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Samar Allouch

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UNIVERSITÉ PARIS XIII - SORBONNE PARIS NORD  
École Doctorale Sciences, Technologies, Santé Galilée

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# **Analyse mathématique des forces d'adhésions dans le contexte de la motilité cellulaire**

**Mathematical analysis of adhesion forces in the context of  
cell motility**

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THÈSE DE DOCTORAT

présentée par

**Samar Allouch**

Laboratoire Analyse, Géométrie et Applications (LAGA)

pour l'obtention du grade de

DOCTEUR EN MATHÉMATIQUES

soutenue le 14 décembre 2022 à un jury d'examen composé de :

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## **Résumé :**

Dans cette thèse, un modèle mathématique d'adhésion dans le contexte de la motilité cellulaire a été étudié. Dans ce cadre, nous étudions un système couplé formé par une équation structurée en âge et une équation intégrale du Volterra. Nos travaux sont divisés en deux parties. Dans une première partie, nous étudions l'équation intégrale avec une force extérieure  $f$  à variation bornée. Nous introduisons la régularisée de la fonction  $f$  ainsi qu'une nouvelle variable et nous reformulons la version régularisée de cette équation. Nous dérivons une nouvelle estimation a priori et nous établissons la convergence de ce système par rapport au paramètre asymptotique  $\varepsilon$ . De plus, nous présentons un cas particulier où un principe de comparaison spécifique à ce genre d'équations intégrales peut être appliqué. Dans une deuxième partie, nous étudions le comportement asymptotique en temps long de la solution de l'équation intégrale. Nous construisons un développement asymptotique à  $N$ -termes de l'opérateur à retardement intégral dans le cas d'un noyau constant en temps et dépend seulement du variable d'âge. De plus, nous affaiblissons les hypothèses sur le taux du mort, de sorte que nous autorisons la décroissance polynomiale de la densité des liens, et nous améliorons heuristiquement les taux de convergence déjà prouver.

**Mots clés :** équation intégrale de Volterra, equation de renouvellement, source terme à variation bornée, principe de comparaison, problème à perturbation singulière, développement asymptotique.

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## **Abstract :**

In this thesis, a mathematical model of adhesion in the context of cell motility has been studied. In this framework, we study a coupled system formed by an age-structured equation and a Volterra integral equation. Our work is divided into two parts. In a first part, we study the integral equation with an external force  $f$  with bounded variation. We introduce the regularized function  $f$  and a new variable and reformulate the regularized version of this equation. We derive a new a priori estimate and establish the convergence of this system with respect to the asymptotic parameter  $\varepsilon$ . Moreover, we present a particular case where a comparison principle specific to this kind of integral equations can be applied. In a second part, we study the asymptotic behavior in long time of the solution of the integral equation. We construct an  $N^{\text{th}}$ -order asymptotic approximation of the delay integral operator in the case of a kernel depending only on the age variable. Moreover, we weaken the assumptions on the death rate, so that we allow polynomial decay of the linkages density, and we heuristically improve the already proven convergence rates.

**Keywords :** integral equation, renewal problem, load bounded variation, comparison principle, Volterra equation, singular perturbation problem, asymptotic expansion.



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*C'est ce que nous pensons déjà  
connaître qui nous empêche  
souvent d'apprendre.*

---

Claude Bernard



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## ADHESION MODELING IN THE CONTEXT OF CELL MOTILITY

This first chapter introduces our research topic, which is cell motility. It depicts a cell's physiological ability to migrate freely and independently of one site to another. Mostly, the movement of cells means the life of organisms. It is the heart of many essential biological processes. The development of certain physiological phenomena such as embryogenesis, the formation of tissues and organs and the immune response requires cell motility. Indeed, immune cells cannot fight off harmful bacteria without being able to migrate between tissues and organs. Consequently, macrophages are unable to go to the location of the injury to aid in its healing. However, scientists think that this movement can also indicate the organism's demise. For instance, in the field of cancerology, this process of migration entails the spread of cancer cells throughout the body, leading to the development of new metastases ([LVA04]; [CTK10]). A deeper comprehension of tumor cell migration might make it easier to find cures for cancer spread. Generally, the immune system's cells must move, either completely or mostly. Cell migration involves numerous actors. However, the extracellular matrix (ECM), focal points, and cytoskeleton are the three most important parts of them. The extracellular matrix (ECM) is a substrate that surrounds the cell and contains adhesion proteins (collagen, proteoglycans, elastin and glycoproteins) that allow the cell to move. The focal points are dynamic sites that guarantee the cell's adhesion to the ECM and substrate. The cytoskeleton is a dynamic filamentous network formed by three types of filaments (actin microfilaments, intermediate filaments and microtubules). The main component thought to be the driver of cell movement is actin. Actin near the membrane can polymerize and depolymerize to allow the cell to extend forward and re-



tract backward.

## 1.1 Biological context : Cell motility

The ability of a cell to produce movement either spontaneously or in response to an external stimulus (chemotaxis) is known as motility. It's important to distinguish between mobility and motility. While an organism's motility is a biological property that characterizes its capability to cause movement, an organism's mobility refers to its capacity to be in motion. The motion of a cell is like the motion of climbing : Find a hold with the arm outstretched, hold on to it, move your body, then release the previous hold. Cell migration can take a variety of forms, including amoebic, lamellipod, and vibratory migration. Our research focuses on a mathematical study of a lamellipodium cell migration mechanism. Located at the front of the cell, the lamellipodia is a lamelliform membrane protrusion. It drives cell migration. Actin filaments, which have a diameter of 5 to 9 nm, are the fine protein fibers that constitute it. These filaments ensure the traction of the cell toward the front and the retraction of the cell away from the back by polymerizing and depolymerizing, respectively [PB03].

### 1.1.1 Cell migration phases

Cell migration onto substrates is a fundamental and crucial process that necessitates the coordination of numerous cellular processes that take place in a cycle. This mechanism consists of three crucial phases (see figure 1.1) :

- 1) First, the cell chooses its migratory direction by chemotaxis. The cell membrane is extended during the actin polymerization process near the front of the cell, where lamellipodia (broad, flat projections) and filopodia (elongated and narrow) are also formed. This step is called protrusion. During this stage, the cell provides a vast family of proteins that interact with actin directly and start the polymerization of it. Depending on how they work, these proteins are divided into numerous categories. The two most important actin nucleators are formins and Arp2/3 complex ([SG10]; [PB02]). The role of the Arp2/3 complex in actin polymerization and the protrusions of lamellipodia has been demonstrated in several in vitro studies of actin polymerization dynamics [PCC01].
- 2) Migration's second stage is adherence. At the front of the cell, the lamellipodium develops new adhesion sites during this phase in order to adhere to the substrate. These

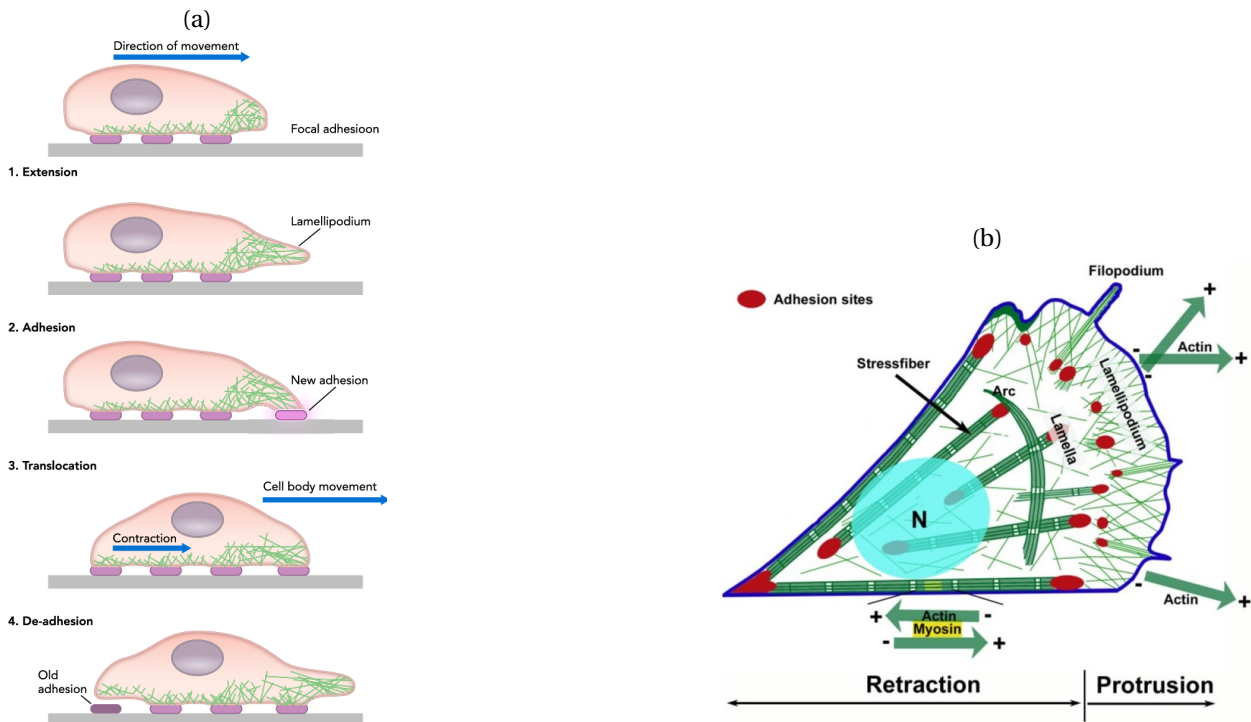


Figure 1.1: (a) Model showing the stages of cell migration. Figure reproduced from [Tsc13]. (b) Diagram showing the different actin structures present in a cell: stress fibers, lamellipod and filopod. Figure reproduced from [Tac+18, Chapter 21].

focal adhesions are responsible for the cell's attachment to the extracellular matrix and for producing the traction forces required for the cell to advance. This is called translocation.

- 3) Retraction is the final stage. The development of new focus points at the cell's front coincides with the disappearance of those in the back. The cell membrane and cell body can retract thanks to a contractile force produced by stress fibers inside the cell.

### 1.1.2 Adhesion process and its significance for cell migration

In many biological processes, cellular adhesion plays a dynamic role. From the embryonic development phase, adhesion proves its importance in the formation and functioning of

tissues. The adhesion phenomenon is crucial to cell survival. It enables cell division, proliferation, and organization within tissues ([Gum96]). Additionally, it permits the transmission of information between two cells in a relationship (cell-cell adhesion, [Bra02]). Alternatively, in the case of cell-substrate adhesions, the cell attaches to a protein of the extracellular matrix and involves the creation of focal adhesions and then the polymerization of actin filaments ([SG12]). So there are two sorts of cell adhesions: intercellular adhesions and cell-matrix adhesions. Understanding the mechanics of adhesion between the cell and its substrate is essential to comprehending the motility process. Our research focuses on the mathematical analysis of the model describing the adhesions between cells and their substrate. In the following, we present some essential molecular structures related to cell migration.

### **1.1.3 The lamellipod: actin polymerization factory**

The lamellipod is a very thin membrane extension that forms at the front of migrating cells during the protrusion phase. It has a thickness of 20 nm and a width of a few micrometers. It is a two-dimensional structure in the form of thin sheets made up of a complex and dynamic network of actin microfilaments. A first model of cell migration that took into account the lamellipodium was developed by Abercrombie and colleagues in 1970 [AHP71]. Numerous studies have been carried out since the 1970s to the present to better understand the structure and function of these membrane protrusions in a variety of biological processes, notably in cell migration. In 1999, Svitkina and Borisy presented a very clear electron microscopic image of the actin filaments, the main part of the lamellipod [SB99]. They observed that these filaments are distributed in the form of Y junctions that polymerize at the front and depolymerize at the back of the lamellipod. It depicts a treadmilling process that happens in the lamellipodium. The lamellipod's presence is essential for controlling cell migration and accelerating it. Hu et al. presented this finding in 2012 [HK12]. They demonstrated the significance of lamellipod development in the situation of cell migration in contact with a substrate. This membrane structure is essential for the organization and alignment of cell-extracellular matrix adhesions, which control the direction of movement.

### **1.1.4 Actin filaments**

Actin filaments are one of the components of the eukaryotic cytoskeleton. The cytoskeleton is a cellular structure formed by a network of protein filaments. Actin filaments are created

during the polymerization of actin-G, a monomeric form of actin, through the action of nucleators like the Arp2/3 complex and formin. They are polar, flexible filaments with a diameter of around 7 nm. These filaments are polarized and have two distinct ends : a barbed end (noted (+)) and a pointed end (noted (-)). Actin does not polymerize at the same rate at the ends of these filaments. At the plus end, it moves more quickly than at the minus end. The quantity of actin-G at each end determines this. During the treadmilling phenomenon that takes place in the lamellipod, we observe the accumulation of actin monomers at the barbed end and their dissipation at the pointed end. Therefore, actin polymerization regulation and location are crucial for cell motility. The Arp2/3 complex is a key component in actin polymerization.

### 1.1.5 The Arp2/3 complex

In 1994, the Arp2/3 complex is considered to be the eukaryotic cell's first actin nucleator [Mac+94]. It is a molecular device made of seven proteins, only two of which are bound to actin : Arp2 and Arp3. This complex has a specific property that allows it to move along the pre-existing actin filaments. It enables the elongation of the actin filaments as it travels toward the barbed end. In addition, by leaving the pointed end, it enables the cross-linking of these filaments [MHP98]. The lamellipod can be extended using this protein motor, since it can produce the protrusion force needed.

## 1.2 Mathematical modeling of cell migration : Adhesion model

Cell migration is crucial for the growth and functioning of the organism, (embryonic development, wound healing, immunity ...). Various models of cell migration intend to assemble, analyse and weight various mechanisms related to cell motility. Abercrombie [AHP71] put the first brick in place by creating a model that outlined the four phases of the reptation : polarization, protrusion, adhesion, and contraction. Later, different models were developed. They describe the role of the different actors involved in each step of cell migration. Mogilner and Oster [MO96], presented a model for the protrusion phase. They made it simpler to understand the protrusion forces brought on by actin polymerization. In addition, there are other studies that discuss, for example : the role of actin [ZM12], the extracellular matrix's participation in migration [SRC12], the creation of focal adhesions [SBK05], and the retraction mechanism [SR05]. In [OS10; OSS08], the authors suggest a model of the

actin cytoskeleton dynamics in the lamellipodium : the lamellipodium is represented by a two-dimensional structure made up of two actin filament families that cross and orient in opposite directions (see figure 1.2). The positions of these actin filaments, denoted by the superscripts + and – according to their orientations, are the key unknowns presented in this model.  $F^-(t, \alpha, s)$  indicates the position of the filaments in the anti-clockwise direction, and  $F^+(t, \alpha, s)$  that of the filaments in the clockwise direction. The balance of forces acting on the cell membrane is the basis of this model :

$$(1.1) \quad \underbrace{\mu^B \partial_s^2 (\eta^\pm \partial_s^2 F^\pm)}_{\text{bending}} - \underbrace{\partial_s (\eta^\pm \lambda^\pm \partial_s F^\pm)}_{\text{inextensibility}} + \eta^\pm \underbrace{\mu^A D_t^\pm F^\pm}_{\text{substrate adhesion}} \pm \underbrace{\partial_s (\eta^+ \eta^- \mu_\pm^T (\varphi - \varphi_0) \partial_s F^{\pm, \perp})}_{\text{twisting}} \pm \eta^+ \eta^- \underbrace{\mu_\pm^S (D_t^+ F^+ - D_t^- F^-)}_{\text{inter-filament adhesion}} = 0$$

where  $s$  denotes the actin filaments' arc length and  $\alpha$  their angular position. The stationary distributions of the filament lengths for each family are represented by the variables  $\eta^\pm$ , respectively. The parameter  $\mu^B$ ,  $\mu^S$ ,  $\mu^T$  and  $\mu^A$  represent respectively the bending parameter, the friction term with the substrate, the stiffness term of cross-link and the friction term due to the stretching elasticity of cross-link. The angle between the filaments is determined by  $\varphi = \arccos(\partial_s F^+(\alpha^+, s^+(\alpha^+, \alpha^-, t), t) \cdot \partial_s F^-(\alpha^-, s^-(\alpha^+, \alpha^-, t), t))$  and  $\varphi_0$  is the equilibrium angle. The speed of transmitting the protrusion force from the F-actin filaments to the substrate is represented by the total derivative operator  $D_t^\pm := \partial_t - v^\pm \partial_s$  and  $v^\pm$  are the polymerization speed. (1.1) describes the mechanical behavior of these two families of filaments. We are interested in the mathematical analysis of a system that models the adhesion terms (term in red in (1.1)).

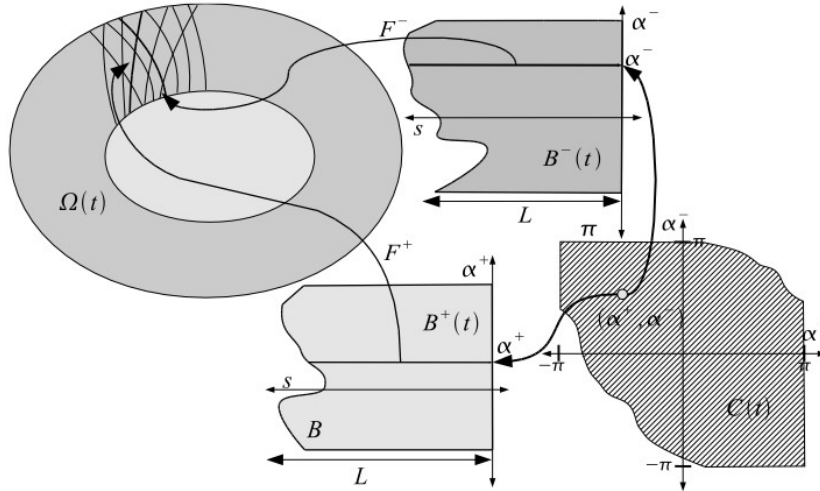


Figure 1.2: The mechanical structure of the lamellipodium.

### 1.3 Mathematical model of adhesion

In 2011, Oelz and Milišić [MO11] have studied a model concerning the terms of adhesions in (1.1). They chose to focus on a single adhesion site in order to reduce the previous problem, which appeared very complicated. According to their model, The adhesion site is seen as a point subjected to a sum of adhesive forces  $f(t)$  that allows it to move along the substrate (see figure 1.3). This adhesion site is described by its position  $z_\varepsilon(t)$  at time  $t$ .

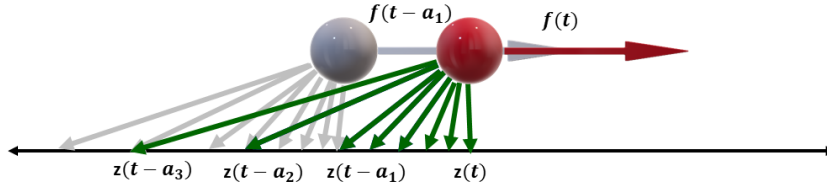


Figure 1.3: The position of an adhesion point that is changing on a substrate while being affected by a force  $f(t)$ .

The position  $z_\varepsilon(t)$  is a solution of a minimization problem :

$$(1.2) \quad z_\varepsilon(t) := \operatorname{argmin}_{w \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon} \int_{\mathbb{R}_+} |w - z_\varepsilon(t - \varepsilon a)|^2 \rho_\varepsilon(a, t) da - f(t)w \right\}$$

They assume that the bonds' stiffness is large and behaves like  $1/\varepsilon$ , where the non-dimensional parameter  $\varepsilon$  represents the typical age of adhesions in respect to the longest possible lifetime of monomers in a filament. The Euler-Lagrange equation related to the minimization problem (1.2) is used to get the equation satisfied by the position  $z_\varepsilon(t)$ . It satisfies the integral equation

$$(1.3) \quad \begin{cases} \frac{1}{\varepsilon} \int_{\mathbb{R}_+} (z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)) \rho_\varepsilon(a, t) da = f(t), & t > 0 \\ z_\varepsilon(t) = z_p(t), & t \leq 0 \end{cases}$$

The density  $\rho_\varepsilon$  represents the age distribution of bonds that attach and detach from the substrate over time. It satisfies the renewal equation

$$(1.4) \quad \begin{cases} (\varepsilon \partial_t + \partial_a + \zeta_\varepsilon(a, t)) \rho_\varepsilon(a, t) = 0, & (a, t) \in \mathbb{R}_+ \times (0, T) \\ \rho_\varepsilon(0, t) = \beta_\varepsilon(t) \left( 1 - \int_{\mathbb{R}_+} \rho_\varepsilon(a, t) da \right), & (a, t) \in \{0\} \times (0, T) \\ \rho_\varepsilon(a, 0) = \rho_I(a), & (a, t) \in \mathbb{R}_+ \times \{0\}. \end{cases}$$

where  $\beta_\varepsilon \in \mathbb{R}_+$  (resp.  $\zeta_\varepsilon \in \mathbb{R}_+$ ) is the kinetic on-rate (resp. off-rate) function. The system (1.4) presents an age-structured model that is comparable to Perthame's standard model

[Per07], but with a boundary condition at  $a = 0$  that shows a saturation effect. No new bonds are formed when the total population  $\mu_{0,\varepsilon} := \int_{\mathbb{R}_+} \rho_\varepsilon(a, t) da$  is near to 1, but when the total population is low, a bond of age 0 is born. In [MO11], the authors have demonstrated existence and uniqueness results of a solution of the weakly coupled system (1.3;1.4), for a fixed  $\varepsilon$ . In this case, the death rate  $\zeta_\varepsilon$  is a known function independent of the position and satisfies assumptions of regularity, positivity and boundedness. Moreover, they demonstrated convergence when  $\varepsilon$  goes to zero under the assumption that the off-rate  $\zeta_\varepsilon$  is not necessarily non-decreasing passed a certain age  $a_0 \geq 0$ . The strategy outlined to prove these findings is first studying the renewal equation (1.4), and then using it to investigate the integral equation (1.3). They introduced a novel entropy associated with (1.4) since there is a saturation effect in the non-local boundary condition for  $a = 0$ . It has the following form

$$\mathcal{H}[u](t) := \left| \int_0^\infty u(a, t) da \right| + \int_0^\infty |u(a, t)| da,$$

it is a Lyapunov functional that allowed them to show the strong convergence when  $\varepsilon$  goes to zero of  $\rho_\varepsilon$  to a limit  $\rho_0$ , satisfying

$$(1.5) \quad \begin{cases} \partial_a \rho_0 + \zeta_0(a, t) \rho_0 = 0, & t > 0, a > 0, \\ \rho_0(a = 0, t) = \beta_0(t) \left( 1 - \int_0^\infty \rho_0(t, \bar{a}) d\bar{a} \right), & t > 0, \end{cases}$$

They have also demonstrated that  $z_\varepsilon$  converges to  $z_0$  solution of

$$(1.6) \quad \begin{cases} \mu_{1,0}(t) \partial_t z_0 = f(t), & t > 0, \\ z_0(t = 0) := z_p(0), \end{cases}$$

where  $\mu_{1,0}(t) := \int_0^\infty a \rho_0(a, t) da$ . The convergence of  $z_\varepsilon$  is based on a comparison principle specific to this kind of Volterra integral equations [GLS09]. Applying this principle and demonstrating convergence required two presumptions : the positivity of the kernel  $\rho_\varepsilon$  and the monotonicity of  $\zeta_\varepsilon$  for a large age. The following theorem serves as a summary of these findings :

**Theorem 1.** Under well-chosen assumptions ([MO11]) and for any fixed  $\varepsilon > 0$ , there exists a unique solution  $(z_\varepsilon, \rho_\varepsilon)$  of the coupled system (1.3;1.4) in  $C^0(\mathbb{R}_+) \times (C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2))$ . Moreover for any  $T > 0$  we have

$$\|z_\varepsilon - z_0\|_{C^0([0, T])} + \|\rho_\varepsilon - \rho_0\|_{C^0([0, T]; L^1(\mathbb{R}_+))} \rightarrow 0 \text{ when } \varepsilon \rightarrow 0.$$

where  $z_0$  and  $\rho_0$  are the solutions of (1.6) and (1.5) respectively.

The authors in [MO15] have added a new elongation variable  $u_\varepsilon$  that is given by

$$(1.7) \quad u_\varepsilon(a, t) = \begin{cases} \frac{z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)}{\varepsilon}, & \text{if } t > \varepsilon a \\ \frac{z_\varepsilon(t) - z_p(t - \varepsilon a)}{\varepsilon}, & \text{if } t \leq \varepsilon a \end{cases}$$

to enhance stability and convergence results as  $\varepsilon$  goes to zero. This variable has the form of a discrete derivative of  $z_\varepsilon$ . The benefit of changing the variable  $z_\varepsilon$  to  $u_\varepsilon$  is that it allows the integral equation (1.3) to become an age-structured equation with a non-local integral source term:

$$(1.8) \quad \begin{cases} \varepsilon \partial_t u_\varepsilon + \partial_a u_\varepsilon = \frac{1}{\mu_{0,\varepsilon}} \left( \varepsilon \partial_t f + \int_0^\infty \zeta_\varepsilon u_\varepsilon \rho_\varepsilon da \right), & (t, a) \in (\mathbb{R}_+)^2, \\ u_\varepsilon(a = 0, t) = 0, & (t, a) \in \mathbb{R}_+ \times \{0\}, \\ u_\varepsilon(a, t = 0) = \frac{z_\varepsilon(0) - z_p(-\varepsilon a)}{\varepsilon} =: u_{I,\varepsilon}(a), & (t, a) \in \{0\} \times \mathbb{R}_+, \end{cases}$$

where  $\mu_{0,\varepsilon}(t) := \int_0^\infty \rho_\varepsilon(\tilde{a}, t) d\tilde{a}$ . In the regular case, when the external force  $f$  is a locally Lipschitz function on  $\mathbb{R}_+$ , there is an equivalence relation between  $z_\varepsilon$  and  $u_\varepsilon$ . If we know  $z_\varepsilon$  as a solution of (1.3), we can calculate  $u_\varepsilon$  and the other way around. The variable  $u_\varepsilon$  has a positive effect on the results proved in [MO11]. It made it possible to derive the convergence results when  $\varepsilon$  tends to zero without having to assume that  $\zeta_\varepsilon$  is necessarily increasing passed a certain age  $a_0$ . Moreover, it allows showing the global existence and uniqueness of the solution for the strong coupling system (1.3;1.4) where  $\zeta_\varepsilon$  also depends on the elongation  $u_\varepsilon$  (see [MO15, Section 7]). In this thesis, we are particularly interested in the weakly coupled system (1.3;1.4).

## 1.4 The renewal equation

The equation (1.4) studied in this research work represents an age-structured model. The classical form of age structured equation has the form

$$(1.9) \quad \begin{cases} \partial_t n(a, t) + \partial_a n(a, t) = 0, & (a, t) \in (\mathbb{R}_+)^2, \\ n(a = 0, t) = \int_0^{+\infty} \beta(a') n(a', t) da', & (a, t) \in \{0\} \times \mathbb{R}_+, \\ n(a, t = 0) = n^0(a), & (a, t) \in \mathbb{R}_+ \times \{0\}, \end{cases}$$

where  $n(a, t)$  represents the population density of age  $a$  at time  $t$  and  $\beta \geq 0$  is the birth rate. (1.9) has been widely studied by Magal and Ruan [AMR08], Diekmann and Metz [MD14],



Perthame [Per07], Doumic and Gabriel [DG10], Grwiazda and Wiedemann [GW16] and Gabriel [Gab14]. Several approaches have been developed to study the asymptotic behavior of the age-structured model. For example, the use of the Volterra integral equation formulation [Ian95], the semigroup theory [PR19; Web+85] and the generalized entropy method [MMP05; Per07]. The method developed by Perthame and al in [MMP05] is based on the definition of the population growth rate  $n$  in terms of an eigenvalue problem (see section A.1 in appendix A). Under specific assumptions on the birth rate ( $\beta$  is a nonnegative bounded rate on  $\mathbb{R}_+$  and  $\int_{\mathbb{R}_+} \beta(a) da > 1$ ) and using the Krein-Rutman theorem, they prove the existence of a unique solution  $(\lambda, N, \phi)$  of the eigenvalue problem and its dual. By establishing a General Relative Entropy (GRE) inequality, they proved that solutions to (1.9) satisfy the long time asymptotics

$$\int_{\mathbb{R}_+} \phi(a) \left| n(a, t) e^{-\lambda t} - m_0 N(a) \right| da \rightarrow 0, \quad \text{when } t \rightarrow +\infty$$

for some real number  $m_0 > 0$  and suitable function  $N$  and  $\phi$ . The GRE method can be applied to the Volterra integral equation (1.3) in the case where the kernel  $\rho$  is constant in time and the source term  $f$  is null :

$$(1.10) \quad z(t) - \int_0^t z(t-a) k(a) da = \int_0^t x_p(t-a) k(a) da$$

where  $k(a) := \frac{\rho(a)}{\mu_0}$  and  $\mu_0 := \int_{\mathbb{R}_+} \rho(a) da$  is the total population. Indeed, if we define

$$(1.11) \quad n(a, t) = \begin{cases} z(t-a) & \text{if } t \geq a \\ z_p(t-a) & \text{otherwise.} \end{cases}$$

It is easy to see that if  $z$  solves (1.10), then  $n$  solves the following equation

$$(1.12) \quad \begin{cases} \partial_t n(a, t) + \partial_a n(a, t) = 0, & (a, t) \in (\mathbb{R}_+)^2, \\ n(a=0, t) = \int_0^{+\infty} k(a') n(a', t) da', & (a, t) \in \{0\} \times \mathbb{R}_+, \\ n(a, t=0) = n^0(a) = z_p(-a), & (a, t) \in \mathbb{R}_+ \times \{0\}, \end{cases}$$

Conversely, if  $n$  is a solution of (1.12), then setting  $z(t) = n(0, t)$  solves (1.10). Indeed,

$$\begin{aligned} z(t) = n(0, t) &= \int_{\mathbb{R}_+} n(a, t) k(a) da = \int_0^t n(0, t-a) k(a) da + \int_t^{+\infty} n^0(a-t) k(a) da \\ &= \int_0^t z(t-a) k(a) da + \int_t^{+\infty} z_p(t-a) k(a) da. \end{aligned}$$

If  $k$  is a nonnegative and bounded kernel on  $\mathbb{R}_+$ , and if moreover it satisfies the hypothesis,

$$(1.13) \quad \int_{\mathbb{R}_+} k(a) da = 1 \quad \text{and} \quad \int_{\mathbb{R}_+} ak(a) da < +\infty$$

we can prove the existence of a triple  $(\lambda = 0, N, \phi)$  solution of the eigenvalue problem associated to (1.12). In this case, we can prove the GRE inequality and study the long time behavior of the equation (1.10). In contrast, the GRE method does not apply to the renewal equation (1.4) because of the non-local boundary term, which contains opposite signs to the total population. To study the asymptotic behavior of the solution of (1.4), the authors in [MO11] have introduced a new entropy which leads them to prove the convergence of the solution when  $t$  grow large.

The second equation (1.3) studied in this work has the form of a Volterra integral equation with a non-convolutive kernel. To study this equation, the authors in [MO11] use the results presented in [GLS09, Chapter 9]. In their case, they showed the existence of a unique continuous solution of (1.3) on  $\mathbb{R}_+$ . This result is based on the fact that the kernel of (1.3) is of bounded continuous type, which implies according to [GLS09, Theorem 5.4] that the resolvent associated to this kernel has the same type as it. They use a Generalized Gronwall Lemma (see Theorem 53 in appendix A) and the positivity of the kernel to show the convergence when  $\varepsilon$  goes to zero.

## 1.5 The thesis's primary goals

During this three years of research, we attempted to enhance the findings from [MO11; MO15]. Our work is divided into two parts. First, we extend the approach of [MO11; MO15], which relies on the Lipschitz property of the source term  $f$ , to the case where  $f$  is of bounded variation in  $(0, T)$ . In this case, the convergence of the solution of (1.3) is not shown by a comparison principle as in [MO11]. This principle is only applicable in a certain situation where the kernel  $\rho_\varepsilon$  is only dependent on age. Our objective is to demonstrate the convergence of  $z_\varepsilon$  when  $\varepsilon$  goes to zero, by using the elongation variable  $u_\varepsilon$ . The difficulty in our case is that the derivative of the function  $f \in BV((0, T))$  lacks sufficient regularity to directly apply the framework of [MO15], which is based on the derivative's regularity. Instead of  $f$ , we will define  $f_\delta$ , the regularized function of  $f$  ([Zie89, Section 5.3]), and apply the methodology of [MO15] to the regular function  $u_\varepsilon^\delta$  solution of (1.8). The work's plan is divided in two steps. In the first step, we fix the parameter  $\varepsilon > 0$  and demonstrate the convergence of  $u_\varepsilon^\delta$  when  $\delta$  approaches to zero. On  $u_\varepsilon^\delta$ , a uniform a priori estimate is proved with regard to

$\delta$  and  $\varepsilon$ . This ensures the passage to the limit with respect to the regularization variable  $\delta$  and the ability to show the existence and uniqueness of a solution  $u_\varepsilon$  satisfying (1.8). Then, we proceed to the limit when  $\varepsilon$  goes to 0 and prove the following result:

**Theorem 3.** Under some assumptions, one has

$$z_\varepsilon \rightarrow z_0 \text{ strongly in } L^\infty((0, T)) \text{ as } \varepsilon \rightarrow 0,$$

where  $z_0$  is defined as

$$z_0(t) = z_p(0) + \int_0^t f(\tilde{t}) / \mu_{1,0}(\tilde{t}) d\tilde{t}.$$

Finally, we provide an error estimate that makes use of the comparison principle under the assumption that the kernel  $\rho_\varepsilon$  depends only on the age variable.

In the second part of this thesis, we study the asymptotic behavior in long time of the problem's solution, defining the location of an adhesion point with the substrate. We are interested in the study of the problem

$$(1.14) \quad \begin{cases} \frac{1}{\varepsilon} \int_{\mathbb{R}_+} (X_\varepsilon(t) - X_\varepsilon(t - \varepsilon a)) \rho_\varepsilon(a, t) da = f(t), & t > 0, \\ X_\varepsilon(t) = X_p(t), & t \leq 0, \end{cases}$$

where  $\rho_\varepsilon$  is the linkages' density satisfying

$$(1.15) \quad \begin{cases} (\varepsilon \partial_t + \partial_a + \zeta(a, t)) \rho_\varepsilon(a, t) = 0, & (a, t) \in \mathbb{R}_+ \times (0, T), \\ \rho_\varepsilon(0, t) = \beta(t) \left( 1 - \int_{\mathbb{R}_+} \rho_\varepsilon(a, t) da \right), & (a, t) \in \{0\} \times (0, T), \\ \rho_\varepsilon(a, 0) = \rho_I(a), & (a, t) \in \mathbb{R}_+ \times \{0\}. \end{cases}$$

The birth rate  $\beta \in \mathbb{R}_+$  and death rate  $\zeta \in \mathbb{R}_+^2$ , respectively, are taken independently of  $\varepsilon$  because we do not yet know how they will behave over a long time ( $t \rightarrow +\infty$ ). It turns out that the analysis of the long time asymptotics of  $X_\varepsilon$  solution of (1.14) depends crucially on the long time asymptotics of the kernel  $\rho_\varepsilon$  satisfying (1.15). Due to this complexity, we begin by researching a simpler case where  $X_\varepsilon$  fulfills (1.14) with a kernel that is independent of time and the parameter  $\varepsilon$  and satisfies

$$(1.16) \quad \begin{cases} (\partial_a + \zeta(a)) \rho(a) = 0 & a > 0, \\ \rho(0) = \beta \left( 1 - \int_{\mathbb{R}_+} \rho(a) da \right) & a = 0, \end{cases}$$

The goal of investigating this first case is to create an asymptotic approximation of order  $N \geq 1$  of the solution  $X_\varepsilon$  satisfying (1.14) with a kernel that satisfies (1.16). This approximation takes the following form:

$$(1.17) \quad \tilde{X}_{\varepsilon,N} = X_{outer}(t) + X_{inner}(\tau) + O(\varepsilon^N),$$

where  $\tau = t/\varepsilon$  is the stretched variable. The  $X_{outer}$  component approximates the solution outside the origin. It is made up of a number of macroscopic correctors in the power of  $\varepsilon$ . The part  $X_{inner}$  is an approximation of the solution close to the origin, formed by microscopic correctors. The initial condition of the first order differential equation solutions that make up  $X_{outer}$  is established by stating that :

$$\lim_{\tau \rightarrow +\infty} X_{inner}(\tau) = 0.$$

A first result is shown :

**Theorem 5.** According to the Assumptions that the higher order moment  $\mu_{N+1} := \int_{\mathbb{R}_+} a^{N+1} \rho(a) da$  is bounded, an error estimate was obtained leading to :

$$\|X_\varepsilon - \tilde{X}_{\varepsilon,N}\|_{C([0,T])} \lesssim \varepsilon^N.$$

The second objective of this part is to investigate how the  $X_\varepsilon$  solution of (1.14) behaves asymptotically with regard to the perturbation parameter  $\varepsilon$ . Studying the asymptotic behavior of the density  $\rho_\varepsilon$  solution of (1.15) is necessary before moving on to this conclusion. In [Mil18], Milišić improved the error estimate obtained in [MO11]. He showed the existence of an initial layer in time. This initial layer noted  $\tau_0$  is written as the solution of the problem,

$$(1.18) \quad \begin{cases} (\partial_t + \partial_a + \zeta(a,0)) \tau_0(a,t) = 0, & a > 0, t > 0, \\ \tau_0(0,t) = -\beta(0) \int_{\mathbb{R}_+} \tau_0(a,t) da, & a = 0, t > 0, \\ \tau_0(a,0) = \rho_I(a) - \rho_0(a,0), & a > 0, t = 0. \end{cases}$$

The convergence rate achieved in [MO11] has improved as a result of the insertion of this term. He came up with the following error estimate :

**Theorem 7.** Under suitable assumptions (see [Mil18]), one has for every fixed time  $t > 0$  :

$$\|\rho_\varepsilon(\cdot, t) - \rho_0(\cdot, t) - \tau_0(\cdot, t/\varepsilon)\|_{L^1(\mathbb{R}_+)} \leq o_\varepsilon(1).$$

The Grönwall Lemma must be used in order for this result to hold, which is implied by the fact that  $\zeta$  admits a strictly positive lower bound  $\zeta_{\min} > 0$ . In our work here, we are interested in weakening the assumptions on  $\zeta$ . We assume that there exists a non-increasing function  $m \in L^1(\mathbb{R}_+; (1+a)^3)$  s.t.

$$(1.19) \quad \zeta(a, t) \geq -\frac{m'(a)}{m(a)}, \quad a.e. a \in \mathbb{R}_+.$$

The Grönwall lemma does not apply in this situation. One needs to add another term to the asymptotic expansion of  $\rho_\varepsilon$ . We introduce  $\rho_1$ , the first order macroscopic solution of :

$$(1.20) \quad \begin{cases} (\partial_a + \zeta(a, t)) \rho_1(a, t) = -\partial_t \rho_0(a, t), & a > 0, t > 0, \\ \rho_1(0, t) = -\beta(t) \int_{\mathbb{R}_+} \rho_1(a, t) da, & a = 0, t > 0. \end{cases}$$

Furthermore, it can be shown that for any  $t > 0$ , one has

$$\mathcal{H}[\rho_\varepsilon(\cdot, t) - \rho_0(\cdot, t) - \varepsilon \rho_1(\cdot, t) - \tau_0(\cdot, t/\varepsilon)] \lesssim o_\varepsilon(1).$$

This result is in turn used to extend the results of [8] to this new framework :

**Theorem 9.** Under some Assumptions (cf. chapter 3), if  $\rho_\varepsilon$  is a solution of (1.4),  $\rho_0$  solves (1.5),  $X_\varepsilon$  is a solution of (1.14) and  $X_0$  solves (1.6) then

$$\|\rho_\varepsilon - \rho_0\|_{L^1((0, T); L^1(\mathbb{R}_+, (1+a)))} \leq o_\varepsilon(1), \quad \|X_\varepsilon - X_0\|_{C([0, T])} \leq o_\varepsilon(1).$$

## 1.6 The structure of the manuscript

Following to what have been mentioned, the rest of the manuscript will be divided into two chapters :

- Chapter 2 can be seen as an extension of the paper [MO11] to the case of loads of bounded variation ( $f \in BV((0, T))$ ). We introduce a careful analysis of different definitions of BV functions with respect to the boundary of the time domain  $(0, T)$ . Thus, we present the main results obtained in this work. The findings in this chapter are the focus of the article [AM21].
- Chapter 3 is dedicated to the results obtained on the study of the long time behavior of the solutions of the Volterra integral equation (1.14). The body of an article to be published [AM22] is made up of the results reported in this chapter.

## FRICION MEDIATED BY TRANSIENT ELASTIC LINKAGES : EXTENSION TO LOADS OF BOUNDED VARIATION

### Abstract

In this work, we are interested in the convergence of a system of integro-differential equations with respect to an asymptotic parameter  $\varepsilon$ . It appears in the context of cell adhesion modelling [OSS08; OS10]. We extend the framework from [MO11; MO15], strongly depending on the hypothesis that the external load  $f$  is in  $\text{Lip}([0, T])$  to the case where  $f \in \text{BV}(0, T)$  only. We show how results presented in [MO15] naturally extend to this new setting, while only partial results can be obtained following the comparison principle introduced in [MO11].

### 2.1 Introduction

Cell motility plays a central role in several important phenomenons in biology : cancer cell migration, neutrophils' extravasation, chemotaxis, etc. The present paper fits in the modelling framework presented in [OSS08; Sfa+18; Man+15]. The adhesive dynamics of actin filaments are at the heart of the project : they contribute to lamellipodium's stabilization and allow the cell to attach to the substrate or the surrounding tissue. This paper contributes to the better mathematical understanding of a minimal model introduced first in [MO11], its aim is to extend results already obtained in [MO11; MO15] to the case of *stiffer* external loads. More precisely, we are interested in the motion of a single binding site, linked to a

one-dimensional substrate and subjected to an external force  $f$ . As in [MO11; MO15], the position of this binding site, denoted  $z_\varepsilon$ , solves a Volterra integral equation

$$(2.1) \quad \begin{cases} \frac{1}{\varepsilon} \int_0^\infty (z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)) \rho_\varepsilon(a, t) da = f(t), & t \geq 0, \\ z_\varepsilon(t) = z_p(t), & t < 0. \end{cases}$$

The kernel  $\rho_\varepsilon$  above solves a non-local age-structured problem :

$$(2.2) \quad \begin{cases} \varepsilon \partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \zeta_\varepsilon \rho_\varepsilon = 0, & t > 0, a > 0, \\ \rho_\varepsilon(a = 0, t) = \beta_\varepsilon(t) \left( 1 - \int_0^\infty \rho_\varepsilon(t, \tilde{a}) d\tilde{a} \right), & t > 0, \\ \rho_\varepsilon(a, t = 0) = \rho_{I, \varepsilon}(a), & a \geq 0, \end{cases}$$

where  $\beta_\varepsilon \in \mathbb{R}_+$  (resp.  $\zeta_\varepsilon \in \mathbb{R}_+$ ) is the kinetic on-rate (resp. off-rate) function. These possibly depend on the dimensionless parameter  $\varepsilon > 0$ . The past positions are stored in the Lipschitz function  $z_p(t) \in \mathbb{R}$ , prescribed for every  $t < 0$ .

Various mathematical issues related to this system have already been investigated [MO11; MO15; MO16]. In [MO11], the authors have introduced a specific Lyapunov functional in order to study the convergence of (2.2) when  $\varepsilon$  goes to 0. Indeed, due to the saturation effect in the non-local boundary condition in (2.2), neither the Generalized Relative Entropy [Per07; Gab13] nor more generic comparison principles [GLS09] do apply. Then, concerning (2.1), under the assumptions that the force  $f$  is Lipschitz on  $\mathbb{R}$ , and because the kernel  $\rho_\varepsilon$  in (2.1) is non-negative, an extension of Gronwall's Lemma to integral equations, shows convergence of  $z_\varepsilon$  towards  $z_0$  the solution of (2.3), the limit equation associated to (2.1). These two steps show that

$$\|z_\varepsilon - z_0\|_{C^0([0, T])} + \|\rho_\varepsilon - \rho_0\|_{C^0([0, T]; L^1(\mathbb{R}_+))} \rightarrow 0$$

where  $z_0$  is given by

$$(2.3) \quad \begin{cases} \mu_{1,0}(t) \partial_t z_0(t) = f(t), & t > 0 \\ z_0(0) = z_p(0) & t = 0 \end{cases}$$

where  $\mu_{1,0}(t) := \int_0^\infty a \rho_0(a, t) da$ , and  $\rho_0$  solves :

$$(2.4) \quad \begin{cases} \partial_a \rho_0 + \zeta_0(a, t) \rho_0 = 0, & t > 0, a > 0, \\ \rho_0(a = 0, t) = \beta_0(t) \left( 1 - \int_0^\infty \rho_0(t, \tilde{a}) d\tilde{a} \right), & t > 0. \end{cases}$$

In [MO15], the authors weakened some assumptions concerning the off-rate  $\zeta_\varepsilon$ , by assuming that  $\zeta_\varepsilon$  is not necessarily non-decreasing passed a certain age  $a_0$ . Then, they introduce a new variable  $u_\varepsilon$  related to  $z_\varepsilon$  which transforms (2.1) into a transport problem with a non-local source term :

$$(2.5) \quad \begin{cases} \varepsilon \partial_t u_\varepsilon + \partial_a u_\varepsilon = \frac{1}{\mu_{0,\varepsilon}(t)} \left( \varepsilon \partial_t f + \int_0^\infty \zeta_\varepsilon(\tilde{a}, t) u_\varepsilon(\tilde{a}, t) \rho_\varepsilon(\tilde{a}, t) d\tilde{a} \right), & t > 0, a > 0, \\ u_\varepsilon(a = 0, t) = 0, & t > 0, \\ u_\varepsilon(a, t = 0) = u_{I,\varepsilon}(a) := \frac{z_\varepsilon(0) - z_p(-\varepsilon a)}{\varepsilon}, & a \geq 0, \end{cases}$$

where  $\mu_{0,\varepsilon}(t) := \int_0^\infty \rho_\varepsilon(\tilde{a}, t) d\tilde{a}$  and according to (2.1), it holds that

$$(2.6) \quad z_\varepsilon(0) = \frac{1}{\mu_{0,\varepsilon}(0)} \left( \int_0^\infty z_p(-\varepsilon a) \rho_{I,\varepsilon}(a) da + \varepsilon f(0) \right).$$

If  $f \in \text{Lip}(\mathbb{R})$ , systems (2.1) and (2.5) are equivalent. Nevertheless, (2.5) admits a stability result that allows to show a weak-\* convergence of  $u_\varepsilon/(1+a)$  towards  $u_0/(1+a)$  in  $L^\infty(\mathbb{R}_+ \times (0, T))$ , where  $u_0$  is the solution of the limit problem

$$(2.7) \quad \begin{cases} \partial_a u_0 = \frac{1}{\mu_{0,0}} \int_0^\infty \zeta_0 u_0 \rho_0 d\tilde{a}, & t > 0, a > 0, \\ u_0(a = 0, t) = 0, & t > 0, \end{cases}$$

which in turn provides the strong convergence of  $z_\varepsilon$  in  $C([0, T])$  towards  $z_0$  solving (2.3).

In our analysis, however, when  $f \in \text{BV}((0, T))$ , the derivative of  $f$  is neither a function nor it is bounded, since it is a Radon measure. Therefore, we cannot apply directly results from [MO11]. Instead, defining  $f_\delta$  to be a specific regularization of  $f$  [Zie89, Section 5.3] provides a *regular* function  $u_\varepsilon^\delta$  solving (2.5). To do this, we use the framework already established in [MO15]. Then we show that  $u_\varepsilon^\delta$  satisfies certain *a priori* estimates that are uniform with respect to both  $\delta$  and  $\varepsilon$ . These provide necessary compactness in order to pass to the limit with the regularization parameter  $\delta$  and give existence and uniqueness of a weak solution  $u_\varepsilon$  associated to (2.5) with a load  $f \in \text{BV}((0, T))$ . The *a priori* estimates holding also in this weaker framework, we can consider convergence with respect to  $\varepsilon$  and prove consistency with the formal limit system. We show that, in the BV framework, the equivalence between (2.5) and (2.1) still holds. For the particular case when the kernel  $\rho_\varepsilon$  is independent on time and on  $\varepsilon$  and under suitable hypotheses, we show error estimates to be compared with [MO11], the comparison principle being applied to the integral of the error's modulus.

In order to clarify the interplay between parameters  $\varepsilon$  and  $\delta$ , we make the following remarks : in the previous literature [MO11; MO15], not only existence (and uniqueness) but



also convergence results were *strongly* related to the Lipschitz regularity of the load  $f$ . This motivates the present work, since it is not clear that the convergence occurs with respect to  $\varepsilon$  in this weaker framework. This explains also why we first regularize the problem with the parameter  $\delta$ , make  $\delta$  tend to zero, and then consider the convergence with respect to  $\varepsilon$ .

The outline of this chapter is as follows : in Section 2.2, collecting various results from the literature on BV-functions in one space dimension, we introduce the framework used in the rest of the paper. We make the link with the Riemann-Stieltjes integral, through a careful analysis of different definitions of BV-functions with respect to the boundary of the time domain  $(0, T)$ . In Section 2.3, we recall some results concerning (2.2) already established in [MO11]. Then, in Section 2.4, we establish uniform (with respect to  $\varepsilon$ ) *a priori* estimates for the regularized system in  $u_\varepsilon$ . After that, in Section 2.5, we show the weak convergence of  $u_\varepsilon$  towards  $u_0$ , the solution of the limit problem. This implies strong convergence of  $z_\varepsilon$  in  $L^\infty(0, T)$  as stated in Theorem 25. We establish, in Section 2.6, a specific comparison principle for Volterra equations when the density  $\rho$  is constant in time and does not depend on  $\varepsilon$ .

## 2.2 Notations and main assumptions

We denote  $L_t^p L_a^q := L^p((0, T); L^q(\mathbb{R}_+))$  for any real  $(p, q) \in [1, \infty]^2$  and

$$(2.8) \quad X_T := \left\{ g \in L_{loc}^\infty((0, T) \times \mathbb{R}_+); \sup_{t \in (0, T)} \|g(t, a) w(a)\|_{L_a^\infty} < \infty \right\}$$

where  $w(a) := (1 + a)^{-1}$ . The space  $\text{Lip}(I)$  is the set of Lipschitz functions on the interval  $I$ .

**Assumptions 1.** For any  $T > 0$  possibly infinite, we assume that :

i) The past condition  $z_p$  is  $L_{z_p}$ -Lipschitz on  $\mathbb{R}_-$  i.e. :

$$|z_p(a_2) - z_p(a_1)| \leq L_{z_p} |a_2 - a_1|, \quad \forall (a_2, a_1) \in \mathbb{R}_- \times \mathbb{R}_-.$$

ii) The function  $\beta_\varepsilon(t)$  is in  $L^\infty(0, T)$  and  $\zeta_\varepsilon(a, t)$  is in  $L^\infty(\mathbb{R}_+ \times (0, T))$ .

iii) For limit functions  $\beta_0 \in L_t^\infty$  and  $\zeta_0 \in L_t^\infty L_a^\infty$  it holds that

$$\|\zeta_\varepsilon - \zeta_0\|_{L_{a,t}^\infty} \rightarrow 0 \quad \text{and} \quad \|\beta_\varepsilon - \beta_0\|_{L_t^\infty} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ .

iv) There are upper and lower bounds such that

$$0 < \zeta_{\min} \leq \zeta_\varepsilon(a, t) \leq \zeta_{\max} \quad \text{and} \quad \beta_{\min} \leq \beta_\varepsilon(a, t) \leq \beta_{\max},$$

for all  $\varepsilon > 0$ ,  $a \geq 0$  and  $t > 0$ .

**Assumptions 2.** *The initial condition  $\rho_{I,\varepsilon} \in L_a^\infty(\mathbb{R}_+)$  satisfies*

i) *Positivity*

$$\rho_{I,\varepsilon}(a) \geq 0, \text{ a.e. in } \mathbb{R}_+,$$

*moreover, one has also that the total initial population satisfies*

$$0 < \int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a) da < 1;$$

ii) *Boundedness of higher moments,*

$$0 < \int_{\mathbb{R}_+} a^p \rho_{I,\varepsilon}(a) da < c_p, \text{ for } p = 1, 2,$$

*where  $c_p$  are positive constants depending only on  $p$ .*

Next, we introduce definitions of functions with bounded variation in one dimension, as well as some related properties.

**Definition 1.** Let  $f : (0, T) \rightarrow \mathbb{R}$  be a Lebesgue measurable function. The pointwise variation (or Jordan variation) of  $f$  on  $(0, T)$  is

$$(2.9) \quad \text{pvar}(f, (0, T)) := \sup_P \text{var}(f, P)$$

where  $\text{var}(f, P) := \sum_{k=1}^n |f(t_k) - f(t_{k-1})|$  and  $P = \{0 < t_0 < \dots < t_n < T\}$  is a partition of  $(0, T)$ .

Moreover, we denote  $BPV((0, T)) := \{f \in \mathcal{L}((0, T)), \text{ s.t. } \text{pvar}(f, (0, T)) < +\infty\}$ , the space of measurable functions with pointwise bounded variation, see for example, [AFP00, section 3.2], [Leo09, chapter 2] and [HPR19, section 2.2, 2.3]. The pointwise variation of  $f$  is clearly dependent on the value of  $f$  at each point of the domain, and it differs from one a.e.-representative of  $f$  to another. For this reason, for every measurable function  $f$ , one defines the essential pointwise variation :

$$(2.10) \quad \text{epvar}(f, (0, T)) := \inf \{ \text{pvar}(g, (0, T)) ; f(t) = g(t) \text{ a.e. } t \in (0, T) \}$$

In [Leo09, Chapter 6], another functional space is defined :

**Definition 2.** Given an open interval  $(0, T) \subset \mathbb{R}$ , the space of functions with bounded variation  $BV((0, T))$  is defined as the space of all functions  $f \in L^1((0, T))$  for which there exists a signed Radon measure  $\mu_f$  such that

$$(2.11) \quad \int_{(0, T)} f \phi' dt = - \int_{(0, T)} \phi d\mu_f, \quad \forall \phi \in C_c^1((0, T))$$

for all  $\phi \in C_c^1((0, T))$ . The measure  $\mu_f$  is called the weak or distributional derivative of  $f$ .

**Remark 1.**

i) We define the total variation of  $f \in L^1((0, T))$  by

$$(2.12) \quad \|Df\|((0, T)) = \sup \left\{ - \int_{(0, T)} f \phi' dt, \phi \in C_c^1((0, T)), |\phi|_\infty \leq 1 \right\}.$$

Moreover,  $f \in BV((0, T))$  if  $\|Df\|((0, T)) < +\infty$ .

ii) Definitions (2.9) and (2.12) are not equivalent. For instance, the Dirichlet indicatrix function  $\chi_{\mathbb{Q} \cap [0, 1]}$  is not of pointwise bounded variation in  $(0, 1)$  in the sense of Definition 1 but is well-defined in the sense of Definition 2. The equivalence between the two definitions holds up to a.e. equality. Moreover, every integrable function  $f : (0, T) \rightarrow \mathbb{R}$  such that  $\text{pvar}(f, \Omega) < +\infty$ , is in  $BV((0, T))$  and  $\|Df\|((0, T)) \leq \text{pvar}(f, (0, T))$ . On the other hand, if  $f$  belongs to  $BV((0, T))$ , then  $f$  admits a right continuous representative  $\bar{f}$  with bounded pointwise variation such that

$$\text{pvar}(\bar{f}, (0, T)) = \|Df\|((0, T)).$$

For more details, see, e.g., [Leo09, theorem 7.3] and [HPR19].

iii) Under the norm

$$\|f\|_{BV} := \|f\|_{L^1} + \text{epvar}(f, (0, T)) < \infty$$

$BV((0, T))$  is a Banach space.

Next, we provide existence of the left and right limits of functions with bounded variation [HPR19, Proposition 2.2].

**Lemma 1.** Let  $f \in BV((0, T))$ , Then both the limits

$$f(0^+) = \lim_{s \rightarrow 0, s > 0} f(s) \quad \text{and} \quad f(T^-) = \lim_{s \rightarrow T, s < T} f(s) \quad \text{exist.}$$

Additionally, if  $f$  is integrable, the left and right limits are as follows :

**Lemma 2.** Suppose that  $f \in BV((0, T))$ , then

$$f(0^+) = \lim_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_0^\rho f(t) dt, \quad f(T^-) = \lim_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_{T-\rho}^T f(t) dt.$$

Next, we present a result used in the proof of Proposition 3, which relates the pointwise variation to the Lebesgue measure :

$$\lambda(f, h, \Omega) := \int_{\{t \in \Omega : t+h \in \Omega\}} |f(t+h) - f(t)| dt,$$

**Lemma 3.** If  $f$  is in BPV((0, T)), then  $\lambda(f, h, (0, T))/|h|$  is bounded. Moreover,

$$\lambda(f, h, (0, T)) \leq |h| \text{pvar}(f, (0, T)).$$

For the proof, we can see [Leo09, Theorem 2.20].

Finally, in [HPR19], the authors add a new notion of variation containing the boundary value in order to expand the total variation of  $f$  to  $[0, T]$ . This variation is defined as

$$(2.13) \quad \text{varw}(f) := \sup_{\substack{\phi \in C_c^1([0, T]) \\ |\phi|_\infty \leq 1}} \left\{ \phi(T)f(T^-) - \phi(0)f(0^+) - \int_{(0, T)} f \phi' dt \right\}$$

Moreover, by summarizing the results of [HPR19, Proposition 2.3, 2.6 and 2.7] all notions of variations coincide :

$$\text{epvar}(f, (0, T)) = \|Df\|((0, T)) = \text{varw}(f)$$

The previous result allows to extend Lemma 3 to BV((0, T)) functions :

**Lemma 4.** If  $f$  is in BV((0, T)), then  $\lambda(f, h, (0, T))/|h|$  is bounded. Moreover,

$$\lambda(f, h, (0, T)) \leq |h| \|Df\|((0, T))$$

**Proof.** By taking the infimum over almost every equal measurable functions, one has

$$\inf_{f=\tilde{f} \text{ a.e.}} \lambda(\tilde{f}, h, (0, T)) \leq |h| \inf_{f=\tilde{f} \text{ a.e.}} \text{pvar}(\tilde{f}, (0, T)) = |h| \text{epvar}(f, (0, T)) = |h| \|Df\|((0, T))$$

Since the left-hand side is a Lebesgue integral one has :

$$\inf_{f=\tilde{f} \text{ a.e.}} \lambda(\tilde{f}, h, (0, T)) = \lambda(f, h, (0, T))$$

which ends the proof. ■

### 2.2.1 Data regularization

**Theorem 11.** For every  $f \in BV((0, T))$ , there exists a sequence of smooth functions  $(f_\delta)_\delta$  in  $C^\infty((0, T))$  such that

$$\lim_{\delta \rightarrow 0} \int_{(0, T)} |f_\delta - f| dt = 0 \quad \text{and} \quad \lim_{\delta \rightarrow 0} \int_{(0, T)} |f'_\delta| dt = \|Df\|((0, T)).$$

Although the proof is classical (see for instance [Zie89, Theorem 5.3.3 p.225]), we need the explicit form of  $f_\delta$  in the rest of the paper. For this reason, we present in Section B the proof of Theorem 11.

**Lemma 5.** Let  $f \in BV((0, T)) \cap L^\infty((0, T))$ . Then the regularization function  $f_\delta$  defined as (B.5) is bounded in  $(0, T)$ .

Next, we compare the left and right limits of  $f$  and its approximation  $f_\delta$  on the boundary :

**Lemma 6.** Let  $f \in BV((0, T))$  and  $f_\delta$  defined as (B.5), then

$$f_\delta(0^+) = f(0^+) \quad \text{and} \quad f_\delta(T^-) = f(T^-).$$

First we need the following result :

**Proposition 1.** Let  $f \in BV((0, T))$ . For every  $\delta > 0$ , and  $t_0 \in \{0, T\}$ ,

$$(2.14) \quad \lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_{I_\tau \cap (0, T)} |f_\delta - f| dt = 0,$$

where  $I_\tau = \{t \in \mathbb{R} : |t - t_0| < \tau\}$ .

**Proof.** For a fixed  $t_0 \in \{0, T\}$  and  $t \in I_\tau \cap (0, T)$ , we have

$$f_\delta(t) - f(t) = \sum_{i=0}^{\infty} [\chi_{\delta_i} * (\phi_i f) - \phi_i f]$$

by the definition of  $\text{supp } \phi_i$  (see (B.1)), we have  $1/(j_0 + i + 1) < \tau < 1/(j_0 + i - 1)$  then  $i > 1/\tau - j_0 - 1$ . Since,  $\mathbb{R}$  is an archimedean space then

$$(2.15) \quad \forall \tau > 0, \exists ! i_0 := \left\lfloor \frac{1}{\tau} \right\rfloor - j_0 \quad \text{s.t.} \quad i_0 \leq \frac{1}{\tau} < i_0 + 1$$

which implies by using (B.3) that

$$\begin{aligned} \int_{I_\tau \cap (0, T)} |f_\delta - f| dt &= \sum_{i=i_0}^{\infty} \int_{I_\tau \cap (0, T)} |\chi_{\delta_i} * (\phi_i f) - \phi_i f| dt \leq \sum_{i=i_0}^{\infty} \delta 2^{-i} \\ &\leq \delta 2^{-i_0} \sum_{i=0}^{\infty} 2^{-i} = 2^{j_0+1} \delta 2^{-\lfloor \frac{1}{\tau} \rfloor} \end{aligned}$$

then

$$\frac{1}{\tau} \int_{I_\tau \cap (0, T)} |f_\delta - f| dt \leq C \delta \frac{2^{-\lfloor \frac{1}{\tau} \rfloor}}{\tau}$$

using again (2.15), we have

$$\frac{2^{-\lfloor \frac{1}{\tau} \rfloor}}{\tau} = \frac{\exp(-\lfloor \frac{1}{\tau} \rfloor \ln 2)}{\tau} \leq \frac{2 \exp(-\frac{1}{\tau} \ln 2)}{\tau}$$

Finally, we conclude that

$$\lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_{I_\tau \cap (0, T)} |f_\delta - f| dt = \lim_{\tau \rightarrow 0^+} \frac{2 \exp(-\frac{1}{\tau} \ln 2)}{\tau} = 0. \quad \blacksquare$$

**Proof of Lemma 6** According to the Lemma 2,

$$\lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_0^\tau |f_\delta(t) - f_\delta(0^+)| dt = 0 \quad \text{and} \quad \lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_0^\tau |f(t) - f(0^+)| dt = 0.$$

Moreover, we have, thanks to Proposition 1

$$\lim_{\tau \rightarrow 0^+} \frac{1}{\tau} \int_0^\tau |f_\delta - f| dt = 0.$$

Thus, for all  $\varepsilon' > 0$ , there exist  $\delta' > 0$  such that  $0 < \tau < \delta'$  implies

$$\begin{aligned} |f_\delta(0^+) - f(0^+)| &= \frac{1}{\tau} \int_0^\tau |f_\delta(0^+) - f(0^+)| dt \\ &\leq \frac{1}{\tau} \int_0^\tau |f_\delta(0^+) - f_\delta(t)| dt + \frac{1}{\tau} \int_0^\tau |f_\delta(t) - f(t)| dt + \frac{1}{\tau} \int_0^\tau |f(t) - f(0^+)| dt \leq 3\varepsilon' \end{aligned}$$

which proves the required result. Similarly, we can prove that  $f_\delta(T^-) = f(T^-)$ .  $\blacksquare$

In the previous setting, the weak derivative of  $f \in BV((0, T))$  defines a linear continuous form on  $C((0, T))$ . In the next section, we show how to *extend* this measure on functions in  $C([0, T])$ .

### 2.2.2 Definition and basic properties of Stieltjes integral

The Riemann–Stieltjes integral (RS–integral) is a generalization of the Riemann integral. Let  $\mathcal{P}$  a tagged partition of  $[0, T]$ , defined as

$$(2.16) \quad \mathcal{P} := \{(\xi_i, [t_{i-1}, t_i]) : 1 \leq i \leq n\}$$

where  $0 = t_1 \leq \dots \leq t_n = T$ , and on each interval  $[t_{i-1}, t_i]$  we choose a single value  $\xi_i$ , for  $i \in \{1, \dots, n\}$ .

**Definition 3.** For any function  $f, g : [0, T] \rightarrow \mathbb{R}$  and a partition  $\mathcal{P}$ , we define the Riemann-Stieltjes sum by

$$S(f, dg, \mathcal{P}, [0, T]) := \sum_i f(\xi_i)[g(t_i) - g(t_{i-1})].$$

Moreover, the RS-integral of  $f$  with respect to  $g$

$$(RS). \int_{[0, T]} f(t) dg(t)$$

exists and has a value  $I \in \mathbb{R}$ , if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$ , such that the mesh size  $\max_i(t_i - t_{i-1}) < \delta$  and for every  $\xi_i$  in  $[t_i, t_{i+1}]$ ,

$$|S(f, dg, \mathcal{P}, [0, T]) - I| < \varepsilon.$$

**Lemma 7.** Suppose that  $f$  is continuous on  $[0, T]$  and  $g$  is of bounded pointwise variation on  $[0, T]$ , then

$$\left| \int_{[0, T]} f dg \right| \leq \|f\|_{\infty} \text{pvar}(g, [0, T])$$

In the following Theorem we see that a Riemann-Stieltjes integral can be used to describe any bounded linear functional on  $C([0, T])$  (see [BC09, Theorem 7.1.1] and [Kre89, Theorem 4.4-1] for more details) :

**Theorem 13.** Let  $\Gamma_f \in (C([0, T]))'$ , then there exist  $g \in BPV([0, T])$  such that

$$\Gamma_f(\varphi) = \int_{[0, T]} \varphi dg, \quad \forall \varphi \in C([0, T]).$$

**Theorem 15.** (Integration by parts). If one of the integrals  $\int_{[0, T]} f dg$  and  $\int_{[0, T]} g df$  exists, then the other exists as well, and we have

$$\int_{[0, T]} f dg + \int_{[0, T]} g df = [fg(t)]_{t=0}^{t=T}.$$

Moreover, if  $f \in C^1([0, T])$  and  $g \in BPV([0, T])$ , then  $df = f' dt$  in the second term of the left-hand side.

For the proof, see [MST19, Theorem 5.52] and [Ane12, Lemma 2].

**Lemma 8.** Let  $f \in BV((0, T))$ . Then there exists  $g \in BPV([0, T])$  s.t

$$[f\varphi]_{t=0^+}^{t=T^-} - \int_{(0,T)} f\varphi' dx = \int_{[0,T]} \varphi dg, \quad \forall \varphi \in C^1([0, T])$$

s.t.  $f(t) = g(t)$ , *a.e.*  $t \in (0, T)$

**Proof.** We regularize  $f \in BV((0, T))$  by  $f_\delta \in C^\infty((0, T))$  as in Theorem 11, then we have :

$$\int_{t_k}^{s_k} f'_\delta \varphi dt + \int_{t_k}^{s_k} f_\delta \varphi' dt = [f_\delta \varphi]_{t=t_k}^{t=s_k} =: L_k, \quad \forall \varphi \in C^1([0, T])$$

where  $s_k \rightarrow T^-$  and  $t_k \rightarrow 0^+$ . We define :

$$I_k := \int_{t_k}^{s_k} f'_\delta \varphi dt, \quad J_k := \int_{t_k}^{s_k} f_\delta \varphi' dt.$$

Thanks to Lebesgue's Theorem, one has that

$$\lim_{k \rightarrow \infty} I_k = I := \int_0^T f'_\delta \varphi dt, \quad \lim_{k \rightarrow \infty} J_k = J := \int_0^T f_\delta \varphi' dt$$

and thanks to Lemma 6 and the continuity of  $\varphi$ ,

$$L_k = L := f(T^-)\varphi(T) - f(0^+)\varphi(0), \quad \forall k \in \mathbb{N}$$

So that we have :

$$\int_0^T f'_\delta \varphi dt + \int_0^T f_\delta \varphi' dt = [f\varphi]_{t=0^+}^{t=T^-}$$

If we set

$$\mathcal{I}_{f_\delta}(\varphi) := \int_0^T f'_\delta \varphi dt,$$

it is a linear continuous form on  $C([0, T])$ , since one has :

$$(2.17) \quad |\mathcal{I}_{f_\delta}(\varphi)| \leq \|f'_\delta\|_{L^1((0,T))} \|\varphi\|_{L^\infty} \leq \{\|Df\|((0, T)) + \delta\} \|\varphi\|_{L^\infty((0,T))}$$

where we used estimates from the proof of [Zie89, Theorem 5.3.3]. Since it is a continuous linear form on  $C([0, T])$ , by Theorem 13 there exists  $h_\delta \in BPV([0, T])$  s.t.

$$\mathcal{I}_{f_\delta}(\varphi) = \int_{[0,T]} \varphi dh_\delta, \quad \forall \varphi \in C([0, T])$$

in the Stieljes' sense. But by using the integration by parts from Theorem 15, we have that

$$(2.18) \quad [f\varphi]_{t=0^+}^{t=T^-} - \int_{(0,T)} f_\delta \varphi' dt = \int_{[0,T]} \varphi dh_\delta = [h_\delta \varphi]_{t=0}^{t=T} - \int_{[0,T]} h_\delta \varphi' dt, \quad \forall \varphi \in C^1([0, T])$$



which implies that

$$\int_{(0,T)} f_\delta \varphi' dt = \int_{(0,T)} h_\delta \varphi' dt, \quad \forall \varphi \in \mathcal{D}((0, T))$$

and then we can apply [Leo09, Lemma 7.4] and conclude that there exists  $c \in \mathbb{R}$ , s.t.

$$f_\delta = h_\delta + c.$$

Then setting  $g_\delta := h_\delta + c$  provides a function s.t.

$$[f_\delta \varphi]_0^T - \int_0^T f_\delta \varphi' dt = \int_0^T \varphi dg_\delta, \quad \forall \varphi \in C^1([0, T])$$

and s.t.

$$f_\delta(t) = g_\delta(t), \quad a.e. t \in (0, T).$$

Thanks to (2.17),  $\mathcal{I}_{f_\delta}$  is a linear continuous form on  $C([0, T])$  uniformly bounded with respect to  $\delta$ . It can be identified via the Riesz representation theorem as a Radon measure  $\mu_\delta$  on  $[0, T]$ . Therefore, there exist  $\mu \in M^1([0, T])$  and a sub-sequence  $\mu_{\delta_k}$  such that

$$\mu_{\delta_k} \xrightarrow{*} \mu$$

in  $\sigma(M^1([0, T]), C([0, T]))$  with respect to the weak-\* topology. By Theorem 13, there exists  $h \in \text{BPV}([0, T])$  s.t.

$$\mu(\varphi) = \int_0^T \varphi dh$$

where the left side is a Radon measure and the right hand side is the Riemann-Stieltjes integral. Because  $f_\delta$  tends to  $f$  in the  $L^1(0, T)$  topology, one has then that

$$[f\varphi]_{t=0^+}^{t=T^-} - \int_{(0,T)} f\varphi' dt = \int_{[0,T]} \varphi dh, \quad \forall \varphi \in C^1([0, T])$$

then using again integration by parts from Theorem 15, one concludes that

$$f(t) = h(t) + \tilde{c}, \quad a.e. t \in (0, T),$$

and setting  $g = h + \tilde{c}$  ends the proof. ■

**Corollary 1.** There exists a sub-sequence  $(f_{\delta_k})_{k \in \mathbb{N}}$ , s.t.

$$I_{f_{\delta_k}}(\varphi) := \int_0^T \varphi f'_{\delta_k} dt \rightarrow \int_0^T \varphi dg, \quad \forall \varphi \in C([0, T]),$$

when  $k \rightarrow \infty$ .

## 2.3 Mathematical background for the linkages' density

We list here some of the results proved in [MO11] used in the next sections of the paper.

**Theorem 17.** Let Assumptions 1 and 2 hold, then for every fixed  $\varepsilon > 0$  there is a unique solution  $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$  of the problem (2.2). It satisfies (2.2) in the sense of characteristics, namely

$$(2.19) \quad \rho_\varepsilon(a, t) = \begin{cases} \beta_\varepsilon(t - \varepsilon a) \left( 1 - \int_{\mathbb{R}_+} \rho_\varepsilon(\tilde{a}, t - \varepsilon a) d\tilde{a} \right) \exp\left(-\int_0^a \zeta_\varepsilon(\tilde{a}, t - \varepsilon(a - \tilde{a})) d\tilde{a}\right), & \forall a < \frac{t}{\varepsilon} \\ \rho_{I, \varepsilon}\left(a - \frac{t}{\varepsilon}\right) \exp\left(-\frac{1}{\varepsilon} \int_0^t \zeta_\varepsilon\left(\frac{\tilde{t}-t}{\varepsilon} + a, \tilde{t}\right) d\tilde{t}\right), & \forall a \geq \frac{t}{\varepsilon} \end{cases}$$

Moreover, it is a weak solution as well since it satisfies (cf [MO11, Lemma 2.1])

$$(2.20) \quad \int_0^T \int_0^{+\infty} \rho_\varepsilon(a, t) (\varepsilon \partial_t \varphi + \partial_a \varphi - \zeta_\varepsilon \varphi) da dt - \varepsilon \int_0^{+\infty} \rho_\varepsilon(a, T) \varphi(a, T) da \\ + \int_0^T \rho_\varepsilon(a=0, t) \varphi(a=0, t) dt + \varepsilon \int_0^{+\infty} \rho_{I, \varepsilon}(a) \varphi(a, t=0) da = 0$$

for every  $T > 0$  and test function  $\varphi \in \mathcal{C}^\infty(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$ . Now we define the moments of  $\rho_\varepsilon$  which we denote by  $\mu_{p, \varepsilon}(t) := \int_{\mathbb{R}_+} a^p \rho_\varepsilon(a, t) da$ , with  $p = 1, 2$ .

**Lemma 9.** Let Assumptions 1 and 2 hold, then the unique solution  $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$  of (2.2) satisfies

- i)  $\rho_\varepsilon(a, t) \geq 0$  for a.e.  $(a, t)$  in  $\mathbb{R}_+^2$ ,
- ii)  $\mu_{0, \min} \leq \mu_{0, \varepsilon}(t) < 1$ ,  $\forall t \in \mathbb{R}_+$ , where,  $\mu_{0, \min} < \min\left(\mu_{0, \varepsilon}(0), \frac{\beta_{\min}}{\beta_{\min} + \zeta_{\max}}\right)$ ,
- iii)  $\mu_{p, \min} \leq \mu_{p, \varepsilon}(t) \leq k$ , where,  $\mu_{p, \min} = \min\left(\mu_{p, \varepsilon}(0), \frac{\mu_{p-1, \min}}{\zeta_{\max}}\right)$ .

The authors provide a Liapunov functional that reads :

$$\mathcal{H}[u] := \left| \int_0^\infty u(a) da \right| + \int_0^\infty |u(a)| da,$$

thanks to which they obtain the following convergence result for  $\rho_\varepsilon$  :

**Lemma 10.** Let  $\zeta_{\min} > 0$  be the lower bound to  $\zeta_\varepsilon(a, t)$  according to Assumptions 1, and setting  $\hat{\rho}_\varepsilon := \text{one}$  has

(2.21)

$$\mathcal{H}[(\rho_\varepsilon - \rho_0)(\cdot, t)] \leq \mathcal{H}[\rho_{I,\varepsilon} - \rho_0(\cdot, 0)] \exp\left(\frac{-\zeta_{\min} t}{\varepsilon}\right) + \frac{2}{\zeta_{\min}} \left\| \|R_\varepsilon\|_{L^1_a(\mathbb{R}_+)} + |M_\varepsilon| \right\|_{L^1_t(\mathbb{R}_+)}$$

with  $R_\varepsilon := -\varepsilon \partial_t \rho_0 - \rho_0(\zeta_\varepsilon - \zeta_0)$  and  $M_\varepsilon := (\beta_\varepsilon - \beta_0)(1 - \int_0^\infty \rho_0 da)$ .

This ensures the convergence of  $\rho_\varepsilon$  that reads :

**Theorem 19.** Let  $\rho_\varepsilon$  the solution of the system (2.2) and let  $\rho_0$  given by (2.4), then

$$\rho_\varepsilon \rightarrow \rho_0 \text{ in } C^0((0, \infty); L^1(\mathbb{R}_+)) \text{ as } \varepsilon \rightarrow 0,$$

where the convergence with respect to time is meant in the sense of uniform convergence on compact subintervals.

## 2.4 Existence, uniqueness and stability

Using the regularized function  $f_\delta$  introduced in Theorem 11, we consider an approximation of (2.1): we denote by  $z_\varepsilon^\delta := z_\varepsilon^\delta(t)$  the function solving

$$(2.22) \quad \begin{cases} \frac{1}{\varepsilon} \int_0^\infty (z_\varepsilon^\delta(t) - z_\varepsilon^\delta(t - \varepsilon a)) \rho_\varepsilon(a, t) da = f_\delta(t), & t \leq 0, \\ z_\varepsilon^\delta(t) = z_p(t), & t < 0. \end{cases}$$

We also define  $u_\varepsilon^\delta(a, t)$ , an approximation of the elongation variable  $u_\varepsilon$ , defined as the mild solution of

$$(2.23) \quad \begin{cases} \varepsilon \partial_t u_\varepsilon^\delta + \partial_a u_\varepsilon^\delta = \frac{1}{\mu_{0,\varepsilon}} \left( \varepsilon f'_\delta + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da \right), & t > 0, a > 0, \\ u_\varepsilon^\delta(a = 0, t) = 0, & t > 0, \\ u_\varepsilon^\delta(a, t = 0) = u_{I,\varepsilon}^\delta(a), & a \geq 0. \end{cases}$$

where

$$(2.24) \quad u_{I,\varepsilon}^\delta(a) := \frac{z_\varepsilon^\delta(0^+) - z_p(-\varepsilon a)}{\varepsilon}$$

and

$$(2.25) \quad z_\varepsilon^\delta(0^+) = \frac{1}{\mu_{0,\varepsilon}(0)} \left( \varepsilon f_\delta(0^+) + \int_0^\infty z_p(-\varepsilon a) \rho_{I,\varepsilon}(a) da \right).$$

More precisely,  $u_\varepsilon^\delta$  is a solution of system (2.23) in the sense of characteristics, namely

$$(2.26) \quad u_\varepsilon^\delta(a, t) = \begin{cases} \int_0^a h(t - \varepsilon \tilde{a}) d\tilde{a}, & \text{if } t > \varepsilon a \\ \int_0^{t/\varepsilon} h(t - \varepsilon \tilde{a}) d\tilde{a} + u_{I,\varepsilon}^\delta(a - t/\varepsilon), & \text{if } t \leq \varepsilon a, \end{cases}$$

where

$$h(t) := \frac{1}{\mu_{0,\varepsilon}} \left( \varepsilon \partial_t f_\delta + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da \right).$$

By arguments similar to [MO11, Lemma 3], it is as well a weak solution of (2.23) *i.e.*

$$(2.27) \quad \begin{aligned} & - \int_0^T \int_0^\infty u_\varepsilon^\delta (\varepsilon \partial_t \varphi + \partial_a \varphi) da dt + \left[ \int_0^\infty u_\varepsilon^\delta(s, a) \varphi(s, a) da \right]_{s=0}^{s=T} \\ & = \int_0^T \frac{1}{\mu_{0,\varepsilon}} \left( \varepsilon f'_\delta + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da \right) \left( \int_0^\infty \varphi(t, \tilde{a}) d\tilde{a} \right) dt \end{aligned}$$

for any function  $\varphi \in C_c^\infty([0, T] \times \mathbb{R}_+)$ . Although, problem (2.22) can be defined for weaker data (typically  $L^1((0, T))$  or the space of Radon measures  $M((0, T))$ ), the elongation problem (2.23), requires to give a meaning to the time derivative of  $f$ , which is more restrictive. Nevertheless, as we are mainly interested in convergence results,  $f \in BV((0, T))$  seems the weakest possible regularity to our knowledge.

**Theorem 21.** Let Assumptions 1 hold, and let  $\rho_\varepsilon$  be the unique solution of (2.2) then the system (2.23) has a unique solution  $u_\varepsilon^\delta \in X_T$ .

We are in the framework of [MO15, Theorem 6.1], but for sake of self-containtness, we recall in an abridged version the proof hereafter.

**Proof.** A Banach fixed point Theorem is used to prove this result. We define the mapping  $\phi(v) = u$  such that by Duhamel's principle

$$(2.28) \quad u(a, t) = \begin{cases} \int_0^a G(t - \varepsilon \tilde{a}) d\tilde{a}, & \text{if } t > \varepsilon a \\ \int_0^{t/\varepsilon} G(t - \varepsilon \tilde{a}) d\tilde{a} + u_{I,\varepsilon}^\delta(a - t/\varepsilon), & \text{if } t \leq \varepsilon a, \end{cases}$$

where

$$G(t) := \frac{1}{\mu_{0,\varepsilon}} \left( \varepsilon f'_\delta + \int_0^\infty \zeta_\varepsilon(a, t) v(a, t) \rho_\varepsilon(a, t) da \right).$$

As in [MO15], a simple computation shows that

$$\|u\|_{X_T} \leq \|G\|_{L^\infty((0,T))} \frac{T}{T+\varepsilon} + \left\| \frac{u_{I,\varepsilon}^\delta(\cdot)}{1+a} \right\|_{L^\infty(\mathbb{R}_+)}$$

Moreover, since  $\partial_t f_\delta \in L^\infty((0,T))$

$$\|G\|_{L^\infty((0,T))} \leq \frac{1}{\mu_{0,\min}} \varepsilon \|f'_\delta\|_{L^\infty((0,T))} + \frac{\zeta_{\max}(1+k)}{\mu_{0,\min}} \|v\|_{X_T}$$

where  $k$  is the upper bound of  $\mu_{1,\varepsilon}$  proved in Lemma 9. Furthermore, by the same argument we can prove that  $\phi$  is a contraction. Indeed, if  $u_i = \phi(v_i)$  for  $i \in \{1,2\}$

$$\|u_2 - u_1\|_{X_T} \leq C \frac{T}{T+\varepsilon} \|v_2 - v_1\|_{X_T}$$

for a constant  $C > 0$ . Then we can choose  $T < \varepsilon/C$  and we obtain the existence of a local solution in time of (2.23), by Banach-Picard's fixed point theorem. As the contraction time does not depend on the initial data, we shall extend the same result by continuation. This shows existence and uniqueness in  $X_T$  for any  $T > 0$ .  $\blacksquare$

**Lemma 11.** If Assumptions 1 holds, then the solution of system (2.23) satisfies the uniform a priori estimates

$$(2.29) \quad \int_0^\infty \rho_\varepsilon(a,t) |u_\varepsilon^\delta(a,t)| da \leq \int_0^t |f'_\delta| d\tilde{t} + \int_{\mathbb{R}_+} \rho_{I,\varepsilon} |u_{I,\varepsilon}^\delta| da \\ \leq C \left( \|f\|_{\text{BV}((0,T))}, \|(1+a)\rho_I\|_{L^1(\mathbb{R}_+)}, \|z'_p\|_{L^\infty(\mathbb{R}_-)} \right)$$

where  $C$  is independent on  $\varepsilon$  and on  $\delta$ .

**Proof.** Again, we proceed as in [MO15, Lemma 5.1], multiplying (2.23) by  $\text{sgn}(u_\varepsilon^\delta)$ , testing against  $\rho_\varepsilon$ , and integrating with respect to  $a$  gives :

$$\frac{d}{dt} \int_{\mathbb{R}_+} |u_\varepsilon^\delta| \rho_\varepsilon da + \int_{\mathbb{R}_+} |u_\varepsilon^\delta| \zeta_\varepsilon \rho_\varepsilon da \leq \varepsilon |f'_\delta| + \int_{\mathbb{R}_+} |u_\varepsilon^\delta| \zeta_\varepsilon \rho_\varepsilon da,$$

the rigorous proof relies on arguments exposed in [MO11, Lemma 3.1] and is left to the reader. Finally, after integration with respect to time, we conclude that

$$\int_0^\infty \rho_\varepsilon(a,t) |u_\varepsilon^\delta(a,t)| da \leq \int_0^t |f'_\delta| dt + \int_{\mathbb{R}_+} \rho_{I,\varepsilon} |u_{I,\varepsilon}^\delta| da,$$

since

$$\begin{aligned}
 |u_{I,\varepsilon}^\delta(a)| &\leq \left| \frac{z_\varepsilon^\delta(0^+) - z_p(0)}{\varepsilon} \right| + \left| \frac{z_p(0) - z_p(-\varepsilon a)}{\varepsilon} \right| \\
 &\leq \frac{1}{\mu_{0,\varepsilon}(0)} \left| f_\delta(0^+) + \frac{1}{\varepsilon} \int_0^\infty (z_p(-\varepsilon a) - z_p(0)) \rho_{I,\varepsilon}(a) . da \right| + \left| \frac{z_p(0) - z_p(-\varepsilon a)}{\varepsilon} \right| \\
 &\leq \frac{1}{\mu_{0,\min}} \left( \|z'_p\|_{L^\infty(\mathbb{R}_-)} \mu_{1,\varepsilon}(0) + f_\delta(0^+) \right) + \|z'_p\|_{L^\infty(\mathbb{R}_-)} a \\
 &\leq \max \left\{ \frac{1}{\mu_{0,\min}} \|z'_p\|_{L^\infty(\mathbb{R}_-)} \mu_{1,\varepsilon}(0) + f_\delta(0^+), \|z'_p\|_{L^\infty(\mathbb{R}_-)} \right\} (1 + a),
 \end{aligned}$$

the result follows. ■

In order to establish the convergence of  $u_\varepsilon^\delta$  in  $X_T$ , for a fixed  $\varepsilon$ , we introduce an intermediate variable  $w$  defined as

$$(2.30) \quad w(a, t) := u_\varepsilon^\delta(a, t) - \frac{f_\delta(t)}{\mu_{0,\varepsilon}(t)}$$

It satisfies

$$(2.31) \quad \begin{cases} \varepsilon \partial_t w + \partial_a w = \frac{1}{\mu_{0,\varepsilon}} \left( \varepsilon \frac{f_\delta \partial_t \mu_{0,\varepsilon}}{\mu_{0,\varepsilon}} + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon da \right), & t > 0, a > 0, \\ w(a=0, t) = \frac{-f_\delta(t)}{\mu_{0,\varepsilon}(t)}, & t > 0, \\ w(a, t=0^+) = u_{I,\varepsilon}^\delta(a) - \frac{f_\delta(0^+)}{\mu_{0,\varepsilon}(0)}, & a \geq 0. \end{cases}$$

The following crucial result holds :

**Lemma 12.** For a fixed  $\delta$  and  $\varepsilon$ , and under the Assumptions 1, the unknowns  $w$  and  $u_\varepsilon^\delta$ , are uniformly bounded in  $X_T$  with respect to  $\delta$  and  $\varepsilon$ .

**Proof.** Using arguments from [MO11, Lemma 2.1], one can show that  $w$  defined as

$$(2.32) \quad w(a, t) := \begin{cases} w(0, t - \varepsilon a) + \int_0^a G_w(t - \varepsilon \tilde{a}) d\tilde{a}, & \text{if } t > \varepsilon a \\ w(a - t/\varepsilon, 0^+) + \int_0^{t/\varepsilon} G_w(t - \varepsilon \tilde{a}) d\tilde{a}, & \text{if } t \leq \varepsilon a. \end{cases}$$

is a weak solution of (2.31). In the latter definition  $G_w(t) := \left\{ \varepsilon \frac{f_\delta \partial_t \mu_{0,\varepsilon}}{\mu_{0,\varepsilon}^2} + \frac{1}{\mu_{0,\varepsilon}} \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon^\delta da \right\}$ .

A simple computation shows that

$$\|w\|_{X_T} \leq \|G_w\|_{L^\infty((0,T))} + \|w(0, \cdot)\|_{L^\infty((0,T))} + \left\| \frac{w(\cdot, 0)}{1+a} \right\|_{L^\infty(\mathbb{R}_+)}.$$

It remains to estimate  $\|G_w\|_{L^\infty((0,T))}$ . For every fixed  $\varepsilon$ ,  $\mu_{0,\varepsilon}$  is a Lipschitz continuous function. Indeed,  $\mu_{0,\varepsilon}$  satisfies

$$\varepsilon \partial_t \mu_{0,\varepsilon} - \beta_\varepsilon (1 - \mu_{0,\varepsilon}) + \int_{\mathbb{R}_+} \zeta_\varepsilon \rho_\varepsilon da = 0$$

and then

$$\|\varepsilon \partial_t \mu_{0,\varepsilon}\|_{L_t^\infty} \leq \|\beta_\varepsilon\|_{L_t^\infty} + \|\zeta_\varepsilon\|_{L_{t,a}^\infty} \|\rho_\varepsilon\|_{L_t^\infty L_a^1},$$

which shows, by using the result of Lemma 11 that  $G_w$ , and also  $w$ , are uniformly bounded in  $L^\infty((0, T))$  with respect to  $\delta$  and  $\varepsilon$ . Indeed,

$$\|G_w\|_{L^\infty((0,T))} \leq \frac{\|\varepsilon \partial_t \mu_{0,\varepsilon}\|_{L^\infty((0,T))} \|f_\delta\|_{L^\infty((0,T))}}{\mu_{0,\min}^2} + \frac{\zeta_{\max}}{\mu_{0,\min}} \int_{\mathbb{R}_+} \rho_\varepsilon |u_\varepsilon^\delta| da < +\infty.$$

Finally, we have that

$$\|u_\varepsilon^\delta\|_{X_T} \leq \|w\|_{X_T} + \frac{\|f_\delta\|_{L^\infty((0,T))}}{\mu_{0,\min}} < +\infty,$$

which ends the proof. ■

Previous stability estimates allow to show :

**Theorem 23.** Under Assumption 1, one has for any fixed  $\varepsilon > 0$ ,

$$u_\varepsilon^\delta \rightharpoonup u_\varepsilon \text{ weakly-}^* \text{ in } X_T$$

as  $\delta \rightarrow 0$ , where  $u_\varepsilon$  solves the weak problem

$$(2.33) \quad \begin{aligned} & - \int_0^T \int_0^\infty u_\varepsilon (\varepsilon \partial_t + \partial_a) \varphi da dt + \varepsilon \left[ \int_0^\infty u_\varepsilon(a, s) \varphi(a, s) da \right]_{s=0}^{s=T} \\ & = \varepsilon \int_0^T \frac{\int_0^\infty \varphi(\tilde{a}, t) d\tilde{a}}{\mu_{0,\varepsilon}} dg + \int_0^T \frac{\int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon da}{\mu_{0,\varepsilon}} \left( \int_0^\infty \varphi(\tilde{a}, t) d\tilde{a} \right) \end{aligned}$$

for any  $\varphi \in C_c^1([0, T] \times \mathbb{R}_+)$ .

**Proof.** The uniform bound on  $u_\varepsilon^\delta$  in  $X_T$ , proved in Lemma 12, implies that

$$\frac{u_\varepsilon^\delta}{1+a} \xrightarrow{*} \frac{u_\varepsilon}{1+a}$$

in  $L^\infty((0, T) \times \mathbb{R}_+)$  in the weak-\* sense and the limit function  $u_\varepsilon$  belongs  $X_T$ . For every  $\psi \in L^\infty((0, T) \times \mathbb{R}_+)$ , we have  $\zeta_\varepsilon(1+a)\rho_\varepsilon\psi \in L^1((0, T) \times \mathbb{R}_+)$  and then

$$\int_0^T \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon^\delta \psi \, da \, dt \rightarrow \int_0^T \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \psi \, da \, dt.$$

By Corollary 1, the first term of right-hand in (2.27) tends to  $\int_0^T \int_0^\infty \varphi(t, \tilde{a}) d\tilde{a} / \mu_{0,\varepsilon} dg$  as  $\delta \rightarrow 0$ , for any  $\varphi \in \mathcal{C}_c^1([0, T] \times \mathbb{R}_+)$ . Regarding the second term of the right-hand side in (2.27), one has that  $\zeta_\varepsilon \rho_\varepsilon / \mu_{0,\varepsilon} \in L^1((0, T) \times \mathbb{R}_+)$  and this leads, thanks again to the weak-\* convergence above to write :

$$(2.34) \quad \int_0^t \frac{1}{\mu_{0,\varepsilon}} \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon^\delta \, da \, d\tilde{t} \rightarrow \int_0^t \frac{1}{\mu_{0,\varepsilon}} \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \, da \, d\tilde{t} \text{ as } \delta \rightarrow 0,$$

which ends the proof.  $\blacksquare$

The latter theorem allows to prove a convergence result when returning to the  $z_\varepsilon$  variable :

**Proposition 2.** Under the same assumptions as above, it holds that

$$z_\varepsilon^\delta \rightarrow z_\varepsilon \text{ strongly in } L^\infty((0, T)) \text{ as } \delta \rightarrow 0,$$

where  $z_\varepsilon$  satisfies,

$$(2.35) \quad z_\varepsilon(t) = z_\varepsilon(0^+) + \int_0^t \frac{\varepsilon}{\mu_{0,\varepsilon}} dg + \int_0^t \frac{1}{\mu_{0,\varepsilon}} \int_0^\infty \zeta_\varepsilon u_\varepsilon \rho_\varepsilon \, da \, d\tilde{t}$$

which is also a solution of (2.1).

Before showing this result, we make some comments : if  $u_\varepsilon^\delta$  is a solution of (2.23) then  $z_\varepsilon^\delta$  defined as

$$(2.36) \quad z_\varepsilon^\delta(t) := z_\varepsilon^\delta(0^+) + \int_0^t \frac{1}{\mu_{0,\varepsilon}(\tilde{t})} \left( \varepsilon f'_\delta(\tilde{t}) + \int_0^\infty \zeta_\varepsilon u_\varepsilon^\delta \rho_\varepsilon \, da \right) d\tilde{t}$$

solves (2.22). Conversely, if  $z_\varepsilon^\delta$  solves (2.22) then  $u_\varepsilon^\delta$ , given by

$$(2.37) \quad u_\varepsilon^\delta(a, t) = \begin{cases} \frac{z_\varepsilon^\delta(t) - z_\varepsilon^\delta(t - \varepsilon a)}{\varepsilon}, & \text{if } t > \varepsilon a \\ \frac{z_\varepsilon^\delta(t) - z_p(t - \varepsilon a)}{\varepsilon}, & \text{if } t \leq \varepsilon a \end{cases}$$



is a solution of (2.23). For more details see [MO15, Lemma 4.1 and Lemma 4.2].

**Proof.** First, by using (2.34) in the proof of Theorem 23, and Lemma 6, we have that  $f_\delta(0^+) = f(0^+)$  and  $z_\varepsilon^\delta$  given by (2.36) converge strongly in  $L^\infty((0, T))$  to  $z_\varepsilon$  which verifies (2.35). Using [MO15, Lemma 4.2], if  $z_\varepsilon^\delta$  is defined as (2.36) it solves (2.22). Multiplying (2.22) by a test function  $\varphi \in L^1(0, T)$  gives :

$$(2.38) \quad \frac{1}{\varepsilon} \int_0^T \int_0^{+\infty} (z_\varepsilon^{\delta k}(t) - z_\varepsilon^{\delta k}(t - \varepsilon a)) \rho_\varepsilon(a, t) \varphi(t) da dt = \int_0^T f_{\delta k}(t) \varphi(t) dt$$

As  $z_\varepsilon^{\delta k}$  converges strongly in  $L^\infty(0, T)$  to  $z_\varepsilon$ , the difference  $z_\varepsilon^{\delta k}(t) - z_\varepsilon^{\delta k}(t - \varepsilon a)$  converges almost everywhere for any fixed  $(a, t)$  in  $\mathbb{R}_+ \times (0, T)$  towards  $z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)$ . Thanks to the  $L^\infty$  bounds on  $u_\varepsilon^{\delta k}/(1+a)$ , and the bounds in  $L^1(\mathbb{R}_+ \times (0, T))$  on the first moment of  $\rho_\varepsilon$ , there exists an integrable majorizing function  $g(a, t)$  on  $\mathbb{R}_+ \times (0, T)$  s.t.

$$\left| z_\varepsilon^{\delta k}(t) - z_\varepsilon^{\delta k}(t - \varepsilon a) \right| |\varphi(t)| \rho_\varepsilon(a, t) \leq g(a, t)$$

uniformly for every  $k$ . Thus, one can apply the Lebesgue's Theorem in the left-hand side of (2.38). Since  $f_\delta$  converges in  $L^1(0, T)$  the convergence occurs in (2.38) for every  $\varphi \in \mathcal{D}(0, T)$  and thus almost everywhere in  $(0, T)$  and thus  $z_\varepsilon$  solves (2.1).  $\blacksquare$

## 2.5 Weak convergence when $\varepsilon$ goes to zero

Next, we prove the weak convergence of  $u_\varepsilon$  from which we deduce the strong convergence of  $z_\varepsilon$ .

**Theorem 25.** Under the same assumptions as above, one has

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly-}^* \text{ in } X_T$$

as  $\varepsilon \rightarrow 0$ , where  $u_0$  satisfies (2.7) and

$$\int_0^\infty u_0(a, t) \rho_0(a, t) da = f(t) \text{ a.e } t \in (0, T).$$

Furthermore, it also holds that

$$z_\varepsilon \rightarrow z_0 \text{ strongly in } L^\infty((0, T)) \text{ as } \varepsilon \rightarrow 0.$$

**Proof.** The proof follows the same steps as in [MO15, Theorem 6.2]. First, by Lemma 12,  $u_\varepsilon^\delta$  is uniformly bounded in  $X_T$  with respect to  $\delta$  and  $\varepsilon$ , and therefore  $u_\varepsilon$  is uniformly bounded

in  $X_T$  with respect to  $\varepsilon$ , then  $u_\varepsilon$  is weakly convergent to  $u_0$  in  $X_T$ . On the other hand, Theorem 3.2 and Lemma 3.4 imply that

$$(1+a)\rho_\varepsilon \rightarrow (1+a)\rho_0$$

strongly in  $L^1((0, T) \times \mathbb{R}_+)$ . These arguments justify that for every  $\psi \in L^\infty((0, T) \times \mathbb{R}_+)$  one has

$$\int_0^T \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \psi \, da \, dt \rightarrow \int_0^T \int_0^\infty \zeta_0 \rho_0 u_0 \psi \, da \, dt.$$

Indeed, one has

$$\begin{aligned} & \int_0^T \int_0^\infty \{\zeta_\varepsilon \rho_\varepsilon u_\varepsilon - \zeta_0 \rho_0 u_0\} \psi \, da \, dt = \int_0^T \int_0^\infty (\zeta_\varepsilon - \zeta_0) \rho_\varepsilon u_\varepsilon \psi \, da \, dt \\ & + \int_0^T \int_0^\infty \zeta_0 (\rho_\varepsilon - \rho_0) u_\varepsilon \psi \, da \, dt + \int_0^T \int_0^\infty \zeta_0 \rho_0 (u_\varepsilon - u_0) \psi \, da \, dt \end{aligned}$$

As  $\zeta_\varepsilon \rightarrow \zeta_0$  by Assumptions 1, and thanks to the weak convergence of  $u_\varepsilon$ , both terms on the right-hand side tend to zero as  $\varepsilon \rightarrow 0$ . Note that this implies the weak convergence of  $\int_{\mathbb{R}_+} \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \, da$  in  $L^1((0, T))$ , since we can choose  $\psi \in L^\infty((0, T))$ . Moreover, thanks to Lemma 7 we obtain that

$$\left| \int_{[0, T]} \frac{\int_0^{+\infty} \varphi \, da}{\mu_{0, \varepsilon}} \, dg \right| \leq \frac{C \|\varphi\|_{L_{t, a}^\infty}}{\mu_{0, \min}} \text{pvar}(g, [0, T]) \leq C \|f\|_{\text{BV}((0, T))}.$$

As in [MO15, Theorem 6.2], passing to the limit in the weak formulation (2.33) we obtain

$$-\int_0^T \int_0^\infty u_0 \partial_a \psi \, da \, dt = \int_0^T \int_0^\infty \frac{\zeta_0 \rho_0 u_0 \psi}{\mu_{0,0}} \, da \, dt$$

which implies that  $u_0$  satisfies

$$(2.39) \quad \begin{cases} \partial_a u_0 = \frac{1}{\mu_{0,0}} \int_0^\infty \zeta_0(\tilde{a}, t) u_0(\tilde{a}, t) \rho_0(\tilde{a}, t) \, d\tilde{a}, & t > 0, a > 0, \\ u_0(a=0, t) = 0, & t > 0, \end{cases}$$

Similarly, we have the weak convergence of  $\int_0^\infty u_\varepsilon(t, a) \rho_\varepsilon(t, a) \, da$  towards  $\int_0^\infty u_0(t, a) \rho_0(t, a) \, da$  in  $L^1((0, T))$ . Hence, one concludes that  $u_0$  satisfies also

$$\int_0^\infty u_0(t, a) \rho_0(t, a) \, da = f(t), \quad a.e. \, t \in (0, T).$$

As the right-hand side of (2.39) does not depend on age, one has that  $u_0 = \gamma(t) \cdot a$ , where in order to satisfy the last compatibility condition implies that

$$\gamma(t) \int_{\mathbb{R}_+} a \rho_0(a, t) \, da = f(t), \quad a.e. \, t \in (0, T).$$

Thus  $u_0(a, t) = f(t)/\mu_{1,0}(t)a$  for almost every  $t \in (0, T)$  and every  $a \in \mathbb{R}_+$ . Using again the weak convergence of  $\int_{\mathbb{R}_+} \zeta_\varepsilon \rho_\varepsilon u_\varepsilon da$  in  $L^1((0, T))$  combined with the strong convergence of  $\mu_{0,\varepsilon}$  allows to pass to the limit in the third term of (2.35). Moreover,

$$\begin{aligned} |z_\varepsilon(0^+) - z_p(0)| &= \frac{1}{\mu_{0,\varepsilon}(0)} \left| \int_0^\infty (z_p(-\varepsilon a) - z_p(0)) \rho_{I,\varepsilon}(a) da + \varepsilon f(0^+) \right| \\ &\leq \frac{\varepsilon k \|z_p'\|_{L^\infty(\mathbb{R}^-)}}{\mu_{0,\min}} + \frac{\varepsilon |f(0^+)|}{\mu_{0,\min}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \end{aligned}$$

where  $k$  is the constant from Lemma 9. All together this provides that  $z_0$  solves :

$$(2.40) \quad z_0(t) = z_p(0) + \int_0^t \frac{f(\tau)}{\mu_{1,0}(\tau)} \frac{\int_{\mathbb{R}_+} \zeta_0(a, \tau) \rho_0(a, \tau) da}{\mu_{0,0}(\tau)} d\tau$$

but because  $\rho_0$  solves (2.4), one has that  $a\rho_0$  solves :

$$\partial_a(a\rho_0) - \rho_0 + a\zeta_0(a, t)\rho_0 = 0,$$

which after integration in time shows that

$$\frac{\int_{\mathbb{R}_+} a\zeta_0(a, t)\rho_0(a, t) da}{\mu_{0,0}(t)} = 1$$

and this shows in turn that (2.40) reduces to

$$z_0(t) = z_0(0) + \int_0^t \frac{f(\tau)}{\mu_{1,0}(\tau)} d\tau$$

which is the integrated version of (2.3). ■

## 2.6 A comparison principle

In this section, we give error estimates between  $z_\varepsilon$  and  $z_0$ , the solution of the problem

$$(2.41) \quad \begin{cases} \frac{1}{\varepsilon} \int_0^\infty (z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)) \varrho(a) da = f(t), & t \geq 0, \\ z_\varepsilon(t) = z_p(t), & t < 0, \end{cases}$$

where  $\varrho$  is constant in time and satisfies

$$(2.42) \quad \begin{cases} \partial_a \varrho + \zeta(a)\varrho = 0, & a > 0, \\ \varrho(0) = \beta \left( 1 - \int_0^\infty \varrho(\bar{a}) d\bar{a} \right), \end{cases}$$

where the data of (2.41) and (2.42) satisfy

- Assumptions 3.** *i)*  $f \in \text{BV}((0, T))$ ,  
*ii)*  $z_p \in \text{Lip}(\mathbb{R}_-)$ ,  
*iii)*  $\beta \in \mathbb{R}_+^*$ ,  
*iv)*  $\zeta \in L^\infty(\mathbb{R}_+)$  such that:  $0 < \zeta_{\min} \leq \zeta(a) \leq \zeta_{\max}$ , *a.e.*  $a \in \mathbb{R}_+$ .

Setting  $\hat{z}_\varepsilon(t) := z_\varepsilon(t) - z_0(t)$ , it solves :

$$\hat{z}_\varepsilon(t) = \frac{1}{\mu_0} \int_0^{\frac{t}{\varepsilon}} \hat{z}_\varepsilon(t - \varepsilon a) \varrho(a) da + \tilde{h}_\varepsilon(t)$$

where

(2.43)

$$\begin{aligned} \tilde{h}_\varepsilon(t) &= \frac{\varepsilon}{\mu_0} \int_0^{t/\varepsilon} \left( \frac{z_0(t) - z_0(t - \varepsilon a)}{\varepsilon} - a \partial_t z_0(t) \right) \varrho(a) da \\ &+ \frac{\varepsilon}{\mu_0} \int_{t/\varepsilon}^{+\infty} \left( \frac{z_0(t) - z_0(0)}{\varepsilon} - a \partial_t z_0(t) \right) \varrho(a) da + \frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} (z_p(t - \varepsilon a) - z_p(0)) \varrho(a) da \end{aligned}$$

where  $\mu_0 := \int_0^{+\infty} \varrho(a) da$  then

$$(2.44) \quad |\hat{z}_\varepsilon(t)| \leq \frac{1}{\mu_0} \int_0^{\frac{t}{\varepsilon}} |\hat{z}_\varepsilon(t - \varepsilon a)| \varrho(a) da + |\tilde{h}_\varepsilon(t)|.$$

Then integrating in time and setting

$$\hat{Z}_\varepsilon(t) := \int_0^t |\hat{z}_\varepsilon(\tau)| d\tau$$

one has that :

$$\hat{Z}_\varepsilon(t) = \int_0^t |\hat{z}_\varepsilon(\tau)| d\tau \leq \frac{1}{\mu_0} \int_0^t \int_0^{\frac{\tau}{\varepsilon}} |\hat{z}_\varepsilon(\tau - \varepsilon a)| \varrho(a) da d\tau + \int_0^t |\tilde{h}_\varepsilon(\tau)| d\tau$$

then, we change the order of integration and the domain of integration becomes  $D' := \{(a, \tau) \in (0, t/\varepsilon) \times (\varepsilon a, t)\}$ . We use the change of variable  $\tilde{t} = \tau - \varepsilon a$  in order to write :

$$\begin{aligned} \int_0^t \int_0^{\frac{\tau}{\varepsilon}} |\hat{z}_\varepsilon(\tau - \varepsilon a)| \varrho(a) da d\tau &= \int_0^{\frac{t}{\varepsilon}} \int_{\varepsilon a}^t |\hat{z}_\varepsilon(\tau - \varepsilon a)| d\tau \varrho(a) da \\ &= \int_0^{\frac{t}{\varepsilon}} \int_0^{t - \varepsilon a} |\hat{z}_\varepsilon(\tilde{t})| d\tilde{t} \varrho(a) da = \int_0^{\frac{t}{\varepsilon}} \hat{Z}_\varepsilon(t - \varepsilon a) \varrho(a) da \end{aligned}$$

So that finally  $\hat{Z}_\varepsilon$  solves :

$$\hat{Z}_\varepsilon(t) \leq \frac{1}{\mu_0} \int_0^{\frac{t}{\varepsilon}} \hat{Z}_\varepsilon(t - \varepsilon a) \varrho(a) da + \int_0^t \tilde{h}_\varepsilon(\tau) d\tau = \int_0^t \hat{Z}_\varepsilon(\tilde{a}) K_\varepsilon(t - \tilde{a}) d\tilde{a} + \int_0^t \tilde{h}_\varepsilon(\tau) d\tau$$

where  $K_\varepsilon(\tilde{a}) := \frac{1}{\varepsilon\mu_0}\varrho\left(\frac{\tilde{a}}{\varepsilon}\right)$  is the kernel of the integral operator. We use a comparison principle [GLS09, the Generalised Gronwall Lemma 8.2 p. 257] and construct a majorizing function  $U_\varepsilon$  of the form  $U_\varepsilon(t) = \varepsilon(K_0 + K_1 t)$  where  $K_0$  and  $K_1$  are suitably chosen, such that  $U_\varepsilon \geq |\hat{Z}_\varepsilon|$  and  $U_\varepsilon \sim \varepsilon$ . The following two lemmas are required in order to apply this comparison principle :

**Lemma 13.** The Volterra kernel  $K_\varepsilon$  satisfies :

$$\|K_\varepsilon\|_{L^\infty(0,T)} := \operatorname{ess\,sup}_{t \in (0,T)} \int_0^t |K_\varepsilon(\tilde{a})| d\tilde{a} < 1.$$

**Proof.** To prove this result, we need to show that

$$(2.45) \quad 0 \leq \int_0^t |K_\varepsilon(\tilde{a})| d\tilde{a} = \frac{\int_0^{\frac{t}{\varepsilon}} \varrho(a) da}{\int_0^{+\infty} \varrho(a) da} < 1.$$

The kernel  $\varrho$  solves (2.42), thus it can be explicitly computed as

$$\varrho(a) = \frac{\beta}{1 + \beta I} \exp\left(-\int_0^a \zeta(s) ds\right)$$

one has the lower bound :

$$\varrho(a) \geq \frac{\beta}{1 + \beta I} \exp(-\zeta_{\max} a) > 0, \quad \forall a \in \mathbb{R}_+.$$

This in turn shows that

$$\int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da > 0,$$

which is equivalent to the claim. ■

**Lemma 14.** Consider the expectation value of a given density  $\varrho$  with respect to the tail  $a > t/\varepsilon$ ,

$$(2.46) \quad A_1[\varrho](t) := \frac{\int_0^{+\infty} a\varrho(a + \frac{t}{\varepsilon}) da}{\int_0^{+\infty} \varrho(a + \frac{t}{\varepsilon}) da}$$

then under Assumptions 3, one has

$$A_1[\varrho](t) \leq \frac{\zeta_{\max}}{\zeta_{\min}^2}.$$

**Proof.** Setting

$$(2.47) \quad q(a, t) := \frac{\varrho(a + \frac{t}{\varepsilon})}{\varrho(\frac{t}{\varepsilon})},$$

it solves

$$\partial_a q(a, t) + \zeta(a + t/\varepsilon) \cdot q(a, t) = 0, \quad q(0, t) = 1.$$

This problem admits an explicit solution of the form

$$(2.48) \quad q(a, t) = \exp\left(-\int_0^a \zeta(\bar{a} + t/\varepsilon) d\bar{a}\right) = \exp\left(-\int_{t/\varepsilon}^{a+t/\varepsilon} \zeta(\hat{a}) d\hat{a}\right).$$

Then, we shall rewrite (2.46) as :

$$A_1[\varrho](t) := \frac{\int_0^{+\infty} a q(a, t) da}{\int_0^{+\infty} q(a, t) da}$$

by using hypothesis iv) from Assumptions 3, one has

$$\exp(-\zeta_{\max} a) \leq q(a, t) \leq \exp(-\zeta_{\min} a)$$

This gives :

$$\int_0^{+\infty} q(a, t) da \geq \int_0^{+\infty} \exp(-\zeta_{\max} a) da = \frac{1}{\zeta_{\max}}$$

and

$$\int_0^{+\infty} a q(a, t) da \leq \int_0^{+\infty} a \exp(-\zeta_{\min} a) da = \frac{1}{\zeta_{\min}^2}$$

which shows the final upper bound. ■

**Proposition 3.** Under Assumptions 3, for  $0 < t < T$  one has the estimates :

$$\tilde{H}_\varepsilon(t) := \int_0^t |\tilde{h}_\varepsilon(\tau)| d\tau \leq \varepsilon^2 C_1$$

where  $C_1$  depends on  $\mu_2, \mu_1, \|\partial_t z_0\|_{\text{BV}((0, T))}$  and on  $\|z_p\|_{\text{Lip}(\mathbb{R}_-)}$  but not on  $\varepsilon$ .

**Proof.** Recalling the definition of  $\tilde{h}_\varepsilon$  in (2.43), we split  $\int_0^t |\tilde{h}_\varepsilon(\tau)| d\tau$  into three parts. First, we define

$$I_1 := \frac{\varepsilon}{\mu_0} \int_0^t \int_0^{\tau/\varepsilon} \left| \frac{z_0(\tau) - z_0(\tau - \varepsilon a)}{\varepsilon} - a \partial_t z_0(\tau) \right| \varrho(a) da d\tau$$

Since  $\partial_t z_0 \in \text{BV}((0, T))$ , then  $I_1$  can be written in the form

$$I_1 = \frac{1}{\mu_0} \int_0^t \int_0^{\tau/\varepsilon} \left| \int_{\tau - \varepsilon a}^{\tau} (\partial_t z_0(\tilde{t}) - \partial_t z_0(\tau)) d\tilde{t} \right| \varrho(a) da d\tau$$

by switching the integration order between  $\tau$  and  $a$ , and using the change of variable  $\tilde{t} = \tau + h$ , we get that

$$\begin{aligned}
 (2.49) \quad I_1 &\leq \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_{\varepsilon a}^t \int_{\tau-\varepsilon a}^\tau |\partial_t z_0(\tilde{t}) - \partial_t z_0(\tau)| d\tilde{t} d\tau \rho(a) da \\
 &\leq \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_{\varepsilon a}^t \int_{-\varepsilon a}^0 |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| dh d\tau \rho(a) da \\
 &\leq \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_{-\varepsilon a}^0 \int_{\varepsilon a}^t |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| d\tau dh \rho(a) da
 \end{aligned}$$

and thus applying Lemma 3, one has the estimate of the inner integral of the latter right-hand side :

$$\int_{\varepsilon a}^t |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| d\tau \leq |h| \|\partial_t z_0\|_{BV}, \quad \forall h \in (-\varepsilon a, 0)$$

which implies that

$$I_1 \leq \frac{\|\partial_t z_0\|_{BV}}{\mu_0} \int_0^{t/\varepsilon} \int_{-\varepsilon a}^0 |h| dh \rho(a) da \leq \frac{\varepsilon^2 \|\partial_t z_0\|_{BV}}{2\mu_0} \int_0^{t/\varepsilon} a^2 \rho(a) da.$$

Next, we set:

$$\begin{aligned}
 I_2 &= \frac{\varepsilon}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} \left| \frac{z_0(\tau) - z_0(0)}{\varepsilon} - a \partial_t z_0(\tau) \right| \rho(a) da d\tau \\
 &\leq \frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} \left| \int_0^\tau \partial_t z_0(\tilde{t}) d\tilde{t} - \tau \partial_t z_0(\tau) \right| \rho(a) da d\tau + \frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} |\tau - \varepsilon a| |\partial_t z_0(\tau)| \rho(a) da d\tau \\
 &=: I_{2,1} + I_{2,2}
 \end{aligned}$$

As in the estimates of  $I_1$ , first, one switches the order of integration and then one integrates on

$$(2.50) \quad D := \{(a, \tau) \in (0, t/\varepsilon) \times (0, \varepsilon a)\} \cup \{(a, \tau) \in (t/\varepsilon, +\infty) \times (0, t)\},$$

and one makes the change of variable  $\tilde{t} = \tau + h$  in order to obtain

$$\begin{aligned}
 I_{2,1} &= \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_0^{\varepsilon a} \int_{-\tau}^0 |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| dh d\tau \rho(a) da \\
 &\quad + \frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} \int_0^t \int_{-\tau}^0 |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| dh d\tau \rho(a) da \\
 &= \frac{1}{\mu_0} \int_0^{t/\varepsilon} \int_{-\varepsilon a}^0 \int_{-\varepsilon a}^{\varepsilon a} |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| d\tau dh \rho(a) da \\
 &\quad + \frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} \int_{-t}^0 \int_{-h}^t |\partial_t z_0(\tau + h) - \partial_t z_0(\tau)| d\tau dh \rho(a) da
 \end{aligned}$$

also, by using Lemma 4, we get that

$$\begin{aligned}
 I_{2,1} &\leq \frac{\|\partial_t z_0\|_{BV}}{\mu_0} \int_0^{t/\varepsilon} \int_{-\varepsilon a}^0 |h| dh \varrho(a) da + \frac{\|\partial_t z_0\|_{BV}}{\mu_0} \int_{t/\varepsilon}^{+\infty} \int_{-t}^0 |h| dh \varrho(a) da \\
 &\leq \frac{\varepsilon^2 \|\partial_t z_0\|_{BV}}{2\mu_0} \int_0^{t/\varepsilon} a^2 \varrho(a) da + \frac{\|\partial_t z_0\|_{BV}}{2\mu_0} \int_{t/\varepsilon}^{+\infty} t^2 \varrho(a) da \\
 &\leq \frac{\varepsilon^2 \|\partial_t z_0\|_{BV}}{2\mu_0} \int_0^{t/\varepsilon} a^2 \varrho(a) da + \frac{\varepsilon^2 \|\partial_t z_0\|_{BV}}{2\mu_0} \int_{t/\varepsilon}^{+\infty} a^2 \varrho(a) da \\
 &\leq \frac{\varepsilon^2 \mu_2 \|\partial_t z_0\|_{BV}}{\mu_0}.
 \end{aligned}$$

The second term  $I_{2,2}$  is estimated in the same way as  $I_{2,1}$ . We have

$$\begin{aligned}
 I_{2,2} &= \frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} |\tau - \varepsilon a| |\partial_t z_0(\tau)| \varrho(a) da d\tau \\
 &= \frac{1}{\mu_0} \left\{ \int_0^{t/\varepsilon} \int_0^{\varepsilon a} + \int_{t/\varepsilon}^{+\infty} \int_0^t |\tau - \varepsilon a| |\partial_t z_0(\tau)| d\tau \varrho(a) da \right\} \\
 &\leq \frac{\varepsilon^2 \|\partial_t z_0\|_{\infty}}{2\mu_0} \int_0^{t/\varepsilon} a^2 \varrho(a) da + \frac{\varepsilon^2 \|\partial_t z_0\|_{\infty}}{2\mu_0} \int_{t/\varepsilon}^{+\infty} a^2 \varrho(a) da \leq \frac{\varepsilon^2 \mu_2 \|\partial_t z_0\|_{\infty}}{\mu_0}.
 \end{aligned}$$

Finally, by similar computations, one has

$$\frac{1}{\mu_0} \int_0^t \int_{\tau/\varepsilon}^{+\infty} |z_p(\tau - \varepsilon a) - z_p(0)| \varrho(a) da d\tau \leq \varepsilon^2 \|z_p\|_{\text{Lip}(\mathbb{R}_-)} \frac{\mu_2}{\mu_0}$$

■

**Theorem 27.** Under Assumptions 3,  $z_\varepsilon$  tends to  $z_0$ , the solution of (2.3), strongly in  $L^1(0, T)$  as  $\varepsilon$  goes to zero. Moreover, there exists a generic constant  $C$  depending only on the data of the problem but not on  $\varepsilon$ , such that :

$$\|z_\varepsilon - z_0\|_{L^1(0, T)} \leq \varepsilon C.$$

**Proof.** We have proved in Lemma 13 that the Volterra kernel  $K_\varepsilon$  is non-positive and bounded (with a bound strictly less than one) in the sense of [GLS09, Definition 2.2 p. 227 and Proposition 2.7 p.231] so the Generalised Gronwall Lemma applies (see for instance Theorem 53 in appendix A). First observe, by using Proposition 3, that

$$\hat{Z}_\varepsilon(t) - \int_0^t \hat{Z}_\varepsilon(\tilde{a}) K_\varepsilon(\tilde{a}) d\tilde{a} \leq \tilde{h}_{1,\varepsilon}(t) + \tilde{h}_{2,\varepsilon}(t)$$



We construct a function  $U_\varepsilon$  which satisfies,

$$(2.51) \quad U_\varepsilon(t) - \int_0^t U_\varepsilon(\tilde{a}) K_\varepsilon(\tilde{a}) d\tilde{a} \geq \tilde{h}_{1,\varepsilon} + \tilde{h}_{2,\varepsilon}$$

We split the integral operator applied to  $U_\varepsilon$  in two parts

$$\begin{aligned} U_\varepsilon(t) - \int_0^t U_\varepsilon(\tilde{a}) K_\varepsilon(\tilde{a}) d\tilde{a} &= U_\varepsilon(t) - \frac{1}{\mu_0} \int_0^{t/\varepsilon} U_\varepsilon(t - \varepsilon a) \rho(a) da \\ &= \underbrace{\frac{1}{\mu_0} \int_0^{+\infty} (U_\varepsilon(t) - U_\varepsilon(t - \varepsilon a)) \rho(a) da}_{:=H_{1,\varepsilon}} + \underbrace{\frac{1}{\mu_0} \int_{t/\varepsilon}^{+\infty} U_\varepsilon(t - \varepsilon a) \rho(a) da}_{:=H_{2,\varepsilon}} \end{aligned}$$

and we shall specify  $U_\varepsilon$  such that  $H_{1,\varepsilon} \geq \tilde{H}_\varepsilon(t)$  and  $H_{2,\varepsilon} \geq 0$ . To this end we set

$$(2.52) \quad U_\varepsilon(t) := \varepsilon(K_0 + K_1 t), \quad \forall t \in \mathbb{R}$$

with constants  $K_0$  and  $K_1$  to be specified. One has obviously that

$$H_{1,\varepsilon}(t) = \frac{\varepsilon^2 K_1 \mu_1}{\mu_0} \geq \varepsilon^2 C_1 \geq \tilde{H}_\varepsilon(t)$$

a.e. on  $\mathbb{R}_+$ , provided that  $K_1$  is chosen as

$$K_1 > \frac{\mu_0}{\mu_1} C_1.$$

Using (2.52) and the change of variable  $\tilde{a} = -t/\varepsilon + a$ , we obtain that

$$H_{2,\varepsilon} = \int_0^{+\infty} (K_0 - \varepsilon K_1 \tilde{a}) \rho(\tilde{a} + t/\varepsilon) d\tilde{a} = (K_0 - \varepsilon K_1 A_\varepsilon[\rho](t)) \int_{\mathbb{R}_+} \rho\left(\frac{t}{\varepsilon} + a\right) da$$

We are in the hypotheses of Lemma 14 :  $A_\varepsilon[\rho](t)$ , the expectation of a given density  $\rho$  with respect to the tail  $a > t/\varepsilon$  is bounded by a positive constant  $A_{\max}$

$$A_\varepsilon[\rho](t) := \frac{\int_{\mathbb{R}_+} a \rho\left(\frac{t}{\varepsilon} + a\right) da}{\int_{\mathbb{R}_+} \rho\left(\frac{t}{\varepsilon} + a\right) da} \leq A_{\max}.$$

Therefore it suffices to chose  $K_0 > \varepsilon K_1 A_{\max}$  in order to obtain that  $H_{2,\varepsilon} \geq 0$ . These computations show that  $U_\varepsilon$  is a super-solution. Then the comparison principle implies that, for all  $0 \leq t \leq T$ ,

$$0 \leq \hat{Z}_\varepsilon(t) = \int_0^t |z_\varepsilon(s) - z_0(s)| ds \leq U_\varepsilon(t) = \varepsilon(K_0 + K_1 t) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

hence  $\hat{Z}_\varepsilon \rightarrow 0$  in  $C([0, T])$ , which ends the proof since  $\|z_\varepsilon - z_0\|_{L^1(0, T)} \leq \|\hat{Z}_\varepsilon\|_{C([0, T])}$ . ■

## 2.7 A simple example

We construct *by hand* solutions of problems (2.1) and (2.3) when the load  $f$  is explicitly defined as

$$(2.53) \quad f(t) := \begin{cases} 1/2 & \text{if } 0 < t \leq \frac{1}{3}, \\ 1 & \text{if } \frac{1}{3} < t \leq \frac{2}{3}, \\ 3/2 & \text{if } \frac{2}{3} < t < 1, \end{cases}$$

and the kernel  $\rho$  is a simple exponential (see more precise statements below). So defined  $f$  is of course of bounded variation on  $(0, 1)$ . The solution  $z_\varepsilon$  solving (2.1) and its limit  $z_0$  show different regularities (see Figure 2.1) : the adhesive approximation is rougher than the limit solution. This is an interesting feature of our approach.

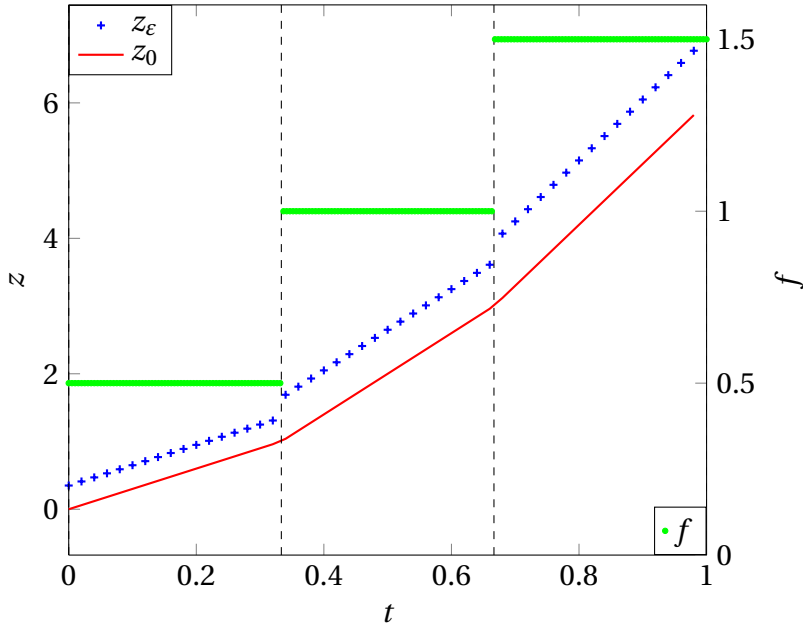


Figure 2.1: The solutions  $z_\varepsilon$  and  $z_0$  as a function of time, for a fixed  $\varepsilon = 10^{-1}$  on the left y-axis. The load  $f$  on the right y-axis.

**Assumptions 4.** *i) the load  $f$  is defined in (2.53),*  
*ii) the on and off rates are constants defined as :*

$$\zeta_\varepsilon = \zeta_0 = \zeta, \quad \beta_\varepsilon = \beta_0 = \beta.$$

*iii) the initial condition is at equilibrium :*

$$\rho_{I,\varepsilon} = \rho_0 = \frac{\beta\zeta}{\beta + \zeta} e^{-\zeta a}.$$

**Lemma 15.** Under Assumptions 4, one has that  $\mu_{0,\varepsilon} = \mu_{0,0} = \beta/(\beta + \zeta)$ ,  $\mu_{1,\varepsilon} = \mu_{1,0} = \beta/(\zeta(\beta + \zeta))$  and  $\rho_0(a) = \mu_{0,0}\zeta e^{-\zeta a}$ . Then the solution  $z_\varepsilon$  of (2.1) is BPV([0, 1]) and it is explicitly given by

$$z_\varepsilon(t) = \int_0^t \frac{f}{\mu_{1,0}} ds + \varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{1}{\mu_{0,0}} \int_0^{+\infty} z_p(-\varepsilon a) \rho_0 da$$

and hence,

$$z_\varepsilon(t) - z_0(t) = \varepsilon \frac{f(t)}{\mu_{0,0}} + \int_{-\infty}^0 z'_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds$$

with  $z_0(t) = z_p(0) + \int_0^t f(s) ds / \mu_{1,0}$  is a continuous function in  $[0, 1]$ . Note that the last term is an  $\varepsilon$  order term according to Assumption 1, indeed it holds that

$$\left| \int_{-\infty}^0 z'_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds \right| \leq \frac{\varepsilon}{\zeta} \|z_p\|_{W^{1,\infty}(\mathbb{R}_-)}.$$

**Proof.** The specific choice of data and kernel allows to rephrase (2.1) as

$$z_\varepsilon(t) - \frac{\zeta}{\varepsilon} \int_0^t z_\varepsilon(s) \exp\left(-\frac{\zeta(t-s)}{\varepsilon}\right) ds = \varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(-\frac{\zeta(t-s)}{\varepsilon}\right) ds.$$

Next, setting

$$q_\varepsilon(t) = z_\varepsilon(t) \exp\left(\frac{\zeta t}{\varepsilon}\right), \quad t \geq 0.$$

Then we can rewrite (1) for all  $t \geq 0$  as

$$(2.54) \quad q_\varepsilon(t) - \frac{\zeta}{\varepsilon} \int_0^t q_\varepsilon(s) ds = \varepsilon \exp\left(\frac{\zeta t}{\varepsilon}\right) \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds.$$

By differentiating (2.54) in time and using (2.6), we prove that  $q_\varepsilon$  solve the equation

$$(2.55) \quad \begin{cases} \dot{q}_\varepsilon(t) - \frac{\zeta}{\varepsilon} q_\varepsilon(t) = \frac{\exp\left(\frac{\zeta t}{\varepsilon}\right)}{\mu_{0,0}} (\zeta f(t) + \varepsilon f'(t)), & t > 0, \\ q_\varepsilon(0^+) = \varepsilon f(0) / \mu_{0,0} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds \end{cases}$$

and therefore  $q_\varepsilon$  is explicitly given by

$$q_\varepsilon(t) = \exp\left(\frac{\zeta t}{\varepsilon}\right) \left( \varepsilon \frac{f(t)}{\mu_{0,0}} + \frac{\zeta}{\varepsilon} \int_{-\infty}^0 z_p(s) \exp\left(\frac{\zeta s}{\varepsilon}\right) ds + \int_0^t \frac{f(s)}{\mu_{1,0}} ds \right)$$

which gives the formula of  $z_\varepsilon$ . Moreover, it is clear that  $z_\varepsilon$  is of bounded variation since  $f$  is it and  $z_0$  is an absolutely continuous function. ■

## FRICITION MEDIATED BY TRANSIENT ELASTIC LINKAGES : ASYMPTOTIC EXPANSIONS AND FAT TAILS

### Abstract

**W**e construct an asymptotic expansion for the integral delay operator already introduced in [MO11; MO15] and show how to improve convergence rates already obtained in the latter papers. Moreover, we weaken one of the major hypotheses made on the off-rates in our previous works : we do not assume exponential decay of the linkages' density. Instead, polynomial decrease is allowed, leading to stronger adhesions and slower motions of adhesion sites.

### 3.1 Introduction

This work is a continuation of a series of works related to the mathematical study of adhesion forces in the context of cell motility (see [AM21], [MO11], [MO15], [MO16] and [MO17]). Cell adhesion and migration play a crucial role in many biological phenomena such as : embryonic development, inflammatory responses, wound healing and tumor metastasis. The adhesion model discussed further has been designed at the scale of a single binding site [OS09, Chapter 5]. Lately, it has also been used at the mesoscopic cell scale ([Gre+18], [MS21]).

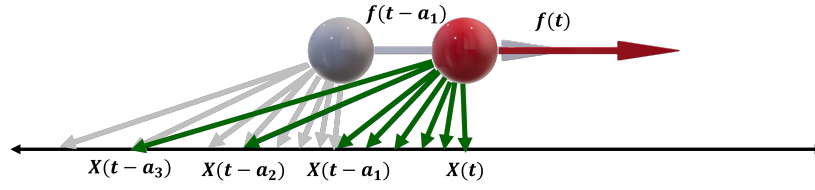


Figure 3.1: The position of the moving binding site at time  $t$  and time  $t - a_1$  with respective linkages to the substrate.

The cell is modelled as a point particle whose position on the real line is denoted  $X_\varepsilon$  and depends on  $t$  (see fig.3.1). The position  $X_\varepsilon$  is obtained solving a force balance equation

$$(3.1) \quad \begin{cases} \frac{1}{\varepsilon} \int_{\mathbb{R}_+} (X_\varepsilon(t) - X_\varepsilon(t - \varepsilon a)) \rho_\varepsilon(a, t) da = f(t), & t > 0, \\ X_\varepsilon(t) = X_p(t), & t \leq 0, \end{cases}$$

where  $f$  is an external force and the left-hand side represents a continuum of elastic forces with respect to past positions. The density of linkages  $\rho_\varepsilon$  is a solution of an age-structured model

$$(3.2) \quad \begin{cases} (\varepsilon \partial_t + \partial_a + \zeta(a, t)) \rho_\varepsilon(a, t) = 0, & (a, t) \in \mathbb{R}_+ \times (0, T), \\ \rho_\varepsilon(0, t) = \beta(t) \left( 1 - \int_{\mathbb{R}_+} \rho_\varepsilon(a, t) da \right), & (a, t) \in \{0\} \times (0, T), \\ \rho_\varepsilon(a, 0) = \rho_I(a), & (a, t) \in \mathbb{R}_+ \times \{0\}. \end{cases}$$

In the latter system,  $\beta \in \mathbb{R}_+$  (resp.  $\zeta \in \mathbb{R}_+$ ) is the kinetic on-rate (resp. off-rate) function and the speed of linkage turnover is represented by the small parameter known as  $\varepsilon > 0$  ([MO11], [MO15], and [MO16]). Under the assumption that the death rate  $\zeta$  admits a strictly positive lower bound  $\zeta_{\min}$ , in [MO11], the authors studied rigorously the asymptotic limit of the systems (3.1) and (3.2) when  $\varepsilon$  goes to zero. They obtained the convergence results

$$\|X_\varepsilon - X_0\|_{C^0([0, T])} + \|\rho_\varepsilon - \rho_0\|_{C^0((0, T); L^1(\mathbb{R}_+))} \rightarrow 0,$$

where the limit  $X_0$  solves

$$(3.3) \quad \begin{cases} \mu_{1,0}(t) X_0' = f(t), & t > 0, \\ X_0(t=0) = X_p(0), & t = 0, \end{cases}$$

where  $\mu_{1,0}(t)$  denotes the first moment of  $\rho_0$ , namely  $\mu_{1,0}(t) := \int_0^\infty a\rho_0(a, t)da$  and  $\rho_0$  satisfies :

$$(3.4) \quad \begin{cases} \partial_a \rho_0 + \zeta(a, t)\rho_0 = 0, & t > 0, a > 0, \\ \rho_0(a = 0, t) = \beta(t) \left(1 - \int_0^\infty \rho_0(t, \tilde{a}) d\tilde{a}\right), & t > 0. \end{cases}$$

We underline that, to some extent, the  $\varepsilon$  scaling can be associated with the long time behavior of solutions of (3.1) and (3.2) [MS21, Theorem 4.4].

In [MO11], using the Lyapunov functional

$$(3.5) \quad \mathcal{H}[u] := \left| \int_0^\infty u(a)da \right| + \int_0^\infty |u(a)|da,$$

the authors have proved the convergence of  $\rho_\varepsilon$  towards  $\rho_0$ . In the same article they showed as well the convergence of the position  $X_\varepsilon$  thanks to a comparison principle specific to Volterra equations [GLS09, Chapter 9, Section 8]. Here as well, the main goal is to study the asymptotic behavior of solutions of the coupled problem (3.1)-(3.2) under two major constrains :

- 1) increase the order of approximation with respect to  $\varepsilon$ ,
- 2) weaken hypotheses on  $\zeta$  allowing  $\rho_\varepsilon$  and  $\rho_0$  to have *fat tails* with respect to the age variable.

More precisely, we aim at constructing the  $N^{th}$ -order asymptotic approximation of the solution  $X_\varepsilon$  satisfying (3.1) as

$$(3.6) \quad \tilde{X}_{\varepsilon, N} = X_{outer}(t) + X_{inner}(\tau) + O(\varepsilon^N),$$

where  $\tau = t/\varepsilon$  is the stretched variable. The construction of this asymptotic development is done in two steps. First, we construct  $X_{outer}$  containing a series of macroscopic correctors in power of  $\varepsilon$ . These correctors are valid away from  $t = 0$ . Next, we construct  $X_{inner}$  containing microscopic correctors that correct the fast variation near the boundary layer at  $t = 0$ .

We start by studying problem (3.1) with a given non-negative density of linkages  $\varrho$  such that

$$\mu_{N+1} := \int_0^{+\infty} (1+a)^{N+1} \varrho(a) da < +\infty$$

and such that it is not compactly supported (cf. Hypotheses 5 for a more precise definition). In this case, we construct an expansion (3.6) for any fixed integer  $N$ . The macroscopic part  $X_{outer}$  is formed by correctors that solve first order differential equations (see (3.16)). The

initial condition of these correctors is then determined by matching inner and outer expansion. An error estimate was obtained, leading to

$$\|X_\varepsilon - \tilde{X}_{\varepsilon,N}\|_{C([0,T])} \lesssim \varepsilon^N.$$

Finally, we consider the general case where the density also depends on  $\varepsilon$  and solves (3.2). We are mainly concerned with the asymptotic behavior of  $X_\varepsilon$  as the perturbation parameter  $\varepsilon$  approaches zero. First, we study the asymptotic behavior of  $\rho_\varepsilon$  when  $\varepsilon$  goes to 0. The novelty here, compared to [MO11], is that we weakened the assumptions on the death rate  $\zeta$ . Namely we assume that there exists a non-increasing function  $m \in L^1(\mathbb{R}_+; (1+a)^3)$  such that

$$(3.7) \quad \zeta(a, t) \geq -\frac{m'(a)}{m(a)}, \quad a.e. a \in \mathbb{R}_+$$

This hypothesis allows  $\zeta$  to go to zero for large  $a$  and allows  $\rho$  to have fat tails. In comparison, in [MO11], the hypothesis on  $\zeta$  was stronger :  $\zeta(a, t) \geq \zeta_{\min} > 0$ . This allowed to use of Gronwall's Lemma and get *a priori* estimates :

$$\|\rho_\varepsilon(\cdot, t) - \rho_0(\cdot, t)\|_{L^1(\mathbb{R}_+)} \leq \mathcal{H}[\rho_I(\cdot) - \rho_0(\cdot, 0)] \exp(-\zeta_{\min} t/\varepsilon) + o_\varepsilon(1).$$

Showing an exponential decay in time and age of the initial layer near to  $t = 0$  [Mil18]. In our case, if  $\zeta$  satisfies the condition (3.7), we cannot use Gronwall's Lemma to establish the convergence when  $\varepsilon$  tends to 0. For this sake, we enrich the asymptotic expansion of  $\rho_\varepsilon$  with supplementary terms. We introduce  $\rho_1$ , the first order macroscopic solution of :

$$(3.8) \quad \begin{cases} (\partial_a + \zeta(a, t)) \rho_1(a, t) = -\partial_t \rho_0(a, t), & a > 0, t > 0, \\ \rho_1(0, t) = -\beta(t) \int_{\mathbb{R}_+} \rho_1(a, t) da, & a = 0, t > 0, \end{cases}$$

and  $\tau_0$  the initial layer approximation solving :

$$(3.9) \quad \begin{cases} (\partial_t + \partial_a + \zeta(a, 0)) \tau_0(a, t) = 0, & a > 0, t > 0, \\ \tau_0(0, t) = -\beta(0) \int_{\mathbb{R}_+} \tau_0(a, t) da, & a = 0, t > 0, \\ \tau_0(a, 0) = \rho_I(a) - \rho_0(a, 0), & a > 0, t = 0. \end{cases}$$

This enhances the earlier error estimates. Indeed, for any  $t > 0$ , one has

$$\mathcal{H}[\rho_\varepsilon(\cdot, t) - \rho_0(\cdot, t) - \varepsilon \rho_1(\cdot, t) - \tau_0(\cdot, t/\varepsilon)] \lesssim o_\varepsilon(1),$$

leading to :

$$\|\rho_\varepsilon - \rho_0\|_{L^1(\mathbb{R}_+ \times (0,T), (1+a))} \leq o_\varepsilon(1).$$

This result is in turn used to establish the strong convergence of  $X_\varepsilon$  towards  $X_0$  solving (3.3) extending the results from [MO15] to this new framework.

The chapter's outline is structured as follows : in section 3.2, we list some assumptions and notations which will be used throughout this paper. In section 3.3, we analyze the case where the kernel in (3.1) is fixed and depends on the age variable. Moreover, we construct the asymptotic expansion of  $X_\varepsilon$  and show error estimates. Finally, in section 3.4, we analyze the asymptotic behavior of  $\rho_\varepsilon$  and  $X_\varepsilon$  solutions of (3.2) and (3.1) respectively when  $\varepsilon$  tends to 0. Then we extend results from [MO15] to our setting and conclude.

## 3.2 Notations

Before presenting our main result, we list some notations and assumptions that will be used in this paper. In the rest of the paper, we'll use some notations for the functional spaces, for instance  $L_t^p L_a^q := L^p((0, T); L^q(\mathbb{R}_+))$  for any real  $(p, q) \in [1, \infty]^2$ ,  $V_T := L^\infty(\mathbb{R}_+ \times (0, T), (1 + a)^{-1})$ , and The space  $\text{Lip}(\mathbb{R}_+)$  is the set of Lipschitz functions on  $\mathbb{R}_+$ . Hereafter, in the following sections, capital letters  $(X_i)_{i \in \mathbb{N}}$  denote the macroscopic correctors defined on  $[0, T]$ , and the microscopic correctors  $(x_{i,j})_{(i,j) \in \mathbb{N}^2}$  or  $(w_k)_{k \in \mathbb{N}}$  defined on  $\mathbb{R}$  are then renamed with a tilde when rescaled with respect to  $\varepsilon$  :  $\tilde{x}_{i,j}(t) = x_{i,j}(t/\varepsilon)$ .

## 3.3 The linkages' density is constant in time

In this section, we begin to study the simple model of the problem (3.1) with a kernel  $\rho$  constant in time. We assume that the data of the problem satisfies the following assumptions :

**Assumptions 5.** *Assume that :*

- i) *the source term is such that  $f \in C^N(\mathbb{R}_+)$ .*
- ii) *the past condition  $X_p \in C^{N+1}(\mathbb{R}_+)$ .*
- iii) *for all  $a \in \mathbb{R}_+$ , there exists  $M \subset (a, \infty)$ ,  $M$  compact and  $|M| > 0$  such that  $\rho(\tilde{a}) > 0$  for almost every  $\tilde{a} \in M$ .*
- iv) *moreover*

$$\mu_{N+1} := \int_{\mathbb{R}_+} (1+a)^{N+1} \rho(a) da < \infty.$$



### 3.3.1 Construction of the expansion

First, we start with the construction of the terms forming the  $N^{th}$ -order approximation of  $X_\varepsilon$  solution of (3.1) for a fixed kernel as

$$(3.10) \quad \tilde{X}_{\varepsilon,N} := \underbrace{\sum_{i=0}^{N-1} \varepsilon^i X_i(t)}_{\text{outer expansion}} + \underbrace{Y_N(t) + Z_N(t) + W_N(t)}_{\text{inner expansion}},$$

where these terms are set later on. Define the operator  $\mathcal{L}_\varepsilon : C([0, T]) \rightarrow C([0, T])$  that maps  $X$  to

$$\mathcal{L}_\varepsilon[X](t) := \frac{1}{\varepsilon} \left\{ \mu_0 X(t) - \int_0^{\frac{t}{\varepsilon}} X(t - \varepsilon a) \varrho(a) da \right\}.$$

Then problem (3.1) can be rephrased as :

$$(3.11) \quad \mathcal{L}_\varepsilon[X_\varepsilon](t) = f(t) + \frac{1}{\varepsilon} \int_{\frac{t}{\varepsilon}}^\infty X_p(t - \varepsilon a) \varrho(a) da, \quad \forall t > 0$$

and we aim at constructing  $\tilde{X}_{\varepsilon,N}$  such that it satisfies

$$(3.12) \quad \mathcal{L}_\varepsilon[\tilde{X}_{\varepsilon,N}] = f(t) + \frac{1}{\varepsilon} \int_{\frac{t}{\varepsilon}}^\infty X_p(t - \varepsilon a) \varrho(a) da + O(\varepsilon^N).$$

**Proposition 4.** Assume that Hypotheses 5 hold, let the sequence of functions  $(X_i)_{i \in \{0, \dots, N-1\}}$  be given and for all  $i \in \{0, \dots, N-1\}$  assume that  $X_i \in W^{N+1, \infty}([0, T])$ , then one has the expansion :

$$\begin{aligned} \mathcal{L}_\varepsilon[X_i](t) &= \sum_{k=1}^{N-i} \frac{\varepsilon^{k-1} (-1)^{k+1}}{k!} X_i^{(k)}(t) (\mu_k - \Xi_{0,k}(t)) + \frac{1}{\varepsilon} \int_{\frac{t}{\varepsilon}}^\infty \varrho(a) da X_i(t) \\ &\quad + \varepsilon^{N-i} \mathcal{R}_i^{N+1-i}, \end{aligned}$$

where

$$\begin{cases} \mu_k := \int_{\mathbb{R}_+} a^k \varrho(a) da, \\ \Xi_{0,k}(t) := \int_{\frac{t}{\varepsilon}}^\infty a^k \varrho(a) da =: \xi_{0,k} \left( \frac{t}{\varepsilon} \right), \\ \left| \mathcal{R}_i^{N+1-i} \right| \leq \frac{1}{(N+1-i)!} \left\| X_i^{(N+1-i)} \right\|_\infty \int_{\mathbb{R}_+} a^{N+1-i} \varrho(a) da, \end{cases}$$

**Proof.** One writes :

$$\mathcal{L}_\varepsilon[X_i](t) = \frac{1}{\varepsilon} \int_0^{\frac{t}{\varepsilon}} (X_i(t) - X_i(t - \varepsilon a)) \varrho(a) da + \frac{1}{\varepsilon} \int_{\frac{t}{\varepsilon}}^\infty \varrho(a) da X_i(t)$$

then using the Taylor expansion :

$$\begin{aligned} X_i(t - \varepsilon a) &= \sum_{k=0}^{N-i} \frac{\varepsilon^k a^k}{k!} (-1)^k X_i^{(k)}(t) \\ &\quad + \frac{(-\varepsilon a)^{N+1-i}}{(N-i)!} \int_0^1 X_i^{(N+1-i)}(t - s\varepsilon a) (1-s)^{N-i} ds \end{aligned}$$

so that the first term above becomes :

$$\begin{aligned} &\frac{1}{\varepsilon} \int_0^{\frac{t}{\varepsilon}} (X_i(t) - X_i(t - \varepsilon a)) \varrho(a) da \\ &= \sum_{k=1}^{N-i} \int_0^{\frac{t}{\varepsilon}} \frac{\varepsilon^{k-1} a^k}{k!} \varrho(a) da (-1)^{k+1} X_i^{(k)}(t) + \varepsilon^{N-i} \mathcal{R}_i^{N+1-i} \\ &= \sum_{k=1}^{N-i} \frac{(-1)^{k+1}}{k!} X_i^{(k)}(t) \left( \varepsilon^{k-1} \mu_k - \frac{1}{\varepsilon} \int_{\frac{t}{\varepsilon}}^{\infty} (\varepsilon a)^k \varrho(a) da \right) + \varepsilon^{N-i} \mathcal{R}_i^{N+1-i} \end{aligned}$$

which provides the result. ■

**Proposition 5.** [Outer expansions]

Under the same hypothesis as above, the zeroth-order macroscopic limit is given by

$$(3.13) \quad \mu_1 X_0^{(1)} = f,$$

and at any order  $\ell \in \{2, \dots, N\}$ , we have :

$$(3.14) \quad \mu_1 X'_{\ell-1} = \sum_{k=2}^{\ell} (-1)^k \frac{\mu_k}{k!} X_{\ell-k}^{(k)}.$$

**Proof.** The result proved in Proposition 4 leads to :

$$(3.15) \quad \begin{aligned} \mathcal{L}_\varepsilon \left[ \sum_{i=0}^{N-1} \varepsilon^i X_i \right] &= \sum_{i=0}^{N-1} \varepsilon^i \sum_{k=1}^{N-i} \frac{\varepsilon^{k-1} (-1)^{k+1}}{k!} X_i^{(k)}(t) (\mu_k - \Xi_{0,k}(t)) \\ &\quad + \sum_{i=0}^{N-1} \varepsilon^{i-1} \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da X_i(t) + S_{N,0}, \end{aligned}$$

where we set  $S_{N,0} := \varepsilon^N \sum_{i=0}^{N-1} \mathcal{R}_i^{N+1-i}$  and  $|S_{N,0}| \leq \max_{i \in \{0, \dots, N-1\}} \{\|X_i\|_{W^{N+1,\infty}(0,T)} \mu_{N+1-i}\}$ . Considering the first sum gives :

$$\begin{aligned} &\sum_{i=0}^{N-1} \varepsilon^i \sum_{k=1}^{N-i} \frac{\varepsilon^{k-1} (-1)^{k+1}}{k!} X_i^{(k)}(t) (\mu_k - \Xi_{0,k}(t)) \\ &= \sum_{k=1}^N \sum_{\ell=k}^N \varepsilon^{\ell-1} \frac{(-1)^{k+1}}{k!} X_{\ell-k}^{(k)}(t) (\mu_k - \Xi_{0,k}(t)) \\ &= \sum_{\ell=1}^N \varepsilon^{\ell-1} \sum_{k=1}^{\ell} \frac{(-1)^{k+1}}{k!} X_{\ell-k}^{(k)}(t) (\mu_k - \Xi_{0,k}(t)) \end{aligned}$$

Separating powers of  $\varepsilon$  and considering that terms containing functions  $\Xi_{0,k}$  belong to the initial layer (these depend only on the microscopic variable  $t/\varepsilon$ ) provides :

$$(3.16) \quad \sum_{k=1}^{\ell} \mu_k \frac{X^{\ell-k}(t)}{k!} (-1)^{k+1} = \begin{cases} 0 & \text{if } \ell \neq 1, \\ f & \text{otherwise,} \end{cases}$$

and by relating the lowest derivative with the highest index to the rest of the correctors, we establish macroscopic nested ODEs (3.13) and (3.14). ■

**Remark 2.** The initial conditions of the macroscopic correctors  $X_i$  are to be defined later (cf Theorem 29).

**Proposition 6.** [Inner expansion] It is threefold.

- The first part accounts for terms containing  $\Xi_{0,k}$  in the first sum of (3.15) :

$$(3.17) \quad Y_N(t) := \sum_{m=1}^N \varepsilon^m \sum_{q=1}^m \sum_{k=1}^q \frac{(-1)^{k+1}}{k!(m-q)!} X_{q-k}^{(k+m-q)}(0) \tilde{x}_{m-q,k}(t)$$

where  $\tilde{x}_{j,k} := x_{j,k}(t/\varepsilon)$  and the microscopic correctors solve :

$$(3.18) \quad \mathcal{L}_1[x_{j,k}](t) = \xi_{j,k}(t) := t^j \int_t^{\infty} a^k \varrho(a) da,$$

and  $\mathcal{L}_1$  is the operator  $\mathcal{L}_\varepsilon$  taken for  $\varepsilon$  set to 1.

- The second part corrects the second sum in (3.15) and reads :

$$(3.19) \quad Z_N(t) := - \sum_{m=1}^N \varepsilon^m \sum_{q=0}^{m-1} \frac{1}{(m-q)!} X_q^{(m-q)}(0) \tilde{x}_{m-q,0} - \sum_{i=0}^{N-1} \varepsilon^i X_i(0) \tilde{x}_{0,0}.$$

- The last part concerns the remainders related to the past source term in (3.12)

$$W_N(t) := \sum_{i=0}^N \frac{\varepsilon^i}{i!} X_p^{(i)}(0) \tilde{w}_i(t),$$

where  $\tilde{w}_i(t) := w_i(t/\varepsilon)$  and  $(w_\ell)_\ell$  solve for  $\ell \in \mathbb{N}$ ,

$$(3.20) \quad \begin{cases} \int_{\mathbb{R}_+} (w_\ell(t) - w_\ell(t-a)) \varrho(a) da = 0, & t > 0, \\ w_\ell(t) = t^\ell, & t \leq 0. \end{cases}$$

**Proof.** First, we begin by constructing the first part of the initial layer  $Y_N$ . We consider the second term in (3.15) and we use Taylor's expansion :

$$X_{\ell-k}^{(k)}(t) = \sum_{j=0}^{N-k} X_{\ell-k}^{(k+j)}(0) \frac{t^j}{j!} + \varepsilon^N \mathcal{R}_{k,l}^N,$$

where

$$\mathcal{R}_{k,l}^N := \frac{\varepsilon^{-k+1}}{(N-k)!} \left(\frac{t}{\varepsilon}\right)^{N-k+1} \int_0^1 (1-s)^{N-k} X_{\ell-k}^{(N-k+1)}(st) ds,$$

which implies that

$$\begin{aligned} (3.21) \quad & \sum_{\ell=1}^N \varepsilon^{\ell-1} \sum_{k=1}^{\ell} \frac{(-1)^k}{k!} X_{\ell-k}^{(k)}(t) \Xi_{0,k}(t) \\ &= \sum_{\ell=1}^N \varepsilon^{\ell-1} \sum_{k=1}^{\ell} \frac{(-1)^k}{k!} \Xi_{0,k}(t) \left\{ \sum_{j=0}^{N-k} X_{\ell-k}^{(k+j)}(0) \frac{t^j}{j!} + \mathcal{R}_{k,l}^N \right\} \\ &= \sum_{\ell=1}^N \sum_{k=1}^{\ell} \sum_{j=0}^{N-k} \frac{(-1)^k}{k!j!} \varepsilon^{\ell+j-1} X_{\ell-k}^{(k+j)}(0) \left(\frac{t}{\varepsilon}\right)^j \Xi_{0,k}(t) + S_{N,1} =: I + S_{N,1} \end{aligned}$$

where

$$(3.22) \quad S_{N,1} := \sum_{\ell,k} \varepsilon^{\ell+N-k} \frac{(-1)^k}{k!} \Xi_{0,k}(t) \mathcal{R}_{k,l}^N,$$

that can be estimated as :

$$(3.23) \quad |S_{N,1}| \leq C \varepsilon^N \mu_{N+1}$$

The first triple sum can be decomposed thanks to Proposition 8 as

$$\begin{aligned} I &:= \sum_{m=1}^N \varepsilon^{m-1} \left( \sum_{q=1}^m \sum_{k=1}^q \frac{(-1)^k}{k!(m-q)!} X_{q-k}^{(k+m-q)}(0) \Xi_{m-q,k}(t) \right) \\ &+ \sum_{m=N+1}^{2N-1} \varepsilon^{m-1} \left( \sum_{q=m+1-N}^N \sum_{k=1}^{q+N-m} \frac{(-1)^k}{k!(m-q)!} X_{q-k}^{(k+m-q)}(0) \Xi_{m-q,k}(t) \right) \\ &=: I_1 + O(\varepsilon^N) \end{aligned}$$

In order to compensate  $I_1$ , we define microscopic correctors  $\tilde{x}_{j,k}$  as (3.18) and set  $Y_N$  as in (3.17).

Now, we need to correct the third term in (3.15), which we do with the same technique as above :

$$\begin{aligned}
& \sum_{i=0}^{N-1} \varepsilon^{i-1} X_i(t) \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da \\
&= \sum_{i=0}^{N-1} \varepsilon^{i-1} \left\{ \sum_{j=0}^{N-i} \frac{t^j}{j!} X_i^{(j)}(0) + \frac{t^{N+1-i}}{(N-i)!} \int_0^1 X_i^{N+1-i}(st)(1-s)^{N-i} ds \right\} \int_{\frac{t}{\varepsilon}}^{\infty} \varrho(a) da \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{N-i} \frac{\varepsilon^{i+j-1}}{j!} X_i^{(j)}(0) \frac{t^j}{\varepsilon^j} \Xi_{0,0}(t) + S_{N,2} = \sum_{i=0}^{N-1} \sum_{j=0}^{N-i} \frac{\varepsilon^{i+j-1}}{j!} \Xi_{j,0}(t) X_i^{(j)}(0) + S_{N,2} \\
&= \sum_{m=1}^N \varepsilon^{m-1} \sum_{q=0}^{m-1} \frac{1}{(m-q)!} \Xi_{m-q,0}(t) X_q^{(m-q)}(0) + \sum_{i=0}^{N-1} \varepsilon^{i-1} \Xi_{0,0}(t) X_i(0) + S_{N,2}
\end{aligned}$$

where  $\Xi_{j,0}(t) := r_{\varepsilon}^1 j, 0(t/\varepsilon)$  and

$$(3.24) \quad S_{N,2}(t) := \sum_{i=0}^{N-1} \varepsilon^{i-1} \frac{t^{N+1-i}}{N-i!} \int_0^1 X_i^{N+1-i}(st)(1-s)^{N-i} ds \Xi_{0,0}(t),$$

and one has :

$$(3.25) \quad |S_{N,2}| \leq C \varepsilon^N \int_{\mathbb{R}_+} (1+a)^{N+1} \varrho(a) da \sup_{i \in \{0, \dots, N\}} \|X_i\|_{W^{N+1-i, \infty}(0, t)}.$$

It suffices then to add the correction  $Z_N$  defined as in (3.19). Lastly, it remains to correct the terms of the past. To find them, we need to develop  $X_p(t - \varepsilon a)$  around 0, which is stated as :

$$\begin{aligned}
X_p(t - \varepsilon a) &= \sum_{i=0}^N \varepsilon^i X_p^{(i)}(0) \frac{(t/\varepsilon - a)^i}{i!} \\
&\quad + \varepsilon^{N+1} \frac{(t/\varepsilon - a)^{N+1}}{N!} \int_0^1 X_p^{(N+1)}(s(t - \varepsilon a))(1-s)^N ds,
\end{aligned}$$

and it involves

$$\int_{\frac{t}{\varepsilon}}^{+\infty} X_p(t - \varepsilon a) \varrho(a) da = \sum_{i=0}^N \varepsilon^i X_p^{(i)}(0) \int_{\frac{t}{\varepsilon}}^{+\infty} \frac{(t/\varepsilon - a)^i}{i!} \varrho(a) da + S_{N,3},$$

where

$$(3.26) \quad S_{N,3} := \varepsilon^{N+1} \int_{\frac{t}{\varepsilon}}^{+\infty} \frac{(t/\varepsilon - a)^{N+1}}{N!} \int_0^1 X_p^{(N+1)}(s(t - \varepsilon a))(1-s)^N ds \varrho(a) da,$$

and

$$\begin{aligned}
(3.27) \quad |S_{N,3}| &\leq \varepsilon^N \frac{1}{N!} \|X_p^{(N)}\|_{\infty} \int_{\frac{t}{\varepsilon}}^{+\infty} \left(\frac{t}{\varepsilon} - a\right)^N \varrho(a) da \\
&\leq \varepsilon^N \frac{1}{N!} \|X_p^{(N)}\|_{\infty} \sum_{k=0}^{N+1} C_k^{N+1} \int_{\frac{t}{\varepsilon}}^{+\infty} \left(\frac{t}{\varepsilon}\right)^k (-a)^{N-k+1} \varrho(a) da \leq c \varepsilon^{N+1} \mu_{N+1}
\end{aligned}$$

Which then yields that past-correctors should be added as :

$$W_N(t) := \sum_{i=0}^N \varepsilon^i \frac{X_p^{(i)}(0)}{i!} \tilde{w}_i(t),$$

where  $\tilde{w}_i(t) := w_i(t/\varepsilon)$  and  $w_i$  satisfies (3.20). ■

**Lemma 16.** If  $\mu_{j+k+1} < \infty$ , then one has :

$$x_{j,k}(0) = \begin{cases} \frac{\mu_k}{\mu_0}, & \text{if } j = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and  $x_{j,k}(t) \rightarrow \mu_{j+1+k}/((j+1)\mu_1)$  when  $t \rightarrow \infty$ .

**Proof.** The resolvent associated to (3.18), satisfies :

$$r(t) - (r \star k)(t) = k(t),$$

where

$$(3.28) \quad k(a) := \varrho(a)/\mu_0, \quad (r \star k)(t) := \int_0^t r(t-\tau)k(\tau)d\tau$$

and it can be decomposed [GLS09, Theorem 7.4.1, p.201] as

$$r(t) = \frac{\mu_0}{\mu_1} + \gamma(t),$$

where the function  $\gamma \in L^1(\mathbb{R}_+)$ . Moreover, the resolvent being defined the solution  $x_{j,k}$  is computed explicitly and reads :

$$x_{j,k} = \xi_{j,k} + \xi_{j,k} \star r = \xi_{j,k} + \xi_{j,k} \star (\mu_0/\mu_1 + \gamma) = \xi_{j,k} + \xi_{j,k} \star \gamma + \frac{\mu_0}{\mu_1} \int_0^t \xi_{j,k}(s)ds.$$

Thus the leading term in  $x_{j,k}$  when  $t$  grows large is the last integral. Indeed

$$\frac{\mu_0}{\mu_1} \int_{\mathbb{R}_+} t^j \int_t^\infty a^k k(a) da dt = \frac{1}{\mu_1} \int_{\mathbb{R}_+} \left( \int_0^a t^j dt \right) a^k \varrho(a) da = \frac{1}{(j+1)} \frac{\mu_{j+k+1}}{\mu_1}$$

and one has :

$$x_{j,k} - \frac{1}{(j+1)} \frac{\mu_{j+k+1}}{\mu_1} \in L^1(\mathbb{R}_+).$$

which we define as the formal expression  $x_{j,k}(t) \rightarrow \mu_{j+1+k}/((j+1)\mu_1)$  when  $t \rightarrow \infty$ . ■

**Lemma 17.** Under the same assumptions as in the previous Lemma, the microscopic functions  $w_\ell$  are discontinuous at  $t = 0$ , for all  $\ell \geq 0$  :

$$w_\ell(0^+) = (-1)^\ell \frac{\mu_\ell}{\mu_0}, \quad w_\ell(0^-) = 0,$$

and  $w_\ell(t) \rightarrow ((-1)^\ell \mu_{\ell+1}) / ((\ell + 1)\mu_1)$  when  $t \rightarrow \infty$ .

**Proof.** Using (3.20), we can easily show the discontinuity of the correctors  $w_\ell$  at  $t = 0$ . By the same arguments as proof of Lemma 16, one has that :

$$\begin{aligned} \lim_{t \rightarrow \infty} w_\ell(t) &= \frac{1}{\mu_1} \int_{\mathbb{R}_+} \int_t^\infty (t-a)^\ell \varrho(a) da dt = \frac{1}{\mu_1} \int_{\mathbb{R}_+} \left( \int_0^a (t-a)^\ell dt \right) \varrho(a) da \\ &= \frac{(-1)^\ell}{(\ell+1)} \frac{\mu_{\ell+1}}{\mu_1}, \end{aligned}$$

which ends the proof. ■

**Lemma 18.** Under the previous results, we obtain the error estimate

$$|X_\varepsilon(0^+) - \tilde{X}_{\varepsilon,N}(0^+)| \lesssim \varepsilon^N.$$

**Proof.** By definition, one has :

$$\tilde{X}_{\varepsilon,N}(0^+) = \sum_{i=0}^{N-1} \varepsilon^i X_i(0) + Y_N(0) + Z_N(0) + W_N(0^+),$$

then one has

$$Y_N(0) := \frac{1}{\mu_0} \sum_{\ell=1}^N \varepsilon^\ell \sum_{k=1}^{\ell} \frac{(-1)^{k+1}}{k!} X_{\ell-k}^{(k)}(0) \mu_k = \varepsilon \frac{f(0)}{\mu_0}$$

where we used that  $\tilde{x}_{j,k}(0) = 0$  for all  $j \neq 0$  and (3.16). By definition,

$$Z_N(0) = - \sum_{i=0}^{N-1} \varepsilon^i X_i(0) \tilde{x}_{0,0}(0) = - \sum_{i=0}^{N-1} \varepsilon^i X_i(0),$$

and compensates the first terms of the sum, then

$$W_N(0) = \frac{1}{\mu_0} \sum_{i=0}^N \varepsilon^i \frac{X_p^{(i)}(0)}{i!} (-1)^i \mu_i.$$

Then one observes that

$$X_\varepsilon(0) = \varepsilon \frac{f(0)}{\mu_0} + \int_{\mathbb{R}_+} X_p(-\varepsilon a) \frac{\varrho(a)}{\mu_0} da$$

and so the Taylor expansion of the last term ends the proof. ■

### 3.3.2 Matching inner and outer expansions

So far, the initial conditions of the outer expansion are not defined. For this sake, we write the inner expansion's limit when  $t \rightarrow \infty$ . This gives :

$$\lim_{t \rightarrow \infty} Y_N(t) = \sum_{q=1}^m \sum_{k=1}^q \frac{(-1)^{k+1}}{(m-q+1)!k!} X_{q-k}^{(k+m-q)}(0) \mu_{m-q+k+1} =: s_1,$$

together with :

$$\lim_{t \rightarrow \infty} Z_N(t) = - \sum_{m=1}^N \varepsilon^m \sum_{q=0}^{m-1} X_q^{(m-q)} \frac{\mu_{m-q+1}}{(m-q+1)! \mu_1} - \sum_{i=0}^{N-1} \varepsilon^i X_i(0)$$

and

$$\lim_{t \rightarrow \infty} W_N(t) = \sum_{i=0}^N \varepsilon^i \frac{X_p^{(i)}(0)}{(i+1)!} (-1)^i \frac{\mu_{i+1}}{\mu_1} =: s_3.$$

As we do not want the inner expansion to interfere with the outer expansion, we gather the powers of  $\varepsilon$  and define the initial conditions of the outer expansion so that

$$\lim_{t \rightarrow \infty} (Y_N(t) + Z_N(t) + W_N(t)) = 0,$$

which then gives :

**Theorem 29.** The macroscopic Ansatz should be given the initial conditions : for  $m = 0$ ,  $X_0(0) = X_p(0)$  while for  $m \in \{1, \dots, N-1\}$

$$\begin{aligned} \mu_1 X_m(0) &= (-1)^m X_p^{(m)}(0) \frac{\mu_{m+1}}{((m+1)!)} - \sum_{q=0}^{m-1} X_q^{(m-q)}(0) \frac{\mu_{m-q+1}}{(m-q+1)!} \\ &+ \sum_{q=1}^m \sum_{k=1}^q \frac{(-1)^{k+1}}{(m-q+1)!k!} X_{q-k}^{(k+m-q)}(0) \mu_{m-q+k+1}. \end{aligned}$$

### 3.3.3 Error estimates

In this section, we give an error estimate between  $X_\varepsilon$ , the solution of

$$(3.29) \quad \begin{cases} \frac{1}{\varepsilon} \int_{\mathbb{R}_+} (X_\varepsilon(t) - X_\varepsilon(t - \varepsilon a)) \rho(a) da = f(t), & t > 0, \\ X_\varepsilon(t) = X_p(t) & t \leq 0, \end{cases}$$

and the asymptotic expansion  $\tilde{X}_{\varepsilon,N}$  given by (3.10). This result is based on the application of a comparison principle [GLS09, the Generalised Gronwall Lemma 8.2 p. 257] and the construction of a super solution  $U_N$  such that  $U_N \geq |X_\varepsilon - \tilde{X}_{\varepsilon,N}|$  and  $U_N \lesssim \varepsilon^N$ . The following lemma is required in order to apply the latter comparison principle :



**Lemma 19.** Under the Assumptions 5,  $K_\varepsilon(\tilde{a}) := \frac{1}{\varepsilon\mu_0} \varrho\left(\frac{\tilde{a}}{\varepsilon}\right)$  satisfies :

$$\|K_\varepsilon\|_{L^\infty(\mathbb{R}_+)} := \operatorname{ess\,sup}_{t \in \mathbb{R}_+} \int_0^t |K_\varepsilon(\tilde{a}, t)| d\tilde{a} < 1.$$

**Proof.** We need to show that for every  $t \in \mathbb{R}_+$ ,

$$(3.30) \quad 0 \leq \int_0^t |K_\varepsilon(\tilde{a}, t)| d\tilde{a} = \frac{\int_0^{\frac{t}{\varepsilon}} \varrho(a) da}{\int_{\mathbb{R}_+} \varrho(a) da} < 1.$$

which is equivalent to show that for every  $t \in \mathbb{R}_+$ ,  $\int_{\frac{t}{\varepsilon}}^\infty \varrho(a) da > 0$ . But, by definition, for every fixed  $t$  and  $\varepsilon$  there exists a compact set  $M \subset (t/\varepsilon, \infty)$  such that  $\varrho(a) > 0$  for almost every  $a \in M$  so that

$$\int_{\frac{t}{\varepsilon}}^\infty \varrho(a) da \geq \int_M \varrho(a) da > 0.$$

■

**Theorem 31.** According to Assumptions 5, it holds that :

$$\|X_\varepsilon - \tilde{X}_{\varepsilon, N}\|_{C([0, T])} \lesssim \varepsilon^N.$$

where  $\tilde{X}_{\varepsilon, N}$  is defined in (3.10) and  $X_\varepsilon$  solving (3.29).

**Proof.** First, we consider the zero order approximation (*i.e.*  $N = 1$ ). We denote  $\hat{X}_1 := X_\varepsilon - \tilde{X}_{\varepsilon, 1}$ , it solves :

$$\mathcal{L}_\varepsilon[\hat{X}_1] = S_1 := \sum_{i=1}^3 S_{1, i},$$

where  $|S_1| \leq \varepsilon K_1$  (see the estimations (3.23), (3.25) and (3.27) for  $N = 1$ ). Now it remains to construct a super-solution  $U_1$  such that

$$\mathcal{L}_\varepsilon[|\hat{X}_1|] \leq \mathcal{L}_\varepsilon[U_1], \quad \text{and } |\hat{X}_1(0)| \leq U_1(0),$$

in order to obtain  $|\hat{X}_1(t)| \leq U_1(t)$ , for all  $t > 0$ . We set

$$U_1(t) := \varepsilon(c_1 + tc_2 - \varepsilon c_3 \tilde{w}_1(t)),$$

where  $\tilde{w}_1$  solves (3.20) with  $\ell = 1$ . As the resolvent associated to (3.20) is non-negative, applying the comparison principle [GLS09, Propostion 8.1 and Lemma 8.2],  $\tilde{w}_1(t) \leq 0$  for

$t > 0$ . Then

$$\begin{aligned}\mathcal{L}_\varepsilon[U_1] &= c_1 \Xi_{0,0}(t) + \varepsilon c_2 \mu_1 + c_2 \int_{\frac{t}{\varepsilon}}^{\infty} (t - \varepsilon a) \varrho(a) da - \varepsilon c_3 \mathcal{L}_\varepsilon[\tilde{w}_1] \\ &= c_1 \Xi_{0,0}(t) + \varepsilon c_2 \mu_1 + \varepsilon c_2 \int_{\frac{t}{\varepsilon}}^{\infty} \left( \frac{t}{\varepsilon} - a \right) \varrho(a) da - \varepsilon c_3 \int_{\frac{t}{\varepsilon}}^{\infty} \left( \frac{t}{\varepsilon} - a \right) \varrho(a) da \geq \varepsilon c_2 \mu_1.\end{aligned}$$

The last inequality being true when  $c_1 \geq 0$  and  $c_2 = c_3$ . Then, one tunes  $c_2 \geq K_1/\mu_1$  so that

$$\mathcal{L}_\varepsilon[|\hat{X}_1|] \leq |S_1| \leq \varepsilon K_1 \leq \varepsilon \mu_1 c_2 \leq \mathcal{L}_\varepsilon[U_1],$$

and the constant  $c_1$  is set such that  $|\hat{X}_1(0)| \leq \varepsilon c_1 \leq \varepsilon c_1 + \varepsilon^2 c_3 \frac{\mu_1}{\mu_0} = U_1(0)$ .

More generally, for any  $N$ , one sets  $U_N := \varepsilon^N (c_1 + c_2 t - \varepsilon c_3 \tilde{w}_1)$  and the result follows the same : choosing  $c_1 := |\hat{X}_N(0)|$ ,  $c_2 := K_N/\mu_1$  where  $|S_N| \leq \varepsilon^N K_N$  and  $c_3 = c_2$ . ■

## 3.4 The general case

**Assumptions 6.** Assume the kernel  $\rho_\varepsilon$  solves (3.2) and that the data satisfies :

a) the off-rate  $\zeta$  is in  $C([0, T]; L^\infty(\mathbb{R}_+))$  and there exists a non-increasing  $m \in L^1(\mathbb{R}_+; (1+a)^3)$  such that

$$-\frac{m'(a)}{m(a)} \leq \zeta(a, t) \leq \zeta_{\max}, \quad a.e. a \in \mathbb{R}_+.$$

b) the birth-rate  $\beta \in C(\mathbb{R}_+)$  is such that

$$0 < \beta_{\min} \leq \beta(t) \leq \beta_{\max}.$$

c) the initial condition  $\rho_I \in BV(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, (1+a)^3)$  satisfies

$$\rho_I \leq c m(a), \text{ for almost every } a \in \mathbb{R}_+.$$

Whereas  $X_\varepsilon$  solves (3.1) and the data satisfy

d) the source term  $f$  belongs to  $C^2(\mathbb{R}_+)$ .

### 3.4.1 The linkages' density problem $\rho_{\varepsilon,t,a}$

Now we assume that  $\rho_{\varepsilon,t,a}$  solves the  $\varepsilon, t, a$ -dependent problem (3.2), and we introduce the first order asymptotic approximation of  $\rho_{\varepsilon,t,a}$  :

$$(3.31) \quad \tilde{\rho}_\varepsilon(a, t) := \rho_0(a, t) - \tilde{\tau}_0(a, t) - \varepsilon \rho_1(a, t)$$

where  $\rho_0$  is the zeroth order macroscopic limits given by (3.4),  $\rho_1$  is the first order macroscopic limits given by (3.8) and  $\tilde{\tau}_0 := \tau_0(a, t/\varepsilon)$  where the initial layer  $\tau_0$  solve (3.9).

### 3.4.1.1 The outer expansion of $\rho_\varepsilon$

**Proposition 7.** Let Assumptions 6 hold, then there exists generic constants  $c_j > 0$  and  $c_{j,1} > 0$ , such that  $\rho_0$  and  $\rho_1$  the solution of (3.4) and (3.8) respectively satisfies

$$|\rho_j(a, t)| \leq c_j(1+a)^{2j} m(a), \text{ where } j \in \{0, 1\}.$$

In a generic way, for all  $j \in \{0, 1\}$  and  $k \in \mathbb{N}$ , if  $\zeta \in W^{k,\infty}(\mathbb{R}_+ \times \mathbb{R}_+)$  then :

$$\left| \partial_{t^k}^k \rho_j(a, t) \right| \leq c_{j,k}(1+a)^{2j+k} m(a).$$

**Proof.** First, for  $j = 0$ ,  $\rho_0$  is explicitly given by

$$\rho_0(a, t) = \rho_0(0, t) \exp\left(-\int_0^a \zeta(\tilde{a}, t) d\tilde{a}\right) \leq \rho_0(0, t) \frac{m(a)}{m(0)} \leq \frac{\beta_{\max} \zeta_{\max}}{\zeta_{\max} + \beta_{\min}} \frac{m(a)}{m(0)}$$

which gives  $c_0$ . Similarly,  $\partial_t \rho_0(a, t)$  it is explicit and reads :

$$\begin{aligned} \partial_t \rho_0(a, t) &= \partial_t \rho_0(0, t) \exp\left(-\int_0^a \zeta(\tilde{a}, t) d\tilde{a}\right) \\ &\quad - \int_0^a \exp\left(-\int_\tau^a \zeta(\tilde{a}, t) d\tilde{a}\right) \partial_t \zeta(\tau, t) \rho_0(\tau, t) d\tau, \end{aligned}$$

where

$$\partial_t \rho_0(0, t) = \frac{g(t)}{1 + \beta(t) \int_0^{+\infty} \exp\left(-\int_\tau^a \zeta(\tilde{a}, t) d\tilde{a}\right) da}$$

and

$$\begin{aligned} g(t) &= \beta'(t) (1 - \mu_0(t)) \int_0^{+\infty} \exp\left(-\int_\tau^a \zeta(\tilde{a}, t) d\tilde{a}\right) da \\ &\quad - \int_0^{+\infty} \int_0^a \exp\left(-\int_\tau^a \zeta(\tilde{a}, t) d\tilde{a}\right) \partial_t \zeta(\tau, t) \rho_0(\tau, t) d\tau da. \end{aligned}$$

So that

$$\begin{aligned} |\partial_t \rho_0(a, t)| &\leq |\partial_t \rho_0(0, t)| \frac{m(a)}{m(0)} + m(a) \|\zeta\|_{W^{1,\infty}} \int_0^a \frac{\rho_0(\tau, t)}{m(\tau)} d\tau \leq (k_1 + k_2 a) m(a) \\ &\leq c'(1+a) m(a) \end{aligned}$$

where

$$k_1 := C \left( \frac{\zeta_{\max}}{\beta_{\max} + \zeta_{\max}}, \|\beta\|_{W^{1,\infty}}, \|m\|_{L^1(\mathbb{R}_+)}, \|\zeta\|_{W^{1,\infty}} \right), \quad k_2 = c_0 \frac{\|\zeta\|_{W^{1,\infty}}}{m(0)}$$

and  $c_{0,1} := \max(k_1, k_2)$ . Now, for  $j = 1$ ,  $\rho_1$  can be given explicitly by

$$\rho_1(a, t) = \rho_1(0, t) \exp\left(-\int_0^a \zeta(\tilde{a}, t) d\tilde{a}\right) - \int_0^a \exp\left(-\int_\tau^a \zeta(\tilde{a}, t) d\tilde{a}\right) \partial_t \rho_0(\tau, t) d\tau,$$

where

$$\rho_1(0, t) = \frac{h(t)}{1 + \beta(t) \int_0^{+\infty} \exp\left(-\int_\tau^a \zeta(\tilde{a}, t) d\tilde{a}\right) da}$$

such that

$$\begin{aligned} |h(t)| &= \left| \int_{\mathbb{R}_+} \int_0^a \exp\left(-\int_\tau^a \zeta(\tilde{a}, t) d\tilde{a}\right) \partial_t \rho_0(\tau, t) d\tau da \right| \\ &\leq \int_{\mathbb{R}_+} \int_0^a \frac{m(a)}{m(\tau)} |\partial_t \rho_0(\tau, t)| d\tau da \leq c_{0,1} \int_{\mathbb{R}_+} (1+a)^2 m(a) da, \end{aligned}$$

and finally, we obtain that

$$|\rho_1(a, t)| \leq k'_1 \frac{m(a)}{m(0)} + \int_0^a c_{0,1} (1+\tau) m(a) d\tau \leq \max(k'_1, c_{0,1}) (1+a)^2 m(a)$$

where

$$k'_1 := C \left( \frac{\zeta_{\max}}{\beta_{\max} + \zeta_{\max}}, \int_{\mathbb{R}_+} (1+a)^2 m(a) da \right).$$

Similarly, we can prove that

$$|\partial_t \rho_1| \leq c_{1,1} (1+a)^3 m(a),$$

and the generic way can be deduced by induction. ■

### 3.4.1.2 The initial layer

One considers the problem (3.9) and defines  $x(t) := \tau_0(0, t)$ , then by using Duhamel's principle this problem can be rewritten as

$$(3.32) \quad \begin{cases} x + k \star x = f, \\ k(a) = \beta \exp\left(-\int_0^a \zeta(\tau, 0) d\tau\right), \\ f(t) := -\beta \int_t^\infty \rho_I(a-t) \exp\left(-\int_a^{a-t} \zeta(\tau, 0) d\tau\right) da. \end{cases}$$

As a consequence of the Paley-Wiener theorem (for instance see Theorem 51 in appendix A) and the fact that  $k$  is a decreasing function of  $a$ , one gets :

**Theorem 33.** If  $k$  is a decreasing non-negative kernel such that  $k \in L^1(\mathbb{R}_+)$ , then the resolvent associated to (3.9) satisfies :

$$r + r \star k = k$$

and  $r \in L^1(\mathbb{R}_+)$ .

**Proof.** The result is proved (if  $k$  decreasing) in [GLS09, p.264]. ■

**Corollary 2.** Let assumptions 6 hold. If moreover,  $\tau_0(\cdot, 0) \in L^1(\mathbb{R}_+, (1+a)^2)$  and that there exists a constant  $c > 0$  such that

$$\tau_0(a, 0) \leq cm(a),$$

then  $x = \tau_0(0, \cdot) \in L^1_t(\mathbb{R}_+; (1+t)^2) \cap L^\infty(\mathbb{R}_+)$  and

$$\tau_0 \in L^1(\mathbb{R}_+ \times \mathbb{R}_+; (1+a)),$$

and there exists another constant  $c' > 0$  such that

$$\tau_0(a, t) \leq c' m(a).$$

**Proof.** Using the Assumption 6 on  $\zeta$  and on the data, one has

$$|f(t)| \leq \beta \int_t^\infty \tau_0(a-t, 0) \frac{m(a)}{m(a-t)} da \leq c \int_t^\infty m(a) da$$

which is bounded since  $m \in L^1(\mathbb{R}_+)$  and it is also integrable because so is the first moment of  $m$ . Writing then that

$$x = f - r \star f$$

and since  $L^1$  is an algebra for the convolution,  $x \in L^1(\mathbb{R}_+)$ . Because,  $f$  is bounded and  $r \in L^1$ ,  $x \in L^\infty(\mathbb{R}_+)$  as well. Then in a similar way, using Duhamel's principle, one has :

$$\tau_0(a, t) := \begin{cases} \tau_0(0, t-a) \exp\left(-\int_0^a \zeta(\tilde{a}, 0) d\tilde{a}\right), & \text{if } t > a, \\ \tau_0(a-t, 0) \exp\left(-\int_{a-t}^a \zeta(\tilde{a}, 0) d\tilde{a}\right), & \text{otherwise,} \end{cases}$$

so that

$$|\tau_0(a, t)| \leq \begin{cases} \|x\|_{L^\infty} \frac{m(a)}{m(0)}, & \text{if } t > a, \\ \frac{|\tau_0(a-t, 0)|}{m(a-t)} m(a) \leq cm(a), & \text{otherwise,} \end{cases}$$

which gives  $c'$  in the last estimates of the claim. Next we consider :

$$\begin{aligned}
 \int_{\mathbb{R}_+} |\tau_0(a, t)| da &\leq \int_0^t |\tau_0(0, t-a)| \exp\left(-\int_0^a \zeta(\tilde{a}, 0) d\tilde{a}\right) da \\
 &\quad + \int_t^\infty |\tau_0(a-t, 0)| \exp\left(-\int_{a-t}^a \zeta(\tilde{a}, 0) d\tilde{a}\right) da \\
 &\leq \int_0^t |\tau_0(0, t-a)| \frac{m(a)}{m(0)} da + \int_t^\infty \tau_0(a-t, 0) \frac{m(a)}{m(a-t)} da \\
 &= \frac{1}{m(0)} \left\{ (|\tau_0(0, \cdot)| \star m)(t) + \int_{\mathbb{R}_+} \frac{|\tau_0(\tilde{a}, 0)|}{m(\tilde{a})} m(\tilde{a}+t) d\tilde{a} \right\} \\
 &= \frac{1}{m(0)} (|x| \star m)(t) + c \int_t^\infty m(a) da
 \end{aligned}$$

Then using that  $L^1(\mathbb{R}_+)$  is an algebra for the convolution, and that  $m \in L^1(\mathbb{R}_+, (1+a))$ , one concludes that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \tau_0(a, t) da dt < \frac{\|m\|_{L^1(\mathbb{R}_+, (1+a))}}{m(0)} (\|\tau_0(0, \cdot)\|_{L^1(\mathbb{R}_+)} + cm(0))$$

the same holds for the first moment as well.

Setting  $q_1(a, t) := t\tau_0(a, t)$  and  $y(t) := q_1(0, t)$ , one has

$$\begin{aligned}
 y(t) &= -\beta(0) \int_0^t y(t-a) \exp\left(-\int_0^a \zeta(s, 0) ds\right) da \\
 &\quad - \beta(0) \int_0^t ax(t-a) \exp\left(-\int_0^a \zeta(s, 0) ds\right) da \\
 &\quad - \beta \int_t^\infty t\tau_0(a-t, 0) \exp\left(-\int_{a-t}^a \zeta(s) ds\right) da
 \end{aligned}$$

*i.e.*

$$y + k \star y = f_x$$

it remains to check whether  $f_x \in L^1(\mathbb{R}_+)$ . Assuming that  $\int_{\mathbb{R}_+} (1+a)^2 m(a) da < \infty$  shows the claim. As above,

$$\begin{aligned}
 \int_0^t |q_1(a, t)| da &\leq \int_0^t (|q_1(0, t-a)| + a|x(t-a)|) \exp\left(-\int_0^a \zeta(s, 0) ds\right) da \\
 &\leq |q_1(0, \cdot)| \star \exp\left(-\int_0^{\cdot} \zeta(s, 0) ds\right) + |x(0, \cdot)| \star (\cdot) \exp\left(-\int_0^{\cdot} \zeta(s, 0) ds\right)
 \end{aligned}$$

together with

$$\int_t^\infty |q_1(a, t)| da \leq t \int_t^\infty \tau_0(a-t, 0) \exp\left(-\int_{a-t}^a \zeta(\tau, 0) d\tau\right) da \leq tc \int_t^\infty m(a) da$$

both left-hand sides are then  $L^1_t$  functions in time, provided that  $m \in L^1_a(\mathbb{R}_+, (1+a)^2)$ . For the next step, one works similarly and obtains that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} t^2 \tau_0(a, t) da dt < \infty$$

since  $m$  is in  $L^1_a(\mathbb{R}_+, (1+a)^3)$ . ■

**Corollary 3.** Under the Assumptions 6, one has  $x(t) := \tau_0(0, t) \rightarrow 0$  when  $t$  grows large and the first moment can be estimated as follows :

$$\int_{\mathbb{R}_+} (1+a) |\tau_0(a, t)| da \leq o_{1/t}(1) + c \int_t^\infty (1+a) m(a) da,$$

where  $o_{1/t}(1)$  means small when  $t$  grows large.

**Proof.** Using Lyapunov's functional (3.5), one has that

$$\int_{\mathbb{R}_+} |\tau_0(a, t)| da \leq \infty$$

which, thanks to the boundary condition, provides that  $x(t)$  is bounded on  $\mathbb{R}_+$ . This shows using Duhamel's principle that  $\tau_0 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ . Moreover, defining the discrete differences

$$D_t^h \tau_0(a, t) := \frac{\tau_0(a, t+h) - \tau_0(a, t)}{h}$$

it solves the problem :

$$(3.33) \quad \begin{cases} (\partial_t + \partial_a + \zeta(a, 0)) D_t^h \tau_0 = 0, & a > 0, t > 0, \\ D_t^h \tau_0(0, t) = -\beta(0) \int_{\mathbb{R}_+} D_t^h \tau_0(a, t) da, & a = 0, t > 0, \\ D_t^h \tau_0(a, 0) = \frac{\tau_0(a, h) - \tau_0(a, 0)}{h}, & a > 0, t = 0. \end{cases}$$

If the initial data is regular enough (for instance  $\tau_0(\cdot, 0) \in BV(\mathbb{R}_+)$  since  $\rho_I$  is too), one has that

$$\left| D_t^h x(t) \right| = \left| D_t^h \tau_0(0, t) \right| \leq \beta(0) \int_{\mathbb{R}_+} \left| D_t^h \tau_0(a, t) \right| da \leq \beta(0) \mathcal{H}[D_t^h \tau_0(a, t)]$$

similarly as [Mil18, Theorem 3.2], by using Gronwall's Lemma, we get that

$$\mathcal{H}[D_t^h \tau_0(\cdot, t)] \leq \mathcal{H}[D_t^h \tau_0(\cdot, 0)],$$

it suffices then to prove that the initial term  $\mathcal{H}[D_t^h \tau_0(\cdot, 0)]$  is bounded. Indeed, we have

$$\mathcal{H}[D_t^h \tau_0(a, 0)] = \int_{\mathbb{R}_+} \left| \frac{\tau_0(a, h) - \tau_0(a, 0)}{h} \right| da + \left| \frac{\mu_{\tau_0}(h) - \mu_{\tau_0}(0)}{h} \right| := I_1 + I_2$$

where  $\mu_{\tau_0}(t) = \int_{\mathbb{R}_+} \tau_0(a, t) da$ . For the first term, we split the integral in two parts

$$\begin{aligned} I_1 &= \int_0^h \left| \frac{\tau_0(a, h) - \tau_0(a, 0)}{h} \right| da + \int_h^{+\infty} \left| \frac{\tau_0(a, h) - \tau_0(a, 0)}{h} \right| da \\ &= \frac{1}{h} \int_0^h \left| \tau_0(0, h-a) \exp\left(-\int_0^a \zeta(\tilde{a}, 0) d\tilde{a}\right) - \tau_0(a, 0) \right| da \\ &\quad + \frac{1}{h} \int_h^{+\infty} \left| \tau_0(a-h, 0) \exp\left(-\int_{a-h}^a \zeta(\tilde{a}, 0) d\tilde{a}\right) - \tau_0(a, 0) \right| da \\ &= I_{1,1} + I_{1,2} \end{aligned}$$

where we used the method of characteristics. It easy to see that the first term

$$I_{1,1} \lesssim \frac{1}{h} \int_0^h da \left( |\beta(0)| \|\tau_0\|_{L_t^\infty L_a^1} + \|\tau_0(a, 0)\|_{L^1(\mathbb{R}_+)} \right) < \infty.$$

Now, for  $I_{1,2}$ , one splits the integral adding and subtracting intermediate terms

$$\begin{aligned} I_{1,2} &\leq \frac{1}{h} \int_h^{+\infty} \left| (\tau_0(a-h, 0) - \tau_0(a, 0)) \exp\left(-\int_{a-h}^a \zeta(\tilde{a}, 0) d\tilde{a}\right) \right| da \\ &\quad + \frac{1}{h} \int_h^{+\infty} \left| \tau_0(a, 0) \exp\left(-\int_{a-h}^a \zeta(\tilde{a}, 0) d\tilde{a}\right) - \tau_0(a, 0) \right| da \\ &\lesssim TV(\tau_0(\cdot, 0)) + C \|\tau_0(a, 0)\|_{L^1(\mathbb{R}_+)} \end{aligned}$$

where  $TV$  denotes total variation of  $\tau_0(\cdot, 0)$  [ABD13]. For the second term  $I_2$  noting that

$$|\partial_t \mu_{\tau_0}(t)| \leq (\zeta_{\max} + \beta_{\max}) \|\tau_0\|_{L_t^\infty L_a^1},$$

one obtains

$$I_2 \leq (\zeta_{\max} + \beta_{\max}) \|\tau_0\|_{L_t^\infty L_a^1},$$

and finally, we obtain that  $\mathcal{H}[D_t^h \tau_0(a, t)] < \infty$ , for all  $t > 0$ , which shows that  $x \in Lip(\mathbb{R}_+)$ .

Since  $x \in L^1(\mathbb{R}_+)$  (see Corollary 2), this implies that  $\lim_{t \rightarrow +\infty} x(t) = 0$ . Now, we consider

$$\begin{aligned} J(t) &:= \int_{\mathbb{R}_+} (1+a) |\tau_0(a, t)| da \leq \int_0^t |x(t-a)| (1+a) m(a) da \\ &= \left( \int_0^{t/2} + \int_{t/2}^t \right) |x(t-a)| (1+a) m(a) da =: J_1 + J_2. \end{aligned}$$

For every  $\delta > 0$  there exists  $\eta_1$  such that  $t > \eta_1$ , implying that

$$\sup_{s \in (\frac{t}{2}, t)} |x(s)| < \frac{\delta}{\left(2 \int_{\mathbb{R}_+} (1+a) m(a) da\right)}$$



which shows that  $J_1 < \delta/2$ . On the other hand, there exists  $\eta_2$  such that  $t > \eta_2$  which implies that

$$J_2 \leq \|x\|_{L^\infty(\mathbb{R}_+)} \int_{t/2}^t (1+a)m(a)da < \delta/2,$$

by Lebesgue's Theorem (since the integral of  $(1+a)m(a)$  is finite). These arguments show that  $J(t)$  vanishes when  $t$  grows large. On the other hand :

$$\int_t^\infty (1+a)|\tau_0(a,t)|da \leq c \int_t^\infty (1+a)m(a)da,$$

which is an initial layer. ■

### 3.4.1.3 Error estimates for the linkage's density

We define the difference :

$$\hat{\rho}_\varepsilon(a,t) := \rho_\varepsilon(a,t) - \tilde{\rho}_\varepsilon(a,t)$$

where  $\tilde{\rho}_\varepsilon$  is defined by (3.31). We obtain :

**Theorem 35.** Under the Assumptions 6, one has that

$$(3.34) \quad \mathcal{H}[\hat{\rho}_\varepsilon](t) \leq o_\varepsilon(1), \quad a.e. t \in \mathbb{R}_+$$

where  $\mathcal{H}[u](t) := \int_{\mathbb{R}_+} |u(a)| da + \left| \int_{\mathbb{R}_+} u(a) da \right|$ .

**Proof.** We start by writing the system satisfied by the error  $\hat{\rho}_\varepsilon$  :

$$(3.35) \quad \begin{cases} (\varepsilon \partial_t + \partial_a + \zeta(a,t)) \hat{\rho}_\varepsilon(a,t) = -\varepsilon (\zeta(a,t) - \zeta(a,0)) \tilde{\tau}_0(a,t) - \varepsilon^2 \partial_t \rho_1 \\ \hat{\rho}_\varepsilon(0,t) = -\beta(t) \int_{\mathbb{R}_+} \hat{\rho}_\varepsilon(a,t) da - (\beta(t) - \beta(0)) \int_{\mathbb{R}_+} \tilde{\tau}_0(a,t) da \\ \hat{\rho}_\varepsilon(a,0) = -\varepsilon \rho_1(a,0) \end{cases}$$

Then following the same steps as in [MO11], one has that

$$\begin{aligned} & \varepsilon \frac{d}{dt} \mathcal{H}[\hat{\rho}_\varepsilon](t) + \int_{\mathbb{R}_+} \zeta(a,t) \left\{ |\hat{\rho}_\varepsilon(a,t)| + \hat{\rho}_\varepsilon(a,t) \operatorname{sgn} \left( \int_{\mathbb{R}_+} \hat{\rho}_\varepsilon(\tilde{a},t) d\tilde{a} \right) \right\} da \\ & \leq 2\varepsilon^2 \int_{\mathbb{R}_+} |\partial_t \rho_1(a,t)| da + 2 \int_{\mathbb{R}_+} |\zeta(a,t) - \zeta(a,0)| |\tilde{\tau}_0(a,t)| da \\ & + |\beta(t) - 2\beta(0)| \int_{\mathbb{R}_+} |\tilde{\tau}_0(a,t)| da \end{aligned}$$

which after integration in time provides :

$$\begin{aligned} \mathcal{H}[\hat{\rho}_\varepsilon](t) &\leq \mathcal{H}[\hat{\rho}_\varepsilon](0) + 2\varepsilon \int_0^t \int_{\mathbb{R}_+} |\partial_t \rho_1(a, s)| \, da \, ds \\ &\quad + 2 \int_0^{\frac{t}{\varepsilon}} \int_{\mathbb{R}_+} |\zeta(a, \varepsilon \tilde{t}) - \zeta(a, 0)| |\mathfrak{r}_0(a, \tilde{t})| \, da \, d\tilde{t} \\ &\quad + 2 \int_0^{\frac{t}{\varepsilon}} |\beta(\varepsilon \tilde{t}) - 2\beta(0)| \int_{\mathbb{R}_+} |\mathfrak{r}_0(a, \tilde{t})| \, da \, d\tilde{t}. \end{aligned}$$

Now, here the crucial point is that, thanks to Lebesgue's Theorem, the last two terms of the right-hand side do tend to zero as  $\varepsilon$  goes to zero.  $\blacksquare$

**Corollary 4.** Let Assumptions 6 hold, then one has

$$\int_{\mathbb{R}_+} (1+a) |\rho_\varepsilon(a, t) - \rho_0(a, t)| \, da \leq o_\varepsilon(1) + \int_{\frac{t}{\varepsilon}}^{\infty} (1+a) m(a) \, da.$$

**Proof.** Considering the system solved by the difference  $e(a, t) = \rho_\varepsilon(a, t) - \rho_0(a, t)$

$$(3.36) \quad \begin{cases} (\varepsilon \partial_t + \partial_a + \zeta(a, t)) e(a, t) = \varepsilon \partial_t \rho_0(a, t), & a > 0, t > 0, \\ e(0, t) = -\beta(t) \int_{\mathbb{R}_+} (\rho_\varepsilon(\tilde{a}, t) - \rho_0(\tilde{a}, t) - \tilde{\mathfrak{r}}_0(\tilde{a}, t)) \, da \\ \quad - \beta(t) \int_{\mathbb{R}_+} \tilde{\mathfrak{r}}_0(a, t) \, da, & a = 0, t > 0, \\ e(a, 0) = \rho_I(a) - \rho_0(a, 0), & a > 0, t = 0. \end{cases}$$

It satisfies (3.36) in the sense of characteristics, namely

$$(3.37) \quad e(a, t) = \begin{cases} e(0, t - \varepsilon a) \exp(-\int_{-a}^0 \zeta(a+s, t + \varepsilon s) \, ds) + \\ \quad + \varepsilon \int_{-a}^0 \partial_t \rho_0(a+s, t + \varepsilon s) \exp(-\int_s^0 \zeta(a+\tau, t + \varepsilon \tau) \, d\tau) \, ds, & a < t/\varepsilon, \\ e(a - t/\varepsilon, 0) \exp(-\int_{t/\varepsilon}^0 \zeta(a+s, t + \varepsilon s) \, ds) + \\ \quad + \varepsilon \int_{-t/\varepsilon}^0 \partial_t \rho_0(a+s, t + \varepsilon s) \exp(-\int_s^0 \zeta(a+\tau, t + \varepsilon \tau) \, d\tau) \, ds, & a > t/\varepsilon. \end{cases}$$

One has that

$$(3.38) \quad \begin{aligned} \int_0^{+\infty} (1+a) e(a, t) \, da &= \int_0^{t/\varepsilon} (1+a) e(a, t) \, da + \int_{t/\varepsilon}^{+\infty} (1+a) e(a, t) \, da \\ &:= I_1 + I_2. \end{aligned}$$

We treat each term separately because they correspond to the two cases of Duhamel's formula (3.37) :

$$\begin{aligned} I_1(t) &\leq \int_0^{\frac{t}{\varepsilon}} (1+a) |e(0, t-\varepsilon a)| \exp\left(-\int_{-a}^0 \zeta(a+s, t+\varepsilon s) ds\right) da \\ &\quad + \varepsilon \int_0^{\frac{t}{\varepsilon}} (1+a) \int_{-a}^0 \partial_t \rho_0(a+s, t+\varepsilon s) \exp\left(-\int_s^0 \zeta(a+\tau, t+\varepsilon \tau) d\tau\right) ds da. \end{aligned}$$

Gathering previous results one has that :

$$|e(0, t-\varepsilon a)| \lesssim o_\varepsilon(1) + \int_{\frac{t}{\varepsilon}-a}^\infty m(a) da,$$

which thanks to Proposition 7 gives that

$$\begin{aligned} I_1(t) &\leq o_\varepsilon(1) \int_0^{\frac{t}{\varepsilon}} (1+a) m(a) da + \int_0^{\frac{t}{\varepsilon}} (1+a) m(a) \int_{\frac{t}{\varepsilon}-a}^\infty m(\tilde{a}) d\tilde{a} da \\ &\quad + \varepsilon c_{0,1} \int_{\mathbb{R}_+} (1+a)^3 m(a) da, \end{aligned}$$

using similar argument as in the proof of Corollary 3, one shows that

$$\int_0^{\frac{t}{\varepsilon}} (1+a) m(a) da \int_{\frac{t}{\varepsilon}-a}^\infty m(\tilde{a}) d\tilde{a} da \leq o_\varepsilon(1),$$

since  $(1+a)m(a)$  is integrable and by Lebesgue's Theorem  $\int_t^\infty m(a) da \rightarrow 0$  as  $t$  grows large. On the other hand,

$$\begin{aligned} I_2 &\leq \int_{\frac{t}{\varepsilon}}^\infty (1+a) \left| e\left(a - \frac{t}{\varepsilon}, 0\right) \right| \exp\left(-\int_{-\frac{t}{\varepsilon}}^0 \zeta(a+s, t+\varepsilon s) ds\right) da \\ &\quad + \varepsilon \int_{\frac{t}{\varepsilon}}^\infty (1+a) \int_{-\frac{t}{\varepsilon}}^0 |\partial_t \rho_0(a+s, t+\varepsilon s)| \exp\left(-\int_s^0 \zeta(a+\tau, t+\varepsilon \tau) d\tau\right) ds da \\ &\leq c \int_{\frac{t}{\varepsilon}}^\infty (1+a) m(a) da + \varepsilon \int_{\mathbb{R}_+} (1+a)^3 m(a) da. \end{aligned}$$

■

### 3.4.2 Error estimates for the position

**Theorem 37.**

Under the Assumptions 6, if  $\rho_\varepsilon$  is a solution of (3.2),  $\rho_0$  solves (3.4),  $X_\varepsilon$  is a solution of (3.1) and  $X_0$  solves (3.13) then

$$\|\rho_\varepsilon - \rho_0\|_{L^1((0,T);L^1(\mathbb{R}_+, (1+a)))} \leq o_\varepsilon(1), \quad \|X_\varepsilon - X_0\|_{C([0,T])} \leq o_\varepsilon(1).$$

**Proof.** Setting  $u_\varepsilon(a, t) := X_\varepsilon(t) - X_\varepsilon(t - \varepsilon a)$  where  $X_\varepsilon(t) = X_p(t)$  when  $t < 0$ , one has that it solves :

$$(3.39) \quad \begin{cases} (\varepsilon \partial_t + \partial_a) u_\varepsilon = \partial_t X_\varepsilon = \frac{1}{\mu_{0,\varepsilon}} \left\{ \varepsilon \partial_t f + \int_{\mathbb{R}_+} u_\varepsilon(a, t) \zeta(a, t) \rho_\varepsilon(a, t) da \right\}, & a > 0, t > 0, \\ u_\varepsilon(0, t) = 0, & a = 0, t > 0, \\ u_\varepsilon(a, 0) = u_I(a) := \frac{X_\varepsilon(0) - X_p(-\varepsilon a)}{\varepsilon}, & a > 0, t = 0, \end{cases}$$

since problem (3.1) can be expressed in a integro-differential equation :

$$\mu_{0,\varepsilon}(t) \partial_t X_\varepsilon = \varepsilon \partial_t f + \int_{\mathbb{R}_+} \left( \frac{X_\varepsilon(t) - X_\varepsilon(t - \varepsilon a)}{\varepsilon} \right) \zeta(a, t) \rho_\varepsilon(a, t) da.$$

Following [MO15, Theorem 6.1], one has that

$$\int_{\mathbb{R}_+} |u_\varepsilon(a, t)| \rho_\varepsilon(a, t) da \leq \int_0^t |\partial_t f(\tau)| d\tau + |f(0)| + L_{X_p} \mu_{1,\max} = c_1$$

Moreover, one has also the bound :

$$\|\partial_t X_\varepsilon\|_{L^\infty(0,T)} \leq \frac{1}{\mu_{0,\min}} \{ \varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \zeta_{\max} c_1 \} =: c_2$$

which provides thanks to Ascoli-Arzelà that there exists a converging sub-sequence  $X_\varepsilon$  in  $C([0, T])$ . Moreover, if  $t > \varepsilon a$

$$|u_\varepsilon(a, t)| \leq \frac{1}{\varepsilon} \int_{t-\varepsilon a}^t |\partial_t X_\varepsilon(\tau)| d\tau \leq c_2 a$$

whereas if  $t < \varepsilon a$ , by similar arguments,

$$|u_\varepsilon(a, t)| \leq \frac{t}{\varepsilon} c_2 + \left| \frac{f(0)}{\mu_{0,\min}} \right| + L_{X_p} \frac{\mu_{1,\max}}{\mu_{0,\min}}$$

Thus one has :  $u_\varepsilon(a, t)/(1+a) \in L^\infty(\mathbb{R}_+ \times 0, T)$  uniformly with respect to  $\varepsilon$ . These results provide that  $u_\varepsilon$  weak-\* converges in  $L^\infty(\mathbb{R}_+ \times (0, T); (1+a)^{-1})$  to  $u_0$  weak solution of

$$(3.40) \quad \begin{cases} \partial_a u_0 = \partial_t X_0 = \frac{1}{\mu_{0,0}} \int_{\mathbb{R}_+} u_0 \zeta(a, t) \rho_0(a, t) da \\ u_0(0, t) = 0 \end{cases}$$

which shows that  $u_0(a, t) = aX_0(t)$ . Since (3.1) reads as

$$A_\varepsilon := \int_0^T \int_{\mathbb{R}_+} u_\varepsilon(a, t) \rho_\varepsilon(a, t) da \varphi(t) dt = \int_0^T f(t) \varphi(t) dt, \quad \forall \varphi \in L^1(0, T),$$

and since

$$\rho_\varepsilon \rightarrow \rho_0 \text{ in } L^1(\mathbb{R}_+ \times (0, T); (1+a)), \quad u_\varepsilon \xrightarrow{*} u_0 \text{ in } L^\infty(\mathbb{R}_+ \times (0, T); (1+a)^{-1})$$

one concludes that

$$A_\varepsilon \rightarrow \int_0^T \int_{\mathbb{R}_+} u_0(a, t) \rho_0(a, t) da \varphi(t) dt = \int_0^T \mu_{0,1}(t) X_0(t) dt = \int_0^T f(t) \varphi(t) dt.$$

■

## CONCLUSION AND PERSPECTIVES

In this thesis, we are interested in the study of a weakly coupled model formed by two equations : the first one is a Volterra integral equation that describes the position of an adhesion point in contact with a substrate and subject to an external force. The second is a renewal equation that describes the distribution in ages of the bonds that can be created or broken during cell migration. We have restricted our attention to the study of the following system with a source term  $f$  with bounded variation in an open interval  $(0, T)$  :

$$(4.1) \quad \begin{cases} \frac{1}{\varepsilon} \int_0^\infty (z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)) \rho_\varepsilon(a, t) da = f(t), & t \geq 0, \\ z_\varepsilon(t) = z_p(t), & t < 0, \end{cases}$$

where the kernel  $\rho_\varepsilon$  solves a non-local age-structured problem :

$$(4.2) \quad \begin{cases} \varepsilon \partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \zeta_\varepsilon \rho_\varepsilon = 0, & t > 0, a > 0, \\ \rho_\varepsilon(a = 0, t) = \beta_\varepsilon(t) \left( 1 - \int_0^\infty \rho_\varepsilon(t, \tilde{a}) d\tilde{a} \right), & t > 0, \\ \rho_\varepsilon(a, t = 0) = \rho_{I, \varepsilon}(a), & a \geq 0, \end{cases}$$

where  $\beta_\varepsilon \in \mathbb{R}_+$  (resp.  $\zeta_\varepsilon \in \mathbb{R}_+$ ) is the kinetic on-rate (resp. off-rate) function.

In this case, we have first demonstrated the existence and uniqueness of a solution of the system (4.1) as well as the convergence to a solution of a macroscopic friction law using methods similar to those used in [MO15]. Additionally, we have presented a comparison

principle related to the system (4.1) but in the case where the density  $\rho_\varepsilon$  is constant in time, in accordance with the methodologies of [MO11].

Next, we studied the asymptotic behavior of the couple(4.1,4.2) under weaker assumptions than in [MO11; MO15]. In this part, we were able to increase the order of the approximation with respect to  $\varepsilon$  in the case where the density  $\rho$  depends only on the age. In the general case, where  $\rho_\varepsilon$  also depends on time, we were initially enhancing the asymptotic expansion of  $\rho_\varepsilon$  proved in [Mil18] before extending the methodology of [MO15] to the framework of our study.

In the following, we present many questions and perspectives related to our framework.

## 4.1 Measure solutions to the renewal equation (4.2)

In this thesis, the data of (4.2) was restricted  $L^1$  initial conditions and bounded and continuous boundary data. Several approaches can be used to study the long time asymptotics of this kind of equation. In particular, In [MMP05] the authors have proposed the generalized relative entropy method (GREM). This method uses an entropy inequality which shows the convergence of the solution towards an asymptotic profile when time goes to infinity. Recently, Gwiazda and Wiedemann [GW16], extended the GRE method to the case of the renewal equation with measure-valued initial data. In this case, in order to establish the GREM inequality and show the convergence, they approximate a measured valued solution by a sequence of regularized solutions. First, GREM applies to (4.1) only when the source term is zero and the past data is present. Then concerning the possible application of GREM to (4.2), the non-local birth term  $\rho_\varepsilon(0, t) = \beta(t)(1 - \mu_{0,\varepsilon})$ , contains the opposite sign in front of the total population. In particular, neither Perron–Frobenius nor Krein–Rutman arguments are at hand. To study the long time asymptotics of (4.2), the authors in [MO11; MO15] have introduced a specific entropy function. We would like to define measure solutions to (4.2) and prove the convergence when  $\varepsilon$  goes to zero, the extension of [MO11] to this framework seems to be an interesting perspective.

## 4.2 Asymptotic expansion of $z_\varepsilon$ : $\rho_\varepsilon$ is time dependent

In Chapter 3, we considered an asymptotic expansion of  $z_\varepsilon$  solution of (4.1) when the density  $\rho_\varepsilon$  is time independent. The extension of these results to the general case where  $\rho_\varepsilon$  satisfies (4.2) is a challenging problem. The dependence of the asymptotic development of  $z_\varepsilon$  on that of  $\rho_\varepsilon$  precludes us from using the same approach as in the case of the con-

stant kernel. Indeed, the main obstacle encountered in this direction is that microscopic problems correcting initial layers start being dependent on some macroscopic parameter. Results of continuous dependence of the resolvent on this parameter are the key missing argument in order to extend asymptotic expansions to the general time dependent case.

### 4.3 Study of the delayed harmonic problem

In this thesis work, we are interested in the long time behavior of the solution  $z_\varepsilon$  of (4.1) which describes the position of a single adhesion point at time  $t$ . Another problem we could be interested is the long time asymptotics of problem (4.1) after the addition of the position variable  $x$  belonging to a bounded and regular domain  $\Omega \subset \mathbb{R}^N$ . The asymptotic when  $\varepsilon$  goes to zero was already considered in [MO17] using compactness results obtained on  $\partial_t z_\varepsilon$  among others quantities. The weakly coupled system considered read :

$$(4.3) \quad \begin{cases} \frac{1}{\varepsilon} \int_0^\infty (z_\varepsilon(x, t) - z_\varepsilon(x, t - \varepsilon a)) \rho_\varepsilon(x, a, t) da - \Delta_x z_\varepsilon = f(x, t), & t \geq 0, x \in \Omega, \\ z_\varepsilon(x, t) = 0, & t \in \mathbb{R}_+, x \in \partial\Omega, \\ z_\varepsilon(x, t) = z_p(x, t), & t < 0, x \in \Omega, \end{cases}$$

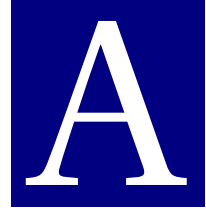
where the density  $\rho_\varepsilon(x, a, t)$  is the solution of :

$$(4.4) \quad \begin{cases} \varepsilon \partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \zeta_\varepsilon \rho_\varepsilon = 0, & x \in \Omega, a > 0, t > 0, \\ \rho_\varepsilon(x, a = 0, t) = \beta_\varepsilon(x, t) \left( 1 - \int_0^\infty \rho_\varepsilon(x, \tilde{a}, t) d\tilde{a} \right), & x \in \Omega, a = 0, t > 0, \\ \rho_\varepsilon(x, a, t = 0) = \rho_{I, \varepsilon}(x, a), & a \geq 0, x \in \Omega, t = 0. \end{cases}$$

Following ideas of Chapter 3, it could be of interest to consider error estimates for  $z_\varepsilon$  and be able to quantify errors with the asymptotic expansion. The main reason is that even when  $\rho$  is constant in time there is no concept of resolvent at hand.







## SOME RESULTS IN RELATION TO THE RENEWAL EQUATION

### A.1 Age-structured model : the renewal equation

The age-structured equation, also called the Mckendrick-Von Foerster equation (or renewal equation) is of the form

$$(A.1) \quad \begin{cases} \partial_t n(a, t) + \partial_a n(a, t) + d(a)n(a, t) = 0, & (a, t) \in (\mathbb{R}_+)^2, \\ n(a = 0, t) = \int_0^{+\infty} \beta(a')n(a', t) da', & (a, t) \in \{0\} \times \mathbb{R}_+, \\ n(a, t = 0) = n^0(a), & (a, t) \in \mathbb{R}_+ \times \{0\}, \end{cases}$$

where  $n(a, t)$  represents the population density of age  $a$  at time  $t$ ,  $\beta \geq 0$  is the birth rate and  $d \geq 0$  the death rate. The name of this equation refers to the non-local boundary term in  $a = 0$  which describes the births of new individuals. In the following, we present some results related to the generalized entropy method (for more detail, see [Per07; Gab14]). The fundamental tool used by Perthame [Per07] to study the asymptotic behavior of the solution of (A.1) is to solve an eigenvalue problem : found  $N \in L^1(\mathbb{R}_+)$  positive and not identically null, and  $\lambda \in \mathbb{R}$  such that

$$(A.2) \quad \begin{cases} \lambda N(a) + \partial_a N(a) + d(a)N(a) = 0, & a \geq 0, \\ N(0) = \int_0^{+\infty} \beta(a)N(a) da, \\ N(\cdot) > 0, \quad \int_{\mathbb{R}_+} N(a) da = 1, \end{cases}$$

and the dual equation

$$(A.3) \quad \begin{cases} \partial_a \phi(a) - (\lambda + d(a)) \phi(a) = -\phi(0) \beta(a), & a \geq 0, \\ \phi(\cdot) > 0, \quad \int_{\mathbb{R}_+} N(a) \phi(a) da = 1, \end{cases}$$

In this case, a necessary condition to prove the behavior of the solutions of (A.1) is

$$(A.4) \quad \int_0^{+\infty} \beta(a) \exp\left(-\int_0^{+\infty} d(a') da'\right) da < 1$$

Biologically, condition (A.4) shows that the populations will not survive, but on the contrary, they will die out and disappear.

**Theorem 39.** Suppose that  $\beta$  and  $d$  are two continuous, positive, non-identically null functions on  $\mathbb{R}_+$  and satisfying (A.4). Then there is a unique solution  $(\lambda, N, \phi)$  to the problem (A.2)-(A.3) such that

$$\int_{\mathbb{R}_+} N(a) da = \int_{\mathbb{R}_+} N(a) \phi(a) da = 1.$$

By studying the equation (A.1), we expect the population to grow exponentially fast in time. Indeed, we define

$$\tilde{n}(a, t) := n(a, t) \exp(-\lambda a).$$

The function  $\tilde{n}$  satisfies

$$(A.5) \quad \begin{cases} \partial_t \tilde{n}(a, t) + \partial_a \tilde{n}(a, t) + (\lambda + d(a)) \tilde{n}(a, t) = 0, & (a, t) \in (\mathbb{R}_+)^2, \\ \tilde{n}(a = 0, t) = \int_0^{+\infty} \beta(a') \tilde{n}(a', t) da', & (a, t) \in \{0\} \times \mathbb{R}_+, \\ \tilde{n}(a, t = 0) = n^0(a), & (a, t) \in \mathbb{R}_+ \times \{0\}, \end{cases}$$

Under the same assumptions of the previous theorem, we can assert the following results

**Theorem 41.** Under the assumptions that the initial data  $n^0$  satisfying

$$n^0 \in L^1(\mathbb{R}_+, \phi(a) da) \quad \text{and} \quad \exists C_0, |n_0(a)| \leq C_0 N(a)$$

there is a unique solution in distribution sense  $\tilde{n} \in C(\mathbb{R}_+; L^1(\mathbb{R}_+, \phi(a) da))$  to (A.5), and we have

i) the maximum principle

$$|\tilde{n}(a, t)| \leq C_0 N(a), \quad \forall t \geq 0,$$

ii) the comparison principle

$$n_1^0 \leq n_2^0 \implies \tilde{n}_1(a, t) \leq \tilde{n}_2(a, t),$$

iii) the conservation law and the  $L^1(\phi(a) da)$  contraction principle hold,

$$\int_0^{+\infty} \tilde{n}(a, t) \phi(a) da = \int_0^{+\infty} n^0(a) \phi(a) da,$$

$$\int_0^{+\infty} |\tilde{n}(a, t)| \phi(a) da \leq \int_0^{+\infty} |n^0(a)| \phi(a) da.$$

### A.1.1 Entropy and convergence

To obtain the convergence of the solution  $\tilde{n}$  and show that it approaches the steady state  $N$  solution of (A.2), the author in [Per07] uses the general relative entropy method. He introduces a Lyapunov function  $\mathcal{H}$  defined on  $\mathbb{R}_+$  by

$$(A.6) \quad \mathcal{H}[n] := \int_0^{+\infty} \phi(a) N(a) \mathcal{H} \left[ \frac{n(a)}{N(a)} \right] da$$

for all  $n \in L^1(\mathbb{R}_+)$ .

**Theorem 43.** Under the same assumptions as above, then

i) for all convex functions  $\mathcal{H} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\mathcal{H}(0) = 0$ , and for all  $t > 0$ ,

$$\mathcal{H}[\tilde{n}(a, t)] \leq \mathcal{H}[n^0(a)],$$

ii) for the probability measure  $d\mu(a) = \beta(a) \frac{N(a)}{N(0)} da$ , and for all convex functions  $\mathcal{H}$ ,

$$\begin{aligned} & \int_0^{+\infty} \left[ \int_0^{+\infty} \mathcal{H} \left( \frac{\tilde{n}(a, t)}{N(a)} \right) d\mu(a) - \mathcal{H} \left( \int_0^{+\infty} \frac{\tilde{n}(a, t)}{N(a)} d\mu(a) \right) \right] dt \\ & \leq \int_0^{+\infty} \phi(a) N(a) \mathcal{H} \left[ \frac{n^0(a)}{N(a)} \right] da \end{aligned}$$

By using the theorem 43, we can prove the following result which gives the exponential convergence of  $\tilde{n}$  in norm  $L^1$ .

**Theorem 45.** Under assumptions above, and

$$\exists \mu_0 > 0, \quad \text{s.t.} \quad \beta(a) \geq \mu_0 \frac{\phi(a)}{\phi(0)}$$

the solution of (A.1) satisfies

$$\int_0^{+\infty} \left| n(a, t) e^{-\lambda t} - \kappa_0 N(a) \right| \phi(a) da \leq e^{-\mu_0 t} \int_0^{+\infty} \left| n^0(a) - \kappa_0 N(a) \right| \phi(a) da$$

where  $\kappa_0 := \int_0^{+\infty} \phi(a) n^0(a) da$ .

To prove this result, it is sufficient to apply the theorem 43 for  $\mathcal{H}[x] = x$  and the Grönwall lemma (for more details, see [Per07; Gab14]).

## A.2 Volterra integral equation with convolution kernel

In this section, we present some main results to study the linear Volterra equations of the type

$$(A.7) \quad x(t) - \int_0^t x(t-a)k(a)da = f(t)$$

where  $k$  is the kernel and  $f$  the forcing function. This equation can be reformulated using the convolution by,

$$(A.8) \quad x(t) = k \star x(t) + f(t),$$

where  $(k \star x)(t) := \int_0^t k(a)x(t-a)da$  is the convolution product of  $k$  and  $x$ . In order to find a solution of (A.8), we must first find a solution of the resolvent  $r$  equation associated to the kernel  $k$ . We recall some of the results presented in [GLS09] without proof.

**Theorem 47.** Let  $k \in L^1_{loc}(\mathbb{R}_+)$ , then there exists a solution  $r \in L^1_{loc}(\mathbb{R}_+)$  of the resolvent equation :

$$(A.9) \quad r - k \star r = k$$

this solution is unique and depends continuously on  $k$  in the topology of  $L^1_{loc}(\mathbb{R}_+)$ .

**Lemma 20.** If  $r \in L^1_{loc}(\mathbb{R}_+)$  is the resolvent of  $k \in L^1_{loc}(\mathbb{R}_+)$  and  $\sigma \in \mathbb{C}$ , then the function  $e^{\sigma t}r(t)$  is the resolvent of  $e^{\sigma t}k(t)$ .

**Lemma 21.** Let  $r$  the resolvent of  $k$ . If the function  $\exp(-\sigma t)k(t)$  belongs to  $L^1(\mathbb{R}_+)$  for some  $\sigma \in \mathbb{R}$ , then there is a constant  $c$  such that the function  $\exp(-ct)r(t)$  belongs to  $L^1(\mathbb{R})$ .

By using the resolvent we can prove the solution of (A.8).

**Theorem 49.** Let  $k \in L^1_{loc}(\mathbb{R}_+)$ . Then for every  $f \in L^1_{loc}(\mathbb{R}_+)$ , there exists a unique solution  $x \in L^1_{loc}(\mathbb{R}_+)$  of (A.8). This solution is given by the variation of constant formula :

$$x(t) = f(t) + (r \star f)(t), \quad t \in \mathbb{R}_+$$

where  $r$  is the resolvent of  $k$ . If moreover,  $f \in L^p_{loc}(\mathbb{R}_+)$  then  $x$  also belongs to it.

One of the methods used to determine the explicit solution of the resolvent equation (A.9) is the inversion of the Laplace transform. In fact, If the Laplace transform  $\hat{k}(z)$  exists in some half plane  $\Re z > \sigma$  (for some sufficiently large  $\sigma$ ) then, by Lemma 21,  $\hat{r}(z)$  exists in some other half plane  $\Re z > c$ . By applying the convolution theorem for Laplace transform, (A.9) may be transformed and becomes :

$$\hat{r}(z) = \frac{\hat{k}(z)}{1 - \hat{k}(z)}, \quad \Re z \geq \max(\sigma, c)$$

then the inversion of the Laplace transform gives  $r$ .

**Remark 3.** If  $k \in L^1(\mathbb{R}_+)$  this does not imply that its resolvent  $r$  also belongs to  $L^1(\mathbb{R}_+)$ . We need more conditions on the kernel  $k$  for this result to be true.

**Theorem 51.** [Half Line Paley-Wiener] Let  $k \in L^1(\mathbb{R}_+)$ , then the resolvent  $r$  of  $k$  satisfies :

$$r \in L^1(\mathbb{R}_+)$$

if and only if

$$\hat{k}(z) \neq 1, \quad \forall z \in \mathbb{C}; \Re z \geq 0.$$

**Theorem 53.** [A Generalized Gronwall Lemma] Let  $k$  be a positive kernel in  $L^1_{loc}(0, T)$ , assume that  $x, f \in L^1_{loc}(0, T)$  and suppose that

$$x(t) \leq (k \star x)(t) + f(t), \quad a.e. t \in (0, T).$$

Then

$$x(t) \leq y(t), \quad a.e. t \in (0, T)$$

where  $y$  is the solution of the composition equation  $y(t) = (k \star y)(t) + f(t)$ .

**Corollary 5.** Let  $k \in L^1(\mathbb{R}_+)$  be nonnegative and have a nonnegative convolution resolvent  $r$ . Then  $r \in L^1(\mathbb{R}_+)$ .

This result is valid only for integral equations with a convolution kernel (A.8). Moreover, if the kernel  $k$  is integrable and admits a bounded first moment  $\int_{\mathbb{R}_+} ak(a)da < +\infty$ , we can find the following result which characterizes the resolvent  $r$  associated to  $k$  :

**Theorem 55.** Let  $k$  be a positive and integrable kernel such that  $\int_{\mathbb{R}_+} ak(a)da < +\infty$ , then there exists a function  $v \in L^1(\mathbb{R}_+)$ , s.t.

$$r(t) = v(t) + \int_0^t v(s)ds$$

where  $r$  is the positive resolvent associated to  $k$  and one has:

$$\int_{\mathbb{R}_+} v(s)ds = \frac{1}{\int_{\mathbb{R}_+} ak(a)da}.$$

Moreover, if  $\int_{\mathbb{R}_+} a^2k(a)da < +\infty$ , then there exists  $\gamma := v - \int_t^{+\infty} v(a)da$  and  $\gamma \in L^1(\mathbb{R}_+)$  s.t.

$$r(t) = \gamma(t) + \frac{1}{\int_{\mathbb{R}_+} k(a)ada}$$

The results presented above can be applied to the equation (1.3) in a particular case. We can rewrite the integral equation (1.3) for  $\varepsilon = 1$  and a constant kernel  $\varrho$  as:

$$(A.10) \quad \mu_0 z(t) - \int_0^t z(t-a)\varrho(a)da = f(t) + \int_t^{+\infty} z_p(t-a)\varrho(a)da$$

By dividing by  $\mu_0$  and defining  $\tilde{f}(t) := f(t) + \int_t^{+\infty} z_p(t-a)\varrho(a)da$ , we can rewrite (A.10) in the convolution form

$$(A.11) \quad z(t) - \int_0^t z(t-a)k(a)da = \tilde{f}(t)$$

then by using Theorem 49, we can prove an exact formula of the solution of (A.10) in terms of the resolvent  $r$  associated to the kernel  $k$ . In the case of (1.3), the existence of the unique solution requires that the kernel is of bounded continuous type and the source term is continuous (see [GLS09, Theorem 5.4]).





## AUXILIARY PROOFS TO CHAPTER 2

In this appendix, we list some results related to functions with bounded variation. The domain  $\Omega$  is an open set of  $\mathbb{R}^n$ .

### B.1 Proof of Theorem 11

As in the proof of [Leo09, Theorem 14.9], the aim is to show that for every  $\delta > 0$ , there exists a sequence  $\{f_\delta\}_\delta$  in  $C^\infty(\Omega)$  such that

$$\int_{\Omega} |f - f_\delta| \, dt < \delta \quad \text{and} \quad |Df_\delta|(\Omega) < \|Df\|(\Omega) + \delta.$$

Since the total variation  $\|Df\|(\Omega)$  is bounded,

$$\lim_{j \rightarrow +\infty} \|Df\|(\Omega \setminus \{t, \text{dist}(t, \partial\Omega) > 1/j, |t| < j\}) = 0$$

then for fixed  $\delta > 0$ , there exists a  $j_0 \in \mathbb{N}$  such that for all  $j \geq j_0$ ,

$$\|Df\|(\Omega \setminus \{t, \text{dist}(t, \partial\Omega) > 1/j, |t| < j\}) \leq \delta.$$

For  $i \in \mathbb{N}$ , we define the subdomain  $\Omega_i$  of  $\Omega$  by

$$\Omega_i := \{t \in \Omega, \text{dist}(t, \partial\Omega) > 1/j_0 + i, |t| < j_0 + i\}$$

such that  $\Omega_i \subset\subset \Omega_{i+1}$  and  $\bigcup_{i=0}^{\infty} \Omega_i = \Omega$ . Let  $W_0 = \Omega_1$  and  $W_i = \Omega_{i+1} \setminus \bar{\Omega}_{i-1}$ , where  $\Omega_{-1} = \Omega_0 := \emptyset$ , and let  $\{\phi_i\}$  be a partition of the unity subordinate to the covering  $\{W_i\}_{i \in \mathbb{N}}$

$$\phi_i \in C_0^\infty(W_i), \quad 0 \leq \phi_i \leq 1, \quad \sum_{i=0}^{\infty} \phi_i = 1.$$

For  $i \in \mathbb{N}$ , the idea is to find  $\delta_i > 0$  so small that

$$(B.1) \quad \text{supp } \chi_{\delta_i} * (\phi_i f) \subset W_i$$

which allows to write

$$(B.2) \quad \|\chi_{\delta_i} * (\phi_i f) - \phi_i f\|_{L^1(\Omega)} = \|\chi_{\delta_i} * (\phi_i f) - \phi_i f\|_{L^1(W_i)}$$

and then from the local approximation Theorem (see Theorem 2 p.125 in [EG92]),

$$\|\chi_{\delta_i} * (\phi_i f) - \phi_i f\|_{L^1(W_i)} < C\delta$$

and also by convenience, we can take  $C = \frac{1}{2^i}$ . Hence we conclude that

$$(B.3) \quad \int_{\Omega} |\chi_{\delta_i} * (\phi_i f) - \phi_i f| \, dt < \delta 2^{-i}$$

$$(B.4) \quad \int_{\Omega} |\chi_{\delta_i} * (f\phi'_i) - f\phi'_i| \, dt < \delta 2^{-i}.$$

for a positive mollifiers  $\chi_{\delta}$  defined as

$$\chi_{\delta}(t) := \frac{1}{\delta} \chi\left(\frac{t}{\delta}\right), \quad \chi(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{if } |t| > 2 \end{cases}$$

Define

$$(B.5) \quad f_{\delta} = \sum_{i=0}^{\infty} \chi_{\delta_i} * (\phi_i f).$$

By the construction of  $\{W_i\}$ , we have  $W_i \cap W_{i+1} \neq \emptyset$  and  $W_i \cap W_{i+1} \cap W_{i+2} = \emptyset$  which give that for every  $t \in \Omega$ ,

$$\#\{i \in \mathbb{N} : \chi_{\delta_i} * (\phi_i f)(t) \neq 0\} \leq 2$$

and since the finite sum of infinitely differentiable functions is infinitely differentiable, we conclude that  $f_{\delta} \in C^{\infty}(\Omega)$  and

$$\int_{\Omega} |f_{\delta} - f| \, dt \leq \sum_{i=1}^{\infty} \int_{\Omega} |\chi_{\delta_i} * (\phi_i f) - \phi_i f| \, dt < \sum_{i=1}^{\infty} \frac{\delta}{2^i} \leq \delta,$$

since  $f = \sum_{i=0}^{\infty} f\phi_i$ . Thus  $f_{\delta} \rightarrow f$  in  $L^1(\Omega)$  as  $\delta \rightarrow 0$ . Using the theorem of Lower semi-continuity of variation measure (see chap 5, [EG92]), we have

$$(B.6) \quad \|Df\|(\Omega) \leq \liminf_{\delta \rightarrow 0} |Df_{\delta}|(\Omega).$$

it remains to show that

$$\limsup_{\delta \rightarrow 0} |Df_\delta|(\Omega) \leq \|Df\|(\Omega).$$

Let  $\psi \in C_c^\infty(\Omega)$  be such that  $|\psi|_\infty \leq 1$ . Since  $\phi_i f \in \text{BV}((0, T))$  for every  $i \in \mathbb{N}$ , we have by Lemma 14.10 in [Leo09] that

$$(B.7) \quad \int_{\Omega} \chi_{\delta_i} * (\phi_i f) \psi' \, dt = \int_{\Omega} \phi_i f (\chi_{\delta_i} * \psi)' \, dt$$

let  $\psi_{\delta_i} = \chi_{\delta_i} * \psi$ , and using the fact that support of  $\psi$  is compact and that the partition of unity is locally finite, we have that

$$\begin{aligned} \int_{\Omega} f_\delta \psi' \, dt &= \sum_{i=1}^{+\infty} \int_{\Omega} \chi_{\delta_i} * (\phi_i f) \psi' \, dt \\ &= \sum_{i=1}^{+\infty} \int_{\Omega} \phi_i f \psi'_{\delta_i} \, dt \\ &= \sum_{i=1}^{+\infty} \int_{\Omega} f((\phi_i \psi_{\delta_i})' - \phi_i' \psi_{\delta_i}) \, dt := I_1 + I_2 \end{aligned}$$

since  $\text{supp}(\phi_i \psi_{\delta_i}) \subset W_i$  and  $|\phi_i \psi_{\delta_i}|_\infty \leq 1$ , it follows that

$$\begin{aligned} I_1 &= \int_{\Omega} f (\phi_i \psi_{\delta_i})' \, dt + \sum_{i=2}^{+\infty} \int_{\Omega} f (\phi_i \psi_{\delta_i})' \, dt \\ &\leq \|Df\|(\Omega) + \sum_{i=2}^{+\infty} \|Df\|(W_i) \\ &\leq \|Df\|(\Omega) + 3\|Df\|(\Omega \setminus \Omega_1) \\ &\leq \|Df\|(\Omega) + 3\delta \end{aligned}$$

since each  $t \in \Omega$  belongs at most two of the sets  $U_j$ . On the other hand, by Fubini's theorem we have

$$\begin{aligned} I_2 &= - \sum_{i=1}^{+\infty} \int_{\Omega} \chi_{\delta_i} * (f \phi_i') \psi \, dt \\ &= - \sum_{i=1}^{+\infty} \int_{\Omega} (\chi_{\delta_i} * (f \phi_i') - f \phi_i') \psi \, dt \end{aligned}$$

by using the fact that  $\sum_{i=1}^{+\infty} \phi_i' = 0$ . We now use (B.4) and the fact that  $|\psi|_\infty \leq 1$ , to conclude that  $I_2 \leq \delta$  and then we obtain that

$$\int_{\Omega} f_\delta \psi' \, dt \leq \|Df\|(\Omega) + 3\delta.$$

By taking the supremum and passing to the limit when  $\delta \rightarrow 0$ , we obtain that

$$(B.8) \quad \limsup_{\delta \rightarrow 0} \|Df_\delta\|(\Omega) \leq \|Df\|(\Omega).$$

Finally, (B.6) and (B.8) together concludes the proof.

**B.1.1 Proof of Lemma 5:**

Let  $f \in \text{BV}((0, T)) \cap L^\infty(\Omega)$ . By using (B.5) and (B.1) we can write for all  $t$  in  $\Omega$ ,

$$\begin{aligned} |f_\delta(t)| &= \left| \sum_{i=0}^{\infty} \chi_{\delta_i} * (f\phi_i)(t) \right| \\ &\leq \sum_{i=0}^{\infty} \|f\|_{L^\infty} \int_{W_i} \phi_i(t-t') \chi_{\delta_i}(t') \, dt' \\ &\leq \sum_{i=0}^{\infty} \|f\|_{L^\infty} \int_{W_i} \chi_{\delta_i}(t') \, dt' \end{aligned}$$

since  $0 \leq \phi_i \leq 1$ . On the other hand, we have  $W_i \cap W_{i-1} = \Omega_i \setminus \Omega_{i-1}$  and  $W_{i-1} \cap W_i \cap W_{i+1} = \emptyset$ , which implies that

$$|f_\delta(t)| \leq \|f\|_{L^\infty(\Omega)} \int_{W_{i-1}} \chi_{\delta_i}(t') \, dt' + \|f\|_{L^\infty(\Omega)} \int_{W_i} \chi_{\delta_{i-1}}(t') \, dt' \leq 2\|f\|_{L^\infty(\Omega)} \|\chi\|_{L^1(\mathbb{R})}$$

which ends the proof.

### AUXILIARY PROOFS TO CHAPTER 3

#### Some summations over integers :

In the following, we present some auxiliary results related to the subsection 3.3.2 of chapter 3.

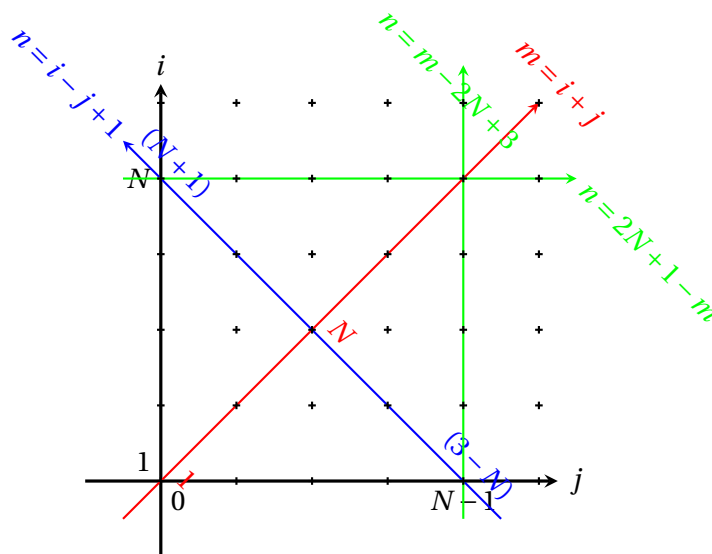


Figure C.1: The index change from  $(i, j)$  to  $(m, n)$  (here as an example  $N = 5$ ).

**Proposition 8.** If one has

$$S := \sum_{i=1}^N \sum_{k=1}^i \sum_{j=0}^{N-k} a_{i,j,k}$$

where  $a_{i,j,k}$  is a sequence of real numbers, then this sum is in fact equal to

$$S = \left( \sum_{m=1}^N \sum_{q=1}^m \sum_{k=1}^q + \sum_{m=N+1}^{2N-1} \sum_{q=m+1-N}^N \sum_{k=1}^{q+N-m} \right) a_{q,m-q,k}$$

**Proof.** We separate the case where  $i < N$  and the case where  $i = N$ . In the first case, one has, by the following computation, that

$$s := \sum_{i=1}^{N-1} \sum_{k=1}^i \sum_{j=0}^{N-k} a_{i,j,k} = \sum_{i=1}^{N-1} \sum_{j=0}^{N-i-1} \sum_{k=1}^i a_{i,j,k} + \sum_{i=1}^{N-1} \sum_{j=N-i}^{N-1} \sum_{k=1}^{N-j} a_{i,j,k},$$

because

$$\begin{cases} 0 \leq j \leq N-i-1, & k \in \{1, \dots, i\} \Rightarrow k \in \{1, \dots, \min(i, N-j)\}, \\ N-i \leq j \leq N-1, & k \in \{1, \dots, N-j\} \Rightarrow k \in \{1, \dots, \min(i, N-j)\}. \end{cases}$$

One has that

$$s = \sum_{i=1}^{N-1} \sum_{j=0}^{N-1} \sum_{k=1}^{\min(i, N-j)} a_{i,j,k},$$

then a simple check shows that

$$s' := \sum_{k=1}^N \sum_{j=0}^{N-k} a_{N,j,k} = \sum_{j=0}^{N-1} \sum_{k=1}^{N-j} a_{N,j,k},$$

and then one remarks simply that  $\min(i = N, N-j) = N-j$  which gathering the terms gives that

$$S = s + s' = \sum_{i=1}^N \sum_{j=0}^{N-1} \sum_{k=1}^{\min(i, N-j)} a_{i,j,k}.$$

The previous sum can also be rewritten as :

$$S = \left( \sum_{m=1}^N \sum_{n=3-m}^{m+1} + \sum_{m=N+1}^{2N-1} \sum_{n=m-2N+3}^{2N-m+1} \right) \sum_{k=1}^{\min(i, N-j)} a_{i(m,n), j(m,n), k} \chi_{\{m, n \in \mathbb{N}^2\}}(m, n),$$

where again  $m := i + j$  and  $n := i - j + 1$  and the inverse transform should provide integer values  $i(m, n)$  and  $j(m, n)$ . Indeed, for the first sum one proceeds as above. When  $m > N$ , one needs to bound the summation on  $n$  in an interval depending on  $m$  (see Fig. C.1). Indeed, when  $i = N$ , we write :

$$n = N - j + 1, \quad m = N + j, \quad \Rightarrow n = 2N - m + 1,$$

while if  $j = N - 1$ ,

$$n = i - N + 2, \quad m = i + N - 1, \quad \Rightarrow n = m - 2N + 3,$$

a simple check shows that  $n \leq 2N - 1 \Leftrightarrow m - 2N + 3 \leq 2N - m + 1$ .

Then since the indicatrix function is not zero when  $[n + m - 1]_2 = 0$ , there exists  $q \in \mathbb{Z}$  such that

$$n + m - 1 = 2q \Leftrightarrow n = 1 + 2q - m$$

so that the summation with respect to  $n$  can be exchanged with a summation over  $q$ . When  $n \in \{3 - m, m + 1\}$ ,  $q \in \{1, m\}$ , and similarly when  $n \in \{m - 2N + 3, 2N - m + 1\}$ ,  $q \in \{m + 1 - N, N\}$ . Moreover  $q = i$  and  $j = m - q$ . Thus the previous sum becomes :

$$S = \left( \sum_{m=1}^N \sum_{q=1}^m + \sum_{m=N+1}^{2N-1} \sum_{q=m+1-N}^N \right) \sum_{k=1}^{\min(q, N-(m-q))} a_{q, m-q, k}$$

Since  $\min(q, q + N - m) = q + \min(0, N - m) = q$  as soon as  $m \leq N$  this gives the first term. If  $m \in \{N + 1, \dots, 2N - 1\}$ , then  $N - m < -1$  and  $\min(q, q + N - m) = q + N - m$  which ends the proof. ■

**Proposition 9.** In the same way as above

$$S' := \sum_{i=0}^{N-1} \sum_{j=1}^{N-i} a_{i,j} = \sum_{m=1}^N \sum_{q=0}^{m-1} a_{q, m-q}$$

**Proof.** Again we perform the change of variables  $m = i + j$  and  $n = i - j + 1$  and we proceed as above. ■





## BIBLIOGRAPHY

- [AHP71] Michael Abercrombie, Joan E. M. Heaysman, and Susan M. Pegrum.  
“The locomotion of fibroblasts in culture. IV. Electron microscopy of the leading lamella.”  
In: *Experimental cell research* 67 2 (1971), pp. 359–67.
- [Web+85] Glenn F Webb et al.  
*Theory of nonlinear age-dependent population dynamics*.  
CRC Press, 1985.
- [Kre89] Erwin Kreyszig.  
*Introductory functional analysis with applications*. English.  
Paperback edition.  
New York etc.: John Wiley & Sons, 1989.  
ISBN: 0-471-50459-9.
- [Zie89] William P. Ziemer.  
*Weakly differentiable functions. Sobolev spaces and functions of bounded variation*. English.  
Vol. 120.  
Grad. Texts Math.  
Berlin etc.: Springer-Verlag, 1989.  
ISBN: 0-387-97017-7.  
DOI: [10.1007/978-1-4612-1015-3](https://doi.org/10.1007/978-1-4612-1015-3).
- [EG92] Lawrence C. Evans and Ronald F. Gariepy.  
*Measure theory and fine properties of functions*. English.  
Boca Raton: CRC Press, 1992,  
Pp. viii + 268.  
ISBN: 0-8493-7157-0.
- [Mac+94] Laura M Machesky et al.

- “Purification of a cortical complex containing two unconventional actins from *Acanthamoeba* by affinity chromatography on profilin-agarose.”  
In: *The Journal of cell biology* 127.1 (1994), pp. 107–115.
- [Ian95] Mimmo Iannelli.  
“Mathematical theory of age-structured population dynamics”.  
In: *Giardini editori e stampatori in Pisa* (1995).
- [Gum96] Barry M Gumbiner.  
“Cell adhesion: the molecular basis of tissue architecture and morphogenesis”.  
In: *Cell* 84.3 (1996), pp. 345–357.
- [MO96] Alexander Mogilner and George Oster.  
“The physics of lamellipodial protrusion”.  
In: *European biophysics journal* 25.1 (1996), pp. 47–53.
- [MHP98] R Dyche Mullins, John A Heuser, and Thomas D Pollard.  
“The interaction of Arp2/3 complex with actin: nucleation, high affinity pointed end capping, and formation of branching networks of filaments”.  
In: *Proceedings of the National Academy of Sciences* 95.11 (1998), pp. 6181–6186.
- [SB99] Tatyana M Svitkina and Gary G Borisy.  
“Arp2/3 complex and actin depolymerizing factor/cofilin in dendritic organization and treadmilling of actin filament array in lamellipodia”.  
In: *The Journal of cell biology* 145.5 (1999), pp. 1009–1026.
- [AFP00] Luigi Ambrosio, Nicola Fusco, and Diego Pallara.  
*Functions of bounded variation and free discontinuity problems*. English.  
Oxford: Clarendon Press, 2000,  
Pp. xviii + 434.  
ISBN: 0-19-850245-1/hbk.
- [PCC01] Dominique Pantaloni, Christophe Le Clainche, and Marie-France Carlier.  
“Mechanism of actin-based motility”.  
In: *Science* 292.5521 (2001), pp. 1502–1506.
- [Bra02] Vania MM Braga.  
“Cell–cell adhesion and signalling”.  
In: *Current opinion in cell biology* 14.5 (2002), pp. 546–556.
- [PB02] Thomas D Pollard and Christopher C Beltzner.  
“Structure and function of the Arp2/3 complex”.

- In: *Current opinion in structural biology* 12.6 (2002), pp. 768–774.
- [PB03] Thomas D Pollard and Gary G Borisy.  
“Cellular motility driven by assembly and disassembly of actin filaments”.  
In: *Cell* 112.4 (2003), pp. 453–465.
- [LVA04] Anja Lambrechts, Marleen Van Troys, and Christophe Ampe.  
“The actin cytoskeleton in normal and pathological cell motility”.  
In: *The international journal of biochemistry & cell biology* 36.10 (2004), pp. 1890–1909.
- [MMP05] P. Michel, S. Mischler, and B. Perthame.  
“General relative entropy inequality: an illustration on growth models”.  
In: *J. Math. Pures Appl. (9)* 84.9 (2005), pp. 1235–1260.  
ISSN: 0021-7824.  
DOI: [10.1016/j.matpur.2005.04.001](https://doi.org/10.1016/j.matpur.2005.04.001).  
URL: <http://dx.doi.org/10.1016/j.matpur.2005.04.001>.
- [SBK05] Tom Shemesh, Alexander D Bershadsky, and Michael M Kozlov.  
“Force-driven polymerization in cells: actin filaments and focal adhesions”.  
In: *Journal of Physics: Condensed Matter* 17.47 (2005), S3913.
- [SR05] J Victor Small and Guenter P Resch.  
“The comings and goings of actin: coupling protrusion and retraction in cell motility”.  
In: *Current opinion in cell biology* 17.5 (2005), pp. 517–523.
- [Per07] Benoît Perthame.  
*Transport equations in biology*. English.  
Basel: Birkhäuser, 2007,  
Pp. ix + 198.  
ISBN: 3-7643-7841-7/pbk.
- [AMR08] Pierre Auger, Pierre Magal, and Shigui Ruan.  
*Structured population models in biology and epidemiology*.  
Vol. 1936.  
Springer, 2008.
- [OSS08] D. Oelz, C. Schmeiser, and V. Small.  
“Modelling of the Actin-cytoskeleton in symmetric lamellipodial fragments”.  
In: *Cell Adhesion and Migration* 2 (2008), pp. 117–126.

- [BC09] John J. Benedetto and Wojciech Czaja.  
*Integration and modern analysis*. English.  
Basel: Birkhäuser, 2009,  
Pp. xix + 575.  
ISBN: 978-0-8176-4306-5/hbk; 978-0-8176-4656-1/ebook.
- [GLS09] G. Gripenberg, S.-O. Londen, and O. Staffans.  
*Volterra integral and functional equations*. English.  
Reprint of the 1990 hardback ed. Vol. 34.  
Encycl. Math. Appl.  
Cambridge: Cambridge University Press, 2009.  
ISBN: 978-0-521-10306-0.
- [Leo09] Giovanni Leoni.  
*A first course in Sobolev spaces*. English.  
Vol. 105.  
Providence, RI: American Mathematical Society (AMS), 2009,  
Pp. xvi + 607.  
ISBN: 978-0-8218-4768-8/hbk.
- [CTK10] Dilki C. Wickramarachchi, Argyrios N Theofilopoulos, and Dwight H Kono.  
“Immune pathology associated with altered actin cytoskeleton regulation”.  
In: *Autoimmunity* 43.1 (2010), pp. 64–75.
- [DG10] Marie Doumic Jauffret and Pierre Gabriel.  
“Eigenelements of a general aggregation-fragmentation model”.  
In: *Mathematical Models and Methods in Applied Sciences* 20.05 (2010), pp. 757–783.
- [OS10] D. Oelz and C. Schmeiser.  
“Derivation of a Model for Symmetric Lamellipodia with Instantaneous Cross-Link Turnover”.  
In: *Archive for Rational Mechanics and Analysis* 198.3 (2010), pp. 963–980.  
DOI: [10.1007/s00205-010-0304-z](https://doi.org/10.1007/s00205-010-0304-z).  
URL: <http://www.scopus.com/inward/record.url?eid=2-s2.0-77958564584&partnerID=40&md5=1e31d2ee56da093cc78a4336cd6e646e>.
- [SG10] André Schönichen and Matthias Geyer.  
“Fifteen formins for an actin filament: a molecular view on the regulation of human formins”.

- In: *Biochimica et Biophysica Acta (BBA)-Molecular Cell Research* 1803.2 (2010), pp. 152–163.
- [MO11] Vuk Milišić and Dietmar Oelz.  
 “On the asymptotic regime of a model for friction mediated by transient elastic linkages”. English.  
 In: *J. Math. Pures Appl.* (9) 96.5 (2011), pp. 484–501.  
 ISSN: 0021-7824.
- [Ane12] Dragi Anevski.  
 “Riemann-Stieltjes integrals”.  
 In: *Mathematical Sciences, Lund University, Lund, Sweden* (2012).
- [HK12] Xiaohua Hu and Jeffrey R Kuhn.  
 “Actin filament attachments for sustained motility in vitro are maintained by filament bundling”.  
 In: *PLoS One* 7.2 (2012), e31385.
- [SRC12] Daniela K Schlüter, Ignacio Ramis-Conde, and Mark AJ Chaplain.  
 “Computational modeling of single-cell migration: the leading role of extracellular matrix fibers”.  
 In: *Biophysical journal* 103.6 (2012), pp. 1141–1151.
- [SG12] Ulrich S Schwarz and Margaret L Gardel.  
 “United we stand—integrating the actin cytoskeleton and cell–matrix adhesions in cellular mechanotransduction”.  
 In: *Journal of cell science* 125.13 (2012), pp. 3051–3060.
- [ZM12] Jie Zhu and Alex Mogilner.  
 “Mesoscopic model of actin-based propulsion”.  
 In: *PLoS computational biology* 8.11 (2012), e1002764.
- [ABD13] Jürgen Appell, Józef Banas, and Nelson José Merentes Díaz.  
 “Bounded variation and around”.  
 In: *Bounded Variation and Around*.  
 de Gruyter, 2013.
- [Gab13] Pierre Gabriel.  
*Équations structurées en dynamique des populations*.  
 Licence.  
 ENS Cachan, Dec. 2013,

- P. 20.  
URL: <https://cel.archives-ouvertes.fr/cel-00918869>
- [Tsc13] Daniel J Tschumperlin.  
“Fibroblasts and the ground they walk on”.  
In: *Physiology* 28.6 (2013), pp. 380–390.
- [Gab14] Pierre Gabriel.  
“Équations structurées en dynamique des populations”.  
In: (2014).
- [MD14] Johan A Metz and Odo Diekmann.  
*The dynamics of physiologically structured populations*.  
Vol. 68.  
Springer, 2014.
- [Man+15] A. Manhart et al.  
“An extended Filament Based Lamellipodium Model produces various moving cell shapes in the presence of chemotactic signals”.  
In: *J. Theoret. Biol.* 382 (2015), pp. 244–258.  
ISSN: 0022-5193.  
DOI: [10.1016/j.jtbi.2015.06.044](https://doi.org/10.1016/j.jtbi.2015.06.044).  
URL: <http://dx.doi.org/10.1016/j.jtbi.2015.06.044>.
- [MO15] Vuk Milišić and Dietmar Oelz.  
“On a structured model for load-dependent reaction kinetics of transient elastic linkages mediating nonlinear friction”. English.  
In: *SIAM J. Math. Anal.* 47.3 (2015), pp. 2104–2121.  
ISSN: 0036-1410; 1095-7154/e.
- [GW16] Piotr Gwiazda and Emil Wiedemann.  
“Generalized entropy method for the renewal equation with measure data”.  
In: *arXiv preprint arXiv:1604.07657* (2016).
- [MO16] V. Milišić and D. Oelz.  
“Tear-off versus global existence for a structured model of adhesion mediated by transient elastic linkages”.  
In: *Commun. Math. Sci.* 14.5 (2016), pp. 1353–1372.  
ISSN: 1539-6746.  
DOI: [10.4310/CMS.2016.v14.n5.a7](https://doi.org/10.4310/CMS.2016.v14.n5.a7).  
URL: <http://dx.doi.org/10.4310/CMS.2016.v14.n5.a7>.

- [MO17] V. Milisic and D. Oelz.  
“From memory to friction : convergence of regular solutions towards the heat equation with a non-constant coefficient”.  
In: *ArXiv e-prints* (June 2017).  
arXiv: [1706.02650](https://arxiv.org/abs/1706.02650) [math.AP].
- [Gre+18] B. Grec et al.  
“A 1D model of leukocyte adhesion coupling bond dynamics with blood velocity”.  
In: *J. Theor. Biol.* 452 (Sept. 2018), pp. 35–46.
- [Mil18] V. Milisic.  
“Initial layer analysis for a linkage density in cell adhesion mechanisms”.  
In: *CIMPA School on Mathematical Models in Biology and Medicine*.  
Vol. 62.  
ESAIM Proc. Surveys.  
EDP Sci., Les Ulis, 2018,  
Pp. 108–122.
- [Sfa+18] Nikolaos Sfakianakis et al.  
“Modelling cell-cell collision and adhesion with the filament based lamellipodium model”.  
In: *arXiv preprint arXiv:1809.07852* (2018).
- [Tac+18] Gérard Tachdjian et al.  
*Toute l'UE 2-Cours+ QCM: La cellule et les tissus*.  
Elsevier Health Sciences, 2018.
- [HPR19] Martin Heida, Robert I. A. Patterson, and D. R. Michiel Renger.  
“Topologies and measures on the space of functions of bounded variation taking values in a Banach or metric space”. English.  
In: *J. Evol. Equ.* 19.1 (2019), pp. 111–152.  
ISSN: 1424-3199; 1424-3202/e.
- [MST19] Giselle Antunes Monteiro, Antonín Slavík, and Milan Tvrdý.  
*Kurzweil-Stieltjes integral. Theory and applications*. English.  
Vol. 15.  
Hackensack, NJ: World Scientific, 2019,  
Pp. xviii + 382.  
ISBN: 978-981-4641-77-7/hbk; 978-981-4641-79-1/ebook.



- [PR19] Antoine Perasso and Quentin Richard.  
“Implication of age-structure on the dynamics of Lotka Volterra equations”.  
In: *differential and integral equations* 32.1/2 (2019), pp. 91–120.
- [AM21] Samar Allouch and Vuk Milišić.  
“Friction mediated by transient elastic linkages: extension to loads of bounded variation”.  
In: *accepted JIE* (2021).
- [MS21] Vuk Milišić and Christian Schmeiser.  
“Asymptotic limits for a nonlinear integro-differential equation modelling leukocytes’ rolling on arterial walls”.  
In: *Nonlinearity* 35.2 (2021), p. 843.
- [AM22] Samar Allouch and Vuk Milisic.  
“Friction mediated by transient elastic linkages : asymptotic expansions and fat tails”.  
In: *arXiv preprint arXiv:2209.14666* (2022).
- [OS09] D. Oelz and C. Schmeiser.  
“How do cells move? mathematical modelling of cytoskeleton dynamics and cell migration”.  
In: *Cell mechanics: from single scale-based models to multiscale modelling*.  
Ed. by A. Chauviere, L. Preziosi, and C. Verdier.  
Chapman and Hall / CRC Press, to appear, 2009.