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Santiago Martinchich

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# Contributions to the study of discretized Anosov flows

*Contributions à l'étude des flots d'Anosov discrétisés*

**Thèse de doctorat de l'université Paris-Saclay et de  
l'Universidad de la República (Uruguay)**

École doctorale n° 574, mathématiques Hadamard (EDMH)  
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Thèse préparée au **Laboratoire de mathématiques d'Orsay** (Université Paris-Saclay) et au **Centro de Matemática** (Universidad de la República, Uruguay), sous la direction de **Sylvain CROVISIER**, directeur de recherches au CNRS, et la direction de **Rafael POTRIE**, professeur titulaire à l'Universidad de la República.

**Thèse soutenue à Montevideo, le 14 février 2023, par**

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## Composition du jury

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**Title :** Contributions to the study of discretized Anosov flows

**Keywords :** Dynamical systems, partial hyperbolicity, Anosov flows, global stability, center foliation, attractors

**Abstract :** This thesis aims to make a contribution to the study of a somehow large class of partially hyperbolic diffeomorphisms denoted as *discretized Anosov flows*.

This class is shown to comprise whole connected components of partially hyperbolic diffeomorphisms with one dimensional center in any dimension.

Several general properties of discretized Anosov flows are proven. These properties include dynamical coherence, uniqueness of invariant foliations, plaque expansivity and unique integrability of the center bundle. In particular, this permits to

establish the equivalence with other similar notion appearing on the literature.

A characterization of discretized Anosov flows is given under some general circumstances : the class is shown to coincide with the partially hyperbolic diffeomorphisms that individually fix each leaf of a one-dimensional center foliation.

Regarding further dynamical properties, a result on uniqueness of attractor is shown.

Several of this results are also seen to happen for the related class of partially hyperbolic diffeomorphisms admitting a uniformly compact center foliation.

**Titre :** Contributions à l'étude des flots d'Anosov discrétisés

**Mots clés :** Systèmes dynamiques, hyperbolicité partielle, flots d'Anosov, stabilité globale, feuilletage central, attracteurs

**Résumé :** Cette thèse a pour but de contribuer à l'étude d'une classe assez large de difféomorphismes partiellement hyperboliques dénomés *flots d'Anosov discrétisés*.

On montre que, en toute dimension, cette classe est constituée de composantes connexes entières de l'ensemble des difféomorphismes partiellement hyperboliques avec fibré central unidimensionnel.

Plusieurs propriétés générales des flots d'Anosov discrétisés sont prouvées. Ces propriétés comprennent la cohérence dynamique, l'unicité des feuilletages invariantes, la expansivité par plaques et l'intégrabilité unique du fibré central. En particulier, cela permet d'établir l'équivalence avec

d'autres notions similaires apparaissant dans la littérature.

Une caractérisation des flots d'Anosov discrétisés est donnée dans certaines circonstances générales : on montre que la classe coïncide avec les difféomorphismes partiellement hyperboliques qui fixent individuellement chacune des feuilles d'un feuilletage central unidimensionnel.

En ce qui concerne d'autres propriétés dynamiques, on montre un résultat sur l'unicité de l'attracteur.

Une bonne partie de ces résultats s'appliquent également à la classe des difféomorphismes partiellement hyperboliques admettant un feuilletage central uniformément compact.



**Título :** Contribuciones al estudio de los flujos de Anosov discretizados

**Palabras claves :** Sistemas dinámicos, hiperbolicidad parcial, flujos de Anosov, estabilidad global, foliación central, atractores

**Resumen :** Esta tesis tiene como objetivo realizar un aporte al estudio de una clase relativamente amplia de difeomorfismos parcialmente hiperbólicos denominados *flujos de Anosov discretizados*.

Se demuestra que esta clase comprende componentes conexas enteras de difeomorfismos parcialmente hiperbólicos con foliación central unidimensional en cualquier dimensión ambiente.

Varias propiedades generales de los flujos de Anosov discretizados son demostradas. Entre ellas la coherencia dinámica, unicidad de foliaciones invariantes, expansividad por placas e integrabilidad única del fibrado central. En particular, esto permite establecer la equivalencia con otras nociones

similares que aparecen en la literatura.

Una caracterización de los flujos de Anosov discretizados es obtenida bajo ciertas condiciones generales : se muestra que la clase de flujos de Anosov discretizados coincide con la de los parcialmente hiperbólicos que dejan invariante cada hoja de una foliación central unidimensional.

En cuanto a otras propiedades dinámicas, un resultado sobre unicidad de atractores es demostrado.

Varios de estos resultados son obtenidos igualmente para la clase de difeomorfismos parcialmente hiperbólicos que admiten una foliación central uniformemente compacta.

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# Chapter 1

## Introduction and presentation of results

### 1.1 Introduction (English)

#### 1.1.1 Differentiable dynamics, hyperbolicity and partial hyperbolicity

The classical theory of dynamical systems deals with the asymptotic behavior of systems that evolve in time with respect to a prescribed deterministic rule governing its evolution.

In the case of differentiable dynamics, this rule is typically given by a diffeomorphism  $f : M \rightarrow M$  in the case of a *discrete dynamical system*, or by an ordinary differential equation  $\dot{X} = F(X)$  defining a flow  $X_t : M \rightarrow M$  in the case of a *continuous dynamical system*.

A paradigmatic example of these systems presenting a global form of rich and *chaotic* behavior is given by *Anosov systems* (also called *globally uniformly hyperbolic systems*). The study of this type of systems traces back at least to the pioneering work of D.V. Anosov and S. Smale (see [A67] and [S67]).

Roughly speaking, a diffeomorphism  $f : M \rightarrow M$  is called an *Anosov diffeomorphism* if the tangent bundle admits an  $f$ -invariant decomposition  $TM = E^s \oplus E^u$  such that vectors in  $E^s$  and  $E^u$  are uniformly contracted by forward and backward iterates of  $f$ , respectively.

A flow without singularities  $X_t : M \rightarrow M$  is called an *Anosov flow* if it preserves an invariant decomposition  $TM = E^s \oplus E^c \oplus E^u$  such that vectors in  $E^s$  and  $E^u$  are uniformly contracted by forward and backward iterates of  $X_t$ , respectively, and the bundle  $E^c$  is the direction tangent to the flow  $X_t$ .

A natural extension of uniformly hyperbolic systems is given by the notion



of *partial hyperbolicity*.

**Definition.** A diffeomorphism  $f : M \rightarrow M$  on a closed Riemannian manifold  $M$  is called *partially hyperbolic* if there exists a  $Df$ -invariant continuous decomposition of the tangent bundle  $TM$  into three non-trivial subbundles

$$TM = E^s \oplus E^c \oplus E^u$$

such that vectors in  $E^s$  and  $E^u$  are uniformly contracted by forward and backward iterates of  $f$ , respectively, and vectors in  $E^c$  experience an intermediate behavior. See Chapter 2 for a precise definition.

The definition of partially hyperbolic diffeomorphisms traces back at least to [BP74] and [HPS77]. It is worth mentioning that many other definitions of *partial hyperbolicity* exist. Typically, all of these notions involve some kind of invariant decomposition of the tangent space (of the whole manifold or an invariant part of it) into subbundles satisfying some sort of *dominated* behavior. Plus, in most of the cases, with at least one of the subbundles being uniformly expanded or contracted. See for example [CP15] as a reference.

As it is also the case for Anosov systems, partial hyperbolicity is a  $C^1$  open property that can be checked in finite iterates (as a consequence of the *cone criterion*, for instance). Moreover, it appears naturally in the study of *robust* dynamical properties. That is, properties that remain unchanged by small perturbations of the system. This is the case in at least two relevant scenarios: *robust transitivity* and *stable ergodicity*. See for example [BPSW01], [CHHU18] and [HP18].

Partially hyperbolic diffeomorphisms constitute a rich class of dynamical systems with enough rigidity so that a certain ‘classification’ goal may be sought:

**Problem.** *Develop a framework for classifying (at least in dimension 3) partially hyperbolic diffeomorphisms. Determine which dynamical properties can occur for these systems and identify which ones of them are robust. Explore the interaction between partially hyperbolic diffeomorphisms and the geometry and topology of the underlying manifold, in particular determining which manifolds and isotopy classes admit partially hyperbolic systems.*

One way to approach this problem is to study the invariant structures preserved by these kind of maps such as the *invariant foliations* that naturally arise on them. One can expect to obtain topological or dynamical consequences from the behavior of these invariant structures, and viceversa.

In dimension 3, the classification problem has been particularly pursued. The classical examples of partially hyperbolic diffeomorphisms in dimension 3 are:

- *Deformations of Anosov diffeomorphisms.*
- *Partially hyperbolic skew-products.*
- *Perturbations of time one maps of Anosov flows.*

Briefly, the first class consists of partially hyperbolic diffeomorphisms homotopic to an Anosov map. The second one, of partially hyperbolic diffeomorphisms such that  $E^c$  integrates to an invariant foliation  $\mathcal{W}^c$  by compact leaves that induces a fiber bundle structure in  $M$  (see Section 1.1.6). And the third one, diffeomorphisms that are sufficiently close to the time 1 map of an Anosov flow.

Despite having been conjectured to account for every partially hyperbolic system in dimension 3 (see *Pujal's conjecture* in [BW05]), the list of classical examples have been recently joined by new types of examples that challenge the classification enterprise. In [HHU16] the first non-dynamically coherent examples were built. Later in [BPP16], [BGP16] and [BGHP17] new striking examples emerged. See also [BFP20].

In higher dimension, one way to approach the study is to restrict to the case where  $\dim(E^c) = 1$ . Under this assumption the above list of classical examples is essentially the same (one should only add the possibility of taking product with an Anosov diffeomorphism).

The class of *discretized Anosov flows* is conceived as a natural extension of the third type of classical examples. The main goal of this thesis is to establish several general properties for this class of systems in any dimension.

### 1.1.2 Discretized Anosov flows

We denote by  $\text{PH}(M)$  the set of partially hyperbolic diffeomorphism in  $M$  and by  $\text{PH}_{c=1}(M)$  the ones such that  $\dim(E^c) = 1$ .

**Definition 1.1.1.** We say that  $f \in \text{PH}_{c=1}(M)$  is a *discretized Anosov flow* if there exist an orientable foliation  $\mathcal{W}^c$  whose leaves are  $C^1$  submanifolds tangent to  $E^c$  and a continuous function  $\tau : M \rightarrow \mathbb{R}_{>0}$  such that

$$f(x) = \varphi_{\tau(x)}^c(x)$$

for every  $x \in M$ , where  $\varphi_t^c : M \rightarrow M$  denotes a unit speed flow whose orbits are the leaves of  $\mathcal{W}^c$ .

The prototypical example of a discretized Anosov flow is the time 1 map of an Anosov flow and all its sufficiently small  $C^1$ -perturbations. The latter is a consequence of [HPS77] and will be revisited in this text.

The term *discretized Anosov flow* was coined in [BFFP19] and derives from the fact that the flow  $\varphi_t^c$  needs to be a *topological Anosov flow* (see Definition 3.7.1). Hence  $f$  can be thought of as a discretization of the topological Anosov flow  $\varphi_t^c$ .

Discretized Anosov flows have been profusely studied in the literature, though not always under the same name. For example, in [BD96] the first examples of robustly transitive diffeomorphisms isotopic to the identity were obtained. These examples are constructed arbitrary close to the time 1 map on any Anosov flow. In particular, they are discretized Anosov flows.

Regarding stable ergodicity, in [GPS94] it was shown that time 1 maps of geodesic flows in closed surfaces of constant negative curvature are  $C^2$  stably ergodic. These were the first non-Anosov examples of stably ergodic diffeomorphisms constructed in the literature. Again, these examples are discretized Anosov flows (as well as their perturbations).

In [BW05] discretized Anosov flows and partially hyperbolic skew-products were shown to be quite ubiquitous among (dynamically coherent) partially hyperbolic diffeomorphisms in dimension 3. This led to the consolidation of the classical examples in dimension 3.

More recently, in [BFFP19] discretized Anosov were shown to account for every dynamically coherent homotopic to the identity partially hyperbolic diffeomorphism of many 3-manifolds. And in [FP22] (see also [FP21]) that in most 3-manifolds discretized Anosov flows are accessible and ergodic whenever they preserve a volume form.

Other recent dynamical results involving discretized Anosov flows are [AVW15]’s rigidity results, [BFT20] regarding measures of maximal entropy, [DWX21] and [BG21] on centralizers rigidity for partially hyperbolic diffeomorphisms, and the ‘invariant principle’ shown in [CP22].

One feature that has proven to be quite useful in the treatment of partially hyperbolic diffeomorphisms is the presence of invariant foliations tangent to the invariant subbundles. In [HPS77] it was shown that the bundles  $E^s$  and  $E^u$  uniquely integrate to  $f$ -invariant foliations  $\mathcal{W}^s$  and  $\mathcal{W}^u$ , respectively.

In contrast, the bundles  $E^s \oplus E^c$  and  $E^s \oplus E^c$  may or may not be integrable. Whenever they integrate to  $f$ -invariant foliations ( $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$ , respectively) the map  $f$  is called *dynamically coherent*. If this is the case then  $\mathcal{W}^c := \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  (the foliation given by the connected components of the intersection of  $\mathcal{W}^{cs}$  leaves and  $\mathcal{W}^{cu}$  leaves) gives rise to an  $f$ -invariant foliation tangent to the center bundle  $E^c$ .

Recall that the *non-wandering set*  $\Omega(f)$  of  $f$  is the set of points  $x$  in  $M$  such that for every neighborhood  $U$  of  $x$  there exists  $N > 0$  such that  $f^N(U) \cap U \neq \emptyset$ .

On our first theorem we state some general properties satisfied by every

discretized Anosov flow:

**Theorem.** *Let  $f$  be a discretized Anosov flow. Let  $\varphi_t^c$  and  $\mathcal{W}^c$  denote the flow and center foliation appearing in the definition of  $f$ , respectively. Then:*

1. *(Topological Anosov flow). The flow  $\varphi_t^c$  is a topological Anosov flow (see Definition 3.7.1).*
2. *(Dynamical coherence). The map  $f$  is dynamically coherent, admitting a center-stable foliation  $\mathcal{W}^{cs}$  and a center-unstable foliation  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .*
3. *(Uniqueness of cs and cu foliations). The foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  are the only  $f$ -invariant foliations tangent to  $E^s \oplus E^c$  and  $E^c \oplus E^u$ , respectively.*
4. *(Completeness of leaves). The leaves of  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  satisfy that  $\mathcal{W}^{cs}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^s(y)$  and  $\mathcal{W}^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^u(y)$  for every  $x \in M$ .*
5. *(Topology of leaves) The leaves of  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  are homeomorphic to either planes or cylinders (see definitions in Section 3.7). The former contain no compact center leaves while the latter contain exactly one.*

It is worth pointing out that in dimension 3 the above theorem was mostly known. Indeed, once (2) is proven then (1), (4) and (5) follow from [BW05, Theorem 2]. In addition, once (1) is proven then (2) has already appeared in [BFP20, Proposition G.2] and (3) follows from [BFFP19] (see [BG21, Lemma 1.1]). Our goal is to synthesize the theory in dimension 3 and to generalize it to higher dimensions where lesser has been stated.

In [BFFP19], [BFP20], [BG21] and [GM22] a map  $f \in \text{PH}_{c=1}(M)$  is called a ‘discretized Anosov flow’ if it satisfies that there exist a topological Anosov flow  $\varphi_t : M \rightarrow M$  and a continuous function  $\tau : M \rightarrow \mathbb{R}_{>0}$  such that  $f(x) = \varphi_{\tau(x)}(x)$  for every  $x$  in  $M$ .

In [BFT20] a diffeomorphism  $f \in \text{PH}_{c=1}(M)$  is called ‘flow-type’ if it satisfies Definition 1.1.1 and in addition is dynamically coherent and admits at least one compact leaf of  $\mathcal{W}^c$ .

Similar yet not a priori identical notions were also studied in [BW05], [BG09] and [BG10].

The question regarding the relationship between these notions arised naturally. The above theorem (add also Proposition 3.7.3 item (4) stated in Section 3.7) shows that all these definitions are equivalent and refer to the same class of partially hyperbolic systems:

**Corollary.** *The definition of discretized Anosov flow given in [BFFP19], [BFP20], [BG21] and [GM22], and the definition of flow-type partially hyperbolic diffeomorphism in [BFT20], are all equivalent to Definition 1.1.1.*

*Moreover, the class of partially hyperbolic diffeomorphisms studied in [BW05, Theorem 2.], [BG09] and [BG10] are also discretized Anosov flows as in Definition 1.1.1.*

### 1.1.3 Global stability

The next result shows that discretized Anosov flows constitute a somehow large class of partially hyperbolic diffeomorphisms with one-dimensional center:

**Theorem.** *The set of discretized Anosov flows is a  $C^1$  open and closed subset of  $\text{PH}_{c=1}(M)$ .*

In other words, the class of discretized Anosov flows comprises whole connected components of  $\text{PH}_{c=1}(M)$ .

Recall that two pairs of partially hyperbolic diffeomorphisms and invariant center foliations  $(f, \mathcal{W}_f^c)$  and  $(g, \mathcal{W}_g^c)$  are said to be *leaf-conjugate* if there exists a homeomorphism  $h : M \rightarrow M$  taking leaves of  $\mathcal{W}_f^c$  to leaves of  $\mathcal{W}_g^c$  and such that  $h \circ f(W) = g \circ h(W)$  for every leaf  $W \in \mathcal{W}_f^c$ . Leaf-conjugacy gives sense to a classification framework for (dynamically coherent) partially hyperbolic diffeomorphism modulo its center behavior (two systems being considered equivalent whenever they are leaf-conjugate).

The proof of the above theorem shows also that leaf-conjugacy is preserved along connected components of discretized Anosov flows:

**Corollary.** *Two discretized Anosov flows in the same  $C^1$  connected component of  $\text{PH}_{c=1}(M)$  are leaf-conjugate.*

It is worth mentioning the background context for the above statements. A classical result from [HPS77] gives conditions for the stability of *normally hyperbolic foliations* (that is, foliations that are tangent to a center bundle):

**Theorem** (Hirsch-Pugh-Shub). *Suppose  $f \in \text{PH}(M)$  admits an  $f$ -invariant center foliation  $\mathcal{W}_f^c$ . If  $(f, \mathcal{W}_f^c)$  is plaque expansive then there exists a  $C^1$  neighborhood  $\mathcal{U}(f)$  of  $f$  so that every  $g \in \mathcal{U}(f)$  admits a  $g$ -invariant center foliation  $\mathcal{W}_g^c$  such that  $(f, \mathcal{W}_f^c)$  and  $(g, \mathcal{W}_g^c)$  are leaf-conjugate.*

A pair  $(f, \mathcal{W}^c)$  is called *plaque expansive* (or  $\delta$ -*plaque expansive*) if there exists  $\delta > 0$  so that every pair of sequences  $(x_n)_{n \in \mathbb{Z}}$  and  $(y_n)_{n \in \mathbb{Z}}$  satisfying that  $x_{n+1} \in \mathcal{W}_\delta^c(f(x_n))$ ,  $y_{n+1} \in \mathcal{W}_\delta^c(f(y_n))$  and  $d(x_n, y_n) < \delta$  for every  $n \in \mathbb{Z}$  must also satisfy  $y_0 \in \mathcal{W}_{loc}^c(x_0)$ . This condition is shown in [HPS77] to be

satisfied whenever  $\mathcal{W}^c$  is a  $C^1$  foliation or  $f$  acts by isometries on  $\mathcal{W}^c$  (as it is the case for the time 1 map of an Anosov flow).

To show the  $C^1$  openness and closeness of discretized Anosov flows a certain ‘uniform version’ of the above stability theorem is needed:

**Theorem.** *Suppose  $f_0 \in \text{PH}_{c=1}(M)$ . For every  $\delta > 0$  there exists a  $C^1$  neighborhood  $\mathcal{U}(f_0)$  of  $f_0$  such that, if some  $f \in \mathcal{U}(f_0)$  admits a center foliation  $\mathcal{W}_f^c$  so that  $(f, \mathcal{W}_f^c)$  is  $\delta$ -plaque expansive, then every  $g \in \mathcal{U}(f_0)$  admits a  $g$ -invariant center foliation  $\mathcal{W}_g^c$  such that  $(f, \mathcal{W}_f^c)$  and  $(g, \mathcal{W}_g^c)$  are leaf-conjugate.*

The key ‘uniform’ part in the above theorem is that the size of  $\mathcal{U}(f_0)$  is fixed beforehand. Then if  $(f_n)_n$  is a sequence in  $\text{PH}_{c=1}(M)$  converging to  $f_0$  it is enough to show that a pair  $(f_N, \mathcal{W}_{f_N}^c)$  is  $\delta$ -plaque expansive for some  $f_N \in \mathcal{U}(f_0)$  to induce a center foliation  $\mathcal{W}_{f_0}^c$  for  $f_0$ , which in addition satisfies that  $(f_0, \mathcal{W}_{f_0}^c)$  is leaf-conjugate to  $(f_N, \mathcal{W}_{f_N}^c)$  (and in fact leaf-conjugate to a pair  $(f, \mathcal{W}_f^c)$  for every  $f \in \mathcal{U}(f_0)$ ).

A similar uniform stability theorem was originally noted in [BFP20] in a different but related context (for  $C^1$  leaf-immersions and branching foliations). In [BFP20] the  $C^1$  open and closeness of the class of collapsed Anosov flows in dimension 3 is shown. The  $C^1$  openness and closeness of discretized Anosov flows in dimension 3 essentially follows from [BFP20] once the equivalence between the definitions of discretized Anosov flow given in [BFP20] and Definition 1.1.1 has been shown (see Chapter 5 for more details).

The fact that leaf-conjugacy persists among connected components of discretized Anosov flows can be seen as a ‘global stability’ result where a plaque expansive system  $(f, \mathcal{W}^c)$  induces leaf-conjugacy among its whole  $C^1$  partially hyperbolic connected component.

This has also been shown to be true in [FPS14] whenever  $f$  is a hyperbolic linear automorphism of the torus  $\mathbb{T}^n$  (seen as a partially hyperbolic diffeomorphism), and generalized in [Pi19] for linear Anosov automorphisms on nilmanifolds. Moreover, we show (see Section 1.1.6 below) that this phenomenon also happens for partially hyperbolic skew-products with one-dimensional center.

It is natural to ask whether this is true in general:

**Question.** *Suppose  $f \in \text{PH}(M)$  admits an  $f$ -invariant center foliation  $\mathcal{W}^c$  such that  $(f, \mathcal{W}^c)$  is plaque expansive. Does every  $g$  in the  $C^1$  partially hyperbolic connected component of  $f$  admits a  $g$ -invariant center foliation  $\mathcal{W}_g^c$  such that  $(g, \mathcal{W}_g^c)$  is plaque expansive and leaf-conjugate to  $(f, \mathcal{W}^c)$ ?*

Since discretized Anosov flows constitute entire connected components of  $\text{PH}_{c=1}(M)$  many other natural questions arise. One may ask which are the

properties that are preserved in whole connected components and which ones are not. Which connected components contain the time 1 map of an Anosov flow, how different are the ones that do not, etc.

As mentioned above, one such property that is preserved among connected components of discretized Anosov flows is leaf-conjugacy. Another one is the unique integrability of the center bundle. That is, the property that modulo reparametrizations there exists a unique local  $C^1$  curve tangent to  $E^c$  through every point of  $M$  (see Section 5.4 for more details on this definition):

**Proposition.** *Suppose  $f$  is a discretized Anosov flow such that  $E_f^c$  is uniquely integrable. Then  $E_g^c$  is uniquely integrable for every  $g$  in the same  $C^1$  connected component of  $\text{PH}_{c=1}(M)$  as  $f$ .*

In particular, every discretized Anosov flow in the same connected component of the time 1 map of an Anosov flow has a uniquely integrable center bundle.

In Example 5.4.3 an example of a discretized Anosov flow  $f$  such that  $E^c$  is not uniquely integrable is given. This is obtained by a simple modification of a construction given in [HHU16] that leads to a 2-torus tangent to  $E^s \oplus E^c$  entirely made of points of non-unique integrability for  $E^c$ . The center flow  $\varphi_t^c$  on this example is orbit equivalent to the suspension of a linear Anosov diffeomorphism  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  on the 2-torus, yet by the proposition above the map  $f$  is not in the same connected component as the time 1 map of the suspension of  $A$ . One concludes the following.

**Corollary.** *There exists  $C^1$  connected components of discretized Anosov flows that do not contain the time 1 map of an Anosov flow.*

Still, these connected components seem a priori very particular. The following question emerge naturally.

**Question.** *Suppose  $f$  is a discretized Anosov flow. Does any of the following conditions: i)  $f$  is transitive, ii)  $E_f^c$  is uniquely integrable or iii)  $\varphi_t^c$  is not orbit equivalent to a suspension flow; implies that  $f$  lies in the same connected component of  $\text{PH}_{c=1}(M)$  than the time 1 map of an Anosov flow?*

#### 1.1.4 Center fixing characterization

A characterization of discretized Anosov flows, which also serves as an alternative definition for the class, is the following:

**Proposition.** *Suppose  $f \in \text{PH}_{c=1}(M)$ . The following are equivalent:*

- (i) *The map  $f$  is a discretized Anosov flow.*

(ii) There exists a center foliation  $\mathcal{W}^c$  and a constant  $L > 0$  such that  $f(x) \in \mathcal{W}_L^c(x)$  for every  $x \in M$ .

It is natural to ask whether the bounded displacement hypothesis in (ii) is needed. That is, whether discretized Anosov flows can be characterized as the class of partially hyperbolic diffeomorphisms individually fixing each leaf of a one-dimensional center foliation:

**Question.** Suppose  $f \in \text{PH}_{c=1}(M)$  admits a center foliation  $\mathcal{W}^c$  such that  $f(W) = W$  for every leaf  $W \in \mathcal{W}^c$ . Is  $f$  a discretized Anosov flow?

A positive answer to this question is given whenever  $f$  is transitive and dynamically coherent:

**Theorem.** Suppose  $f \in \text{PH}_{c=1}(M)$  is a transitive dynamically coherent map such that  $f(W) = W$  for every leaf  $W$  in the center foliation  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ . Then  $f$  is a discretized Anosov flow.

In fact, the above theorem is true if one replaces the hypothesis ‘ $f$  transitive’ with the more general ‘ $\mathcal{W}^c$  transitive’. See Chapter 6 for more details.

### 1.1.5 Uniqueness of attractor

The results presented in this section are the product of a joint work with N. Guelman (see [GM22]).

Exploring further on the dynamical properties of discretized Anosov flows we focus our attention on the problem of finiteness and uniqueness of *quasi-attractors* and *quasi-repellers*.

Given a map  $f : M \rightarrow M$  one denotes by  $\mathcal{R}(f) \subset M$  the *chain recurrent set* of  $f$ . That is, the union of all points  $x \in M$  such that there exists a non-trivial  $\epsilon$ -pseudo orbit from  $x$  to  $x$  for every  $\epsilon > 0$ . It coincides with the complement of all points contained in a wandering region of the form  $U \setminus f(\overline{U})$  for some open set  $U$  such that  $\overline{f(U)} \subset U$ . One considers  $\mathcal{R}(f)$  divided in equivalent classes, called *chain recurrence classes*. These classes are given by the relation  $x \sim y$  if and only if for every  $\epsilon > 0$  there exist a non-trivial  $\epsilon$ -pseudo orbit from  $x$  to  $y$ , and from  $y$  to  $x$ .

A *quasi-attractor* is a chain recurrence class  $A$  for which there exists a base of neighborhoods  $\{U_i\}_i$  (i.e.  $A \subset U_i$  and  $A = \bigcap_i U_i$ ) such that  $\overline{f(U_i)} \subset U_i$  for every  $i$ . Every homeomorphism in a compact metric space admits at least one quasi-attractor. See [CP15] for a reference.

Since quasi-attractors are pairwise disjoint compact sets saturated by  $\mathcal{W}^u$ -leaves, each one of them contains at least one minimal set for the foliation  $\mathcal{W}^u$ . We call a minimal set of  $\mathcal{W}^u$  a *minimal unstable laminations*.



Thus, uniqueness (resp. finiteness) of minimal unstable laminations implies uniqueness (resp. finiteness) of quasi-attractors.

Finiteness of minimal unstable laminations is shown in [CPS17] for a  $C^1$ -open and dense subset of partially hyperbolic diffeomorphisms with one-dimensional center. Here we aim to a more global (non-perturbative) study involving uniqueness/finiteness results for whole classes of examples.

Discretized Anosov flows with arbitrary number of attractors and repellers can be constructed by perturbing the time 1 map of an Anosov diffeomorphism's suspension  $\varphi_t : M \rightarrow M$ . As  $M$  fibers over the circle and  $\varphi_1$  preserves fibers one can perturb  $\varphi_1$  so that it becomes Morse-Smale or even a dynamics with infinitely many quasi-attractors in the base (see Example 7.2.1 for details).

Recall that two flows are said to be *orbit equivalent* whenever there exists a homeomorphism taking orbits of one into orbits of the other and preserving its orientation. We obtain:

**Theorem.** *Let  $f$  be a discretized Anosov flow and let  $\varphi_t^c$  be the center flow of  $f$ . Suppose  $\varphi_t^c$  is transitive and not orbit equivalent to a suspension. Then  $f$  has a unique minimal unstable lamination.*

**Corollary.** *Any  $f$  as in the previous theorem has at most one quasi-attractor.*

It is worth pointing out that we look at minimal unstable laminations and quasi-attractors but the results have obvious analogous statements for minimal stable laminations and quasi-repellers.

Note that the above statements apply to every  $f \in \text{PH}_{c=1}(M)$  in the same connected component of  $\text{PH}_{c=1}(M)$  than the time 1 map of a transitive Anosov flow  $\varphi_t : M \rightarrow M$  that is not orbit equivalent to a suspension flow. This includes, for example, the time 1 map of any geodesic flow in the unitary tangent bundle of a closed surface of negative curvature.

The non-wandering set of the topological Anosov flow  $\varphi_t^c$  obtained as the center flow of a discretized Anosov flow admits the same type of *spectral decomposition*

$$\Omega(\varphi_t^c) = \Lambda_1 \cup \dots \cup \Lambda_N$$

into *basic pieces*  $\{\Lambda_i\}_{1 \leq i \leq N}$  as is the case for classical Anosov flows. See Lemma 7.4.2.

In case  $\varphi_t^c$  is not transitive the problem of uniqueness and finiteness reduces to study the behavior of  $\varphi_t^c$  in restriction to its attracting basic pieces. We obtain:

**Theorem.** *Let  $f$  be a discretized Anosov flow. Suppose  $\Lambda$  is an attracting basic piece of the center flow  $\varphi_t^c$ . If  $\varphi_t^c|_\Lambda : \Lambda \rightarrow \Lambda$  is not orbit equivalent to a suspension then  $\Lambda$  contains a unique minimal unstable lamination of  $f$ .*

**Corollary.** *Let  $f$  be a discretized Anosov flow. Suppose that all the attracting basic pieces  $\Lambda_1, \dots, \Lambda_k$  of  $\varphi_t^c$  satisfy that  $\varphi_t^c|_{\Lambda_i} : \Lambda_i \rightarrow \Lambda_i$  is not orbit equivalent to a suspension. Then  $f$  has exactly  $k$  minimal unstable laminations (and exactly  $k$  quasi-attractors). Moreover, each one of them is contained in one of the attracting basic pieces  $\Lambda_1, \dots, \Lambda_k$ .*

In addition to the above statements concerning discretized Anosov flows, a result on uniqueness of attractors is also shown for certain *partially hyperbolic skew-products*. See Theorem 7.1.5.

### 1.1.6 Partially hyperbolic skew-products, uniformly compact center foliations and quasi-isometrically center action

The center foliation of a partially hyperbolic diffeomorphism is said to be *uniformly compact* if its leaves are compact and their volume is uniformly bounded in  $M$ . In particular, this includes *partially hyperbolic skew-products* (also called *fibred partially hyperbolic diffeomorphism*) where the center foliation induces a fiber bundle structure on  $M$ :

**Definition 1.1.2.** We say that  $f \in \text{PH}(M)$  is a *partially hyperbolic skew-product* if there exists a continuous fiber bundle  $\pi : M \rightarrow B$  whose fibers are  $C^1$  compact submanifold tangent to  $E^c$  forming an  $f$ -invariant center foliation  $\mathcal{W}^c$ .

An analogous statement to the  $C^1$  openness and closeness of discretized Anosov flows is satisfied for this class of systems:

**Theorem.** *The set of diffeomorphisms in  $\text{PH}_{c=1}(M)$  admitting an invariant uniformly compact center foliation form a  $C^1$  open and closed subset of  $\text{PH}_{c=1}(M)$ .*

Again, the above theorem shows that the diffeomorphisms in  $\text{PH}_{c=1}(M)$  admitting an invariant uniformly compact center foliation comprise whole connected components of  $\text{PH}_{c=1}(M)$ . Moreover, two maps in the same connected component need also be leaf-conjugate (see Corollary 5.1.4).

A key property for discretized Anosov flows turns out to be that bounded segments inside  $\mathcal{W}^c$ -leaves do not get arbitrarily long for past or future iterates of  $f$ . This property sets an essential bridge between the class of discretized Anosov flows and that of partially hyperbolic systems admitting a uniformly compact center foliation.

**Definition 1.1.3.** Suppose  $f \in \text{PH}(M)$  admits an  $f$ -invariant center foliation  $\mathcal{W}^c$ . We say that  $f$  *acts quasi-isometrically on  $\mathcal{W}^c$*  if there exist constants  $l, L > 0$  such that

$$f^n(\mathcal{W}_l^c(x)) \subset \mathcal{W}_L^c(f^n(x))$$

for every  $x$  in  $M$  and  $n \in \mathbb{Z}$ .

It is immediate to check that the above property is satisfied by every discretized Anosov flow as well as by systems admitting a uniformly compact center foliation (see Remark 3.4.2 and Remark 3.4.3). We obtain that some properties shown for discretized Anosov flows extend to systems acting quasi-isometrically on a center foliation:

**Theorem.** *Suppose  $f \in \text{PH}_{c=1}(M)$  acts quasi-isometrically on an  $f$ -invariant center foliation  $\mathcal{W}^c$ . Then the following properties hold:*

1. *(Dynamical coherence). The map  $f$  is dynamically coherent. Moreover, it admits a center-stable foliation  $\mathcal{W}^{cs}$  and a center-unstable foliation  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .*
2. *(Uniqueness of foliations). The foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  are the only  $f$ -invariant foliations tangent to  $E^s \oplus E^c$  and  $E^c \oplus E^u$ , respectively.*
3. *(Completeness of leaves). The leaves of  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  satisfy that  $\mathcal{W}^{cs}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^s(y)$  and  $\mathcal{W}^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^u(y)$  for every  $x \in M$ .*

Item (1) of the above theorem was shown in [BB16, Theorem 1] for systems admitting a uniformly compact center (and for any center dimension). Nevertheless, the proof given in this text is independent.

Item (2) shows that  $\mathcal{W}^c$  is the only  $f$ -invariant center foliation where  $f$  acts quasi-isometrically. For uniformly compact center foliations this gives a partial answer to [BB16, Question 8.4.] (the general question is for any center dimension).

An analogous result on unique integrability of the center bundle is also satisfied in this context:

**Proposition.** *Suppose  $f \in \text{PH}_{c=1}(M)$  admits a uniformly compact center foliation such that  $E^c$  is uniquely integrable. Then every systems in the same  $C^1$  connected component of  $f$  in  $\text{PH}_{c=1}(M)$  has a uniquely integrable center bundle.*

In particular, the above proposition shows that if  $f = A \times \text{Id} : N \times S^1 \rightarrow N \times S^1$  is the product of an Anosov diffeomorphism  $A : N \rightarrow N$  and the identity map on the circle  $\text{Id} : S^1 \rightarrow S^1$ , then the center bundle is uniquely integrable for every system in the same  $C^1$  connected component as  $f$  in  $\text{PH}_{c=1}(N \times S^1)$ .

## 1.2 Introduction (Français)

### 1.2.1 Dynamique différentiable, hyperbolicité et hyperbolicité partielle

La théorie classique des systèmes dynamiques étudie le comportement asymptotique de systèmes évoluant dans le temps par rapport à une règle déterministe prescrite qui gouverne leur évolution.

Dans le cas de dynamiques différentiables, cette règle est typiquement donnée par un difféomorphisme  $f : M \rightarrow M$  dans le cas d'un *système dynamique discret*, ou par une équation différentielle ordinaire  $\dot{X} = F(X)$  définissant un flot  $X_t : M \rightarrow M$  dans le cas d'un *système dynamique continu*.

Un exemple paradigmatique de ces systèmes présentant un comportement global riche et *chaotique* est donné par les systèmes de *Anosov* (également appelés *systèmes globalement uniformément hyperboliques*). L'étude de ce type de systèmes remonte au moins aux travaux pionniers de D.V. Anosov et S. Smale (voir [A67] et [S67]).

En termes succincts, un difféomorphisme  $f : M \rightarrow M$  est appelé *difféomorphisme d'Anosov* si le fibré tangente admet une décomposition  $f$ -invariante  $TM = E^s \oplus E^u$ , telle que les vecteurs dans  $E^s$  et  $E^u$  sont uniformément contractés par les itérations futures et passées de  $f$ , respectivement.

Un flot  $X_t : M \rightarrow M$  sans singularité est appelé *flot d'Anosov* s'il préserve une décomposition invariante par la différentielle du flot  $TM = E^s \oplus E^c \oplus E^u$ , telle que les vecteurs dans  $E^s$  et  $E^u$  sont uniformément contractés par des itérés positifs et négatifs de  $X_t$ , respectivement, et le fibré  $E^c$  est la direction tangente au flot  $X_t$ .

Une extension naturelle des systèmes uniformément hyperboliques est donnée par la notion d'*hyperbolicité partielle*.

**Definition.** Un difféomorphisme  $f : M \rightarrow M$  sur une variété Riemannienne fermée  $M$  est appelé *partiellement hyperbolique* s'il existe une décomposition continue et invariante par  $Df$  du fibré tangente  $TM$  en trois sous-fibrés non triviaux

$$TM = E^s \oplus E^c \oplus E^u$$

telles que les vecteurs dans  $E^s$  et  $E^u$  sont uniformément contractés par les itérations futures et passées de  $f$ , respectivement, et les vecteurs dans  $E^c$  subissent un comportement intermédiaire. Voir le chapitre 2 pour une définition précise.

La définition des difféomorphismes partiellement hyperboliques remonte au moins à [BP74] et [HPS77]. Il est bon de mentionner qu'il existe de nombreuses autres définitions de l'*hyperbolicité partielle*. Typiquement, toutes ces

notions impliquent une certaine décomposition invariante de l'espace tangent (de la variété entière ou d'une partie invariante de celle-ci) en sous-fibrés satisfaisant une forme de *domination*. De plus, dans la plupart des cas, on demande qu'au moins un des sous-fibrés soit uniformément dilatée ou contractée. Voir par exemple [CP15] comme référence.

Tout comme dans le cas des systèmes d'Anosov, l'hyperbolicité partielle est une propriété  $C^1$  ouverte qui peut être vérifiée en un nombre fini d'itérations (comme conséquence du *critère de champs de cônes*, par exemple). De plus, elle apparaît naturellement dans l'étude des propriétés dynamiques robustes, autrement dit, des propriétés qui restent inchangées face à de petites perturbations du système. Cela est le cas dans au moins deux scénarios significatifs : la *transitivité robuste* et l'*ergodicité stable*. Voir, par exemple, [BPSW01], [CHHU18] et [HP18].

Les difféomorphismes partiellement hyperboliques constituent une classe riche de systèmes dynamiques avec une rigidité suffisante pour qu'un certain type de 'classification' puisse être attendu :

**Problème.** *Développer un contexte de classification (au moins en dimension 3) pour les difféomorphismes partiellement hyperboliques. Déterminer quelles propriétés dynamiques peuvent être données pour ces systèmes et identifier ceux qui sont robustes. Explorer l'interaction entre les difféomorphismes partiellement hyperboliques et la géométrie et la topologie de la variété sous-jacente, en particulier déterminer quelles variétés et classes d'isotopie admettent des systèmes partiellement hyperboliques.*

Une façon d'aborder ce problème est d'étudier les structures invariantes préservées par de telles applications, telles que les feuilletages invariants qui y apparaissent naturellement. On peut espérer obtenir des conséquences topologiques ou dynamiques du comportement de ces structures invariantes, et vice versa.

En dimension 3, le problème de la classification a été particulièrement étudié. Les exemples classiques de difféomorphismes partiellement hyperboliques en dimension 3 sont :

- *Las déformations des difféomorphismes d'Anosov.*
- *Les produits fibrés partiellement hyperboliques.*
- *Las perturbations du temp 1 d'un flot d'Anosov.*

Succinctement, la première classe comprend les difféomorphismes partiellement hyperboliques homotopes à un difféomorphisme d'Anosov, la seconde les difféomorphismes partiellement hyperboliques tels que  $E^c$  s'intègre

en un feuilletage invariant  $\mathcal{W}^c$  par feuilles compactes qui induit une structure de fibré sur  $M$  (voir la Section 1.1.6), et la troisième, les difféomorphismes suffisamment proches du temps 1 d'un flot d'Anosov.

Bien que la liste précédente ait été conjecturée comme couvrant tous les systèmes partiellement hyperboliques de dimension 3 (voir la *conjecture de Pujals* dans [BW05]), la liste des exemples classiques a récemment été complétée par de nouveaux types d'exemples qui défient leur tentatives de classification. Dans [HHU16], les premiers exemples non-dynamiquement cohérents ont été construits. Plus tard, dans [BPP16], [BGP16] et [BGHP17], de nouveaux exemples surprenants sont apparus. Voir également [BFP20].

En dimension supérieure, une façon d'aborder l'étude est de se limiter au cas où  $\dim(E^c) = 1$ . Sous cette hypothèse, la liste d'exemples classiques mentionnée auparavant est essentiellement la même (il suffit d'ajouter la possibilité de réaliser un produit avec un difféomorphisme d'Anosov).

La classe des *flots d'Anosov discrétisés* est conçue comme une extension naturelle du troisième type d'exemples classiques. L'objectif principal de cette thèse est d'établir plusieurs propriétés générales pour cette classe de systèmes en toute dimension.

## 1.2.2 Flots d'Anosov discrétisés

On désignera par  $\text{PH}(M)$  l'ensemble des difféomorphismes partiellement hyperboliques sur  $M$  et par  $\text{PH}_{c=1}(M)$  ceux tels que  $\dim(E^c) = 1$ .

**Definition 1.2.1.** On dit que  $f \in \text{PH}_{c=1}(M)$  est un *flot d'Anosov discrétisé* s'il existe un feuilletage orientable  $\mathcal{W}^c$  dont les feuilles sont des sous-variétés  $C^1$  de  $M$  tangentes à  $E^c$ , et une fonction continue  $\tau : M \rightarrow \mathbb{R}_{>0}$  telle que

$$f(x) = \varphi_{\tau(x)}^c(x)$$

pour chaque  $x \in M$ , où  $\varphi_t^c : M \rightarrow M$  désigne un flot de vitesse unitaire dont les orbites sont les feuilles de  $\mathcal{W}^c$ .

L'exemple prototypique d'un flot d'Anosov discrétisé est le temps 1 d'un flot d'Anosov et toutes ses perturbations  $C^1$  suffisamment petites. Ce dernier point est une conséquence de [HPS77] et sera revu dans ce texte.

Le terme *flot d'Anosov discrétisé* a été introduit dans [BFFP19] et découle du fait que le flot  $\varphi_t^c$  est nécessairement un flot d'Anosov topologique (voir la Définition 3.7.1). Autrement dit,  $f$  peut être considéré comme une discrétisation du flot d'Anosov topologique  $\varphi_t^c$ .

Les flots d'Anosov discrétisés ont été largement étudiés dans la littérature, mais pas toujours sous le même nom. Un de ces cas est celui de [BD96] où les premiers exemples de difféomorphismes robustement transitifs isotopiques

à l'identité ont été produits. Ces exemples sont construits arbitrairement proches du temps 1 de tout flot d'Anosov. En particulier, ce sont des flots d'Anosov discrétisés.

En ce qui concerne l'ergodicité stable, il est démontré dans [GPS94] que les temps 1 de flots géodésiques sur des surfaces fermées de courbure négative constante sont  $C^2$  stablement ergodiques. Ceux-ci ont été les premiers exemples de difféomorphismes stablement ergodiques non-Anosov construits dans la littérature. Encore une fois, ces exemples sont des flots d'Anosov discrétisés (ainsi que leurs perturbations).

Dans [BW05], il a été montré que les flots d'Anosov discrétisés et les produits fibrés partiellement hyperboliques sont assez abondants parmi les difféomorphismes partiellement hyperboliques (dynamiquement cohérents) en dimension 3. Cela a conduit à la consolidation des exemples classiques en dimension 3.

Plus récemment, il a été montré dans [BFFP19] que les flots d'Anosov discrétisés représentent tous les difféomorphismes partiellement hyperboliques dynamiquement cohérents dans de nombreuses 3-variétés. Et dans [FP22] (voir aussi [FP21]) que dans la plupart des 3-variétés les flots d'Anosov discrétisés sont accessibles et ceux qui préservent une forme de volume sont ergodiques.

D'autres résultats dynamiques récents concernant des flots d'Anosov discrétisés sont les résultats de rigidité de [AVW15], [BFT20] sur les mesures d'entropie maximale, [DWX21] et [BG21] relatifs à la rigidité des centralisateurs, et le 'principe d'invariance' démontré dans [CP22].

Une caractéristique qui s'est avérée très utile dans l'étude des difféomorphismes partiellement hyperboliques est la présence de feuilletages invariants tangents aux sous-fibrés invariants. Dans [HPS77], il a été montré que les fibrations  $E^s$  et  $E^u$  sont uniquement intégrables à des feuilletages  $f$ -invariants  $\mathcal{W}^s$  et  $\mathcal{W}^u$ , respectivement.

Par contraste, les fibrés  $E^s \oplus E^c$  et  $E^s \oplus E^c$  peuvent ou non être intégrables. Lorsqu'ils s'intègrent en des feuilletages  $f$ -invariants (notés  $\mathcal{W}^{cs}$  et  $\mathcal{W}^{cu}$ , respectivement), le difféomorphisme  $f$  est qualifié de *dynamiquement cohérent*. Lorsque c'est le cas, alors le feuilletage  $\mathcal{W}^c := \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  (formé par les composantes connexes de l'intersection des feuilles de  $\mathcal{W}^{cs}$  et de  $\mathcal{W}^{cu}$ ) donne lieu à un feuilletage  $f$ -invariant tangent au fibré central  $E^c$ .

Rappelons que l'ensemble non errant  $\Omega(f)$  de  $f$  est l'ensemble des points  $x$  dans  $M$  tels que pour tout voisinage  $U$  de  $x$  il existe  $N > 0$  tel que  $f^N(U) \cap U \neq \emptyset$ .

Dans ce premier théorème, on établit quelques propriétés générales satisfaites par tout flot d'Anosov discrétisé :

**Théorème.** *Soit  $f$  un flot d'Anosov discrétisé. Soit  $\varphi_t^c$  et  $\mathcal{W}^c$  le flot et le feuilletage central apparaissant dans la définition de  $f$ , respectivement. Alors :*

1. *(Flot d'Anosov topologique). Le flot  $\varphi_t^c$  est un flot d'Anosov topologique (voir la Définition 3.7.1).*
2. *(Cohérence dynamique). La fonction  $f$  est dynamiquement cohérente, admettant un feuilletage centre stable  $\mathcal{W}^{cs}$  et un feuilletage centre instable  $\mathcal{W}^{cu}$  tels que  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .*
3. *(Unicité des feuilletages cs et cu). Les feuilletages  $\mathcal{W}^{cs}$  et  $\mathcal{W}^{cu}$  sont les uniques feuilletages  $f$ -invariants tangents à  $E^s \oplus E^c$  et  $E^c \oplus E^u$ , respectivement.*
4. *(Complétude des feuilles). Les feuilles de  $\mathcal{W}^{cs}$  et  $\mathcal{W}^{cu}$  satisfont  $\mathcal{W}^{cs}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^s(y)$  et  $\mathcal{W}^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^u(y)$  pour tout  $x \in M$ .*
5. *(Topologie des feuilles) Les feuilles de  $\mathcal{W}^{cs}$  et  $\mathcal{W}^{cu}$  sont homéomorphes à des plans ou des cylindres. Les premiers ne contiennent aucune feuille centrale compacte, alors que les deuxièmes en contiennent exactement une.*

Il est bon de noter qu'en dimension 3, le théorème ci-dessus était en grande partie connu. En effet, une fois que (2) est prouvé, alors (1), (4) et (5) découlent de [BW05, Theorem 2]. D'autre part, une fois que (1) est prouvé, alors (2) est déjà apparu dans [BFP20, Proposition G.2] et (3) découle de [BFFP19] (voir [BG21, Lemma 1.1]). Notre objectif a été de synthétiser la théorie en dimension 3 et de la généraliser aux dimensions supérieures où moins de résultats avaient été établis dans la littérature.

[BFFP19], [BFP20], [BG21] et [GM22] appellent 'flot d'Anosov discrétisé' toute difféomorphisme  $f \in \text{PH}_{c=1}(M)$  telle qu'il existe un flot d'Anosov topologique  $\varphi_t : M \rightarrow M$  et une fonction continue  $\tau : M \rightarrow \mathbb{R}_{>0}$  tels que  $f(x) = \varphi_{\tau(x)}(x)$  pour tout  $x \in M$ .

Dans [BFT20], un difféomorphisme  $f \in \text{PH}_{c=1}(M)$  est appelé 'diffeomorphisme de type flot' s'il satisfait la Définition 1.2.1 et s'il est aussi dynamiquement cohérent et admet au moins une feuille compacte dans  $\mathcal{W}^c$ .

Des notions similaires mais a priori non identiques ont également été étudiées dans [BW05], [BG09] et [BG10].

Le problème de la relation entre ces notions s'est posé naturellement. Avec le théorème précédent (et aussi la Proposition 3.7.3 item (4) énoncé dans la Section 3.7) on obtient que toutes ces définitions sont équivalentes et se réfèrent à la même classe de systèmes partiellement hyperboliques :



**Corollaire.** *La définition de flot d’Anosov discrétisé qui se trouve dans [BFFP19], [BFP20], [BG21] et [GM22], et la définition de difféomorphisme partiellement hyperbolique de type flot dans [BFT20], sont équivalentes à la définition 1.1.1.*

*De plus, la classe des difféomorphismes partiellement hyperboliques étudiés dans [BW05, Théorème 2.], [BG09] et [BG10] sont également des flots d’Anosov discrétisés comme dans la définition 1.1.1.*

### 1.2.3 Stabilité globale

Le résultat suivant montre que les flots d’Anosov discrétisés constituent, d’une certaine manière, une large classe de difféomorphismes partiellement hyperboliques à centre unidimensionnel :

**Théorème.** *L’ensemble des flots d’Anosov discrétisés est un sous-ensemble  $C^1$  ouvert et fermé de  $\text{PH}_{c=1}(M)$ .*

Autrement dit, la classe des flots d’Anosov discrétisés constitue des composantes connexes entières de  $\text{PH}_{c=1}(M)$ .

Deux difféomorphismes partiellement hyperboliques et feuilletages centraux invariants respectifs  $(f, \mathcal{W}_f^c)$  et  $(g, \mathcal{W}_g^c)$  sont dits *conjugés par feuilles* s’il existe un homéomorphisme  $h : M \rightarrow M$  qui envoie les feuilles de  $\mathcal{W}_f^c$  sur les feuilles de  $\mathcal{W}_g^c$  de telle sorte que  $h \circ f(W) = g \circ h(W)$  pour chaque feuille  $W \in \mathcal{W}_f^c$ . La conjugaison par les feuilles donne sens à une façon de classer les difféomorphismes partiellement hyperboliques (dynamiquement cohérents) modulo le comportement central : deux systèmes sont considérés comme équivalents s’ils sont conjugés par les feuilles.

La preuve du théorème précédent montre également que la conjugaison par feuilles est préservée le long des composantes connexes des flots d’Anosov discrétisés :

**Corollaire.** *Deux flots d’Anosov discrétisés dans la même composante connexe  $C^1$  de  $\text{PH}_{c=1}(M)$  sont conjugés par feuilles.*

Il vaut la peine de mentionner le contexte dans lequel s’inscrivent les énoncés précédents.

Un résultat classique de [HPS77] donne des conditions pour la stabilité des feuilletages normalement hyperboliques (c’est-à-dire des feuilletages qui sont tangents au fibré central d’un difféomorphisme partiellement hyperbolique) :

**Théorème (Hirsch-Pugh-Shub).** *Supposons que  $f \in \text{PH}(M)$  admet un feuilletage central  $f$ -invariant  $\mathcal{W}_f^c$ . Si la paire  $(f, \mathcal{W}_f^c)$  est expansive par plaques, alors il existe  $\mathcal{U}(f)$  un voisinage  $C^1$  de  $f$  qui satisfait que chaque  $g \in \mathcal{U}(f)$  admet un feuilletage central  $g$ -invariant  $\mathcal{W}_g^c$  tel que  $(f, \mathcal{W}_f^c)$  et  $(g, \mathcal{W}_g^c)$  sont conjugés par feuilles.*

Une paire  $(f, \mathcal{W}^c)$  est appelé *expansive par plaques* (ou  $\delta$ -*expansive par plaques*) s'il existe  $\delta > 0$  tel que toute paire de séquences  $(x_n)_{\mathbb{Z}}$  et  $(y_n)_{\mathbb{Z}}$  satisfaisant  $x_{n+1} \in \mathcal{W}_\delta^c(f(x_n))$ ,  $y_{n+1} \in \mathcal{W}_\delta^c(f(y_n))$  et  $d(x_n, y_n) < \delta$  pour chaque  $n \in \mathbb{Z}$  satisfait également que  $y_0 \in \mathcal{W}_{loc}^c(x_0)$ . [HPS77] montre que cette condition est satisfaite chaque fois que  $\mathcal{W}^c$  est une feuilletage  $C^1$  ou lorsque  $f$  est une isométrie entre chaque feuille  $W$  de  $\mathcal{W}^c$  et son image  $f(W)$  (ceci est le cas pour le temps 1 d'un flot d'Anosov, par exemple).

Afin de prouver la propriété  $C^1$  ouverte et fermée des flots d'Anosov discrétisés, on se sert d'une certaine 'version uniforme' du théorème de stabilité précédent :

**Théorème.** *Supposons que  $f_0 \in \text{PH}_{c=1}(M)$ . Pour chaque  $\delta > 0$  il existe un  $C^1$  voisinage  $\mathcal{U}(f_0)$  de  $f_0$  tel que, si un certain  $f \in \mathcal{U}(f_0)$  admet un feuilletage central  $\mathcal{W}_f^c$  tel que  $(f, \mathcal{W}_f^c)$  est  $\delta$ -expansive par plaques, alors chaque  $g \in \mathcal{U}(f_0)$  admet un feuilletage central  $g$ -invariante  $\mathcal{W}_g^c$  telle que  $(f, \mathcal{W}_f^c)$  et  $(g, \mathcal{W}_g^c)$  sont conjugués par feuilles.*

La partie 'uniforme' du théorème précédent porte sur la taille de  $\mathcal{U}(f_0)$ , fixée à l'avance. Par conséquence, si  $(f_n)_n$  est une suite dans  $\text{PH}_{c=1}(M)$  convergeant vers  $f_0$ , il suffit de montrer qu'une paire  $(f_N, \mathcal{W}_{f_N}^c)$  est  $\delta$ -expansive par plaques pour un certain  $f_N \in \mathcal{U}(f_0)$  pour induire un feuilletage central  $\mathcal{W}_{f_0}^c$  pour  $f_0$ , qui satisfait ensuite que  $(f_0, \mathcal{W}_{f_0}^c)$  est conjuguée par feuilles à  $(f_N, \mathcal{W}_{f_N}^c)$  (et en effet conjuguée par feuilles à une paire  $(f, \mathcal{W}_f^c)$  pour tout  $f \in \mathcal{U}(f_0)$ ).

Un théorème de stabilité uniforme similaire a été observé originellement dans [BFP20] dans un contexte différent mais voisin (pour les *immersions*  $C^1$  par feuilles et pour les *feuilletages branchés*). [BFP20] en déduit qu'une classe de difféomorphismes appelé des *flots d'Anosov effondrés* a la propriété d'être  $C^1$  ouverte et fermée. La propriété  $C^1$  ouverte et fermée des flots d'Anosov discrétisés en dimension 3 est essentiellement déduite de [BFP20] (voir le Chapitre 5 pour plus de détails).

Le fait que la conjugaison par feuilles persiste parmi les composantes connexes des flots d'Anosov discrétisés peut être considéré comme un résultat de 'stabilité globale' où une paire expansive par plaques  $(f, \mathcal{W}^c)$  induit conjugaison par feuilles dans toute sa composante connexe  $C^1$  de systèmes partiellement hyperboliques.

Ce phénomène a également été observé dans [FPS14] pour tout automorphisme linéaire hyperbolique  $f$  sur le tore  $\mathbb{T}^n$  (vu comme un difféomorphisme partiellement hyperbolique), et a été généralisé dans [Pi19] pour des automorphismes linéaires hyperboliques dans des nilvariétés. De plus, il est montré dans ce texte (voir Section 1.1.6) que ce comportement se produit également

pour les produits fibrés partiellement hyperboliques avec une centrale unidimensionnelle.

Il est naturel de se demander si cela est vrai en général :

**Question.** *Supposons que  $f \in \text{PH}(M)$  admet un feuilletage central  $f$ -invariant  $\mathcal{W}^c$  tel que  $(f, \mathcal{W}^c)$  est expansive par plaques. Est-ce que chaque  $g$  dans la même composante connexe des difféomorphismes  $C^1$  partiellement hyperbolique que  $f$  admet un feuilletage central  $g$ -invariant  $\mathcal{W}_g^c$  tel que  $(g, \mathcal{W}_g^c)$  est expansive par plaques et conjugué par feuilles à  $(f, \mathcal{W}^c)$  ?*

Puisque les flots d’Anosov discrétisés constituent des composantes connexes entières de  $\text{PH}_{c=1}(M)$ , plusieurs autres questions naturelles peuvent être posées. On peut se demander quelles propriétés sont préservées dans la totalité de la composante connexe et lesquelles ne le sont pas. Quelles composantes connexes contiennent le temps 1 d’un flot d’Anosov, quelles sont les différences entre celles qui ne le contiennent pas, etc.

Comme mentionné antérieurement, une des propriétés qui est conservée à travers les composantes connexes des flots d’Anosov discrétisés est la conjugaison par feuilles. Une autre est l’intégrabilité unique du fibré central, c’est à dire la propriété selon laquelle, a reparamétrisation près, il existe une unique courbe  $C^1$  locale tangente à  $E^c$  passant par chaque point de  $M$  (voir la Section 1.1.6 pour plus de détails sur cette définition) :

**Proposition.** *Soit  $f$  un flot d’Anosov discrétisé tel que  $E_f^c$  est uniquement intégrable. Alors  $E_g^c$  est uniquement intégrable pour tout  $g$  dans la même  $C^1$  composante connexe de  $\text{PH}_{c=1}(M)$  que  $f$ .*

En particulier, tout flot d’Anosov discrétisé dans la même composante connexe que le temps 1 d’un flot d’Anosov a un fibré central uniquement intégrable.

Dans l’Exemple 5.4.3, on donne un exemple d’un flot d’Anosov discrétisé  $f$  tel que  $E^c$  n’est pas uniquement intégrable. Ceci est obtenu à partir d’une simple modification d’une construction donnée dans [HHU16] conduisant à un 2-tore tangent à  $E^s \oplus E^c$  entièrement composé de points d’intégrabilité non-unique pour  $E^c$ . Le flot central  $\varphi_t^c$  dans cet exemple est orbitalement équivalent à la suspension d’un difféomorphisme linéaire d’Anosov  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  sur le 2-tore, cependant par la proposition ci-dessus la fonction  $f$  n’est pas dans la même composante connexe que le temps 1 de la suspension de  $A$ . On obtient la conclusion suivante.

**Corollaire.** *Il existe des composantes connexes de flots d’Anosov discrétisés qui ne contiennent pas le temps 1 d’un flot d’Anosov.*

Ces composantes connexes semblent encore très particulières. La question suivante se pose naturellement.

**Question.** *Si l'une des conditions suivantes est remplie : (i)  $f$  est transitif, (ii)  $E_f^c$  est uniquement intégrable ou (iii)  $\varphi_t^c$  n'est pas orbitalement équivalent à un flot de suspension ; alors est-ce que  $f$  se trouve dans la même composante connexe de  $\text{PH}_{c=1}(M)$  que le temps 1 d'un flot d'Anosov ?*

#### 1.2.4 Feuilles centrales fixes

Une caractérisation des flots d'Anosov discrétisés, qui sert également de définition alternative pour la classe, est la suivante :

**Proposition.** *Soit  $f \in \text{PH}_{c=1}(M)$ . Ces affirmations sont équivalentes :*

- (i) *Le difféomorphisme  $f$  est un flot d'Anosov discrétisé.*
- (ii) *Il existe un feuilletage central  $W^c$  et une constante  $L > 0$  tels que  $f(x) \in \mathcal{W}_L^c(x)$  pour tout  $x \in M$ .*

Il est naturel de se demander si l'hypothèse de déplacement limité dans (ii) est nécessaire. Plus précisément, on cherche à savoir si les flots d'Anosov discrétisés peuvent être caractérisés comme la classe des difféomorphismes partiellement hyperboliques qui fixent individuellement chaque feuille d'un feuilletage central unidimensionnel :

**Question.** *Supposons que  $f$  dans  $\text{PH}_{c=1}(M)$  admet un feuilletage central  $W^c$  tel que  $f(W) = W$  pour chaque feuille  $W$  dans  $W^c$ . Est-ce que  $f$  est un flot d'Anosov discrétisé ?*

Une réponse affirmative à cette question est obtenue lorsque  $f$  est transitif et dynamiquement cohérent :

**Théorème.** *Soit  $f \in \text{PH}_{c=1}(M)$  transitif et dynamiquement cohérent tel que  $f(W) = W$  pour chaque feuille  $W$  du feuilletage central  $W^c = W^{cs} \cap W^{cu}$ . Alors  $f$  est un flot d'Anosov discrétisé.*

En fait, le théorème précédent est vrai si on remplace l'hypothèse '  $f$  transitif ' par la plus générale '  $W^c$  transitif '. Pour plus de détails, voir le Chapitre 6.

#### 1.2.5 Unicité de l'attracteur

Les résultats présentés dans cette section sont le produit d'un travail en collaboration avec N. Guelman (voir [GM22]).

En s'intéressant à d'autres propriétés dynamiques des flots d'Anosov discrétisés, on examine le problème de la finitude et de l'unicité des *quasi-attracteurs* et *quasi-répulseurs*.

Étant donné une application  $f : M \rightarrow M$ , on désigne par  $\mathcal{R}(f) \subset M$  l'ensemble récurrent par chaînes de  $f$ , c'est-à-dire, l'ensemble des points  $x$  de  $M$  tels qu'il existe une  $\epsilon$ -pseudo orbite non triviale de  $x$  à  $x$  pour tout  $\epsilon > 0$ . Cet ensemble coïncide avec le complément des points qui appartiennent à une certaine région piégeante de la forme  $U \setminus f(\overline{U})$  pour un ouvert  $U$  tel que  $f(\overline{U}) \subset U$ . On considère  $\mathcal{R}(f)$  divisé en classes d'équivalence appelées *classes de récurrence de chaîne*. Ces classes sont données par la relation  $x \sim y$  si et seulement si pour chaque  $\epsilon > 0$  il existe une  $\epsilon$ -pseudo orbite non triviale de  $x$  à  $y$ , et une autre de  $y$  à  $x$ .

Un *quasi-attracteur* est une classe de récurrence par chaînes  $A$  pour laquelle il existe une base de voisinages  $\{U_i\}_i$  (c'est-à-dire que  $A \subset U_i$  et  $A = \bigcap_i U_i$ ) telle que  $f(\overline{U_i}) \subset U_i$  pour chaque  $i$ . Tout homéomorphisme sur un espace métrique compact admet au moins un quasi-attracteur. Voir [CP15] pour une référence.

Puisque les quasi-attracteurs sont des ensembles compacts disjoints deux à deux saturés par des feuilles de  $\mathcal{W}^u$ , chacun d'eux contient au moins un ensemble minimal pour le feuilletage  $\mathcal{W}^u$ . Une union compacte de feuilles de  $\mathcal{W}^u$ , minimal pour l'inclusion, est appelée *lamination instable minimale*. Ainsi, l'unicité (resp. la finitude) des laminations instables minimales implique l'unicité (resp. la finitude) des quasi-attracteurs.

La finitude des laminations instables minimales est obtenue dans [CPS17] pour un sous-ensemble  $C^1$ -ouvert et dense des difféomorphismes partiellement hyperboliques de fibré central unidimensionnel. Ici, on se propose de poursuivre une étude plus globale (non-perturbative) comprenant des résultats d'unicité/finitude pour des classes entières d'exemples.

Les flots d'Anosov discrétisés avec un nombre arbitraire d'attracteurs et de répulseurs peuvent être construits en perturbant le temps 1 de la suspension  $\varphi_t : M \rightarrow M$  d'un difféomorphisme d'Anosov. Puisque  $M$  fibre sur le cercle et que  $\varphi_1$  préserve les fibres, il est possible de perturber  $\varphi_1$  pour qu'il préserve toujours les fibres mais devienne Morse-Smale ou qu'il possède une infinité de quasi-attracteurs pour la dynamique dans la base (voir l'Exemple 7.2.1 pour les détails).

Rappelons que deux flots sont dits *orbitalement équivalents* s'il existe un homéomorphisme qui envoie les orbites de l'un en orbites de l'autre en préservant leur orientation. On obtient le résultat suivant :

**Théorème.** *Soit  $f$  un flot d'Anosov discrétisé et soit  $\varphi_t^c$  le flot central de  $f$ . Supposons que  $\varphi_t^c$  soit transitif et non orbitalement équivalent à une suspension. Alors  $f$  a une unique lamination instable minimale.*

**Corollaire.** *es difféomorphismes comme dans le théorème précédent ont au plus un quasi-attracteur.*

Il convient de noter que les énoncés ci-dessus sont formulés pour des laminations minimales instables et des quasi-attracteurs mais que, naturellement, les mêmes énoncés ont des analogues pour des laminations minimales stables et des quasi-répulseurs.

Il est à noter que les résultats précédents s'appliquent à tout  $f \in \text{PH}_{c=1}(M)$  dans la même composante connexe de  $\text{PH}_{c=1}(M)$  que le temps 1 d'un flot d'Anosov transitif  $\varphi_t : M \rightarrow M$  qui n'est pas orbitalement équivalent à un flot de suspension. Cela inclut, par exemple, tout temps 1 d'un flot géodésique sur le fibré tangente unitaire d'une surface fermée à courbure négative.

L'ensemble non errant du flot d'Anosov topologique  $\varphi_t^c$  obtenu comme le flot central d'un flot d'Anosov discrétisé admet le même type de *décomposition spectrale*

$$\Omega(\varphi_t^c) = \Lambda_1 \cup \dots \cup \Lambda_N$$

en pièces basiques  $\{\Lambda_i\}_{1 \leq i \leq N}$  que les flots d'Anosov classiques. Voir le Lemme 7.4.2.

Dans le cas où  $\varphi_t^c$  n'est pas transitif, le problème d'unicité et de finitude se réduit à l'étude du comportement de  $\varphi_t^c$  en restriction à ses pièces basiques de type attracteurs. On obtient :

**Théorème.** *Soit  $f$  un flot d'Anosov discrétisé. Supposons que  $\Lambda$  est une pièce basique de type attracteur du flot central  $\varphi_t^c$ . Si  $\varphi_t^c|_{\Lambda}$  n'est pas orbitalement équivalent à une suspension alors  $\Lambda$  contient une unique lamination instable minimale pour  $f$ .*

**Corollaire.** *Soit  $f$  un flot d'Anosov discrétisé. Supposons que toutes les pièces basiques attracteur  $\Lambda_1, \dots, \Lambda_k$  de  $\varphi_t^c$  satisfont que  $\varphi_t^c|_{\Lambda_i}$  n'est pas orbitalement équivalent à une suspension. Alors  $f$  a exactement  $k$  laminations instables minimales (et exactement  $k$  quasi-attracteurs). De plus, chacune des laminations instables minimales est contenue dans l'une des pièces basiques de type attracteur  $\Lambda_1, \dots, \Lambda_k$ .*

Il est bon de mentionner enfin que, en plus des résultats ci-dessus pour les flots d'Anosov discrétisés, des conclusions similaires d'unicité de quasi-attracteur sont obtenues pour certains *produits fibrés partiellement hyperboliques*. Voir le Théorème 7.1.5.

### 1.2.6 Produits fibrés partiellement hyperboliques, feuilletages centraux uniformément compacts et action quasi-isométrique dans le central

Le feuilletage central d'un difféomorphisme partiellement hyperbolique est dit *uniformément compact* si ses feuilles sont compactes et si le volume de

chaque feuille est uniformément borné dans  $M$ . En particulier, ceci est le cas du feuilletage central des *produits fibrés partiellement hyperboliques* induisant une structure fibrée dans  $M$  :

**Definition 1.2.2.** On dit que  $f \in \text{PH}(M)$  est un *produit fibré partiellement hyperbolique* s'il existe un espace fibré continu  $\pi : M \rightarrow B$  dont les fibres sont des sous-variétés compactes  $C^1$  tangentes à  $E^c$  qui forment un feuilletage central  $\mathcal{W}^c$  invariant par  $f$ .

Un résultat analogue à la propriété  $C^1$  ouverte et fermée des flots d'Anosov discrétisés est valable pour cette classe de systèmes :

**Théorème.** *L'ensemble des difféomorphismes dans  $\text{PH}_{c=1}(M)$  qui admettent un feuilletage central invariant uniformément compact est un sous-ensemble  $C^1$  ouvert et fermé de  $\text{PH}_{c=1}(M)$ .*

A nouveau, le théorème précédent montre que les applications dans  $\text{PH}_{c=1}(M)$  qui admettent un feuilletage central invariant uniformément compact constituent des composantes connexes entières de  $\text{PH}_{c=1}(M)$ . De plus, deux applications dans la même composante connexe sont conjuguées par feuilles (voir Corollaire 5.1.4).

Une propriété clé des flots d'Anosov discrétisés s'avère être que les segments bornés dans les feuilles de  $\mathcal{W}^c$  ne deviennent pas arbitrairement longs par les itérations passées ou futures de  $f$ . Cette propriété établit un pont essentiel entre la classe des flots d'Anosov discrétisés et celle des systèmes partiellement hyperboliques admettant un feuilletage central uniformément compact.

**Definition 1.2.3.** Supposons que  $f$  dans  $\text{PH}(M)$  admet un feuilletage central  $f$ -invariant  $\mathcal{W}^c$ . On dit que  $f$  agit *quasi-isométriquement* sur  $\mathcal{W}^c$  s'il existe des constantes  $l, L > 0$  telles que

$$f^n(\mathcal{W}_l^c(x)) \subset \mathcal{W}_L^c(f^n(x))$$

pour chaque  $x \in M$  et  $n \in \mathbb{Z}$ .

Il est immédiat de constater que la propriété précédente est satisfaite pour tout flot d'Anosov discrétisé, ainsi que pour tout système admettant un feuilletage central uniformément compact (voir Remarque 3.4.2 et Remarque 3.4.3). On obtient que certaines propriétés démontrées pour les flots d'Anosov discrétisés s'étendent automatiquement aux systèmes agissant de manière quasi-isométrique sur un feuilletage central :

**Théorème.** *Supposons que  $f \in \text{PH}_{c=1}(M)$  agit de manière quasi-isométrique sur le feuilletage central  $f$ -invariant  $\mathcal{W}^c$ . Les propriétés suivantes sont alors satisfaites :*

1. (Cohérence dynamique). Le difféomorphisme  $f$  est dynamiquement cohérent. De plus, il admet un feuilletage centre stable  $\mathcal{W}^{cs}$  et un feuilletage centre instable  $\mathcal{W}^{cu}$  tels que  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .
2. (Unicité des feuilletages). Les feuilletages  $\mathcal{W}^{cs}$  et  $\mathcal{W}^{cu}$  sont les uniques feuilletages  $f$ -invariants tangents à  $E^s \oplus E^c$  et  $E^c \oplus E^u$ , respectivement.
3. (Complétude des feuilles). Les feuilles de  $\mathcal{W}^{cs}$  et  $\mathcal{W}^{cu}$  satisfont  $\mathcal{W}^{cs}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^s(y)$  et  $\mathcal{W}^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^u(y)$  pour chaque  $x \in M$ .

Il est bon de mentionner que le point (1) du théorème ci-dessus a été prouvé dans [BB16, Theorem 1] pour les systèmes admettant un feuilletage central uniformément compact (et pour toute dimension centrale). Cependant, la preuve donnée dans ce texte est indépendante.

Le point (2) montre que  $\mathcal{W}^c$  est le seul feuilletage central  $f$ -invariant où  $f$  agit de manière quasi-isométrique. Pour des feuilletages centraux uniformément compacts, cela donne une réponse partielle à la question [BB16, Question 8.4.] (la question générale est posée pour toute dimension centrale).

Un résultat analogue sur l'intégrabilité unique du fibré central est également valable dans ce contexte :

**Proposition.** *Supposons que  $f$  dans  $\text{PH}_{c=1}(M)$  admet un feuilletage central uniformément compact tel que  $E^c$  est uniquement intégrable. Alors, chaque application dans la même composante connexe  $C^1$  de  $f$  dans  $\text{PH}_{c=1}(M)$  a un fibré central uniquement intégrable.*

Notamment, la proposition précédente montre que si  $f = A \times \text{Id} : N \times S^1 \rightarrow N \times S^1$  est le produit d'un difféomorphisme d'Anosov  $A : N \rightarrow N$  et de la fonction d'identité sur le cercle  $\text{Id} : S^1 \rightarrow S^1$ , alors le fibré central est uniquement intégrable pour chaque application dans la même  $C^1$  composante connexe que  $f$  dans  $\text{PH}_{c=1}(N \times S^1)$ .

## 1.3 Introducción (Español)

### 1.3.1 Dinámica diferenciable, hiperbolicidad e hiperbolicidad parcial

La teoría clásica de sistemas dinámicos estudia el comportamiento asintótico de sistemas que evolucionan en el tiempo con respecto a una regla determinista prescrita que gobierna su evolución.

En el caso de dinámicas diferenciables, esta regla viene dada típicamente por un difeomorfismo  $f : M \rightarrow M$  en el caso de un *sistema dinámico discreto*, o por una ecuación diferencial ordinaria  $\dot{X} = F(X)$  que define un flujo  $X_t : M \rightarrow M$  en el caso de un *sistema dinámico continuo*.



Un ejemplo paradigmático de estos sistemas que presentan una forma global de comportamiento rica y *caótica* viene dado por los sistemas de *Anosov* (también llamados *sistemas hiperbólicos globalmente uniformes*). El estudio de este tipo de sistemas se remonta al menos a los trabajos pioneros de D.V. Anosov y S. Smale (véase [A67] y [S67]).

A grandes rasgos, un difeomorfismo  $f : M \rightarrow M$  se denomina *difeomorfismo de Anosov* si el fibrado tangente admite una descomposición  $f$ -invariante  $TM = E^s \oplus E^u$ , de forma tal que los vectores en  $E^s$  y  $E^u$  son uniformemente contraídos por iterados futuros y pasados de  $f$ , respectivamente.

Un flujo sin singularidades  $X_t : M \rightarrow M$  se denomina *flujo de Anosov* si preserva una descomposición invariante por el diferencial del flujo  $TM = E^s \oplus E^c \oplus E^u$ , de forma tal que los vectores en  $E^s$  y  $E^u$  son uniformemente contraídos por iterados positivos y negativos de  $X_t$ , respectivamente, y el fibrado  $E^c$  es la dirección tangente al flujo  $X_t$ .

Una extensión natural de los sistemas uniformemente hiperbólicos está dada por la noción de *hiperbolicidad parcial*.

**Definición.** Un difeomorfismo  $f : M \rightarrow M$  en una variedad Riemanniana cerrada  $M$  se denomina *parcialmente hiperbólico* si existe una descomposición continua y  $Df$  invariante del fibrado tangente  $TM$  en tres subfibrados no triviales

$$TM = E^s \oplus E^c \oplus E^u$$

de forma tal que los vectores en  $E^s$  y  $E^u$  son contraídos uniformemente por iterados futuros y pasados de  $f$ , respectivamente, y los vectores en  $E^c$  experimentan un comportamiento intermedio. Ver el capítulo 2 para una definición precisa.

La definición de difeomorfismo parcialmente hiperbólico se remonta al menos a [BP74] y [HPS77]. Vale la pena mencionar que existen muchas otras definiciones de *hiperbolicidad parcial*. Típicamente, todas estas nociones implican algún tipo de descomposición invariante del espacio tangente (de toda la variedad o de una parte invariante de ella) en subfibrados que satisfacen algún tipo de *dominación*. Además, en la mayoría de los casos, con al menos uno de los subfibrados uniformemente expandido o contraído. Véase por ejemplo [CP15] como referencia.

Al igual que en el caso de sistemas de Anosov, la hiperbolicidad parcial es una propiedad abierta  $C^1$  que puede comprobarse en finitos iterados (como consecuencia del *criterio de conos*, por ejemplo). Además, aparece de forma natural en el estudio de propiedades dinámicas robustas. Es decir, propiedades que permanecen inalteradas ante pequeñas perturbaciones del

sistema. Este es el caso en al menos dos escenarios relevantes: *transitividad robusta* y *ergodicidad estable*. Ver, por ejemplo, [BPSW01], [CHHU18] y [HP18].

Los difeomorfismos parcialmente hiperbólicos constituyen una rica clase de sistemas dinámicos con suficiente rigidez para que un cierto tipo de ‘clasificación’ pueda ser esperado:

**Problema.** *Desarrollar un contexto de clasificación (al menos en dimensión 3) para los difeomorfismos parcialmente hiperbólicos. Determinar qué propiedades dinámicas pueden darse para estos sistemas e identificar cuáles de ellas son robustas. Explorar la interacción entre los difeomorfismos parcialmente hiperbólicos y la geometría y topología de la variedad subyacente, en particular determinar qué variedades y clases de isotopía admiten sistemas parcialmente hiperbólicos.*

Una manera de abordar este problema es estudiar las estructuras invariantes preservadas por este tipo de mapas, como las *foliaciones invariantes* que aparecen naturalmente en ellos. Uno podría esperar obtener consecuencias topológicas o dinámicas del comportamiento de estas estructuras invariantes, y viceversa.

En dimensión 3, el problema de clasificación ha sido particularmente tratado. Los ejemplos clásicos de difeomorfismos parcialmente hiperbólicos en dimensión 3 son:

- *Deformaciones de difeomorfismos de Anosov*
- *Productos fibrados parcialmente hiperbólicos*
- *Perturbaciones del tiempo 1 de un flujo de Anosov*

Brevemente, la primera clase consiste en difeomorfismos parcialmente hiperbólicos homotópicos a un mapa de Anosov. La segunda, difeomorfismos parcialmente hiperbólicos tales que  $E^c$  integra a una foliación invariante  $W^c$  por hojas compactas que induce una estructura de fibrado en  $M$  (ver Sección 1.1.6). Y la tercera, difeomorfismos suficientemente próximos al tiempo 1 de un flujo de Anosov.

A pesar de haberse conjeturado que la lista anterior cubría todos los sistemas parcialmente hiperbólicos de dimensión 3 (ver la *conjetura de Pujals* en [BW05]), a la lista de ejemplos clásicos se le han unido recientemente nuevos tipos de ejemplos que desafían la tentativa de clasificación. En [HHU16] se construyeron los primeros ejemplos no dinámicamente coherentes. Más tarde, en [BPP16], [BGP16] y [BGHP17] nuevos ejemplos sorprendentes emergieron. Ver también [BFP20].

En dimensión mayor, una forma de abordar el estudio es restringirse al caso en que  $\dim(E^c) = 1$ . Bajo este supuesto la lista anterior de ejemplos clásicos es esencialmente la misma (sólo hay que añadir la posibilidad de tomar producto con un difeomorfismo de Anosov).

La clase de *flujos de Anosov discretizados* es concebida como una extensión natural del tercer tipo de ejemplos clásicos. El objetivo principal de esta tesis es establecer diversas propiedades generales para esta clase de sistemas en cualquier dimensión.

### 1.3.2 Flujos de Anosov discretizados

Denotamos por  $\text{PH}(M)$  al conjunto de difeomorfismos parcialmente hiperbólicos en  $M$  y por  $\text{PH}_{c=1}(M)$  a aquellos tales que  $\dim(E^c) = 1$ .

**Definición 1.3.1.** Decimos que  $f \in \text{PH}_{c=1}(M)$  es un *flujo de Anosov discretizado* si existe una foliación orientable  $\mathcal{W}^c$  cuyas hojas son  $C^1$  subvariedades de  $M$  tangentes a  $E^c$ , y una función continua  $\tau : M \rightarrow \mathbb{R}_{>0}$  tal que

$$f(x) = \varphi_{\tau(x)}^c(x)$$

para cada  $x \in M$ , donde  $\varphi_t^c : M \rightarrow M$  denota un flujo de velocidad unitaria cuyas órbitas son las hojas de  $\mathcal{W}^c$ .

El ejemplo prototípico de un flujo de Anosov discretizado es el tiempo 1 de un flujo de Anosov y todas sus perturbaciones  $C^1$  suficientemente pequeñas. Esto último es una consecuencia de [HPS77] y será revisado en este texto.

El término “flujo de Anosov discretizado” fue acuñado en [BFFP19] y se deriva del hecho de que el flujo  $\varphi_t^c$  es necesariamente un *flujo de Anosov topológico* (ver la Definición 3.7.1). Es decir,  $f$  puede considerarse como una discretización del flujo de Anosov topológico  $\varphi_t^c$ .

Los flujos de Anosov discretizados han sido profusamente estudiados en la literatura, aunque no siempre bajo el mismo nombre. Un ejemplo de ello es [BD96] donde se obtuvieron los primeros ejemplos de difeomorfismos robustamente transitivos isotópicos a la identidad. Estos ejemplos se construyen arbitrariamente cerca del tiempo 1 de cualquier flujo de Anosov. En particular, son flujos de Anosov discretizados.

En cuanto a la ergodicidad estable, en [GPS94] se demostró que los tiempo 1 de flujos geodésicos en superficies cerradas de curvatura negativa constante son  $C^2$  establemente ergódicos. Estos fueron los primeros ejemplos no Anosov de difeomorfismos establemente ergódicos construidos en la literatura. También en este caso estos ejemplos son flujos de Anosov discretizados (así como sus perturbados).

En [BW05] se demostró que los flujos de Anosov discretizados y los productos fibrados parcialmente hiperbólicos son particularmente ubíquos entre los difeomorfismos parcialmente hiperbólicos (dinámicamente coherentes) en dimensión 3. Esto condujo a la consolidación de los ejemplos clásicos en dimensión 3.

Más recientemente, en [BFFP19] se demostró que los flujos de Anosov discretizados representan todos los difeomorfismos parcialmente hiperbólicos dinámicamente coherentes en numerosas 3-variedades. Y en [FP22] (ver también [FP21]) que en la mayoría de las 3-variedades los flujos de Anosov discretizados son accesibles y ergódicos siempre que preserven una forma de volumen.

Otros resultados dinámicos recientes que involucran flujos de Anosov discretizados son los resultados de rigidez obtenidos en [AVW15], [BFT20] sobre medidas de entropía maximal, [DWX21] y [BG21] respecto a rigidez de centralizadores, y el ‘principio de invarianza’ obtenido en [CP22].

Una característica que ha demostrado ser muy útil en el estudio de difeomorfismos parcialmente hiperbólicos es la presencia de foliaciones invariantes tangentes a los subfibrados invariantes. En [HPS77] se demostró que los fibrados  $E^s$  y  $E^u$  son únicamente integrables a foliaciones  $f$ -invariantes  $\mathcal{W}^s$  y  $\mathcal{W}^u$ , respectivamente.

En contraste, los fibrados  $E^s \oplus E^c$  y  $E^u \oplus E^c$  pueden o no ser integrables. Siempre que integran en foliaciones  $f$ -invariantes ( $\mathcal{W}^{cs}$  y  $\mathcal{W}^{cu}$ , respectivamente) el mapa  $f$  se denomina *dinámicamente coherente*. Si este es el caso entonces  $\mathcal{W}^c := \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  (la foliación dada por las componentes conexas de la intersección de las hojas de  $\mathcal{W}^{cs}$  y  $\mathcal{W}^{cu}$ ) da lugar a una foliación  $f$ -invariante tangente al fibrado central  $E^c$ .

Recordemos que el conjunto no errante  $\Omega(f)$  de  $f$  es el conjunto de puntos  $x$  en  $M$  tal que para cada entorno  $U$  de  $x$  existe  $N > 0$  tal que  $f^N(U) \cap U \neq \emptyset$ .

En nuestro primer teorema establecemos algunas propiedades generales satisfechas por todo flujo de Anosov discretizado:

**Teorema.** *Sea  $f$  un flujo de Anosov discretizado. Sean  $\varphi_t^c$  y  $\mathcal{W}^c$  el flujo y la foliación central que aparecen en la definición de  $f$ , respectivamente. Entonces:*

1. *(Flujo de Anosov topológico). El flujo  $\varphi_t^c$  es un flujo de Anosov topológico (ver Definición 3.7.1).*
2. *(Coherencia dinámica). El mapa  $f$  es dinámicamente coherente, admitiendo una foliación centro estable  $\mathcal{W}^{cs}$  y una foliación centro inestable  $\mathcal{W}^{cu}$  tales que  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .*

3. (Unicidad de las foliaciones  $cs$  y  $cu$ ). Las foliaciones  $\mathcal{W}^{cs}$  y  $\mathcal{W}^{cu}$  son las únicas foliaciones  $f$ -invariantes tangentes a  $E^s \oplus E^c$  y  $E^c \oplus E^u$ , respectivamente.
4. (Complejitud de las hojas). Las hojas de  $\mathcal{W}^{cs}$  y  $\mathcal{W}^{cu}$  cumplen que  $\mathcal{W}^{cs}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^s(y)$  y  $\mathcal{W}^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^u(y)$  para todo  $x \in M$ .
5. (Topología de las hojas) Las hojas de  $\mathcal{W}^{cs}$  y  $\mathcal{W}^{cu}$  son homeomorfas a planos o cilindros. Las primeras no contienen hojas centrales compactas mientras que las segundas contienen exactamente una.

Cabe señalar que en dimensión 3 el teorema anterior era conocido en su mayor parte. En efecto, una vez demostrado (2) entonces (1), (4) y (5) se deducen de [BW05, Teorema 2]. Por otro lado, una vez que (1) es demostrado entonces (2) ya ha aparecido en [BFP20, Proposición G.2] y (3) se sigue de [BFFP19] (ver [BG21, Lemma 1.1]). Nuestro objetivo ha sido sintetizar la teoría en dimensión 3 y generalizarla a dimensiones superiores donde menos resultados habían sido establecidos en la literatura.

En [BFFP19], [BFP20], [BG21] y [GM22] se denomina ‘flujo de Anosov discretizado’ a todo mapa  $f \in \text{PH}_{c=1}(M)$  tal que existe un flujo de Anosov topológico  $\varphi_t : M \rightarrow M$  y una función continua  $\tau : M \rightarrow \mathbb{R}_{>0}$  tales que  $f(x) = \varphi_{\tau(x)}(x)$  para toda  $x$  en  $M$ .

En [BFT20] un difeomorfismo  $f \in \text{PH}_{c=1}(M)$  se denomina ‘tipo flujo’ si satisface la Definición 1.3.1 y además es dinámicamente coherente y admite al menos una hoja compacta de  $\mathcal{W}^c$ .

Nociones similares pero no a priori idénticas fueron estudiadas también en [BW05], [BG09] y [BG10].

El problema respecto a la relación entre estas nociones se planteaba de forma natural. Con el teorema anterior (añadir también la Proposición 3.7.3 ítem (4) enunciada en la Sección 3.7) se obtiene que todas estas definiciones son equivalentes y refieren a la misma clase de sistemas parcialmente hiperbólicos:

**Corolario.** *La definición de flujo discretizado de Anosov dada en [BFFP19], [BFP20], [BG21] y [GM22], y la definición de difeomorfismo parcialmente hiperbólico de tipo flujo en [BFT20], son equivalentes a la Definición 1.1.1.*

*Además, la clase de difeomorfismos parcialmente hiperbólicos estudiados en [BW05, Teorema 2.], [BG09] y [BG10] son también flujos discretizados de Anosov como en la Definición 1.1.1.*

### 1.3.3 Estabilidad global

El siguiente resultado muestra que los flujos de Anosov discretizados constituyen, en cierto forma, una clase amplia de difeomorfismos parcialmente hiperbólicos con central unidimensional:

**Teorema.** *El conjunto de flujos de Anosov discretizados es un subconjunto  $C^1$  abierto y cerrado de  $\text{PH}_{c=1}(M)$ .*

En otras palabras, la clase de flujos de Anosov discretizados constituye componentes conexas enteras de  $\text{PH}_{c=1}(M)$ .

Dos difeomorfismos parcialmente hiperbólicos y respectivas foliaciones centrales invariantes  $(f, \mathcal{W}_f^c)$  y  $(g, \mathcal{W}_g^c)$  se denominan *conjugados por hojas* si existe un homeomorfismo  $h : M \rightarrow M$  que lleva hojas de  $\mathcal{W}_f^c$  en hojas de  $\mathcal{W}_g^c$  de forma tal que  $h \circ f(W) = g \circ h(W)$  para cada hoja  $W \in \mathcal{W}_f^c$ . La conjugación por hojas da sentido una manera de clasificar los difeomorfismos parcialmente hiperbólicos (dinámicamente coherentes) módulo comportamiento central: dos sistemas se consideran equivalentes si son conjugados por hojas.

La demostración del teorema anterior muestra además que la conjugación por hojas es preservada a lo largo de las componentes conexas de flujos de Anosov discretizados:

**Corolario.** *Dos flujos de Anosov discretizados en la misma componente conexa  $C^1$  de  $\text{PH}_{c=1}(M)$  son conjugados por hojas.*

Vale la pena mencionar el contexto de fondo para los enunciados mencionados anteriormente.

Un resultado clásico de [HPS77] da condiciones para la estabilidad de *foliaciones normalmente hiperbólicas* (es decir, foliaciones que son tangentes al fibrado central de un difeomorfismo parcialmente hiperbólico):

**Teorema** (Hirsch-Pugh-Shub). *Supongamos que  $f \in \text{PH}(M)$  admite una foliación central  $f$ -invariante  $\mathcal{W}_f^c$ . Si el par  $(f, \mathcal{W}_f^c)$  es expansivo por placas entonces existe  $\mathcal{U}(f)$  un entorno  $C^1$  de  $f$  que satisface que todo  $g \in \mathcal{U}(f)$  admite una foliación central  $g$ -invariante  $\mathcal{W}_g^c$  tal que  $(f, \mathcal{W}_f^c)$  y  $(g, \mathcal{W}_g^c)$  son conjugados por hojas.*

Un par  $(f, \mathcal{W}^c)$  se denomina *expansivo por placas* (o  $\delta$ -*expansivo por placas*) si existe  $\delta > 0$  tal que todo par de sucesiones  $(x_n)_{n \in \mathbb{Z}}$  y  $(y_n)_{n \in \mathbb{Z}}$  que satisfacen  $x_{n+1} \in \mathcal{W}_\delta^c(f(x_n))$ ,  $y_{n+1} \in \mathcal{W}_\delta^c(f(y_n))$  y  $d(x_n, y_n) < \delta$  para cada  $n \in \mathbb{Z}$  también satisface que  $y_0 \in \mathcal{W}_{loc}^c(x_0)$ . En [HPS77] se demuestra que esta condición se cumple siempre que  $\mathcal{W}^c$  sea una foliación  $C^1$  o en caso que  $f$  sea una isometría entre cada hoja  $W$  de  $\mathcal{W}^c$  y su imagen  $f(W)$  (este es el caso para el tiempo 1 de un flujo de Anosov, por ejemplo).

Para demostrar la propiedad  $C^1$  abierta y cerrada de los flujos de Anosov discretizados de Anosov se necesita una cierta ‘versión uniforme’ del teorema de estabilidad anterior:

**Teorema.** *Supongamos  $f_0 \in \text{PH}_{c=1}(M)$ . Para cada  $\delta > 0$  existe un  $C^1$  entorno  $\mathcal{U}(f_0)$  de  $f_0$  tal que, si algún  $f \in \mathcal{U}(f_0)$  admite una foliación central  $\mathcal{W}_f^c$  tal que  $(f, \mathcal{W}_f^c)$  es  $\delta$ -expansivo por placas, entonces cada  $g \in \mathcal{U}(f_0)$  admite una foliación central  $g$ -invariante  $\mathcal{W}_g^c$  tal que  $(f, \mathcal{W}_f^c)$  y  $(g, \mathcal{W}_g^c)$  son conjugados por hojas.*

La parte ‘uniforme’ clave en el teorema anterior es que el tamaño de  $\mathcal{U}(f_0)$  está fijado de antemano. Entonces si  $(f_n)_n$  es una sucesión en  $\text{PH}_{c=1}(M)$  convergente a  $f_0$  basta con demostrar que un par  $(f_N, \mathcal{W}_{f_N}^c)$  es  $\delta$ -expansivo por placas para algún  $f_N \in \mathcal{U}(f_0)$  para inducir una foliación central  $\mathcal{W}_{f_0}^c$  para  $f_0$ , que además satisface que  $(f_0, \mathcal{W}_{f_0}^c)$  es conjugado por hojas a  $(f_N, \mathcal{W}_{f_N}^c)$  (y de hecho conjugado por hojas a un par  $(f, \mathcal{W}_f^c)$  para toda  $f \in \mathcal{U}(f_0)$ ).

Un teorema de estabilidad uniforme similar fue observado originalmente en [BFP20] en un contexto diferente pero relacionado (para  $C^1$  *inmersiones por hojas* y *foliaciones ramificadas*). En [BFP20] se muestra la propiedad  $C^1$  abierta y cerrada de la clase de *flujos de Anosov colapsados* en dimensión 3. La propiedad  $C^1$  abierta y cerrada de los flujos de Anosov discretizados en dimensión 3 se deduce esencialmente de [BFP20] (ver el Capítulo 5 para más detalles).

El hecho de que la conjugación por hojas persista dentro de las componentes conexas de flujos de Anosov discretizados puede verse como un resultado de ‘estabilidad global’ donde un sistema expansivo por placas  $(f, \mathcal{W}^c)$  induce conjugación por hojas en toda su  $C^1$  componente conexa de parcialmente hiperbólicos.

Este fenómeno también ha sido observado en [FPS14] para todo automorfismo lineal hiperbólico  $f$  en toro  $\mathbb{T}^n$  (visto como un difeomorfismo parcialmente hiperbólico), y ha sido generalizado en [Pi19] para automorfismos lineales de hiperbólicos en nilvariedad. Más aún, se demuestra en este texto (ver Sección 1.1.6) que este comportamiento también ocurre para productos fibrados parcialmente hiperbólicos con central unidimensional.

Es natural preguntarse si esto es cierto en general:

**Pregunta.** *Supongamos que  $f \in \text{PH}(M)$  admite una foliación central  $f$ -invariante  $\mathcal{W}^c$  tal que  $(f, \mathcal{W}^c)$  es expansivo por placas. ¿Cada  $g$  en la misma  $C^1$  componente conexa de parcialmente hiperbólicos que  $f$  admite una foliación central  $g$ -invariante  $\mathcal{W}_g^c$  tal que  $(g, \mathcal{W}_g^c)$  es expansivo por placas y conjugado por hojas a  $(f, \mathcal{W}^c)$ ?*

Puesto que los flujos de Anosov discretizados constituyen componentes conexas enteras de  $\text{PH}_{c=1}(M)$  muchas otras preguntas naturales pueden ser

planteadas. Uno puede preguntarse cuáles son las propiedades que se conservan en componentes conexas enteras y cuáles no. Qué componentes conexas contienen el tiempo 1 de un flujo de Anosov, qué diferencias hay entre las que no lo contienen, etc.

Como fue mencionado anteriormente, una de las propiedades que es conservada a lo largo de las componentes conexas de los flujos de Anosov discretizados es la conjugación por hojas. Otra de ellas es la integrabilidad única del fibrado central. Es decir, la propiedad de que módulo reparametrizaciones existe una única curva local  $C^1$  tangente a  $E^c$  por cada punto de  $M$  (ver la Sección 1.1.6 para más detalles sobre esta definición):

**Proposición.** *Sea  $f$  un flujo de Anosov discretizado tal que  $E_f^c$  es únicamente integrable. Entonces  $E_g^c$  es únicamente integrable para toda  $g$  en la misma componente conexa  $C^1$  de  $\text{PH}_{c=1}(M)$  que  $f$ .*

En particular, todo flujo de Anosov discretizado en la misma componente conexa que el tiempo 1 de un flujo de Anosov tiene fibrado central únicamente integrable.

En el Ejemplo 5.4.3 se da un ejemplo de un flujo de Anosov discretizado  $f$  tal que  $E^c$  no es únicamente integrable. Esto se obtiene a partir de una modificación simple de una construcción dada en [HHU16] que conduce a un 2-toro tangente a  $E^s \oplus E^c$  enteramente compuesto por puntos de integrabilidad no única para  $E^c$ . El flujo central  $\varphi_t^c$  en este ejemplo es orbitalmente equivalente a la suspensión de un difeomorfismo lineal de Anosov  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  en el 2-toro, sin embargo por la proposición anterior el mapa  $f$  no está en la misma componente conexa que el tiempo 1 de la suspensión de  $A$ . Se concluye lo siguiente.

**Corollary.** *Existen componentes conexas de flujos de Anosov discretizados que no contienen el tiempo 1 de un flujo de Anosov.*

Con todo, estas componentes conexas parecen aún muy particulares. La siguiente pregunta surge de forma natural.

**Pregunta.** *Sea  $f$  un flujo de Anosov discretizado. ¿Alguna de las siguientes condiciones: i)  $f$  es transitivo, ii)  $E_f^c$  es únicamente integrable o iii)  $\varphi_t^c$  no es orbitalmente equivalente a un flujo suspensión; implica que  $f$  se encuentra en la misma componente conexa de  $\text{PH}_{c=1}(M)$  que el tiempo 1 de un flujo de Anosov?*

### 1.3.4 Caracertización por hojas centrales fijas

Una caracterización de los flujos de Anosov discretizados, que también sirve como definición alternativa para la clase, es la siguiente:



**Proposición.** Sea  $f \in \text{PH}_{c=1}(M)$ . Las siguientes afirmaciones son equivalentes:

- (i) El mapa  $f$  es un flujo de Anosov discretizado.
- (ii) Existe una foliación central  $\mathcal{W}^c$  y una constante  $L > 0$  tales que  $f(x) \in \mathcal{W}_L^c(x)$  para todo  $x \in M$ .

Es natural preguntarse si la hipótesis de desplazamiento acotado en (ii) es necesaria. Es decir, si los flujos de Anosov discretizados pueden caracterizarse como la clase de difeomorfismos parcialmente hiperbólicos que fijan individualmente cada hoja de una foliación central unidimensional:

**Pregunta.** Supongamos que  $f$  en  $\text{PH}_{c=1}(M)$  admite una foliación central  $\mathcal{W}^c$  tal que  $f(W) = W$  para cada hoja  $W \in \mathcal{W}^c$ . ¿Es  $f$  un flujo de Anosov discretizado?

Una respuesta afirmativa a esta pregunta se obtiene siempre que  $f$  sea transitivo y dinámicamente coherente:

**Teorema.** Sea  $f \in \text{PH}_{c=1}(M)$  transitivo y dinámicamente coherente tal que  $f(W) = W$  para cada hoja  $W$  en la foliación central  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ . Entonces  $f$  es un flujo de Anosov discretizado.

De hecho, el teorema anterior es cierto si se sustituye la hipótesis ‘ $f$  transitivo’ por la más general ‘ $\mathcal{W}^c$  transitiva’. Para más detalles, ver el Capítulo 6.

### 1.3.5 Unicidad de atractores

Los resultados presentados en esta sección son el producto de un trabajo en colaboración con N. Guelman (ver [GM22]).

Profundizando en las propiedades dinámicas de los flujos de Anosov discretizados, centramos nuestra atención en el problema de la finitud y unicidad de *cuasi-atractores* y *cuasi-repulsores*.

Dado un mapa  $f : M \rightarrow M$  se denota por  $\mathcal{R}(f) \subset M$  al *conjunto recurrente por cadenas* de  $f$ . Es decir, la unión de todos los puntos  $x$  en  $M$  tal que existe una  $\epsilon$ -pseudo órbita no trivial de  $x$  a  $x$  para todo  $\epsilon > 0$ . Este conjunto coincide con el complemento de los puntos que pertenecen a alguna *región atrapante* de la forma  $U \setminus f(\bar{U})$  para cierto abierto  $U$  tal que  $f(\bar{U}) \subset U$ . Se considera  $\mathcal{R}(f)$  dividido en clases de equivalencia denominadas *clases de recurrencia por cadenas*. Estas clases vienen dadas por la relación  $x \sim y$  si y sólo si para cada  $\epsilon > 0$  existe una  $\epsilon$ -pseudo órbita no trivial de  $x$  a  $y$ , y otra de  $y$  a  $x$ .

Un *cuasi-atractor* es una clase de recurrencia por cadenas  $A$  para la que existe una base de entornos  $\{U_i\}_i$  (es decir,  $A \subset U_i$  y  $A = \bigcap_i U_i$ ) tal que  $\overline{f(U_i)} \subset U_i$  para cada  $i$ . Todo homeomorfismo en un espacio métrico compacto admite al menos un cuasi-atractor. Ver [CP15] para una referencia.

Dado que los cuasi-atractores son conjuntos compactos disjuntos dos a dos y saturados por hojas de  $W^u$ , cada uno de ellos contiene al menos un conjunto minimal para la foliación  $W^u$ . Llamamos a un conjunto minimal para  $W^u$  de *laminación inestable minimal*. Así, la unicidad (resp. finitud) de las laminaciones inestables minimales implica la unicidad (resp. finitud) de los cuasi-atractores.

Finitud de laminaciones inestables minimales se obtiene en [CPS17] para un subconjunto  $C^1$ -abierto y denso de los difeomorfismos parcialmente hiperbólicos de central unidimensional. Aquí pretendemos un estudio más global (no perturbativo) que implique resultados de unicidad/finitud para clases enteras de ejemplos.

Flujos de Anosov discretizados con un número arbitrario de atractores y repulsores pueden construirse perturbando el tiempo 1 de la suspensión  $\varphi_t : M \rightarrow M$  de un difeomorfismo de Anosov. Puesto que  $M$  fibra sobre el círculo y  $\varphi_1$  preserva las fibras es posible perturbar  $\varphi_1$  de modo que aún lleve fibras en fibras pero que se convierta en Morse-Smale o incluso en una dinámica con infinitos cuasi-atractores en la base (ver el Ejemplo 7.2.1 para más detalles).

Recordar que dos flujos se dicen *orbitalmente equivalentes* si existe un homeomorfismo que lleva órbitas de uno en órbitas del otro preservando su orientación. Se obtiene el siguiente resultado:

**Teorema.** *Sea  $f$  un flujo de Anosov discretizado y sea  $\varphi_t^c$  el flujo central de  $f$ . Supongamos que  $\varphi_t^c$  es transitivo y no orbitalmente equivalente a una suspensión. Entonces  $f$  tiene una única laminación minimal inestable.*

**Corolario.** *Todo  $f$  como en el teorema anterior tiene a lo sumo un cuasi-atractor.*

Vale la pena señalar que los enunciados anteriores están formulados para laminaciones minimales inestables y cuasi-atractores pero que, naturalmente, los mismos tienen enunciados análogos para laminaciones minimales estables y cuasi-repulsores.

Notar que los resultados anteriores aplica para todo  $f \in \text{PH}_{c=1}(M)$  en la misma componente conexa de  $\text{PH}_{c=1}(M)$  que el tiempo 1 de un flujo de Anosov transitivo  $\varphi_t : M \rightarrow M$  que no sea orbitalmente equivalente a un flujo suspensión. Esto incluye, por ejemplo, todo tiempo 1 de un flujo geodésico en el fibrado tangente unitario de una superficie cerrada de curvatura negativa.

El conjunto no errante del flujo de Anosov topológico  $\varphi_t^c$  obtenido como flujo central de un flujo de Anosov discretizado admite el mismo tipo de *descomposición espectral*

$$\Omega(\varphi_t^c) = \Lambda_1 \cup \dots \cup \Lambda_N$$

en *piezas básicas*  $\{\Lambda_i\}_{1 \leq i \leq N}$  como es el caso para flujos de Anosov clásicos. Ver el Lema 7.4.2.

En el caso de que  $\varphi_t^c$  no sea transitivo el problema de unicidad y finitud se reduce a estudiar el comportamiento de  $\varphi_t^c$  en restricción a sus piezas básicas atractoras. Obtenemos:

**Teorema.** *Sea  $f$  un flujo de Anosov discretizado. Supongamos que  $\Lambda$  es una pieza básica atractora del flujo central  $\varphi_t^c$ . Si  $\varphi_t^c|_\Lambda : \Lambda \rightarrow \Lambda$  no es orbitalmente equivalente a una suspensión entonces  $\Lambda$  contiene una única laminación minimal inestable para  $f$ .*

**Corolario.** *Sea  $f$  un flujo de Anosov discretizado. Supongamos que todas las piezas básicas atractoras  $\Lambda_1, \dots, \Lambda_k$  de  $\varphi_t^c$  satisfacen que  $\varphi_t^c|_{\Lambda_i} : \Lambda_i \rightarrow \Lambda_i$  no es orbitalmente equivalente a una suspensión. Entonces  $f$  tiene exactamente  $k$  laminaciones minimales inestables (y exactamente  $k$  cuasi-atractores). Además, cada una de las laminaciones minimales inestables está contenida en una de las piezas básicas atractoras  $\Lambda_1, \dots, \Lambda_k$ .*

Vale la pena mencionar por último que, además de los enunciados mencionados anteriormente para flujos de Anosov discretizados, resultados similares de unicidad de cuasi-atractores son obtenidos para ciertos *productos fibrados parcialmente hiperbólicos*. Ver Teorema 7.1.5.

### 1.3.6 Productos fibrados parcialmente hiperbólicos, foliaciones centrales uniformemente compactas y acción cuasi-isométrica en la central

La foliación central de un difeomorfismo parcialmente hiperbólico se denomina *uniformemente compacta* si sus hojas son compactas y el volumen de cada hoja está uniformemente acotado en  $M$ . En particular, este es el caso para la foliación central de los *productos fibrados parcialmente hiperbólicos* donde la misma induce una estructura de fibrado en  $M$ :

**Definición 1.3.2.** Decimos que  $f \in \text{PH}(M)$  es un *producto fibrado parcialmente hiperbólico* si existe un fibrado continuo  $\pi : M \rightarrow B$  cuyas fibras son  $C^1$  subvariedades compactas tangentes a  $E^c$  que forman una foliación central  $f$ -invariante  $\mathcal{W}^c$ .

Un resultado análogo a la propiedad  $C^1$  abierta y cerrada de los flujos de Anosov discretizados se cumple para esta clase de sistemas:

**Teorema.** *El conjunto de difeomorfismos en  $\text{PH}_{c=1}(M)$  que admiten una foliación central invariante uniformemente compacta es un subconjunto  $C^1$  abierto y cerrado de  $\text{PH}_{c=1}(M)$ .*

Nuevamente, el teorema anterior muestra que los mapas en  $\text{PH}_{c=1}(M)$  que admiten una foliación central invariante uniformemente compacta constituyen componentes conexas enteras de  $\text{PH}_{c=1}(M)$ . Por otra parte, dos mapas en la misma componente conexa también son conjugados por hojas (ver Corolario 5.1.4).

Una propiedad clave para los flujos de Anosov discretizados resulta ser que los segmentos acotados dentro de hojas de  $\mathcal{W}^c$  no se hacen arbitrariamente largos para iterados pasados o futuros de  $f$ . Esta propiedad establece un puente esencial entre la clase de flujos de Anosov discretizados y la de sistemas parcialmente hiperbólicos que admiten una foliación central uniformemente compacta.

**Definición 1.3.3.** Supongamos que  $f$  en  $\text{PH}(M)$  admite una foliación central  $f$ -invariante  $\mathcal{W}^c$ . Decimos que  $f$  actúa cuasi-isométricamente en  $\mathcal{W}^c$  si existen constantes  $l, L > 0$  tales que

$$f^n(\mathcal{W}_l^c(x)) \subset \mathcal{W}_L^c(f^n(x))$$

para cada  $x$  en  $M$  y  $n \in \mathbb{Z}$ .

Es inmediato comprobar que la propiedad anterior es satisfecha por todo flujo de Anosov discretizado, así como por todo sistema que admiten una foliación central uniformemente compacta (ver Observación 3.4.2 y Observación 3.4.3). Obtenemos que algunas propiedades mostradas para flujos de Anosov discretizados se extienden automáticamente a sistemas que actúan cuasi-isométricamente sobre una foliación central:

**Teorema.** *Supongamos que  $f \in \text{PH}_{c=1}(M)$  actúa cuasi-isométricamente en la foliación central  $f$ -invariante  $\mathcal{W}^c$ . Entonces se cumplen las siguientes propiedades:*

1. *(Coherencia dinámica). El mapa  $f$  es dinámicamente coherente. Además, admite una foliación centro estable  $\mathcal{W}^{cs}$  y una foliación centro inestable  $\mathcal{W}^{cu}$  tales que  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .*
2. *(Unicidad de las foliaciones). Las foliaciones  $\mathcal{W}^{cs}$  y  $\mathcal{W}^{cu}$  son las únicas foliaciones  $f$ -invariantes tangentes a  $E^s \oplus E^c$  y  $E^c \oplus E^u$ , respectivamente.*

3. (Compleitud de las hojas). Las hojas de  $\mathcal{W}^{cs}$  y  $\mathcal{W}^{cu}$  cumplen que  $\mathcal{W}^{cs}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^s(y)$  y  $\mathcal{W}^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x)} \mathcal{W}^u(y)$  para todo  $x \in M$ .

Vale la pena mencionar que el punto (1) del teorema anterior fue demostrado en [BB16, Theorem 1] para sistemas que admiten un central uniformemente compacto (y para cualquier dimensión central). Sin embargo, la prueba dada en este texto es independiente.

El punto (2) muestra que  $\mathcal{W}^c$  es la única foliación central  $f$ -invariante donde  $f$  actúa cuasi-isométricamente. Para foliaciones centrales uniformemente compactas esto da una respuesta parcial a la pregunta [BB16, Question 8.4.] (la pregunta general es para cualquier dimensión central).

Un resultado análogo sobre integrabilidad única del fibrado central también se cumple en este contexto:

**Proposición.** *Supongamos que  $f \in \text{PH}_{c=1}(M)$  admite una foliación central uniformemente compacta tal que  $E^c$  es únicamente integrable. Entonces cada sistema en la misma  $C^1$  componente conexa de  $f$  en  $\text{PH}_{c=1}(M)$  tiene fibrado central únicamente integrable.*

En particular, la proposición anterior muestra que si  $f = A \times \text{Id} : N \times S^1 \rightarrow N \times S^1$  es el producto de un difeomorfismo de Anosov  $A : N \rightarrow N$  y el mapa identidad en el círculo  $\text{Id} : S^1 \rightarrow S^1$ , entonces el fibrado central es únicamente integrable para cada sistema en la misma componente conexa  $C^1$  que  $f$  en  $\text{PH}_{c=1}(N \times S^1)$ .

## Chapter 2

# Preliminaries

**Partially hyperbolic diffeomorphisms.** A  $C^1$ -diffeomorphism  $f : M \rightarrow M$  in a closed Riemannian manifold  $M$  is called *partially hyperbolic* if it preserves a continuous splitting  $TM = E^s \oplus E^c \oplus E^u$ , with non-trivial *stable bundle*  $E^s$  and *unstable bundle*  $E^u$ , such that for some positive integer  $\ell > 0$  it satisfies

$$\|Df_x^\ell v^s\| < \frac{1}{2}\|v^s\|, \quad \|Df_x^{-\ell} v^u\| < \frac{1}{2}\|v^u\| \quad \text{and} \\ \|Df_x^\ell v^s\| < \|Df_x^\ell v^c\| < \|Df_x^\ell v^u\|$$

for every  $x \in M$  and unit vectors  $v^\sigma \in E^\sigma(x)$  for  $\sigma \in \{s, c, u\}$ . Modulo changing the constant  $\ell > 0$ , the property of being partially hyperbolic is independent of the Riemannian metric in  $M$ .

**Invariant manifolds.** If  $f$  is a partially hyperbolic diffeomorphism it is known since [HPS77] that the bundles  $E^s$  and  $E^u$  uniquely integrate to  $f$ -invariant foliations. We denote these foliations as  $\mathcal{W}^s$  and  $\mathcal{W}^u$ , respectively. It is a well-known fact that the leaves of  $\mathcal{W}^s$  and  $\mathcal{W}^u$  are homeomorphic to  $\mathbb{R}^{\dim(E^s)}$  and  $\mathbb{R}^{\dim(E^u)}$ , respectively.

The bundles  $E^s \oplus E^c$  and  $E^c \oplus E^u$  may or may not be integrable. Whenever they integrate to  $f$ -invariant foliations ( $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$ , respectively) we say that  $f$  is *dynamically coherent*. If this is the case then  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  is an  $f$ -invariant foliation whose leaves are tangent to  $E^c$ .

*Notations.* Whenever a foliation  $\mathcal{W}^\sigma$  tangent to  $E^\sigma$  is well defined for  $\sigma \in \{s, c, u, cs, cu\}$  we will denote by  $\mathcal{W}_\delta^\sigma(x)$  the ball of radius  $\delta > 0$  and center  $x$  inside the leaf  $\mathcal{W}^\sigma(x)$  with respect to the intrinsic metric induced by the Riemannian metric in  $M$ . In this context, if  $A$  is any subset of  $M$  we will denote by  $\mathcal{W}^\sigma(A)$  the saturation of  $A$  by  $\mathcal{W}^\sigma$ -leaves, that is, the set  $\bigcup_{y \in A} \mathcal{W}^\sigma(y)$ . We will also denote by  $\mathcal{W}_\delta^\sigma(A)$  the set  $\bigcup_{y \in A} \mathcal{W}_\delta^\sigma(y)$ .

For every  $\sigma \in \{s, c, u, cs, cu\}$  an invariant foliation  $\mathcal{W}^\sigma$  has, by definition,  $C^1$  leaves that are tangent to the continuous bundle  $E^\sigma$ . From this type

of regularity it is immediate to check the following property that we will implicitly use several times along the text: For every  $R > 0$  and  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $d(x, y) < \delta$  then  $d_H(\mathcal{W}_R^\sigma(x), \mathcal{W}_R^\sigma(y)) < \epsilon$ , where  $d_H$  denotes the Hausdorff distance among subsets of  $M$ .

**Invariant cone fields.** We say that  $\mathcal{C}$  is a continuous cone field in the Riemannian manifold  $M$  if there exists a continuous splitting  $TM = E \oplus F$  such that for every  $x \in M$  the cone  $\mathcal{C}(x) \subset T_x M$  is given by  $\mathcal{C}(x) = \{v = v_E + v_F \in T_x M : \|v_E\|_E \geq \|v_F\|_F\}$  for some continuous norms  $\|\cdot\|_E$  and  $\|\cdot\|_F$  in  $E$  and  $F$ , respectively (not necessarily the ones induced by the underlying Riemannian metric). In this context we say that  $\mathcal{C}$  has dimension  $\dim(E)$ . We define the interior of the cone by  $\text{int } \mathcal{C}(x) = \{v = v_E + v_F \in T_x M : \|v_E\|_E > \|v_F\|_F\} \cup \{0\}$  for every  $x \in M$ .

We say that  $\mathcal{C}$  is  $f$ -invariant if for some  $N > 0$  one has  $Df^N \mathcal{C}(x) \subset \text{int } \mathcal{C}(f^N(x))$  for every  $x \in M$ . If this is the case, we say that  $\mathcal{C}$  is uniformly expanded by  $f$  if  $\|f^N(v)\| > \|v\|$  for every  $v \in \mathcal{C} \setminus \{0\}$ .

If  $f : M \rightarrow M$  is a partially hyperbolic diffeomorphism one can check that there exists  $\mathcal{C}^u$  and  $\mathcal{C}^{cu}$  continuous cone fields of dimension  $\dim(E^u)$  and  $\dim(E^{cu})$ , respectively, that are  $f$ -invariant and such that  $E^u$  is uniformly expanded by  $f$  and  $E^u(x) = \bigcap_{n \geq 0} Df^n(\mathcal{C}^u(f^{-n}(x)))$  and  $E^{cu}(x) = \bigcap_{n \geq 0} Df^n(\mathcal{C}^{cu}(f^{-n}(x)))$  for every  $x \in M$ . Analogously for  $f^{-1}$ -invariant cone fields  $\mathcal{C}^s$  and  $\mathcal{C}^{cs}$ .

In fact, the *cone criterion* gives us a kind of reciprocal of the above: A  $C^1$  diffeomorphism  $f : M \rightarrow M$  is partially hyperbolic whenever there exists an  $f$ -invariant cone field  $\mathcal{C}^u$  uniformly expanded by  $f$  and a  $f^{-1}$ -invariant cone field  $\mathcal{C}^s$  uniformly expanded by  $f^{-1}$ . As a consequence, it is immediate to check that  $\text{PH}(M)$  is  $C^1$  open in  $\text{Diff}^1(M)$ . See for example [CP15].

**Continuous flows.** We say that a map  $\varphi : M \times \mathbb{R} \rightarrow M$  is a *continuous flow* if it is continuous and satisfies that  $x \mapsto \varphi(x, t)$  is a homeomorphism for every  $t \in \mathbb{R}$  and  $\varphi(x, t + t') = \varphi(\varphi(x, t), t')$  for every  $x \in M$  and  $t, t' \in \mathbb{R}$ . As is usual, we denote a continuous flow as above by  $\varphi_t : M \rightarrow M$  and the point  $\varphi(x, t)$  by  $\varphi_t(x)$  for every  $x \in M$  and  $t \in \mathbb{R}$ .

**Anosov flows and topological Anosov flows.** A  $C^1$  flow  $\varphi_t : M \rightarrow M$  (that is, such that  $(x, t) \mapsto \varphi_t(x)$  is a  $C^1$  map) is called an *Anosov flow* if there exists a continuous  $D\varphi_t$ -invariant splitting  $TM = E^s \oplus E^c \oplus E^u$  such that  $E^c$  is the bundle generated by  $\frac{\partial \varphi_t}{\partial t}|_{t=0}$  and such that for some  $t_0 \neq 0$  the map  $f = \varphi_{t_0}$  is a partially hyperbolic diffeomorphism with respect to the decomposition  $TM = E^s \oplus E^c \oplus E^u$ . If  $\varphi_t$  is an Anosov flow it is immediate to check that  $g = \varphi_{t_1}$  is a partially hyperbolic diffeomorphism for every  $t_1 \neq 0$ .

The definition of *topological Anosov flow* that will be treated in this text is given in Definition 3.7.1.

**Stable saturation of center curves.** The following will be used several times along the text.

**Lemma 2.0.1.** *Suppose  $f \in \text{PH}_{c=1}(M)$ . There exists  $\delta > 0$  such that for every  $C^1$  arc  $\eta \subset M$  tangent to  $E^c$  with  $\text{length}(\eta) < \delta$  the set  $\mathcal{W}_\delta^s(\eta)$  is a  $C^1$  submanifold tangent to  $E^s \oplus E^c$ .*

A proof of the above lemma can be found in [BBI04, Proposition 3.4.] (it is stated for *absolute* partially hyperbolic diffeomorphism but the proof does not use this fact). See also [HPS77, Theorem 6.1] and [BB16, Remark 4.7.].

**Quasi-attractors and minimal unstable laminations.** We say that  $A \subset M$  is a *minimal unstable lamination* if it is a minimal set of the foliation  $\mathcal{W}^u$ . That is, if it is a  $\mathcal{W}^u$ -saturated compact set such that  $\overline{\mathcal{W}^u(x)} = A$  for every  $x \in A$ . Minimal unstable laminations are minimal, with respect to the inclusion, among non-empty compact  $\mathcal{W}^u$ -saturated sets. Note that in this definition of minimal unstable lamination we are not asking for it to be  $f$ -invariant.

Given  $\epsilon > 0$ , a  $\epsilon$ -pseudo orbit for  $f$  is a sequence  $(x_n)_{n \in \mathbb{Z}}$  such that  $d(x_{n+1}, f(x_n)) < \epsilon$  for every  $n$ . The *chain recurrent set* of  $f$ , denoted by  $\mathcal{R}(f) \subset M$ , is the union of all points  $x \in M$  such that there exists a non-trivial  $\epsilon$ -pseudo orbit from  $x$  to  $x$  for every  $\epsilon > 0$ . It coincides with the complement of all points contained in a wandering region of the form  $U \setminus f(\overline{U})$  for some open set  $U$  such that  $f(\overline{U}) \subset U$ .

One considers  $\mathcal{R}(f)$  divided in equivalent classes, called *chain recurrence classes*, given by the relation  $x \sim y$  if and only if there exists a non-trivial  $\epsilon$ -pseudo orbit from  $x$  to  $y$  and another from  $y$  to  $x$  for every  $\epsilon > 0$ .

A *quasi-attractor* is a chain recurrence class  $A$  for which there exists a basis of neighborhoods  $\{U_i\}_i$  (i.e.  $A \subset U_i$  and  $A = \bigcap_i U_i$ ) such that  $\overline{f(U_i)} \subset U_i$  for every  $i$ . Quasi-attractors always exists for homeomorphisms in compact metric spaces. A good reference for the notions of chain recurrence classes and quasi-attractors is [CP15].



## Chapter 3

# Discretized Anosov flows

### 3.1 Introduction

In this chapter we prove several general properties of discretized Anosov flows and establish the equivalence with other a priori different notions appearing in the literature. In particular, we cover the statements given in Section 1.1.2 of the introduction.

### 3.2 Definition and first properties

**Definition 3.2.1.** We say that  $f \in \text{PH}_{c=1}(M)$  is a *discretized Anosov flow* if there exist a continuous flow  $\varphi_t^c : M \rightarrow M$ , with  $\frac{\partial \varphi_t^c}{\partial t}|_{t=0}$  a continuous vector field without singularities, and a continuous function  $\tau : M \rightarrow \mathbb{R}$  satisfying

$$f(x) = \varphi_{\tau(x)}^c(x)$$

for every  $x \in M$ .

Note that Definition 3.2.1 is slightly more general than the one given in the introduction since it does not ask for  $\varphi_t^c$  to generate a center foliation or to be parametrized by arc-length. The former is derived as a consequence in the next proposition while the latter can always be achieved by reparametrizing the flow  $\varphi_t^c$  as seen in Remark 3.2.3.

Moreover, in contrast with the definition given in [BFFP19], we do not ask for  $\varphi_t^c$  to be a topological Anosov flow (see Definition 3.7.1). This is derived as a consequence in Proposition 3.7.2.

**Proposition 3.2.2.** *If  $f$  is a discretized Anosov flow then:*

- (i) *The vector field  $\frac{\partial \varphi_t^c}{\partial t}|_{t=0}$  generates the bundle  $E^c$  and the flow lines of  $\varphi_t^c$  form a center foliation  $\mathcal{W}^c$  whose leaves are fixed by  $f$ .*

(ii) The function  $\tau$  has no zero and is  $C^1$  restricted to each leaf of  $\mathcal{W}^c$ .

*Proof.* Let  $F$  be the one-dimensional bundle generated by  $\frac{\partial \varphi_t^c}{\partial t}|_{t=0}$ . In order to show (i) let us see that  $F = E^c$ . This has essentially been done for  $\dim(M) = 3$  in [BFFP19, Proposition G.2.] and the arguments are equally valid in any dimension. We will briefly reproduce them for the sake of completeness.

We claim first that it is enough to show that  $F$  is never contained in  $E^s$  nor  $E^u$ . Indeed, if  $F(x)$  is not contained in  $E^s(x)$  nor  $E^u(x)$  for every  $x \in M$  then the angle formed by  $F$  and  $E^s$  is bounded away from zero by a positive constant independent of the point in  $M$ . As a consequence, for every  $x \in M$  the subspace  $Df^n(F(f^{-n}(x)))$  gets arbitrarily close to  $E^{cu}(x)$  as  $n$  tends to  $+\infty$ . As  $F$  is  $Df$ -invariant (see justification below) we deduce that  $F(x)$  needs to be contained in  $E^{cu}(x)$ . Arguing analogously for backwards iterates using the never-zero angle between  $F$  and  $E^u$  one obtains that  $F(x)$  has to be contained in  $E^{cs}(x)$  for every  $x$  in  $M$ . We conclude that  $F$  coincides everywhere with  $E^c = E^{cs} \cap E^{cu}$ .

The bundle  $F$  needs to be  $Df$ -invariant as every small piece of  $\varphi_t^c$ -orbit through a point  $x \in M$  is sent by  $f$  to a  $C^1$  curve that is a reparametrization of a small piece of  $\varphi_t^c$ -orbit through  $f(x)$ . Thus  $F(f(x))$  that is generated by  $\frac{\partial \varphi_t^c}{\partial t}|_{t=0}(f(x))$  coincides with  $DfF(f(x))$  that is generated by  $\frac{\partial f \varphi_t^c}{\partial t}|_{t=0}(x)$ .

It remains to see now that  $F$  is never contained in  $E^s$  nor  $E^u$ . Without loss of generality suppose by contradiction that  $F(x)$  is contained in  $E^u(x)$  for some  $x$ . Note that  $F(f^{-n}(x))$  is then contained in  $E^u(f^{-n}(x))$  for every  $n \geq 0$ .

Let  $\mathcal{C}^u$  be a continuous  $f$ -invariant unstable cone field such that  $Df^N \mathcal{C}^u \subset \text{int } \mathcal{C}^u$  for some  $N > 0$  and  $\bigcap_{n \geq 0} Df^n(\mathcal{C}^u(f^{-n}y)) = E^u(y)$  for every  $y \in M$  (see the preliminaries for more details). Since for every  $n \geq 0$  a piece of  $\varphi_t^c$ -orbit containing  $f^{-n}(x)$  is tangent to  $\mathcal{C}^u$  we obtain in the limit with  $n$  that at least a piece  $\eta$  of  $\varphi_t^c$ -orbit containing  $x$  is contained in  $\mathcal{W}^u(x)$ .

As  $\tau : M \rightarrow \mathbb{R}_{>0}$  is continuous it has some positive upper bound so there exists  $L > 0$  such that every forward iterate of  $\eta$  has length less than  $L$ . This contradicts the fact that  $f$  expands uniformly the length of any  $C^1$  arc tangent to  $E^u$ . This ends the proof of  $F(x) = E^c(x)$  for every  $x \in M$ .

It follows that the flow lines of  $\varphi_t^c$  are tangent to  $E^c$  and consequently they form a center foliation  $\mathcal{W}^c$  whose leaves are fixed by  $f$ . Property (i) is settled.

Since  $f$  is  $C^1$  and preserves the bundle  $E^c$  it is immediate to check that the function  $\tau$  needs to be  $C^1$  restricted to each leaf of  $\mathcal{W}^c$ . In order to end (ii) it remains to show that  $\tau$  has no zeros. For this we will use a similar argument as in [BG09, Lema 1.2.] or [BFP20, Proposition 5.14.].

Let us suppose by contradiction that  $\tau(x) = 0$  for some  $x \in M$  and

consider  $U$  a small  $\varphi_t^c$  flow box neighborhood of  $x$ . By the continuity of  $f$  there exists  $\epsilon > 0$  such that  $B_{2\epsilon}(x) \subset U$  and  $f(B_{2\epsilon}(x)) \subset U$ .

We claim that  $\epsilon$  can be considered small enough so that  $y$  and  $f(y)$  need to lie in the same  $\varphi_t^c$ -plaque of  $U$  for every  $y \in B_\epsilon(x)$ . Indeed, let  $l > 0$  be a constant smaller than the distance between  $B_\epsilon(x)$  and  $M \setminus U$  and let  $C > 0$  be a constant larger than  $\|\frac{\partial \varphi_t^c}{\partial t}|_{t=0}(y)\|$  for every  $y \in M$ . By the continuity of  $\tau$  we can consider  $\epsilon$  small enough so that  $\tau(y)C < l$  for every  $y \in B_\epsilon(x)$ . It follows that the center arc  $[y, f(y)]_c$  from  $y$  to  $f(y)$  along  $\varphi_t^c$  needs to have length less than  $l$  for every  $y \in B_\epsilon(x)$ . Hence,  $[y, f(y)]_c$  needs to be contained in  $U$  for every  $y \in B_\epsilon(x)$  and this proves the claim.

As  $f$  contracts distances inside  $\mathcal{W}^s$ -leaves for large enough forward iterates there exists  $\delta > 0$  such that  $\mathcal{W}_\delta^s(x)$  and  $f^n(\mathcal{W}_\delta^s(x))$  for every  $n \geq 0$  are contained in  $B_\epsilon(x)$ . Moreover, for every  $y \in \mathcal{W}_\delta^s(x) \setminus \{x\}$  the sequence  $f^n(y)$  tends to  $x$ . This contradicts the fact that by the previous claim every point in  $\{f^n(y)\}_{n \geq 0}$  must lie in the same  $\varphi_t^c$ -plaque of  $U$  than  $y$  (which is at positive distance from  $x$ ).  $\square$

Note that because of (ii) in the previous proposition one can always assume that  $\tau$  is positive (modulo inverting the time of  $\varphi_t^c$  if needed).

The next remark shows that Definition 3.2.1 can be seen as independent of reparametrizations of the flow  $\varphi_t^c$ . In particular, one can always assume that  $\varphi_t^c$  has been parametrized by arc-length.

**Remark 3.2.3.** Suppose  $f$  is a discretized Anosov flow such that  $f(x) = \varphi_{\tau(x)}^c(x)$  for every  $x \in M$  as in Definition 3.2.1. Let  $\alpha : M \rightarrow \mathbb{R}_{>0}$  be a continuous function. If  $\tilde{\varphi}_t$  is the reparametrization of  $\varphi_t^c$  generated by the continuous vector field  $\alpha \frac{\partial \varphi_t^c}{\partial t}|_{t=0}$  then there exists  $\tilde{\tau} : M \rightarrow M$  continuous such that  $f(x) = \tilde{\varphi}_{\tilde{\tau}(x)}(x)$  for every  $x \in M$ .

*Proof.* Let  $\mathcal{W}^c$  be the foliation by flow lines of  $\varphi_t^c$ . As  $\frac{\partial \varphi_t^c}{\partial t}|_{t=0}(x) \neq 0$  for every  $x \in M$  then  $\alpha \frac{\partial \varphi_t^c}{\partial t}|_{t=0}$  is a continuous vector field without singularities restricted to each leaf of  $\mathcal{W}^c$ . It follows that it uniquely integrates inside each leaf of the one-dimensional foliation  $\mathcal{W}^c$ . The flow  $\tilde{\varphi}_t : M \rightarrow M$  obtained in this way has the same flow lines as  $\varphi_t^c$ .

Moreover, there exists  $r : M \times \mathbb{R} \rightarrow \mathbb{R}$  continuous such that  $\varphi_t^c(x) = \tilde{\varphi}_{r(x,t)}(x)$  for every  $x \in M$  and  $t \in \mathbb{R}$ . Then  $\tilde{\tau}(x) = r(x, \tau(x))$  satisfies that  $f(x) = \tilde{\varphi}_{\tilde{\tau}(x)}(x)$ .  $\square$

As a consequence of Proposition 3.2.2 and Remark 3.2.3 one obtains:

**Corollary 3.2.4.** *Definition 1.1.1 and Definition 3.2.1 are equivalent.*

### 3.3 Fixed center foliation and bounded displacement along center

As pointed out in Proposition 3.2.2, an immediate consequence of Definition 3.2.1 is that discretized Anosov flows fix the leaves of a one dimensional center foliation  $\mathcal{W}^c$ . In Chapter 6 we will see that this is enough for characterizing discretized Anosov flows under some general circumstances.

For the moment, we can show that *center fixing* property characterizes discretized Anosov flows provided a uniformly bounded displacement along center leaves is satisfied. We point out that item (ii) in the following proposition can be seen as an alternative definition for discretized Anosov flows.

**Proposition 3.3.1.** *Suppose  $f \in \text{PH}_{c=1}(M)$ . The following are equivalent:*

- (i) *The map  $f$  is a discretized Anosov flow.*
- (ii) *There exists a center foliation  $\mathcal{W}^c$  and a constant  $L > 0$  such that  $f(x) \in \mathcal{W}_L^c(x)$  for every  $x \in M$ .*

*Proof.* Suppose  $f$  is a discretized Anosov flow. Let  $\varphi_t^c$  be the flow appearing in the definition of  $f$  such that  $f(x) = \varphi_{\tau(x)}^c(x)$  for every  $x \in M$ . Proposition 3.2.2 shows that  $f$  fixes the leaves of the center foliation  $\mathcal{W}^c$  given by the flow lines of  $\varphi_t^c$ . If  $T > 0$  denotes an upper bound for  $\tau$  and  $C > 0$  an upper bound for  $y \mapsto \|\frac{\partial \varphi_t^c}{\partial t}|_{t=0}(y)\|$  it follows that  $f(x) \in \mathcal{W}_{TC}^c(x)$  for every  $x$  in  $M$ . Thus (i) implies (ii).

Let us see that (ii) implies (i). Suppose that there exists  $L > 0$  such that  $f(x) \in \mathcal{W}_L^c(x)$  for every  $x \in M$ . In particular,  $f(W) = W$  for every leaf  $W \in \mathcal{W}^c$ .

Note first that, by transverse hyperbolicity, every compact leaf of  $\mathcal{W}^c$  of length less than  $2L$  can not be accumulated by compact leaves of  $\mathcal{W}^c$  of length less than  $2L$ . Then the number of compact leaves of length less than  $2L$  needs to be finite.

Let  $U \subset M$  denote the union of leaves of  $\mathcal{W}^c$  with length larger or equal to  $2L$ . For every  $x \in U$  let  $[x, f(x)]_c$  denote the center segment in  $\mathcal{W}_L^c(x)$  joining  $x$  with  $f(x)$ . It is immediate to check that  $[x, f(x)]_c$  varies continuously in the Hausdorff topology for every  $x$  in  $U$ .

Essentially the same argument used to show (ii) in Proposition 3.2.2 shows that  $f$  has no fixed points in  $U$ : If  $x$  is a fixed point of  $f$  consider  $U_\epsilon(x) \subset U$  a small foliation box neighborhood of  $\mathcal{W}^c$  containing  $x$  such that  $\mathcal{W}_L^c(y) \cap U_\epsilon(x)$  has only one connected component for every  $y \in U_\epsilon(x)$ . For  $\delta > 0$  small enough, if  $y \in \mathcal{W}_\delta^s(x) \setminus \{x\}$  then  $f^n(y) \in U_\epsilon(x)$  for every  $n \geq 0$  and  $\lim_n f^n(y) = x$ . However,  $f^n(y) \in \mathcal{W}_L^c(f^{n-1}(y))$  and  $f^n(y) \in U_\epsilon$  implies that  $f^n(y)$  must lie in the center plaque  $\mathcal{W}_L^c(y) \cap U_\epsilon(x)$  for every  $n \geq 0$ . This gives us a

contradiction with  $\lim_n f^n(y) = x$  and ends the proof that  $f$  has no fixed points in  $U$ .

As  $f$  has no fixed points in  $U$  then for every  $x \in U$  we can define  $X^c(x)$  to be the unit vector in  $E^c(x)$  pointing inwards to the segment  $[x, f(x)]_c$ . As  $[x, f(x)]_c$  varies continuously with  $x$  in  $U$  it follows that  $X^c$  is a continuous vector field in  $U$ .

Let  $\varphi_t : U \rightarrow U$  be the flow whose orbits are the leaves of  $\mathcal{W}^c$  in  $U$  and such that  $\frac{\partial \varphi_t}{\partial t}|_{t=0}$  is equal to  $X^c$ . Let us define  $\tau(x)$  to be the length of  $[x, f(x)]_c$  for every  $x$  in  $U$ . Clearly  $f(x) = \varphi_{\tau(x)}^c(x)$  for every  $x \in U$ . It remains to see that  $X^c$ ,  $\varphi_t^c$  and  $\tau$ , which are a priori defined only in  $U$ , extend well to  $M$ . That is, that they extend well to the union of compact center leaves of length less than  $2L$ .

Let  $\eta$  be a compact center leaf of length less than  $2L$ . For every  $x \in \eta$  consider  $V_x$  a small  $\mathcal{W}^c$ -box neighborhood containing  $x$  so that if  $V_x \cap V_y \neq \emptyset$  then  $\mathcal{W}^c|_{V_x \cup V_y}$  is orientable. We can suppose that for every  $x$  the neighborhood  $V_x$  is small enough so that it is disjoint from every other compact center leaf of length less than  $2L$ .

Consider  $V$  be the neighborhood of  $\eta$  that is the union of the elements of  $\{V_x\}_{x \in \eta}$ . It follows that  $\mathcal{W}^c|_V$  is orientable since any orientation given to  $\eta$  can be extended to an orientation on each  $V_x$  and this orientations coincide in  $V_x \cap V_y$  whenever  $V_x \cap V_y \neq \emptyset$ . Then, as the set  $U \cap V$  is connected, it follows that the orientation induced by  $X^c$  in  $\mathcal{W}^c|_U$  can be extended to  $\mathcal{W}^c|_{U \cap V}$ . Now that the a priori orientation issue has been ruled out, it follows immediately that  $X^c$  and  $\varphi_t^c$  extend continuously to  $\eta$ .

It remains to extend  $\tau$  continuously to  $\eta$  so that  $f(x) = \varphi_{\tau(x)}^c(x)$  for every  $x \in \eta$ . To this end, for every  $x$  in  $\eta$  let us denote by  $[x, f(x)]_c$  the center segment from  $x$  to  $f(x)$  such that  $X^c(x)$  points inwards in  $[x, f(x)]_c$ . Note that it may be the case that if  $x_n \rightarrow x$  with  $(x_n)_n \subset U$  then  $[x_n, f(x_n)]_c$  ‘turns around’  $\eta$  many times so that  $[x_n, f(x_n)]_c$  accumulates in the Hausdorff topology to  $\eta$  instead of  $[x, f(x)]_c$ .

However, since  $\mathcal{W}^c$  is a continuous foliation tangent to a continuous sub-bundle there exists  $\epsilon > 0$  such that if  $d(y, x) < \epsilon$  then  $\varphi_t^c(y)$  is in  $V_{\varphi_t^c(x)}$  for every  $t \in [0, L]$  and  $x \in \eta$ . It follows that the ‘number of turns’ (measured, for example, as the number of connected component of  $[x_n, f(x_n)]_c \cap V_x$  minus 1) needs to be constant for  $x_n$  close enough to  $x$ . As this integer number varies continuously with  $x$  in  $\eta$  it has to be a constant  $N$  independent of the point  $x$ . Hence by defining  $\tau$  in  $\eta$  as

$$\tau(x) = \text{length}[x, f(x)]_c + N \text{length } \eta$$

it follows that  $\tau$  extends continuously to  $\eta$ .

By doing the above for every center leaf  $\eta$  of length less than  $2L$  it follows that  $\tau$  is well defined and continuous in  $M$ , and that

$$f(x) = \varphi_{\tau(x)}^c(x)$$

is satisfied for every  $x \in M$ . This settles (ii) implies (i).  $\square$

### 3.4 Dynamical coherence and quasi-isometrical action on the center foliation

A key property for discretized Anosov flows turns out to be that segments inside  $\mathcal{W}^c$  do not get arbitrarily long for past and future iterates of  $f$ . We will use this fact to show that every discretized Anosov flow is dynamically coherent.

It is worth noting that this property sets an essential bridge between the class of discretized Anosov flows and that of partially hyperbolic systems admitting a uniformly compact center foliation.

The following definition is valid for any center dimension.

**Definition 3.4.1.** A partially hyperbolic diffeomorphism  $f$  admitting an  $f$ -invariant center foliation  $\mathcal{W}^c$  is said to *act quasi-isometrically on  $\mathcal{W}^c$*  if there exist constants  $l, L > 0$  such that

$$f^n(\mathcal{W}_l^c(x)) \subset \mathcal{W}_L^c(f^n(x))$$

for every  $x$  in  $M$  and  $n \in \mathbb{Z}$ .

The following is immediate to check.

**Remark 3.4.2.** Every discretized Anosov flow acts quasi-isometrically on the center foliation  $\mathcal{W}^c$  given by the flow lines of the flow  $\varphi_t^c$  as in Definition 3.2.1. Indeed, since  $f(x) = \varphi_{\tau(x)}^c(x)$  for every  $x \in M$  then  $f$  acts quasi-isometrically on  $\mathcal{W}^c$  with constants  $l = \min \|\frac{\partial \varphi_t^c}{\partial t}|_{t=0}\|, \min \tau$  and  $L = \max \|\frac{\partial \varphi_t^c}{\partial t}|_{t=0}\|, \max \tau$ .

**Remark 3.4.3.** Every partially hyperbolic diffeomorphism admitting an invariant uniformly compact center foliation  $\mathcal{W}^c$  acts quasi-isometrically on  $\mathcal{W}^c$ . Indeed, it is enough to show that under these circumstances the diameter of every center leaf is bounded and then set  $L > 0$  larger than this bound.

To show that the diameter of every leaf of  $\mathcal{W}^c$  is bounded one can argue as follows. Let  $\delta, \epsilon > 0$  be such that for every  $x \in M$  the set  $\mathcal{W}_\delta^c(x)$  has volume less than  $\epsilon$ . Suppose by contradiction that there exist center leaves with arbitrarily large diameter. It follows that for every  $N > 0$  one can find  $N$  points in the same center leaf such that any two points are separated

more than  $2\delta$ . Then the volume of the center leaf containing these points is larger than  $N\epsilon$ . This contradicts the fact that center leaves have a uniformly bounded volume.

By Remark 3.4.2 the next proposition shows that discretized Anosov flows and partially hyperbolic skew-products are dynamically coherent. Moreover, it shows that the stable and unstable saturations of center leaves are complete subsets of  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  leaves, respectively.

**Proposition 3.4.4** (Dynamical coherence). *Suppose  $f \in \text{PH}_{c=1}(M)$  acts quasi-isometrically on a center foliation  $\mathcal{W}^c$ . Then  $f$  is dynamically coherent, admitting center-stable foliation  $\mathcal{W}^{cs}$  and center-unstable foliation  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ . Moreover,  $\mathcal{W}^{cs}(x) = \mathcal{W}^s(\mathcal{W}^c(x))$  and  $\mathcal{W}^{cu}(x) = \mathcal{W}^u(\mathcal{W}^c(x))$  for every  $x \in M$ .*

Proposition 3.4.4 will be derived from the following lemma that may be of independent interest.

**Lemma 3.4.5.** *Suppose  $f \in \text{PH}_{c=D}(M)$  for some  $D > 0$  admits an invariant center foliation  $\mathcal{W}^c$ . Let  $y$  be a point in  $\mathcal{W}^s(x)$  for some  $x \in M$  and suppose  $\eta \subset \mathcal{W}^c(y)$  is a  $C^1$  curve through  $y$  such that  $\{\text{length}(f^n\eta)\}_{n \geq 0}$  is bounded. Then  $\eta$  is contained in  $\mathcal{W}^s(\mathcal{W}^c(x))$ .*

*Proof.* Let  $x \in M$ ,  $y \in \mathcal{W}^s(x)$  and  $\eta : [0, 1] \rightarrow \mathcal{W}^c(y)$  be a  $C^1$  curve with  $\eta(0) = y$ . Suppose that  $\{\text{length}(f^n\eta)\}_{n \geq 0}$  is bounded by some constant  $L > 0$ .

Let  $\delta > 0$  be as in Lemma 2.0.1 so that  $\mathcal{W}_\delta^s(\mathcal{W}_\delta^c(z))$  is a  $C^1$  submanifold tangent to  $E^s \oplus E^c$  for every  $z \in M$ . Recall that the bundles  $E^s$ ,  $E^c$  and  $E^u$  vary continuously in  $M$ . By taking  $\delta$  small enough we can ensure that for every  $z$  and  $z'$  in  $M$  such that  $d(z, z') < \frac{\delta}{2}$  the sets  $\mathcal{W}_\delta^u(z')$  and  $\mathcal{W}_\delta^s(\mathcal{W}_\delta^c(z))$  intersect, and that this intersection takes place in a unique point.

We claim that there exists a constant  $\delta' > 0$  such that if  $d(z, z') < \delta'$  and  $\gamma : [0, 1] \rightarrow \mathcal{W}^c(z')$  is a curve of length at most  $L$  with  $\gamma(0) = z'$  then there exists a continuous curve  $H^{su}\gamma : [0, 1] \rightarrow \mathcal{W}^c(z)$  such that  $\mathcal{W}_\delta^u(\gamma(t)) \cap \mathcal{W}_\delta^s(H^{su}\gamma(t)) \neq \emptyset$  for every  $t \in [0, 1]$  and  $H^{su}\gamma(0) \in \mathcal{W}_\delta^c(z)$ . Note that if this claim is true then  $H^{su}\gamma$  is a particular choice of continuation by center holonomy of  $\gamma$  along  $\mathcal{W}^c(z)$  that is uniquely determined by the properties  $\mathcal{W}_\delta^u(\gamma(t)) \cap \mathcal{W}_\delta^s(H^{su}\gamma(t)) \neq \emptyset$  for every  $t \in [0, 1]$  and  $H^{su}\gamma(0) \in \mathcal{W}_\delta^c(z)$ .

Let us prove the claim. As  $\mathcal{W}^c$  is a foliation tangent to a continuous bundle we can consider  $\delta' > 0$  so that whenever  $z$  and  $z'$  are points in  $M$  satisfying  $d(z, z') < \delta'$  and  $\gamma : [0, 1] \rightarrow \mathcal{W}^c(z')$  is a curve of length at most  $L$  with  $\gamma(0) = z'$ , then there exists a continuous curve  $H\gamma : [0, 1] \rightarrow \mathcal{W}^c(z)$  with  $H\gamma(0) = z$  and  $d(\gamma(t), H\gamma(t)) \leq \frac{\delta}{2}$  for every  $t \in [0, 1]$ .

The curve  $H\gamma$  is an auxiliary curve used to define  $H^{su}\gamma$ . Indeed, we can consider  $P^u(t)$  as the intersection point of  $\mathcal{W}_\delta^u(\gamma(t))$  and  $\mathcal{W}_\delta^s(\mathcal{W}_\delta^c(H\gamma(t)))$  for every  $t \in [0, 1]$ . Then  $H^{su}\gamma(t)$  can be defined as the unique point in  $\mathcal{W}_\delta^c(H\gamma(t))$  such that  $P^u(t)$  is contained in  $\mathcal{W}_\delta^s(H^{su}\gamma(t))$ . This proves the claim.

Let  $N > 0$  be such that  $d(f^n(x), f^n(y)) < \delta'$  for every  $n \geq N$ . For simplicity, let  $\gamma$  denote the curve  $f^N \circ \eta$ . Then  $H^{su}(f^N \circ \gamma)$  is well defined for every  $n \geq 0$ . Moreover, as  $f$  preserves  $\mathcal{W}^s$ ,  $\mathcal{W}^c$  and  $\mathcal{W}^u$ -leaves, the special choice of  $H^{su}$  gives us the following invariance: the curve  $H^{su}(f^n \circ \gamma)$  coincides with the curve  $f^n \circ H^{su}\gamma$  for every  $n \geq 0$ .

In particular,  $f^n(P^u(t))$  lies in  $\mathcal{W}_\delta^u(f^n \circ \gamma(t))$  for every  $t \in [0, 1]$  and  $n \geq 0$ . Iterating  $n$  times backwards yields that  $P^u(t)$  lies in  $\mathcal{W}_{C(1/2)^{n/\ell}\delta}^u(\gamma(t))$  for some constants  $\ell \in \mathbb{Z}^+$  and  $C > 0$  given by the partial hyperbolicity of  $f$ .

It follows that  $P^u(t) = \gamma(t)$  for every  $t \in [0, 1]$ . That is,  $f^N \circ \eta$  is contained in  $\mathcal{W}^s(\mathcal{W}^c(f^N(x)))$ . Then  $\eta$  is contained in  $\mathcal{W}^s(\mathcal{W}^c(x))$ .  $\square$

*Proof of Proposition 3.4.4.* Suppose  $f \in \text{PH}_{c=1}(M)$  acts quasi-isometrically on a center foliation  $\mathcal{W}^c$ .

Given  $x \in M$  and  $y \in \mathcal{W}^s(\mathcal{W}^c(x))$  let us see first that  $\mathcal{W}^c(y)$  is contained in  $\mathcal{W}^s(\mathcal{W}^c(x))$ . Indeed, as  $f$  acts quasi-isometrically on  $\mathcal{W}^c$  for every  $l > 0$  there exists  $L > 0$  such every  $f$ -iterate of  $\mathcal{W}_l^c(y)$  is bounded in length by  $L$ . By Lemma 3.4.5 it follows that  $\mathcal{W}_l^c(y) \subset \mathcal{W}^s(\mathcal{W}^c(x))$ . Since this happens for every  $l > 0$  it follows that  $\mathcal{W}^c(y) \subset \mathcal{W}^s(\mathcal{W}^c(x))$ .

By Lemma 2.0.1 for every  $x \in M$  the set  $\mathcal{W}^s(\mathcal{W}^c(x))$  is a  $C^1$  injectively immersed submanifold tangent to  $E^s \oplus E^c$ . As  $\mathcal{W}^s(\mathcal{W}^c(x))$  is saturated by  $\mathcal{W}^s$  and  $\mathcal{W}^c$  leaves it follows that its intrinsic metric is complete and that, if  $y \in \mathcal{W}^s(\mathcal{W}^c(x))$ , then  $\mathcal{W}^s(\mathcal{W}^c(y)) = \mathcal{W}^s(\mathcal{W}^c(x))$ .

Then  $\{\mathcal{W}^s(\mathcal{W}^c(x))\}_{x \in M}$  defines a partition of  $M$  whose elements are the leaves of an  $f$ -invariant foliation tangent to  $E^s \oplus E^c$  and subfoliated by leaves of  $\mathcal{W}^s$  and  $\mathcal{W}^c$ . Thus a center-stable invariant foliation  $\mathcal{W}^{cs}$  whose leaves are *complete* (meaning that  $\mathcal{W}^{cs}(x) = \mathcal{W}^s(\mathcal{W}^c(x))$  for every  $x \in M$ ). The same arguments show that the sets  $\{\mathcal{W}^u(\mathcal{W}^c(x))\}_{x \in M}$  define an invariant center-unstable foliation with complete leaves.  $\square$

Note that acting quasi-isometrically on a center foliation is preserved under finite lifts and finite powers. One can build other examples of quasi-isometrically center actions as follows:

**Example 3.4.6.** Let  $\varphi_t : M \rightarrow M$  be an Anosov flow,  $\pi : N \rightarrow M$  be a finite cover of  $M$  and  $\tilde{\varphi}_t : N \rightarrow N$  be the lift of  $\varphi_t$  to  $N$ . Note that  $\tilde{\varphi}_t$  is also an Anosov flow in  $N$ . One can define  $f : N \rightarrow N$  as the composition of the time 1 map of  $\tilde{\varphi}_t$  with a non-trivial deck transformation of order  $k > 1$ . It follows



that  $f$  is a partially hyperbolic diffeomorphism acting quasi-isometrically on the center (in fact, isometrically) that is not a discretized Anosov flow or a partially hyperbolic skew-product, but such that the power  $g = f^k$  is a discretized Anosov flow.

A construction from [BPP16] gives an example of a system  $f \in \text{PH}_{c=1}(M^3)$  acting quasi-isometrically on an  $f$ -invariant center foliation  $\mathcal{W}^c$  such that  $f^k$  is not a discretized Anosov for every  $k \neq 0$  nor  $\mathcal{W}^c$  is uniformly compact. This is done via a *h-transversality surgery* over the time 1 map of a non-transitive Anosov flow. One can easily check from its construction that this example is not transitive.

One more type of examples of partially hyperbolic diffeomorphisms acting quasi-isometrically on a center foliation can be constructed by taking the product  $f \times A : M \times N \rightarrow M \times N$  of a discretized Anosov flow  $f : M \rightarrow M$  and an Anosov map  $A : N \rightarrow N$ .

In view of the above known examples of quasi-isometrically center actions we may ask the following:

**Question 3.4.7.** *Suppose  $f \in \text{PH}_{c=1}(M)$  acts quasi-isometrically on a center foliation  $\mathcal{W}^c$ . If  $\mathcal{W}^c$  is transitive (i.e. has a dense leaf) then does there exist  $k \in \mathbb{Z}^+$  such that  $f^k$  is a discretized Anosov flow?*

**Remark 3.4.8** (Relation with the notion of ‘neutral center’). In [Z17] and [BZ20] the notions of partially hyperbolic diffeomorphisms that are *neutral along center* and *topologically neutral along center* were introduced. In [BZ20] a positive answer to Question 3.4.7 in dimension 3 is obtained for these class of systems.

A partially hyperbolic diffeomorphism  $f$  is called *neutral along center* if there exists  $C > 1$  satisfying  $1/C < \|Df^n|_{E^c(x)}\| < C$  for any  $x \in M$  and  $n \in \mathbb{Z}$ . And is called *topologically neutral along center* if for any  $\epsilon > 0$  there exists  $\delta > 0$  so that any  $C^1$  center curve  $\sigma$  of length bounded by  $\delta$  has all its images  $f^n(\sigma), n \in \mathbb{Z}$ , bounded in length by  $\epsilon$ . One easily checks that if  $f$  is neutral, then it is topologically neutral.

By [RHRHU07, Corollary 7.6] topologically neutral systems admit a center foliation and it is immediate that they act quasi-isometrically on it. However, acting quasi-isometrically on a center foliation is strictly more general. It is easy to see that being topologically neutral along center forbids the existence of a hyperbolic periodic point. In particular, every discretized Anosov flow with a hyperbolic periodic point is an example of a quasi-isometrically action on  $\mathcal{W}^c$  that is not topologically neutral along center. This includes Examples 7.2.1 and 7.2.2.

### 3.5 Uniqueness of $cs$ and $cu$ foliations

The goal of this section is to show uniqueness of invariant center-stable and center-unstable foliation for discretized Anosov flows, and more generally for partially hyperbolic systems acting quasi-isometrically on a one-dimensional center foliation:

**Proposition 3.5.1.** *Suppose  $f \in \text{PH}_{c=1}(M)$  acts quasi-isometrically on an  $f$ -invariant center foliation  $\mathcal{W}^c$ . Let  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  be the foliations given by Proposition 3.4.4. Then  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  are the only  $f$ -invariant foliations tangent to  $E^s \oplus E^c$  and  $E^c \oplus E^u$ , respectively.*

We will rely on the following lemma.

**Lemma 3.5.2.** *Suppose  $f \in \text{PH}(M)$  admits an  $f$ -invariant center-stable foliation  $\mathcal{W}^{cs}$ . If  $\eta$  is a  $C^1$  curve that is not contained in a leaf of  $\mathcal{W}^{cs}$  then  $\lim_{n \rightarrow +\infty} \text{length}(f^n \circ \eta) = \infty$ .*

*Proof.* Let  $\delta > 0$  be a constant as in Lemma 2.0.1. As the invariant bundles vary continuously in  $M$  we can suppose that  $\delta$  is small enough so that at scale  $\delta$  the invariant bundles are nearly constant (more precisely, one can consider for example a constant  $\delta = \delta(f) > 0$  and a metric in  $M$  as in Lemma 4.2.1). In particular,  $\delta > 0$  is such that for every  $0 < \delta' \leq \delta$ , if  $x, y \in M$  satisfy  $d(x, y) < \delta'$ , then  $\mathcal{W}_{2\delta'}^\sigma(x)$  and  $\mathcal{W}_{2\delta'}^\sigma(y)$  intersect and the intersection point is unique for every  $(\sigma, \sigma') \in \{(cs, u), (cu, s)\}$ .

Suppose  $\eta$  is a  $C^1$  curve that is not contained in a leaf of  $\mathcal{W}^{cs}$ . Let us see that that  $\lim_{n \rightarrow +\infty} \text{length}(f^n \circ \eta) = \infty$ . Note that it is enough to show this for  $\text{length}(\eta) < \delta/4$  since otherwise one can divide  $\eta$  in finite pieces of length less than  $\delta/4$  and argue from there. Then, suppose from now on that  $\text{length}(\eta) < \delta/4$ .

Let  $x$  be a point in  $\eta$ . For every  $y \in \mathcal{W}_{\delta/4}^u(x)$  let  $D^{cs}(y)$  be the intersection of  $\mathcal{W}_\delta^{cs}(y)$  with  $\mathcal{W}_\delta^u(\mathcal{W}_{\delta/4}^{cs}(x))$ . It follows that  $D := \bigcup_{y \in \mathcal{W}_{\delta/4}^u(x)} D^{cs}(y)$  is an open subset of  $M$  that is subfoliated by  $u$ -plaques and  $cs$ -plaques. The latter being the plaques  $\{D^{cs}(y)\}_{y \in \mathcal{W}_{\delta/4}^u(x)}$ . Analogously  $f^n(D)$  is subfoliated by  $u$ -plaques and the  $cs$ -plaques  $\{f^n D^{cs}(y)\}_{y \in \mathcal{W}_{\delta/4}^u(x)}$  for every  $n > 0$ .

Note that, since  $\text{length}(\eta) < \delta/4$  then  $\eta$  is contained in  $D$ . Informally, forwards iterates of  $f$  will separate indefinitely the  $cs$ -plaques of  $D$ . If  $\eta$  is not contained in a unique  $cs$ -plaque this will force the length of  $\eta$  to increase indefinitely.

We will work with the intrinsic metric in  $D$  and in its forward iterates  $\{f^n D\}_{n > 0}$ . Given  $D^{cs}(y)$  and  $D^{cs}(y')$  two different  $cs$ -plaques in  $D$  let us denote  $d_u(D^{cs}(y), D^{cs}(y'))$  the infimum length among all unstable arcs inside  $u$ -plaques of  $D$  joining  $D^{cs}(y)$  and  $D^{cs}(y')$ . Analogously for every  $f^n D$ .

Note that, as backwards iterates of  $f$  contract distances uniformly inside  $\mathcal{W}^u$ -leaves, then for every pair of disjoint  $cs$ -plaques  $D^{cs}(y)$  and  $D^{cs}(y')$  in  $D$  there exists  $N > 0$  such that  $d_u(f^n D^{cs}(y), f^n D^{cs}(y')) > 2\delta$  for every  $n \geq N$ .

Moreover, we claim that if for some  $n > 0$  one has that the distance  $d_u(f^n D^{cs}(y), f^n D^{cs}(y'))$  is greater than  $\delta$  and  $\mathcal{W}_\delta^u(f^n D^{cs}(y))$  is contained in  $f^n D$  then in the intrinsic metric of  $f^n D$  every point of  $D^{cs}(y)$  is at distance greater than  $\delta/2$  from every other point in  $f^n D^{cs}(y')$ . Indeed, by contradiction, if  $z \in f^n D^{cs}(y)$  and  $z' \in f^n D^{cs}(y')$  are at distance less than  $\delta/2$  and  $\mathcal{W}_\delta^u(f^n D^{cs}(y)) \subset f^n D$  then  $\mathcal{W}_\delta^u(z)$  intersects  $\mathcal{W}_\delta^u(z')$  and this intersection point needs to be a point in  $f^n D^{cs}(y')$  since  $\mathcal{W}_\delta^u(z)$  is contained in  $f^n D^{cs}(y')$ . It follows that  $d_u(f^n D^{cs}(y), f^n D^{cs}(y')) < \delta$  and we get to a contradiction. This proves the claim.

Finally, given any constant  $L > 0$ , let  $K > 0$  be an integer larger than  $L/2\delta$ . As  $\eta$  is not contained in  $\mathcal{W}^{cs}(x)$  then there exist  $K$  different  $cs$ -plaques in  $D$  intersecting  $\eta$ . Let us denote them as  $D^{cs}(y_1), \dots, D^{cs}(y_K)$ . There exists  $N > 0$  such that  $d_u(f^n D^{cs}(y_i), f^n D^{cs}(y_j)) > 2\delta$  for every  $n \geq N$  and  $i \neq j$ .

Moreover, for every  $1 \leq i \leq K$  there exist  $\epsilon_i$  such that  $\mathcal{W}_{\epsilon_i}^u(D^{cs}(y_i))$  is contained in  $D$ . By taking  $N$  larger, if needed, one can ensure that  $\mathcal{W}_\delta^u(f^n D^{cs}(y_i))$  is contained in  $f^n D$  for every  $n \geq N$ .

It follows that  $\text{length}(f^n \circ \eta) > L$  for every  $n \geq N$  since  $f^n \circ \eta$  must contain at least  $K$  disjoint subsegments of length at least  $\delta/2$ , each one of them corresponding to an intersection of  $f^n \circ \eta$  with  $f^n D^{cs}(y_i)$  for every  $1 \leq i \leq K$ .  $\square$

**Remark 3.5.3.** From Lemma 3.5.2 one can easily justify that every  $f \in \text{PH}(M)$  admitting an  $f$ -invariant center-stable foliation  $\mathcal{W}^{cs}$  satisfies that the leaves of  $\mathcal{W}^{cs}$  are saturated by leaves of  $\mathcal{W}^s$ .

Indeed, for every  $x \in M$  and  $y \in \mathcal{W}^s(x)$  one can join  $x$  and  $y$  by a  $C^1$  curve  $\eta$  contained in  $\mathcal{W}^s(x)$ . Since  $\eta$  gets contracted uniformly by forward iterates of  $f$  it follows that  $\eta$  must be contained in  $\mathcal{W}^{cs}(x)$ . Then  $\mathcal{W}^s(x) \subset \mathcal{W}^{cs}(x)$ .

*Proof of Proposition 3.5.1.* Suppose  $f \in \text{PH}_{c=1}(M)$  acts quasi-isometrically on an  $f$ -invariant center foliation  $\mathcal{W}^c$ . By Proposition 3.4.4 there exist  $f$ -invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  whose leaves are characterized as  $\mathcal{W}^{cs}(x) = \mathcal{W}^s(\mathcal{W}^c(x))$  and  $\mathcal{W}^{cu}(x) = \mathcal{W}^u(\mathcal{W}^c(x))$  for every  $x \in M$ .

Suppose  $\mathcal{W}_1^{cs}$  is an  $f$ -invariant center-stable foliation. As  $f$  acts quasi-isometrically in  $\mathcal{W}^c$  then by Lemma 3.5.2 the leaf  $\mathcal{W}^c(y)$  needs to be contained in  $\mathcal{W}_1^{cs}(x)$  for every  $x \in M$  and  $y \in \mathcal{W}_1^{cs}(x)$ . Moreover, as pointed out in Remark 3.5.3 the leaf  $\mathcal{W}^s(y)$  must also be contained in  $\mathcal{W}_1^{cs}(x)$  for every  $y \in \mathcal{W}_1^{cs}(x)$ . It follows that  $\mathcal{W}^{cs}(x) = \mathcal{W}^s(\mathcal{W}^c(x))$  needs to be a subset of  $\mathcal{W}_1^{cs}(x)$  for every  $x \in M$ .

For the intrinsic leaf metric induced by the Riemannian metric in  $M$  each leaf of  $\mathcal{W}^{cs}$  and  $\mathcal{W}_1^{cs}$  is a complete metric space. This implies that the boundary of  $\mathcal{W}^{cs}(x)$  in  $\mathcal{W}_1^{cs}(x)$  needs to be empty. We conclude that  $\mathcal{W}^{cs}(x) = \mathcal{W}_1^{cs}(x)$  for every  $x \in M$ .

Analogously for  $f$ -invariant center-unstable foliations.  $\square$

### 3.6 Center flow and flow center foliation

Suppose  $f$  is a discretized Anosov flow of the form  $f(x) = \varphi_{\tau(x)}^c(x)$  as in Definition 3.2.1. By Proposition 3.2.2 the flow lines of  $\varphi_t^c : M \rightarrow M$  form a center foliation  $\mathcal{W}^c$  whose leaves are fixed by  $f$ .

By Remark 3.4.2, Proposition 3.4.4 and Proposition 3.5.1 we can deduce the following characterization of  $\mathcal{W}^c$ .

**Remark 3.6.1.** The foliation  $\mathcal{W}^c$  is:

- The only foliation tangent to  $E^c$  that is the intersection of  $f$ -invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$ .
- The only  $f$ -invariant foliation tangent to  $E^c$  such that  $f$  acts quasi-isometrically on it.

Moreover, if  $f$  is of the form  $f(x) = \phi_{\rho(x)}^c(x)$  for every  $x \in M$  as in Definition 3.2.1, for some other flow  $\phi_t^c : M \rightarrow M$  and continuous function  $\rho : M \rightarrow \mathbb{R}$ , then  $\phi_t^c$  needs to be a reparametrization of  $\varphi_t^c$ . That is:

**Remark 3.6.2.** The flow  $\varphi_t^c : M \rightarrow M$  is, modulo reparametrizations, the only flow satisfying Definition 3.2.1.

In light of the above remarks, we will designate from now on  $\mathcal{W}^c$  as the *flow center foliation of  $f$* . And the flow  $\varphi_t^c : M \rightarrow M$  as the *center flow of  $f$* .

In view of Remark 3.2.3, if not otherwise stated we may implicitly assume from now on that the center flow  $\varphi_t^c : M \rightarrow M$  is parametrized by arc-length.

It would be interesting to know if, in general, the flow center foliation of a discretized Anosov is the only  $f$ -invariant center foliation. Or at least if it is the only center foliation whose leaves are individually fixed by  $f$ . We do not have a general proof for either of these statements.

### 3.7 Topological Anosov flows

**Definition 3.7.1.** We say that a flow  $\varphi_t : M \rightarrow M$  is a *topological Anosov flow* if it is a continuous flow, with  $\frac{\partial \varphi_t}{\partial t}|_{t=0}$  a continuous vector field without singularities, such that it preserves two topologically transverse continuous foliations  $\mathcal{F}^{ws}$  and  $\mathcal{F}^{wu}$  satisfying the following:

- (i) The foliation  $\mathcal{F}^{ws} \cap \mathcal{F}^{wu}$  is the foliation given by the orbits of  $\varphi_t$ .
- (ii) Given  $x$  in  $M$  and  $y \in \mathcal{F}^{ws}(x)$  (resp.  $y \in \mathcal{F}^{wu}(x)$ ) there exists an increasing continuous reparametrization  $h : \mathbb{R} \rightarrow \mathbb{R}$  satisfying that  $d(\varphi_t(x), \varphi_{h(t)}(y)) \rightarrow 0$  as  $t \rightarrow +\infty$  (resp.  $t \rightarrow -\infty$ ).
- (iii) There exists  $\epsilon > 0$  such that for every  $x \in M$  and  $y \in \mathcal{F}_\epsilon^{ws}(x)$  (resp.  $y \in \mathcal{F}_\epsilon^{wu}(x)$ ), with  $y$  not in the same orbit as  $x$ , and for every increasing continuous reparametrization  $h : \mathbb{R} \rightarrow \mathbb{R}$  with  $h(0) = 0$ , there exists  $t \leq 0$  (resp.  $t \geq 0$ ) such that  $d(\varphi_t(x), \varphi_{h(t)}(y)) > \epsilon$ .

It is worth noting that Definition 3.7.1 is a priori more restrictive than other definitions of topological Anosov flows appearing in the literature since we are asking for  $\frac{\partial \varphi_t}{\partial t}|_{t=0}$  to be a continuous vector field.

It has been a long standing problem to determine whether in general every topological Anosov flow is orbit equivalent to an Anosov flow. Just recently in [Sh21] every transitive topological Anosov flow in dimension 3 (for a more general definition of topological Anosov flow that covers Definition 3.2.1) has been shown to be orbit equivalent to a smooth Anosov flow.

The following is the main goal of this section.

**Proposition 3.7.2.** *Let  $f$  be a discretized Anosov flow and  $\varphi_t^c$  be the center flow of  $f$ . Then  $\varphi_t^c : M \rightarrow M$  is a topological Anosov flow.*

*Proof.* The map  $f$  is of the form  $f(x) = \varphi_{\tau(x)}^c(x)$  for some  $\tau : M \rightarrow \mathbb{R}$  continuous. By Proposition 3.2.2 the function  $\tau$  has constant sign. Without loss of generality we can assume that  $\tau$  is positive, otherwise we can argue analogously using  $f^{-1}$  instead of  $f$ .

The flow  $\varphi_t^c$  is a continuous flow with  $\frac{\partial \varphi_t^c}{\partial t}|_{t=0}$  a continuous vector field. By Proposition 3.4.4 the map  $f$  is dynamically coherent with center-stable foliation  $\mathcal{W}^{cs}$  and center-unstable foliation  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  is the flow center foliation of  $f$ . Hence property (i) in the definition of topological Anosov flow is immediately satisfied for  $\mathcal{F}^{ws} = \mathcal{W}^{cs}$  and  $\mathcal{F}^{wu} = \mathcal{W}^{cu}$ .

Let us see property (ii). Suppose  $x$  and  $y$  are points in  $M$  such that  $y$  belongs to  $\mathcal{W}^{cs}(x)$ . By Proposition 3.4.4 the leaf  $\mathcal{W}^{cs}(x)$  coincides with  $\mathcal{W}^s(\mathcal{W}^c(x))$ . Then  $y$  belongs to  $\mathcal{W}^s(z)$  for some  $z \in \mathcal{W}^c(x)$ .

Let us assume first that  $z = x$ . Consider  $\gamma_y : \mathbb{R} \rightarrow M$  the continuous curve in  $\mathcal{W}^c(y)$  such that  $\gamma_y(0) = y$  and  $\gamma_y(t) \in \mathcal{W}^s(\varphi_t^c(x))$  for every  $t$ . The curve  $\gamma_y$  is the transport by center holonomy of  $y$  along stable transversals with respect to the  $\varphi_t^c$ -orbit of  $x$ .

The key property to note is that  $\gamma_y(\tau(x)) = f(y)$  for every  $y \in \mathcal{W}^s(x)$ . In fact, more generally, if  $y' = \gamma_y(t)$  for some  $t \in \mathbb{R}$  then one has that

$$f(y') = \gamma_y(\tau(\varphi_t^c(x))). \quad (3.7.1)$$

This property follows immediately from the continuity of  $\tau$  and local product structure of the foliations  $\mathcal{W}^c$  and  $\mathcal{W}^s$  restricted to  $\mathcal{W}^{cs}(x)$ . For more details, one can see Section 3.9 for a precise characterization of discretized Anosov flows in terms of center holonomy.

Let  $R > 0$  denote a constant such that  $\gamma_y(t) \in \mathcal{W}_R^s(\varphi_t^c(x))$  for every  $t \in [0, \tau(x)]$ . This constant exists since the stable distance  $d_s(\gamma_y(t), \varphi_t^c(x))$  varies continuously with  $t$ . Then, as  $f$  contracts distances uniformly inside stable leaves, it follows from (3.7.1) that  $\lim_{t \rightarrow +\infty} d(\gamma_y(t), \varphi_t^c(x)) = 0$ . Defining  $h_y : \mathbb{R} \rightarrow \mathbb{R}$  as the increasing reparametrization such that  $\varphi_{h_y(t)}^c(y) = \gamma_y(t)$  for every  $t$  we obtain (ii) for the case  $z = x$ .

If  $z$  is different from  $x$  consider some  $t_0 > 0$  and  $h : (-\infty, t_0] \rightarrow \mathbb{R}$  continuous and increasing so that  $h(0) = 0$  and  $y' = \varphi_{h(t_0)}^c(y)$  lies in  $\mathcal{W}^s(x')$  for  $x' = \varphi_{t_0}^c(x)$ . Defining as above  $h_{y'} : \mathbb{R} \rightarrow \mathbb{R}$  so that  $\varphi_{h_{y'}(t)}^c(y') = \varphi_t^c(x')$  for every  $t$ , then the function  $h$  can be extended to  $h : \mathbb{R} \rightarrow \mathbb{R}$  by the formula  $h(t) = h(t_0) + h_{y'}(t - t_0)$  for every  $t > t_0$ . It follows that  $\lim_{t \rightarrow +\infty} d(\varphi_t^c(x), \varphi_{h(t)}^c(y)) = 0$  as above.

In the case  $y$  lies in  $\mathcal{W}^{cu}(x)$  one argues analogously for  $\varphi_t^c$ -past iterations. This settles property (ii).

Finally, let us see property (iii). As the bundles  $E^c$  and  $E^u$  vary continuously there exists a small constant  $\epsilon > 0$  such that for every  $z$  and  $z'$  satisfying  $z' \in \mathcal{W}_\epsilon^{cu}(z)$  it follows that  $\mathcal{W}_{2\epsilon}^c(z')$  and  $\mathcal{W}_{2\epsilon}^u(z)$  intersect and that this intersection point is unique.

Let  $x$  and  $y$  be points in  $M$  such that  $y \in \mathcal{W}_\epsilon^{cu}(x)$ . Suppose that  $h : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing continuous reparametrization with  $h(0) = 0$  such that  $d(\varphi_t^c(x), \varphi_{h(t)}^c(y)) \leq \epsilon$  for every  $t \geq 0$ . Let  $y'$  denote the intersection  $\mathcal{W}_{2\epsilon}^c(y)$  and  $\mathcal{W}_{2\epsilon}^u(x)$  and let  $\gamma_{y'}(t) = \mathcal{W}_{2\epsilon}^c(\varphi_{h(t)}^c(y)) \cap \mathcal{W}_{2\epsilon}^u(\varphi_t^c(x))$  for every  $t \geq 0$ . The curve  $\gamma_{y'}$  is no other than the transport by center holonomy of  $y'$  along unstable transversals with respect to the  $\varphi_t^c$ -orbit of  $x$ . In analogy with (3.7.1) it follows that  $f(y') = \gamma_{y'}(\tau(x))$ , so  $f(y')$  lies in  $\mathcal{W}_{2\epsilon}^u(f(x))$ . Inductively,  $f^n(y')$  lies in  $\mathcal{W}_{2\epsilon}^u(f^n(x))$  for every  $n > 0$ . Iterating  $n$  times backwards and taking limit with  $n$  we conclude that  $y'$  needs to coincide with  $x$ . Then  $y$  lies in  $\mathcal{W}_{2\epsilon}^c(x)$  and, in particular, lies in the  $\varphi_t^c$ -orbit of  $x$ .

In the case  $x$  and  $y$  are points such that  $y \in \mathcal{W}_\epsilon^{cs}(x)$  one can argue analogously for past iterates of  $f$  and  $\varphi_t^c$ . Property (iii) is settled.  $\square$

Let us end this section with a statement showing that some classical properties of Anosov flows are satisfied (by means of the same type of arguments) by the topological Anosov flows arising as center foliations of discretized Anosov flows. Some of these properties will be needed later in the text. For the sake of completeness we will sketch their proofs.

We say that a leaf of a foliation of dimension  $d > 0$  is a *plane* if it is

homeomorphic to  $\mathbb{R}^d$ , and that it is a *cylinder* if it is homeomorphic to a fiber bundle over the circle whose fibers are homeomorphic to  $\mathbb{R}^{d-1}$ . We say that two foliations  $\mathcal{W}$  and  $\mathcal{W}'$  have *global product structure* if  $\mathcal{W}(x)$  and  $\mathcal{W}'(y)$  intersect for every pair  $x$  and  $y$ , and this intersection is a unique point.

**Proposition 3.7.3.** *Suppose  $f$  is a discretized Anosov flow. Let  $\varphi_t^c : M \rightarrow M$  and  $\mathcal{W}^c$  be the center flow and flow center foliation of  $f$ , respectively. Let  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  denote the center-stable and center-unstable foliations such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ . Then:*

1. *Every leaf of  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  is a plane or a cylinder.*
2. *If a leaf  $\mathcal{W}^{cs}(x)$  is a plane then  $\mathcal{W}^c$  and  $\mathcal{W}^s$  restricted to  $\mathcal{W}^{cs}(x)$  have global product structure. Analogously for  $\mathcal{W}^{cu}$ -leaves.*
3. *If a leaf  $\mathcal{W}^{cs}(x)$  is a cylinder then  $\mathcal{W}^c$  restricted to  $\mathcal{W}^{cs}(x)$  contains a unique compact leaf  $L$  and the omega limit set under  $\varphi_t^c$  of every point  $y$  in  $\mathcal{W}^{cs}(x)$  is  $L$ . Analogously for  $\mathcal{W}^{cu}$ -leaves and alpha limit sets.*
4. *There exists at least one compact leaf of  $\mathcal{W}^c$ .*

*Proof.* Let  $x$  be a point in  $M$ . For every  $y \in \mathcal{W}^s(x)$  we can define  $\gamma_y : \mathbb{R} \rightarrow M$  as the continuous curve in  $\mathcal{W}^c(y)$  such that  $\gamma_y(0) = y$  and  $\gamma_y(t) \in \mathcal{W}^s(\varphi_t^c(x))$  for every  $t$ . The curve  $\gamma_y$  is a transport by center holonomy of  $y$  with respect to the  $\varphi_t^c$ -orbit of  $x$ . As in the previous proposition, note the key property:  $\gamma_y(\tau(x)) = f(y)$  for every  $y \in \mathcal{W}^s(x)$ .

If  $x$  is a periodic point for  $\varphi_t^c$  of period  $t_x > 0$  let us denote  $H(y) \in \mathcal{W}^s(x)$  to the point  $\gamma_y(t_x)$  for every  $y \in \mathcal{W}^s(x)$ . For some  $N > 0$  large enough  $H^N : \mathcal{W}^s(x) \rightarrow \mathcal{W}^s(x)$  is a contraction with  $x$  the unique fixed point. In this case  $\mathcal{W}^{cs}$  is a cylinder and it is immediate to check that the  $\varphi_t^c$ -omega limit of every point in  $\mathcal{W}^{cs}(x)$  is the orbit of  $x$ .

If  $x$  is not periodic for  $\varphi_t^c$  but some point  $y$  in  $\mathcal{W}^s(x)$  is periodic then we can argue as above and conclude that  $\mathcal{W}^{cs}(x)$  is a cylinder and that the  $\varphi_t^c$ -omega limit of every point in  $\mathcal{W}^{cs}(x)$  is the orbit of  $y$ .

If none of the points in  $\mathcal{W}^s(x)$  is periodic for  $\varphi_t^c$  then for every  $y \in \mathcal{W}^s(x)$  the point  $\gamma_y(t)$  lies in  $\mathcal{W}^s(x)$  if and only if  $t = 0$ , otherwise a contraction  $H^N : \mathcal{W}^s(x) \rightarrow \mathcal{W}^s(x)$  as above can be constructed and some  $\varphi_t^c$ -periodic point in  $\mathcal{W}^s(x)$  should be found. It follows that  $\bigcup_{y \in \mathcal{W}^s(x)} \gamma_y(t) = \mathcal{W}^s(\varphi_t^c(x))$  for every  $t$  and, since  $\mathcal{W}^{cs}(x) = \mathcal{W}^s(\mathcal{W}^c(x))$  by Proposition 3.4.4, then  $\mathcal{W}^{cs}(x)$  is a plane and  $\mathcal{W}^c$  and  $\mathcal{W}^s$  have a global product structure inside  $\mathcal{W}^{cs}(x)$ .

Properties (1), (2) and (3) are settled. Let us see that  $\varphi_t^c$  must have at least one periodic orbit and this will settle the last property.

For some  $x$  in  $M$  let  $z$  be a point in the  $\varphi_t^c$ -omega limit of  $x$ . Consider  $D$  a small  $C^1$  disc transverse to  $\mathcal{W}^c$  and containing  $z$  in its interior. Let  $D$  be



such that the leaves of  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  intersect  $D$  in  $C^1$  discs. For every  $z' \in D$  let  $w^s(z')$  and  $w^u(z')$  denote the connected components of  $\mathcal{W}^{cu}(z') \cap D$  and  $\mathcal{W}^{cs}(z') \cap D$  containing  $z'$ , respectively.

Let  $D' \subset D$  be such that if  $z', z'' \in D'$  then  $w^s(z') \cap w^u(z'') \neq \emptyset$  and  $w^u(z') \cap w^s(z'') \neq \emptyset$ . For every  $z' \in D'$  let  $\pi^u(z')$  denote the point in  $w^s(z')$  such that  $w^u(z') \cap w^s(z) = \pi^u(z')$ .

Let  $t_x > 0$  be a time such that  $\varphi_{t_x}^c(x)$  lies in  $D'$  close to  $z$  and let  $T_x > t_x$  be a large enough time so that  $\varphi_{T_x}^c(x)$  lies also in  $D'$ , is close to  $z$  and the Poincaré return map  $P$  from  $w^s(\varphi_{t_x}^c(x))$  to  $D'$  is well defined. Then  $\pi^u \circ P$  needs to be a contraction if  $T_x$  is large enough. Let  $z'$  denote the fixed point of this contraction. It follows that  $P(z')$  lies in  $w^u(z')$  so there exists some positive time  $t_{z'}$  close to  $T_x - t_x$  such that  $\varphi_{t_{z'}}^c(z')$  lies in  $\mathcal{W}^u(z')$ . By (3) it follows that  $\mathcal{W}^{cu}(z')$  has to be a cylinder leaf and, as a consequence, it has to contain a periodic orbit for  $\varphi_t^c$ .  $\square$

### 3.8 Equivalence with other definitions

Discretized Anosov flows have been richly studied in the literature, though not always under this name. Without trying to be exhaustive, it is worth establishing that many of these classes studied before are in fact discretized Anosov flows as in Definition 3.2.1. This is one of the primary goals of this chapter.

In [BFFP19], [BFP20], [BG21] and [GM22] a map  $f \in \text{PH}_{c=1}(M)$  was called a ‘discretized Anosov flow’ if it satisfied the following: there exist a topological Anosov flow  $\varphi_t : M \rightarrow M$  and a continuous function  $\tau : M \rightarrow \mathbb{R}_{>0}$  such that  $f(x) = \varphi_{\tau(x)}(x)$  for every  $x$  in  $M$ .

As a direct consequence of Proposition 3.7.2 and Proposition 3.2.2 item (ii) we obtain:

**Corollary 3.8.1.** *The definition of discretized Anosov flow given in [BFP20], [BFFP19], [BG21] and [GM22] is equivalent with Definition 3.2.1.*

It is worth noting the following two other classes of systems studied before that are also discretized Anosov flows.

**Remark 3.8.2.** Partially hyperbolic diffeomorphisms on 3-manifolds were investigated in the seminal article [BW05]. The statement of [BW05, Theorem 2. items 1. and 2.] can be paraphrased as the following criterion for detecting discretized Anosov flows (in particular, using Proposition 3.3.1 to conclude):

*Suppose  $f \in \text{PH}_{c=1}(M^3)$  is transitive and dynamically coherent with invariant foliations  $\mathcal{W}^{cs}$ ,  $\mathcal{W}^{cu}$  and  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ . Then  $f^n$  is a discretized*



Anosov flow for some  $n > 0$  if and only if there exists a periodic compact leaf  $\eta \in \mathcal{W}^c$  and every center leaf through  $\mathcal{W}_{loc}^s(\eta)$  is also periodic by  $f$ .

**Remark 3.8.3.** In [BG09] and [BG10] diffeomorphisms in  $\text{PH}_{c=1}(M)$  that are Axiom A and admit a center foliation tangent to an Anosov vector field  $X^c$  were studied. In [BG09] it is shown that these systems can be written as  $f(x) = X_{\tau(x)}^c(x)$  for some  $\tau : M \rightarrow \mathbb{R}^+$  continuous. It follows that, in particular, they are all discretized Anosov flows.

Finally, we can establish the equivalence with the notion of *flow-type partially hyperbolic diffeomorphism*. In [BFT20] a diffeomorphism  $f \in \text{PH}_{c=1}(M)$  is called *flow-type* if it is dynamically coherent with orientable center foliation  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  admitting a compact leaf and such that  $f$  can be written as  $f(x) = \varphi_{\tau(x)}^c(x)$  for every  $x \in M$ , where  $\varphi_t^c$  is a flow of unit positive speed along the leaves of  $\mathcal{W}^c$  and  $\tau : M \rightarrow \mathbb{R}_{>0}$  is some continuous function.

As a consequence of what we have seen so far we get the following:

**Corollary 3.8.4.** *The definition of flow-type partially hyperbolic diffeomorphism as given in [BFT20] is equivalent with Definition 3.2.1 of a discretized Anosov flow.*

*Proof.* It is immediate to check that every flow-type partially hyperbolic diffeomorphism is a discretized Anosov flow as in Definition 3.2.1.

Conversely, suppose  $f$  is a discretized Anosov flow and let  $\varphi_t^c$  and  $\mathcal{W}^c$  denote the center flow and flow center foliation of  $f$ , respectively. Proposition 3.4.4 shows that every discretized Anosov is dynamically coherent with center-stable foliation  $\mathcal{W}^{cs}$  and center-unstable foliation  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ . Moreover, modulo reparametrization and inverting the time of  $\varphi_t^c$ , Proposition 3.2.2 and Remark 3.2.3 show that  $f$  can be written down as  $f(x) = \varphi_{\tau(x)}^c(x)$  where  $\varphi_t^c$  is parametrized by arc-length and  $\tau : M \rightarrow \mathbb{R}$  is continuous and positive. Finally, Proposition 3.7.3 shows that  $\mathcal{W}^c$  has a compact leaf. One concludes that  $f$  is a flow-type partially hyperbolic diffeomorphism.  $\square$

### 3.9 Characterization in terms of center holonomy

Let us end this chapter by pointing out a characterization of discretized Anosov flows in terms of center holonomy maps.

Recall the definition of a holonomy map for a foliation:

**Remark 3.9.1** (Holonomy map along a curve). Suppose  $\mathcal{W}$  is a foliation with  $C^1$  leaves tangent to a continuous subbundle in the compact Riemannian manifold  $M$ . The construction that follows is standard to check.

Suppose  $x$  in  $M$ ,  $y$  in  $\mathcal{W}(x)$  and  $\gamma : [0, 1] \rightarrow \mathcal{W}(x)$  a  $C^1$  curve such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . Suppose  $D_x$  and  $D_y$  are  $C^1$  discs transverse to  $\mathcal{W}$ , containing  $x$  and  $y$ , respectively. Let  $\delta > 0$  be a constant such that every ball in  $M$  of radius  $2\delta$  is contained in a foliation box neighborhood of  $\mathcal{W}$ .

Every small enough  $C^1$  disc  $D'_x \subset D_x$  containing  $x$  has the property that for every  $z \in D'_x$  there exists a  $C^1$  curve  $\gamma_z : [0, 1] \rightarrow \mathcal{W}(z)$  such that  $\gamma_z(0) = z$ ,  $\gamma_z(1) \in D_y$  and  $d(\gamma(t), \gamma_z(t)) < \delta$  for every  $t \in [0, 1]$ . Moreover, the point  $\gamma_z(1)$  in  $D_y$  is independent of the choice of such a  $\gamma_z$ . In particular, there exists a well defined *holonomy map along  $\gamma$*

$$H : D'_x \rightarrow D_y$$

given by  $H(z) = \gamma_z(1)$  for every  $z \in D'_x$ .

Furthermore, one can chose the curves  $\gamma_z$  so that  $z \mapsto \gamma_z$  varies continuously in the  $C^1$  topology as  $z$  varies continuously in  $D'_x$ .

The following characterizes discretized Anosov flows in terms of center holonomy:

**Proposition 3.9.2.** *Suppose  $f \in \text{PH}_{c=1}(M)$ . The following are equivalent:*

- (i) *The map  $f$  is a discretized Anosov flow.*
- (ii) *The bundle  $E^c$  integrates to an  $f$ -invariant foliation  $\mathcal{W}^c$  such that for every  $x \in M$  there exist:*
  - *A curve  $\gamma : [0, 1] \rightarrow \mathcal{W}^c(x)$  with  $\gamma(0) = x$  and  $\gamma(1) = f(x)$ ,*
  - *A  $C^1$  disc  $D$  transverse to  $\mathcal{W}^c$  with  $x \in D$  such that the  $\mathcal{W}^c$  holonomy map  $H$  along  $\gamma$  is well defined from  $D$  to  $f(D)$  and satisfies*

$$H(y) = f(y)$$

*for every  $y \in D$ .*

*Proof.* Suppose that  $f$  is a discretized Anosov flow. By Definition 3.2.1, Proposition 3.2.2 and Remark 3.2.3 the map  $f$  can be written down as  $f(x) = \varphi_{\tau(x)}^c(x)$ , where  $\tau : M \rightarrow \mathbb{R}_{>0}$  is continuous and  $\varphi_t^c : M \rightarrow M$  is a unit speed flow whose flow lines coincide with the leaves of the flow center foliation  $\mathcal{W}^c$  of  $f$ .

Given  $x \in M$  let  $\gamma : [0, 1] \rightarrow \mathcal{W}^c(x)$  be the reparametrization of the piece of  $\varphi_t^c$  orbit from  $x$  to  $f(x)$  so that  $\|\frac{\partial \gamma}{\partial t}\| = \frac{1}{\tau(x)}$  for every  $t \in [0, 1]$ . Let  $D_x$  be a  $C^1$  disc containing  $x$  and transverse to  $\mathcal{W}^c$ . Then  $f(D_x)$  contains  $f(x)$  and is also a  $C^1$  disc transverse to  $\mathcal{W}^c$ .

Let  $\delta > 0$  be a constant such that every ball of radius  $2\delta$  is contained in a foliation box neighborhood of  $\mathcal{W}^c$ . As in Remark 3.9.1, let  $D'_x \subset D_x$  be such that  $x \in D'_x$  and the holonomy map along  $\gamma$

$$H : D'_x \rightarrow f(D_x)$$

is well defined.

For every  $z \in D'_x$  let  $\gamma_z : [0, 1] \rightarrow \mathcal{W}^c$  denote the piece of  $\varphi_t^c$  orbit from  $z$  to  $f(z)$  reparametrized so that  $\|\frac{\partial(\gamma_y)_t}{\partial t}\| = \frac{1}{\tau(y)}$  for every  $t \in [0, 1]$ .

We can assume that  $D'_x$  is small enough so that  $d(\gamma_z(t), \gamma(t)) < \delta$  for every  $z \in D'_x$  and  $t \in [0, 1]$ . It follows that  $f(z) = \gamma_z(1)$  for every  $z \in D'_x$ . This shows that (i) implies (ii).

Conversely, suppose that (ii) is satisfied. In particular,  $f$  individually fixes each leaf of  $\mathcal{W}^c$ . Given  $x \in M$  let  $\gamma$  and  $D$  be as in (ii). Let us see that locally in a neighborhood of  $x$  the condition  $f(w) \in \mathcal{W}_L^c(w)$  is satisfied for some  $L > 0$ .

Let  $\delta > 0$  be such that every ball of radius  $2\delta$  is contained in a foliation box neighborhood of  $\mathcal{W}^c$ . As in Remark 3.9.1 let  $D' \subset D$  be a  $C^1$  disc containing  $x$  so that its closure is a subset of  $D$  and such that for every  $y \in D'$  a  $C^1$  holonomy curve  $\gamma_y : [0, 1] \rightarrow \mathcal{W}^c(y)$  with the following properties is well defined:  $\gamma_y(0) = y$ ,  $\gamma_y(1) = f(y)$  and  $d(\gamma(t), \gamma_y(t)) < \delta$  for every  $t \in [0, 1]$ .

Moreover, the curves  $\gamma_y$  can be considered so that  $y \mapsto \gamma_y$  varies continuously with  $y$ . Then  $y \mapsto \text{length}(\gamma_y)$  varies continuously and as consequence there exists  $K > 0$  a constant larger than  $\sup_{y \in D'} \text{length}(\gamma_y)$ .

Let  $U$  be a foliation box neighborhood of  $\mathcal{W}^c$  obtained as  $\bigcup_{z \in f(D')} \mathcal{W}_{\epsilon_1}^c(z)$  for some small  $\epsilon_1 > 0$ . Let  $\epsilon_2 > 0$  be such that  $f(\mathcal{W}_{\epsilon_2}^c(y))$  is a subset of  $\mathcal{W}_{\epsilon_1}^c(f(y))$  for every  $y \in D'$  and let  $U'$  be the neighborhood  $\bigcup_{y \in D'} \mathcal{W}_{\epsilon_2}^c(y)$ . It follows that  $f(w)$  lies in  $\mathcal{W}_{K+\epsilon_1+\epsilon_2}^c(w)$  for every  $w \in U'$ .

Let us rename  $U'$  as  $U_x$  and  $K + \epsilon_1 + \epsilon_2$  as  $L_x$  to highlight the dependence on the point  $x$ . We conclude that for every  $x \in M$  there exists a neighborhood  $U_x$  and a constant  $L_x$  so that  $f(w) \in \mathcal{W}_{L_x}^c(y)$  for every  $w \in U_x$ . By taking a finite subcover  $\{U_{x_i}\}_{i \in I}$  of the cover  $\{U_x\}_{x \in M}$  of  $M$  it follows that  $f(w) \in \mathcal{W}_L^c(w)$  for every  $w \in M$  and  $L = \max_{x \in I} L_{x_i}$ . Then (i) follows as a consequence of Proposition 3.3.1.  $\square$

## Chapter 4

# Continuation of normally hyperbolic foliations revisited

### 4.1 Introduction

In this chapter we revisit the stability of normally hyperbolic foliations of [HPS77] (see also [PSW12]). The main goal is to show Theorem 4.2.3 which guarantees that, in a certain sense, the continuation of a normally hyperbolic foliation can be carried out along sets of uniform size in  $\text{PH}_{c=1}(M)$ . The immediate antecedent for this result is [BFP20, Theorem 4.1] (see also [BFP20, Section 4.1] and [BFP20, Appendix B]).

Everything in this chapter is independent from the previous one.

### 4.2 Statements

From now on throughout this chapter let  $M$  be a closed (compact and without boundary) Riemannian manifold.

Suppose  $\mathcal{C}^1$  and  $\mathcal{C}^2$  are continuous cone fields in  $M$  of complementary dimension. Given constants  $\epsilon, \delta > 0$  we will say that the metric in  $M$  and the cone fields  $(\mathcal{C}^1, \mathcal{C}^2)$  are  $\epsilon$ -nearly euclidean at scale  $\delta$  if for every  $x \in M$  the exponential map  $\exp_x : T_x M \rightarrow M$  restricted to  $B_\delta(0) \subset T_x M$  is a diffeomorphism onto its image  $B_\delta(x) \subset M$  satisfying that, if one identifies  $T_x M$  isometrically with the euclidean space  $\mathbb{R}^n$  by a linear map  $A : T_x M \rightarrow \mathbb{R}^n$ , then

$$|\|A(\exp_x^* v_y)\| - 1| < \epsilon$$

and

$$|\angle(A(\exp_x^* v_y^1), A(\exp_x^* v_y^2)) - \frac{\pi}{2}| < \epsilon\pi$$

for every  $y, y' \in B_\delta(x)$ , every unit vector  $v_y$  in  $T_y M$  and every unit vectors

$v_y^1 \in \mathcal{C}^1(y)$  and  $v_{y'}^2 \in \mathcal{C}^2(y')$ ; where in this context, if  $y = \exp_x(z)$  and  $v \in T_y M$ , then  $\exp_x^* v$  denotes the vector in  $T_z(T_x M) \simeq T_x M$  that is sent to  $v$  by the differential of  $\exp_x$  at  $z$ .

Informally, for  $\epsilon > 0$  small the property of being  $\epsilon$ -nearly euclidean at scale  $\delta$  indicates that in restriction to balls of radius  $\delta$  the metric is close to being euclidean and the cone fields are fairly narrow, almost constant and almost pairwise orthogonal.

**Lemma 4.2.1.** *Suppose  $f_0 \in \text{PH}_{c=D}(M)$  for some  $D > 0$ . There exists a Riemannian metric in  $M$ , a constant  $\delta(f_0) > 0$  and for every  $\delta$  with  $0 < \delta \leq \delta(f_0)$  a  $C^1$ -neighborhood  $\mathcal{U}_\delta(f_0) \subset \text{PH}_{c=D}(M)$  of  $f_0$  such that:*

(P1) *There exists a constant  $\kappa > 1$  such that  $\max\{\|Df_x\|, \|Df_x^{-1}\|\} < \kappa$  for every  $x \in M$  and every  $f \in \mathcal{U}_\delta(f_0)$ .*

(P2) *There exists a constant  $0 < \lambda < 1$  such that  $\|Df|_{E^s(x)}\| < \lambda$  and  $\|Df^{-1}|_{E^u(x)}\| < \lambda$  for every  $x \in M$  and every  $f \in \mathcal{U}_\delta(f_0)$ .*

(P3) *There exist continuous cone fields  $\mathcal{C}^s$ ,  $\mathcal{C}^{cs}$ ,  $\mathcal{C}^u$  and  $\mathcal{C}^{cu}$  on  $M$  such that for every  $f \in \mathcal{U}_\delta(f_0)$  and  $x \in M$ :*

1. *The dimension of  $\mathcal{C}^\sigma$  is equal to  $\dim(E^\sigma)$  and the bundle  $E_f^\sigma(x)$  is contained in  $\mathcal{C}^\sigma(x)$  for every  $x \in M$  and  $\sigma \in \{s, cs, u, cu\}$ .*
2. *The cones  $\mathcal{C}^s$  and  $\mathcal{C}^{cs}$  are  $f^{-1}$ -invariant and satisfy  $E_f^\sigma(x) = \bigcap_{n \geq 0} Df^{-n} \mathcal{C}_{f^n(x)}^\sigma$  for every  $x \in M$  and  $\sigma \in \{s, cs\}$ .*
3. *The cones  $\mathcal{C}^u$  and  $\mathcal{C}^{cu}$  are  $f$ -invariant and satisfy that  $E_f^\sigma(x) = \bigcap_{n \geq 0} Df^n \mathcal{C}_{f^{-n}(x)}^\sigma$  for every  $x \in M$  and  $\sigma \in \{u, cu\}$ .*

(P4) *The metric and the cone fields  $(\mathcal{C}^s, \mathcal{C}^{cu})$  and  $(\mathcal{C}^{cs}, \mathcal{C}^u)$  are  $\frac{1}{16}$ -nearly euclidean at scale  $20\delta$ .*

(P5) *The  $C^0$  distance  $d_0(f, g)$  is smaller than  $\frac{\delta}{64\kappa^2}(1 + \lambda + \lambda^2 + \dots)^{-1}$  and smaller than  $\frac{1}{10}(\lambda^{-1} - 1)$  for every  $f, g \in \mathcal{U}_\delta(f_0)$ .*

*Proof.* Let us start by considering  $\mathcal{U}$  a  $C^1$  open neighborhood of  $f_0$  contained in  $\text{PH}_{c=D}(M)$ . If  $\mathcal{U}$  is small enough then property (P1) is automatically satisfied for some constant  $\kappa > 1$ .

By [Gou07] there exists a constant  $0 < \lambda < 1$  and an *adapted metric*  $g_1$  in  $M$  such that  $f_0$  satisfies  $\|Df_0|_{E^s(x)}\| < \lambda$  and  $\|Df_0^{-1}|_{E^u(x)}\| < \lambda$  for every  $x \in M$ .

Let  $g_2$  be the metric that makes the subbundles  $E_{f_0}^s$ ,  $E_{f_0}^c$  and  $E_{f_0}^u$  pairwise orthogonal and coincides with  $g_1$  in restriction to each of them. Note that since the invariant bundles of  $f_0$  vary a priori only continuously with respect

to the point in  $M$  we can not guarantee that  $g_1$  has better regularity than continuous. Nevertheless, if we consider  $g$  a  $C^\infty$  metric close enough to  $g_2$  we can ensure that  $\|Df_0|_{E^s(x)}\| < \lambda$  and  $\|Df_0^{-1}|_{E^u(x)}\| < \lambda$  is still satisfied for every  $x \in M$  and that the pairwise angles between the subbundles  $E_{f_0}^s$ ,  $E_{f_0}^c$  and  $E_{f_0}^u$  lie in  $(\pi/2 - \pi/64, \pi/2 + \pi/64)$ .

Since the invariant bundles vary continuously in the  $C^1$  topology we can shrink  $\mathcal{U}$ , if necessary, so that (P2) is satisfied for every  $f \in \mathcal{U}$  with respect to the same constant  $\lambda$  and such that the pairwise angles between the subbundles  $E_f^s$ ,  $E_f^c$  and  $E_f^u$  also lie in  $(\pi/2 - \pi/64, \pi/2 + \pi/64)$  for every  $f \in \mathcal{U}$ .

In order to obtain (P3) and (P4) let  $\bar{\mathcal{C}}^s$ ,  $\bar{\mathcal{C}}^{cs}$ ,  $\bar{\mathcal{C}}^u$  and  $\bar{\mathcal{C}}^{cu}$  be invariant cone fields, given by the partial hyperbolicity of  $f_0$ , satisfying that  $E_{f_0}^\sigma(x) = \bigcap_{n \geq 0} Df_0^{-n} \bar{\mathcal{C}}_{f_0^n(x)}^\sigma$  for every  $x \in M$  and  $\sigma \in \{s, cs\}$ , and that  $E_{f_0}^\sigma(x) = \bigcap_{n \geq 0} Df_0^n \bar{\mathcal{C}}_{f_0^{-n}(x)}^\sigma$  for every  $x \in M$  and  $\sigma \in \{u, cu\}$ .

Let us define  $\mathcal{C}_x^s = Df_0^{-N} \bar{\mathcal{C}}_{f_0^N(x)}^s$ ,  $\mathcal{C}_x^u = Df_0^N \bar{\mathcal{C}}_{f_0^{-N}(x)}^u$ ,  $\mathcal{C}_x^{cu} = Df_0^N \bar{\mathcal{C}}_{f_0^{-N}(x)}^{cu}$  and  $\mathcal{C}_x^{cs} = Df_0^{-N} \bar{\mathcal{C}}_{f_0^N(x)}^{cs}$  for  $N > 0$  large enough so that the angle between every vector of  $\mathcal{C}_x^\sigma$  and  $E_{f_0}^\sigma(x)$  is less than  $\pi/64$ , for every  $x \in M$  and every  $\sigma \in \{s, u, cs, cu\}$ .

By shrinking  $\mathcal{U}$  even more in the  $C^1$  topology, if necessary, one obtains that (1), (2) and (3) of property (P3) need to be fulfilled by every  $f \in \mathcal{U}$ .

Moreover, it is not difficult to check that for every  $x \in M$  there exists  $\delta_x > 0$  such that for every  $z \in B_{\delta_x}(x)$  the exponential map  $\exp_z : T_z M \rightarrow M$  restricted to  $B_{\delta_x}(0) \subset T_z M$  is a diffeomorphism onto its image  $B_{\delta_x}(z) \subset M$  and, if one identifies isometrically  $T_z M$  with euclidean  $\mathbb{R}^n$  by a linear map  $A : T_z M \rightarrow \mathbb{R}^n$ , then

$$|\|A(\exp_x^* v_y)\| - 1| < \frac{1}{16}$$

and

$$|\angle(A(\exp_x^* v_y^\sigma), A(\exp_x^* v_{y'}^{\sigma'})) - \frac{\pi}{2}| < \frac{\pi}{16}$$

for every  $y, y' \in B_{\delta_x}(z)$ , every unit vector  $v_y$  in  $T_y M$  and every unit vectors  $v_y^\sigma \in \mathcal{C}_y^\sigma$  and  $v_{y'}^{\sigma'} \in \mathcal{C}_{y'}^{\sigma'}$  for every pair  $(\sigma, \sigma') \in \{(s, cu), (cs, u)\}$ .

By taking a finite subcover  $\{B_{\delta_{x_i}}(x_i)\}_{1 \leq i \leq k}$  of  $M$  it follows that  $\delta(f_0) = \frac{1}{20} \min\{\delta_{x_i}\}_{1 \leq i \leq k}$  guarantees that property (P4) is satisfied by every  $f \in \mathcal{U}$  for  $\delta = \delta(f_0)$ .

Given  $0 < \delta \leq \delta(f_0)$ , properties (P1), ..., (P4) are still fulfilled for every  $f \in \mathcal{U}$ . It is enough now to shrink  $\mathcal{U}$  in the  $C^0$  topology even more, if necessary, to a neighborhood  $\mathcal{U}_\delta(f_0)$  so that property (P5) is satisfied for every  $f, g \in \mathcal{U}_\delta(f_0)$ .  $\square$

**Remark 4.2.2.** It is worth pointing out that, according to the order in which each property of Lemma 4.2.1 was proven, it follows that modulo shrinking

$\delta(f_0)$  one can consider the metric in  $M$  and the cone fields  $\mathcal{C}^s$ ,  $\mathcal{C}^{cs}$ ,  $\mathcal{C}^u$  and  $\mathcal{C}^{cu}$  so that  $(\mathcal{C}^s, \mathcal{C}^{cu})$  and  $(\mathcal{C}^{cs}, \mathcal{C}^u)$  are  $\epsilon'$ -nearly euclidean at scale  $\delta'$  for any prescribed  $\epsilon' = \epsilon(\lambda, \kappa) > 0$  and  $\delta'(\lambda, \kappa) > 0$  depending on the constants  $\kappa$  and  $\lambda$  of property (P1) and (P2), respectively. This will be used in Section 5.2 where a narrower version of property (P4) is needed.

Suppose  $E$  is a continuous subbundle of  $TM$ . If  $N$  is a connected manifold of dimension  $\dim(E)$  we say that  $\eta : N \rightarrow M$  is a *complete  $C^1$  immersion tangent to  $E$*  if  $\eta$  is a (not necessarily injective)  $C^1$  map such that  $D_x\eta(T_xN) = E(\eta(x))$  for every  $x \in N$  and such that the pull-back metric in  $N$  is complete. Moreover, if  $L \subset M$  denotes the image of  $\eta$  we say that  $L$  is a *complete  $C^1$  immersed submanifold tangent to  $E$* .

**Theorem 4.2.3.** *(Uniform continuation of normally hyperbolic foliations). Suppose  $f_0 \in \text{PH}_{c=1}(M)$ . Consider a metric in  $M$  and a constant  $\delta(f_0) > 0$  as in Lemma 4.2.1. Then for every  $\delta$  with  $0 < \delta \leq \delta(f_0)$  a  $C^1$  neighborhood  $\mathcal{U}_\delta(f_0)$  as in Lemma 4.2.1 satisfies the following properties.*

*For every pair  $f$  and  $g$  in  $\mathcal{U}_\delta(f_0)$ , if  $\mathcal{W}^c$  is an  $f$ -invariant center foliation, then there exists*

- *A map  $h : M \rightarrow M$  continuous, surjective and  $\delta$ -close to identity,*
- *A homeomorphism  $\rho : M \rightarrow M$  so that for every leaf  $L \in \mathcal{W}^c$ , one has that  $\rho(L) = L$  and the map  $\rho|_L : L \rightarrow L$  is a  $C^1$  diffeomorphism that is  $\delta$ -close to the identity on  $L$ ,*

*such that*

1. *For every leaf  $L \in \mathcal{W}^c$  the set  $h(L)$  is a complete  $C^1$  immersed submanifold tangent to  $E_g^c$ . Furthermore, the map  $h|_L : L \rightarrow M$  is  $C^1$  with respect to the inner differentiable structure of  $L$ , the derivative  $D(h|_L)|_{E_f^c}$  varies continuously in  $M$  and satisfies  $D(h|_L)_x(E_f^c(x)) = E_g^c(h(x))$  and  $\frac{1}{2} < \|D(h|_L)_x|_{E_f^c(x)}\| < 2$  for every  $x \in L$ .*
2. *The equation  $h \circ \rho \circ f(x) = g \circ h(x)$  is satisfied for every  $x \in M$ . In particular,  $h \circ f(L) = g \circ h(L)$  for every  $L \in \mathcal{W}^c$ .*

From the proof of Theorem 4.2.3 we will also derive the following.

**Theorem 4.2.4** *(Uniform continuation of complete  $C^1$  center immersions). Suppose  $f_0 \in \text{PH}_{c=1}(M)$ . Consider a metric in  $M$  and a constant  $\delta(f_0) > 0$  as in Lemma 4.2.1. Then for every  $\delta$  with  $0 < \delta \leq \delta(f_0)$  a  $C^1$  neighborhood  $\mathcal{U}_\delta(f_0)$  as in Lemma 4.2.1 satisfies the following properties.*

If  $f$  and  $g$  are maps in  $\mathcal{U}_\delta(f_0)$  then for every  $\eta : \mathbb{R} \rightarrow M$  a complete  $C^1$  immersion tangent to  $E_f^c$  there exists a sequence  $\{\gamma_n : \mathbb{R} \rightarrow M\}_{n \in \mathbb{Z}}$  of complete  $C^1$  immersions tangent to  $E_g^c$  such that

$$d(f^n \circ \eta(t), \gamma_n(t)) < \delta \quad (4.2.1)$$

and

$$\gamma_{n+1} \text{ is a reparametrization of } g \circ \gamma_n \quad (4.2.2)$$

for every  $t \in \mathbb{R}$  and  $n \in \mathbb{Z}$ .

Moreover, if  $\{\gamma'_n : \mathbb{R} \rightarrow M\}_{n \in \mathbb{Z}}$  is another sequence of complete  $C^1$  immersions tangent to  $E_g^c$  satisfying (4.2.1) and (4.2.2), then  $\gamma'_n$  is a reparametrization of  $\gamma_n$  for every  $n \in \mathbb{Z}$ .

### 4.3 Plaque expansivity and leaf-conjugacies

It is worth noting in this section some consequences of Theorem 4.2.3 before getting into its proof.

**Remark 4.3.1.** (Leaf-conjugacy) Note that if  $h$  is injective then  $h(\mathcal{W}^e)$  is a  $g$ -invariant center foliation and  $h$  is a homeomorphism taking leaves of  $\mathcal{W}^e$  into leaves of  $g$  such that

$$h \circ f(L) = g \circ h(L)$$

for every  $L$  in  $\mathcal{W}^c$ . That is,  $(f, \mathcal{W}_f^c)$  and  $(g, \mathcal{W}_g^c)$  are *leaf-conjugate*.

As detailed in Lemma 4.3.3 below, a sufficient condition for  $h$  to be injective is given by the following property. Note that by *metric* in  $M$  we will always mean a *Riemannian metric* in  $M$ .

**Definition 4.3.2.** Suppose  $f \in \text{PH}(M)$  admits an  $f$ -invariant center foliation  $\mathcal{W}^c$ . Assume that a metric in  $M$  has been fixed. We say that  $(f, \mathcal{W}^c)$  is  *$\delta$ -plaque expansive* if every pair of  $\delta$ -pseudo orbits  $(x_n)_n$  and  $(y_n)_n$  satisfying

- $x_{n+1} \in \mathcal{W}_\delta^c(f(x_n))$  for every  $n \in \mathbb{Z}$
- $y_{n+1} \in \mathcal{W}_\delta^c(f(y_n))$  for every  $n \in \mathbb{Z}$ ,
- $d(x_n, y_n) < 2\delta$  for every  $n \in \mathbb{Z}$ ,

also satisfy  $y_0 \in \mathcal{W}_{3\delta}^c(x_0)$ .

**Lemma 4.3.3.** *In the context of Theorem 4.2.3, if  $f \in \mathcal{U}_\delta(f_0)$  is  $\delta$ -plaque expansive then  $h$  is a homeomorphism and  $(f, \mathcal{W}^c)$  and  $(g, h(\mathcal{W}^c))$  are leaf-conjugate.*



*Proof.* Suppose  $h(x_0) = h(y_0) = z_0$  for some  $x_0, y_0 \in M$ . The orbit of  $z_0$  by  $g$  defines two  $\delta$ -pseudo orbits for  $f$  with ‘jumps’ in  $\mathcal{W}^c$ -plaques as follows.

Since  $h \circ \rho \circ f = g \circ h$  then  $x_1 = \rho(f(x_0))$  and  $y_1 = \rho(f(y_0))$ , and inductively  $x_{n+1} = \rho(f(x_n))$  and  $y_{n+1} = \rho(f(y_n))$  for every  $n \in \mathbb{Z}$ , satisfy

$$g^n(z_0) = h(x_n) = h(y_n)$$

for every  $n \in \mathbb{Z}$ .

As  $h$  and  $\rho$  are  $\delta$ -close to the identity, the sequences  $(x_n)_{n \in \mathbb{Z}}$  and  $(y_n)_{n \in \mathbb{Z}}$  satisfy  $x_{n+1} \in \mathcal{W}_\delta^c(f(x_n))$ ,  $y_{n+1} \in \mathcal{W}_\delta^c(f(y_n))$  and  $d(x_n, y_n) < 2\delta$ .

If  $f$  is  $\delta$ -plaque expansive the above implies that  $y_0$  belongs to  $\mathcal{W}_{3\delta}^c(x_0)$ . By (1) in Theorem 4.2.3 and property (P4) in Lemma 4.2.1 the image by  $h$  of  $\mathcal{W}_{3\delta}^c(x_0)$  is a  $C^1$  arc tangent to  $E_g^c$  and  $h$  restricted to  $\mathcal{W}_{3\delta}^c(x_0)$  is a  $C^1$  diffeomorphism over its image. As  $y_0$  belongs to  $\mathcal{W}_{3\delta}^c(x_0)$  and  $h(x_0) = h(y_0)$  it follows that  $x_0 = y_0$ .

This proves the global injectivity of  $h$ . By Remark 4.3.1 one concludes that  $(f, \mathcal{W}_f^c)$  and  $(g, h(\mathcal{W}^c))$  are leaf-conjugate.  $\square$

It is important to note that, in contrast with the usual definition of plaque-expansivity (as given in the introduction and below), the notion of  $\delta$ -plaque expansivity is sensible to the metric one chooses for  $M$ .

Note also that, for  $\delta > 0$  small, if  $(f, \mathcal{W}^c)$  is  $\delta$ -plaque expansive with respect to some metric, then  $(f, \mathcal{W}^c)$  is  $\delta'$ -plaque expansive with respect to the same metric for every  $0 < \delta' \leq \delta$ .

Recall that  $(f, \mathcal{W}^c)$  is called *plaque expansive* if for some metric and some  $\delta > 0$  every pair of sequences  $(x_n)_{n \in \mathbb{Z}}$  and  $(y_n)_{n \in \mathbb{Z}}$  satisfying that  $x_{n+1} \in \mathcal{W}_\delta^c(f(x_n))$ ,  $y_{n+1} \in \mathcal{W}_\delta^c(f(y_n))$  and  $d(x_n, y_n) < \delta$  for every  $n \in \mathbb{Z}$  must also satisfy  $y_0 \in \mathcal{W}_{loc}^c(x_0)$ . Note that here  $\mathcal{W}_{loc}^c(x)$  should be understood as  $\mathcal{W}_\epsilon^c(x)$  for some small  $\epsilon > 0$  independent of  $x \in M$ .

It is immediate to check that:

**Remark 4.3.4.** If  $(f, \mathcal{W}^c)$  is  $\delta$ -plaque expansive with respect to some metric then  $(f, \mathcal{W}^c)$  is plaque expansive.

*Proof.* It is enough to consider  $0 < \delta' \leq \delta$  so that  $\mathcal{W}_{\delta'}^c(x) \subset \mathcal{W}_{loc}^c(x)$  for every  $x \in M$ . Then  $(f, \mathcal{W}^c)$  being  $\delta'$ -plaque expansive automatically implies that  $(f, \mathcal{W}^c)$  is plaque expansive.  $\square$

Conversely, the following is also satisfied.

**Lemma 4.3.5.** *Suppose  $(f, \mathcal{W}^c)$  is a plaque expansive system in  $M$ . Given a metric in  $M$  there exists  $\delta > 0$  such that  $(f, \mathcal{W}^c)$  is  $\delta$ -plaque expansive with respect to that metric.*

*Proof.* Suppose  $(f, \mathcal{W}^c)$  is plaque expansive. Then for some metric in  $M$  and some small  $\epsilon > 0$  there exists  $\delta > 0$  such that every pair of sequences  $(x_n)_{n \in \mathbb{Z}}$  and  $(y_n)_{n \in \mathbb{Z}}$  satisfying  $x_{n+1} \in \mathcal{W}_\delta^c(f(x_n))$ ,  $y_{n+1} \in \mathcal{W}_\epsilon^c(f(y_n))$  and  $d(x_n, y_n) < \delta$  for every  $n \in \mathbb{Z}$  must also satisfy  $y_0 \in \mathcal{W}_\epsilon^c(x_0)$ . Note that  $\delta > 0$  can be considered as small as wanted so that the previous property remains to be true. At first, let  $\delta$  be smaller than  $\epsilon$ .

Suppose we consider another metric in  $M$  and let us denote by  $d'$  the distance induced by this new metric (in contrast with  $d$  for the first one). As  $M$  is a compact manifold there exists  $C \geq 1$  such that  $\frac{1}{C}d'(x, y) \leq d(x, y) \leq Cd'(x, y)$  for every  $x, y \in M$ .

Let  $d_c$  and  $d'_c$  denote the distances inside center leaves with respect to  $d$  and  $d'$ , respectively. Note that we can consider  $C$  so that it also satisfies  $\frac{1}{C}d'_c(x, y) \leq d_c(x, y) \leq Cd'_c(x, y)$  for every  $x$  and  $y$  in the same center leaf.

Finally, suppose  $\delta$  smaller, if needed, so that for every  $0 < \delta' \leq C\delta$  if  $d'_c(x, y) < C\delta$  and  $d'(x, y) < \delta'$  then  $d'_c(x, y) < (3/2)\delta'$ .

Under this conditions it is immediate to check that  $f$  needs to be  $\frac{\delta}{C}$ -plaque expansive with respect to the new metric. Indeed, let  $(x_n)_{n \in \mathbb{Z}}$  and  $(y_n)_{n \in \mathbb{Z}}$  be such that  $d'_c(x_{n+1}, f(x_n)) < \delta/C$ ,  $d'_c(y_{n+1}, f(y_n)) < \delta/C$  and  $d'(x_n, y_n) < 2\delta/C$  for every  $n \in \mathbb{Z}$ . It follows that  $d_c(x_{n+1}, f(x_n)) < \delta$ ,  $d_c(y_{n+1}, f(y_n)) < \delta$  and  $d(x_n, y_n) < 2\delta$  for every  $n \in \mathbb{Z}$ . Then  $y_0$  lies in  $\mathcal{W}_\epsilon^c(x_0)$ . Since  $\mathcal{W}_\epsilon^c(x_0)$  is a subset of  $\mathcal{W}_\delta^c(x_0)$  then  $d_c(x_0, y_0) < \delta$ . Which in turns implies  $d'_c(x_0, y_0) < C\delta$ . As  $d'(x_0, y_0) < 2\delta/C$  then from the last constraint imposed to  $\delta$  it follows that  $d'_c(x_0, y_0) < 3\delta/C$ .  $\square$

Note that from the proof of the previous lemma one can also deduce the following.

**Lemma 4.3.6.** *Consider two distinct metrics in  $M$ . Given  $\delta > 0$  there exists  $C > 0$  such that, if  $(f, \mathcal{W}^c)$  is  $C\delta$ -plaque expansive with respect to the first metric, then  $(f, \mathcal{W}^c)$  is  $\delta$ -plaque expansive with respect to the second one.*

Recall that in the introduction a simplified version of Theorem 4.2.3 was stated:

**Theorem 4.3.7.** *Suppose  $f_0 \in \text{PH}_{c=1}(M)$ . There is a constant  $\delta > 0$  and a  $C^1$  neighborhood  $\mathcal{U}(f_0)$  of  $f_0$  such that, if some  $f \in \mathcal{U}(f_0)$  admits a center foliation  $\mathcal{W}_f^c$  so that  $(f, \mathcal{W}_f^c)$  is  $\delta$ -plaque expansive, then every  $g \in \mathcal{U}(f_0)$  admits a  $g$ -invariant center foliation  $\mathcal{W}_g^c$  such that  $(f, \mathcal{W}_f^c)$  and  $(g, \mathcal{W}_g^c)$  are leaf-conjugate.*

As a corollary of the discussion above one obtains:

**Corollary 4.3.8.** *Theorem 4.3.7 follows from Theorem 4.2.3.*

*Proof.* The statement of Theorem 4.3.7 presupposes a metric in  $M$ . In parallel, let us consider  $\delta(f_0) > 0$  and the metric in  $M$  as in Lemma 4.2.1.

Let  $C > 0$  be as in Lemma 4.3.6 so that, if  $(f, \mathcal{W}^c)$  is  $\delta(f_0)C$ -plaque expansive with respect to the first metric, then  $(f, \mathcal{W}^c)$  is  $\delta(f_0)$ -plaque expansive with respect to the second one. It is enough to consider now  $\delta := \delta(f_0)C$  and  $\mathcal{U} := \mathcal{U}_{\delta(f_0)}(f_0)$ . The rest follows by Theorem 4.2.3.  $\square$

We recover also the classical stability statement for normally hyperbolic foliations (see [HPS77, Theorem 7.1]):

**Corollary 4.3.9.** *Suppose  $(f, \mathcal{W}^c)$  is a plaque expansive system in  $\text{PH}_{c=1}(M)$ . There exists a neighborhood  $\mathcal{U} \subset \text{PH}_{c=1}(M)$  of  $f$  such that every  $g \in \mathcal{U}$  admits a  $g$ -invariant center foliation  $\mathcal{W}_g^c$  such that  $(g, \mathcal{W}_g^c)$  is plaque expansive and leaf conjugate to  $(f, \mathcal{W}^c)$ .*

*Proof.* Let  $(f, \mathcal{W}^c)$  be a plaque expansive system in  $\text{PH}_{c=1}(M)$ . For  $f = f_0$  consider the metric in  $M$  and the constant  $\delta(f) > 0$  given by Lemma 4.2.1.

By Lemma 4.3.5 there exists  $\delta > 0$  such that  $(f, \mathcal{W}^c)$  is  $\delta$ -plaque expansive (with respect to the metric we have just fixed). We can suppose that  $\delta$  is smaller than  $\delta(f)$ .

Let  $\delta' > 0$  be such that  $3\delta' < \delta$ . Let  $\mathcal{U}_{\delta'}(f)$  be the  $C^1$  neighborhood of  $f$  given by Lemma 4.2.1 (for  $f = f_0$ ). If  $g$  is a system in  $\mathcal{U}_{\delta'}(f)$  then by Lemma 4.3.3 the map  $h$  given by Theorem 4.2.3 is a homeomorphism and  $(f, \mathcal{W}^c)$  and  $(g, h(\mathcal{W}^c))$  are leaf conjugate. Let  $\mathcal{W}^{c,g}$  denote  $h(\mathcal{W}^c)$ .

Always with respect to the metric in  $M$  given by Lemma 4.3.5 suppose that  $(x'_n)_{n \in \mathbb{Z}}$  and  $(y'_n)_{n \in \mathbb{Z}}$  are  $\delta'$  pseudo-orbits for  $g$  so that  $x'_{n+1} \in \mathcal{W}_{\delta'}^{c,g}(g(x'_n))$ ,  $y'_{n+1} \in \mathcal{W}_{\delta'}^{c,g}(g(y'_n))$  and  $d(x'_n, y'_n) < 2\delta'$  for every  $n \in \mathbb{Z}$ . Let us see that  $y'_0$  must lie in  $\mathcal{W}_{3\delta'}^{c,g}(x'_0)$ .

Consider  $x_n = h^{-1}(x'_n)$  and  $y_n = h^{-1}(y'_n)$  for every  $n \in \mathbb{Z}$ . Let  $\rho$  be the map given by Theorem 4.2.3. As  $\rho \circ f(x_n) = h^{-1} \circ g(x'_n)$  and  $\rho$  is  $\delta'$  close to the identity it follows that  $h^{-1} \circ g(x'_n)$  lies in  $\mathcal{W}_{\delta'}^c(f(x_n))$ . Moreover, as  $1/2 < \|Dh|_{E^c}\| < 2$  and  $x'_{n+1} \in \mathcal{W}_{\delta'}^{c,g}(g(x'_n))$  then  $h^{-1} \circ g(x'_n)$  lies in  $\mathcal{W}_{2\delta'}^c(x_{n+1})$ . It follows that

$$x_{n+1} \in \mathcal{W}_{3\delta'}^c(f(x_n))$$

for every  $n \in \mathbb{Z}$ . Analogously for  $(y_n)_{n \in \mathbb{Z}}$ .

Moreover, as  $d(h^{-1}(x'_n), x'_n) < \delta'$  and  $d(h^{-1}(y'_n), y'_n) < \delta'$  because  $h$  is  $\delta'$ -close to the identity, then  $d(x'_n, y'_n) < 2\delta'$  implies

$$d(x_n, y_n) < 4\delta'$$

for every  $n \in \mathbb{Z}$ .

As  $3\delta' < \delta$  and  $4\delta' < 2\delta$  it follows from the  $\delta$ -plaque expansivity of  $(f, \mathcal{W}^c)$  that  $y_0$  needs to lie in  $\mathcal{W}_{3\delta}^c(x_0)$ . Then  $x'_0$  needs to lie in  $\mathcal{W}_{6\delta}^c(x'_0)$  because of  $1/2 < \|Dh|_{E^c}\| < 2$ . Since  $d(x'_0, y'_0) < 2\delta'$  and because at scale  $20\delta(f)$  the center bundles are almost constant (property (P4) in Lemma 4.2.1) it follows that  $y'_0$  needs to lie in  $\mathcal{W}_{3\delta'}^{c,g}(x'_0)$ .

This shows that  $(g, \mathcal{W}^{c,g})$  is  $\delta'$ -plaque expansive. Then  $(g, \mathcal{W}^{c,g})$  is plaque expansive.  $\square$

**Remark 4.3.10.** Note that from the proof of the previous corollary the following statement can also be deduced: If  $(f, \mathcal{W}^c)$  in  $\text{PH}_{c=1}(M)$  is plaque expansive and a metric as in Lemma 4.2.1 (for  $f_0 = f$ ) has been fixed, then the  $C^1$  neighborhood  $\mathcal{U} \subset \text{PH}_{c=1}(M)$  of  $f$  given by Corollary 4.3.9 can be chosen so that there exists  $\delta' > 0$  such that  $(g, \mathcal{W}_g^c)$  is  $\delta'$ -plaque expansive for every  $g \in \mathcal{U}$  (with respect to the metric that has been fixed).

**Remark 4.3.11.** Suppose  $f_0 \in \text{PH}_{c=1}(M)$  is the limit of a sequence  $f_n$  in  $\text{PH}_{c=1}(M)$  such that  $(f_n, \mathcal{W}_{f_n}^c)$  is plaque expansive for some invariant center manifold  $\mathcal{W}_{f_n}^c$ . Consider a metric in  $M$ , a constant  $\delta(f_0) > 0$  and, for every  $0 < \delta \leq \delta(f_0)$ , a neighborhood  $\mathcal{U}_\delta(f_0)$  as in Lemma 4.2.1.

As  $(f_n, \mathcal{W}_{f_n}^c)$  is  $\delta'$ -plaque expansive for every small enough  $\delta' > 0$  we can consider  $\delta_n > 0$  the largest constant such that  $(f_n, \mathcal{W}_{f_n}^c)$  is  $\delta'$ -plaque expansive for every  $\delta' \in (0, \delta_n)$ .

A key point worth noting is that, a priori, we can not rule out that for every  $0 < \delta \leq \delta(f_0)$  and  $f_n \in \mathcal{U}_\delta(f_0)$  the constant  $\delta_n$  may be smaller than  $\delta$ . Thus a priori we can not conclude that  $f_0$  has to admit a center foliation and that there exists a leaf-conjugacy with some  $(f_n, \mathcal{W}_{f_n}^c)$ . To show the  $C^1$  openness and closeness of discretized Anosov flows and of partially hyperbolic skew-products with one-dimensional center (Theorem 5.1.1 and Theorem 5.1.3 in the next chapter) an extra argument will be needed.

## 4.4 Stability of unique integrability for plaque expansive systems

It is also worth noting the following consequences of Theorem 4.2.3 and Theorem 4.2.4.

**Lemma 4.4.1.** *In the context of Theorem 4.2.3 and Theorem 4.2.4 suppose  $0 < \delta \leq \delta_0(f_0)$  and  $f, g \in \mathcal{U}_\delta(f_0)$ . If  $E_f^c$  is uniquely integrable then for every  $C^1$  curve  $\gamma$  tangent to  $E_g^c$  there exists  $L \in \mathcal{W}_f^c$  such that  $\gamma \subset h(L)$ .*

*Proof.* Suppose  $\gamma : (0, 1) \rightarrow M$  is a  $C^1$  curve tangent to  $E_g^c$ . By a little abuse of notation we denote both the curve and its image by  $\gamma$ . By Peano's

existence theorem we can extend  $\gamma$ , if needed, and redefine its domain so that  $\gamma : \mathbb{R} \rightarrow M$  is a complete  $C^1$  immersion tangent to  $E_g^c$ . Let us see that  $\gamma$  needs to be contained in  $h(L)$  for some leaf  $L \in \mathcal{W}^c$ .

By Theorem 4.2.4 (with the names of  $f$  and  $g$ , and the etas and gammas, interchanged) there exists a sequence  $\eta_n : \mathbb{R} \rightarrow M$  of complete  $C^1$  immersions tangent to  $E_f^c$  such that  $\eta_{n+1}$  is a reparametrization of  $f \circ \eta_n$  for every  $n \in \mathbb{Z}$  and

$$d(g^n \circ \gamma(t), \eta_n(t)) < \delta \quad (4.4.1)$$

for every  $t \in \mathbb{R}$  and  $n \in \mathbb{Z}$ .

Since  $E_f^c$  is uniquely integrable the key observation to note is that each  $\eta_n$  needs to be the  $C^1$  parametrization of a leaf of  $\mathcal{W}^c$  (as these are the only  $C^1$  curves tangent to  $E_f^c$ ). If  $L$  denote the leaf of  $\mathcal{W}_f^c$  whose parametrization is  $\eta_0 : \mathbb{R} \rightarrow M$ , let us see that  $\gamma$  must be contained in the continuation  $h(L)$  of  $L$ .

On the one hand, as  $\eta_{n+1}$  is a reparametrization of  $f \circ \eta_n$  then (4.4.1) implies that  $g^n \circ \gamma$  can be reparametrized to a  $C^1$  curve  $\gamma_n$  satisfying that

$$d(f^n \circ \eta_0(t), \gamma_n(t)) < \delta$$

for every  $t \in \mathbb{R}$  and  $n \in \mathbb{Z}$ . It is immediate to check that, in addition, the curve  $\gamma_{n+1}$  is a reparametrization of  $g \circ \gamma_n$  for every  $n \in \mathbb{Z}$ .

On the other hand, since  $h \circ f^n(L) = g^n \circ h(L)$  for every  $n \in \mathbb{Z}$  and  $h$  is  $\delta$ -close to the identity the curves  $\gamma'_n := h \circ \eta_n$  satisfy that  $\gamma'_{n+1}$  is a reparametrization of  $g \circ \gamma'_n$  for every  $n \in \mathbb{Z}$  and

$$d(f^n \circ \eta_0(t), \gamma'_n(t)) < \delta$$

for every  $t \in \mathbb{R}$  and  $n \in \mathbb{Z}$ .

By the uniqueness part of Theorem 4.2.4 (for  $f$  and  $g$  not interchanged) it follows that  $\gamma$  is a reparametrization of  $h \circ \eta_0$ . In particular,  $\gamma$  is contained in  $h(L)$  for  $L \in \mathcal{W}_f^c$  the image of  $\eta_0$ . □

As an immediate consequence of Lemma 4.4.1 one gets the following.

**Corollary 4.4.2.** *In the context of Theorem 4.2.3, if  $h$  is a homeomorphism and  $E_f^c$  is uniquely integrable then  $E_g^c$  is uniquely integrable.*

*Proof.* If  $h$  is a homeomorphism then  $h(\mathcal{W}_f^c)$  is a center foliation for  $g$ . If  $\gamma$  is a  $C^1$  curve tangent to  $E_g^c$  then by Lemma 4.4.1 it has to be contained in a leaf of  $h(\mathcal{W}_f^c)$ . We conclude that through every point of  $M$  there exists a unique  $C^1$  curve tangent to  $E_g^c$ , modulo reparametrizations. □

As a consequence of the previous corollary one can show the following proposition.

**Proposition 4.4.3.** *Let  $(f, \mathcal{W}^c)$  be a plaque expansive system in  $\text{PH}_{c=1}(M)$ . There exists a  $C^1$  neighborhood  $\mathcal{U} \subset \text{PH}_{c=1}(M)$  of  $f$  such that, if  $E_g^c$  is uniquely integrable for some  $g \in \mathcal{U}$ , then  $E_{g'}$  is uniquely integrable for every  $g' \in \mathcal{U}$ .*

*Proof.* Suppose  $(f, \mathcal{W}^c)$  is a plaque expansive system in  $\text{PH}_{c=1}(M)$ . Consider  $\delta(f) > 0$  and a metric in  $M$  as in Lemma 4.2.1 for  $f_0 = f$ .

Consider  $\mathcal{U} \subset \text{PH}_{c=1}(M)$  a  $C^1$  neighborhood of  $f$  and  $\delta' > 0$  given by Corollary 4.3.9 and Remark 4.3.10 so that every  $g \in \mathcal{U}$  is  $\delta'$ -plaque expansive.

We can suppose without loss of generality that  $\delta' \leq \delta(f)$ . Consider  $\mathcal{U}_{\delta'}(f) \subset \text{PH}_{c=1}(M)$  the  $C^1$  neighborhood of  $f$  given by Lemma 4.2.1 with respect to the metric already fixed. Consider  $\mathcal{U}' = \mathcal{U}_{\delta'}(f) \cap \mathcal{U}$ . Let us see that if  $\mathcal{U}'$  contains a systems with uniquely integrable center bundle then every system in  $\mathcal{U}'$  has this property.

Suppose  $E_g^c$  is uniquely integrable for some  $g \in \mathcal{U}'$  and let  $\mathcal{W}_g^c$  denote the corresponding center foliation for  $g$ . As  $\mathcal{U}' \subset \mathcal{U}_{\delta'}(f)$  then for  $g' \in \mathcal{U}'$  we can consider  $h : M \rightarrow M$  given by Theorem 4.2.3 so that  $h(L)$  is a complete  $C^1$  immersion tangent to  $E_{g'}^c$  for every  $L \in \mathcal{W}_g^c$ . Since  $(g, \mathcal{W}_g^c)$  is  $\delta'$ -plaque expansive then  $h$  needs to be a homeomorphism (Lemma 4.3.3). By Corollary 4.4.2 we conclude that  $E_{g'}^c$  has to be uniquely integrable.  $\square$

## 4.5 Proof of Theorem 4.2.3

From now on throughout this subsection let us fix a metric in  $M$ , a  $C^1$  open set  $\mathcal{U}_\delta(f_0)$  and a pair of partially hyperbolic diffeomorphisms  $f, g \in \mathcal{U}_\delta(f_0)$  as in the hypothesis of Theorem 4.2.3. Let  $\mathcal{C}^\sigma$  denote the invariant cone fields given by Lemma 4.2.1 for every  $\sigma \in \{s, u, cs, cu\}$ . Note that  $f$  and  $g$  satisfy properties (P1), ..., (P5) from Lemma 4.2.1. We will refer to properties (P1), ..., (P5) implicitly referring to the ones from Lemma 4.2.1.

Informally, for every leaf  $L$  of  $\mathcal{W}^c$  we will consider  $U(L)$  an ‘unfolded’  $\delta$ -wide tubular neighborhood of  $L$  (see next subsection for the formal construction) and a manifold  $V(L)$  which is the disjoint union of the manifolds  $U(f^n(L))$  for every integer  $n$ . As  $f$  and  $g$  are  $C^0$  close enough we will be able to ‘lift’ the map  $g$  to  $V(L)$  in a neighborhood of  $\bigcup_n f^n L$ , sending points of each connected component  $U(f^n L)$  to the ‘next’ connected component  $U(f^{n+1} L)$ . By ‘transverse hyperbolicity’ and the constraints imposed by Lemma 4.2.1 there will exist a non empty set  $L'$  in  $U(L)$  whose points are exactly those ones whose  $g$  orbit remains in  $V(L)$  for every backwards and forwards iterate. We will call  $L'$  the *continuation* of  $L$ .

The set of points in  $U(L)$  whose whole  $g$  backwards orbit remains in  $V(L)$  will be obtained as the limit set in  $n$  of the ‘ $cu$ -strips’  $\mathcal{W}_\delta^u(f^{-n}L)$  iterated  $n$  times forwards by  $g$ . As  $\mathcal{W}_\delta^u(f^{-n}L)$  is tangent to the  $cu$ -cone and  $g$  contracts uniformly this cone for positive iterates then the limit set would be a  $C^1$  submanifold tangent to  $E_g^c \oplus E_g^u$ . The same argument shows that the points whose  $g$  forwards orbit is well defined in  $V(L)$  is a  $C^1$  strip tangent  $E_g^s \oplus E_g^c$ . Hence  $L'$ , the intersection of both sets, would be tangent to  $E_g^c$ .

Once the continuation of every center leaf has been constructed it will remain to define the maps  $h$  and  $\rho$  that coherently identify each leaf  $L$  with its continuation  $L'$  so that the identity  $h \circ \rho \circ f = g \circ h$  holds.

### Part 1: Good cover of every center leaf

For every leaf  $L$  of  $\mathcal{W}^c$  let us consider the set which is the disjoint union of the balls  $\{B_\delta(x)\}_{x \in L}$ . Namely  $\bigcup_{x \in L} \{(y, x) : y \in B_\delta(x)\}$ . On this set let us identify two points  $(y, x)$  and  $(y', x')$  if and only if  $y = y'$  and  $x' \in L_{4\delta}(x)$ . (Recall the notation  $L_r(x)$  for the points in the leaf  $L$  at intrinsic distance less than  $r$  from  $x$ ). We denote by  $U(L)$  the space obtained after this identification.

The space  $U(L)$  has a natural differential structure and a projection  $\pi : U(L) \rightarrow M$ , defined explicitly by  $\pi(y, x) = x$ , which is a local diffeomorphism at any point. Moreover,  $U(L)$  can be given the structure of abstract Riemannian manifold by taking pull-back of the structure in  $M$  by the restrictions of  $\pi$  to the sets  $B_\delta(x)$ .

As informally stated before, let  $V(L)$  be the manifold which is the disjoint union of the manifolds  $U(f^n(L))$  for every integer  $n$ . Note that in the case  $L$  is fixed by  $f$  then  $V(L)$  has only one connected component. Otherwise  $V(L)$  has countable connected components, namely  $\{U(f^n L)\}_{n \in \mathbb{Z}}$ . Note also that the projection  $\pi : V(L) \rightarrow M$  is well defined as it is well defined on each connected component.

For every  $\epsilon < \delta$  let us denote by  $U_\epsilon(L)$  the subset of  $U(L)$  given by the points at distance less than  $\epsilon$  from  $L$ . That is,  $U_\epsilon(L) = \bigcup_{x \in L} B_\epsilon(x) \subset U(L)$ . Accordingly let  $V_\epsilon(L)$  be the subset of  $V(L)$  whose connected components are  $\{U_\epsilon(f^n L)\}_{n \in \mathbb{Z}}$ .

Recall that the  $C^0$  distance  $d_0(f, g)$  is smaller than  $\frac{\delta_2}{2}$  by property (P5). Recall also that by property (P1) there exists a constant  $\kappa > 1$  satisfying that  $\max\{\|Dg'_x\|, \|Dg'^{-1}_x\|\} < \kappa$  for every  $g' \in \mathcal{U}_\delta$ . Let us fix from now on  $\delta_1 = \frac{\delta}{2\kappa}$ .

**Claim 4.5.1.** *For every leaf  $L$  of  $\mathcal{W}^c$  the maps  $f$  and  $g$  lift to maps*

$$f, g : V_{\delta_1}(L) \rightarrow V(L)$$

such that the connected component  $U_{\delta_1}(f^n L)$  is sent by  $f$  and by  $g$  inside  $U_\delta(f^{n+1}L)$ , and is sent by  $f^{-1}$  and  $g^{-1}$  inside  $U_\delta(f^{n-1}L)$ , for every  $n \in \mathbb{Z}$ .

*Proof.* First of all, note that  $f$  lifts directly to  $\bigcup_{n \in \mathbb{Z}} f^n L \subset V_{\delta_1}(L)$ .

For  $y \in U_{\delta_1}(f^n L)$  let  $x$  be a point in  $f^n L \subset U_{\delta_1}(f^n L)$  such that  $d(x, y) < \delta_1$ . Let us denote  $y' = \pi(y)$  and  $x' = \pi(x)$ . Since  $y' \in B_{\delta_1}(x')$  then  $d(f(y'), f(x')) < \kappa \delta_1$ . Moreover, as  $f$  and  $g$  are  $\frac{\delta}{2}$ -close by property (P5) then  $d(f(y'), g(y')) < \delta/2$ . We conclude that  $f(y')$  and  $g(y')$  lie  $B_\delta(f(x'))$  as  $\delta_1 + \delta/2 < \delta$ .

As  $\pi$  is bijective from  $B_\delta(f(x'))$  to  $B_\delta(f(x))$  then  $f(y)$  and  $g(y)$  can be lifted to  $B_\delta(f(x))$  to points  $f(y')$  and  $g(y')$ , respectively. In this way, it is easy to check that  $f$  and  $g$  are well defined  $C^1$  maps from  $V_{\delta_1}(L) \rightarrow V(L)$ .  $\square$

The proof of the theorem is going to show that  $h(L)$ , the *continuation of  $L$* , will be the projection by  $\pi$  of set of points in  $U_{\delta_1}(L)$  whose  $g$  orbit in  $V(L)$  is well defined for every future and past iterate (see Remark 4.5.5).

*Notations.* We will denote by  $E^\sigma$  and  $E^{\sigma,g}$  the  $f$ -invariant and  $g$ -invariant bundles in  $M$ , respectively, for every  $\sigma \in \{s, c, u, cs, cu\}$ . Analogously for the  $f$  and  $g$ -invariant foliations  $\mathcal{W}^\sigma$  and  $\mathcal{W}^{\sigma,g}$ . Note that we can lift these bundles and leaves to  $V(L)$ . Let us denote these lifted bundles as  $\tilde{E}^\sigma$  and  $\tilde{E}^{\sigma,g}$ , and the lifted foliations as  $\tilde{\mathcal{W}}^\sigma$  and  $\tilde{\mathcal{W}}^{\sigma,g}$ , respectively. Note that they are (locally) invariant wherever  $f$  and  $g$  are well defined. The same for the  $f$  and  $g$ -invariant cone-fields  $\mathcal{C}^\sigma$  lifting to cone-fields  $\tilde{\mathcal{C}}^\sigma$ .

## Part 2: Graph transform for $cu$ -strips

Let us fix from now on the constant  $\delta_2 > 0$  such that  $\delta_2 = \frac{\delta_1}{2} = \frac{\delta}{4\kappa}$ .

For every leaf  $L$  of  $\mathcal{W}^c$  and every  $\epsilon \leq \delta_2$  let us define  $U_\epsilon^{su}(L)$  in  $U(L)$  as

$$U_\epsilon^{su}(L) = \tilde{\mathcal{W}}_\epsilon^{s,g}(\tilde{\mathcal{W}}_\epsilon^u(L)).$$

Note that the unstable plaques are considered with respect to  $f$  and the stable plaques with respect to  $g$ . This is not essential but will make some arguments simpler.

Recall that by Lemma 2.0.1 the sets  $\tilde{\mathcal{W}}_\epsilon^u(L)$  are  $C^1$  submanifolds tangent to  $\tilde{E}^{cu}$ . These sets are what we call *cu-strips*. As the  $g$  stable local manifolds  $\tilde{\mathcal{W}}_\epsilon^{s,g}(x)$  are transverse to  $\tilde{\mathcal{W}}_\epsilon^u(L)$  for every  $x \in \tilde{\mathcal{W}}_\epsilon^u(L)$  it is easy to check that  $U_\epsilon^{su}(L)$  is an open subset of  $U(L)$ .

Note that  $U_\epsilon^{su}(L)$  is a subset of  $U_{2\epsilon}(L)$  since every point in  $U_\epsilon^{su}(L)$  can be joined to a point in  $L$  by a concatenation of an  $g$ -stable and an  $f$ -unstable arc of lengths less than  $\epsilon$ . Moreover, by property (P4) it follows that  $U_{\epsilon/2}(L)$  is contained in  $U_\epsilon^{su}(L)$ .



Let us define  $V_\epsilon^{su}(L)$  as the subset of  $V_{\delta_1}(L)$  which is the union of the sets  $U_\epsilon^{su}(f^n L)$  for every integer  $n$ . And let us define

$$\pi^s : V_{\delta_2}^{su}(L) \rightarrow \bigcup_{n \in \mathbb{Z}} \tilde{W}_{\delta_2}^u(f^n L)$$

the projection along local stable  $g$ -plaques.

Let  $\delta_3 > 0$  be the constant  $\delta_3 = \frac{\delta_2}{4\kappa}$ . Recall that the  $C^0$  distance  $d_0(f, g)$  is smaller than  $\frac{\delta_2}{2}$  by (P5). By the same arguments as in Claim 4.5.1, the image by  $g$  of  $U_{2\delta_3}(f^n L)$  is a subset of  $U_{\delta_2}(f^{n+1} L)$  for every  $n \in \mathbb{Z}$ . Since  $U_{\delta_3}^{su}(f^{n+1} L)$  is contained in  $U_{2\delta_3}(f^n L)$  then:

**Remark 4.5.2.** The map  $g$  from  $V_{\delta_3}^{su}(L)$  to  $V_{\delta_2}^{su}(L)$  is well defined.

Let us consider the set of continuous functions

$$\Pi^{cu}(L) = \{ \xi : \bigcup_{n \in \mathbb{Z}} \tilde{W}_{\delta_3}^u(f^n L) \rightarrow V_{\delta_2}^{su}(L) \text{ such that } \pi^s \circ \xi = \text{id} \}.$$

Note that if  $V(L)$  has many connected components then  $\xi \in \Pi^{cu}(L)$  is given by functions  $\xi|_{\tilde{W}_{\delta_3}^u(f^n L)} : \tilde{W}_{\delta_3}^u(f^n L) \rightarrow U_{\delta_2}^{su}(f^n L)$  for each  $n \in \mathbb{Z}$ .

Given two maps  $\xi, \xi'$  in  $\Pi^{cu}(L)$  we can define a distance between them

$$d(\xi, \xi') = \sup d_s(\xi(x), \xi'(x))$$

where  $d_s$  denotes the distance inside the plaque  $\tilde{W}_{\delta_2}^{s,g}(x)$  and the supremum is taken over all  $x$  in  $\bigcup_{n \in \mathbb{Z}} \tilde{W}_{\delta_3}^u(f^n L)$ .

The *zero-section* is the function  $\xi^0$  in  $\Pi^{cu}(L)$  defined by  $\xi^0(x) = x$  for every  $x$ . For every  $\xi$  in  $\Pi^{cu}(L)$  we denote by  $\text{graph}(\xi)$  the set which is the image of  $\xi$ . For simplicity, let us denote from now on by  $\delta'$  the  $C^0$  distance  $d_0(f, g)$ .

**Claim 4.5.3** (Graph transform).

1. The image by  $g$  of  $\text{graph}(\xi^0)$  induces a new map  $g\xi^0$  in  $\Pi^{cu}(L)$  such that  $\text{graph}(g\xi^0) \subset g \text{graph}(\xi^0)$  and  $d(\xi^0, g\xi^0) < 2\delta'$ .
2. Moreover, for every  $\xi$  in  $\Pi^{cu}(L)$  such that  $d(\xi^0, \xi) < \delta_3$  the image by  $g$  of  $\text{graph}(\xi)$  induces a new map  $g\xi$  in  $\Pi^{cu}(L)$  such that  $\text{graph}(g\xi) \subset g \text{graph}(\xi)$  and  $d(\xi^0, g\xi) < 2\delta' + \lambda d(\xi^0, \xi)$ .
3. Finally, for every  $\xi, \xi'$  in  $\Pi^{cu}(L)$  with  $d(\xi, \xi') < \delta_3$  and  $d(\xi', \xi^0) < \delta_3$  we have  $d(g\xi, g\xi') < \lambda d(\xi, \xi')$ .

*Proof.* Let us start by looking at the image by  $g$  of the zero section  $\xi^0$ . Recall that  $\text{graph}(\xi^0)$  is the union in  $n \in \mathbb{Z}$  of the  $C^1$  submanifolds  $\tilde{W}_{\delta_3}^u(f^n L)$ . Let

$n$  be any fixed integer. Let us see that the image by  $\pi^s \circ g$  of  $\tilde{W}_{\delta_3}^u(f^n L)$  covers  $\tilde{W}_{\delta_3}^u(f^{n+1} L)$ , and that  $\pi^s \circ g$  restricted to  $\tilde{W}_{\delta_3}^u(f^n L)$  is injective. As a consequence the map  $g\xi^0$  at any point  $y \in \tilde{W}_{\delta_3}^u(f^{n+1} L)$  will be unambiguously defined as the unique point in the image by  $g$  of  $\tilde{W}_{\delta_3}^u(f^n L)$  whose projection by  $\pi^s$  is  $y$ . It will be clear from the construction that  $g\xi^0(y)$  defined in this way will vary continuously with  $y$ .

Note that by property (P2) the set  $\tilde{W}_{\lambda^{-1}\delta_3}^u(f^{n+1} L)$  is contained in the image by  $f$  of  $\tilde{W}_{\delta_3}^u(f^n L)$ . Thus for every  $y \in \tilde{W}_{\lambda^{-1}\delta_3}^u(f^{n+1} L)$  there exists  $y'$  in  $\tilde{W}_{\delta_3}^u(f^n L)$  such that  $f(y') = y$ .

As  $\tilde{W}_{\delta_3}^u(f^n L)$  is a  $C^1$  submanifold tangent to the cone field  $\tilde{\mathcal{C}}^{cu}$  it follows that its image by  $g$  is also a  $C^1$  submanifold tangent to  $\tilde{\mathcal{C}}^{cu}$ . By property (P4) it follows that  $\pi^s \circ g$  has to be injective restricted to  $\tilde{W}_{\delta_3}^u(f^n L)$ .

Since  $f(y')$  and  $g(y')$  are at distance less than  $\delta' = d_0(f, g)$  then again by property (P4) it follows that  $\tilde{W}_{2\delta'}^{s,g}(g(y'))$  and  $\tilde{W}_{2\delta'}^{cu}(y)$  intersect. In particular  $\pi^s \circ g(y')$  and  $y$  need to be at distance less than  $2\delta'$  for the intrinsic metric of  $\tilde{W}_{\delta_2}^u(f^{n+1} L)$ . We conclude that  $\pi^s \circ g \circ f^{-1}$  is a well defined continuous and injective function from  $\tilde{W}_{\lambda^{-1}\delta_3}^u(f^{n+1} L)$  to  $\tilde{W}_{\delta_2}^u(f^{n+1} L)$  that is  $2\delta'$ -close to the identity.

For every  $y \in \tilde{W}_{\delta_3}^u(f^{n+1} L)$  the ball of radius  $10\delta'$  in  $\tilde{W}_{\delta_2}^u(f^{n+1} L)$  is contained in  $\tilde{W}_{\lambda^{-1}\delta_3}^u(f^n L)$  by property (P5). By a standard topology argument using that  $\pi^s \circ g \circ f$  is  $2\delta'$ -close to the identity we obtain that  $y$  needs to be in the image of this ball. So the image by  $\pi^s \circ g$  of  $\tilde{W}_{\delta_3}^u(f^n L)$  covers  $\tilde{W}_{\delta_3}^u(f^{n+1} L)$  as we wanted to prove. This settles (1).

In order to see (2) suppose  $\xi$  is not the zero section but  $d(\xi^0, \xi) < \delta_3$ . For simplicity let  $d$  denote  $d(\xi^0, \xi)$ . For every  $w$  in  $\tilde{W}_{\delta_3}^u(f^n L)$  the point  $\xi(w)$  lies in  $\tilde{W}_d^{s,g}(w)$  so  $g \circ \xi(w)$  needs to lie in  $\tilde{W}_{\lambda d}^{s,g}(g(w))$ . Moreover, as seen before, the point  $g(w)$  lies in  $\tilde{W}_{2\delta'}^{s,g}(\pi^s \circ g(w))$ . It follows that  $g \circ \xi(w)$  lies in  $\tilde{W}_{2\delta' + \lambda d}^{s,g}(\pi^s \circ g(w))$ .

As the image of  $\pi^s \circ g \circ \xi$  coincides with that of  $\pi^s \circ g$  it follows that graph  $g \circ \xi$  defines a function  $g\xi$  in  $\Pi^{cu}(L)$  such that  $d(\xi^0, g\xi) < 2\delta' + \lambda d(\xi^0, \xi)$ . This proves (2).

Finally, (3) follows immediately from the previous arguments.  $\square$

*Notations.* Let us denote  $g(g\xi^0)$  by  $g^2\xi^0$  and, inductively,  $g(g^n\xi^0)$  by  $g^{n+1}\xi^0$  for every  $n > 0$ .

From (1) and (2) of the previous claim it follows that  $d(\xi^0, g\xi^0) < 2\delta'$ , then  $d(\xi^0, g^2\xi^0) < 2\delta' + \lambda 2\delta' = 2\delta'(1 + \lambda)$ , and inductively

$$d(\xi^0, g^n\xi^0) < 2\delta'(1 + \lambda + \dots + \lambda^{n-1})$$

for every  $n > 0$ . Note that  $g^n\xi^0 \in \Pi^{cu}(L)$  is well defined for every  $n > 0$  since

$\delta' = d_0(f, g)$  satisfies  $\delta'(1 + \lambda + \dots) < \frac{\delta}{64\kappa^2} < \delta_3/2$  by property (P5).

Moreover, by (3) of the previous claim it follows from  $d(\xi^0, g\xi^0) < 2\delta'$  that  $d(g\xi^0, g^2\xi^0) < 2\delta'\lambda$ , and inductively

$$d(g^n\xi^0, g^{n+1}\xi^0) < 2\delta'\lambda^n$$

for every  $n > 0$ .

Hence we obtain a well defined limit function  $\xi^\infty \in \Pi^{cu}(L)$  given by

$$\xi^\infty(x) := \lim_n g^n\xi^0(x)$$

for every  $x \in \bigcup_{n \in \mathbb{Z}} \tilde{W}_{\delta_3}^u(f^n L)$ . Clearly  $\xi^\infty$  satisfies  $d(\xi^0, \xi^\infty) \leq 2\delta'(1 + \lambda + \dots) < \delta_3/2$ .

Moreover, note that  $g\xi^\infty = \xi^\infty$  since the image by  $g$  of  $g^n\xi^0(x)$  coincides with  $g^{n+1}\xi^0(\pi^s \circ f(x))$  and the image by  $g$  of  $\lim_n g^n\xi^0(x)$  coincides with  $\lim_n g^{n+1}\xi^0(\pi^s \circ f(x))$ . In particular

$$\text{graph } g\xi^\infty \subset g(\text{graph } \xi^\infty).$$

As  $g^{-1}$  expands  $g$ -stable arcs uniformly then the points in  $\text{graph}(\xi^\infty)$  are precisely the points in  $V_{\delta_3}(L)$  whose  $g$  backwards orbit is well defined for every past iterate in  $V_{\delta_3}(L)$ .

**Claim 4.5.4.** *The set  $\text{graph } \xi^\infty$  is a  $C^1$ -submanifold tangent to  $\tilde{E}_g^{cu}$ .*

*Proof.* We will make a local argument near every  $x$  in  $L$ . Let us consider the local exponential map  $\exp_x : B_\delta^{cu} \times B_\delta^s \subset T_x M \rightarrow M$  where  $B_\delta^{cu}$  and  $B_\delta^s$  denote the balls of center  $x$  and radius  $\delta$  in  $\tilde{E}^{cu}(x)$  and  $\tilde{E}^s(x)$ , respectively.

Let  $\tilde{E}^{cu}$  and  $\tilde{\mathcal{C}}^{cu}$  denote the pull-back by  $\exp_x$  of the bundle  $\tilde{E}_g^{cu}$  and the cone field  $\tilde{\mathcal{C}}^{cu}$ .

Let  $S_n \subset B_\delta^{cu} \times B_\delta^s$  denote the preimage by  $\exp_x$  of  $\text{graph } g^n\xi^0$  for every  $n > 0$ . Since  $\text{graph } g^n\xi^0$  is a  $C^1$  submanifold tangent to the cone field  $Dg^n\mathcal{C}^{cu}$  then by property (P4) there exists  $\epsilon > 0$  small enough so that the sets  $\{z\} \times B_\delta^s(0)$  intersects  $S_n$  and this intersection point is a unique point for every  $z \in B_\epsilon^{cu}$ .

This defines  $C^1$  functions

$$\psi_n : B_\epsilon^{cu} \rightarrow B_\delta^s$$

for every  $n > 0$  given by  $\psi_n(z) := (\{z\} \times B_\delta^s) \cap S_n$ .

For every  $z \in B_\epsilon^{cu}$  it is immediate to check that the limit  $\psi_\infty(z) := \lim_n \psi_n(z)$  exists and defines a function  $\psi_\infty : B_\epsilon^{cu} \rightarrow B_\delta^s$ . Moreover, by property (P3)(3) the sequence  $D(\psi_n)_z(\tilde{E}^{cu}(x))$  needs to converge uniformly

to  $\tilde{E}^{cu}(\psi_\infty(z))$  for  $z \in B_\epsilon^{cu}$ .

We obtain that  $\psi_\infty$  is of class  $C^1$  and that  $D_z\psi_\infty(\tilde{E}^{cu}(x))$  is equal to  $\tilde{E}_g^{cu}(\psi_\infty(z))$  for every  $z \in B_\epsilon^{cu}$  by the following standard fact from multivariable calculus that is a consequence of Arzelà-Ascoli's theorem:

*If  $\psi_n : U \subset \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$  is a sequence of  $C^1$  maps defined in an open subset  $U \subset \mathbb{R}^{d_1}$  such that:*

- *The limit  $\psi_\infty(x) := \lim_n \psi_n(x)$  exists for every  $x \in U$ .*
- *The limit  $A(x) := \lim_n D_x\psi_n$  given by the rule  $(A(x))_{ij} = \lim_n (D_x\psi_n)_{ij}$  exists for every  $x \in U$ , varies continuously with  $x$  and  $\sup_{x \in U} \|D_x\psi_n - A(x)\| \xrightarrow{n} 0$ .*

*Then  $\psi_\infty : U \subset \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$  is a  $C^1$  map and  $D_x\psi_\infty(x) = A(x)$  for every  $x \in U$ .*

This proves that  $\text{graph } \xi^\infty$  is a  $C^1$ -submanifold tangent to  $\tilde{E}_g^{cu}$ . □

### Part 3: Construction of $h$ and $\rho$

For every leaf  $L$  of  $\mathcal{W}^c$  we have constructed a limit map  $\xi^\infty$  in  $\Pi^{cu}(L)$  such that  $d(\xi^0, \xi^\infty) < \delta_3/2$ . As this limit map corresponds to a limit graph for  $cu$ -strips let us rename it as  $\xi_{cu}^\infty$ . And let us also rename by  $\xi_{cu}^0$  the zero-section  $\xi^0$ .

Analogously as before one can define neighborhoods  $U_{\delta_3}^{su}(f^n L)$  for every  $n \in \mathbb{Z}$ , a map  $\pi^u$ , a family of maps  $\Pi^{cs}(L)$  and a limit map  $\xi_{cs}^\infty$  for  $cs$ -strips satisfying analogous properties than the  $cu$  ones (interchanging the roles of  $g$  and  $g^{-1}$ ).

Following Claim 4.5.4 we obtain that the intersection

$$\text{graph}(\xi_{cs}) \cap \text{graph}(\xi_{cu})$$

defines a  $C^1$  manifold in  $V(L)$  that is  $g$ -invariant and tangent to  $\tilde{E}_g^c$ . Let us denote by  $L'$  the connected component of this intersection that lies in  $U(L)$ , and in general let us denote by  $(f^n L)'$  the one that lies in  $U(f^n L)$ .

**Remark 4.5.5.** Note that from the properties of  $\xi_{cs}^\infty$  and  $\xi_{cu}^\infty$  (see, in particular, the discussion before Claim 4.5.4) the points in  $L'$  are characterized as the points in  $V(L)$  for which its  $g$ -orbit is well defined for every future and past iterate.

The projection  $\pi(L')$  in  $M$  is going to be  $h(L)$ , the continuation of  $L$ . Let us see how we can construct  $h : M \rightarrow M$  and  $\rho : M \rightarrow M$  so that the properties detailed in the statement of the theorem are verified.

For every  $L$  in  $\mathcal{W}^c$  let us start by defining a map  $h_1$  from  $L$  to  $L'$  in  $U(L)$ . For every  $x \in L$  we define  $h_1(x) \in L'$  by

$$h_1(x) := \xi_{cs}^\infty \circ \pi^u \circ \xi_{cu}^\infty(x).$$

In other words,  $h_1(x)$  is the unique point in  $L'$  satisfying that  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(x)$  and  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(h_1(x))$  intersect. As  $L'$  is tangent to  $\tilde{E}^{c,g}$  then  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(L')$  is tangent to  $\tilde{E}^{cu,g}$  by Lemma 2.0.1. This justifies why the intersection of  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(L')$  with  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(x)$  is a unique point.

It is immediate that  $h_1$  is continuous. Moreover, by property (P4) it is easy to check that for every  $x, y \in L$ :

$$d_L(x, y) = \delta_3 \quad \text{implies} \quad \delta_3/2 < d_{L'}(h_1(x), h_1(y)) < 2\delta_3 \quad (4.5.1)$$

In particular,  $h_1$  continuous and (4.5.1) imply that  $h_1$  from  $L$  to  $L'$  is also surjective.

Let us see now what happens when we iterate by  $g$ . Since  $g \text{ graph}(\xi_{cs}) \subset \text{graph}(\xi_{cs})$  and  $g^{-1} \text{ graph}(\xi_{cu}) \subset \text{graph}(\xi_{cu})$  it follows that

$$gL' = (fL)'$$

Given  $x$  in  $L$  the point  $h_1(x)$  lies in  $L'$  and the point  $f(x)$  lies in  $fL$ . Then  $g \circ h_1(x)$  and  $h_1 \circ f(x)$  both lie in  $gL' = (fL)'$ . We want to justify that the distance between  $g \circ h_1(x)$  and  $h_1 \circ f(x)$  inside  $(fL)'$  needs to be small.

Indeed, note first that  $d(f(x), g(x)) \leq \delta'$  (recall that  $\delta'$  denotes  $d_0(f, g)$ ). Then, on the one hand  $h_1 \circ f(x)$  is given as the unique point in  $(fL)'$  such that  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(f(x))$  and  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(h_1 \circ f(x))$  intersect. On the other hand,  $h_1(x)$  is given as the unique point in  $L'$  such that  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(x)$  and  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(h_1(x))$  intersect, and then by the  $g$ -invariance of the foliations  $\mathcal{W}^{s,g}$  and  $\mathcal{W}^{cu,g}$  one obtains that  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(g(x))$  intersects  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(g \circ h_1(x))$ . That is,  $g \circ h_1(x)$  is given as the unique point (unique by the same reasons as before) such that  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(g(x))$  and  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(g \circ h_1(x))$  intersect.

By property (P4) one can derive the following two properties. If two points  $z$  and  $w$  satisfy  $d(z, w) \leq \delta'$  then  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(z)$  and  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(w)$  are at Hausdorff distance less than  $\frac{3\delta'}{2}$ . And if two points  $w$  and  $z$  lie in  $(fL)'$  at distance not smaller than  $2\delta'$  then  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(z)$  and  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(w)$  are Hausdorff distance greater than  $\frac{3\delta'}{2}$ .

By applying the two properties above together with the properties that  $d(f(x), g(x)) \leq \delta'$ , that  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(f(x))$  has no trivial intersection with  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(h_1 \circ f(x))$  and that  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(g(x))$  has no trivial intersection with  $\tilde{\mathcal{W}}_{\delta_3}^{u,g}(g \circ h_1(x))$ , one obtains that

$$g \circ h_1(x) \in (fL)'_{2\delta'}(h_1 \circ f(x)) \quad (4.5.2)$$

for every  $x \in L$ .

A priori  $h_1$  from  $L$  to  $L'$  may not be injective. However, by a ‘regulating’ process we can rely on  $h_1$  to construct the desired  $C^1$  diffeomorphism  $h$  from  $L$  to  $L'$ . Let  $\gamma : \mathbb{R} \rightarrow L$  and  $\gamma' : \mathbb{R} \rightarrow L'$  be parametrizations by arc-length and let  $\Psi_1 : \mathbb{R} \rightarrow \mathbb{R}$  denote the map

$$\Psi_1(t) = \gamma'^{-1} \circ h_1 \circ \gamma(t).$$

We can assume that  $L$  and  $L'$  are parametrized with the same orientation, that is, such that  $\lim_{t \rightarrow +\infty} \Psi_1(t) = +\infty$ . Note that by (4.5.1) it follows that

$$\frac{\delta_3}{2} < \Psi_1(t + \delta_3) - \Psi_1(t) < 2\delta_3$$

for every  $t \in \mathbb{R}$ . If we define  $\Psi : \mathbb{R} \rightarrow \mathbb{R}$  as

$$\Psi(t) = \frac{1}{\delta_3} \int_{t - \frac{\delta_3}{2}}^{t + \frac{\delta_3}{2}} \Psi_1(s) ds$$

it follows that the derivative  $D\Psi(t)$  exists everywhere, varies continuously with  $t$  and satisfies  $\frac{1}{2} < D\Psi(t) < 2$ . Defining  $h$  as

$$h(x) = \gamma' \circ \Psi \circ \gamma^{-1}(x)$$

for every  $x \in L$  we conclude that  $h|_L : L \rightarrow L'$  is a  $C^1$  diffeomorphism satisfying

$$\frac{1}{2} < \|Dh(v^c)\| < 2$$

for every unit vector  $v^c$  in  $\tilde{E}^c$ .

Moreover by (4.5.1) it follows that

$$h(x) \in L'_{2\delta_3}(h_1(x))$$

for every  $x \in L$ . Since  $g \circ h_1(x)$  lies in  $(fL)'_{2\delta'}(h_1 \circ f(x))$  by (4.5.2) and  $2\delta' + 2\delta_3 < \delta$  then

$$g \circ h(x) \in (fL)'_{\delta}(h \circ f(x))$$

for every  $x \in L$ .

If we define  $\rho : L \rightarrow L$  as

$$\rho(x) = h^{-1} \circ g \circ h \circ f^{-1}(x)$$

it follows that  $\rho$  is a  $C^1$  diffeomorphism that is  $\delta$ -close to the identity and

satisfies

$$h \circ \rho \circ f(x) = g \circ h(x)$$

for every  $x \in L$ .

It remains to ‘descend’  $h$  and  $\rho$  to  $M$ . By a little abuse of notation let us denote by  $h$  and  $\rho$  the maps in  $M$  given by  $\pi \circ h \circ \pi^{-1}$  and  $\pi \circ \rho \circ \pi^{-1}$ , respectively.

All the properties claimed for  $h$  and  $\rho$  are immediately satisfied except maybe for the ones contained in the following two claims which may require further justification.

**Claim 4.5.6.** *The map  $h : M \rightarrow M$  is continuous, surjective and  $\delta$ -close to the identity.*

*Proof.* It is easy to check that  $h$  is  $\delta$ -close to the identity: Since  $\tilde{W}_{\delta_3}^{s,g}(x)$  and  $\tilde{W}_{\delta_3}^{u,g}(h_1(x))$  intersect for every  $x$  it follows that  $h_1$  is  $2\delta_3$  close to the identity. Since  $h(x)$  lies in  $L_{2\delta_3}(h_1(x))$  we conclude that  $h$  is  $4\delta_3 < \delta$  close to the identity in  $M$ .

The remaining of the proof is devoted to show that  $h$  is continuous. The surjectivity of  $h$  is a direct consequence of  $h$  continuous and  $\delta$ -close to the identity.

Note first that  $h_1$  also descends naturally to  $h_1 : M \rightarrow M$  and that if  $h_1$  is continuous in  $M$  then  $h$  will also be continuous as the regulating process has to preserve continuity. So we will show the continuity of  $h_1$ .

The map  $h_1$  has been defined by means of maps  $\xi_{cs}^\infty$ ,  $\pi^u$  and  $\xi_{cu}^\infty$  depending on the ‘unfolded’ tubular neighborhoods  $U(L)$  for each leaf  $L \in \mathcal{W}^c$ . These neighborhoods are a priori disjoint for different leaves of  $\mathcal{W}^c$ . We need to somehow merge them in  $M$  to be able to compare them.

For every  $x \in M$  let  $L(x)$  denote the leaf of  $\mathcal{W}^c$  through  $x$ . Let us define the map  $\xi_{cu,x}^\infty : \mathcal{W}_{\delta_3}^u(L_{\delta_3}(x)) \subset M \rightarrow M$  as the map such that  $\xi_{cu,x}^\infty|_{\tilde{W}_{\delta_3}^u(L_{\delta_3}(x))}$  in  $U_{\delta_3}^{su}(L(x))$  is a lift of it. Analogously we define the map  $\xi_{cs,x}^\infty : \mathcal{W}_{\delta_3}^s(L_{\delta_3}(x)) \rightarrow M$  for every  $x \in M$ .

Let  $\pi_x^u : B_{\delta_3/2}(x) \rightarrow \mathcal{W}_{\delta_3}^s(L_{\delta_3}(x))$  be such that  $\pi_x^u(z)$  is the intersection of  $\mathcal{W}_{\delta_3}^{u,g}(z)$  with  $\mathcal{W}_{\delta_3}^s(L_{\delta_3}(x))$  for every  $z \in B_{\delta_3/2}(x)$ . Again,  $\pi_x^u|_{B_{\delta_3/2}(x)}$  in  $U_{\delta_3}^{su}(L(x))$  is a lift of  $\pi_x^u$ .

For every  $x \in M$  we have

$$h_1(x) = \xi_{cs,x}^\infty \circ \pi_x^u \circ \xi_{cu,x}^\infty(x).$$

Let us see that if  $x_n \xrightarrow{n} x$  then

$$\xi_{cs,x_n}^\infty \circ \pi_{x_n}^u \circ \xi_{cu,x_n}^\infty(x_n) \xrightarrow{n} \xi_{cs,x}^\infty \circ \pi_x^u \circ \xi_{cu,x}^\infty(x).$$

It will be enough to show that:

1. If  $x_n \xrightarrow{n} x$  and  $y_n \xrightarrow{n} y$  then  $\pi_{x_n}^u(y_n) \xrightarrow{n} \pi_x^u(y)$ .
2. If  $x_n \xrightarrow{n} x$  and  $z_n \xrightarrow{n} z$  with  $z_n \in \mathcal{W}_{\delta_3}^u(\mathcal{W}_{\delta_3}^c(x_n))$  and  $z \in \mathcal{W}_{\delta_3}^u(\mathcal{W}_{\delta_3}^c(x))$ , then  $\xi_{cu,x_n}^\infty(z_n) \xrightarrow{n} \xi_{cu,x}^\infty(z)$ . Analogous property hold for *cs* maps.

Let us see first why (1) and (2) are enough for proving  $h_1$  continuous. Indeed, if  $x_n \xrightarrow{n} x$  in  $M$  then by (2) for  $x_n = z_n$  and  $x = z$  it follows that  $\xi_{cu,x_n}^\infty(x_n) \xrightarrow{n} \xi_{cu,x}^\infty(x)$ . Then  $\pi_{x_n}^u \circ \xi_{cu,x_n}^\infty(x_n)$  converges with  $n$  to  $\pi_x^u \circ \xi_{cu,x}^\infty(x)$  by (1). And again by (2) for *cs* maps with  $z_n = \pi_{x_n}^u \circ \xi_{cu,x_n}^\infty(x_n)$  and  $z = \pi_x^u \circ \xi_{cu,x}^\infty(x)$  we conclude that  $\xi_{cs,x_n}^\infty \circ \pi_{x_n}^u \circ \xi_{cu,x_n}^\infty(x_n)$  converges with  $n$  to  $\xi_{cs,x}^\infty \circ \pi_x^u \circ \xi_{cu,x}^\infty(x)$ .

The proof of (1) is immediate by the regularity of the foliations  $\mathcal{W}^{u,g}$  and  $\mathcal{W}^{cs}$ .

The remaining is devoted to showing (2). Informally, the key property we will use is that, by the regularity of  $\mathcal{W}^c$ , for every  $R > 0$  and  $\mu > 0$  the sets  $L_R(x)$  and  $L_R(x_n)$  are at Hausdorff distance less than  $\mu$  for every large enough  $n$ . This will enable us to ‘lift’ to  $U(L(x))$  long pieces of the leaf  $L(x_n)$  and to ‘see’ in  $U(L(x))$  the first iterates of the *cu* graph transform for  $L(x_n)$ .

Suppose from now on  $x_n \xrightarrow{n} x$  and  $z_n \xrightarrow{n} z$  with  $z_n \in \mathcal{W}_{\delta_3}^u(\mathcal{W}_{\delta_3}^c(x_n))$  and  $z \in \mathcal{W}_{\delta_3}^u(\mathcal{W}_{\delta_3}^c(x))$ . Given  $\epsilon > 0$  let us see that  $d(\xi_{cu,x_n}^\infty(z_n), \xi_{cu,x}^\infty(z)) < \epsilon$  for every  $n$  large enough. The proof for *cs* maps is analogous.

Let us assume without loss of generality that  $\epsilon < \delta_3/2$ . Recall the inclusions  $U_{\delta_3/2}(L(y)) \subset U_{\delta_3}^{su}(L(y)) \subset U_{2\delta_3}^{su}(L(y)) \subset U_{\delta_2}(L(y))$  for every  $y \in M$ .

For every  $n$  large enough the point  $x_n$  lies in  $B_{\delta_3/2}(x) \subset M$  so we can lift it to  $B_{\delta_3/2}(x) \subset U_{\delta_3/2}(L(x))$ . For simplicity, let us call these lifts of  $x_n$  with the same name,  $x_n$ .

For every  $y \in M$  recall that  $\tilde{\mathcal{W}}_{\delta_3}^u(L(y))$  denotes the *cu*-strip in  $U_{\delta_3}^{su}(L(y))$ . For every  $R > 0$  let  $\tilde{\mathcal{W}}_{\delta_3}^u(L_R(y))$  denote the ‘truncated’ *cu*-strip that is the set  $\bigcup_{z \in L_R(y)} \tilde{\mathcal{W}}_{\delta_3}^u(z)$ . By the regularity of the foliations  $\mathcal{W}^c$  and  $\mathcal{W}^u$  the following is immediate to check:

**Remark 4.5.7.** Suppose  $R > 0$  and  $\mu > 0$ . For every  $x_n$  close enough to  $x$  the projection to  $M$  of the truncated *cu*-strip  $\tilde{\mathcal{W}}_{\delta_3}^u(L_R(x_n))$  can be lifted to be a subset of  $U_{2\delta_3}^{su}(L(x))$  that is at Hausdorff distance less than  $\mu$  from  $\tilde{\mathcal{W}}_{\delta_3}^u(L_R(x))$  and such that  $x_n$  lifts close to  $x$  in  $U_{2\delta_3}^{su}(L(x))$ .

Recall that for every  $y \in M$  the map  $\xi_{cu}^\infty$  in  $\Pi^{cu}(L(y))$  is defined as a limit of the maps  $g^k \xi_{cu}^0$  for  $\xi_{cu}^0$  the zero-section in  $\Pi^{cu}(L(y))$ . Let us denote the zero section as  $\xi_{cu,y}^0$  and the maps  $g^k \xi^0$  as  $\xi_{cu,y}^k$  to highlight the dependence on the point  $y$ .



Recall that by Claim 4.5.3 it follows that

$$d(\xi_{cu,y}^k, \xi_{cu,y}^\infty) < 2\delta'(\lambda^k + \lambda^{k+1} + \dots)$$

for every  $k > 0$ . Hence for every  $\mu > 0$  there exists  $K > 0$  independent of  $y$  such that

$$d(\xi_{cu,y}^k, \xi_{cu,y}^\infty) < \mu$$

for every  $k \geq K$ .

Let  $K_0 > 0$  be such that

$$2\delta'(\lambda^{K_0} + \lambda^{K_0+1} + \dots) < \epsilon/6. \quad (4.5.3)$$

For every  $w \in M$  and  $z \in \tilde{W}_{\delta_3}^u(L(w))$  let  $z^{-1}$  denote the point in the  $cu$ -strip of  $f^{-1}L(z)$  such that  $g(z^{-1}) = \xi_{cu,w}^1(z)$ . Analogously, let  $z^{-k}$  be the point in the  $cu$ -strip of  $f^{-k}L(w)$  such that  $g^k(z^{-k}) = \xi_{cu,w}^k(z)$ . From the construction of  $\xi_w^k$  it follows that  $z^{-k-1}$  is contained in  $\tilde{W}_{2\delta'}^{cu}(f^{-1}(w^{-k}))$  for  $\delta' = d_0(f, g)$ . That is,  $(z^{-k})_{k \geq 0}$  is a backwards  $2\delta'$ -pseudo orbit for  $f$  with jumps in local  $cu$ -plaques.

Recall that the maximal expansion possible for  $df^{-1}$  is given by a constant  $\kappa > 1$ . It follows that, independently of  $w$ , for every  $K > 0$  there exists a constant  $R(K) > 0$  (in terms of  $\kappa$  and  $\delta'$ ) such that  $z^{-k}$  lies in the truncated  $cu$ -strip  $\tilde{W}_{\delta_3}^u(L_{R(K)}(f^{-k}(w)))$  for every  $k \in \{0, \dots, K\}$ . This is satisfied for every  $w \in M$  and  $z \in \tilde{W}_{\delta_3}^u(L(w))$ .

Let  $N > 0$  be such that, by Remark 4.5.7 for  $R_0 = R(K_0)$  and  $\mu_0 = \delta_3/100$ , the truncated  $cu$ -strip  $\tilde{W}_{\delta_3}^u(L_{R_0}(f^{-k}(x_n)))$  can be projected to  $M$  and then lifted to  $U_{2\delta_3}^{su}(L(f^{-k}(x)))$  so that it gets at Hausdorff distance less than  $\mu_0$  from  $\tilde{W}_{\delta_3}^u(L_{R_0}(f^{-k}(x)))$  for every  $k \in \{0, \dots, K_0\}$  and every  $n \geq N$ . For simplicity, let us call these projection-lifts to  $U_{2\delta_3}^{su}(L(f^{-k}(x)))$  of the truncated  $cu$ -strips  $\tilde{W}_{\delta_3}^u(L_{R_0}(f^{-k}(x_n)))$  with the same name  $\tilde{W}_{\delta_3}^u(L_{R_0}(f^{-k}(x_n)))$ .

It follows that the maps  $\xi_{cu,x_n}^\infty$  can also be lifted to  $U_{2\delta_3}^{su}(L(x))$  for every  $n \geq N$ . The domains of such maps being contained in the truncated  $cu$ -strips  $\tilde{W}_{\delta_3}^u(L_{R_0}(x_n))$ . Analogously, the maps  $\xi_{cu,z_n}^\infty$  can be lifted to  $U_{2\delta_3}^{su}(L(x))$  with domain contained in  $\tilde{W}_{\delta_3}^u(L_{R_0}(z_n))$ . Again, for simplicity let us call these lifted maps with the same names  $\xi_{cu,x_n}^\infty$  and  $\xi_{cu,z_n}^\infty$ .

We have to show that  $d(\xi_{cu,x_n}^\infty, \xi_{cu,x}^\infty) < \epsilon$  is satisfied in  $U_{2\delta_3}^{su}(L(x))$  for every  $n \geq N$ .

Note that, modulo taking  $N$  larger, for every  $n \geq N$  the set  $\tilde{W}_{\delta_3}^{s,g}(z_n)$  intersects  $\tilde{W}_{\delta_3}^u(L(x))$  and that this intersection point is unique. Let us call it  $w_n$ .

As  $z_n \xrightarrow{n} z$  then  $w_n \xrightarrow{n} z$ . Since  $\xi_{cu,x}^\infty$  is continuous in  $\tilde{W}_{\delta_3}^u(L(x))$  it follows that  $d(\xi_{cu,x}^\infty(z), \xi_{cu,x}^\infty(w_n)) < \epsilon/2$  for every  $n \geq N$ , by taking  $N$  larger

if needed. It remains to show that  $d(\xi_{cu,x_n}^\infty(z_n), \xi_{cu,x}^\infty(w_n)) < \epsilon/2$  for every  $n \geq N$ .

The points  $z_n^{-k}$  are well defined points in  $U_{2\delta_3}^{su}(f^{-k}L(x))$  satisfying that  $g^k(z_n^{-k}) = \xi_{cu,x_n}^k(z_n)$  for every  $k \in \{0, \dots, K_0\}$ . The points  $w_n^{-k}$  are well defined points in  $U_{2\delta_3}^{su}(f^{-k}L(x))$  satisfying that  $g^k(w_n^{-k}) = \xi_{cu,x}^k(w_n)$  for every  $k \in \{0, \dots, K_0\}$ . The key point to note is that the above implies

$$w_n^{-k} \in \tilde{\mathcal{W}}_{4\delta_3}^{s,g}(z_n^{-k}) \quad (4.5.4)$$

for every  $k \in \{0, \dots, K_0\}$  and  $n \geq N$ . This is because, as  $w_n$  lies in  $\tilde{\mathcal{W}}_{\delta_3}^{s,g}(z_n)$ , then  $\xi_{cu,x_n}^k(z_n)$  and  $\xi_{cu,x}^k(w_n)$  lie in the same  $\tilde{\mathcal{W}}^{s,g}$ -plaque of  $U_{2\delta_3}^{su}(L(x))$ . Then  $z_n^{-1}$  and  $w_n^{-1}$ , which are two points in  $U_{2\delta_3}^{su}(f^{-1}L(x))$  satisfying that  $g(z_n^{-1})$  and  $g(w_n^{-1})$  are in the same  $\mathcal{W}^{s,g}$  plaque of  $U_{2\delta_3}^{su}(L(x))$ , need to lie in the same  $\tilde{\mathcal{W}}^{s,g}$  plaque too. Inductively,  $z_n^{-k}$  and  $w_n^{-k}$  need to lie in the same  $\tilde{\mathcal{W}}^{s,g}$ -plaque of  $U_{2\delta_3}^{su}(f^{-k}L(x))$ . As all of these  $\tilde{\mathcal{W}}^{s,g}$ -plaques have diameter less than  $4\delta_3$  then (4.5.4) follows.

For  $k = K_0$  in (4.5.4) it follows that  $w_n^{-K_0}$  lies in  $\tilde{\mathcal{W}}_{4\delta_3}^{s,g}(z_n^{-K_0})$ . Then  $g^{K_0}(z_n^{-K_0})$  lies in  $\tilde{\mathcal{W}}_{4\delta_3\lambda^{K_0}}^{s,g}(g^{K_0}(z_n^{-K_0}))$ . Recall that  $g^{K_0}(z_n^{-K_0}) = \xi_{cu,x_n}^{K_0}(z_n)$  and  $g^{K_0}(w_n^{-K_0}) = \xi_{cu,x}^{K_0}(w_n)$ . Using (4.5.3) and the fact that  $2\delta' < 4\delta_3$  by property (P5) it follows that

$$d(\xi_{cu,x_n}^{K_0}(z_n), \xi_{cu,x}^{K_0}(w_n)) < \epsilon/6$$

for every  $n \geq N$ .

Again by (4.5.3) it follows that

$$d(\xi_{cu,x}^{K_0}(z), \xi_{cu,x}^\infty(w_n)) < \epsilon/6 \quad \text{and} \quad d(\xi_{cu,x_n}^{K_0}(z_n), \xi_{cu,x_n}^\infty(z_n)) < \epsilon/6$$

for every  $n \geq N$ .

By triangular inequality (two times) we conclude that

$$d(\xi_{cu,x}^\infty(z_n), \xi_{cu,x}^\infty(w_n)) < \epsilon/2$$

for every  $n \geq N$ .

This shows that  $h$  is continuous and ends the proof of the claim.  $\square$

**Claim 4.5.8.** *The map  $\rho : M \rightarrow M$  is a homeomorphism.*

*Proof.* Recall that we have already seen that  $\rho(L) = L$  and  $\rho|_L : L \rightarrow L$  is injective and  $\delta$ -close to the identity for every leaf  $L \in \mathcal{W}^c$ . And that  $h \circ \rho \circ f = g \circ h$  is satisfied.

It remains to show that  $\rho$  is continuous as  $\rho$  continuous, injective and  $\delta$  close to the identity implies  $\rho$  homeomorphism.

Suppose  $x_n \xrightarrow{n} x$  in  $M$ . Let us see that  $\rho \circ f(x_n) \xrightarrow{n} \rho \circ f(x)$ . As  $f$  is continuous this implies  $\rho$  continuous.

As  $x_n \xrightarrow{n} x$  then  $g \circ h(x_n) \xrightarrow{n} g \circ h(x)$  by the continuity of  $h$  and  $g$ . Since  $h \circ \rho \circ f = g \circ h$  it follows that  $h \circ \rho \circ f(x_n)$  has limit  $h \circ \rho \circ f(x)$ .

As  $x_n \xrightarrow{n} x$  and  $\rho \circ f(x_n)$  lies in  $\mathcal{W}_\delta^c(f(x_n))$  for every  $n$  it follows that every accumulation point of the sequence  $\rho \circ f(x_n)$  must lie in  $\mathcal{W}_\delta^c(\rho \circ f(x))$ . Because of  $\frac{1}{2} < \|Dh|_{E^c}\| < 2$  the map  $h$  is injective restricted to  $\mathcal{W}_\delta^c(\rho \circ f(x))$ . Hence the only way that  $h \circ \rho \circ f(x_n)$  has limit  $h \circ \rho \circ f(x)$  is that  $\rho \circ f(x_n)$  converges to  $\rho \circ f(x)$ . This shows the continuity of  $\rho$  and ends the proof of the claim and of Theorem 4.2.3.  $\square$

## 4.6 Proof of Theorem 4.2.4: Continuation of complete $C^1$ center immersions

Suppose  $\eta : \mathbb{R} \rightarrow M$  is a complete  $C^1$  immersion tangent to  $E_f^c$  as in the hypothesis of Theorem 4.2.4. As in the proof of Theorem 4.2.3 one can construct an abstract manifold  $U(\eta)$ , informally an ‘unfolded neighborhood’ of  $\eta$ , given as the disjoint union of the sets  $\{B_\delta(\eta(t))\}_{t \in \mathbb{R}}$  with the points in  $B_\delta(\eta(t))$  and  $B_\delta(\eta(s))$  identified if and only if the piece of  $\eta$ -orbit from  $\eta(t)$  to  $\eta(s)$  has length less than  $4\delta$ . Then in  $U(\eta)$  there exists a natural projection  $\pi : U(\eta) \rightarrow M$  which is a local diffeomorphism at any point and  $U(\eta)$  can be given the structure of abstract Riemannian manifold by taking pull-back of the structure in  $M$  by the restrictions of  $\pi : U(\eta) \rightarrow M$  to the sets  $B_\delta(\eta(t))$  for  $t \in \mathbb{R}$ .

Analogously as for Theorem 4.2.3 one can construct a manifold  $V(\eta)$  whose connected components are  $U(f^n \circ \eta)$  for every  $n \in \mathbb{Z}$ , so that  $g$  can be ‘lifted’ to  $V(\eta)$  (sending points from one connected component  $U(f^n \circ \eta)$  to the next one  $U(f^{n+1} \circ \eta)$  for every  $n \in \mathbb{Z}$ ) so that the graph transform method (Lemma 4.5.3) can be performed in  $V(\eta)$ .

One obtains the existence of a sequence  $\gamma_n : \mathbb{R} \rightarrow M$  of complete  $C^1$  immersions tangent to  $E_g^c$  satisfying (4.2.1) and (4.2.2) of Theorem 4.2.4 by exactly the same arguments already seen for the continuation of  $\mathcal{W}^c$ -leaves in the proof of Theorem 4.2.3.

For the uniqueness part of Theorem 4.2.3, modulo reparametrizations, note that by the same arguments showing that the continuation  $h(L)$  of a center leaf  $L$  is characterized as the only points in  $U(L)$  for which its  $g$  orbit is well defined for every backwards and forwards iterate (see Remark 4.5.5) it follows that the image of every lift of  $\gamma_0$  to  $U(\eta)$  coincides with the points in  $U(\eta)$  whose  $g$ -orbit is well defined in  $V(\eta)$  for every backwards and forwards iterate (in particular, there exists a unique lift).

Then if  $\gamma'_n : \mathbb{R} \rightarrow M$  is another sequence of complete  $C^1$  immersions tangent to  $E_g^c$  satisfying (4.2.1) and (4.2.2) one deduces that the lift of  $\gamma'_0$  to  $U(\eta)$  has to have the same image as the lift of  $\gamma_0$ . It follows that  $\gamma'_0$  is a reparametrization of  $\gamma_0$  and, by (4.2.2), that  $\gamma'_n$  is a reparametrization of  $\gamma_n$  for every  $n \in \mathbb{Z}$ .

# Chapter 5

## Global stability

### 5.1 Introduction

Recall that  $\text{PH}_{c=1}(M)$  denotes the set of partially hyperbolic diffeomorphisms with one-dimensional center in the closed manifold  $M$ . The set  $\text{PH}_{c=1}(M)$  is an open subset of  $\text{Diff}^1(M)$  for the  $C^1$  topology.

The main goal of this chapter is to show that discretized Anosov flows constitute a  $C^1$  open and closed class of diffeomorphisms in  $\text{PH}_{c=1}(M)$ . As a consequence, it also follows that leaf-conjugacy is preserved among whole connected components of discretized Anosov flows.

**Theorem 5.1.1.** *The set of discretized Anosov flows is a  $C^1$  open and closed subset of  $\text{PH}_{c=1}(M)$ .*

**Corollary 5.1.2.** *Two discretized Anosov flows in the same  $C^1$  connected component of  $\text{PH}_{c=1}(M)$  are leaf-conjugate.*

Recall that an  $f$ -invariant center foliation  $\mathcal{W}^c$  is called *uniformly compact* if every leaf of  $\mathcal{W}^c$  is compact and the leaf volume function  $x \mapsto \text{vol}(\mathcal{W}^c(x))$  is bounded in  $M$ . Analogous statements are shown to happen for these type of systems:

**Theorem 5.1.3.** *The set of diffeomorphisms in  $\text{PH}_{c=1}(M)$  admitting an invariant uniformly compact center foliation form a  $C^1$  open and closed subset of  $\text{PH}_{c=1}(M)$ .*

**Corollary 5.1.4.** *Suppose  $f \in \text{PH}_{c=1}(M)$  admits an invariant uniformly compact center foliation  $\mathcal{W}_f^c$  and  $g \in \text{PH}_{c=1}(M)$  lies in the same  $C^1$  connected component of  $\text{PH}_{c=1}(M)$  as  $f$ . Then  $g$  admits an invariant uniformly compact center foliation  $\mathcal{W}_g^c$  such that  $(f, \mathcal{W}_f^c)$  and  $(g, \mathcal{W}_g^c)$  are leaf-conjugate.*

The main example of a uniformly compact center foliation is given by the center foliation  $\mathcal{W}^c$  of a *partially hyperbolic skew-product*. In this case

the leaves of  $W^c$  are the fibers of a fiber bundle structure in  $M$  and the leaf-volume function is, in fact, continuous. It follows immediately from Corollary 5.1.4 that partially hyperbolic skew-products with one-dimensional center constitute a  $C^1$  open and closed class in  $\text{PH}_{c=1}(M)$ .

Finally, in the last section of this chapter we show that unique integrability of the center bundle is also preserved among whole connected components of discretized Anosov flows and of partially hyperbolic diffeomorphisms admitting a uniformly compact center foliation.

By constructing an example of a discretized Anosov flow with non-uniquely integrable center bundle (see Example 5.4.3) it follows that there exists  $C^1$  connected components of discretized Anosov flows that do not contain the time 1 map of an Anosov flow.

## 5.2 Global stability of discretized Anosov flows

As shown below, Theorem 5.1.1 and Corollary 5.1.2 are immediate consequences of the following.

**Proposition 5.2.1.** *For every  $f_0 \in \text{PH}_{c=1}(M)$  there exists a  $C^1$ -neighborhood  $\mathcal{U}$  of  $f_0$  such that, if  $f \in \mathcal{U}$  is a discretized Anosov flow, then every  $g \in \mathcal{U}$  is also a discretized Anosov flow.*

*Moreover, if  $W_f^c$  and  $W_g^c$  denote the flow center foliations of  $f$  and  $g$ , respectively, then  $(f, W_f^c)$  is plaque expansive and  $(f, W_f^c)$  and  $(g, W_g^c)$  are leaf-conjugate.*

Let us first mention how Theorem 5.1.1 follows from Proposition 5.2.1.

*Proof of Theorem 5.1.1 assuming Proposition 5.2.1.* Suppose  $f_0$  is a discretized Anosov flow. By Proposition 5.1.1 there exists a neighborhood  $\mathcal{U}$  of  $f_0$  such that every element of  $\mathcal{U}$  is a discretized Anosov flow. This proves the open property.

Suppose now that  $(f_n)$  is a sequence of discretized Anosov flows converging to  $f_0 \in \text{PH}_{c=1}(M)$ . Let  $\mathcal{U}$  be as in Proposition 5.2.1. For some large  $N$  the map  $f_N$  lies in  $\mathcal{U}$  and, as a consequence,  $f_0$  is also a discretized Anosov flow. This proves the closed property.  $\square$

Let us see now how Corollary 5.1.2 is also immediate from Proposition 5.2.1.

*Proof of Corollary 5.1.2 assuming Proposition 5.2.1.* Suppose  $f$  and  $g$  are discretized Anosov flows in the same  $C^1$  connected component of  $\text{PH}_{c=1}(M)$ . Let  $W_f^c$  and  $W_g^c$  denote their flow center foliations, respectively.

Let  $\{f_t\}_{t \in [0,1]}$  be a path in  $\text{PH}_{c=1}(M)$  joining  $f_0 = f$  with  $f_1 = g$ . For every  $f_t$  let  $\mathcal{U}(f_t)$  be a  $C^1$  neighborhood as in Proposition 5.2.1. By taking a finite cover of  $\{f_t\}_{t \in [0,1]}$  by open sets  $\mathcal{U}(f_t)$  one can argue inductively to show that  $(f, \mathcal{W}_f^c)$  and  $(g, \mathcal{W}_g^c)$  are leaf-conjugate.  $\square$

The rest of the section is devoted to prove Proposition 5.2.1. We will crucially use Theorem 4.2.3 and Lemma 4.3.3 from Chapter 4. And we will assume familiarity with the terminology used in Section 4.3.

### 5.2.1 Uniform plaque expansivity for discretized Anosov flows

The notion of  $\delta$ -plaque expansivity was introduced in Definition 4.3.2. For every neighborhood  $\mathcal{U}_\delta(f_0)$  as in Lemma 4.2.1 we will consider a subset of  $\mathcal{U}_\delta(f_0)$  satisfying a stronger version of property (P4). This will allow us to show in Proposition 5.2.4 that any discretized Anosov on this new neighborhood has to be  $\delta$ -plaque expansive.

As discussed in Remark 4.2.2, the next lemma can be derived from the proof of Lemma 4.2.1.

**Lemma 5.2.2.** *In the setting of Lemma 4.2.1, one can furthermore require that the following reinforcement of property (P4) is satisfied:*

(P4') *The metric and the cone fields  $(\mathcal{C}^s, \mathcal{C}^{cu})$  and  $(\mathcal{C}^{cs}, \mathcal{C}^u)$  are  $\frac{(1-\lambda)}{1000}$ -nearly euclidean at scale  $\kappa 20\delta$ .*

It is immediate to check the following consequence from (P4').

**Lemma 5.2.3.** *Suppose  $f_0 \in \text{PH}_{c=1}(M)$ . Consider a metric in  $M$ , a constant  $0 < \lambda < 1$  and a  $C^1$ -neighborhood  $\mathcal{U}_\delta(f_0) \subset \text{PH}_{c=1}(M)$  of  $f_0$  as in Lemma 5.2.2. Then for every*

- *map  $f \in \mathcal{U}_\delta(f_0)$ ,*
- *pair of points  $x \in M$  and  $x' \in \mathcal{W}_f^s(x)$  with  $d_s(x, x') = 10\delta$ ,*
- *pair of  $C^1$  curves  $\eta$  and  $\eta'$  of length less than  $20\delta$ , tangent to  $\mathcal{C}^{cs} \cap \mathcal{C}^{cu}$  with  $x \in \eta$  and  $x' \in \eta'$ ,*

*it follows that*

$$d(\eta, \eta') > \lambda(10\delta),$$

*where  $d(\eta, \eta')$  denotes the infimum distance between points in  $\eta$  and  $\eta'$ .*

The following is the goal of this subsection.

**Proposition 5.2.4** (Uniform plaque expansivity for discretized Anosov flows). *Suppose  $f_0 \in \text{PH}_{c=1}(M)$ . Consider a metric in  $M$  and a  $C^1$  neighborhood  $\mathcal{U}_\delta(f_0) \subset \text{PH}_{c=1}(M)$  of  $f_0$  as in Lemma 5.2.2. If  $f$  is a discretized Anosov flow in  $\mathcal{U}_\delta(f_0)$  and  $\mathcal{W}^c$  is the flow center foliation of  $f$  then  $(f, \mathcal{W}^c)$  is  $\delta$ -plaque expansive.*

*Proof of Proposition 5.2.4.* By Definition 3.2.1, Proposition 3.2.2 and Remark 3.2.3 the map  $f$  can be written down as  $f(x) = \varphi_{\tau(x)}^c(x)$  for  $\tau : M \rightarrow \mathbb{R}_{>0}$  continuous and  $\varphi_t^c : M \rightarrow M$  a unit speed flow whose flow lines are the flow center foliation  $\mathcal{W}^c$  of  $f$ .

The following is a key claim showing that, even taking into account possible ‘backwards jumps’, every  $\delta$ -pseudo orbit ‘advances forward’ in the direction of the flow. Roughly speaking this allows us to bring into play the expansivity of the topological Anosov flow  $\varphi_t^c$  to obtain expansivity for pairs of  $\delta$ -pseudo orbits belonging to different center leaves.

**Claim 5.2.5.** *The function  $\tau$  is always larger than  $10\delta$ .*

*Proof.* Suppose by contradiction that  $\tau(x) < 10\delta$  for some  $x \in M$ .

For every  $y \in \mathcal{W}_{11\delta}^s(x)$  let  $\gamma_y : [0, 1] \rightarrow \mathcal{W}^c(y)$  be the constant speed reparametrization of the piece of  $\varphi_t^c$ -orbit from  $y$  to  $f(y)$ . Note that by the continuity of  $\tau$  the length of  $\gamma_y$  varies continuously with  $y$ .

By property (P4’) the image of  $\gamma_y$  needs to be a segment from  $\mathcal{W}_{11\delta}^s(x)$  to  $\mathcal{W}_{12\delta}^s(f(x))$ , contained in  $B_{20\delta}(x)$  and whose length does not surpass  $12\delta$ . In particular, the image of  $\gamma_y$  is contained in  $\mathcal{W}_{20\delta}^c(y)$  for every  $y \in \mathcal{W}_{11\delta}^s(x)$ .

Let us fix  $y_0$  in  $\mathcal{W}_{11\delta}^s(x)$  such that  $d_s(x, y_0) = 10\delta$ . It follows that  $\gamma_{y_0}$  is a curve joining  $y_0$  to  $f(y_0)$ , where  $d(f(x), f(y_0)) \leq d_s(f(x), f(y_0)) < \lambda(10\delta)$ .

By Lemma 5.2.3 the sets  $\mathcal{W}_{20\delta}^c(x)$  and  $\mathcal{W}_{20\delta}^c(y_0)$  must be at distance greater than  $\lambda 10\delta$ . However, we have just shown that  $f(x) \in \mathcal{W}_{20\delta}^c(x)$  and  $f(y_0) \in \mathcal{W}_{20\delta}^c(y_0)$  are at distance less than  $\lambda 10\delta$ . This gives us a contradiction and proves Claim 5.2.5. See Figure 5.1 for a schematic idea of the argument used.

Recall that by Proposition 3.4.4 the discretized Anosov flow  $f$  is dynamically coherent with center-stable foliation  $\mathcal{W}^{cs}$  and center-unstable foliation  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ . As stated in the next claim, dynamical coherence let us obtain  $\delta$ -plaque expansivity by checking  $2\delta$ -plaque expansivity inside  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  leaves.

**Claim 5.2.6.** *Suppose the following statement is true: For every  $(x_n)_{n \geq 0}$  and  $(y_n)_{n \geq 0}$  forward  $2\delta$ -pseudo orbits such that  $x_{n+1} \in \mathcal{W}_{2\delta}^c(f(x_n))$ ,  $y_{n+1} \in \mathcal{W}_{2\delta}^c(f(y_n))$  and  $y_n \in \mathcal{W}_{4\delta}^{cu}(x_n)$  for every  $n \geq 0$ , then  $y_0 \in \mathcal{W}_{8\delta}^c(x_0)$ .*



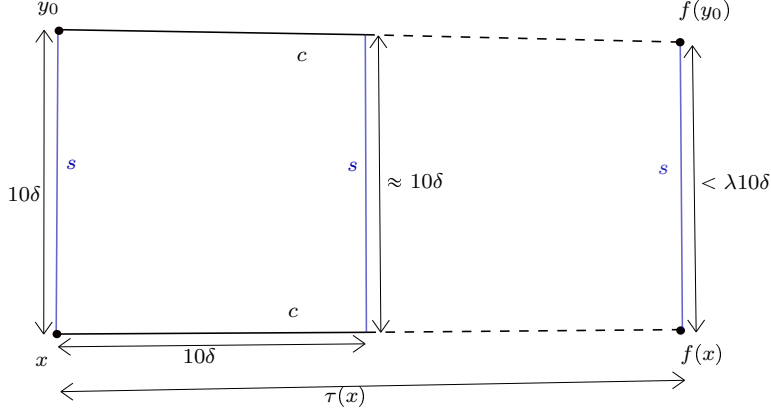


Figure 5.1: At small scale the bundles  $E^\sigma$ ,  $\sigma \in \{s, c, u\}$ , are nearly parallel and pairwise disjoint for every  $f$  near  $f_0$ . Hence  $\tau$  must be greater than  $10\delta$  to be able to ‘see’ the contraction of rate  $\lambda \in (0, 1)$ . As a consequence, if  $\delta' < 10\delta$  then every pair of  $\delta'$ -pseudo orbits for  $f$  advance in the same direction as the center flow and therefore should eventually separate because of the expansivity of the center flow.

*Suppose that is also true the analogous statement for backwards  $2\delta$ -pseudo orbits inside  $W^{cs}$  leaves. Then  $(f, W^c)$  is  $\delta$ -plaque expansive.*

*Proof.* Let  $(x_n)_n$  and  $(y_n)_n$  be a pair of  $\delta$ -pseudo orbits satisfying  $x_{n+1} \in W_\delta^c(f(x_n))$ ,  $y_{n+1} \in W_\delta^c(f(y_n))$  and  $d(x_n, y_n) < 2\delta$  for every  $n \in \mathbb{Z}$ . Let us see that  $y_0 \in W_{3\delta}^c(x_0)$ .

All along the proof of the claim we will implicitly use that, by property (P4'), at scale  $\kappa 20\delta$  the invariant bundles are nearly pairwise orthogonal. It will be clear on each case that what is stated follows directly from property (P4'). And we will implicitly use that by dynamical coherence  $cs$  (resp  $cu$ ) discs are subfoliated by  $c$  and  $s$  (resp  $u$ ) discs and that  $cs$  and  $cu$  discs intersect in  $c$  discs.

As  $d(x_n, y_n) < 2\delta$  one can consider  $y'_n$  the intersection of  $W_{3\delta}^s(y_n)$  with  $W_{3\delta}^{cu}(x_n)$ .

Moreover,  $y'_n \in W_{3\delta}^{cu}(x_n)$  implies  $f(y'_n) \in W_{\kappa 3\delta}^{cu}(f(x_n))$  and  $y'_n \in W_{3\delta}^s(y_n)$  implies  $f(y'_n) \in W_{\kappa 3\delta}^s(f(y_n))$ . It follows that  $f(y'_n)$  is the intersection point of  $W_{\kappa 3\delta}^s(f(y_n))$  with  $W_{\kappa 3\delta}^{cu}(f(x_n))$ .

The point  $x_{n+1}$  lies in  $W_\delta^c(f(x_n)) \subset W_{\kappa 3\delta}^{cu}(f(x_n))$  and the point  $y'_{n+1}$  is given by the intersection of  $W_{3\delta}^s(y_{n+1})$  with  $W_{3\delta}^{cu}(x_{n+1}) \subset W_{\kappa 3\delta}^{cu}(f(x_n))$ . We obtain that  $f(y'_n)$  and  $y'_{n+1}$  are both contained in  $W_{\kappa 3\delta}^{cu}(f(x_n))$ . And both contained in  $W_{\kappa 3\delta}^s(W_\delta^c(f(y_n)))$  since  $y_{n+1} \in W_\delta^c(f(y_n))$ . It follows that  $f(y'_n)$  and  $y'_{n+1}$ , which lie in the intersection of  $W_{\kappa 3\delta}^s(W_\delta^c(f(y_n))) \subset W^{cs}(f(y_0))$  and  $W_{\kappa 3\delta}^{cu}(f(x_n))$ , are in the same local center manifold. Since  $y_{n+1} \in W_\delta^c(f(y_n))$

it follows that  $y'_{n+1} \in \mathcal{W}_{2\delta}^c(f(y'_n))$ .

Then  $(y'_n)_{n \geq 0}$  is a forward  $2\delta$ -pseudo orbit with jumps in center plaques, as well as  $(x_n)_{n \geq 0}$ , and they satisfy  $y'_n \in \mathcal{W}_{3\delta}^{cu}(x_n)$  for every  $n \geq 0$ . By the assumption of the statement it follows that  $y'_0$  lies  $\mathcal{W}_{8\delta}^c(x_0)$ . Which in turns imply  $y_0 \in \mathcal{W}_{11\delta}^{cs}(x_0)$  as  $y'_0 \in \mathcal{W}_{3\delta}^c(y_0)$ .

By defining analogously  $(y''_n)_{n \leq 0}$  a backward  $2\delta$ -pseudo orbit as the intersection of  $\mathcal{W}_{3\delta}^u(y_n)$  with  $\mathcal{W}_{3\delta}^{cs}(x_n)$  for every  $n \leq 0$  we conclude that  $y_0$  lies  $\mathcal{W}_{11\delta}^{cu}(x_0)$ . It follows that  $y_0 = y'_0 = y''_0$ , and then that  $y_0$  lies in  $\mathcal{W}_{8\delta}^c(x_0)$ . As we are at scale  $\kappa 20\delta$  then  $d(x_0, y_0) < 2\delta$  and  $y_0 \in \mathcal{W}_{8\delta}^c(x_0)$  imply that  $y_0$  lies in  $\mathcal{W}_{3\delta}^c(x_0)$ . This proves Claim 5.2.6.

Suppose  $(x_n)_{n \geq 0}$  and  $(y_n)_{n \geq 0}$  are two forward  $2\delta$ -pseudo orbits such that  $x_{n+1} \in \mathcal{W}_{2\delta}^c(f(x_n))$ ,  $y_{n+1} \in \mathcal{W}_{2\delta}^c(f(y_n))$  and  $y_n \in \mathcal{W}_{4\delta}^{cu}(x_n)$  for every  $n \geq 0$ . Let us see that  $y_0$  needs to lie in  $\mathcal{W}_{8\delta}^c(x_0)$ . It will be clear that in a similar fashion one can show the analogous statement for backwards  $2\delta$ -pseudo orbits if  $y_0$  is a point in  $\mathcal{W}_{4\delta}^{cs}(x_0)$ . By Claim 5.2.6 this will end the proof of Proposition 5.2.4.

Suppose by contradiction that  $y_0$  does not belong to  $\mathcal{W}_{8\delta}^c(x_0)$ . As  $y_n \in \mathcal{W}_{4\delta}^{cu}(x_n)$  we can consider  $x'_n \in \mathcal{W}_{5\delta}^c(x_n)$  such that  $y_n \in \mathcal{W}_{5\delta}^u(x'_n)$  for every  $n \geq 0$ . It follows that  $y_n \neq x'_n$  for every  $n \geq 0$ .

Note that  $y_0 \in \mathcal{W}_{5\delta}^u(x'_0)$  implies that  $d_u(f^{-n}(y_0), f^{-n}(x'_0))$  tends to 0 with  $n$ . Since  $\{f^{-n}(y_0)\}_{n \geq 0}$  are points in  $\mathcal{W}^c(y_0)$  and  $\{f^{-n}(x'_0)\}_{n \geq 0}$  are points in  $\mathcal{W}^c(x_0)$  it follows that  $\mathcal{W}^c(x_0)$  and  $\mathcal{W}^c(y_0)$  can not be both compact leaves. As the conditions for  $(x_n)_{n \geq 0}$  and  $(y_n)_{n \geq 0}$  are symmetric let us assume without loss of generality that  $\mathcal{W}^c(x_0)$  is not compact.

For every pair of different points  $z, z' \in \mathcal{W}^c(x_0)$  let  $[z, z']_c$  and  $[z, z')_c$  denote the closed and half-open center segments from  $z$  to  $z'$  inside  $\mathcal{W}^c(x_0)$ , respectively.

Let  $K_n$  be the sequence such that  $x'_n$  lies in  $[f^{K_n}(x_0), f^{K_n+1}(x_0)]_c$  for every  $n \geq 0$ . By Claim 5.2.5 it follows that  $K_n \xrightarrow{n} +\infty$ .

For every  $n$  the point  $x'_n$  is a point in  $[f^{K_n}(x_0), f^{K_n+1}(x_0)]_c$ . Hence  $f^{-K_n}(x'_n)$  is a point in  $[x_0, f(x_0)]_c$ . Moreover, since  $y_n$  lies in  $\mathcal{W}_{5\delta}^u(x'_n)$  it follows that  $f^{-K_n}(y_n)$  lies in  $\mathcal{W}_{\lambda^{K_n} 5\delta}^u(f^{-K_n}(x'_n))$ . And as  $y_n \neq x'_n$ , then  $f^{-K_n}(y_n)$  is not contained in  $[x_0, f(x_0)]_c$ . It follows that

$$f^{-K_n}(y_n) \in \mathcal{W}_{\lambda^{K_n} 5\delta}^u([x_0, f(x_0)]_c) \setminus [x_0, f(x_0)]_c \quad (5.2.1)$$

for every  $n$ .

As  $\lambda$  is a constant in  $(0, 1)$  the sequence  $\lambda^{-K_n} 5\delta$  tends to 0 with  $n$ . And since  $\mathcal{W}^c(y_0)$  contains  $y_n$  for every  $n$  and is an invariant leaf by  $f$  it follows that  $f^{-K_n}(y_n)$  is a sequence contained in  $\mathcal{W}^c(y_0)$ . Hence if the following claim

is true one gets a contradiction with (5.2.1), ending the proof of Proposition 5.2.4.

**Claim 5.2.7.** *There exists  $\epsilon > 0$  such that  $\mathcal{W}_\epsilon^u([x_0, f(x_0)]_c) \setminus [x_0, f(x_0)]_c$  is disjoint from  $\mathcal{W}^c(y_0)$ .*

*Proof.* Note that a priori one can not rule out that  $\mathcal{W}^c(x_0)$  and  $\mathcal{W}^c(y_0)$  may be the same leaf. That is why we will show that  $\mathcal{W}^c(y_0)$  is disjoint from  $\mathcal{W}_\epsilon^u([x_0, f(x_0)]_c) \setminus [x_0, f(x_0)]_c$  and not simply disjoint from  $\mathcal{W}_\epsilon^u([x_0, f(x_0)]_c)$ .

Recall from Proposition 3.7.3 the topological description of the center-unstable leaves of a discretized Anosov flow in terms of plane leaves, cylinder leaves, etc.

If  $\mathcal{W}^{cu}(x_0)$  is a plane leaf the claim follows straightforwardly from Proposition 3.7.3 since in that case the foliations  $\mathcal{W}^u$  and  $\mathcal{W}^c$  need to have a global product structure inside  $\mathcal{W}^{cu}(x_0)$ .

If  $\mathcal{W}^{cu}(x_0)$  is a cylinder leaf then Proposition 3.7.3 shows that the alpha-limit of  $y_0$  by the center flow  $\varphi_t^c$  coincides with the unique compact leaf  $L$  of  $\mathcal{W}^c$  contained in  $\mathcal{W}^{cu}(x_0)$ . Moreover, as  $\mathcal{W}^c(x_0)$  is not compact, then  $L \neq \mathcal{W}^c(x_0)$ .

In case  $\mathcal{W}^c(y_0)$  coincides with  $L$  it is enough to consider  $\epsilon > 0$  smaller than the Hausdorff distance between the compact and disjoint sets  $\mathcal{W}^c(y_0)$  and  $[x_0, f(x_0)]_c$ .

In case  $\mathcal{W}^c(y_0)$  does not coincide with  $L$  then  $\mathcal{W}^c(y_0)$  is not compact and it is immediate to check that the omega-limit of  $y_0$  in the intrinsic metric of  $\mathcal{W}^{cu}(x_0)$  needs to be empty. This follows from the fact that for every  $R > 0$  the point  $f^n(y_0)$  can not be contained in  $\mathcal{W}_R^u(L)$  for arbitrarily large  $n > 0$ . Indeed, if  $f^n(y_0)$  lies in  $\mathcal{W}_R^u(L)$  for arbitrarily large  $n > 0$  then  $y_0 = f^{-n} \circ f^n(y_0)$ , which is not contained in  $L$ , would be at arbitrarily small distance from the compact leaf  $L$  getting to a contradiction.

It follows that for some  $T > 0$  the set  $\mathcal{W}^c(y_0) \setminus \mathcal{W}_T^c(x_0)$  is at positive distance from the compact set  $[x_0, f(x_0)]_c$  in the intrinsic metric of  $\mathcal{W}^{cu}(x_0)$ . Say  $d > 0$ .

If  $\mathcal{W}_T^c(y_0)$  is disjoint from  $[x_0, f(x_0)]_c$  it is enough to consider  $d > \epsilon > 0$  so that  $\epsilon$  is smaller than the Hausdorff distance between  $\mathcal{W}_T^c(y_0)$  and  $[x_0, f(x_0)]_c$ .

If  $\mathcal{W}_T^c(y_0)$  is not disjoint from  $[x_0, f(x_0)]_c$  then for some  $x_1, x_2 \in \mathcal{W}^c(x_0)$  satisfying  $[x_0, f(x_0)]_c \subset (x_1, x_2)_c$  the center segment  $[x_1, x_2]_c$  needs to be contained in  $\mathcal{W}_T^c(y_0)$  since the endpoints of  $\mathcal{W}_T^c(y_0)$  are far from  $[x_0, f(x_0)]_c$ . It is enough to consider in this case  $d > \epsilon > 0$  so that  $\epsilon$  is smaller than the Hausdorff distance between  $\mathcal{W}_T^c(x_0) \setminus (x_1, x_2)_c$  and  $[x_0, f(x_0)]_c$ .

This proves Claim 5.2.7 and ends the proof of Proposition 5.2.4. □

### 5.2.2 Proof of Proposition 5.2.1

Suppose  $f_0 \in \text{PH}_{c=1}(M)$ . Consider a metric in  $M$ , a constant  $\delta(f_0) > 0$  and, for some  $0 < \delta \leq \delta(f_0)$ , a  $C^1$  neighborhood  $\mathcal{U}_\delta(f_0) \subset \text{PH}_{c=1}(M)$  of  $f_0$  as in Lemma 5.2.2.

Suppose  $f$  and  $g$  are maps in  $\mathcal{U}_\delta(f_0)$  such that  $f$  is a discretized Anosov flow. Let us see that  $g$  needs to be also a discretized Anosov flow.

Suppose  $f$  is of the form  $f(x) = \varphi_{\tau(x)}^c(x)$  and let  $\mathcal{W}^c$  denote the flow center foliation whose leaves are the flow lines of  $\varphi_t^c$ .

Let  $h : M \rightarrow M$  and  $\rho : M \rightarrow M$  be as in Theorem 4.2.3. By Proposition 5.2.4 the system  $(f, \mathcal{W}^c)$  is  $\delta$ -plaque expansive (in particular it is plaque expansive, see Remark 4.3.4). By Remark 4.3.1 and Lemma 4.3.3 it follows that  $h$  is a homeomorphism and that  $h(\mathcal{W}^c) = \mathcal{W}_g^c$  is a  $g$ -center foliation such that  $(f, \mathcal{W}^c)$  and  $(g, \mathcal{W}_g^c)$  are leaf conjugate. In particular,  $g(W') = W'$  for every leaf  $W' \in \mathcal{W}_g^c$ .

By Proposition 3.3.1 there exists  $L > 0$  such that  $f(x) \in \mathcal{W}_L^c(x)$  for every  $x \in M$ . By Theorem 4.2.3 the maps  $h$  and  $\rho$  satisfy  $\frac{1}{2} < \|Dh|_{E_f^c}\| < 2$  and  $h \circ \rho \circ f = g \circ h$ . Moreover,  $\rho(W) = W$  and  $\rho$  is a  $\delta$ -close to the identity map inside  $W$  for every leaf  $W \in \mathcal{W}^c$ . Then  $g(x) \in \mathcal{W}_{g, 2(L+\delta)}^c(x)$  for every  $x \in M$ .

By denoting  $L' = 2(L + \delta)$  we obtain that  $g$  individually fixes each leaf of the center foliation  $\mathcal{W}_g^c$  satisfying

$$g(x) \in \mathcal{W}_{g, L'}^c(x)$$

for every  $x \in M$ . By Proposition 3.3.1 we conclude that  $g$  is a discretized Anosov flow. Moreover, it is immediate to check from the proof of Proposition 3.3.1 that  $\mathcal{W}_g^c$  needs to be the flow center foliation of  $g$ . This ends the proof of Proposition 5.2.1.

## 5.3 Global stability of uniformly compact center foliations

By means of the same type of arguments already used in Section 5.2 for the case of discretized Anosov flows, Theorem 5.1.3 and Corollary 5.1.4 are proven once we show the following.

**Proposition 5.3.1.** *Suppose  $f_0 \in \text{PH}_{c=1}(M)$ . There exists a  $C^1$ -neighborhood  $\mathcal{U} \subset \text{PH}_{c=1}(M)$  of  $f_0$  such that, if  $f \in \mathcal{U}$  admits a uniformly compact center foliation, then every  $g \in \mathcal{U}$  also admits a uniformly compact center foliation.*

*Moreover, if  $\mathcal{W}_f^c$  and  $\mathcal{W}_g^c$  denote the uniformly compact center foliations of  $f$  and  $g$ , respectively, then  $(f, \mathcal{W}_f^c)$  and  $(g, \mathcal{W}_g^c)$  are leaf-conjugate.*

*Proof.* Suppose  $f_0 \in \text{PH}_{c=1}(M)$ . Consider a metric in  $M$ , a constant  $\delta(f_0) > 0$  and a  $C^1$  neighborhood  $\mathcal{U}_{\delta_0}(f_0) \subset \text{PH}_{c=1}(M)$  of  $f_0$  as in Lemma 5.2.2.

Suppose there exists  $f$  in  $\mathcal{U}_{\delta_0}(f_0)$  admitting a uniformly compact center foliation  $\mathcal{W}_f^c$ . Let us see that every  $g \in \mathcal{U}_{\delta_0}(f_0)$  admits a uniformly compact center foliation  $\mathcal{W}_g^c$  such that  $(f, \mathcal{W}_f^c)$  and  $(g, \mathcal{W}_g^c)$  are leaf-conjugate. By Lemma 4.3.3 it is enough to show that  $(f, \mathcal{W}^c)$  is  $\delta$  plaque-expansive as in Definition 4.3.2.

By Proposition 3.4.4 (see also [BB16, Theorem 1]) the map  $f$  is dynamically coherent admitting  $f$ -invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .

Note that, as it was shown in Claim 5.2.6 during the proof of Proposition 5.2.4, in order to show that  $(f, \mathcal{W}^c)$  is  $\delta$ -plaque expansive it is enough to show that the following property is satisfied (together with its analogous version for backwards orbits and  $cs$ -leaves): if  $(x_n)_{n \geq 0}$  and  $(y_n)_{n \geq 0}$  are two forward  $2\delta$ -pseudo orbits such that  $x_{n+1} \in \mathcal{W}_{2\delta}^c(f(x_n))$ ,  $y_{n+1} \in \mathcal{W}_{2\delta}^c(f(y_n))$  and  $y_n \in \mathcal{W}_{4\delta}^{cu}(x_n)$  for every  $n \geq 0$ , then  $y_0 \in \mathcal{W}_{8\delta}^c(x_0)$ .

Suppose by contradiction that in the context above the point  $y_0$  does not belong to  $\mathcal{W}_{8\delta}^c(x_0)$ . Again, as in the proof of Proposition 5.2.4 the fact that  $y_n$  lies in  $\mathcal{W}_{4\delta}^{cu}(x_n)$  allows us to consider  $x'_n \in \mathcal{W}_{5\delta}^c(x_n)$  such that  $y_n \in \mathcal{W}_{5\delta}^u(x'_n)$  for every  $n \geq 0$ . As  $y_0 \notin \mathcal{W}_{8\delta}^c(x_0)$  it follows that  $y_n \neq x'_n$  for every  $n \geq 0$ .

By defining  $w_n = f^{-n}(x'_n)$  and  $z_n = f^{-n}(y_n)$  we obtain that  $w_n$  and  $z_n$  are points contained in  $\mathcal{W}^c(x_0)$  and  $\mathcal{W}^c(y_0)$ , respectively, satisfying that  $\lim_{n \rightarrow +\infty} d(w_n, z_n) = 0$ . By considering  $w_\infty$  an accumulation point of  $(w_n)_{n \geq 0}$  and  $U(w_\infty)$  a small  $\mathcal{W}^c$ -foliation box neighborhood of  $w_\infty$  we obtain that there exists a subsequence  $(z_{n_k})_{k \geq 0}$  tending to  $w_\infty$  such that each  $z_{n_k}$  corresponds to a different center plaque in  $U(w_\infty)$ . As  $(z_{n_k}) \subset \mathcal{W}^c(y_0)$  and  $\mathcal{W}^c(y_0)$  is compact we get to a contradiction since  $\mathcal{W}^c(y_0)$  cannot contain infinitely many disjoint plaques of  $U(w_\infty)$ .  $\square$

Let us end this section with a small parenthesis. Whether there exists  $f$  in  $\text{PH}(M)$  admitting a compact center foliation with non uniformly bounded volume of leaves is still unknown. Partial non-existence results have been given in [C15], [G12] and [DMM20] (not exclusively for the one-dimensional center scenario).

Assuming one-dimensional center it is worth noting that the second part of the proof of Proposition 5.3.1 only uses that  $\mathcal{W}^c$  is compact and that  $f$  is dynamically coherent. Moreover, by Theorem 4.2.3 (1), whenever  $h$  is a homeomorphism the volume of a compact center leaf  $L$  and its continuation  $h(L)$  differ at most by a constant factor depending only on the  $C^1$  neighborhood  $\mathcal{U}_{\delta(f_0)}(f_0)$ . Thus the following statement follows from the proof of Proposition 5.3.1.

**Proposition 5.3.2.** *Suppose  $f \in \text{PH}_{c=1}(M)$  is a dynamically coherent system admitting  $f$ -invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  is a non-uniformly compact center foliation.*

*There exists a  $C^1$  neighborhood  $\mathcal{U} \subset \text{PH}_{c=1}(M)$  of  $f$  satisfying that every  $g \in \mathcal{U}$  admits a non-uniformly compact center foliation  $\mathcal{W}_g^c$  such that  $(f, \mathcal{W}^c)$  and  $(g, \mathcal{W}_g^c)$  are leaf-conjugate.*

Proposition 5.3.2 could potentially be useful for bringing into play perturbative techniques to the existence problem of non-uniformly compact center foliations with one-dimensional center.

## 5.4 Unique integrability of the center bundle

Given  $f \in \text{PH}_{c=1}(M)$  it follows from Peano's existence theorem that through every point of  $M$  there exists at least one local  $C^1$  curve tangent to  $E^c$ .

We say that  $E^c$  is *uniquely integrable* if through every point of  $M$  there exists a unique  $C^1$  local curve tangent to  $E^c$  modulo reparametrizations. That is, if for every  $\eta : (-\delta, \delta) \rightarrow M$  and  $\gamma : (-\epsilon, \epsilon) \rightarrow M$  a pair of  $C^1$  curves tangent to  $E^c$  with  $\eta(0) = \gamma(0)$  there exists  $\delta' > 0$  such that  $\eta(-\delta', \delta')$  is a subset of  $\gamma(-\epsilon, \epsilon)$ .

It turns out that unique integrability of the center bundle persists along whole connected components of discretized Anosov flows and of systems admitting a uniformly compact center foliation (at least for one-dimensional center):

**Proposition 5.4.1.** *Suppose  $f \in \text{PH}_{c=1}(M)$  is a discretized Anosov flow or admits a uniformly compact center foliation. If  $E^c$  is uniquely integrable then every diffeomorphism in the same  $C^1$  connected component of  $f$  in  $\text{PH}_{c=1}(M)$  has a uniquely integrable center bundle.*

*Proof.* Suppose  $\{f_t\}_{t \in [0,1]} \subset \text{PH}_{c=1}(M)$  is a  $C^1$  path of partially hyperbolic diffeomorphisms joining  $f_0 = f$  with  $f_1 = g$ .

Suppose first that  $f_0$  is a discretized Anosov flow such that  $E_{f_0}^c$  is uniquely integrable. Let us see that  $E_{f_1}^c$  is also uniquely integrable.

By Theorem 5.1.1 every  $f_t$  is a discretized Anosov flow. By Proposition 5.2.1, if  $\mathcal{W}_{f_t}^c$  denotes the flow center foliation  $f_t$  then  $(f_t, \mathcal{W}_{f_t}^c)$  is plaque expansive. By Proposition 4.4.3 there exists a  $C^1$  neighborhood  $\mathcal{U}(f_t) \subset \text{PH}_{c=1}(M)$  of  $f_t$  such that, if  $\mathcal{U}(f_t)$  contains a system with uniquely integrable center bundle, then every system in  $\mathcal{U}(f_t)$  has a uniquely integrable center bundle. As  $E_{f_0}^c$  is uniquely integrable, the above implies that  $E_{f_1}^c$  is also uniquely integrable.

In case  $f_0$  is a system admitting a uniformly compact center foliation the argument is analogous using Theorem 5.1.3 and Proposition 5.3.1 in the place of Theorem 5.1.1 and Proposition 5.2.1.  $\square$

**Corollary 5.4.2.** *Every discretized Anosov flow in the same  $C^1$  connected component of  $\text{PH}_{c=1}(M)$  as the time 1 map of an Anosov flow has a uniquely integrable center bundle.*

*Proof.* Let  $\varphi_t : M \rightarrow M$  be an Anosov flow. We can approximate  $\frac{\partial \varphi_t}{\partial t}|_{t=0}$  by a  $C^\infty$  vector field  $X$  so that, if  $X_t$  denotes the flow generated by  $X$ , then  $f := \varphi_1$  and  $g := X_1$  are  $C^1$ -close (in particular, such that they are in the same  $C^1$  partially hyperbolic connected component). Since  $g$  is a discretized Anosov flow with uniquely integrable center bundle (because  $X$  is  $C^\infty$ ) it follows that  $f$  and every systems in the  $C^1$  connected component of  $\text{PH}_{c=1}(M)$  containing  $f$  has a uniquely integrable center bundle.  $\square$

In [HHU16] two types of partially hyperbolic diffeomorphisms in  $\mathbb{T}^3$  are built. Ones which are non-dynamically coherent and ones which are dynamically coherent but such that  $E^c$  is not uniquely integrable. The following sketches how a discretized Anosov flow with non-uniquely integrable center bundle can be obtained as a simple modification of the second type of examples.

**Example 5.4.3** (Example of a discretized Anosov flow with non-uniquely integrable center bundle). We will start by giving a brief description a dynamically coherent example from [HHU16]. For more details see [HHU16] itself.

The aforementioned partially hyperbolic diffeomorphism  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  can be considered homotopic to  $A \times \text{Id}$ , where  $\mathbb{T}^3$  is identified with  $\mathbb{T}^2 \times S^1$  and  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  is a linear hyperbolic automorphism with eigenvalues  $0 < \lambda < 1$  and  $1/\lambda$ .

Denote by  $E_A^s$  the contracting eigenspace of  $A$  and by  $e_s$  a unit vector in  $E_A^s$ . And identify  $S^1$  with  $\mathbb{R}/2\mathbb{Z}$ . Then the map  $f$  can be taken to be of the form

$$f(x, \theta) = (Ax + v(\theta)e_s, \Psi(\theta))$$

for suitable  $v : S^1 \rightarrow \mathbb{R}$  and  $\Psi : S^1 \rightarrow S^1$  such that  $v$  is positive in  $(-1, 0) \subset S^1$  and negative in  $(0, 1) \subset S^1$ , and  $\Psi$  is a Morse-Smale map with  $-1$  and  $0$  as only fixed points that in addition satisfy  $\Psi'(0) < \lambda < 1 < \Psi'(-1) < 1/\lambda$ .

The sets  $\mathbb{T}^2 \times \{-1\}$  and  $\mathbb{T}^2 \times \{0\}$  are two invariant tori that are fixed by  $f$ , with  $f$  acting as  $A$  on each of them. The torus  $\mathbb{T}^2 \times \{0\}$  is a *cu*-torus (it is saturated by  $\mathcal{W}^c$  and  $\mathcal{W}^u$ -leaves) and the torus  $\mathbb{T}^2 \times \{-1\}$  is a repelling *su*-torus (it is saturated by  $\mathcal{W}^s$  and  $\mathcal{W}^u$ -leaves).

The construction given by [HHU16] shows the following. The map  $f$  admits an  $f$ -invariant foliation by circles  $\mathcal{W}^c$ . Each of these circles is homotopic to a horizontal circle and intersects in a unique point each torus  $\mathbb{T}^2 \times \theta$  for every  $\theta \in S^1$ . Thus  $f$  is a partially hyperbolic skew-product where  $\mathcal{W}^c$  is a foliation by circles that gives to  $M$  a structure of fiber bundle.

Moreover, it can be seen that the bundle  $E^c$  is uniquely integrable outside of the  $cu$ -torus  $\mathbb{T}^2 \times \{0\}$ . However, remarkably, through each point of  $\mathbb{T}^2 \times \{0\}$  there exists more than one local  $C^1$  curve tangent to  $E^c$ . Namely, through each point  $y$  of  $\mathbb{T}^2 \times \{0\}$  one can consider the center arc corresponding to the leaf  $\mathcal{W}^c(y)$ , but also all the center arcs that are a concatenation of a piece of arc of  $\mathcal{W}^c$ , a center arc through  $y$  contained in the  $cu$  torus  $\mathbb{T}^2 \times \{0\}$  and a third piece of  $\mathcal{W}^c$  arc. See Figure 5.2.

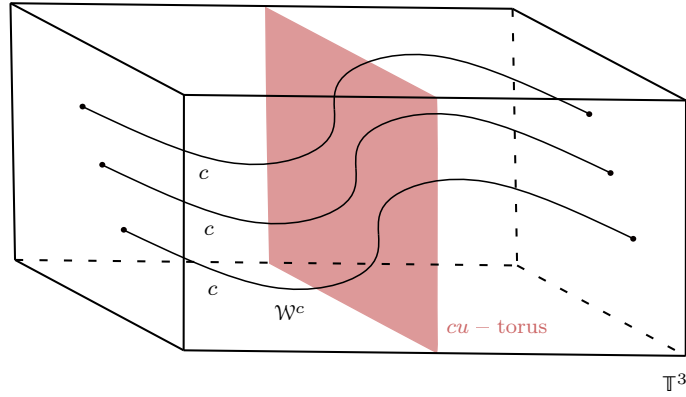


Figure 5.2:

The simple modification of the example proceeds as follows. Let  $F : \mathbb{T}^2 \times \mathbb{R} \rightarrow \mathbb{T}^2 \times \mathbb{R}$  be the lift of  $f$  such that  $F(x, -1) = (Ax, -1)$  for every  $x \in \mathbb{T}^2$ . It is immediate to check that  $F$  commutes with the elements of the group  $\Gamma = \{(x, \tilde{\theta}) \mapsto (A^n x, \tilde{\theta} + 2n)\}_{n \in \mathbb{Z}}$ . Indeed, if  $\gamma : \mathbb{T}^2 \times \mathbb{R} \rightarrow \mathbb{T}^2 \times \mathbb{R}$  denotes the generator of  $\Gamma$  given by  $\gamma(x, \tilde{\theta}) = (Ax, \tilde{\theta} + 2)$ , then it is enough to show that  $F \circ \gamma = \gamma \circ F$  in restriction to  $\mathbb{T}^2 \times \{-1\}$ . On the one hand one has  $F \circ \gamma(x, -1) = F(Ax, \tilde{\theta} + 2) = (A^2 x, \tilde{\theta} + 2)$  since  $F$  commutes with  $(x, \tilde{\theta}) \mapsto (x, \tilde{\theta} + 2)$ .

As a consequence,  $F$  descends to a partially hyperbolic diffeomorphism  $g : N \rightarrow N$  in  $N = (\mathbb{T}^2 \times \mathbb{R})/\Gamma$ .

Let  $\tilde{\mathcal{W}}^c$  denote the lift of  $\mathcal{W}^c$  to  $\mathbb{T}^2 \times \mathbb{R}$  and  $\mathcal{W}_g^c$  the descended one to  $N$ . It follows that  $\mathcal{W}_g^c$  is a  $g$ -invariant center foliation for  $g$ . Since  $F(\tilde{\mathcal{W}}^c(x, -1)) = \tilde{\mathcal{W}}^c(Ax, -1)$  for every  $x \in \mathbb{T}^2$  the leaves of  $\mathcal{W}_g^c$  are individually fixed by  $g$  (that is,  $g(W) = W$  for every  $W \in \mathcal{W}_g^c$ ).

Moreover, for every  $z \in N$  the point  $g(z)$  lies in  $\mathcal{W}_{g,L}^c(z)$  for  $L > 0$  any



constant larger than the maximum length of a leaf in  $W^c$ . By Proposition 3.3.1 it follows that  $g$  is a discretized Anosov flow.

Finally, the property of non-unique integrability of the center bundle is preserved along the  $cu$ -torus that is the projection of  $\mathbb{T}^2 \times \{0\}$  to  $N$  since this is a local property that is preserved by lifts and quotients. Hence  $E_g^c$  is not uniquely integrable.

From Corollary 5.4.2 it is immediate to conclude the following.

**Corollary 5.4.4.** *The  $C^1$  connected component of  $\text{PH}_{c=1}(M)$  containing the discretized Anosov flow given by Example 5.4.3 does not contain the time 1 map of an Anosov flow.*

The conclusion is that unique integrability versus non-unique integrability of the center bundle provides a way for distinguishing between different connected components of discretized Anosov flows and of partially hyperbolic systems in general. The following questions arise naturally.

**Question 5.4.5.** *Is it possible to connect (via a  $C^1$ -path of discretized Anosov flows) every discretized Anosov flow with uniquely integrable center bundle to the time 1 map of an Anosov flow? Are there examples of discretized Anosov flows with a non uniquely integrable center bundle which are transitive or such that the center flow is not orbit equivalent to a suspension flow?*

More generally, one may ask whether there exist examples of  $C^1$ -connected components of partially hyperbolic diffeomorphisms containing both systems with uniquely integrable and non-uniquely integrable center bundle.

One can put the above into an even more general framework. For every  $f \in \text{PH}_{c=1}(M)$  one can consider all the  $C^1$  curves tangent to the center bundle  $E_f^c$ . This gives rise to a kind of *branched center foliation* in  $M$ . One may ask whether the structure of this normally hyperbolic branched foliation is preserved:

**Question 5.4.6.** *Suppose  $f, g \in \text{PH}_{c=1}(M)$  lie in the same  $C^1$  connected component of  $\text{PH}_{c=1}(M)$ . Does there exist a homeomorphism  $h : M \rightarrow M$  such that for every  $C^1$  curve  $\eta : (0, 1) \rightarrow M$  tangent to  $E_f^c$  the curve  $h \circ \eta : (0, 1) \rightarrow M$  is (or can be reparametrized to be) a  $C^1$  curve tangent to  $E_g^c$ ?*

It is worth noting [BFP20, Question 2.] for a similar question in the context of *collapsed Anosov flow* and [HPS77, Section 7.] for other related questions.

## Chapter 6

# Center fixing characterization

### 6.1 Introduction

The following question motivates this chapter. It is worth noting [G12, Question 1.3.] where a similar question has been posed.

**Question 6.1.1.** *Suppose  $f \in \text{PH}_{c=1}(M)$  admits a center foliation  $\mathcal{W}^c$  such that  $f(W) = W$  for every leaf  $W \in \mathcal{W}^c$ . Is  $f$  a discretized Anosov flow?*

We will give a positive answer to Question 6.1.1 in two cases: 1) whenever  $\mathcal{W}^c$  has a dense leaf and  $f$  is dynamically coherent and 2) whenever  $f$  is transitive and  $\dim(M) = 3$ .

Recall that a partially hyperbolic diffeomorphism  $f$  is *dynamically coherent* if there exists  $f$ -invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  tangent to  $E^s \oplus E^c$  and  $E^c \oplus E^u$ , respectively. We say that  $f$  is a *dynamically coherent center fixing map* if the center foliation  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  (given by the connected components of the intersection between  $\mathcal{W}^{cs}$ -leaves and  $\mathcal{W}^{cu}$ -leaves) satisfies that  $f(W) = W$  for every leaf  $W \in \mathcal{W}^c$ . Note that by Proposition 3.4.4 every discretized Anosov flow is a dynamically coherent center fixing map.

We say that  $\mathcal{W}^c$  is *transitive* if it has a dense leaf. The following is the main goal of this chapter.

**Theorem 6.1.2.** *Suppose  $f \in \text{PH}_{c=1}(M)$  is a dynamically coherent center fixing map such that  $\mathcal{W}^c$  is transitive. Then  $f$  is a discretized Anosov flow.*

Because of the center fixing property one can replace the hypothesis ‘ $\mathcal{W}^c$  transitive’ by ‘ $f$  transitive’ in the above theorem. See Remark 6.7.5.

Furthermore, in dimension 3 we can show that the dynamical coherent hypothesis is not needed provided  $f$  is transitive. For this, we will give a dynamical coherence criterion for maps that are center fixing or admit a compact center foliation in dimension 3 (Proposition 6.8.1 below).

The aforementioned criterion combined with [DMM20] (also [G12]), [B13] and Theorem 6.1.2 (or [BW05]) allows us to give a statement characterizing both discretized Anosov flows and partially hyperbolic skew-products in dimension 3:

**Theorem 6.1.3.** *Suppose  $f \in \text{PH}_{c=1}(M^3)$  is transitive and admits an  $f$ -invariant center foliation  $\mathcal{W}^c$ .*

1. *If  $f(W) = W$  for every  $W \in \mathcal{W}^c$  then  $f$  is a discretized Anosov flow.*
2. *If  $W$  is compact for every  $W \in \mathcal{W}^c$  then, modulo double cover,  $f$  is a partially hyperbolic skew-product.*

The proof of Theorem 6.1.2 will take place up until Section 6.7. The proof of Theorem 6.1.3 is left to Section 6.8.

## 6.2 Outline of the proof of Theorem 6.1.2

Let us see in this section an outline of the proof of Theorem 6.1.2. It is worth pointing out that an starting point for this proof is an unpublished strategy proposed by A. Gogolev and R. Potrie.

One begins by defining  $\rho : M \rightarrow \mathbb{R}_{\geq 0}$  the *center displacement function* that measures the distance from  $x$  to  $f(x)$  inside  $\mathcal{W}^c(x)$  for every  $x$  in  $M$ . By Proposition 3.3.1 it is enough to show that  $\rho$  is bounded in  $M$ . This will be the goal for the remaining of the proof.

In some sense, the demonstration progresses then by showing an increasingly number of properties that every discretized Anosov flow must satisfy. Until one reaches no other possibility than  $f$  being a discretized Anosov flow.

Since it is enough to show the theorem in the case that  $\mathcal{W}^c$  is orientable (Lemma 6.3.3), one can define the *center flow*  $\varphi_t^c : M \rightarrow M$  as one of the two flows by arc-length whose orbits are the leaves of  $\mathcal{W}^c$ .

By showing that  $\rho$  is a lower semicontinuous function (Proposition 6.3.7) one obtains that there exists a residual subset  $Y \subset M$  that is saturated by leaves of  $\mathcal{W}^c$  and such that every point on it is a continuity point for  $\rho$  (Proposition 6.3.11). This marks the end of Section 6.3.

In Section 6.4 one shows first that there is no self-recurrence of the center foliation inside leaves of  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  (Proposition 6.4.4). Then this is used to obtain the key property that the center flow  $\varphi_t^c$  is an *expansive* flow (Proposition 6.4.7). And it allows us to show that the function  $f$  has no fixed points in non-compact leaves of  $\mathcal{W}^c$  (Proposition 6.4.8).

Modulo inverting the time of  $\varphi_t^c$ , a connection argument in  $Y$  gives that  $f(x)$  and  $\varphi_1^c(x)$  must lie in the same connected component of  $\mathcal{W}^c(x) \setminus x$  for

every  $x \in Y$ . That is, forward iterates of  $f$  and  $\varphi_t^c$  move points in the same direction inside  $\mathcal{W}^c$  leaves (Lemma 6.4.10).

Combined with classical arguments from expansive systems, the above allows us to construct local stable and unstable sets for  $\varphi_t^c$  of uniform size inside stable and unstable leaves of  $f$ , respectively, at any point of  $M$  (Proposition 6.5.2). This is the main goal of Section 6.5.

In Section 6.6 the local stable and unstable sets of  $\varphi_t^c$  are used to show that  $\rho$  must be continuous at every point  $x$  such that  $\mathcal{W}^c(x)$  is not compact (Proposition 6.6.1 via the key Lemma 6.6.7).

The remaining of the proof, developed in Section 6.7, is devoted to show that the continuity of  $\rho$  in the union of non-compact leaves of  $\mathcal{W}^c$  (whose complement is a countable union of compact leaves of  $\mathcal{W}^c$  by Lemma 6.3.1) implies that  $\rho$  is bounded in  $M$ . As was already mentioned, one then concludes from this that  $f$  must be a discretized Anosov flow as a consequence of Proposition 3.3.1.

### 6.3 Center flow and the center displacement function

This section initiates the proof of Theorem 6.1.2 which will take place up to Section 6.7.

Let us suppose from now on that  $f \in \text{PH}_{c=1}(M)$  admits  $f$ -invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  satisfies that  $f(W) = W$  for every  $W \in \mathcal{W}^c$ . Let us suppose also that  $\mathcal{W}^c$  has a dense leaf. The goal is to show that  $f$  must be a discretized Anosov flow.

There are two types of leaves of  $\mathcal{W}^c$ . The compact ones, that we will call *circles*, and the non compact ones, that we will call *lines*. As shown in the next lemma, most of the leaves of  $\mathcal{W}^c$  are lines.

**Lemma 6.3.1.** *For every  $L > 0$  the number of circle leaves of  $\mathcal{W}^c$  whose length is less than  $L$  is finite. In particular, the set  $\{W \in \mathcal{W}^c : W \text{ is compact}\}$  has at most countably many elements.*

*Proof.* The proof of this lemma is immediate by transverse hyperbolicity in a neighborhood of each compact center leaf of  $\mathcal{W}^c$  (every such leaf is an  $f$ -invariant compact submanifold that is normally hyperbolic).

Suppose by contradiction that for some  $L_0 > 0$  there exists an infinite number  $\{\mathcal{W}^c(x_n)\}_{n \geq 0}$  of distinct circle leaves of  $\mathcal{W}^c$  whose length is less than  $L_0$ .

Modulo subsequence one can suppose that the sequence  $(x_n)_n$  has limit  $x \in M$  and that the sequence of lengths  $(\text{length } \mathcal{W}^c(x_n))_n$  converges to some

constant  $L_1 > 0$ . It follows that  $\mathcal{W}^c(x)$  is compact and that  $\text{length}(\mathcal{W}^c(x)) \leq L_1$ .

Let  $\epsilon > 0$  be a small constant so that  $\mathcal{W}_\epsilon^s(y) \cap \mathcal{W}_\epsilon^s(z) = \emptyset$  for every  $y, z \in \mathcal{W}^c(x)$  such that  $y \neq z$ . And small enough so that  $\mathcal{W}_\epsilon^u(y) \cap \mathcal{W}_\epsilon^u(z) = \emptyset$  for every  $y, z \in \mathcal{W}_\epsilon^s(\mathcal{W}^c(x))$  such that  $y \neq z$ . Let  $\kappa > 0$  be such that  $\max\{\|Df_x\|, \|Df_x^{-1}\|\} < \kappa$  for every  $x \in M$ . Let  $U := \mathcal{W}_{\kappa^{-1}\epsilon}^u(\mathcal{W}_{\kappa^{-1}\epsilon}^s(\mathcal{W}^c(x)))$ .

Note that  $f(U) \subset U$  and that  $f^{-1}(U) \subset U$ . Since  $f$  contracts indefinitely stable and unstable discs for forwards and backwards iterates, respectively, it follows that  $\mathcal{W}^c(x) = \bigcap_{k \in \mathbb{Z}} f^k(U)$ .

For every  $n$  large enough the leaf  $\mathcal{W}^c(x_n)$  is contained in  $U$ . It follows from  $f(\mathcal{W}^c(x_n)) = \mathcal{W}^c(x_n)$  that  $\mathcal{W}^c(x_n) \subset \bigcap_{k \in \mathbb{Z}} f^k(U)$ . That is,  $\mathcal{W}^c(x_n) = \mathcal{W}^c(x)$  for every  $n$  large enough. This gives us a contradiction.  $\square$

The next two lemmas will allow us to reduce the problem to the case where  $\mathcal{W}^c$  is orientable and  $f$  preserves the orientation of its leaves. Recall that  $f$  is being supposed to be a dynamically coherent center fixing map. Then:

**Lemma 6.3.2.** *If  $f^n$  is a discretized Anosov flow for some  $n > 1$  then  $f$  is a discretized Anosov flow.*

*Proof.* Suppose that  $f^n$  is a discretized Anosov flow for some  $n > 1$ .

Note first that, by Proposition 3.2.2 item (i), if  $W$  is a leaf of  $\mathcal{W}^c$  that is not compact then  $f^n$  has no fixed points in  $W$ . As a consequence,  $f$  has no fixed points in  $W$  either. In particular,  $f$  preserves the orientation of  $W$ .

By Proposition 3.3.1 there exists  $L > 0$  such that  $f^n(x) \in \mathcal{W}_L^c(x)$  for every  $x \in M$ . Note that, if  $\mathcal{W}^c(x)$  is not compact, then the fact that  $f$  preserves the orientation of  $\mathcal{W}^c(x)$  implies that  $f(x)$  lies in the center interval  $[x, f^n(x)]$  joining  $x$  with  $f^n(x)$ . In particular,  $f(x) \in \mathcal{W}_L^c(x)$  for every  $x \in M$  such that  $\mathcal{W}^c(x)$  is compact.

By Lemma 6.3.1 there are at most countably many compact leaves of  $\mathcal{W}^c$ . Given  $x$  in a compact leaf  $W$  of  $\mathcal{W}^c$  one can consider a sequence  $x_n$  converging to  $x$  so that  $\mathcal{W}^c(x_n)$  is not compact for every  $n$ . As  $f(x_n)$  belongs to  $\mathcal{W}_L^c(x_n)$  for every  $n$  and the sequence  $f(x_n)$  tends to  $f(x)$  one obtains that  $f(x)$  must lie in  $\mathcal{W}_L^c(x)$ .

We have shown that  $f(x) \in \mathcal{W}_L^c(x)$  for every  $x \in M$ . By Proposition 3.3.1 we conclude that  $f$  is a discretized Anosov flow.  $\square$

The above lemma has the following immediate consequence:

**Lemma 6.3.3.** *To show that  $f$  is a discretized Anosov flow we can assume that the foliation  $\mathcal{W}^c$  is orientable and that  $f$  preserves the orientation of  $\mathcal{W}^c$ -leaves.*

*Proof.* Consider a double cover  $\tilde{M}$  of  $M$  so that  $\mathcal{W}^c$  lifts to an orientable foliation  $\tilde{\mathcal{W}}^c$ . Let  $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$  be a lift of  $f$ . A priori  $\tilde{f}$  may not individually fix every leaf of  $\tilde{\mathcal{W}}^c$ . However,  $\tilde{f}^2$  does. Moreover,  $\tilde{f}^4$  preserves the orientations of these leaves.

Let  $g := \tilde{f}^4$  and suppose that  $g$  is a discretized Anosov flow. It follows from Proposition 3.3.1 that there exists  $L > 0$  so that  $g(x)$  lies in  $\tilde{\mathcal{W}}_L^c(\tilde{x})$  for every  $\tilde{x} \in \tilde{M}$ . Then  $f^4(x)$  lies in  $\mathcal{W}_L^c(x)$  for every  $x \in M$ . It follows that  $f^4$  is a discretized Anosov flow. By Lemma 6.3.2,  $f$  itself is a discretized Anosov flow. This concludes the proof.  $\square$

By Lemma 6.3.3 we can assume, and we will do so from now on, that the foliation  $\mathcal{W}^c$  is orientable and that  $f$  preserves the orientation of  $\mathcal{W}^c$  leaves.

In particular, this allows us to consider the following:

**Definition 6.3.4** (Center flow). Let  $\varphi_t^c : M \rightarrow M$  be a flow by arc-length whose orbits are the leaves of  $\mathcal{W}^c$ .

We will also work with the following function:

**Definition 6.3.5** (Center displacement function). Let us define  $\rho : M \rightarrow \mathbb{R}$  to be

$$\rho(x) := d_c(x, f(x))$$

for every  $x \in M$ .

**Remark 6.3.6.** It is immediate from Proposition 3.3.1 that if  $\rho$  is bounded then  $f$  is a discretized Anosov flow.

Recall that a real valued function  $F$  in  $M$  is called *lower semicontinuous* if for every sequence  $x_n$  converging to  $x$  one has that  $\liminf_n F(x_n) \geq F(x)$ .

**Proposition 6.3.7.** *The function  $\rho$  is lower semicontinuous.*

*Proof.* Suppose  $(x_n)$  is a sequence converging to  $x$ . In case  $\liminf_n \rho(x_n) = +\infty$  then there is nothing to prove. Otherwise, one can consider a subsequence so that the limit inferior is in fact a limit. By a slight abuse of notation let us denote this subsequence also  $(x_n)$ .

For every  $n$  let  $\gamma_n : [0, 1] \rightarrow \mathcal{W}^c(x_n)$  denote a  $C^1$  curve parametrized by arc-length so that  $\text{length}(\gamma_n) = \rho(x_n)$ . The sequence  $\dot{\gamma}_n(0)$  accumulates in a unitary vector  $v_c$  in  $E^c(x)$ . Up to taking a subsequence (in case  $-v_c$  is also an accumulation point), let us suppose that  $\dot{\gamma}_n(0)$  converges to  $v_c$ .

Let  $\gamma : [0, 1] \rightarrow \mathcal{W}^c(x)$  denote the  $C^1$  curve parametrized by arc-length so that  $\dot{\gamma}(0) = v_c$ . It follows that  $\gamma_n$  converges with  $n$  to  $\gamma$  in the  $C^1$  topology. In particular,  $\text{length}(\gamma_n)$  converges to  $\text{length}(\gamma)$ .

Since  $\gamma_n(1)$  converges with  $n$  to  $\gamma(1)$  and  $\gamma_n(1) = f(x_n)$  for every  $n$  it follows that  $\gamma(1) = f(x)$ . As  $\gamma$  is a  $C^1$  curve in  $\mathcal{W}^c(x)$  joining  $x$  to  $f(x)$  it follows that  $\rho(x) \leq \text{length}(\gamma)$ . Since  $\text{length}(\gamma_n) = \rho(x_n)$  converges with  $n$  to  $\text{length}(\gamma)$  we obtain that  $\rho(x) \leq \lim_n \rho(x_n)$  as desired.  $\square$

**Remark 6.3.8.** It is worth noting that we can not expect to show that, in general,  $\rho$  has to be continuous at every point of  $M$ . For example, if  $f$  is the time 1 map of an Anosov flow  $\varphi_t : M \rightarrow M$  that is parametrized by arc-length, then  $\rho$  will only be continuous in the complement of the set of periodic orbits of period smaller than 2.

**Definition 6.3.9.** Let us denote by  $X \subset M$  the set of continuity points of  $\rho$  in  $M$ . Namely

$$X = \{x \in M \mid \rho \text{ is continuous at } x\}.$$

Let us denote by  $Y \subset X$  the set of continuity points  $x$  such that  $\mathcal{W}^c(x)$  is a line. Namely

$$Y = \{x \in X \mid \mathcal{W}^c(x) \text{ is a line}\}.$$

**Remark 6.3.10.** By the semicontinuity of  $\rho$  it follows from classical arguments (see for example Lemma 7.3.3) that  $X$  is a *residual subset* of  $M$  (meaning that it is a countable intersection of open and dense subsets of  $M$ ).

Recall that a topological space is *locally path connected* if every point has a local basis made of path connected open sets.

**Proposition 6.3.11.** *The set  $Y$  is residual in  $M$ , saturated by leaves of  $\mathcal{W}^c$  and locally path connected.*

*Proof.* By Lemma 6.3.1 the set  $\{W \in \mathcal{W}^c : W \text{ is compact}\}$  has countably many elements. Moreover, each of these elements is a nowhere dense subset of  $M$ . Since  $X$  is residual in  $M$  it follows that  $Y = X \setminus (\bigcup_{W \in \mathcal{W}^c, W \text{ is compact}} W)$  is also residual in  $M$ .

Let us see now that  $Y$  is saturated by leaves of  $\mathcal{W}^c$ . Fix  $\delta > 0$ . The goal will be to see that  $\mathcal{W}_\delta^c(x)$  is a subset of  $Y$  for every  $x \in Y$ .

Let  $x \in Y$ . The leaf  $\mathcal{W}^c(x)$  is a line. Let  $[x, f(x)]_c$  denote the segment joining  $x$  with  $f(x)$  inside  $\mathcal{W}^c(x)$ . Let  $U$  be an open  $\mathcal{W}^c$ -tubular neighborhood of  $[x, f(x)]_c$ . More precisely,  $U$  is the image of a certain homeomorphism over its image  $\Psi : \mathbb{R}^{\dim(M)-1} \times \mathbb{R} \rightarrow M$  such that  $\Psi(p \times \mathbb{R})$  is contained in a leaf of  $\mathcal{W}^c$  for every  $p \in \mathbb{R}^{\dim(M)-1}$  and  $[x, f(x)]_c \subset \Psi(0 \times \mathbb{R})$ .

The open tubular neighborhood  $U$  can be considered ‘long and thin’ enough, and a ball  $B_{\epsilon_x}(x) \subset U$  for some  $\epsilon_x > 0$  considered small enough, so that for every  $y \in B_{\epsilon_x}(x)$  one has that

- (i)  $\mathcal{W}_{\rho(x)+2\delta}^c(y)$  is contained in a center plaque of  $U$ ,
- (ii)  $f(\mathcal{W}_{2\delta}^c(y))$  is contained in a center plaque of  $U$  and
- (iii)  $|\rho(y) - \rho(x)| < \delta$ .

We claim first that every point in  $B_{\epsilon_x}(x)$  is a continuity point of  $\rho$ . Indeed, for every  $y \in B_{\epsilon_x}(x)$  it follows from (iii) that the set  $\mathcal{W}_{\rho(y)+\delta}^c(y)$  is a subset of  $\mathcal{W}_{\rho(x)+2\delta}^c(y)$ . By (i) the set  $\mathcal{W}_{\rho(x)+2\delta}^c(y)$  is contained in a center plaque of  $U$ , and  $f(y)$  lies in  $\mathcal{W}_{\rho(y)+\delta}^c(y)$  by definition of  $\rho$ , so  $y$  and  $f(y)$  must lie in the same center plaque of  $U$ .

For every  $y \in B_{\epsilon_x}(x)$  let  $[y, f(y)]_c$  be the center segment in  $U$  joining  $y$  with  $f(y)$ . Since  $\mathcal{W}_{\rho(y)+\delta}^c(y) \subset U$  it follows that  $\rho(y) = \text{length}[y, f(y)]_c$  for every  $y \in B_{\epsilon_x}(x)$ .

Given a sequence  $(y_n)$  in  $B_{\epsilon_x}(x)$  converging to  $y \in B_{\epsilon_x}(x)$ , the center plaque of  $U$  containing  $y_n$  approaches the one containing  $y$ . Since  $f(y_n)$  tends to  $f(y)$  it follows that  $[y_n, f(y_n)]_c$  converges to  $[y, f(y)]_c$  in the Hausdorff topology. Then  $\rho(y_n)$  converges to  $\rho(y)$ . This proves the claim.

Suppose now that  $z$  is a point in  $\mathcal{W}_\delta^c(x)$  and  $(z_n)$  is a sequence converging to  $z$ . Let  $(x_n)$  be a sequence converging to  $x$  so that  $z_n$  belongs to  $\mathcal{W}_\delta^c(x_n)$  for every  $n$ . For every  $n$  large enough the point  $x_n$  lies in  $B_{\epsilon_x}(x)$ , so  $x_n$  and  $f(x_n)$  lie in the same center plaque of  $U$  as it was seen above.

Modulo dropping the first iterates of the sequence suppose without loss of generality that  $x_n$  lies in  $B_{\epsilon_x}(x)$  for every  $n$ . Then by (i) and (ii) one has that  $\mathcal{W}_\delta^c(x_n) \subset U$  and  $f(\mathcal{W}_\delta^c(x_n)) \subset U$  for every  $n$ .

Since  $[x_n, f(x_n)]_c$  is contained in a center plaque of  $U$  it follows that  $\mathcal{W}_\delta^c(x_n) \cup [x_n, f(x_n)]_c \cup f(\mathcal{W}_\delta^c(x_n))$  is a center segment that is also contained in the same center plaque of  $U$ . Since  $z_n$  lies in  $\mathcal{W}_\delta^c(x_n)$  and  $f(z_n)$  lies in  $f(\mathcal{W}_\delta^c(x_n))$  one obtains that  $z_n$  and  $f(z_n)$  also lie in the same center plaque of  $U$  for every  $n$ .

Let  $[z_n, f(z_n)]_c$  denote the center segment in  $U$  joining  $z_n$  with  $f(z_n)$  for every  $n$ . Since  $x_n$  lies in  $B_{\epsilon_x}(x)$  we have shown above that  $\mathcal{W}_{\rho(x_n)+\delta}^c(x_n)$  is contained in a center plaque of  $U$ . So  $z_n \in \mathcal{W}_\delta^c(x_n)$  implies that  $\mathcal{W}_{\rho(z_n)}^c(z_n)$  is contained in the same center plaque of  $U$ . This shows that  $\rho(z_n)$  is equal to the length of  $[z_n, f(z_n)]_c$  for every  $n$ .

As above, the sequence  $[z_n, f(z_n)]_c$  needs to converge to  $[z, f(z)]_c$  in the Hausdorff topology. One obtains that  $\rho(z_n)$  converges to  $\rho(z)$ . That is,  $z$  is a continuity point of  $\rho$ .

We have shown that for every  $x \in Y$  the set  $\mathcal{W}_\delta^c(x)$  is a subset of  $Y$  for some uniform  $\delta > 0$ . We conclude that  $Y$  is saturated by leaves of  $\mathcal{W}^c$ .

Locally, the set  $Y$  is an open subset of  $M$  minus, at most, countably many center plaques corresponding to circle leaves of  $\mathcal{W}^c$ . It is immediate from this



that  $Y$  is locally path connected. □

## 6.4 Expansivity of the center flow and $u$ -recurrent points

### 6.4.1 There are no $s$ or $u$ -recurrent points

The goal of this subsection is to show that a certain ‘bad’ type of points, denoted as  $s$  and  $u$ -recurrent points, can not occur.

Let us start by pointing out the following lemmas.

**Lemma 6.4.1.** *There are at most finitely many line leaves of  $\mathcal{W}^c$  having fixed points. Moreover, in every such a leaf the fixed points lie in a bounded interval of the leaf.*

*Proof.* Given  $x \in M$  let  $U_x$  be a small  $\mathcal{W}^c$  foliation box neighborhood of  $x$ . By transverse hyperbolicity (see for example Lemma 6.3.1 for more details) the set  $U_x$  contains at most one center plaque  $I_x$  such that  $f(I_x) \cap I_x \neq \emptyset$ . Consider a finite subcover  $\{U_{x_1}, \dots, U_{x_N}\}$  of  $M$ . Every fixed point of  $f$  must lie in  $I_{x_1} \cup \dots \cup I_{x_N}$ . This proves the lemma. □

**Lemma 6.4.2.** *Two distinct circle leaves of  $\mathcal{W}^c$  can not intersect the same leaf of  $\mathcal{W}^s$ . And every circle leaf of  $\mathcal{W}^c$  intersects every leaf of  $\mathcal{W}^s$  in at most one point.*

*Proof.* Suppose by contradiction that two distinct circle leaves  $W, W' \in \mathcal{W}^c$  intersect the same leaf of  $\mathcal{W}^s$  in points  $x \in W$  and  $y \in W'$ . By iterating forwards one obtains that  $d(f^n(x), f^n(y))$  tends to zero. By the center fixing property  $f^n(x)$  and  $f^n(y)$  are points in  $W$  and  $W'$ , respectively, for every  $n$ . This contradicts the fact that the disjoint compact sets  $W$  and  $W'$  are at positive distance from each other.

Given a circle leaf  $W \in \mathcal{W}^c$  there exists  $\delta > 0$  such that  $\bigcup_{x \in W} \mathcal{W}_\delta^s(x) \setminus \{x\}$  is disjoint from  $W$ , otherwise  $W$  would not be compact. If one supposes, by contradiction, that  $W$  intersects the same leaf of  $\mathcal{W}^s$  in two different points  $x$  and  $y$  then  $d(f^n(x), f^n(y))$  will be smaller than  $\delta$  for some large enough  $n > 0$ . Since  $f^n(x)$  and  $f^n(y)$  are points in  $W$  this contradicts that  $\mathcal{W}_\delta^s(f^n(x)) \setminus \{f^n(x)\}$  is disjoint from  $W$ . □

**Notation.** From now on, given  $x \in M$  and  $y \in \mathcal{W}^c(x)$  such that  $\mathcal{W}^c(x)$  is a line let  $(x, y)_c$  denote the open center segment from  $x$  to  $y$  inside  $\mathcal{W}^c(x)$ . In case  $\mathcal{W}^c(x)$  is a circle let  $(x, y)_c$  denote the center segment joining  $x$  and  $y$ , and containing  $\varphi_t^c(x)$  for every small enough positive  $t$ .

Moreover, in case  $\mathcal{W}^c(x)$  is a line let  $(x, -\infty)_c$  and  $(x, +\infty)_c$  denote the connected component of  $\mathcal{W}^c(x) \setminus x$  containing negative and positive  $\varphi_t^c$ -iterates of  $x$ , respectively.

Analogously let us define the closed  $[x, y]_c$  and half-open  $[x, y)_c$  and  $(x, y]_c$  center segments.

**Definition 6.4.3.** We say that a point  $x$  in  $M$  is a *s-recurrent point* if for some  $\epsilon > 0$  there exists a sequence  $(y_n)$  so that  $y_n \in \mathcal{W}^c(x) \cap (\mathcal{W}_\epsilon^s(x) \setminus \{x\})$  for every  $n$  and  $(y_n)$  converges to  $x$ . Analogously we define *u-recurrent points*.

It follows immediately from the definition of *s* and *u*-recurrent points that circle leaves of  $\mathcal{W}^c$  do not contain any of them. Let us see that line leaves of  $\mathcal{W}^c$  do not contain them either.

**Proposition 6.4.4.** *There are no s or u-recurrent points in M.*

*Proof.* Let us see that there are no *u*-recurrent points in  $M$ . To show that there are no *s*-recurrent points the reasoning is analogous.

Suppose by contradiction that  $x$  is a *u*-recurrent point. There exists  $\epsilon > 0$  and  $y_n \in \mathcal{W}^c(x) \cap (\mathcal{W}_\epsilon^u(x) \setminus \{x\})$  such that  $y_n \xrightarrow{n} x$ .

As  $\mathcal{W}^c(x)$  is a line it follows that  $y_n \rightarrow \infty$  in  $\mathcal{W}^c(x)$ , meaning that for every  $R > 0$  the points  $y_n$  do not lie in  $\mathcal{W}_R^c(x)$  for every  $n$  large enough. Modulo subsequence, suppose without loss of generality that  $y_n$  tends to  $+\infty$  in  $\mathcal{W}^c(x)$ . It will be clear from the proof that if  $y_n$  tends to  $-\infty$  the arguments are analogous.

By Lemma 6.4.1 there exists  $p \in \mathcal{W}^c(x)$  the ‘last’ fixed point of  $f$  in  $\mathcal{W}^c(x)$  (if any fixed point exists) so that  $(p, +\infty)_c$  has no fixed points. In case  $\mathcal{W}^c(x)$  has no fixed points of  $f$  let  $p = -\infty$ . Note that being a *u*-recurrent point is clearly a  $\mathcal{W}^c$ -saturated property. We can assume then, without loss of generality, that  $x$  lies in  $(p, +\infty)_c$ .

Note that the half-open center segment  $[x, f(x))_c$  is a fundamental domain for  $f$  restricted to  $(p, +\infty)_c$ . So there exists  $x_n \in [x, f(x))_c$  and  $k_n \in \mathbb{Z}$  such that  $y_n = f^{k_n}(x_n)$  for every  $n$ . Note that either  $k_n \rightarrow -\infty$  or  $k_n \rightarrow +\infty$  as  $n$  tends to  $+\infty$ .

Consider  $\mathcal{W}_\delta^u([f^{-1}(x), f^2(x)]_c)$  for some  $\delta > 0$  small so that  $\mathcal{W}_\delta^u(w) \cap \mathcal{W}_\delta^u(w') = \emptyset$  for every  $w, w' \in [f^{-1}(x), f^2(x)]_c$  such that  $w \neq w'$ . For  $\delta' > 0$  small enough let  $H^c : \mathcal{W}_{\delta'}^u(x) \rightarrow \mathcal{W}_{\delta'}^u(f(x))$  be a *center holonomy map* so that  $[z, H^c(z)]_c$  is a center segment in  $\mathcal{W}_\delta^u([x, f(x)]_c)$  for every  $z \in \mathcal{W}_{\delta'}^u(x)$ . Note that  $\delta' > 0$  can be considered small enough so that  $f([z, H^c(z)]_c)$  lies in  $\mathcal{W}_\delta^u([f(x), f^2(x)]_c)$  and  $f^{-1}([z, H^c(z)]_c)$  lies in  $\mathcal{W}_\delta^u([f^{-1}(x), x]_c)$  for every  $z \in \mathcal{W}_{\delta'}^u(x)$ . And small enough so that for every  $z \in \mathcal{W}_{\delta'}^u(x)$  the center segment  $I_z$  from  $\mathcal{W}_\delta^u(f^{-1}(x))$  to  $\mathcal{W}_\delta^u(f(x))$  containing  $z$  and contained in  $\mathcal{W}_\delta^u([f^{-1}(x), f^2(x)]_c)$  is well defined. Let  $U$  be the set  $\bigcup_{z \in \mathcal{W}_{\delta'}^u(x)} I_z$ . Note that

$U$  is a foliation box neighborhood for the foliations  $\mathcal{W}^c$  and  $\mathcal{W}^u$  restricted to  $\mathcal{W}^{cu}(x)$ .

In  $U$  consider  $\pi^u : U \rightarrow \mathcal{W}_{\delta'}^u(x)$  the projection along centers so that  $\pi^u(I_z) = z$  for every  $z \in \mathcal{W}_{\delta'}^u(x)$ . And for every  $n$  let  $H_n^c : \mathcal{W}_{\delta'}^u(x) \rightarrow \mathcal{W}_{\delta}^u(x_n)$  be such that  $H_n^c(z)$  is the intersection of  $I_z$  with  $\mathcal{W}_{\delta}^u(x_n)$  for every  $z \in \mathcal{W}_{\delta'}^u(x)$ .

Let  $\delta'' > 0$  be such that  $\mathcal{W}_{\delta''}^u(z)$  is contained in  $U$  for every  $z$  in the segment  $[f^{-1}(x), f^2(x)]_c$ . In particular, for every  $n$  the inverse map  $(H_n^c)^{-1}$  is well defined from  $\mathcal{W}_{\delta''}^u(x_n)$  to  $\mathcal{W}_{\delta'}^u(x)$ .

Consider  $N$  large enough so that  $y_N$  lies in  $\mathcal{W}_{\delta'}^u(x)$ . And so that  $f^{-|k_N|}$  contracts distances enough so that  $\mathcal{W}_{\delta'}^u(w)$  is sent inside  $\mathcal{W}_{\delta''/2}^u(f^{-|k_N|}(w))$  for every  $w \in M$ .

In case  $k_N > 0$  the map  $(H_N^c)^{-1} \circ f^{-k_N}$  is a continuous map that sends  $\mathcal{W}_{\delta'}^u(x)$  strictly inside itself. In case  $k_N < 0$  then  $f^{k_N} \circ H_N^c$  is a continuous map that sends  $\mathcal{W}_{\delta'}^u(x)$  strictly inside itself. Covering both scenarios at the same time, let  $z_0$  denote the fixed point of this map. In the first case  $f^{-k_N}(z_0)$  lies in  $I_{z_0}$  and in the second case  $f^{k_N}(z_0)$  lies in  $I_{z_0}$ . In both cases  $f^2(z_0)$  and  $f^{-2}(z_0)$  do not lie in  $I_{z_0}$ . This implies that  $\mathcal{W}^c(z_0)$  must be compact because  $f|_{\mathcal{W}^c(z_0)} : \mathcal{W}^c(z_0) \rightarrow \mathcal{W}^c(z_0)$  is an orientation preserving homeomorphism and in case  $\mathcal{W}^c(z_0)$  is a line the fact that  $f^2(z_0)$  and  $f^{-2}(z_0)$  do not lie in  $I_{z_0}$  would imply that  $f^n(z_0)$  does not lie in  $I_{z_0}$  either for every  $|n| \geq 2$ .

Consider now  $\delta_1 > 0$  so that  $\mathcal{W}_{\delta_1}^u([f^{-1}(x), f^2(x)]_c)$  is disjoint from  $\mathcal{W}^c(z_0)$ . One can replicate the argument above to obtain  $z_1$  in  $\mathcal{W}_{\delta_1'}^u(x)$ , for some  $\delta_1' > 0$ , so that  $\mathcal{W}^c(z_1)$  is compact.

As  $\mathcal{W}^c(z_0)$  and  $\mathcal{W}^c(z_1)$  are two distinct compact leaves of  $\mathcal{W}^c$  that intersect the same  $\mathcal{W}^u$ -leaf we get to a contradiction with Lemma 6.4.2.  $\square$

### 6.4.2 The center flow is expansive

The goal of this subsection is to show that  $\varphi_t^c : M \rightarrow M$  is an expansive flow.

**Definition 6.4.5.** Given a non-singular flow  $\phi_t : X \rightarrow X$  in a metric space  $X$  and a constant  $\epsilon > 0$  the flow  $\phi_t$  is said to be  $\epsilon$ -expansive if for every  $x, y \in X$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  an increasing homeomorphism with  $h(0) = 0$  satisfying

$$d(\phi_t(x), \phi_{h(t)}(y)) < \epsilon \quad \forall t \in \mathbb{R}$$

one has that  $y$  lies in a piece  $\eta$  of the  $\phi_t$ -orbit of  $x$  satisfying  $\eta \subset B_\epsilon(x)$ .

**Remark 6.4.6.** Let us fix from now on  $\epsilon_0 > 0$  a small constant so that at scale  $10\epsilon_0$  one has local product structure and almost constant invariant bundles. To be more precise, one can fix a metric in  $M$  and consider  $\epsilon_0$  to be equal to  $\delta(f_0)$  as given by Lemma 4.2.1.

In particular,  $\epsilon_0$  and the metric in  $M$  are such that if  $d(x, y) = \epsilon$  for some  $0 < \epsilon \leq \epsilon_0$  then  $\mathcal{W}_{2\epsilon}^\sigma(x)$  intersect  $\mathcal{W}_{2\epsilon}^{\sigma'}(y)$ , and this intersection point is unique, for every  $(\sigma, \sigma') \in \{(s, cu), (cs, u)\}$ .

Recall that  $\varphi_t^c : M \rightarrow M$  denotes a unit speed flow whose orbits are the leaves of  $\mathcal{W}^c$ .

**Proposition 6.4.7.** *The flow  $\varphi_t^c$  is  $\epsilon_0$ -expansive.*

*Proof.* We claim first that, by dynamical coherence, it is enough to show that  $\varphi_t^c$  is  $2\epsilon_0$ -expansive in restriction to  $\mathcal{W}^{cu}$  and  $\mathcal{W}^{cs}$  leaves for their intrinsic topology.

Let us show the claim. Assume that  $\varphi_t^c$  is  $2\epsilon_0$ -expansive in restriction to  $\mathcal{W}^{cu}$  and  $\mathcal{W}^{cs}$  leaves and suppose that  $x$  and  $y$  are two points in  $M$  such that there exists  $h : \mathbb{R} \rightarrow \mathbb{R}$  an increasing homeomorphism satisfying  $h(0) = 0$  and  $d(\phi_t(x), \phi_{h(t)}(y)) < \epsilon_0$  for every  $t \in \mathbb{R}$ . It follows that  $\mathcal{W}_{2\epsilon_0}^u(\phi_{h(t)}(y))$  must intersect  $\mathcal{W}_{2\epsilon_0}^{cs}(\phi_t(x))$  for every  $t \in \mathbb{R}$ , and this intersection point is unique.

Let  $y_{cs} := \mathcal{W}_{2\epsilon_0}^u(y) \cap \mathcal{W}_{2\epsilon_0}^{cs}(x)$ . It follows that there exists an increasing homeomorphism  $h_{cs} : \mathbb{R} \rightarrow \mathbb{R}$  satisfying that  $h_{cs}(0) = 0$  and that  $d_{cs}(\phi_t(x), \phi_{h_{cs}(t)}(y_{cs})) < 2\epsilon_0$  for every  $t \in \mathbb{R}$ , where  $d_{cs}$  denotes the intrinsic distance in  $\mathcal{W}^{cs}$  leaves and where  $\phi_{h_{cs}(t)}(y_{cs})$  is the intersection of  $\mathcal{W}_{2\epsilon_0}^u(\phi_{h(t)}(y))$  and  $\mathcal{W}_{2\epsilon_0}^{cs}(\phi_t(x))$  for every  $t \in \mathbb{R}$ .

Since  $\varphi_t^c$  is  $2\epsilon_0$ -expansive inside  $\mathcal{W}^{cs}$  leaves it follows that  $y_{cs}$  must lie in a local piece of  $\varphi_t^c$ -orbit of  $x$ . Analogously, using that  $\varphi_t^c$  is  $2\epsilon_0$ -expansive inside  $\mathcal{W}^{cu}$  leaves it follows that the point  $y_{cu} := \mathcal{W}_{2\epsilon_0}^s(y) \cap \mathcal{W}_{2\epsilon_0}^{cu}(x)$  lies also in a local piece of  $\varphi_t^c$ -orbit of  $x$ . As a consequence of both facts,  $y$  itself must lie in a local piece of  $\varphi_t^c$ -orbit of  $x$ . This shows the claim.

Let us see that  $\varphi_t^c$  is  $2\epsilon_0$ -expansive inside  $\mathcal{W}^{cu}$  leaves. For  $\mathcal{W}^{cs}$  leaves the reasoning is analogous. Consider  $x \in M$ ,  $y \in \mathcal{W}^{cu}(x)$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  an increasing homeomorphism such that  $h(0) = 0$  and  $d_{cu}(\varphi_t^c(x), \varphi_{h(t)}^c(y)) < 2\epsilon_0$  for every  $t \in \mathbb{R}$ . Let  $y_u$  denote the intersection of  $\mathcal{W}_{4\epsilon_0}^c(y)$  with  $\mathcal{W}_{4\epsilon_0}^u(x)$ . There exists  $h_u : \mathbb{R} \rightarrow \mathbb{R}$  an increasing homeomorphism satisfying that  $h_u(0) = 0$  and  $\varphi_{h_u(t)}^c(y_u)$  equal to the intersection of  $\mathcal{W}_{4\epsilon_0}^c(\varphi_{h(t)}^c(y))$  with  $\mathcal{W}_{4\epsilon_0}^u(\varphi_t^c(x))$  for every  $t \in \mathbb{R}$ . If we show that  $y_u = x$  then we show that  $y$  lies in a local piece of the  $\varphi_t^c$ -orbit of  $x$ .

For simplicity in the notation, let us rename  $y = y_u$  and  $h = h_u$ . Suppose by contradiction that  $x \neq y$ . As a consequence, the point  $\varphi_{h(t)}^c(y)$  lies in  $\mathcal{W}_{4\epsilon_0}^u(\varphi_t^c(x)) \setminus \varphi_t^c(x)$  for every  $t \in \mathbb{R}$ .

Let  $N > 0$  be such that  $f^{-N}$  contracts distances inside  $\mathcal{W}^u$  leaves. For every  $n \geq N$  there exists  $t_n \in \mathbb{R}$  so that  $\varphi_{t_n}^c(x) = f^n(x)$ . One obtains that  $z_n := f^{-n}(\varphi_{h(t_n)}^c(y))$  is a sequence in  $\mathcal{W}^c(y) \cap \mathcal{W}_{4\epsilon_0}^u(x) \setminus x$  converging to  $x$ . Analogously, there exists  $t'_n \in \mathbb{R}$  so that  $f^n(y) = \varphi_{h(t'_n)}^c(y)$  for every  $n \geq N$

and then  $w_n := f^{-n}(\varphi_{t'_n}^c(x))$  is a sequence in  $\mathcal{W}^c(x) \cap \mathcal{W}_{4\epsilon_0}^u(y) \setminus y$  converging to  $y$ .

Let  $\delta_n > 0$  be such that  $\mathcal{W}_{\delta_n}^u(z_n)$  is a subset of  $\mathcal{W}_{4\epsilon_0}^u(x) \setminus x$  for every  $n \geq N$ . Consider a compact center segment  $[y, z_n]_c$  in  $\mathcal{W}^c(y)$  joining  $y$  with  $z_n$ . For each  $n$ , by taking  $w_{k_n}$  close enough to  $y$  one can construct a center segment  $[w_{k_n}, w'_n]_c$  close enough to  $[y, z_n]_c$  so that  $w'_n$  lies in  $\mathcal{W}_{\delta_n}^u(z_n)$ .

Since  $z_n$  converges to  $x$  then  $w'_n$  converges to  $x$  as well. Moreover,  $w'_n$  lies in  $\mathcal{W}^c(x) \cap \mathcal{W}_{4\epsilon_0}^u(x)$  for every  $n$  and is different from  $x$  since  $\mathcal{W}_{\delta_n}^u(z_n)$  is disjoint from  $\{x\}$ . One obtains that  $x$  is a  $u$ -recurrent point. By Proposition 6.4.4 this gives us a contradiction. □

### 6.4.3 No fixed points in lines of $\mathcal{W}^c$

Another consequence of the non-existence of  $s$  and  $u$ -recurrent points is the following statement (which is the only goal of this subsection).

**Proposition 6.4.8.** *If  $x$  is a fixed point of  $f$  then  $\mathcal{W}^c(x)$  is compact.*

*Proof.* Suppose by contradiction that  $x$  is a fixed point in a leaf  $\mathcal{C} \in \mathcal{W}^c$  that is not compact. Recall that  $(x, +\infty)_c$  denotes the connected component of  $\mathcal{W}^c(x) \setminus \{x\}$  containing positive iterates of  $x$  by  $\varphi_t^c$ . By Lemma 6.4.1 we can suppose, without loss of generality, that  $(x, +\infty)_c$  has no fixed points of  $f$ .

Since  $f$  preserves the orientations of  $\mathcal{W}^c$  leaves it follows that  $(x, +\infty)_c$  is invariant by  $f$ . As a consequence, either for every  $y \in (x, +\infty)_c$  the sequence  $f^n(y)$  tends to  $x$  as  $n$  tends to  $+\infty$  or for every  $y \in (x, +\infty)_c$  it tends to  $x$  as  $n$  tends to  $-\infty$ . Suppose without loss of generality the later, otherwise the argument is analogous with  $\mathcal{W}^s(\mathcal{C})$  in the place of  $\mathcal{W}^u(\mathcal{C})$  and  $f^{-1}$  in the place of  $f$ .

The proof now continues with a series of claims.

**Claim 1.** The leaf  $\mathcal{W}^u(x)$  intersects  $\mathcal{C}$  only in  $x$ .

*Proof.* Suppose by contradiction that  $y$  is a point in  $(\mathcal{W}^u(x) \setminus \{x\}) \cap \mathcal{C}$ . Since  $f$  contracts distances indefinitely on  $\mathcal{W}^u$  leaves and  $x$  is fixed by  $f$  it follows that the sequence  $f^{-n}(y)$  tends to  $x$  inside  $\mathcal{W}^u(x)$  as  $n$  tends to  $+\infty$ . By the center fixing property  $f^{-n}(y)$  lies in  $\mathcal{C}$  for every  $n$ . One obtains that  $x$  is a  $u$ -recurrent point and this contradicts Proposition 6.4.4. This proves the first claim.

**Claim 2.** The interval  $[y, f(y)]_c$  is not contained in  $\mathcal{W}^u(\mathcal{C})$  for every  $y \in \mathcal{W}^u(x) \setminus x$ .

*Proof.* Suppose by contradiction that  $[y, f(y)]_c$  is contained in  $\mathcal{W}^u(\mathcal{C})$  for some  $y \in \mathcal{W}^u(x) \setminus x$ . Let  $\gamma : [0, 1] \rightarrow [y, f(y)]_c$  be a homeomorphism over its image such that  $\gamma(0) = y$  and  $\gamma(1) = f(y)$ . One can consider  $p^u \gamma : [0, 1] \rightarrow \mathcal{C}$  the unique continuous curve such that  $p^u \gamma(0) = x$  and  $\gamma(t) \in \mathcal{W}^u(p^u \gamma(t))$  for every  $t \in [0, 1]$ . The curve  $p^u \gamma$  can be seen as a projection by unstable holonomy of  $\gamma$  to  $\mathcal{C}$ .

On the one hand, it is immediate to check that  $p^u \gamma$  from  $[0, 1]$  to  $\mathcal{C}$  is a local homeomorphism. Thus a homeomorphism over its image since  $\mathcal{C}$  is a line. In particular,  $p^u \gamma(1)$  needs to be different from  $x$ . On the other hand,  $y$  lies in  $\mathcal{W}^u(x)$  and, as  $x$  is fixed by  $f$ , the point  $f(y)$  lies also in  $\mathcal{W}^u(x)$ . As a consequence,  $p^u \gamma(1)$  lies in  $\mathcal{W}^u(x) \cap \mathcal{C}$  and by the previous claim this implies that  $p^u \gamma(1) = x$ . Since also  $p^u \gamma(0) = x$  one gets to a contradiction with the injectivity of  $p^u \gamma$ . This shows the second claim.

It follows from the previous claim that for every  $y \in \mathcal{W}^u(x) \setminus x$  the supremum  $t_y := \sup\{t \mid \varphi_s^c(y) \in \mathcal{W}^u(\mathcal{C}) \forall s \in [0, t]\}$  is finite. Indeed, if  $\mathcal{W}^c(y)$  is compact or  $\mathcal{W}^c(y)$  is a line such that  $\varphi_1^c(y)$  and  $f(y)$  lie in the same connected component of  $\mathcal{W}^c(y) \setminus y$  the existence of  $t_y$  finite follows directly from  $[y, f(y)]_c$  not contained in  $\mathcal{W}^u(\mathcal{C})$ . In the case that  $\mathcal{W}^c(y)$  is a line such that  $\varphi_1^c(y)$  and  $f(y)$  lie in different connected components of  $\mathcal{W}^c(y) \setminus y$  the existence of  $t_y$  finite follows directly from the fact that  $[f^{-1}(y), y]_c$ , which is equal to  $f^{-1}([y, f(y)]_c)$ , is not contained in  $\mathcal{W}^u(\mathcal{C})$  because  $[y, f(y)]_c$  is not contained in  $\mathcal{W}^u(\mathcal{C})$  and  $\mathcal{W}^u(\mathcal{C})$  is  $f$ -invariant.

Let  $\partial_{cu} \mathcal{W}^u(\mathcal{C})$  denote the boundary of  $\mathcal{W}^u(\mathcal{C})$  as a subset of  $\mathcal{W}^{cu}(\mathcal{C})$ . It follows that  $\varphi_{t_y}^c(y)$  lies in  $\partial_{cu} \mathcal{W}^u(\mathcal{C})$  for every  $y \in \mathcal{W}^u(x) \setminus x$ . Let  $p^c : \mathcal{W}^u(x) \setminus \{x\} \rightarrow \partial_{cu} \mathcal{W}^u(\mathcal{C})$  denote the map  $p^c(y) := \varphi_{t_y}^c(y)$ . Note that, since  $\mathcal{W}^u(\mathcal{C})$  is saturated by  $\mathcal{W}^u$  leaves, then  $\partial_{cu} \mathcal{W}^u(\mathcal{C})$  is the union of  $\mathcal{W}^u$  leaves.

Fix from now on  $y_0 \in \mathcal{W}^u(x) \setminus \{x\}$  and  $x_0 := p^c(y_0)$ . We will see in the following claim that  $p^c$  needs to be a homeomorphism from the connected component of  $\mathcal{W}^u(x) \setminus \{x\}$  containing  $y_0$  to the leaf  $\mathcal{W}^u(x_0)$ .

If this third claim is true and  $\dim(E^u) \geq 2$  we get to an immediate contradiction since  $\mathcal{W}^u(x) \setminus \{x\}$  is homeomorphic to  $\mathbb{R}^{\dim(E^u)} \setminus \{0\}$  and  $\mathcal{W}^u(x_0)$  homeomorphic to  $\mathbb{R}^{\dim(E^u)}$ . And if this third claim is true and  $\dim(E^u) = 1$  a contradiction arrives as follows: Note that  $p^c \circ f(y) = f \circ p^c(y)$  for every  $y \in \mathcal{W}^u(x) \setminus \{x\}$ . As a consequence,  $\mathcal{W}^u(x_0)$  is invariant by  $f^2$  since the connected component of  $\mathcal{W}^u(x) \setminus \{x\}$  containing  $y_0$  is invariant by  $f^2$ . Since  $f^{-2}$  induces a contraction on  $\mathcal{W}^u(x_0)$  it follows that  $\mathcal{W}^u(x_0)$  has a fixed point  $p \in \mathcal{W}^u(x_0)$  for  $f^2$ . As a consequence,  $q := (p^c)^{-1}(p)$  is a fixed point of  $f^2$  in  $\mathcal{W}^u(x) \setminus \{x\}$ . We get to a contradiction since  $x$  and  $q$  would be two fixed points of  $f^2$  in the same leaf of  $\mathcal{W}^u$ .

It remains to show the following claim.

**Claim 3.** The map  $p^c$  is a homeomorphism from the connected component of  $\mathcal{W}^u(x) \setminus \{x\}$  containing  $y_0$  to the leaf  $\mathcal{W}^u(x_0)$ .

*Proof.* Let us see first that  $p^c$  is injective. Indeed, suppose by contradiction that there exists  $y, y' \in \mathcal{W}^u(x) \setminus \{x\}$  such that  $p^c(y) = p^c(y')$  and  $y \neq y'$ . Suppose without loss of generality that  $y' \in (y, +\infty)_c$ . Let  $\gamma_{y,y'} : [0, 1] \rightarrow (y, y')_c$  be a homeomorphism over its image such that  $\gamma_{y,y'}(0) = y$  and  $\gamma_{y,y'}(1) = y'$ . As above, one can consider  $p^u \gamma_{y,y'} : [0, 1] \rightarrow \mathcal{C}$  the continuous curve such that  $p^u \gamma_{y,y'}(0) = x$  and  $\gamma_{y,y'}(t) \in \mathcal{W}^u(p \gamma_{y,y'}(t))$  for every  $t \in [0, 1]$ . The curve  $p^u \gamma_{y,y'}$  is a local homeomorphism, thus a homeomorphism over its image. But  $p^u \gamma_{y,y'}(0) = p^u \gamma_{y,y'}(1)$  by the first claim. This gives us a contradiction and shows that  $p^c$  is injective.

The continuity of  $p^c$  and  $(p^c)^{-1}$  is immediate from the regularity of  $\mathcal{W}^c$ . Moreover, for every  $y \in \mathcal{W}^u(x) \setminus \{x\}$  there exists  $\epsilon > 0$  such that  $p^c(\mathcal{W}_\epsilon^u(y))$  is sent by  $p^c$  to a neighborhood of  $p^c(y)$  in  $\mathcal{W}^u(y)$ . As a consequence the connected component of  $\mathcal{W}^u(x) \setminus \{x\}$  containing  $y_0$  has its image by  $p^c$  inside  $\mathcal{W}^u(x_0)$ .

It remains to show that  $p^c$  is surjective over  $\mathcal{W}^c(x_0)$ . Let  $\gamma : [0, 1] \rightarrow [x_0, y_0]_c$  be a homeomorphism over its image such that  $\gamma(0) = x_0$  and  $\gamma(1) = y_0$ . As before, one can consider  $p^u \gamma : [0, 1] \rightarrow \mathcal{C}$  the continuous curve such that  $p^u \gamma(0) = x$  and  $\gamma(t) \in \mathcal{W}^u(p \gamma(t))$  for every  $t \in [0, 1]$ . The curve  $p^u \gamma$  is a local homeomorphism. In particular the image of  $p^u \gamma$  lies in  $[x, +\infty)_c$ .

Suppose  $x_1 \in \mathcal{W}^u(x_0)$ . Let us see that  $x_1$  is in the image of  $p^c$ . The center segment  $(y_0, x_0)_c$  is a subset of  $\mathcal{W}^u(\mathcal{C})$ . By unstable holonomy inside  $\mathcal{W}^u(\mathcal{C})$  one can see that there exists  $z_1$  in  $\mathcal{W}_{loc}^c(x_1)$  such that  $(z_1, x_1)_c$  is contained in  $\mathcal{W}^u(\mathcal{C})$ . And such that for some  $z \in (x_0, y_0)_c$  one has that  $\mathcal{W}^u(z) \cap (z_1, x_1)_c \neq \emptyset$ . Let  $w$  denote a point in this intersection. It follows from the previous paragraph that  $\mathcal{W}^u(w) \cap [x, +\infty)_c \neq \emptyset$ . Let  $w'$  be a point on this intersection.

Let  $U$  be a small foliation box of the foliation  $\mathcal{W}^c$  restricted to  $\mathcal{W}^{cu}(x)$  such that  $x$  lies in  $U$  and  $U \subset \mathcal{W}^u(\mathcal{C})$ . Recall that for every  $y \in [x, +\infty)_c$  the sequence  $f^{-n}(y)$  tends to  $x$  as  $n$  tends to  $+\infty$ . Then, for  $N > 0$  large enough  $f^{-N}(w')$  lies in  $U$ . And since  $f^{-1}$  contracts distances in  $\mathcal{W}^u$  leaves and  $w \in \mathcal{W}^u(w')$  then  $N$  can be considered large enough so that  $f^{-N}(w')$  also lies in  $U$ .

Let  $w'' \in \mathcal{W}_{loc}^u(x)$  be such that  $[w'', f^{-N}(w')]_c$  is a center segment in a plaque of  $U$ . It follows that  $(w'', f^{-N}(w'))_c \cup [f^{-N}(w'), f^{-N}(x_1)]_c$  is a center segment contained in  $\mathcal{W}^u(\mathcal{C})$  joining  $w''$  with  $f^{-N}(x_1)$ . Namely, the center segment  $(w'', f^{-N}(x_1))_c$ . That is,  $f^{-N}(x_1)$  lies in the image of  $p^c$ . It follows from  $p^c \circ f = f \circ p^c$  that  $x_1$  itself is in the image of  $p^c$ . This shows that  $p^c$  is surjective over  $\mathcal{W}^u(x_0)$ . This ends the proof of the third claim and of

Proposition 6.4.8.

□

#### 6.4.4 Coherent behavior of $f$ and $\varphi_t^c$ in $Y$

Informally, this subsection shows in Lemma 6.4.10 that, modulo inverting the time of  $\varphi_t^c$ , forwards iterates of  $f$  and  $\varphi_t^c$  displace points of  $Y$  in the same direction along the leaves of  $\mathcal{W}^c$ .

**Remark 6.4.9.** Note that, since  $\mathcal{W}^c$  has a dense leaf then  $\varphi_t^c : M \rightarrow M$  is a transitive flow. It follows by classical arguments<sup>1</sup> that there exists a residual subset of points in  $M$  whose backwards and forwards orbit by  $\varphi_t^c$  is dense in  $M$ .

Since the intersection of two residual sets in  $M$  is non empty and the subset  $Y \subset M$  is residual by Proposition 6.3.11 then there exists a point in  $Y$  whose backwards and forwards orbit by  $\varphi_t^c$  is dense in  $M$ .

Suppose  $x_0 \in Y$ . It follows from Proposition 6.4.8 that  $f(x_0)$  lies in one of the two connected components of  $\mathcal{W}^c(x_0) \setminus x_0$ . Modulo inverting the time of  $\varphi_t^c : M \rightarrow M$  let us suppose from now on that  $\varphi_1^c(x_0)$  and  $f(x_0)$  lie in the same connected component of  $\mathcal{W}^c(x_0) \setminus x_0$ .

Recall that by Proposition 6.3.11 the set  $Y \subset M$  is a residual,  $\mathcal{W}^c$ -saturated and locally path connected subset of  $M$ . The transitive hypothesis on  $\mathcal{W}^c$  combined with Proposition 6.3.11 allows us to make connection argument to obtain the following.

**Lemma 6.4.10.** *For every  $x \in Y$  the points  $f(x)$  and  $\varphi_1^c(x)$  lie in the same connected component of  $\mathcal{W}^c(x) \setminus x$ .*

*Proof.* Let  $x$  be any point in  $Y$ . Since  $Y$  contains a dense leaf  $L$  of  $\mathcal{W}^c$  it follows that we can join  $x$  with  $x_0$  by a curve  $\gamma \subset Y$ . Indeed, such a curve  $\gamma$  can be constructed as the concatenation of a local curve  $\gamma_1 \subset Y$  joining  $x$  with a point  $y \in L$ , a curve  $\gamma_2$  contained in  $L$  joining  $y$  with another point  $z \in L$ , and a local curve  $\gamma_3 \subset Y$  joining  $x_0$  with  $z$ .

By Proposition 6.4.8 the function  $\rho$  has no zero in  $\gamma$ . Since  $\rho|_\gamma : \gamma \rightarrow \mathbb{R}$  is continuous and  $\rho(x_0) > 0$  it follows that  $\rho(x) > 0$ . That is,  $f(x)$  and  $\varphi_1(x)$  lie in the same connected component of  $\mathcal{W}^c(x) \setminus \{x\}$ .

□

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<sup>1</sup>Indeed, given a countable base of  $M$  by open sets  $\{U_n\}_{n \geq 0}$  it follows from the transitivity of  $\varphi_t^c$  that  $V_n = \bigcup_{t \leq 0} \varphi_t^c(U_n)$  is open and dense for every  $n \geq 0$ . Then  $\bigcap_{n \geq 0} V_n$  is a residual subset of  $M$  whose points have a forwards dense  $\varphi_t^c$ -orbit.



## 6.5 Uniform size of stable and unstable sets of $\varphi_t^c$

The goal of this section is to show Proposition 6.5.2 (stated below) relating local stable and unstable leaves of  $f$  with local stable and unstable sets of  $\varphi_t^c$ .

**Definition 6.5.1.** Given  $\epsilon > 0$  and  $x \in M$  let us consider  $S_\epsilon(x)$  the *forwards  $\epsilon$ -stable set of  $x$*  for the flow  $\varphi_t^c$ , that is, the set of points  $y \in M$  such that there exists a homeomorphism  $h : [0, +\infty) \rightarrow [0, +\infty)$  satisfying

$$d(\varphi_t^c(x), \varphi_{h(t)}^c(y)) \leq \epsilon,$$

for every  $t \geq 0$ . Analogously we define  $U_\epsilon(x)$  the *backwards  $\epsilon$ -stable set of  $x$* .

**Proposition 6.5.2** (Uniform size of stable and unstable sets). *For every  $\epsilon > 0$  there exists  $\delta > 0$  such that for every  $x \in M$  one has  $\mathcal{W}_\delta^s(x) \subset S_\epsilon(x)$  and  $\mathcal{W}_\delta^u(x) \subset U_\epsilon(x)$ .*

The proof of Proposition 6.5.2 is done in incremental stages of generality. First, Lemma 6.5.3 for points in  $Y$  and a point-dependent  $\delta$ . Second, Lemma 6.5.6 for points in  $Y$  and uniform  $\delta$  along backwards or forwards orbits of  $\varphi_t^c$ . Finally, Proposition 6.5.2 for any point in  $M$  and uniform  $\delta$  via passing to the limit.

**Lemma 6.5.3.** *Suppose  $x \in Y$ . For every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\mathcal{W}_\delta^s(x) \subset S_\epsilon(x)$  and  $\mathcal{W}_\delta^u(x) \subset U_\epsilon(x)$ .*

*Proof.* Suppose  $\epsilon > 0$ . Let us see that there exists  $\delta > 0$  such that  $\mathcal{W}_\delta^s(x) \subset S_\epsilon(x)$ . To show that there exists  $\delta > 0$  so that  $\mathcal{W}_\delta^u(x)$  is contained in  $U_\epsilon(x)$  one argues analogously.

Let  $\delta' > 0$  be such that for every  $z$  and  $z'$  in the center segment  $[x, f(x)]_c$  joining  $x$  with  $f(x)$  one has that  $\mathcal{W}_{\delta'}^s(z) \cap \mathcal{W}_{\delta'}^s(z') = \emptyset$  if  $z \neq z'$ . Let us consider  $\delta' > 0$  small enough so that  $f^n(\mathcal{W}_{\delta'}^s(z))$  is contained in  $\mathcal{W}_\epsilon^s(f^n(z))$  for every  $z \in [x, f(x)]_c$  and  $n \geq 0$ .

Since  $x$  is a point in  $Y$  then  $\rho$  is continuous in  $x$ . One can consider  $\delta > 0$  small enough so that for every  $y \in \mathcal{W}_\delta^s(x)$  one has that  $[y, f(y)]_c$  is a subset of  $\mathcal{W}_{\delta'}^s([x, f(x)]_c)$ . Then  $[f^n(y), f^{n+1}(y)]_c$  is a subset of  $\mathcal{W}_\epsilon^s([f^n(x), f^{n+1}(x)]_c)$  for every  $n > 0$ .

Given  $y \in \mathcal{W}_\delta^s(x)$  one can consider  $h_0$  from  $I_0 := [0, \rho(x)] \subset \mathbb{R}$  to  $\mathbb{R}$  an increasing homeomorphism over its image such that  $h_0(0) = 0$  and  $\varphi_{h_0(t)}^c(y)$  lies in  $\mathcal{W}_\epsilon^s(\varphi_t^c(x))$  for every  $t \in I_0$ . Then one can consider  $h_1$  from  $I_1 := [\rho(x), \rho(x) + \rho(f(x))] \subset \mathbb{R}$  to  $\mathbb{R}$  so that  $h_0(\rho(x)) = h_1(\rho(x))$  and  $\varphi_{h_1(t)}^c(y)$  lies in  $\mathcal{W}_\epsilon^s(\varphi_t^c(x))$  for every  $t \in I_1$ . And inductively, for every  $n > 0$ , a function  $h_n$  defined in  $I_n := [\rho(x) + \dots + \rho(f^{n-1}(x)), \rho(x) + \dots + \rho(f^n(x))]$  so that

$h_{n-1}$  and  $h_n$  take the same value in  $\rho(x) + \dots + \rho(f^{n-1}(x))$  and such that  $\varphi_{h_n(t)}^c(y)$  lies in  $\mathcal{W}_\epsilon^s(\varphi_t(x))$  for every  $t \in I_n$ .

By Proposition 6.4.8 it follows that  $[0, +\infty) = \bigcup_{n \geq 0} I_n$ . Since  $h_n$  and  $h_{n+1}$  coincide in  $I_n \cap I_{n+1}$  for every  $n \geq 0$  it follows that  $h : [0, +\infty) \rightarrow [0, +\infty)$  defined as  $h(x) = h_n(x)$  for every  $x \in I_n$  is a well defined homeomorphism over its image. Moreover, by Proposition 6.4.8 and Lemma 6.4.10 the image of  $t \mapsto \varphi_{h(t)}^c(y)$ , namely  $[y, f(y)]_c \cup [f(y), f^2(y)]_c \cup \dots$ , needs to be equal to  $[y, +\infty)_c$ . This shows that  $h$  is surjective, thus a homeomorphism.

One concludes that  $d(\varphi_t^c(x), \varphi_{h(t)}^c(y)) \leq \epsilon$  for every  $t \geq 0$ , for  $h : [0, +\infty) \rightarrow [0, +\infty)$  the homeomorphism constructed above. That is,  $y$  lies in  $S_\epsilon(x)$ .  $\square$

The next one is a technical lemma that will be used in this section. For simplicity, from now on whenever we refer to the stable distance  $d_s(x, y)$  between two points  $x, y \in M$  we will implicitly mean that both points lie in the same leaf of  $\mathcal{W}^s$ . Recall the constant  $\epsilon_0 > 0$  from Remark 6.4.6.

**Lemma 6.5.4.** *Let  $\epsilon > 0$  be a constant such that  $\epsilon \leq \epsilon_0$  and  $I \subset \mathbb{R}$  be an interval containing 0. Suppose that  $(x_n)$  and  $(y_n)$  are two sequences in  $M$  such that there exist increasing and continuous maps  $h_n : I \rightarrow \mathbb{R}$  satisfying that  $h_n(0) = 0$  and  $d_s(\varphi_t^c(x_n), \varphi_{h_n(t)}^c(y_n)) \leq \epsilon$  for every  $t \in I$ .*

*If  $x_n \xrightarrow{n} x$  and  $y_n \xrightarrow{n} y$  then there exists  $h : I \rightarrow \mathbb{R}$  increasing and continuous such that  $h(0) = 0$  and  $d_s(\varphi_t^c(x), \varphi_{h(t)}^c(y)) \leq \epsilon$  for every  $t \in I$ . Moreover, such a map  $h : I \rightarrow \mathbb{R}$  is unique and is given by  $h(t) = \lim_n h_n(t)$  for every  $t \in I$ .*

*Proof.* Note that, by taking  $t = 0$ , one gets that  $x_n$  lies in  $\mathcal{W}^s(y_n)$  and  $d_s(x_n, y_n) \leq \epsilon$  for every  $n$ . As a consequence,  $y$  lies in  $\mathcal{W}^s(x)$  and  $d_s(x, y) \leq \epsilon$ .

Let us suppose first that  $I = [a, b]$  for some  $a, b \in \mathbb{R}$  such that  $a \leq 0 \leq b$ . Let  $T^+$  denote the supremum of the points  $s \in [0, b]$  such that there exists  $h_s : [0, s] \rightarrow \mathbb{R}$  increasing and continuous satisfying that  $h_s(0) = 0$  and  $d_s(\varphi_t^c(x), \varphi_{h_s(t)}^c(y)) \leq 2\epsilon$  for every  $t \in [0, s]$ .

Note that, since  $d_s(x, y) \leq \epsilon$ , then  $T^+ > 0$ . Moreover, note that if  $s \in [0, T^+)$  then the increasing and continuous map  $h_s$  as above is unique. And that if  $s' \in [0, T^+)$  and  $s \leq s'$  then  $h_{s'}$  coincides with  $h_s$  in  $[0, s]$ .

We claim that  $T^+$  needs to be a maximum. Indeed, one can define  $h : [0, T^+] \rightarrow \mathbb{R}$  increasing and continuous as  $h(t) = h_s(t)$  for every  $t \in [0, T^+)$  and every  $s \in [0, T^+)$  so that  $t < s$ , and define  $h(T^+)$  as  $\lim_t h(t)$  for  $t \in [0, T^+)$ . Then if  $(s_n)$  is a sequence in  $[0, T^+)$  converging to  $T^+$  one has that  $d_s(\varphi_{s_n}^c(x), \varphi_{h_{s_n}(s_n)}^c(y)) \leq \epsilon$  for every  $n$  implies that  $d_s(\varphi_{T^+}^c(x), \varphi_{h(T^+)}^c(y)) \leq \epsilon$ .

Moreover, note that if we show that  $d_s(\varphi_t^c(x), \varphi_{h(t)}^c(y)) \leq \epsilon$  for every  $t \in [0, T^+]$  then  $T^+ = b$ . This is because, if  $T^+ < b$  and  $d_s(\varphi_{T^+}^c(x), \varphi_{h(T^+)}^c(y)) \leq$

$\epsilon$ , then one can extend  $h$  to an interval strictly larger than  $[0, T^+]$ , and still contained in  $I$ , so that  $d_s(\varphi_t^c(x), \varphi_{h_s(t)}^c(y)) \leq 2\epsilon$  for every  $t$  on this interval, contradicting that  $T^+$  is a maximum.

Note that the continuity of  $\varphi_t^c$  implies that  $\lim_n \varphi_t^c(x_n) = \varphi_t^c(x)$  for every  $t \in [0, T^+]$ . We claim that  $\lim_n \varphi_{h_n(t)}^c(y_n) = \varphi_{h(t)}^c(y)$  and  $d_s(\varphi_t^c(x), \varphi_{h(t)}^c(y)) \leq \epsilon$  for every  $t \in [0, T^+]$ .

Indeed, let us start by fixing  $t \in [0, \epsilon_0] \cap [0, T^+]$ . The sequence  $\varphi_{h_n(t)}^c(y_n)$  lies in  $B_{2\epsilon_0}(x)$  for every  $n$  large enough. Let  $w$  be an accumulation point of this sequence. It follows from  $d_s(\varphi_t^c(x_n), \varphi_{h_n(t)}^c(y_n)) \leq \epsilon \leq \epsilon_0$  for every  $n$  that  $d_s(\varphi_t^c(x), w) \leq \epsilon_0$ . Moreover, since  $t \in [0, \epsilon_0]$  then  $\varphi_{h_n(t)}^c(y_n)$  lies in  $\mathcal{W}_{2\epsilon_0}^c(y_n)$ . And since the sequence  $(y_n)$  converges to  $y$  then  $\mathcal{W}_{2\epsilon_0}^c(y_n)$  converges to  $\mathcal{W}_{2\epsilon_0}^c(y)$  in the Hausdorff topology. One obtains that  $w$  lies in the intersection of  $\mathcal{W}_{2\epsilon_0}^c(y)$  and  $\mathcal{W}_{2\epsilon_0}^s(\varphi_t^c(x))$ .

Since at scale  $10\epsilon_0$  this intersection can only happen in one point (recall the definition of  $\epsilon_0$  in Remark 6.4.6) it follows that  $w$  is unique, independently of the subsequence. That is,  $\lim_n \varphi_{h_n(t)}^c(y_n) = w$ . Moreover, since  $\varphi_{h(t)}^c(y)$  lies in the intersection of  $\mathcal{W}_{2\epsilon_0}^c(y)$  and  $\mathcal{W}_{2\epsilon_0}^s(\varphi_t^c(x))$  it follows that  $w = \varphi_{h(t)}^c(y)$ . This shows that  $\lim_n \varphi_{h_n(t)}^c(y_n) = \varphi_{h(t)}^c(y)$  and  $d_s(\varphi_t^c(x), \varphi_{h(t)}^c(y)) \leq \epsilon_0$  for every  $t \in [0, \epsilon_0]$ .

One argues analogously to show that for every  $t \in [\epsilon_0, 2\epsilon_0] \cap [0, T^+]$  one has that  $\lim_n \varphi_{h_n(t)}^c(y_n) = \varphi_{h(t)}^c(y)$  and  $d_s(\varphi_t^c(x), \varphi_{h(t)}^c(y)) \leq \epsilon_0$ . Inductively one obtains that this happens for every  $t \in [0, T^+]$ . This shows the claim.

Analogously, via auxiliary  $T^- \in [a, 0]$  and showing that  $T^- = a$  one can extend  $h$  to  $h : [a, b] \rightarrow \mathbb{R}$  increasing and continuous satisfying that  $\lim_n \varphi_{h_n(t)}^c(y_n) = \varphi_{h(t)}^c(y)$  and  $d_s(\varphi_t^c(x), \varphi_{h(t)}^c(y)) \leq \epsilon_0$  for every  $t \in [a, b]$ .

By construction, such a  $h : I \rightarrow \mathbb{R}$  is unique and it follows from the limit  $\lim_n \varphi_{h_n(t)}^c(y_n) = \varphi_{h(t)}^c(y)$  and the continuity of  $\varphi_t^c$  that  $\lim_n h_n(t) = h(t)$  for every  $t \in [a, b]$ . This shows the lemma for  $I$  closed and bounded.

In case  $I \subset \mathbb{R}$  is another type of interval it is enough to write  $I$  as the increasing union of closed bounded intervals  $I_k$ . For such  $I_k$  there exists  $h^{(k)} : I_k \rightarrow \mathbb{R}$  increasing and continuous such that  $h^{(k)}(0) = 0$  and  $d_s(\varphi_t^c(x), \varphi_{h^{(k)}(t)}^c(y)) \leq \epsilon$  for every  $t \in I_k$ . Since  $h^{(k)}(t) = \lim_n h_n(t)$  for every  $t \in I_k$  then  $h : I \rightarrow \mathbb{R}$  is well defined by  $h(t) := \lim_n h_n(t)$  and satisfies that  $d_s(\varphi_t^c(x), \varphi_{h^{(k)}(t)}^c(y)) \leq \epsilon$  for every  $t \in I$ .  $\square$

The next one is a classical argument from expansive systems, though adapted to our context for simplicity. Recall that, by Proposition 6.4.7, the flow  $\varphi_t^c$  is  $\epsilon_0$ -expansive.

**Lemma 6.5.5.** *There is no pair of sequences  $(x_n)$  and  $(y_n)$  in  $M$  such that:*

1. *The point  $y_n$  lies in  $\mathcal{W}_{\delta_n}^s(x_n)$  for some sequence  $\delta_n \xrightarrow{n} 0$ .*

2. There exist homeomorphisms  $h_n : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\varphi_{h_n(t)}^c(y_n)$  lies in  $\mathcal{W}^s(\varphi_t^c(x_n))$  and

$$d_s(\varphi_t^c(x_n), \varphi_{h_n(t)}^c(y_n)) \leq \epsilon_0 \quad (6.5.1)$$

for every  $t \in [0, +\infty)$ .

3. There exists  $\delta > 0$  and  $t_n \in [0, +\infty)$  such that

$$d_s(\varphi_{t_n}^c(x_n), \varphi_{h_n(t_n)}^c(y_n)) = \delta \quad (6.5.2)$$

for every  $n$ .

Analogously for  $u$  in the place of  $s$  and  $(-\infty, 0]$  in the place of  $[0, +\infty)$ .

*Proof.* Suppose by contradiction that such a pair of sequences exists. Up to taking a subsequence suppose that the sequence  $z_n := \varphi_{t_n}^c(x_n)$  converges to a point  $z_\infty$  and that the sequence  $w_n := \varphi_{h_n(t_n)}^c(y_n)$  converges to a point  $w_\infty$ .

Since  $w_n \in \mathcal{W}^s(z_n)$  and  $d_s(z_n, w_n) = \delta$  for every  $n$  it follows that  $w_\infty \in \mathcal{W}^s(z_\infty)$  and  $d_s(z_\infty, w_\infty) = \delta$ . In particular,  $w_\infty$  is different from  $z_\infty$  and does not lie in  $\mathcal{W}_{\epsilon_0}^c(z_\infty)$ . To get to a contradiction, let us see that the  $\varphi_t^c$ -orbit of  $z_\infty$  and  $w_\infty$  do not  $\epsilon_0$ -separate, contradicting the expansivity of  $\varphi_t^c$ .

For every  $n$  let  $g_n : [-t_n, +\infty) \rightarrow \mathbb{R}$  be the increasing and continuous map given by  $g_n(t) = h_n(t + t_n) - h_n(t_n)$  for every  $t \in [-t_n, +\infty)$ . We claim that  $\varphi_{g_n(t)}^c(w_n)$  lies in  $\mathcal{W}^s(\varphi_t^c(z_n))$  and that  $d_s(\varphi_t^c(z_n), \varphi_{g_n(t)}^c(w_n)) \leq \epsilon_0$  for every  $t \in [-t_n, +\infty)$ . Indeed, note that  $h_n(t_n)$  is the time it takes  $y_n$  to get to  $w_n$  by the flow  $\varphi_t^c$ . It follows that, for every  $t \in [-t_n, +\infty)$ , one has that  $\varphi_{g_n(t)}^c(w_n) = \varphi_{h_n(t+t_n)}^c \circ \varphi_{-h_n(t_n)}^c(w_n) = \varphi_{h_n(t+t_n)}^c(y_n)$ . Since  $\varphi_{t+t_n}^c(x_n) = \varphi_t^c(z_n)$  then  $d_s(\varphi_t^c(z_n), \varphi_{g_n(t)}^c(w_n)) \leq \epsilon_0$  for every  $t \in [-t_n, +\infty)$  by (6.5.1). This proves the claim.

Since  $\delta_n \xrightarrow{n} 0$  it follows that  $t_n$  tends to  $+\infty$ . Let  $T > 0$ . There exists  $N$  such that  $t_n > T$  for every  $n \geq N$ . Let  $H_n : [-T, +\infty) \rightarrow \mathbb{R}$ , for every  $n \geq N$ , be the increasing and continuous map given by the restriction of  $g_n$  to  $[-T, +\infty)$ .

By Lemma 6.5.4 for  $I = [-T, +\infty)$  it follows that there exists  $h_T : [-T, +\infty) \rightarrow \mathbb{R}$  increasing and continuous such that  $h(0) = 0$  and such that  $d_s(\varphi_t^c(z_\infty), \varphi_{h_T(t)}^c(w_\infty)) \leq \epsilon_0$  for every  $t \in [-T, +\infty)$ .

Moreover, Lemma 6.5.4 tells us that  $h_T(t)$  is equal to  $\lim_n H_n(t)$  for every  $t \in [-T, +\infty)$ . This shows that if one performs the above construction for another  $T' > 0$  such that  $T' > T$  then the map  $h_{T'}$  coincides with  $h_T$  in  $[-T, \infty)$ . As a consequence, there exists a well defined increasing and continuous map  $h : \mathbb{R} \rightarrow \mathbb{R}$  (given by  $h(t) = h_T(t)$  for every  $T > 0$  such that  $t \geq -T$ ) such that  $h(0) = 0$  and such that  $d_s(\varphi_t^c(z_\infty), \varphi_{h(t)}^c(w_\infty)) \leq \epsilon_0$  for

every  $t \in \mathbb{R}$ . This contradicts that the flow  $\varphi_t^c$  is  $\epsilon_0$ -expansive by Proposition 6.4.7.

□

Using the previous lemmas one can show the following.

**Lemma 6.5.6.** *Suppose  $x \in Y$ . For every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\mathcal{W}_\delta^s(\varphi_t(x)) \subset S_\epsilon(\varphi_t(x))$  for every  $t \leq 0$  and  $\mathcal{W}_\delta^u(\varphi_t(x)) \subset U_\epsilon(\varphi_t(x))$  for every  $t \geq 0$ .*

*Proof.* Suppose  $\epsilon > 0$ . Given  $x \in Y$ , let us see that there exists  $\delta > 0$  such that  $\mathcal{W}_\delta^s(\varphi_t^c(x))$  is contained in  $S_\epsilon(\varphi_t^c(x))$  for every  $t \leq 0$ . To show that there exists  $\delta > 0$  so that  $\mathcal{W}_\delta^s(\varphi_t^c(x))$  is contained in  $U_\epsilon(\varphi_t^c(x))$  for every  $t \geq 0$  one can argue analogously.

Suppose by contradiction that there exists  $\delta_n > 0$  converging to 0 with  $n$  and  $T_n \geq 0$  so that the point  $x_n := \varphi_{-T_n}^c(x)$  satisfies that  $\mathcal{W}_{\delta_n}^s(x_n)$  is not contained in  $S_\epsilon(x_n)$ . Let  $z_n \in \mathcal{W}_{\delta_n}^s(x_n)$  be such that  $z_n$  is not in  $S_\epsilon(x_n)$ . Without loss of generality, let us assume that  $\epsilon \leq \epsilon_0$ .

Since  $x \in Y$  it follows from Lemma 6.5.3 that there exists  $\delta_x > 0$  such that  $\mathcal{W}_{\delta_x}^s(x) \subset S_\epsilon(x)$ . Let  $\delta > 0$  be a constant such that  $\delta < \delta_x$ .

For every  $n$  let  $\gamma_n \subset \mathcal{W}_{\delta_n}^s(x_n)$  be an arc (embedding of  $[0, 1]$ ) joining  $x_n$  with  $z_n$ . Let us fix for the points in  $\gamma_n$  the following order: two points  $w, w' \in \gamma_n$  satisfy that  $w \leq_{\gamma_n} w'$  if and only if  $w$  lies in the subsegment of  $\gamma_n$  joining  $x_n$  with  $w'$ .

By the continuity of the flow  $\varphi_t^c$ , every  $w \in \gamma_n$  close enough to  $x_n$  satisfies the following property (called property  $(\mathcal{P})$  from now on): There exists  $h_w : [0, T_n] \rightarrow \mathbb{R}$  increasing and continuous such that  $h_w(0) = 0$  and  $d_s(\varphi_t^c(x_n), \varphi_{h_w(t)}^c(w)) \leq \delta$  for every  $t \in [0, T_n]$ .

We claim that property  $(\mathcal{P})$  is a closed property in  $\gamma_n$ . Indeed, suppose that  $(w_k)$  is a sequence in  $\gamma_n$  converging to  $w \in \gamma_n$  and that every point in  $w_k$  satisfies property  $(\mathcal{P})$ . Then by Lemma 6.5.4 (applied to the pair of sequences in  $k$  which are the constant sequence  $x_n$  and the sequence  $(w_k)$ ) it follows that there exists  $h_w : [0, T_n] \rightarrow \mathbb{R}$  increasing and continuous so that  $w$  satisfies property  $(\mathcal{P})$ . This proves the claim.

Moreover, given  $w \in \gamma_n$  that satisfies property  $(\mathcal{P})$ , one has that in particular  $d_s(\varphi_{T_n}^c(x_n), \varphi_{h_w(T_n)}^c(w)) \leq \delta$ . Then from  $\delta < \delta_x$  and  $\mathcal{W}_{\delta_x}^s(x) \subset S_\epsilon(x)$  it follows that  $h_n : [0, T_n] \rightarrow \mathbb{R}$  can be extended to  $h_w : [0, +\infty) \rightarrow \mathbb{R}$  so that  $d_s(\varphi_t^c(x_n), \varphi_{h_w(t)}^c(w)) \leq \epsilon$  for every  $t \in [0, +\infty)$ .

Let  $y_n \in \gamma_n$  be the maximum (with respect to the fixed order in  $\gamma_n$ ) of the  $w'$  in  $\gamma_n$  such that every  $w$  in the subsegment of  $\gamma_n$  joining  $x_n$  with  $w'$  satisfies property  $(\mathcal{P})$ . Note that  $y_n$  is a maximum since property  $(\mathcal{P})$  is close

in  $\gamma_n$ . Moreover, note that since  $z_n$  is not in  $S_\epsilon(x_n)$  then by the remark made in the last paragraph it follows that  $y_n \neq z_n$ .

Finally, note that, again by the continuity of the flow  $\varphi_t^c$ , if  $w \in \gamma_n$  satisfies property (P) and  $d_s(\varphi_t^c(x_n), \varphi_{h_w(t)}^c(w))$  is strictly less than  $\delta$  for every  $t \in [0, T_n]$  then every  $w'$  in a neighborhood of  $w$  in  $\gamma_n$  satisfies property (P). As a consequence, it follows from  $y_n$  being a maximum that for some  $t_n \in [0, T_n]$  one has that  $d_s(\varphi_{t_n}^c(x_n), \varphi_{h_n(t_n)}^c(y_n)) = \delta$ .

We have found two sequences  $(x_n)$  and  $(y_n)$  as in the hypothesis of Lemma 6.5.5. This gives us a contradiction. □

*Proof of Proposition 6.5.2.* Suppose  $\epsilon > 0$ . Let  $\bar{x}$  be a point in  $Y$  such that its backwards and forwards orbit by  $\varphi_t^c$  is dense in  $M$  (as pointed out in Remark 6.4.9 the transitivity of  $\varphi_t^c$  guarantees this).

By Lemma 6.5.6 there exists  $\delta > 0$  such that  $\mathcal{W}_\delta^s(y) \subset S_\epsilon(y)$  for every  $y$  in the  $\varphi_t^c$ -backwards orbit of  $\bar{x}$  and  $\mathcal{W}_\delta^u(y) \subset U_\epsilon(y)$  for every  $y$  in the  $\varphi_t^c$ -forwards orbit of  $\bar{x}$ .

Let  $x$  be a point in  $M$ . Let us see that  $\mathcal{W}_\delta^s(x) \subset S_\epsilon(x)$ . To show that  $\mathcal{W}_\delta^u(x) \subset U_\epsilon(x)$  the reasoning is analogous.

Since the backwards orbit by  $\varphi_t^c$  of  $\bar{x}$  is dense in  $M$  there exists  $t_n \xrightarrow{n} -\infty$  so that  $y_n := \varphi_{t_n}(\bar{x})$  converges to  $x$ . Let  $x'$  be a point in  $\mathcal{W}_\delta^s(x)$ . There exists  $y'_n \in \mathcal{W}_\delta^s(y_n)$  converging to  $x'$ .

As  $\mathcal{W}_\delta^s(y_n)$  is a subset of  $S_\epsilon(y_n)$  then for every  $y'_n$  there exists a homeomorphism  $h_n : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$d_s(\varphi_t^c(y_n), \varphi_{h_n(t)}^c(y'_n)) \leq \epsilon$$

for every  $t \in [0, +\infty)$ .

By Lemma 6.5.4 for  $I := [0, +\infty)$  the functions  $h_n$  converge pointwise to an increasing and continuous map  $h : [0, +\infty) \rightarrow \mathbb{R}$  satisfying that

$$d_s(\varphi_t^c(x), \varphi_{h(t)}^c(x')) \leq \epsilon$$

for every  $t \in [0, +\infty)$ . That is,  $x'$  lies in  $S_\epsilon(x)$ . This shows that  $\mathcal{W}_\delta^s(x) \subset S_\epsilon(x)$  as wanted. □

**Remark 6.5.7.** It is worth mentioning that the proofs of Lemma 6.4.10 and Proposition 6.5.2 are the key points where the transitivity of  $\mathcal{W}^c$  is used.

## 6.6 Continuity of $\rho$ in lines of $\mathcal{W}^c$

The goal of this section is to show the following.

**Proposition 6.6.1.** *The function  $\rho$  is continuous at every  $x$  in  $M$  such that  $\mathcal{W}^c(x)$  is a line.*

We will use the next definitions on this section.

**Definition 6.6.2.** We say that  $x$  is a *s-continuity point* of  $\rho$  if there exists  $\delta > 0$  such that  $\rho$  restricted to  $\mathcal{W}_\delta^s(x)$  is continuous. Analogously we define a *u-continuity point* of  $\rho$ .

**Definition 6.6.3** (Center holonomy maps). Suppose  $x \in M$  such that  $\mathcal{W}^c(x)$  is a line. Suppose  $\gamma : [0, 1] \rightarrow [x, f(x)]_c$  is a homeomorphism such that  $\gamma(0) = x$  and  $\gamma(1) = f(x)$ . The following is standard from foliation theory: For every  $\delta > 0$  small enough the *center holonomy map between s-transversals*

$$H_{s,x,\delta}^c : \mathcal{W}_\delta^s(x) \rightarrow \mathcal{W}^s(f(x))$$

is well defined as  $H_{s,x,\delta}^c(y) := \gamma_y(1)$  for every  $y \in \mathcal{W}_\delta^s(x)$ , where  $\gamma_y : [0, 1] \rightarrow M$  is the unique continuous curve such that  $\gamma_y(0) = y$  and  $\gamma_y(t) \in \mathcal{W}^s(\gamma(t))$  for every  $t \in [0, 1]$ . Moreover, the map  $H_{s,x,\delta}^c$  does not depend on the choice of the homeomorphism  $\gamma : [0, 1] \rightarrow [x, f(x)]_c$ .

Analogously one defines  $H_{u,x,\delta}^c : \mathcal{W}_\delta^u(x) \rightarrow \mathcal{W}^u(f(x))$  the *center holonomy map between u-transversals*.

**Remark 6.6.4.** Note that in the above definition the dynamical coherence hypothesis is crucially used.

**Remark 6.6.5.** It is immediate to check that if  $\mathcal{W}^c(x)$  is a line for some  $x \in M$  then  $x$  is a *s-continuity point* of  $\rho$  if and only if  $H_{s,x,\delta}^c$  is equal to the restriction of  $f$  to  $\mathcal{W}_\delta^s(x)$  for some  $\delta > 0$ . Analogously,  $x$  is a *u-continuity point* of  $\rho$  if and only if  $H_{u,x,\delta}^c$  coincides with  $f$  in  $\mathcal{W}_\delta^u(x)$  for some  $\delta > 0$ .

Dynamical coherence allows us to get the following.

**Lemma 6.6.6.** *Suppose  $x \in M$  such that  $\mathcal{W}^c(x)$  is a line. If  $x$  is a *s* and *u-continuity point* of  $\rho$  then  $\rho$  is continuous at  $x$ .*

*Proof.* Let  $\gamma : [0, 1] \rightarrow \mathcal{W}^c(x)$  be a homeomorphism from  $[0, 1]$  to the center segment  $[x, f(x)]_c$  so that  $\gamma(0) = x$  and  $\gamma(1) = f(x)$ . Since  $x$  is a *s* and *u-continuity point* of  $\rho$  there exists  $\delta > 0$  such that  $H_{s,x,\delta}^c(y) = f(y)$  and  $H_{u,x,\delta}^c(z) = f(z)$  for every  $y \in \mathcal{W}_\delta^s(x)$  and  $z \in \mathcal{W}_\delta^u(x)$ . That is, for every  $y \in \mathcal{W}_\delta^s(x)$  there exists  $\gamma_y^s : [0, 1] \rightarrow \mathcal{W}^c(y)$  satisfying  $\gamma_y^s(0) = y$  and  $\gamma_y^s(t) \in \mathcal{W}^s(\gamma(t))$  for every  $t \in [0, 1]$ . Analogously, for every  $z \in \mathcal{W}_\delta^u(x)$  there exists  $\gamma_z^u : [0, 1] \rightarrow \mathcal{W}^c(y)$  satisfying  $\gamma_z^u(0) = z$  and  $\gamma_z^u(t) \in \mathcal{W}^u(\gamma(t))$  for every  $t \in [0, 1]$ .

Recall  $\epsilon_0 > 0$  the small constant fixed in Remark 6.4.6. One can consider  $\delta > 0$  small enough so that for every  $y \in \mathcal{W}_\delta^s(x)$  and  $z \in \mathcal{W}_\delta^u(x)$  one has  $\gamma_y^s(t) \in \mathcal{W}_{\epsilon_0}^s(\gamma(t))$  and  $\gamma_z^u(t) \in \mathcal{W}_{\epsilon_0}^u(\gamma(t))$  for every  $t \in [0, 1]$ .

Suppose  $(x_n)$  is a sequence converging to  $x$  and let  $\delta_n$  denote the distance  $d(x, x_n)$  for every  $n$ . Suppose, modulo subsequence, that  $\delta_n$  is smaller than  $\delta/2$  for every  $n$ . Let  $x_n^s$  be the point of intersection of  $\mathcal{W}_{2\delta_n}^s(x)$  and  $\mathcal{W}_{2\delta_n}^{cu}(x_n)$ . And  $x_n^u$  the point of intersection of  $\mathcal{W}_{2\delta_n}^u(x)$  and  $\mathcal{W}_{2\delta_n}^{cs}(x_n)$ .

For simplicity in the notation, for every  $n$  let  $\gamma_n^s$  and  $\gamma_n^u$  denote the curves  $\gamma_{x_n^s}^s$  and  $\gamma_{x_n^u}^u$ , respectively. It follows that  $\mathcal{W}_{4\delta_n}^{cu}(\gamma_n^s(t))$  intersects  $\mathcal{W}_{4\delta_n}^{cs}(\gamma_n^u(t))$  for every  $t \in [0, 1]$ . Moreover, by dynamical coherence, one can construct a homeomorphism over its image  $\gamma_n : [0, 1] \rightarrow \mathcal{W}^c(x_n)$  such that  $\gamma_n(0) = x_n$  and  $\gamma_n(t)$  lies in the intersection of  $\mathcal{W}_{4\delta_n}^{cu}(\gamma_n^s(t))$  and  $\mathcal{W}_{4\delta_n}^{cs}(\gamma_n^u(t))$  for every  $t \in [0, 1]$ .

Again, by dynamical coherence, it follows from  $f(x_n^s) = \gamma_n^s(1)$  and  $f(x_n^u) = \gamma_n^u(1)$  that  $f(x_n)$  lies in the intersection of  $\mathcal{W}_{4\delta_n}^{cu}(\gamma_n^s(1))$  and  $\mathcal{W}_{4\delta_n}^{cs}(\gamma_n^u(1))$ . One can chose  $\gamma_n$  so that, in addition to the properties from the last paragraph, it satisfies that  $\gamma_n(1) = f(x_n)$ .

Since the sequences  $(x_n^s)$  and  $(x_n^u)$  converge to  $x$  it is immediate that  $\gamma_n^s$  and  $\gamma_n^u$  converge in the  $C^0$  topology to  $\gamma$ . As a consequence, since  $\delta_n$  tends to 0, then  $\gamma_n$  converges in the  $C^0$  topology to  $\gamma$  as well.

The maps  $\gamma_n : [0, 1] \rightarrow \mathcal{W}^c(x_n)$  form a sequence of homeomorphisms over its image joining  $x_n$  with  $f(x_n)$  and converging  $C^0$  to  $\gamma$ . One obtains that  $\lim_n \text{length}(\gamma_n) = \text{length}(\gamma)$ . Since  $\rho(x) = \text{length}(\gamma)$ , then  $\liminf_n \rho(x_n) \leq \rho(x)$ . By Proposition 6.3.7 one has the converse inequality. It follows that  $\lim_n \rho(x_n) = \rho(x)$ . This shows that  $x$  is a continuity point of  $\rho$ . □

The following is a key lemma relating stable and unstable sets of  $\varphi_t^c$  with  $s$  and  $u$ -continuity points.

**Lemma 6.6.7.** *Suppose  $x \in M$  such that  $\mathcal{W}^c(x)$  is a line. If for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\mathcal{W}_\delta^s(x) \subset S_\epsilon(x)$  then  $x$  is a  $s$ -continuity point of  $\rho$ . Analogously, if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\mathcal{W}_\delta^u(x) \subset U_\epsilon(x)$  then  $x$  is a  $u$ -continuity point of  $\rho$ .*

*Proof.* Suppose that for every  $\epsilon > 0$  there exists  $\delta_\epsilon > 0$  such that  $\mathcal{W}_{\delta_\epsilon}^s(x) \subset S_\epsilon(x)$ . Let us see that  $x$  needs to be a  $s$ -continuity point of  $\rho$ . The symmetric statement for unstable local leaves of  $f$  and backwards stable sets of  $\varphi_t^c$  follows by analogous reasons.

Let  $\delta > 0$  be such that  $H_{s,x,\delta}^c : \mathcal{W}_\delta^s(x) \rightarrow \mathcal{W}^s(f(x))$  is well defined. Then  $H_{s,x,\delta'}^c : \mathcal{W}_{\delta'}^s(x) \rightarrow \mathcal{W}^s(f(x))$  is also well defined for every  $0 < \delta' < \delta$ . By Remark 6.6.5 it is enough to show that  $H_{s,x,\delta'}^c = f|_{\mathcal{W}_{\delta'}^s(x)}$  for some  $\delta' > 0$ .



Consider a sequence  $\epsilon_n > 0$ ,  $n \geq 1$ , converging to 0 and let  $\delta_n > 0$  be such that  $\mathcal{W}_{\delta_n}^s(x) \subset S_{\epsilon_n}(x)$  for every  $n$ . Without loss of generality suppose  $\delta_n$  is smaller than  $\delta$  for every  $n$ . Then  $H_{s,x,\delta_n}^c : \mathcal{W}_{\delta_n}^s(x) \rightarrow \mathcal{W}^s(f(x))$  is well defined. Suppose also that  $\epsilon_n$  is, for every  $n$ , smaller than the constant  $\epsilon_0 > 0$  fixed in Remark 6.4.6.

Suppose by contradiction that  $f$  restricted to  $\mathcal{W}_{\delta_n}^s(x)$  is different from  $H_{s,x,\delta_n}^c$  for every  $n$ . Let  $y_n \in \mathcal{W}_{\delta_n}^s(x)$  be such that  $H_{s,x,\delta_n}^c(y_n) \neq f(y_n)$ .

Recall that a leaf  $W$  of  $\mathcal{W}^s$  can intersect at most one compact leaf of  $\mathcal{W}^c$ , and that this intersection can happen in at most one point. Since the sequence  $y_n$  lies in  $\mathcal{W}_{\delta_n}^s(x)$  it follows that  $\mathcal{W}^c(y_n)$  is compact for at most one of the  $y_n$ . Modulo subsequence, we can suppose that  $\mathcal{W}^c(y_n)$  is a line for every  $n$ .

Since  $y_n$  lies in  $S_{\epsilon_n}(x)$  for every  $n$  one can consider a homeomorphism  $H_n : [0, +\infty) \rightarrow [0, +\infty)$  such that  $d(\varphi_t^c(x), \varphi_{H_n(t)}^c(y_n)) \leq \epsilon_n$  for every  $t \in [0, +\infty)$ . The points  $\varphi_t^c(x)$  and  $\varphi_{H_n(t)}^c(y_n)$  may not be in a same leaf of  $\mathcal{W}^s$ . However by dynamical coherence they lie in the same leaf of  $\mathcal{W}^{cs}$  and  $d_{cs}(\varphi_t^c(x), \varphi_{H_n(t)}^c(y_n)) \leq 2\epsilon_n$  for every  $t \in [0, +\infty)$ .

It follows that one can consider for every  $n$  the well-defined homeomorphism  $h_n : [0, +\infty) \rightarrow [0, +\infty)$  satisfying that  $\varphi_{h_n(t)}^c(y_n)$  is equal to the intersection of  $\mathcal{W}_{4\epsilon_n}^s(\varphi_t^c(x))$  and  $\mathcal{W}_{4\epsilon_n}^c(\varphi_{H_n(t)}^c(y_n))$  for every  $t \in [0, +\infty)$ . In particular,  $\varphi_{h_n(t)}^c(y_n)$  lies in  $\mathcal{W}^s(\varphi_t^c(x))$  and  $d_s(\varphi_t^c(x), \varphi_{h_n(t)}^c(y_n)) \leq 4\epsilon_n$  for every  $t \in [0, +\infty)$ .

Note that from the above construction one obtains that  $H_{s,x,\delta_n}^c(y_n)$  is equal to  $\varphi_{h_n(\rho(x))}^c(y_n)$  for every  $n$ . Consider  $t_n > 0$  such that  $\varphi_{h_n(t_n)}^c(y_n) = f(y_n)$ . The fact that  $H_{s,x,\delta_n}^c(y_n)$  is different from  $f(y_n)$  translates to the fact that  $t_n$  is larger than  $\rho(x)$  for every  $n$ .

Since  $\mathcal{W}^c(x)$  is a line it follows that  $\varphi_{t_n}^c(x)$  is different from  $f(x)$  for every  $n$ . Since  $\varphi_{t_n}^c(x)$  is a point in  $\mathcal{W}_{\epsilon_n}^s(x)$  and the sequence  $(\epsilon_n)$  converges to 0 it follows that  $f(x)$  that the points  $\varphi_{t_n}^c(x)$  converge to  $f(x)$  with  $n$ . That is,  $f(x)$  is accumulated by points lying in  $\mathcal{W}^c(x) \cap (\mathcal{W}_{\epsilon_0}^s(f(x)) \setminus f(x))$ . Since  $\mathcal{W}^c(x) = \mathcal{W}^c(f(x))$  one obtains that  $f(x)$  is a  $s$ -recurrent point. This contradicts Proposition 6.4.4.  $\square$

We are now able to show Proposition 6.6.1.

*Proof of Proposition 6.6.1.* Suppose  $x \in M$  such that  $\mathcal{W}^c(x)$  is a line. By Proposition 6.5.2 it follows that for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\mathcal{W}_{\delta}^s(x) \subset S_{\epsilon}(x)$  and  $\mathcal{W}_{\delta}^u(x) \subset U_{\epsilon}(x)$ . By Lemma 6.6.7 one obtains that  $x$  is a  $s$  and  $u$ -continuity point of  $\rho$ . By Lemma 6.6.6 the function  $\rho$  is continuous at  $x$ .  $\square$

## 6.7 The function $\rho$ is bounded in $M$

This section ends the proof of Theorem 6.1.2. Up until now we have shown that the function  $\rho$  is continuous at every  $x$  in  $M$  such that  $\mathcal{W}^c(x)$  is a line. We will see in this section how to show from this that  $\rho$  is bounded in  $M$ .

Given  $\mathcal{C} \in \mathcal{W}^c$  a circle leaf it follows from Lemma 6.4.2 that  $\mathcal{W}^c(y)$  is a line for every  $y$  in  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$ . In particular, the center segment  $[y, f(y)]_c$  is well defined for every  $y \in \mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  and by Proposition 6.6.1 the function  $\rho$  is continuous in restriction to  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$ .

Recall that the stable saturation  $\mathcal{W}^s(\mathcal{C})$  of  $\mathcal{C}$  is contained in the center-stable leaf  $\mathcal{W}^{cs}(\mathcal{C})$  but that a priori  $\mathcal{W}^s(\mathcal{C})$  may be a proper subset of  $\mathcal{W}^{cs}(\mathcal{C})$  (the *completeness* problem).

**Lemma 6.7.1.** *Suppose  $\mathcal{C} \in \mathcal{W}^c$  is a circle leaf. For every  $y \in \mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  one has that  $[y, f(y)]_c$  is contained in  $\mathcal{W}^s(\mathcal{C})$ .*

*Proof.* Since  $\rho$  is continuous in restriction to  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  one has that for every  $y \in \mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  the compact center segment  $[y, f(y)]_c$  varies continuously with  $y$  in the Hausdorff topology.

Let  $A$  denote the set of points  $y \in \mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  such that  $[y, f(y)]_c$  is contained in  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$ . Let  $B$  denote its complement in  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  so that  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  is equal to the disjoint union  $A \cup B$ . The goal is to show that  $B$  is empty.

Let  $\partial_{cs}\mathcal{W}^s(\mathcal{C}) \subset \mathcal{W}^{cs}(\mathcal{C})$  denote the boundary of  $\mathcal{W}^s(\mathcal{C})$  in  $\mathcal{W}^{cs}(\mathcal{C})$ . Since  $\mathcal{W}^s(\mathcal{C})$  is saturated by leaves of  $\mathcal{W}^s$  the set  $\partial_{cs}\mathcal{W}^s(\mathcal{C})$  is a union of leaves of  $\mathcal{W}^s$ . It follows that  $y \in \mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  is in  $B$  if and only if  $[y, f(y)]_c \cap \partial_{cs}\mathcal{W}^s(\mathcal{C}) \neq \emptyset$ .

Since  $y \mapsto [y, f(y)]_c$  varies continuously with  $y$  in  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  it is immediate to check that both  $A$  and  $B$  are open subsets of  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  (for this, note that if  $y \in B$  then  $[y, f(y)]_c$  is transverse to  $\mathcal{W}^s(f(y))$  at  $f(y)$ ).

As  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  is the union of the disjoint open sets  $A$  and  $B$  it follows that  $A$  and  $B$  comprise whole connected components of  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$ . Note that if  $\dim(E^s) \geq 2$  then  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  has only one connected component and if  $\dim(E^s) = 1$  it may have two. We will cover both scenarios simultaneously.

Suppose by contradiction that  $B$  is not empty. For every  $y \in B$  the center segment  $[y, f(y)]_c$  intersects  $\partial_{cs}\mathcal{W}^s(\mathcal{C})$ . Since  $\mathcal{W}^s(\mathcal{C})$  is  $f$ -invariant it follows that  $f^{-1} \circ [y, f(y)]_c = [f^{-1}(y), y]_c$  also intersects  $\partial_{cs}\mathcal{W}^s(\mathcal{C})$ . One can then consider  $y^+$  and  $y^-$  the ‘first time’ that  $\mathcal{W}^c(y)$  leaves  $\mathcal{W}^s(\mathcal{C})$  in both directions. That is,  $y^-$  and  $y^+$  are the only points in  $\mathcal{W}^c(y) \cap \partial_{cs}\mathcal{W}^s(\mathcal{C})$  such that there exists a center segment  $(y^-, y^+)_c$  contained in  $\mathcal{W}^s(\mathcal{C})$  and satisfying  $y \in (y^-, y^+)_c$ . In other words,  $(y^-, y^+)_c$  is the connected component of  $\mathcal{W}^c(y) \cap \mathcal{W}^s(\mathcal{C})$  containing  $y$ .

It is immediate to check (by transversality again) that the functions  $y \mapsto y^+$  and  $y \mapsto y^-$  are continuous from  $B$  to  $\partial_{cs}\mathcal{W}^s(\mathcal{C})$ . Moreover, if  $z$  is a point

in  $\partial_{cs}\mathcal{W}^s(\mathcal{C})$  that is in the image of  $y \mapsto y^+$ , then every  $z' \in \mathcal{W}^s(z)$  need to be also in the image of  $y \mapsto y^+$ . This is because, by stable holonomy, one can transport  $(y, y^+)_c$  to a center segment in  $\mathcal{W}^s(\mathcal{C})$  such that one of its endpoints is  $z'$ .

Moreover, for every  $y \in B$  it is immediate to check that there exists  $\epsilon > 0$  such that  $w^+$  lies in  $\mathcal{W}^s(y^+)$  for every  $w \in \mathcal{W}_\epsilon^{cs}(y)$ . That is, the function that assigns to every  $y \in B$  the stable leaf  $\mathcal{W}^s(y^+)$  is locally constant. Combined with the information from the previous paragraph one obtains that the image by  $y \mapsto y^+$  of  $B$  is exactly one or two leaves of  $\mathcal{W}^s$ , whether  $B$  has one or two connected components, respectively.

Let  $V$  be one of the leaves of  $\mathcal{W}^s$  in the image of  $y \mapsto y^+$ . Since  $B$  is  $f$ -invariant and has at most two connected components then  $V$  is invariant by  $f^2$ . It follows that  $f^2$  induces a contraction in  $V$ . As a consequence  $f^2$  has a fixed point in  $V$  and this fixed point is unique. Let  $y_0 \in B$  be such that  $y_0^+$  is the fixed point of  $f^2$  in  $V$ .

On the one hand, since  $\mathcal{W}^s(\mathcal{C})$  and  $\partial_{cs}\mathcal{W}^s(\mathcal{C})$  are  $f$ -invariant then the image of  $[y_0^-, y_0^+]_c$  by  $f^2$  is a center segment whose interior lies in  $\mathcal{W}^s(\mathcal{C})$  and its end-points lie in  $\partial_{cs}\mathcal{W}^s(\mathcal{C})$ . Since  $y_0^+$  is fixed by  $f^2$  it follows that  $[y_0^-, y_0^+]_c$  is invariant by  $f^2$ .

On the other hand,  $[y_0^-, y_0^+]_c$  contains the point  $y_0$  which is a point in  $\mathcal{W}^s(x_0)$  for some  $x_0 \in \mathcal{C}$ . By iterating forwards by  $f^2$  one obtains that the orbit of  $y_0$  needs to get arbitrarily close to  $\mathcal{C}$ . Since  $[y_0^-, y_0^+]_c$  is  $f^2$ -invariant this contradicts the fact that  $[y_0^-, y_0^+]_c$  and  $\mathcal{C}$  are disjoint compact sets that are at a positive distance from each other. This shows that the set  $B$  needs to be empty and ends the proof of the lemma.  $\square$

**Lemma 6.7.2.** *Suppose  $\mathcal{C} \in \mathcal{W}^c$  is a circle leaf. There exists  $\delta > 0$  such that  $\rho$  restricted to  $\mathcal{W}_\delta^s(\mathcal{C})$  is bounded.*

*Proof.* The set  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  has one or two connected components. Without loss of generality let us suppose that it has one. Otherwise, one should only repeat the argument below separately on each connected component.

Suppose  $y$  in  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$ . By Lemma 6.4.2 the leaf  $\mathcal{W}^s(y)$  intersects  $\mathcal{C}$  in a unique point. Let us call it  $p^s y$ .

Let  $\gamma_y : [0, 1] \rightarrow \mathcal{W}^c(y)$  be the  $C^1$  curve of constant speed such that  $\gamma_y(0) = y$  and  $\gamma_y(1) = f(y)$ . By Lemma 6.7.1 the center segment  $[y, f(y)]_c$  is contained in  $\mathcal{W}^s(\mathcal{C})$ . It follows that there exists  $p^s \gamma_y : [0, 1] \rightarrow \mathcal{C}$  the (unique) continuous curve such that  $p^s \gamma_y(0) = p^s y$  and  $\gamma_y(t) \in \mathcal{W}^s(p^s \gamma_y(t))$  for every  $t \in [0, 1]$ .

By Proposition 6.6.1 the function  $\rho$  is continuous in restriction to  $\mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$ . That is,  $\gamma_y$  varies continuously with  $y \in \mathcal{W}^s(\mathcal{C}) \setminus \mathcal{C}$  in the  $C^1$  topology. At the same time, if  $y'$  varies continuously in  $\mathcal{W}^s(p^s y) \setminus p^s y$  one has that  $f(y')$  varies

continuously in  $\mathcal{W}^s(f(p^s y)) \setminus f(p^s y)$ . One obtains that  $p^s \gamma_y$  needs to be a reparametrization of  $p^s \gamma_{y'}$  for every  $y'$  in  $\mathcal{W}^s(y)$ .

Given  $x$  in  $\mathcal{C}$  and  $y \in \mathcal{W}^s(x) \setminus x$  let  $\gamma_x : [0, 1] \rightarrow \mathcal{C}$  be the constant speed reparametrization of  $p^s \gamma_y$ . From the above paragraph one has that the definition of  $\gamma_x$  is independent of the point  $y$  one chooses in  $\mathcal{W}^s(x) \setminus x$ .

It is now immediate to check that  $z \mapsto \gamma_z$  varies continuously in the  $C^1$  topology as  $z$  varies in  $\mathcal{W}^s(\mathcal{C})$ . Since this implies that  $z \mapsto \text{length}(\gamma_z)$  varies continuously with  $z \in \mathcal{W}^s(\mathcal{C})$  it follows that  $\rho$  is continuous in a neighborhood of  $\mathcal{C}$  in  $\mathcal{W}^s(\mathcal{C})$ . □

**Proposition 6.7.3.** *For every circle leaf  $\mathcal{C} \in \mathcal{W}^c$  there exists a neighborhood  $U$  of  $\mathcal{C}$  such that  $\rho|_U : U \rightarrow \mathbb{R}$  is bounded.*

*Proof.* Let  $\mathcal{C} \in \mathcal{W}^c$  be a circle leaf. It follows from Lemma 6.7.2 that there exists  $\delta > 0$  and  $L > 0$  such that  $\rho$  restricted to  $\mathcal{W}_\delta^s(\mathcal{C})$  is bounded by  $L$ .

From the regularity of  $\mathcal{W}^c$  one can define an unstable holonomy along center transversals as follows: There exists  $\delta_L > 0$  such that, if  $y$  is a point in  $M$  and  $z$  a point in  $\mathcal{W}_{\delta_L}^u(y)$ , then for every curve  $\gamma : [0, 1] \rightarrow \mathcal{W}^c(y)$  such that  $\gamma(0) = y$  and  $\text{length} \gamma \leq L$  there exists a unique curve  $p^u \gamma : [0, 1] \rightarrow \mathcal{W}^c(z)$  given by  $p^u \gamma(0) = z$  and  $p^u \gamma(t) \in \mathcal{W}^u(\gamma(t))$  for every  $t \in [0, 1]$ , and this curve satisfies that  $\text{length} p^u \gamma \leq 2L$ .

Let us see that  $\rho$  is bounded by  $2L$  in  $U = \mathcal{W}_{\delta_L}^u(\mathcal{W}_\delta^s(\mathcal{C}))$ . Since  $U$  is a neighborhood of  $\mathcal{C}$  this will show the proposition.

Given  $z$  in  $U$  there exists  $x \in \mathcal{C}$  and  $y \in \mathcal{W}_\delta^s(x)$  such that  $z \in \mathcal{W}_{\delta_L}^u(y)$ . Let us suppose first that for every  $w$  in  $\mathcal{W}_{\delta_L}^u(y)$  the center leaf  $\mathcal{W}^c(w)$  is not compact. We can join then  $y$  with  $z$  by a curve  $\eta : [0, 1] \rightarrow \mathcal{W}_{\delta_L}^u(y)$  satisfying that  $\eta(0) = y$ ,  $\eta(1) = z$  and  $\mathcal{W}^c(\eta(s))$  is a line for every  $s \in [0, 1]$ .

By Proposition 6.6.1 one has that  $\rho$  is continuous at every point in the image of  $\eta$ . Let  $\gamma : [0, 1] \rightarrow \mathcal{W}^c(y)$  be a homeomorphism from  $[0, 1]$  to  $[y, f(y)]_c$  such that  $\gamma(0) = y$  and  $\gamma(1) = f(y)$ . Since  $\rho$  is continuous in the image of  $\eta$  it follows that for every  $s \in [0, 1]$  there exists  $\gamma_s : [0, 1] \rightarrow \mathcal{W}^c(\eta(s))$  joining  $\eta(s) = \gamma_s(0)$  and  $f(\eta(s)) = \gamma_s(1)$ , and satisfying that  $\gamma_0 = \gamma$  and  $\gamma_s(t) \in \mathcal{W}^u(\gamma(t))$  for every  $t \in [0, 1]$ . In particular,  $f(z) = \gamma_1(1)$ .

One has that  $\text{length} \gamma \leq L$  since  $\rho$  is bounded by  $L$  in  $\mathcal{W}_\delta^s(\mathcal{C})$ . Then by the election of  $\delta_L$  it follows that  $\text{length} \gamma_1 \leq 2L$ . Since  $\gamma_1$  is a curve in  $\mathcal{W}^c(z)$  joining  $z = \gamma_1(0)$  with  $f(z) = \gamma_1(1)$  one obtains that  $\rho(z) \leq 2L$ .

In case  $\mathcal{W}_{\delta_L}^u(y)$  intersects a compact leaf of  $\mathcal{W}^c$  one can argue as follows. By Lemma 6.3.1 all but countably many  $y' \in \mathcal{W}_\delta^s(x)$  satisfy that  $\mathcal{W}_{\delta_L}^u(y')$  does not intersect a compact leaf of  $\mathcal{W}^c$ . One can consider then  $(y_n)$  a sequence in  $\mathcal{W}_\delta^s(x)$  converging to  $y$  such that for every  $w \in \mathcal{W}_{\delta_L}^u(y_n)$  the center leaf

$\mathcal{W}^c(w)$  is not compact. And consider  $z_n \in \mathcal{W}_{\delta_L}^u(y_n)$ , for every  $n$ , so that the sequence  $(z_n)$  converges to  $z$ .

By the arguments above one has that  $\rho(z_n) \leq 2L$  for every  $n$ . By the semicontinuity of  $\rho$  (see Proposition 6.3.7) it follows that  $\rho(z) \leq 2L$ .  $\square$

As pointed out in Remark 6.3.6, the following ends the proof of Theorem 6.1.2 as a consequence of Proposition 3.3.1.

**Corollary 6.7.4.** *The function  $\rho$  is bounded in  $M$*

*Proof.* Suppose  $x \in M$ . If  $\mathcal{W}^c(x)$  is a line then by Proposition 6.6.1 the function  $\rho$  is continuous at  $x$ . In particular, it is bounded in a neighborhood of  $x$ . If  $\mathcal{W}^c(x)$  is a circle, then by Proposition 6.7.3 the function  $\rho$  is bounded on a neighborhood of  $\mathcal{W}^c(x)$ . By compactness of  $M$  one obtains that  $\rho$  is bounded in  $M$ .  $\square$

By now, it is worth also pointing out the following.

**Remark 6.7.5.** Note that if  $f$  is a center fixing map and  $x$  is a point whose  $f$ -orbit is dense in  $M$  then  $\mathcal{W}^c(x)$  is also dense in  $M$ . Thus the hypothesis ‘ $\mathcal{W}^c$  transitive’ in Theorem 6.1.2 can be replaced by ‘ $f$  transitive’.

## 6.8 Compact center foliations and center fixing in dimension 3

The goal of this section is to show Theorem 6.1.3 stated in the introduction of this chapter.

By Theorem 6.1.2 it is enough show dynamical coherence (Proposition 6.8.1 below) in order to show Theorem 6.1.3 item (1). Alternatively, it is worth mentioning that once dynamical coherence is shown one can also use [BW05, Theorem 2] (see Remark 3.8.2). Indeed, by Lemma 6.8.3 there exists at least one compact leaf  $\gamma$  of  $\mathcal{W}^c$  (in fact, the union of such leaves is dense in  $M$ ). Moreover, for every  $x \in \mathcal{W}_{loc}^s(\gamma)$  the leaf  $\mathcal{W}^c(x)$  is fixed by  $f$  (in particular, periodic). By [BW05, Theorem 2] (see Remark 3.8.2) one obtains that  $f^n$  is a discretized Anosov flow for some  $n > 0$ . Then by Lemma 6.3.2 the map  $f$  is itself a discretized Anosov flow.

To show Theorem 6.1.3 item (2) one argues as follows. Once dynamical coherence is proved it follows from [DMM20] (also [G12] in case  $E^c$  is uniquely integrable) that the center foliation  $\mathcal{W}^c$  is uniformly compact. Then by [B13] one concludes that, modulo double cover,  $(f, \mathcal{W}^c)$  is a partially hyperbolic skew product. Alternatively, one could try to use [BW05, Theorem 1].

Recall that the *non-wandering set* of  $f$ , denoted by  $\Omega(f)$ , is the set of all  $x$  in  $M$  such that for every neighborhood  $U$  of  $x$  there exists  $k > 0$  satisfying  $f^k(U) \cap U \neq \emptyset$ . It is immediate to check that if  $f$  is transitive then  $\Omega(f) = M$ .

In conclusion, the above discussion justifies that Theorem 6.1.3 is proven once the following proposition has been shown.

**Proposition 6.8.1.** *Suppose  $f \in \text{PH}_{c=1}(M^3)$  with  $\Omega(f) = M^3$  admits an invariant center foliation  $\mathcal{W}^c$  satisfying one of the following conditions:*

1.  $f(W) = W$  for every  $W \in \mathcal{W}^c$ .
2.  $W$  is compact for every  $W \in \mathcal{W}^c$ .

*Then  $f$  is dynamically coherent with invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .*

The proof of Proposition 6.8.1 occupies the rest of this section and is done in the next two lemmas.

**Lemma 6.8.2.** *Suppose  $f \in \text{PH}_{c=1}(M^3)$  admits an  $f$ -invariant center foliation  $\mathcal{W}^c$ . Suppose that the set  $\{W \in \mathcal{W}^c \mid W \text{ compact and } f^n(W) = W \text{ for some } n \neq 0\}$  is dense in  $M$ . Then  $f$  is dynamically coherent and admits  $f$ -invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .*

*Proof.* We first claim that it is enough to show that there exists  $\delta > 0$  so that for every  $x$  and  $y$  in  $M$ , if  $y \in \mathcal{W}_\delta^s(x)$  then  $\mathcal{W}_\delta^c(y) \subset \mathcal{W}_{2\delta}^s(\mathcal{W}_{2\delta}^c(x))$ . And if  $y \in \mathcal{W}_\delta^u(x)$  then  $\mathcal{W}_\delta^c(y) \subset \mathcal{W}_{2\delta}^u(\mathcal{W}_{2\delta}^c(x))$ .

Indeed, suppose such a  $\delta$  exists. For every  $x \in M$  one can define  $\mathcal{W}^{cs}(x)$  as the set of all points in  $M$  that can be joined to  $x$  by a finite concatenation of  $\mathcal{W}^s$  and  $\mathcal{W}^c$  arcs. In this way  $\{\mathcal{W}^{cs}(x) : x \in M\}$  defines a partition of  $M$ . On each element of this partition one can consider the distance  $d(y, z) := \inf_\gamma \text{length}(\gamma)$  where  $\gamma$  varies among all finite concatenations of  $\mathcal{W}^s$  and  $\mathcal{W}^c$  arcs joining  $y$  to  $z$ .

By shrinking  $\delta$ , if necessary, one can ensure by Lemma 2.0.1 that the set  $\mathcal{W}_{2\delta}^s(\mathcal{W}_{2\delta}^c(x))$  is a  $C^1$  submanifold tangent to  $E^s \oplus E^c$  for every  $x \in M$ . As  $\mathcal{W}_\delta^c(y)$  is contained in  $\mathcal{W}_{2\delta}^s(\mathcal{W}_{2\delta}^c(x))$  for every  $y \in \mathcal{W}_\delta^s(x)$  it follows that for some  $\epsilon, \epsilon' > 0$  independent of  $x$  the ball  $B_\epsilon(x) \subset \mathcal{W}^{cs}(x)$  with respect to  $d$  is an open subset of the  $C^1$  submanifold  $\mathcal{W}_{2\delta}^s(\mathcal{W}_{2\delta}^c(x))$  that contains the ball  $B_{\epsilon'}(x) \subset \mathcal{W}_{2\delta}^s(\mathcal{W}_{2\delta}^c(x))$  with respect to the inner metric in  $\mathcal{W}_{2\delta}^s(\mathcal{W}_{2\delta}^c(x))$  induced by the Riemannian metric of  $M$ .

In this way one obtains that each element of  $\mathcal{W}^{cs}$  is a  $C^1$  submanifold tangent to  $E^s \oplus E^c$ , saturated by  $\mathcal{W}^s$  and  $\mathcal{W}^c$  leaves and whose inner metric is complete. Hence  $\mathcal{W}^{cs}$  is an  $f$ -invariant center-stable foliation. Analogously

one constructs  $\mathcal{W}^{cu}$  an  $f$ -invariant center-unstable foliation. The property  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  follows immediately. This proves the claim.

It remains to show that there exists  $\delta > 0$  such that for every  $x, y \in M$  with  $y \in \mathcal{W}_\delta^s(x)$  then  $\mathcal{W}_\delta^c(y) \subset \mathcal{W}_{2\delta}^s(\mathcal{W}_{2\delta}^c(x))$ . For  $cu$  discs the arguments are analogous.

The key point to note is that two distinct leaves of  $\mathcal{W}^c$  that are compact and periodic can not intersect the same leaf of  $\mathcal{W}^s$ . Indeed, suppose by contradiction that two such leaves  $W, W' \in \mathcal{W}^c$  contain points  $x \in W$  and  $y \in W'$  that belong to the same leaf of  $\mathcal{W}^s$ . One can consider  $N > 0$ , a multiple of the periods of  $W$  and  $W'$ , so that  $f^N(W) = W$  and  $f^N(W') = W'$ . On the one hand,  $d(f^{kN}(x), f^{kN}(y))$  tends to 0 as  $k \rightarrow +\infty$  because  $x$  and  $y$  belong to the same stable leaf. On the other hand,  $f^{kN}(x)$  lies in  $W$  and  $f^{kN}(y)$  in  $W'$  so for every  $k > 0$  the distance between both points can not be smaller than the positive distance between the disjoint compact sets  $W$  and  $W'$ . This gives us a contradiction.

Consider from now on a metric in  $M$  and  $\delta > 0$  small enough so that the bundles  $E^s$ ,  $E^c$  and  $E^u$  are almost constant and pairwise orthogonal at scale  $\delta$ . For a precise construction of such a metric and constant see for example Lemma 4.2.1. And consider  $\delta > 0$  small enough so that by Lemma 2.0.1 the set  $\mathcal{W}_{4\delta}^u(\mathcal{W}_{4\delta}^c(x))$  is a  $C^1$  submanifold tangent to  $E^c \oplus E^u$  for every  $x \in M$ . In particular, let us consider the above so that for every  $x, y \in M$  such that  $d(x, y) < 3\delta$  the set  $\mathcal{W}_{4\delta}^s(y)$  intersects  $\mathcal{W}_{4\delta}^u(\mathcal{W}_{4\delta}^c(x))$  and this intersection point is unique.

For every  $y$  such that  $d(x, y) < 3\delta$  let  $\pi_x^s(y)$  denote the intersection of  $\mathcal{W}_{4\delta}^s(y)$  with  $\mathcal{W}_{4\delta}^u(\mathcal{W}_{4\delta}^c(x))$ . It is immediate to check that  $\pi^s(y)$  varies continuously with  $y$ . For every  $x \in M$  let  $D(x)$  denote  $\mathcal{W}_{2\delta}^u(\mathcal{W}_{2\delta}^c(x))$ . By Lemma 2.0.1 it is a  $C^1$  disc tangent to  $E^c \oplus E^u$  for every  $x \in M$ . The set  $D(x) \setminus \mathcal{W}_{2\delta}^c(x)$  has two connected components. Let us denote them by  $D^+(x)$  and  $D^-(x)$ .

Suppose by contradiction that there exists  $x_0, y_0 \in M$  such that  $y_0$  lies in  $\mathcal{W}_\delta^s(x_0)$  and  $\mathcal{W}_\delta^c(y_0)$  is not contained in  $\mathcal{W}_{2\delta}^s(\mathcal{W}_{2\delta}^c(x_0))$ . Then there exists  $z_0 \in \mathcal{W}_\delta^c(y_0)$  such that  $\pi^s(z_0)$  is not in  $\mathcal{W}_{2\delta}^c(x_0)$ . Suppose without loss of generality that  $\pi^s(z_0)$  lies in  $D^+(x_0)$ . See Figure 6.1.

On the one hand, there exists  $\epsilon > 0$  small so that  $\pi^s(B_\epsilon(z_0))$  is entirely contained in  $D^+(x_0)$ . On the other hand, since  $\pi^s(y_0) = x_0$  one can consider  $y_1$  as close as wanted to  $y_0$  so that  $\pi^s(y_1)$  lies in  $D^-(x_0)$  and  $\mathcal{W}_\delta^c(y_1)$  intersects  $B_\epsilon(z_0)$ . In particular, for such a  $y_1$  there exists an arc  $\gamma \subset \mathcal{W}_\delta^c(y_1)$  joining  $y_1$  with a point  $z_1 \in B_\epsilon(z_0)$ .

As  $\{W \in \mathcal{W}^c \mid W \text{ compact and } f^n(W) = W \text{ for some } n \neq 0\}$  is dense in  $M$  we can approximate  $\gamma$  and  $\mathcal{W}_{2\delta}^c(x_0)$  by center arcs contained in compact periodic leaves of  $\mathcal{W}^c$ .

By construction  $\pi^s(\gamma)$  is an arc in  $D(x_0)$  joining a point in  $D^+(x_0)$  with



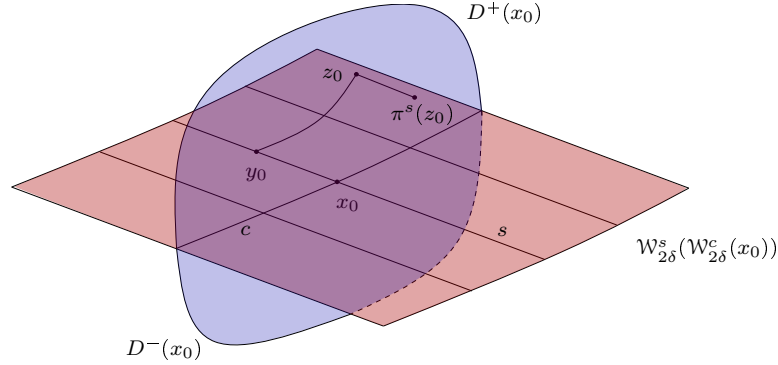


Figure 6.1:

a point in  $D^-(x_0)$ . In particular,  $\pi^s(\gamma)$  intersects  $\mathcal{W}_{2\delta}^c(x_0)$ . One can then approximate  $\gamma$  by an arc  $\gamma'$  contained in a periodic compact leaf of  $\mathcal{W}^c$  so that the  $\pi^s(\gamma')$  continues to satisfy the same property, namely that  $\pi^s(\gamma')$  intersects  $\mathcal{W}_{2\delta}^c(x_0)$  and has each of its endpoints in a different connected component of  $D(x_0) \setminus \mathcal{W}_{2\delta}^c(x_0)$ .

By approximating  $\mathcal{W}_{2\delta}^c(x_0)$  close enough by a center arc  $\eta$  contained in a periodic compact leaf of  $\mathcal{W}^c$  one obtains that  $\pi^s(\gamma')$  and  $\pi^s(\eta)$  must intersect. This gives us a contraction with the aforementioned fact that one can not join two different compact periodic leaves of  $\mathcal{W}^c$  by an arc contained in a leaf of  $\mathcal{W}^s$ .  $\square$

The criterion above combined with the following lemma ends the proof of Proposition 6.8.1.

**Lemma 6.8.3.** *In the setting of Proposition 6.8.1 the set  $\{W \in \mathcal{W}^c \mid W \text{ is compact and } f^n(W) = W \text{ for some } n \neq 0\}$  is dense in  $M$ .*

*Proof.* Note first that the set of fixed points of  $f^2$ , denoted by  $\text{Fix}(f^2) \subset M$ , has empty interior in  $M$ . This follows immediately from the fact that, if  $x$  is a fixed point of  $f^2$ , then every  $y$  in  $\mathcal{W}_{loc}^s(x) \setminus \{x\}$  can not be a fixed point of  $f^2$  because its forward  $f^2$ -orbit must tend to  $x$ .

As a consequence of  $\text{Fix}(f^2)$  having empty interior in  $M$  it is enough to show that  $\{W \in \mathcal{W}^c \mid W \text{ compact and } f^n(W) = W \text{ for some } n \neq 0\}$  is dense in  $M \setminus \text{Fix}(f^2)$ .

Suppose from now on that  $x_0$  is a point in  $M \setminus \text{Fix}(f^2)$ . Let us see that for every  $\epsilon > 0$  small enough there exists  $x \in B_\epsilon(x_0)$  and  $k > 0$  such that  $f(\mathcal{W}_\epsilon^c(x)) \cup f^2(\mathcal{W}_\epsilon^c(x))$  is disjoint from  $\mathcal{W}_\epsilon^c(x)$  and  $f^k(x) \in \mathcal{W}_\epsilon^c(x)$ . This immediately implies that  $\mathcal{W}^c(x)$  needs to be compact and periodic (see next paragraph) and shows that  $x_0$  can be approximated by periodic compact leaves of  $\mathcal{W}^c$ .



Indeed, in case every leaf of  $\mathcal{W}^c$  is compact then  $\mathcal{W}^c(x)$  is automatically compact and periodic and there is no more to say. In case  $f(W) = W$  for every leaf  $W \in \mathcal{W}^c$  let us suppose by contradiction that  $\mathcal{W}^c(x)$  is not compact. Then  $f : \mathcal{W}^c(x) \rightarrow \mathcal{W}^c(x)$  is a homeomorphism of the line. In case this homeomorphism preserves the orientation of  $\mathcal{W}^c(x)$  then  $f(\mathcal{W}_\epsilon^c(x))$  disjoint from  $\mathcal{W}_\epsilon^c(x)$  impedes  $f^k(x)$  from lying in  $\mathcal{W}_\epsilon^c(x)$  for some  $k > 0$  and gives us a contradiction. In case the homeomorphism inverts the orientation of  $\mathcal{W}^c(x)$  then  $f(\mathcal{W}_\epsilon^c(x))$  disjoint from  $\mathcal{W}_\epsilon^c(x)$  implies that  $\mathcal{W}_\epsilon^c(x)$  is disjoint from the unique fixed point of  $f$  in  $\mathcal{W}^c(x)$ . Let us denote by  $p$  this fixed point. Since  $f^k(x)$  lies in  $\mathcal{W}_\epsilon^c(x)$  it follows that  $k$  needs to be even because odd iterates of  $x$  and  $x$  itself must lie in different connected component of  $\mathcal{W}^c(x) \setminus \{p\}$ . The fact that  $f^2 : \mathcal{W}^c(x) \rightarrow \mathcal{W}^c(x)$  preserves the orientation of  $\mathcal{W}^c(x)$  and  $f^2(\mathcal{W}_\epsilon^c(x))$  is disjoint from  $\mathcal{W}_\epsilon^c(x)$  prevents  $f^k(x)$  from lying in  $\mathcal{W}_\epsilon^c(x)$  for some  $k > 0$  even and gives us a contradiction.

Let  $\epsilon > 0$  be small enough so that  $f(B_{2\epsilon}(x_0))$  and  $f^2(B_{2\epsilon}(x_0))$  are disjoint from  $B_{2\epsilon}(x_0)$ . And small enough so that at scale  $\epsilon$  the bundles are almost constant and the distances inside the invariant manifolds  $\mathcal{W}^\sigma$ ,  $\sigma \in \{s, c, u\}$  are nearly the same as in the manifold. For a precise construction see for example the scale and metric considered in property (P4) of Lemma 4.2.1.

Inside  $B_\epsilon(x_0)$  let  $U$  be a  $\mathcal{W}^c$ -foliation box neighborhood containing  $x_0$  that is obtained as  $U := \mathcal{W}_\delta^c(D)$  for  $\delta > 0$  some small constant and  $D$  some  $C^1$  disc transverse to  $\mathcal{W}^c$  and nearly tangent to  $E^s \oplus E^u$ . In particular, let  $\delta > 0$  be such that  $\delta/2$  is smaller than the constant given by Lemma 2.0.1.

Let us consider  $0 < \delta' < \delta$  and  $0 < \epsilon' < \epsilon$  such that  $10\epsilon' < \delta'$  and such that for every  $y \in B_{\epsilon'}(x_0)$  the set  $\mathcal{W}_{\delta'}^s(\mathcal{W}_{\delta'}^u(y))$  is contained in  $U$ .

We claim that for every  $y \in B_{\epsilon'}(x_0)$  the set  $\mathcal{W}_{\delta'}^s(\mathcal{W}_{\delta'}^u(y))$  intersects every center plaque of  $U$  in at most one point. This is a consequence of Lemma 2.0.1. Indeed, suppose that  $w, w' \in \mathcal{W}_{\delta'}^s(\mathcal{W}_{\delta'}^u(y))$  are points in the same center plaque of  $U$ . Then  $w' \in \mathcal{W}_\delta^c(w)$ . Let  $z, z' \in \mathcal{W}_{\delta'}^u(y)$  be such that  $w \in \mathcal{W}_{\delta'}^s(z)$  and  $w' \in \mathcal{W}_{\delta'}^s(z')$ . As  $w' \in \mathcal{W}_{\delta'}^c(w)$  then both  $z$  and  $z'$  lie in  $\mathcal{W}_{\delta'}^s(\mathcal{W}_{\delta'}^c(w))$ . As  $\mathcal{W}_{\delta'}^s(\mathcal{W}_{\delta'}^c(w))$  is  $C^1$  and tangent to  $E^s \oplus E^c$  it follows that  $\mathcal{W}_{\delta'}^s(\mathcal{W}_{\delta'}^c(w))$  intersects  $\mathcal{W}_{\delta'}^u(y)$  in at most one point. That is,  $z = z'$ . Then  $w = w'$ . This proves the claim.

Let  $\pi^c : U \rightarrow D$  denote the projection along center plaques. It is immediate to check that  $\pi^c$  needs to be continuous. The previous paragraph then implies that  $\pi^c$  from  $\mathcal{W}_{\delta'}^s(\mathcal{W}_{\delta'}^u(y))$  to  $D$  is a homeomorphism onto its image for every  $y \in B_{\epsilon'}(x_0)$ .

Since  $x_0 \in \Omega(f)$  there exists  $k > 0$  such that  $f^k(B_{\epsilon'}(x_0)) \cap B_{\epsilon'}(x_0) \neq \emptyset$ . Moreover, such a  $k$  can be considered arbitrarily large. Let us fix such a  $k$  large enough so that  $\mathcal{W}_{2\delta'}^u(f(x)) \subset f^k(\mathcal{W}_{\delta'}^u(x))$  and  $f^k(\mathcal{W}_{\delta'}^s(x)) \subset \mathcal{W}_{\delta'/2}^s(f(x))$  for every  $x \in M$ .

Let us fix from now on  $y$  a point in  $B_{\epsilon'}(x_0)$  so that  $f^k(y) \in B_{\epsilon'}(x_0)$ . It follows that there exists a sub arc  $\gamma_y^u$  in  $\mathcal{W}_{\delta'}^u(y)$  such that  $f^k(\gamma_y^u) = \mathcal{W}_{2\delta'}^u(f^k(y))$ .

Then  $f^k(\mathcal{W}_{\delta'}^s(\gamma_y^u))$  is a subset of  $\mathcal{W}_{\delta'/2}^s(\mathcal{W}_{2\delta'}^u(f^k(y)))$ . Consider  $R \subset D$  the closure of the image by  $\pi^c$  of  $\mathcal{W}_{\delta'}^s(\gamma_y^u)$ . It follows that  $R$  is a topological disc in  $D$ . Its boundary can be viewed as a rectangle. Two of its opposite sides,  $\Gamma_2$  and  $\Gamma_4$ , correspond to the projection by  $\pi^c$  of the two  $s$ -arcs  $\mathcal{W}_{\delta'/2}^s(y_1)$  and  $\mathcal{W}_{\delta'/2}^s(y_2)$  for  $y_1, y_2$  each one of the two endpoints of  $\gamma_y^u$  in  $\mathcal{W}^u(y)$ . The other two sides,  $\Gamma_1$  and  $\Gamma_3$ , correspond to the projection by  $\pi^c$  of the two segments formed by the endpoints of  $\mathcal{W}_{\delta'}^s(y')$  as  $y'$  varies in  $\gamma_y^u$ .

It follows that  $h = \pi^c \circ f^k \circ (\pi^c)^{-1}$  is a well defined continuous map from  $R$  to  $D$ . We claim that it is enough to show that  $h$  has a fixed point. Indeed, if  $p$  denotes a fixed point for  $h$  then  $o := (\pi^c)^{-1}(p)$  satisfies that  $f^k(o)$  and  $o$  are in the same center plaque of  $U$ . Since  $f(\mathcal{W}_\epsilon^c(x)) \cup f^2(\mathcal{W}_\epsilon^c(x))$  is disjoint from  $\mathcal{W}_\epsilon^c(x)$ , because  $f(B_{2\epsilon}(x_0)) \cup f^2(B_{2\epsilon}(x_0))$  is disjoint from  $B_{2\epsilon}(x_0)$ , one obtains that  $\mathcal{W}^c(o)$  needs to be compact and periodic as discussed before.

The existence of a fixed point for  $h$  follows by a classic Lefschetz's index argument. Let  $\Gamma$  denote the boundary of  $R$ . The closed curve  $\Gamma$  is the union  $\Gamma_1 \cup \dots \cup \Gamma_4$  of the sides of  $R$  as explained above. Since  $f^k(\mathcal{W}_{\delta'}^s(\gamma_y^u))$  is a subset of  $\mathcal{W}_{\delta'/2}^s(\mathcal{W}_{2\delta'}^u(f^k(y)))$  and  $d(y, f^k(y)) < \epsilon'$  for  $10\epsilon' < \delta'$  it follows that  $h$  sends the rectangle  $R$  to a new rectangle  $h(R)$  that 'crosses'  $R$  so that  $\Gamma_1$  and  $\Gamma_3$  do not intersect  $h(R)$  and  $h(\Gamma_2)$  and  $h(\Gamma_4)$  lie in two different connected components of  $R \setminus h(R)$  that are adjacent to  $\Gamma_2$  and  $\Gamma_4$ , respectively (see Figure 6.2 and Figure 6.3). This is enough for finding a fixed point for  $h$ . For the sake of completeness we will reproduce this classical argument for finding a fixed point under these hypothesis.

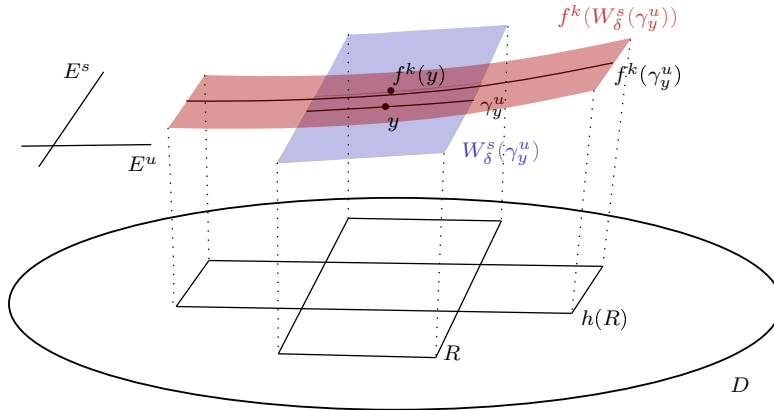


Figure 6.2:

Let  $t \mapsto \Gamma(t)$  be an homeomorphism from the circle  $S^1$  to  $\Gamma$ . We can consider a nullhomotopy  $\{\Gamma^{(s)}\}_{s \in [0,1]}$  of  $\Gamma$  inside  $R$  as follows. Let us identify

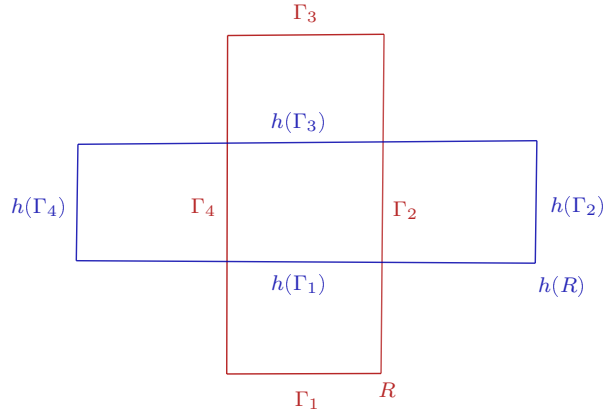


Figure 6.3:

$R$  homeomorphically with  $[0, 1] \times [0, 1]$ . Then let  $\Gamma^{(0)}$  be equal to  $\Gamma$  and let  $\Gamma^{(s)}$ , varying continuously with  $s \in [0, 1]$ , be such that the image of  $\Gamma^{(s)}$  is the boundary of the square  $[0, 1 - s] \times [0, 1 - s]$ .

Suppose by contradiction that  $h$  has no fixed points in  $R$ . Let us identify  $D$  with the euclidean plane  $\mathbb{R}^2$ . The no fixed points assumption implies that the continuous family of maps  $\rho_s : S^1 \rightarrow S^1$  given by

$$\rho_s(t) := \frac{h(\Gamma^{(s)}(t)) - \Gamma^{(s)}(t)}{\|h(\Gamma^{(s)}(t)) - \Gamma^{(s)}(t)\|}$$

is well defined for every  $s \in [0, 1]$ .

On the one hand, from the way the sides  $\Gamma_1, \dots, \Gamma_4$  are mapped by  $h$  it is an immediate computation to check that  $\rho_0 : S^1 \rightarrow S^1$  has index different from 0. On the other hand, if  $y_0$  denote the point that is the image of  $\Gamma^{(1)}$ , then  $h(y_0) \neq y_0$  and one can consider a small ball  $B$  containing  $y_0$  so that  $h(B) \cap B = \emptyset$ . This immediately implies that for every  $s$  close enough to 0 (so that  $\Gamma^{(s)} \subset B$ ) the map  $\rho_s : S^1 \rightarrow S^1$  must have index 0. As the index of a continuous family of maps from  $S^1$  to  $S^1$  is an invariant of the family one gets to a contradiction. Hence  $h$  must have a fixed point on  $R$ .  $\square$

# Chapter 7

## Uniqueness of attractor

### 7.1 Introduction

This chapter deepens on the study of the dynamics of discretized Anosov flows. In particular, on the problem of uniqueness or finiteness of *quasi-attractors* and *quasi-repellers*.

Recall that quasi-attractors are pairwise disjoint compact sets saturated by  $W^u$ -leaves. It follows that each one of them contains at least one minimal set for the foliation  $W^u$ . We call a minimal set for  $W^u$  a *minimal unstable lamination*. Thus, uniqueness (resp. finiteness) of minimal unstable laminations implies uniqueness (resp. finiteness) of quasi-attractors.

In [CPS17] finiteness of minimal unstable laminations was shown to happen for a  $C^1$ -open and dense subset of partially hyperbolic diffeomorphisms with one-dimensional center. In this chapter we aim to a more global (non-perturbative) study involving uniqueness and finiteness results for whole classes of examples.

It is worth pointing out that we focus on minimal unstable laminations and quasi-attractors but the results have obvious analogous statements for minimal stable laminations and quasi-repellers.

#### 7.1.1 Uniqueness of attractor for discretized transitive Anosov flows

Discretized Anosov flows with arbitrary number of attractors and repellers can be obtained by perturbing the time 1 map of an Anosov diffeomorphism's suspension  $\varphi_t : M \rightarrow M$ . Indeed, as  $M$  fibers over the circle and  $\varphi_1$  preserves fibers one can perturb  $\varphi_1$  so that it becomes Morse-Smale or even a dynamics with infinitely many quasi-attractors in the base (see Example 7.2.1 for details).

In [BG10] examples of Axiom A discretized Anosov flows with a proper attractor and a proper repeller have been built as a discretization of any transitive Anosov flow  $\varphi_t : M \rightarrow M$  provided that  $\dim(M) = 3$ .

Recall that two flows are said to be *orbit equivalent* whenever there exists a homeomorphism taking orbits of one into orbits of the other and preserving its orientation. We obtain:

**Theorem 7.1.1.** *Let  $f$  be a discretized Anosov flow and let  $\varphi_t^c$  be the center flow of  $f$ . Suppose  $\varphi_t^c$  is transitive and not orbit equivalent to a suspension. Then  $f$  has a unique minimal unstable lamination.*

**Corollary 7.1.2.** *Any  $f$  as in Theorem 7.1.1 has at most one quasi-attractor.*

Theorem 7.1.1 is already known from [HU19] for discretized Anosov flows in a  $C^1$ -neighborhood of the time 1 map of a transitive Anosov flow that is not orbit equivalent to a suspension.

We point out that the proof given here relies on a different approach. The main inspiration for it comes from [BG09] where it was shown that every discretized Anosov flow that is Axiom A and satisfies the hypothesis of Theorem 7.1.1 admits a unique attractor. By generalizing the arguments in [BG09] (see also [G02]) we are able to remove the ‘Axiom A’ hypothesis and to obtain not only uniqueness of quasi-attractor but also of minimal unstable lamination.

In the case when  $f$  is chain-transitive the statement of Corollary 7.1.2 gives us no new information but uniqueness of minimal unstable lamination may give. It implies, for example, that the supports of all  $u$ -Gibbs measures have non-trivial intersection since the support of any such a measure is a  $\mathcal{W}^u$ -saturated compact set. In [HU19, Theorem 1.2] more precise consequences are obtained.

## 7.1.2 Finiteness of attractors for discretized non-transitive Anosov flows

Recall that the center flow  $\varphi_t^c : M \rightarrow M$  of a discretized Anosov flow  $f$  is a *topological Anosov flow* (see Proposition 3.7.2). Most of the classical properties of Anosov flows are valid also in the context of topological Anosov flows (see for example [Ba05] and the references therein).

In particular, if the center flow  $\varphi_t^c$  is not transitive then the non-wandering set of  $\varphi_t^c$  admits a decomposition  $\Omega(\varphi_t^c) = \Lambda_1 \cup \dots \cup \Lambda_K$  in disjoint *basic pieces*  $\{\Lambda_i\}_{1 \leq i \leq K}$  that are compact,  $\varphi_t^c$ -invariant and such that  $\varphi_t^c|_{\Lambda_i} : \Lambda_i \rightarrow \Lambda_i$  is transitive. Moreover, some of them,  $\Lambda_1, \dots, \Lambda_k$ , are *attracting basic pieces* such that its whole basin  $\mathcal{F}^{ws}(\Lambda_1) \cup \dots \cup \mathcal{F}^{ws}(\Lambda_k)$  is an open and dense

subset of  $M$ . For the sake of completeness we detail these facts in Lemma 7.4.2.

In case the center flow  $\varphi_t^c$  is not transitive the problem of uniqueness or finiteness of attractors reduces to the study of the behavior of  $\varphi_t^c$  on its attracting basic pieces.

For instance, the time 1 map of the Franks-Williams's non-transitive Anosov flow [FW80] can be perturbed to obtain arbitrary number of quasi-attractors (see Example 7.2.2). The unique attractor  $\Lambda$  in this example satisfies that  $\varphi_t^c|_\Lambda : \Lambda \rightarrow \Lambda$  is orbit equivalent to a suspension so one can essentially perform, in a neighborhood of  $\Lambda$ , the same type of perturbation mentioned above for the time 1 map of an Anosov's suspension.

On the other hand, the arguments for obtaining Theorem 7.1.1 are also valid in restriction to any non-suspension basic attracting piece. We obtain:

**Theorem 7.1.3.** *Let  $f$  be a discretized Anosov flow and  $\Lambda$  be an attracting basic piece of its center flow  $\varphi_t^c$ . If  $\varphi_t^c|_\Lambda : \Lambda \rightarrow \Lambda$  is not orbit equivalent to a suspension then  $\Lambda$  contains a unique minimal unstable lamination of  $f$ .*

**Corollary 7.1.4.** *Let  $f$  be a discretized Anosov flow. Suppose that all the attracting basic pieces  $\Lambda_1, \dots, \Lambda_k$  of  $\varphi_t^c$  satisfy that  $\varphi_t^c|_{\Lambda_i} : \Lambda_i \rightarrow \Lambda_i$  is not orbit equivalent to a suspension. Then  $f$  has exactly  $k$  minimal unstable laminations (and exactly  $k$  quasi-attractors). Moreover, each one of them is contained in one of the attracting basic pieces  $\Lambda_1, \dots, \Lambda_k$ .*

Discretized non-transitive Anosov flows in the hypothesis of Corollary 7.1.4 can be constructed using the techniques from [FW80] (see also [BBY17]). We briefly sketch their construction in Example 7.2.3.

### 7.1.3 Uniqueness of attractor for partially hyperbolic skew-products

Recall that  $f : M \rightarrow M$  is a *partially hyperbolic skew-product* if it admits an  $f$ -invariant center foliation  $\mathcal{W}^c$  such that  $M$  is a fiber bundle with  $M/\mathcal{W}^c$  as base and the leaves of  $\mathcal{W}^c$  as fibers. If  $\dim(E^c) = 1$ , we say that  $(M, \mathcal{W}^c)$  is the *trivial bundle* if  $\mathcal{W}^c$  is topologically equivalent to the foliation  $\{\cdot\} \times S^1$  in  $M/\mathcal{W}^c \times S^1$ . We say that  $(M, \mathcal{W}^c)$  is a *virtually trivial bundle* if it is the trivial bundle modulo finite cover. In [BW05] one can find examples of non-trivial skew-products that are virtually trivial and not virtually trivial.

The proof of Theorem 5.1.1 will follow from the more general statements of Proposition 7.3.1 and Proposition 7.3.3 (see Section 7.3.1). As a consequence of these propositions we recover also the uniqueness of minimal unstable lamination result of [HP14] when the bundle is non-trivial in dimension 3 and we extend it to any dimension:

**Theorem 7.1.5.** *Suppose  $f \in \text{PH}_{c=1}(M)$  is a partially hyperbolic skew-product such that the induced dynamics in the space of center leaves,  $F : M/\mathcal{W}^c \rightarrow M/\mathcal{W}^c$ , is transitive. If  $(M, \mathcal{W}^c)$  is not a virtually trivial bundle then  $f$  admits a unique minimal unstable lamination and a unique quasi-attractor.*

In fact, Theorem 7.1.5 is still valid if we exchange the hypothesis ‘skew-product’ for ‘ $\mathcal{W}^c$  uniformly compact’. We will precise this in Section 7.3.1.

In dimension 3, examples of partially hyperbolic skew-products with a proper attractor and a proper repeller such that  $\mathcal{W}^c$  is given by the fibers of a non-trivial bundle over  $M/\mathcal{W}^c = \mathbb{T}^2$  are constructed in [Sh14].

One more time, it is worth noting the marked correspondence between skew-products and discretized Anosov flows. In this case, concerning the uniqueness and existence results. The trivial bundle case, the uniqueness of minimal unstable lamination result of [HP14] for 3-nilmanifolds that are not  $\mathbb{T}^3$  (extended in Theorem 7.1.5) and the examples of [Sh14] mirror the suspension case, Theorem 7.1.1 and the examples of [BG10], respectively.

Notice that the hypothesis ‘ $F : M/\mathcal{W}^c \rightarrow M/\mathcal{W}^c$  transitive’ in Theorem 7.1.5 is somehow natural since in this setting  $F$  is a *topological Anosov homeomorphism* that preserves two topologically transverse contracting/expanding continuous foliations  $\mathcal{W}^{cs}|_{M/\mathcal{W}^c}$  and  $\mathcal{W}^{cu}|_{M/\mathcal{W}^c}$ . A potential Theorem B’, in analogy with Theorem 7.1.3, would involve dealing with attracting basic pieces of a non-transitive  $F$ .

#### 7.1.4 Uniqueness and finiteness of attractors for the classical examples in dimension 3 and beyond

As was already mentioned, the ‘classical examples’ (in the sense of Pujals’s conjecture and [BW05]) of partially hyperbolic diffeomorphisms in dimension 3 are skew-products, *deformations of Anosov diffeomorphisms* (those that are homotopic to Anosov in  $\mathbb{T}^3$ ) and discretized Anosov flows.

For deformations of Anosov diffeomorphisms uniqueness of minimal stable and unstable lamination is proved in [Po13]. Existence of a proper quasi-attractor is unknown (see [Po13, Question 2]).

Theorem 7.1.1 and Theorem 7.1.3 complete, in a certain sense, the uniqueness and finiteness problem for the classical examples in dimension 3 modulo the structure of  $\mathcal{W}^c$ . In particular, the existence of infinitely many minimal unstable laminations is always associated with a region (the whole manifold or some proper attracting region) where  $\mathcal{W}^c$  ‘looks like’ a suspension flow.

Recall that, beyond the classical examples, the first non-dynamically coherent examples were obtained in [HHU16]. These examples detected the existence of a periodic torus tangent to  $E^s \oplus E^c$  or  $E^c \oplus E^u$  as a possible

obstruction for dynamically coherence. Notice that such a torus is necessarily an attractor or a repeller. In [HP19] it was shown that examples with this type of tori have periodic regions homeomorphic to  $\mathbb{T}^2 \times (0, 1)$  in the complement of these tori. In these regions  $E^c$  integrates to  $f$ -invariant ‘interval fibers’ transverse to  $\mathbb{T}^2 \times \{\cdot\}$  and the dynamics is of the type ‘Anosov times identity’. So, essentially, the existence of minimal unstable laminations or quasi-attractors for this type of examples is similar to the trivial skew-product and suspension’s of Anosov map scenarios.

More recently, the realm of classical examples has been enlarged with new challenging examples (as [BPP16], [BGP16] and [BGHP17]). It is natural to ask if results of uniqueness or finiteness of minimal unstable laminations and quasi-attractors are also valid for whole classes of these new examples.

## 7.2 Examples

In this section we briefly outline some examples. We give in Example 7.2.1 and Example 7.2.2 the construction of discretized Anosov flows with arbitrary number of quasi-attractors, even infinitely many. The center flow  $\varphi_t^c$  on these examples is orbit equivalent to a suspension flow in the whole manifold or in restriction to an attracting basic piece of  $\varphi_t^c$ . Then in Example 7.2.3 we show an example of an Anosov flow in the hypothesis of Corollary 7.1.4.

We say that a flow  $\varphi_t : X \rightarrow X$  is a *suspension flow* if there exists a homeomorphism  $g : Y \rightarrow Y$  such that the flow  $\varphi_t$  is the projection of the flow in  $Y \times \mathbb{R}$  generated by the vector field  $\frac{\partial}{\partial t} = (0, 1)$  into the quotient  $X = Y \times \mathbb{R} / \sim$  given by  $(y, t + 1) \sim (g(y), 1)$ .

Notice that for a suspension flow the space  $X$  has the structure of a bundle over the circle  $S^1$  with fibers that are homeomorphic to  $Y$ . Moreover, the flow  $\varphi_t$  takes fibers to fibers and the time 1 map  $\varphi_1 : X \rightarrow X$  leaves invariant each fiber (it projects as the identity on the base) and acts on each of them as the map  $g$ .

**Example 7.2.1** (Perturbing the time 1 map of an Anosov’s suspension). Let us consider  $\varphi_t : M \rightarrow M$  to be the suspension of an Anosov diffeomorphism  $g : N \rightarrow N$ .

We can perturb the time 1 map of  $\varphi_t$  in order to get a partially hyperbolic map  $f$  that still preserves fibers but acts like a Morse-Smale in the base. Indeed, taking coordinates  $x = (y, t)$ , we can consider  $f$  explicitly as  $f(y, t) = \varphi_{\tau(y,t)}(y, t)$  with  $\tau(y, t) = 1 + \alpha \sin(2\pi kt)$  for any  $\alpha \in (0, 1)$ . In this case,  $f$  has  $k$  proper attractors and  $k$  proper repellers.

Further, for a suitable 1-periodic map  $h : \mathbb{R} \rightarrow (-1, 1)$ , the discretization  $\tau(y, t) = 1 + h(t)$  can produce infinite number of quasi-attractors and



quasi-repellers. It is sufficient for  $h$  to have infinite zeros (countably or uncountably), each of them accumulated by positive and negative values.

In analogy with the above example one can perform the following one:

**Example 7.2.2.** (Perturbing the time 1 map of the Franks-Williams's example) Consider  $\varphi_t : M \rightarrow M$  the Franks-Williams's example of a non-transitive Anosov flow ([FW80]). Let  $\Lambda$  be the unique basic attracting piece for  $\varphi_t$ .

In this particular flow one has that  $\varphi_t|_\Lambda : \Lambda \rightarrow \Lambda$  is conjugate to the suspension of a derived from Anosov map  $g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  restricted to its unique attractor  $Y \subset \mathbb{T}^2$ . In particular, the time 1 map  $\varphi_1|_\Lambda$  leaves invariant the fibers of this bundle structure acting on each of them as  $g|_Y : Y \rightarrow Y$ . Notice that each fiber is already a minimal unstable lamination for  $\varphi_1$ .

Moreover, as the construction of  $\varphi_t$  involves performing a surgery far from the suspension of  $g|_Y : Y \rightarrow Y$ , in fact, there exists a neighborhood  $V$  of  $Y$  with  $g(V) \subset V$  such that  $\varphi_t$  is conjugate to the suspension of  $g|_V : V \rightarrow V$  in a neighborhood  $U$  of  $\Lambda$ .

So we can perturb  $\varphi_1$  in a smaller bundle neighborhood  $U' \subset U$  of  $\Lambda$  to obtain a discretization  $f(x) = \varphi_{\tau(x)}(x)$  that acts with arbitrary number of quasi-attractors in the base and leaves unchanged the dynamics of  $\varphi_1$  outside  $U$ . This can be done as in the previous example by taking  $\tau(y, t) = 1 + h(t)$  for a suitable 1-periodic diffeomorphism  $h : \mathbb{R} \rightarrow \mathbb{R}$  in the neighborhood  $U'$  and glueing it with the constant  $\tau = 1$  outside  $U$ .

This construction produces an arbitrary number of quasi-attractors for  $f$ , each one of them homeomorphic to  $Y$ . On the fibered neighborhood  $U'$  of  $\Lambda$  the map  $f$  acts as  $h$  in the base  $S^1$  and as  $g$  on the fibers near  $\Lambda$ .

Let us finish this section with a brief sketch on how to construct an Anosov flow in the hypothesis of Corollary 7.1.4:

**Example 7.2.3.** Let  $S$  be a negatively curved hyperbolic closed surface. Let  $\varphi_t : T^1S \rightarrow T^1S$  be the geodesic flow on the unitary tangent bundle of  $S$ .

Consider  $\alpha$  and  $\beta$  two simple, closed, oriented and disjoint geodesics in  $S$ . Let us see them as periodic orbits  $\alpha, \beta : [0, 1] \rightarrow T^1S$  of the flow  $\varphi_t$ .

It is a standard procedure to make a DA-type perturbation of the vector field  $\frac{\partial \varphi_t}{\partial t}|_{t=0}$  in a neighborhood of  $\alpha$  in order to transform  $\alpha$  into a repelling periodic orbit for a new flow  $\psi_t : T^1S \rightarrow T^1S$  such that  $\frac{\partial \varphi_t}{\partial t}|_{t=0}$  and  $\frac{\partial \psi_t}{\partial t}|_{t=0}$  coincide outside a small neighborhood of  $\alpha$ .

By considering then  $T$  a small toroidal neighborhood of  $\alpha$  such that  $\psi_t$  points inward into  $T^1S \setminus T$  along the boundary  $\partial T$  we obtain that the maximal invariant set of  $\psi_t|_{T^1S \setminus T}$  is a connected attracting hyperbolic set  $\Lambda \subset T^1S \setminus T$ . By cutting out  $T$  from  $T^1S$  and gluing back adequately another copy of  $\psi_t|_{T^1S \setminus T}$  with the inverse orientation one can obtain a non-transitive

Anosov flow with  $\Lambda$  as its unique attracting basic piece (see the techniques in [FW80] and [BBY17] for all the details).

We claim now that  $\psi_t|_\Lambda$  is not orbit equivalent to a suspension. Suppose by contradiction that it is. Then we can consider  $\rho : \Lambda \rightarrow S^1$  such that  $\lim_{t \rightarrow +\infty} \widetilde{\rho \circ \varphi_t}(x) = +\infty$  and  $\lim_{t \rightarrow -\infty} \widetilde{\rho \circ \varphi_t}(x) = -\infty$  for every  $x \in \Lambda$ , where  $t \mapsto \widetilde{\rho \circ \varphi_t}(x) : \mathbb{R} \rightarrow \mathbb{R}$  is any lift of  $t \mapsto \rho \circ \varphi_t(x) : \mathbb{R} \rightarrow S^1$ .

We can extend  $\rho$  to a small open  $\varphi_t$ -forward invariant neighborhood  $U$  of  $\Lambda$  such that  $\lim_{t \rightarrow +\infty} \widetilde{\rho \circ \varphi_t}(x) = +\infty$  continues to be valid for every  $x \in U$  (see the proof of Theorem 7.1.3 for details on how to construct such an  $U$ ). By considering an adapted metric such that  $\varphi_t$  contracts distances inside strong stable leaves for all future iterates we can take  $U$  of the form  $\bigcup_{x \in \Lambda} \mathcal{W}_\delta^s(x)$  for some  $\delta > 0$ . In particular,  $\varphi_t$  points inwards to  $U$  in every point of  $\partial U$ .

We can extend  $\rho$  continuously to  $T^1S \setminus T$  by setting  $\rho(y) = \rho(\varphi_{t_y}(y))$  for every  $y \in T^1S \setminus (T \cup U)$  where  $t_y$  is the unique non-negative time such that  $\varphi_{t_y}(y) \in \partial U$ .

Now,  $\beta : [0, 1] \rightarrow \Lambda$  is freely homotopic to its inverse  $\beta^{-1} : [0, 1] \rightarrow \Lambda$  in  $T^1S$  by the homotopy  $\beta_s$  with  $s \in [0, 1]$  that for each  $t$  takes  $\dot{\beta}(t)$  and rotates it clockwise  $s\pi$ . As  $\beta_s$  coincides with  $\beta$  in the base  $S$ , we can consider  $T$  sufficiently close to  $\alpha$  so that this homotopy takes place inside  $T^1S \setminus T$ . This homotopy gives an homotopy between the curve  $t \mapsto \widetilde{\rho \circ \beta}(t) : \mathbb{R} \rightarrow \mathbb{R}$  that lifts  $t \mapsto \rho \circ \beta(t) : \mathbb{R} \rightarrow S^1$  and the curve  $t \mapsto \widetilde{\rho \circ \beta^{-1}}(t) : \mathbb{R} \rightarrow \mathbb{R}$  that lifts  $t \mapsto \rho \circ \beta^{-1}(t) : \mathbb{R} \rightarrow S^1$ . This is an impossible homotopy since  $\lim_{t \rightarrow +\infty} \widetilde{\rho \circ \beta}(t) = +\infty$  and  $\lim_{t \rightarrow +\infty} \widetilde{\rho \circ \beta^{-1}}(t) = -\infty$ . We get to a contradiction and the claim is proved.

## 7.3 Uniqueness of attractor for discretized Anosov flows such that $\varphi_t^c$ is transitive

### 7.3.1 Proof of Theorem 7.1.1 and Theorem 7.1.5 assuming Proposition 7.3.1 and Proposition 7.3.3

Theorem 7.1.1 and Theorem 7.1.5 will be a consequence of the following more general statements. Together, the following two propositions can be seen as an obstruction to the existence of more than one minimal unstable lamination (or more than one attracting region) for certain partially hyperbolic systems with one-dimensional center.

Recall that  $f$  acts *quasi-isometrically on  $\mathcal{W}^c$*  if there exist some constants  $l, L > 0$  such that

$$f^n(\mathcal{W}_l^c(x)) \subset \mathcal{W}_L^c(f^n(x)) \quad (7.3.1)$$

for every  $x$  in  $M$  and  $n \in \mathbb{Z}$ . We say that  $f$  acts *quasi-isometrically in the*

future on  $\mathcal{W}^c$  if (7.3.1) is verified for every  $n \geq 0$ . And quasi-isometrically in the past on  $\mathcal{W}^c$  if it is satisfied for every  $n \leq 0$ .

**Proposition 7.3.1.** *Suppose  $f \in \text{PH}_{c=1}(M)$  is dynamically coherent admitting  $f$ -invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$ . Suppose that  $\mathcal{W}^{cs}$  is minimal and that  $f$  acts quasi-isometrically in the future on  $\mathcal{W}^c := \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ .*

*Then there exists  $L > 0$  such that every minimal unstable lamination  $A \subset M$  satisfies that*

$$\mathcal{W}_L^c(x) \cap A \neq \emptyset$$

*for every  $x \in M$ . In particular,  $A$  intersects every leaf of  $\mathcal{W}^c$ .*

**Remark 7.3.2.** In Proposition 7.3.1 the hypothesis ‘ $\mathcal{W}^{cs}$  minimal’ can be replaced by ‘ $f$  chain-transitive’ or ‘ $f$  volume preserving’ since the latter ones imply the former one (see for example [BW05, Lemma 1.1]).

We say that a one-dimensional center manifold  $\mathcal{W}^c$  admits a *global section* if there exists a codimension one closed submanifold  $N \subset M$  transverse to the leaves of  $\mathcal{W}^c$  such that  $\mathcal{W}_L^c(x) \cap N \neq \emptyset$  for every  $x \in M$  and some constant  $L > 0$ .

**Proposition 7.3.3.** *Suppose  $f \in \text{PH}_{c=1}(M)$  is dynamically coherent. Suppose that the foliation  $\mathcal{W}^c$  is orientable and that there exists  $L > 0$  such that  $\mathcal{W}_L^c(x) \cap A \neq \emptyset$  for every minimal unstable lamination  $A$ .*

*If  $M$  has more than one minimal unstable lamination then  $\mathcal{W}^c$  admits a global section.*

**Remark 7.3.4.** Notice that Proposition 7.3.3 does not include the hypothesis ‘ $f$  acts quasi-isometrically on  $\mathcal{W}^c$ ’. In fact, it is derived as a consequence along the proof (see Lemma 5.3.).

*Proof of Theorem 7.1.1 assuming Propositions 7.3.1 and 7.3.3.* Suppose  $f$  is a discretized Anosov flow such that its center flow  $\varphi_t^c : M \rightarrow M$  is transitive and not orbit equivalent to a suspension.

From Proposition 3.4.4 we know that  $f$  is dynamically coherent admitting  $f$ -invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$  is the flow center foliation of  $f$  (whose leaves are the orbits of  $\varphi_t^c$ ). Moreover,  $f$  acts quasi-isometrically on  $\mathcal{W}^c$  (see Remark 3.4.2). And the leaves of  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  are the *weak-stable* and *weak-unstable* leaves of the topological Anosov flow  $\varphi_t^c$ , respectively (see Proposition 3.7.2).

Since  $\varphi_t^c$  is transitive we claim that  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  are minimal foliations. Indeed, suppose by contradiction that there exists  $\Lambda^{cu} \subset M$  a compact proper subset of  $M$  that is saturated by leaves of  $\mathcal{W}^{cu}$ . Let  $N > 0$  be such that  $f^N$  contracts distances from the first iterate on leaves of  $\mathcal{W}^s$  and let  $L' > 0$  be

such that  $f^N(x) \in \mathcal{W}_{L'}^c(x)$  for every  $x \in M$ . Let  $\epsilon, \delta > 0$  small be such that  $\mathcal{W}_\epsilon^s(\Lambda^{cu})$  contains  $\mathcal{W}_{L'}^c(x)$  for every  $x \in \mathcal{W}_\delta^s(\Lambda^{cu})$ . And such that  $M \setminus \mathcal{W}_\epsilon^s(\Lambda^{cu})$  has non-empty interior.

Let  $x$  be a point whose  $\varphi_t^c$ -forward orbit  $o^+(x)$  is dense in  $M$ . Without loss of generality we can suppose that  $x$  lies in  $\mathcal{W}_\delta^s(\Lambda^{cu})$  since  $\mathcal{W}_\delta^s(\Lambda^{cu})$  has non-empty interior. And that  $f(x)$  and  $\varphi_1^c(x)$  lie in the same connected component of  $\mathcal{W}^c(x) \setminus \{x\}$  (otherwise, one just reverses the time of  $\varphi_t^c$ ).

For every  $y, z \in \mathcal{W}^c(x)$  let  $[y, z]_c$  denote the center segment in  $\mathcal{W}^c(x)$  joining  $y$  and  $z$ . It follows that the forwards orbit  $o^+(x)$  of  $x$  is equal to  $\bigcup_{k \geq 0} [f^{kN}(x), f^{(k+1)N}(x)]_c$ . Since  $[f^{kN}(x), f^{(k+1)N}(x)]_c$  is a subset of  $\mathcal{W}_{L'}^c(f^{kN}(x))$  for every  $k \geq 0$  one obtains from the election of  $\delta$  and  $\epsilon$  that  $o^+(x)$  is contained in  $\mathcal{W}_\epsilon^s(\Lambda^{cu})$ . This contradicts that  $o^+(x)$  is dense in  $M$  since  $\mathcal{W}_\epsilon^s(\Lambda^{cu})$  was chosen so that  $M \setminus \mathcal{W}_\epsilon^s(\Lambda^{cu})$  has non-empty interior.

Analogously in the case of a compact proper subset of  $M$  that is saturated by leaves of  $\mathcal{W}^{cs}$ . This proves the claim.

Combining Proposition 7.3.1 and Proposition 7.3.3 one obtains that  $f$  can not admit more than one minimal unstable lamination, otherwise  $\varphi_t^c$  would have a global section, and as a consequence it would be orbit equivalent to a suspension flow.  $\square$

Recall that a one-dimensional center manifold  $\mathcal{W}^c$  is *uniformly compact* if every leaf of  $\mathcal{W}^c$  is compact and the leaf length function  $x \mapsto \text{length}(\mathcal{W}^c(x))$  is bounded in  $M$ . In case  $f$  is a skew-product, it is immediate that  $\mathcal{W}^c$  is uniformly compact (in fact, the leaf length function is continuous in  $M$ ). We will prove Theorem 7.1.5 in its more general version for the case when  $\mathcal{W}^c$  is an  $f$ -invariant uniformly compact foliation.

*Proof of Theorem 7.1.5 assuming Propositions 7.3.1 and 7.3.3.* Let  $f : M \rightarrow M$  be a partially hyperbolic diffeomorphism with  $\dim(E^c) = 1$  admitting an  $f$ -invariant uniformly compact center foliation  $\mathcal{W}^c$  such that the induced dynamics in the space of center leaves,  $F : M/\mathcal{W}^c \rightarrow M/\mathcal{W}^c$ , is transitive. Suppose that  $M$  admits more than one minimal unstable lamination. We are going to see that under this hypothesis  $(M, \mathcal{W}^c)$  has to be a virtually trivial bundle.

From [BB16, Theorem 1] the map  $f$  is dynamically coherent admitting center-stable and center-unstable foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$ , respectively, such that  $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$ . As  $F : M/\mathcal{W}^c \rightarrow M/\mathcal{W}^c$  is transitive the foliation  $\mathcal{W}^{cs}$  has to be minimal, otherwise a proper minimal set for  $\mathcal{W}^{cs}$  would project to  $M/\mathcal{W}^c$  into a proper repeller for  $F$ .

Furthermore, as the length of center leaves is bounded, then  $f$  automatically acts quasi-isometrically on  $\mathcal{W}^c$  as pointed out in Remark 3.4.3.

Suppose first that  $\mathcal{W}^c$  is orientable. By combining Propositions 7.3.1 and 7.3.3 we obtain that  $\mathcal{W}^c$  admits a global section.

Let us denote the global section of  $\mathcal{W}^c$  as  $N \subset M$ . Let  $\alpha : N \rightarrow N$  denote the first return map of  $\mathcal{W}^c$  to  $N$ , modulo fixing an orientation for  $\mathcal{W}^c$ .

For every  $x \in N$  let  $k(x) \in \mathbb{Z}^+$  be the smallest positive integer such that  $\alpha^{k(x)}(x) = x$ . As  $x \mapsto \text{length}(\mathcal{W}^c(x))$  is bounded in  $M$  there exists some constant  $k \in \mathbb{Z}^+$  such that  $k(x) \leq k$  for every  $x$ . By taking  $K = k!$  we obtain that  $\alpha^K = \text{id}$ .

Let us consider a metric in  $M$  such that every center segment  $[x, \alpha(x)]_c$  is of length  $\frac{1}{K}$  and let  $\phi^c : M \rightarrow M$  denote the flow by arc-length whose flow lines are the leaves of  $\mathcal{W}^c$ . Then the map  $p : N \times S^1 \rightarrow M$  given by  $(x, \theta) \mapsto \phi_\theta^c(x)$  is a  $K : 1$  covering map sending circles of the form  $\{\cdot\} \times S^1$  to leaves of the foliation  $\mathcal{W}^c$ . We conclude that  $(M, \mathcal{W}^c)$  is a virtually trivial bundle.

In the case  $\mathcal{W}^c$  is not orientable we can argue as above after taking an orientable double cover for  $\mathcal{W}^c$ . Indeed, we can lift  $f, \mathcal{W}^c$  and all the minimal unstable laminations to an orientable double cover  $\tilde{M}$ . The quasi-isometrically action of  $f$  on  $\mathcal{W}^c$  remains valid on the lifted dynamics.

We claim that the minimality of  $\mathcal{W}^{cs}$  also remains valid on the lift. Indeed, if we suppose that the lift of  $\mathcal{W}^{cs}$  is not minimal then there exist  $\tilde{x}$  and  $\tilde{x}'$  lifts of a point  $x \in M$  such that  $\overline{\mathcal{W}^{cs}(\tilde{x})}$  and  $\overline{\mathcal{W}^{cs}(\tilde{x}')}$  are minimal proper subsets of  $\tilde{M}$ . Then  $\tilde{M}$  coincides with the disjoint union  $\overline{\mathcal{W}^{cs}(\tilde{x})} \cup \overline{\mathcal{W}^{cs}(\tilde{x}'')}$  and we get to a contradiction. This proves the claim.

We obtain that the lifted dynamics verifies Proposition 7.3.1 and Proposition 7.3.3. Then, as argued above,  $\tilde{M}$  and the lift of  $\mathcal{W}^c$  form a virtually trivial bundle. We conclude that  $(M, \mathcal{W}^c)$  is also a virtually trivial bundle.  $\square$

### 7.3.2 Proof of Proposition 7.3.1

Let  $f : M \rightarrow M$  be as in the hypothesis of Proposition 7.3.1.

Recall that for every  $r > 0$  and  $x \in M$  we denote by  $\mathcal{W}_r^s(\mathcal{W}_r^c(x))$  the set  $\bigcup_{y \in \mathcal{W}_r^c(x)} \mathcal{W}_r^s(y)$ . As a consequence of  $f$  acting quasi-isometrically in the future on  $\mathcal{W}^c$  it was shown in Proposition 3.4.4 that  $\bigcup_{r \geq 0} \mathcal{W}_r^s(\mathcal{W}_r^c(x)) = \mathcal{W}^{cs}(x)$  for every  $x$  in  $M$  (the proposition is stated for quasi-isometrically action in the past and future but the proof of this fact only uses that  $f$  acts quasi-isometrically in the future).

**Lemma 7.3.5.** *There exists  $R > 0$  such that  $\mathcal{W}_R^s(\mathcal{W}_R^c(x)) \cap \mathcal{W}^u(y) \neq \emptyset$  for every  $x$  and  $y$  in  $M$ .*

*Proof.* By contradiction, suppose there exist  $R_n \xrightarrow{n} \infty$  and sequences  $\{x_n\}_n$  and  $\{y_n\}_n$  in  $M$  such that  $\mathcal{W}_{R_n}^s(\mathcal{W}_{R_n}^c(x_n)) \cap \mathcal{W}^u(y_n) = \emptyset$  for every  $n$ . Then,

as we are dealing with leaves of foliations tangent to continuous bundles, by taking accumulation points  $x$  and  $y$  of the sequences  $\{x_n\}_n$  and  $\{y_n\}_n$  we obtain that  $\mathcal{W}^{cs}(x) \cap \mathcal{W}^u(y) = \emptyset$ . This contradicts that  $\mathcal{W}^{cs}(x)$  is dense in  $M$ .  $\square$

As  $f$  acts quasi-isometrically in the future on  $\mathcal{W}^c$  there exists  $L > 0$  such that  $f^n(\mathcal{W}_R^c(x))$  is contained in  $\mathcal{W}_L^c(f^n(x))$  for every  $n \geq 0$  and  $x \in M$ .

Proposition 7.3.1 is a direct consequence of the following lemma.

**Lemma 7.3.6.** *For every  $x$  and  $y$  in  $M$  we have that  $\mathcal{W}_L^c(x) \cap \overline{\mathcal{W}^u(y)} \neq \emptyset$ .*

*Proof.* Let us fix  $x$  and  $y$  arbitrary points in  $M$ . For every  $n \geq 0$  we have that  $\mathcal{W}_R^s(\mathcal{W}_R^c(f^{-n}(x))) \cap \mathcal{W}^u(f^{-n}(y)) \neq \emptyset$ . Then, as  $f$  contracts distances uniformly to the future inside stable leaves, there exists  $r_n \xrightarrow{n} 0$  such that the image of  $\mathcal{W}_R^s(\mathcal{W}_R^c(f^{-n}(x)))$  by  $f^n$  is contained in  $\mathcal{W}_{r_n}^s(\mathcal{W}_L^c(x))$ . We obtain that  $\mathcal{W}_{r_n}^s(\mathcal{W}_L^c(x)) \cap \mathcal{W}^u(y) \neq \emptyset$  for every  $n \geq 0$  and then  $\mathcal{W}_L^c(x) \cap \overline{\mathcal{W}^u(y)} \neq \emptyset$ .  $\square$

### 7.3.3 First part of the proof of Proposition 7.3.3: The sets $(A, A')_c$ and $(A', A)_c$

From now on let  $f : M \rightarrow M$  and  $L > 0$  be as in the hypothesis of Proposition 7.3.3 and suppose that there exist  $A$  and  $A'$  different minimal unstable laminations in  $M$ . We will see that under this hypothesis  $\mathcal{W}^c$  has to admit a global section.

The goal of this subsection is to show that the sets  $(A, A')_c$  and  $(A', A)_c$  defined below are disjoint open subsets of  $M$  that ‘separate’ the disjoint and closed subsets  $[A]_c$  and  $[A']_c$  (see Proposition 7.3.12).

Let us fix from now on an orientation for  $\mathcal{W}^c$  and denote  $\phi^c : M \times \mathbb{R} \rightarrow M$  a non-singular flow that parameterizes the leaves of  $\mathcal{W}^c$ .

**Notation.** For  $x$  and  $y$  in the same center leaf we will say that  $x \leq y$  if  $y = \phi_t^c(x)$  for some  $t \geq 0$ . If this is the case, let  $(x, y)_c$  and  $[x, y]_c$  denote the open and closed center segments from  $x$  to  $y$ .

Let us define the sets:

$$\begin{aligned} [A]_c &= \bigcup \{[x, y]_c : x \in A, y \in A, [x, y]_c \cap A' = \emptyset\}, \\ (A, A')_c &= \bigcup \{(x, y)_c : x \in A, y \in A', (x, y)_c \cap (A \cup A') = \emptyset\}. \end{aligned}$$

Notice that the center segments in the definition of  $[A]_c$  may be singletons. We define analogously the sets  $[A']_c$  and  $(A', A)_c$ . By an abuse of notation, we will consider this sets both as subsets of  $M$  and as an abstract collection center segments.

The following remark is a direct consequence from the definitions.

**Remark 7.3.7.** The manifold  $M$  is equal to the disjoint union  $[A]_c \cup (A, A')_c \cup [A']_c \cup (A', A)_c$ .

Let us first point out that:

**Lemma 7.3.8.** *The map  $f$  acts quasi-isometrically on  $\mathcal{W}^c$ .*

*Proof.* Let  $d > 0$  be the distance between the disjoint minimal unstable laminations  $A$  and  $A'$ .

We claim that, as every center segment of length  $2L > 0$  intersects every minimal unstable lamination, then  $f^n(\mathcal{W}_d^c(x))$  can not have length larger than  $2L$  for any  $x \in M$  and  $n \in \mathbb{Z}$ .

By contradiction, if the length of  $f^n(\mathcal{W}_d^c(x))$  is larger than  $2L$  for some  $x \in M$  and  $n \in \mathbb{Z}$  then  $f^n(\mathcal{W}_d^c(x))$  intersects both minimal unstable laminations  $f^n(A)$  and  $f^n(A')$ . Then  $\mathcal{W}_d^c(x)$  has to intersect both  $A$  and  $A'$ . This gives us a contradiction and the claim is proved. We obtain that  $f$  acts quasi-isometrically on  $\mathcal{W}^c$  with constants  $d, 2L > 0$ .  $\square$

As a consequence of the previous lemma it follows from Proposition 3.4.4 that for every  $x \in M$  the center unstable leaf  $\mathcal{W}^{cu}(x)$  is equal to  $\mathcal{W}^u(\mathcal{W}^c(x))$ , the unstable saturation of the center leaf  $\mathcal{W}^c(x)$ . Moreover, we will be able to make ‘long’ transports by unstable holonomy of any center segment as stated in the in Lemma 7.3.9 below.

We say that a curve  $x^u : [0, 1] \rightarrow M$  is an *unstable curve* if it is a  $C^1$  curve tangent to the bundle  $E^u$ . In the setting of the following lemma we say that  $\{[x^u(t), y^u(t)]_c\}_{t \in [0, 1]}$  is the *transport by unstable holonomy* of the center segment  $[x, y]_c$  along the unstable curve  $x^u$ .

**Lemma 7.3.9.** *Let  $[x, y]_c$  be a center segment in  $M$  and  $x^u : [0, 1] \rightarrow M$  be an unstable curve such that  $x^u(0) = x$ . Then there exist a unique unstable curve  $y^u : [0, 1] \rightarrow M$  and unique center segments  $[x^u(t), y^u(t)]_c$  varying continuously with  $t \in [0, 1]$  in the Hausdorff topology and satisfying that  $[x^u(0), y^u(0)]_c = [x, y]_c$ .*

*Proof.* As  $f$  acts quasi-isometrically on  $\mathcal{W}^c$  there exists  $R > 0$  such that the length of  $f^{-n}([x, y]_c)$  is less than  $R$  for every  $n \geq 0$ .

By dynamical coherence the leaves of  $\mathcal{W}^{cu}$  are subfoliated by leaves of  $\mathcal{W}^c$  and  $\mathcal{W}^u$  having local product structure. Then, as  $\mathcal{W}^c$  is tangent to a continuous bundle, there exist small constants  $\epsilon, \delta > 0$  such that for every center segment  $[x', y']_c$  of length less than  $R$  and every point  $x'' \in \mathcal{W}_\delta^u(x')$  there exists a unique center segment  $[x'', y'']_c$  with  $y'' \in \mathcal{W}_\epsilon^u(y')$  such that  $[x'', y'']_c$  is contained in an unstable  $\epsilon$ -neighborhood of  $[x', y']_c$  of the form  $\bigcup_{z \in [x', y']_c} \mathcal{W}_\epsilon^u(z)$ . In other words, the transport by unstable holonomy for

center segments of length at most  $R$  is well defined along any unstable curve of a certain small length  $\delta > 0$ .

Since  $f$  contracts unstable distances in the past, then  $f^{-n_0}(x^u)$  will have length less than  $\delta$  for some  $n_0 \geq 0$ . Then the transport by unstable holonomy of  $[f^{-n_0}(x), f^{-n_0}(y)]_c$  along  $f^{-n_0}(x^u)$  is well defined and, iterating it  $n_0$  to the future, the transport by unstable holonomy of  $[x, y]_c$  along  $x^u$  is also well defined.  $\square$

For every  $x \in A$  let us define  $S(x)$  to be the ‘first point’ of  $A'$  in  $W^c(x)$  in the direction of the flow  $\phi^c$ . That is,  $S(x)$  is such that  $(x, S(x))_c$  is a center segment in  $[A]_c \cup (A, A')_c$ . Let us define  $l_S(x)$  as the length of the arc  $[x, S(x)]_c$ .

**Lemma 7.3.10.** *The function  $l_S : A \rightarrow \mathbb{R}$  is lower semicontinuous and continuous in a residual subset of  $A$ .*

*Proof.* Since  $l_S$  is bounded from above by the constant  $2L > 0$  then, for every sequence  $\{x_n\}_n \subset A$  that converges to a point  $x$  in  $A$ , any accumulation point  $y$  of  $S(x_n)$  lies in  $W^c(x)$ . Since  $A'$  is closed,  $y$  is a point in  $A'$ . Then  $[x, S(x)]_c$  has to be contained in  $[x, y]_c$  for any such an accumulation point  $y$ . This implies that  $l_S(x) \leq \liminf_n l_S(x_n)$  and we obtain that  $l_S$  is lower semicontinuous.

It is a well-known result that semicontinuous functions are continuous in a residual set. For the sake of completeness we outline the proof of this fact for  $l_S$ : Consider the sets  $F_m = \{x \in A : \exists x_n \xrightarrow{n} x \text{ s.t. } \liminf_n l_S(x_n) \geq l_S(x) + \frac{1}{m}\}$  for every  $m$  in  $\mathbb{Z}^+$ . The set of continuity points of  $l_S$  coincides with  $A \setminus \bigcup_m F_m$ . It is direct to prove that each  $F_m$  is a closed nowhere dense subset of  $A$ . Then  $A \setminus \bigcup_m F_m$  is a residual set in  $A$  by Baire category theorem.  $\square$

For a continuity point  $x$  of  $l_S$  every sequence  $\{x_n\}_n \subset A$  converging to  $x$  verifies that the center segments  $[x_n, S(x_n)]_c$  converges in the Hausdorff topology to  $[x, S(x)]_c$ . For a discontinuity point this is not the case, however, we will see in the following lemma that the failure of continuity is not that dramatic. To show this, we will crucially use that the behavior of  $S$  near a continuity point can be extended by unstable holonomy to any point of  $A$  thanks to Lemma 7.3.9.

**Lemma 7.3.11.** *Let  $\{x_n\}_n \subset A$  be a sequence converging to a point  $x \in A$ . Up to taking a subsequence, suppose that  $\{S(x_n)\}_n$  converges to a point  $y \in A'$ . Then  $y$  lies in  $W^c(x)$ , the center segments  $[x_n, S(x_n)]_c$  converge in the Hausdorff topology to  $[x, y]_c$  and  $[S(x), y]_c$  is a center segment (possibly degenerate to a point) contained in  $[A']_c$ .*



*Proof.* We claim first that the lemma is true for every  $x \in A$  in a neighborhood of a continuity point of  $l_S$ .

Indeed, let  $z \in A$  be a continuity point of  $l_S$  and consider  $U_{S(z)}$  a small neighborhood of  $S(z)$  at a positive distance from  $A$ . We can suppose that  $U_{S(z)}$  is a foliation box of  $\mathcal{W}^c$ , that is, that  $U_{S(z)}$  is the image of a homeomorphism  $h : D \times [0, 1] \rightarrow U_{S(z)}$  such that  $D$  is a compact disc of dimension  $\dim(M) - 1$  and  $h(\{x\} \times [0, 1])$  is a center segment for every  $x \in D$ . Let us denote by  $D_1$  the disc  $h(D \times \{1\})$ .

Since  $z$  is a continuity point of  $l_S$  we can consider  $\delta > 0$  such that for every  $x \in A \cap B_\delta(z)$  we have that  $S(x)$  lies in the interior of  $U_{S(z)}$ . In particular, the center segment  $[x, S(x)]_c$  does not cross the disc  $D_1$ .

If  $\{x_n\}_n \subset A$  is a sequence converging to a point  $x \in A \cap B_\delta(z)$ , then any accumulation point  $y$  of  $\{S(x_n)\}_n$  has to lie in  $U_{S(z)}$ . Up to a subsequence, let us assume that  $S(x_n) \xrightarrow{n} y$ . Then, as each  $[x_n, S(x_n)]_c$  does not intersect  $D_1$ , the segments  $[x_n, S(x_n)]_c$  need to converge in the Hausdorff topology to  $[x, y]_c$  and the whole segment  $[S(x), y]_c$  has to be contained  $U_{S(z)}$  (see Figure 7.1). As  $S(x)$  and  $y$  are in  $A'$  and  $U_{S(z)}$  is disjoint from  $A$  we conclude that  $[S(x), y]_c$  is a center segment in  $[A']_c$ . This proves the first claim.

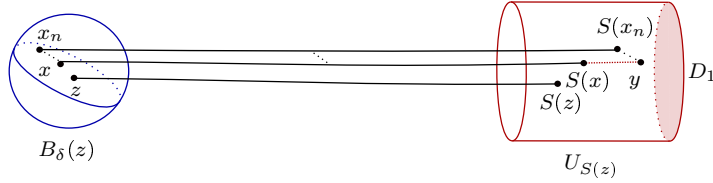


Figure 7.1:

Let us see now that the lemma is true for every point in  $\hat{x} \in A$ . We will use as an auxiliary construction a continuity point  $z$  for  $l_S$  and the neighborhoods  $B_\delta(z)$  and  $U_{S(z)}$  as in the previous claim.

Let  $\{\hat{x}_n\}_n \subset A$  be a sequence converging to the point  $\hat{x} \in A$ . Suppose, up to taking a subsequence, that  $S(\hat{x}_n)$  converges to a point  $\hat{y}$ . As  $A$  is  $\mathcal{W}^u$ -minimal and  $z \in A$  we can consider  $x \in \mathcal{W}^u(\hat{x}) \cap B_\delta(z)$  and  $x^u : [0, 1] \rightarrow M$  an unstable arc such that  $x^u(0) = x$  and  $x^u(1) = \hat{x}$ . We can consider also unstable arcs  $\{x_n^u : [0, 1] \rightarrow M\}_n$  converging uniformly to  $x^u$  such that  $x_n^u(0) = x_n$  lies in  $\mathcal{W}^u(\hat{x}_n) \cap B_\delta(z)$  and  $x_n^u(1) = \hat{x}_n$ .

Let us denote  $y_n = S(x_n)$  for every  $n$ . Notice that  $y_n \in U_{S(z)}$  since  $x_n \in B_\delta(z)$ . As the sequence  $x_n$  converges to  $x$  then by the first claim, up to taking a subsequence,  $[x_n, y_n]_c$  converges in the Hausdorff topology to a center segment  $[x, y]_c$  such that  $[S(x), y]_c$  is in  $[A']_c$ .

Consider now  $\{[x^u(t), y^u(t)]_c\}_{t \in [0, 1]}$  the transport by unstable holonomy

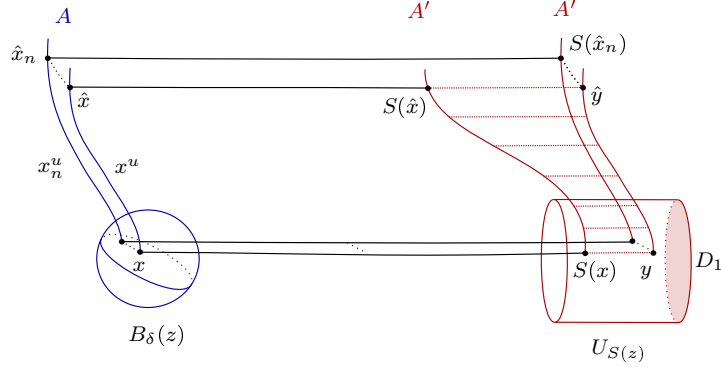


Figure 7.2:

of  $[x, y]_c$  along  $x^u$  such that  $[x^u(0), y^u(0)]_c = [x, y]_c$  (see Lemma 7.3.9). Consider also  $\{[x_n^u(t), y_n^u(t)]_c\}_{t \in [0, 1]}$  the transport by unstable holonomy of  $[x_n^u(0), y_n^u(0)]_c = [x_n, y_n]_c$  along  $x_n^u$  for every  $n$ . Notice that, as  $S(x_n) = y_n$  and  $A$  and  $A'$  are  $\mathcal{W}^u$ -invariant, then  $S(x_n^u(t)) = y_n^u(t)$  for every  $t \in [0, 1]$ .

The foliations  $\mathcal{W}^c$  and  $\mathcal{W}^u$  have  $C^1$  leaves tangent to continuous subbundles of  $M$ . So as  $x_n^u$  converges uniformly to  $x^u$  we have that  $[x_n^u(t), y_n^u(t)]_c$  has to converge in the Hausdorff topology to  $[x^u(t), y^u(t)]_c$  for every  $t \in [0, 1]$ . In particular, the sequence  $[x_n^u(1), y_n^u(1)]_c = [\hat{x}_n, S(\hat{x}_n)]_c$  needs to converge to  $[x^u(1), y^u(1)]_c$ . As  $x^u(1) = \hat{x}$  and  $S(\hat{x}_n) \xrightarrow{n} \hat{y}$  we obtain that  $[x^u(1), y^u(1)]_c = [\hat{x}, \hat{y}]_c$ . Then the sequence  $[\hat{x}_n, S(\hat{x}_n)]_c$  converges to  $[\hat{x}, \hat{y}]_c$ .

Finally, from the first claim,  $[S(x), y]_c = [S(x^u(0)), y^u(0)]_c$  is a center segment in  $[A']_c$ . This property is preserved by unstable holonomy since  $A$  and  $A'$  are  $\mathcal{W}^u$ -saturated so  $[S(x^u(t)), y^u(t)]_c$  is a center segment in  $[A']_c$  for every  $t \in [0, 1]$ . We conclude that  $[S(x^u(1)), y^u(1)]_c = [S(\hat{x}), \hat{y}]_c$  needs to be a center segment in  $[A']_c$  (see Figure 7.2) and this ends the proof of the lemma.  $\square$

We are now able to prove:

**Proposition 7.3.12.** *The sets  $(A, A')_c$  and  $(A', A)_c$  are disjoint open subsets of  $M$ . The sets  $[A]_c$  and  $[A']_c$  are disjoint closed subsets of  $M$ .*

*Proof.* The sets  $(A, A')_c$  and  $(A', A)_c$  are disjoint by definition. For proving that  $(A', A)_c$  is open let us see that its complement,  $[A] \cup (A, A')_c \cup [A']_c$ , is closed. The proof of  $(A, A')_c$  open is analogous.

Let  $\{v_n\}_n$  be a sequence in  $[A]_c \cup (A, A')_c \cup [A']_c$  converging to a point  $v$ . The sequence  $\{v_n\}_n$  lies infinitely many times in  $[A]_c \cup (A, A')_c$  or  $(A, A')_c \cup [A']_c$ . Suppose without loss of generality that it is the former. So, up to a

subsequence, there exist  $x_n \in A$  and  $y_n = S(x_n) \in A'$  such that  $v_n$  lies in the center segment  $[x_n, y_n]_c$  for every  $n$ .

Then by Lemma 7.3.11, up to taking a converging subsequence such that  $x_n \xrightarrow{n} x$  and  $y_n \xrightarrow{n} y$ , the sequence  $[x_n, y_n]_c$  converges in the Hausdorff topology to the center segment  $[x, y]_c$  and  $[S(x), y]_c$  is in  $[A']_c$ .

Then  $[x, y]_c = [x, S(x)]_c \cup [S(x), y]_c$  is a center segment in  $[A] \cup (A, A')_c \cup [A']_c$ . As the limit point  $v$  needs to lie in  $[x, y]_c$  this proves that  $[A] \cup (A, A')_c \cup [A']_c$  is closed.

The sets  $[A]_c$  and  $[A']_c$  are disjoint by definition. Let us see that  $[A]_c$  is closed. The proof of  $[A']_c$  closed is analogous.

Let  $\{w_n\}_n$  be a sequence in  $[A]_c$  converging to a point  $w$ . Suppose that each  $w_n$  is contained in a segment  $[x_n, z_n]_c$  in  $[A]_c$  and consider  $y_n = S(x_n)$  for every  $n$ . Then by Lemma 7.3.11, up to taking a converging subsequence such that  $x_n \xrightarrow{n} x$  and  $y_n \xrightarrow{n} y$ , the sequence  $[x_n, y_n]_c$  converges in the Hausdorff topology to the center segment  $[x, y]_c$  such that  $[S(x), y]_c$  is in  $[A']_c$ .

Up to taking another subsequence if necessary the sequence  $\{z_n\}_n \subset A$  converges to a point  $z \in A$  contained in  $[x, y]_c$ . The sequence  $[x_n, z_n]_c$  converges in the Hausdorff topology to  $[x, z]_c$  so the point  $w$  needs to lie in  $[x, z]_c$  as it is the limit of points  $w_n$  in  $[x_n, z_n]_c$ . Moreover, since  $[S(x), y]_c \cap A = \emptyset$ , then  $[x, z]_c$  needs to be contained in  $[x, S(x)]_c$ . We deduce that  $[x, z]_c$  is a center segment in  $[A]_c$  containing  $w$ . This proves that  $[A]_c$  is closed in  $M$ .  $\square$

Let us end this subsection with a small parenthesis:

**Remark 7.3.13.** Let us compare with the examples in [BG10] of Axiom A discretized Anosov flows having a proper attractor  $\Lambda$  and a proper repeller  $\Lambda'$  such that  $\mathcal{W}^c$  is not topologically conjugate to a suspension.

In these examples,  $\Lambda$  and  $\Lambda'$  are minimal unstable and stable laminations, respectively, and Proposition 7.3.1 is verified: there exist  $L > 0$  such that  $\mathcal{W}_L^c(x)$  intersects  $\Lambda$  and  $\Lambda'$  for every  $x \in M$ .

If one is tempted to imitate the present proof with  $\Lambda$  and  $\Lambda'$  in the place of  $A$  and  $A'$ , it fails at the following point: By considering analogously the sets  $(\Lambda, \Lambda')_c$  and  $(\Lambda', \Lambda)_c$  of center segments from  $\Lambda$  to  $\Lambda'$  and from  $\Lambda'$  to  $\Lambda$ , respectively, the property that is not verified is that  $(\Lambda, \Lambda')_c$  and  $(\Lambda', \Lambda)_c$  are open. Indeed, there exist segments in  $(\Lambda, \Lambda')_c$  accumulated by segments from  $(\Lambda', \Lambda)_c$ , and vice versa. The basin of  $\Lambda$  or  $\Lambda'$  does not decomposes in two connected components, rather  $\Lambda$  and  $\Lambda'$  are geometrically intertwined in such a fashion that its basins have a unique connected component.

### 7.3.4 Second part of the proof of Proposition 7.3.3: Construction of a global section

We can conclude from the previous subsection that  $M$  decomposes as the disjoint union

$$M = [A]_c \cup (A, A')_c \cup [A']_c \cup (A', A)_c,$$

where  $(A, A')_c$  and  $(A', A)_c$  are open subsets of  $M$ , and  $[A]_c$  and  $[A']_c$  are closed subsets of  $M$ . From Proposition 7.3.1 there exists  $L > 0$  such that  $\mathcal{W}_L^c(x) \cap A \neq \emptyset$  and  $\mathcal{W}_L^c(x) \cap A' \neq \emptyset$  for every  $x \in M$ . Let us see in this subsection that this is sufficient for showing that  $\mathcal{W}^c$  has to admit a global section.

Consider  $\theta : M \rightarrow [0, 1]$  continuous such that  $\theta^{-1}(0) = [A]_c$  and  $\theta^{-1}(1) = [A']_c$ . Define next  $\rho : M \rightarrow S^1$  such that

$$\rho(x) = \begin{cases} \frac{1}{2}\theta(x) \pmod{1} & \text{if } x \in [A]_c \cup (A, A')_c \cup [A']_c \\ 1 - \frac{1}{2}\theta(x) \pmod{1} & \text{if } x \in [A']_c \cup (A', A)_c \cup [A]_c \end{cases}$$

**Remark 7.3.14.** The function  $\rho : M \rightarrow S^1$  is well-defined and continuous.

*Proof.* If  $x$  is a point belonging both to  $[A]_c \cup (A, A')_c \cup [A']_c$  and  $[A']_c \cup (A', A)_c \cup [A]_c$  then  $x \in [A]_c = \theta^{-1}(0)$  or  $x \in [A']_c = \theta^{-1}(1)$ . In both cases,  $\frac{1}{2}\theta(x)$  and  $1 - \frac{1}{2}\theta(x)$  take the same value (mod 1). We obtain that  $\rho$  is well defined.

Since  $\rho$  is a continuous function restricted to each closed subset  $[A]_c \cup (A, A')_c \cup [A']_c$  and  $[A']_c \cup (A', A)_c \cup [A]_c$  (they are closed as they are the complement of  $(A', A)_c$  and  $(A, A')_c$ , respectively), and since the union of both closed subsets is  $M$ , then  $\rho$  is continuous.  $\square$

Recall that  $\phi^c : M \times \mathbb{R} \rightarrow M$  denotes a flow whose flow lines are the leaves of  $\mathcal{W}^c$ . Let us assume that  $\phi^c$  is parametrized by arclength. Let  $p : \tilde{M} \rightarrow M$  be the universal cover of  $M$  and  $\tilde{\phi}^c : \tilde{M} \times \mathbb{R} \rightarrow \tilde{M}$  be the lift of  $\phi^c$  to  $\tilde{M}$ . Consider  $\tilde{\rho} : \tilde{M} \rightarrow \mathbb{R}$  to be a lift of  $\rho : M \rightarrow S^1$ , that is, such that  $\pi \circ \tilde{\rho} = \rho \circ p$ .

As  $A$  and  $A'$  intersect every center segment of length  $2L > 0$  then for every  $\tilde{x}$  in  $\tilde{M}$ :

$$\tilde{\rho} \circ \tilde{\phi}^c(\tilde{x}, 4L) - \tilde{\rho} \circ \tilde{\phi}^c(\tilde{x}, 0) > 1. \quad (7.3.2)$$

Notice that for a given  $x$  in  $M$  the difference considered in (7.3.2) is independent of the lift  $\tilde{x}$  of  $x$ . Informally, it measures how much ‘winds around’  $S^1$  the image by  $\rho$  of the center segment  $[x, x + 4L]_c$ .

Now an argument of Schwartzman (see [Sc57] and [BG09]) allows us to conclude the proof of Proposition 7.3.3. We reproduce it for the sake of completeness.

**Proposition 7.3.15.** *Let  $M$  be a smooth manifold and  $\phi : M \times \mathbb{R} \rightarrow M$  be a flow tangent to a continuous vector field  $X_\phi$  in  $M$  and satisfying (7.3.2) for a certain continuous function  $\rho : M \rightarrow S^1$  and some constant  $L > 0$ . Then  $\phi$  admits a smooth global section.*

*Proof.* Let  $p : \tilde{M} \rightarrow M$  be the universal cover of  $M$ . Consider  $X_\psi$  a smooth vector field  $C^0$ -close to  $X_\phi$  and  $\mu : M \rightarrow S^1$  a smooth map  $C^0$ -close to  $\rho : M \rightarrow S^1$ . Let  $\tilde{\psi} : \tilde{M} \rightarrow \tilde{M}$  be the lift to  $\tilde{M}$  of the flow  $\psi : M \rightarrow M$  tangent to  $X_\psi$  and  $\tilde{\mu} : \tilde{M} \rightarrow \mathbb{R}$  be such that  $\tilde{\pi} \circ \tilde{\mu} = \mu \circ p$ . Then, if  $X_\psi$  and  $\mu$  are close enough to  $X_\phi$  and  $\rho$ , respectively, we still have

$$\tilde{\mu} \circ \tilde{\psi}(x, 4L) - \tilde{\mu} \circ \tilde{\psi}(x, 0) > 1,$$

for every  $x \in \tilde{M}$ .

Let us consider now the smooth map  $\tilde{\lambda} : \tilde{M} \rightarrow \mathbb{R}$  given by

$$\tilde{\lambda}(\tilde{x}) = \frac{1}{4L} \int_0^{4L} \tilde{\mu} \circ \tilde{\psi}(\tilde{x}, t) dt.$$

We claim that  $\tilde{\lambda}$  projects to a map  $\lambda : M \rightarrow S^1$ . Indeed, if  $\tilde{x}$  and  $\tilde{y}$  are two points in  $\tilde{M}$  such that  $x = p(\tilde{x}) = p(\tilde{y})$  then there exists an integer  $n$  such that  $\tilde{\mu}(\tilde{y}) = \tilde{\mu}(\tilde{x}) + n$ . Furthermore,  $n$  satisfies that  $\tilde{\mu} \circ \tilde{\psi}(\tilde{y}, t) = \tilde{\mu} \circ \tilde{\psi}(\tilde{x}, t) + n$  for every  $t$ . This implies that  $\tilde{\lambda}(\tilde{y}) = \tilde{\lambda}(\tilde{x}) + n$ . We deduce that

$$\lambda(x) := \tilde{\lambda}(\tilde{x}) \pmod{1}$$

is well defined independently of the lift  $\tilde{x}$ . This proves the claim.

Moreover, for any  $\tilde{x}$  in  $\tilde{M}$  we have:

$$\frac{\partial}{\partial t} \tilde{\lambda} \circ \tilde{\psi}(\tilde{x}, t)|_{t=0} = \frac{1}{4L} (\tilde{\mu} \circ \tilde{\psi}(\tilde{x}, 4L) - \tilde{\mu} \circ \tilde{\psi}(\tilde{x}, 0)) > \frac{1}{4L} > 0. \quad (7.3.3)$$

This proves that  $\lambda : M \rightarrow S^1$  is a submersion such that the orbits of  $\psi$  are transverse to the fibers. We obtain that  $N = \lambda^{-1}(0)$  is a submanifold of  $M$  that is a global section for the flow  $\psi$ .

Moreover, since (7.3.3) gives us a positive lower bound (which only depends on the a priori constant  $L > 0$ ) for the angle between the vector field  $X_\psi$  and the fibers of  $\lambda : M \rightarrow S^1$  then we can consider  $X_\psi$  to be  $C^0$ -close enough to  $X_\phi$  so that  $\phi : M \times \mathbb{R} \rightarrow M$  is also transverse to the fibers and  $N$  is a global section for  $\phi$ .  $\square$

## 7.4 Finiteness of attractors for certain discretized Anosov flows such that $\varphi_t^c$ is not transitive

Given a discretized Anosov flow  $f$  with center flow  $\varphi_t^c : M \rightarrow M$  we have shown in Proposition 3.7.2 that  $\varphi_t^c : M \rightarrow M$  is a topological Anosov flow (see Definition 3.7.1). In fact, an a priori stronger property than the ones stated in Definition 3.7.1 is satisfied by  $\varphi_t^c : M \rightarrow M$ :

**Remark 7.4.1.** It is immediate from the proof of Proposition 3.7.2 that the following ‘uniform’ form of contraction and expansion is satisfied by the center flow  $\varphi_t^c : M \rightarrow M$  of a discretized Anosov:

There exists constants  $\epsilon > 0$ ,  $C > 0$  and  $\lambda \in (0, 1)$  satisfying that for every  $x \in M$  and  $y \in \mathcal{W}_\delta^s(x)$  there exists a homeomorphism  $h_y : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\varphi_{h_y(t)}^c(y)$  lies in  $\mathcal{W}^s(\varphi_t^c(x))$  and

$$d_s(\varphi_t^c(x), \varphi_{h_y(t)}^c(y)) < Ce^{\lambda t} d_s(x, y)$$

for every  $t \geq 0$ .

Analogously for  $u$  in the place of  $s$  and backwards orbits of  $\varphi_t^c$ .

Using the property stated in the above remark and the local product structure given by the weak-stable and weak-unstable foliations of  $\varphi_t^c$  (which coincides with center stable  $\mathcal{W}^{cs}$  and center-unstable  $\mathcal{W}^{cu}$  of  $f$ ) one can show that  $\varphi_t^c$  needs to satisfy the *shadowing property* and that the *spectral decomposition* stated in the following lemma is satisfied by means of the same type of arguments used for classical Anosov flows. See for example [FH19] as a reference.

Recall that the *non-wandering set* of a flow  $\varphi_t : M \rightarrow M$  is the set  $\Omega(\varphi_t) := \{x \in M \mid \text{for every neighborhood } U \text{ of } x \text{ and } T > 0 \text{ there exists } t > T \text{ s.t. } \varphi_t(U) \cap U \neq \emptyset\}$ .

**Lemma 7.4.2.** *The topological Anosov flow  $\varphi_t^c : M \rightarrow M$  obtained as the center flow of a discretized Anosov flow  $f$  satisfies that the non-wandering set  $\Omega(\varphi_t^c)$  admits a decomposition*

$$\Omega(\varphi_t^c) = \Lambda_1 \cup \dots \cup \Lambda_N,$$

where  $\{\Lambda_i\}_{1 \leq i \leq N}$  are compact, disjoint and  $\varphi_t^c$ -invariant subsets of  $M$  such that  $\varphi_t^c|_{\Lambda_i} : \Lambda_i \rightarrow \Lambda_i$  is transitive for every  $1 \leq i \leq N$ .

Moreover, some of them,  $\Lambda_1, \dots, \Lambda_k$ , are attracting basic pieces satisfying that their joint basin of attraction  $\mathcal{F}^{ws}(\Lambda_1) \cup \dots \cup \mathcal{F}^{ws}(\Lambda_k)$  is an open and dense subset of  $M$  and that  $\mathcal{F}^{cu}(\Lambda_i) = \Lambda_i$  for every  $1 \leq i \leq k$ .

We are now able to show Theorem 7.1.3:

*Adapting the proof of Theorem 7.1.1 to show Theorem 7.1.3.* Let  $f$  be a discretized Anosov flow such that its center flow  $\varphi_t^c : M \rightarrow M$  is not transitive.

Let  $\Lambda$  be an attracting basic piece of  $\varphi_t^c$ . Since the flow  $\varphi_t^c|_\Lambda : \Lambda \rightarrow \Lambda$  is transitive one obtains that  $\mathcal{W}^{cs}(x) \cap \Lambda$  is dense in  $\Lambda$  for every  $x$  in  $\Lambda$ , otherwise  $\overline{\mathcal{W}^{cs}(x) \cap \Lambda}$  would be a proper repeller for  $\varphi_t^c|_\Lambda$ . So  $\mathcal{W}^{cs}|_\Lambda$  is minimal in  $\Lambda$ . As  $f$  acts quasi-isometrically on  $\mathcal{W}^c$  then Proposition 7.3.1 adapts identically and we get that there exists  $L > 0$  such that every minimal unstable lamination  $A$  in  $\Lambda$  verifies that  $\mathcal{W}_L^c(x) \cap A \neq \emptyset$  for every  $x \in \Lambda$ .

Suppose now that there exist two different minimal unstable laminations  $A$  and  $A'$  in  $\Lambda$ . We want to show that  $\varphi_t^c|_\Lambda$  needs to be orbit equivalent to a suspension. This will conclude the proof of Theorem 7.1.3.

We can analogously define the sets  $[A]_c$ ,  $(A, A')_c$ ,  $[A']_c$  and  $(A', A)_c$  as in the proof of Proposition 7.3.3. The proof that  $(A, A')_c$  and  $(A', A)_c$  are open and that  $[A]_c$  and  $[A']_c$  are closed in  $\Lambda$  works analogously. This allows us to define a continuous function  $\rho : \Lambda \rightarrow S^1$  such that

$$|\widetilde{\rho \circ \varphi_{4L}}(x) - \widetilde{\rho \circ \varphi_0}(x)| > 1, \quad (7.4.1)$$

where  $t \mapsto \widetilde{\rho \circ \varphi_t^c}(x) : \mathbb{R} \rightarrow \mathbb{R}$  is any lift of  $t \mapsto \rho \circ \varphi_t^c(x)$  for every  $x \in \Lambda$ .

We can extend now  $\rho$  to a small open  $\varphi_t^c$ -forward invariant neighborhood  $U$  of  $\Lambda$  in the following way: We can cover  $\Lambda$  by  $B_{\delta_1}(x_1) \cup \dots \cup B_{\delta_j}(x_j)$  such that  $x_i \in \Lambda$  and  $|\rho(x) - \rho(x_i)| < 1/10$  for every  $x \in \Lambda \cap B_{\delta_i}(x_i)$ . By Tietze extension theorem we can extend  $\rho|_{\Lambda \cap B_{\delta_i}(x_i)}$  to  $\rho_i : B_{\delta_i}(x_i) \rightarrow S^1$  such that we still have  $|\rho_i(x) - \rho(x_i)| < 1/10$  for every  $x \in B_{\delta_i}(x_i)$ . Then by taking a partition of unity  $\{\tau_i : B_{\delta_i}(x_i) \rightarrow [0, 1]\}_i$  subordinated to  $\{B_{\delta_i}(x_i)\}_i$  the functions  $\{\rho_i\}_i$  can be interpolated in order to obtain an extension of  $\rho$  to  $B_{\delta_1}(x_1) \cup \dots \cup B_{\delta_j}(x_j)$ . Finally, we can take  $V \subset B_{\delta_1}(x_1) \cup \dots \cup B_{\delta_j}(x_j)$  such that  $\varphi_t^c(V) \subset B_{\delta_1}(x_1) \cup \dots \cup B_{\delta_j}(x_j)$  for every  $t \geq 0$  and then define  $U = \bigcup_{t \geq 0} \varphi_t^c(V)$ .

This construction of  $U$  gives us that (7.4.1) continues to be valid for every  $x \in U$ . The argument of Schwartzman also works well restricted to  $U$ : by taking smooth approximations  $\mu$  and  $X_\psi$  of  $\rho$  and  $\frac{\partial \varphi_t^c}{\partial t}|_{t=0}$ , respectively, we can define the function  $\lambda : U \rightarrow S^1$  as  $\lambda(x) = \frac{1}{4L} \int_0^L \mu \circ \psi_t(x) dt \pmod{1}$  and obtain that  $\frac{\partial}{\partial t} \widetilde{\lambda \circ \psi_t}(x)|_{t=0} > \frac{1}{4L} > 0$  for every  $x \in U$ . Then  $\lambda^{-1}(0)$  gives us a global forward section for  $\varphi_t^c|_U$ . This global forward section gives us a global section for  $\varphi_t^c|_\Lambda$ .  $\square$

*Proof of Corollary 7.1.4.* Let  $f$  be a discretized Anosov flow such that its center flow  $\varphi_t^c$  is not transitive. Let  $\Lambda_1, \dots, \Lambda_k$  be the attracting basic pieces of  $\varphi_t^c$  and suppose that  $\varphi_t^c$  is not orbit equivalent to a suspension restricted to any of these pieces.

Recall that  $\mathcal{W}^{cs}(\Lambda_1) \cup \dots \cup \mathcal{W}^{cs}(\Lambda_k)$  is an open and dense  $\mathcal{W}^{cs}$ -saturated subset of  $M$ . As  $f$  acts quasi-isometrically on  $\mathcal{W}^c$  then  $\mathcal{W}^{cs}(x) = \mathcal{W}^s(\mathcal{W}^c(x))$  for every  $x$  in  $M$  by Proposition 3.4.4. As each  $\Lambda_i$  is  $\mathcal{W}^c$ -saturated, then  $\mathcal{W}^{cs}(\Lambda_1) \cup \dots \cup \mathcal{W}^{cs}(\Lambda_k)$  coincides with  $\mathcal{W}^s(\Lambda_1) \cup \dots \cup \mathcal{W}^s(\Lambda_k)$ .

We claim that there exists  $R > 0$  such that

$$\mathcal{W}^u(x) \cap (\mathcal{W}_R^s(\Lambda_1) \cup \dots \cup \mathcal{W}_R^s(\Lambda_k)) \neq \emptyset$$

for every  $x$  in  $M$ . Indeed, let  $V_1^u, \dots, V_j^u$  be a finite collection of  $\mathcal{W}^u$ -foliation boxes such that  $\bigcup_i V_i^u = M$ . For every  $1 \leq i \leq j$  there exist  $R_i > 0$  such that  $\mathcal{W}_{R_i}^s(\Lambda_1) \cup \dots \cup \mathcal{W}_{R_i}^s(\Lambda_k)$  intersects every  $\mathcal{W}^u$ -plaque in  $V_i^u$ . The claim follows from taking  $R = \max\{R_1, \dots, R_j\}$ .

As a consequence of the previous claim we obtain that

$$\overline{\mathcal{W}^u(x)} \cap (\Lambda_1 \cup \dots \cup \Lambda_k) \neq \emptyset$$

for every  $x$  in  $M$ . Indeed, as  $\mathcal{W}^u(f^{-n}(x))$  intersects  $(\mathcal{W}_R^s(\Lambda_1) \cup \dots \cup \mathcal{W}_R^s(\Lambda_k))$  for every  $n \geq 0$  then  $\mathcal{W}^u(x) = f^n(\mathcal{W}^u(f^{-n}(x)))$  is at distance 0 from  $\Lambda_1 \cup \dots \cup \Lambda_k$ . We deduce that every minimal unstable lamination for  $f$  intersects  $\Lambda_1 \cup \dots \cup \Lambda_k$ .

Moreover, as each attracting basic piece is compact and  $\mathcal{W}^u$ -saturated, then every minimal unstable lamination of  $f$  has to be contained in one of the attracting basic pieces.

Finally, by Theorem 7.1.3, each attracting basic piece  $\Lambda_i$  contains a unique minimal unstable lamination. We conclude that  $f$  admits exactly  $k$  minimal unstable laminations and that each one of them is contained in one of the attracting basic pieces  $\Lambda_1, \dots, \Lambda_k$  of  $\varphi_i^c$ .  $\square$



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