_{i=0}^n \sum_{\substack{F \subset E \\ |F|=i}} \delta^F \Phi^\#$$

When  $|E| = \infty$ , as it is the case for  $\mathcal{P}$ , this series does not converge in general but we expect its finite truncations to provide a good ansatz. If  $\Phi$  is an averaged quantity (like  $\bar{V}$ ), the  $j$ -point intensities  $(\lambda_i)_{i \geq 1}$  of the point process  $\mathcal{P}$  (see Definition 3.2) should provide an accurate measure for the size of the coefficients in the decomposition and for the error term (see [DG20c, Lemma 1.2 p.10] for the complete treatment of a toy model) : for all  $n \geq 1$ ,

$$\left| \sum_{\substack{F \subset \mathcal{P} \\ |F|=n}} \delta^F \Phi^\# \right| \lesssim \lambda_n \quad (3.8)$$

as it involves a sum over  $n$ -tuples and

$$|\Phi^\mathcal{P} - \mathcal{C}^{(n)} \Phi^\#| \lesssim \lambda_{n+1} . \quad (3.9)$$

### Heuristic of the proof

Using the cluster formalism, we are now in position to explain (informally) our strategy of proof. Note that we will stay rather hazy regarding the estimates involving the multi-point intensities  $(\lambda_i)_{i \geq 1}$  at this stage (note that we always have  $\lambda_1 = \lambda$  and one can keep in mind the hard-core Poisson case where  $\lambda_i \sim \lambda^i$  for all  $i \geq 1$  to follow the scalings).

If we had a finite number of particles, i.e.  $|\mathcal{P}| < \infty$ , one would naturally define

the averaged settling speed  $\bar{V}$  as an arithmetic mean via

$$\bar{V} = \mathbb{E} \left[ \frac{1}{|\mathcal{P}|} \sum_{x \in \mathcal{P}} V_x \right] = \mathbb{E} \left[ \frac{1}{|\mathcal{P}|} \sum_{x \in \mathcal{P}} \int_{B(x)} \phi \cdot e \right] \quad (3.10)$$

recalling that the velocity fields  $\phi$  is given by the system (3.2). One can actually reformulate this expression using an energy identity (see (3.52)) and obtain the rigorous definition (3.4) from Proposition 3.1 where  $\bar{V} \propto \mathbb{E} [|\nabla \phi|^2]$ . This last formula still holds even when  $|\mathcal{P}| = \infty$ .

We do not apply the cluster expansion directly to  $\bar{V}$  but to  $\phi$  instead and plug it in the definition (3.10). This could be surprising since  $\phi$ , unlike  $\bar{V}$ , is not an averaged quantity and we thus do not expect its cluster expansion to be fully accurate. However, using  $\phi$  allows us to rely efficiently on PDE estimates which will be crucial for the reformulation and control of the cluster error.

For a finite set  $E$  of particles, we define  $\phi^E$  solving a system similar to (3.2) (replacing  $\mathcal{P}$  by  $E$ , see Lemma 3.2). Noting that  $\phi^\emptyset = 0$ , we have  $\delta^x \phi^\# = \phi^{\{x\}} =: \phi^x$  for all  $x \in \mathcal{P}$  so that the first order cluster expansion writes  $\mathcal{C}^{(1)} \phi^\# = \sum_{x \in \mathcal{P}} \phi^x$ . Plugging this expansion in the expression (3.10) of  $\bar{V}$ , we obtain

$$\begin{aligned} \bar{V} &= \mathbb{E} \left[ \frac{1}{|\mathcal{P}|} \sum_{x \in \mathcal{P}} \int_{B(x)} \sum_{y \in \mathcal{P}} \phi^y \cdot e \right] + \mathbb{E} \left[ \frac{1}{|\mathcal{P}|} \sum_{B(x)} \int_{B(x)} (\phi - \mathcal{C}^{(1)} \phi^\#) \cdot e \right] \\ &= \mathbb{E} \left[ \frac{1}{|\mathcal{P}|} \sum_{x \in \mathcal{P}} \int_{B(x)} \phi^x \cdot e \right] \\ &\quad + \mathbb{E} \left[ \frac{1}{|\mathcal{P}|} \sum_{x, y \in \mathcal{P}}^{\neq} \int_{B(x)} \phi^y \cdot e \right] + \mathbb{E} \left[ \int_{\mathcal{I}} (\phi - \mathcal{C}^{(1)} \phi^\#) \cdot e \right] \end{aligned}$$

separating, for  $x \in \mathcal{P}$ , the rigid part on  $B(x)$  in the first order cluster expansion,  $\sum_y \phi^y = \phi^x + \sum_{y \neq x} \phi^y$ . Noting that the one particle problems only differs by a translation, i.e.  $\phi^x = \phi^\circ(\cdot - x)$  with  $\phi^\circ := \phi^{\{0\}}$  which is deterministic (see Lemma 3.4), this yields

$$\bar{V} = \int_B \phi^\circ \cdot e + \mathbb{E} [R^1]$$

with

$$\begin{aligned} R^1 &= |\mathcal{P}|^{-1} \sum_{x, y \in \mathcal{P}}^{\neq} \int_{B(x)} \phi^y \cdot e + \int_{\mathcal{I}} (\phi - \mathcal{C}^{(1)} \phi^\#) \cdot e \\ &= R_1^1 + R_2^1. \end{aligned} \quad (3.11)$$

We expect that

$$\mathbb{E} [R^1] \lesssim \lambda_1 + \text{lower order terms} \quad (3.12)$$

and this would conclude the proof of Theorem 3.1.

We have separated the remainder term in two part in (3.11). For the first one  $R_1^1$ , the dependence on  $\mathcal{P}$  is fully explicit. We will then be able to estimate it using multi-point intensities so that  $\mathbb{E} [R_1^1] \lesssim \frac{\lambda_2}{\lambda} \sim \lambda$  (as we are dividing a sum over pairs by the total number of points).

For the second one  $R_2^2$ , the dependence on  $\mathcal{P}$  remains implicit because of  $\phi$ . Using the PDE solved by  $\phi - \mathcal{C}^{(1)}\phi^\#$  and the rigidity on the particles, one can establish the following energy estimate

$$\mathbb{E} [|\nabla(\phi - \mathcal{C}^{(1)}\phi^\#)|^2] \lesssim \lambda_2 + \text{lower order terms} \quad (3.13)$$

to control the cluster error, see Lemma 3.9 and (3.82) for a rigorous version. Yet,  $R_2^2$  involves the function not its gradient. Using again the equations, we can reformulate  $R_2^2$  in a form better suited to use the energy estimate, cf Remark 3.13. This reformulation is obtained testing the equation of  $\mathcal{C}^{(1)}\phi^\#$  (for which  $\mathbb{1}_{\mathcal{I}^e}$  appears as a source term) with  $\phi - \mathcal{C}^{(1)}\phi^\#$  and vice versa, see (3.81).

The main difficulty to establish the remainder estimate (3.12) (and (3.13)) is summability, that will be dealt with through a so-called renormalization procedure (which amounts to reformulating a non absolutely converging integral into an absolutely converging one using suitable cancellations). Treating for instance  $R_1^1$  and using again that  $\phi^x = \phi^\circ(\cdot - x)$ , we have

$$\frac{1}{|\mathcal{P}|} \sum_{x,y \in \mathcal{P}}^{\neq} \int_{B(x)} \phi^y \cdot e = \frac{1}{|\mathcal{P}|} \sum_{x,y \in \mathcal{P}}^{\neq} \int_B \phi^{y-x} \cdot e \sim \sum_z \int_B \phi^z \cdot e$$

where we have cancelled out one sum with  $|\mathcal{P}|^{-1}$  and we are now summing over  $(z = x - y)_{x,y \in \mathcal{P}}$ . As previously discussed, the contribution of a single particle is long range and its decays  $|\int_B \phi^z| \sim \langle z \rangle^{2-d}$  is not summable. But  $\langle z \rangle^{2-d-\gamma}$  would be for  $\gamma > 2$ . As we are estimating  $\mathbb{E} [R_1^1]$  (and not just  $R_1^1$ ), this extra decay will be provided by our mixing assumption, see (3.84) and Remark 3.15.

Thanks to our approximation assumption, for  $L \geq 1$ , we indeed have  $|\mathcal{P}_L| < \infty$  so that this formal computations can be made rigorous (in particular, all the sums are finite) and we recover our original process passing to the limit  $L \rightarrow \infty$ .

As this strategy is very straightforward, we expect it to hold at any order (truncating the cluster expansion accordingly). This is left for future work as the renormalization procedure is more involved at higher order, see Section 3.4.

### 3.1.4 Relation to other works

The mathematical literature regarding the sedimentation problem is very scarce. Most previous works focused on the dynamical model described in Remark 3.1 : in [JO04], Jabin and Otto identified the regime in which particles do not significantly interact while in [Höf18] and its improvements [Mec19] and [HS21], Höfer, Mecherbet and Schubert studied the mean-field limit. In those works, the volume fraction  $\varphi$  is supposed to go to zero (with a certain scaling in the total number of particles) and rely on the deterministic method of reflections (which allows to propagate some minimal distance through the flow ; the dynamic fails as soon as there is a collision between particles). See also [Mec21] for the study of a sinking droplet.

Regarding our stationary statistical approach, it was first studied in [Glo21] by Gloria on a scalar version of the sedimentation problem. The velocity  $\phi$  from problem (3.2) is reminiscent of the standard corrector in stochastic homogenization (for linear elliptic equation in divergence form, see for instance [GO17]). Borrowing ideas and tools from this field, he gave a rigorous meaning to the (scalar) mean settling speed and studied its fluctuation. In [DG22b], Duerinckx and Gloria improved and extended this work to the present setting with a focus on the notion of hyperuniformity and its effect on fluctuations (see Remark 3.2).

Our work should also be put in perspective with the ones on dilute expansion of effective models. Keeping our model (3.2) but assuming that the particles have the same density as the fluid, one could be interested in the effective rheology of the mixture fluid + particles defining for instance an effective viscosity (in particular so-called active fluids have received a lot of attention in recent years, see the introduction of [Gir22] for a more detailed review on this topic). The dilute expansion of the effective viscosity is known as Einstein's formula (at first order) or Batchelor-Green (at second order). The justification of these formulas has received a lot of attention in the past year. The earlier works [Sán85], [HM12] obtained Einstein formula in the case of periodic array of particles. [HW20], [NS20] extended this result to the disorder setting using the method of reflections. The Batchelor-Green correction (second order) was captured in [GH20] (extended by [GM22]) assuming the convergence of some mean field quantities. In all these works, the minimal distance between the particles is supposed to be large enough. This condition was relaxed to some extent using a non-concentration condition on the particles in [GH21]. In all cases, the results state that the velocity field of the particle model can be approximated (with an error term controlled in  $\varphi$ ) by a dilute effective solution  $u$  solving a whole space Stokes problem (without any particles) with an effective dilute viscosity (yielding an effective stress tensor  $\sigma_{\text{eff}}(u, p) := \nu(\varphi)D(u) - p\text{Id}$ ) defined as a power law of  $\varphi$ ,  $\nu(\varphi) := \text{Id} + \varphi\nu^{(1)}$  for instance in the Einstein case (with  $\nu^{(1)}$  explicitly given).

In parallel, Gloria and Duerinckx approached this problem through a (stochastic) homogenization perspective [DG21a; DG21b; Due22]. Instead of directly looking for an effective dilute model they decomposed the approach, first defining an effective model (with a notion of effective viscosity valid even in a non-dilute setting) then expanding the effective viscosity in a dilute regime. They used several approaches to justify this dilute expansion (energy method,  $\ell^1 - \ell^2$  estimates, implicit or explicit renormalization) which are summarized in the memoir [DG20c]. In [Gér21], Gérard-Varet used a similar approach tackling this time the expansion through a fine analysis of Calderón-Zygmund kernels (a notable difference also lies in the decomposition of multi-point densities for which the multi-point correlation approach introduced by Duerinckx and Gloria, see Definition 3.3, allows a more convenient treatment of high orders). These techniques no longer require the inter-particle distance to be large, only positive. In exchange, the particles need to be distributed along a stationary ergodic point process (for homogenization to hold) where the previous techniques only required some mean field quantities to converge (these convergences hold in the stationary ergodic case).

Our work can be seen as an extension of Duerinckx and Gloria works in the sedimentation setting (using the renormalization method) and share the same assumptions. The notable difference with the effective viscosity case is that it can be defined and expanded under the minimal homogenization assumption (stationary and qualitative ergodicity) when we require a (weak) quantitative mixing assumption for  $\bar{V}$  to be well defined and for the expansion to hold.

### 3.1.5 Extensions and future works

Regarding our assumptions, first we have chosen our particles to be spherical to simplify the exposition. This condition can be relaxed to any bounded smooth (at least  $C^2$ ) shape. We use this smoothness assumption for our boundary estimate (see Lemma 3.7). Hence, it is unclear if we can assume our shapes to be only locally Lipschitz. We also have taken our shapes to be identical. This condition can be relaxed to distribution of (smooth) bounded shape (independent of our underlying point process) : we would essentially see the averaged shape.

Second, our separation assumption is more stringent : we need a positive uniform lower bound on our inter-particle distance to define the mean settling speed  $\bar{V}$ . However, several recent works [Due22], [DG20c, Theorem 1], [GG22] have managed to relax this condition for the problem of effective viscosity (or effective conductance) using a fine treatment of clusters. One could try to adapt these ideas in our context.

Finally, as explained in Remark 3.2, we have always assumed that  $d \geq 3$  even if  $\bar{V}$  can be defined  $d = 2$  under the additional hyper-uniformity assumption. One could investigate the behavior of our expansion for  $d = 2$  (one should expect some

sort of blow up as the single particle problem is not well defined anymore for  $d = 2$ ).

Regarding future works, with Duerinckx and Gloria, we are currently working on the extension Theorem 3.1 to the second order as it would fully justify Batchelor's expansion, see Section 3.4.

Although not very physical, one could be interested in the dilation case (which include the periodic case)  $\mathcal{P}^\ell = \ell\mathcal{P}$  for  $\ell \geq 1$  and look for an asymptotic expansion of the corresponding map  $\ell^{-1} \mapsto \bar{V}(\mathcal{P}^\ell)$  as  $\ell \rightarrow \infty$ . In a future work, we would like to see if the fixed-point approach developed in Chapter 2 can be adapted to this setting.

## 3.2 Preliminaries

### 3.2.1 Notations

We summarize here the various notations that we use in this chapter.

- We use the notation  $A \lesssim B$  (resp.  $A \gtrsim B$ ) for  $A \leq C \times B$  (resp.  $A \geq \frac{1}{C} \times B$ ) with a constant  $C$  that depends only on the dimension  $d$  and on controlled parameters appearing in different assumptions. Note that the value of the constant  $C$  is allowed to change from one line to another. We write  $A \sim B$  when  $A \lesssim B$  and  $B \lesssim A$ . In addition, we write  $A \ll B$  (resp.  $A \gg B$ ) for  $A \leq \frac{1}{C} \times B$  (resp.  $A \geq C \times B$ ) for some sufficiently large constant  $C$ . We add subscripts to  $C, \lesssim, \gtrsim, \ll, \gg$  to indicate the dependence on other parameters.
- The ball of radius  $R$  centered at  $x$  in  $\mathbb{R}^d$  is denoted by  $B_R(x)$  and we simply write  $B(x) := B_1(x)$ ,  $B_R := B_R(0)$  and  $B: B_1(0)$ . Similarly, we define  $Q_R(x) = x + [-\frac{R}{2}, \frac{R}{2}]^d$  the cube centered at  $x$  of size  $R$  and its short-hand notations.
- We also denote the slightly enlarge unit ball  $B_+ := B_{1+\rho}$  for any  $0 < \rho \leq \frac{3}{2}$ .
- For  $x \in \mathbb{R}^d$ , we use the Japanese bracket notation  $\langle x \rangle^2 := 1 + |x|^2$ .
- For  $L \geq 1$ , we identify the cube  $Q_L$  with the flat periodic torus  $\mathbb{R}^d/L\mathbb{Z}^d$ . We denote by  $|\cdot|_L$  its corresponding distance as well as the associated Japanese bracket  $\langle \cdot \rangle_L^2 := 1 + |\cdot|_L^2$ . We denote by  $B_R^L(x)$  the periodic ball (i.e. associated to the periodic Euclidean distance on  $Q_L$ ) of size  $R$  centered at  $x$ ,  $Q_R^L(x)$  the periodic cube (i.e. associated to the periodic  $L^\infty$ -distance on  $Q_L$ ) of size  $R$  centered at  $x$  and their associated short-hand notations.
- As there is no ambiguity, we use the notation  $|E| = \text{Leb}(E)$  for the volume of a measurable  $E$  sets in  $\mathbb{R}^d$  and  $|H| = \#H$  for the cardinal of a countable set  $H$ .

- For a vector valued function  $u = (u_i)_{1 \leq i \leq d}$ , recalling that in coordinates its gradient writes  $[\nabla u]_{ij} = \partial_j u_i$ , we denote its symmetric gradient  $D(u)$  by  $[D(u)]_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ . For  $A = [A]_{ij}$  and  $B = [B]_{ij}$ , we write  $A : B = A_{ij} B_{ij}$  for the usual scalar product on tensors.
- For a Borel  $U \subset \mathbb{R}^d$  and a function  $u$ , we use the short-hand notation  $\int_U u := |U|^{-1} \int_U u$  and we define  $\mathcal{R}_U(\psi) := \int_U \psi + (\int_U \nabla \psi - D(\psi))(\cdot - \int_U x \, dx)$ , the projection on rigid motion on  $U$ .
- We denote by  $\sigma(u, p) := 2D(u) - p \text{Id}$  the Stokes stress tensor,  $\nu$  the outward unit normal vector at particles boundaries and  $\mathbb{R}_{\text{skew}}^{d \times d}$  the set of skew-symmetric  $d \times d$  matrices.
- We denote by  $t_+ = \max\{t, 0\}$  the positive part of  $t \in \mathbb{R}$  and by  $a \wedge b := \min\{a, b\}$  the minimum between  $a, b \in \mathbb{R}$ .
- For all  $n \geq 1$  and countable set  $H$ , we add the superscript  $\neq$  on sums over  $n$ -tuples of *distinct* points in  $H$  : 
$$\sum_{\substack{x_1, \dots, x_n \in H \\ \neq}} := \sum_{\substack{(x_1, \dots, x_n) \in H^n \\ x_i \neq x_j, \forall i \neq j}}$$
.

### 3.2.2 Point processes

We recall statistical notions of random point processes which will be needed for our analysis ; in particular, multi-point intensities, multi-point correlations and their relation with mixing assumption as well as a good notion of finite volume approximation.

#### Statistics

**Definition 3.1** We say that a point process  $\mathcal{P}$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  is strongly mixing if, for all  $U, V \subset \mathbb{R}^d$  and for any  $A \in \sigma(\mathcal{P} \cap U)$  and  $B \in \sigma(\mathcal{P} \cap V)$ ,

$$\mathbb{P}[A \cap B] - \mathbb{P}[A] \mathbb{P}[B] \xrightarrow{\text{dist}(U, V) \rightarrow \infty} 0 .$$

One can quantify this notion saying that  $\mathcal{P}$  is  $\gamma$ -algebraically mixing if there exists  $\gamma > 0$  s.t., for all  $U, V, A, B$  as above,

$$|\mathbb{P}[A \cap B] - \mathbb{P}[A] \mathbb{P}[B]| \lesssim (1 + \text{dist}(U, V))^{-\gamma} .$$

Both of these notions imply ergodicity. □

As stated in the heuristic, we need to define an appropriate notion of multi-point intensities for a point process. We also recall some useful properties (see [DG20c, Lemma 1.1 p.8] for more details).



**Definition 3.2** Let  $\mathcal{P} \in \Omega$  be a stationary random point process. For  $j \geq 1$ , we define its  $j$ -point density  $f_j$  (also known as  $j$ th-order reduced/factorial moment measure, see [Chi+13, Section 4.3.2 p.121]) as the non-negative function defined by the relation

$$\int_{(\mathbb{R}^d)^j} \chi f_j = \mathbb{E} \left[ \sum_{x_1, \dots, x_j \in \mathcal{P}}^{\neq} \chi(x_1, \dots, x_j) \right], \quad \forall \chi \in C_c^\infty((\mathbb{R}^d)^j). \quad (3.14)$$

Note that  $f_j$  is symmetric as the points are indistinguishable. Intuitively,  $f_j dx_1 \dots dx_j$  is the probability to get  $j$ -points at positions  $(x_1, \dots, x_j)$ .

We then define the  $j$ -point intensity  $\lambda_j$  as the (locally averaged) maximum of  $f_j$

$$\lambda_j := \sup_{z_1, \dots, z_j \in \mathbb{R}^d} \int_{Q_3(z_1) \times \dots \times Q_3(z_j)} f_j. \quad (3.15)$$

We also define the aggregated quantity

$$\bar{\lambda}_j := \max_{\sum_i j_i = j} \prod_i \lambda_{j_i}.$$

Note that the 1-point intensity is nothing but the standard intensity of the point process  $\lambda_1 = \bar{\lambda}_1 = \lambda := \mathbb{E} [|\mathcal{P} \cap Q|]$  (we can remove the supremum by stationarity).

Since  $\mathcal{P}$  is hardcore,

$$\lambda_{j+1}(\mathcal{P}) \leq 3^{-d} \lambda_j(\mathcal{P}) \quad (3.16)$$

and  $\lambda < 3^{-d}$ . Note that we also have

$$|\mathcal{P} \cap Q_R| \lesssim R^d \quad \forall R \geq 1. \quad (3.17)$$

Moreover, if  $\mathcal{P}$  is stationary and strongly mixing, we have

$$\lambda_j(\mathcal{P}) = \bar{\lambda}_j(\mathcal{P}) \quad (3.18)$$

which implies in particular that  $\lambda_j(\mathcal{P}) \geq \lambda(\mathcal{P})^j$  (see also Remark 3.6 below if  $\mathcal{P}$  only satisfies a truncated mixing).  $\square$

**Remark 3.3** For  $\mathcal{P}$  hard-core and mixing, combining (3.16) and (3.18), we have  $\lambda^j \lesssim \lambda_j \lesssim \lambda$ . One can actually construct points processes for which  $\lambda_j$  spans this entire range. Indeed, if  $\mathcal{P}$  is a hard-core Poisson point process, one has  $\lambda_j \sim \lambda^j$  and the diluteness of  $\mathcal{P}$  is entirely characterized by  $\lambda \ll 1$ . However, for any  $\beta \in [0, 1]$ , there exist point processes s.t.  $\lambda_2 \sim \lambda(\mathcal{P})^{1+\beta}$  (this construction can be generalized to  $k \geq 2$ , see [DG20c, Section 5.1 p.121]).  $\square$

Following [DG20c, Section 4.3.1 p.74] (see also Mayer's expansion in the lit-

erature or joint cumulants), we also need to introduce a notion of multi-point correlations to efficiently exploit the mixing of the point process in our framework.

**Definition 3.3** Let  $\mathcal{P} \in \Omega$  be a stationary point process. For all  $j \geq 1$ , we introduce its  $j$ -point correlation function  $h_j$  from the following decomposition of its  $j$ -point density  $f_j$ ,

$$f_j(x_1, \dots, x_j) = \sum_{\pi} \prod_{H \in \pi} h_{|H|}(x_H) \quad (3.19)$$

where  $\pi$  runs over all the partitions of the index set  $\{1, \dots, j\}$ ,  $H$  runs over all cells of the partition  $\pi$  and we have set  $x_H := (x_{i_1}, \dots, x_{i_{|H|}})$  for  $H = \{i_1, \dots, i_{|H|}\}$ . For  $j = 1, 2, 3, 4$ , this reads

$$f_1(x) = h_1(x) = \lambda, \quad (3.20)$$

$$f_2(x, y) = h_2(x, y) + \lambda^2, \quad (3.21)$$

$$f_3(x, y, z) = h_3(x, y, z) + \lambda(h_2(x, y) + h_2(y, z) + h_2(x, z)) + \lambda^3, \quad (3.22)$$

$$\begin{aligned} f_4(x, y, z, u) &= h_4(x, y, z, u) + \lambda(h_3(x, y, z) + h_3(x, y, u) + h_3(y, z, u)) \\ &\quad + h_2(x, y)h_2(z, u) + h_2(x, z)h_2(y, u) + h_2(x, u)h_2(y, z) \\ &\quad + \lambda^2(h_2(x, y) + h_2(x, z) + h_2(x, u) + h_2(y, z) + h_2(y, u) + h_2(z, u)) \\ &\quad + \lambda^4. \end{aligned} \quad (3.23)$$

Inverting recursively these relations,  $h_j$  is a polynomial in  $(f_i)_{i \leq j}$  and, as  $f_j$ ,  $h_j$  is a symmetric function. In particular, this yields the following bound

$$\sup_{z_1, \dots, z_j \in \mathbb{R}^d} \int_{Q_3(z_1) \times \dots \times Q_3(z_j)} |h_j| \lesssim_j \bar{\lambda}_j. \quad (3.24)$$

□

Moreover, if  $\mathcal{P}$  is  $\gamma$ -algebraically mixing,  $h_j$  inherits this decay (see [DG20c, Lemma 4.2 p.74]) and for all  $j \geq 1$  and for all  $z_1, \dots, z_j \in \mathbb{R}^d$ , we have

$$\int_{Q_3(z_1) \times \dots \times Q_3(z_j)} |h_j| \lesssim_j \min_{k \neq l} \langle z_k - z_l \rangle^{-\gamma}. \quad (3.25)$$

This motivates to introduce a truncated version of the  $\gamma$ -algebraically mixing requiring only the decay of a finite number of correlation functions.

**Definition 3.4** Let  $n \geq 2$ . We say that the stationarity random point process  $\mathcal{P}$  is  $\gamma$ -algebraically mixing to order  $n$  if, for all  $2 \leq j \leq n$  and  $z_1, \dots, z_j \in \mathbb{R}^d$ , we

have

$$\int_{Q_3(z_1) \times \dots \times Q_3(z_j)} |h_j| \lesssim \min_{k \neq l} \langle z_k - z_l \rangle^{-\gamma}. \quad (3.26)$$

□

### Periodic approximation

With these notations at hand, we are now in position to rigorously define our notion of periodic approximation for a point process.

**Definition 3.5 (Periodic approximation)** Let  $\gamma > 0$  and  $n \geq 2$ . For a stationary ergodic random point process  $\mathcal{P}$  which is  $\gamma$ -algebraically mixing to order  $n$  on  $(\Omega, \mathcal{F}, \mathbb{P})$ , we say that it admits a periodic approximation to order  $n$  if there exists a family  $(\mathcal{P}_L)_{L \geq 1}$  of random process constructed on  $(\Omega, \mathcal{F}, \mathbb{P})$  satisfying the following properties :

- *Periodicity in law* : For all  $L \geq 1$ , the point process  $\mathcal{P}_L$  is defined on the periodic torus  $Q_L$  and is stationary with respect to (continuous) shifts on the latter.
- *Stabilization* : For any compact set  $K \subset \mathbb{R}^d$ , the restriction of the point set on  $K$  converges to  $\mathcal{P}$  i.e.  $\mathcal{P}_L \cap K \xrightarrow[L \rightarrow \infty]{a.s.} \mathcal{P} \cap K$ .
- *Truncated mixing to order  $n$*  : Defining  $f_{j,L}$  and  $h_{j,L}$  as in Definitions 3.2 and 3.3 (replacing the whole space  $\mathbb{R}^d$  by the torus  $Q_L$  and the cube  $Q_3$  by its periodized version  $Q_3^L$ ), we assume that, for all  $2 \leq j \leq n$ ,

$$\int_{Q_3^L(z_1) \times \dots \times Q_3^L(z_j)} |h_{j,L}| \lesssim \min_{k \neq l} \langle z_k - z_l \rangle_L^{-\gamma}. \quad (3.27)$$

Note that for all  $L$ , the condition  $\mathcal{P}_L \in \Omega$  already implies that  $\mathcal{P}_L$  is hardcore.

We also define  $\lambda_{j,L}$  and  $\bar{\lambda}_{j,L}$  as Definitions 3.2 and 3.3 (adapted on  $Q_L$ ). □

Such an approximation should exist as soon as  $\mathcal{P}$  is defined through a constructive procedure replacing the whole space by  $Q_L$ . For instance, the Poisson process that can be constructed sequentially adding points chosen independently and uniformly in space. Its hardcore variants are then obtained by thinning, i.e. by deleting points which are too close (either sequentially or at the end of the procedure), see for instance the so-called Matérn's hard-core point processes type I, II, III [Mat86] or Penrose graphical construction of the random parking measure [Pen01]. These examples satisfy our mixing assumptions for any  $\gamma > 0$ , their correlations actually decay exponentially (this could be efficiently captured through so-called multi-scales functional inequalities developed by Duerinckx and Gloria, see [DG20b, Section 3.3 and 3.4]).

**Remark 3.4** One could also define a naive approximation of  $\mathcal{P}$  on the torus setting, for all  $L \geq 3$ ,  $\widehat{\mathcal{P}}_L := \mathcal{P} \cap Q_{L-2}$  and periodizing it on  $Q_L$ . Note that we have trimmed the points too close to the border to avoid cutting a particle when periodizing.

The main difference with Definition 3.5 is that  $\widehat{\mathcal{P}}_L$  thus defined is not stationary anymore. This destroys the structure of our estimates in the sequel creating divergent boundary terms as  $L \rightarrow \infty$  (see Remark 3.17).  $\square$

**Remark 3.5** Even if the bounds (3.15), (3.24), (3.26) and (3.27) only control local averages of  $(f_{j,(L)})_{j \geq 1}$  and  $(|h_{j,(L)}|)_{j \geq 1}$ , we can treat them as pointwise bounds in our computation.

For instance, defining for  $j \geq 1$

$$\omega_j(x_1, \dots, x_j) := \bar{\lambda}_{j,L} \wedge \min_{k \neq l} \langle x_k - x_l \rangle_L^{-\gamma},$$

the bounds (3.24) and (3.27) rewrites

$$\int_{Q_3^L(x_1) \times \dots \times Q_3^L(x_j)} |h_{j,L}| \lesssim \omega_j(x_1, \dots, x_j). \quad (3.28)$$

This implies

$$\int_{Q_L^j} g |h_{j,L}| \lesssim \int_{Q_L^j} g \omega_k \quad (3.29)$$

for  $g$  s.t.  $\sup_{Q_3^L(x_1) \times \dots \times Q_3^L(x_j)} g \lesssim g(x_1, \dots, x_j)$ .

We will typically apply (3.29) for  $g = \prod_i \prod_{k \neq l} \langle x_k \rangle_L^{\alpha_k} \langle x_k - x_l \rangle_L^{\alpha_{k,l}}$  with  $\alpha_k, \alpha_{k,l} \in \mathbb{R}$ .

(3.29) is a direct consequence of Fubini and the hypothesis on  $g$  and  $h_j$  writing

$$\begin{aligned} \int_{Q_L^j} g |h_{j,L}| &= \int_{Q_L^j} \int_{Q_3^L(x_1) \times \dots \times Q_3^L(x_j)} g |h_{j,L}| dx_1 \dots dx_j \\ &\lesssim \int_{Q_L^j} \sup_{Q_3^L(x_1) \times \dots \times Q_3^L(x_j)} g \omega_j dx_1 \dots dx_j. \quad \square \end{aligned}$$

This notion of approximation is constructed, inter alia, so that we have convergence of the multi-point intensities.

**Lemma 3.1** *Let  $\mathcal{P}$  be a stationary ergodic  $\gamma$ -algebraically mixing to order  $n$  and  $(\mathcal{P}_L)_{L \geq 1}$  its approximation (see Definition 3.5). Then, for all  $1 \leq j \leq n$ , we have  $\lambda_{i,L} \xrightarrow{L \rightarrow \infty} \lambda_i$  and  $\bar{\lambda}_{i,L} \xrightarrow{L \rightarrow \infty} \bar{\lambda}_i = \lambda_i$  (see Remark 3.6).*

PROOF By definition, the  $(\bar{\lambda}_{i,L})_{i>0}$  are products of the  $(\lambda_{i,L})_{i>0}$  so we only need to prove the convergence of the latter. For shortness, we only display the proofs of convergence for  $\lambda_L$  and  $\lambda_{2,L}$ . The ones for  $(\lambda_{i,L})_{i>2}$  works similarly by induction (see [DG20c, proof of Lemma 4.2 p.75] for a proof in the same spirit).

We start with the convergence  $\lambda_L \rightarrow \lambda$ . We recall that, by stationarity,  $\lambda_L = \mathbb{E}[|\mathcal{P}_L \cap Q|]$  and  $\lambda = \mathbb{E}[|\mathcal{P} \cap Q|]$ . Indeed, as  $\mathcal{P}_L$  is stationary, the measure on the torus (resp.  $\mathbb{R}^d$ )  $A \mapsto \mathbb{E}[|\mathcal{P}_{(L)} \cap A|]$  is invariant by translation, hence proportional to the Lebesgue measure on the torus (resp.  $\mathbb{R}^d$ ). The convergence of  $\lambda_L$  then follows by the z property  $\mathcal{P}_{L|Q} \rightarrow \mathcal{P}|_Q$  (and the dominated convergence theorem recalling that  $|\mathcal{P}_L \cap Q| \lesssim 1$  by the hardcore property).

For the convergence  $\bar{\lambda}_{2,L} \rightarrow \bar{\lambda}_2$ , we set for all  $x, y \in Q_L$

$$\bar{f}_{2,L}(x-y) := \int_{Q_3^L(x) \times Q_3^L(y)} f_{2,L} = |Q_3|^{-2} \mathbb{E} \left[ \sum_{a,b \in \mathcal{P}_L} \mathbf{1}_{Q_3^L(x) \times Q_3^L(y)}(a,b) \right]$$

and for all  $x, y \in \mathbb{R}^d$

$$\bar{f}_2(x-y) := \int_{Q_3(x) \times Q_3(y)} f_{2,L} = |Q_3|^{-2} \mathbb{E} \left[ \sum_{a,b \in \mathcal{P}} \mathbf{1}_{Q_3(x) \times Q_3(y)}(a,b) \right]$$

(which are functions of  $x-y$  and not  $(x,y)$  by stationarity) so that  $\lambda_{2,L} = \sup_{Q_L} \bar{f}_{2,L}$  and  $\lambda_2 = \sup_{\mathbb{R}^d} \bar{f}_2$ . The convergence of  $\lambda_{2,L} \rightarrow \lambda_2$  then amounts to the uniform convergence of  $\bar{f}_{2,L} \rightarrow \bar{f}_2$ .

By the stabilization hypothesis, we have that for all  $R > 0$  s.t.  $Q_R + Q_3 \subset Q_L$ ,  $\sup_{Q_R} |\bar{f}_{2,L} - \bar{f}_2| \rightarrow 0$  as  $\mathcal{P}_{L|Q_R+Q_3} \rightarrow \mathcal{P}|_{Q_R+Q_3}$  (and using the dominated convergence theorem).

From the decomposition (3.21)  $f_{2,L}$ , we have  $\bar{f}_{2,L}(z) = \lambda_L^2 + \bar{h}_{2,L}$  (with  $\bar{h}_{2,L}$  defined as  $\bar{f}_{2,L}$ ). From our mixing assumption,  $\sup_{Q_L \setminus Q_R} |h_{2,L}| \lesssim R^{-\gamma}$  hence  $|\lambda_L^2 - \sup_{Q_L \setminus Q_R} \bar{f}_{2,L}| \lesssim R^{-\gamma}$ . Similarly,  $|\lambda^2 - \sup_{\mathbb{R}^d \setminus Q_R} \bar{f}_2| \lesssim R^{-\gamma}$ .

Combining the previous convergences, we fix  $\varepsilon > 0$  and we write

$$\lambda_{2,L} = \max\left\{ \sup_{Q_R} \bar{f}_{2,L}, \sup_{Q_L \setminus Q_R} \bar{f}_{2,L} \right\}, \quad \lambda_2 = \max\left\{ \sup_{Q_R} \bar{f}_2, \sup_{\mathbb{R}^d \setminus Q_R} \bar{f}_2 \right\}.$$

We first take  $R$  s.t.  $|\lambda_L^2 - \sup_{Q_L \setminus Q_R} \bar{f}_{2,L}| + |\lambda^2 - \sup_{\mathbb{R}^d \setminus Q_R} \bar{f}_2| \leq \varepsilon$  and then  $L > R$  s.t.  $\sup_{Q_R} |\bar{f}_{2,L} - \bar{f}_2| + |\lambda_L^2 - \lambda^2| \leq \varepsilon$ . For this choice of  $L$  and  $R$ , it yields  $|\lambda_{2,L} - \lambda_2| \lesssim \varepsilon$  and this concludes the proof.  $\blacksquare$

**Remark 3.6** If  $\mathcal{P}$  is stationary and  $\gamma$ -algebraically mixing to order  $n$  (see Defini-

tion 3.4), then for all  $1 \leq j \leq n$ ,

$$\lambda_j(\mathcal{P}) = \bar{\lambda}_j(\mathcal{P})$$

(see Definition 3.2).

However, this property only holds for processes defined on the whole space. In particular, even if  $\mathcal{P}$  admits a periodic approximation to order  $n$  (see Definition 3.5), in general  $(\lambda_{j,L})_{(1 \leq j \leq n)} \neq (\bar{\lambda}_{j,L})_{(1 \leq j \leq n)}$ .  $\square$

PROOF For shortness, we only display the proof that  $\lambda_3(\mathcal{P}) = \bar{\lambda}_3(\mathcal{P})$  assuming that  $n \geq 3$ . The general case works similarly, it is a combination of the arguments of [DG20c, proof of Lemma 1.1 (ii) p.9] with the one of [DG20c, proof of Lemma 4.2 p.75].

Fix  $z_1, z_2, z_3 \in \mathbb{R}^d$ . From the decompositions (3.21) and (3.22) of the multi-point densities in multi-point correlations, we have

$$\begin{aligned} & \left| \int_{Q_3(z_1) \times Q_3(z_2) \times Q_3(z_3)} f_3 - \left( \int_{Q_3(z_1) \times Q_3(z_2)} f_2 \right) \left( \int_{Q_3(z_3)} f_1 \right) \right| \\ &= \left| \int_{Q_3(z_1) \times Q_3(z_2) \times Q_3(z_3)} h_3(x, y, z) + \lambda(h_2(x, z) + h_2(y, z)) \, dx \, dy \, dz \right| \\ &\lesssim \langle \text{dist}(\{z_1, z_2\}, z_3) \rangle^{-\gamma} \end{aligned}$$

by definition (3.26) of  $\gamma$ -algebraic mixing.

By stationarity,  $\int_{Q_3(z_1) \times Q_3(z_2)} f_2 = \int_{Q_3(z_1+u) \times Q_3(z_2+u)} f_2$  for all  $u \in \mathbb{R}^d$ . Hence, by definition (3.15) of the multipoint intensities

$$\lambda_3(\mathcal{P}) \geq \lim_{|u| \rightarrow \infty} \int_{Q_3(z_1+u) \times Q_3(z_2+u) \times Q_3(z_3)} f_3 = \left( \int_{Q_3(z_1) \times Q_3(z_2)} f_2 \right) \left( \int_{Q_3(z_3)} f_1 \right)$$

so that  $\lambda_3(\mathcal{P}) \geq \lambda_2(\mathcal{P})\lambda(\mathcal{P})$  taking the suprema over  $(z_1, z_2)$  and  $z_3$ .

Proceeding similarly, we get  $\lambda_2(\mathcal{P}) \geq \lambda(\mathcal{P})^2$  so that  $\lambda_3(\mathcal{P}) \geq \lambda(\mathcal{P})^3$  and therefore  $\lambda_3(\mathcal{P}) = \bar{\lambda}_3(\mathcal{P})$  by definition of the latter.

This strategy fails in the bounded case as we can not send the shift  $u$  to infinity ( $|u|_L \lesssim L$ ).  $\blacksquare$

### 3.2.3 Finite volume approximation

#### Definitions and properties

With the periodic finite volume approximation of  $\mathcal{P}$  (see Definition 3.5), we turn to the corresponding finite volume periodic approximation of (3.2), the fluid

velocity (and pressure) generated by the particles. As we want to use the cluster formalism, we define a periodic analogous of (3.2) for any finite subset of particles.

**Lemma 3.2** *Let  $L > 1$ ,  $e \in \mathbb{R}^d$  and let  $E \subset Q_L$  be a finite hard-core set (i.e.  $\min_{\substack{x,y \in E \\ x \neq y}} |x - y| \geq 3$ ) and set  $\mathcal{I}_L^E := \cup_{x \in E} B^L(x)$ . Then, there exists a unique periodic couple  $(\phi_L^E, \Pi_L^E) \in (H^1(Q_L))^d \times L^2(Q_L)$  solution of*

$$\left\{ \begin{array}{ll} -\nabla \cdot \sigma(\phi_L^E, \Pi_L^E) = -\alpha_{|E|,L} e & \text{in } Q_L \setminus \mathcal{I}_L^E \\ \nabla \cdot \phi_L^E = 0 & \text{in } Q_L \setminus \mathcal{I}_L^E \\ D(\phi_L^E) = 0 & \text{in } \mathcal{I}_L^E \\ \Pi_L^E = 0 & \text{in } \mathcal{I}_L^E \\ e|B| + \int_{\partial B^L(x)} \sigma(\phi_L^E, \Pi_L^E) \nu = 0 & \forall x \in E \\ \int_{\partial B^L(x)} \Theta \nu \cdot \sigma(\phi, \Pi) \nu = 0 & \forall \Theta \in \mathbb{R}_{skew}^{d \times d}, \forall x \in E \\ \int_{Q_L} \phi_L^E = 0 \\ \int_{Q_L} \Pi_L^E = 0 \end{array} \right. \quad (3.30)$$

defining the back-flow constant

$$\alpha_{|E|,L} := \frac{|\mathcal{I}_L^E|}{|Q_L \setminus \mathcal{I}_L^E|} = \frac{|E|L^{-d}|B|}{1 - |E|L^{-d}|B|} \quad (3.31)$$

and the Newtonian stress tensor  $\sigma(\phi_L^E, \Pi_L^E) := 2D(\phi_L^E) - \Pi_L^E \text{Id}$  (with its short-hand notation  $\sigma_L^E := \sigma(\phi_L^E, \Pi_L^E)$ ).

$(\phi_L^E, \Pi_L^E)$  are smooth in  $Q_L \setminus \mathcal{I}_L^E$ . In addition, they satisfy the following equation in the whole torus  $Q_L$

$$-\nabla \cdot \sigma(\phi_L^E, \Pi_L^E) = -\alpha_{|E|,L} e \mathbf{1}_{Q_L \setminus \mathcal{I}_L^E} - \sum_{x \in E} \delta_{\partial B^L(x)} \sigma_L^E \nu \quad \text{in } Q_L \quad (3.32)$$

and we have

$$\nabla \cdot \phi_L^E = 0 \quad \text{in } Q_L. \quad (3.33)$$

The weak formulation of (3.32) then implies that for all  $\chi \in (H^1(Q_L))^d$  s.t.  $\int_{Q_L} \chi = 0$  and  $\nabla \cdot \chi = 0$  in  $Q_L$ , we have

$$\int_{Q_L} \nabla \phi_L^E : \nabla \chi = \int_{Q_L} 2D(\phi_L^E) : D(\chi) = \alpha_{|E|,L} \int_{\mathcal{I}_L^E} e \cdot \chi - \sum_{x \in E} \int_{\partial B^L(x)} \chi \cdot \sigma_L^E \cdot \nu. \quad (3.34)$$

This yields the following energy estimate (as  $d \geq 3$ )

$$\|D(\phi_L^E)\|_{L^2(Q_L)} \sim \|\nabla \phi_L^E\|_{L^2(Q_L)} \lesssim |E|^{\frac{1}{2} + \frac{1}{d}}. \quad (3.35)$$

□

PROOF For Dirichlet boundary conditions, these results are standard and can be found in [Gal11, Section IV and V] (reference textbook) or [BF13, Theorem IV.5.1 p.274] (less complete but more accessible). However, as our boundary conditions are not standard, we provide the necessary adaptations.

First, we assume that we already have existence and smoothness of  $(\phi_L^E, \Pi_L^E)$  solution of (3.30) and we prove (3.32) and (3.34). Let  $\chi = (\chi_j)_{1 \leq j \leq d}$  be a smooth  $Q_L$ -periodic test function in  $Q_L$ . Testing (3.30) in  $Q_L \setminus \mathcal{I}_L^E$  with  $\chi$ , we have

$$\int_{Q_L \setminus \mathcal{I}_L^E} -\nabla \cdot \sigma_L^E \cdot \chi = -\alpha_{|E|,L} \int_{Q_L \setminus \mathcal{I}_L^E} e \cdot \chi.$$

Integrating by parts (in coordinates) and using the symmetry of  $\sigma_L^E$ , the LHS rewrites

$$\begin{aligned} - \int_{Q_L \setminus \mathcal{I}_L^E} \partial_i \sigma_{Lij}^E \chi_j &= \int_{Q_L \setminus \mathcal{I}_L^E} \sigma_{Lij}^E \partial_i \chi_j - \int_{\partial(Q_L \setminus \mathcal{I}_L^E)} \sigma_{Lij}^E \chi_j \nu_i \\ &= \int_{Q_L \setminus \mathcal{I}_L^E} \sigma_{Lij}^E \partial_j \chi_i + \sum_{x \in E} \int_{\partial B^L(x)} \chi_i \sigma_{Lij}^E \nu_j \end{aligned}$$

As  $\sigma_L^E = 0$  on  $\mathcal{I}_L^E$  ( $D(\phi_L^E)$  and  $\Pi_L^E$  vanish there by construction), this yields

$$\int_{Q_L} \sigma_L^E : \nabla \chi = -\alpha_{|E|,L} \int_{Q_L} \mathbf{1}_{Q_L \setminus \mathcal{I}_L^E} e \cdot \chi - \sum_{x \in E} \int_{\partial B^L(x)} \chi \cdot \sigma_L^E \cdot \nu$$

which is the weak form of (3.32).

Adding to  $\chi$  the constraints  $\nabla \cdot \chi = 0$  in  $Q_L$  and  $\int_{Q_L} \chi = 0$ , we obtain (3.34). Indeed,

$$\int_{Q_L} \sigma_L^E : \nabla \chi = \int_{Q_L} 2D(\phi_L^E) : \nabla \chi - \int_{Q_L} \Pi_L^E \text{Id} : \nabla \chi = \int_{Q_L} D\phi_L^E : D\chi$$

as  $\text{Id} : \nabla \chi = \nabla \cdot \chi = 0$ . Moreover, integrating by parts, we have

$$\int_{Q_L} \partial_i \phi_{Lj}^E \partial_j \chi_i = - \int_{Q_L} \phi_{Lj}^E \partial_j \partial_i \chi_i = \int_{Q_L} \partial_j \phi_{Lj}^E \partial_i \chi_i$$

so that

$$2 \int_{Q_L} D(\phi_L^E) : D(\chi) = \int_{Q_L} \nabla \phi_L^E : \nabla \chi + \int_{Q_L} (\nabla \cdot \phi_L^E)(\nabla \cdot \chi) \quad (3.36)$$

which yields (3.34) (see also (3.33)).



Assuming in addition that  $D(\chi) = 0$  in  $\mathcal{I}_L^E$ , i.e.  $\chi$  is rigid in the inclusions, we have

$$\begin{aligned} \sum_{x \in E} \int_{\partial B^L(x)} \chi \cdot \sigma_L^E \cdot \nu &= \sum_{x \in E} \int_{\partial B^L(x)} \mathcal{R}_{B^L(x)} \chi \cdot \sigma_L^E \cdot \nu \\ &= \sum_{x \in E} \int_{B^L(x)} \chi \cdot \int_{\partial B^L(x)} \sigma_L^E \cdot \nu = - \sum_{x \in E} \int_{B^L(x)} \chi \cdot e |B| \end{aligned}$$

where we have used the boundary conditions on  $\sigma_L^E$  (see (3.30)). Hence, (3.34) rewrites

$$\int_{Q_L} \nabla \phi_L^E : \nabla \chi = (1 + \alpha_{|E|,L}) \int_{\mathcal{I}_L^E} e \cdot \chi. \quad (3.37)$$

This motivates to introduce  $\phi_L^E$  as the unique Lax-Milgram solution in

$$\mathcal{H}_L^E := \{ \chi \in (H_{\text{per}}^1(Q_L))^d \mid \nabla \cdot \chi = 0 \text{ in } Q_L \setminus \mathcal{I}_L^E, D(\chi) = 0 \text{ in } \mathcal{I}_L^E \text{ and } \int_{Q_L} \chi = 0 \}$$

which satisfies (3.37) for all  $\chi \in \mathcal{H}_L^E$ . We endow  $\mathcal{H}_L^E$  with the norm  $\|\nabla \cdot\|_{L^2(Q_L)}$  which is equivalent to the standard  $H^1$ -norm (by Poincaré-Wirtinger's inequality). Note that, since we are working with divergence free fields, we could also use  $\|D \cdot\|_{L^2(Q_L)}$  (see (3.36)). The coercivity of the bilinear form (LHS of (3.37)) is clear. We claim that for all  $\chi \in \mathcal{H}_L^E$

$$\|\chi\|_{L^1(\mathcal{I}_L^E)} \lesssim |E|^{\frac{1}{2} + \frac{1}{d}} \|\nabla \chi\|_{L^2(Q_L)} \quad (3.38)$$

which implies the continuity of the linear form (RHS of (3.37)) as  $\alpha_{|E|,1} \lesssim 1$  (by the hard-core condition (3.17)) and yields the energy estimate (3.35).

Estimate (3.38) is a consequence of the (periodic) Sobolev's inequality (see [BO13] for instance) which only holds as  $d \geq 3$ . Indeed, setting  $2^* := \frac{2d}{d-2}$  (s.t.  $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$ ) and  $2^{*'} := \frac{2d}{d+2}$  (s.t.  $\frac{1}{2^*} + \frac{1}{2^{*'}} = 1$ ), we combine Hölder's and Sobolev's inequalities and obtain

$$\|\chi\|_{L^1(\mathcal{I}_L^E)} \leq |\mathcal{I}_L^E|^{\frac{1}{2^{*'}}} \|\chi\|_{L^{2^*}(Q_L)} \lesssim |E|^{\frac{1}{2} + \frac{1}{d}} \|\nabla \chi\|_{L^2(Q_L)}.$$

This yields (3.35) taking  $\chi = \phi_L^E$ . Note that by scaling, the constant given by Sobolev inequality is independent of  $L$ .

We then reconstruct the pressure by standard functional analytic arguments. For all  $\chi \in C_c^\infty(Q_L \setminus \mathcal{I}_L^E)$  s.t.  $\nabla \cdot \chi = 0$ , we have

$$\langle -\nabla \cdot 2D(\phi_L^E) + \mathbf{1}_{Q_L \setminus \mathcal{I}_L^E} (1 + \alpha_{|E|,L}), \chi \rangle_{H^{-1}(Q_L \setminus \mathcal{I}_L^E), H_0^1(Q_L \setminus \mathcal{I}_L^E)} = 0$$

testing (3.37) with  $\chi - \int_{Q_L \setminus E} \chi$ . By de Rham decomposition (see for instance [BF13, Theorem IV.2.4 p.243]), this yields the existence and uniqueness (up to constant) of  $\Pi_L^E \in L^2(Q_L \setminus \mathcal{I}_L^E)$  satisfying (3.30). We extend it by zero in  $\mathcal{I}_L^E$  and we fix the constant setting  $\int_{Q_L} \Pi_L^E = 0$ .

To upgrade the regularity of  $(\phi_L^E, \Pi_L^E)$  outside the inclusions, we see (3.30) as an exterior Stokes problem on  $Q_L \setminus \mathcal{I}_L^E$  with an inhomogeneous Dirichlet boundary condition  $\phi_L^E|_{\mathcal{I}_L^E}$  and a smooth source term  $\alpha_{|E|,L} \mathbf{1}_{Q_L \setminus \mathcal{I}_L^E}$ . [Gal11, Theorem IV.5.1 p.276] or [BF13, Theorem IV.5.8 p.281] then yields that for all  $k \geq 0$ ,

$$\|\phi_L^E\|_{H^{k+2}(Q_L \setminus \mathcal{I}_L^E)} + \|\Pi_L^E\|_{H^{k+1}(Q_L \setminus \mathcal{I}_L^E)} \lesssim \alpha_{|E|,L} + \|\phi_L^E|_{\mathcal{I}_L^E}\|_{H^{k+\frac{3}{2}}(\partial\mathcal{I}_L^E)} \lesssim 1 \quad (3.39)$$

We only need to control  $\|\phi_L^E|_{\mathcal{I}_L^E}\|_{H^{k+\frac{3}{2}}(\partial\mathcal{I}_L^E)}$ . Using first a trace estimate and then the fact that  $\phi_L^E$  is rigid hence affine on (each connected component) of  $\mathcal{I}_L^E$ , we have

$$\|\phi_L^E|_{\mathcal{I}_L^E}\|_{H^{k+\frac{3}{2}}(\partial\mathcal{I}_L^E)} \lesssim \|\phi_L^E\|_{H^{k+2}(\mathcal{I}_L^E)} = \|\phi_L^E\|_{H^1(\mathcal{I}_L^E)}$$

We control this last norm thanks to the energy estimate (3.35) and proceeding as in (3.38) (noting that  $2^* > 2$ ) for the  $L^2$  part

$$\|\phi_L^E\|_{L^2(\mathcal{I}_L^E)} \lesssim_{|E|} \|\nabla \phi_L^E\|_{L^2(Q_L)} \lesssim_{|E|} 1. \quad (3.40)$$

This concludes the proof of the regularity upgrade.

For (3.33), we note that by construction, we already have  $\nabla \cdot \phi_L^E = 0$  in  $Q_L \setminus \mathcal{I}_L^E$ . We also have  $\nabla \cdot \phi_L^E = 0$  in  $\mathcal{I}_L^E$  since  $D(\phi_L^E) = 0$ . Indeed,  $\nabla \cdot \phi_L^E = \text{Tr}(\nabla \phi_L^E) = 0$  as the symmetric part of the matrix vanishes. This yields in  $Q_L$

$$\nabla \cdot \phi_L^E = \nabla \cdot (\phi_L^E \mathbf{1}_{\mathcal{I}_L^E}) + \nabla \cdot (\phi_L^E \mathbf{1}_{Q_L \setminus \mathcal{I}_L^E}) + \partial_{\partial\mathcal{I}_L^E} \llbracket \phi_L^E \cdot \nu \rrbracket = \partial_{\partial\mathcal{I}_L^E} \llbracket \phi_L^E \cdot \nu \rrbracket$$

but the jump necessarily vanishes as  $\phi_L^E \in H^1(Q_L)$  which concludes the proof of (3.33).  $\blacksquare$

**Remark 3.7 (Back-flow)**  $\alpha_{|E|,L}$  is called *back-flow* and appears naturally for the well-posedness of the equation as we are working on the torus. It is the analogous in fluid mechanics of the (uniform) background charge density in electrostatic (i.e. Laplace equation) which guaranties the total charge neutrality of the system. Note that with Dirichlet boundary conditions (i.e. grounding the system in electrostatic), this term would not appear (in the fluid case, it would be absorbed in the pressure).

Testing (3.32) with  $e$  (s.t.  $|e| = 1$ ) and using the boundary conditions of  $\sigma_L^E$ ,

we have

$$0 = -\alpha_{|E|,1}|Q_L \setminus \mathcal{I}_L^E| + \sum_{x \in E} |B|$$

which justifies the expression (3.31) of  $\alpha_{|E|,L}$ .

Note that if  $E$  is a random set,  $\alpha_{|E|,L}$  is a random variable.  $\square$

As usual for the Stokes equation, the pressure, as a consequence of incompressibility, is fully determined by the velocity field. Hence, in the sequel we will sometimes call  $\phi_L^E$  the solution to (3.30).

**Remark 3.8** Lemma 3.2 is tailored for the set  $E$  s.t.  $|E| \lesssim 1$ . In this case, the energy estimate (3.35) yields  $\|\nabla \phi_L^E\|_{L^2(Q_L)} \lesssim 1$ , independently of  $L$  thanks to Sobolev inequality (only valid as  $d \geq 3$ ).

We could have obtained the cruder energy estimate

$$\|\nabla \phi_L^E\|_{L^2(Q_L)} \lesssim L|E|^{\frac{1}{2}} \quad (3.41)$$

replacing the estimate (3.38) for  $\chi \in \mathcal{H}_L^E$  by

$$\|\chi\|_{L^1(\mathcal{I}_L^E)} \leq |\mathcal{I}_L^E|^{\frac{1}{2}} \|\chi\|_{L^2(Q_L)} \lesssim |E|^{\frac{1}{2}} L \|\nabla \chi\|_{L^2(Q_L)}$$

obtained using Cauchy-Schwarz' and Poincaré-Wirtinger's inequalities (as  $\int_{Q_L} \chi = 0$ ). This approach holds in any dimension.

If  $|E| \sim L^d$ , the upper bounds of (3.35) and (3.41) coincide and yield

$$\|\nabla \phi_L^E\|_{L^2(Q_L)} \lesssim L^{1+\frac{d}{2}} \quad (3.42)$$

which blows up as  $L \rightarrow \infty$ .  $\square$

As it appears in the boundary condition, the problem (3.30) depends non-linearly on the particle set  $E$ . As it will be useful in the sequel, we define its linear analogous  $\gamma_L^E$ . By linearity, we start by defining the single particle problem.

**Definition 3.6** Let  $L > 1$  and  $e \in \mathbb{R}^d$ . We define  $(\gamma_L^\circ, \Sigma_L^\circ) \in (H^1(\mathbb{R}^d))^d \times L^2(Q_L)$  as the unique periodic couple solving

$$\left\{ \begin{array}{ll} -\nabla \cdot \sigma(\gamma_L^\circ, \Sigma_L^\circ) = e(\mathbf{1}_{B^L} - L^{-d}|B|) & \text{in } Q_L \\ \nabla \cdot \phi_L^\circ = 0 & \text{in } Q_L \\ \int_{Q_L} \phi_L^\circ = 0 \\ \int_{Q_L} \Sigma_L^\circ = 0 \end{array} \right. . \quad (3.43)$$

The solution  $(\gamma_L^\circ, \Sigma_L^\circ)$  is smooth in  $Q_L$ .

Defining the (periodic) Stokeslet (Green kernel for Stokes flow as known as Oseen tensor)  $\mathcal{G}_L$  by

$$\begin{cases} -\nabla \cdot \sigma(\mathcal{G}_L, \mathcal{S}_L) = \delta - L^{-d} & \text{in } Q_L \\ \nabla \cdot \mathcal{G}_L = 0 & \text{in } Q_L \\ \int_{Q_L} \mathcal{G}_L = 0 \\ \int_{Q_L} \mathcal{S}_L = 0 \end{cases}, \quad (3.44)$$

we have the representation formula

$$\gamma_L^\circ = \int_{Q_L} \mathcal{G}_L(\cdot - y) e \mathbf{1}_{B^L}(y) dy = \int_{B^L(\cdot)} \mathcal{G}_L.$$

For all  $x \in Q_L$ , we then set  $\gamma_L^x := \phi_L^\circ(\cdot - x)$ . Since  $\gamma_L^x$  is a (moving) average of  $\mathcal{G}_L$ , we will also abusively call it a Stokeslet. For  $i \in \{0, 1\}$ , the standard decay of  $\mathcal{G}_L$

$$|\nabla^i \mathcal{G}_L(x)| \lesssim |x|_L^{2-(d+i)}$$

then implies that,

$$|\nabla^i \gamma_L^x(0)| \lesssim \langle x \rangle_L^{2-(d+i)}. \quad (3.45)$$

Let  $E \subset Q_L$  be a finite hard-core set. We finally define the linear analogous of  $\phi_L^E$  as the solution of the following problem

$$\begin{cases} -\nabla \cdot \sigma(\gamma_L^\circ, \Sigma_L^\circ) = e \left( \mathbf{1}_{\mathcal{I}_L^E} - L^{-d} |\mathcal{I}_L^E| \right) & \text{in } Q_L \\ \nabla \cdot \phi_L^\circ = 0 & \text{in } Q_L \\ \int_{Q_L} \phi_L^\circ = 0 \\ \int_{Q_L} \Sigma_L^\circ = 0 \end{cases}. \quad (3.46)$$

By linearity, we have  $\gamma_L^E := \sum_{x \in E} \gamma_L^x$ .  $\square$

PROOF Existence, uniqueness and smoothness of the Stokeslet are similar to Lemma 3.2. In particular, it is still necessary that  $d \geq 3$ .

The representation formula is a direct consequence of the definition of the Stokeslet  $\mathcal{G}_L$ . Its decay is standard folklore in the literature.  $\blacksquare$

**Remark 3.9** For any finite subset  $E$  of  $Q_L$  and following [Höf21, (2.3) p.87] or [DG22b, Remark 1.1],  $\nabla \phi_L^E$  is the orthogonal projection of  $(1 + \alpha_{|E|,L}) \nabla \gamma_L^E$  in  $H := \{\nabla \phi \mid \phi \in H_{\text{per}}^1(\mathbb{R}^d), \nabla \cdot \phi = 0\}$  onto its subspace  $\{\nabla \phi \in H \mid D(\phi) = 0 \text{ in } \mathcal{I}_L^E\}$ .

Quantitatively, this yields the control

$$\|\nabla \phi_L^E\|_{L^2(Q_L)} \lesssim \|\nabla \gamma_L^E\|_{L^2(Q_L)} = \left\| \sum_{x \in E} \nabla \gamma_L^x \right\|_{L^2(Q_L)} \quad (3.47)$$

which is well suited to unravel stochastic cancellation when  $E$  is random. Otherwise, it is equivalent to the energy estimate (3.35) adapting its proof to  $\gamma_L^E$  to obtain

$$\|\nabla \gamma_L^E\|_{L^2(Q_L)} \lesssim |E|^{\frac{1}{2} + \frac{1}{d}}. \quad (3.48)$$

Indeed, testing the equation (3.46) of  $\gamma_L^E$  with  $\chi$  s.t.  $\nabla \cdot \chi = 0$ ,  $\int_{Q_L} \chi = 0$  and  $D(\chi) = 0$  in  $\mathcal{I}_L^E$ , we obtain

$$\int_{Q_L} \nabla \gamma_L^E : \nabla \chi = \int_{Q_L} \mathbb{1}_{\mathcal{I}_L^E} e \cdot \chi = (1 + \alpha_{|E|,L})^{-1} \int_{Q_L} \nabla \phi_L^E : \nabla \chi$$

recognizing the RHS of (3.37), the Lax-Milgram formulation of  $\phi_L^E$ . Taking  $\chi = \phi_L^E$ , this yields (3.47) by Cauchy-Schwarz' inequality (and since  $0 \leq \alpha_{|E|,L} \leq 1$ ). Note that these equalities also hold replacing the gradient  $\nabla$  by its symmetric part  $D$ .  $\square$

We now focus on the case  $E = \mathcal{P}_L$  and  $|E| = 1$ . Note that if  $|E| = 0$ ,  $\phi_L^\emptyset \equiv 0$ .

**Definition 3.7** Let  $L > 1$ ,  $e \in \mathbb{R}^d$  s.t.  $|e| = 1$ ,  $\mathcal{P} \in \Omega$  and its periodic approximation  $\mathcal{P}_L$  (see Definition 3.5). As  $\mathcal{P}_L$  is finite (see (3.17)), using Lemma 3.2, we define

$$\phi_L := \phi_L^{\mathcal{P}_L} \quad (3.49)$$

and the associated (random) quantities  $\mathcal{I}_L$ ,  $\alpha_L$  and  $\sigma_L$  as well as  $\gamma_L := \gamma_L^{\mathcal{P}_L}$ . Setting

$$\lambda_L := \mathbb{E} \left[ \frac{|\mathcal{P}_L|}{|Q_L|} \right] = \mathbb{E} [L^{-d} |\mathcal{P}_L|] , \quad (3.50)$$

we define the average settling speed (in the direction  $e$ ) via

$$V_L := (L^d |B| \lambda_L)^{-1} \int_{\mathcal{I}_L} \phi_L \cdot e \quad (3.51)$$

and its mean  $\bar{V}_L := \mathbb{E}[V_L]$ .  $\square$

**Remark 3.10** The definition (3.51) can be rewritten as

$$V_L = \frac{1 - L^{-d} |B| |\mathcal{P}_L|}{|B| \lambda_L} \int_{Q_L} |\nabla \phi_L|^2 \quad (3.52)$$

testing the Lax-Milgram formulation of  $\phi_L$  with  $\phi_L$  itself and noting that  $1 + \alpha_L = (1 - L^{-d} |B| |\mathcal{P}_L|)^{-1}$ .

From the heuristic definition, it should be more natural to define the average settling speed by

$$\widehat{V}_L = (|B| |\mathcal{P}_L|)^{-1} \int_{\mathcal{I}_L} \phi_L \cdot e$$

( $|B|$  is just a volume normalization factor).

Choosing the definition (3.51) will simplify our analysis. Since  $|\mathcal{P}_L|$  is a random quantity, by replacing it by its expectation, we avoid treating the correlation between  $|\mathcal{P}_L|$  and  $\int_{\mathcal{I}_L} \phi_L \cdot e$  (the latter being the main challenge).

This bias actually disappears as  $L \rightarrow \infty$  thanks to ergodicity as

$$\lim_{L \rightarrow \infty} \frac{|\mathcal{P}_L \cap Q_L|}{|Q_L|} = \lambda = \lim_{L \rightarrow \infty} \lambda_L . \quad (3.53)$$

Indeed, for the first equality, for any  $R > 0$ ,  $\lim_{L \rightarrow \infty} |\mathcal{P}_L \cap Q_R| \rightarrow |\mathcal{P} \cap Q_R|$  by the stabilization property (see Definition 3.5) and  $\lim_{R \rightarrow \infty} \frac{|\mathcal{P} \cap Q_R|}{|Q_R|} \rightarrow \lambda$  by ergodicity see [DV08, Corolary 12.2.V p.201] or [Chi+13, (4.15) p.115]. For  $\lambda_L \rightarrow \lambda$ , see Lemma 3.1.  $\square$

**Lemma 3.3** *Let  $L > 1$  and  $e \in \mathbb{R}^d$ . For all  $x \in Q_L$ , we define  $\phi_L^x := \phi_L^{\{x\}}$  and the associated  $\sigma_L^x$  (see Lemma 3.2).*

*Setting  $\phi_L^\circ := \phi_L^{\{0\}}$ , we note that  $\phi_L^x := \phi_L^\circ(\cdot - x)$  for all  $x \in Q_L$ . It yields the cancellation property*

$$\int_{Q_L} \phi_L^x dx = 0 . \quad (3.54)$$

*Moreover, for  $i \in \{0, 1\}$  and for all  $x, y \in Q_L$ , we have the following decay*

$$\|\nabla^i \phi_L^y\|_{L^2(B^L(x))} \lesssim \langle y - x \rangle_L^{2-(d+i)} \quad (3.55)$$

*similar to the decay of the Stokeslet (3.45).*  $\square$

PROOF (3.54) is a direct consequence of  $\phi_L^x := \phi_L^\circ(\cdot - x)$  writing  $\int_{Q_L} \phi_L^x dx = \int_{Q_L} \phi_L^\circ(\cdot - x) dx = 0$  by definition (3.30) of  $\phi_L^\circ$ .

Since  $\phi_L^x$  is a shift of  $\phi^\circ$ , it is enough to establish (3.55) for  $x = 0$ .

For  $|y| \leq 3$  and  $i \in \{0, 1\}$ , we have  $\|\nabla^i \phi_L^y\|_{L^2(B^L)} \leq \|\nabla^i \phi_L^y\|_{L^2(B_4^L)} \lesssim 1$  proceeding as in (3.40) for  $i = 0$  and using the energy estimate (3.35) for  $i = 1$  and (3.55) is satisfied.

For  $|y| > 3$ , using the equation (3.32) for  $\phi_L^x$  in  $Q_L$  and  $\mathcal{G}_L$ , we have the following representation formula for all  $z \in B$ ,

$$\phi_L^y(z) = - \int_{Q_L} \alpha_{1,L} \mathcal{G}_L(z - x) e \mathbf{1}_{Q_L \setminus B^L(y)}(x) dx - \int_{\partial B^L(y)} \mathcal{G}_L(z - x) \sigma_L^y \nu dx .$$

Since  $\int_{Q_L} \mathcal{G}_L = 0$  (see (3.44)) and  $\sigma_L^y = \sigma_L^\circ(\cdot - y)$ , we get

$$\phi_L^y(z) = \alpha_{1,L} \int_B \mathcal{G}_L(z - x - y) e dx - \int_{\partial B^L} \mathcal{G}_L(z - x - y) \sigma_L^\circ \nu dx$$

Noting that for all  $x, y \in B^L$ ,  $|z - x - y| \geq ||y| - (|z| + |x|)| \geq \frac{|y|}{3}$  as  $|z| + |x| \leq 2 \leq \frac{2}{3}|y|$ , the pointwise decay of  $\mathcal{G}_L$  (see (3.45)) then implies

$$|\phi_L^y(z)| \lesssim |y|^{2-d} \left( L^{-d} + \int_{\partial B^L} |\sigma_L^\circ| \right) \quad (3.56)$$

noting that  $\alpha_{1,L} \lesssim L^{-d}$ . Combining Cauchy-Schwarz' inequality, trace estimates and the regularity estimation (3.39), we have

$$\int_{\partial B^L} |\sigma_L^\circ| \lesssim \|D(\phi_L^\circ) + \Pi_L^\circ\|_{L^2(\partial B)} \lesssim \|\phi_L^\circ\|_{H^{\frac{3}{2}}(Q_L \setminus B)} + \|\Pi_L^\circ\|_{H^{\frac{1}{2}}(Q_L \setminus B)} \lesssim 1.$$

Plugging this estimate in (3.56) yields (3.55) for  $i = 0$ . We obtain the decay of the gradient similarly.  $\blacksquare$

### Convergence

The objects previously defined are only finite volume approximation. We now define their limiting object and show convergence of the quantities of interest. We start with the single particle problem and recall some of its properties.

**Lemma 3.4** *Let  $e \in \mathbb{R}^d$ . We define the single particle solution and its associated pressure  $(\phi^\circ, \Pi^\circ) \in (H^1(\mathbb{R}^d)) \times L^2_{loc}(\mathbb{R}^d)$  as the unique (up to constant for the pressure) solution of*

$$\left\{ \begin{array}{ll} -\nabla \cdot \sigma(\phi^\circ, \Pi^\circ) & = 0 & \text{in } \mathbb{R}^d \setminus B \\ \nabla \cdot \phi^\circ & = 0 & \text{in } \mathbb{R}^d \setminus B \\ D(\phi^\circ) & = 0 & \text{in } B \\ \Pi^\circ & = 0 & \text{in } B \\ e|B| + \int_{\partial B} \sigma(\phi^\circ, \Pi^\circ) \nu & = 0 \\ \int_{\partial B} \Theta \nu \cdot \sigma(\phi^\circ, \Pi^\circ) \nu & = 0 & \forall \Theta \in \mathbb{R}^{d \times d}_{skew} \\ (\phi^\circ(x), \Pi^\circ(x)) & \xrightarrow{|x| \rightarrow \infty} (0, 0) \end{array} \right. \quad (3.57)$$

Note that we have to impose that the fluid is at rest at infinity as the domain is unbounded.

The solution  $(\phi^\circ, \Pi^\circ)$  is smooth in  $\mathbb{R}^d \setminus B$ .

For all  $x \in \mathbb{R}^d$ , we define  $\phi^x := \phi^\circ(\cdot - x)$ . Similarly, we define  $(\gamma^\circ, \Sigma^\circ) \in (H^1(\mathbb{R}^d))^d \times L^2_{loc}(\mathbb{R}^d)$  as the unique solution of

$$\left\{ \begin{array}{ll} -\nabla \cdot \sigma(\gamma^\circ, \Sigma^\circ) & = e \mathbf{1}_B & \text{in } \mathbb{R}^d \\ \nabla \cdot \phi_L^\circ & = 0 & \text{in } \mathbb{R}^d \\ (\gamma^\circ, \Sigma^\circ) & \xrightarrow{|x| \rightarrow \infty} 0 \end{array} \right. \quad (3.58)$$

and for all  $x \in \mathbb{R}^d$ ,  $\gamma^x := \gamma^\circ(\cdot - x)$ .  $\square$

PROOF The proof is standard (see [Gal11, Chap.5]) and similar to the periodic case (see Lemma 3.2 and its proof) with the torus  $Q_L$  replaced by the whole space  $\mathbb{R}^d$ . In particular,  $\phi^\circ$  is the unique solution of the following Lax-Milgram problem

$$\int_{\mathbb{R}^d} \nabla \phi^\circ : \nabla \chi = \int_B e \cdot \chi \quad (3.59)$$

for all  $\chi \in \{\chi \in (H^1(\mathbb{R}^d))^d \mid \nabla \cdot \chi = 0 \text{ in } \mathbb{R}^d \setminus B, D(\chi) = 0 \text{ in } B\}$  and continuity of the linear form follows from the standard Sobolev inequality for such  $\chi$  :  $\|\chi\|_{L^{2^*}(\mathbb{R}^d)} \lesssim \|\nabla \chi\|_{H^1(\mathbb{R}^d)}$ . In particular, it is necessary here that  $d \geq 3$ .  $\blacksquare$

Since our particles are spherical, the problem is radial and  $\phi^\circ$  could even be written explicitly. In particular for  $d = 3$ , we recover the formula  $\int_B \phi^\circ \cdot e = \frac{2}{9}$  which is standard in the physics literature, see [Bat72, Eq (2.1) p.248].

**Lemma 3.5** For all  $x \in \mathbb{R}^d$ ,  $\phi_L^x \xrightarrow[L \rightarrow \infty]{H_{\text{loc}}^1(\mathbb{R}^d)} \phi^x$  (see Lemmas 3.3 and 3.4).  $\square$

PROOF Fix a compact set  $K$ ,  $x \in \mathbb{R}^d$  and  $L > 1$  s.t.  $K \subset Q_L$  and  $x \in Q_L$ . From the energy estimate (3.35) and proceeding similarly to (3.38), we have  $\|\phi_L^x\|_{H^1(K)} \lesssim_K 1$ . Hence,  $\phi_L^x$  converges weakly in  $H^1(K)$  and strongly in  $L^2(K)$  (up to extraction, by Rellich theorem). This limit is indeed  $\phi^x$  by uniqueness as we can pass to the limit in the Lax-Milgram formulation (3.59).

The convergence is also strong in  $H^1(K)$ . Indeed, in  $B(x)$ ,  $\phi_L^x$  is rigid i.e.  $\phi_L^x = \mathcal{R}_B \phi_L^x$  so that the weak convergence is in fact pointwise. In  $K \setminus B(x)$ ,  $\|\phi_L^x\|_{H^2(K \setminus B(x))} \lesssim 1$  using the regularity estimate (3.39) and we conclude thanks to Sobolev embedding (again).  $\blacksquare$

**Remark 3.11** Using similar arguments, we also have  $\phi_L^E \rightarrow \phi^E$  for all finite subsets  $E \subset Q_L$  s.t.  $|E| \lesssim 1$ .

This does not apply to  $\phi_L$  as  $|\mathcal{P}_L|$  is unbounded as  $L \rightarrow \infty$ .  $\square$

We have constructed our approximation Definition 3.5 so that we have convergence of the mean settling speed.

**Lemma 3.6** Let  $\mathcal{P}$  be as in Proposition 3.1. The approximation of the mean settling speed (see Definition 3.7 and (3.4)) is consistent :

$$\bar{V}_L \xrightarrow[L \rightarrow \infty]{} \bar{V}. \quad (3.60)$$

$\square$



PROOF This result is already contained in [DG22b, Theorem 1]. We recall the main ideas of the proof with our notations.

Using the rewriting (3.52) of  $V_L$  and the stationarity of  $\mathcal{P}_L$  (which implies the stationarity of  $\phi_L$ ), we have

$$\bar{V}_L = \mathbb{E} \left[ \frac{1 - L^{-d}|B||\mathcal{P}_L|}{|B|\lambda_L} |\nabla \phi_L|^2 \right] .$$

As  $\frac{1-L^{-d}|B||\mathcal{P}_L|}{|B|\lambda_L} \xrightarrow[L \rightarrow \infty]{a.s.} \alpha^{-1}$  (see (3.53)), we only need to prove that  $\nabla \phi_L \xrightarrow[L \rightarrow \infty]{L^2(\Omega)} \nabla \phi$  (see [DG22b, proof of Theorem 1, Step 3]).

For that purpose, we first claim that  $\mathbb{E} [|\nabla \phi_L(0)|^2] \lesssim 1$  (uniformly in  $L$ ). Indeed, thanks to the estimate (3.47) and stationarity  $\mathbb{E} [|\nabla \phi_L(0)|^2] \lesssim \mathbb{E} [|\nabla \gamma_L(0)|^2]$ . We combine this bound with  $\mathbb{E} [|\nabla \gamma_L|^2] \lesssim 1$  (see (3.86) in the sequel) obtained from our quantitative mixing assumption and the point-wise decay (3.45) of  $\gamma_L^x$ .

Using the mixing assumption, we also define  $\nabla \gamma := \lim_{R \rightarrow \infty} \sum_{x \in \mathcal{P} \cap Q_R} \nabla \gamma^x$ . Thanks to the stabilization property, we have  $\nabla \gamma_L \xrightarrow[L \rightarrow \infty]{L^2(\Omega)} \nabla \gamma$ .

Using the uniform bound, we can extract a weak limit in  $L^2(\Omega)$  of  $\nabla \phi_L$  which is in fact  $\nabla \phi$  by uniqueness. From Remark 3.9 and stationarity, we have

$$\mathbb{E} [|\nabla \phi_L|^2] = \mathbb{E} [(1 + \alpha_L) \nabla \gamma_L : \nabla \phi_L] .$$

This identity upgrades the convergence  $\nabla \phi_L \rightharpoonup \nabla \phi$  from weak to strong in  $L^2(\Omega)$  (as the norm also converges) since  $\alpha_L \xrightarrow[L \rightarrow \infty]{a.s.} \alpha$  and  $\nabla \gamma_L \rightarrow \nabla \gamma$ .  $\blacksquare$

### 3.2.4 Trace estimates

To conclude this preliminary section, we recall some deterministic local boundary estimates on Stokes equation which we will profusely use in the sequel.

**Lemma 3.7** *Let  $z \in \mathbb{R}^d$ .*

*For any  $\psi \in H^1(B(z))^d$ , we have*

$$\int_{\partial B(z)} |\psi - \mathcal{R}_{B(z)} \psi|^2 \lesssim \int_{B(z)} |D(\psi)|^2 \quad (3.61)$$

where for all  $U \subset \mathbb{R}^d$ ,  $\mathcal{R}_U \psi := \int_U \psi + (\int_U \nabla \psi - D(\psi))(\cdot - \int_U x \, dx)$ .

We use the short-hand notation  $B_+ := B_{1+\rho}$ . For any pair  $(\psi, \Sigma) \in H^1(B_+(z))^d \times L^2(B_+(z))$  locally satisfying the following relations around  $B(z)$

$$\begin{cases} -\nabla \cdot \sigma(\psi, \Sigma) = \beta & B_+(z) \setminus B(z) \\ -\nabla \cdot \psi = 0 & B_+(z) \setminus B(z) \\ D(\psi) = 0 & B(z) \end{cases}$$

where  $\beta \in \mathbb{R}^d$  is a constant vector, there exists  $c \in \mathbb{R}^d$  s.t.

$$\min_{c \in \mathbb{R}^d} \int_{\partial B(z)} |\sigma(\psi, \Sigma) - c \text{Id}|^2 \lesssim \int_{B_+(z)} |D(\psi)|^2 + |\beta|^2. \quad (3.62)$$

□

PROOF This is a straightforward adaptation of [DG20c, Lemma 2.3 p.24]. Estimate (3.61) follows from a combination of the trace estimate  $\|\psi - \mathcal{R}_{B(z)}\psi\|_{L^2(\partial B(z))} \lesssim \|\psi - \mathcal{R}_{B(z)}\psi\|_{H^1(B(z))}$  with Korn's inequality.

As in (3.39), (3.62) is a consequence of the local regularity for Stokes equation seeing the rigidity constraint as an inhomogeneous Dirichlet boundary condition combined with the fact that  $\psi$  is rigid hence affine in  $B$  to deal with the boundary terms. ■

**Corollary 3.1** For  $\chi \in H^1(B(z))^d$  s.t.  $\nabla \cdot \chi = 0$  in  $B(z)$  and  $(\psi, \Sigma)$  as in Lemma 3.7, we have

$$\left| \int_{\partial B(z)} (\text{Id} - \mathcal{R}_{B(z)}) \chi \sigma(\psi, \Sigma) \nu \right|^2 \lesssim \left( \int_{B(z)} |D(\chi)|^2 \right) \left( \int_{B_+(z)} |D(\psi)|^2 + |\beta|^2 \right) \quad (3.63)$$

□

PROOF As  $\nabla \cdot \chi = 0$ ,  $\int_{\partial B(z)} \chi \nu = 0$  by the divergence theorem. Hence, for any  $c \in \mathbb{R}^d$ ,

$$\int_{\partial B(z)} (\text{Id} - \mathcal{R}_{B(z)}) \chi \sigma(\psi) \nu = \int_{\partial B(z)} (\text{Id} - \mathcal{R}_{B(z)}) \chi (\sigma(\psi) - c \text{Id}) \nu$$

and (3.63) then follows from Cauchy-Schwarz and (3.61) and (3.62). ■

## 3.3 1<sup>st</sup> order expansion : proof of Theorem 3.1

### 3.3.1 Strategy

As we have assumed that  $\mathcal{P}$  admits a periodic approximation  $\mathcal{P}_L$  (see Definition 3.5), we obviously start by obtaining an expansion of the approximate mean settling speed  $\bar{V}_L$  (see Definition 3.7) similar to (3.6) and then try to pass to the limit  $L \rightarrow \infty$ .

Following the heuristic, we plug  $\mathcal{C}^{(1)} \phi_L^\# = \sum_{x \in \mathcal{P}_L} \phi_L^x$ , the 1<sup>st</sup> order cluster expansion of  $\phi_L$  (see (3.7)), into the definition (3.51) of the averaged settling

speed  $V_L$  and obtain

$$\begin{aligned} L^d |B| \lambda_L V_L &= \sum_{x \in \mathcal{P}_L} \int_{B^L(x)} (\mathcal{C}^{(1)} \phi_L^\# + \phi_L - \mathcal{C}^{(1)} \phi_L^\#) \cdot e \\ &= \sum_{x \in \mathcal{P}_L} \int_{B^L(x)} \phi_L^x \cdot e + \sum_{x, y \in \mathcal{P}_L}^{\neq} \int_{B^L(x)} \phi_L^y \cdot e + \int_{\mathcal{I}_L} (\phi_L - \mathcal{C}^{(1)} \phi_L^\#) \cdot e \end{aligned}$$

where for a given  $x \in \mathcal{P}_L$ , we have separated the rigid part on the particle  $B^L(x)$  in the expansion  $\mathcal{C}^{(1)} \phi_L^\# = \phi_L^x + \sum_{y \neq x} \phi_L^y$ . Note that from the definition of  $\phi_L^x$  (see Lemma 3.3),  $\sum_{x \in \mathcal{P}_L} \int_{B^L(x)} \phi_L^x \cdot e = |\mathcal{P}_L| \int_B \phi_L^\circ$ . This computation motivates the following definition.

**Definition 3.8** Let  $L \geq 1$  and  $\mathcal{P}_L$  be the approximation of  $\mathcal{P}$  as in Theorem 3.1. Recalling the definition (3.51) of the averaged settling speed  $V_L$ , we have the decomposition

$$V_L = V_L^1 + R_L^1 \quad (3.64)$$

where

$$V_L^1 := \lambda_L^{-1} (L^{-d} |\mathcal{P}_L|) \int_B \phi_L^\circ \cdot e \quad (3.65)$$

and

$$R_L^1 := L^{-d} \lambda_L^{-1} \sum_{x, y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \phi_L^y \cdot e + L^{-d} (|B| \lambda_L)^{-1} \int_{\mathcal{I}_L} r_L^{(1)} \cdot e \quad (3.66)$$

with  $r_L^{(1)} := \phi_L - \mathcal{C}^{(1)} \phi_L^\#$ . □

Our key claim is that we can control the expectation of this remainder uniformly in  $L$ .

**Proposition 3.2** *Let  $L \geq 1$ ,  $\mathcal{P}$  as in Theorem 3.1 with its periodic approximation  $\mathcal{P}_L$ . We have the following estimate of the remainder term (3.66)*

$$|\mathbb{E} [R_L^1]| \lesssim \lambda + \frac{\lambda_2^{1-\frac{2}{\gamma}}}{\lambda} + \frac{\lambda_3^{1-\frac{2}{\gamma}}}{\lambda} + o_L(1). \quad (3.67)$$

□

We obtain (3.6) combining the decomposition  $\bar{V}_L = \mathbb{E} [V_L] = \mathbb{E} [V_L^1] + \mathbb{E} [R_L^1]$  from Definition 3.8 with the control of the remainder from Proposition 3.2 and passing to the limit  $L \rightarrow \infty$  as  $\bar{V}_L \rightarrow \bar{V}$  (see Lemma 3.6),  $\bar{\lambda}_{i,L} \rightarrow \bar{\lambda}_i = \lambda_i$  for  $i = 1, 2, 3$  (see Lemma 3.1) and  $\mathbb{E} [V_L^1] = \int_B \phi_L^\circ \cdot e \rightarrow \int_B \phi^\circ \cdot e$  (see Lemma 3.5).

### 3.3.2 Control of the remainder : proof of Proposition 3.2

We split the proof into two step. First, we establish an upper bound on  $|\mathbb{E}[R_L^1]|$  via purely deterministic arguments. Then, we use our stochastic assumptions to estimate this bound in terms of the multi-points intensities  $(\lambda_i)_{i \geq 1}$ .

#### Deterministic bounds

In the definition (3.66) of  $R_L^1$ , the first term is already in a good form for the next step. We only need to control the term involving the cluster expansion error  $r_L^{(1)}$ . For that purpose, we start by establishing its equation.

**Lemma 3.8** *Let  $L \geq 1$ . For  $\phi_L$  given by Definition 3.7, we set its 1<sup>st</sup> order cluster expansion  $\mathcal{C}^{(1)}\phi_L^\# := \sum_{x \in \mathcal{P}_L} \phi_L^x$  (see (3.7) and Lemma 3.3) and the corresponding 1<sup>st</sup> order error  $r_L^{(1)} := \phi_L - \mathcal{C}^{(1)}\phi_L^\#$ . They solve the following equations*

$$-\nabla \cdot \sigma(\mathcal{C}^{(1)}\phi_L^\#) = -\alpha_{1,L}|\mathcal{P}_L|e + \alpha_{1,L}\mathbf{1}_{\mathcal{I}_L}e - \sum_{x \in \mathcal{P}_L} \delta_{\partial B(x)}\sigma_L^x \nu \quad \text{in } Q_L \quad (3.68)$$

and

$$-\nabla \cdot \sigma(r_L^{(1)}) = -\mu_{1,L}e + \Lambda_{1,L}\mathbf{1}_{\mathcal{I}_L}e - \sum_{x \in \mathcal{P}_L} \delta_{\partial B(x)}(\sigma_L - \sigma_L^x)\nu \quad \text{in } Q_L \quad (3.69)$$

with  $\mu_{1,L} := \alpha_L - \alpha_{1,L}|\mathcal{P}_L|$  and  $\Lambda_{1,L} := \alpha_L - \alpha_{1,L}$ . □

PROOF We obtain these equations by direct computations using the equations for  $\phi_L$  and  $\phi_L^x$ , see (3.32). ■

We then obtain the following (deterministic) energy estimate for  $r_L^{(1)}$ .

**Lemma 3.9** *Let  $L \geq 1$  and  $r_L^{(1)}$  given by Lemma 3.8. We have the following energy estimate*

$$\int_{Q_L} |D(r_L^{(1)})|^2 \lesssim \mathcal{E}_L^1, \quad (3.70)$$

where

$$\begin{aligned} \mathcal{E}_L^1 := & |\Lambda_{1,L}| \int_{Q_L} |D(\gamma_L)|^2 + L^{-d} \sum_x \int_{B_+(x)} |D(\sum_{y \in \mathcal{P}_L \setminus \{x\}} \phi_L^y)|^2 \\ & + (L^{-d}|\mathcal{P}_L|)^3 + L^{-d}|\mathcal{P}_L|\mu_{1,L}^2. \quad (3.71) \end{aligned}$$

□

PROOF Testing the equation (3.69) of  $r_L^{(1)}$  with  $r_L^{(1)}$  itself (see (3.34)), we have

$$\int_{Q_L} |D(r_L^{(1)})|^2 = \Lambda_{1,L} \int_{\mathcal{I}_L} r_L^{(1)} \cdot e - \sum_{x \in \mathcal{P}_L} \int_{\partial B(x)} (\text{Id} - \mathcal{R}_{B(x)}) r_L^{(1)} (\sigma_L - \sigma^x) \nu \quad (3.72)$$

as  $\int_{Q_L} r_L^{(1)} = 0$  by construction and  $\int_{\partial B(x)} \mathcal{R}(\sigma_L - \sigma^x) \nu = 0$ , for any rigid motion  $\mathcal{R}$ , using the stress boundary conditions.

Isolating the terms which are rigid in  $B(x)$ , we write  $r_L^{(1)} = (\phi_L - \phi_L^x) - \sum_{y \neq x} \phi_L^y$  and we have

$$(\text{Id} - \mathcal{R}_{B(x)}) r_L^{(1)} = -(\text{Id} - \mathcal{R}_{B(x)}) \sum_{y \neq x} \phi_L^y. \quad (3.73)$$

Adding and subtracting the missing terms to reconstruct  $\sigma(r_L^{(1)})$  in the last term of the RHS of (3.72), we obtain

$$\int_{Q_L} |D(r_L^{(1)})|^2 \leq |\Lambda_{1,L} \int_{\mathcal{I}_L} r_L^{(1)} \cdot e| + \left| \sum_{x \in \mathcal{P}_L} \int_{\partial B(x)} \left( (\text{Id} - \mathcal{R}_{B(x)}) \sum_{y \neq x} \phi_L^y \right) \left( \sum_{y \neq x} \sigma_L^y \right) \nu \right| \quad (3.74)$$

$$+ \left| \sum_{x \in \mathcal{P}_L} \int_{\partial B(x)} \left( (\text{Id} - \mathcal{R}_{B(x)}) \sum_{y \neq x} \phi_L^y \right) \sigma(r_L^{(1)}) \nu \right| \quad (3.75)$$

$$\leq I_1 + I_2 + I_3$$

For all  $\varepsilon > 0$ , we claim that

$$I_1 \lesssim \varepsilon^{-1} |\Lambda_{1,L}| \int_{Q_L} |D(\gamma_L)|^2 + \varepsilon \int_{Q_L} |D(r_L^{(1)})|^2, \quad (3.76)$$

$$I_2 \lesssim \sum_{x \in \mathcal{P}_L} \int_{B_+(x)} |D(\sum_{y \neq x} \phi_L^y)|^2 + |\mathcal{P}_L| (L^{-d} |\mathcal{P}_L|)^2 \quad (3.77)$$

and

$$I_3 \lesssim \varepsilon^{-1} \sum_{x \in \mathcal{P}_L} \int_{B(x)} |D(\sum_{y \neq x} \phi_L^y)|^2 + \varepsilon \int_{Q_L} |D(r_L^{(1)})|^2 + |\mathcal{P}_L| \mu_{1,L}^2. \quad (3.78)$$

Plugging (3.76) to (3.78) into (3.74), this yields (3.70) choosing  $\varepsilon$  small enough to absorb  $\int_{Q_L} |D(r_L^{(1)})|^2$  into the LHS.

For (3.76), using the linear solution  $\gamma_L$  (see Definition 3.6) and proceeding as

in Remark 3.9, we use that  $r_L^{(1)}$  has zero average to write

$$\int_{\mathcal{I}_L} r_L^{(1)} \cdot e = \int_{Q_L} r_L^{(1)} \cdot e (\mathbf{1}_{\mathcal{I}_L} - L^{-d} |\mathcal{I}_L|) = \int_{Q_L} D(r_L^{(1)}) : D(\gamma_L) .$$

We then conclude using Cauchy-Schwarz' and Young's inequalities noting that  $|\Lambda_{1,L}| \lesssim 1$ .

Estimate (3.77) follows from the trace estimate of Corollary 3.1 and Young's inequality noting that for all  $x \in \mathcal{P}_L$

$$-\nabla \cdot \sum_{y \neq x} \sigma_L^y = -\alpha_{1,L} (|\mathcal{P}_L| - 1) + \alpha_{1,L} \sum_{y \neq x} \mathbf{1}_{B(y)} - \sum_{y \neq x} \delta_{\partial B(y)} \sigma_L^y \nu ,$$

$B(x) \subset B_+(x)$  and  $|\alpha_{1,L} (|\mathcal{P}_L| - 1)| \lesssim (L^{-d} |\mathcal{P}_L|)$ .

For (3.78), we use again the trace estimate of Corollary 3.1 with the equation (3.69) of  $r_L^{(1)}$  and Young's inequality to obtain

$$\begin{aligned} I_3 &\lesssim \sum_{x \in \mathcal{P}_L} \left( \int_{B(x)} |D(\sum_{y \neq x} \phi_L^y)|^2 \right)^{\frac{1}{2}} \left( \int_{B_+(x)} |D(r_L^{(1)})|^2 + \mu_{1,L}^2 \right)^{\frac{1}{2}} \\ &\lesssim \varepsilon^{-1} \sum_{x \in \mathcal{P}_L} \int_{B(x)} |D(\sum_{y \neq x} \phi_L^y)|^2 + \varepsilon \sum_{x \in \mathcal{P}_L} \int_{B_+(x)} |D(r_L^{(1)})|^2 + \varepsilon |\mathcal{P}_L| \mu_{1,L}^2 . \end{aligned}$$

This yields (3.78) noting that  $\sum_{x \in \mathcal{P}_L} \int_{B_+(x)} |D(r_L^{(1)})|^2 \leq \int_{Q_L} |D(r_L^{(1)})|^2$  since the union  $\cup_{x \in \mathcal{P}_L} B_+(x) \subset Q_L$  is disjoint by the hardcore condition.  $\blacksquare$

Thanks to the energy estimate (3.70), we claim that

$$|B| \lambda_L |\mathbb{E} [R_L^1]| \lesssim \mathbb{E} [\mathcal{E}_L^1] + \mathcal{F}_L^1 \quad (3.79)$$

where

$$\begin{aligned} \mathcal{F}_L^1 &:= |\mathbb{E} \left[ \left( 1 + \frac{\Lambda_{1,L}}{1 + \alpha_{1,L}} \right) L^{-d} \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \phi_L^y \cdot e \right]| \\ &\quad + |L^{-d} \mathbb{E} \left[ \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{\partial B(x)} (\text{Id} - \mathcal{R}_{B(x)}) \phi_L^y \cdot \sigma^x \nu \right]| \\ &\quad + |B| \lambda_L |\mathbb{E} \left[ \frac{|\Lambda_{1,L}|}{1 + \alpha_{1,L}} V_L^1 \right]| . \quad (3.80) \end{aligned}$$

Even if this step is fully deterministic, we already have to take the expectation in

order to keep stochastic cancellations : it is only the expectation of  $R_L^1$  that we can control, not its  $L^1(\Omega)$ -norm.

We start by reformulating the cluster expansion error as

$$(1+\alpha_{1,L}) \int_{\mathcal{I}_L} r_L^{(1)} \cdot e = \Lambda_{1,L} \int_{\mathcal{I}_L} \mathcal{C}^{(1)} \phi_L^\# \cdot e - \sum_{x \in \mathcal{P}_L} \int_{\partial B(x)} \left( (\text{Id} - \mathcal{R}_{B(x)}) \sum_{y \neq x} \phi_L^y \right) \cdot \sigma_L \nu. \quad (3.81)$$

To prove this, using (3.34), we test the equation (3.68) of  $\mathcal{C}^{(1)} \phi_L^\#$  with  $r_L^{(1)}$  and the equation (3.69) for  $r_L^{(1)}$  with  $\mathcal{C}^{(1)} \phi_L^\#$ . We have chosen  $\mathcal{C}^{(1)} \phi_L^\#$  as  $\mathbb{1}_{\mathcal{I}_L}$  appears as a source term in (3.68). Indeed, as  $\nabla \cdot r_L^{(1)} = 0$  and  $\int_{Q_L} r_L^{(1)} = 0$ , we have

$$\begin{aligned} \int_{Q_L} D(\mathcal{C}^{(1)} \phi_L^\#) : D(r_L^{(1)}) &= \alpha_{1,L} \int_{\mathcal{I}_L} r_L^{(1)} \cdot e - \sum_{x \in \mathcal{P}_L} \int_{\partial B(x)} r_L^{(1)} \cdot \sigma_L^x \nu \\ &= (1 + \alpha_{1,L}) \int_{\mathcal{I}_L} r_L^{(1)} \cdot e - \sum_{x \in \mathcal{P}_L} \int_{\partial B(x)} (\text{Id} - \mathcal{R}_{B(x)}) r_L^{(1)} \cdot \sigma_L^x \nu \end{aligned}$$

using the boundary condition for  $\sigma_L^x$ . Conversely,

$$\begin{aligned} \int_{Q_L} D(r_L^{(1)}) : D(\mathcal{C}^{(1)} \phi_L^\#) &= \Lambda_{1,L} \int_{\mathcal{I}_L} \mathcal{C}^{(1)} \phi_L^\# \cdot e \\ &\quad - \sum_{x \in \mathcal{P}_L} \int_{\partial B(x)} (\text{Id} - \mathcal{R}_{B(x)}) \mathcal{C}^{(1)} \phi_L^\# \cdot (\sigma_L - \sigma_L^x) \nu. \end{aligned}$$

Combining both and removing the rigid motions on  $B(x)$  in the boundary terms (see (3.73)), this yields (3.81).

We then plug (3.81) into the definition (3.66) of  $R_L^1$ . Decomposing  $\int_{\mathcal{I}_L} \mathcal{C}^{(1)} \phi_L^\# \cdot e$

to recognize  $V_L^1$  and writing  $\sigma_L = \mathcal{C}^{(1)}\sigma_L^\# + \sigma_L - \mathcal{C}^{(1)}\sigma_L^\#$ , we have

$$\begin{aligned} L^d|B|\lambda_L R_L^1 &= \frac{\Lambda_{1,L}}{1 + \alpha_{1,L}} L^d|B|\lambda_L V_L^1 + \left(1 + \frac{\Lambda_{1,L}}{1 + \alpha_{1,L}}\right) \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \phi_L^y \cdot e \\ &\quad + \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{\partial B(x)} (\text{Id} - \mathcal{R}_{B(x)}) \phi_L^y \cdot \sigma_L^x \nu \\ &\quad + \sum_{x \in \mathcal{P}_L} \int_{\partial B(x)} \left( (\text{Id} - \mathcal{R}_{B(x)}) \sum_{y \neq x} \phi_L^y \right) \cdot \left( \sum_{z \neq x} \sigma_L^z \right) \nu \\ &\quad - \sum_{x \in \mathcal{P}_L} \int_{\partial B(x)} \left( (\text{Id} - \mathcal{R}_{B(x)}) \sum_{y \neq x} \phi_L^y \right) \cdot \sigma(r_L^{(1)}) \nu \end{aligned}$$

The control (3.79) of  $R_L^1$  by  $\mathcal{E}_L^1$  and  $\mathcal{F}_L^1$  then follows by Young's inequality and trace estimates proceeding as in (3.77) and (3.78).

### Stochastic estimates

We shall now prove the following bounds on  $\mathbb{E}[\mathcal{E}_L^1]$  and  $\mathcal{F}_L^1$  (see (3.71) and (3.80))

$$\mathbb{E}[\mathcal{E}_L^1] \lesssim \lambda(\lambda + \lambda_2^{1-\frac{2}{\gamma}}) + \lambda_2 + \lambda_3^{1-\frac{2}{\gamma}} + \lambda^3 + \lambda^5 + o_L(1) \quad (3.82)$$

$$\mathcal{F}_L^1 \lesssim \lambda_2^{1-\frac{2}{\gamma}} + \lambda^2 + o_L(1). \quad (3.83)$$

As  $\mathbb{E}[\mathcal{E}_L^1]$  and  $\mathcal{F}_L^1$  control  $|\mathbb{E}[R_L^1]|$  by (3.79), this yields the claimed control (3.67) of  $\mathbb{E}[R_L^1]$  (recalling that, by the hard-core assumption (see (3.16)),  $\lambda < 1$  so that  $\lambda^2 \leq \bar{\lambda}_2 = \lambda_2 < 1$  to simplify the upper bounds). It is crucial here that we are taking expectations : the control of these quantities uniformly in  $L$  is only possible thanks to the random cancellations (in the form of our mixing assumptions), see Remark 3.15.

To establish the estimates (3.82) and (3.83), we rely on the following lemma which controls the expectation of all the quantities involved.

**Lemma 3.10** *Let  $L \geq 1$ ,  $\mathcal{P}_L \in \Omega$  be a  $Q_L$  periodic stationary hard-core  $\gamma$ -algebraically mixing to order 3 point process with  $\gamma > 2$  (see Definitions 3.4 and 3.5) and  $(\bar{\lambda}_{i,L})_{1 \leq i \leq 3}$  be its associated multi-point intensities. We then have*



the following controls of expectations :

$$|L^{-d}\mathbb{E}\left[\sum_{x,y\in\mathcal{P}_L}^{\neq}\int_{B(x)}\phi_L^y\cdot e\right]|\lesssim\bar{\lambda}_{2,L}^{1-\frac{2}{\gamma}},\quad(3.84)$$

$$|L^{-d}\mathbb{E}\left[\sum_{x,y\in\mathcal{P}_L}^{\neq}\int_{\partial B(x)}(\text{Id}-\mathcal{R}_{B(x)})\phi_L^y\cdot\sigma^x\nu\right]|\lesssim\bar{\lambda}_{2,L}^{1-\frac{1}{\gamma}},\quad(3.85)$$

$$\mathbb{E}\left[|D(\gamma_L)|^2(0)\right]\lesssim\lambda_L+\bar{\lambda}_{2,L}^{1-\frac{2}{\gamma}},\quad(3.86)$$

and

$$|L^{-d}\mathbb{E}\left[\sum_{x\in\mathcal{P}_L}\int_{B_+(x)}|D(\sum_{y\neq x}\phi_L^y)|^2\right]|\lesssim\bar{\lambda}_{2,L}+\bar{\lambda}_{3,L}^{1-\frac{2}{\gamma}}.\quad(3.87)$$

Lemma 3.10 is not enough to prove the bounds (3.82) and (3.83). Indeed, as  $\Lambda_{1,L}$  is a random quantity itself and from the definition (3.71) and (3.80) of  $\mathcal{E}_L^1$  and  $\mathcal{F}_L^1$ , we also have to handle the random product terms  $\mathbb{E}\left[\left(1+\frac{\Lambda_{1,L}}{1+\alpha_{1,L}}\right)L^{-d}\sum_{x,y\in\mathcal{P}_L}\int_{B(x)}\phi_L^y\cdot e\right]$ ,  $\mathbb{E}\left[\frac{|\Lambda_{1,L}|}{1+\alpha_{1,L}}V_L^1\right]$  and  $\mathbb{E}\left[|\Lambda_{1,L}|f_{Q_L}|D(\gamma_L)|^2\right]$ . For that purpose, we need the following variance estimates.

**Lemma 3.11** *Let  $L \geq 1$ ,  $\mathcal{P}_L \in \Omega$  be a  $Q_L$  periodic stationary hard-core  $\gamma$ -algebraically mixing to order 4 point process with  $\gamma > 2$  (see Definitions 3.4 and 3.5). We then have the following controls of variances :*

$$\text{Var}\left[|D(\gamma_L)|^2(0)\right]\lesssim L^{(4-\gamma)_+}(1+\ln^2(L)\mathbf{1}_{\gamma=4})\quad(3.88)$$

and

$$\text{Var}\left[L^{-d}\sum_{x,y\in\mathcal{P}_L}^{\neq}\int_{B(x)}\phi_L^y\cdot e\right]\lesssim L^{(4-\gamma)_+}(1+\ln(L)^2\mathbf{1}_{\gamma=4}).\quad(3.89)$$

**Lemma 3.12** *Let  $L \geq 1$ ,  $\mathcal{P}_L \in \Omega$  be a  $Q_L$  periodic stationary hard-core  $\gamma$ -algebraically mixing to order 2 point process with  $\gamma > 2$  (see Definitions 3.4 and 3.5). We then have the following controls of variances :*

$$\text{Var}\left[L^{-d}|\mathcal{P}_L|\right]\lesssim L^{-(d\wedge\gamma)}(1+\ln(L)\mathbf{1}_{\gamma=d})\quad(3.90)$$

and

$$\text{Var}\left[|\Lambda_{1,L}|\right]\lesssim L^{-(d\wedge\gamma)}(1+\ln(L)\mathbf{1}_{\gamma=d}).\quad(3.91)$$

□

Note that the convergence rate to zero (as  $L \rightarrow \infty$ ) in Lemma 3.12 is better than the blow up rate in Lemma 3.11 (which only diverge for  $2 < \gamma \leq 4$ ).

We now prove the bound (3.82) on  $\mathbb{E} [\mathcal{E}_L^1]$  based on the above lemmas. First, we get rid of the lower order terms and claim

$$\mathbb{E} [(L^{-d}|\mathcal{P}_L|)^3 + L^{-d}|\mathcal{P}_L|\mu_{1,L}^2] \lesssim \lambda^3 + \lambda^5 + o_L(1) , \quad (3.92)$$

using dominated convergence as we have the a.s convergence  $L^{-d}|\mathcal{P}_L| \rightarrow \lambda$  and  $\mu_{1,L} \rightarrow \theta\alpha$  as  $L \rightarrow \infty$  and these random variables are bounded.

As Lemma 3.10 already provides the bound (3.87), there only remains to control

$$\mathbb{E} \left[ |\Lambda_{1,L}| \int_{Q_L} |D(\gamma_L)|^2 \right] = \mathbb{E} [|\Lambda_{1,L}| |D(\gamma_L)|^2(0)]$$

(by stationarity). If the two random variables were uncorrelated, we would write

$$\mathbb{E} [|\Lambda_{1,L}| |D(\gamma_L)|^2(0)] = \mathbb{E} [|\Lambda_{1,L}|] \times \mathbb{E} [|D(\gamma_L)|^2(0)] \quad (3.93)$$

$$\lesssim \lambda(\lambda + \lambda_2^{1-\frac{2}{\gamma}}) + o_L(1) \quad (3.94)$$

combining the expectation estimate (3.86) on  $D(\gamma_L)$  (and Lemma 3.1 so that  $(\lambda_L, \bar{\lambda}_{2,L}) \rightarrow (\lambda, \lambda_2)$ ) with the convergence  $\mathbb{E} [|\Lambda_{1,L}|] \xrightarrow{L \rightarrow \infty} \alpha \lesssim \lambda$ . As the two random variables are correlated, (3.93) only holds up to a covariance correction

$$\mathbb{E} [|\Lambda_{1,L}| |D(\gamma_L)|^2(0)] = \mathbb{E} [|\Lambda_{1,L}|] \times \mathbb{E} [|D(\gamma_L)|^2(0)] + \text{Cov}(|\Lambda_{1,L}|, |D(\gamma_L)|^2(0)) .$$

But this correction vanishes : we have

$$|\text{Cov}(|\Lambda_{1,L}|, |D(\gamma_L)|^2(0))|^2 \leq \text{Var} [|D(\gamma_L)|^2(0)] \text{Var} [|\Lambda_{1,L}|] \xrightarrow{L \rightarrow \infty} 0$$

combining the estimates (3.88) and (3.91) (in any case, as  $\gamma > 2$ ). Hence, the bound (3.94) still holds and we conclude that

$$\mathbb{E} [|\Lambda_{1,L}| |D(\gamma_L)|^2(0)] \lesssim \lambda(\lambda + \bar{\lambda}_2^{1-\frac{2}{\gamma}}) + o_L(1) . \quad (3.95)$$

The estimate (3.83) on  $\mathcal{F}_L^1$  is obtained similarly (noting that  $V_L^1$  is proportional to  $L^{-d}|\mathcal{P}_L|$ ).

**Remark 3.12** Instead of relying on a covariance estimate, we could also treat the product terms directly using Cauchy-Schwarz. For instance, in the case of (3.95), we could write

$$\mathbb{E} [|\Lambda_{1,L}| |D(\gamma_L)|^2(0)]^2 \leq \mathbb{E} [|\Lambda_{1,L}|^2] \mathbb{E} [|D(\gamma_L)|^4(0)] .$$

This would then require to keep track of the dependency of the 4<sup>th</sup> moment  $\mathbb{E}[|D(\gamma_L)|^4(0)]$  on  $(\lambda_i)_{1 \leq i \leq 4}$  (which could be cumbersome).

Our method capitalizes on the expectation estimates already obtained in Lemma 3.10 and allows for a cruder treatment of the 4<sup>th</sup> moments.  $\square$

**Remark 3.13** It is crucial to reformulate the cluster expansion error  $\int_{\mathcal{I}_L} r_L^{(1)} \cdot e$  in (3.81) before applying the energy estimate (3.70) on  $r_L^{(1)}$  to obtain the sharp control (3.67) on  $|\mathbb{E}[R_L^1]|$ . Not doing so would have led to a coarser bound (essentially worse by a square root).

Indeed, as in (3.76) or Remark 3.9, one could be tempted to use

$$\int_{\mathcal{I}_L} r_L^{(1)} \cdot e = \int_{Q_L} D(r_L^{(1)}) : D(\gamma_L) .$$

Using Cauchy-Schwartz inequality, this would then lead to

$$\lambda_L^{-1} L^{-d} |\mathbb{E} \left[ \int_{\mathcal{I}_L} r_L^{(1)} \cdot e \right]| \lesssim \lambda_L^{-1} |\mathbb{E}[\mathcal{E}_L^1]|^{\frac{1}{2}} \mathbb{E} \left[ \int_{Q_L} |D(\gamma_L)|^2 \right]^{\frac{1}{2}} \lesssim \left( \frac{\lambda_2}{\lambda} \right)^{\frac{1}{2}} + o_L(1) .$$

where, to track scaling more easily, we have assumed that  $\lambda_3^{1-\frac{2}{\gamma}} \leq \lambda_2 \leq \lambda_2^{1-\frac{2}{\gamma}} \leq \lambda < 1$  so that (3.82) and (3.86) becomes  $|\mathbb{E}[\mathcal{E}_L^1]| \lesssim \lambda_2 + o_L(1)$  and  $\mathbb{E} \left[ \int_{Q_L} |D(\gamma_L)|^2 \right] \lesssim \lambda + o_L(1)$ .

In comparison, the other term in the definition (3.66) of  $R_L^1$  can be estimated

$$\lambda_L^{-1} L^{-d} \mathbb{E} \left[ \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \phi_L^y \cdot e \right] \lesssim \frac{\lambda_2^{1-\frac{2}{\gamma}}}{\lambda} + o_L(1)$$

using the explicit renormalization (3.84) which is, in fact, the true leading order in the sharp bound (3.67).  $\square$

### 3.3.3 Proof of the renormalization lemmas : Lemmas 3.10 to 3.12

#### Expectations

In view of our mixing assumption of Definition 3.4, we need the following deterministic estimates before turning to the proof of the expectation estimates in Lemma 3.10.

**Lemma 3.13** *Let  $\bar{\lambda} \leq 1$ ,  $\gamma > 2$  and  $\beta \in \{1, 2\}$ . We have*

$$\int_{Q_L} \langle z \rangle^{\beta-d} (\langle z \rangle^{-\gamma} \wedge \bar{\lambda}) dz \lesssim_{\gamma} \bar{\lambda}^{1-\frac{\beta}{\gamma}} \quad (3.96)$$

and

$$\iint_{Q_L \times Q_L} \langle x \rangle^{1-d} \langle y \rangle^{1-d} (\langle x-y \rangle_L^{-\gamma} \wedge \bar{\lambda}) dx dy \lesssim \bar{\lambda}^{1-\frac{2}{\gamma}}. \quad (3.97)$$

□

We recall that we are using the Japanese bracket notation  $\langle \cdot \rangle$ , see Section 3.2.1.

PROOF For (3.96), we set  $R > 0$  s.t.  $\langle R \rangle^{-\gamma} = \bar{\lambda}$ . Since  $\bar{\lambda} \leq 1$ ,  $R \geq 1$  and  $R \sim \langle R \rangle$ . Using polar coordinates, we have

$$\int_{Q_L} \langle z \rangle^{\beta-d} (\langle z \rangle^{-\gamma} \wedge \bar{\lambda}) dz \lesssim \bar{\lambda} \int_0^R \langle r \rangle^{\beta-d} r^{d-1} dr + \int_R^\infty \langle r \rangle^{\beta-d-\gamma} r^{d-1} dr \lesssim \bar{\lambda} R^\beta + R^{\beta-\gamma}$$

since  $\gamma > \beta$ . Plugging in  $R \sim \bar{\lambda}^{-\frac{1}{\gamma}}$ , this yields (3.96).

For (3.97), we set  $I := \iint_{Q_L \times Q_L} \langle x \rangle^{1-d} \langle y \rangle^{1-d} (\langle x-y \rangle_L^{-\gamma} \wedge \bar{\lambda}) dx dy$  and, as before,  $\langle R \rangle^{-\gamma} = \bar{\lambda}$ . We split the integral

$$\begin{aligned} I &\lesssim R^{-\gamma} \iint_{Q_L^2} \langle x \rangle^{1-d} \langle y \rangle^{1-d} \mathbf{1}_{|x-y|_L \leq R} dx dy \\ &\quad + \iint_{Q_L^2} \langle x \rangle^{1-d} \langle y \rangle^{1-d} \langle x-y \rangle_L^{-\gamma} \mathbf{1}_{|x-y|_L \geq R} dx dy \\ &\lesssim I_1 + I_2 \end{aligned}$$

and shall prove that each term is  $\lesssim R^{2-\gamma}$  which yields (3.97).

For  $I_1$ , we claim that

$$\int_{|y-x|_L \leq R} \langle y \rangle^{1-d} dy \lesssim R \mathbf{1}_{|x| \leq 2R} + R^d \langle x \rangle^{1-d} \mathbf{1}_{|x| \geq 2R}. \quad (3.98)$$

Indeed, either  $|x| \leq 2R$  and we have  $\int_{|y-x|_L \leq R} \langle y \rangle^{1-d} dy \leq \int_{|y| \leq 3R} \langle y \rangle^{1-d} dy \lesssim R$  or  $|x| \geq 2R$  and we have  $\langle y \rangle^{1-d} \lesssim \langle x \rangle^{1-d}$  for all  $y \in B_R^L(x)$  as  $|y|_L \geq ||x|_L - |y-x|_L| \geq |x| - R \geq \frac{|x|}{2}$ .

Integrating (3.98) yields

$$I_1 \lesssim R^{1-\gamma} \int_{|x| \leq 2R} \langle x \rangle^{1-d} dx + R^{d-\gamma} \int_{|x| \geq 2R} \langle x \rangle^{2(1-d)} \lesssim R^{2-\gamma}.$$

For  $I_2$ , we set  $J(x) := \int_{\substack{y \in Q_L \\ |y| \leq |x|}} \langle y \rangle^{1-d} \langle x-y \rangle_L^{-\gamma} \mathbf{1}_{|x-y|_L \geq R} dy$ . As  $x$  and  $y$  play symmetric roles, we have

$$I_2 = 2 \int_{Q_L} \langle x \rangle^{1-d} J(x) dx . \quad (3.99)$$

We claim that

$$J(x) \lesssim \mathbf{1}_{|x| \geq \frac{R}{2}} \times \begin{cases} \langle x \rangle^{1-d} R^{d-\gamma} & \gamma > d \\ \langle x \rangle^{1-d} \ln \left( \frac{2\langle x \rangle}{R} \right) & \gamma = d \\ \langle x \rangle^{1-\gamma} & \gamma < d \end{cases} . \quad (3.100)$$

This yields  $I_2 \lesssim R^{2-\gamma}$  in every case (as  $\gamma > 2$ ) plugging this bound into (3.99).

To prove (3.100), we first remark that we can introduce  $\mathbf{1}_{|x| \geq \frac{R}{2}}$  at no cost since  $|x-y|_L \geq R$  and  $|y| \leq |x|$  imply that  $R \leq |x-y|_L \leq |x|_L + |y|_L \leq 2|x|$ . Then, splitting whether  $y$  is comparable to  $x$  or not, we have

$$J(x) \leq \int_{|y| \leq \frac{|x|}{2}} \langle y \rangle^{1-d} \langle x-y \rangle_L^{-\gamma} dy + \int_{\frac{|x|}{2} \leq |y| \leq |x|} \langle y \rangle^{1-d} \langle x-y \rangle_L^{-\gamma} \mathbf{1}_{|x-y|_L \geq R} dy . \quad (3.101)$$

For the first term in the RHS of (3.101), we write

$$\int_{|y| \leq \frac{|x|}{2}} \langle y \rangle^{1-d} \langle x-y \rangle_L^{-\gamma} dy \lesssim \langle x \rangle^{-\gamma} \int_{|y| \leq \frac{|x|}{2}} \langle y \rangle^{1-d} dx \lesssim \langle x \rangle^{1-\gamma}$$

as for  $|y| \leq \frac{|x|}{2}$ , we have  $|x-y|_L \geq |x|_L - |y|_L \geq \frac{|x|}{2}$ . This term is controlled by the bound in (3.100) in any case.

For the second term in the RHS of (3.101), we can directly write

$$\begin{aligned} \int_{\frac{|x|}{2} \leq |y| \leq |x|} \langle y \rangle^{1-d} \langle x-y \rangle_L^{-\gamma} \mathbf{1}_{|x-y|_L \geq R} dy \\ \lesssim \langle x \rangle^{1-d} \int_{\frac{|x|}{2} \leq |y| \leq |x|} \langle x-y \rangle_L^{-\gamma} \mathbf{1}_{|x-y|_L \geq R} dy . \end{aligned}$$

Then, as  $|\cdot|_L \leq |\cdot|$ , we note that  $\{y \in Q_L \mid \frac{|x|}{2} \leq |y| \leq |x|, |x-y|_L \geq R\} \subset \{y \in$

$Q_L \mid R \leq |x - y|_L \leq 2|x|$  and we can estimate

$$\begin{aligned} \int_{\frac{|x|}{2} \leq |y| \leq |x|} \langle x - y \rangle_L^{-\gamma} \mathbf{1}_{|x-y|_L \geq R} dy &\lesssim \int_{B_{2|x|}^L(x) \setminus B_R^L(x)} \langle x - y \rangle_L^{-\gamma} dy \\ &\lesssim \int_R^{2|x|} r^{d-1-\gamma} dr \lesssim \begin{cases} R^{d-\gamma} & \gamma > d \\ \ln\left(\frac{2|x|}{R}\right) & \gamma = d \\ \langle x \rangle^{d-\gamma} & \gamma < d \end{cases} \end{aligned}$$

■

which then leads to (3.100).

**Remark 3.14** We also have that for all  $\delta \in (2, \gamma)$ ,

$$\iint_{Q_L \times Q_L} \langle x \rangle^{1-d} \langle y \rangle^{1-d} (\langle x - y \rangle_L^{-\gamma} \wedge \bar{\lambda}) dx dy \lesssim_{\delta, \gamma} \bar{\lambda}^{1-\frac{\delta}{\gamma}}. \quad (3.102)$$

This is a slightly weaker result than (3.97) but the proof is shorter.

Indeed, using the same notations, we take  $p, q$  s.t.  $\frac{1}{p} + \frac{1}{q} = 1$  and  $a, b$  s.t.  $1 + \frac{1}{q} = \frac{1}{a} + \frac{1}{b}$  (to be fixed later) and we have

$$\begin{aligned} I &\leq \|\langle \cdot \rangle^{1-d}\|_{L^p(Q_L)} \|\langle \cdot \rangle^{1-d} * (\langle \cdot \rangle^{-\gamma} \wedge \bar{\lambda})\|_{L^q(Q_L)} \\ &\leq \|\langle \cdot \rangle^{1-d}\|_{L^p(\mathbb{R}^d)} \|\langle \cdot \rangle^{1-d}\|_{L_w^b(\mathbb{R}^d)} \|\langle \cdot \rangle^{-\gamma} \wedge \bar{\lambda}\|_{L^a(\mathbb{R}^d)} \end{aligned}$$

by Hölder's inequality and Young's inequality (in weak  $L^p$  spaces). From a direct computation,  $\|\langle \cdot \rangle^{1-d}\|_{L^p(\mathbb{R}^d)} \lesssim 1$  for  $p > \frac{d}{d-1}$  and  $\|\langle \cdot \rangle^{1-d}\|_{L_w^b(\mathbb{R}^d)} \lesssim 1$  for  $b \geq \frac{d}{d-1}$ .

Proceeding similarly to (3.96), we obtain  $\|\langle \cdot \rangle^{-\gamma} \wedge \bar{\lambda}\|_{L^a(\mathbb{R}^d)} \lesssim \bar{\lambda}^{1-\frac{d}{a\gamma}}$  for  $a > \frac{d}{\gamma}$ . Hence, with all the exponents in those ranges, we have obtained

$$I \lesssim \bar{\lambda}^{1-\frac{d}{a\gamma}}.$$

To optimize this bound, we would like to take  $a$  as big as possible. Hence, we chose the smallest  $b$  i.e.  $b = \frac{d}{d-1}$ . The condition on  $p$  can be rewritten  $\frac{1}{p} < 1 - \frac{1}{d}$  which is equivalent to  $\frac{1}{a} > \frac{2}{d}$  as  $\frac{1}{a} = 1 + \frac{1}{d} - \frac{1}{p}$ . We can then pick any  $a$  s.t.  $2 < \frac{d}{a} < \gamma$  (which exists as  $\gamma > 2$ ). This yields (3.102) setting  $\frac{d}{a} = \delta$ . □

With these results at hand, we can turn to our expectation estimates.

**PROOF (LEMMA 3.10)** For (3.84), we use the short-hand notation  $E_1 := L^{-d} \mathbb{E} \left[ \sum_{x, y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \phi_L^y \cdot e \right]$ . We start by reformulating it and we call this step

renormalization (as this rewriting would even allow us to pass to the limit  $L \rightarrow \infty$  in the renormalized formula (3.103) below which we could not by taking absolute values, see Remark 3.15). By definition of  $f_{2,L}$ ,

$$E_1 = L^{-d} \iint_{Q_L \times Q_L} \int_{B(x)} \phi_L^y \cdot e f_{2,L}(x, y) dx dy$$

Decomposing  $f_{2,L} = h_{2,L} + \lambda_L^2$  (see (3.21)) and using the cancellation property (3.54)  $\int_{Q_L} \phi_L^y dy = 0$ , we have

$$E_1 = L^{-d} \iint_{Q_L \times Q_L} \int_{B(x)} \phi_L^y \cdot e h_{2,L}(x, y) dx dy . \quad (3.103)$$

Using the decay (3.55) of  $\phi_L^y$ , we then obtain

$$\begin{aligned} |E_1| &\lesssim L^{-d} \iint_{Q_L \times Q_L} \langle x - y \rangle_L^{2-d} |h_{2,L}|(x, y) dx dy \\ &\lesssim L^{-d} \iint_{Q_L \times Q_L} \langle x - y \rangle_L^{2-d} (\langle x - y \rangle_L^{-\gamma} \wedge \bar{\lambda}_{2,L}) dx dy \end{aligned}$$

where we have used Remark 3.5 to smuggle in the decay of  $h_{2,L}$ . By a change of variables and using Lemma 3.13 to compute the upper bounds, this yields

$$|E_1| \lesssim \int_{Q_L} \langle z \rangle^{2-d} (\langle z \rangle^{-\gamma} \wedge \bar{\lambda}_2) dz \lesssim \bar{\lambda}_{2,L}^{-1-\frac{2}{\gamma}}$$

which concludes the proof of (3.84).

For (3.85), we set  $E_2 := L^{-d} \mathbb{E} \left[ \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{\partial B(x)} (\text{Id} - \mathcal{R}_{B(x)}) \phi_L^y \cdot \sigma^x \nu \right]$  and proceed similarly. After renormalization, we get

$$E_2 = L^{-d} \iint_{Q_L \times Q_L} \int_{\partial B(x)} (\text{Id} - \mathcal{R}_B(x)) \phi_L^y \cdot \sigma_L^x \nu h_{2,L}(x, y) dd dy .$$

This implies

$$|E_2| \lesssim L^{-d} \iint_{Q_L \times Q_L} \langle x - y \rangle_L^{1-d} |h_{2,L}|(x, y) dx dy$$

by combining the trace estimate of Lemma 3.7 with the decay  $\|D(\phi_L^y)\|_{L^2(B(x))} \lesssim \langle x - y \rangle^{1-d}$  (see (3.55)). It yields (3.85) using Remark 3.5 and Lemma 3.13 as above.

For (3.86), we set  $E_3 := \mathbb{E} [|D(\gamma_L)|^2(0)]$ . Recalling that  $\gamma_L = \sum_{x \in \mathcal{P}_L} \gamma_L^x$ , we

take the square of the sum, sort it and obtain

$$\begin{aligned} E_3 &= \mathbb{E} \left[ \sum_{x \in \mathcal{P}_L} |D(\gamma_L^x)|^2(0) + \sum_{\substack{x, y \in \mathcal{P}_L \\ x \neq y}} D(\gamma_L^x) : D(\gamma_L^y)(0) \right] \\ &= \int_{Q_L} |D(\gamma_L^x)|^2(0) \lambda_L dx + \iint_{Q_L \times Q_L} D(\gamma_L^x) : D(\gamma_L^y)(0) f_{2,L}(x, y) dx dy \\ &= E_{3,1} + E_{3,2} \end{aligned}$$

by Definition 3.2 of  $\lambda_L$  and  $f_{2,L}$ .

Recalling that  $D(\gamma_L^x)(0) \lesssim \langle x \rangle_L^{1-d}$  (see (3.45)), this yields

$$|E_{3,1}| \lesssim \lambda_L \int_{Q_L} \langle x \rangle^{2(1-d)} dx \lesssim \lambda_L \quad (3.104)$$

by direct computation since  $d \geq 3$ .

$E_{3,2}$  need to be renormalized. As  $\int_{Q_L} D(\gamma_L^x)(0) dx = \int_{Q_L} D(\gamma_L^\circ)(-x) dx = 0$  (the gradient of a periodic function has mean zero),

$$E_{3,2} = \iint_{Q_L \times Q_L} D(\gamma_L^x) : D(\gamma_L^y)(0) h_{2,L}(x, y) dx dy$$

Using again the decay (3.45) of  $D(\gamma_L^x)$ , the one of  $h_{2,L}$  (with Remark 3.5), we get

$$|E_{3,2}| \lesssim \iint_{Q_L \times Q_L} \langle x \rangle_L^{1-d} \langle y \rangle_L^{1-d} (\langle x - y \rangle_L^{-\gamma} \wedge \bar{\lambda}_{2,L}) dx dy \lesssim \bar{\lambda}_{2,L}^{1-\frac{2}{\gamma}}$$

estimating the integral with Lemma 3.13. Combined with (3.104), this concludes the proof of (3.86).

For (3.87), we set  $E_4 := L^{-d} \mathbb{E} \left[ \sum_{x \in \mathcal{P}_L} \int_{B_+(x)} |D(\sum_{y \neq x} \phi_L^y)|^2 \right]$ . Sorting the squared sum and using the Definition 3.2 of  $f_{2,L}$  and  $f_{3,L}$ , we have

$$\begin{aligned} E_4 &= L^{-d} \mathbb{E} \left[ \sum_{\substack{x, y \in \mathcal{P}_L \\ x \neq y}} \int_{B_+(x)} |D(\phi_L^y)|^2 + \sum_{\substack{x, y, z \in \mathcal{P}_L \\ x \neq y, z}} \int_{B_+(x)} D(\phi_L^y) : D(\phi_L^z) \right] \\ &= L^{-d} \iint_{Q_L^2} \int_{B_+(x)} |D(\phi_L^y)|^2 f_{2,L}(x, y) dx dy \\ &\quad + L^{-d} \iiint_{Q_L^3} \int_{B_+(x)} D(\phi_L^y) : D(\phi_L^z) f_{3,L}(x, y, z) dx dy dz \\ &= E_{4,1} + E_{4,2} . \end{aligned}$$



For  $E_{4,1}$ , we directly use the decay (3.55) of  $\phi_L^y$  and Remark 3.5 to write

$$|E_{4,1}| \lesssim L^{-d} \iint_{Q_L^2} \langle x - y \rangle_L^{2(1-d)} \lambda_{2,L} dx dy \lesssim \bar{\lambda}_{2,L} . \quad (3.105)$$

$E_{4,2}$  needs renormalization. Recalling the decomposition (3.22) of  $f_{3,L}$ , we have

$$E_{4,2} = L^{-d} \iiint_{Q_L^3} \int_{B_+(x)} D(\phi_L^y) : D(\phi_L^z) (h_{3,L}(x, y, z) + \lambda_L h_{2,L}(y, z)) dx dy dz .$$

Indeed, as  $\int_{Q_L} D(\phi_L^y) dy = 0$ , we can remove the terms which do not depend on  $y$  in the decomposition (3.22) of  $f_{3,L}$  and, as  $\int_{Q_L} D(\phi_L^z) dz$ , we can proceed likewise with  $z$ . From the decay (3.55) of  $\phi_L^{y/z}$ , the decay of  $h_{2/3,L}$  and Remark 3.5, we obtain

$$|E_{4,2}| \lesssim L^{-d} \iiint_{Q_L^3} \langle y - x \rangle_L^{1-d} \langle z - x \rangle_L^{1-d} \\ ((\langle y - z \rangle_L^{-\gamma} \wedge \bar{\lambda}_{3,L}) + \lambda (\langle y - z \rangle_L^{-\gamma} \wedge \bar{\lambda}_{2,L})) dx dy dz \quad \blacksquare$$

Getting rid of the integral in  $x$  by a change of variable, we apply Lemma 3.13 to the effect that  $|E_{4,2}| \lesssim \bar{\lambda}_{3,L}^{-1-\frac{2}{\gamma}} + \lambda_L \bar{\lambda}_{2,L}^{-1-\frac{2}{\gamma}} \lesssim \bar{\lambda}_{3,L}^{-1-\frac{2}{\gamma}}$  (as  $\lambda_L \bar{\lambda}_{2,L} \leq \bar{\lambda}_{3,L} < 1$ ). Combined with the estimate (3.105) on  $E_{4,1}$  this concludes the proof of (3.87).

**Remark 3.15** We would not be able to control the random quantities appearing in Lemma 3.10 without expectation and without a careful treatment of cancellations that constitute the crucial renormalization step. Let's illustrate this on (3.84).

If we don't take the expectation, directly comparing the sum over the hardcore point process with an integral and then using the decay (3.55) of  $\phi_L^y$ , we could write

$$\left| L^{-d} \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \phi_L^y \cdot e \right| \lesssim L^{-d} \iint_{Q_L \times Q_L} \langle x - y \rangle_L^{2-d} .$$

Computing this upper bound, this would yield

$$\left| L^{-d} \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \phi_L^y \cdot e \right| \lesssim L^2$$

and this bound diverges as  $L \rightarrow \infty$ .

Now, taking the expectation but using the triangle inequality right away, we would have

$$\begin{aligned} \left| L^{-d} \mathbb{E} \left[ \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \phi_L^y \cdot e \right] \right| &\lesssim L^{-d} \mathbb{E} \left[ \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} |\phi_L^y| \right] \\ &= L^{-d} \iint_{Q_L^2} \int_{B(x)} |\phi_L^y| f_{2,L}(x,y) dx dy \end{aligned}$$

Here,  $\int_{Q_L} |\phi_L^y| dx > 0$  and we can not replace  $f_{2,L}$  by  $h_{2,L}$  as in (3.103). There is no extra decay and we can only appeal to  $\int_{Q_3^L(x) \times Q_3^L(y)} f_{2,L} \lesssim \lambda_{2,L}$  (see Definition 3.2). This would have lead to the estimate

$$\left| L^{-d} \mathbb{E} \left[ \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} |\phi_L^y| \right] \right| \lesssim \int_{Q_L} \langle z \rangle^{2-d} \lambda_{2,L} dz \lesssim L^2 \lambda_{2,L}$$

which also diverges as  $L \rightarrow \infty$  ( $\lambda_{2,L} \rightarrow \lambda_2$  by Lemma 3.1).  $\square$

**Remark 3.16** The stochastic cancellations improve the energy estimate on  $\phi_L$ .

By deterministic arguments, almost surely, we only have

$$\int_{Q_L} |\nabla \phi_L|^2 \lesssim L^2$$

from energy estimate (3.35) as  $\mathcal{P}_L \sim_{a.s.} L^d$  ( $L^{-d} \mathcal{P}_L \rightarrow \lambda$  a.s by (3.53)), see also Remark 3.8. The RHS blows up as  $L \rightarrow \infty$  (at the same rate as in Remark 3.15 without renormalization).

The renormalization removes this blow up. Indeed, recalling that  $\nabla \phi_L$  is a projection of  $\nabla \gamma_L$  (see Remark 3.9), the inequality (3.47) reads  $\int_{Q_L} |\nabla \phi_L|^2 \lesssim \int_{Q_L} |\nabla \gamma_L|^2$ . Taking the expectation and using stationarity, we obtain  $\mathbb{E} [|\nabla \phi_L|^2] \leq \mathbb{E} [|\nabla \gamma_L|^2]$ . This finally yields

$$\mathbb{E} [|\nabla \phi_L|^2] \lesssim 1 \tag{3.106}$$

using the estimate (3.86) on  $\gamma_L$  (the proof adapts straightforwardly replacing  $D(\gamma_L)$  by  $\nabla \gamma_L$ ) and the convergence of  $\lambda_L, \bar{\lambda}_{2,L}$  (see Lemma 3.1).  $\square$

**Remark 3.17** We can also illustrate here why the naive periodization procedure of Remark 3.4 does not allow to properly renormalize these random quantities.

We treat again (3.84) but with  $\widehat{\mathcal{P}}_L = \mathcal{P} \cap Q_{L-2}$ . By Definition 3.2 for  $f_2$ , we

have

$$L^{-d}\mathbb{E}\left[\sum_{x,y\in\widehat{\mathcal{P}}_L}^{\neq}\int_{B(x)}\phi_L^y\cdot e\right]=\iint_{Q_L^2}\mathbf{1}_{Q_{L-2}}(x,y)\int_{B(x)}\phi_L^y\cdot e f_2\,dx\,dy.$$

Decomposing  $f_2 = h_2 + \lambda^2$  (see (3.21)) and using the cancellation (3.54)  $\int_{Q_L}\phi_L^y\,dy$ , we obtain

$$\begin{aligned} L^{-d}\mathbb{E}\left[\sum_{x,y\in\widehat{\mathcal{P}}_L}^{\neq}\int_{B(x)}\phi_L^y\cdot e\right]&=L^{-d}\iint_{Q_{L-2}^2}\int_{B(x)}\phi_L^y h_2\cdot e\,dx\,dy \\ &\quad -L^{-d}\iint_{Q_{L-2}\times Q_L\setminus Q_{L-2}}\int_{B(x)}\phi_L^y\cdot e\lambda^2\,dx\,dy \\ &=I_{\text{bulk}}-I_{\text{boundary}}. \end{aligned}$$

Compared to the rewriting (3.103), when  $P_L$  was the periodization in law of  $\mathcal{P}$ , we have an extra boundary term  $I_{\text{boundary}}$  which can not be controlled. Indeed, the bulk term  $I_{\text{bulk}}$  is bounded ( $\lesssim \lambda_2^{1-\frac{2}{\gamma}}$ ) proceeding as with (3.103). But,

$$|I_{\text{boundary}}|\lesssim L^{-d}|Q_L\setminus Q_{L-2}|\int_{Q_L}\langle z\rangle^{2-d}\,dz\lesssim\lambda^2 L$$

which diverges as  $L\rightarrow\infty$ . □

### Variations

For the following proofs, we stop keeping track of the dependence on the multi-point intensities  $(\lambda_i)_{i\geq 1}$  in the bounds.

Once again, in view of our mixing assumption (see Definition 3.4), we need the following explicit rate of decay for the convolution of (decaying) polynomials before turning to the proof of the variances estimates of Lemma 3.11.

**Lemma 3.14** *Let  $L > 1$ ,  $\beta \in \{1, 2\}$  and  $\delta > \beta$ . Then, for all  $x \in Q_L$ ,*

$$\int_{Q_L}\langle y\rangle^{\beta-d}\langle x-y\rangle_L^{-\delta}\,dy\lesssim\langle x\rangle^{\beta-\delta\wedge d}(1+\ln\langle x\rangle\mathbf{1}_{\delta=d}). \quad (3.107)$$

□

PROOF Set  $I_L(x) := \int_{Q_L}\langle y\rangle^{\beta-d}\langle x-y\rangle_L^{-\delta}\,dy$ .

If  $x = 0$ ,  $I_L(0) = \int_{Q_L}\langle y\rangle^{-d-(\delta-\beta)}\,dy \lesssim 1$  and (3.107) holds.

For  $x \neq 0$ , we partition the torus into dyadic annuli of size  $\sim |x|$ . Setting  $n$  s.t.  $2^n \leq \frac{L}{|x|} < 2^{n+1}$  and for all  $k$ ,  $C_k^L(x) := \{y \in Q_L \mid 2^k \frac{|x|}{2} < |y - x|_L \leq 2^{k+1} \frac{|x|}{2}\}$ , we obtain

$$\begin{aligned} I_L(x) &= \int_{|y-x|_L \leq \frac{|x|}{2}} \langle y \rangle^{\beta-d} \langle x-y \rangle_L^{-\delta} dy + \sum_{k=0}^{n-1} \int_{C_k^L(x)} \langle y \rangle^{\beta-d} \langle x-y \rangle_L^{-\delta} dy \\ &\quad + \int_{\{|y-x|_L > 2^n \frac{|x|}{2}\} \cap Q_L} \langle y \rangle^{\beta-d} \langle x-y \rangle_L^{-\delta} dy \\ &= J_{-1} + \sum_{k=0}^{n-1} J_k + J_n . \end{aligned}$$

We, then, estimate each term. As  $|y|_L \geq |x|_L - |y-x|_L \geq \frac{|x|}{2}$ , we have

$$\begin{aligned} J_{-1} &\lesssim \langle x \rangle^{\beta-d} \int_{|y-x|_L \leq \frac{|x|}{2}} \langle x-y \rangle_L^{-\delta} dy \lesssim \langle x \rangle^{\beta-d} \int_1^{\frac{|x|}{2}} r^{d-1-\delta} dr \\ &\lesssim \langle x \rangle^{\beta-\delta \wedge d} (1 + \ln \langle x \rangle \mathbf{1}_{\delta=d}) . \end{aligned}$$

For all  $0 \leq k \leq n$ ,

$$J_k \lesssim 2^{-k\delta} \langle x \rangle^{-\delta} \int_{C_k^L(x)} \langle y \rangle^{\beta-d} dy \lesssim 2^{k(\beta-\delta)} \langle x \rangle^{\beta-\delta}$$

as  $\int_{C_k^L(x)} \langle y \rangle^{\beta-d} dy \lesssim \int_{|y| \leq (2^{k+1}+2) \frac{|x|}{2}} \langle y \rangle^{\beta-d} dy \lesssim 2^{k\beta} \langle x \rangle^\beta$ . And finally

$$J_n \lesssim 2^{-n\delta} \langle x \rangle^{-\delta} \int_{Q_L} \langle y \rangle^{\beta-d} dy \lesssim 2^{n(\beta-\delta)} \langle x \rangle^{\beta-\delta}$$

as  $\int_{Q_L} \langle y \rangle^{\beta-d} dy \lesssim L^\beta \lesssim 2^{\beta(n+1)} \langle x \rangle^\beta$ .

Summing these estimates, this yields (3.107) as  $\beta - \delta < 0$  implies that  $\sum_{k=0}^n 2^{k(\beta-\delta)} \lesssim 1$ .  $\blacksquare$

With this at hand, we can now turn to the proof of our (potentially slightly) diverging variances.

PROOF (LEMMA 3.11) For (3.88), we set  $V := \text{Var}[|D(\gamma_L)(0)|^2]$  and  $F(x) := D(\gamma_L^x)(0)$ . As we already control the corresponding expectation from (3.86), we have

$$V = \mathbb{E}[|D(\gamma_L)(0)|^4] - \mathbb{E}[|D(\gamma_L)(0)|^2]^2 \lesssim 1 + \mathbb{E}[|D(\gamma_L)(0)|^4] .$$

Recalling  $D(\gamma_L) = \sum_{x \in \mathcal{P}_L} F(x)$ , we take the 4<sup>th</sup> power and sort the sums so that

$$\begin{aligned} \left| \sum_{x \in \mathcal{P}_L} F(x) \right|^4 &= \sum_{x,y,z,u \in \mathcal{P}_L} F(x)F(y)F(z)F(u) \\ &= \sum_{\substack{x,y,z,u \in \mathcal{P}_L \\ \neq}} F(x)F(y)F(z)F(u) + 6 \sum_{\substack{x,y,z \in \mathcal{P}_L \\ \neq}} F(x)^2 F(y)F(z) \\ &\quad + 4 \sum_{x,y \in \mathcal{P}_L} F(x)^3 F(y) + 3 \sum_{x,y \in \mathcal{P}_L} F(x)^2 F(y)^2 + \sum_{x \in \mathcal{P}_L} F(x)^4 . \end{aligned}$$

By the definition (3.14) of the  $(f_j)_{1 \leq j \leq 4}$ , this yields

$$\begin{aligned} V &\lesssim 1 + \left| \iiint_{Q_L^4} F(x)F(y)F(z)F(u) f_{4,L}(x,y,z,u) dx dy dz du \right| \\ &\quad + \left| \iiint_{Q_L^3} F(x)^2 F(y)F(z) f_{3,L}(x,y,z) dx dy dz \right| \end{aligned} \quad (3.108)$$

$$\begin{aligned} &\quad + \left| \iint_{Q_L^2} F(x)^3 F(y) f_{2,L}(x,y) dx dy \right| \\ &\quad + \left| \int_{Q_L^2} F(x)^2 F(y)^2 f_{2,L}(x,y) dx dy \right| + \left| \int_{Q_L} F(x)^4 \lambda_L dx \right| \\ &\lesssim 1 + |I_4| + |I_3| + |I_{2,1}| + |I_{2,2}| + |I_1| . \end{aligned} \quad (3.109)$$

$I_4$  needs to be renormalized. Plugging the decomposition (3.23) of  $f_{4,L}$  in terms of  $(h_j)_{1 \leq j \leq 4}$  in  $I_4$ , we only keep the terms which depend on the four variables :  $h_4$  and the products  $h_2 \otimes h_2$ . Indeed, since  $\int_{Q_L} F(x) dx = 0$ , we can remove all the terms independent of  $x$  and similarly with  $y, z$  and  $u$ . Moreover, as the variables plays symmetric role, we obtain

$$\begin{aligned} I_4 &= \iiint_{Q_L^4} F(x)F(y)F(z)F(u) h_{4,L}(x,y,z,u) dx dy dz du \\ &\quad + 3 \left( \iint_{Q_L^2} F(x)F(y) h_{2,L}(x,y) dx dy \right)^2 . \end{aligned}$$

Using the decays (3.26) of  $h_{2,L}$  and  $h_{4,L}$  (with Remark 3.5) in the form

$$\int_{Q_3^L(x) \times Q_3^L(y)} |h_{2,L}| \lesssim \langle x - y \rangle_L^{-\gamma}$$

and

$$\int_{Q_3^L(x) \times \dots \times Q_3^L(u)} |h_{4,L}| \lesssim \langle x - y \rangle_L^{-\gamma} \wedge \langle z - u \rangle_L^{-\gamma} \leq \langle x - y \rangle_L^{-\frac{\gamma}{2}} \langle z - u \rangle_L^{-\frac{\gamma}{2}}$$

(as  $a \wedge b \leq a^{\frac{1}{2}} b^{\frac{1}{2}}$  for  $a, b \in \mathbb{R}$ ) with the decay  $|F(x)| \lesssim \langle x \rangle^{1-d}$  (see (3.45)), we obtain

$$|I_4| \lesssim \left( \iint_{Q_L^2} \langle x \rangle^{1-d} \langle y \rangle^{1-d} \langle x - y \rangle_L^{-\frac{\gamma}{2}} dx dy \right)^2 + \left( \iint_{Q_L^2} \langle x \rangle^{1-d} \langle y \rangle^{1-d} \langle x - y \rangle_L^{-\gamma} dx dy \right)^2.$$

Using Lemma 3.14 to estimate these upper bounds, this yields

$$\begin{aligned} |I_4| &\lesssim \left( \int_{Q_L} \langle x \rangle^{2-d-d\wedge\frac{\gamma}{2}} (1 + \ln \langle x \rangle) \mathbf{1}_{\gamma=2d} dx \right)^2 + \left( \int_{Q_L} \langle x \rangle^{2-d-d\wedge\gamma} (1 + \ln \langle x \rangle) \mathbf{1}_{\gamma=d} dx \right)^2 \\ &\lesssim \left( L^{(2-\frac{\gamma}{2})+} (1 + \ln(L) \mathbf{1}_{\gamma=4}) \right)^2 + 1. \end{aligned} \quad (3.110)$$

For  $I_3$ , we proceed similarly. We use the decomposition (3.22) of  $f_{3,L}$  in  $(h_j)_{1 \leq j \leq 3}$  and only keep the terms depending both on  $y$  and  $z$ . Note that here we can not remove the terms independent of  $x$  as  $\int_{Q_L} F^2(x) dx > 0$ . We obtain

$$I_3 = \iiint_{Q_L^3} F(x)^2 F(y) F(z) (h_{3,L}(x, y, z) + \lambda_L h_{2,L}(y, z)) dx dy dz.$$

Using this time the bounds (3.26) on  $h_{3,L}$  and  $h_{2,L}$  in the form  $\lesssim \langle y - z \rangle_L^{-\gamma}$  (see Remark 3.5) and the decay (3.45) of  $F(x)$ , this yields

$$|I_3| \lesssim \left( \int_{Q_L} \langle x \rangle^{2(1-d)} dx \right) \left( \iint_{Q_L^2} \langle y \rangle^{1-d} \langle z \rangle^{1-d} \langle y - z \rangle_L^{-\gamma} dy dz \right) \lesssim 1 \quad (3.111)$$

by Lemma 3.14 as in (3.110).

For  $I_{2,1}$ , proceeding similarly, we have

$$I_{2,1} = \iint_{Q_L^2} F(x)^3 F(y) h_{2,L}(x, y) dx dy.$$

So that

$$|I_{2,1}| \lesssim \iint_{Q_L^2} \langle x \rangle^{3(1-d)} \langle y \rangle^{1-d} \langle x-y \rangle_L^{-\gamma} dx dy \lesssim 1. \quad (3.112)$$

For  $I_{2,2}$  and  $I_1$ , there is no need for renormalization. We can directly write

$$I_{2,2} \lesssim \lambda_{2,L} \left( \int_{Q_L} \langle x \rangle^{2(1-d)} dx \right) \lesssim 1 \quad (3.113)$$

and

$$|I_1| \lesssim \int_{Q_L} \langle x \rangle^{4(1-d)} dx \lesssim 1. \quad (3.114)$$

Combining the intermediate bounds (3.114), (3.112), (3.113), (3.111), (3.110) and (3.109), this concludes the proof of (3.88).

For (3.89), we set  $W := \text{Var} \left[ L^{-d} \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \phi_L^y \cdot e \right]$  and  $G(x,y) := \int_{B(x)} \phi_L^y \cdot e$ . Note that from the definition of  $\phi_L^x$  (see Lemma 3.3)  $G(x,y) = G(0,y-x) = G(x-y,0)$  hence  $\int_{Q_L} G(x,y) dx = \int_{Q_L} G(x,y) dy = 0$  (see (3.54)) and  $|G(x,y)| \lesssim \langle x-y \rangle_L^{2-d}$ .

We proceed similarly to (3.88). As we already control the expectation from (3.84), we have

$$W \lesssim 1 + L^{-2d} \mathbb{E} \left[ \left( \sum_{x,y \in \mathcal{P}_L}^{\neq} G(x,y) \right)^2 \right].$$

Squaring and sorting the sum and by definition (3.14) of the  $(f_{j,L})_{1 \leq j \leq 4}$ , we obtain

$$W \lesssim 1 + \left| L^{-2d} \iiint_{Q_L^4} G(x,y) G(z,u) f_{4,L} dx dy dz du \right| \quad (3.115)$$

$$+ \left| L^{-2d} \iiint_{Q_L^3} G(x,y) (G(x,z) + G(y,z) + G(z,x) + G(z,y)) f_{3,L} dx dy dz \right| \quad (3.116)$$

$$+ \left| L^{-2d} \iint_{Q_L^2} G(x,y) (G(x,y) + G(y,x)) f_{2,L} dx dy \right| \quad (3.117)$$

$$\lesssim 1 + |J_4| + |J_3| + |J_2| \quad (3.118)$$

We need to renormalize  $J_4$ . Using the cancellation property of  $G$ , we only keep the terms depending on the four variables in the decomposition (3.23) of  $f_{4,L}$  so

that

$$\begin{aligned}
J_4 &= L^{-2d} \iiint_{Q_L^4} G(x, y)G(z, u)h_{4,L} dx dy dz du \\
&\quad + L^{-2d} \iiint_{Q_L^4} G(x, y)G(z, u)h_{2,L}(x, y)h_{2,L}(z, u) dx dy dz du \\
&\quad + L^{-2d} \iiint_{Q_L^4} G(x, y)G(z, u)h_{2,L}(x, z)h_{2,L}(y, u) dx dy dz du \\
&\quad + L^{-2d} \iiint_{Q_L^4} G(x, y)G(z, u)h_{2,L}(x, u)h_{2,L}(y, z) dx dy dz du \\
&= J_{4,1} + J_{4,2} + J_{4,3} + J_{4,4} .
\end{aligned}$$

Using the decay of  $h_{2,L}$ ,  $h_{4,L}$  and  $G$ , we have

$$\begin{aligned}
|J_{4,1}| &\lesssim L^{-2d} \iiint_{Q_L^4} \langle x - y \rangle_L^{2-d-\frac{\gamma}{2}} \langle z - u \rangle_L^{2-d-\frac{\gamma}{2}} dx dy dz du \\
&\lesssim \left( \int_{Q_L} \langle x \rangle^{2-d-\frac{\gamma}{2}} \right)^2 \lesssim \left( L^{(2-\frac{\gamma}{2})_+} (1 + \ln(L) \mathbf{1}_{\gamma=4}) \right)^2 .
\end{aligned}$$

Similarly, we obtain

$$|J_{4,2}| \lesssim L^{-2d} \iiint_{Q_L^4} \langle x - y \rangle_L^{2-d-\gamma} \langle z - u \rangle_L^{2-d-\gamma} dx dy dz du \lesssim \left( \int_{Q_L} \langle x \rangle^{2-d-\gamma} \right)^2 \lesssim 1 .$$

Besides, using Lemma 3.14, we obtain

$$\begin{aligned}
|J_{4,3}| &\lesssim L^{-2d} \iiint_{Q_L^4} \langle x - y \rangle_L^{2-d} \langle z - u \rangle_L^{2-d} \langle x - z \rangle_L^{-\gamma} \langle y - u \rangle_L^{-\gamma} dx dy dz du \\
&\lesssim L^{-d} \iiint_{Q_L^3} \langle y \rangle^{2-d} \langle y - u \rangle_L^{-\gamma} \langle z \rangle_L^{2-d} \langle z - u \rangle^{-\gamma} dz dy du \\
&\lesssim L^{-d} \int_{Q_L} \langle u \rangle^{2(2-\gamma \wedge d)} (1 + \ln \langle u \rangle \mathbf{1}_{\gamma=d})^2 du \\
&\lesssim \begin{cases} L^{-([\!2(\gamma-2)\!] \wedge d)} (1 + \ln(L) \mathbf{1}_{2(\gamma-2)=d}) & \gamma < d \\ (1 + \ln(L)^2 \mathbf{1}_{\gamma=d}) L^{-d} L^{(4-d)_+} (1 + \ln(L) \mathbf{1}_{d=4}) & \gamma \geq d \end{cases} \\
&\lesssim 1 .
\end{aligned}$$

Finally,  $J_{4,4}$  is treated as  $J_{4,3}$  as  $z$  and  $u$  play symmetric roles. We have obtained

$$|J_4| \lesssim L^{(4-\gamma)_+} (1 + \ln(L)^2 \mathbf{1}_{\gamma=4}) . \quad (3.119)$$



For  $J_3$ , we proceed similarly. We only estimate

$$J_{3,1} := L^{-2d} \iiint_{Q_L^3} G(x, y)G(x, z)f_{3,L} dx dy dz ,$$

the other terms are treated similarly permuting the variables (as  $f_{3,L}$  is symmetric). Plugging in the decomposition (3.22) of  $f_{3,L}$ , we only keep the terms depending both on  $y$  and  $z$  so that

$$J_{3,1} = L^{-2d} \iiint_{Q_L^3} G(x, y)G(x, z)(h_{3,L} + \lambda_L h_{2,L}(y, z)) dx dy dz$$

Using the decays, this yields

$$|J_{3,1}| \lesssim L^{-d} \iint_{Q_L^2} \langle y \rangle^{2-d} \langle z \rangle^{2-d} \langle y - z \rangle_L^{-\gamma} dy dz$$

which we estimate using Lemma 3.14 so that

$$\begin{aligned} |J_{3,1}| &\lesssim L^{-d} \int_{Q_L} \langle y \rangle^{4-d-d\wedge\gamma} (1 + \ln \langle x \rangle \mathbf{1}_{\gamma=d}) dy \\ &\lesssim L^{-d} \begin{cases} L^{(4-\gamma)_+} (1 + \ln(L) \mathbf{1}_{\gamma=4}) & \gamma < d \\ 1 + (L \mathbf{1}_{d=3} + \ln(L) \mathbf{1}_{d=4}) (1 + \ln(L) \mathbf{1}_{\gamma=d}) & \gamma \geq d \end{cases} \lesssim 1 . \end{aligned}$$

We have obtained

$$|J_3| \lesssim 1 . \quad (3.120)$$

For  $J_2$ , we directly write

$$\begin{aligned} |J_2| &\lesssim L^{-2d} \iint_{Q_L^2} \langle x - y \rangle_L^{2(2-d)} \lambda_{2,L} dx dy \lesssim L^{-d} \int_{Q_L} \langle z \rangle^{2(2-d)} dz \\ &\lesssim L^{-d} (1 + L \mathbf{1}_{d=3} + \ln(L) \mathbf{1}_{d=4}) \lesssim 1 . \end{aligned} \quad (3.121)$$

■

Combining (3.118) with (3.119), (3.120), (3.121), this concludes the proof of (3.89).

We conclude this section by proving the rate of convergence of our vanishing variances.

PROOF (LEMMA 3.12) For (3.90), as  $|\mathcal{P}_L| = \sum_{x \in \mathcal{P}_L} \mathbf{1}_{Q_L}(x)$ , we expand and sort the square in the variance and use the definition (3.14) of  $f_{1,L} = \lambda_L$  and  $f_{2,L}$  to

write

$$\begin{aligned} \text{Var} [L^{-d}|\mathcal{P}_L|] &= L^{-2d}\mathbb{E} \left[ \sum_{x \in \mathcal{P}_L} \mathbb{1}_{Q_L}^2(x) \right] + L^{-2d}\mathbb{E} \left[ \sum_{\substack{x,y \in \mathcal{P}_L \\ x \neq y}} \mathbb{1}_{Q_L}(x)\mathbb{1}_{Q_L}(y) \right] \\ &\quad - \mathbb{E} [L^{-d}|\mathcal{P}_L|]^2 \\ &= L^{-2d} \int_{Q_L} \lambda_L + L^{-2d} \iint_{Q_L^2} f_{2,L} - \lambda_L^2 = L^{-d}\lambda_L + L^{-2d} \iint_{Q_L^2} h_{2,L} . \end{aligned}$$

Then, using the decay (3.27) of  $h_{2,L}$  (with Remark 3.5), we have

$$L^{-2d} \iint_{Q_L^2} |h_{2,L}| \lesssim L^{-d} \int_{Q_L} \langle z \rangle^{-\gamma} dz \lesssim L^{-d} \int_1^{2L} r^{d-1-\gamma} dz \lesssim \begin{cases} L^{-d} & \gamma > d \\ L^{-d} \ln(L) & \gamma = d \\ L^{-\gamma} & \gamma < d \end{cases} .$$

Plugging this bound into the previous equality then yields (3.90).

(3.91) is a corollary of (3.90). Indeed, for a r.v  $X$  and a real-valued Lipschitz function  $f$

$$\begin{aligned} \text{Var} [f(X)] &= \inf_{c \in \mathbb{R}} \mathbb{E} [|f(X) - c|^2] \leq \mathbb{E} [|f(X) - f(\mathbb{E}[X])|^2] \\ &\lesssim \mathbb{E} [|X - \mathbb{E}[X]|^2] = \text{Var} [X] . \end{aligned}$$

This applies to  $\Lambda_{1,L}$  which is a rational fraction of  $L^{-d}|\mathcal{P}_L|$  by its definition (see Lemma 3.8), away from its poles (as  $0 \leq L^{-d}|\mathcal{P}_L| \leq c_d < 1$  from the hard-core property (3.17)). ■

## 3.4 2<sup>nd</sup> order expansion : strategy, issues and how to overcome them

### 3.4.1 Expected results

Following the heuristic, we proceed as in Section 3.3.1 at next order. Hence, for a given  $x \in \mathcal{P}_L$ , we rewrite the 2<sup>nd</sup> order cluster expansion of  $\phi_L$

$$\mathcal{C}^{(2)}\phi_L^\# = \sum_{y \in \mathcal{P}_L} \phi_L^y + \frac{1}{2} \sum_{\substack{y,z \in \mathcal{P}_L \\ y \neq z}} \delta^{\{y,z\}} \phi_L^\# = \phi_L^x + \sum_{y \in \mathcal{P}_L \setminus \{x\}} \delta^y \phi_L^{\{x,\#\}} + \frac{1}{2} \sum_{\substack{y,z \in \mathcal{P}_L \setminus \{x\} \\ y \neq z}} \delta^{\{y,z\}} \phi_L^\#$$

singling out the part which is rigid on the particle  $B^L(x)$ . We then plug this expansion in the formula of average settling speed  $V_L$ . This extends the expansion of Definition 3.8 in the following form.

**Definition 3.9** Let  $L \geq 1$  and  $\mathcal{P}_L$  be the approximation of  $\mathcal{P}$  as in Theorem 3.1. Recalling the definition (3.51) of the average settling speed  $V_L$ , we have the decomposition

$$V_L = V_L^1 + V_L^2 + R_L^2 \quad (3.122)$$

where  $V_L^1$  is given as before by (3.65),

$$V_L^2 := \lambda_L^{-1} L^{-d} \sum_{x,y \in \mathcal{P}_L}^{\neq} \int_{B(x)} \delta^{\{y\}} \phi_L^{\{x\} \cup \#} \cdot e \quad (3.123)$$

and

$$R_L^2 := L^{-d} \lambda_L^{-1} \frac{1}{2} \sum_{x,y,z \in \mathcal{P}_L}^{\neq} \int_{B(x)} \delta^{\{y,z\}} \phi_L^{\#} \cdot e + L^{-d} (|B| \lambda_L)^{-1} \int_{\mathcal{I}_L} r_L^{(2)} \cdot e \quad (3.124)$$

with  $r_L^{(2)} := \phi_L - \mathcal{C}^{(2)} \phi_L^{\#}$ . □

Using the same method as in Proposition 3.2, we expect the following control on the remainder.

**Conjecture 3.1** Let  $L \geq 1$  and  $\mathcal{P}$  be stationary ergodic  $\gamma$ -algebraically mixing point to order 6 with its periodic approximation  $\mathcal{P}_L$ . We then have the following estimate of the 2<sup>nd</sup> order remainder term  $R_L^2$  (see (3.124))

$$|\mathbb{E} [R_L^2]| \lesssim \lambda_2^{1-\frac{2}{\gamma}} + \frac{\lambda_3^{1-\frac{2}{\gamma}}}{\lambda} + \frac{\lambda_4^{1-\frac{2}{\gamma}}}{\lambda} + o_L(1) . \quad (3.125)$$

□

We also expect to be able to pass to the limit  $L \rightarrow \infty$  in the definition (3.123) of  $V_L^2$ .

**Conjecture 3.2** Let  $L \geq 1$ ,  $\mathcal{P}$  as in Theorem 3.1 with its periodic approximation  $\mathcal{P}_L$ . We can define the second order of the expansion of  $\bar{V}$  by

$$\bar{V}^{(2)} := \lim_{L \rightarrow \infty} \mathbb{E} [V_L^2] \quad (3.126)$$

where  $V_L^2$  is given by (3.123).

Moreover,

$$|\bar{V}^{(2)}| \lesssim \frac{\lambda_2^{1-\frac{2}{\gamma}}}{\lambda} . \quad (3.127)$$

□

This would then yield the following result.

**Conjecture 3.3** Let  $\mathcal{P}$  be stationary ergodic  $\gamma$ -algebraically mixing point to order 6 which can be periodically approximated (to order 6). We have the following expansion of  $\bar{V}$

$$|\bar{V} - \bar{V}^{(1)} - \bar{V}^{(2)}| \lesssim \lambda_2^{1-\frac{2}{\gamma}} + \frac{\lambda_3^{1-\frac{2}{\gamma}}}{\lambda} + \frac{\lambda_4^{1-\frac{2}{\gamma}}}{\lambda} \quad (3.128)$$

where  $\bar{V}^{(1)} = f_B \phi^\circ \cdot e$  as in Theorem 3.1 and  $\bar{V}^{(2)}$  is given by Conjecture 3.2.  $\square$

### 3.4.2 Difficulties

The proofs of the 2<sup>nd</sup> order Conjectures 3.1 and 3.2 follow the same structure as the 1<sup>st</sup> order Proposition 3.2. The algebra of the deterministic part can be adapted but the main issue appears in the stochastic estimates. Indeed, it is far less clear how to perform the necessary renormalizations.

We illustrate this difficulty on the bound

$$|\mathbb{E} [V_L^2]| \lesssim \frac{\bar{\lambda}_{2,L}^{1-\frac{2}{\gamma}}}{\lambda_L} \quad (3.129)$$

which is needed for the control (3.127) of  $\bar{V}^{(2)}$  and can be seen as a toy-model for the control of the remainder  $\mathbb{E} [R_L^2]$ . From the definition (3.123) of  $V_L^2$ , we need to control

$$\mathbb{E} \left[ \sum_{x,y \in \mathcal{P}_L} \int_{B(x)} \delta^{\{y\}} \phi_L^{\{x\} \cup \#} \right] = \iint_{Q_L^2} \int_{B(x)} \delta^{\{y\}} \phi_L^{\{x\} \cup \#} f_{2,L}(x,y) dx dy .$$

We observe the same phenomenon as in Remark 3.15. As  $\delta^{\{y\}} \phi_L^{\{x\}, \#} \lesssim \langle x-y \rangle_L^{2-d}$ , this decay is not sufficient and we need to rely on the mixing for extra decay replacing  $f_{2,L}$  by  $h_{2,L}$ . To do so, our main observation was that  $\int_{Q_L} \phi_L^y dy = 0$ . But here, we are dealing with  $\delta^y \phi_L^{x, \#} = \phi_L^{\{x,y\}} - \phi_L^x$  and if  $\iint_{Q_L^2} \delta^{x,y} \phi_L^{\#} dx dy = 0$ , generically  $\int_{Q_L} \delta^{x,y} \phi_L^{\#} dx \neq 0$  for fixed  $y$ . Hence,  $f_{2,L}$  can not be replaced by  $h_{2,L}$  here and our bound seems to diverge as in Remark 3.15.

To overcome this difficulty, the solution is to borrow a strategy developed by Duerinckx and Gloria in [DG20c] breaking up  $\delta^{x,y} \phi_L^{\#}$  into elementary contributions which either decay better or can be renormalized. This decomposition combines nicely with the one of the multi-point densities in multi-point correlations (see Definition 3.3). It can be represented via a graph formalism which we will be needed to perform the variance estimates (leading to rather involved computa-

tions). The adaptation of these tools in our framework is left for future work with these authors.

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## Résumé

Cette thèse porte sur le développement en régimes dilués des propriétés effectives de milieux aléatoires et justifie rigoureusement plusieurs formules communément utilisées en physique. Dans ce manuscrit, ces propriétés effectives sont obtenues par la théorie de l'homogénéisation stochastique. On montre que le type de dilution a un impact sur les ordres élevés de ces développements et on justifie les formules de Clausius-Mossotti et de Batchelor issues respectivement de l'électrostatique et de la mécanique des fluides.

Les différents modèles de dilution étudiés (dilatation, effacement aléatoire ou dilution générale) ainsi que quelques outils généraux du domaine (développements en cluster, intensités à plusieurs particules) sont présentés dans un premier chapitre introductif. Le deuxième chapitre est consacré à l'étude d'un problème de conductivité effective avec dilution par dilatation. Grâce à une nouvelle méthode par point fixe, on montre que cette conductivité effective dépend de manière analytique du paramètre de dilatation. Le troisième et dernier chapitre étudie la notion de vitesse effective de sédimentation de particules dans un fluide visqueux. En utilisant une approche par cluster, on en fournit un développement au premier ordre avec contrôle quantitatif de l'erreur.

**Mots clés :** EDP, homogénéisation, milieu aléatoire, homogénéisation stochastique, processus de points, Clausius-Mossotti, sédimentation, Batchelor, développement dilué

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## Abstract

This thesis focuses on the expansion of some effective properties of random media in dilute regimes and the rigorous justification of several formulas commonly used in physics. These effective properties are obtained from the theory of stochastic homogenization. We show that the type of dilution plays an important role on the high orders of these developments and we justify the Clausius-Mossotti and Batchelor formulas originating respectively from electrostatic and fluid mechanics.

The different dilution models under consideration (dilation, random deletion, or general dilution) as well as some general tools of the field (cluster developments, multi-particle intensities) are presented in the first chapter. The second chapter is devoted to the study of an effective conductivity problem with dilution by dilation. Using a new fixed-point approach, we show that this effective conductivity depends analytically on the dilation parameter. The third and final chapter study the notion of effective sedimentation speed for particles in a viscous fluid. Using the cluster expansion, we provide a first order expansion of this effective speed with a quantitative control of the error.

**Keywords:** PDE, homogenization, stochastic homogenization, points processes, Clausius-Mossotti, Batchelor, sedimentation, dilute expansion

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