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Thèse de doctorat en Mathématiques

de

Université de Paris
Ecole Doctorale de Sciences Mathématiques de Paris Centre (ED 386)
Institut de Mathématiques de Jussieu-Paris Rive Gauche (UMR 7586)

Surfaces in 3D contact sub-Riemannian manifolds and controllability of nonlinear ODEs

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Abstract

The four chapters of this thesis present original results about smooth surfaces in a 3-dimensional contact sub-Riemannian manifold, and properties involving controllability in geometric control theory. They are preceded by an introduction, which gives an overview of the results and the previous literature.

The topic of smooth surfaces is studied from two viewpoints. First, given a surface in a 3-dimensional contact sub-Riemannian manifold, we investigate the metric structure induced on the surface, in the sense of length spaces. We define a new metric coefficient at any characteristic point, which determines locally the characteristic foliation of the surface, and we identify some global conditions for the induced distance to be finite. In particular, we prove that the induced distance is finite for surfaces with the topology of a sphere embedded in a tight coorientable distribution, with isolated characteristic points.

Second, we study a new canonical stochastic process on such surfaces. Precisely, employing the Riemannian approximations with respect to the Reeb vector field of the sub-Riemannian manifold, we obtain a second order partial differential operator on the surface arising as a limit of Laplace–Beltrami operators. The stochastic process associated with this limiting operator moves along the characteristic foliation induced on the surface by the contact distribution. For this stochastic process we show that elliptic characteristic points are inaccessible, while hyperbolic characteristic points are accessible from the separatrices. We illustrate this process with examples, and we recognise some well-known stochastic processes appearing on certain surfaces embedded in the canonical model spaces for sub-Riemannian structures on 3-dimensional Lie groups.

Concerning controllability, we show that a control system on a connected manifold satisfying the local controllability property is controllable, as it was somehow implicitly expected. Herein we say that a control systems satisfies the local controllability property if the attainable sets from any initial state are a neighbourhood of the respective initial states, while a system is controllable if the attainable set from every state is the entire state manifold. After the dissertation took place, we had been informed that our self-contained proof is alternative to the combination of two previous results by Kevin Grasse.

Finally, we show that a bilinear control system is approximately controllable if and only if it is controllable in $\mathbb{R}^n \setminus \{0\}$. We approach this property by looking at the foliation made by the orbits of the system, and by showing that there does not exist a codimension-one foliation in $\mathbb{R}^n \setminus \{0\}$ with dense leaves that are everywhere transversal to the radial direction. The proposed geometric approach allows to extend the result to homogeneous systems that are angularly controllable.

Keywords: contact geometry, sub-Riemannian manifold, Stochastic process, controllability, local controllability, approximate controllability, bilinear control systems, foliations, length space, Riemannian approximation, Gaussian curvature, Heisenberg group

Resumé

Les quatre chapitres de cette thèse contiennent des résultats originaux relatifs aux surfaces dans une variété sous-riemanienne de contact de dimension trois, et à certaines propriétés concernant la contrôlabilité en théorie géométrique du contrôle. Ils sont précédés par une introduction, qui donne un aperçu de ces résultats et de la littérature antérieure.

Nous avons étudié le sujet des surfaces de deux points de vue. En premier lieu, étant donnée une surface dans une variété sous-riemannienne de contact de dimension trois, nous examinons la structure métrique induite sur la surface, au sens des espaces de longueur. Nous définissons un nouveau coefficient métrique en tout point caractéristique de la surface, et nous identifions des conditions globales pour que la distance induite soit finie. En particulier, nous montrons que la distance induite est finie pour des surfaces avec la topologie d'une sphère, plongées dans une distribution coorientable tendue, et avec des points caractéristiques isolés.

En second lieu, nous étudions un nouveau processus stochastique sur des telles surfaces. Précisément, en utilisant l'approximation riemanienne par rapport au champ de Reeb de la structure sous-riemanienne, nous obtenons un opérateur différentiel d'ordre deux sur la surface résultant de la limite d'opérateurs de Laplace–Beltrami. Le processus stochastique associé avec cet opérateur se déplace le long du feuilletage caractéristique induit sur la surface par la distribution de contact. Pour ce processus stochastique nous montrons que les points caractéristiques elliptiques sont inaccessibles, tandis que les points caractéristiques hyperboliques sont accessibles à travers les séparatrices. Nous illustrons ce processus avec des exemples, et nous reconnaissons des processus stochastiques classiques qui apparaissent sur certaines surfaces plongées dans les espaces modèles de structure sous-riemanienne sur les groupes de Lie de dimension trois.

Quant à la contrôlabilité, nous montrons qu'un système qui est localement controllable dans une variété connexe est contrôlable. Ci-dessus nous disons qu'un système de contrôle est localement controllable si les ensembles atteignables à partir de tout état sont un voisinage de l'état de départ, tandis que le système est contrôlable si les ensembles atteignables à partir de tout état coïncident avec la variété entière. Après la soutenance, nous avons été informé que notre démonstration indépendante est alternative a deux résultats precedents par Kevin Grasse.

Pour conclure, nous montrons qu'un système de contrôle bilinéaire est contrôlable de façon approchée si et seulement s'il est contrôlable en $\mathbb{R}^n \setminus \{0\}$. Nous étudions ce problème en analysant le feuilletage défini par les orbites du système, et en montrant qu'il n'existe pas de feuilletage de codimension un en $\mathbb{R}^n \setminus \{0\}$ dont les feuilles sont denses et partout transversales à la direction radiale. L'approche géométrique ainsi proposée permet d'étendre ce résultat aux systèmes homogènes qui sont contrôlables angulairement.

Mots-clés : géométrie de contact, variété sous-riemanienne, processus stochastique, contrôlabilité, contrôlabilité locale, contrôlabilité approchée, système de contrôle bilinéaire, feuilletage, espace de longueur, approximation riemanienne, courbure de Gauss, groupe d'Heisenberg

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Chapter 1

Introduction

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In this chapter we present some original results about controllability in geometric control theory, and about properties of smooth surfaces in a three-dimensional contact sub-Riemannian manifold. The results are accompanied by the relevant literature and are proven in the forthcoming chapters.

Precisely, in Section 1.1 we introduce control systems, and we present some new relations:

- between controllability and local controllability (in Subsection 1.1.1),
- between controllability and approximate controllability (in Subsection 1.1.3).

The former result is discussed in Chapter 6, following my paper [BCFS21] (joint work with U. Boscain, V. Franceschi and M. Sigalotti), currently submitted for review. The latter is developed in Chapter 5 and published in [CS21] (joint work with M. Sigalotti).

Next, in Section 1.2 we present some new metric properties of surfaces embedded in 3D contact sub-Riemannian manifolds. Indeed, we present:

- a new metric invariant $\widehat{K}$ at the characteristic points (in Subsection 1.2.2),
- an analysis of the length distance induced on surfaces (in Subsection 1.2.3).

These results are discussed more extensively in Chapter 3 and published in my paper [BBC21] (joint work with D. Barilari and U. Boscain).

Finally, in Section 1.3 we discuss the properties of a new canonical stochastic process defined on such surfaces. These results are discussed in Chapter 4 and published in my paper [BBCH21] (joint work with D. Barilari U. Boscain and K. Habermann).

Notations. In what follows M is a smooth n-dimensional manifold. We denote the vector fields of M by capital letters such as X, Y and Z. The notations $X(f) = Xf$ and $X(q)$ indicate, respectively, the derivative of a smooth function f in $C^\infty(M)$ with respect to X, and the derivation based at a point q in M defined by X. The $C^\infty(M)$-module of vector fields of M is denoted $\Gamma(TM)$, which with the Lie bracket $[\cdot,\cdot]$ is a Lie algebra. The flow at time t of a vector field X is denoted by e^{tX}. For notational simplicity we assume that the vector fields are complete, i.e., their flows are defined for all t in $\mathbb{R}$.

1.1 Control systems associated with a family of vector fields

Let $\mathcal{F} \subset \Gamma(TM)$ be any set of smooth vector fields on a manifold M. Let Ω be a set of indices for the family $\mathcal{F}$, i.e., $\mathcal{F} = \{X_u \mid u \in \Omega\}$. The *control system* associated to the family $\mathcal{F}$ is the system

$$\dot{p}(t) = X_{u(t)}(p(t)), \qquad p \in M,\ X_{u(t)} \in \mathcal{F}, \tag{C}$$

where as *control* we use maps $u \in \mathcal{U}_{pc}$, where $\mathcal{U}_{pc} = \bigcup_{T\geq 0}\{u : [0,T] \to \Omega \mid u \text{ piecewise constant}\}$. Following the vocabulary of control systems, an element $u \in \Omega$ is called a *control parameter*, Ω is the *space of control parameters*, points in M are *states*, and M is the *state space*.

Since (for the moment) we use piecewise constant controls, the differential equation (C) is defined in the classical sense, up to a discrete set of times in which the control is not continuous. Therefore, once a control u is fixed, the solution of (C) is determined by the initial conditions. Let us note $\phi(t,p,u)$ the value at time t of the solution of (C) starting from p. Precisely, consider a control $u\colon [0,T] \to \Omega$ such that there exists a partition $0 = t_0 < t_1 < \cdots < t_k = T$ and control parameters $u_1,\dots,u_k \in \Omega$ satisfying

$$u(t) = u_i, \qquad \forall t \in (t_{i-1}, t_i),\ i = 1,\dots,k.$$

If $t \in [t_{i-1}, t_i]$ for a certain $i = 1,\dots,k$, then

$$\phi(t,p,u) = e^{(t-t_{i-1})X_{u_i}} \circ e^{(t_{i-1}-t_{i-2})X_{u_{i-1}}} \circ \cdots \circ e^{t_1 X_{u_1}}(p),$$

namely $\phi(t,p,u)$ is constructed by concatenating the flows of the vector fields in $\mathcal{F}$ indexed by the control u. The *attainable set* $\mathcal{A}_p$ from a state p in M for system (C) is the set of points reached by solutions of (C) starting from p using positive times; precisely,

$$\mathcal{A}_p = \{e^{t_k X_k} \circ \cdots \circ e^{t_1 X_1}(p) \mid k \in \mathbb{N},\ t_1,\dots,t_k \geq 0,\ X_1,\dots,X_k \in \mathcal{F}\}.$$

Similarly, the set of points reached using positive and negative times is called the *orbit* $\mathcal{O}_p$ of a state x; precisely,

$$\mathcal{O}_p = \{e^{t_k X_k} \circ \cdots \circ e^{t_1 X_1}(p) \mid k \in \mathbb{N},\ t_1,\dots,t_k \in \mathbb{R},\ X_1,\dots,X_k \in \mathcal{F}\}.$$

Attainable sets are of greater interest than orbits from the point of view of control theory, since they are obtained following strictly the vector fields in $\mathcal{F}$. Indeed, to follow the flow of a vector field $X \in \mathcal{F}$ for negative time is equivalent to follow the flow of $-X$, but $-X$ need not to be in the family $\mathcal{F}$. Note that orbits and attainable sets coincide if the family $\mathcal{F}$ is symmetric, i.e., if $-\mathcal{F} = \mathcal{F}$.

Sometimes we want to use measurable, essentially bounded functions as controls, instead of piecewise constant maps. When this is the case, we implicitly assume that Ω is a subset of $\mathbb{R}^m$, for some $m \in \mathbb{N}$, and that a smooth function $X : M \times \Omega \to TM$ parametrises $\mathcal{F}$, i.e., $\mathcal{F} = \{X(\cdot,u) \mid u \in \Omega\}$. To adhere to the notation in system (C), we continue to write $X_u = X(\cdot,u)$ for all u in Ω. We denote the set of essentially bounded controls as

$$\mathcal{U}_\infty = \bigcup_{T\geq 0} L^\infty([0,T],\Omega).$$

Fixed a control u in $\mathcal{U}_\infty$, the non-autonomous differential equation (C) is well-posed in the space of absolutely continuous functions. More precisely, for any given initial condition p in M, there exist

$$
\begin{tikzcd}
 & \text{ST-local controllability} \arrow[dr, Rightarrow] & & \\
\text{STL-local controllability} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{local controllability} \arrow[r, bend right, Rightarrow, "\text{Thm. 1.1}"'] & \text{controllability} \arrow[l, bend right, Rightarrow] \\
 & \text{L-local controllability} \arrow[ur, Rightarrow] & &
\end{tikzcd}
$$

Table 1.1: Relations between different types of local controllability. As discusses in Section 6.2, the missing arrows cannot be added to the scheme. The only arrow that needs to be justified here is the one representing the fact that local controllability implies controllability. This is the object of Theorem 1.1.

$T > 0$ and a neighbourhood V of p such that $\phi(t, q, u)$ is defined for $(t, q) \in [0, T] \times V$ and absolutely continuous with respect to time. Moreover, for $t \in [0, T]$, the flow $\phi(t, \cdot, u)$ restricted to V is a local diffeomorphism (see, e.g., [Jea17, Thm. 6.2] or [Son98, Thm. 1]).

In what follows $\mathcal{U}$ denotes one of the control set presented above. (Observe that $U_{pc} \subset \mathcal{U}_\infty$.) The attainable set from a state p in M is written in full generality as

$$\mathcal{A}_p = \{\phi(T, p, u) \mid T \geq 0,\ u \in \mathcal{U},\ \phi(\cdot, p, u) \text{ is defined on } [0, T]\}.$$

System (C) is said to be *controllable* if the attainable set from any state in M coincides with the entire state space, i.e.,

$$\mathcal{A}_p = M, \qquad \forall\, p \in M.$$

1.1.1 Local controllability and controllability

System (C) is said to be *local controllability* if, for each $p \in M$ the attainable set $\mathcal{A}_p$ contains a neighbourhood of p, i.e.,

$$p \in \operatorname{Int} \mathcal{A}_p, \qquad \forall\, p \in M. \tag{LC}$$

Observe that sometimes in the literature the expression *local controllability* is used with a different meaning (see, for example, [Cor07, Definition 3.2]).

The notion of local controllability has been studied extensively in the literature, especially in the stronger forms of *small-time local controllability* (ST-local controllability) (for which the attainable set $\mathcal{A}_x$ is replaced by the set of points attainable from x within an arbitrarily small positive time) and *localized local controllability* (L-local controllability) (for which one considers the set of points attainable from x by admissible trajectories that stay in an arbitrarily small neighborhood of x). A combination of the two constraints yields the notion of *small-time localized local controllability* (STL-local controllability). Table 1.1 contains a scheme of the implications that can be directly deduced from the above definitions, and we refer to Section 6.2 for a detailed description of these different types of local controllability and the relations between them.

Folklore has it that controllability can be deduced from suitable versions of local controllability: for example, in [CLH$^+$07, Section 12.3] it is stated (without proof) that STL-local controllability implies controllability. Another example are linear systems for which controllability is known to be equivalent to ST-local controllability. The question whether ST-local controllability implies controllability for a more general control system was formulated for instance in [BS83, Sec. 3]. The purpose of our study is to prove that the weakest version of local controllability is sufficient to deduce controllability, giving in particular a positive answer to the question just mentioned.

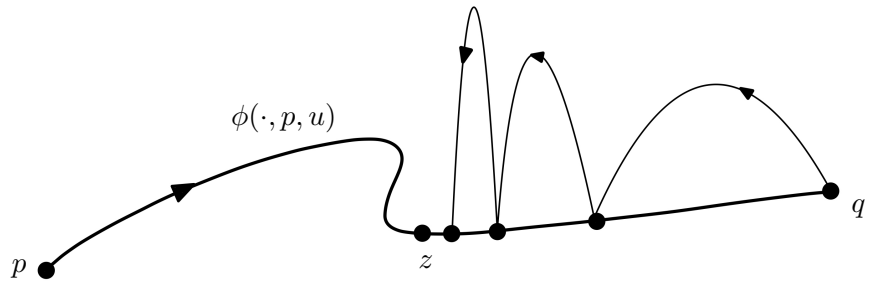

Figure 1.1: When retracing back the trajectory $\phi(\cdot, p, u)$ the reachable sets might get smaller and smaller and collapse to a point z before attaining x, since a priori their size is not lower semi-continuous. Lemma 6.2 shows that this situation cannot happen, proving a key step for the proof of Theorem 1.1.

Theorem 1.1 [BCFS21, Thm. 1]

Assume that M is connected and that system (C) is equipped with controls in $\mathcal{U}_{pc}$ or $\mathcal{U}_\infty$. If system (C) satisfies the local controllability property, then it is controllable.

We mention that if L-local controllability is known, then controllability can be shown with a simpler proof than what is proposed here for Theorem 1.1 (see Section 6.5) but still does not follow immediately from the definitions, since reachability is not a symmetric property. Indeed, the fact that one can reach an open neighborhood of a given initial state does not imply that any point in the neighborhood can be steered back to the initial state.

Let us mention that it is hard to find testable conditions for local controllability to hold. Indeed, in the literature it is more common to find conditions for ST-local controllability since those can be deduced from Lie algebraic arguments (see, e.g., [KN21] and references therein). It should be noticed that such conditions do not usually provide local controllability at *every* point, since they typically require that the point at which local controllability is studied is an equilibrium of one of the admissible vector fields. Finally, we note that the interest in ST-local controllability is motivated, for example, by its relation with the continuity of the optimal time function, as explained in [Sus87].

Our proof of Theorem 1.1 is based on the following property: if system (C) satisfies the local controllability property, then for all p, q in M one has that $p \in \mathcal{A}_q$ if and only if $q \in \mathcal{A}_p$ (see Lemma 6.2). This property is shown by proving that the trajectories of (C) can be retraced back by finding a control driving their endpoints to their starting points. More precisely, assume $q \in \mathcal{A}_p$ and consider a control u such that $q = \phi(T, p, u)$. For t in a left neighbourhood of T, the states $\phi(t, p, u)$ can be reached from q due to local controllability. By repeating this argument and concatenating controls, one can find smaller and smaller times $t \geq 0$ such that $\phi(t, p, u)$ can be reached from q. In order to reach $p = \phi(0, p, u)$ one has to show that the sequence of times t found following such a procedure eventually attains zero, unlike the situation depicted in Figure 1.1.

Remark 1.2. The argument of the proof of Theorem 1.1 generalises to more general classes of controls, provided that the control system remains well-posed in the space of absolutely continuous functions, in the sense expressed in Section 1.1. Here we decided to use essentially bounded controls rather than to stick with piecewise constant controls in order to show that the differences which arise between controls in $\mathcal{U}_\infty$ and $\mathcal{U}_{pc}$ do not affect Theorem 1.1.

The fact that local controllability implies global controllability was already treated in [KS83] for a compact state manifold M. For the noncompact case, in [Gra85] the author shows that local controllability implies global one for piecewise constant controls. A related result in [Gra95] shows that local controllability via bounded measurable controls implies the local controllability via piecewise

constant controls, therefore extending the previous result to control systems with bounded measurable controls. We are grateful to Kevin Grasse for attracting our attention on these papers after the first version of our proof was presented on arXiv.

1.1.2 Orbits, distributions and Lie-brackets

In this section we recall some important classical properties of orbits and attainable sets. First, orbits are immersed submanifolds of M. Precisely, an *immersed k-dimension submanifold* is a subset $S \subset M$ with a structure of smooth k-dimensional manifold (not necessarily with the topology inherited by M) such that the pushforward of the inclusion $i : S \hookrightarrow M$ satisfies $\dim i_*(T_pS) = k$ for all $p \in S$, i.e., the inclusion is an *immersion*.

Theorem 1.3 (Orbit theorem, [Sus73])**.** *For every $q \in M$, the orbit $\mathcal{O}_q$ is a connected, immersed submanifold of M. Moreover, for all $p \in \mathcal{O}_q$,*

$$T_p\mathcal{O}_q = \operatorname{span}_{\mathbb{R}}\big\{(e_*^{t_kX_k} \circ \cdots \circ e_*^{t_1X_1}Y)(p) \mid k \in \mathbb{N},\ t_1, \dots, t_k \in \mathbb{R},\ X_1, \dots, X_k, Y \in \mathcal{F}\big\}. \tag{1.1}$$

It follows from the orbit theorem that the orbits define a *foliation*, i.e., a partition of X in connected, immersed submanifolds (called *leaves*), possibly of varying dimension. Furthermore, the foliation described by the orbits is a *smooth* foliation, i.e., for any p in M and vector $v \in T_pM$ tangent to the leaf through p can be extended to a smooth vector field everywhere tangent to the leafs of the foliation. Indeed, the fact that foliations arising as orbits of control systems as in (C) are smooth is a consequence of the orbit theorem and formula (1.1). Conversely, any smooth foliation is the orbit partition of a control system. For this, it suffices to take as family of admissible vector fields the collection of all vector fields that are everywhere tangent to the leaves.

Remark 1.4. In some texts the term foliation describes what we shall call here a *regular foliation*, i.e., a foliation which admits locally around each point a chart (U, x) such that $x(U)$ can be written as $x(U) = V' \times V'' \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, for a fixed $k \in \mathbb{N}$, and such that the intersection of a leaf with U is either empty or the countable union of sets of the form $x^{-1}(V' \times \{c\})$ for $c \in V''$. Such a chart is called a *foliated* chart, and k is the dimension of the foliation.

We will call a *distribution* D the arbitrary assignment, for each point $p \in M$, of a linear subspace $D_p \subset T_pM$. Given a vector field $X\colon M \to TM$ we say that X is a section of D if $X_p \in D_p$ for all $p \in M$. We denote by $\Gamma(D)$ the set of sections of D. A distribution D is *smooth* if, for all points $p \in M$, there exist smooth sections $X_1, \dots, X_k$ of D such that $(X_1)_p, \dots, (X_k)_p$ is a basis of D_p. In such a case, we call k the *rank* of D in p.

Now, define $\operatorname{Lie}\mathcal{F}$ to be the smallest Lie subalgebra of $\Gamma(TM)$ containing $\mathcal{F}$. Precisely,

$$\operatorname{Lie}\mathcal{F} = \operatorname{span}_{C^\infty(U)}\big\{[X_1, \dots [X_{k-1}, X_k] \dots] \mid k \in \mathbb{N},\ X_1, \dots, X_k \in \mathcal{F}\big\}.$$

For any family $\mathcal{G}$ of vector fields, denote by $D^{\mathcal{G}}$ the smooth distribution defined by assigning, for all $p \in M$, the subspace $D^{\mathcal{G}}_p = \operatorname{span}\{X_p \mid X \in \mathcal{G}\}$. Since the Lie bracket of two vector fields X and Y can be expressed as

$$[X, Y]_p = \frac{d}{dt}\Big|_{t=0} e^{-\sqrt{t}Y} \circ e^{-\sqrt{t}X} \circ e^{\sqrt{t}Y} \circ e^{\sqrt{t}X}(p),$$

formula (1.1) for the tangent space of an orbit implies that

$$D^{\mathcal{F}}_p \subset D^{\operatorname{Lie}\mathcal{F}}_p \subset T_p\mathcal{O}_q, \qquad \forall\, q \in M,\ p \in \mathcal{O}_q. \tag{1.2}$$

We say that the control system (C) is Lie-determined if the last inclusion in (1.2) is an equality, i.e.,

$$D^{\operatorname{Lie}\mathcal{F}} = T\mathcal{O}. \tag{1.3}$$

Under some rather general hypotheses, system (C) is Lie-determined. In this regard, let us recall some additional definitions. As before, let $\mathcal{G}$ be a family of vector fields of M.

(i) $\mathcal{G}$ is *analytic* if M is an analytic manifold and, for all $X \in \mathcal{G}$, X is analytic.

(ii) $\mathcal{G}$ is *locally finitely generated* if for every $q \in M$ there exist an open neighbourhood U of q and finite vector fields $X_1, \dots, X_k$ in $\mathcal{G}$ such that $\mathcal{G}|_U \subset \operatorname{span}_{C^\infty(U)}\{X_1|_U, \dots, X_k|_U\}$.

(iii) $D^{\mathcal{G}}$ is *$\mathcal{G}$-invariant* if for all X in $\mathcal{G}$ one has $e_*^{tX}(D_p^{\mathcal{G}}) = D^{\mathcal{G}}_{e^{tX}p}$ for all $p \in M$ and t.

The theorem of Nagano [Nag66], the theorem of Hermann [Her62], and the theorem of Stefan-Sussmann [Sus73, Ste74] state that if, respectively, $\mathcal{F}$ is analytic, $\operatorname{Lie}\mathcal{F}$ is locally finitely generated, $D^{\operatorname{Lie}\mathcal{F}}$ is $\operatorname{Lie}\mathcal{F}$-invariant, then system (C) is Lie-determined. We recall here also the renewed Frobenius theorem [Fro77] stating that if $D^{\operatorname{Lie}\mathcal{F}}$ is a constant rank distribution, then system (C) is Lie-determined and the foliation described by the orbits is regular. For additional details to the subject we refer to [Lav18].

Under the assumption that system (C) is Lie-determined, it can be shown that orbits cannot be of a greater dimension than attainable sets. This is the content of Krener's theorem.

Theorem 1.5 (Krener's theorem [Kre74])**.** *Assume that* (C) *is Lie-determined. Then, for every p in M, the attainable set $\mathcal{A}_p$ has a nonempty interior in the orbit $\mathcal{O}_p$.*

The family $\mathcal{F}$ is said to satisfy the *Lie algebra rank condition at $p \in M$* if the evaluation at p of the Lie algebra generated by $\mathcal{F}$ has maximal dimension, i.e.,

$$D_p^{\operatorname{Lie}\mathcal{F}} = T_pM. \tag{1.4}$$

Remark 1.6. When M is a real-analytic manifolds and the vector fields in $\mathcal{F}$ are real-analytic, the following holds true: if system (C) satisfies the local reachability property, then $\mathcal{F}$ satisfies the Lie-algebra rank condition at any point. See for instance [SJ72, Thm. 3.1].

1.1.3 Approximate controllability for bilinear control systems

A first result obtained studying approximate controllability is that, if system (C) is approximately controllable, then the foliation described by its orbits is a regular foliation of M.

Lemma 1.7 ([CS21, Lem. 4])**.** *Assume that system* (C) *is approximately controllable. Then, the orbits of* (C) *form a regular foliation of M with dense leaves.*

The proof of this lemma, which is given in Chapter 5, follows from the lower semi-continuity of the dimension of the orbits. Lemma 1.7 provides no direct information about controllability. Indeed, there might be only one orbit and still the attainable sets might not coincide with M. However, with the additional hypothesis that (C) is Lie-determined, one deduces the following.

Corollary 1.8 ([CS21, Cor. 5])**.** *Assume that system* (C) *is Lie-determined and approximately controllable. Then, exactly one of the following alternatives holds:*

(a) $\mathcal{F}$ satisfies the Lie algebra rank condition at all points in M; hence, system (C) *is controllable.*

(b) There exists an integer k with $0 < k < n$ such that the orbits of (C) *form a regular k-dimensional foliation of N with dense leaves.*

Corollary 1.8 turns out to be useful if one can exclude the existence of a regular foliation of M with the properties described in *(b)*; this might be possible thanks to the particular form of system (C) or some topological properties of M. This is what we managed to do for the control systems of the form

$$\dot{p} = A(t)p, \qquad p \in \mathbb{R}^n \setminus \{0\},\ A(t) \in \mathcal{M} \tag{BL}$$

for piecewise constant controls $A : [0, +\infty) \to \mathcal{M}$ taking values in a subset $\mathcal{M}$ of the space $M_n(\mathbb{R})$ of $n \times n$ matrices with real coefficients, $n \geq 1$. By a slight abuse of notation, we refer to control systems such as (BL) as *bilinear control systems*, although the latter term usually denotes systems for which $M(t) = A + u^1(t)B_1 + \cdots + u^m(t)B_m$ for some fixed $A, B_1, \dots, B_m \in M_n(\mathbb{R})$, with control $t \mapsto (u^1(t), \dots, u^m(t))$ taking values in some subset Ω of $\mathbb{R}^m$. For an introduction to bilinear control system we refer to [CK00, Ell09]. Precisely, we proved the following result.

Theorem 1.9 [CS21, Thm. 1]

Consider the bilinear control system (BL) of $\mathbb{R}^n \setminus \{0\}$. System (BL) is approximately controllable if and only if it is controllable.

The proof, which is given in Chapter 5, is as follows. First, we deduce from [BS20, BV13] that, if the projection of (BL) onto $\mathbb{RP}^{n-1}$ is approximately controllable, then the orbits of (BL) are transversal to the radial direction. Corollary 1.8 applies since (BL) is Lie-determined due to Nagano theorem. If (BL) is not controllable, then the orbit foliation has codimension one with leaves transversal to the radial direction. Next, we show in Lemma 5.5 that such a foliation cannot have dense leaves, giving the desired result. As a byproduct of the above method, we extend Theorem 1.9 to angularly controllable homogenous control systems; see Corollary 5.6.

This result shows that the a priori weaker notion of approximate controllability implies controllability with no additional assumption, other than the finite-dimensional system being bilinear. This is useful when topological arguments lead directly to approximate controllability results, as it is the case for constructions based on control sets, whose definition involve the closure of attainable sets [CK00]. For example, in [CRS21] the authors use Theorem 1.9 to conclude that their sufficient condition for approximate controllability, expressed in terms of the Floquet spectrum of the bilinear system, actually yields controllability (see [CRS21, Corollary 3.19]).

Another possible way of applying Theorem 1.9 is the following: if for a bilinear system one is able to identify vector fields compatible with (BL) in the sense of [AS02, Definition 8.4] that, when added to the admissible ones, lead to approximate controllability, then controllability of (BL) follows, without the need of checking the Lie algebra rank condition. Such an extension argument by compatible vector fields is, e.g., at the core of the results in [Che05], which can therefore be improved by our result. In particular, Theorem 1.9 implies that the hypotheses ii) and iii) on the existence of stable and antistable equilibria can be dropped from [Che05, Theorem 4.3]. Similarly, Propositions 3.3 and 3.6 from the same paper can be strengthened by replacing in their conclusions approximate and practical controllability by controllability.

The result in Theorem 1.9 is in sharp contrast with the case of bilinear systems in infinite-dimension: when the controlled operators B_i appearing in the representation $M(t) = A+u^1(t)B_1+\cdots+u^m(t)B_m$ are bounded, these systems cannot be controllable (see [BMS82, Theorem 3.6] and also [BCC20] for recent extensions), while there exist some criteria for approximate controllability (see, e.g., [Kha10, Chapters 4 and 9], [CFK17] and [BCS14]).

For general finite-dimensional systems (to which the notions of controllability and approximate controllability straightforwardly extend), while controllability clearly implies approximate controllability, the converse may fail to hold. A standard example can be provided using the irrational winding of a line in the torus $\mathbb{T}^n$, $n \geq 2$.

On the other hand, the equivalence stated in Theorem 1.9 is known to hold for some classes of control systems. This is the case for linear control systems in $\mathbb{R}^n$, i.e., systems of the form

$$\dot{x} = Ax + Bu(t), \qquad u : [0, +\infty) \to \mathbb{R}^m, \quad x \in \mathbb{R}^n, \tag{1.5}$$

with $A \in M_n(\mathbb{R})$ and $B \in M_{n\times m}(\mathbb{R})$. Indeed, if system (1.5) is approximately controllable, then the attainable set from the origin $\mathcal{A}_0$ is dense in $\mathbb{R}^n$. Since $\mathcal{A}_0$ is a linear space (and in particular it is closed), it follows that $\mathcal{A}_0 = \mathbb{R}^n$, which is well known to be equivalent to the controllability of (1.5) due to the linear structure of the system.

Few other classes of control systems for which approximate controllability implies controllability are known: closed quantum systems on S^{n-1} [BGRS15, Theorem 17]; right-invariant control systems on simple Lie groups (as it follows from [JS72, Lemma 6.3] and [Smi42, Note at p. 312]); control systems obtained by projecting onto $\mathbb{RP}^{n-1}$ systems of the form of (BL) [BS20, Proposition 44].

By proving Theorem 1.9, we are able to identify bilinear control systems as a new class of systems for which approximate controllability and controllability are equivalent. The proposed proof of Theorem 1.9 (which can be found in Section 5.2), works first by deducing from [BS20] and [BV13] that, if the projection of (BL) onto $\mathbb{RP}^{n-1}$ is approximately controllable, then the orbits of (BL) are transversal to the radial direction, and then by proving that there does not exist a codimension one foliation in $\mathbb{R}^n \setminus \{0\}$ with dense leaves which are transversal to the radial direction.

As a byproduct of this demonstration strategy, we can extend Theorem 1.9 to angularly controllable homogenous control systems (see Corollary 5.6).

1.2 Sub-Riemannian manifolds

Let $\mathcal{F}$ be a family of vector fields on a smooth manifold M. Assume that the distribution defined by $\mathcal{F}$ has constant rank, i.e., there exists $k \in N$ such that

$$\dim D_p^{\mathcal{F}} = k \qquad \forall p \in M. \tag{1.6}$$

We say that a locally Lipschitz curve $\gamma\colon I \to M$ is *horizontal* with respect to $\mathcal{F}$ if $\dot{\gamma}(t) \in D^{\mathcal{F}}$ for almost every $t \in I$. Recall that a Lipschitz curve admits a derivative almost everywhere; e.g., see [Hei04, p. 18]. We call a curve *admissible* if it is Lipschitz and horizontal. In other words, a curve is admissible if and only if for any sufficiently small intervals $I' \subset I$ there exist $X_1, \dots, X_k$ in $\mathcal{F}$ and measurable, essentially bounded functions $u^1, \dots, u^k\colon I' \to \mathbb{R}$ such that

$$\dot{\gamma}(t) = u^1(t)X_1(\gamma(t)) + \cdots + u^k(t)X_k(\gamma(t)), \qquad \text{for almost every } t \in I'. \tag{1.7}$$

This is a version of the control equation (C), with essentially bounded controls and symmetric control parameters. Now, assume that a smooth scalar product g has been chosen on the distribution $D^{\mathcal{F}}$. In this case, the *sub-Riemannian length* of an admissible curve γ is defined as

$$L_{sR}(\gamma) = \int_I \sqrt{g(\dot{\gamma}, \dot{\gamma})}. \tag{1.8}$$

Ultimately, one would like to define the *sub-Riemannian distance* between two points p and q in M as

$$d_{sR}(p, q) = \inf\{L_{sR}(\gamma) \mid \gamma : [a, b] \to M \text{ admissible}, \gamma(a) = p,\ \gamma(b) = q\}. \tag{1.9}$$

A property which is required is that the topology defined by d_{sR} should coincide with the topology of M. This is the case when the family $\mathcal{F}$ to satisfy the Lie algebra rank condition at every point. The latter property depends uniquely on the distribution $D^{\mathcal{F}}$, motivating the following definition. A smooth constant rank distribution D is said to be *bracket-generating* if, for any family $\mathcal{F}$ of vector fields with $D = D^{\mathcal{F}}$, the family $\mathcal{F}$ satisfies the Lie algebra rank condition at every point. Finally, a *sub-Riemannian structure* on a manifold M consists of a smooth, bracket-generating distribution D, and a smooth scalar product g defined on D. The triple (M, D, g) is a *sub-Riemannian manifold.*

The above definition of sub-Riemannian manifold is not the most general one can give, but it is sufficient for the purposes of this thesis. For a general introduction to sub-Riemannian geometry we refer the reader to the monographs [ABB20], [LD17], [Jea14], [Mon02] and [Rif14]. In what follows it is assumed that M is a three-dimensional manifold and D is a distribution of rank two.

1.2.1 Surfaces in 3D contact manifolds

From now on, the manifold M is supposed to be 3-dimensional. The distribution D is said to be *coorientable* if there exists a one-form ω on M such that

$$\ker \omega_p = D_p, \qquad \forall p \in M. \tag{1.10}$$

Under the assumption that D is coorientable, the pair (M, D) is called a *contact manifold* if any one-form ω satisfying locally (1.10) satisfies also $\mathrm{d}\omega|_D \neq 0$, or equivalently

$$\omega \wedge \mathrm{d}\omega \neq 0. \tag{1.11}$$

Such one-form ω is called a *contact form*. One can verify that if (M, D) is a contact manifold then, for every p in M and $X_1, X_2 \in \Gamma(D)$, one has

$$\operatorname{span}\{X_1(p), X_2(p)\} = D_p \implies [X_2, X_1]_p \notin D_p. \tag{1.12}$$

In this section we recall some relevant facts about contact manifolds, and we refer to [Etn03] and [Gei08] for an introduction to the subject.

Let S be an embedded surface in M. A point p in S is a *characteristic point* if the tangent space T_pS coincides with the distribution D_p. The set of characteristic points of S is the *characteristic set*, noted $\Sigma(S)$. The characteristic set is closed due to the lower semi-continuity of the rank, and it cannot contain open sets due to the contact condition. Moreover, since the distribution D is contact, the set $\Sigma(S)$ is contained in a 1-dimensional submanifold of S (see Lemma 3.5) and, generically, it is composed of isolated points (see [Gei08, Par. 4.6]).

Outside the characteristic set, the intersection $TS \cap D$ is a one-dimensional distribution and defines (due to Frobenius theorem) a regular one-dimensional foliation on $S \setminus \Sigma(S)$. This foliation extends to a smooth foliation (cf. Subsection 1.1.2) of S by adding a singleton at every characteristic point. The resulting foliation is the *characteristic foliation* of S. Characteristic foliations of surfaces in 3D contact manifolds are studied in numerous references; in this regard we refer to [Gir91, Gir00, Ben83]. See Figure 1.3 (at the end of the current chapter) for a graphical representation of the characteristic foliation on a sphere in the Heisenberg group.

We call characteristic vector fields the vector fields of S whose orbit partition coincides with the characteristic foliation of S. Precisely, given an open set U in S, a vector field X is a *characteristic vector field* of S in U if, for all q in U,

$$\operatorname{span}_{\mathbb{R}} X(q) = \begin{cases} \{0\}, & \text{if } q \in \Sigma(S), \\ T_qS \cap D_q, & \text{otherwise}, \end{cases} \tag{1.13}$$

and satisfies the condition

$$\operatorname{div} X(p) \neq 0, \qquad \forall\, p \in \Sigma(S) \cap U. \tag{1.14}$$

Notice that $\operatorname{div} X(p)$ is well-defined since $X(p) = 0$, i.e., p is a characteristic point, and it is independent on the volume form; in particular $\operatorname{div} X(p) = \operatorname{tr} DX(p)$. Under some classical hypothesis, one can assure the existence of a global characteristic vector field as recalled in the following lemma. Moreover, these hypotheses always hold locally, in an open neighbourhood of any point.

Lemma 1.10 ([Gei08, Par. 4.6]). *Assume that S is orientable and that D is coorientable. Then, S admits a global characteristic vector field; moreover, the characteristic vector fields of S are the vector fields X for which there exists a volume form Ω of S such that*

$$\Omega(X, Y) = \omega(Y) \qquad \textit{for all } \ Y \in TS. \tag{1.15}$$

Indeed, in [Gei08] it is shown that, if a vector field satisfies (1.15), then it satisfies (1.13) and (1.14). Reciprocally, a vector field $\bar{X}$ satisfying (1.13) is a multiple of any other vector field X satisfying (1.15) for some function ϕ with $\phi|_{S\setminus\Sigma(S)} \neq 0$; additionally, if (1.14) holds, then $\phi|_{\Sigma(S)} \neq 0$; thus, $\bar{X}$ satisfies (1.15) with $\frac{1}{\phi}\Omega$ as volume form of S.

Remark 1.11. Formula (1.15) means that the characteristic vector fields are dual to the restrictions of the contact forms $\omega|_S$ with respect to the volume forms Ω of S. Since the volume forms of S are proportional by nowhere-zero functions, the same holds for the characteristic vector fields. In particular, if X is a characteristic vector field, then also $-X$ is a characteristic vector field.

Let us provide another way to find, locally, an explicit expression for a local characteristic vector field. Any point in S admits a neighbourhood U in M in which there exists an orthonormal frame (X_1, X_2) for $D|_U$, and a submersion u of class C^2 for which S is a level set, i.e.,

$$S \cap U = \{q \in U : u(q) = 0\}, \quad \text{and } \mathrm{d}u \neq 0 \text{ on } S \cap U\ . \tag{1.16}$$

Observe that for any point $p \in U \cap S$, one has

$$p \in \Sigma(S) \quad \text{if and only if} \quad X_1 u(p) = X_2 u(p) = 0. \tag{1.17}$$

Here we used that a vector V is in TS if and only if $Vu = 0$. Now, the vector field X_u defined by

$$X_u = (X_1 u)X_2 - (X_2 u)X_1, \tag{1.18}$$

is a characteristic vector field of S. Indeed, X_u satisfies (1.13) since it follows from the definition that, for all q in S, the vector $X_u(q) \in T_qS \cap D_q$, and $X_u(p) = 0$ if and only if $p \in \Sigma(S)$, due to (1.17). Moreover, X_u satisfies (1.14) since the divergence of X_u at the characteristic points is nonzero due to the contact condition (1.12) and the following expression

$$\operatorname{div} X_u(p) = X_2 X_1 u(p) - X_1 X_2 u(p) = [X_2, X_1]u(p), \qquad \forall\, p \in \Sigma(S).$$

In the literature, the study of surfaces in three-dimensional contact manifolds has found a lot of interest since, amongst others, the characteristic foliations of surfaces provide an important invariant used to classify contact structures. Moreover, the following theorem holds.

Theorem 1.12 (Giroux, [Gei08, Thm. 2.5.22 and 2.5.23])**.** *Let S_i be closed surfaces in contact three-dimensional manifolds (M_i, D_i), $i = 0, 1$, with D_i cooriented. Assume that and $\phi : S_0 \to S_1$ is a diffeomorphism between the respective characteristic foliations. Then there is a contactomorphism $\psi : N(S_0) \to N(S_1)$ of suitable neighbourhoods $N(S_i)$ of S_i, i.e., $\psi_* D_1|_{N(S_1)} = D_0|_{N(S_0)}$, with $\psi|_{S_0} = \phi$.*

Tight contact structures

Roughly, a distribution is *tight* if it does not admit an *overtwisted disk*, i.e., an embedding of a disk with horizontal boundary such that the distribution does not twists along the boundary. The notion of tight distribution will be necessary to state Theorem 1.16. However, a reader who is satisfied with the above definition might skip ahead.

To define an overtwisted disk, let us first consider an embedding of $\Delta = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ in M, and denote $\Gamma = \partial\Delta$. Let Γ be horizontal with respect to the contact distribution D, i.e., $T\Gamma \subset D$. Then, the normal bundle $N\Gamma = TM|_\Gamma / T\Gamma$ can be decomposed in two ways: the first with respect to the tangent space of Δ, i.e.,

$$N\Gamma \cong {}^{TM}\!/_{T\Delta} \oplus {}^{T\Delta}\!/_{T\Gamma}, \tag{1.19}$$

and the second with respect to the contact distribution D, i.e.,

$$N\Gamma \cong {}^{TM}\!/_{D} \oplus {}^{D}\!/_{T\Gamma}. \tag{1.20}$$

A frame (Y_1, Y_2) of $N\Gamma$ is called a *surface frame* if it respects the splitting (1.19), i.e., $Y_1 \in TM \setminus T\Delta$ and $Y_2 \in T\Delta \setminus T\Gamma$; similarly, it is called a *contact frame* if it respects the splitting (1.20). Since the contact distribution is cooriented near Δ, both bundles (1.19) and (1.20) are trivial, thus one can always find contact and surface frames.

The Thurston–Bennequin invariant of Γ, denoted by $\mathrm{tb}(\Gamma)$, is the number of twists of a contact frame of Γ with respect to a surface frame: the right-handed twists are counted positively, and the left-handed twists negatively (cf. for instance [Gei08, Def. 3.5.4]). Note that $\mathrm{tb}(\Gamma)$ is independent of

the orientation of Γ. The requirement that the distribution D does not twist along the boundary of Δ is equivalent to $\mathrm{tb}(\partial\Delta) = 0$, i.e., the Thurston–Bennequin invariant of $\partial\Delta$ being zero.

An embedded disk Δ in a cooriented contact manifold (M, D) with smooth boundary $\partial\Delta$ is an *overtwisted disk* if $\partial\Delta$ is a horizontal curve of D, $\mathrm{tb}(\partial\Delta) = 0$, and there is exactly one characteristic point in the interior of the disk. Note that the elimination lemma of Giroux allows to remove the condition that there is only one characteristic point in the interior of the overtwisted disk, as discussed for instance in [Gei08, Def. 4.5.2].

1.2.2 The Riemannian approximation

The construction of Riemannian approximation of a sub-Riemannian structure is a key tool used in this thesis, and their use to define sub-Riemannian geometric invariants is a well-known technique. For a general description of the properties of the Riemannian approximation in the Heisenberg group we refer to [CDPT07]. In this thesis, we use the Riemannian approximation for two purposes. First to associate with each characteristic point a real number $\widehat{K}$, as explained below in Theorem 1.13. Second, as explained in Section 1.3, to construct a canonical stochastic process on S.

Assume that (M, D, g) is a three-dimensional manifold equipped with a cooriented distribution D of rank two. Under this assumption, one can fix a vector field X_0 transverse to the distribution, i.e.,

$$\mathrm{span}\{D, X_0\} = TM.$$

Once this choice has been made, one can extend the sub-Riemannian metric g to a Riemannian metric g^{X_0} by defining X_0 to be unitary and orthogonal to D. The Riemannian metrics $g^{\varepsilon X_0}$, for $\varepsilon > 0$, are the *Riemannian approximations* of (D, g) with respect to X_0. Precisely, for every $\varepsilon > 0$, one has

$$\langle D, X_0\rangle_{g^{\varepsilon X_0}} = 0, \quad |X_0|_{g^{\varepsilon X_0}} = \frac{1}{\varepsilon}, \qquad \forall \varepsilon > 0.$$

To simplify the notation, once it is clear which transversal vector field has been chosen we might drop the dependance on X_0 in the superscript, writing $g^\varepsilon = g^{\varepsilon X_0}$.

Let K^{X_0} be the Gaussian curvature of S with respect to g^{X_0}. Fixed a characteristic point $p \in \Sigma(S)$, the coefficient $\widehat{K}$ will be defined as the limit for $\varepsilon \to 0$ of $K^{\varepsilon X_0}$, using a suitably normalisation of the Lie bracket structure on the distribution. Precisely, let B^{X_0} be the bilinear form $B^{X_0} : D \times D \to \mathbb{R}$ defined by

$$B^{X_0}(X, Y) = \alpha \qquad \text{if} \qquad [X, Y] = \alpha X_0 \mod D.$$

Since D is endowed with the metric g, the bilinear form B^{X_0} admits a well-defined determinant.

Theorem 1.13 [BBC21, Thm. 1.1]

Let S be a C^2 surface embedded in a 3D contact sub-Riemannian manifold. Let p be a characteristic point of S, and let X_0 be a vector field transverse to the distribution D in a neighbourhood of p. Then, in the notations defined above, the limit

$$\widehat{K}_p = \lim_{\varepsilon \to 0} \frac{K_p^{\varepsilon X_0}}{\det B_p^{\varepsilon X_0}} \tag{1.21}$$

is finite and independent on the vector field X_0.

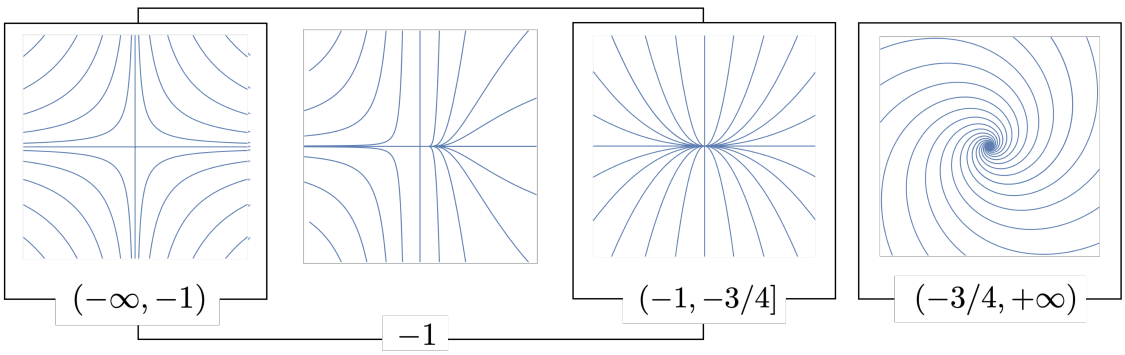

Figure 1.2: The qualitative picture for the characteristic foliation at an *isolated* characteristic point, for the corresponding values of $\widehat{K}$. Left to right, we recognise a saddle, a saddle-node, a node, and a focus.

Notice that in the previous literature the Riemannian approximation is employed to define sub-Riemannian geometric invariants *outside* of the characteristic set. For instance, in [BTV17] the authors defined the sub-Riemannian Gaussian curvature at a point $x \in S \setminus \Sigma(S)$ as

$$\mathcal{K}_S(x) = \lim_{\varepsilon \to 0} K_x^{\varepsilon X_0}, \tag{1.22}$$

and they proved that a Gauss-Bonnet type theorem holds; here the authors worked in the setting of the Heisenberg group, and with X_0 equals to the Reeb vector field of the Heisenberg group. This construction is extended in [WW20] to the affine group and to the group of rigid motions of the Minkowski plane, and in [Vel20] to a general sub-Riemannian manifold. In the latter, the author linked $\mathcal{K}_S$ with the curvature introduced in [DV16], and, when $\Sigma(S) = \emptyset$, they proved a Gauss-Bonnet theorem by Stokes formula. A Gauss-Bonnet theorem (in a different setting) was also proven in [ABS08]. We finally notice that the invariant $\mathcal{K}_S$ also appears in [Lee13], where it is called curvature of transversality. An expression for $\mathcal{K}_S$ is provided also in Proposition 1.18.

As we shall see, the coefficient $\widehat{K}_p$ determines the qualitative behaviour of the characteristic foliation near a characteristic point p. Following the terminology of contact geometry (cf. for instance [Gei08, Par. 4.6]), given a characteristic point $p \in \Sigma(S)$ and a characteristic vector field X, the point p is called *non-degenerate* if $\det DX(p) \neq 0$. Furthermore, p is called *elliptic* if $\det DX(p) > 0$, and *hyperbolic* if $\det DX(p) < 0$. In the theory of dynamical systems, saddles and hyperbolic points are, respectively, what we call hyperbolic characteristic points and non-degenerate characteristic points.

Proposition 1.14 ([BBC21, Prop. 1.2])**.** *Let S be a C^2 surface embedded in a 3D contact sub-Riemannian manifold. Given a characteristic point p in $\Sigma(S)$, let X be a characteristic vector field X near p. Then,* $\operatorname{tr} DX(p) \neq 0$ *and*

$$\widehat{K}_p = -1 + \frac{\det DX(p)}{(\operatorname{tr} DX(p))^2}. \tag{1.23}$$

Thus, p is hyperbolic if and only if $\widehat{K}_p < -1$, and p it is elliptic if and only if $\widehat{K}_p > -1$.

This equality links $\widehat{K}_p$ to the eigenvalues of $DX(p)$, which determine the qualitative behaviour of the characteristic foliation around the characteristic point p. This relation is made explicit in Corollary 3.10 for a non-degenerate characteristic point, and in Corollary 3.12 for a degenerate characteristic point. Moreover, equation (1.23) shows that $\widehat{K}_p$ is independent on the sub-Riemannian metric, and depends only on the line field defined by D on S.

1.2.3 Induced distance on surfaces

The study of the geometry of submanifolds S of an ambient manifold M with a given geometric structure is a classical subject. A familiar example, whose study goes back to Gauss, is that of a surface S embedded in the Euclidean space $\mathbb{R}^3$. In such a case, S inherits its natural Riemannian structure by restricting the metric tensor to the tangent space of S. The distance induced on S by this metric tensor is not the restriction of the distance of $\mathbb{R}^3$ to points on S, but rather the length space structure induced on S by the ambient space.

Things are less straightforward for a smooth 3-manifold M endowed with a contact sub-Riemannian structure (D, g). Indeed, for a two-dimensional submanifold S, the intersection $T_xS \cap D_x$ is one-dimensional for most points x in S; thus, $TS \cap D$ is not a bracket-generating distribution and there is no well-defined sub-Riemannian distance induced by (M, D, g) on S. This fact is indeed more general, as already observed in [Gro96, Sec. 0.6.B]. Nevertheless, one can still define a distance on S following the length space viewpoint: the sub-Riemannian distance d_{sR} defines the *length* of any continuous curve $\gamma : [0, 1] \to M$ as

$$L_{sR}(\gamma) = \sup \left\{ \sum\nolimits_{i=1}^{N} d_{sR}(\gamma(t_i), \gamma(t_{i+1})) \mid 0 = t_0 \leq \ldots \leq t_N = 1 \right\},$$

and one can define $d_S : S \times S \to [0, +\infty]$ by

$$d_S(x, y) = \inf\{L_{sR}(\gamma) \mid \gamma : [0, 1] \to S,\ \gamma(0) = x,\ \gamma(1) = y\}.$$

The space (S, d_S) is called a *length space*, and d_S the *induced distance* defined by (M, d_{sR}). (In the theory of length metric spaces, the induced distance d_S is called intrinsic distance, emphasising that it depends uniquely on lengths of curves in S, see [BBI01].) We stress that the induced distance d_S is not the restriction $d_{sR}|_{S\times S}$ of the sub-Riemannian distance to S.

We started the investigation by looking for necessary and sufficient conditions on the surface S so that the induced distance d_S is finite, i.e., $d_S(x, y) < +\infty$ for all points x, y in S; this is equivalent to (S, d_S) being a metric space. This can be formulated through the use of the characteristic foliation of S. Precisely, consider a continuous curve $\gamma : [0, 1] \to S$. Its length is finite, i.e., $L_{sR}(\gamma) < +\infty$, if and only if γ is a reparametrisation of a curve $\bar{\gamma}$ horizontal with respect to D; in such case, the length of γ coincides with the sub-Riemannian length of $\bar{\gamma}$. We refer to [BBI01, Ch. 2] and [ABB20, Sec. 3.3] for more details. In conclusion, the distance $d_S(x, y)$ between two points x and y in S is finite if and only if there exists a finite-length continuous concatenation of leaves of the characteristic foliation of S connecting them.

From a local point of view, for the induced distance to be finite one needs characteristic points to be accessible from their complement. In this regard, we were able to prove that the one-dimensional leaves of the characteristic foliation of S which converge to a characteristic point have finite length. Precisely, let ℓ be a leaf of the characteristic foliation of S; we say that a point p in S is a *limit point* of ℓ if there exists a point q in ℓ and a characteristic vector field X of S such that

$$e^{tX}(q) \to p \qquad \text{for } t \to +\infty, \tag{1.24}$$

where e^{tX} is the flow of X. In such case, we denote the semi-leaf $\ell_X^+(q) = \{e^{tX}(q) \mid t \geq 0\}$. With the above definition, a leaf can have at most two limit points: one for each extremity. Finally, notice that a limit point of a leaf must be a zero of the corresponding characteristic vector field X, i.e., a characteristic point of S.

Proposition 1.15 ([BBC21, Thm. 1.3])**.** *Let S be a C^2 surface embedded in a 3D contact sub-Riemannian manifold, and let p be a limit point of a one-dimensional leaf ℓ. Let $x \in \ell$, and X be a characteristic vector field such that $e^{tX}(x) \to p$ for $t \to +\infty$. Then, the length of $\ell_X^+(x)$ is finite.*

This result is a consequence of the sub-Riemannian structure being contact. Indeed, for a non-contact distribution this conclusion is false; for instance, in [ZZ95, Lem. 2.1] the authors prove that the length of the semi-leaves of the characteristic foliation of a Martinet surface converging to an elliptic point is infinite.

On the global side, we determine some conditions for the induced metric d_S to be finite under the assumption that there exists a global characteristic vector field of S. In such a case, for a compact, connected surface S with isolated characteristic points, in Proposition 3.16 we show that d_S is finite in the absence of the following classes of leaves in the characteristic foliation of S: nontrivial recurrent trajectories, periodic trajectories, and sided contours. Those conditions are satisfied by spheres in coorientable, tight contact spaces,

Theorem 1.16 [BBC21, Thm. 1.4]

Let (M, D, g) be a tight coorientable sub-Riemannian contact structure, and let S be a C^2 embedded surface with isolated characteristic points, homeomorphic to a sphere. Then the induced distance d_S is finite.

We stress that the property that any characteristic point is isolated is a generic property for surfaces in contact manifolds. Here by a generic property we mean a property satisfied on an open and dense subset of the set of all coorientable contact distributions for the C^∞ topology (once the surface is fixed), and on an open and dense subset of the set of all C^2 surfaces for the C^2 topology (once the contact distributions is fixed). Example 3.5.4 and Example 3.5.5 in the Heisenberg distribution show that, if S is not a topological sphere, then S presents possibly nontrivial recurrent trajectories or periodic trajectories, cases in which d_S is not finite. Moreover, if one removes the hypothesis of the contact structure being tight, then a sphere S might present a periodic trajectory, hence the induced distance d_S would not be finite. The compactness hypothesis is also important, as one can see in Example 3.5.1.

When considering hypersurfaces in contact structures, usually one needs to demand the absence of characteristic points to obtain results that generalize from Riemannian ones. Consider, for instance, the result in [TY04, Thm. 1.1] describing the induced distance on hypersurfaces of sub-Riemannian manifolds in dimension greater than three, or the result in [RR21] about the small-time asymptotic of the heat content surfaces in the 3D case. On the contrary, here the finiteness of the distance in the 3D case needs the presence of characteristic points; this is mainly due to the integrability of the characteristic vector field and of the global nature of the question. Finally, we are left to observe that for many surfaces the induced distance is not finite. For instance, the induced distance on a surface without characteristic points is not finite, and the property of not having characteristic points do not change with small perturbations of the surface or of the contact structure in the appropriate topologies.

1.3 Stochastic processes on sub-Riemannian surfaces

Let (M, D, g) be a contact, cooriented sub-Riemannian space. We shall assume that the distribution D is *free*, i.e., globally generated by a pair of vector fields (X_1, X_2) which constitute an orthonormal frame for D oriented with respect to the volume form vol_g on D defined by g. Moreover, we choose the contact form ω to be normalised so that

$$\mathrm{d}\omega|_D = \mathrm{vol}_g\,. \tag{1.25}$$

Associated with such a contact form ω, we have a canonical choice of a vector field everywhere transversal to D: this is the *Reeb* vector field X_0, which is the unique vector field on M satisfying

$$\mathrm{d}\omega(X_0, \cdot) \equiv 0, \quad \text{and} \quad \omega(X_0) \equiv 1. \tag{1.26}$$

Let g^ε, for $\varepsilon > 0$, be the Riemannian approximations of (M, D, g) with respect to the Reeb vector field X_0, as described in Section 1.2.2. Recall that g^ε is defined by requiring $(X_1, X_2, \varepsilon X_0)$ to be a global orthonormal frame for g^ε. Let S be a surface embedded in M, and assume that S is globally defined by a submersion u in the sense of equation (1.16). Namely, one has

$$S = \{x \in M : u(x) = 0\}, \qquad \text{and } \mathrm{d}u \neq 0 \text{ on } S. \tag{1.27}$$

This implies that the characteristic vector field X_u introduced in (1.18) is defined globally on S. Let $\widehat{X}_S$ be the vector field on $S \setminus \Sigma(S)$ defined by $\widehat{X}_S = X_u/|X_u|_g$. Explicitly,

$$\widehat{X}_S = \frac{(X_1 u)X_2 - (X_2 u)X_1}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \ . \tag{1.28}$$

Even though $\widehat{X}_S$ is expressed in terms of X_1, X_2 and u, it only depends on the sub-Riemannian manifold (M, D, g), the embedded surface S and a choice of sign. Let $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ be the function given by

$$b = \frac{X_0 u}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \ . \tag{1.29}$$

Similarly to the vector field $\widehat{X}_S$, the function b can be understood intrinsically. Indeed, let $\widehat{X}_S^\perp$ be such that $(\widehat{X}_S, \widehat{X}_S^\perp)$ is an oriented orthonormal frame for $D|_{S\setminus\Sigma(S)}$. Then, the function b is uniquely defined by requiring $b\widehat{X}_S^\perp - X_0$ to be a vector field on $S \setminus \Sigma(S)$. Finally, define

$$\Delta_0 = \widehat{X}_S^2 + b\widehat{X}_S \ , \tag{1.30}$$

which is a second order partial differential operator on $S \setminus \Sigma(S)$. The operator Δ_0 is invariant under multiplications of u by nonzero functions. As stated in the theorem below, it arises as the limiting operator of the Laplace–Beltrami operators Δ_ε of the Riemannian approximations g^ε, for $\varepsilon \to 0$.

Theorem 1.17 ([BBCH21, Thm. 1.1])**.** *For any twice differentiable function $f \in C_c^2(S\setminus\Sigma(S))$ compactly supported in $S \setminus \Sigma(S)$, the functions $\Delta_\varepsilon f$ converge uniformly on $S \setminus \Sigma(S)$ to $\Delta_0 f$ as $\varepsilon \to 0$.*

Following the definition in Balogh, Tyson and Vecchi [BTV17] for surfaces in the Heisenberg group, the intrinsic Gaussian curvature $\mathcal{K}_S$ of a surface in a general three-dimensional contact sub-Riemannian manifold is defined as the limit of the Gaussian curvatures with respect to the Riemannian metrics g_ε, as described in (1.22). In the following proposition we derive an expression for $\mathcal{K}_S$, employing the same orthogonal frame exhibited to prove Theorem 1.17.

Proposition 1.18 ([BBCH21, Prop. 1.2])**.** *Uniformly on compact subsets of $S \setminus \Sigma(S)$, we have*

$$\mathcal{K}_S := \lim_{\varepsilon \to 0} K^\varepsilon = -\widehat{X}_S(b) - b^2 \ .$$

We now consider the *canonical stochastic process* on $S \setminus \Sigma(S)$ whose generator is $\frac{1}{2}\Delta_0$. Assuming that it starts at a certain point then, up to explosion, the process moves along the unique leaf of the characteristic foliation picked out by the starting point. This follows from the fact that the vector field $\widehat{X}_S$ is tangent to the characteristic foliation of S. As shown by the next theorem and the following proposition, for this stochastic process, elliptic characteristic points are inaccessible, while hyperbolic characteristic points are accessible from the separatrices.

Theorem 1.19 [BBCH21, Thm. 1.3]

The set of elliptic characteristic points in a surface S embedded in M is inaccessible for the stochastic process with generator $\frac{1}{2}\Delta_0$ on $S \setminus \Sigma(S)$.

In Section 4.3.3, we discuss an example of a surface in the Heisenberg group whose induced stochastic process is killed in finite time if started along the separatrices of the characteristic point. Indeed, this phenomena always occurs in the presence of a hyperbolic characteristic point.

Theorem 1.20 [BBCH21, Prop. 1.4]

Suppose that the surface S embedded in M has a hyperbolic characteristic point. Then the stochastic process having generator $\frac{1}{2}\Delta_0$ and started on the separatrices of the hyperbolic characteristic point reaches that characteristic point with positive probability.

Sections 4.3 and 4.4 are devoted to illustrating the various behaviours shown by the canonical stochastic process induced on the surface S. Besides illustrating Proposition 1.20, we show in Theorem 1.21 below that three classes of familiar stochastic processes arise when considering a natural choice for the surface S in the three classes of model spaces for three-dimensional sub-Riemannian structures, which are the Heisenberg group in $\mathbb{R}^3$, and the special unitary group $\mathrm{SU}(2)$ and the special linear group $\mathrm{SL}(2,\mathbb{R})$ equipped with sub-Riemannian contact structures with scalar products differing by a constant multiple. In all these cases, the orthonormal frame (X_1, X_2) for the distribution D is formed by two left-invariant vector fields which together with the Reeb vector field X_0 satisfy, for some $\kappa \in \mathbb{R}$, the commutation relations $[X_2, X_1] = X_0$, $[X_1, X_0] = \kappa X_2$, and $[X_2, X_0] = -\kappa X_1$, with $\kappa = 0$ in the Heisenberg group, $\kappa > 0$ in $\mathrm{SU}(2)$ and $\kappa < 0$ in $\mathrm{SL}(2,\mathbb{R})$. Associated with each of these Lie groups and their Lie algebras, we have the group exponential map exp for which we identify a left-invariant vector field with its value at the origin.

Theorem 1.21 ([BBCH21, Thm. 1.5])**.** *Fix $\kappa \in \mathbb{R}$. For $\kappa \neq 0$, let $k \in \mathbb{R}$ with $k > 0$ be such that $|\kappa| = 4k^2$. Set $I = (0, \frac{\pi}{k})$ if $\kappa > 0$ and $I = (0, \infty)$ otherwise. In the model space for three-dimensional sub-Riemannian structures corresponding to κ, we consider the embedded surface S parameterised as*

$$S = \{\exp(r\cos\theta X_1 + r\sin\theta X_2) : r \in I \text{ and } \theta \in [0, 2\pi)\} .$$

Then, the limiting operator Δ_0 on S is given by $\Delta_0 = \frac{\partial^2}{\partial r^2} + b(r)\frac{\partial}{\partial r}$, where

$$b(r) = \begin{cases} 2k\cot(kr) & \text{if } \kappa = 4k^2 \\ \frac{2}{r} & \text{if } \kappa = 0 \\ 2k\coth(kr) & \text{if } \kappa = -4k^2 \end{cases} .$$

The stochastic process induced by the operator $\frac{1}{2}\Delta_0$ moving along the leaves of the characteristic foliation of S is a Bessel process of order 3 if $\kappa = 0$, a Legendre process of order 3 if $\kappa > 0$ and a hyperbolic Bessel process of order 3 if $\kappa < 0$.

The stochastic processes we recover here are all related to one-dimensional Brownian motion by the same type of Girsanov transformation, with only the sign of a parameter distinguishing between them. For the details, see Revuz and Yor [RY99, p. 357]. Let us recall here that a Bessel process of order 3 arises by conditioning a one-dimensional Brownian motion started on the positive real line to never hit the origin, whereas a Legendre process of order 3 is obtained by conditioning a Brownian motion started inside an interval to never hit either endpoint of the interval. The examples making up Theorem 1.21 can be considered as model cases for our setting, and all of them illustrate Theorem 1.19.

Finally, notice that the limiting operator we obtain on the leaves is not the Laplacian associated with the metric structure restricted to the leaves as the latter has no drift term. However, the operator Δ_0 restricted to a leaf can be considered as a weighted Laplacian. For a smooth measure $\mu = h^2\,\mathrm{d}x$ on an interval I of the Euclidean line $\mathbb{R}$, the weighted Laplacian applied to a scalar function u yields

$$\mathrm{div}_\mu\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{2h'(x)}{h(x)}\frac{\partial f}{\partial x} .$$

In the model cases above, we have

$$h(r) = \begin{cases} \sin(kr) & \text{if } \kappa = 4k^2 \\ r & \text{if } \kappa = 0 \\ \sinh(kr) & \text{if } \kappa = -4k^2 \end{cases} .$$

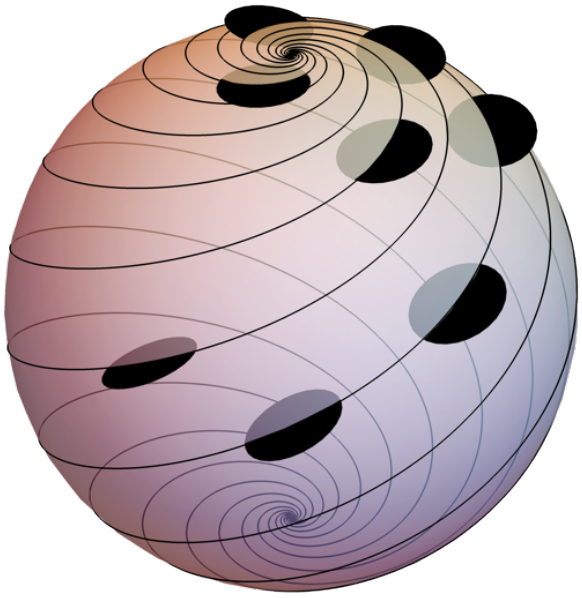

Figure 1.3: The characteristic foliation defined by the Heisenberg distribution $(\mathbb{R}^3, \ker(dz+\frac{1}{2}(ydx-xdy))$ on an Euclidean sphere centred at the origin: any horizontal curve connecting points on different spirals goes though one of the characteristic points, at the North or the South pole. The sub-Riemannian length of the leaves spiralling around the characteristic points is finite because of Proposition 1.15. Thus, the induced distance d_S is finite: this is a particular case of Theorem 1.16. The canonical stochastic process started outside the characteristic points never hits neither the north pole nor the south pole, and it induces a one-dimensional process on the unique leaf of the characteristic foliation picked out by the starting point, due to Theorem 1.19.

Chapter 2

Introduction

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Dans ce chapitre, nous présentons des résultats originaux sur la contrôlabilité en théorie du contrôle géométrique et sur les propriétés des surfaces lisses dans une variété sous-riemannienne de contact tridimensionnelle. Les résultats sont prouvés dans les prochains chapitres, uniquement en anglais.

Précisément, dans la Section 2.1 nous introduisons les systèmes de contrôle, et nous présentons quelques nouvelles relations :

- entre contrôlabilité et contrôlabilité locale (au Paragraphe 2.1.1),
- entre contrôlabilité et contrôlabilité approchée (au Paragraphe 2.1.3).

Le premier résultat est discuté au Chapitre 6, suite à mon article [BCFS21] (travail en commun avec U. Boscain, V. Franceschi et M. Sigalotti), actuellement soumis à révision. Ce dernier est développé au Chapitre 5 et publié dans [CS21] (travail en commun avec M. Sigalotti).

Ensuite, dans la Section 2.2 nous présentons quelques nouvelles propriétés métriques des surfaces plongées dans des variétés sous-riemanniennes de contact 3D. En effet, nous vous présentons :

- un nouvel invariant métrique $\widehat{K}$ aux points caractéristiques (au Paragraphe 2.2.2),
- une analyse de la distance de longueur induite sur les surfaces (au Paragraphe 2.2.3).

Ces résultats sont discutés plus en détail au Chapitre 3 et publiés dans mon article [BBC21] (travail en collaboration avec D. Barilari et U. Boscain).

Enfin, dans la Section 2.3 nous discutons des propriétés d'un nouveau processus stochastique canonique défini sur de telles surfaces. Ces résultats sont discutés au Chapitre 4 et publiés dans mon article [BBCH21] (travail en collaboration avec D. Barilari U. Boscain et K. Habermann).

Notations. Dans ce qui suit, M est une variété lisse de dimension n . Nous désignons les champs vectoriels de M par des lettres majuscules telles que X, Y et Z. Les notations $X(f) = Xf$ et $X(q)$ indiquent respectivement la dérivée d'une fonction lisse f dans $C^\infty(M)$ par rapport à X, et la dérivation basée sur un point q dans M défini par X. Le $C^\infty(M)$-module des champs vectoriels de M est noté $\Gamma(TM)$, qui avec le crochet de Lie $[\cdot,\cdot]$ est un Lie algèbre. Le flux à l'instant t d'un champ vectoriel X est noté e^{tX}. Pour simplifier la notation, nous supposons que les champs vectoriels sont complets, c'est-à-dire que leurs flux sont définis pour tout t dans $\mathbb{R}$.

2.1 Systèmes de contrôle associés à une famille de champs vectoriels

Soit $\mathcal{F} \subset \Gamma(TM)$ un ensemble quelconque de champs de vecteurs lisses sur une variété M. Soit Ω un ensemble d'indices pour la famille $\mathcal{F}$, c'est-à-dire $\mathcal{F} = \{X_u \mid u \in \Omega\}$. Le *système de contrôle* associé à la famille $\mathcal{F}$ est le système

$$\dot{p}(t) = X_{u(t)}(p(t)), \qquad p \in M,\ X_{u(t)} \in \mathcal{F}, \tag{C}$$

où par *contrôle* nous utilisons des fonctions $u \in \mathcal{U}_{pc} = \bigcup_{T\geq 0}\{u : [0,T] \to \Omega \mid u \text{ constante par morceaux}\}$. Suivant le vocabulaire des systèmes de contrôle, un élément $u \in \Omega$ est appelé un *paramètre de contrôle*, Ω est l'*espace des paramètres de contrôle*, les points dans M sont *états*, et M est l'*espace des états*.

Puisque (pour le moment) nous utilisons des contrôles constants par morceaux, l'équation différentielle (C) est définie au sens classique, jusqu'à un ensemble discret de temps où le contrôle n'est pas continu. Ainsi, une fois qu'un contrôle u est fixé, la solution de (C) est déterminée par les conditions initiales. Notons $\phi(t,p,u)$ la valeur à l'instant t de la solution de (C) à partir de p. Précisément, considérons un contrôle $u\colon [0,T] \to \Omega$ tel qu'il existe une partition $0 = t_0 < t_1 < \cdots < t_k = T$ et des paramètres de contrôle $u_1, \ldots, u_k \in \Omega$ satisfaisant

$$u(t) = u_i, \qquad \forall t \in (t_{i-1}, t_i),\ i = 1, \ldots, k.$$

Si $t \in [t_{i-1}, t_i]$ pour un certain $i = 1, \ldots, k$, alors

$$\phi(t,p,u) = e^{(t-t_{i-1})X_{u_i}} \circ e^{(t_{i-1}-t_{i-2})X_{u_{i-1}}} \circ \cdots \circ e^{t_1 X_{u_1}}(p),$$

à savoir $\phi(t,p,u)$ est construit en concaténant les flux des champs de vecteurs dans $\mathcal{F}$ indexés par le contrôle u. L'*ensemble atteignable* $\mathcal{A}_p$ d'un état p dans M pour le système (C) est l'ensemble des points atteints par les solutions de (C) commençant à partir de p en utilisant des temps positifs ; précisément,

$$\mathcal{A}_p = \{e^{t_k X_k} \circ \cdots \circ e^{t_1 X_1}(p) \mid k \in \mathbb{N},\ t_1, \ldots, t_k \geq 0,\ X_1, \ldots, X_k \in \mathcal{F}\}.$$

De même, l'ensemble des points atteints en utilisant des temps positifs et négatifs est appelé la *orbite* $\mathcal{O}_p$ d'un état x ; précisément,

$$\mathcal{O}_p = \{e^{t_k X_k} \circ \cdots \circ e^{t_1 X_1}(p) \mid k \in \mathbb{N},\ t_1, \ldots, t_k \in \mathbb{R},\ X_1, \ldots, X_k \in \mathcal{F}\}.$$

Les ensembles atteignables sont plus intéressants que les orbites du point de vue de la théorie du contrôle, puisqu'ils sont obtenus en suivant strictement les champs de vecteurs dans $\mathcal{F}$. En effet, suivre le flux d'un champ de vecteurs $X \in \mathcal{F}$ pour un temps négatif équivaut à suivre le flux de $-X$, mais $-X$ n'est pas forcement dans la famille $\mathcal{F}$. Notez que les orbites et les ensembles atteignables coïncident si la famille $\mathcal{F}$ est symétrique, c'est-à-dire si $-\mathcal{F} = \mathcal{F}$.

Parfois, nous voulons utiliser par contrôles des fonctions mesurables essentiellement limitées, au lieu des fonctions constantes par morceaux. Lorsque c'est le cas, nous supposons implicitement que Ω est un sous-ensemble de $\mathbb{R}^m$, pour un certain $m \in \mathbb{N}$, et qu'une fonction lisse $X : M \times \Omega \to TM$ paramétrise $\mathcal{F}$, c'est-à-dire $\mathcal{F} = \{X(\cdot,u) \mid u \in \Omega\}$. Pour respecter la notation dans le système (C),

$$\begin{tikzcd}
 & \text{ST-contrôlabilité locale} \arrow[dr, Rightarrow] & & \\
\text{STL-contrôlabilité locale} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{contrôlabilité locale} \arrow[r, Rightarrow, bend right, "\text{Thm. 2.1}"'] & \text{contrôlabilité} \arrow[l, Rightarrow, bend right] \\
 & \text{L-contrôlabilité locale} \arrow[ur, Rightarrow] & &
\end{tikzcd}$$

Table 2.1: Relations entre différents types de contrôlabilité locale. Comme discuté dans la Section 6.2, les flèches manquantes ne peuvent pas être ajoutées au schéma. La seule flèche à justifier ici est celle représentant le fait que la contrôlabilité locale implique la contrôlabilité. C'est l'objet du Théorème 2.1.

nous continuons à écrire $X_u = X(\cdot, u)$ pour tout u dans Ω. Nous désignons l'ensemble des contrôles essentiellement limités par

$$\mathcal{U}_\infty = \bigcup_{T \geq 0} L^\infty([0,T], \Omega).$$

Fixé un contrôle u dans $\mathcal{U}_\infty$, l'équation différentielle non autonome (C) est bien posée dans l'espace des fonctions absolument continues. Plus précisément, pour toute condition initiale p dans M, il existe $T > 0$ et un voisinage V de p tel que $\phi(t, q, u)$ soit défini pour $(t, q) \in [0, T] \times V$ et absolument continue par rapport au temps, et tel que, pour tout $t \in [0, T]$, le flot $\phi(t, \cdot, u)$ restreint à V est un difféomorphisme local (voir, par exemple, [Jea17, Thm. 6.2] ou [Son98, Thm. 1]).

Dans ce qui suit, $\mathcal{U}$ désigne l'un des ensembles de contrôle présentés ci-dessus. (Remarquez que $U_{pc} \subset \mathcal{U}_\infty$.) L'ensemble atteignable à partir d'un état p dans M s'écrit en toute généralité sous la forme

$$\mathcal{A}_p = \{\phi(T, p, u) \mid T \geq 0,\ u \in \mathcal{U},\ \phi(\cdot, p, u) \text{ défini sur } [0, T]\}.$$

Le système (C) est dit *contrôlable* si l'ensemble atteignable à partir de n'importe quel état de M coïncide avec l'ensemble de l'espace d'états, c'est-à-dire,

$$\mathcal{A}_p = M, \qquad \forall\, p \in M.$$

2.1.1 Contrôlabilité locale et contrôlabilité

Le système (C) est dit *contrôlable localement* si, pour chaque $p \in M$ l'ensemble atteignable $\mathcal{A}_p$ contient un voisinage de p, i.e.,

$$p \in \operatorname{Int} \mathcal{A}_p, \qquad \forall\, p \in M. \tag{LC}$$

Remarquez que parfois dans la littérature l'expression *contrôlabilité locale* est utilisée avec un sens différent (voir, par exemple, [Cor07, Definition 3.2]).

La notion de contrôlabilité locale a été largement étudiée dans la littérature, en particulier dans les formes plus fortes de *contrôlabilité locale en petit temps* (ST-contrôlabilité locale) (pour laquelle l'ensemble atteignable $\mathcal{A}_x$ est remplacé par l'ensemble des points atteignables à partir de x dans un temps positif arbitrairement petit) et *contrôlabilité locale localisée* (L-contrôlabilité locale) (pour lequel on considère l'ensemble des points atteignables à partir de x par des trajectoires admissibles qui restent dans un voisinage arbitrairement petit de x). Une combinaison des deux contraintes donne la notion de *contrôlabilité locale localisée en petit temps* (STL-contrôlabilité locale). La Table 2.1 contient un schéma des implications qui peuvent être directement déduites des définitions ci-dessus, et nous renvoyons à la section 6.2 pour une description détaillée de ces différents types de contrôlabilité locale et les relations entre elles.

Le folklore veut que la contrôlabilité puisse être déduite de versions appropriées de la contrôlabilité locale : par exemple, dans [CLH$^+$07, Section 12.3], il est indiqué (sans preuve) que la STL-contrôlabilité

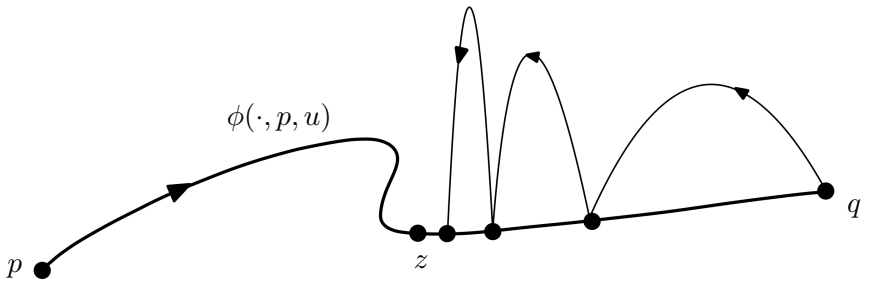

Figure 2.1: En remontant la trajectoire $\phi(\cdot, p, u)$ les ensembles atteignables peuvent devenir de plus en plus petits et s'effondrer à un point z avant d'atteindre x, car a priori leur taille n'est pas semi-continue inférieurement. Le Lemme 6.2 montre que cette situation ne peut pas arriver, prouvant une étape clé pour la preuve du Théorème 2.1.

locale implique la contrôlabilité. Un autre exemple sont les systèmes linéaires pour lesquels la contrôlabilité est connue pour être équivalente à la ST-contrôlabilité locale. La question de savoir si la ST-contrôlabilité locale implique contrôlabilité pour un système de contrôle plus général a été formulée par exemple dans [BS83, Sec. 3]. Le but de notre étude est de prouver que la version la plus faible de la contrôlabilité locale est suffisante pour déduire la contrôlabilité, en donnant notamment une réponse positive à la question qui vient d'être évoquée.

Theorem 2.1 [BCFS21, Thm. 1]

Supposons que M est connecté et que le système (C) a comme contrôles $\mathcal{U}_{pc}$ ou $\mathcal{U}_{\infty}$. Si le système (C) satisfait la propriété de contrôlabilité locale, alors il est contrôlable.

Nous observons que si la L-contrôlabilité locale est connue, alors la contrôlabilité peut être montrée avec une preuve plus simple que ce qui est proposé ici pour le Théorème 2.1 (voir Section 6.5) mais ne s'en suit pas immédiatement des définitions, puisque l'accessibilité n'est pas une propriété symétrique. En effet, le fait que l'on puisse atteindre un voisinage ouvert d'un état initial donné n'implique pas que tout point du voisinage puisse être ramené à l'état initial.

Il est a mentionner qu'il est difficile de trouver des conditions testables pour la contrôlabilité locale. En effet, dans la littérature, il est plus courant de trouver des conditions de ST-contrôlabilité locale puisque celles-ci peuvent être déduites d'arguments d'algébre de Lie (voir, par exemple, [KN21] et ses références). Il convient de noter que de telles conditions ne fournissent généralement pas de contrôlabilité locale en *chaque* point, car elles nécessitent généralement que le point auquel la contrôlabilité locale est étudiée soit un équilibre de l'un des champs de vecteurs admissibles. Enfin, notons que l'intérêt de la ST-contrôlabilité locale est motivé, par exemple, par sa relation avec la continuité de la fonction de temps optimal, comme expliqué dans [Sus87].

Notre démonstration du Théorème 2.1 est basée sur la propriété suivante : si le système (C) vérifie la propriété de contrôlabilité locale, alors pour tout p, q dans M un a que $p \in \mathcal{A}_q$ si et seulement si $q \in \mathcal{A}_p$ (voir le Lemme 6.2). Cette propriété est démontrée en prouvant que les trajectoires de (C) peuvent être retracées en trouvant un contrôle conduisant le point final à le point de départ. Plus précisément, supposons $q \in \mathcal{A}_p$ et considérons un contrôle u tel que $q = \phi(T, p, u)$. Pour t dans un voisinage gauche de T, les états $\phi(t, p, u)$ peuvent être atteints à partir de q grâce à la contrôlabilité locale. En répétant cet argument et en concaténant les contrôles, on peut trouver des temps $t \geq 0$ de plus en plus petits tels que $\phi(t, p, u)$ est atteint depuis q. Pour atteindre $p = \phi(0, p, u)$ il faut montrer que la séquence de temps t trouvée en suivant une telle procédure finit par atteindre zéro, contrairement à la situation décrite dans la Figure 2.1.

Remark 2.2. L'argument de la preuve du Théorème 2.1 se généralise à des classes de contrôles plus générales, à condition que le système de contrôle reste bien posé dans l'espace des fonctions absolument continues, au sens exprimé à la Section 2.1. Ici, nous avons décidé d'utiliser des contrôles essentiellement bornés plutôt que de nous en tenir à des contrôles constants par morceaux afin de montrer que les différences qui surviennent entre les contrôles dans $\mathcal{U}_\infty$ et $\mathcal{U}_{pc}$ n'affectent pas le Théorème 2.1.

Le fait que la contrôlabilité locale implique la contrôlabilité globale a déjà été traité dans [KS83] pour un espace d'états compact M. Pour le cas non compact, dans [Gra85] l'auteur montre que la contrôlabilité locale implique une contrôlabilité globale pour les contrôles constants par morceaux. Un ultérieur résultat dans [Gra95] montre que la contrôlabilité locale via des contrôles mesurables essentiellement limités implique la contrôlabilité locale via des contrôles constants par morceaux, étendant ainsi le résultat précédent aux systèmes de contrôle avec des contrôles essentiellement limités. Nous sommes reconnaissants à Kevin Grasse d'avoir attiré notre attention sur ces articles après la présentation de la première version de notre preuve sur arXiv.

2.1.2 Orbites, distributions et crochets de Lie

Dans cette section, nous rappelons quelques propriétés classiques importantes des orbites et des ensembles atteignables. Premièrement, les orbites sont des sous-variétés immergées de M. Précisément, une *sous-variété k-dimensionnelle immergée* est un sous-ensemble $S \subset M$ avec une structure de variété lisse k-dimensionnelle (pas nécessairement avec la topologie héritée par M) telle que le pushforward de l'inclusion $i : S \hookrightarrow M$ vérifie $\dim i_*(T_pS) = k$ pour tout $p \in S$, c'est-à-dire que l'inclusion est une *immersion*.

Theorem 2.3 (Théorème des orbites, [Sus73])**.** *Pour chaque $q \in M$, l'orbite $\mathcal{O}_q$ est une sous-variété connexe et immergée de M. De plus, pour tout $p \in \mathcal{O}_q$,*

$$T_p\mathcal{O}_q = \operatorname{span}_{\mathbb{R}}\big\{(e_*^{t_kX_k} \circ \cdots \circ e_*^{t_1X_1}Y)(p) \mid k \in \mathbb{N},\ t_1, \dots, t_k \in \mathbb{R},\ X_1, \dots, X_k, Y \in \mathcal{F}\big\}. \tag{2.1}$$

Il découle du théorème des orbites que les orbites définissent un *feuilletage*, c'est-à-dire une partition de X en sous-variétés connectées et immergées (appelées *feuilles*), éventuellement de dimension variable. De plus, le feuilletage décrit par les orbites est un feuilletage *lisse*, c'est-à-dire que pour tout p dans M et vecteur $v \in T_pM$ tangent à la feuille par p peut être étendu à un champ vectoriel lisse partout tangent aux feuilles du feuilletage. En effet, le fait que les feuilletages apparaissant comme des orbites de systèmes de contrôle comme dans (C) sont lisses est une conséquence du théorème des orbite et de la formule (2.1). Inversement, tout feuilletage lisse est la partition d'orbite d'un système de contrôle. Pour cela, il suffit de prendre comme famille de champs de vecteurs admissibles l'ensemble de tous les champs de vecteurs partout tangents aux feuilles.

Remark 2.4. Dans certains textes le terme feuilletage décrit ce que nous appellerons ici un *feuilletage régulier*, c'est-à-dire un feuilletage qui admet localement autour de chaque point une carte (U, x) telle que $x(U)$ puisse s'écrit $x(U) = V' \times V'' \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, pour un $k \in \mathbb{N}$ fixé, et tel que l'intersection d'une feuille avec U est soit vide, soit l'union dénombrable d'ensembles de la forme $x^{-1}(V' \times \{c\})$ pour $c \in V''$. Une telle carte est appelée une carte *folié*, et k est la dimension du feuilletage.

On appellera une *distribution* D l'affectation arbitraire, pour chaque point $p \in M$, d'un sous-espace linéaire $D_p \subset T_pM$. Étant donné un champ vectoriel $X \colon M \to TM$ on dit que X est une section de D si $X_p \in D_p$ pour tout $p \in M$. On note $\Gamma(D)$ l'ensemble des sections de D. Une distribution D est *lisse* si, pour tout point $p \in M$, il existe des sections lisses $X_1, \dots, X_k$ de D telles que $(X_1)_p, \dots, (X_k)_p$ est une base de D_p. Dans un tel cas, on appelle k le *rang* de D dans p.

Maintenant, définissons $\operatorname{Lie}\mathcal{F}$ comme étant la plus petite sous-algèbre de Lie de $\Gamma(TM)$ contenant $\mathcal{F}$. Précisément,

$$\operatorname{Lie}\mathcal{F} = \operatorname{span}_{C^\infty(U)}\big\{[X_1, \dots [X_{k-1}, X_k] \dots] \mid k \in \mathbb{N},\ X_1, \dots, X_k \in \mathcal{F}\big\}.$$

Pour toute famille $\mathcal{G}$ de champs de vecteurs, notons $D^{\mathcal{G}}$ la distribution lisse définie en affectant, pour tout $p \in M$, le sous-espace $D_p^{\mathcal{G}} = \operatorname{span}\{X_p \mid X \in \mathcal{G}\}$. Étant donné que le crochet de Lie de deux champs vectoriels X et Y peut être exprimé comme

$$[X,Y]_p = \frac{d}{dt}\Big|_{t=0} e^{-\sqrt{t}Y} \circ e^{-\sqrt{t}X} \circ e^{\sqrt{t}Y} \circ e^{\sqrt{t}X}(p),$$

la formule (2.1) pour l'espace tangent d'une orbite implique que

$$D_p^{\mathcal{F}} \subset D_p^{\operatorname{Lie}\mathcal{F}} \subset T_p\mathcal{O}_q, \qquad \forall\, q \in M,\ p \in \mathcal{O}_q. \tag{2.2}$$

On dit que le système de contrôle (C) est Lie-déterminé si la dernière inclusion dans (2.2) est une égalité, c'est-à-dire,

$$D^{\operatorname{Lie}\mathcal{F}} = T\mathcal{O}. \tag{2.3}$$

Sous certaines hypothèses assez générales, le système (C) est Lie-déterminé. A cet égard, rappelons quelques définitions complémentaires. Comme précédemment, soit $\mathcal{G}$ une famille de champs de vecteurs de M.

(i) $\mathcal{G}$ est *analytique* si M est une variété analytique et, pour tout $X \in \mathcal{G}$, X est analytique.

(ii) $\mathcal{G}$ est *localement finitement générée* si pour tout $q \in M$ il existe un voisinage ouvert U de q et un nombre fini des champs vectoriels $X_1, \ldots, X_k$ dans $\mathcal{G}$ tel que $\mathcal{G}|_U \subset \operatorname{span}_{C^\infty(U)}\{X_1|_U, \ldots, X_k|_U\}$.

(iii) $D^{\mathcal{G}}$ est *$\mathcal{G}$-invariant* si pour tout X dans $\mathcal{G}$ on a $e_*^{tX}(D_p^{\mathcal{G}}) = D_{e^{tX}p}^{\mathcal{G}}$ pour tout $p \in M$ et t.

Le théorème de Nagano [Nag66], le théorème d'Hermann [Her62] et le théorème de Stefan-Sussmann [Sus73, Ste74] indiquent que si, respectivement, $\mathcal{F}$ est analytique, $\operatorname{Lie}\mathcal{F}$ est localement finitement générée, $D^{\operatorname{Lie}\mathcal{F}}$ est $\operatorname{Lie}\mathcal{F}$-invariante, alors le système (C) est Lie-déterminé. Nous rappelons ici aussi le célèbre théorème de Frobenius [Fro77] indiquant que si $D^{\operatorname{Lie}\mathcal{F}}$ est une distribution de rang constant, alors le système (C) est Lie-déterminé et le feuilletage décrit par les orbites est régulier. Pour plus de détails sur le sujet, nous nous référons à [Lav18].

Sous l'hypothèse que le système (C) est Lie-déterminé, on peut montrer que les orbites ne peuvent pas être d'une dimension supérieure aux ensembles atteignables. C'est le contenu du théorème de Krener.

Theorem 2.5 (Théorème de Krener [Kre74])**.** *Supposons que* (C) *est Lie-déterminé. Alors, pour tout p dans M, l'ensemble atteignable $\mathcal{A}_p$ a un intérieur non vide dans l'orbite $\mathcal{O}_p$.*

On dit que la famille $\mathcal{F}$ est *crochet-générante en $p \in M$* si l'évaluation en p de l'algèbre de Lie générée par $\mathcal{F}$ a une dimension maximale, c'est-à-dire,

$$D_p^{\operatorname{Lie}\mathcal{F}} = T_pM. \tag{2.4}$$

Remark 2.6. Lorsque M est une variété analytique réelle et que les champs de vecteurs dans $\mathcal{F}$ sont analytiques réels, ce qui suit est vrai : si le système (C) satisfait la propriété d'accessibilité locale, alors $\mathcal{F}$ satisfait la condition de rang de l'algèbre de Lie à tout moment. Voir par exemple [SJ72, Thm. 3.1].

2.1.3 Contrôlabilité approchée pour les systèmes de contrôle bilinéaires

Un premier résultat obtenu en étudiant la contrôlabilité approchée est que, si le système (C) est contrôlable de façon approchée, alors le feuilletage décrit par ses orbites est un feuilletage régulier.

Lemma 2.7 ([CS21, Lem. 4])**.** *Supposons que le système* (C) *est contrôlable de façon approchée. Alors, les orbites de* (C) *forment un feuilletage régulier de M avec des feuilles denses.*

La preuve de ce lemme, qui est donnée au Chapitre 5, découle de la semi-continuité inférieure de la dimension des orbites. Le lemme 2.7 ne fournit aucune information directe sur la contrôlabilité. En effet, il pourrait n'y avoir qu'une seule orbite et pourtant les ensembles atteignables pourraient ne pas coïncider avec M. Cependant, avec l'hypothèse supplémentaire que (C) est déterminé par Lie, on en déduit ce qui suit.

Corollary 2.8 ([CS21, Cor. 5])**.** *Supposons que system* (C) *est Lie-déterminé et contrôlable de façon approchée. Alors, exactement une des alternatives suivantes vaut :*

(a) $\mathcal{F}$ *est crochet-générante en tout point de* M *; donc, le système* (C) *est contrôlable.*

(b) Il existe un entier k *avec* $0 < k < n$ *tel que les orbites de* (C) *forment un feuilletage régulier* k*-dimensionnel de* N *avec des feuilles denses.*

Le Corollaire 2.8 s'avère utile si l'on peut exclure l'existence d'un feuilletage régulier de M avec les propriétés décrites dans *(b)* ; cela pourrait être possible grâce à la forme particulière du système (C) ou à certaines propriétés topologiques de M. C'est ce que nous avons réussi à faire pour les systèmes de contrôle de la forme

$$\dot{p} = A(t)p, \qquad p \in \mathbb{R}^n \setminus \{0\},\ A(t) \in \mathcal{M} \tag{BL}$$

pour les contrôles constants par morceaux $A\ : [0, +\infty) \to \mathcal{M}$ prenant des valeurs dans un sous-ensemble $\mathcal{M}$ de l'espace $M_n(\mathbb{R})$ de $n \times n$ des matrices à coefficients réels, $n \geq 1$. Par un léger abus de notation, on désigne les systèmes de contrôle tels que (BL) par *systèmes de contrôle bilinéaires*, bien que ce dernier terme désigne généralement des systèmes pour lesquels $M(t) = A + u^1(t)B_1 + \cdots + u^m(t)B_m$ pour certains $A, B_1, \ldots, B_m \in M_n(\mathbb{R})$, avec le contrôle $t \mapsto (u^1(t), \ldots, u^m(t))$ prenant des valeurs dans un sous-ensemble Ω de $\mathbb{R}^m$. Pour une introduction aux systèmes de contrôle bilinéaire, nous nous référons à [CK00, Ell09]. Précisément, nous avons démontré le résultat suivant.

Theorem 2.9 [CS21, Thm. 1]

Considérons le système de contrôle bilinéaire (BL) de $\mathbb{R}^n \setminus \{0\}$. Le système (BL) est contrôlable de façon approchée si et seulement s'il est contrôlable.

La preuve, qui est donnée au Chapitre 5, est la suivante. Premièrement, on déduit de [BS20, BV13] que, si la projection de (BL) sur $\mathbb{RP}^{n-1}$ est contrôlable de façon approchée, alors les orbites de (BL) sont transversales à la direction radiale. Le Corollaire 2.8 s'applique puisque (BL) est Lie-déterminé en raison du théorème de Nagano. Si (BL) n'est pas contrôlable, alors le feuilletage des orbites est de codimension un avec des feuilles transversales à la direction radiale. Ensuite, nous montrons dans le Lemme 5.5 qu'un tel feuilletage ne peut pas avoir de feuilles denses, donnant le résultat recherché. En tant que sous-produit de la méthode ci-dessus, nous étendons le Théorème 2.9 aux systèmes de contrôle homogènes angulairement contrôlables ; voir Corollaire 5.6.

Ce résultat montre que la notion a priori plus faible de contrôlabilité approchée implique la contrôlabilité sans hypothèse supplémentaire, autre que le système de dimension finie étant bilinéaire. Ceci est utile lorsque des arguments topologiques conduisent directement à des résultats pour la contrôlabilité approchée, comme c'est le cas pour les constructions basées sur des ensembles de contrôle, dont la définition implique la fermeture d'ensembles atteignables [CK00]. Par exemple, dans [CRS21] les auteurs utilisent le Théorème 2.9 pour conclure que leur condition suffisante pour la contrôlabilité approchée, exprimée en termes de spectre de Floquet du système bilinéaire, donne en fait la contrôlabilité (voir [CRS21, Corollary 3.19]).

Une autre manière possible d'appliquer le Théorème 2.9 est la suivante : si pour un système bilinéaire on est capable d'identifier des champs de vecteurs compatibles avec (BL) au sens de [AS02, Definition 8.4] qui, ajoutées à celles admissibles, conduisent à une contrôlabilité approchée, puis la contrôlabilité de

(BL) s'ensuit, sans qu'il soit nécessaire de vérifier la condition crochet-générante. Un tel argument d'extension par des champs de vecteurs compatibles est, par exemple, au cœur des résultats dans [Che05], qui peuvent donc être améliorés par notre résultat. En particulier, le Théorème 2.9 implique que les hypothèses ii) et iii) sur l'existence d'équilibres stables et antistables peuvent être supprimées de [Che05, Théorème 4.3]. De même, les Propositions 3.3 et 3.6 du même article peuvent être renforcées en remplaçant dans leurs conclusions la contrôlabilité approchée et la contrôlabilité pratique par la contrôlabilité.

Le résultat du Théorème 2.9 contraste fortement avec le cas des systèmes bilinéaires en dimension infinie : lorsque les opérateurs contrôlés B_i apparaissant dans la représentation $M(t) = A + u^1(t)B_1 + \cdots + u^m(t)B_m$ sont bornés, ces systèmes ne peuvent pas être contrôlables (voir [BMS82, Theorem 3.6] et aussi [BCC20] pour les extensions récentes), alors qu'il existe des critères de contrôlabilité approchée (voir, par exemple, [Kha10, Chapitres 4 et 9], [CFK17] et [BCS14]).

Pour les systèmes de contrôle de dimension finie (auxquels s'étendent directement les notions de contrôlabilité et de contrôlabilité approchée), alors que la contrôlabilité implique clairement la contrôlabilité approchée, l'inverse peut ne pas être vrai sans hypothèses ultérieures. Un exemple standard peut être fourni en utilisant l'enroulement irrationnel d'une ligne dans le tore $\mathbb{T}^n$, $n \geq 2$.

D'autre part, l'équivalence énoncée dans le Théorème 2.9 est connue pour certaines classes de systèmes de contrôle. C'est le cas des systèmes de contrôle linéaire en $\mathbb{R}^n$, c'est-à-dire des systèmes de la forme

$$\dot{x} = Ax + Bu(t), \qquad u : [0, +\infty) \to \mathbb{R}^m, \quad x \in \mathbb{R}^n, \tag{2.5}$$

avec $A \in M_n(\mathbb{R})$ et $B \in M_{n\times m}(\mathbb{R})$. En effet, si le système (2.5) est approximativement contrôlable, alors l'ensemble atteignable depuis l'origine $\mathcal{A}_0$ est dense en $\mathbb{R}^n$. Puisque $\mathcal{A}_0$ est un espace linéaire (et en particulier il est fermé), il s'ensuit que $\mathcal{A}_0 = \mathbb{R}^n$, ce qui est bien connu pour être équivalent à la contrôlabilité de (2.5) en raison de la structure linéaire du système.

Peu d'autres classes de systèmes de contrôle pour lesquels la contrôlabilité approximative implique la contrôlabilité sont connues : systèmes quantiques fermés sur S^{n-1} [BGRS15, Theorem 17] ; systèmes de contrôle invariants à droite sur des groupes de Lie simples (comme il résulte de [JS72, Lemme 6.3] et [Smi42, Note at p. 312]) ; systèmes de contrôle obtenus en projetant sur $\mathbb{RP}^{n-1}$ systèmes de la forme de (BL) [BS20, Proposition 44].

En démontrant le Théorème 2.9, nous pouvons identifier les systèmes de contrôle bilinéaires comme une nouvelle classe de systèmes pour lesquels contrôlabilité approchée et contrôlabilité sont équivalentes. La preuve proposée du Théorème 2.9 (que l'on peut trouver dans la Section 5.2), fonctionne d'abord en déduisant de [BS20] et [BV13] que, si la projection de (BL) sur $\mathbb{RP}^{n-1}$ est contrôlable de façon approchée, alors les orbites de (BL) sont transversales à la radiale direction, puis en prouvant qu'il n'existe pas de feuilletage de codimension un dans $\mathbb{R}^n \setminus \{0\}$ avec des feuilles denses transversales à la direction radiale.

En tant que sous-produit de cette stratégie de démonstration, nous pouvons étendre le Théorème 2.9 aux systèmes de commande homogènes angulairement contrôlables (voir Corollaire 5.6).

2.2 Variétés sous-riemanniennes

Soit $\mathcal{F}$ une famille de champs de vecteurs sur une variété lisse M. Supposons que la distribution définie par $\mathcal{F}$ soit de rang constant, c'est-à-dire qu'il existe $k \in N$ tel que

$$\dim D_p^{\mathcal{F}} = k \qquad \forall p \in M. \tag{2.6}$$

On dit qu'une courbe localement Lipschitz $\gamma\colon I \to M$ est *horizontale* par rapport à $\mathcal{F}$ si $\dot{\gamma}(t) \in D^{\mathcal{F}}$ pour presque tout $t \in I$. Rappelons qu'une courbe Lipschitzienne admet une dérivée presque partout ; par exemple, voir [Hei04, p. 18]. On appelle une courbe *admissible* si elle est Lipschitz et horizontale. En d'autres termes, une courbe est admissible si et seulement si pour tout intervalle suffisamment

petit $I' \subset I$ il existe $X_1, \dots, X_k$ dans $\mathcal{F}$ et des fonctions mesurables et essentiellement limitées $u^1, \dots, u^k \colon I' \to \mathbb{R}$ telles que

$$\dot{\gamma}(t) = u^1(t)X_1(\gamma(t)) + \cdots + u^k(t)X_k(\gamma(t)), \qquad \text{pour presque tout } t \in I'. \tag{2.7}$$

Il s'agit d'une version de l'équation de contrôle (C), avec des contrôles essentiellement limitées et des paramètres de contrôle symétriques. Supposons maintenant qu'un produit scalaire lisse g ait été choisi sur la distribution $D^{\mathcal{F}}$. Dans ce cas, la *longueur sous-riemannienne* d'une courbe admissible γ est définie comme

$$L_{sR}(\gamma) = \int_I \sqrt{g(\dot{\gamma}, \dot{\gamma})}. \tag{2.8}$$

Au final, on voudrait définir la *distance sous-riemannienne* entre deux points p et q dans M comme

$$d_{sR}(p, q) = \inf\{L_{sR}(\gamma) \mid \gamma : [a, b] \to M \text{ admissible}, \ \gamma(a) = p, \ \gamma(b) = q\}. \tag{2.9}$$

Une propriété requise est que la topologie définie par d_{sR} doit coïncider avec la topologie de M. C'est le cas lorsque la famille $\mathcal{F}$ est crochet-générante. Cette dernière propriété dépend uniquement de la distribution $D^{\mathcal{F}}$, motivant la définition suivante. Une distribution lisse de rang constant D est dite *crochet-générante* si, pour toute famille $\mathcal{F}$ de champs vectoriels avec $D = D^{\mathcal{F}}$, la famille $\mathcal{F}$ est crochet-générante en tout point. Enfin, une *structure sous-riemannienne* sur une variété M est constituée d'une distribution génératrice de parenthèses lisse D et d'un produit scalaire lisse g défini sur D. Le triplet (M, D, g) est une *variété sous-riemannienne*.

La définition ci-dessus de la variété sous-riemannienne n'est pas la plus générale que l'on puisse donner, mais elle est suffisante pour les besoins de cette thèse. Pour une introduction générale à la géométrie sous-riemannienne, nous renvoyons le lecteur aux monographies [ABB20], [LD17], [Jea14], [Mon02] et [Rif14]. Dans ce qui suit, on suppose que M est une variété tridimensionnelle et D est une distribution de rang deux.

2.2.1 Surfaces dans les variétés de contact 3D

Désormais, la variété M est supposée tridimensionnelle. La distribution D est dite *coorientable* s'il existe une 1-forme ω sur M telle que

$$\ker \omega_p = D_p, \qquad \forall p \in M. \tag{2.10}$$

Sous l'hypothèse que D est coorientable, la paire (M, D) est appelée une *variété de contact* si une 1-forme ω qui satisfait localement (2.10) satisfait aussi $\mathrm{d}\omega|_D \neq 0$, ou de manière équivalente

$$\omega \wedge \mathrm{d}\omega \neq 0. \tag{2.11}$$

Un telle 1-forme ω est appelé une *forme de contact*. On peut vérifier que si (M, D) est une variété de contact alors, pour tout p dans M et $X_1, X_2 \in \Gamma(D)$, on a

$$\operatorname{span}\{X_1(p), X_2(p)\} = D_p \implies [X_2, X_1]_p \notin D_p. \tag{2.12}$$

Dans cette section, nous rappelons quelques faits pertinents sur les variétés de contact, et nous nous référons à [Etn03] et [Gei08] pour une introduction au sujet.

Soit S une surface plongée dans M. Un point p dans S est un *point caractéristique* si l'espace tangent T_pS coïncide avec la distribution D_p. L'ensemble des points caractéristiques de S est l'*ensemble caractéristiques*, noté $\Sigma(S)$. L'ensemble caractéristique est fermé en raison de la semi-continuité inférieure du rang, et il ne peut pas contenir d'ensembles ouverts en raison de la condition de contact. De plus, comme la distribution D est contact, l'ensemble $\Sigma(S)$ est contenu dans une sous-variété

unidimensionnelle de S (voir Lemme 3.5) et , génériquement, il est composé de points isolés (voir [Gei08, Par. 4.6]).

En dehors de l'ensemble caractéristique, l'intersection $TS \cap D$ est une distribution unidimensionnelle et définit (dû au théorème de Frobenius) un feuilletage unidimensionnel régulier sur $S \setminus \Sigma(S)$. Ce feuilletage se prolonge en un feuilletage lisse (cf. Subsection 2.1.2) de S en ajoutant un singleton en chaque point caractéristique. Le feuilletage résultant est le *feuilletage caractéristique* de S. Les feuilletages caractéristiques des surfaces dans les variétés de contact 3D sont étudiés dans de nombreuses références; à cet égard, nous nous référons à [Gir91, Gir00] et [Ben83]. Voir Figure 2.3 (à la fin du présent chapitre) pour une représentation graphique du feuilletage caractéristique sur une sphère du groupe de Heisenberg.

On appelle champs caractéristiques les champs de vecteurs de S dont la partition en orbites coïncide avec le feuilletage caractéristique de S. Précisément, étant donné un ouvert U dans S, un champ vectoriel X est un *champ caractéristique* de S dans U si, pour tout q dans U,

$$\operatorname{span}_{\mathbb{R}} X(q) = \begin{cases} \{0\}, & \text{si } q \in \Sigma(S), \\ T_q S \cap D_q, & \text{autrement,} \end{cases} \tag{2.13}$$

et satisfait la condition

$$\operatorname{div} X(p) \neq 0, \qquad \forall\, p \in \Sigma(S) \cap U. \tag{2.14}$$

Remarquez que $\operatorname{div} X(p)$ est bien défini puisque $X(p) = 0$, c'est-à-dire que p est un point caractéristique, et qu'il est indépendant de la forme du volume ; en particulier $\operatorname{div} X(p) = \operatorname{tr} DX(p)$. Sous certaines hypothèses classiques, on peut assurer l'existence d'un champ de vecteurs caractéristique global comme rappelé dans le lemme suivant. De plus, ces hypothèses sont toujours valables localement, dans un voisinage ouvert de tout point.

Lemma 2.10 ([Gei08, Par. 4.6])**.** *Supposons que S est orientable et que D est coorientable. Alors, S admet un champ caractéristique global ; de plus, les champs caractéristiques de S sont les champs de vecteurs X pour lesquels il existe une forme de volume Ω de S telle que*

$$\Omega(X, Y) = \omega(Y) \qquad \textit{pour tour } \ Y \in TS. \tag{2.15}$$

En effet, dans [Gei08] on montre que, si un champ vectoriel vérifie (2.15), alors il vérifie (2.13) et (2.14). Réciproquement, un champ vectoriel $\bar{X}$ satisfaisant (2.13) est un multiple de tout autre champ vectoriel X satisfaisant (2.15) pour des fonction ϕ avec $\phi|_{S\setminus\Sigma(S)} \neq 0$; de plus, si (2.14) est vrai, alors $\phi|_{\Sigma(S)} \neq 0$; ainsi, $\bar{X}$ satisfait (2.15) avec $\frac{1}{\phi}\Omega$ comme forme de volume de S.

Remark 2.11. La formule (2.15) signifie que les champs caractéristiques sont duaux des restrictions des formes de contact $\omega|_S$ par rapport aux formes de volume Ω de S. Puisque les formes de volume de S sont proportionnelles par des fonctions nulle nul part, il en va de même pour les champs caractéristiques. En particulier, si X est un champ caractéristique, alors $-X$ est également un champ caractéristique.

Donnons une autre façon de trouver, localement, une expression explicite pour un champ caractéristique local. Tout point de S admet un voisinage U de M dans lequel il existe un repère orthonormé (X_1, X_2) pour $D|_U$, et une submersion u de la classe C^2 pour laquelle S est un ensemble de niveau, c'est-à-dire,

$$S \cap U = \{q \in U : u(q) = 0\}, \quad \text{et } \mathrm{d}u \neq 0 \text{ sur } S \cap U\ . \tag{2.16}$$

Remarquons que pour tout point $p \in U \cap S$, on a

$$p \in \Sigma(S) \quad \text{si et seulement si} \quad X_1 u(p) = X_2 u(p) = 0. \tag{2.17}$$

Ici nous avons utilisé qu'un vecteur V est dans TS si et seulement si $Vu = 0$. Maintenant, le champ vectoriel X_u défini par

$$X_u = (X_1 u) X_2 - (X_2 u) X_1, \tag{2.18}$$

est un champ caractéristique de S. En effet, X_u satisfait (2.13) puisqu'il découle de la définition que, pour tout q dans S, le vecteur $X_u(q)\ dans T_qS \cap D_q$, et $X_u(p) = 0$ si et seulement si $p \in \Sigma(S)$, grâce à (2.17). De plus, X_u satisfait (2.14) puisque la divergence de X_u aux points caractéristiques est non nulle en raison de la condition de contact (2.12) et l'expression suivante

$$\operatorname{div} X_u(p) = X_2X_1u(p) - X_1X_2u(p) = [X_2, X_1]u(p), \qquad \forall\, p \in \Sigma(S).$$

Dans la littérature, l'étude des surfaces dans les variétés de contact tridimensionnelles a trouvé beaucoup d'intérêt puisque, entre autres, les feuilletages caractéristiques des surfaces fournissent un invariant important utilisé pour classer les structures de contact. De plus, le théorème suivant est valable.

Theorem 2.12 (Giroux, [Gei08, Thm. 2.5.22 and 2.5.23])**.** *Soient S_i des surfaces fermées dans des variétés de contact tridimensionnelles (M_i, D_i), $i = 0, 1$, avec D_i coorientées. Supposons que et $\phi : S_0 \to S_1$ soit un difféomorphisme entre les feuilletages caractéristiques respectifs. Alors il existe un contactomorphisme $\psi : N(S_0) \to N(S_1)$ entre voisinages convenables $N(S_i)$ de S_i, c'est-à-dire $\psi_* D_1|_{N(S_1)} = D_0|_{N(S_0)}$, avec $\psi|_{S_0} = \phi$.*

Structures de contact tendues

En gros, une distribution est *tendues* si elle n'admet pas de *disque vrillé*, c'est-à-dire un plongement d'un disque avec une frontière horizontale telle que la distribution ne se tord pas le long de la frontière. La notion de distribution tendue sera nécessaire pour énoncer le Théorème 2.16. Cependant, un lecteur qui est satisfait de la définition ci-dessus pourrait sauter ce paragraphe.

Pour définir un disque vrillé, considérons d'abord un plongement de $\Delta = \{x \in \mathbb{R}^2 \ : |x| \leq 1\}$ dans M, et notons $\Gamma = \partial\Delta$. Soit Γ horizontal par rapport à la distribution de contact D, c'est-à-dire $T\Gamma \subset D$. Ensuite, le fibré normal $N\Gamma = TM|_\Gamma / T\Gamma$ peut être décomposé de deux façons : la première par rapport à l'espace tangent de Δ, c'est-à-dire,

$$N\Gamma \cong {}^{TM}\!/_{T\Delta} \oplus {}^{T\Delta}\!/_{T\Gamma}, \tag{2.19}$$

et la seconde par rapport à la distribution des contact D, c'est-à-dire,

$$N\Gamma \cong {}^{TM}\!/_{D} \oplus {}^{D}\!/_{T\Gamma}. \tag{2.20}$$

Un repère (Y_1, Y_2) de $N\Gamma$ est appelé de *surface* s'il respecte le découpage (2.19), c'est-à-dire, $Y_1 \in TM \setminus T\Delta$ et $Y_2 \in T\Delta \setminus T\Gamma$; de même, il est appelé de *contact* s'il respecte le découpage (2.20). Puisque la distribution de contact est coorientée près de Δ, les deux fibrés (2.19) et (2.20) sont triviaux, donc on peut toujours trouver des repères de contact et de surface.

L'invariant de Thurston–Bennequin de Γ, noté $\mathrm{tb}(\Gamma)$, est le nombre de torsions d'un repère de contact de Γ par rapport à un repère de surface : les torsions à droite sont comptés positivement, et les torsions à gauche négativement (cf. par exemple [Gei08, Def. 3.5.4]). Notez que $\mathrm{tb}(\Gamma)$ est indépendant de l'orientation de Γ. L'exigence que la distribution D ne se torde pas le long de la frontière de Δ est équivalente à $\mathrm{tb}(\partial\Delta) = 0$, c'est-à-dire l'invariant de Thurston–Bennequin de $\partial\Delta$ étant nul.

Un disque plongé Δ dans une variété de contact coorientée (M, D) à bord lisse $\partial\Delta$ est un *disque vrillé* si $\partial\Delta$ est une courbe horizontale de D, $\mathrm{tb}(\partial\Delta) = 0$, et il y a exactement un point caractéristique à l'intérieur du disque. Notons que le lemme d'élimination de Giroux permet de supprimer la condition qu'il n'y ait qu'un seul point caractéristique à l'intérieur du disque vrillé, comme discuté par exemple dans [Gei08, Def. 4.5.2].

2.2.2 Approximation riemannienne

La construction d'approximations riemanniennes d'une structure sous-riemannienne est un outil clé utilisé dans cette thèse, et leur utilisation pour définir des invariants géométriques sous-riemanniens est une technique bien connue. Pour une description générale des propriétés de l'approximation riemannienne dans le groupe de Heisenberg, nous renvoyons à [CDPT07]. Dans cette thèse, nous utilisons l'approximation riemannienne pour deux raisons. D'abord associer à chaque point caractéristique un nombre réel $\widehat{K}$, comme expliqué ci-dessous dans le Théorème 2.13. Deuxièmement, comme expliqué dans la section 2.3, construire un processus stochastique canonique sur S.

Supposons que (M, D, g) est une variété tridimensionnelle munie d'une distribution coorientée D de rang deux. Sous cette hypothèse, on peut fixer un champ vectoriel X_0 transverse à la distribution, c'est-à-dire,

$$\operatorname{span}\{D, X_0\} = TM.$$

Une fois ce choix fait, on peut étendre la métrique sous-riemannienne g à une métrique riemannienne g^{X_0} en définissant X_0 comme unitaire et orthogonal à D. Les métriques riemanniennes $g^{\varepsilon X_0}$, pour $\varepsilon > 0$, sont les *approximations riemanniennes* de (D, g) par rapport à X_0. Précisément, pour chaque $\varepsilon > 0$, on a

$$\langle D, X_0 \rangle_{g^{\varepsilon X_0}} = 0, \quad |X_0|_{g^{\varepsilon X_0}} = \frac{1}{\varepsilon}, \qquad \forall \varepsilon > 0.$$

Pour simplifier la notation, une fois qu'il est clair quel champ vectoriel transversal a été choisi, nous pouvons supprimer la dépendance à X_0 dans l'exposant, en écrivant $g^{\varepsilon} = g^{\varepsilon X_0}$.

Soit K^{X_0} la courbure gaussienne de S par rapport à g^{X_0}. Fixé un point caractéristique $p \in \Sigma(S)$, le coefficient $\widehat{K}$ sera défini comme la limite pour $\varepsilon \to 0$ de $K^{\varepsilon X_0}$, en utilisant une normalisation appropriée de la structure des crochets de Lie sur la distribution. Précisément, soit B^{X_0} la forme bilinéaire $B^{X_0} : D \times D \to \mathbb{R}$ définie par

$$B^{X_0}(X, Y) = \alpha \qquad \text{if} \qquad [X, Y] = \alpha X_0 \mod D.$$

Comme D est muni de la métrique g, la forme bilinéaire B^{X_0} admet un déterminant bien défini.

Theorem 2.13 [BBC21, Thm. 1.1]

Soit S une surface C^2 plongée dans une variété sous-riemannienne de contact 3D. Soit p un point caractéristique de S, et soit X_0 un champ vectoriel transverse à la distribution D au voisinage de p. Alors, dans les notations définies ci-dessus, la limite

$$\widehat{K}_p = \lim_{\varepsilon \to 0} \frac{K_p^{\varepsilon X_0}}{\det B_p^{\varepsilon X_0}} \tag{2.21}$$

est fini et indépendant du champ vectoriel X_0.

Remarquez que dans la littérature précédente l'approximation riemannienne est employée pour définir des invariants géométriques sous-riemanniens *en dehors* de l'ensemble de caractéristiques. Par exemple, dans [BTV17] les auteurs ont défini la courbure gaussienne sous-riemannienne en un point $x \in S \setminus \Sigma(S)$ comme

$$\mathcal{K}_S(x) = \lim_{\varepsilon \to 0} K_x^{\varepsilon X_0}, \tag{2.22}$$

et ils ont prouvé qu'un théorème de type Gauss-Bonnet est vrai ; ici, les auteurs ont travaillé dans le cadre du groupe de Heisenberg, et avec X_0 égal au champ vectoriel de Reeb du groupe de Heisenberg. Cette construction est étendue dans [WW20] au groupe affine et au groupe des mouvements rigides du

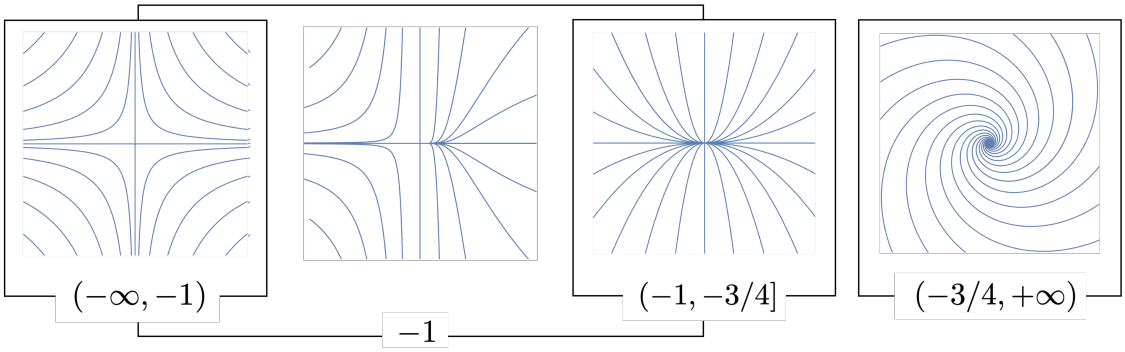

Figure 2.2: L'image qualitative du feuilletage caractéristique en un point caractéristique *isolé*, pour les valeurs correspondantes de $\widehat{K}$. De gauche à droite, on reconnaît une selle, une selle-nœud, un nœud et un foyer.

plan de Minkowski, et dans [Vel20] à une variété sous-riemannienne générale. Dans ce dernier, l'auteur a lié $\mathcal{K}_S$ avec la courbure introduite dans [DV16], et, lorsque $\Sigma(S) = \emptyset$, ils ont prouvé un Théorème de Gauss-Bonnet par la formule de Stokes. Un théorème de Gauss-Bonnet (dans un cadre différent) a également été prouvé dans [ABS08]. On remarque enfin que l'invariant $\mathcal{K}_S$ apparaît aussi dans [Lee13], où il est appelé courbure de transversalité. Une expression pour $\mathcal{K}_S$ est également fournie dans la Proposition 2.18.

Comme nous le verrons, le coefficient $\widehat{K}_p$ détermine le comportement qualitatif du feuilletage caractéristique près d'un point caractéristique p. Suivant la terminologie de la géométrie de contact (cf. par exemple [Gei08, Par. 4.6]), étant donné un point caractéristique $p \in \Sigma(S)$ et un champ caractéristique X, le point p est appelé *non dégénéré* si $\det DX(p) \neq 0$. De plus, p est appelé *elliptique* si $\det DX(p) > 0$, et *hyperbolique* si $\det DX(p) < 0$. Dans la théorie des systèmes dynamiques, les selles et les points hyperboliques sont respectivement ce que nous appelons points caractéristiques hyperboliques et points caractéristiques non dégénérés.

Proposition 2.14 ([BBC21, Prop. 1.2])**.** *Soit S une surface C^2 plongée dans une variété sous-riemannienne de contact 3D. Étant donné un point caractéristique p dans $\Sigma(S)$, soit X un champ caractéristique X proche de p. Alors,* $\operatorname{tr} DX(p) \neq 0$ *et*

$$\widehat{K}_p = -1 + \frac{\det DX(p)}{(\operatorname{tr} DX(p))^2}. \tag{2.23}$$

Ainsi, p est hyperbolique si et seulement si $\widehat{K}_p < -1$, et p est elliptique si et seulement si $\widehat{K}_p > -1$.

Cette égalité relie $\widehat{K}_p$ aux valeurs propres de $DX(p)$, qui déterminent le comportement qualitatif du feuilletage caractéristique autour du point caractéristique p. Cette relation est explicitée dans le Corollaire 3.10 pour un point caractéristique non dégénéré, et dans le Corollaire 3.12 pour un point caractéristique dégénéré. De plus, l'équation (2.23) montre que $\widehat{K}_p$ est indépendant de la métrique sous-riemannienne, et ne dépend que du champ linéaire défini par D sur S.

2.2.3 Distance induite sur les surfaces

L'étude de la géométrie des sous-variétés S d'une variété ambiante M équipée d'une structure géométrique donnée est un sujet classique. Un exemple familier, dont l'étude remonte à Gauss, est celui d'une surface S plongée dans l'espace euclidien $\mathbb{R}^3$. Dans un tel cas, S hérite de sa structure riemannienne naturelle en restreignant le tenseur métrique à l'espace tangent de S. La distance induite

sur S par ce tenseur métrique n'est pas la restriction de la distance de $\mathbb{R}^3$ aux points sur S, mais plutôt la structure d'espace des longueurs induite sur S par l'espace ambiant.

Les choses sont moins simples pour une 3-variété lisse M dotée d'une structure sous-riemannienne de contact (D, g). En effet, pour une sous-variété bidimensionnelle S, l'intersection $T_xS \cap D_x$ est unidimensionnelle pour la plupart des points x dans S ; ainsi, $TS \cap D$ n'est pas une distribution génératrice de parenthèses et il n'y a pas de distance sous-riemannienne bien définie induite par (M, D, g) sur S. Ce fait est en effet plus général, comme déjà observé dans [Gro96, Sec. 0.6.B]. Néanmoins, on peut encore définir une distance sur S suivant le point de vue espace longueur : la distance sous-riemannienne d_{sR} définit la *longueur* de toute courbe continue $\gamma : [0, 1]$ M comme

$$L_{sR}(\gamma) = \sup \left\{ \sum\nolimits_{i=1}^{N} d_{sR}(\gamma(t_i), \gamma(t_{i+1})) \mid 0 = t_0 \leq \ldots \leq t_N = 1 \right\},$$

et on peut définir $d_S : S \times S \to [0, +\infty]$ par

$$d_S(x, y) = \inf\{L_{sR}(\gamma) \mid \gamma : [0, 1] \to S,\ \gamma(0) = x,\ \gamma(1) = y\}.$$

L'espace (S, d_S) est appelé un *espace de longueur*, et d_S la *distance induite* définie par (M, d_{sR}). (Dans la théorie des espaces métriques de longueur, la distance induite d_S est appelée distance intrinsèque, en soulignant qu'elle dépend uniquement des longueurs des courbes dans S, voir [BBI01].) Nous soulignons que la la distance induite d_S n'est pas la restriction $d_{sR}|_{S\times S}$ de la distance sous-riemannienne à S.

Nous avons commencé l'investigation en cherchant les conditions nécessaires et suffisantes sur la surface S pour que la distance induite d_S soit finie, c'est-à-dire $d_S(x, y) < +\infty$ pour tout points x, y dans S ; cela équivaut à (S, d_S) étant un espace métrique. Cela peut être formulé en utilisant le feuilletage caractéristique de S. Précisément, considérons une courbe continue $\gamma : [0, 1] \to S$. Sa longueur est finie, c'est-à-dire $L_{sR}(\gamma) < +\infty$, si et seulement si γ est une reparamétrisation d'une courbe $\bar{\gamma}$ horizontale par rapport à D ; dans ce cas, la longueur de γ coïncide avec la longueur sous-riemannienne de $\bar{\gamma}$. Nous nous référons à [BBI01, Ch. 2] et [ABB20, Sec. 3.3] pour plus de détails. En conclusion, la distance $d_S(x, y)$ entre deux points x et y dans S est finie si et seulement s'il existe une concaténation continue de longueur finie de feuilles du feuilletage caractéristique de S les reliant.

D'un point de vue local, pour que la distance induite soit finie, il faut que des points caractéristiques soient accessibles depuis leur complément. A cet égard, nous avons pu prouver que les feuilles unidimensionnelles du feuilletage caractéristique de S qui convergent vers un point caractéristique ont une longueur finie. Précisément, soit ℓ une feuille du feuilletage caractéristique de S ; on dit qu'un point p dans S est un *point limite* de ℓ s'il existe un point q dans ℓ et un champ vectoriel caractéristique X de S tel que

$$e^{tX}(q) \to p \qquad \text{for } t \to +\infty, \tag{2.24}$$

où e^{tX} est le flux de X. Dans ce cas, on note la demi-feuille $\ell_X^+(q) = \{e^{tX}(q) \mid t \geq 0\}$. Avec la définition ci-dessus, une feuille peut avoir au plus deux points limites : un pour chaque extrémité. Enfin, notez qu'un point limite d'une feuille doit être un zéro du champ de vecteurs caractéristique correspondant X, c'est-à-dire un point caractéristique de S.

Proposition 2.15 ([BBC21, Thm. 1.3])**.** *Soit S une surface C^2 plongée dans une variété sous-riemannienne de contact 3D, et soit p un point limite d'une feuille unidimensionnelle ℓ. Soit $x \in \ell$, et X un champ caractéristique tel que $e^{tX}(x) \to p$ pour $t \to +\infty$. Alors, la longueur de $\ell_X^+(x)$ est finie.*

Ce résultat est une conséquence du fait que la structure sous-riemannienne est en contact. En effet, pour une distribution non de contact cette conclusion peut-être fausse ; par exemple, dans [ZZ95, Lem. 2.1] les auteurs prouvent que la longueur des demi-feuilles du feuilletage caractéristique

d'une surface de Martinet convergeant vers un point elliptique est infinie.

Du côté global, nous déterminons certaines conditions pour que la métrique induite d_S soit finie sous l'hypothèse qu'il existe un champ caractéristique global de S. Dans un tel cas, pour une surface compacte et connexe S avec des points caractéristiques isolés, dans la Proposition 3.16 nous montrons que d_S est fini en l'absence des classes suivantes de feuilles dans le feuilletage caractéristique de S : trajectoires récurrentes non triviales, trajectoires périodiques et contours d'un côté. Ces conditions sont satisfaites par des sphères dans des espaces de contact étroits et coorientables.

Theorem 2.16 [BBC21, Thm. 1.4]

Soit (M, D, g) une structure de contact sous-riemannienne tendue coorientable, et soit S une surface plongée C^2 avec des points caractéristiques isolés, homéomorphe à une sphère. Alors la distance induite d_S est finie.

Nous soulignons que la propriété selon laquelle tout point caractéristique est isolé est une propriété générique pour les surfaces dans les variétés de contact. Ici, par propriété générique, nous entendons une propriété satisfaite sur un sous-ensemble ouvert et dense de l'ensemble de toutes les distributions de contact coorientables pour la topologie C^∞ (une fois la surface fixée), et sur un sous-ensemble ouvert et dense de la ensemble de toutes les surfaces C^2 pour la topologie C^2 (une fois les distributions de contact fixées). Example 3.5.4 et Example 3.5.5 dans la distribution de Heisenberg montrent que, si S n'est pas une sphère topologique, alors S présente éventuellement trajectoires récurrentes non triviales ou trajectoires périodiques, cas où d_S n'est pas fini. De plus, si l'on enlève l'hypothèse que la structure de contact est serrée, alors une sphère S pourrait présenter une trajectoire périodique, donc la distance induite d_S ne serait pas finie. L'hypothèse de compacité est également importante, comme on peut le voir dans Exemple 3.5.1.

Lorsque l'on considère des hypersurfaces dans des structures de contact, il faut généralement exiger l'absence de points caractéristiques pour obtenir des résultats généralisant à partir de ceux riemanniens. Considérons, par exemple, le résultat dans [TY04, Thm. 1.1] décrivant la distance induite sur les hypersurfaces de variétés sous-riemanniennes de dimension supérieure à trois, ou le résultat dans [RR21] sur le petit temps asymptotique des surfaces d'enthalpie dans le cas 3D. Au contraire, ici la finitude de la distance dans le cas 3D nécessite la présence de points caractéristiques ; ceci est principalement dû à l'intégrabilité du champ caractéristique et au caractère global de la question. Enfin, il nous reste à observer que pour de nombreuses surfaces la distance induite n'est pas finie. Par exemple, la distance induite sur une surface sans points caractéristiques n'est pas finie, et la propriété de ne pas avoir de points caractéristiques ne change pas avec de petites perturbations de la surface ou de la structure de contact dans les topologies appropriées.

2.3 Processus stochastiques sur des surfaces sous-riemanniennes

Soit (M, D, g) un espace sous-riemannien de contact coorienté. On supposera que la distribution D est *libre*, c'est-à-dire globalement engendrée par un couple de champs de vecteurs (X_1, X_2) qui constituent un repère orthonormé pour D orienté par rapport à la forme de volume vol_g sur D définie par g. De plus, nous choisissons la forme de contact ω normalisée pour que

$$\mathrm{d}\omega|_D = \mathrm{vol}_g\,. \tag{2.25}$$

Associé à une telle forme de contact ω, on a un choix canonique d'un champ vectoriel partout transversal à D : c'est le champ vectoriel de textitReeb X_0, qui est l'unique champ vectoriel sur M satisfaisant

$$\mathrm{d}\omega(X_0, \cdot) \equiv 0, \quad \text{et} \quad \omega(X_0) \equiv 1. \tag{2.26}$$

Soit g^ε, pour $\varepsilon > 0$, les approximations riemanniennes de (M, D, g) par rapport au champ vectoriel de Reeb X_0, comme décrit dans la section 2.2.2. Rappelez-vous que g^ε est défini en exigeant que $(X_1, X_2, \varepsilon X_0)$ soit un repère orthonormé global pour g^ε. Soit S une surface plongée dans M, et supposons que S est globalement définie par une submersion u au sens de l'équation (2.16). A savoir, on a

$$S = \{x \in M : u(x) = 0\}, \quad \text{et} \quad \mathrm{d}u \neq 0 \text{ on } S. \tag{2.27}$$

Ceci implique que le champ caractéristique X_u introduit dans (2.18) est défini globalement sur S. Soit $\widehat{X}_S$ le champ vectoriel sur $S \setminus \Sigma(S)$ défini par $\widehat{X}_S = X_u/|X_u|_g$. Explicitement,

$$\widehat{X}_S = \frac{(X_1 u)X_2 - (X_2 u)X_1}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \ . \tag{2.28}$$

Même si $\widehat{X}_S$ est exprimé en termes de X_1, X_2 et u, il ne dépend que de la variété sous-riemannienne (M, D, g), de la surface plongée S et un choix de signe. Soit $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ la fonction définie par

$$b = \frac{X_0 u}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \ . \tag{2.29}$$

Comme pour le champ vectoriel $\widehat{X}_S$, la fonction b peut être comprise intrinsèquement. En effet, soit $\widehat{X}_S^\perp$ tel que $(\widehat{X}_S, \widehat{X}_S^\perp)$ soit un repère orthonormé orienté pour $D|_{S\setminus\Sigma(S)}$. La fonction b est définie de manière unique en exigeant que $b\widehat{X}_S^\perp - X_0$ soit un champ vectoriel sur $S \setminus \Sigma(S)$. Enfin, définissez

$$\Delta_0 = \widehat{X}_S^2 + b\widehat{X}_S \ , \tag{2.30}$$

qui est un opérateur différentiel partiel du second ordre sur $S \setminus \Sigma(S)$. L'opérateur Δ_0 est invariant par multiplications de u par des fonctions non nulles. Comme indiqué dans le théorème ci-dessous, il apparaît comme l'opérateur limite des opérateurs de Laplace–Beltrami Δ_ε des approximations riemanniennes g^ε, pour $\varepsilon \to 0$.

Theorem 2.17 ([BBCH21, Thm. 1.1])**.** *Pour toute fonction deux fois dérivable $f \in C_c^2(S \setminus \Sigma(S))$ avec support compact dans $S \setminus \Sigma(S)$, les fonctions $\Delta_\varepsilon f$ convergent uniformément sur $S \setminus \Sigma(S)$ vers $\Delta_0 f$ pour $\varepsilon \to 0$.*

Suivant la définition de Balogh, Tyson et Vecchi [BTV17] pour les surfaces du groupe de Heisenberg, la courbure gaussienne intrinsèque $\mathcal{K}_S$ d'une surface dans une variété sous-riemannienne de contact tridimensionnelle générale est définie comme la limite de les courbures gaussiennes par rapport aux métriques riemanniennes g_ε, comme décrit dans (2.22). Dans la proposition suivante, nous dérivons une expression pour $\mathcal{K}_S$, en utilisant le même repère orthogonal présenté pour prouver Théorème 2.17.

Proposition 2.18 ([BBCH21, Prop. 1.2])**.** *Uniformément sur des sous-ensembles compacts de $S\setminus\Sigma(S)$, on a*

$$\mathcal{K}_S := \lim_{\varepsilon \to 0} K^\varepsilon = -\widehat{X}_S(b) - b^2 \ .$$

Considérons maintenant le *processus stochastique canonique* sur $S \setminus \Sigma(S)$ avec générateur $\frac{1}{2}\Delta_0$. Une fois qu'il commence à un certain point, à moins qu'il diverge, le processus se déplace le long de la feuille unique de la foliation caractéristique repérée par le point de départ. Cela découle du fait que le champ de vecteurs $\widehat{X}_S$ est tangent au feuilletage caractéristique de S. Comme le montre le théorème suivant et la proposition suivante, pour ce processus stochastique les points caractéristiques elliptiques sont inaccessibles, tandis que les points caractéristiques hyperboliques sont accessibles depuis les séparatrices.

Theorem 2.19 [BBCH21, Thm. 1.3]

L'ensemble des points caractéristiques elliptiques d'une surface S plongée dans M est inaccessible pour le processus stochastique avec générateur $\frac{1}{2}\Delta_0$ sur $S \setminus \Sigma(S)$.

Dans la section 4.3.3, nous discutons d'un exemple de surface du groupe d'Heisenberg dont le processus stochastique induit est arrêté en temps fini s'il démarre le long des séparatrices du point caractéristique. En effet, ce phénomène se produit toujours en présence d'un point caractéristique hyperbolique.

Theorem 2.20 [BBCH21, Prop. 1.4]

Supposons que la surface S plongée dans M ait un point caractéristique hyperbolique. Alors le processus stochastique de générateur $\frac{1}{2}\Delta_0$ si commencé sur les séparatrices du point caractéristique hyperbolique atteint ce point caractéristique avec une probabilité positive.

Les sections 4.3 et 4.4 sont consacrées à illustrer les différents comportements qu'on retrouve dans le processus stochastique canonique induit sur la surface S. En plus d'illustrer la Proposition 2.20, nous montrons dans le Théorème 2.21 ci-dessous que trois classes de processus stochastiques familiers apparaissent lorsque l'on considère un choix naturel pour la surface S dans les trois classes d'espaces modèles pour les structures sous-riemanniennes tridimensionnelles, qui sont le groupe d'Heisenberg dans $\mathbb{R}^3$, et le groupe unitaire spécial $\mathrm{SU}(2)$ et le groupe linéaire spécial $\mathrm{SL}(2,\mathbb{R})$ équipé de structures de contact sous-riemanniennes avec des produits scalaires différant par un multiple constant. Dans tous ces cas, le repère orthonormé (X_1, X_2) pour la distribution D est formé de deux champs vectoriels invariants à gauche qui, avec le champ vectoriel de Reeb X_0, satisfont, pour un certain $\kappa \in R$, les relations suivantes entre crochets de Lie

$$[X_2, X_1] = X_0, \quad [X_1, X_0] = \kappa X_2, \quad \text{et} \quad [X_2, X_0] = -\kappa X_1,$$

avec $\kappa = 0$ dans le groupe Heisenberg, $\kappa > 0$ dans $\mathrm{SU}(2)$ et $\kappa < 0$ en $\mathrm{SL}(2,\mathbb{R})$. Associés à chacun de ces groupes de Lie et à leurs algèbres de Lie, nous avons l'application exponentielle de groupe exp pour laquelle nous identifions un champ vectoriel invariant à gauche avec sa valeur à l'origine.

Theorem 2.21 ([BBCH21, Thm. 1.5])**.** *Choisissons $\kappa \in \mathbb{R}$. Pour $\kappa \neq 0$, soit $k \in \mathbb{R}$ avec $k > 0$ tel que $|\kappa| = 4k^2$. Définissez $I = (0, \frac{\pi}{k})$ si $\kappa > 0$ et $I = (0, \infty)$ sinon. Dans l'espace modèle des structures tridimensionnelles sous-riemanniennes correspondant à κ, on considère la surface plongée S paramétrée par*

$$S = \{\exp(r\cos\theta X_1 + r\sin\theta X_2) : r \in I \text{ et } \theta \in [0, 2\pi)\}\ .$$

Alors, l'opérateur limitant Δ_0 sur S est donné par $\Delta_0 = \frac{\partial^2}{\partial r^2} + b(r)\frac{\partial}{\partial r}$, où

$$b(r) = \begin{cases} 2k\cot(kr) & \text{if } \kappa = 4k^2 \\ \frac{2}{r} & \text{if } \kappa = 0 \\ 2k\coth(kr) & \text{if } \kappa = -4k^2 \end{cases}.$$

Le processus stochastique induit par l'opérateur $\frac{1}{2}\Delta_0$ se déplaçant le long des feuilles du feuilletage caractéristique de S est un processus de Bessel d'ordre 3 si $\kappa = 0$, un processus de Legendre d'ordre 3 si $\kappa > 0$ et un processus de Bessel hyperbolique d'ordre 3 si $\kappa < 0$.

Les processus stochastiques que nous récupérons ici sont tous liés au mouvement brownien unidimensionnel par le même type de transformation de Girsanov, seul le signe d'un paramètre les distinguant. Pour les détails, voir Revuz et Yor [RY99, p. 357]. Rappelons ici qu'un processus de Bessel d'ordre 3 naît en conditionnant un mouvement brownien unidimensionnel commencé sur la droite réelle positive à ne

jamais atteindre l'origine, alors qu'un processus de Legendre d'ordre 3 est obtenu en conditionnant un mouvement brownien commencé à l'intérieur un intervalle pour ne jamais atteindre l'une des extrémités de l'intervalle. Les exemples composant Théorème 2.21 peuvent être considérés comme des cas modèles pour notre cadre, et tous illustrent Théorème 2.19.

Enfin, notons que l'opérateur limite que nous obtenons sur les feuilles n'est pas le Laplacien associé à la structure métrique restreinte aux feuilles car cette dernière n'a pas de terme de drift. Cependant, l'opérateur Δ_0 restreint à une feuille peut être considéré comme un Laplacien pondéré. Pour une mesure lisse $\mu = h^2\,\mathrm{d}x$ sur un intervalle I de la droite euclidienne $\mathbb{R}$, le Laplacien pondéré appliqué à une fonction scalaire u donne

$$\mathrm{div}_\mu\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{2h'(x)}{h(x)}\frac{\partial f}{\partial x}\ .$$

Dans les cas modèles ci-dessus, nous avons

$$h(r) = \begin{cases} \sin(kr) & \text{if } \kappa = 4k^2 \\ r & \text{if } \kappa = 0 \\ \sinh(kr) & \text{if } \kappa = -4k^2 \end{cases}\ .$$

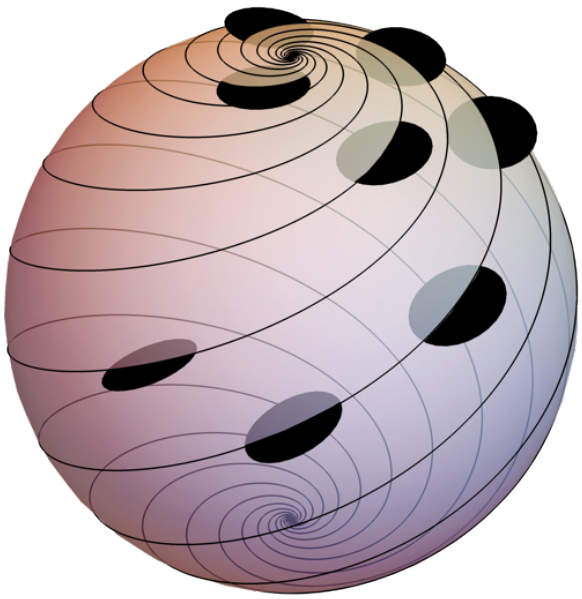

Figure 2.3: Le feuilletage caractéristique défini par la distribution de Heisenberg $(\mathbb{R}^3, \ker(dz + \frac{1}{2}(ydx - xdy))$ sur une sphère euclidienne centrée à l'origine : toute courbe horizontale qui relie des points sur des différentes spirales passe par l'un des points caractéristiques, au pôle Nord ou au pôle Sud. La longueur sous-riemannienne des feuilles en spirale autour des points caractéristiques est finie à cause de la Proposition 2.15. Ainsi, la distance induite d_S est finie : c'est un cas particulier du Théorème 2.16. Le processus stochastique canonique démarré en dehors des points caractéristiques n'atteint jamais ni le pôle nord ni le pôle sud, et il induit un processus unidimensionnel sur l'unique feuille du feuilletage caractéristique repéré par le point de départ, grâce au Théorème 2.19.

Chapter 3

On the induced geometry on surfaces in 3D contact sub-Riemannian manifolds

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This chapter describes the results in the paper [BBC21], joint work with Davide Barilari and Ugo Boscain, and it is currently provisionally accepted for publication in ESAIM: Control, Optimisation and Calculus of Variations.

First, in Section 3.1 we discuss the limit for $\varepsilon \to 0$ of the Gaussian curvature K^ε of a surface S in the Riemannian approximations $g^{\varepsilon X_0}$; this will give the asymptotic presented in Theorem 1.13 at the characteristic points. Next, we focus on the characteristic foliation of a surface S and its local properties around a characteristic point, and we prove Proposition 1.14 (pictured in Figure 1.2 for isolated characteristic points), and Proposition 1.15. Next, in Section 3.3 we discuss the global features of the characteristic foliations which might prevent the intrinsic distance d_S to be finite, and we prove Proposition 3.16. In Section 3.4 we show that such conditions are satisfied by spheres with the hypothesis that the distribution D is contact, proving Theorem 1.16. Finally, in Section 3.5 we discuss some examples.

3.1 Riemannian approximations and Gaussian curvature

Following the notations introduced in the introduction, let M be a smooth 3-dimensional manifold, and (D, g) be a smooth contact sub-Riemannian structure on M. As in Subsection 1.2.2, assume having fixed a vector field X_0 transverse to the distribution, and extend the sub-Riemannian metric g to a family of Riemannian metrics $g^\varepsilon = g^{\varepsilon X_0}$, $\varepsilon > 0$, for which X_0 is transversal to D and with norm $1/\varepsilon$. Let $\overline{\nabla}^\varepsilon$ be the Levi-Civita connection of (M, g^ε). Let us express this connection locally, in a domain equipped with an orthonormal frame (X_1, X_2) of D oriented with respect to vol_g; thus, $(\varepsilon X_0, X_1, X_2)$ is an orthonormal basis of g^ε. Due to the Koszul formula, for all $i, j, k = 0, 1, 2$, one has

$$\big\langle \overline{\nabla}^\varepsilon_{X_i} X_j, X_k \big\rangle_{g^\varepsilon} = \frac{1}{2}\Big(- \langle X_i, [X_j, X_k]\rangle_{g^\varepsilon} + \langle X_k, [X_i, X_j]\rangle_{g^\varepsilon} + \langle X_j, [X_k, X_i]\rangle_{g^\varepsilon}\Big).$$

This identity enables us to describe $\overline{\nabla}^\varepsilon$ using the frame (X_0, X_1, X_2), which is independent from ε. This is done using the Lie bracket structure of the frame, i.e., the C^∞ functions c^k_{ij} called the *structure constants* of the frame (X_0, X_1, X_2) defined by

$$[X_j, X_i] = c^1_{ij} X_1 + c^2_{ij} X_2 + c^0_{ij} X_0 \qquad \text{for } i, j = 0, 1, 2. \tag{3.1}$$

It follows immediately from Koszul formula that

$$\begin{aligned}
\overline{\nabla}^\varepsilon_{X_0} X_0 &= \frac{c^0_{01}}{\varepsilon^2} X_1 + \frac{c^0_{02}}{\varepsilon^2} X_2 && \forall \varepsilon > 0 \qquad (3.2)\\
\overline{\nabla}^\varepsilon_{X_0} X_1 &= -c^0_{01} X_0 + \frac{1}{2}\left(c^1_{02} - c^2_{01} + \frac{c^0_{12}}{\varepsilon^2}\right) X_2\\
\overline{\nabla}^\varepsilon_{X_0} X_2 &= -c^0_{02} X_0 + \frac{1}{2}\left(c^2_{01} - c^1_{02} - \frac{c^0_{12}}{\varepsilon^2}\right) X_1\\
\overline{\nabla}^\varepsilon_{X_1} X_0 &= c^1_{01} X_1 + \frac{1}{2}\left(c^1_{02} + c^2_{01} + \frac{c^0_{12}}{\varepsilon^2}\right) X_2\\
\overline{\nabla}^\varepsilon_{X_1} X_1 &= -c^1_{01}\varepsilon^2 X_0 + c^1_{12} X_2\\
\overline{\nabla}^\varepsilon_{X_1} X_2 &= \frac{1}{2}\left(-c^1_{02}\varepsilon^2 - c^2_{01}\varepsilon^2 - c^0_{12}\right) X_0 - c^1_{12} X_1\\
\overline{\nabla}^\varepsilon_{X_2} X_0 &= \frac{1}{2}\left(c^2_{01} + c^1_{02} - \frac{c^0_{12}}{\varepsilon^2}\right) X_1 + c^2_{02} X_2\\
\overline{\nabla}^\varepsilon_{X_2} X_1 &= \frac{1}{2}\left(-c^2_{01}\varepsilon^2 - c^1_{02}\varepsilon^2 + c^0_{12}\right) X_0 + c^2_{12} X_2\\
\overline{\nabla}^\varepsilon_{X_2} X_2 &= -c^2_{02}\varepsilon^2 X_0 - c^2_{12} X_1.
\end{aligned}$$

Now, let S be an embedded surface in M. The *Gaussian curvature* $K^\varepsilon = K^{\varepsilon X_0}$ of S in (M, g^ε) is defined by the Gauss formula

$$K^\varepsilon = K^\varepsilon_{\mathrm{ext}} + \det(\mathrm{II}^\varepsilon), \tag{3.3}$$

where, given a frame (X, Y) of TS, the *extrinsic curvature* $K^\varepsilon_{\mathrm{ext}}$ is

$$K^\varepsilon_{\mathrm{ext}} = \frac{\big\langle \overline{\nabla}^\varepsilon_X \overline{\nabla}^\varepsilon_Y Y - \overline{\nabla}^\varepsilon_Y \overline{\nabla}^\varepsilon_X Y - \overline{\nabla}^\varepsilon_{[X,Y]} Y, X\big\rangle_{g^\varepsilon}}{|X|^2_{g^\varepsilon}|Y|^2_{g^\varepsilon} - \langle X, Y\rangle^2_{g^\varepsilon}}, \tag{3.4}$$

and the determinant $\det \mathrm{II}^\varepsilon$ of the second fundamental form is

$$\det \mathrm{II}^\varepsilon = \frac{\big\langle \mathrm{II}^\varepsilon(X, X), \mathrm{II}^\varepsilon(Y, Y)\big\rangle_{g^\varepsilon} - \big\langle \mathrm{II}^\varepsilon(X, Y), \mathrm{II}^\varepsilon(X, Y)\big\rangle_{g^\varepsilon}}{|X|^2_{g^\varepsilon}|Y|^2_{g^\varepsilon} - \langle X, Y\rangle^2_{g^\varepsilon}}. \tag{3.5}$$

In this last formula, the *second fundamental form* II^ε of S is defined as the projection of the Levi-Civita connection $\overline{\nabla}^\varepsilon$ on the orthogonal to the tangent space of the surface. Both quantities (3.4) and (3.5) are independent on the frame (X, Y) of TS chosen to compute them.

3.1.1 Proof of Theorem 1.13

To prove the theorem, we explicitly compute the asymptotic of the quantities in limit (1.21). Let us fix a characteristic point p, and, in a neighbourhood of p, let us fix an oriented orthonormal frame (X_1, X_2) of D and a submersion u defining S in the sense of (1.16).

The determinant of the bilinear form $B_p^{\varepsilon X_0}$ is homogeneous in ε, and satisfies

$$\det B_p^{\varepsilon X_0} = \frac{\det B_p^{X_0}}{\varepsilon^2} = \frac{B_p^{X_0}(X_1, X_2)^2}{\varepsilon^2} = \frac{(c_{12}^0(p))^2}{\varepsilon^2}, \tag{3.6}$$

where c_{12}^0 is defined in (3.1). Therefore, in order to prove the convergence of the limit in (1.21), it suffices to show that the Gaussian curvature $K_p^{\varepsilon X_0}$ at p diverges at most as $1/\varepsilon^2$.

Let us start with the computation of the determinant (3.5) of the second fundamental form at a characteristic point. It is convenient to write the second fundamental form as

$$\mathrm{II}^\varepsilon(X, Y) = \langle \overline{\nabla}_X^\varepsilon Y, N^\varepsilon \rangle N^\varepsilon.$$

where N^ε is the Riemannian unitary gradient of u, i.e.,

$$N^\varepsilon = \frac{(X_1 u)X_1 + (X_2 u)X_2 + \varepsilon(X_0 u)X_0}{\sqrt{(X_1 u)^2 + (X_2 u)^2 + \varepsilon(X_0 u)^2}}.$$

At the characteristic point p, the gradient $N^\varepsilon(p)$ simplifies to

$$N^\varepsilon(p) = \varepsilon \ \mathrm{sign}(X_0 u) \ X_0(p). \tag{3.7}$$

To compute (3.5) one needs to choose a frame of TS; we will use the frame (F_1, F_2) with

$$F_i = (X_0 u)X_i - (X_i u)X_0 \qquad \text{for } i = 1, 2. \tag{3.8}$$

This frame is well-defined for $X_0 u \neq 0$; in particular, it is suited to calculate the Gaussian curvature at the characteristic points. Recall that the horizontal Hessian of u is

$$\mathrm{Hess}_H(u) = \begin{pmatrix} X_1X_1u & X_1X_2u \\ X_2X_1u & X_2X_2u \end{pmatrix}. \tag{3.9}$$

Lemma 3.1. *Let $p \in S$ be a characteristic point. Then, in the previous notations, for every $\varepsilon > 0$, the determinant (3.5) of the second fundamental form in p is*

$$\det \mathrm{II}^\varepsilon(p) = \frac{1}{\varepsilon^2}\left(\frac{\det \mathrm{Hess}_H\, u(p)}{(X_0 u(p))^2} - \frac{(c_{12}^0(p))^2}{4}\right) + O(1).$$

Proof. Let p be a characteristic point. Because $X_1 u(p) = X_2 u(p) = 0$, one can show that,

$$\overline{\nabla}_{F_i}^\varepsilon F_j(p) = \Big((X_0 u)^2 \overline{\nabla}_{X_i}^\varepsilon X_j + (X_0 u)(X_i X_0 u)X_j - (X_0 u)(X_i X_j u)X_0\Big)\Big|_p,$$

for $i, j = 1, 2$. Using formula (3.7) for N^ε, one finds that only the component along X_0 plays a role in the second fundamental form in p. Thus, using the covariant derivatives in (3.2),

$$\langle \overline{\nabla}_{F_i}^\varepsilon F_j, N^\varepsilon \rangle\big|_p = -\frac{|X_0 u(p)|}{\varepsilon}\left(X_i X_j u + (X_0 u)\frac{c_{ij}^0}{2} + (X_0 u)\varepsilon^2 \frac{c_{0i}^j + c_{0j}^i}{2}\right)\Bigg|_p,$$

for $i, j = 1, 2$. This, together with $\big(|F_1|^2|F_2|^2 - \langle F_1, F_2\rangle^2\big)\big|_p = (X_0 u(p))^4$, gives the result. □

Next, the extrinsic curvature (3.4) is the sectional curvature of the plane T_pS in M, which is known when X_0 is the Reeb vector field and $\varepsilon = 1$; this can be found for instance in [BBL20, Prop. 14]. In our setting, the resulting expression for $\varepsilon \to 0$ is the following.

Lemma 3.2. *Let $p \in S$ be a characteristic point. Then, for every $\varepsilon > 0$,*

$$K^{\varepsilon}_{\text{ext}}(p) = -\frac{3}{4\varepsilon^2}(c^0_{12}(p))^2 + O(1).$$

Proof. To compute the extrinsic curvature we use the frame (X_1, X_2) of TM, which coincides with $T_pS = D_p$ at the characteristic point p. Then, to compute

$$K^{\varepsilon}_{\text{ext}}(p) = \langle \overline{\nabla}^{\varepsilon}_{X_1}\overline{\nabla}^{\varepsilon}_{X_2}X_2 - \overline{\nabla}^{\varepsilon}_{X_2}\overline{\nabla}^{\varepsilon}_{X_1}X_2 - \overline{\nabla}^{\varepsilon}_{[X_1,X_2]}X_2, X_1\rangle\Big|_p$$

it suffices to use the expressions (3.2). □

Remark 3.3. Following the proof of Lemma 3.1 and Lemma 3.2, the exact expressions for $\det \mathrm{II}^{\varepsilon}(p)$ and $K^{\varepsilon}_{\text{ext}}(p)$ at a characteristic point p are, for all $\varepsilon > 0$,

$$\begin{aligned}
\det \mathrm{II}^{\varepsilon}(p) = &+ \frac{1}{\varepsilon^2}\left(\frac{\det \mathrm{Hess}_H f}{(X_0u)^2} - \frac{(c^0_{12})^2}{4}\right)\Big|_p + \varepsilon^2\left(c^1_{01}c^2_{02} - \frac{\left(c^1_{02}+c^2_{01}\right)^2}{4}\right)\Big|_p \\
&+ \frac{1}{X_0u(p)}\left(c^2_{02}X_1X_1u + c^1_{01}X_2X_2u - \frac{c^2_{01}+c^1_{02}}{2}(X_2X_1u + X_1X_2u)\right)\Big|_p, \\
K^{\varepsilon}_{\text{ext}}(p) = &- \frac{3}{4}\frac{(c^0_{12}(p))^2}{\varepsilon^2} - \varepsilon^2\left(c^1_{01}c^2_{02} - \frac{(c^2_{01}+c^1_{02})^2}{4}\right)\Big|_p \\
&+ \left(X_2(c^1_{12}) - X_1(c^2_{12}) - (c^1_{12})^2 - (c^2_{12})^2 + c^0_{12}\frac{c^2_{01}-c^1_{02}}{2}\right)\Big|_p.
\end{aligned}$$

If one chooses as transversal vector field the Reeb vector field of the contact sub-Riemannian manifold, then one recognises the first and the second functional invariants of the sub-Riemannian structure, defined in [ABB20, Ch. 17]. Finally, notice that these expressions are still valid for non-contact distributions.

Proof of Theorem 1.13. In the previous notations, due to the Gauss formula (3.3), Lemma 3.1 and Lemma 3.2, the Gaussian curvature at a characteristic point p satisfies

$$K^{\varepsilon}_p = K^{\varepsilon X_0}_p = \frac{(c^0_{12}(p))^2}{\varepsilon^2}\left(-1 + \frac{\det \mathrm{Hess}_H\, u(p)}{[X_2, X_1]u(p)^2}\right) + O(1).$$

Here we have used that $c^0_{12}(p)X_0u(p) = [X_2, X_1]u(p)$ at p, which holds due to definition (3.1) and $X_1u(p) = X_2u(p) = 0$. Using formula (3.6) for the determinant of $B^{\varepsilon X_0}_p$, one finds that

$$\frac{K^{\varepsilon X_0}_p}{\det B^{\varepsilon X_0}_p} = -1 + \frac{\det \mathrm{Hess}_H\, u(p)}{[X_2, X_1]u(p)^2} + O(\varepsilon^2), \tag{3.10}$$

which shows that the limit (1.21) is finite. Moreover, $\widehat{K}_p$ is independent of X_0 because the transversal vector field X_0 is absent in the constant term of equation (3.10). □

Formula (3.10) is useful to compute $\widehat{K}_p$ explicitly, as it contains only derivatives of the submersion u; thus, let us enclose it with the following corollary.

Corollary 3.4. *Let p be a characteristic point of S. Let u be a local submersion of class C^2 describing S, and let (X_1, X_2) be a local oriented orthonormal frame of D. Then,*

$$\widehat{K}_p = -1 + \frac{\det \mathrm{Hess}_H\, u(p)}{[X_2, X_1]u(p)^2}. \tag{3.11}$$

Note that both $\det \mathrm{Hess}_H\, u(p)$ and $[X_2, X_1]u(p)$ calculated at the characteristic point p are invariant with respect to the frame (X_1, X_2). Moreover, we emphasise that their ratio, which appears in (3.11), is independent on the choice of u.

3.2 Local study near a characteristic point

In this section, we prove Proposition 1.14, and we discuss the local qualitative behaviour of the characteristic foliation near $\Sigma(S)$ in relation to the metric coefficient $\widehat{K}$; next, we estimate the length of a semi-leaf converging to a point, proving Proposition 1.15.

However, let us begin by mentioning a fact which was already known in the literature, but whose proof is straightforward with setting explained in Subsection 1.2.1. For a more general discussion on the size of the characteristic set, we refer to [Bal03] and references therein.

Lemma 3.5 ([BBC21, Lem. 2.1])**.** *The characteristic set $\Sigma(S)$ of a surface S of class C^2 is contained in a 1-dimensional submanifold of S of class C^1.*

Proof. It suffices to show that for every point p in $\Sigma(S)$ there exists a neighbourhood V of p such that $V \cap \Sigma(S)$ is contained in an embedded C^1 curve. Let us fix a point p in $\Sigma(S)$, and a neighbourhood U of p in M equipped with a frame (X_1, X_2) and a function u with the properties described above. Because of (1.17), the characteristic points in $V = U \cap S$ are the solutions of the system $X_1 u = X_2 u = 0$.

Due to the implicit function theorem, it suffices to show that $d_p(X_1 u) \neq 0$ or $d_p(X_2 u) \neq 0$. Thanks to the contact condition, we have that $[X_2, X_1]u(p) \neq 0$. As a consequence, since $X_2 X_1 u(p) = [X_2, X_1]u(p) + X_1 X_2 u(p)$, at least one of the following is true: $X_2 X_1 u(p) \neq 0$, or $X_1 X_2 u(p) \neq 0$. Assume that the first is true; then $d_p(X_1 u)(X_2) = X_2 X_1 u(p) \neq 0$. The other case being similar, the lemma is proved. □

Let us fix a characteristic point p in $\Sigma(S)$, and a characteristic vector field X. Since $X(p) = 0$, there exists a well-defined linear map $DX(p) : T_pS \to T_pS$. Indeed, let e^{tX} be the flow of X. The pushforward of the flow gives, for every x in S, a family of linear maps $e^{tX}_* : T_xS \to T_{e^{tX}(x)}S$. Since $e^{tX}(p) = p$ for all t, then the preceding gives the linear flow $e^{tX}_* : T_pS \to T_pS$, whose infinitesimal generator is the differential $DX(p)$.

Definition 3.6. A characteristic point $p \in \Sigma(S)$ is *non-degenerate* if, given a characteristic vector field X of S, the differential $DX(p)$ is invertible. Otherwise, p is called *degenerate*.

Remark 3.7. Condition (1.14) in the definition of characteristic vector field ensures that the degeneracy of a characteristic point is independent on the choice of characteristic vector field.

Since T_pS coincides with D_p at the characteristic point p, we can endow T_pS with a metric; thus, $DX(p)$ admits a well-defined determinant and trace. Now, let X be the vector field $X = a_1 X_1 + a_2 X_2$, where (X_1, X_2) is an orthonormal oriented frame of D and $a_i \in C^1(S)$, for $i = 1, 2$. Then, in the frame defined by (X_1, X_2) one has

$$DX = \begin{pmatrix} X_1 a_1 & X_2 a_1 \\ X_1 a_2 & X_2 a_2 \end{pmatrix}, \tag{3.12}$$

and the formulas for the determinant and the trace are

$$\det DX = (X_1 a_1)(X_2 a_2) - (X_1 a_2)(X_2 a_1), \tag{3.13}$$

$$\operatorname{tr} DX = \operatorname{div} X = (X_1 a_1) + (X_2 a_2). \tag{3.14}$$

3.2.1 Proof of Proposition 1.14

Let us fix a characteristic point p in $\Sigma(S)$. We claim that the right-hand side of (1.23) is independent on the choice of the characteristic vector field X. Indeed, due to Remark 1.11 any two characteristic vector fields are multiples by nonzero functions, thus, at characteristic point p, their differentials are multiples by nonzero scalars; precisely, if $Y = \phi X$, for ϕ in $C^1(S)$, then one has $DY(p) = \phi(p) DX(p)$. Thus, the claim follows because both determinant and trace-squared are homogenous of the degree two.

Thus, we fix a local submersion u defining S near p, and the characteristic vector field $X_f = (X_1u)X_2 - (X_2u)X_1$ defined in (1.18). Using expression (3.12) for the differential of a vector field, we get

$$DX_f(p) = \begin{pmatrix} -X_1X_2u(p) & -X_2X_2u(p) \\ X_1X_1u(p) & X_2X_1u(p) \end{pmatrix}.$$

Thus, using expressions (3.13) and (3.14) for the determinant and the trace, we find that $\det DX_f(p) = \det \mathrm{Hess}_H\, u(p)$, and $\operatorname{tr} DX_f(p) = [X_2, X_1]u(p)$. In conclusion,

$$\frac{\det DX_f(p)}{\operatorname{tr} DX_f(p)^2} = \frac{\det \mathrm{Hess}_H\, u(p)}{[X_2, X_1]u(p)^2},$$

which, together with Corollary 3.4, gives the desired result.

□

The eigenvalues of the linearisation $DX(p)$ of a characteristic vector field X can be written as a function of $\widehat{K}_p$ by rearranging equation (1.23), as in the following corollary.

Corollary 3.8. *In the hypothesis of Proposition 1.14, let $\lambda_+(X,p)$ and $\lambda_-(X,p)$ be the two eigenvalues of $DX(p)$. Then*

$$\lambda_\pm(X,p) = \operatorname{tr} DX(p)\left(\frac{1}{2} \pm \sqrt{-\frac{3}{4} - \widehat{K}_p}\right). \tag{3.15}$$

Proof. Let us note $\lambda_\pm = \lambda_\pm(X,p)$, and $\alpha = \operatorname{tr} DX(p)$. Equation (1.23) reads

$$\widehat{K}_p = -1 + \frac{\lambda_+\lambda_-}{\alpha^2}.$$

Using that $\lambda_+ + \lambda_- = \alpha$, equation (1.23) implies that the eigenvalues satisfy the equation $z^2 - \alpha z + \alpha^2(\widehat{K}_p + 1) = 0$, which implies (3.15). □

Remark 3.9. It is possible to choose canonically a characteristic vector field with trace 1. Indeed, in the notations used to define X_u in (1.18), let us define the characteristic vector field

$$X_S = \frac{(X_1u)X_2 - (X_2u)X_1}{Zf}, \tag{3.16}$$

where Z is the Reeb vector field of the contact form ω of D normalised as in (1.25). Recall that the Reeb vector field was defined in (1.26) as the unique vector field satisfying $\omega(Z) = 1$ and $d\omega(Z,\cdot) = 0$. The vector field X_S is a characteristic vector field in a neighbourhood of p because it is a nonzero multiple of X_f near $\Sigma(S)$, since $Zu(p) = [X_2, X_1]u(p) \neq 0$. Using the latter, one can verify that $\operatorname{div} X_S(p) = \operatorname{tr} DX_S(p) = 1$.

It is worth mentioning that the vector field X_S is independent on u and on the frame (X_1, X_2), i.e., it depends uniquely on S and (M, D, g). Moreover, the norm of X_S satisfies $|X_S|_g^{-1} = |p_S|$, where p_S is the *degree of transversality* defined in [Lee13]; in the case of the Heisenberg group, p_S coincides with the *imaginary curvature* introduced in [AF07, AF08].

Expression (3.15) for the eigenvalues of the linearisation $DX(p)$ implies the following relations between the eigenvalues and the metric coefficient $\widehat{K}_p$:

(i) $\widehat{K}_p < -1$ if and only if $\lambda_\pm \in \mathbb{R}^*$ with different signs;

(ii) $\widehat{K}_p = -1$ if and only if $\lambda_- = 0$ and $\lambda_+ \in \mathbb{R}^*$;

(iii) $-1 < \widehat{K}_p \leq -3/4$ if and only if $\lambda_\pm \in \mathbb{R}^*$ with same sign;

(iv) $-3/4 < \widehat{K}_p$ if and only if $\Re(\lambda_\pm) \neq 0 \neq \Im(\lambda_\pm)$ and $\lambda_- = \overline{\lambda}_+$.

Notice that the characteristic point p is degenerate if and only if $\widehat{K}_p = -1$, which is case *(ii)*.

Assume that p is a non-degenerate characteristic point. Then, the linear dynamical system defined by $DX(p)$ is a saddle, a node, and a focus respectively in case *(i)*, *(iii)* and *(iv)*. In these cases there exists a local C^1-diffeomorphism near p which sends the flow of X to the flow of $DX(p)$ in $\mathbb{R}^2$, i.e., the flows are C^1-*conjugate*, as proven by Hartman in [Har60]. For this theorem to hold, one needs the characteristic vector field X to be of class C^2. For this reason, in the following corollary we assume the surface S to be of class C^3.

Corollary 3.10. *Assume that the surface S is of class C^3, and let p be a non-degenerate characteristic point in $\Sigma(S)$. Then, $\widehat{K}_p \neq -1$, and the characteristic foliation of S in a neighbourhood of p is C^1-conjugate to*

- *a saddle if and only if $\widehat{K}_p < -1$;*
- *a node if and only if $-1 < \widehat{K}_p \leq -3/4$;*
- *a focus if and only if $-3/4 < \widehat{K}_p$.*

The cases are depicted, respectively, in the first, third and fourth image in Figure 1.2 in Subsection 1.2.2.

Remark 3.11. For surfaces of class C^2, i.e., with characteristic vector fields of class C^1, one can use the Hartman-Grobman theorem, by which one recovers a C^0-conjugation to the corresponding linearisation. However, under this hypothesis, a node and a focus become indistinguishable. For the Hartman-Grobman theorem we refer to [Per12, Par. 2.8]. Finally, for a C^∞ surface some informations can be found in [GHR03].

Next, if p is a degenerate characteristic point, then we are in case *(ii)*. Thus, $\widehat{K}_p = -1$, and the differential $DX(p)$ has a zero eigenvalue with multiplicity one. In this situation, the qualitative behaviour of the characteristic foliation does not depend uniquely on the linearisation, but also on the nonlinear dynamic along a *center manifold*, i.e., an embedded curve $\mathcal{C} \subset S$ with the same regularity as X, invariant with respect to the flow of X, and tangent to the zero eigenvector of $DX(p)$. The analogue of Corollary 3.10 is the following.

Corollary 3.12. *Assume that the surface S is of class C^2, and let p be a degenerate characteristic point in $\Sigma(S)$. Then, $\widehat{K}_p = -1$, and the characteristic foliation in a neighbourhood centred at p is C^0-conjugate at the origin to the orbits of a system of the form*

$$\begin{cases} \dot{u} = \phi(u) \\ \dot{v} = v \end{cases}, \tag{3.17}$$

for a function ϕ with $\phi(0) = \phi'(0) = 0$. If p is isolated, then the characteristic foliation described in (3.17) *at the origin is either a saddle, a saddle-node, or a node; those cases are depicted, respectively, in the first, second, and third image in Figure 1.2.*

The proof of Corollary 3.12 follows from considerations on the center manifold of the dynamical system defined by X, which we recall in Section 3.6.

Remark 3.13. A node and a focus are not distinguishable by a conjugation C^0. However, the center manifold of the characteristic point p is an embedded curve of class C^1, thus it does not spiral around p. Therefore, the existence of a center manifold gives further properties then what is expressed in Corollary 3.12.

To justify the last sentence of Corollary 3.12 let us get a sense of the qualitative properties of a system as (3.17). The line $\{v = 0\}$, parametrised by u, is a center manifold of (3.17), and the function ϕ determines the dynamic of (3.17); this illustrates the fact that the nonlinear terms on a center manifold

determine the dynamic near a degenerate characteristic point.

The equilibria of (3.17) occur only in $\{v = 0\}$, i.e., on a center manifold, and a point $(u, 0)$ is an equilibrium if and only if $\phi(u) = 0$. Thus, if the characteristic point p is isolated, then $u_0 = 0$ is an isolated zero of ϕ. In such case, let us note $\phi^+ = \phi|_{u>0}$ and $\phi^- = \phi|_{u<0}$, and without loss of generality let us suppose that the signs of ϕ^+ and ϕ^- are constant.

- If $\phi^+ > 0$ and $\phi^- < 0$, then the origin is a topological node.
- If $\phi^+ < 0$ and $\phi^- > 0$, then the origin is a a topological saddle.
- If ϕ^+ and ϕ^- have the same sign, then the two half spaces $\{u > 0\}$ and $\{u < 0\}$ have two different behaviours: one is a node, and the other one is a saddle. This gives the characteristic foliation called saddle-node.

Remark 3.14. For an isolated characteristic point, combining Corollary 3.10 and Corollary 3.12, we obtain the four characteristic foliations depicted in Figure 1.2.

3.2.2 Proof of Proposition 1.15

In this section we prove the finiteness of the sub-Riemannian length of a semi-leaf converging to a point. Since we are interested in a local property, it is not restrictive to assume the existence of a global characteristic vector field X of S.

Let ℓ be a one-dimensional leaf of the characteristic foliation of S, and $x \in \ell$ such that $e^{tX}(x) \to p$ as $t \to +\infty$. The limit point p has to be an equilibrium of X, i.e., $X(p) = 0$, hence p is a characteristic point of S. Let U be a small open neighbourhood of p in S for which we have a coordinate chart $\Phi : U \to B \subset \mathbb{R}^2$ with $\Phi(p) = 0$, where B is the open unit ball. Let y be the point of last intersection between $\ell_X^+(x)$ and the boundary ∂U. Since $L_{sR}(\ell_X^+(x)) = L_{sR}(\ell|_{[x,y]}) + L_{sR}(\ell_X^+(y))$ and $L_{sR}(\ell|_{[x,y]})$ is finite, it suffices to show that $L_{sR}(\ell_X^+(y))$ is finite. We claim that there exists a constant $C > 0$ such that

$$\frac{1}{C}|V|_{\mathbb{R}^2} \leq |V|_g \leq C|V|_{\mathbb{R}^2} \qquad \forall\, V \in D \cap TS|_U, \tag{3.18}$$

where we have dropped Φ_* in the notation. Indeed, let $\tilde{g}$ be any Riemannian extension of g on the surface S (for example $\tilde{g} = g^{X_0}|_S$). Since $\tilde{g}$ is an extension, one has $|v|_g = |v|_{\tilde{g}}$ for all v in $D \cap TS$. Equivalence (3.18) follows from the local equivalence of $\tilde{g}$ with the pullback by Φ of the Euclidean metric of $\mathbb{R}^2$. Now, inequality (3.18) implies that

$$L_{sR}(\ell_X^+(y)) = \int_0^{+\infty} |X(e^{tX}(y))|_g \, dt \leq C \int_0^{+\infty} |X(e^{tX}(y))|_{\mathbb{R}^2} \, dt. \tag{3.19}$$

At this point the proof of the finiteness of the sub-Riemannian length of $\ell_X^+(y)$ differs depending on whether p is a non-degenerate or a degenerate characteristic point.

First, assume that p is a non-degenerate characteristic point. Since p is non-degenerate, then the set of point w with $e^{tX}(w) \to p$ for $t \to +\infty$ form a manifold, called the *stable manifold* at p for the dynamical system defined by X. In our case, since $e^{tX}(y) \to p$ for $t \to +\infty$, the semi-leaf $\ell_X^+(y)$ is contained in the stable manifold at p. Moreover, the stable manifold convergence property, precisely stated in [Per12, Par. 2.8], shows that each trajectory inside the stable manifold converges to p sub-exponentially in t. Precisely, if α satisfies $|\Re(\lambda_\pm(p, X))| > \alpha$, then there exists constants $C, t_0 > 0$ such that

$$|e^{tX}(y) - p|_{\mathbb{R}^2} \leq Ce^{-\alpha t} \qquad \forall\, t > t_0. \tag{3.20}$$

Since $X(p) = 0$, for all $t > 0$ one has

$$\left|X(e^{tX}(y))\right|_{\mathbb{R}^2} = \left|X(e^{tX}(y)) - X(p)\right|_{\mathbb{R}^2} \leq \sup_B ||DX(x)|| \; |e^{tX}(y) - p|_{\mathbb{R}^2}.$$

Due to the inequality (3.19) and (3.20), this shows that $L_{sR}(\ell_X^+(y))$ is finite.

Next, assume that p is a degenerate characteristic point. As we said in the introduction of Corollary 3.12, there exists a center manifold $\mathcal{C}$ at p for the dynamical system defined by X. The asymptotic approximation property of the center manifold, recalled in Proposition 3.26, shows that if a trajectory converges to p, then it approximates any center manifold exponentially fast. Precisely, since $e^{tX}(y) \to p$, then there exist constants $C, \alpha, t_0 > 0$ and a trajectory $e^{tX}(z)$ contained in $\mathcal{C}$, such that

$$|e^{tX}(y) - e^{tX}(z)|_{\mathbb{R}^2} \leq Ce^{-\alpha t} \qquad \forall\, t \geq t_0. \tag{3.21}$$

The triangle inequality implies that

$$|X(e^{tX}(y))|_{\mathbb{R}^2} \leq |X(e^{tX}(y)) - X(e^{tX}(z))|_{\mathbb{R}^2} + |X(e^{tX}(z))|_{\mathbb{R}^2}. \tag{3.22}$$

Due to inequality (3.19), to prove that $L_{sR}(\ell_X^+(y))$ is finite, it suffices to show that the two terms on the right-hand side of (3.22) are integrable for $t \geq 0$. Thanks to (3.21) and

$$|X(e^{tX}(y)) - X(e^{tX}(z))|_{\mathbb{R}^2} \leq \sup_B ||DX||\ |e^{tX}(y) - e^{tX}(z)|_{\mathbb{R}^2},$$

the first term in (3.22) is integrable. Next, because $e^{tX}(z)$ is a regular parametrisation of a bounded interval inside a C^1 embedded curve (the center manifold $\mathcal{C}$), then its derivative $|X(e^{tX}(z))|_{\mathbb{R}^2}$ is integrable. □

Remark 3.15. Let X be a characteristic vector field of a compact surface S. If the ω-limit set with respect to X of a non-periodic leaf ℓ contains more then one point, then $L_{sR}(\ell_X^+) = +\infty$. Therefore, if a leaf ℓ does not converge to a point in any of its extremities, then the points in ℓ have infinite distance from the points in $S \setminus \ell$.

In particular, if the characteristic set of a surface S is empty, then the induced distance d_S is not finite. For a discussion on non-characteristic domains we refer to [DGN06, Ch. 3].

3.3 Global study of the characteristic foliation

The main goal of this section is to identify a sufficient condition for the induced distance d_S to be finite. As explained in the introduction, this is done by excluding the existence of certain leaves in the characteristic foliation of S, as in Proposition 3.16. In this section we assume the existence of a global characteristic vector field X of S.

The leaves the characteristic foliation of S are precisely the orbits of the dynamical system defined by X, therefore we are going to call them *trajectories*, stressing that they are parametrised by the flow of X. Moreover, the vector field X enables us to use the notions of ω-limit set and α-limit set of a point y in S, which are, respectively,

$$\omega(y, X) = \Big\{ q \in S \ \Big|\ \exists\, t_n \to +\infty \text{ such that } \ e^{t_n X}(y) \to q \Big\}, \qquad \alpha(y, X) = \omega(y, -X).$$

The points y in a leaf ℓ have the same limit sets, thus one can define $\omega(\ell, X)$ and $\alpha(\ell, X)$.

Proposition 3.16. *Let S be a compact, connected surface C^2 embedded in a contact sub-Riemannian structure. Assume that S has isolated characteristic points, and that the characteristic foliation of S is described by a global characteristic vector field of S which does not contain any of the following trajectories:*

- *nontrivial recurrent trajectories,*

- *periodic trajectories,*

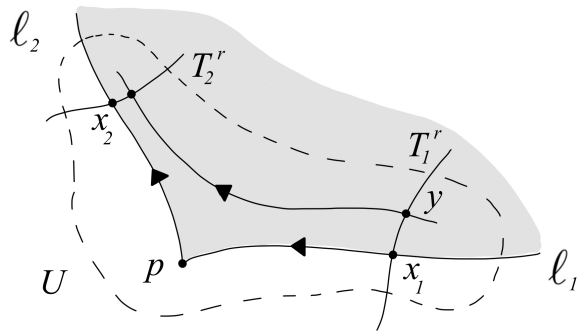

Figure 3.1: The illustration of a right-sided hyperbolic sector

- *sided contours.*

Then, d_S is finite.

Let us give a formal definition of these objects. A *periodic* trajectory is a leaf of the characteristic foliation homeomorphic to a circle. A periodic trajectory has infinite distance from its complementary, hence it is necessary to exclude its presence for d_S to be finite.

Next, a leaf ℓ is *recurrent* if $\ell \subset \omega(\ell, X)$ and $\ell \subset \alpha(\ell, X)$. A *nontrivial* recurrent trajectory is a recurrent trajectory which is not an equilibrium nor a periodic trajectory. Because the ω-limit and the α-limit set of a nontrivial recurrent trajectory contains more then one point, then, due to Remark 3.15, those trajectories have infinite distance from their complementary.

Lastly, a *sided contour* is either a left-sided or right-sided contour. A right-sided contour (resp. left-sided) is a family of points $p_1, \dots, p_s$ in $\Sigma(S)$ and trajectories $\ell_1, \dots, \ell_s$ such that:

- for all $j = 1, \dots, s$, we have $\omega(\ell_j, X) = p_j = \alpha(\ell_{j+1}, X)$ (where $\ell_{s+1} = \ell_1$);
- for every $j = 1, \dots, s$, there exists a neighbourhood U_j of p_j such that U_j is a right-sided hyperbolic sector (resp. left-sided) for p_j with respect to ℓ_j and ℓ_{j+1}.

Let us give a precise definition of a hyperbolic sector. Note that, given a non-characteristic point $x \in S$, and a curve T going through x and transversal to the flow of X, the orientation defined by X defines the right-hand and the left-hand connected component of $T \setminus \{x\}$, denoted T^r and T^l respectively.

Definition 3.17. Let p be a characteristic point, and ℓ_1 and ℓ_2 be two trajectories such that $\omega(\ell_1, X) = p = \alpha(\ell_2, X)$. A neighbourhood U of p homeomorphic to a disk is a *right-sided hyperbolic sector* (resp. left-sided) with respect to ℓ_1 and ℓ_2 iu, for every point $x_i \in \ell_i \cap U$, for $i = 1, 2$, there exists a curve T_i going through x_i and transversal to the flow of X such that:

- for every point $y \in T_1^r$ (resp. T_1^l) the positive semi-trajectory $\ell_X^+(y)$ starting from y intersects T_2^r (resp. T_2^l) before leaving U;
- the point of first intersection of $\ell_X^+(y)$ and T_2^r (resp. T_2^l) converges to x_2, for $y \to x_1$.

Note that a right-sided hyperbolic sector for X is a left-sided hyperbolic sector for $-X$. An illustration of hyperbolic sector can be found in Figure 3.1, an example of sided contours can be found in Figure 3.4, and for the general theory we refer to [ABZ96, Par. 2.3.5].

3.3.1 Topological structure of the characteristic foliation

Now, assume that S does not contain any nontrivial recurrent trajectories. To prove Proposition 3.16 we are going to use the topological structure of a flow. We resume here the relevant theory, following the exposition in [ABZ96, Par. 3.4].

The *singular trajectories* of the characteristic foliation of S are precisely the following:

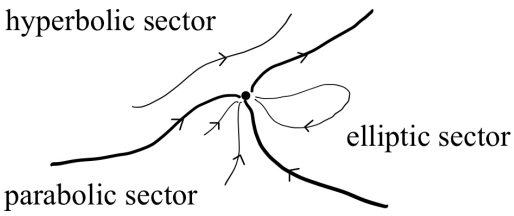

Figure 3.2: The sectors of an isolated equilibrium of a dynamical system.

- characteristic points;
- separatrices of characteristic points (see [ABZ96, Par. 2.3.3]);
- isolated periodic trajectories;
- periodic trajectories which contain in every neighbourhood both periodic and non-periodic trajectories.

The union of the singular trajectories is noted $ST(S)$, and it is closed. The open connected components of $S \setminus ST(S)$ are called *cells*. The leaves of the characteristic foliation of S contained in the same cell have the same behaviour, as shown in the following proposition.

Proposition 3.18 ([ABZ96, Par. 3.4.3])**.** *Assume that the flow of X has a finite number of singular trajectories. Let R be a cell filled by non-periodic trajectories; then:*

(i) R is homeomorphic to a disk, or to an annulus;

(ii) the trajectories contained in R have all the same ω-limit and α-limit sets;

(iii) the limit sets of any trajectory in R belongs to ∂R;

(iv) each connected component of ∂R contains points of the ω-limit or α-limit sets.

Using this proposition, we show the following lemma.

Lemma 3.19. *Let S be surface satisfying the hypothesis of Proposition 3.16 . Then, for every cell R of the characteristic foliation of S, we have that*

$$d_S(x, y) < +\infty \qquad \forall x, y \in R \cup \partial R.$$

Proof. Since the surface S is compact and the characteristic points in $\Sigma(S)$ are isolated, there is a finite number of characteristic points. Moreover, there are no periodic trajectories. This implies that there is a finite number of singular trajectories, hence we can apply Proposition 3.18.

Let R be a cell of the characteristic foliation of S, and let Γ be one of the connected components of the boundary ∂R (of which there are either one or two, due to Proposition 3.18). The curve Γ is the union of characteristic points and separatrices. If all characteristic points have a hyperbolic sector towards R (right-sided or left-sided), then Γ would be a sided contour, which is excluded. Therefore, there exists a characteristic point $p \in \Gamma$ without a hyperbolic sector towards R. As shown in [ALGM73, Par. 8.18], around an isolated equilibrium there are only the three kinds of sectors depicted in Figure 3.2. Since there is no elliptic sector due to Remark 3.14, the point p has a parabolic sector towards R.

Due to Proposition 3.18, the point p is the ω-limit or the α-limit of every trajectory in R. Then, for every point $x \in R$, there exists a semi-leaf $\ell_X^+(x)$ or $\ell_{-X}^+(x)$ starting from x and converging to p. Due to Proposition 1.15, this semi-leaf has finite sub-Riemannian length, hence $d_S(x, p)$ is finite.

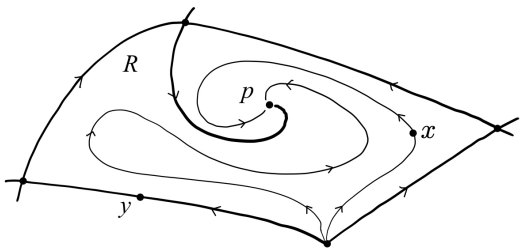

Figure 3.3: How to connect the points of a cell with the points in the boundary.

Next, for every point $y \in \Gamma$, note that

$$d_S(x, y) \leq d_S(x, p) + d_S(p, y).$$

We have already proven that $d_S(x, p)$ is finite, and the same holds for $d_S(p, y)$. Indeed, one can find a horizontal curve of finite length connecting p and y using a concatenation of the separatrices contained in Γ.

If the boundary of R has a second connected component, then the above argument holds also for the other connected component because it suffices to repeat the above argument for it. Thus, we have shown that

$$d_S(x, y) < +\infty \qquad \forall\ x \in R,\ y \in \partial R,$$

which implies the statement of the lemma. □

Lemma 3.20. *Let S be surface satisfying the hypothesis of Proposition 3.16. Then, for every x in S, there exists an open neighbourhood U of x such that, for all y in U,*

$$d_S(x, y) < +\infty.$$

Proof. Let x be a point of S. If x does not belong to the union of the singular trajectories, then it is in the interior of a cell R. Thus, due to Lemma 3.19, one can choose $U = R$. Otherwise, the point x belongs to a separatrix, or it is a characteristic point of S.

Assume that x belongs to a separatrix Γ. Then, there exists a neighbourhood U of x which is divided by Γ in two connected components. Those two connected components are contained in some cell R_1 and R_2, which contain Γ in their boundary. For every $y \in U$, then either $y \in R_i$, for $i = 1, 2$, or $y \in \Gamma$. If $y \in R_i$, then it suffices to apply Lemma 3.19. Otherwise, if $y \in \Gamma$, the separatrix Γ itself connects x and y.

Finally, assume that x is a characteristic point. Due to Corollary 3.10, Remark 3.11, and Corollary 3.12, there exists a neighbourhood U of x in which the characteristic foliation of S is topologically conjugate to a saddle, a node or a saddle-node. Thus, one can repeat the same argument as before: for every $y \in U$, if y belongs to a cell then one applies Lemma 3.19; otherwise, if y belongs to a separatrix one can connect x and y directly. □

The proof of Proposition 3.16 is a corollary of Lemma 3.20.

Proof of Proposition 3.16 . The property of having finite distance is an equivalence relation on the points of S. Because of Lemma 3.20, the equivalence classes are open. Thus, because S is connected, there is only one class. □

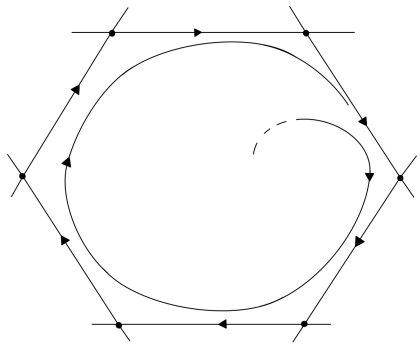

Figure 3.4: An embedded polygon which bounds a right-sided contour

3.4 Spheres in a tight contact distribution

In this section we prove Theorem 1.16, i.e., in a tight coorientable contact distribution the topological spheres have finite induced distance. This is done by showing that the hypothesis of Proposition 3.16 are satisfied in this setting.

An overtwisted disk, precisely defined at the end of Section 1.2.1, is an embedding of a disk with horizontal boundary such that the distribution does not twist along the boundary. A contact distribution is called *overtwisted* if it admits a overtwisted disk, and it is called *tight* if it is non-overtwisted.

Remark 3.21. Note that if the boundary of a disk is a periodic trajectory of its characteristic foliation, then the disk is overtwisted. Indeed, since a periodic trajectory does not contain characteristic points, then the plane distribution never coincides with the tangent space of the disk, thus the distribution can't perform any twists.

Lemma 3.22. *Let (M, D) be a tight contact 3-manifold, and S an embedded surface with the topology of a sphere. Then, the characteristic foliation of S does not contain periodic trajectories.*

Proof. Assume that the characteristic foliation of S has a periodic trajectory ℓ. Then, because ℓ does not have self-intersections, the leaf ℓ divides S in two topological half-spheres Δ_1 and Δ_2. The disks Δ_i, for $i = 1, 2$, are overtwisted, which contradicts the hypothesis that the distribution is tight because Remark 3.21. □

Now, let us discuss the sided contours.

Lemma 3.23. *Let (M, D) be a tight contact 3-manifold, and $S \subset M$ an embedded surface with the topology of a sphere. Then the characteristic foliation of S does not contain sided contours.*

Proof. Assume that the characteristic foliation presents a sided contour Γ. Its complementary $S \setminus \Gamma$ has two connected components, which are topologically half-spheres. Let us call Δ the component on the same side of Γ, i.e., if Γ is right-sided (resp. left-sided) then Δ is on the right (resp. left). For instance, if Γ is right-sided, then the characteristic foliation of Δ looks like that of the polygon in Figure 3.4.

Let p be one of the vertices of Δ, let ℓ_1 and ℓ_2 be the separatrices adjacent to p, and let U be a neighbourhood of p such that we are in the condition of Definition 3.17. Let us fix two points $x_i \in \ell_i \cap U$, for $i = 1, 2$. Due to the definition of hyperbolic sector, in a neighbourhood of x_1 the leaves pass arbitrarily close to x_2.

We are going to give the idea of how to perturb the surface near x_1 and x_2 so that the separatrices ℓ_1 and ℓ_2 are diverted to the same nearby leau, therefore bypassing p. In other words, via a C^∞-small perturbation of S supported in a neighbourhood of x_1 and x_2, we obtain a sphere which contains a sided contour with one less vertex, see Figure 3.5. By repeating such perturbation for every vertex, one

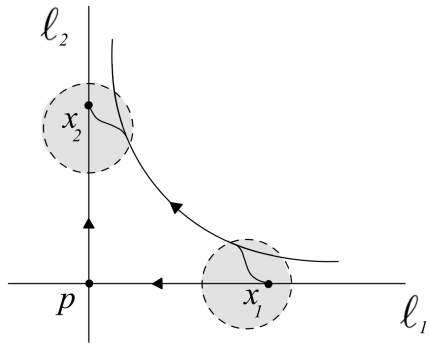

Figure 3.5: The characteristic foliation of the perturbed surface.

obtains a new surface with a periodic trajectory in its characteristic foliation, which is excluded due to Lemma 3.22.

Consider the Heisenberg distribution $(\mathbb{R}^3, \ker(dz + \frac{1}{2}(ydx - xdy))$. Let $\mathcal{P}$ be the vertical plane $\mathcal{P} = \{x = 0\}$, and q a point in $\mathcal{P}$ contained in the y-axis. As one can see in Example 3.5.1, the characteristic foliation of $\mathcal{P}$ is made up of parallel horizontal lines.

Locally, it is possible to rectify the surface S into the plane $\mathcal{P}$ using a contactomorphism of the respective ambient spaces, as explained in the following lines. Due to the rectification theorem of dynamical systems, the characteristic foliation of S in a neighbourhood of x_1 is diffeomorphic to that of a neighbourhood of q in $\mathcal{P}$. A generalisation of a theorem of Giroux [Gei08, Thm. 2.5.23] implies that the C^1-conjugation between the characteristic foliations of the two surfaces can be extended, in a smaller neighbourhood, to a contactomorphism. Precisely, there exists a contactomorphism ψ from a neighbourhood $V \subset M$ of x_1 to a neighbourhood of q in $\mathbb{R}^3$, with $\psi(S) \subset \mathcal{P}$.

For what it has been said above, the image of ℓ_1 by ψ is contained in the y-axis. By creating a small bump in $\mathcal{P}$ after the point q, we will be able to divert the leaf going through q to any other parallel line. Precisely, for any curve $\gamma(t) = (x(t), y(t))$, defining

$$z(t) = \frac{1}{2}\int_{t_1}^{t} x(s)y'(s) - y(s)x'(s)ds \qquad \forall\ t \in [t_1, t_2],$$

we obtain a horizontal curve $(x(t), y(t), z(t))$. Now, let γ be a smooth curve which joins smoothly to the y-axis at its end points $\gamma(t_1) = q$ and $\gamma(t_2)$, and let Ω be the set between γ and the y-axis. One can verify that $z(t_2) = \text{Area}\,(\Omega)$, where the area is a signed area. By choosing an appropriate curve γ, we can connect the y-axis from q to any other parallel line in $\mathcal{P}$ via a horizontal curve (Figure 3.6). Next, by creating a small bump in $\mathcal{P}$ in order to include this horizontal curve one has successfully diverted the leaf. This procedure can be done C^∞-small, provided one wants to connect to parallel lines sufficiently close to the y-axis. Thus, one can make sure that no new characteristic points are created. Finally, this perturbation has to be transposed to a perturbation of S using ψ.

The same argument has to be repeated *mutatis mutandis* in a neighbourhood of x_2, ensuring that one connects x_2 exactly to the leaf coming from x_1. This is possible due to the continuity property of a hyperbolic sector, which ensures that the leaf coming from x_1 intersects the domain of the rectifying contactomorphism of x_2. □

We can finally prove Theorem 1.16.

Proof of Theorem 1.16. The surface S admits a global characteristic vector field, due to Lemma 1.10. Next, a surface with the topology of a sphere doesn't allow flows with nontrivial recurrent trajectories, see [ABZ96, Lem. 2.4]. Indeed, from a nontrivial recurrent trajectory one can construct a closed curve transversal to the flow which does not separate the surface, which contradicts the Jordan curve theorem.

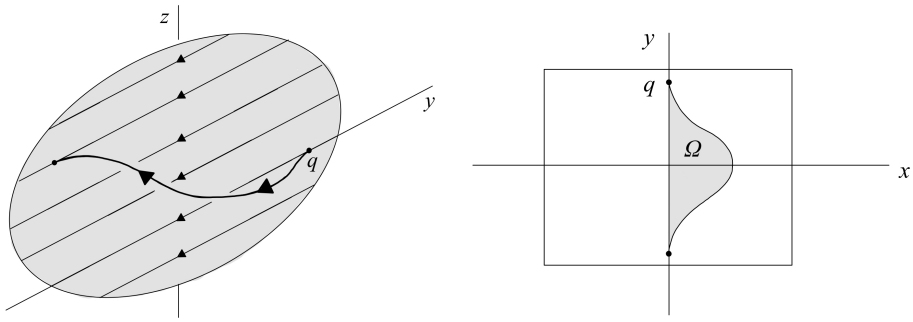

Figure 3.6: The lift to an horizontal curve connecting different leaves.

Then, Lemma 3.22 and Lemma 3.23 imply that the flow of a characteristic vector field of S does not contain periodic trajectories and sided contours, thus the hypothesis of Proposition 3.16 are satisfied. Consequently, d_S is finite. □

3.5 Examples of surfaces in the Heisenberg structure

In this section we present some examples of surfaces in the Heisenberg sub-Riemannian structure, that is the contact, tight, sub-Riemannian structure of $\mathbb{R}^3$ for which (X_1, X_2) is a global orthonormal frame, where

$$X_1 = \partial_x - y/2\ \partial_z, \qquad X_2 = \partial_y + x/2\ \partial_z.$$

If $(u, v) \mapsto (x(u, v), y(u, v), z(u, v))$ is a parametrisation of a surface S, then the characteristic vector field X in coordinates u, v becomes

$$X = -\left(z_v + x_v \frac{y}{2} - y_v \frac{x}{2}\right) \frac{\partial}{\partial u} + \left(z_u + x_u \frac{y}{2} - y_u \frac{x}{2}\right) \frac{\partial}{\partial v}, \tag{3.23}$$

where have used the subscripts to denote a partial derivative. When the surface is the graph of a function $S = \{z = h(x, y)\}$, then in the graph coordinates

$$X = \left(\frac{x}{2} - \partial_y h\right) \frac{\partial}{\partial x} + \left(\partial_x h + \frac{y}{2}\right) \frac{\partial}{\partial y},$$

and, at a characteristic point $p = (x, y, z)$, the metric coefficient $\widehat{K}_p$ is computed by

$$\widehat{K}_p = -3/4 + \partial^2_{xx} h(x, y)\ \partial^2_{yy} h(x, y) - \partial^2_{xy} h(x, y)\ \partial^2_{yx} h(x, y).$$

3.5.1 Planes.

Let us consider affine planes in Heisenberg. Thanks to the left-invariance, it is not restrictive to consider a plane $\mathcal{P}$ going throughout the origin. Thus,

$$\mathcal{P} = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0\} \qquad \text{with } (a, b, c) \neq (0, 0, 0).$$

If $c = 0$, i.e., the plane is vertical, then $\mathcal{P}$ does not contain characteristic points. Every characteristic vector field is parallel to the vector $(b, -a,\ 0)$, therefore the characteristic foliation of $\mathcal{P}$ consists of lines that are parallel to the xy-plane. This implies that points with different z-coordinate are not at finite distance from each other, see Figure 3.7 (left).

Otherwise, if $c \neq 0$, then $\mathcal{P}$ has exactly one characteristic point $p = (-2b/c, 2a/c, 0)$. One has that

$$\widehat{K}_p = -\frac{3}{4}.$$

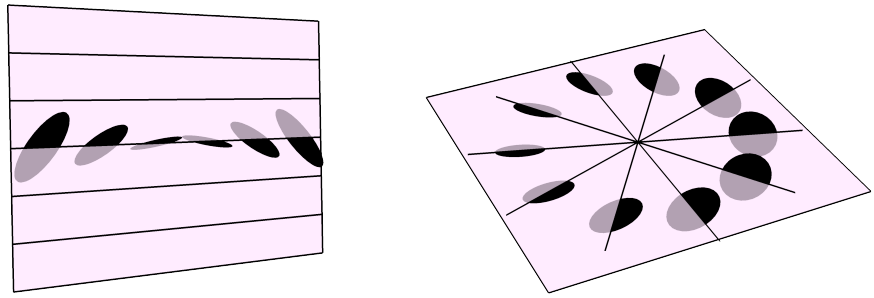

Figure 3.7: The qualitative picture of the characteristic foliation of a vertical plane (left), and of a non-vertical plane (right).

Thus, because of formula (3.15), there is one eigenvalue of multiplicity two. Due to Corollary 3.10, the characteristic foliation of $\mathcal{P}$ has a node at p. An explicit computation of X_S shows that

$$X_S(q) = \frac{q-p}{2} \qquad \forall\, q \in \mathcal{P},$$

which shows that the characteristic foliation of $\mathcal{P}$ is composed of Euclidean half-lines radiating out of p. The metric $d_{\mathcal{P}}$ induced by the Heisenberg group on $\mathcal{P}$ satisfies the following relation: for all $q, q' \in \mathcal{P}$, one has

$$d_{\mathcal{P}}(q,q') = \begin{cases} |(x,y)-(x',y')|_{\mathbb{R}^2}, & \text{if } (q-p) \mathbin{/\!/} (q'-p) \\ d_{\mathcal{P}}(q,p) + d_{\mathcal{P}}(q',p), & \text{otherwise,} \end{cases}$$

where we have written $q = (x,y,z)$ and $q' = (x',y',z')$. This distance is sometimes called British Rail metric. See Figure 3.7 (right).

3.5.2 Ellipsoids

Fix $a, b, c > 0$, and consider the surface $\mathcal{E} = \mathcal{E}_{a,b,c}$ defined by

$$\mathcal{E}_{a,b,c} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \right\}.$$

This surface has exactly two characteristic points $p_1 = (0,0,c)$ and $p_2 = (0,0,-c)$, respectively at the North and the South pole. For both points, one has

$$\widehat{K}_{p_i} = -\frac{3}{4} + \frac{c^2}{a^2 b^2}, \qquad i = 1, 2.$$

Because of Corollary 3.10, the characteristic foliation of $\mathcal{E}$ spirals around the two poles, as in Figure 1.3. Due to Proposition 1.15, the spirals converging to the poles have finite sub-Riemannian length, thus the length distance $d_{\mathcal{S}}$ is finite. Indeed, $d_{\mathcal{S}}$ is realised by the length of the curves joining the points with either the North, or the South pole. Here, the finiteness of d_S is also a particular case of Theorem 1.16.

3.5.3 Symmetric paraboloids

Let $a \in \mathbb{R}$, and consider the paraboloid $\mathcal{P}_a$ with

$$\mathcal{P}_a = \left\{ (x,y,z) \in \mathbb{R}^3 \mid z = a\big(x^2 + y^2\big) \right\}.$$

The origin p is the unique characteristic point of $\mathcal{P}_a$. Note that

$$\widehat{K}_p = -\frac{3}{4} + 4a^4,$$

therefore the characteristic foliation is a focus.

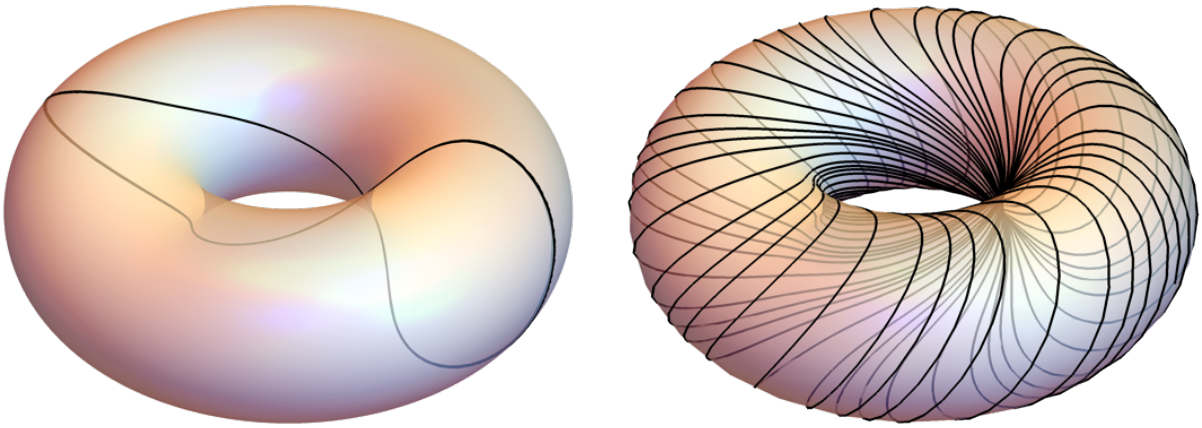

Figure 3.8: A leaf of the characteristic foliation of two Horizontal tori. On the left-hand side the leaf is periodic, and on the right-hand side there is a portion of an everywhere dense leaf .

3.5.4 Horizontal torus

Fix $R > r > 0$, and consider the torus parametrised by

$$\Phi(u, v) = \big((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u\big).$$

This is the torus obtained by revolving a circle of radius $r > 0$ in the xz-plane around a circle of radius $R > r$ surrounding the z-axis. Using formula (3.23), a characteristic vector field X in the coordinates (u, v) is

$$X = \frac{(R + r\cos(u))^2}{2}\frac{\partial}{\partial u} - \frac{r\cos(u)}{2}\frac{\partial}{\partial v}. \tag{3.24}$$

It is immediate to see that the characteristic set is empty. Thus, no point can be a limit point of any leaves of the characteristic foliation; due to Remark 3.15, this implies that the length distance is infinite.

Lemma 3.24. *The characteristic foliation of a horizontal torus is filled either with periodic trajectories, or with everywhere dense trajectories.*

Proof. Using expression (3.24), in the coordinates u, v the trajectories of X satisfy

$$\begin{cases} \dot{u} = (R + r\cos(u))^2/2 \\ \dot{v} = -r\cos(u)/2. \end{cases} \tag{3.25}$$

Because the Heisenberg distribution and the horizontal torus are invariant under rotations around the z-axis, the same applies to the characteristic foliation. Thus, the solutions of (3.25) are v-translations of the solution $\gamma_0(t) = (u(t), v(t))$ with initial condition $\gamma_0(0) = (0, 0)$.

Note that $(r+R)^2/2 \geq \dot{u}(t) \geq (R-r)^2/2$. Thus, there exists a time t_0 in which the trajectory $\gamma_0(t)$, satisfies $u(t_0) = 2\pi$. Define $\alpha_{r,R} = v(t_0)$. If $\alpha_{r,R}/(2\pi) = m/n$ is rational, then $\gamma_0(nt_0) = 0 \pmod{2\pi}$. This shows that $\gamma_0(t)$ is periodic, as every other trajectory. On the other hand, if $\alpha_{r,R}/(2\pi)$ is irrational, then a classical argument shows that $\gamma(t)$ is dense in the torus, see for instance [ABZ96, E.g .2.3.1]. □

See Figure 3.8 for a picture of a leaf in these two cases.

3.5.5 Vertical torus

Fix $R > r > 0$, and consider the torus $\mathcal{T} = \mathcal{T}_{r,R}$ parametrised by

$$\Phi(u, v) = \big(r\sin u, (R + r\cos u)\cos v, (R + r\cos u)\sin v\big).$$

This is the torus obtained by turning a circle of radius r in the xy-plane around a circle of radius R surrounding the x-axis. Due to formula (3.23), a characteristic vector field X in coordinates u, v is

$$\begin{aligned} X =&(R + r\cos u)\Big(\cos v + \frac{r}{2}\sin v \sin u\Big)\frac{\partial}{\partial u} \\ &+ \frac{r}{2}\Big(2\sin u \sin v - R\cos u\cos v - r\cos v\Big)\frac{\partial}{\partial v}. \end{aligned}$$

The characteristic points are critical points of the vector field X. If $\cos v = \sin u = 0$, then (u, v) corresponds to a solution; this gives 4 characteristic points

$$F_\pm = \big(0, 0, \pm(R + r)\big), \qquad V_\pm = \big(0, 0, \pm(R - r)\big).$$

The other critical points of X occur if and only if

$$\tan v = -\frac{2}{r\sin u}, \qquad \cos u = -\frac{4 + r^2}{rR}. \tag{3.26}$$

System (3.26) has solutions if and only if $R > 4$ and $|2r - R| \leq \sqrt{R^2 - 16}$, in which case we have 4 additional characteristic points $S_i(r, R)$, for $i = 1, 2, 3, 4$. Now, the metric coefficient at the characteristic points $F_\pm$ and $V_\pm$ is

$$\begin{aligned} \widehat{K}_{F_\pm} &= -\frac{3}{4} + \frac{1}{r(R + r)}, \\ \widehat{K}_{V_\pm} &= -\frac{3}{4} - \frac{1}{r(R - r)}. \end{aligned}$$

Note that $\widehat{K}_{F_\pm} > -3/4$, thus, due to Corollary 3.10, $F_\pm$ is a focus for all value of r and R. On the other hand, $\widehat{K}_{V_\pm}$ can attain any value between $-\infty$ and $-3/4$; precisely:

- if $R < 4$ or $|2r - R| > \sqrt{R^2 - 16}$, then $\widehat{K}_{V_\pm} < -1$ and $V_\pm$ are saddles.
- if $|2r - R| = \sqrt{R^2 - 16}$, then $\widehat{K}_{V_\pm} = -1$ and $V_\pm$ is a degenerate characteristic point; due to the Poincaré Index theorem, the points $V_\pm$ are saddles.
- if $|2r - R| < \sqrt{R^2 - 16}$, then $-1 < \widehat{K}_{V_\pm} < -3/4$ and $V_\pm$ are nodes.

The values for which $|2r - R| = \sqrt{R^2 - 16}$ are a bifurcation of the dynamical system X, because the number of characteristic point changes from 4 to 8. The characteristic points S_i which appears at this bifurcation are saddles, due to the Poincaré Index theorem. The bifurcation which takes place is the one presented in [Per12, E.g. 4.2.6].

3.6 Appendix on the center manifold theorem

In the language of dynamical systems, a non-degenerate characteristic point p is a *hyperbolic equilibrium* for any characteristic vector field X, i.e., an equilibrium for which the real parts of the eigenvalues of $DX(p)$ are non-zero. For a hyperbolic equilibrium p, the Hartman-Grobman theorem and the Hartman theorem give a conjugation between the flow of X and the flow of $DX(p)$, see [Per12, Par. 2.8] and [Har60].

Let us discuss here the case of a non-hyperbolic equilibrium, i.e., of a degenerate characteristic point. Let E be an open set of $\mathbb{R}^n$ containing the origin, and let X be a vector field in $C^1(E, \mathbb{R}^n)$ with $X(0) = 0$. Due to the Jordan decomposition theorem, we can assume that the linearisation of X at the origin is

$$DX(0) = \begin{pmatrix} C & & \\ & P & \\ & & Q \end{pmatrix},$$

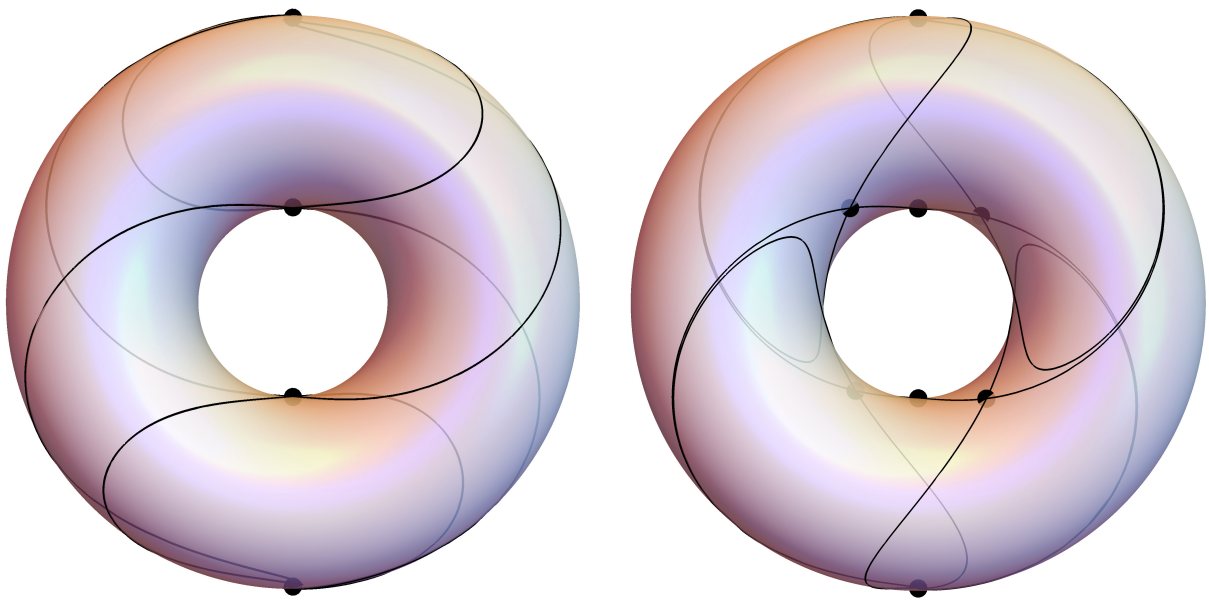

Figure 3.9: The topological skeleton, i.e., the singular trajectories, of the characteristic foliations of two vertical tori: the torus on the left-hand side has four characteristic points, and the torus on the right-hand side has eight characteristic points.

where C is a square $c \times c$ matrix with c complex (generalised) eigenvalues with zero real part, P with p complex (generalised) eigenvalues with positive real part, and Q with q complex (generalised) eigenvalues with negative real part. Thus, the dynamical system $\dot{\gamma} = X(\gamma)$ can be rewritten as

$$\begin{cases} \dot{x} = Cx + F(x, y, z) \\ \dot{y} = Py + G(x, y, z) \\ \dot{z} = Qz + H(x, y, z) \end{cases}$$

for $(x, y, z) \in \mathbb{R}^c \times \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^n$, and for suitable functions F, G and H with $F(0) = G(0) = H(0) = 0$ and $DF(0) = DG(0) = DH(0) = 0$.

The origin is a non-hyperbolic characteristic point if and only if $c \geq 1$. Under these hypotheses, the following theorem shows that there exists an embedded submanifold $\mathcal{C}$ of dimension c, tangent to $\mathbb{R}^c$, and invariant for the flow of X. Such manifold is called a *central manifold* of X at the origin.

Proposition 3.25 ([Per12, Par. 2.12])**.** *Under the previous notations, there exists an open set $U \subset \mathbb{R}^c$ containing the origin, and two functions $h_1 : U \to \mathbb{R}^p$ and $h_2 : U \to \mathbb{R}^q$ of class C^1 with $h_1(0) = h_2(0) = 0$ and $Dh_1(0) = Dh_2(0) = 0$, and such that the map $x \mapsto (x, h_1(x), h_2(x))$ parametrises a submanifold invariant for the flow of X. Moreover, the flow of X is C^0-conjugate to the flow of*

$$\begin{cases} \dot{x} = Cx + F(x, h_1(x), h_2(x)) \\ \dot{y} = Py \\ \dot{z} = Qz. \end{cases} \tag{3.27}$$

In general, the central manifold $\mathcal{C}$ is non-unique. Note that the dynamic of the x-variable in equation (3.27) is simply the restriction of X to the center manifold $\mathcal{C}$. One can show that the trajectory converging to the origin approaches $\mathcal{C}$ exponentially fast: this is the asymptotic approximation property we used in (3.21).

Proposition 3.26 ([Bre07, p. 330])**.** *Under the previous assumptions, let us denote $\mathcal{C}$ a center manifold of the flow of X at the origin. Then, for every trajectory $l(t)$ such that $l(t) \to 0$ as $t \to +\infty$, there exists $\eta > 0$ and a trajectory $\zeta(t)$ in the center manifold $\mathcal{C}$, such that*

$$e^{\eta t}|l(t) - \zeta(t)|_{\mathbb{R}^n} \to 0, \qquad as \quad t \to +\infty.$$

Chapter 4

Stochastic processes on embedded surfaces in sub-Riemannian manifolds

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4.4.3 A unified viewpoint 74

This chapter presents the results in [BBCH21] as a joint work with Davide Barilari, Ugo Boscain and Karen Habermann in the journal Annales de l'Institut Henri Poincaré, Probabilités et Statistiques.

First, in Section 4.1 we prove Theorem 1.17 and Proposition 1.18. Their proofs rely on the expression of $\Delta_\varepsilon|_{S\setminus\Sigma(S)}$ given in Lemma 4.2 in terms of an orthogonal frame for $T(S\setminus\Sigma(S))$. In Section 4.2, we prove Theorem 1.19 and Proposition 1.20 using Lemma 4.3 and Lemma 4.4, which expand the function $b\colon S\setminus\Sigma(S)\to\mathbb{R}$ from (1.29) in terms of the arc length along the integral curves of $\widehat{X}_S$. These results are illustrated in the last two sections with some examples. In Section 4.3, we study quadric surfaces in the Heisenberg group, whereas in Section 4.4, we consider canonical surfaces in $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ equipped with the standard sub-Riemannian contact structures. The examples establishing Theorem 1.21 are discussed in Section 4.3.1, Section 4.4.1 and Section 4.4.2, with a unified viewpoint presented in Section 4.4.3.

Given a coorientable sub-Riemannian contact manifold (M,D,g), in this section we make the choice to normalise the contact form ω using $\mathrm{d}\omega|_D=-\mathrm{vol}_g$; this in order to change the sign of the Reeb vector field. This will ease the references to [BBCH21], where this convention is used.

4.1 Family of Laplace–Beltrami operators on the embedded surface

Let (M, D, g) be a sub-Riemannian contact manifold, and assume that distribution D is free, that is, globally generated by a pair of vector fields X_1 and X_2, which we can choose so that (X_1, X_2) is an oriented orthonormal frame for D. Let (M, g_ε) be the Riemannian approximations constructed (canonically) using the Reeb vector field X_0 associated to the contact sub-Riemannian structure. By the Cartan formula and due to $\mathrm{d}\omega|_D = -\operatorname{vol}_g$, we have

$$\omega([X_1, X_2]) = -\mathrm{d}\omega(X_1, X_2) = 1 \ .$$

Since X_0 is the Reeb vector field, we have that

$$\omega([X_0, X_i]) = -\mathrm{d}\omega(X_0, X_i) = 0 \quad \text{for } i \in \{1, 2\} \ .$$

It follows that the structure constant defined in (3.1) are given by

$$[X_1, X_2] = c_{12}^1 X_1 + c_{12}^2 X_2 + X_0 \ , \tag{4.1}$$

$$[X_0, X_1] = c_{01}^1 X_1 + c_{01}^2 X_2 \ , \tag{4.2}$$

$$[X_0, X_2] = c_{02}^1 X_1 + c_{02}^2 X_2 \ . \tag{4.3}$$

In particular, it becomes clear that vector fields X_1, X_2 and $[X_1, X_2]$ are linearly independent everywhere.

Following the introduction, let Δ_ε be the Laplace–Beltrami operator of the Riemannian manifold $(S, i^* g_\varepsilon)$, where $i\colon S \hookrightarrow M$ is the natural immersion. We now express Δ_ε in terms of two vector fields on the surface S (independent on ε) which are orthogonal for each of the Riemannian approximations. Using these expressions of the Laplace–Beltrami operators Δ_ε where only the coefficients and not the vector fields depend on $\varepsilon > 0$, we prove Theorem 1.17. The orthogonal frame exhibited further allows us to establish Proposition 1.18.

For a vector field X on the manifold M, the property $Xu|_S \equiv 0$ ensures that $X(x) \in T_x S$ for all $x \in S$. Therefore, we see that F_1 and F_2 given by

$$F_1 = \frac{(X_2 u) X_1 - (X_1 u) X_2}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \qquad \text{and} \tag{4.4}$$

$$F_2 = \frac{(X_0 u)(X_1 u) X_1 + (X_0 u)(X_2 u) X_2}{(X_1 u)^2 + (X_2 u)^2} - X_0 \ , \tag{4.5}$$

are indeed well-defined vector fields on $S \setminus \Sigma(S)$ due to (1.17) and because we have $F_1 u|_{S\setminus\Sigma(S)} \equiv 0$ as well as $F_2 u|_{S\setminus\Sigma(S)} \equiv 0$. Here, $S \setminus \Sigma(S)$ is a manifold itself because the characteristic set $\Sigma(S)$ is a closed subset of S. We observe that both F_1 and F_2 remain unchanged if the function u defining the surface S is multiplied by a positive function, whereas F_1 changes sign and F_2 remains unchanged if u is multiplied by a negative function. Since the zero set of the twice differentiable submersion defining S needs to remain unchanged, these are the only two options which can occur. Observe that the vector field F_1 on $S \setminus \Sigma(S)$ is opposite to the vector field $\widehat{X}_S$ defined in (1.28).

Recalling that g_ε is obtained by requiring $(X_1, X_2, \varepsilon X_0)$ to be a global orthonormal frame, we further obtain

$$g_\varepsilon(F_1, F_2) = 0$$

as well as

$$g_\varepsilon(F_1, F_1) = 1 \quad \text{and} \quad g_\varepsilon(F_2, F_2) = \frac{(X_0 u)^2}{(X_1 u)^2 + (X_2 u)^2} + \frac{1}{\varepsilon^2} \ . \tag{4.6}$$

Thus, (F_1, F_2) is an orthogonal frame for $T(S \setminus \Sigma(S))$ for each Riemannian manifold $(S, i^* g_\varepsilon)$. While in general, the frame (F_1, F_2) is not orthonormal it has the nice property that it does not depend on $\varepsilon > 0$,

which aids the analysis of the convergence of the operators Δ_ε in the limit $\varepsilon \to 0$. Since F_1 and F_2 are vector fields on $S \setminus \Sigma(S)$, there exist functions $b_1, b_2 \colon S \setminus \Sigma(S) \to \mathbb{R}$, not depending on $\varepsilon > 0$, such that

$$[F_1, F_2] = b_1 F_1 + b_2 F_2 \ . \tag{4.7}$$

Whereas determining the functions b_1 and b_2 explicitly from (4.4) and (4.5) is a painful task, we can express them nicely in terms of, following the notations in [BK19], the characteristic deviation h and a tensor η related to the torsion. Let $J \colon D \to D$ be the linear transformation induced by the contact form ω by requiring that, for vector fields X and Y in the distribution D,

$$g\left(X, J(Y)\right) = \mathrm{d}\omega(X, Y) \ . \tag{4.8}$$

Under the assumption of the existence of the global orthonormal frame (X_1, X_2) this amounts to saying that

$$J(X_1) = X_2 \quad \text{and} \quad J(X_2) = -X_1 \ . \tag{4.9}$$

For a unit-length vector field X in the distribution D, we use $[X, J(X)]|_D$ to denote the restriction of the vector field $[X, J(X)]$ on M to the distribution D and we set

$$\begin{aligned} h(X) &= -g\left([X, J(X)]|_D, X\right) \ , \\ \eta(X) &= -g\left([X_0, X], X\right) \ , \end{aligned}$$

where the expression for η is indeed well-defined because according to (4.2) and (4.3), the vector field $[X_0, X]$ lies in the distribution D.

Lemma 4.1. *For $b \colon S \setminus \Sigma(S) \to \mathbb{R}$ defined by (1.29), we have*

$$[F_1, F_2] = -\left(bh(F_1) + \eta(F_1)\right) F_1 - bF_2 \ ,$$

that is, $b_1 = -bh(F_1) - \eta(F_1)$ and $b_2 = -b$.

Proof. We first observe that due to (4.9), we can write

$$F_2 = bJ(F_1) - X_0 \ .$$

Using (4.2) and (4.3) as well as (4.8), it follows that

$$\omega\left([F_1, F_2]\right) = \omega\left([F_1, bJ(F_1) - X_0]\right) = -\mathrm{d}\omega(F_1, bJ(F_1)) = -g(F_1, bJ^2(F_1)) = b \ .$$

On the other hand, from (4.4), (4.5) and (4.7), we deduce

$$\omega\left([F_1, F_2]\right) = \omega\left(b_2 F_2\right) = -b_2 \ ,$$

which implies that $b_2 = -b$, as claimed. It remains to determine b_1. From (4.8), we see that

$$g(F_1, J(F_1)) = -\omega([F_1, F_1]) = 0 \ .$$

Together with (4.7) this yields

$$b_1 = g\left([F_1, F_2], F_1\right) = g\left([F_1, bJ(F_1) - X_0], F_1\right) = bg\left([F_1, J(F_1)]|_D, F_1\right) + g\left([X_0, F_1], F_1\right) \ ,$$

and therefore, we have $b_1 = -bh(F_1) - \eta(F_1)$, as required. □

To derive an expression for the Laplace–Beltrami operators Δ_ε of $(S, i^* g_\varepsilon)$ restricted to $S \setminus \Sigma(S)$ in terms of the vector fields F_1 and F_2, it is helpful to consider the normalised frame associated with the orthogonal frame (F_1, F_2). For $\varepsilon > 0$ fixed, we define $a_\varepsilon \colon S \setminus \Sigma(S) \to \mathbb{R}$ by

$$a_\varepsilon = \left(\frac{(X_0 u)^2}{(X_1 u)^2 + (X_2 u)^2} + \frac{1}{\varepsilon^2} \right)^{-\frac{1}{2}} \tag{4.10}$$

and we introduce the vector fields E_1 and $E_{2,\varepsilon}$ on $S \setminus \Sigma(S)$ given by

$$E_1 = F_1 \quad \text{and} \quad E_{2,\varepsilon} = a_\varepsilon F_2 \ . \tag{4.11}$$

In the Riemannian manifold (S, g_ε), this yields the orthonormal frame $(E_1, E_{2,\varepsilon})$ for $T(S \setminus \Sigma(S))$.

Lemma 4.2. *For $\varepsilon > 0$, the operator Δ_ε restricted to $S \setminus \Sigma(S)$ can be expressed as*

$$\Delta_\varepsilon|_{S\setminus\Sigma(S)} = F_1^2 + a_\varepsilon^2 F_2^2 + \left(b - \frac{F_1(a_\varepsilon)}{a_\varepsilon}\right) F_1 - a_\varepsilon^2 \left(bh(F_1) + \eta(F_1)\right) F_2 \ .$$

Proof. Fix $\varepsilon > 0$ and let $\operatorname{div}_\varepsilon$ denote the divergence operator on the Riemannian manifold (S, g_ε) with respect to the corresponding Riemannian volume form. Since $(E_1, E_{2,\varepsilon})$ is an orthonormal frame for $T(S \setminus \Sigma(S))$, we have

$$\Delta_\varepsilon|_{S\setminus\Sigma(S)} = E_1^2 + E_{2,\varepsilon}^2 + (\operatorname{div}_\varepsilon E_1) E_1 + (\operatorname{div}_\varepsilon E_{2,\varepsilon}) E_{2,\varepsilon} \ . \tag{4.12}$$

Let $(\nu_1, \nu_{2,\varepsilon})$ denote the dual to the orthonormal frame $(E_1, E_{2,\varepsilon})$. Proceeding, for instance, in the same way as in [Bar13, Proof of Proposition 11], we show that, for any vector field X on $S \setminus \Sigma(S)$,

$$\operatorname{div}_\varepsilon X = \nu_1([E_1, X]) + \nu_{2,\varepsilon}([E_{2,\varepsilon}, X]) \ .$$

This together with (4.11) and Lemma 4.1 implies that

$$\operatorname{div}_\varepsilon E_1 = \nu_{2,\varepsilon}([a_\varepsilon F_2, F_1]) = -\nu_{2,\varepsilon}(a_\varepsilon[F_1, F_2] + F_1(a_\varepsilon)F_2) = b - \frac{F_1(a_\varepsilon)}{a_\varepsilon}$$

as well as

$$\operatorname{div}_\varepsilon E_{2,\varepsilon} = \nu_1([F_1, a_\varepsilon F_2]) = \nu_1(a_\varepsilon[F_1, F_2] + F_1(a_\varepsilon)F_2) = -a_\varepsilon\left(bh(F_1) + \eta(F_1)\right) \ .$$

The desired result follows from (4.11) and (4.12). □

Note that $\Delta_\varepsilon|_{S\setminus\Sigma(S)}$ in Lemma 4.2 can equivalently be written as

$$\Delta_\varepsilon|_{S\setminus\Sigma(S)} = F_1^2 + a_\varepsilon^2 F_2^2 + \left(b - \frac{F_1\left(a_\varepsilon^2\right)}{2a_\varepsilon^2}\right) F_1 - a_\varepsilon^2 \left(bh(F_1) + \eta(F_1)\right) F_2 \ .$$

Using Lemma 4.2 we can prove Theorem 1.17.

Proof of Theorem 1.17. From (1.29) and (4.10), we obtain that

$$a_\varepsilon^2 = \left(b^2 + \frac{1}{\varepsilon^2}\right)^{-1} = \frac{\varepsilon^2}{\varepsilon^2 b^2 + 1} \ , \tag{4.13}$$

which we use to compute

$$\frac{F_1(a_\varepsilon)}{a_\varepsilon} = \frac{F_1(a_\varepsilon^2)}{2a_\varepsilon^2} = -\frac{\varepsilon^2 b F_1(b)}{\varepsilon^2 b^2 + 1} \ .$$

It follows that

$$a_\varepsilon^2 \le \varepsilon^2 \quad \text{as well as} \quad \left|\frac{F_1(a_\varepsilon)}{a_\varepsilon}\right| \le \varepsilon^2 \left|bF_1(b)\right| \ . \tag{4.14}$$

Since $u \in C^2(M)$ by assumption, both $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ and $F_1(b)\colon S \setminus \Sigma(S) \to \mathbb{R}$ are continuous and therefore bounded on compact subsets of $S \setminus \Sigma(S)$. In a similar way, we argue that the function $b_1 = -bh(F_1) - \eta(F_1)$ is bounded on compact subsets of $S \setminus \Sigma(S)$. Due to (4.14), this implies that, uniformly on compact subsets of $S \setminus \Sigma(S)$,

$$\lim_{\varepsilon\to 0} a_\varepsilon^2 = 0 \ , \quad \lim_{\varepsilon\to 0} \frac{F_1(a_\varepsilon)}{a_\varepsilon} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} a_\varepsilon^2 \left(bh(F_1) + \eta(F_1)\right) = 0 \ . \tag{4.15}$$

Let $f \in C_c^2(S \setminus \Sigma(S))$. We then have $F_1 f, F_2 f \in C_c^1(S \setminus \Sigma(S))$ and $F_1^2 f, F_2^2 f \in C_c^0(S \setminus \Sigma(S))$. Since the expression (1.30) for Δ_0 can be rewritten as

$$\Delta_0 = F_1^2 + bF_1$$

and since the convergence in (4.15) is uniformly on compact subsets of $S \setminus \Sigma(S)$, we deduce from Lemma 4.2 that

$$\lim_{\varepsilon\to 0} \|\Delta_\varepsilon f - \Delta_0 f\|_{\infty, S\setminus\Sigma(S)} = \lim_{\varepsilon\to 0} \left\| a_\varepsilon^2 F_2^2 f - \frac{F_1(a_\varepsilon)}{a_\varepsilon} F_1 f - a_\varepsilon^2 \left(bh(F_1) + \eta(F_1)\right) F_2 f \right\|_{\infty, S\setminus\Sigma(S)} = 0 \ ,$$

that is, the functions $\Delta_\varepsilon f$ indeed converge uniformly on $S \setminus \Sigma(S)$ to $\Delta_0 f$. □

Using the orthonormal frames $(E_1, E_{2,\varepsilon})$, we easily derive the expression given in Proposition 1.18 for the intrinsic Gaussian curvature K_0 of the surface S in terms of the vector field $\widehat{X}_S$ and the function b. Unlike the reasoning presented in [BTV17], which further exploits intrinsic symmetries of the Heisenberg group $\mathbb{H}$, our derivation does not rely on the cancellation of divergent quantities and holds for surfaces in any three-dimensional contact sub-Riemannian manifold, cf. [BTV17, Remark 5.3].

Proof of Proposition 1.18. From Lemma 4.1 and due to (4.7) as well as (4.11), we have

$$[E_1, E_{2,\varepsilon}] = [F_1, a_\varepsilon F_2] = a_\varepsilon [F_1, F_2] + F_1(a_\varepsilon) F_2 = a_\varepsilon b_1 E_1 + \left(-b + \frac{F_1(a_\varepsilon)}{a_\varepsilon}\right) E_{2,\varepsilon} \ .$$

According to the classical formula for the Gaussian curvature of a surface in terms of an orthonormal frame, see e.g. [ABB20, Proposition 4.40], the Gaussian curvature K_ε of the Riemannian manifold (S, g_ε) is given by

$$K_\varepsilon = F_1\left(-b + \frac{F_1(a_\varepsilon)}{a_\varepsilon}\right) - a_\varepsilon F_2\left(a_\varepsilon b_1\right) - \left(a_\varepsilon b_1\right)^2 - \left(-b + \frac{F_1(a_\varepsilon)}{a_\varepsilon}\right)^2 \ . \tag{4.16}$$

We deduce from (4.13) that

$$a_\varepsilon F_2\left(a_\varepsilon\right) = \frac{1}{2} F_2\left(a_\varepsilon^2\right) = -\frac{\varepsilon^4 b F_2(b)}{\left(\varepsilon^2 b^2 + 1\right)^2}$$

as well as

$$F_1\left(\frac{F_1(a_\varepsilon)}{a_\varepsilon}\right) = -F_1\left(\frac{\varepsilon^2 b F_1(b)}{\varepsilon^2 b^2 + 1}\right) = -\frac{\varepsilon^2 F_1\left(b F_1(b)\right)}{\varepsilon^2 b^2 + 1} + \frac{2\varepsilon^4 b^2 \left(F_1(b)\right)^2}{\left(\varepsilon^2 b^2 + 1\right)^2} \ ,$$

which, in addition to (4.14), implies

$$\left|a_\varepsilon F_2\left(a_\varepsilon\right)\right| \le \varepsilon^4 \left|b F_2(b)\right| \quad \text{and} \quad \left|F_1\left(\frac{F_1(a_\varepsilon)}{a_\varepsilon}\right)\right| \le \varepsilon^2 \left|F_1\left(b F_1(b)\right)\right| + 2\varepsilon^4 b^2 \left(F_1(b)\right)^2 \ .$$

By passing to the limit $\varepsilon \to 0$ in (4.16), the desired expression follows. □

Notice that, by construction, the function b and the intrinsic Gaussian curvature K_0 are related by the Riccati-like equation

$$\dot{b} + b^2 + K_0 = 0 \ ,$$

with the notation $\dot{b} = \widehat{X}_S(b)$, which is independent on the convection used to determine the sign of the Reeb vector field.

4.2 Canonical stochastic process on the embedded surface

We study the stochastic process with generator $\frac{1}{2}\Delta_0$ on $S \setminus \Sigma(S)$. After analysing the behaviour of the drift of the process around non-degenerate characteristic points, we prove Theorem 1.19 and Proposition 1.20.

By construction, the process with generator $\frac{1}{2}\Delta_0$ moves along the characteristic foliation of S, that is, along the integral curves of the vector field $\widehat{X}_S$ on $S \setminus \Sigma(S)$ defined in (1.28). Around a fixed

non-degenerate characteristic point $x \in \Sigma(S)$, the behaviour of the canonical stochastic process is determined by how $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ given in (1.29) depends on the arc length along integral curves emanating from x. Since the vector fields X_1, X_2 and the Reeb vector field X_0 are linearly independent everywhere, the function $X_0 u\colon S \to \mathbb{R}$ does not vanish near characteristic points. In particular, we may and do choose the function $u \in C^2(M)$ defining the surface S such that $X_0 u \equiv 1$ in a neighbourhood of x.

Understanding the expression for the horizontal Hessian $\operatorname{Hess} u$ in (3.9) as a matrix representation in the dual frame of (X_1, X_2), and noting that the linear transformation $J\colon D \to D$ defined in (4.8) has the matrix representation

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

we see that

$$(\operatorname{Hess} u)\, J = \begin{pmatrix} X_1 X_2 u & -X_1 X_1 u \\ X_2 X_2 u & -X_2 X_1 u \end{pmatrix} .$$

The dynamics around the characteristic point $x \in \Sigma(S)$ is uniquely determined by the eigenvalues λ_1 and λ_2 of $((\operatorname{Hess} u)(x))J$. Since $x \in \Sigma(S)$ is non-degenerate by assumption both eigenvalues are non-zero, and due to $X_0 u \equiv 1$ in a neighbourhood of x, we further have

$$\lambda_1 + \lambda_2 = \operatorname{Tr}\left(((\operatorname{Hess} u)(x))J\right) = \left(X_1 X_2 u\right)(x) - \left(X_2 X_1 u\right)(x) = \left(X_0 u\right)(x) = 1 . \tag{4.17}$$

Thus, one of the following three cases occurs, where we use the terminology from [Rob95, Section 4.4] to distinguish between them. In the first case, where the eigenvalues λ_1 and λ_2 are complex conjugate, the characteristic point x is of focus type and the integral curves of $\widehat{X}_S$ spiral towards the point x. In the second case, where both eigenvalues are real and of positive sign, we call $x \in \Sigma(S)$ of node type, and all integral curves of $\widehat{X}_S$ approaching x do so tangentially to the eigendirection corresponding to the smaller eigenvalue, with the exception of the separatrices of the larger eigenvalue. In the third case with the characteristic point x being of saddle type, the two eigenvalues are real but of opposite sign, and the only integral curves of $\widehat{X}_S$ approaching x are the separatrices.

Note that an elliptic characteristic point is of focus type or of node type, whereas a hyperbolic characteristic point is of saddle type. Depending on which of theses cases arises, we can determine how the function b depends on the arc length along integral curves of $\widehat{X}_S$ emanating from x. The choice of the function $u \in C^2(M)$ such that $X_0 u \equiv 1$ in a neighbourhood of x fixes the sign of the vector field $\widehat{X}_S$. In particular, an integral curve γ of $\widehat{X}_S$ which extends continuously to $\gamma(0) = x$ might be defined either on the interval $[0, \delta)$ or on $(-\delta, 0]$ for some $\delta > 0$. As the derivation presented below works irrespective of the sign of the parameter of γ, we combine the two cases by writing $\gamma\colon I_\delta \to S$ for integral curves of $\widehat{X}_S$ extended continuously to $\gamma(0) = x$.

The expansion around a characteristic point of focus type is a result of the fact that the real parts of complex conjugate eigenvalues satisfying (4.17) equal $\frac{1}{2}$.

Lemma 4.3. *Let $x \in \Sigma(S)$ be a non-degenerate characteristic point and suppose that $u \in C^2(M)$ is chosen such that $X_0 u \equiv 1$ in a neighbourhood of x. For $\delta > 0$, let $\gamma\colon I_\delta \to S$ be an integral curve of the vector field $\widehat{X}_S$ extended continuously to $\gamma(0) = x$. If the eigenvalues of $((\operatorname{Hess} u)(x))J$ are complex conjugate then, as $s \to 0$,*

$$b(\gamma(s)) = \frac{2}{s} + O(1) .$$

Proof. Since $X_0 u \equiv 1$ in a neighbourhood of x, we may suppose that $\delta > 0$ is chosen small enough such that, for $s \in I_\delta \setminus \{0\}$,

$$b(\gamma(s)) = \frac{1}{\sqrt{\left(\left(X_1 u\right)(\gamma(s))\right)^2 + \left(\left(X_2 u\right)(\gamma(s))\right)^2}} .$$

A direct computation shows

$$\frac{\partial}{\partial s}\left(b(\gamma(s))^{-1}\right) = \widehat{X}_S\left(b(\gamma(s))^{-1}\right) = \left((\operatorname{Hess} u)\left(\gamma(s)\right)\right)\left(J\left(\widehat{X}_S\left(\gamma(s)\right)\right), \widehat{X}_S\left(\gamma(s)\right)\right) .$$

By the Hartman–Grobman theorem, it follows that, for $s \to 0$,

$$\frac{\partial}{\partial s}\left(b(\gamma(s))^{-1}\right) = \left((\operatorname{Hess} u)(x)\right)\left(J\left(\widehat{X}_S\left(\gamma(s)\right)\right), \widehat{X}_S\left(\gamma(s)\right)\right) + O(s) .$$

As complex conjugate eigenvalues of $((\operatorname{Hess} u)(x))J$ have real part equal to $\frac{1}{2}$ and due to $\widehat{X}_S$ being a unit-length vector field, the previous expression simplifies to

$$\frac{\partial}{\partial s}\left(b(\gamma(s))^{-1}\right) = \frac{1}{2} + O(s) . \tag{4.18}$$

Since $(X_1 u)(x) = (X_2 u)(x) = 0$ at the characteristic point x, we further have

$$\lim_{s\to 0} \frac{1}{b(\gamma(s))} = 0 . \tag{4.19}$$

A Taylor expansion together with (4.18) and (4.19) then implies that, as $s \to 0$,

$$\frac{1}{b(\gamma(s))} = \frac{s}{2} + O\left(s^2\right) ,$$

which yields, for $s \to 0$,

$$b(\gamma(s)) = \frac{2}{s}\left(1 + O(s)\right)^{-1} = \frac{2}{s} + O(1) ,$$

as claimed. □

The expansion of the function b around characteristic points of node type or of saddle type depends on along which integral curve of $\widehat{X}_S$ we are expanding. By the discussions preceding Lemma 4.3, all possible behaviours are covered by the next result.

Lemma 4.4. *Fix a non-degenerate characteristic point $x \in \Sigma(S)$. For $\delta > 0$, let $\gamma\colon I_\delta \to S$ be an integral curve of the vector field $\widehat{X}_S$ which extends continuously to $\gamma(0) = x$. Assume $u \in C^2(M)$ is chosen such that $X_0 u \equiv 1$ in a neighbourhood of x and suppose $((\operatorname{Hess} u)(x))J$ has real eigenvalues. If the curve γ approaches x tangentially to the eigendirection corresponding to the eigenvalue λ_i, for $i \in \{1, 2\}$, then, as $s \to 0$,*

$$b(\gamma(s)) = \frac{1}{\lambda_i s} + O(1) .$$

Proof. As in the proof of Lemma 4.3, we obtain, for $\delta > 0$ small enough and $s \in I_\delta \setminus \{0\}$,

$$\widehat{X}_S\left(b(\gamma(s))^{-1}\right) = \left((\operatorname{Hess} u)\left(\gamma(s)\right)\right)\left(J\left(\widehat{X}_S\left(\gamma(s)\right)\right), \widehat{X}_S\left(\gamma(s)\right)\right) .$$

Since γ is an integral curve of the vector field $\widehat{X}_S$, we deduce that

$$\frac{\partial}{\partial s}\left(\frac{1}{b(\gamma(s))}\right) = \left((\operatorname{Hess} u)\left(\gamma(s)\right)\right)\left(J\left(\gamma'(s)\right), \gamma'(s)\right) .$$

By Taylor expansion, this together with (4.19) yields, for $s \to 0$,

$$\frac{1}{b(\gamma(s))} = \left((\operatorname{Hess} u)\left(x\right)\right)\left(J\left(\gamma'(0)\right), \gamma'(0)\right) s + O\left(s^2\right) .$$

By assumption, the vector $\gamma'(0) \in T_xS$ is a unit-length eigenvector of $((\operatorname{Hess} u)(x))J$ corresponding to the eigenvalue λ_i, which has to be non-zero because x is a non-degenerate characteristic point. It follows that

$$((\operatorname{Hess} u)(x))\left(J\left(\gamma'(0)\right), \gamma'(0)\right) = \lambda_i \neq 0 \ ,$$

which implies, for $s \to 0$,

$$b(\gamma(s)) = \frac{1}{\lambda_i s}\left(1 + O(s)\right)^{-1} = \frac{1}{\lambda_i s} + O(1) \ ,$$

as required. □

Remark 4.5. We stress Lemma 4.4 does not contradict the positivity of the function b near the point x ensured by the choice of $u \in C^2(M)$ such that $X_0 u \equiv 1$ in neighbourhood of x. The derived expansion for b simply implies that on the separatrices corresponding to the negative eigenvalue of a hyperbolic characteristic point, the vector field $\widehat{X}_S$ points towards the characteristic point for that choice of u, that is, we have $s \in (-\delta, 0)$. At the same time, we notice that

$$\frac{\partial^2}{\partial s^2} + b\left(\gamma(s)\right)\frac{\partial}{\partial s}$$

remains invariant under a change from s to $-s$. Therefore, in our analysis of the one-dimensional diffusion processes induced on integral curves of $\widehat{X}_S$, we may again assume that the integral curves are parameterised by a positive parameter.

With the classification of singular points for stochastic differential equations given by Cherny and Engelbert in [CE05, Section 2.3], the previous two lemmas provide what is needed to prove Theorem 1.19 and Proposition 1.20. One additional crucial observation is that for a characteristic point of node type both eigenvalues of $((\operatorname{Hess} u)(x))\, J$ are positive and less than one, whereas for a characteristic point of saddle type, the positive eigenvalue is greater than one.

Proof of Theorem 1.19. Fix an elliptic characteristic point $x \in \Sigma(S)$. For $\delta > 0$, let $\gamma\colon [0, \delta] \to S$ be an integral curve of the vector field $\widehat{X}_S$ extended continuously to $x = \lim_{s \downarrow 0} \gamma(s)$. Following Cherny and Engelbert [CE05, Section 2.3], since the one-dimensional diffusion process on γ induced by $\frac{1}{2}\Delta_0$ has unit diffusivity and drift equal to $\frac{1}{2}b$, we set

$$\rho(t) = \exp\left(\int_t^\delta b(\gamma(s))\,\mathrm{d}s\right) \quad \text{for } t \in (0, \delta] \ . \tag{4.20}$$

If the characteristic point x is of node type the real positive eigenvalues λ_1 and λ_2 of $((\operatorname{Hess} u)(x))J$ satisfy $0 < \lambda_1, \lambda_2 < 1$ by (4.17). As x is of focus type or of node type by assumption, Lemma 4.3 and Lemma 4.4 establish the existence of some $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$ such that, as $s \downarrow 0$,

$$b(\gamma(s)) = \frac{1}{\lambda s} + O(1) \ .$$

We deduce, for $\delta > 0$ sufficiently small,

$$\rho(t) = \exp\left(\int_t^\delta \left(\frac{1}{\lambda s} + O\left(1\right)\right)\mathrm{d}s\right) = \exp\left(\frac{1}{\lambda}\ln\left(\frac{\delta}{t}\right) + O(\delta - t)\right) = \left(\frac{\delta}{t}\right)^{\frac{1}{\lambda}}\left(1 + O(\delta - t)\right).$$

Due to $\frac{1}{\lambda} > 1$, this implies that

$$\int_0^\delta \rho(t)\,\mathrm{d}t = \infty \ .$$

According to [CE05, Theorem 2.16 and Theorem 2.17], it follows that the elliptic characteristic point x is an inaccessible boundary point for the one-dimensional diffusion processes induced on the integral curves of $\widehat{X}_S$ emanating from x. Since $x \in \Sigma(S)$ was an arbitrary elliptic characteristic point, the claimed result follows. □

Proof of Proposition 1.20. We consider the stochastic process with generator $\frac{1}{2}\Delta_0$ on $S \setminus \Sigma(S)$ near a hyperbolic point $x \in \Sigma(S)$. Let γ be one of the four separatrices of x parameterised by arc length $s \geq 0$ and such that $\gamma(0) = x$. Let λ_1 be the positive eigenvalue and λ_2 be the negative eigenvalue of $((\operatorname{Hess} u)(x))J$. From the trace property (4.17), we see that $\lambda_1 > 1$. By Lemma 4.4 and Remark 4.5, we have, for $i \in \{1, 2\}$ and as $s \downarrow 0$,

$$b(\gamma(s)) = \frac{1}{\lambda_i s} + O(1) \ .$$

As in the previous proof, for $\delta > 0$ sufficiently small and $\rho\colon (0, \delta] \to \mathbb{R}$ defined by (4.20), we have

$$\rho(t) = \left(\frac{\delta}{t}\right)^{\frac{1}{\lambda_i}} (1 + O(\delta - t)) \ .$$

However, this time, due to $\frac{1}{\lambda_i} < 1$ for $i \in \{1, 2\}$, we obtain

$$\int_0^\delta \rho(t) \, \mathrm{d}t < \infty \ .$$

Using $\frac{1}{\lambda_1} > 0$, we further compute that, on the separatrices corresponding to the positive eigenvalue,

$$\int_0^\delta \frac{1 + \frac{1}{2}|b(\gamma(t))|}{\rho(t)} \, \mathrm{d}t = \int_0^\delta \frac{t^{\frac{1}{\lambda_1} - 1}}{2\lambda_1 \delta^{\frac{1}{\lambda_1}}} (1 + O(t)) \, \mathrm{d}t < \infty$$

and

$$\int_0^\delta \frac{|b(\gamma(t))|}{2} \, \mathrm{d}t = \infty \ .$$

On the separatrices corresponding to the negative eigenvalue, we have, due to $\frac{1}{\lambda_2} < 0$,

$$\int_0^\delta \frac{1 + \frac{1}{2}|b(\gamma(t))|}{\rho(t)} \, \mathrm{d}t = \int_0^\delta \frac{t^{\frac{1}{\lambda_2} - 1}}{2\lambda_2 \delta^{\frac{1}{\lambda_2}}} (1 + O(t)) \, \mathrm{d}t = \infty$$

as well as

$$s(t) = \int_0^t \rho(s) \, \mathrm{d}s = \frac{\lambda_2 \delta^{\frac{1}{\lambda_2}}}{\lambda_2 - 1} t^{1 - \frac{1}{\lambda_2}} (1 + O(t))$$

and

$$\int_0^\delta \frac{1 + \frac{1}{2}|b(\gamma(t))|}{\rho(t)} s(t) \, \mathrm{d}t = \int_0^\delta \frac{1}{2(\lambda_2 - 1)} (1 + O(t)) \, \mathrm{d}t < \infty \ .$$

Hence, as a consequence of the criterions [CE05, Theorem 2.12 and Theorem 2.13], the hyperbolic characteristic point x is reached with positive probability by the one-dimensional diffusion processes induced on the separatrices. Thus, the canonical stochastic process started on the separatrices is killed in finite time with positive probability. □

4.3 Stochastic processes on quadric surfaces in the Heisenberg group

Let $\mathbb{H}$ be the first Heisenberg group, that is, the Lie group obtained by endowing $\mathbb{R}^3$ with the group law, expressed in Cartesian coordinates,

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\right) \ .$$

On $\mathbb{H}$, we consider the two left-invariant vector fields

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \qquad \text{and} \qquad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \ ,$$

and the contact form

$$\omega = \mathrm{d}z - \frac{1}{2}\left(x\,\mathrm{d}y - y\,\mathrm{d}x\right) \ .$$

We note that the vector fields X and Y span the contact distribution D corresponding to ω, that they are orthonormal with respect to the smooth fibre inner product g on D given by

$$g_{(x,y,z)} = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y \ ,$$

and that

$$\mathrm{d}\omega|_D = -\mathrm{d}x \wedge \mathrm{d}y = -\operatorname{vol}_g \ .$$

Therefore, the Heisenberg group $\mathbb{H}$ understood as the three-dimensional contact sub-Riemannian manifold $(\mathbb{R}^3, D, g)$ falls into our setting, with $X_1 = X$, $X_2 = Y$ and the Reeb vector field

$$X_0 = \frac{\partial}{\partial z} = [X_1, X_2] \ .$$

In Section 4.3.1 and in Section 4.3.2, we discuss paraboloids and ellipsoids of revolution admitting one or two characteristic points, respectively, which are elliptic and of focus type. For these examples, the characteristic foliations can be described by logarithmic spirals in $\mathbb{R}^2$ lifted to the paraboloids and spirals between the poles on the ellipsoids, which are loxodromes, also called rhumb lines, on spheres. The induced stochastic processes are the Bessel process of order 3 for the paraboloids and Legendre-like processes for the ellipsoids moving along the leaves of the characteristic foliation. In Section 4.3.3, we consider hyperbolic paraboloids where, depending on a parameter, the unique characteristic point is either of saddle type or of node type, and we analyse the induced stochastic processes on the separatrices.

4.3.1 Paraboloid of revolution

For $a \in \mathbb{R}$, let S be the Euclidean paraboloid of revolution given by the equation $z = a(x^2 + y^2)$ for Cartesian coordinates (x, y, z) in the Heisenberg group $\mathbb{H}$. This corresponds to the surface given by (1.27) with $u\colon \mathbb{R}^3 \to \mathbb{R}$ defined as

$$u(x, y, z) = z - a\left(x^2 + y^2\right) \ .$$

We compute

$$X_0 u \equiv 1 \ , \quad (X_1 u)\,(x, y, z) = -2ax - \frac{y}{2} \quad \text{and} \quad (X_2 u)\,(x, y, z) = -2ay + \frac{x}{2} \ ,$$

which yields

$$((X_1 u)(x, y, z))^2 + ((X_2 u)(x, y, z))^2 = \frac{1}{4}\left(1 + 16a^2\right)\left(x^2 + y^2\right) \ . \tag{4.21}$$

Thus, the origin of $\mathbb{R}^3$ is the only characteristic point on the paraboloid S. It is elliptic and of focus type because $X_0 u \equiv 1$ and

$$(\operatorname{Hess} u)\, J \equiv \begin{pmatrix} \frac{1}{2} & 2a \\ -2a & \frac{1}{2} \end{pmatrix}$$

has eigenvalues $\frac{1}{2} \pm 2a\,\mathrm{i}$. On $S \setminus \Sigma(S)$, the vector field $\widehat{X}_S$ defined by (1.28) can be expressed as

$$\widehat{X}_S = \frac{1}{\sqrt{(1 + 16a^2)\,(x^2 + y^2)}}\left((x - 4ay)\,\frac{\partial}{\partial x} + (y + 4ax)\,\frac{\partial}{\partial y} + 2a\left(x^2 + y^2\right)\frac{\partial}{\partial z}\right) \ . \tag{4.22}$$

Changing to cylindrical coordinates (r, θ, z) for $\mathbb{R}^3 \setminus \{0\}$ with $r > 0$, $\theta \in [0, 2\pi)$, $z \in \mathbb{R}$ and using

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{as well as} \quad \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \ ,$$

the expression (4.22) for the vector field $\widehat{X}_S$ simplifies to

$$\widehat{X}_S = \frac{1}{\sqrt{1+16a^2}}\left(\frac{\partial}{\partial r} + \frac{4a}{r}\frac{\partial}{\partial \theta} + 2ar\frac{\partial}{\partial z}\right) \ .$$

From (4.21), we further obtain that the function $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ defined by (1.29) can be written as

$$b(r, \theta, z) = \frac{1}{\sqrt{1+16a^2}}\frac{2}{r} \ .$$

Characteristic foliation The characteristic foliation induced on the paraboloid S of revolution by the contact structure D of the Heisenberg group $\mathbb{H}$ is described through the integral curves of the vector field $\widehat{X}_S$, cf. Figure 4.1. Its integral curves are spirals emanating from the origin which can be indexed by $\psi \in [0, 2\pi)$ and parameterised by $s \in (0, \infty)$ as follows

$$s \mapsto \left(\frac{s}{\sqrt{1+16a^2}}, 4a\ln\left(\frac{s}{\sqrt{1+16a^2}}\right) + \psi, \frac{as^2}{1+16a^2}\right) \ . \tag{4.23}$$

By construction, the vector field $\widehat{X}_S$ is a unit vector field with respect to each metric induced on the surface S from Riemannian approximations of the Heisenberg group. In particular, it follows that the parameter $s \in (0, \infty)$ describes the arc length along the spirals (4.23).

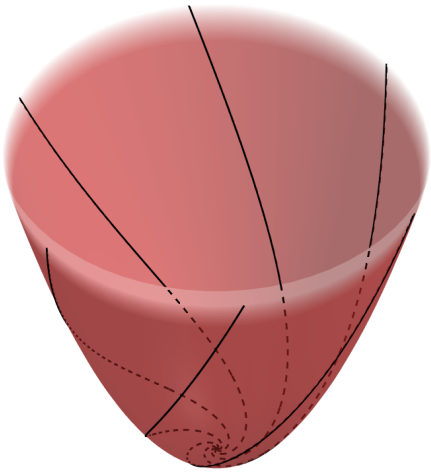

Figure 4.1: Characteristic foliation described by logarithmic spirals

Remark 4.6. The spirals on S defined by (4.23) are logarithmic spirals in $\mathbb{R}^2$ lifted to the paraboloid of revolution. In polar coordinates (r, θ) for $\mathbb{R}^2$, a logarithmic spiral can be written as

$$r = \mathrm{e}^{k(\theta+\theta_0)} \quad \text{for } k \in \mathbb{R} \setminus \{0\} \text{ and } \theta_0 \in [0, 2\pi) \ . \tag{4.24}$$

Therefore, the spirals in (4.23) correspond to lifts of logarithmic spirals (4.24) with $k = \frac{1}{4a}$. The arc length $s \in (0, \infty)$ of a logarithmic spiral (4.24) measured from the origin satisfies

$$s = \sqrt{1 + \frac{1}{k^2}}\, r \ ,$$

which for $k = \frac{1}{4a}$ yields $s = \sqrt{1+16a^2}\, r$. Note that this is the same relation between arc length and radial distance as obtained for integral curves (4.23) of the vector field $\widehat{X}_S$. For further information on logarithmic spirals, see e.g. Zwikker [Zwi63, Chapter 16].

Using the spirals (4.23) which describe the characteristic foliation on the paraboloid of revolution, we introduce coordinates (s, ψ) with $s > 0$ and $\psi \in [0, 2\pi)$ on the surface $S \setminus \Sigma(S)$. The vector field $\widehat{X}_S$ on $S \setminus \Sigma(S)$ and the function $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ are then given by

$$\widehat{X}_S = \frac{\partial}{\partial s} \quad \text{and} \quad b(s, \psi) = \frac{2}{s} \, .$$

Thus, the canonical stochastic process induced on $S \setminus \Sigma(S)$ has generator

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\left(\widehat{X}_S^2 + b\widehat{X}_S\right) = \frac{1}{2}\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} \, .$$

This gives rise to a Bessel process of order 3 which out of all the spirals (4.23) describing the characteristic foliation on S stays on the unique spiral passing through the chosen starting point of the induced stochastic process. In agreement with Theorem 1.19, the origin is indeed inaccessible for this stochastic process because a Bessel process of order 3 with positive starting point remains positive almost surely. It arises as the radial component of a three-dimensional Brownian motion, and it is equal in law to a one-dimensional Brownian motion started on the positive real line and conditioned to never hit the origin. We further observe that the operator Δ_0 coincides with the radial part of the Laplace–Beltrami operator for a quadratic cone, cf. [BN20, BP16] for $\alpha = -2$, where the self-adjointness of Δ_0 is also studied.

As the limiting operator Δ_0 does not depend on the parameter $a \in \mathbb{R}$, the behaviour described above is also what we encounter on the plane $\{z = 0\}$ in the Heisenberg group $\mathbb{H}$, where the spirals (4.23) degenerate into rays emanating from the origin. We note that the stochastic process induced by $\frac{1}{2}\Delta_0$ on the rays differs from the singular diffusion introduced by Walsh [Wal78] on the same type of structure, but that it falls into the setting of Chen and Fukushima [CF15].

4.3.2 Ellipsoid of revolution

For $a, c \in \mathbb{R}$ positive, we study the Euclidean spheroid, also called ellipsoid of revolution, in the Heisenberg group $\mathbb{H}$ given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2c^2} = 1$$

in Cartesian coordinates (x, y, z). To shorten the subsequent expressions, we choose $u\colon \mathbb{R}^3 \to \mathbb{R}$ defining the Euclidean spheroid S through (1.27) to be given by

$$u(x, y, z) = x^2 + y^2 + \frac{z^2}{c^2} - a^2 \, .$$

Proceeding as in the previous example, we first obtain

$$(X_0 u)(x, y, z) = \frac{2z}{c^2}$$

as well as

$$(X_1 u)(x, y, z) = 2x - \frac{yz}{c^2} \quad \text{and} \quad (X_2 u)(x, y, z) = 2y + \frac{xz}{c^2} \, ,$$

which yields

$$((X_1 u)(x, y, z))^2 + ((X_2 u)(x, y, z))^2 = \left(x^2 + y^2\right)\left(4 + \frac{z^2}{c^4}\right) \, . \tag{4.25}$$

This implies the north pole $(0, 0, ac)$ and the south pole $(0, 0, -ac)$ are the only two characteristic points on the spheroid S. We further compute that

$$(X_2 u)X_1 - (X_1 u)X_2 = \left(2y + \frac{xz}{c^2}\right)\frac{\partial}{\partial x} - \left(2x - \frac{yz}{c^2}\right)\frac{\partial}{\partial y} - \left(x^2 + y^2\right)\frac{\partial}{\partial z} \, . \tag{4.26}$$

Using adapted spheroidal coordinates (θ, φ) for $S \setminus \Sigma(S)$ with $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$, which are related to the coordinates (x, y, z) by

$$x = a\sin(\theta)\cos(\varphi)\ , \quad y = a\sin(\theta)\sin(\varphi)\ , \quad z = ac\cos(\theta)\ ,$$

we have

$$\begin{aligned} \frac{a\sin(\theta)}{c}\frac{\partial}{\partial\theta} &= \frac{xz}{c^2}\frac{\partial}{\partial x} + \frac{yz}{c^2}\frac{\partial}{\partial y} - \left(x^2+y^2\right)\frac{\partial}{\partial z} \quad \text{and} \\ \frac{\partial}{\partial\varphi} &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\ . \end{aligned}$$

It follows that (4.26) on the surface $S \setminus \Sigma(S)$ simplifies to

$$(X_2 u)X_1 - (X_1 u)X_2 = \frac{a\sin(\theta)}{c}\frac{\partial}{\partial\theta} - 2\frac{\partial}{\partial\varphi}\ ,$$

whereas (4.25) on $S \setminus \Sigma(S)$ rewrites as

$$((X_1 u)(\theta,\varphi))^2 + ((X_2 u)(\theta,\varphi))^2 = a^2\left(\sin(\theta)\right)^2\left(4 + \frac{a^2\left(\cos(\theta)\right)^2}{c^2}\right)\ .$$

This shows that the vector field $\widehat{X}_S$ on $S \setminus \Sigma(S)$ defined by (1.28) is given as

$$\widehat{X}_S = \frac{1}{\sqrt{4c^2 + a^2\left(\cos(\theta)\right)^2}}\left(\frac{\partial}{\partial\theta} - \frac{2c}{a\sin(\theta)}\frac{\partial}{\partial\varphi}\right)\ . \tag{4.27}$$

For the function $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ defined by (1.29), we further obtain that

$$b(\theta,\varphi) = \frac{2\cot(\theta)}{\sqrt{4c^2 + a^2\left(\cos(\theta)\right)^2}}\ . \tag{4.28}$$

As in the preceding example, in order to understand the canonical stochastic process induced by the operator $\frac{1}{2}\Delta_0$ defined through (1.30), we need to express the vector field $\widehat{X}_S$ and the function b in terms of the arc length along the integral curves of $\widehat{X}_S$. Since both $\widehat{X}_S$ and b are invariant under rotations along the azimuthal angle φ, this amounts to changing coordinates on the spheroid S from (θ, φ) to (s, φ) where $s = s(\theta)$ is uniquely defined by requiring that

$$\frac{\partial}{\partial s} = \frac{1}{\sqrt{4c^2 + a^2\left(\cos(\theta)\right)^2}}\left(\frac{\partial}{\partial\theta} - \frac{2c}{a\sin(\theta)}\frac{\partial}{\partial\varphi}\right) \qquad \text{and} \qquad s(0) = 0\ .$$

This corresponds to

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{\sqrt{4c^2 + a^2\left(\cos(\theta)\right)^2}}\ , \tag{4.29}$$

which together with $s(0) = 0$ yields

$$s(\theta) = \int_0^\theta \sqrt{4c^2 + a^2\left(\cos(\tau)\right)^2}\,\mathrm{d}\tau = \int_0^\theta \sqrt{(4c^2 + a^2) - a^2\left(\sin(\tau)\right)^2}\,\mathrm{d}\tau \quad \text{for } \theta \in (0, \pi)\ .$$

Hence, the arc length s along the integral curves of $\widehat{X}_S$ is given in terms of the polar angle θ as a multiple of an elliptic integral of the second kind. Consequently, the question if θ can be expressed

explicitly in terms of s is open. However, for our analysis, it is sufficient that the map $\theta \mapsto s(\theta)$ is invertible and that (4.28) as well as (4.29) then imply

$$b(s,\varphi) = 2\cot\left(\theta(s)\right)\frac{\mathrm{d}\theta}{\mathrm{d}s}\ .$$

Therefore, using the coordinates (s,φ), the operator $\frac{1}{2}\Delta_0$ on $S\setminus\Sigma(S)$ can be expressed as

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\frac{\partial^2}{\partial s^2} + \left(\cot\left(\theta(s)\right)\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)\frac{\partial}{\partial s}\ ,$$

which depends on the constants $a, c \in \mathbb{R}$ through (4.29). Without the Jacobian factor $\frac{\mathrm{d}\theta}{\mathrm{d}s}$ appearing in the drift term, the canonical stochastic process induced by the operator $\frac{1}{2}\Delta_0$ and moving along the leaves of the characteristic foliation would be a Legendre process, that is, a Brownian motion started inside an interval and conditioned not to hit either endpoint of the interval. The reason for the appearance of the additional factor $\frac{\mathrm{d}\theta}{\mathrm{d}s}$ is that the integral curves of $\widehat{X}_S$ connecting the two characteristic points are spirals and not just great circles. For some further discussions on the characteristic foliation of the spheroid, see the subsequent Remark 4.8.

The emergence of an operator which is almost the generator of a Legendre process moving along the leaves of the characteristic foliation motivates the search for a surface in a three-dimensional contact sub-Riemannian manifold where we do exhibit a Legendre process moving along the leaves of the characteristic foliation induced by the contact structure. This is achieved in Section 4.4.1.

Remark 4.7. The northern hemisphere of the spheroid could equally be defined by the function

$$u(x,y,z) = z - c\sqrt{a^2 - x^2 - y^2}\ .$$

With this choice we have $X_0 u \equiv 1$. We further obtain

$$\left(\left(\operatorname{Hess} u\right)(0,0,ac)\right) J = \begin{pmatrix} \frac{1}{2} & -\frac{c}{a} \\ \frac{c}{a} & \frac{1}{2} \end{pmatrix}\ ,$$

whose eigenvalues are $\frac{1}{2} \pm \frac{c}{a}\,\mathrm{i}$. A similar computation on the southern hemisphere implies that both characteristic points are elliptic and of focus type. Thus, by Theorem 1.19, the stochastic process with generator $\frac{1}{2}\Delta_0$ hits neither the north pole nor the south pole, and it induces a one-dimensional process on the unique leaf of the characteristic foliation picked out by the starting point.

Remark 4.8. With respect to the Euclidean metric $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^3$, we have for the adapted spheroidal coordinates (θ,φ) of $S\setminus\Sigma(S)$ as above that

$$\left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right\rangle = a^2\left(\cos(\theta)\right)^2 + a^2c^2\left(\sin(\theta)\right)^2 \quad \text{and} \quad \left\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi} \right\rangle = a^2\left(\sin(\theta)\right)^2\ .$$

It follows that the angle α formed by the vector field $\widehat{X}_S$ given in (4.27) and the azimuthal direction satisfies

$$\cos\left(\alpha(\theta,\varphi)\right) = -\frac{2c}{\sqrt{a^2\left(\cos(\theta)\right)^2 + a^2c^2\left(\sin(\theta)\right)^2 + 4c^2}}\ .$$

Notably, on spheres, that is, if $c = 1$, the angle α is constant everywhere. Hence, the integral curves of $\widehat{X}_S$ considered as Euclidean curves on an Euclidean sphere are loxodromes, cf. Figure 1.3, which are also called rhumb lines. They are related to logarithmic spirals through stereographic projection. Loxodromes arise in navigation by following a path with constant bearing measured with respect to the north pole or the south pole, see Carlton-Wippern [CW92].

4.3.3 Hyperbolic paraboloid

For $a \in \mathbb{R}$ positive and such that $a \neq \frac{1}{2}$, we consider the Euclidean hyperbolic paraboloid S in the Heisenberg group $\mathbb{H}$ given by (1.27) with $u\colon \mathbb{R}^3 \to \mathbb{R}$ defined as

$$u(x,y,z) = z - axy \ ,$$

for Cartesian coordinates (x,y,z). We compute

$$X_0 u \equiv 1 \ , \quad (X_1 u)\,(x,y,z) = -ay - \frac{y}{2} \quad \text{as well as} \quad (X_2 u)\,(x,y,z) = -ax + \frac{x}{2} \ , \tag{4.30}$$

and further that

$$(\operatorname{Hess} u)\, J \equiv \begin{pmatrix} \frac{1}{2} - a & 0 \\ 0 & \frac{1}{2} + a \end{pmatrix} \ . \tag{4.31}$$

Due to

$$((X_1 u)(x,y,z))^2 + ((X_2 u)(x,y,z))^2 = \left(\frac{1}{2} - a\right)^2 x^2 + \left(\frac{1}{2} + a\right)^2 y^2 \ ,$$

the hyperbolic paraboloid S has the origin of $\mathbb{R}^3$ as its unique characteristic point. By (4.31), this characteristic point is elliptic and of node type if $0 < a < \frac{1}{2}$, and hyperbolic and therefore of saddle type if $a > \frac{1}{2}$. The reason for having excluded the case $a = \frac{1}{2}$ right from the beginning is that it gives rise to a line of degenerate characteristic points.

We note that the x-axis and the y-axis lie in the hyperbolic paraboloid S. From (4.30), we see that the positive and negative x-axis as well as the positive and negative y-axis are integral curves of the vector field $\widehat{X}_S$ on $S \setminus \Sigma(S)$. In the following, we restrict our attention to studying the behaviour of the canonical stochastic process on these integral curves, which nevertheless nicely illustrates Theorem 1.19 and Proposition 1.20.

We start by analysing the one-dimensional diffusion process induced on the positive y-axis γ_y^+, which by symmetry is equal in law to the process induced on the negative y-axis. For all positive $a \in \mathbb{R}$ with $a \neq \frac{1}{2}$, we have

$$\widehat{X}_S|_{\gamma_y^+} = \frac{\partial}{\partial y} \ ,$$

implying that the arc length $s > 0$ along γ_y^+ is given by $s = y$. This yields, for all $s > 0$,

$$b\left(\gamma_y^+(s)\right) = \frac{1}{\left(\frac{1}{2} + a\right) s} \ .$$

Thus, the one-dimensional diffusion process on γ_y^+ induced by $\frac{1}{2}\Delta_0$ has generator

$$\frac{1}{2}\frac{\partial^2}{\partial s^2} + \frac{1}{(1+2a)\, s}\frac{\partial}{\partial s} \ ,$$

which gives rise to a Bessel process of order $1 + \frac{2}{1+2a}$. If started at a point with positive value this diffusion process stays positive for all times almost surely if $1 + \frac{2}{1+2a} > 2$ whereas it hits the origin with positive probability if $1 + \frac{2}{1+2a} < 2$. This is consistent with Theorem 1.19 and Proposition 1.20 because for $a > \frac{1}{2}$ the positive y-axis is a separatrix for the hyperbolic characteristic point at the origin and

$$2 < 1 + \frac{2}{1+2a} \quad \text{if } 0 < a < \frac{1}{2} \qquad \text{as well as} \qquad 2 > 1 + \frac{2}{1+2a} \quad \text{if } a > \frac{1}{2} \ .$$

Some more care is needed when studying the diffusion process induced on the positive x-axis γ_x^+. As before, this process is equal in law to the process induced on the negative x-axis. We obtain

$$\widehat{X}_S|_{\gamma_x^+} = \begin{cases} \frac{\partial}{\partial x} & \text{if } 0 < a < \frac{1}{2} \\ -\frac{\partial}{\partial x} & \text{if } a > \frac{1}{2} \end{cases}$$

as well as, for $x > 0$,

$$b(x,0,0) = \begin{cases} \dfrac{1}{\left(\frac{1}{2}-a\right)x} & \text{if } 0 < a < \frac{1}{2} \\ -\dfrac{1}{\left(\frac{1}{2}-a\right)x} & \text{if } a > \frac{1}{2} \end{cases} .$$

It follows that the one-dimensional diffusion process on γ_x^+ induced by $\frac{1}{2}\Delta_0$ has generator

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{(1-2a)\,x}\frac{\partial}{\partial x} .$$

This yields a Bessel process of order $1 + \frac{2}{1-2a}$. In agreement with Theorem 1.19 and Proposition 1.20, if started at a point with positive value this process never reaches the origin if $0 < a < \frac{1}{2}$ which ensures $1 + \frac{2}{1-2a} > 3$, whereas the process reaches the origin with positive probability if $a > \frac{1}{2}$ as this corresponds to $1 + \frac{2}{1-2a} < 1$.

4.4 Stochastic processes on canonical surfaces in $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$

In Section 4.3.1, we establish that for a paraboloid of revolution embedded in the Heisenberg group $\mathbb{H}$, the operator $\frac{1}{2}\Delta_0$ induces a Bessel process of order 3 moving along the leaves of the characteristic foliation, which is described by lifts of logarithmic spirals emanating from the origin. As discussed in Revuz and Yor [RY99, Chapter VIII.3], the Legendre processes and the hyperbolic Bessel processes arise from the same type of Girsanov transformation as the Bessel process, where these three cases only differ by the sign of a parameter. We further recall that in Section 4.3.2 we encounter a canonical stochastic process which is almost a Legendre process moving along the leaves of the characteristic foliation induced on a spheroid in the Heisenberg group $\mathbb{H}$. This motivates the search for surfaces in three-dimensional contact sub-Riemannian manifolds where the canonical stochastic process is a Legendre process of order 3 or a hyperbolic Bessel process of order 3 moving along the leaves of the characteristic foliation.

We consider surfaces in the Lie groups $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ endowed with standard sub-Riemannian structures. Together with the Heisenberg group, these sub-Riemannian geometries play the role of model spaces for three-dimensional contact sub-Riemannian manifolds. In the first two subsections, we find, by explicit computations, the canonical stochastic processes induced on certain surfaces in these groups, when expressed in convenient coordinates. The last subsection proposes a unified geometric description, justifying the choice of our surfaces.

4.4.1 Special unitary group $\mathrm{SU}(2)$

One obstruction to recovering Legendre processes moving along the characteristic foliation in Section 4.3.2 is that the characteristic foliation of a spheroid in the Heisenberg group is described by spirals connecting the north pole and the south pole instead of great circles. This is the reason for considering S^2 as a surface embedded in $\mathrm{SU}(2) \simeq S^3$ understood as a contact sub-Riemannian manifold because this gives rise to a characteristic foliation on S^2 described by great circles.

The special unitary group $\mathrm{SU}(2)$ is the Lie group of 2×2 unitary matrices of determinant 1, that is,

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z + w\,\mathrm{i} & y + x\,\mathrm{i} \\ -y + x\,\mathrm{i} & z - w\,\mathrm{i} \end{pmatrix} : x,y,z,w \in \mathbb{R} \text{ with } x^2+y^2+z^2+w^2 = 1 \right\} ,$$

with the group operation being given by matrix multiplication. Using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

we identify SU(2) with the unit quaternions, and hence also with S^3, via the map

$$\begin{pmatrix} z + w\,\mathrm{i} & y + x\,\mathrm{i} \\ -y + x\,\mathrm{i} & z - w\,\mathrm{i} \end{pmatrix} \mapsto z I_2 + x\,\mathrm{i}\,\sigma_1 + y\,\mathrm{i}\,\sigma_2 + w\,\mathrm{i}\,\sigma_3 \ .$$

The Lie algebra $\mathfrak{su}(2)$ of SU(2) is the algebra formed by the 2×2 skew-Hermitian matrices with trace zero. A basis for $\mathfrak{su}(2)$ is $\{\frac{\mathrm{i}\sigma_1}{2}, \frac{\mathrm{i}\sigma_2}{2}, \frac{\mathrm{i}\sigma_3}{2}\}$ and the corresponding left-invariant vector fields on the Lie group SU(2) are

$$\begin{aligned} U_1 &= \frac{1}{2}\left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} - w\frac{\partial}{\partial y} + y\frac{\partial}{\partial w}\right) , \\ U_2 &= \frac{1}{2}\left(-y\frac{\partial}{\partial z} + w\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - x\frac{\partial}{\partial w}\right) , \\ U_3 &= \frac{1}{2}\left(-w\frac{\partial}{\partial z} - y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + z\frac{\partial}{\partial w}\right) , \end{aligned}$$

which satisfy the commutation relations $[U_1, U_2] = -U_3$, $[U_2, U_3] = -U_1$ and $[U_3, U_1] = -U_2$. Thus, any two of these three left-invariant vector fields give rise to a sub-Riemannian structure on SU(2). To streamline the subsequent computations, we choose $k \in \mathbb{R}$ with $k > 0$ and equip SU(2) with the sub-Riemannian structure obtained by setting $X_1 = 2kU_1$, $X_2 = 2kU_2$ and by requiring (X_1, X_2) to be an orthonormal frame for the distribution D spanned by the vector fields X_1 and X_2. The appropriately normalised contact form ω for the contact distribution D is

$$\omega = \frac{1}{2k^2}\left(w\,\mathrm{d}z + y\,\mathrm{d}x - x\,\mathrm{d}y - z\,\mathrm{d}w\right)$$

and the associated Reeb vector field X_0 satisfies

$$X_0 = [X_1, X_2] = -4k^2 U_3 = 2k^2\left(w\frac{\partial}{\partial z} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - z\frac{\partial}{\partial w}\right) .$$

In SU(2), we consider the surface S given by the function $u\colon \mathrm{SU}(2) \to \mathbb{R}$ defined by

$$u(x, y, z, w) = w \ .$$

The surface S is isomorphic to S^2 because

$$S = \left\{\begin{pmatrix} z & y + x\,\mathrm{i} \\ -y + x\,\mathrm{i} & z \end{pmatrix} : x, y, z \in \mathbb{R} \text{ with } x^2 + y^2 + z^2 = 1\right\} .$$

We compute

$$(X_0 u)(x, y, z, w) = -2k^2 z \ , \quad (X_1 u)(x, y, z, w) = ky \quad \text{and} \quad (X_2 u)(x, y, z, w) = -kx \ ,$$

which yields

$$((X_1 u)(x, y, z, w))^2 + ((X_2 u)(x, y, z, w))^2 = k^2\left(x^2 + y^2\right) .$$

Due to $x^2 + y^2 + z^2 = 1$, it follows that a point on S is characteristic if and only if $z = \pm 1$. Thus, the characteristic points on S are the north pole $(0, 0, 1)$ and the south pole $(0, 0, -1)$. The vector field $\widehat{X}_S$ on $S \setminus \Sigma(S)$ defined by (1.28) is given as

$$\widehat{X}_S = \frac{k}{\sqrt{x^2 + y^2}}\left(\left(x^2 + y^2\right)\frac{\partial}{\partial z} - xz\frac{\partial}{\partial x} - yz\frac{\partial}{\partial y}\right) , \tag{4.32}$$

and for the function $b\colon S \setminus \Sigma(S) \to \mathbb{R}$ defined by (1.29), we obtain

$$b(x, y, z) = -\frac{2kz}{\sqrt{x^2 + y^2}} \ . \tag{4.33}$$

We now change coordinates for $S \setminus \Sigma(S)$ from (x, y, z) with $x^2 + y^2 + z^2 = 1$ and $z \neq \pm 1$ to (θ, φ) with $\theta \in (0, \frac{\pi}{k})$ and $\varphi \in [0, 2\pi)$ by

$$x = \sin(k\theta)\cos(\varphi) \ , \quad y = \sin(k\theta)\sin(\varphi) \quad \text{and} \quad z = \cos(k\theta) \ .$$

We note that

$$\frac{\partial}{\partial\theta} = k\cos(k\theta)\cos(\varphi)\frac{\partial}{\partial x} + k\cos(k\theta)\sin(\varphi)\frac{\partial}{\partial y} - k\sin(k\theta)\frac{\partial}{\partial z}$$

as well as

$$xz = \sin(k\theta)\cos(k\theta)\cos(\varphi) \ , \quad yz = \sin(k\theta)\cos(k\theta)\sin(\varphi) \quad \text{and} \quad \sqrt{x^2 + y^2} = \sin(k\theta) \ .$$

This together with (4.32) and (4.33) implies that

$$\widehat{X}_S = -\frac{\partial}{\partial\theta} \quad \text{and} \quad b(\theta, \varphi) = -2k\cot(k\theta) \ .$$

We deduce that the integral curves of $\widehat{X}_S$ are great circles on S and that

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\frac{\partial^2}{\partial\theta^2} + k\cot(k\theta)\frac{\partial}{\partial\theta} \ ,$$

which indeed, on each great circle, induces a Legendre process of order 3 on the interval $(0, \frac{\pi}{k})$. These processes first appeared in Knight [Kni69] as so-called taboo processes and are obtained by conditioning Brownian motion started inside the interval $(0, \frac{\pi}{k})$ to never hit either of the two boundary points, see Bougerol and Defosseux [BD19, Section 5.1]. As discussed in Itô and McKean [IM74, Section 7.15], they also arise as the latitude of a Brownian motion on the three-dimensional sphere of radius $\frac{1}{k}$.

4.4.2 Special linear group $\mathrm{SL}(2, \mathbb{R})$

The appearance of the Bessel process on the plane $\{z = 0\}$ in the Heisenberg group $\mathbb{H}$ and of the Legendre processes on a compactified plane in $\mathrm{SU}(2)$ understood as a contact sub-Riemannian manifold suggests that the hyperbolic Bessel processes arise on planes in the special linear group $\mathrm{SL}(2, \mathbb{R})$ equipped with a sub-Riemannian structure. This is indeed the case if we consider the standard sub-Riemannian structures on $\mathrm{SL}(2, \mathbb{R})$ where the flow of the Reeb vector field preserves the distribution and the fibre inner product.

The special linear group $\mathrm{SL}(2, \mathbb{R})$ of degree two over the field $\mathbb{R}$ is the Lie group of 2×2 matrices with determinant 1, that is,

$$\mathrm{SL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : x, y, z, w \in \mathbb{R} \text{ with } xw - yz = 1 \right\} \ ,$$

where the group operation is taken to be matrix multiplication. The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of $\mathrm{SL}(2, \mathbb{R})$ is the algebra of traceless 2×2 real matrices. A basis of $\mathfrak{sl}(2, \mathbb{R})$ is formed by the three matrices

$$p = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad q = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad j = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ,$$

whose corresponding left-invariant vector fields on $\mathrm{SL}(2, \mathbb{R})$ are

$$\begin{aligned} X &= \frac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w} \right) \ , \\ Y &= \frac{1}{2}\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w} \right) \ , \\ K &= \frac{1}{2}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w} \right) \ . \end{aligned}$$

These vector fields satisfy the commutation relations $[X,Y]=K$, $[X,K]=Y$ and $[Y,K]=-X$. For $k\in\mathbb{R}$ with $k>0$, we equip $\mathrm{SL}(2,\mathbb{R})$ with the sub-Riemannian structure obtain by considering the distribution D spanned by $X_1=2kX$ and $X_2=2kY$ as well as the fibre inner product uniquely given by requiring (X_1,X_2) to be a global orthonormal frame. The appropriately normalised contact form corresponding to this choice is

$$\omega=\frac{1}{4k^2}\left(z\,\mathrm{d}x+w\,\mathrm{d}y-x\,\mathrm{d}z-y\,\mathrm{d}w\right)\ ,$$

and the Reeb vector field X_0 associated with the contact form ω satisfies

$$X_0=[X_1,X_2]=4k^2K=2k^2\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}-w\frac{\partial}{\partial z}+z\frac{\partial}{\partial w}\right)\ .$$

The plane in $\mathrm{SL}(2,\mathbb{R})$ passing tangentially to the contact distribution through the identity element is the surface S given as (1.27) by the function $u\colon \mathrm{SL}(2,\mathbb{R})\to\mathbb{R}$ defined by

$$u(x,y,z,w)=y-z\ .$$

Observe that, on S, we have the relation $xw=1+y^2\geq 1$. Therefore, if a point (x,y,z,w) lies on the surface S then so does the point $(-x,y,z,-w)$, and neither x nor w can vanish on S. Thus, the function $u\colon \mathrm{SL}(2,\mathbb{R})\to\mathbb{R}$ induces a surface consisting of two sheets. By symmetry, we restrict our attention to the sheet containing the 2×2 identity matrix, henceforth referred to as the upper sheet. We compute

$$(X_1u)(x,y,z,w)=-k\,(y+z)\quad\text{and}\quad(X_2u)(x,y,z,w)=k\,(x-w)\ ,$$

as well as

$$(X_0u)(x,y,z,w)=2k^2\,(x+w)\ .$$

We note that

$$((X_1u)(x,y,z,w))^2+((X_2u)(x,y,z,w))^2=k^2\,(y+z)^2+k^2\,(x-w)^2$$

vanishes on S if and only if $y=z=0$ and $x=w$. From $xw=1+y^2$, it follows that the surface S admits the two characteristic points $(1,0,0,1)$ and $(-1,0,0,-1)$, that is, one unique characteristic point on each sheet. Following Rogers and Williams [RW00, Section V.36], we choose coordinates (r,θ) with $r>0$ and $\theta\in[0,2\pi)$ on the upper sheet of $S\setminus\Sigma(S)$ such that

$$\begin{aligned}x&=\cosh(kr)+\sinh(kr)\cos(\theta)\ ,\\ w&=\cosh(kr)-\sinh(kr)\cos(\theta)\ ,\quad\text{and}\\ y&=\sinh(kr)\sin(\theta)\ .\end{aligned}$$

On the upper sheet of $S\setminus\Sigma(S)$, we obtain

$$(X_1u)(r,\theta)=-2k\sinh(kr)\sin(\theta)\quad\text{and}\quad(X_2u)(r,\theta)=2k\sinh(kr)\cos(\theta)\ ,$$

which yields

$$\sqrt{((X_1u)(r,\theta))^2+((X_2u)(r,\theta))^2}=2k\sinh(kr)\ ,$$

as well as

$$(X_0u)(r,\theta)=4k^2\cosh(kr)\ .$$

A direct computation shows that on the upper sheet of $S\setminus\Sigma(S)$, we have

$$\widehat{X}_S=\frac{\partial}{\partial r}\quad\text{and}\quad b(r,\theta)=2k\coth(kr)\ ,$$

which implies that

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\frac{\partial^2}{\partial r^2} + k\coth(kr)\frac{\partial}{\partial r} .$$

Hence, we recover all hyperbolic Bessel processes of order 3 as the canonical stochastic processes moving along the leaves of the characteristic foliation of the upper sheet of $S \setminus \Sigma(S)$, and similarly on its lower sheet. For further discussions on hyperbolic Bessel processes, see Borodin [Bor08], Gruet [Gru00], Jakubowski and Wiśniewolski [JW13], and Revuz and Yor [RY99, Exercise 3.19]. As for the Bessel process of order 3 and the Legendre processes of order 3, the hyperbolic Bessel processes of order 3 can be defined as the radial component of Brownian motion on three-dimensional hyperbolic spaces.

4.4.3 A unified viewpoint

The surfaces considered in the last two examples together with the plane $\{z = 0\}$ in the Heisenberg group are particular cases of the following construction.

Let G be a three-dimensional Lie group endowed with a contact sub-Riemannian structure whose distribution D is spanned by two left-invariant vector fields X_1 and X_2 which are orthonormal for the fibre inner product g defined on D. Assume that the commutation relations between X_1, X_2 and the Reeb vector field X_0 are given by, for some $\kappa \in \mathbb{R}$,

$$[X_1, X_2] = X_0 , \quad [X_0, X_1] = \kappa X_2 , \quad [X_0, X_2] = -\kappa X_1 .$$

Under these assumptions the flow of the Reeb vector field X_0 preserves not only the distribution, namely $\mathrm{e}_*^{tX_0} D = D$, but also the fibre inner product g. The examples presented in Section 4.3.1 and in Sections 4.4.1 and 4.4.2 satisfy the above commutation relations with $\kappa = 0$ in the Heisenberg group, and for a parameter $k > 0$, with $\kappa = 4k^2$ in $\mathrm{SU}(2)$ and $\kappa = -4k^2$ in $\mathrm{SL}(2, \mathbb{R})$. These are the three classes of model spaces for three-dimensional sub-Riemannian structures on Lie groups with respect to local sub-Riemannian isometries, see for instance [ABB20, Chapter 17] and [AB12] for more details.

In each of the examples concerned, the surface S that we consider can be parameterised as

$$\begin{aligned} S &= \{\exp(x_1 X_1 + x_2 X_2) : x_1, x_2 \in \mathbb{R}\} \\ &= \{\exp(r\cos\theta X_1 + r\sin\theta X_2) : r \geq 0, \theta \in [0, 2\pi)\} . \end{aligned}$$

Observe that S is automatically smooth, connected, and contains the origin of the group. Under these assumptions, the sub-Riemannian structure is of type $\mathbf{d} \oplus \mathbf{s}$ in the sense of [ABB20, Section 7.7.1], and for θ fixed, the curve $r \mapsto \exp(r\cos\theta X_1 + r\sin\theta X_2)$ is a geodesic parameterised by length. Hence, $r \geq 0$ is the arc length parameter along the corresponding trajectory. It follows that the surface S is ruled by geodesics, each of them having vertical component of the initial covector equal to zero. We refer to [ABB20, Chapter 7] for more details on explicit expressions for sub-Riemannian geodesics in these cases, see also [BR08].

Chapter 5

Approximately controllable finite-dimensional bilinear systems are controllable

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This chapter presents the results in the paper [CS21], joint work with Mario Sigalotti and published in the journal Systems & Control Letters.

We begin in Section 5.1 with the proofs for Lemma 1.7 and Corollary 1.8. Next, in Section 5.2 we specialise in bilinear control systems, proving Theorem 1.9.

5.1 Properties of approximately controllable systems

Consider a control system defined as in (C) by a fixed family $\mathcal{F}$ on vector fields on M. We now prove Lemma 1.7, which shows that if (C) is approximately controllable, then the orbits of (C) form a regular foliation of M.

Proof of Lemma 1.7. As noticed above, the orbits of system (C) form a partition of M in immersed submanifolds. Since the attainable sets are contained in the orbits, the approximate controllability implies that the orbits are dense. Finally, due to the expression of the tangent space of the orbits in Theorem 1.3, the dimension of the orbits is lower semi-continuous, i.e., for all x in M, there exists a neighbourhood $V(x)$ of x such that

$$\dim \mathcal{O}_x \leq \dim \mathcal{O}_y, \qquad \forall\, y \in V(x).$$

Now let $\mathcal{O}_x$ be an orbit of maximal dimension; since $\mathcal{O}_x$ is dense, all other orbits have the same dimension as $\mathcal{O}_x$. Finally, an integral foliation of constant rank is a regular foliation. □

Recall that the family $\mathcal{F}$ is said to satisfy the *Lie algebra rank condition at* $x \in M$ if the evaluation at x of the Lie algebra generated by $\mathcal{F}$ has maximal dimension, i.e., $D_x^{\mathrm{Lie}\,\mathcal{F}} = T_xM$.

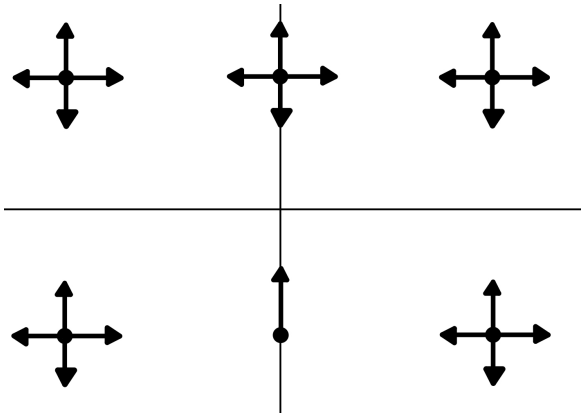

Figure 5.1: The admissible vector fields for the control system introduced in Example 5.2.

Proof of Corollary 1.8. If there exists a point x in M at which $\mathcal{F}$ satisfies the Lie algebra rank condition, then (C) has a single orbit. Indeed, Krener's theorem implies that the interior of $\mathcal{A}_x$ is nonempty. Hence, due to the approximate controllability assumption, the attainable set from any other point intersects $\mathcal{A}_x$, and therefore is contained in $\mathcal{O}_x$. Thus, (C) has a single orbit and, due to the Lie-determinedness property (1.3), every point in M satisfies the Lie algebra rank condition. The controllability follows as a corollary of Krener's theorem (see, e.g., [AS02, Cor. 8.3]).

Otherwise, assume that $\mathcal{F}$ does not satisfy the Lie algebra rank condition at any point. Then, Lemma 1.7 shows that the orbits of (C) form a regular foliation, whose leaves are dense since they contain the attainable sets. Finally, the dimension of the orbits is less than the dimension of M, otherwise $\mathcal{F}$ would satisfy the Lie algebra rank condition due to (1.3). □

Remark 5.1. Corollary 1.8 is useful if one can exclude the existence of a foliation with the properties described in *(b)*; this might be possible thanks to the particular form of system (C) or some topological properties of M. Most of the results in this direction are for codimension-one regular foliations: for example, it is known that even dimensional spheres do not admit codimension-one regular foliations [Dur72]. We recall also that a compact manifold with finite fundamental group has no analytic regular foliations of codimension one [Hae57]; some additional results can be found in [Law74].

The hypothesis of Lie-determinedness in Corollary 1.8 is necessary. Indeed, if we drop the hypothesis that the system is Lie-determined, it is possible to construct an approximately controllable and not controllable system having only one orbit. This is shown in the following example.

Example 5.2. Let $M = \mathbb{R}^2$ and consider the family $\mathcal{F} = \{f_1,\ f_2,\ f_3^+,\ f_3^-\}$ with

$$f_1 = \frac{\partial}{\partial y}, \quad f_2 = -\phi(x,y) f_1, \quad f_3^{\pm} = \pm\phi(x,y)\frac{\partial}{\partial x},$$

where $\phi : \mathbb{R}^2 \to [0, +\infty)$ is a smooth function such that, for all $(x, y) \in \mathbb{R}^2$, $\phi(x, y) = 0$ if and only if $x = 0$ and $y \leq 0$. The four vector fields are illustrated in Figure 5.1. It is not hard to check that this system has a single orbit, is approximately controllable, and still it is not controllable.

Finally, there are examples of controllable systems which are nowhere Lie-determined. For instance, one can consider the following system.

Example 5.3. Let $M = \mathbb{R}^3$, and consider the family $\mathcal{F} = \{f_1^+,\ f_1^-,\ f_2^+\ f_2^-,\ f_3^+,\ f_3^-\}$ with

$$f_1^{\pm} = \pm\frac{\partial}{\partial x}, \quad f_2^{\pm} = \pm\psi(x)\frac{\partial}{\partial y}, \quad f_3^{\pm} = \pm\psi(-x)\frac{\partial}{\partial z},$$

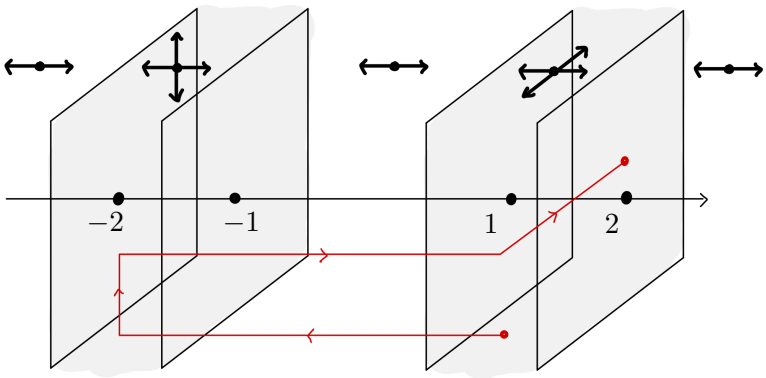

Figure 5.2: The vector fields of the control system of $\mathbb{R}^3$ in Example 5.3.

where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function with support in $\mathbb{R} \setminus [1, 2]$ and such that $\psi|_{(1,2)} > 0$. It can be easily verified that this system is controllable (see Figure 5.2), but the dimension of Lie $\mathcal{F}$ is everywhere less or equal to 2.

5.2 Bilinear control systems

Consider a bilinear control system, as defined in (BL). Given a matrix A in $\mathcal{M}$, let us denote by f_A the associated vector field

$$f_A : x \mapsto Ax, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

Since the vector fields f_A are $\mathbb{R}$-homogenous for each A in $\mathcal{M}$, the set $\mathcal{F}^{\mathcal{M}} = \{f_A \mid A \in \mathcal{M}\}$ is a family of analytical, homogenous vector fields in $\mathbb{R}^n \setminus \{0\}$.

Let us introduce two systems which can be naturally associated with (BL): the projections of system (BL) onto S^{n-1} and $\mathbb{RP}^{n-1}$. First, consider the projection $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $\pi(x) = x/|x|$. For every x in S^{n-1} and every $v \in T_x\mathbb{R}^n$, consider the pushforward $\pi_*(v) = v - \langle x, v\rangle x$, and let us introduce the control system

$$\dot{x} = \pi_*(A(t)x), \qquad A(t) \in \mathcal{M},\ x \in S^{n-1}. \tag{SΣ}$$

Due to homogeneity, the trajectories of (SΣ) are the image of the trajectories of (BL) via π; thus, the orbits $\mathcal{O}^S$ of (SΣ) are projections of the orbits of (BL), that is

$$\mathcal{O}^S_{\pi(y)} = \pi\big(\mathcal{O}_y\big), \qquad \forall y \in \mathbb{R}^n \setminus \{0\}. \tag{5.1}$$

We say that (BL) is *angularly controllable* if (SΣ) is controllable. Similarly, consider the canonical projection $\omega : \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^{n-1}$ and the system

$$\dot{q} = \omega_*(A(t)x), \qquad A(t) \in \mathcal{M},\ q = \omega(x) \in \mathbb{RP}^{n-1}. \tag{$\mathbb{P}\Sigma$}$$

This system is well-defined because $\omega_*(A(t)x)$ depends only on q and not on the choice of the specific $x \in \mathbb{R}^n \setminus \{0\}$ such that $q = \omega(x)$.

5.2.1 Proof of Theorem 1.9

Assume that system (BL) is approximately controllable and that $n \geq 2$, the case $n = 1$ being trivial. System (BL) is Lie-determined due to the analyticity of each linear vector field and the already cited Nagaro theorem [Nag66]. Thus, Corollary 1.8 applies: either system (BL) is controllable, or the partition of $\mathbb{R}^n \setminus \{0\}$ into the orbits of (BL) forms a regular foliation of positive codimension as

described in *(b)*. Let us assume the latter, and show that this leads to a contradiction.

As mentioned in the introduction, Theorem 1.9 holds if one replaces (BL) by ($\mathbb{P}\Sigma$), as shown in [BS20, Prop. 44]. Using this result, let us deduce the following property.

Lemma 5.4. *If* (BL) *is approximately controllable then* (BL) *is angularly controllable.*

Proof. Since the projection of the trajectories of (BL) are trajectories of ($\mathbb{P}\Sigma$), if the former is approximately controllable then the same holds for the latter. Due to [BS20, Prop. 44], if ($\mathbb{P}\Sigma$) is approximately controllable then it is controllable. In [BV13, Thm. 1] the authors show that system ($\mathbb{P}\Sigma$) is controllable if and only if system (SΣ) is controllable. Therefore, system (SΣ) is controllable, meaning that system (BL) is angularly controllable. □

Let us denote by $\mathcal{O} = \{\mathcal{O}_x \mid x \in \mathbb{R}^n \setminus \{0\}\}$ the orbit partition of system (BL). Due to (5.1) and Lemma 5.4, one has

$$\pi_*\big(T_y\mathcal{O}_x\big) = T_{\pi(y)}\mathcal{O}^S_{\pi(x)} = T_{\pi(y)}S^{n-1}, \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\},\ \forall\, y \in \mathcal{O}_x.$$

Therefore, we have that

$$T_y\mathcal{O}_x + \mathbb{R}y = \mathbb{R}^n, \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\},\ \forall\, y \in \mathcal{O}_x, \tag{5.2}$$

which is to say that the orbits are transversal to the radial direction. Additionally, since we assumed to be in case *(b)* of Corollary 1.8, this implies that $\dim T_y\mathcal{O}_x = n-1$. It follows that $\mathcal{O}$ is a codimension-one regular foliation of $\mathbb{R}^n \setminus \{0\}$ transversal to the radial direction and with dense leaves. In the following lemma we show that such a foliation cannot exist.

Lemma 5.5. *Assume that $n \geq 2$. Then, there does not exist a homogenous, codimension-one regular foliation of $\mathbb{R}^n \setminus \{0\}$ transversal to the radial direction and with dense leaves.*

The hypothesis of transversality between the foliation and the radial direction is necessary, as counterexamples can be constructed otherwise. For instance, in [Hec76] the author presents an example of codimension-one regular foliation of $\mathbb{R}^3$ with dense leaves; this construction is presented with additional details in [CN13, Chap. 4], and in this thesis in Example 5.8 for completeness.

Proof of Lemma 5.5. By contradiction, suppose there exists a codimension-one regular foliation $\mathcal{L} = \{L_\alpha \mid \alpha \in A\}$ of $\mathbb{R}^n \setminus \{0\}$ with dense leaves transversal to the radial direction.

Let us first consider the case $n = 2$. Orienting the foliation using the clockwise direction and applying Whitney's theorem (see [ABZ96, Thm. 2.3 at p. 23]), the foliation can be identified with the set of trajectories of a vector field. Using the stereographic projection, the flow of such a vector field can be pushed to the sphere S^2 minus two points. However, a flow with dense trajectories on S^2 minus finitely many points does not exist: see, for instance, [ABZ96, Lem. 2.4 at p. 56].

Assume that $n \geq 3$. Let us fix the point $p = (0, \dots, 0, 1) \in \mathbb{R}^n$, and denote by S^{n-2} the embedded sphere $S^{n-2} \times \{0\} \subset \mathbb{R}^n$. For every θ in S^{n-2}, let $\mathcal{P}_\theta$ be the plane

$$\mathcal{P}_\theta = \operatorname{span}\{p, \theta\},$$

as depicted in Figure 5.3. Because of the transversality between the leaves of $\mathcal{L}$ and the radial direction, the linear subbundle $I_\theta = \mathcal{P}_\theta \cap T\mathcal{L}|_{\mathcal{P}_\theta \setminus \{0\}}$ is a one-dimensional distribution on $\mathcal{P}_\theta \setminus \{0\}$ satisfying

$$I_\theta(x) \oplus \mathbb{R}x = T_x\mathcal{P}_\theta, \qquad \forall\, x \in \mathcal{P}_\theta \setminus \{0\}. \tag{5.3}$$

By definition, the intersection $L_p \cap \mathcal{P}_\theta$ contains the integral curve to I_θ starting at p. The integral curves of I_θ are nothing but the leaves of the one-dimensional foliation defined by the distribution I_θ in $\mathcal{P}_\theta \setminus \{0\}$. We claim that such a foliation is orientable. Indeed, for each x in $\mathcal{P}_\theta \setminus \{0\}$, we can say that a nonzero vector v in $I_\theta(x)$ has a positive orientation if (v, x) is an oriented frame of $\mathcal{P}_\theta$. It follows from

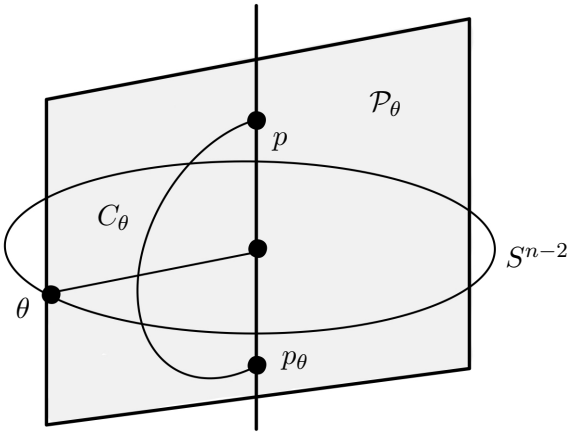

Figure 5.3: A graphic representation of the construction in the proof of Lemma 5.5

the already cited Whitney's theorem that the foliation defined by I_θ is the orbit partition of $\mathcal{P}_\theta \setminus \{0\}$ by the flow of a vector field g_θ everywhere transversal to the radial direction.

Because of the transversality condition and the homogeneity, the flow of g_θ makes a turn around the origin, in the sense that its angular component is monotone with respect to time and the radial component does not converge to zero nor diverge in finite time. Let us choose the vector field g_θ such that, starting from p, one intersects $\mathbb{R}_{>0}\theta$ before $-\mathbb{R}_{>0}\theta$. Define p_θ to be the point of first intersection between the integral curve of g_θ starting at p and the ray $-\mathbb{R}_{>0}p$, and C_θ to be the arc between p and p_θ (see Figure 5.3).

Now, let us define the map $\Phi : S^{n-2} \to -\mathbb{R}_{>0}p$ by $\Phi(\theta) = p_\theta$. The map Φ is well-defined, in the sense that p_θ does not depend on the vector field g_θ (once the latter is chosen with the appropriate orientation). Moreover, the map Φ is continuous, as it follows from the transversality between $I_\theta(-p)$ and $-\mathbb{R}_{>0}p$ for all θ in S^{n-2}. In addition, the image of Φ is contained in the intersection $L_p \cap -\mathbb{R}_{>0}p$, which has empty interior because of the transversality between $T\mathcal{L}$ and the radial direction. Since S^{n-2} is connected ($n > 2$), it follows that Φ is constant. Let us define

$$S = \bigcup_{\theta \in S^{n-2}} C_\theta.$$

By the transversality between $I_\theta(-p)$ and $-\mathbb{R}_{>0}p$ for all θ in S^{n-2} it follows that we can parameterize C_θ as a continuous arc on $[0, 1]$ continuously with respect to θ. Hence, the topology of S as a subset of $\mathbb{R}^n$ is that of $(S^{n-2} \times [0, 1])/\sim$, where $\sim$ is the equivalence relation which identifies the points in $S^{n-2} \times \{0\}$ to a single equivalence class, and analogously for the points in $S^{n-2} \times \{1\}$. That is, S is a topological manifold homeomorphic to the sphere S^{n-1}. In particular, $S \subset L_p$ is closed in $\mathbb{R}^n$, and therefore it is closed in L_p. Since S has the same topological dimension as L_p, we have that S is open in the topology of L_p. Since L_p is connected, we can conclude that $S = L_p$. This is contradicts the assumptions that the leaves are dense. □

Lemma 5.5 shows that the supposition that we are in case *(b)* of Corollary 1.8 leads to a contradiction. Therefore, we are in case *(a)*, and (BL) is controllable. This concludes the proof of Theorem 1.9. □

5.3 Complementary remarks

The result in Lemma 5.5 implies that Theorem 1.9 generalises for control systems which are Lie-determined, homogeneous, and angularly controllable. We say that (C) is *homogeneous* if $X = \mathbb{R}^n \setminus \{0\}$

and for every $x \in X$, $u \in U$, and $\lambda > 0$, $\pi_*(f_u(\lambda x)) = \pi_*(f_u(x))$, where $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ still denotes the canonical projection. Just as in the bilinear case, the projection of such systems on the sphere S^{n-1} is

$$\dot{x} = \pi_*(f_{u(t)}(x)), \qquad f_{u(t)} \in \mathcal{F},\ x \in S^{n-1}, \tag{SC}$$

and we say that (C) is angularly controllable if (SC) is controllable.

Corollary 5.6. *Let $n \geq 2$. Assume that the control system* (C) *is Lie-determined, homogenous, and angularly controllable. Then,* (C) *is approximately controllable if and only if it is controllable.*

Since for $n = 2$ system (C) projects to S^1, in this case the hypothesis of angular controllability can be easily removed.

Remark 5.7. For all n odd, $n = 2k+1$ with $k \geq 1$, the hypothesis of angular controllability in Corollary 5.6 is superfluous. Indeed, if system (C) is Lie-determined, homogenous, and approximately controllable, then Corollary 1.8 implies that either (C) is angularly controllable, or the projection of its orbits forms a nontrivial regular foliation of the even-dimensional sphere S^{2k}. Since even dimensional spheres do not admit nontrivial regular foliations (indeed their tangent spaces do not admit any nontrivial subbundles; see, e.g., [MS74, Problem 9C]), the angular controllability follows.

However, it has not been possible to fully remove the hypothesis of angular controllability in Corollary 5.6. In this regard, let us discuss the case $n = 4$. Due to Corollary 1.8, if (C) is Lie-dermined, homogeneous, approximately controllable, and (SC) is not controllable, then the orbits of (SC) form a regular foliation of S^3 of either dimension one or codimension one. The latter gives a contradiction, since the Novikov compact leaf theorem implies that any codimension-one regular foliation of the sphere S^3 has a compact leaf [Nov65]. The former implies that the orbits of (SC) are given by the flow of a minimal vector field, i.e., a vector field whose orbits are dense. The existence of such flows has been raised as an open question in [Got58] for compact metric spaces, and for the sphere S^3 has been mentioned by Smale in [Sma98] under the name Gottschalk conjecture; further details can be found in [FPW07]. If the Gottschalk conjecture were to be true, it would imply the existence of Lie-determined, homogenous, approximate controllable, yet not controllable systems, showing that Corollary 1.8 fails to hold if we remove the angular controllability hypothesis.

Finally, we mention that the hypotheses of transversality to the radial direction in Lemma 5.5 is necessary. Indeed, the following example gives a sketch of the construction of a smooth foliation of $\mathbb{R}^3$ with dense leaves, from which one can obtain a regular foliation of $\mathbb{R}^3 \setminus \{0\}$ by subtracting the origin.

Example 5.8 (Hector's example, [Hec76]). This example is due to Hector. The first part of the construction is to associate to certain diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$ a foliation ξ_f of the three-dimensional cylinder $D^2 \times \mathbb{R}$, where D^2 is a closed two-dimensional disk; the hypothesis demanded on f is the existence of an interval $[a, b]$ such that $f(x) = x$ for all $x \in [a, b]$. Given such a diffeomorphism f, one first constructs a foliation in the solid cylinder $D^2 \times \mathbb{R} \setminus \{0\} \times [(-\infty, a) \cup (b, +\infty)]$ with the property that (r, θ, z) and $(r, \theta, f(z))$ are in the same leaf, for all $r \in (0, 1]$, $\theta \in S^1$ and $z \in \mathbb{R}$ (see the image on the left in Figure 5.4). Next, one can obtain a foliation on the whole cylinder by performing a C^∞ deformation supported on $\frac{2}{3}D^2 \times \mathbb{R}$ (i.e., not changing a neighbourhood of $S^1 \times \mathbb{R}$) sending a and b to $-\infty$ and $+\infty$ respectively (see the image on the right in Figure 5.4). Denote this foliation by ξ_f.

Now, suppose having two solid cylinders $D_1^2 \times \mathbb{R}$ and $D_2^2 \times \mathbb{R}$ with two respective foliations ξ_{f_1} and ξ_{f_2}. Fix two compact arcs $V_i \subset S^1$, for $i = 1, 2$, and take a diffeomorphism $\phi : V_1 \times \mathbb{R} \to V_2 \times \mathbb{R}$ for which $\phi_*(\xi_{f_2}|_{V_2 \times \mathbb{R}}) = \xi_{f_1}|_{V_1 \times \mathbb{R}}$. The cylinders $D_1^2 \times \mathbb{R}$ and $D_2^2 \times \mathbb{R}$ in can be glued together along ϕ to obtain the foliation ξ_{f_1,f_2} on $(D_1^2 \cup_\phi D_2^2) \times \mathbb{R}$. Observe that the intersection of the leaf of ξ_{f_1,f_2} going thought a point z in S_1 (denoted by $\xi_{f_1,f_2}(z)$) and the vertical line $\{z\} \times \mathbb{R}$ is

$$\xi_{f_1,f_2}(z) \cap \big(\{z\} \times \mathbb{R}\big) = \big\{ f_1^{i_1} \circ f_2^{j_1} \circ \cdots \circ f_1^{i_k} \circ f_2^{j_k}(z) \ \big|\ i_1, j_1, \ldots, i_k, j_k \in \mathbb{Z} \big\}.$$

This can be verified by looking at the fundamental group of the bouquet of two circles $S^1 \wedge S^1$; see Figure 5.5. Notice that $D_1^2 \cup_\phi D_2^2$ is diffeomorphic to a disk, and that the gluing described above can

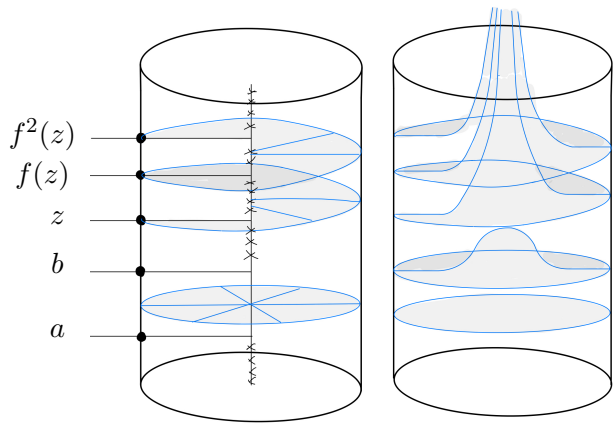

Figure 5.4: On the left-hand side the foliation constructed from the *suspension* by the diffeomorphism $f : \mathbb{R} \to \mathbb{R}$, and on the right-hand side the *whirlwind* defined by f on the whole cylinder.

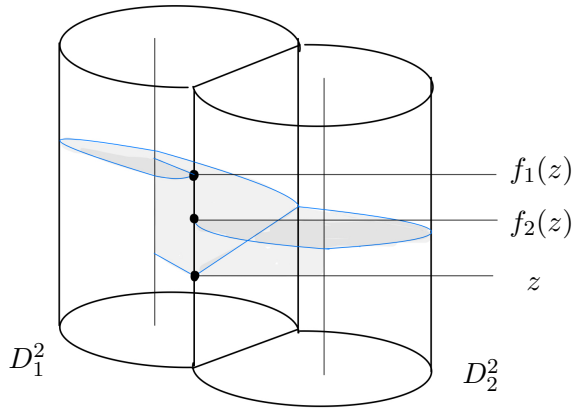

Figure 5.5: A portion of a leaf of the foliation ξ_{f_1,f_2}.

be repeated multiple times to obtain a foliation $\xi_{f_1,\dots,f_k}$ on $D^2 \times \mathbb{R}$. Moreover, for all z in S^1, the intersection of the leaf going through z and the vertical line $\{z\} \times \mathbb{R}$ is

$$\xi_{f_1,\dots,f_k}(z) \cap \big(\{z\} \times \mathbb{R}\big) = \big\{ f_{i_1}^{j_1} \circ \cdots \circ f_{i_n}^{j_n}(z) \;\big|\; i_1, \dots, i_n \in \{1, \dots, k\},\ j_1, \dots, j_n \in \mathbb{Z} \big\}.$$

Observe that if the intersection $\xi_{f_1,\dots,f_k}(z) \cap \big(\{z\} \times \mathbb{R}\big)$ is dense, then the leaf $\xi_{f_1,\dots,f_k}(z)$ is dense in the cylinder $D^2 \times \mathbb{R}$.

Lemma 5.9. *There exists $f_1, f_2, f_3, f_4 : \mathbb{R} \to \mathbb{R}$ diffeomorphism for which the above construction can be applied, and such that, for all z, the set*

$$\big\{ f_{i_1}^{j_1} \circ \cdots \circ f_{i_n}^{j_n}(z) \;\big|\; i_1, \dots, i_n \in \{1, \dots, 4\},\ j_1, \dots, j_n \in \mathbb{Z} \big\}$$

is dense in $\mathbb{R}$.

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ smooth and increasing, with $\phi|_{(-\infty,0]} \equiv 0$ and $\phi|_{[1,+\infty)} \equiv 1$. Define $f_1(x) = x + \alpha\phi(x)$ and $f_2(x) = f_1^{-1}(\alpha + x)$, and $f_3(x) = x + \phi(x)$ and $f_4(x) = f_1^{-1}(x + 1)$. By definition, $f_1 \circ f_2(x) = x + \alpha$ and $f_3 \circ f_4 = x + 1$, therefore

$$(f_1 \circ f_2)^n \circ (f_3 \circ f_4)^m(x) = x + \alpha n + m, \qquad \forall n, m \in \mathbb{Z},\ \forall x \in \mathbb{R}.$$

If α is irrational, then the set of numbers $\{\alpha n + m \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. $\square$

Thus, using the functions constructed in Lemma 5.9, one obtains a foliation of $D^2 \times \mathbb{R}$ for which every leaf is dense. Finally, by sending the boundary S^1 to the infinity of $\mathbb{R}^2$ one has a foliation of $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ with dense leaves.

Chapter 6

Locally controllable control systems are controllable

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This chapter presents the results collected in the paper [BCFS21], joint work with Ugo Boscain, Valentina Franceschi and Mario Sigalotti and currently submitted for publication.

6.1 Introduction

In the notations introduces in the the introduction, let M be a connected smooth manifold. Consider the control system C with piecewise constant controls or essentially bounded controls. The proof of Theorem 1.1 relies on the following lemmas.

Lemma 6.1. *If* (C) *is locally controllable, then it is approximately controllable.*

The proof of this lemma, presented in the next section, relies on the regularity of the flow of (C) for a fixed control, and on the connectedness of M. Lemma 6.1 is a key step in the proof of the following key property.

Lemma 6.2. *Assume* (C) *is locally controllable. Then, for any state x and y in M,*

$$y \in \mathcal{A}_x \implies x \in \mathcal{A}_y.$$

Before discussing the the proof of these lemmas, let us discuss some some the stronger notions of local controllability.

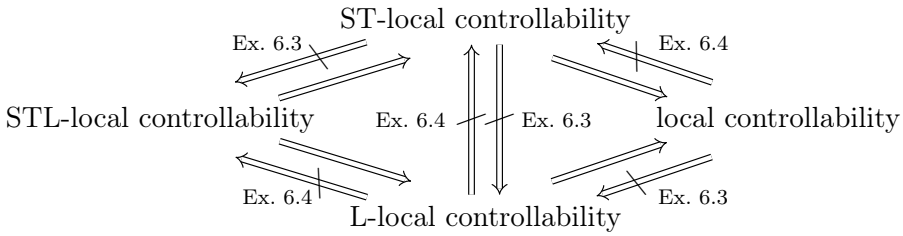

Table 6.1: A diagram summarizing the fact that the weaker forms of local controllability do not imply the stronger.

6.2 On the local forms of controllability

Let us define the set of points attainable from a point x in M at a time $t > 0$ with trajectories of (C) remaining inside of a domain Ω by

$$\mathcal{A}^t_{x,\Omega} = \{\phi(t, x, u) \mid u \in \mathcal{U},\ \phi(\cdot, x, u) \text{ defined on } [0, t] \text{ with values in } \Omega\},$$

and let

$$\mathcal{A}^{\leq T}_{x,\Omega} = \bigcup_{0<t\leq T} \mathcal{A}^t_{x,\Omega}, \qquad \mathcal{A}_{x,\Omega} = \bigcup_{0<t<+\infty} \mathcal{A}^t_{x,\Omega}.$$

Moreover, let us denote $\mathcal{A}^{\leq T}_x = \mathcal{A}^{\leq T}_{x,M}$ and notice that $\mathcal{A}_x = \mathcal{A}^{\leq +\infty}_{x,M}$. System (C) is said to be small-time locally controllable if

$$x \in \operatorname{Int} \mathcal{A}^{\leq T}_x \qquad \forall\, T > 0,\ \forall\, x \in M; \tag{ST-locally controllable}$$

moreover, system (C) is said to be localized locally controllable if

$$x \in \operatorname{Int} \mathcal{A}_{x,\Omega} \qquad \forall\, x \in M,\ \forall\, \Omega \text{ neigh. of } x. \tag{L-locally controllable}$$

Finally, system (C) is said to be small-time localized locally controllable if

$$x \in \operatorname{Int} \mathcal{A}^T_{x,\Omega} \qquad \forall\, T > 0,\ \forall\, x \in M,\ \forall\, \Omega \text{ neigh. of } x. \tag{STL-locally controllable}$$

One recognizes immediately that the implications contained in Table 1.1 can be directly deduced from the above definitions, with the exception of Theorem 1.1. In this section we show that the missing arrows cannot be added to the scheme, provinding the examples summarised in Table 6.1.

Example 6.3 (ST-local controllability $\not\Longrightarrow$ L-local controllability). Recall that a linear control system $\dot{x} = Ax + Bu$ is controllable if and only if it is ST-locally controllable which, in turns, is equivalent to the Kalman condition (see, e.g, [Son98]). On the other hand, a control system is L-locally controllable only if the range $\operatorname{Im}(B)$ of B is the entire state space. Indeed, if $A\bar{x}$ points outside $\operatorname{Im}(B)$ at some $\bar{x} \in \operatorname{Im}(B)$, considering a linear system of coordinates associated with a basis containing $A\bar{x}$ and a set of generators for $\operatorname{Im}(B)$, we have that the component along $A\bar{x}$ of every admissible trajectory staying in a sufficiently small neighborhood of $\bar{x}$ is increasing. Hence, a necessary condition for L-local controllability is that Ax is in $\operatorname{Im}(B)$ for every $x \in \operatorname{Im}(B)$. If such a condition is satisfied, every admissible trajectory starting from $\operatorname{Im}(B)$ cannot exit it. Therefore, L-local controllability can only hold when $\operatorname{Im}(B)$ is maximal. (See [BS83] for more results on controllability and local controllability of control-affine systems with unbounded controls.)

Hence any controllable linear system such that $\operatorname{Im}(B)$ is not maximal is ST-locally controllable without being L-locally controllable. This also proves that a ST-locally controllable system is not necessarily STL-locally controllable, and that a locally controllable system is not necessary L-locally controllable, as indicated in Table 6.1.

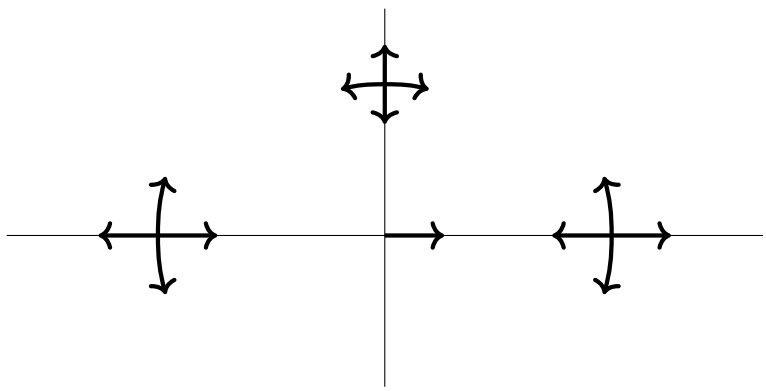

Figure 6.1: An illustration of the admissible vector fields of the control system in Example 6.4, which is L-locally controllable everywhere but is not ST-locally controllable at the origin.

Example 6.4 (L-local controllability $\not\Longrightarrow$ ST-local controllability). We now present an example of a control system in $\mathbb{R}^2$ that is L-locally controllable, but which fails to be ST-locally controllable at the origin. The example shows, in particular, that a L-locally controllable system is not necessarily STL-locally controllable and that a locally controllable system is not necessarily ST-locally controllable, as indicated in Table 6.1. Let us define, for all $x = (x_1, x_2) \in \mathbb{R}^2$,

$$X_0(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_1(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad \text{and} \quad X_2(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and let us consider the control system

$$x' = u_0 X_0(x) + u_1 X_1(x) + u_2 X_2(x), \qquad u_0 \in [0,1],\ u_1, u_2 \in [-1,1].$$

An illustration of this system can be found in Figure 6.1. Outside of the origin this system is STL-locally controllable, since the vector fields X_1 and X_2 are transversal and u_1, u_2 can take both positive and negative values. One can check that the maximal angular velocity is independent of the radius, and that the time needed to complete a semicircle is greater than or equal to π.

6.3 Proofs

Preliminaries

Let us observe that, fixed a control $u \in \mathcal{U}$, the non-autonomous differential equation (C) is well-posed in the space of absolutely continuous functions. Precisely, for a given initial condition $x \in M$, there exist $T > 0$ and a neighborhood W of x such that $\phi(t, y, u)$ is defined for $(t, y) \in [0, T] \times W$ and absolutely continuous with respect to t. Moreover, for any $t \in [0, T]$ the flow $\phi(t, \cdot, u)$ restricted to W is a local diffeomorphism (see, e.g., [Jea17, Thm. 6.2] or [Son98, Thm. 1]).

Let us denote by $\mathcal{F} = \{F(\cdot, u) \mid u \in U\}$ the family of vector fields of M parametrized by F. For a fixed smooth vector field f in $\mathcal{F}$, we denote by $e^{tf}(y)$ the value at time t of the trajectory of $\dot{x} = f(x)$ starting from y, implicitly assuming that such a trajectory is indeed defined between 0 and t.

Given a point x in M, the *controllable set to* x is the set of points which can be steered to x, i.e.,

$$\mathcal{A}_x^- = \{y \in M \mid x \in \mathcal{A}_y\}.$$

Observe that $\mathcal{A}_x^-$ is the attainable set from x for the control system defined by $-F$, whose solutions are the trajectories of (C) followed in the opposite time direction.

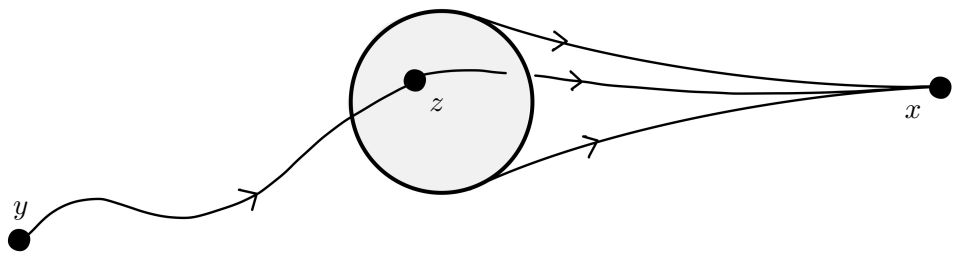

Figure 6.2: If (C) is approximately controllable and the controllable set to a certain state x has nonempty interior, then x can be reached from any other state in M. This is observed in Remark 6.5.

Remark 6.5. Assume that (C) is approximately controllable. If a point x in M satisfies $\operatorname{Int} \mathcal{A}_x^- \neq \emptyset$, then x can be reached from any other point. Indeed, for any y in M, since $\mathcal{A}_y$ is dense in M it intersects the interior of A_x^-. Thus, there exists $z \in A_y \cap A_x^-$. Since $z \in \mathcal{A}_x$ and $z \in \mathcal{A}_y$, system (C) can be steered from y to x (see Figure 6.2).

6.3.1 Approximate controllability of locally controllable systems

We are ready to prove Lemma 6.1.

Proof of Lemma 6.1. Let $x \in M$. We want to show that $\operatorname{cl}(\mathcal{A}_x)$ is open. By connectedness of M, this implies that $\operatorname{cl}(\mathcal{A}_x) = M$, thus proving the lemma.

Let $y \in \operatorname{cl}(\mathcal{A}_x)$. We claim that for every control $u \in \mathcal{U}$ and $t > 0$ such that $\phi(t, y, u)$ is defined, we have that

$$\phi(t, y, u) \in \operatorname{cl}(\mathcal{A}_x). \tag{6.1}$$

This concludes the proof of the lemma. Indeed, from (6.1) it follows that $\mathcal{A}_y \subset \operatorname{cl}(\mathcal{A}_x)$; since $\mathcal{A}_y$ contains y in its interior due to local controllability, this proves that $\operatorname{cl}(\mathcal{A}_x)$ is open.

In order to prove (6.1), fix any neighborhood V of $\phi(t, y, u)$: we show that V has nonempty intersection with $\mathcal{A}_x$. Consider a neighborhood W of y such that the map $\varphi = \phi(t, \cdot, u)|_W$ is a diffeomorphism. In particular, $\varphi(W)$ is a neighborhood of $\phi(t, y, u)$, and the set $W' = \varphi^{-1}(V \cap \varphi(W))$ is a neighborhood of y. Since y is in the closure of $\mathcal{A}_x$, there exists $y_1 \in W' \cap \mathcal{A}_x$. Consider an admissible control steering (C) from x to y_1: by concatenating such a control with u one finds that $\phi(t, y_1, u)$ is in $\mathcal{A}_x$. This implies that $\phi(t, y_1, u)$ belongs to $V \cap \mathcal{A}_x$, proving that $V \cap \mathcal{A}_x$ is nonempty, as required. □

6.3.2 Symmetry of attainable sets of locally controllable systems

Proof of Lemma 6.2. Let x and y in M be such that $y \in \mathcal{A}_x$. We argue by contradiction supposing that $x \notin \mathcal{A}_y$. We claim that this implies the existence of a point z in M (actually $z \in \mathcal{A}_x$) such that

$$z \notin \mathcal{A}_y \quad \text{and} \quad \operatorname{Int} \mathcal{A}_z^- \neq \emptyset. \tag{6.2}$$

This yields a contradiction, since the assertions in (6.2) cannot hold both at the same time due to Remark 6.5. (Notice that Remark 6.5 applies to system (C) because of Lemma 6.1.) The rest of the proof is dedicated to proving the existence of a point z satisfying (6.2).

Consider $u \in \mathcal{U}$ and $T > 0$ such that $\phi(T, x, u) = y$. Define the absolutely continuous curve $\gamma : [0, T] \to M$ by $\gamma(t) = \phi(t, x, u)$. Let

$$\tau = \inf\{t \in [0, T] \mid \gamma(t) \in \mathcal{A}_y\}.$$

We claim that $\gamma([0, T]) \cap \mathcal{A}_y = \gamma((\tau, T])$ (see Figure 1.1). Indeed, $\gamma^{-1}(\gamma([0, T]) \cap \mathcal{A}_y)$ is open since $\mathcal{A}_y$ is open, and its complement is nonempty since it contains zero (we are assuming that $x \notin \mathcal{A}_y$).

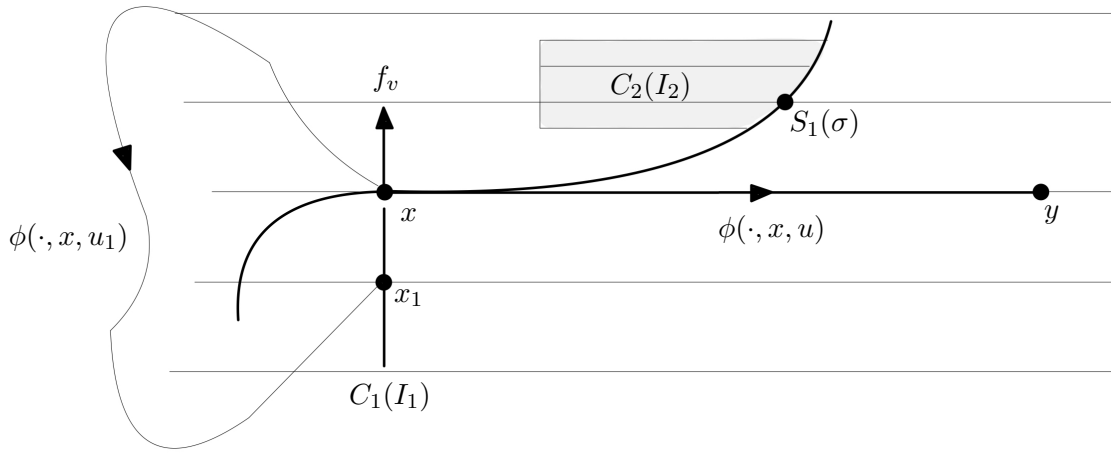

Figure 6.3: A graphic representation of the iterations in the proof of Lemma 6.2. One can steer any point in $C_2(I_2)$ to x by first attaining $S_1(I_1)$, then attaining $C_1(I_1)$ via the control u_1 from which one can reach the initial state x.

Moreover, if a certain $s \in [0, T]$ satisfies $\gamma(s) \in \mathcal{A}_y$, then, for all t in $[s, T]$, one has $\gamma(t) \in \mathcal{A}_y$ since it suffices to concatenate the control steering (C) from y to $\gamma(s)$ with $u|_{[s,t]}$ in order to attain $\gamma(t)$. Up to renaming $\gamma(\tau)$ as x, we can assume that $\tau = 0$. Namely, without loss of generality, we can assume

$$x \notin \mathcal{A}_y \quad \text{and} \quad \phi(t, x, u) \in \mathcal{A}_y \text{ for all } t \in (0, T].$$

Let V be a neighborhood of x contained in $\mathcal{A}_x$. We now construct a parametrization $C_n : I_n \to M$ ($I_n \subset \mathbb{R}^n$) of a n-dimensional embedded submanifold of M satisfying $C_n(I_n) \subset \mathcal{A}_x^-$, and therefore x (or more exactly $\gamma(\tau)$) satisfies (6.2). This will be done by a recursive argument, by constructing a finite sequence of embeddings $C_k : I_k \to M$ ($I_k \subset \mathbb{R}^k$), $k = 1, \ldots, n$, with

$$C_k(s) \in V \quad \text{and} \quad x \in \mathcal{A}_{C_k(s)}, \qquad \text{for all } s \in I_k. \tag{6.3}$$

Let us begin with $k = 1$. Let $v \in U$ (constant) such that $F(x, v) \neq 0$ and denote $f_1 = F(\cdot, v)$. Let I_1 be an open interval of the form $(0, \delta_1)$ such that the map $C_1 \colon I_1 \to M$ defined by $C_1(t) = e^{-tf_1}(x)$ parameterizes an embedded curve. Since the constant control defined by v belongs to $\mathcal{U}$, then one can reach x from any point in $C_1(I_1)$. Moreover, one has that $C_1(I_1)$ is contained in V, up to choosing δ_1 sufficiently small. Thus, C_1 satisfies (6.3) for $k = 1$.

Now, suppose having constructed a k-dimensional parameterization C_k satisfying (6.3), with $1 \leq k \leq n-1$. Fix a point $x_k \in C_k(I_k)$, and consider a control u_k in $\mathcal{U}$ and a time $T_k \geq 0$ such that $x_k = \phi(T_k, x, u_k)$. (We are using here that $x_k \in V \subset \mathcal{A}_x$.) Let W_k be a neighborhood of x such that $\varphi_k = \phi(T_k, \cdot, u_k)|_{W_k}$ is a diffeomorphism. Then $S_k := \varphi_k^{-1} \circ C_k$ parameterizes an embedded submanifold of dimension k containing x. Moreover,

$$x \in \mathcal{A}_{S_k(s)}, \qquad \forall s \in I_k, \tag{6.4}$$

since $x \in \mathcal{A}_{C_k(s)}$, and $C_k(s) \in \mathcal{A}_{S_k(s)}$ using u_k as control. In particular, we have that $S_k(s) \notin \mathcal{A}_y$ for all $s \in I_k$. As a consequence, since $\phi(t, x, u) \in \mathcal{A}_y$ for all $t \in (0, T]$, we have that

$$S_k(I_k) \cap \{\phi(t, x, u) \mid t \in (0, T]\} = \emptyset. \tag{6.5}$$

This implies the existence of $t_k \in [0, T]$ and of $\sigma \in I_k$ with $S_k(\sigma) \in V$ such that $F(S_k(\sigma), u(t_k))$ is transverse to $T_{S_k(\sigma)} S_k(I_k)$. Indeed, if one had $F(S_k(s), u(t)) \in T_{S_k(s)} S_k(I_k)$ for all $s \in I_k$ and $t \in [0, T]$, then, by uniqueness of solutions, $\phi(t, x, u)$ would stay in $S_k(I_k)$, at least for t sufficiently

small. However, this contradicts (6.5). Moreover, σ can be chosen so that $S_k(\sigma)$ is arbitrarily close to x, and in particular so that it belongs to V. Let f_k be the vector field $f_k = F(\cdot, u(t_k))$.

By transversality of $f_k(S_k(\sigma))$ and $T_{S_k(\sigma)}S_k(I_k)$, there exist $\delta_{k+1} > 0$ and an open neighborhood $I'_k \subset I_k$ containing σ such that the map $C_{k+1} : I'_k \times (-\delta_{k+1}, \delta_{k+1}) \to M$ defined by

$$C_{k+1}(s,t) = e^{tf_k} \circ S_k(s), \qquad \forall (s,t) \in I'_k \times (-\delta_{k+1}, \delta_{k+1}),$$

is a parametrization of an embedded submanifold of dimension $k+1$. Moreover, since $C_{k+1}(\sigma, 0) = S_k(\sigma) \in V$, the set $I'_k \times (-\delta_{k+1}, \delta_{k+1})$ can be chosen so that $C_{k+1}(I'_k \times (-\delta_{k+1}, \delta_{k+1})) \subset V$. We are now left to observe that $x \in \mathcal{A}_{C_{k+1}(s)}$ for all $s \in I_{k+1} := I'_k \times (-\delta_{k+1}, 0)$. In fact, starting from $C_{k+1}(s)$ one can reach $S_k(I_k)$ using the (constant) control $u(t_k)$ corresponding to f_k, and x can be reached from $S_k(I_k)$ by (6.4). This concludes the iteration, since $C_{k+1}|_{I_{k+1}}$ satisfies (6.3). □

6.3.3 Conclusion

Once Lemma 6.2 is proven, Theorem 1.1 follows from the following standard argument.

Proof of Theorem 1.1. Assume that (C) is locally controllable. Define the relation $\sim$ on M by saying that $x \sim y$ if and only if $x \in \mathcal{A}_y$. Thanks to Lemma 6.2, $\sim$ is an equivalence relation. Due to the local controllability, the equivalence classes are open. Each class is also closed, since its complement is the union of the other classes, and such an union is open. Due to the connectedness of M, there is only one class and system (C) is controllable. □

6.4 Complementary remarks

As mentioned in Remark 1.2, Lemma 6.2 (and consequently Theorem 1.1) generalises to other classes of controls provided that for any fixed control one still has existence and uniqueness in the class of absolutely continuous functions, and regularity on the initial conditions. In particular, uniqueness in necessary to prove the base case of the induction; on the other hand, regularity is used to define the submanifolds S_k starting from the submanifolds C_k.

In Theorem 1.1 we have shown that if system C satisfies local reachability then it is controllable. The property of local reachability means that one can reach an open neighbourhood of any initial condition. A slight modification of Example 5.2 gives a system which satisfies local reachability at every point *except* for one. The resulting system is not controllable, showing that the lack of local reachability even only at a point might impair global cotrollability.

Example 6.6. Let $M = \mathbb{R}^2$, and denote by V the half-line $V = \{0\} \times \mathbb{R}_{\leq 0}$. Consider the control system defined by the family $\mathcal{F} = \{f_1,\ f_2,\ f_3^+,\ f_3^-\}$ with

$$f_1 = \frac{\partial}{\partial y}, \quad f_2 = -\psi(x,y)\frac{\partial}{\partial y}, \quad f_3^{\pm} = \pm\phi(x,y)\frac{\partial}{\partial x},$$

for two smooth functions $\phi, \psi : \mathbb{R}^2 \to [0, +\infty)$ satisfying, for all $(x,y) \in \mathbb{R}^2$, $\phi(x,y) = 0$ if and only if $(x,y) \in V$, and $\psi(x,y) = 0$ if and only if $x = y = 0$, respectively. For any $p \in \mathbb{R}^2 \setminus V$ one has that $\mathcal{A}_p = \mathbb{R}^2 \setminus V$, while for any $p \in V \setminus \{0\}$ one has that $\mathcal{A}_p = \mathbb{R}^2$. Thus, at these points the system satisfies the local controllability property. However, for $p_0 = 0$ one has $\mathcal{A}_{p_0} = \{p_0\} \cup \mathbb{R}^2 \setminus V$ which does not contain p_0 in its interior.

6.5 A simpler proof when L-local controllability holds

In this appendix we give a simpler proof of the controllability of L-locally controllable systems. We first observe the following.

Proposition 6.7. *If system* (C) *is L-locally controllable, then for any point* $x \in M$ *the set* $\mathcal{A}_x^-$ *has nonempty interior.*

A proof of this proposition can be found below. However, observe that Proposition 6.7 can be directly deduced from [Gra92, Theorem 5.3], since the property of localized local controllability implies, in the terminology of [Gra92], that (C) has the nontangency property.

Proof. The argument mimics the proof of Krener's theorem [Kre74]. Fix x in M. We claim that there exists $f_1 \in \mathcal{F}$ such that $f_1(x) \neq 0$. Indeed, if that were not the case, any solution $\phi(\cdot, x, u)$ with $u \in \mathcal{U}$ would be constant. Let

$$N_1 = \{e^{-tf_1}(x) \mid t \in (0, \delta)\}$$

for $\delta > 0$. If M is one-dimensional, then we have concluded. Otherwise, we claim that there exist $y_1 \in N_1$ and $f_2 \in \mathcal{F}$ such that $f_1(y_1)$ and $f_2(y_1)$ are linearly independent. Indeed, let V_1 be a neighborhood of $e^{-\frac{\delta}{2}f_1}(x)$ not containing x nor $e^{-\delta f_1}(x)$ and assume that every $f \in \mathcal{F}$ is tangent to $N_1 \cap V_1$. Then the trajectories of (C) starting from $N_1 \cap V_1$ and staying in V_1 cannot quit $N_1 \cap V_1$. This contradicts the localized local controllability property.

Thus, define the embedded two-dimensional submanifold

$$N_2 = \{e^{-t_2 f_2} \circ e^{-t_1 f_1}(x) \mid (t_1, t_2) \in I_2 \times (0, \delta_2)\},$$

for a suitable nonempty open subinterval I_2 of $(0, \delta)$ and a suitable $\delta_2 > 0$. If the dimension of M is equal to 2 the proof is concluded, otherwise, reasoning as above, there exist $y_2 =\in N_2$ and $f_3 \in \mathcal{F}$ such that $f_3(y_2)$ is transverse to N_2, i.e.,

$$\dim(\operatorname{span}\{f_3(y_2)\} + T_{y_2}N_2) = \dim(N_2) + 1 = 3.$$

Hence, the differential of the map $(t_1, t_2, t_3) \mapsto e^{-t_3 f_3} \circ e^{-t_2 f_2} \circ e^{-t_1 f_1}(x)$ has full rank in a neighborhood of $(\bar{t}_1, \bar{t}_2, 0)$, where $\bar{t}_1, \bar{t}_2$ are such that $y_2 = e^{-\bar{t}_2 f_2} \circ e^{-t\bar{f}_1}(x)$. We can then iterate the construction up to reaching the dimension of M. □

Now due to Lemma 6.1, L-locally controllable systems are approximately controllable. Moreover, Proposition 6.7 ensures that any point x in M satisfies $\operatorname{Int} \mathcal{A}_x^- \neq \emptyset$. Thus, by Remark 6.5, we deduce that any x in M can be reached from any other point.

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