Study of beta-ensembles and their high-temperature limit

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Viens,
Ecoute ces mots qui vibrent,
Sur les murs du mois de mai

G. Moustaki

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Résumé

Cette thèse porte sur les comportements global et local des β-ensembles lorsque leur dimension tend vers l'infini, d'abord à $\beta > 0$ fixé puis dans la limite haute température, lorsque β tend vers 0. Dans une première partie, nous étudions les ensembles β-*Hermite*, β-*Laguerre* et β-*Jacobi* lorsque $\beta > 0$. Le chapitre 1 introduit des représentations matricielles de ces ensembles et explique la convergence de leurs mesures spectrales vers des mesures d'équilibre. Les chapitres 2 et 3 présentent des classes d'universalité pour les comportements locaux des β-ensembles lorsque leur dimension tend vers l'infini, définies par les processus ponctuels des valeurs propres de certains opérateurs stochastiques. Dans une seconde partie, les chapitres 4, 5 et 6 s'intéressent au comportement des valeurs propres de ces opérateurs stochastiques dans la limite haute température. Le chapitre 5 présente les résultats précédents de Laure Dumaz et Cyril Labbé sur l'*opérateur stochastique d'Airy*. Le Chapitre 6 contient notre contribution de recherche, avec l'étude des plus petites valeurs propres l'*opérateur stochastique de Bessel* dans la limite haute température.

Abstract

This thesis focuses on the global and local behavior of β-ensembles when their size tends to infinity, first at set $\beta > 0$, then in the high temperature limit, when β tends to 0. In a first part, we study the β-*Hermite*, the β-*Laguerre* and the β-*Jacobi* ensembles when $\beta > 0$. Chapter 1 introduces matrix models for these ensembles and explains the convergence of their spectral measures to equilibrium measures. Chapters 2 and 3 present universality classes for the local behaviors of β-ensembles in the large dimension limit, defined using the eigenvalues point processes of stochastic operators. In a second part, we analyze the eigenvalues of these stochastic operators in the high temperature limit. Chapter 5 presents the previous results from Laure Dumaz and Cyril Labbé on the *stochastic Airy operator*. Chapter 6 contains our research contribution, with the study of the smallest eigenvalues *stochastic Bessel operator* in the high temperature limit.

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Introduction

Contexte

Les β-ensembles, ou gas de Coulomb en physique statistique, sont des mesures de probabilité sur un ensemble de points, appelés également valeurs propres du β-ensemble. Ils ont fait l'objet de nombreuses études dans les dernières années (voir par exemple [24], [25], [31], [64], [16], [17], [67], [71], [77], [84], ainsi que leurs références). Le comportement des points d'un β-ensemble est régi par deux forces contraires : les points sont confinés dans un même potentiel V, mais se repoussent deux à deux. Le paramètre positif β influe à la fois sur le potentiel et la répulsion entre les points, il s'interprète en physique statistique comme une température inverse.

Nous nous intéressons au comportement global et local de trois β-ensembles : l'ensemble β-Hermite, l'ensemble β-Laguerre et l'ensemble β-Jacobi. La thèse se divise en deux parties. Dans la première partie, nous synthétisons les résultats de convergence globale et locale existants sur les β-ensembles pour $\beta > 0$. Nous introduisons les objets utiles à la description de ces comportements limites et présentons des heuristiques de convergence. Dans la seconde partie, nous étudions le comportement des valeurs propres des ensembles β-Hermite et β-Laguerre dans la limite haute température, c'est à dire lorsque le paramètre β tend vers 0. Cette partie contient la présentation des résultats existants pour l'ensemble β-Hermite, puis notre contribution au sujet avec l'étude de l'ensemble β-Laguerre.

Statistiques globales et locales

L'étude des β-ensembles à paramètre β fixé se divise en trois chapitres. Le Chapitre 1 introduit l'ensemble β-Hermite, l'ensemble β-Laguerre et l'ensemble β-Jacobi ainsi que des représentations matricielles tridiagonales de ces ensembles. Il établit le lien entre ces β-ensembles et des modèles classiques de matrices aléatoires dans les cas orthogonal ($\beta = 1$), unitaire ($\beta = 2$) et symplectique ($\beta = 4$). Dans le cadre général $\beta > 0$, il énonce la convergence des mesures empiriques associées aux β-ensembles vers des mesures d'équilibre, lorsque la dimension tend vers l'infini.

Le Chapitre 2 se concentre sur le comportement local des valeurs propres de ces β-ensembles lorsque la dimension tend vers l'infini. Ce comportement est décrit par des classes d'universalité, qui correspondent à la convergence vers des processus ponctuels ne dépendant que de la position des valeurs propres dans le spectre de la mesure empirique (aux bords du spectre, ou loin des bords), et non du potentiel V définissant le β-ensemble, ce pour une large classe de potentiels.

La définition des processus ponctuels limites introduits au Chapitre 2 requiert l'introduction d'opérateurs stochastiques. Le Chapitre 3 définit l'opérateur stochastique d'Airy, l'opérateur stochastique de Bessel et l'opérateur Sine_β. Il explique dans quelle mesure ces opérateurs peuvent être interprétés comme la limite en grande dimension des modèles matriciels tridiagonaux introduits au Chapitre 1.

Limite haute température

L'étude locale des β-ensembles à haute température se divise également en trois chapitres. Le Chapitre 4 est consacré à l'introduction de la transformée de Riccati, qui permet de caractériser les valeurs propres d'opérateurs stochastiques à l'aide des temps d'explosions de familles de diffusions.

Le Chapitre 5 suit l'analyse de Dumaz et Labbé [22], qui exploite cette caractérisation pour montrer la convergence des valeurs propres minimales de l'opérateur stochastique d'Airy vers un processus ponctuel de Poisson.

Enfin, le Chapitre 6 représente notre contribution de recherche et conduit l'analyse des valeurs propres minimales de l'opérateur stochastique de Bessel dans la limite haute température. Il prouve leur convergence vers un processus ponctuel limite décrit à l'aide d'équations différentielles stochastiques couplées.

Global and local statistics of β-ensembles

Chapter 1

Global statistics

In this first chapter, we introduce the concept of β-ensembles and study the global statistics of the β-Hermite, β-Laguerre and β-Jacobi ensembles as their size tends to infinity. Section 1.1 defines these three β-ensembles and provides useful random matrix representations. Section 1.2 shows their connection to classical random matrix models for specific values of the parameter β. Section 1.3 presents global convergence of the three β-ensembles to equilibrium measures. Finally, Section 1.4 summarizes the main results of this chapter.

1.1 Introduction to β-ensembles

For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, introduce the *Vandermonde determinant*

$$\Delta(\boldsymbol{\lambda}) = \prod_{i<j} (\lambda_i - \lambda_j).$$

Let $W_n = \{\boldsymbol{\lambda} \in \mathbb{R}^n, \lambda_1 \leqslant \ldots \leqslant \lambda_n\}$, and $V : \mathrm{Dom}(V) \subset \mathbb{R} \to \mathbb{R}$.

Definition 1.1.1 (β-ensemble)**.** *For $\beta > 0$, we call* β-ensemble *of size n and* potential V *the probability measure on the set W_n:*

$$\mathbb{Q}_n^{\beta,V}(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n) = \frac{1}{Z_n^{\beta,V}} |\Delta(\boldsymbol{\lambda})|^\beta \prod_{i=1}^n \exp\left(-\frac{n\beta}{2} V(\lambda_i)\right) \mathbf{1}_{\lambda_i \in \mathrm{Dom}(V)} \mathrm{d}\lambda_i, \quad (1.1.1)$$

where $Z_n^{\beta,V}$ is a normalization constant, the potential V can depend on n, β and typical choices for $\mathrm{Dom}(V)$ *are* $\mathbb{R}$, $(0, +\infty[$ *and* $(0, 1)$.

We can analyze the β-ensemble (1.1.1) as the encounter of two competing forces. The potential V tends to gather the points of the β-ensembles in its minima, while the Vandermonde determinant acts as a repulsive force between any two points. The value of the parameter β tunes the respective importance of those two forces and can also change the shape of the potential V.

From a statistical physics viewpoint, β-ensembles are measures of a one dimensional *Coulomb log-gas* at the inverse temperature β, also called *Dyson parameter*. They have attracted significant interest in recent years (see e.g. [24], [25], [31], [64], [16], [17], [67], [71], [77], [84], and references therein).

1.1.1 The β-Hermite ensemble

The choice of potential $V_H(\lambda) = \lambda^2/2,\ \mathrm{Dom}(V_H) = \mathbb{R}$ defines the *β-Hermite ensemble* on $\mathbb{R}$:

$$\mathbb{H}_n^\beta(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n) = \frac{1}{Z_n^{\beta,V_H}} |\Delta(\boldsymbol{\lambda})|^\beta \prod_{i=1}^n \exp\big(-\frac{n\beta}{4}\lambda_i^2\big)\mathrm{d}\lambda_i. \tag{1.1.2}$$

Adapting earlier work from Trotter [76], Dimitriu and Edelman [24, Theorem 2.12] found a measure on a set of tridiagonal matrices which induces the measure $\mathbb{H}_n^\beta$ on the eigenvalues. Recall that a random variable is said to be χ_r-distributed if it has the following density $f_r(x) = \frac{1}{2^{r/2-1}\Gamma(\frac{r}{2})} x^{r-1}e^{-x^2/2}\mathbf{1}_{x>0}$.

Theorem 1.1.1 (Matrix representation for the β-Hermite ensemble)**.**
Let $(G_i)_{1\leqslant i\leqslant n}$ be a family of independent standard Gaussian random variables. For $\beta > 0$, consider the symmetric *tridiagonal random matrix, with* independent *coefficients in the upper triangular part:*

$$H_n^\beta = \frac{1}{\sqrt{n\beta}} \begin{pmatrix} \sqrt{2}G_1 & \chi_{(n-1)\beta} & & \\ \chi_{(n-1)\beta} & \sqrt{2}G_2 & \chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \sqrt{2}G_n \end{pmatrix}. \tag{1.1.3}$$

The real eigenvalues $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ of H_n^β have the joint density $\mathbb{H}_n^\beta$.

1.1.2 The β-Laguerre ensemble

For $m \geqslant n$, the choice of potential

$$V_L(\lambda) = \lambda - \big(\frac{1}{n}(m-n+1) - \frac{2}{n\beta}\big)\ln\lambda,\ \mathrm{Dom}(V_L) = (0, +\infty[$$

defines the *β-Laguerre ensemble* on $(0, +\infty[$:

$$\mathbb{L}_{n,m}^\beta(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n) = \frac{1}{Z_n^{\beta,V_L}} |\Delta(\boldsymbol{\lambda})|^\beta \prod_{i=1}^n \lambda_i^{\frac{\beta}{2}(m-n+1)-1} \exp\big(-n\frac{\beta}{2}\lambda_i\big)\mathbf{1}_{(0,\infty[}(\lambda_i)\mathrm{d}\lambda_i. \tag{1.1.4}$$

Adapting earlier work from Silverstein [68], Dimitriu and Edelman [24, Theorem 3.4] found a measure on a set of bidiagonal matrices which induces the measure $\mathbb{L}_{n,m}^\beta$ on the eigenvalues.

Theorem 1.1.2 (Matrix representation for the β-Laguerre ensemble)**.**
For $\beta > 0$, consider the bidiagonal random matrix $L_{n,m}^\beta$ with independent *coefficients:*

$$L_{n,m}^\beta = \frac{1}{\sqrt{m\beta}} \begin{pmatrix} \chi_{m\beta} & & & \\ \chi_{(n-1)\beta} & \chi_{(m-1)\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{(m-n+1)\beta} \end{pmatrix}. \tag{1.1.5}$$

The real eigenvalues $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ of $L_{n,m}^\beta (L_{n,m}^\beta)^T$ have the joint density $\mathbb{L}_{n,m}^\beta$.

1.1.3 The β-Jacobi ensemble

For $m_1, m_2 \geqslant n$, the choice of potential

$$V_J(\lambda) = (\frac{1}{n}(m_1-n+1)-\frac{2}{n\beta}) \ln \lambda + (\frac{1}{n}(m_2-n+1)-\frac{2}{n\beta}) \ln(1-\lambda),\ \mathrm{Dom}(V_J) = (0,1)$$

defines the *β-Jacobi ensemble* on $(0,1)$:

$$\mathbb{J}^{\beta}_{n,m_1,m_2}(\mathrm{d}\lambda_1, \dots, \mathrm{d}\lambda_n) = \frac{1}{Z_n^{\beta,V_J}} |\Delta(\boldsymbol{\lambda})|^{\beta} \prod_{i=1}^{n} \lambda_i^{\frac{\beta}{2}(m_1-n+1)-1} (1-\lambda_i)^{\frac{\beta}{2}(m_2-n+1)-1} \mathbf{1}_{(0,1)}(\lambda_i) \mathrm{d}\lambda_i. \tag{1.1.6}$$

Kilip and Nenciu [50, Theorem 2] found a measure on a set of tridiagonal matrices which induces the measure $\mathbb{Q}_n^{\beta,V_J}$ on the eigenvalues.
Let $p_{-1} = p_0 = 0$ and $p_1, \dots, p_{2n-1}$ be independent random variables distributed as

$$p_k \sim \begin{cases} \text{Beta } (\frac{2n-k}{4}\beta, \frac{2n-k-2}{4}\beta + \delta_1 + \delta_2 + 2) & \text{if } k \text{ even} \\ \text{Beta } (\frac{2n-k-1}{4}\beta + \delta_1 + 1, \frac{2n-k-1}{4}\beta + \delta_2 + 1) & \text{if } k \text{ odd} \end{cases}$$

where $\delta_1 = \frac{\beta}{2}(m_1 - n + 1) - 1$ and $\delta_2 = \frac{\beta}{2}(m_2 - n + 1) - 1$. Then, define $\alpha_k,\ \beta_k$:

$$\begin{cases} \alpha_k = p_{2k-2}(1 - p_{2k-3}) + p_{2k-1}(1 - p_{2k-2}) \\ \beta_k = \sqrt{p_{2k-1}(1 - p_{2k-2}) + p_{2k}(1 - p_{2k-1})} \end{cases}.$$

Theorem 1.1.3 (Matrix representation for the β-Jacobi ensemble [26])**.**
For $\beta > 0$, consider the symmetric *tridiagonal random matrix:*

$$J^{\beta}_{n,m_1,m_2} = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \beta_2 & \\ & \ddots & \ddots & \ddots \\ & & \beta_{n-1} & \alpha_n \end{pmatrix}. \tag{1.1.7}$$

The real eigenvalues $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ of J^{β}_{n,m_1,m_2} have the joint density $\mathbb{J}^{\beta}_{n,m_1,m_2}$.

Remark 1.1.1 (About terminology). *The names Hermite, Laguerre and Jacobi refer to the orthogonal polynomials associated to the spectral measures (see Chapter 2, Section 2.2). Depending on the context, the term 'β-ensemble' can also refer to the associated tridiagonal or bidiagonal matrix ensemble. In this spirit, we will refer to the points of a β-ensemble as* eigenvalues.

Remark 1.1.2 (Structured matrices). *One can easily create a matrix representation of $\mathbb{Q}_n^{\beta,V}$ by conjugating $D = \mathrm{diag}(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ has the law $\mathbb{Q}_n^{\beta,V}$, with an orthogonal (resp. unitary, symplectic) matrix. However, the resulting matrices will be full and their coefficients not independent. The advantage of Theorems 1.1.1, 1.1.2 and 1.1.3 is to provide sparse matrices, with independent (but not i.i.d) coefficients in the Hermite and Laguerre cases, called* structured matrices.

We seek to understand the behavior of these three β-ensembles when n tends to infinity. **We will restrict our study to the following regimes of interest:**

$$n/m \underset{n,m\to+\infty}{\longrightarrow} \gamma \in (0,1] \text{ for the } \beta\text{-Laguerre ensemble,} \tag{1.1.8}$$

$$m_1/n \underset{n,m_1\to+\infty}{\longrightarrow} \gamma_1 \in [1,\infty[,\ m_2/n \underset{n,m_2\to+\infty}{\longrightarrow} \gamma_2 \in [1,\infty[\text{ for the } \beta\text{-Jacobi ensemble.} \tag{1.1.9}$$

1.2 Classical random matrix models

There are many historical sources to random matrix theory. In the 1920's, the statistician Wishart studied the empirical covariance matrix of multivariate Gaussian samples [83], a random Hermitian matrix now called Wishart matrix. At the birth of informatics and numerical analysis in the 1940's, Goldstine and Von Neumann [36] introduced random matrices to understand the behavior of algorithms in matrix numerical analysis, such as linear systems resolution, when the data are flawed by errors.
In nuclear physics in the 1950's, Wigner worked to explain the distribution of energy levels in atomic nuclei [81], [82]. In quantum mechanics, energy levels are related to the spectra of Hermitian operators. By randomizing those operators using Gaussian variables and studying the spacing between the eigenvalues of the associated random Hermitian matrices, Wigner was able to fairly account for the empirical observations. The pioneering work of Wigner was then developed in particular by Dyson [28] and Mehta [58].
Between 1960 and 1980 came the works of Marchenko, Pastur, Girko, Bai and Silverstein. Since then, random matrix theory has been a very active field that interacts with a wide range of mathematical branches. Thorough references on the subject include [18], [33], [2], [9] and [59].

This section, inspired by Edelman and Rao's review [29], shows how the β-Hermite, β-Laguerre and β-Jacobi ensembles connect to classical random matrix ensembles: the Gaussian ensembles, the Wishart (empirical covariance) ensembles and the MANOVA (Multivariate Analysis of Variance) ensembles.

1.2.1 Gaussian matrices

For $\beta = 1$ (resp. $\beta = 2$, $\beta = 4$), let $\mathrm{Gauss}_\beta(n, m)$ be the set of $n \times m$ matrices populated with independent and identically distributed standard real normal variables (resp. complex normal variables, quaternionic normal variables). Here, the parameter $\beta \in \{1, 2, 4\}$ encodes the dimension over $\mathbb{R}$ of the space on which the coefficients live. Recall the Frobenius norm of a matrix $M \in \mathcal{M}_{n,m}(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or the field of quaternions $\mathbb{H}$:

$$\|M\|_F = \mathrm{Tr}(MM^*)^{1/2} = \Big(\sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} |m_{i,j}|^2 \Big)^{1/2}.$$

We write $\mathrm{d}M = \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} \mathrm{d}m_{i,j}$ the matrix differential element, with

$$\mathrm{d}m_{i,j} = \begin{cases} \mathrm{d}\Re(m_{i,j})\ \mathrm{d}\Im(m_{i,j}) \text{ if } K = \mathbb{C} \\ \mathrm{d}m_{i,j}^1\ \mathrm{d}m_{i,j}^2\ \mathrm{d}m_{i,j}^3\ \mathrm{d}m_{i,j}^4 \text{ if } K = \mathbb{H} \end{cases}$$

An element $M = (m_{i,j})_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}}$ of $\mathrm{Gauss}_\beta(n, m)$ has a joint element density

$$\frac{1}{(2\pi)^{\beta mn/2}} \exp\big(-\frac{1}{2} \|M\|_F^2 \big)\ \mathrm{d}M.$$

A key property of $\text{Gauss}_1(n,m)$ (resp. $\text{Gauss}_2(n,m)$, $\text{Gauss}_4(n,m)$) is its *orthogonal* (resp. *unitary*, *symplectic*) *invariance*, which is inherited from the invariance of the Frobenius norm. Going forward, we call *orthogonal* the case $\beta = 1$, *unitary* the case $\beta = 2$ and *symplectic* the case $\beta = 4$.

1.2.2 Classical random matrix ensembles

From $\text{Gauss}_\beta(n,m)$ we can construct matrix ensembles that preserve its invariance properties.

Definition 1.2.1 (Gaussian, Wishart and MANOVA ensembles)**.**
Set $\beta \in \{1,2,4\}$. The Gaussian orthogonal/unitary/symplectic ensembles *(abbreviated* GOE/GUE/GSE*) of size n are the ensembles:*

$$G_\beta(n) = \left\{ \frac{M + M^*}{\sqrt{2}}, M \in \text{Gauss}_\beta(n,n) \right\}.$$

For $m \leqslant n$, The Wishart orthogonal/unitary/symplectic ensembles *of size (n,m) are the ensembles:*

$$W_\beta(n,m) = \{MM^*, M \in \text{Gauss}_\beta(n,m)\}.$$

For $m_1, m_2 \leqslant n$, the MANOVA orthogonal/unitary/symplectic ensembles *of size (n, m_1, m_2) are the ensembles:*

$$\text{MAN}_\beta(n,m_1,m_2) = \left\{ \frac{M_1}{M_1 + M_2}, M_1 \in W_\beta(n,m_1), M_2 \in W_\beta(n,m_2) \right\}.$$

The matrix coefficients of the Gaussian ensembles have a simple expression. They are i.i.d $\mathcal{N}(0,2)$ variables on the diagonal and i.i.d $\mathcal{N}(0,1)$ variables above the diagonal.

The following result connects the β-Hermite, β-Laguerre and β-Jacobi ensembles with the Gaussian, Wishart and MANOVA ensembles.

Theorem 1.2.1 (Eigenvalues density of $G_\beta(n)$, $W_\beta(n,m)$ and $\text{MAN}_\beta(n,m_1,m_2)$)**.**
For $\beta \in \{1,2,4\}$,

- *$\mathbb{H}_n^\beta$ is the joint eigenvalues density of $\frac{1}{\sqrt{n\beta}}M$, $M \in G_\beta(n)$,*
- *$\mathbb{L}_{n,m}^\beta$ is the joint eigenvalues density of $\frac{1}{m\beta}M$, $M \in W_\beta(n,m)$,*
- *$\mathbb{J}_{n,m_1,m_2}^\beta$ is the joint eigenvalues density of $M \in \text{MAN}_\beta(n,m_1,m_2)$.*

1.2.3 Matrix distributions for β-ensembles, $\beta \in \{1,2,4\}$

For $\beta \in \{1,2,4\}$, let H_n^β denote the set of self-adjoint matrices (i.e. real symmetric if $\beta = 1$, complex Hermitian if $\beta = 2$ and quaternionic self-dual if $\beta = 4$) and $\mathsf{H}_n^{\beta,+}$ denote the set of definite positive self-adjoint matrices. Another way to construct the β-Hermite and β-Laguerre ensembles for $\beta \in \{1,2,4\}$ is to equip H_n^β and $\mathsf{H}_n^{\beta,+}$ with proper probability measures. Such a result does not seem to exist for the β-Jacobi ensembles.

Theorem 1.2.2 (Matrix distributions for β-ensembles [2, Section 4.1])**.**
Set $\beta \in \{1, 2, 4\}$. The orthogonal (reps. unitary, symplectic) invariant probability measure on H_n^β

$$\mathbb{P}_n^H(\mathrm{d}M) = \frac{1}{Z_n^{V_H}} \exp\Big(-\frac{n\beta}{2}\mathrm{Tr}(V_H(M))\Big)\mathrm{d}M, \text{ where } Z_n^H \text{ is a normalizing constant,}$$

induces the law $\mathbb{H}_n^\beta$ on the eigenvalues of M.
The Wishart distribution *on $\mathsf{H}_n^{\beta,+}$*

$$\mathbb{P}_n^L(\mathrm{d}M) = \frac{1}{Z_n^{V_H}} |\det(M)|^{\frac{\beta}{2}(m-n+1)-1} \exp\Big(-\frac{n\beta}{2}\mathrm{Tr}(M)\Big)\mathrm{d}M, \text{ where } Z_n^L \text{ is a normalizing constant,}$$

induces the law $\mathbb{L}_{n,m}^\beta$ on the eigenvalues of M.

Remark 1.2.1 (Maxwell's theorem)*. The fact that matrices in $G_\beta(n)$ have normal coefficients can be read from $\mathbb{P}_n^H$ and is a consequence of Maxwell's theorem (see [59, Theorem 2.6.3]). A random matrix from H_n^β has in the same time independent entries and a law invariant by conjugacy with respect to orthognal (resp. unitary, symplectic) matrices if and only if it has a Gaussian law with density of the form $M \mapsto \exp\left(a\mathrm{Tr}(M^2) + b\mathrm{Tr}(M) + c\right)$.*

1.3 First order global asymptotics

1.3.1 Convergence of spectral measures

Definition 1.3.1 (Empirical spectral measure)**.**
For a random matrix M of size n with eigenvalues $\lambda_1, \ldots, \lambda_n$, we call $\mu(M)$ its spectral measure, where δ denotes the Dirac measure:

$$\mu(M) = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}.$$

The spectral measures of the β-Hermite, β-Laguerre and β-Jacobi ensembles are $\mu(H_n^\beta)$, $\mu(L_{n,m}^\beta (L_{n,m}^\beta)^T)$ and $\mu(J_{n,m_1,m_2}^\beta)$. We equip the set of probability measures on $\mathbb{R}$ with the weak convergence with respect to continuous and bounded test functions and we denote it $\xrightarrow{\text{weak}}$.
For $\beta \in \{1, 2, 4\}$, the almost sure convergences of $\mu(H_n^\beta)$ and $\mu(L_{n,m}^\beta (L_{n,m}^\beta)^T)$ to deterministic measures were historically established by Wigner [82] and Marchenko and Pastur [56] respectively. Using large deviations, these convergences can be extended to any $\beta > 0$. We state this result for the Hermite case.

Theorem 1.3.1 (Large deviation principle for H_n^β)**.** *[15, Theorem 1.1]*
The empirical measure $\mu_n = \mu(H_n^\beta)$ satisfies a large deviation principle *at scale n^2 with good rate function*

$$\mathcal{I}(\mu) = \mathcal{E}(\mu) - \mathcal{E}\left(\mu_{\mathrm{eq}}\right),$$

where μ_{eq} is the unique probability measure which minimizes:

$$\mathcal{E}(\mu) = \int V_H(x)\mathrm{d}\mu(x) - \iint \log|x-y|\mathrm{d}\mu(x)\mathrm{d}\mu(y), \; V_H(x) = x^2/2.$$

Theorem 1.3.1 implies that, for any open subset O of the set $\mathcal{M}^1(\mathbb{R})$ of measure probabilities on $\mathbb{R}$ and any closed set $F \in \mathcal{M}^1(\mathbb{R})$,

$$-\inf_O \mathcal{I}^V \leqslant \liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}(\mu_n \in O) \leqslant \limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}(\mu_n \in F) \leqslant -\inf_F \mathcal{I}^{V_H}.$$

As a result, for any open neighborhood $\mathcal{U}(\mu_{\mathrm{eq}})$ of μ_{eq},

$$\limsup_{n\to\infty} \frac{1}{n^2} \mathbb{H}_n^\beta\Big(\mu_n \in \mathcal{U}(\mu_{\mathrm{eq}})^c\Big) < 0.$$

A direct application of Borel-Cantelli's lemma then provides the almost-sure convergence of μ_n to μ_{eq}.

Theorem 1.3.1 remains valid for a wide range of potentials V defined on $\mathbb{R}$ that grow fast enough at infinity (see [10], [45] and the textbooks [9], [18]). Its proof was adapted to the β-Laguerre and β-Jacobi ensembles by Hiai and Petz to show large deviation principles for $\mu(L_{n,m}^\beta (L_{n,m}^\beta)^T)$ [40, Theorem 5.5.7] and $\mu(J_{n,m_1,m_2}^\beta)$ [41, Theorem 2.2]. The same argument as in the Hermite case shows the almost sure convergence of $\mu(L_{n,m}^\beta (L_{n,m}^\beta)^T)$ and $\mu(J_{n,m_1,m_2}^\beta)$ to their respective equilibrium measures. Identifying these equilibrium measures yields the following theorems (see [26] for a review), similar to laws of large numbers.

Theorem 1.3.2 (Spectral convergence of the β-Hermite ensemble)**.**

$$\mu(H_n^\beta) \xrightarrow[n\to\infty]{\textit{weak}} \mu_{\mathrm{sc}} \textit{ almost surely,}$$

where μ_{sc} is the semi-circle distribution*:*

$$\mu_{\mathrm{sc}}(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(x)\mathrm{d}x. \tag{1.3.1}$$

Theorem 1.3.3 (Spectral convergence of the β-Laguerre ensemble)**.**
In the regime (1.1.8),

$$\mu(L_{n,m}^\beta (L_{n,m}^\beta)^T) \xrightarrow[n\to\infty]{\textit{weak}} \mu_{\mathrm{MP}}^\gamma \textit{ almost surely,}$$

where μ_{MP}^γ is the Marchenko-Pastur distribution*:*

$$\begin{aligned} \mu_{\mathrm{MP}}^\gamma(\mathrm{d}x) &= \frac{1}{2\pi\gamma x}\sqrt{(\gamma_+ - x)(x-\gamma_-)}\mathbf{1}_{[\gamma_-,\gamma_+]}(x)\mathrm{d}x, \\ \gamma_\pm &= (1 \pm \sqrt{\gamma})^2. \end{aligned} \tag{1.3.2}$$

Theorem 1.3.4 (Spectral convergence of the β-Jacobi ensemble)**.**
In the regime (1.1.9),

$$\mu(J_{n,m_1,m_2}^\beta) \xrightarrow[n\to\infty]{\textit{weak}} \mu_{\mathrm{KM}}^{\gamma_1,\gamma_2} \textit{ almost surely,}$$

where $\mu_{\mathrm{KM}}^{\gamma_1,\gamma_2}$ is the generalized Kesten-McKay distribution*; which arises as the asymptotic distribution of large random d-regular graphs (see [49] and [57]):*

$$\begin{aligned} \mu_{\mathrm{KM}}^{\gamma_1,\gamma_2}(\mathrm{d}x) &= \frac{\gamma_1+\gamma_2}{2\pi}\frac{\sqrt{(x-u_-)(u_+-x)}}{x(1-x)}\mathbf{1}_{]u_-,u_+[}(x)\mathrm{d}x, \\ u_\pm &= \Big(\frac{\sqrt{\gamma_1(\gamma_1+\gamma_2-1)} \pm \sqrt{\gamma_2}}{\gamma_1+\gamma_2}\Big)^2. \end{aligned} \tag{1.3.3}$$

Figures 1, 2 and 3 illustrate Theorems 1.3.2, 1.3.3 and 1.3.4 in the case $\beta = 1$.

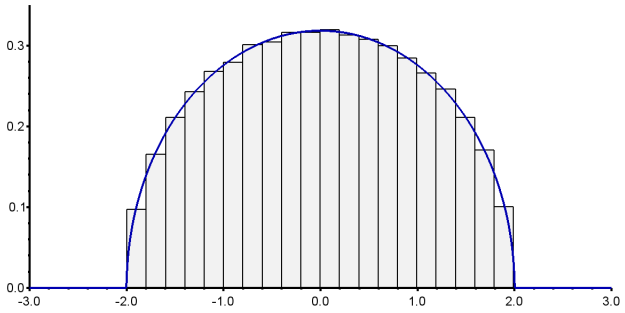

Figure 1: **Empirical eigenvalues density of $M \in G_1(10^4)$ (*black*); semi-circle distribution μ_{sc} (*blue*).**

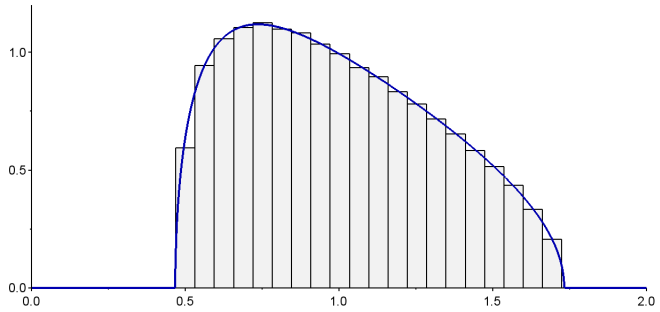

Figure 2: **Empirical eigenvalues density of $M \in W_1(10^3, 10^4)$ (*black*); Marchenko-Pastur distribution $\mu_{\mathrm{MP}}^{0.1}$ (*blue*).**

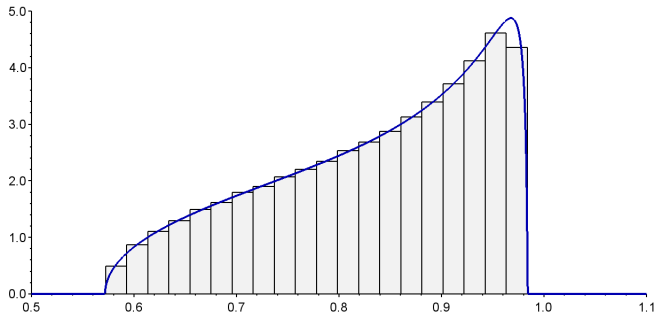

Figure 3: **Empirical eigenvalues density of $M \in M_1(10^3, 10^4, 2{\cdot}10^3)$ (*black*); Kesten-McKay distribution $\mu_{\mathrm{KM}}^{10,2}$ (*blue*).**

1.3.2 The objective method

The convergence of the spectral measures of β-ensembles can also be derived from the convergence of their matrix representations, viewing random matrices as adjacency matrices of randomly rooted graphs. This is an instance of the *objective method* pushed forward by Aldous and Steele in [4]. In this subsection we introduce the notion of randomly rooted graphs, define the Benjamini-Schramm convergence and use it to prove weaker versions of Theorem 1.3.2 and Theroem 1.3.3.

Benjamini-Schramm convergence

The Benjamini-Schramm convergence is defined in [1] by Abért, Thom and Virág. For an integer $D > 0$, let $\mathcal{G}_D$ denote the set of (isomorphism classes of) connected, undirected graphs where every vertex has at most D neighbors. Let $\mathcal{RG}_D$ denote the set of graphs G in $\mathcal{G}_D$ together with a distinguished vertex, called the root of G.
A *random rooted graph* of degree D is a Borel probability distribution on $\mathcal{RG}_D$. We say that a sequence of random rooted graphs G_n defined on $\mathcal{RG}_D$ converges to G if it converges in the weak topology. That is, if for every continous function $f : \mathcal{RG}_D \to \mathbb{R}$, we have

$$\lim_{n\to\infty} \int_{\mathcal{RG}_D} f(x)\ \mathrm{d}\lambda_n(x) = \int_{\mathcal{RG}_D} f(x)\mathrm{d}\lambda(x),$$

where λ_n is the distribution of G_n and λ is the distribution of G. The weak convergence of measures translates to convergence of the measures on the neighborhoods of the root in the following way. For a random rooted graph G, a finite rooted graph α and $k > 0$, let

$$\mathbf{P}(G, \alpha, k) = \mathbb{P}(B(o, k) \cong \alpha)$$

be the probability that the k-ball around the root of G is isomorphic to α. Then G_n converges to G if and only if, for all finite rooted graphs α and $k > 0$, we have

$$\lim_{n\to\infty} \mathbf{P}\left(G_n, \alpha, k\right) = \mathbf{P}(G, \alpha, k).$$

Any finite graph $G \in \mathcal{G}_D$ gives rise to a random rooted graph by assigning the root of G uniformly randomly. We denote the distribution of this random rooted graph by λ_G.

Definition 1.3.2 (Benjamini-Schramm convergence [1, Definition 1])**.**
A sequence of finite graphs $G_n \in \mathcal{G}_D$ is Benjamini-Schramm convergent if λ_{G_n} weakly converges, or, equivalently, if $P\left(G_n, \alpha\right)$ converges for every sample α. The local limit of G_n is defined as the weak limit of λ_{G_n}.

This convergence notion was historically introduced by Benjamini, Schramm [12] and Aldous and Lyons [3].

Tridiagonal matrices and Schrödinger operators

Let us interpret the tridiagonal representations of β-ensembles as adjacency matrices of random graphs. We follow the work of Bordenave, Caputo, Chafaï and Salez in [13] and introduce the symmetric matrix

$$T^{(n)} = \begin{pmatrix} a_1^{(n)} & b_1^{(n)} & 0 & \cdots & 0 \\ b_1^{(n)} & a_2^{(n)} & b_2^{(n)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_{n-2}^{(n)} & a_{n-1}^{(n)} & b_{n-1}^{(n)} \\ 0 & \cdots & \cdots & b_{n-1}^{(n)} & a_n^{(n)} \end{pmatrix}. \tag{1.3.4}$$

This is the adjacency matrix of a weighted finite chain graph with totally ordered vertices $V_n = \{1, 2, \ldots, n\}$. It is also the matrix representation of a linear operator on $A^{(n)}$ on $L^2(V_n)$ defined for all $f \in L^2(V_n)$ and all $i \in V_n$ by

$$\left(A^{(n)} f\right)(i) = \sum_{j \in V_n} T_{ij}^{(n)} f(j) = b_{i-1}^{(n)} f(i-1) + a_i^{(n)} f(i) + b_i^{(n)} f(i+1), \tag{1.3.5}$$

with convention $b_0^{(n)} = b_n^{(n)} = f(0) = f(n+1) = 0$.
We wish to study the $n \to \infty$ limit of these graphs, in the sense of the Benjamini-Schramm weak local convergence of Definition 1.3.2. We assume the coefficients a, b to be random but asymptotically deterministic, which encompasses the β-Hermite and β-Laguerre cases.

Theorem 1.3.5 (Asymptotically deterministic case [13])**.**
Let $G^{(n)}$ be the weighted graph with $n \geqslant 1$ vertices and adjacency operator $A^{(n)}$ given by (1.3.5), where $\left(a_i^{(n)}\right)_{1 \leqslant i \leqslant n}$ and $\left(b_i^{(n)}\right)_{1 \leqslant i \leqslant n}$ are real random vectors such that

$$a_i^{(n)} - a\left(\frac{i}{n}\right) \underset{\substack{n,i \to \infty \\ n-i \to \infty}}{\xrightarrow{\mathrm{d}}} 0 \quad \text{and} \quad b_i^{(n)} - b\left(\frac{i}{n}\right) \underset{\substack{n,i \to \infty \\ n-i \to \infty}}{\xrightarrow{\mathrm{d}}} 0, \quad 1 \leqslant i \leqslant n,$$

with $a, b : [0,1] \to \mathbb{R}$ deterministic and continuous functions which do not depend on n and i. Then, for the Benjamini-Schramm convergence,

$$\lim_{n \to \infty} G^{(n)} = G,$$

where G is the random graph with vertex set $\mathbb{Z}$, rooted at an arbitrary vertex, with adjacency operator acting on $f \in L^2(\mathbb{Z}, \mathbb{R})$ as

$$(Af)(i) = a(U) f(i) + b(U)\big(f(i-1) + f(i+1)\big), \quad i \in \mathbb{Z}, \quad U \sim \mathrm{Uniform}\big([0,1]\big).$$

This limiting operator A is a random real Schrödinger operator acting on $L^2(\mathbb{Z}, \mathbb{R})$ as

$$A(f) = b(U)(\Delta f) + \Phi(U) f,$$

with diffusion coefficient and multiplicative potential

$$b(U) \quad \text{and} \quad \Phi(U) = a(U) + 2b(U),$$

where Δ is the discrete Laplacian acting on $L^2(\mathbb{Z}, \mathbb{R})$ as

$$(\Delta f)(i) = f(i-1) + f(i+1) - 2f(i), \quad i \in \mathbb{Z}.$$

Proof. The argument relies on two probabilistic lemmas proved in the Appendices.

Lemma 1.3.1. *[13] For all $n \geqslant 1$ let i_n be uniformly distributed on $\{1, \dots, n\}$.*

$$\text{Then } i_n \xrightarrow[n\to\infty]{\mathbb{P}} +\infty, \quad n+1-i_n \xrightarrow[n\to\infty]{\mathbb{P}} +\infty, \quad \frac{i_n}{n} \xrightarrow[n\to\infty]{} \text{Uniform}\left([0,1]\right),$$

and we can couple the $(i_n)_{n\geqslant 1}$ by defining them on the same probability space in such a way that for a random variable $U \sim \text{Uniform}\left([0,1]\right)$ on this space,

$$\frac{i_n}{n} \xrightarrow[n\to\infty]{a.s.} U \quad \text{and in particular} \quad \frac{i_n}{n} \xrightarrow[n\to\infty]{\mathbb{P}} U.$$

Lemma 1.3.2. *[13] Let $\left(X^{(n)}\right)_{n\geqslant 1}$ be a sequence of random vectors of $\mathbb{R}^d$ and let $X = (X_1, \dots, X_d)$ be a random vector of $\mathbb{R}^d$. Then*

$$X^{(n)} \xrightarrow[n\to\infty]{\mathbb{P}} X \text{ if and only if } X_i^{(n)} \xrightarrow[n\to\infty]{\mathbb{P}} X_i \text{ for all } i \in \{1, \dots, d\}.$$

Let i_n be the label of a vertex chosen as being a root for $G^{(n)}$, uniformly distributed on $\{1, \dots, n\}$ and independent of $G^{(n)}$. Using Lemma 1.3.1, we can choose the probability space big enough to define $(a_i^{(n)})_{1\leq i\leq n}$ and $(b_i^{(n)})_{1\leq i\leq n}$ for all $n \geqslant 1$, as well as $(i_n)_{n\geqslant 1}$ independent of $\left(G^{(n)}\right)_{n\geqslant 1}$ and such that

$$\frac{i_n}{n} \xrightarrow[n\to\infty]{\mathbb{P}} U$$

for a random variable $U \sim \text{Uniform}\left([0,1]\right)$ also defined on this space. We have:

$$\varepsilon_{n,i} = a_i^{(n)} - a\left(\frac{i}{n}\right) \xrightarrow[\substack{n,i\to\infty \\ n-i\to\infty}]{\text{d}} 0 \text{ and } \quad \varepsilon'_{n,i} = b_i^{(n)} - b\left(\frac{i}{n}\right) \xrightarrow[\substack{n,i\to\infty \\ n-i\to\infty}]{\text{d}} 0.$$

Lemma 1.3.1 and the properties of $i_n, \varepsilon_{n,i}, a$, and b yield:

$$\left(A^{(n)}\right)_{i_n,i_n} = a\left(\frac{i_n}{n}\right) + \varepsilon_{n,i_n} \xrightarrow[n\to\infty]{\mathbb{P}} a(U) + 0 = a(U)$$

and, for $j_n = i_n + 1$ as well as for $j_n = i_n - 1$,

$$\left(A^{(n)}\right)_{i_n,j_n} = b\left(\frac{i_n \wedge j_n}{n}\right) + \varepsilon'_{n,i_n\wedge j_n} \xrightarrow[n\to\infty]{\mathbb{P}} b(U) + 0 = b(U).$$

Moreover, for all vertex j such that $|i_n - j| > 1$, we have $\left(A^{(n)}\right)_{i_n,j} = 0$. Using Lemma 1.3.2, for any depth d, the random rooted graph $\left(G^{(n)}, i_n\right)$ restricted to the ball $B\left(i_n, d\right)$ centered at i_n and of radius d converges in probability as $n \to \infty$ to the same restriction for G rooted at an arbitrary vertex. □

Note that there is a single U in this proof, which provides the randomness of G. This randomness is due only to the nature of the convergence, since the coefficients $a^{(n)}$ and $b^{(n)}$ are asymptotically deterministic.

Mean empirical distribution

For a random matrix M with (random) empirical spectral distribution $\mu(M)$, we define its *mean empirical spectral distribution*

$$(\mathbb{E}\mu(M))(B) = \mathbb{E}\left[\mu(M)(B)\right] = \frac{\mathbb{E}\left|\left\{1 \leqslant k \leqslant n : \lambda_k \in B\right\}\right|}{n}.$$

Theorem 1.3.6 (Spectral solvability of random Schrödinger operator [13])**.** *Let us consider the random Schrödinger operator $A = b(U)\Delta + a(U) + 2b(U)$ of Theorem 1.3.5, where $a, b : [0,1] \mapsto \mathbb{R}$ and $U \sim \text{Uniform}\left([0,1]\right)$. Then its mean spectral distribution is given by the uniform mixture of deformed arcsine laws*

$$\mathbb{E}\mu(A) = \left(\int_0^1 \frac{1_{x \in [a(u)-2b(u),\ a(u)+2b(u)]}}{\pi\sqrt{4b(u)^2 - \left(x - a(u)\right)^2}}\ \mathrm{d}u \right) \mathrm{d}x.$$

Moreover, for special choices of a and b given in the following table, this distribution is the Wigner semicircle law or the Marchenko-Pastur law.

$a(u), u \in [0,1]$	$b(u), u \in [0,1]$	$\mathbb{E}\mu_A$
0	1	Arcsine law $\frac{1}{\pi\sqrt{4-x^2}}\mathbf{1}_{x\in[-2,2]}\mathrm{d}x$
0	$\sqrt{u}$	Wigner semicircle law $\frac{\sqrt{4-x^2}}{2\pi}1_{x\in[-2,2]}\mathrm{d}x$
$1+\gamma-2\gamma u$	$\sqrt{\gamma(1-\gamma u)(1-u)}$	Marchenko-Pastur law $\frac{\sqrt{(x-\gamma_-)(\gamma_+-x)}}{2\pi\gamma x}\mathbf{1}_{x\in[\gamma_-,\gamma_+]}\mathrm{d}x$

Table 1: **Examples of mean empirical spectral distributions [13]**

Theorem 1.3.6 can be proved by noticing that the operator A is the sum of a Laplacian operator, exactly solvable using Fourier analysis, and a diagonal operator multiple of the identity. Specializing a and b and technical calculations gets us to the Arcsine, Wigner and Marchenko-Pastur laws. The continuity of the spectral measure for the weak local convergence of self-adjoint graphs (see [1]) yields the following convergence of mean spectral distributions.

Theorem 1.3.7 (Convergence of the mean empirical spectral distribution [13])**.**

Let $A^{(n)}$ and A be as in Theorem 1.3.5 and $\mathbb{E}\mu_A$ be as in Theorem 1.3.6. Then

$$\mathbb{E}\mu(A^{(n)}) \xrightarrow[n\to\infty]{weak} \mathbb{E}\mu(A).$$

Moreover, for special choices of a and b, $\mathbb{E}\mu_A$ is a Wigner semicircle law or a Marchenko-Pastur law (see Table 1).

Spectral convergence of the β-Hermite and β-Laguerre ensembles

A consequence of Theorem 1.3.7 is the convergence of the mean empirical spectral distributions of the β-Hermite and β-Laguerre ensembles.

Theorem 1.3.8 (Convergence of the β-Hermite ensemble [13])**.**
For $\beta > 0$,

$$\mathbb{E}\mu(H_n^\beta) \xrightarrow[n\to\infty]{weak} \mu_{\mathrm{sc}},$$

where μ_{sc} is the semi-circle distribution from (1.3.1).

Proof. Let $G^{(n)}$ be the random weighted graph with n vertices and adjacency matrix H_n^β. Let us show that the assumptions of Theorem 1.3.5 are satisfied. Keeping the notations of (1.3.4) for the coefficients of H_n^β, we have:

$$a_i^{(n)} \overset{\mathrm{d}}{=} \frac{\mathcal{N}(0,2)}{\sqrt{\beta n}} \xrightarrow[\substack{n,i\to\infty \\ n-i\to\infty}]{\mathrm{d}} 0, \quad b_i^{(n)} \overset{\mathrm{d}}{=} \frac{\chi_{(n-i)\beta}}{\sqrt{\beta n}} \overset{\mathrm{d}}{=} \frac{\chi_{(n-i)\beta}}{\sqrt{(n-i)\beta}} \sqrt{\left(1-\frac{i}{n}\right)}.$$

Now, if $Z \sim \chi_\alpha$ then $\frac{Z}{\sqrt{\alpha}} \xrightarrow[\alpha\to\infty]{\mathbb{P}} 1$, indeed $Z^2 \sim \chi_\alpha^2$ gives $\mathbb{E}\left[Z^2\right] = \alpha$ and $\mathrm{Var}\left[Z^2\right] = 2\alpha$, hence $\frac{Z^2}{\alpha} \xrightarrow[\alpha\to\infty]{L^2} 1$, which gives

$$\frac{\chi_{(n-i)\beta}}{\sqrt{(n-i)\beta}} \xrightarrow[\substack{n,i\to\infty \\ n-i\to\infty}]{\mathrm{d}} 1, \text{ and thus } \quad b_i^{(n)} - \sqrt{\left(1-\frac{i}{n}\right)} \xrightarrow[\substack{n,i\to\infty \\ n-i\to\infty}]{\mathrm{d}} 0.$$

Therefore, by Theorem 1.3.5 with $a = 0$ and $b(u) = \sqrt{(1-u)}, u \in [0,1]$, since $U \sim \mathrm{Uniform}\left([0,1]\right)$ implies $1-U \sim \mathrm{Uniform}\left([0,1]\right)$, we get that $G^{(n)}$ converges as $n \to \infty$ for the weak local convergence to the real Schrödinger operator acting on $f \in L^2(\mathbb{Z}, \mathbb{R})$ as

$$(Af)(i) = \sqrt{U}\Delta(f) + 2\sqrt{U}f(i), \quad i \in \mathbb{Z}.$$

Finally, the convergence of $\mathbb{E}\mu_{H_n^\beta}$ to μ_{sc} follows from Theorem 1.3.7. □

Similar calculations can be performed for the β-Laguerre ensemble via its bidiagonal representation $L_{n,m}^\beta$ from (1.1.5) and yield the following weak convergence.

Theorem 1.3.9 (Convergence of the β-Laguerre ensemble [13])**.**
For $\beta > 0$ and $\gamma \in (0,1]$,

$$\mathbb{E}\mu(L_{n,m}^\beta (L_{n,m}^\beta)^T) \xrightarrow[\substack{n,m\to\infty \\ n/m\to\gamma}]{weak} \mu_{\mathrm{MP}}^\gamma,$$

where μ_{MP}^γ is the Marchenko-Pastur distribution from (1.3.2).

1.4 Summary of results

β-ensemble	β-Hermite	β-Laguerre	β-Jacobi
Eigenvalues density	$\frac{1}{Z_n^{\beta,V_H}}\lvert\Delta(\boldsymbol{\lambda})\rvert^\beta \prod_{i=1}^n \exp\big(-\frac{n\beta}{4}\lambda_i^2\big)$	$\frac{1}{Z_n^{\beta,V_L}}\lvert\Delta(\boldsymbol{\lambda})\rvert^\beta \prod_{i=1}^n \lambda_i^\delta \exp\big(-n\frac{\beta}{2}\lambda_i\big)\mathbf{1}_{\mathbb{R}_+^*}(\lambda_i)$, $\delta = \frac{\beta}{2}(m-n+1)-1$	$\frac{1}{Z_n^{\beta,V_J}}\lvert\Delta(\boldsymbol{\lambda})\rvert^\beta \prod_{i=1}^n \lambda_i^{\delta_1}(1-\lambda_i)^{\delta_2}\mathbf{1}_{(0,1)}(\lambda_i)$, $\delta_1 = \frac{\beta}{2}(m_1-n+1)-1$, $\delta_2 = \frac{\beta}{2}(m_2-n+1)-1$
Matrix model	$\frac{1}{\sqrt{n\beta}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{(n-1)\beta} & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0,2) & \chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix}$	$\frac{1}{\sqrt{m\beta}}\begin{pmatrix} \chi_{m\beta} & & & \\ \chi_{(n-1)\beta} & \chi_{(m-1)\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{(m-n+1)\beta} \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \beta_2 & \\ & \ddots & \ddots & \ddots \\ & & \beta_{n-1} & \alpha_n \end{pmatrix}$ $\alpha_i,\ \beta_i$ defined recursively using Beta variables
Equilibrium measure	$\mu_{\mathrm{sc}}(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(x)\mathrm{d}x$	$\mu_{\mathrm{MP}}^\gamma(\mathrm{d}x) = \frac{\sqrt{(\gamma_+-x)(x-\gamma_-)}}{2\pi\gamma x}\mathbf{1}_{[\gamma_-,\gamma_+]}(x)\mathrm{d}x$ $\gamma_\pm = (1\pm\sqrt{\gamma})^2$	$\mu_{\mathrm{KM}}^{\gamma_1,\gamma_2}(\mathrm{d}x) = \frac{\gamma_1+\gamma_2}{2\pi}\frac{\sqrt{(x-u_-)(u_+-x)}}{x(1-x)}\mathbf{1}_{]u_-,u_+[}(x)\mathrm{d}x$ $u_\pm = \big(\frac{\sqrt{\gamma_1(\gamma_1+\gamma_2-1)}\pm\sqrt{\gamma_2}}{\gamma_1+\gamma_2}\big)^2$

Table 2: **Main properties of the β-Hermite, β-Laguerre and β-Jacobi ensembles.**

Chapter 2

Local statistics

When n tends to infinity, the β-Hermite, β-Laguerre and β-Jacobi ensembles exhibit locally a universal behavior: key limit statistics of the eigenvalues do not depend on the ensemble type but only on the position of the eigenvalues in the spectrum: in the *bulk*, at a *soft edge* or at a *hard edge*, thus forming three *universality classes*. Section 2.1 properly defines these zones of the spectrum. Section 2.2 studies the unitary case $\beta = 2$, for which universality results are obtained using the determinantal representation of the correlation functions. Finally, Section 2.3 presents universality results for general $\beta > 0$.

2.1 Zones of the spectrum

Consider an equilibrium measure $\mu_{\text{eq}} \in \{\mu_{\text{sc}}, \mu_{\text{MP}}^{\gamma}, \mu_{\text{KM}}^{\gamma_1,\gamma_2}\}$ and its support $\text{Supp}(\mu_{\text{eq}}) = [\mu_{\text{eq}}^-, \mu_{\text{eq}}^+]$, where:

$$\mu_{\text{eq}}^- = \inf\{x \in \mathbb{R},\ \mu_{\text{eq}}(x) > 0\},\ \mu_{\text{eq}}^+ = \sup\{x \in \mathbb{R},\ \mu_{\text{eq}}(x) > 0\}.$$

We have $\text{Supp}(\mu_{\text{sc}}) = [-2, 2],\ \text{Supp}(\mu_{\text{MP}}^{\gamma}) = [\gamma^-, \gamma^+],\ \text{Supp}(\mu_{\text{KM}}^{\gamma_1,\gamma_2}) = [u^-, u^+]$.

Definition 2.1.1 (Bulk, soft edge, hard edge)**.** *We call*

- $]\mu_{\text{eq}}^-, \mu_{\text{eq}}^+[$ *the* bulk *of* μ_{eq}*,*
- $\{\mu_{\text{eq}}^-, \mu_{\text{eq}}^+\}$ *the* edges *of* μ_{eq}*,*
- *a* soft edge *an edge where* μ_{eq} *vanishes continuously,*
- *a* hard edge *an edge where* μ_{eq} *diverges.*

At a soft edge, for finite n, there is a non-zero probability of an eigenvalue appearing on both sides of the edge. It applies when the density hits the horizontal axis and there is still 'wiggle room'. Conversely, the hard edge applies when the density meets a constraint, such as a positivity constraint, that forbids any eigenvalue to exist behind the edge, even for finite n.

Remark 2.1.1 (Spectrum zones in a general setting). Definition 2.1.1 holds for the β-Hermite, β-Laguerre and β-Jacobi ensembles. For measures supported on a finite union of intervals, a more general definition of the spectrum zones can be found in [2, Subsection 6.3.3].

2.1.1 The β-Hermite ensemble

The β-Hermite ensemble has two soft edges.

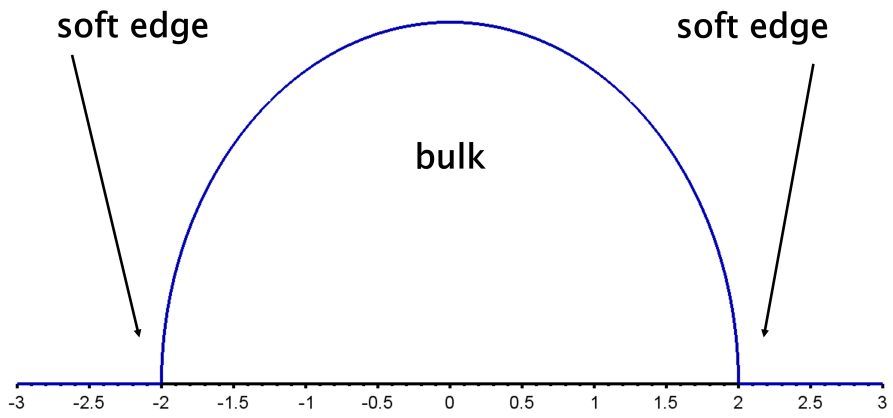

Figure 4: **Semi-circle distribution** μ_{sc}

2.1.2 The β-Laguerre ensemble

The β-Laguerre ensemble has a hard left edge at 0 if the parameter γ of the Marchenko-Pastur distribution is equal to 1 (i.e. $\lim m/n = 1$), and a soft left edge otherwise. The right edge is always a soft edge.

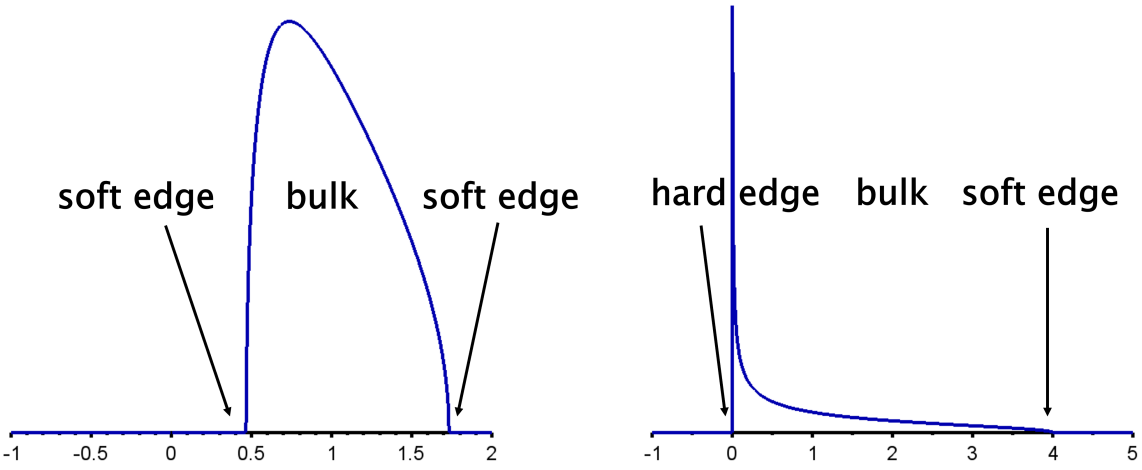

Figure 5: **Marchenko-Pastur distribution** $\mu_{\text{MP}}^{0.1}$ **(*left*) and** μ_{MP}^{1} **(*right*).**

2.1.3 The β-Jacobi ensemble

Appropriate tuning of the parameters can lead to any combination of soft-/hard upper/lower edges. The asymptotics of m_1 (resp. m_2) with respect to n determine the nature of the lower (resp. upper) edge. More precisely, if $m_1 - n \to a_J^1$, with $a_J^1 < \infty$, then the left edge is a hard edge, while if $m_1 - n \to \infty$ then the left edge is a soft edge. The same goes for the right edge.

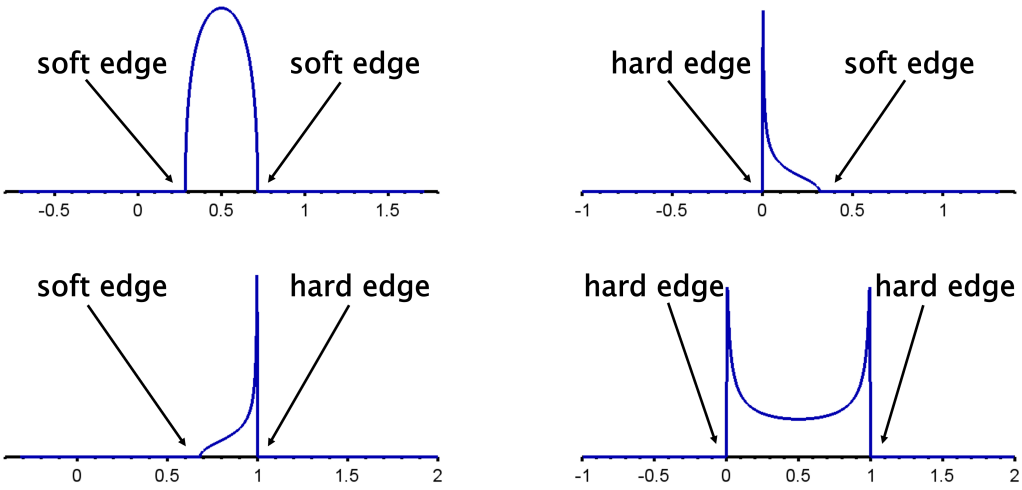

Figure 6: **Kesten-McKay distribution $\mu_{\mathrm{KM}}^{10,10}$ (*upper left*), $\mu_{\mathrm{KM}}^{1,10}$ (*upper right*), $\mu_{\mathrm{KM}}^{10,1}$ (*lower left*) and $\mu_{\mathrm{KM}}^{1,1}$(*lower right*).**

2.1.4 Parameters regimes

Definition 2.1.2 (Soft edge and hard edge regimes)**.**
In our study of the limit local statistics near an edge, we call soft edge regime *any of the regimes:*

$$\begin{aligned} \text{(Laguerre left edge)} \quad & n/m \to \gamma \in (0,1), \\ \text{(Jacobi left edge)} \quad & m_1/n \to \gamma_1 \in (1,\infty[, \\ \text{(Jacobi right edge)} \quad & m_2/n \to \gamma_2 \in (1,\infty[. \end{aligned} \tag{2.1.1}$$

Conversely, we call hard edge regime *any of the regimes:*

$$\begin{aligned} \text{(Laguerre left edge)} \quad & n/m \to \gamma = 1, \\ \text{(Jacobi left edge)} \quad & m_1/n \to \gamma_1 = 1, \\ \text{(Jacobi right edge)} \quad & m_2/n \to \gamma_2 = 1. \end{aligned} \tag{2.1.2}$$

Going forward, we restrict the hard edge regimes to the cases:

$$\begin{aligned} \text{(Laguerre left edge)} \quad & m - n \to a_L > -1, \\ \text{(Jacobi left edge)} \quad & m_1 - n \to a_J^1 > -1, \\ \text{(Jacobi right edge)} \quad & m_2 - n \to a_J^2 > -1. \end{aligned} \tag{2.1.3}$$

2.2 Unitary case $\beta = 2$

We start by studying the limit local statistics of eigenvalues in the unitary case. When $\beta = 2$, the correlation functions admit a useful determinantal representation. For the general theory on correlation functions, we refer the reader to the probabilistic survey [51] and to the handbook [2, Chapter 4].

2.2.1 Determinantal representation

Consider a β-ensemble $\mathbb{Q}_n^{\beta,V}$ from (1.1.1) and extend it to a permutation invariant probability measure $\hat{\mathbb{Q}}_n^{\beta,V}$.

Definition 2.2.1 (Correlation functions)**.**
For $k \in \mathbb{N}$, introduce the k-point correlation function*:*

$$R_n^{(k)}(x_1, \dots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} \hat{\mathbb{Q}}_n^{\beta,V}(x_1, \dots, x_n)\, \mathrm{d}x_{k+1}\mathrm{d}x_{k+2} \dots \mathrm{d}x_n,\ x_1, \dots, x_k \in \mathbb{R}.$$

For any measurable set $A \subset \mathbb{R}$, the quantity $\int_{A^k} R_{n,k}(x_1, \dots, x_k)\ \mathrm{d}x_1 \dots \mathrm{d}x_k$ is equal to the expected number of k-tupels $(\lambda_{i_1}, \dots, \lambda_{i_k})$ of distinct eigenvalues such that $\lambda_{i_j} \in A$ for all $j = 1, \dots, k$. In particular, $R_{n,1}(x)\mathrm{d}x$ is the expected number of particles in $\mathrm{d}x$. The main spectral statistics, such as the probability that a given number of eigenvalues lie in a given set, gap probabilities and eigenvalues spacing, can be expressed in terms of the k-point correlation functions.
In the unitary case, the orthogonal polynomial method provides a determinantal representation for the k-point correlation functions. The β-ensemble rewrites:

$$\mathbb{Q}_n^{2,V}(\mathrm{d}x_1, \dots, \mathrm{d}x_n) = \frac{1}{Z_n^{2,V}} |\Delta(\mathbf{x})|^2 \prod_{i=1}^{n} \exp\big(-nV(x_i)\big)\mathrm{d}x_i. \tag{2.2.1}$$

When all the moments of the measure $e^{-nV(x)}\mathrm{d}x$ are finite, we can define its family of orthogonal polynomials $(\pi_j)_{j\in\mathbb{N}}$:

$$\forall\ i, j \in \mathbb{N},\ \int_{\mathbb{R}} \pi_i(x)\pi_j(x)e^{-nV(x)}\ \mathrm{d}x = c_i c_j \delta_{i,j}.$$

With linear manipulation, the Vandermonde determinant can be expressed as a determinant with the monomials x_j replaced by the polynomials $\pi_j(x)$:

$$\Delta(\mathbf{x}) = \det\Big[\big(\pi_{j-1}(x_i)\big)_{i,j=1,\dots,n}\Big].$$

We now normalize the π_j to obtain an orthonormal system $(\phi_j)_{j\in\mathbb{N}}$ of $L^2(\mathbb{R})$ with respect to the Lebesgue measure. The functions $\phi_j(x) = \frac{1}{c_j} e^{-nV(x)/2}\pi_j(x)$ satisfy:

$$\forall\ i, j \in \mathbb{N},\ \int_{\mathbb{R}} \phi_i(x)\phi_j(x)\mathrm{d}x = \delta_{i,j}.$$

For the unitary Hermite, Laguerre and Jacobi ensembles, the associated orthogonal polynomials are (unsurprisingly) the Hermite, Laguerre and Jacobi polynomials (see [73] for a definition). Finally, we introduce the kernel K_n:

$$K_n(x,y) = \sum_{j=0}^{n-1} \phi_j(x)\phi_j(y),\ x,y \in \mathbb{R}. \tag{2.2.2}$$

The k-point correlation function $R_{n,k}$ admits the following fundamental determinantal representation.

Proposition 2.2.1 (Determinantal representation [51, Lemma 2.8])**.**
Fix $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$, then, for any $k \in \{1,\dots,n\}$,

$$R_n^{(k)}(x_1,\dots,x_k) = \det\left[\left(K_n(x_i,x_j)\right)_{i,j=1,\dots,k}\right].$$

In particular,

$$R_n^{(1)}(x_1) = K_n(x_1,x_1) \quad \text{and} \quad \hat{\mathbb{Q}}_n^{\beta,V}(\mathbf{x}) = \frac{1}{n!}\det\left[\left(K_n(x_i,x_j)\right)_{i,j=1,\dots,n}\right].$$

Remark 2.2.1. *In the orthgonal ($\beta = 1$) and symplectic ($\beta = 4$) settings, the determinant from Proposition 2.2.1 is replaced by a Pfaffian:*

$$R_n^{(k)}(x_1,\dots,x_k) = \mathrm{Pf}\left[\left(K_n(x_i,x_j)\right)_{i,j=1,\dots,k}\right].$$

2.2.2 Unitary universality classes

Fix a reference point x^* and a constant $c_n > 0$. We center the points around x^* and scale by a factor c_n, so that $(x_1,\dots,x_n)$ is mapped to

$$\left(c_n(x_1 - x^*),\dots,c_n(x_n - x^*)\right).$$

These centered and scaled points have the following rescaled k-point correlation functions

$$\frac{1}{c_n^k} R_n^{(k)}\left(x^* + \frac{x_1}{c_n}, x^* + \frac{x_2}{c_n}, \dots, x^* + \frac{x_k}{c_n}\right).$$

Universality at x^* means that, for a suitably chosen sequence $(c_n)_n$, the rescaled k-point correlation functions have a specific limit as $n \to \infty$. The precise limit determines the universality class. In the unitary case, this limit can be expressed as a $k \times k$ determinant

$$\det\left[K(x_i,x_j)\right]_{1\leqslant i,j\leqslant k},$$

where K is the *eigenvalue correlation kernel*. Therefore, the universality comes down to

$$\frac{1}{c_n} K_n\left(x^* + \frac{x}{c_n}, x^* + \frac{y}{c_n}\right) \underset{n\to\infty}{\longrightarrow} K(x,y). \tag{2.2.3}$$

Definition 2.2.2 (Airy and Bessel functions [55, Chapter 5])**.**
The Airy function *$Ai : \mathbb{R} \to \mathbb{R}$ is the unique solution of the differential equation*

$$f''(x) = xf(x)$$

with asymptotics $\frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}}$ *when* $x \to \infty$ *(see [73, Equation 1.81.4] for an exact form).*
The Bessel function J_a *of the first kind and of order* a *is the solution of the differential equation*

$$x^2\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2} + x\frac{\mathrm{d}y(x)}{\mathrm{d}x} + (x^2 - a^2)y(x) = 0$$

which is well-defined at $x = 0$ *(see [73, Equation 1.71.1] for an exact form).*

Definition 2.2.3 (Bulk unitary universality)**.**
A unitary β*-ensemble shows* bulk unitary universality *at a point* x^* *if (2.2.3) occurs with the* sine kernel $K = K^{\mathrm{Sine}}$*:*

$$K^{\mathrm{Sine}}(x,y) = \frac{\sin\pi(x-y)}{\pi(x-y)} = \frac{\sin(\pi x)\sin'(\pi y) - \sin'(\pi x)\sin(\pi y)}{\pi(x-y)},\ x,y \in \mathbb{R}. \tag{2.2.4}$$

Definition 2.2.4 (Soft edge unitary universality)**.**
A unitary β*-ensemble shows* soft edge unitary universality *at a point* x^* *if (2.2.3) occurs with the* Airy kernel $K = K^{\mathrm{Airy}}$*:*

$$K^{\mathrm{Airy}}(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}(y)\,\mathrm{Ai}'(x)}{x-y},\ x,y \in \mathbb{R}. \tag{2.2.5}$$

Definition 2.2.5 (Hard edge unitary universality)**.**
A unitary β*-ensemble shows* hard edge unitary universality *at a point* x^* *if (2.2.3) occurs with the* Bessel kernel $K = K^{\mathrm{Bessel}}$*:*

$$K^{\mathrm{Bessel}}(x,y) = \frac{J_a(\sqrt{x})\sqrt{y}J_a'(\sqrt{y}) - J_a'(\sqrt{x})\sqrt{x}J_a(\sqrt{y})}{2(x-y)},\ x,y \in (0,\infty[. \tag{2.2.6}$$

The properties of Ai and J_a can be found in [55, Chapter 5].

Universality unitary classes were first discovered for classical β-ensembles. In particular, the Hermite, Laguerre and Jacobi unitary ensembles show bulk unitary universality in their bulks and soft/hard edge unitary universality at their edges. These results are treated for instance in [33], [59] for the Hermite and Laguerre unitary ensemble and in [60], [46] for the Jacobi unitary ensemble.

Remark 2.2.2 (Orthognal and symplectic universality classes)*. The Hermite, Laguerre and Jacobi orthogonal and symplectic ensembles also show bulk, soft edge and hard edge universality classes, as Pfaffian point processes instead of determinantal point processes (see [20] for the bulk and [19], [21] for the edges).*

2.3 General case $\beta > 0$

We now extend our quest for universality classes to the general case $\beta > 0$. We notate $\lambda_0^+(\mathbb{Q}_n^{\beta,V}) \geqslant \ldots \geqslant \lambda_k^+(\mathbb{Q}_n^{\beta,V})$ (resp. $\lambda_0^-(\mathbb{Q}_n^{\beta,V}) \leqslant \ldots \leqslant \lambda_k^-(\mathbb{Q}_n^{\beta,V})$) the $(k+1)^{st}$ highest (resp. lowest) eigenvalues of the β-ensemble $\mathbb{Q}_n^{\beta,V}$, for $k \leqslant n$. Large deviation techniques show that the extremal eigenvalues $\lambda_0^+(\mathbb{Q}_n^{\beta,V})$ and $\lambda_0^-(\mathbb{Q}_n^{\beta,V})$ converge to the corresponding edges of the equilibrium measure.

Theorem 2.3.1 (Convergence of the extremal eigenvalues [9, Theorem 2.6.6])**.**
Almost surely,

$$\begin{aligned}
\lambda_0^+(\mathbb{H}_n^\beta) &\underset{n\to\infty}{\longrightarrow} \mu_{\mathrm{sc}}^+ = 2,\\
\lambda_0^-(\mathbb{H}_n^\beta) &\underset{n\to\infty}{\longrightarrow} \mu_{\mathrm{sc}}^- = -2,\\
\lambda_0^+(\mathbb{L}_{n,m}^\beta) &\underset{n\to\infty}{\longrightarrow} \mu_{\mathrm{MP}}^{\gamma,+} = \gamma^+,\\
\lambda_0^-(\mathbb{L}_{n,m}^\beta) &\underset{n\to\infty}{\longrightarrow} \mu_{\mathrm{MP}}^{\gamma,-} = \gamma^-,\\
\lambda_0^+(\mathbb{J}_{n,m_1,m_2}^\beta) &\underset{n\to\infty}{\longrightarrow} \mu_{\mathrm{KM}}^{\gamma_1,\gamma_2,+} = u^+,\\
\lambda_0^-(\mathbb{J}_{n,m_1,m_2}^\beta) &\underset{n\to\infty}{\longrightarrow} \mu_{\mathrm{KM}}^{\gamma_1,\gamma_2,-} = u^-,
\end{aligned}$$

where γ^+, γ^-, u^+ and u^- are from (1.3.2) and (1.3.3).

Like in the orthogonal, unitary and symplectic cases (see Section 2.2), the limit statistics of the eigenvalues for general $\beta > 0$ show three universality classes depending on the zone of the spectrum: bulk universality, soft edge universality and hard edge universality. Since the determinantal or Pfiaffan structures are lost, new objects are needed to describe the local limit of the β-ensembles. These objective limits (see Chapter 1, Subsection 1.3.2) are given by stochastic operators.

2.3.1 Bulk universality

The statement of bulk universality involves the Sine_β *point process*, defined by Valkó and Virág [77]. The same authors later constructed the stochastic Sine_β operator, whose eigenvalues are distributed as this point process [78]. We reserve the introduction of the Sine_β operator for Chapter 3, Section 3.1 and until then we refer to [77, Section 2] for the construction and the properties of the Sine_β *point process.*

Definition 2.3.1 (Bulk universality)**.**
A β-ensemble shows bulk universality *when its eigenvalues, properly centered and rescaled, converge in law to the* Sine_β *point process, with respect to vague topology for the counting measure of the point process.*

Valkó and Virág proved bulk universality for the β-Hermite ensemble [77]. It was extended to the β-Laguerre ensemble by Jacquot and Valkò [44]. Bulk universality for the β-Jacobi ensemble is expected but does not appear to have been proven.

Theorem 2.3.2 (Bulk universality for $\mathbb{H}_n^\beta$ and $\mathbb{L}_{n,m}^\beta$)**.**
Set $\beta > 0$ and $|z| < 2$. Let $c \in (\gamma^-, \gamma^+)$. Then

$$\begin{aligned}
n\sqrt{4-z^2}\,(\lambda(\mathbb{H}_n^\beta) - z) &\xrightarrow[n\to\infty]{\mathrm{d}} \mathsf{Sine}_\beta,\\
\frac{\sqrt{(\gamma^+ - c)(c - \gamma^-)}}{c}\,(\lambda(\mathbb{L}_{n,m}^\beta) - cn) &\xrightarrow[n\to\infty]{\mathrm{d}} \mathsf{Sine}_\beta,
\end{aligned}$$

where $\lambda(\mathbb{H}_n^\beta)$ and $\lambda(\mathbb{L}_{n,m}^\beta)$ denote the point processes associated to $\mathbb{H}_n^\beta$ and $\mathbb{L}_{n,m}^\beta$.

2.3.2 Soft edge universality

The statement of *soft edge universality* involves the *stochastic Airy operator*, notated $\mathcal{A}^\beta$, properly defined in Chapter 3, Section 3.1. For each integer $k \geqslant 0$, let $\Lambda_k(\mathcal{A}^\beta)$ be its $(k+1)^{st}$ lowest eigenvalue.

Definition 2.3.2 (Soft edge universality)**.**
A β-ensemble shows soft edge universality *when its eigenvalues, properly centered and rescaled, converge in law to the lowest eigenvalues of the stochastic Airy operator $\mathcal{A}^\beta$.*

Ramírez, Rider and Virag [64] proved soft edge universality for the β-Hermite and β-Laguerre ensembles, Holcomb and Moreno Flores [42] for the β-Jacobi ensemble. Krishnapur, Rider and Virág [53] extended it to a larger class of β-ensembles for which the potential V from Definition 1.1.1 satisfies convexity assumptions.

Theorem 2.3.3 (Soft edge universality for $\mathbb{H}_n^\beta$, $\mathbb{L}_{n,m}^\beta$ and $\mathbb{J}_{n,m_1,m_2}^\beta$)**.**
Set $\beta > 0$. In the soft edge regime (2.1.1), for any $k \in \mathbb{N}$,

$$\begin{aligned}
\Big(\sigma_n^H(\mu_{\mathrm{sc}}^+ - \lambda_j^+(\mathbb{H}_n^\beta))\Big)_{0\leqslant j\leqslant k} &\xrightarrow[n\to\infty]{\mathrm{d}} (\Lambda_0(\mathcal{A}^\beta),\dots,\Lambda_k(\mathcal{A}^\beta)),\\
\Big(\sigma_n^H(\lambda_j^-(\mathbb{H}_n^\beta) - \mu_{\mathrm{sc}}^-)\Big)_{0\leqslant j\leqslant k} &\xrightarrow[n\to\infty]{\mathrm{d}} (\Lambda_0(\mathcal{A}^\beta),\dots,\Lambda_k(\mathcal{A}^\beta)),\\
\Big(\sigma_{m,n}^L(\mu_{\mathrm{MP}}^{\gamma,+} - \lambda_j^+(\mathbb{L}_{n,m}^\beta))\Big)_{0\leqslant j\leqslant k} &\xrightarrow[n\to\infty]{\mathrm{d}} (\Lambda_0(\mathcal{A}^\beta),\dots,\Lambda_k(\mathcal{A}^\beta)),\\
\Big(\sigma_{m,n}^L(\lambda_j^-(\mathbb{L}_{n,m}^\beta) - \mu_{\mathrm{MP}}^{\gamma,-})\Big)_{0\leqslant j\leqslant k} &\xrightarrow[n\to\infty]{\mathrm{d}} (\Lambda_0(\mathcal{A}^\beta),\dots,\Lambda_k(\mathcal{A}^\beta)),\\
\Big(\sigma_{m_1,m_2,n}^J(\mu_{\mathrm{KM}}^{\gamma_1,\gamma_2,+} - \lambda_j^+(\mathbb{J}_{n,m_1,m_2}^\beta))\Big)_{0\leqslant j\leqslant k} &\xrightarrow[n\to\infty]{\mathrm{d}} (\Lambda_0(\mathcal{A}^\beta),\dots,\Lambda_k(\mathcal{A}^\beta)),\\
\Big(\sigma_{m_1,m_2,n}^J(\lambda_j^-(\mathbb{J}_{n,m_1,m_2}^\beta) - \mu_{\mathrm{KM}}^{\gamma_1,\gamma_2,-})\Big)_{0\leqslant j\leqslant k} &\xrightarrow[n\to\infty]{\mathrm{d}} (\Lambda_0(\mathcal{A}^\beta),\dots,\Lambda_k(\mathcal{A}^\beta)).
\end{aligned}$$

with $\sigma_n^H = n^{2/3}, \sigma_{m,n}^L = \frac{m(\sqrt{nm})^{1/3}}{(\sqrt{n}+\sqrt{m})^{4/3}}$ and $\sigma_{m_1,m_2,n}^J$ explicited in [42, Theorem 4].

The law of $\Lambda_0(\mathcal{A}^\beta)$ is called the *Tracy-Widom distribution* TW_β. Historically, Tracy and Widom found TW_2 [74] and then TW_1, TW_4 [75] as the limiting law of the rescaled largest eigenvalue of an n by n Gaussian unitary (resp. orthogonal, symplectic) matrix.

2.3.3 Hard edge universality

The statement of *hard edge universality* involves the *stochastic Bessel operator* of parameter a, notated $\mathfrak{G}_a^\beta$, properly defined in Chapter 3, Section 3.1. For each integer $k \geqslant 0$, let $\Lambda_k(\mathfrak{G}_a^\beta)$ its $(k+1)^{st}$ lowest eigenvalue.

Definition 2.3.3 (Hard edge universality)**.**
A sequence of β-ensembles $(\mathbb{Q}_n^{\beta,V})_n$ has hard edge universality *when its eigenvalues, properly centered and rescaled, converge in law to the lowest eigenvalues of the stochastic Bessel operator $\mathfrak{G}_a^\beta$.*

Ramírez and Rider [63] proved hard edge universality for the β-Laguerre ensemble, Holcomb and Moreno Flores [42] for the β-Jacobi ensemble. Rider and Waters [65] extended it to a large class of β-ensembles for which the potential V from Definition 1.1.1 satisfies convexity assumptions.
Note that, in the hard edge regime (2.1.3), we have $\mu_{\mathrm{MP}}^{\gamma,-} = 0$, $\mu_{\mathrm{KM}}^{\gamma_1,\gamma_2,-} = 0$ and $\mu_{\mathrm{MP}}^{\gamma_1,\gamma_2,+} = 1$.

Theorem 2.3.4 (Hard edge universality for $\mathbb{L}_{n,m}^{\beta}$ and $\mathbb{J}_{n,m_1,m_2}^{\beta}$)**.**
For $\beta > 0$ and any $k \in \mathbb{N}$,

$$\Big(nm(\lambda_j^-(\mathbb{L}_{n,m}^{\beta}) - \mu_{\mathrm{MP}}^-)\Big)_{0\leqslant j\leqslant k} \xrightarrow[n\to\infty]{\mathrm{d}} (\Lambda_0(\mathfrak{G}_{a_L}^{\beta}), \dots, \Lambda_k(\mathfrak{G}_{a_L}^{\beta})),$$
$$\Big(nm_1(\mu_{\mathrm{KM}}^{\gamma_1,\gamma_2,+} - \lambda_j^+(\mathbb{J}_{n,m_1,m_2}^{\beta}))\Big)_{0\leqslant j\leqslant k} \xrightarrow[n\to\infty]{\mathrm{d}} (\Lambda_0(\mathfrak{G}_{a_J^2}^{\beta}), \dots, \Lambda_k(\mathfrak{G}_{a_J^2}^{\beta})),$$
$$\Big(nm_2(\lambda_j^-(\mathbb{J}_{n,m_1,m_2}^{\beta}) - \mu_{\mathrm{KM}}^{\gamma_1,\gamma_2,-})\Big)_{0\leqslant j\leqslant k} \xrightarrow[n\to\infty]{\mathrm{d}} (\Lambda_0(\mathfrak{G}_{a_J^1}^{\beta}), \dots, \Lambda_k(\mathfrak{G}_{a_J^1}^{\beta})).$$

2.3.4 Summary of results

β-ensemble	Universality	Limit correlation kernel ($\beta = 2$)	Limit point process ($\beta > 0$)
Hermite	bulk	K^{Sine}	Sine_β
Hermite	soft edge	K^{Airy}	lowest eigenvalues of $\mathcal{A}^\beta$
Laguerre	bulk	K^{Sine}	Sine_β
Laguerre	soft edge	K^{Sine}	lowest eigenvalues of $\mathcal{A}^\beta$
Laguerre	hard left edge	K^{Bessel}	lowest eigenvalues of $\mathfrak{G}_{a_L}^{\beta}$
Jacobi	bulk	K^{Sine}	Sine_β
Jacobi	soft edge	K^{Airy}	lowest eigenvalues of $\mathcal{A}^\beta$
Jacobi	hard left edge	K^{Bessel}	lowest eigenvalues of $\mathfrak{G}_{a_J^1}^{\beta}$
Jacobi	hard right edge	K^{Bessel}	lowest eigenvalues of $\mathfrak{G}_{a_J^2}^{\beta}$

Table 3: **Universality and limit objects for the β-ensembles.**

Chapter 3

From matrices to operators

Stochastic operators emerge as objective limits for β-ensembles in the study of universality classes for local eigenvalues statistics (see Chapter 2, Section 2.3). This chapter studies the connection of the Sine_β operator, the stochastic Airy operator and the stochastic Bessel operator to the β-Hermite, β-Laguerre and β-Jacobi ensembles. Section 3.1 properly defines these operators. Section 3.2 explains heuristically how they emerge in the $n \to \infty$ limit of the β-ensembles by interpreting matrices as finite difference schemes. Sections 3.3 and 3.4 make this idea rigorous and prove respectively the hard edge universality for the β-Laguerre ensemble and a Feynman-Kac formula for the stochastic Airy operator at the soft edge. Section 3.5 presents an operator level transition between hard and soft edge.

We now work in the hard edge regime (2.1.3), where the left edge of the β-Laguerre ensemble and both edges of the β-Jacobi ensemble are hard edges. For the the β-Laguerre ensemble, we consider only the case $m - n = a_L > -1$ and we notate $a = a_L$.

3.1 Stochastic Operators

3.1.1 The Sine_β operator

The Sine_β operator was introduced by Valkó and Virág [78] as a type of *Dirac operator*. The definition and properties of Dirac operators can be found in [80], those of the Brownian motion on the hyperbolic plane in [34].

Definition 3.1.1 (Dirac operator [78, Section 3.3])**.**
A Dirac operator τ *is a differential operator of the form*

$$\tau v(t) = R^{-1}(t) J v'(t),$$

where $R(t)$ is an integrable strictly positive definite real 2×2 matrix valued function, $v(t)$ a $\mathbb{R}^2$-valued function and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$R(t)$ has a unique representation in the form of

$$R = \frac{f}{y}\begin{pmatrix} 1 & -x \\ -x & x^2+y^2 \end{pmatrix},$$

where $f = \sqrt{\det R}, y = \frac{f}{R_{1,1}}$ and $x = -\frac{R_{1,2}}{R_{1,1}}$.
The notation $\mathrm{Dir}\,(x+iy, u_0/u_1, f)$ *denotes the Dirac operator τ with ratio of boundary conditions u_0/u_1.*

Definition 3.1.2 (Hyperbolic Brownian motion [78, Definition 3])**.**
In the hyperbolic half-plane model, the standard hyperbolic Brownian motion *is the solution of the stochastic differential equation*

$$\mathrm{d}\mathcal{B} = \Im\mathcal{B}\ \mathrm{d}Z,$$

where Z is standard complex Brownian motion, i.e. $\Im Z, \Re Z$ are independent standard real Brownian motions. If we replace $\mathrm{d}Z$ by $\sigma\mathrm{d}Z$, we get the hyperbolic Brownian motion with variance σ^2.

Theorem 3.1.1 (The Sine_β operator [78, Theorems 25 and 26])**.**
Fix $\beta > 0$. Let $x+iy$ be the hyperbolic Brownian motion 3.1.2 with variance $\frac{4}{\beta}$ in the half plane with initial condition i. Let $q = \lim_{t\to\infty} x(t)$, and use the notation $\tilde{x}(t) = x(-\log(1-t)), \tilde{y}(t) = y(-\log(1-t))$. Then the Sine_β operator

$$\mathrm{Sine}_\beta = \mathrm{Dir}(\tilde{x}+i\tilde{y}, \infty, q)$$

has its spectrum distributed as the Sine_β process.

We refer to [78] for the properties of the Sine_β operator and the proof of bulk universality, which are beyond the scope of this thesis.

3.1.2 Generalized Sturm-Liouville operators

The proper framework to introduce the *stochastic Airy operator* and the *stochastic Bessel operator* is that of generalized *Sturm-Liouville (S-L) operators*. General theory on these operators can be found in [80], its application to the *stochastic Airy operator* and the *stochastic Bessel operator* in [23, Section 2].

Definition 3.1.3 (Generalized Sturm-Liouville operators)**.**
Generalized Sturm-Liouville operators *are differential operators τ of the form*

$$\tau u(x) = \frac{1}{r(x)}\Big(-\big(p_1(x)u'(x) - q_0(x)u(x)\big)' - q_0(x)u'(x) + p_0(x)u(x)\Big), \quad (3.1.1)$$

where u is a real valued function on $[0,L]$ for some $L > 0$ or on $\mathbb{R}_+$(which we consider to be the $L = \infty$ case in the following). We assume that the real functions p_0, p_1, q_0, r are continuous on $[0,\infty[$ and $r(x), p_1(x) > 0$ for $x \geqslant 0$.

We introduce

$$\mathcal{D}_L(\tau) = \Big\{u \in L^2\big([0,L],r\big) : \tau u \in L^2\big([0,L],r\big) \text{ and } u, p_1u' - q_0u \in \mathrm{AC}\big([0,L]\big)\Big\},$$

where $\mathrm{AC}\,([0,L])$ is the set of absolutely continuous real functions on $[0,L]$.

Theorem 3.1.2 (Self-adjoint version of τ [23, Propositions 4 and 5])**.**
Let $L \in (0, \infty]$, assume that τ is of the form (3.1.1) and that it satisfies the following conditions:

- *the solution $u_{\mathbf{d}}$ of the equation $\tau u_{\mathbf{d}} = 0$ with Dirichlet initial conditions $u_{\mathbf{d}}(0) = 0$, $u'_{\mathbf{d}}(0) = 1$ is not in $L^2(\mathbb{R}_+, r)$, i.e. $\int_0^\infty u_{\mathbf{d}}^2(x) r(x) \mathrm{d}x = \infty$,*
- *there is a unique solution u_∞ of the equation $\tau u_\infty = 0$, with initial condition $u_\infty(0) = 1$ that is in $L^2(\mathbb{R}_+, r)$,*
- *$\int_0^\infty \int_0^x u_\infty(x)^2 u_{\mathrm{d}}(y)^2 r(x) r(y) \mathrm{d}y \mathrm{d}x < \infty$.*

Then, there is a self-adjoint τ_L version of the operator on $[0, L]$ with Dirichlet boundary conditions on the domain

$$\mathcal{D}_{L,0}(\tau) = \mathcal{D}_L(\tau) \cap \{u : u(0) = 0, u(L) = 0\}, \tag{3.1.2}$$

where the end condition $u(L) = 0$ is dropped in the case $L = \infty$. We denote this self-adjoint operator by τ_L.
Furthermore, the inverse operator τ_L^{-1} is an integral Hilbert-Schmidt operator in $L^2([0, L], r)$, and it has a bounded pure point spectrum.

3.1.3 The stochastic Airy operator

Stochastic Airy and Bessel operators are randomized versions of known differential operators, which conversely can be seen as '$\beta = \infty$' versions of their stochastic counterparts.

Definition 3.1.4 (Airy operator [30, Definition 3.1])**.**
The Airy operator *is the second order differential operator*

$$\mathcal{A}^\infty = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x, \tag{3.1.3}$$

acting on $f \in L^2((0, \infty[)$, with the boundary conditions

$$f(0) = 0 \text{ and } \lim_{x \to \infty} f(x) = 0. \tag{3.1.4}$$

.

Let $\mathcal{D}(\mathcal{A}^\infty[$ be the subspace of $L^2((0, \infty[)$ of functions f verifying (3.1.4). An eigenvalue/eigenfunction pair of $\mathcal{A}^\infty$ is $(\lambda, \psi) \in \mathbb{R} \times \mathcal{D}(\mathcal{A}^\infty[$ such that $\|\psi\|_2 = 1$ and $\mathcal{A}^\infty \psi = \lambda \psi_k$.

Proposition 3.1.1 (Eigenvalue decomposition for $\mathcal{A}^\infty$ [30, Section 3.1])**.**
The eigenvalue decomposition of $\mathcal{A}^\infty$ is:

$$\forall k \in \mathbb{N}, \ (\mathcal{A}^\infty - \lambda_k) \mathrm{Ai}(-x - \lambda_k) = 0, \tag{3.1.5}$$

with λ_k the $(k+1)^{st}$ zero of Ai *from Definition (2.2.2).*

Definition 3.1.5 (Stochastic Airy operator [64, Section 1])**.**
Set $\beta > 0$ and let B' be a standard white noise on $\mathbb{R}_+$ (see [54, Chapter 1]). The stochastic Airy operator*, known as SAO, is the random differential operator*

$$\mathcal{A}^\beta = \mathcal{A}^\infty + \frac{2}{\sqrt{\beta}}B' \text{ i.e. } \mathcal{A}^\beta = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x + \frac{2}{\sqrt{\beta}}B', \tag{3.1.6}$$

defined on a subset of $L^2(\mathbb{R}_+)$ with Dirichlet boundary conditions at 0.

Proposition 3.1.2 ($\mathcal{A}^\beta$ as a generalized S-L operator [23, Proposition 9])**.**
The operator $\mathcal{A}^\beta$ is a generalized Sturm-Liouville operator of the form (3.1.1) with

$$r(x) = p_1(x) = 1,\ q_0(x) = \frac{2}{\sqrt{\beta}}B(x),\ p_0(x) = x.$$

It satisfies the hypothesis of Theorem 3.1.2 with probability one and we still notate $\mathcal{A}^\beta$ its self-adjoint version on $\mathcal{D}(\mathcal{A}^\beta) = \mathcal{D}_{\infty,0}(\mathcal{A}^\beta)$ from (3.1.2).
If ψ solves the equation $\mathcal{A}^\beta\psi = 0$ with initial conditions $\psi(0) = c_0, \psi'(0) = c_1$, $(c_0, c_1) \neq (0,0)$, then (ψ, ψ') is the strong solution of the stochastic differential equation system

$$\mathrm{d}\psi(x) = \psi'(x)\mathrm{d}x, \quad \mathrm{d}\psi'(x) = \psi(x)\left(\frac{2}{\sqrt{\beta}}\mathrm{d}B + x\mathrm{d}x\right),$$

which is well defined for all times.

Proposition 3.1.3 (Pure point spectrum of $\mathcal{A}_\beta$ [64, Theorem 1.1])**.**
With probability one, for each integer $k \geqslant 0$, the set of eigenvalues of $\mathcal{A}^\beta$ has a well-defined $(k+1)^{st}$ lowest element $\Lambda_k(\mathcal{A}^\beta)$.

Definition 3.1.6 (Eigenvalue problem for $\mathcal{A}^\beta$ [64, Section 1])**.**
An eigenvalue/eigenfunction pair of $\mathcal{A}^\beta$ is $(\lambda, \psi) \in \mathbb{R}\times\mathcal{D}(\mathcal{A}^\beta)$ such that $\|\psi\|_2 = 1$ and $\mathcal{A}^\beta\psi = \lambda\psi$, i.e.

$$\psi''(x) = \frac{2}{\sqrt{\beta}}\psi(x)B'(x) + (x-\lambda)\psi(x), \tag{3.1.7}$$

which holds in the integration by parts sense:

$$\psi'(x) - \psi'(0) = \frac{2}{\sqrt{\beta}}\psi(x)B(x) + \int_0^x -\frac{2}{\sqrt{\beta}}B(y)\psi'(y)\mathrm{d}y + \int_0^x (y-\lambda_k)\psi(y)\mathrm{d}y.$$

3.1.4 The stochastic Bessel operator

Definition 3.1.7 (Bessel operator [30, Definition 3.3])**.**
The Bessel operator *parametrized by $a > -1$ is the first order differential operator*

$$\mathcal{J}_a^\infty = -2\sqrt{x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{a}{\sqrt{x}}, \tag{3.1.8}$$

acting on $L^2([0,1])$ with the boundary conditions

$$f(1) = 0 \text{ and } (\mathcal{J}_a^\infty f)(0) = 0. \tag{3.1.9}$$

Let $\mathcal{D}(\mathcal{J}_a^\infty)$ be the subspace of $L^2([0,1])$ of functions f verifying (3.1.9). The adjoint of the Bessel operator is

$$(\mathcal{J}_a^\infty)^* = 2\sqrt{x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{a+1}{\sqrt{x}},$$

with same domain and boundary conditions.

Proposition 3.1.4 (Singular value decomposition of $\mathcal{J}_a^\infty$ [30, Section 3.1])**.**
The singular value decomposition of $\mathcal{J}_a^\infty$ and $(\mathcal{J}_a^\infty[^*$ is:

$$\begin{aligned}\mathcal{J}_a^\infty\big(J_a(\lambda_k\sqrt{x})\big) &= \lambda_k J_{a+1}(\lambda_k\sqrt{x}),\\ (\mathcal{J}_a^\infty)^*\big(J_{a+1}(\lambda_k\sqrt{x})\big) &= \lambda_k J_a(\lambda_k\sqrt{x}),\end{aligned}$$

for $k\in\mathbb{N}$ and with $0<\lambda_0<\lambda_1<\dots$ the zeros of J_a from Definition (2.2.2).

Definition 3.1.8. *Set $\beta>0$, $a>-1$ and let B' be a standard white noise on $[0,1]$. Define the stochastic operator acting on $L^2([0,1])$:*

$$\mathcal{J}_a^\beta = \mathcal{J}_a^\infty + \frac{2}{\sqrt{\beta}}B' \text{ i.e. } \mathcal{J}_a^\beta = -2\sqrt{x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{a}{\sqrt{x}} + \frac{2}{\sqrt{\beta}}B'. \tag{3.1.10}$$

Through formal manipulations (see [63, Section 1]), $\mathcal{J}_a^\beta(\mathcal{J}_a^\beta)^*$ can be transformed into the *stochastic Bessel operator*.

Definition 3.1.9 (Stochastic Bessel operator [63, Section 1])**.**
Set $\beta>0$, $a>-1$ and let B be a standard Brownian motion on $\mathbb{R}_+$. The stochastic Bessel operator*, known as SBO, is the random differential operator*

$$\mathfrak{G}_a^\beta = -\frac{1}{m_a(x)}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{1}{s_a(x)}\frac{\mathrm{d}}{\mathrm{d}x}\Big), \tag{3.1.11}$$

$$m_a(x) = \exp\big(-(a+1)x - \frac{2}{\sqrt{\beta}}B(x)\big), \quad s_a(x) = \exp\big(ax + \frac{2}{\sqrt{\beta}}B(x)\big). \tag{3.1.12}$$

defined on a subset of $L^2(\mathbb{R}_+, m_a)$ with Dirichlet boundary condition at 0 and Neumann at infinity.

The operator $-\mathfrak{G}_a^\beta$ can be seen as generating a diffusion with random speed and scale measures m_a and s_a. The motion can be built pathwise in the classical mode (see for example [43, Chapter 4]).

Proposition 3.1.5 ($\mathfrak{G}_a^\beta$ as a S-L operator [23, Proposition 7])**.**
The stochastic operator $\mathfrak{G}_a^\beta$ is a Sturm-Liouville operator of the form (3.1.1) with

$$r = m_a,\ p_1 = s_a^{-1},\ p_0 = q_0 = 0.$$

It satisfies the hypothesis of Theorem 3.1.2 with probability one if $a>1$ and we still notate $\mathfrak{G}_a^\beta$ its self-adjoint version on $\mathcal{D}(\mathfrak{G}_a^\beta) = \mathcal{D}_{\infty,0}(\mathfrak{G}_a^\beta)$ from (3.1.2).

If ψ solves the equation $\mathfrak{G}_a^\beta \psi = \lambda\psi$ with deterministic initial conditions $\psi(0) = c_0$, $\psi'(0) = c_1$ then (ψ, ψ') is the unique strong solution of the stochastic differential equation system:

$$\mathrm{d}\psi(x) = \psi'(x)\mathrm{d}x, \quad \mathrm{d}\psi'(x) = \frac{2}{\sqrt{\beta}}\psi'(x)\mathrm{d}B(x) + \left((a + \frac{2}{\beta})\psi'(x) - \lambda e^{-x}\psi(x)\right)\mathrm{d}x, \tag{3.1.13}$$

with the corresponding initial conditions.

Proposition 3.1.6 (Pure point spectrum of $\mathfrak{G}_a^\beta$ [63, Theorem 1])**.**
With probability one, for each integer $k \geqslant 0$, the set of eigenvalues of $\mathfrak{G}_a^\beta$ has a well-defined $(k+1)^{st}$ lowest element $\Lambda_k(\mathfrak{G}_a^\beta)$.

Definition 3.1.10 (Eigenvalue problem for $\mathcal{A}^\beta$ [63, Section 1])**.**
An eigenvalue/eigenfunction pair of $\mathfrak{G}_a^\beta$ is $(\lambda, \psi) \in \mathbb{R}\times\mathcal{D}(\mathfrak{G}_a^\beta)$ such that $\|\psi\|_2 = 1$ and $\mathfrak{G}_a^\beta\psi = \lambda\psi$, i.e. $\psi = \lambda(\mathfrak{G}_a^\beta)^{-1}\psi$, with the explicit inverse form

$$((\mathfrak{G}_a^\beta)^{-1}\psi)(x) = \int_0^\infty \Big(\int_0^{x\wedge y} s_a(\mathrm{d}z)\Big)\psi(y)m_a(dy). \tag{3.1.14}$$

$(\mathfrak{G}_a^\beta)^{-1}$ is non-negative symmetric in $L^2([\mathbb{R}_+, m_a])$ and the Dirichlet condition at the origin is automatic for $\psi = \lambda(\mathfrak{G}_a^\beta)^{-1}\psi$.

3.2 Heuristic of convergence

This section presents heuristic arguments from [30] and [72] as to how the stochastic Airy operator and the stochastic Bessel operator emerge as limits of the matrix representations of β-ensembles, at the soft ad hard edges respectively.

3.2.1 Finite difference schemes

A *finite difference scheme* approximates numerically the solution of a differential equation by replacing differential operators with matrices that mimic their behavior. Let us introduce the $n \times n$ matrices

$$\nabla_n = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}, \ \Delta_n = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \tag{3.2.1}$$

$$F_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & \iddots & & & \\ 1 & & & & \end{pmatrix}, \ S_n = \begin{pmatrix} & & & & -1 \\ & & & 1 & \\ & & -1 & & \\ & \iddots & & & \\ (-1)^n & & & & \end{pmatrix}. \tag{3.2.2}$$

Up to a constant factor, ∇_n encodes a finite difference scheme for the first derivative operator when certain boundary conditions are in place. Likewise, under certain conditions, Δ_n discretizes the second derivative operator, also up to a constant factor. F_n acts as a 'flip' permutation matrix.
In the next two subsections we explain how, properly rescaled at the soft and hard edge respectively, the β-Hermite and β-Laguerre tridiagonal matrix representations can be interpreted as finite difference schemes for $\mathcal{A}^\beta$ and $\mathcal{J}_a^\beta$. Similar arguments for the β-Jacobi ensemble can be found in [30, Section 6].

3.2.2 Soft edge of the β-Hermite ensemble

For $\beta > 0$, recall the tridiagonal representation $H_n^\beta = (h_{i,j}^\beta)_{1\leqslant i,j\leqslant n}$ of the β-Hermite ensemble of size n from (1.1.3). We first consider its low temperature limit $\beta \to \infty$:

$$H_n^\infty = \frac{1}{\sqrt{n}} \begin{pmatrix} 0 & \sqrt{n-1} & & & & \\ \sqrt{n-1} & 0 & \sqrt{n-2} & & & \\ & \sqrt{n-2} & 0 & \sqrt{n-3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \sqrt{2} & 0 & \sqrt{1} \\ & & & & \sqrt{1} & 0 \end{pmatrix}. \tag{3.2.3}$$

This matrix encodes the recurrence relation for Hermite polynomials and its eigenvalues are the roots of the n^{th} polynomial. Properly rescaled, they converge to the zeros of Ai (see [73, Equation 5.5.8]). Heuristically, applying the same rescaling to a well-chosen similar transformation of H_n^∞ should put forward a discretized version of $\mathcal{A}^\infty$.
For β finite or infinite, let D_n be the diagonal matrix of size n whose (i,i) entry is $(n/2)^{-(i-1)/2} \prod_{k=1}^{i-1} h_{k,k+1}^\beta$. With $\beta = \infty$, the conjugate of H_n^∞ by D_n reads:

$$D_n H_n^\infty D_n^{-1} = \begin{pmatrix} 0 & 1 & & & & \\ \frac{n-1}{n} & 0 & 1 & & & \\ & \frac{n-2}{n} & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{2}{n} & 0 & 1 \\ & & & & \frac{1}{n} & 0 \end{pmatrix}. \tag{3.2.4}$$

Applying the appropriate rescaling, we introduce the ∞-Hermite model of size n scaled at the soft edge, where I_n is the identity matrix of size n:

$$H_n^{\infty,\text{soft}} = -n^{2/3}(D_n H_n^\infty D_n^{-1} - 2I_n). \tag{3.2.5}$$

The translation is meant to pull the largest eigenvalues toward the origin, then a scalar factor is applied to 'zoom in' so that the eigenvalues approach distinct limits as $n \to \infty$. Using (3.2.4), we derive the following proposition, which interprets $H_n^{\infty,\text{soft}}$ as a finite difference scheme for the Airy operator $\mathcal{A}^\infty$ on the mesh $x_i = hi,\ i = 1, \ldots, n$, with mesh size $h = n^{-1/3}$.

Proposition 3.2.1 (Finite difference scheme for $\mathcal{A}^\infty$ [30, Theorem 5.2])**.**
$\forall n \in \mathbb{N}$,

$$H_n^{\infty,\text{soft}} = \frac{1}{h^2}\Delta_n + \text{diag}_{-1}(x_1, \ldots, x_{n-1}),$$

with $\text{diag}_{-1}(x_1, \ldots, x_{n-1})$ *the* $n \times n$ *matrix with* $x_1, \ldots, x_{n-1}$ *on the subdiagonal and zeros elsewhere.*
Furthermore, for any fixed $k \in \mathbb{N}$*, the* $(k+1)^{st}$ *least eigenvalue of* $H_n^{\infty,\text{soft}}$ *converges to the* $(k+1)^{st}$ *least eigenvalue of* $\mathcal{A}^\infty$*:*

$$\lambda_k^-(H_n^{\infty,\text{soft}}) \underset{n\to\infty}{\longrightarrow} \Lambda_k(\mathcal{A}^\infty).$$

The convergence of the eigenvalues is a consequence of the eigenvalue decomposition (3.1.5) for $\mathcal{A}^\infty$ and the fact that our choice of rescaling is that of the convergence of the roots of the Hermite polynomials to those of the Airy function. We now extend our argument to the β-Hermite ensemble for general $\beta > 0$. Intuitively, the β-Hermite model of size n scaled at the soft edge is

$$H_n^{\beta,\text{soft}} = -n^{2/3}(D_n H_n^\beta D_n^{-1} - 2I_n), \tag{3.2.6}$$

where

$$D_n H_n^\beta D_n^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{n\beta}}G_1 & 1 & & & \\ \frac{1}{n\beta}\chi^2_{(n-1)\beta} & \frac{\sqrt{2}}{\sqrt{n\beta}}G_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{n\beta}\chi^2_{2\beta} & \frac{\sqrt{2}}{\sqrt{n\beta}}G_{n-1} & 1 \\ & & & \frac{1}{n\beta}\chi^2_{\beta} & \frac{\sqrt{2}}{\sqrt{n\beta}}G_n \end{pmatrix}. \tag{3.2.7}$$

Proposition 3.2.2 (Finite difference scheme for $\mathcal{A}^\beta$ [30, Theorem 6.2])**.**
We have

$$H_n^{\beta,\text{soft}} = H_n^{\infty,\text{soft}} + \frac{2}{\sqrt{\beta}}W_n,$$

with

$$W_n = \frac{1}{\sqrt{h}}\left(\frac{-1}{\sqrt{2}}\right)\begin{pmatrix} G_1 & & & & \\ \tilde{\chi}^2_{(n-1)\beta} & G_2 & & & \\ & \tilde{\chi}^2_{(n-2)\beta} & G_3 & & \\ & & \ddots & \ddots & \\ & & & \tilde{\chi}^2_{\beta} & G_n \end{pmatrix},$$

where $G_1, \ldots, G_n$ *are standard Gaussian random variables, and*

$$\tilde{\chi}^2_{(n-j)\beta} = \frac{1}{\sqrt{2\beta n}}(\chi^2_{(n-j)\beta}) - (n-j)\beta).$$

Furthermore, $\tilde{\chi}^2_{(n-j)\beta}$ *has mean* 0 *and standard deviation* $1 + O(h^2)$ *uniformly in* j *for* j *such that* $x_j = hj \leqslant M$*, where* $M > 0$ *is fixed.*

Since the increment of Brownian motion over an interval $(x, x+h]$ has mean 0 and standard deviation $\sqrt{h}$, a discretization of white noise over this interval should have mean 0 and standard deviation $\frac{\sqrt{h}}{h} = \frac{1}{\sqrt{h}}$, which makes W_n a suitable candidate. Therefore, $H_n^{\beta,\text{soft}}$ seems a reasonable discrete approximation of $\mathcal{A}^\beta$.

3.2.3 Hard edge of the β-Laguerre ensemble

Let us show a similar heuristic for hard edge universality, looking at the left edge of the β-Laguerre ensemble. With $m = n+a$, we can rewrite the bidiagonal matrix representation of the β-Laguerre ensemble introduced in (1.1.5):

$$L_{n,a}^{\beta} = \frac{1}{\sqrt{(n+a)\beta}} \begin{pmatrix} \chi_{(a+n)\beta} & & & & & \\ \chi_{(n-1)\beta} & \chi_{(a+n-1)\beta} & & & & \\ & \ddots & \ddots & & & \\ & & & \chi_{2\beta} & \chi_{(a+2)\beta} & \\ & & & & \chi_{\beta} & \chi_{(a+1)\beta} \end{pmatrix}. \tag{3.2.8}$$

In the low temperature limit $\beta \to \infty$, $L_n^{\beta,a}$ approaches

$$L_{n,a}^{\infty} = \frac{1}{\sqrt{n+a}} \begin{pmatrix} \sqrt{a+n} & & & & \\ \sqrt{n-1} & \sqrt{a+n-1} & & & \\ & \ddots & \ddots & & \\ & & \sqrt{2} & \sqrt{a+2} & \\ & & & \sqrt{1} & \sqrt{a+1} \end{pmatrix}. \tag{3.2.9}$$

The k^{th} least singular value of $L_{n,a}^{\infty}$ squared is the k^{th} least root of the n^{th} Laguerre polynomial with parameter a, which, properly rescaled, converges to k^{th} positive zero of the Bessel function of the first kind J_a (see [73, Section 5.1]). Introduce the ∞-Laguerre model of size n scaled at the hard edge:

$$L_{n,a}^{\infty,\text{hard}} = \frac{2\sqrt{n+a}}{\sqrt{h}} F_n S_n L_{n,a}^{\infty} S_n^{-1} F_n. \tag{3.2.10}$$

Consider the mesh $x_i = (1 - \frac{h}{2}) - h(n-i),\ 1 = 1, \dots, n$ of size $h = \frac{1}{n+(a+1)/2}$.

Proposition 3.2.3 (Finite difference scheme for $\mathcal{J}_a^{\infty}$ [72, Theorem 7.3.6])**.** $\forall n \in \mathbb{N}$,

$$L_{n,a}^{\infty,\text{hard}} = -2\,\text{diag}(\sqrt{x_1}, \dots, \sqrt{x_n})\frac{1}{h}\nabla_n + \frac{a}{2} \begin{pmatrix} \frac{1}{\sqrt{x_1}} & \frac{1}{\sqrt{x_1}} & & & \\ & \frac{1}{\sqrt{x_2}} & \frac{1}{\sqrt{x_2}} & & \\ & & \frac{1}{\sqrt{x_3}} & \ddots & \\ & & & \ddots & \frac{1}{\sqrt{x_{n-1}}} \\ & & & & \frac{1}{\sqrt{x_n}} \end{pmatrix} + E_n,$$

where E_n is upper bidiagonal with entries in rows $\lceil \frac{\epsilon}{h} \rceil, \dots, n$ uniformly $O(h)$, for any fixed $0 < \epsilon < 1$.

Indeed, for $k = 1, \dots, n$, the (k,k) entry of E_n is

$$(2\sqrt{k+a} - 2\sqrt{k+a/2} - \frac{a}{2\sqrt{k+a/2}})\frac{1}{\sqrt{h}}.$$

A series expansion of $\sqrt{k+a}$ about $\sqrt{k+a/2}$ shows that the entry can be bounded in magnitude by $\frac{a^2}{16(\epsilon-h)^{3/2}}h$ for all $k \geqslant \frac{\epsilon}{h}$. Therefore, $L_{n,a}^{\infty,\text{hard}}$ seems a viable candidate for an approximation of a finite difference scheme for the Bessel operator $\mathcal{J}_a^\infty$.

Moving on to the stochastic version, we use the approximation of the χ distribution $\chi_r \approx \sqrt{r} + \frac{1}{\sqrt{2}}G$, valid for large r, to say that $L_{n,a}^\beta \approx \tilde{L}_{n,a}^\beta$, where

$$\tilde{L}_{n,a}^\beta = L_{n,a}^\infty + \frac{1}{\sqrt{2(n+a)\beta}} \begin{pmatrix} G_n & & & & \\ G'_{n-1} & G_{n-1} & & & \\ & G'_{n-2} & G_{n-2} & & \\ & & \ddots & \ddots & \\ & & & G'_1 & G_1 \end{pmatrix},$$

with $G_1, \ldots, G_n, G'_1, \ldots, G'_{n-1}$ independent standard Gaussian random variables. With this approximation we can define the β-Laguerre model of size n scaled at the hard edge:

$$\begin{aligned} L_{n,a}^{\beta,\text{hard}} &= \frac{2\sqrt{n+a}}{\sqrt{h}} F_n S_n L_n^{\beta,a} S_n^{-1} F_n \\ &\approx \frac{2\sqrt{n+a}}{\sqrt{h}} F_n S_n \tilde{L}_n^{\beta,a} S_n^{-1} F_n \\ &\approx L_{n,a}^{\infty,\text{hard}} + \sqrt{\frac{2}{\beta}} \frac{1}{\sqrt{h}} W_n, \end{aligned} \tag{3.2.11}$$

where W_n is a random bidiagonal matrix whose (k,k) entry is G_k and whose $(k,k+1)$ entry is G'_k. The total standard deviation of $\sqrt{\frac{2}{\beta}}\frac{1}{\sqrt{h}}W_n$ over row k is $\frac{2}{\sqrt{\beta}}\frac{1}{\sqrt{h}}$, which is consistent with that of $\frac{2}{\sqrt{\beta}}B'$ from (3.1.10). Therefore, $L_{n,a}^{\beta,\text{hard}}$ makes a viable candidate for a discrete approximation of $\mathcal{J}_a^\beta$.

3.3 Hard edge universality

The following theorem gathers results already stated in Theorem 2.3.4 and Proposition 3.1.6.

Theorem 3.3.1 (Hard edge universality for $\mathbb{L}_{n,a}^\beta$ [63, Theorem 1])**.**
With probability one, when restricted to the positive half-line with Dirichlet conditions at the origin, $\mathfrak{G}_a^\beta$ has a discrete spectrum comprised of single eigenvalues $0 < \Lambda_0(\mathfrak{G}_a^\beta) < \Lambda_1(\mathfrak{G}_a^\beta) < \ldots \uparrow \infty$.

Moreover, for any $k < \infty$, the $(k+1)^{st}$ smallest eigenvalues of the β-Laguerre ensemble converge to the $(k+1)^{st}$ smallest eigenvalues of the stochastic Bessel operator:

$$n(n+a)(\lambda_0^-(\mathbb{L}_{n,a}^\beta), \ldots, \lambda_k^-(\mathbb{L}_{n,a}^\beta)) \xrightarrow[n\to\infty]{\text{d}} (\Lambda_0(\mathfrak{G}_a^\beta), \ldots, \Lambda_k(\mathfrak{G}_a^\beta)).$$

In this section, we explain the proof of Theorem 3.3.1 from Ramírez and Rider. Its structure is threefold. First, we inject the sequence $\left(L^\beta_{n,a}(L^\beta_{n,a})^T\right)_{n\geqslant 0}$ into an integral operator space. Then, we prove the convergence of the operators sequence. Finally, we identify its limit as $\mathfrak{G}^\beta_a$. To lighten the writing, we will denote by $\lambda^n_0 < \lambda^n_1 < \dots$ the smallest eigenvalues of $L^\beta_{n,a}$.
First, notice that for fixed λ, the differential system (3.1.13) has Lipschitz coefficients of linear growth and thus defines a pathwise unique Markov process $x \mapsto (\psi(x), \psi'(x))$ for any $(\psi(0), \psi'(0))$ pair. Two solutions of $\psi = \lambda(\mathfrak{G}^\beta_a)^{-1}\psi$ vanishing at the origin must be constant multiples of one another, hence the simplicity of the corresponding eigenvalues of $\mathfrak{G}^\beta_a$.

3.3.1 Injection into the operator space

Set $\beta > 0$ and $a > 1$. Recall S_n from (3.2.2). As in the heuristic from Subsection 3.2.3, we work with $\hat{L}^\beta_{n,a} = S_n(L^\beta_{n,a})^T S_n^{-1}$, which has the same singular values as $L^\beta_{n,a}$:

$$\hat{L}^\beta_{n,a} = \frac{1}{\sqrt{(n+a)\beta}} \begin{pmatrix} \chi_{(a+1)\beta} & & & \\ -\tilde{\chi}_\beta & \chi_{(a+2)\beta} & & \\ & \ddots & \ddots & \\ & & -\tilde{\chi}_{(n-1)\beta} & \chi_{(a+n)\beta} \end{pmatrix}, \tag{3.3.1}$$

where the notation $\tilde{\chi}$ is meant to emphasize the independence of the processes along the main and lower diagonals. Since we want to prove the convergence to the inverse operator $(\mathfrak{G}^\beta_a)^{-1}$ defined in (3.1.14), we compute the inverse of $\hat{L}^\beta_{n,a}$.

Lemma 3.3.1. *[63, Lemma 4] For any inversible lower bidiagonal matrix $B = (b_{i,j})$, B^{-1} is lower triangular and has the expression*

$$(B^{-1})_{i,j} = \frac{(-1)^{i+j}}{b_{i,i}} \prod_{k=j}^{i-1} \frac{b_{k+1,k}}{b_{k,k}} \text{ for } j < i,$$
$$(B^{-1})_{i,i} = \frac{1}{b_{i,i}}.$$

The proof of Lemma 3.3.1 rests on straightforward calculations, we do not give it here. Define the mesh $x_i = i/n,\ i = 1, \dots, n$. For a matrix $M \in \mathcal{M}_n(\mathbb{R})$, $M = (m_{i,j})_{1\leqslant i,j\leqslant n}$, there is a natural embedding $M \to \mathfrak{M}$ into the space of operators acting on $f \in L^2([0,1])$, which does not change the spectrum:

$$(\mathfrak{M}f)(x) = \sum_{j=1}^{n} m_{i,j} n \int_{x_{j-1}}^{x_j} f(y)\mathrm{d}y \text{ for } x_{i-1} \leqslant x < x_i.$$

Since the proper scaling at the hard edge is $\left(n(n+a)\hat{L}^\beta_{n,a}\ (\hat{L}^\beta_{n,a})^T\right)^{-1}$, using Lemma 3.3.1, we read the action of $\left(n(n+a)\right)^{-1/2}(\hat{L}^\beta_{n,a})^{-1}$ on $f \in L^2([0,1])$:

$$\left((\sqrt{n(n+a)}\hat{L}^\beta_{n,a})^{-1} f\right)(x) = \sum_{j=1}^{i} \frac{\sqrt{\beta n}}{\chi_{(i+a)\beta}} \prod_{k=j}^{i-1} \frac{\tilde{\chi}_{k\beta}}{\chi_{(k+a)\beta}} \int_{x_{j-1}}^{x_j} f(y)\mathrm{d}y.$$

Therefore, we can identify $\big(n(n-a)\big)^{-1/2}(\hat{L}^{\beta}_{n,a})^{-1}$ with the integral operator $K^{\beta,a}_n$ with discrete kernel

$$k^{\beta,a}_n(x,y) = \frac{\sqrt{\beta n}}{\chi_{(i+a)\beta}} \exp\Big(\sum_{k=j}^{i-1} \log\tilde{\chi}_{k\beta} - \log\chi_{(k+a)\beta}\Big)\mathbf{1}_{\mathbf{D}}(x,y) \tag{3.3.2}$$

where $\mathbf{1}_{\mathbf{D}}(x,y) = \mathbf{1}_{x_{i-1}\leqslant x<x_i}\mathbf{1}_{x_{j-1}\leqslant y<x_j}\mathbf{1}_{j\leqslant i}$.
To identify the limit of $K^{\beta,a}_n$ as $n\to\infty$, we need the following lemma.

Lemma 3.3.2. *[63, Lemma 5] There is a Brownian motion B such that, for $x<y$ lying in $(0,1]$, for the Skorohod topology:*

$$\frac{\sqrt{\beta n}}{\chi_{(i+a)\beta}} \xrightarrow[n\to+\infty]{\mathrm{d}} \frac{1}{\sqrt{x}} \tag{3.3.3}$$

and

$$\sum_{k=j}^{i-1} \log\tilde{\chi}_{k\beta} - \log\chi_{(k+a)\beta} \xrightarrow[n\to+\infty]{\mathrm{d}} \frac{a}{2}\log\frac{y}{x} + \int_y^x \frac{db_z}{\sqrt{\beta z}}. \tag{3.3.4}$$

Lemma 3.3.2 tells us that, as $n\to\infty$, $K^{\beta,a}_n$ should approach the integral operator $K^{\beta,a}_\infty$ with kernel

$$k^{\beta,a}_\infty(x,y) = x^{-\frac{1+a}{2}} \exp\Big[\int_y^x \frac{db_z}{\sqrt{\beta z}}\Big] y^{\frac{a}{2}} \mathbf{1}_{y<x}. \tag{3.3.5}$$

3.3.2 Convergence of the operators sequence

The proof of the convergence of the sequence $(K^{\beta,a}_n)_n$ uses bounds on χ variables and notions of operator theory, which we do not prove here.

Lemma 3.3.3. *[63, Lemma 5] Let $T(x) = \frac{1}{\beta}\log\frac{1}{x}$. There exist tight random constants $\kappa_n>0$ and $\kappa'_n>0$ independent of β so that*

$$\sup_{1\leqslant k\leqslant n} \frac{\sqrt{k\beta}}{\chi_{(k+a)\beta}} \leqslant \kappa_n,$$

$$\forall\ 1\leqslant j<i\leqslant n,\ \sum_{k=j}^{i-1}\big(\log\tilde{\chi}_{k\beta} - \log\chi_{(k+a)\beta}\big) - (a/2)\log(\frac{j}{i}) \leqslant \kappa'_n\big(1+T^{3/4}(x_i)+T^{3/4}(x_j)\big).$$

Proposition 3.3.1 (Hilbert-Schmidt operator [66, Chapter 4])**.**
An operator K on the Hilbert space $\big(L^2([0,1]),\|\cdot\|_2\big)$ is said to be Hilbert-Schmidt *if there exists a Hilbertian base $(e_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}}\|Ke_n\|_2$ converges. In this case, we set*

$$\|K\|_{\mathrm{HS}} = \Big(\sum_{n\in\mathbb{N}}\|Ke_n\|_2\Big)^{1/2}.$$

A Hilbert-Schmidt operator is compact. Furthermore, if K is an integral operator with kernel $k\in L^2([0,1]\times[0,1])$ then K is Hilbert-Schmidt and $\|K\|_{\mathrm{HS}} = \|k\|_2$.

Lemma 3.3.4 (Convergence of the operators sequence [63, Lemma 6])**.** *$K_\infty^{\beta,a}$ is almost surely Hilbert-Schmidt. Also, there exists a probability space on which all $K_n^{\beta,a}$ and $K_\infty^{\beta,a}$ are defined, such that any sequence of the operators $K_n^{\beta,a}$ contains a subsequence $K_{n'}^{\beta,a}$ which converges to $K_\infty^{\beta,a}$ in Hilbert-Schmidt norm with probability one, and in particular:*

$$\lim_{n'\to\infty} \int_0^1 \int_0^1 \left|k_{n'}^{\beta,a}(x,y)(\omega) - k_\infty^{\beta,a}(x,y)(\omega)\right|^2 \mathrm{d}x\mathrm{d}y = 0 \text{ almost surely.} \tag{3.3.6}$$

Lemma 3.3.4 is proved in the Appendices. We place ourselves on its advertised probability space to prove the convergence in law of the eigenvalues. Start with the scaled minimal eigenvalue of $L_{n,a}^\beta$, and notate $\|\cdot\|$ the $L^2 \mapsto L^2$ operator norm:

$$\begin{aligned} n(n+a)\lambda_0^n &= \inf_{\|v\|_{l_2}=1} \left\langle v, n(n+a)\hat{L}_{n,a}^\beta (\hat{L}_{n,a}^\beta)^T v \right\rangle \\ &= \Big(\sup_{\|f\|_{L^2}=1} \left\langle f, (K_n^{\beta,a})^T K_n^{\beta,a} f \right\rangle \Big)^{-1} \\ &= \left\| (K_n^{\beta,a})^T K_n^{\beta,a} \right\|^{-1}, \end{aligned}$$

The final equality holds because $(K_n^{\beta,a})^T K_n^{\beta,a}$ is non-negative symmetric.
Since $K_\infty^{\beta,a}$ is almost surely Hilbert-Schmidt, $(K_\infty^{\beta,a})^T K_\infty^{\beta,a}$ is non-negative symmetric and compact so has a well defined maximal eigenvalue $\hat{\Lambda}_0 = \left\|(K_\infty^{\beta,a})^T K_\infty^{\beta,a}\right\|$.
Going forward, we write $\hat{\Lambda}_0 > \hat{\Lambda}_1 > \cdots$ the eigenvalues of $\mathcal{K}_\infty^{\beta,a} = (K_\infty^{\beta,a})^T K_\infty^{\beta,a}$ and $\hat{\Lambda}_0^n > \hat{\Lambda}_0^1 > \ldots$ those of $\mathcal{K}_n^{\beta,a} = (K_n^{\beta,a})^T K_n^{\beta,a}$.
We use Lemma 3.3.4 to choose a subsequence $\big(\phi(n)\big)_{n\geqslant 0}$ along which

$$\left\|K_{\phi(n)}^{\beta,a} - K_\infty^{\beta,a}\right\|_{\mathrm{HS}} \longrightarrow 0.$$

$\mathcal{K}_{\phi(n)}^{\beta,a}$ converges strongly to $\mathcal{K}_\infty^{\beta,a}$ so the norms converge:

$$\|\mathcal{K}_{\phi(n)}^{\beta,a}\| = \hat{\Lambda}_0^{\phi(n)} \underset{n\to\infty}{\longrightarrow} \|\mathcal{K}_\infty^{\beta,a}\| = \hat{\Lambda}_0 \text{ almost surely.}$$

For any sequence of $n(n+a)\lambda_0^n$, there is a subsequence that converges almost surely to $1/\hat{\Lambda}_0$. This implies

$$n(n+a)\lambda_0^n \underset{n\to\infty}{\longrightarrow} \hat{\Lambda}_0^{-1} \text{ almost surely.} \tag{3.3.7}$$

To extend our result to $\lambda_1^n, \lambda_2^n, \ldots$, we need the convergence of the eigenvectors. Define $(f_n)_{n\geqslant 0}$ and f in $L^2([0,1])$ with unit norm by

$$\langle f_n, \mathcal{K}_n^{\beta,a} f_n \rangle = \hat{\Lambda}_0^n, \ \langle f, \mathcal{K}_\infty^{\beta,a} f \rangle = \hat{\Lambda}_0.$$

We have

$$\|\mathcal{K}_{\phi(n)}^{\beta,a} f_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} \|\mathcal{K}_\infty^{\beta,a}\|_{L^2}.$$

Since $f_{\phi(n)}$ is uniformly bounded in $L^2([0,1])$, we can extract a subsequence $(\phi_1(n))_{n\geqslant 0}$ along which it converges weakly to f_∞. Then, for any $\psi \in L^2([0,1])$,

$$\begin{aligned}\Big\langle \psi, \mathcal{K}^{\beta,a}_{\phi_1(n)} f_{\phi_1(n)} - \mathcal{K}^{\beta,a}_{\infty} f_\infty \Big\rangle \\ = \Big\langle \psi, (\mathcal{K}^{\beta,a}_{\phi_1(n)} - \mathcal{K}^{\beta,a}_{\infty}) f_{\phi_1(n)} \Big\rangle + \Big\langle \mathcal{K}^{\beta,a}_{\infty}\psi, f_{\phi_1(n)} - f_\infty \Big\rangle \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

by norm convergence of the first term and boundedness of $\mathcal{K}^{\beta,a}_\infty$ in the second. The weak convergence and the convergence of the norm imply the strong convergence of a subsequence of $(\mathcal{K}^{\beta,a}_n f_n)_{n\geqslant 0}$. Coupled with $\mathcal{K}^{\beta,a}_n \mathcal{K}^{\beta,a}_n f_n = \hat{\Lambda}^n_0 f_n$ and $\hat{\Lambda}^n_0 \to \hat{\Lambda}_0$, this induces a strongly convergent subsequence of $(f_n)_{n\geqslant 0}$, which by continuity converges to f.
Finally, denote by P_{f_n} the projection onto the orthogonal complement of f_n in $L^2([0,1])$. $P_{f_{\phi_1(n)}} \mathcal{K}^{\beta,a}_{\phi_1(n)} P_{f_{\phi_1(n)}}$ converges strongly to $P_f \mathcal{K}^{\beta,a}_\infty P_f$, therefore

$$\hat{\Lambda}^{\phi_1(n)}_1 = \|P_{f_{\phi_1(n)}} \mathcal{K}^{\beta,a}_n P_{f_{\phi_1(n)}}\| \underset{n\to\infty}{\longrightarrow} \|P_f \mathcal{K}^{\beta,a}_\infty P_f\| = \hat{\Lambda}_1 \text{ almost surely.}$$

The same argument as for λ^n_0 shows that

$$n(n+a)\lambda^n_1 \underset{n\to\infty}{\longrightarrow} \hat{\Lambda}_1^{-1} \text{ almost surely,} \tag{3.3.8}$$

and by induction the almost sure convergence extends to any finite number of eigenvalues of $\mathbb{L}^\beta_{n,a}$.

3.3.3 Identification of $(\mathfrak{G}^{\beta,a})^{-1}$

For fixed $k \in \mathbb{N}^*$, we have proved the joint convergence in law of the rescaled k highest eigenvalues of $\mathbb{L}^\beta_{n,a}$ to the k highest eigenvalues of the integral operator $K^{\beta,a}_\infty (K^{\beta,a}_\infty)^T$. Its spectral problem reads

$$\begin{aligned} f(x) &= \lambda \int_x^1 x^{a/2} e^{\int_y^x \frac{\mathrm{d}B(s)}{\sqrt{\beta}s}} y^{-(a+1)} \int_0^y e^{\int_z^y \frac{\mathrm{d}B(s)}{\sqrt{\beta}s}} z^{a/2} f(z)\mathrm{d}z\mathrm{d}y \\ &= \lambda \int_0^1 (xy)^{a/2} \Big(\int_{x\vee y}^1 e^{-2\int_z^1 \frac{\mathrm{d}B(s)}{\sqrt{\beta}s}} z^{-(a+1)}\mathrm{d}z \Big) e^{\int_x^1 \frac{dB(s)}{\sqrt{\beta}s}} e^{\int_y^1 \frac{\mathrm{d}B(s)}{\sqrt{\beta}s}} f(y)\mathrm{d}y, \end{aligned}$$

after integration by parts and where $f \in L^2([0,1])$ and $f(1) = 0$. The substitution $g(x) = x^{-a/2} e^{\int_x^1 \frac{\mathrm{d}B(s)}{\sqrt{\beta}s}} f(x)$ and the time change $\int_x^1 s^{-1/2}\mathrm{d}B(s) = \hat{B}\big(\log(1/x)\big)$ with a new Brownian motion $\hat{B}$ yield the equivalent formulation:

$$g(x) = \lambda \int_0^1 \Big(\int_{x\vee y}^1 e^{-\frac{2}{\sqrt{\beta}}\hat{B}(\log 1/z)} z^{-(a+1)}\mathrm{d}z \Big) g(y) y^a e^{\frac{2}{\sqrt{\beta}}\hat{B}(\log 1/y)} \mathrm{d}y,$$

with $g \in L^2([0,1], \tilde{m}_a)$, for $\tilde{m}_a(\mathrm{d}x) = x^a e^{\frac{2}{\sqrt{\beta}}\hat{B}(\log 1/x)}\mathrm{d}x$. Finally, the change of variables $(x,y) \to (e^{-x}, e^{-y})$ gets us back to the desired form (3.1.14) and transforms the Dirichlet condition at 1 into one at the origin.
We have thus identified $\mathfrak{G}^\beta_a$ as the limit operator and this concludes the proof of Theorem 3.3.1.

3.4 Soft edge universality

Universality at the soft edge as stated in Theorem 2.3.3 was proved by Ramírez, Rider and Virág [64]. We wish here to present the beautiful analysis of Gorin and Shkolnikov [37]. Resorting to combinatorics of high powers of the β-Hermite ensemble matrix representation, they obtain a Feynman-Kac formula for the stochastic Airy operator. We skecth the main ideas of their proof and refer to their article for details. The concepts we use from theory of stochastic integration, such as local times and Brownian bridges, can be found in the textbook [48].

3.4.1 Setup

Consider a standard Brownian motion W, and for each $T > 0$ the following random kernel on $\mathbb{R}_+ \times \mathbb{R}_+$:

$$K(x,y;T) = \frac{1}{\sqrt{2\pi T}} \exp\big(-\frac{(x-y)^2}{2T}\big)$$
$$\cdot \mathbb{E}_{B^{x,y}}\Big(1_{\{\forall t: B^{x,y}(t)\in\mathbb{R}_+\}} \exp\big(-\frac{1}{2}\int_0^T B^{x,y}(t)\mathrm{d}t + \frac{1}{\sqrt{\beta}}\int_0^\infty \mathsf{L}_a(B^{x,y})\mathrm{d}W(a)\big)\Big), \tag{3.4.1}$$

where $B^{x,y}$ is a standard Brownian bridge starting at x at time 0 and ending at y at time T, which is independent of W, and the $\mathsf{L}_a(B^{x,y})$ are the local times accumulated by $B^{x,y}$ at level a on $[0,T]$. The expectation $\mathbb{E}_{B^{x,y}}$ is taken only with respect to $B^{x,y}$.
We define $\mathcal{U}(T), T > 0$ as the integral operators on $\mathbb{R}_+$ with kernels $K(x,y;T)$, and $\mathcal{U}(0)$ as the identity operator. To make statements about multiple operators $\mathcal{U}(T)$, we use the same path of W in (3.4.1).
Let $\mathcal{S}$ denote the set of all locally integrable functions f on $\mathbb{R}_+$ for which there exists a $\delta > 0$ such that $f(x) = O\big(\exp(x^{1-\delta})\big)$ as $x \to \infty$. Further, for any $n \in \mathbb{N}$ and $f \in \mathcal{S}$, write $\pi_n f$ for the vector in $\mathbb{R}^n$ with components

$$(\pi_n f)_i = n^{1/6}\int_{n^{-1/3}(n-i)}^{n^{-1/3}(n-i+1)} f(x)\mathrm{d}x,\ i = 1,2,\dots,n.$$

Finally, recall the β-Hermite ensemble tridiagonal matrix representation H_n^β from (1.1.3), the permutation matrix F_n from (3.2.2) and introduce the $n \times n$ matrix:

$$\mathcal{H}_n^\beta(T) = \frac{1}{2}\Big(\Big(\frac{F_n H_n^\beta F_n^{-1}}{2}\Big)^{\lfloor Tn^{2/3}\rfloor} + \Big(\frac{F_n H_n^\beta F_n^{-1}}{2}\Big)^{\lfloor Tn^{2/3}\rfloor - 1}\Big). \tag{3.4.2}$$

Theorem 3.4.1 (Limit operator for $\mathcal{H}_n^\beta(T)$ [37, Theorem 2.8])**.**
We have the convergence

$$\lim_{n\to\infty} \mathcal{H}_n^\beta(T) = \mathcal{U}(T), \quad T \geqslant 0$$

in the following senses:

(a) Weak convergence: For any $f, g \in \mathcal{S}$ and $T \geqslant 0$,

$$\lim_{n\to\infty} (\pi_n f)^T \mathcal{H}_n^\beta(T)(\pi_n g) = \int_{\mathbb{R}_+} (\mathcal{U}_A(T)f)(x)g(x)\mathrm{d}x$$

in distribution and in the sense of moments.

(b) Convergence of traces: For any $T \geqslant 0$,

$$\lim_{n\to\infty} \mathrm{Trace}\left((\mathcal{H}_n^\beta(T)\right) = \mathrm{Trace}\left(\mathcal{U}(T)\right)$$

in distribution and in the sense of moments.

The convergence of the eigenvalues of $\mathcal{H}_n^\beta(T)$ to those of $\mathcal{U}(T)$ can be deduced from the convergence of the traces (see [37], Corollary 2.10).

Remark 3.4.1. *Theorem 3.4.1 from [37] is stated in a more general setting, defining for any interval $A \subset \mathbb{R}_+$ the kernel $K_A(x, y; T)$ and the restriction $H_n^{\beta,A}$ of H_n^β onto A.*

Remark 3.4.2. *Gorin and Shkolnikov proved Theorem 3.4.1 for a wider range of matrices that have diagonal entries of smaller order than their super/sub-diagonal entries (see Assumption 2.1 in [37]). Gaudreau Lamarre was able to lift this condition by rising the matrices to a different power (see [35, Section 4.3] for a comparison with [37]), thus incorporating the soft edge of the β-Laguerre ensemble.*

Remark 3.4.3. *Work has been done by Waters in [79] toward extending this technique to the hard edge and obtaining a Feynman-Kac formula for the stochastic Bessel operator but remains to be completed.*

Recall the stochastic Airy operator $\mathcal{A}^\beta$ from 3.1.5 and its eigenvalues

$$\Lambda_0(\mathcal{A}^\beta) \leqslant \Lambda_1(\mathcal{A}^\beta) \leqslant \dots$$

Theorem 3.4.1 implies the following result.

Proposition 3.4.1 (Stochastic Airy semigroup [37, Corollary 2.12])**.**
For any $T \geqslant 0$, define $e^{-\frac{T}{2}\mathcal{A}^\beta}$ as the unique operator on $L^2(\mathbb{R}_+)$ with the same orthonormal basis of eigenfunctions as $\mathcal{A}^\beta$ and the corresponding eigenvalues $e^{T\Lambda_0(\mathcal{A}^\beta)/2} \leqslant e^{T\Lambda_1(\mathcal{A}^\beta)/2} \leqslant \dots$
If one couples $e^{-\frac{T}{2}\mathcal{A}^\beta}$ with $\mathcal{U}(T)$ by identifying the Brownian motions B in their respective definitions, then for each $T \geqslant 0$, the operators $e^{-\frac{T}{2}\mathcal{A}^\beta}$ and $\mathcal{U}(T)$ coincide with probability one.

The relationship between $\mathcal{A}^\beta$ and the family of operators $\{\mathcal{U}(T), T > 0\}$ can be viewed as a variant of the Feynman-Kac formula for Schroedinger operators (see [70, Section 6] for the known case where the potential is a deterministic function).

3.4.2 Sketch of the proof of Theorem 3.4.1

This section explains how the Brownian bridges $B^{x,y}$ and the Brownian motion W from the kernel $K(x,y;T)$ from (3.4.1) arise in the study of high powers of the matrix H_n^β. We follow the original proof [37, Section 3] and work with a rescaled and conjugated version of H_n^β:

$$M_n = F_n H_n^\beta F_n^{-1} = \frac{1}{\sqrt{n}} \begin{pmatrix} \mathfrak{a}(1) & \mathfrak{b}(1) & 0 & \cdots & 0 \\ \mathfrak{b}(1) & \mathfrak{a}(2) & \mathfrak{b}(2) & \ddots & \vdots \\ 0 & \mathfrak{b}(2) & \mathfrak{a}(3) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \mathfrak{b}(n-1) \\ 0 & \cdots & 0 & \mathfrak{b}(n-1) & \mathfrak{a}(n) \end{pmatrix},$$

where $\mathfrak{a}(m) \sim \mathcal{N}(0, 2/\beta)$ and $\mathfrak{b}(m) \sim \chi_{\beta m}$ are independent. Note that $\mathfrak{a}(m)$ is of order 1 in m while $\mathfrak{b}(m) = \sqrt{m} + \xi(m)$ with $|\mathbb{E}[\xi(m)]| = o(m^{-1/3})$. Defining $s_\mathfrak{a} = 1$ and $s_\xi = 1/\sqrt{2\beta}$, we have

$$\mathbb{E}(\mathfrak{a}(m)^2) = s_\mathfrak{a}^2,\ \mathbb{E}(\xi(m)^2) = s_\xi^2 + o(1) \text{ and } s_\mathfrak{a}^2/4 + s_\xi^2 = 1/\beta.$$

These estimates on the χ distribution tail can be found in a statistical distribution handbook such as [52]. By definition, for $k \in \mathbb{N}^*$,

$$(M_n)^k [i,i'] = \sum M_n[i_0,i_1] M_n[i_1,i_2] \dots M_n[i_{k-2},i_{k-1}] M_n[i_{k-1},i_k], \quad (3.4.3)$$

where the sum is taken over all sequences of integers $i_0, i_1, \dots, i_k$ in $\{1,2,\dots,n\}$ such that $i_0 = i, i_k = i'$, and $|i_j - i_{j-1}| \leqslant 1$ for all $j = 1,2,\dots,k$.
Take $k = \left\lfloor Tn^{2/3} \right\rfloor$ and assume that k is even (the odd case can be treated similarly). We first consider the sequences in (3.4.3) without 'horizontal' segments $i_{j-1} = i_j$ and therefore suppose that $i - i'$ is even. In (3.4.3), they account for

$$\sum_{\substack{1 \leqslant i_0, i_1, \dots, i_k \leqslant n \\ |i_j - i_{j-1}| = 1 \text{ for all } j \\ i_0 = i,\, i_k = i'}} \prod_{l=1}^{k} \frac{\sqrt{i_l \wedge i_{l-1}}}{\sqrt{n}} \Big(1 + \frac{\xi(i_l \wedge i_{l-1})}{\sqrt{i_l \wedge i_{l-1}}}\Big). \quad (3.4.4)$$

Recall the definition of $\mathcal{H}_n^\beta(T)$ from (3.4.2):

$$\mathcal{H}_n^\beta(T) = \frac{1}{2}\Big(\Big(\frac{M_n}{2}\Big)^{\lfloor Tn^{2/3} \rfloor} + \Big(\frac{M_n}{2}\Big)^{\lfloor Tn^{2/3} \rfloor - 1} \Big). \quad (3.4.5)$$

Write i^* for $\min(i_0, i_1, \dots, i_k)$. The contribution of the sequences of integers with $\frac{n - i^*}{n^{1/3^*}} \to \infty$ to the sum in (3.4.4) becomes negligible in the limit so we restrict our attention to sequences with $\limsup_{n\to\infty} \frac{n-i^*}{n^{1/4}} < \infty$. In particular, we choose i, i' such that

$$x = \lim_{n\to\infty} \frac{n-i}{n^{1/3}} < \infty, \quad y = \lim_{n\to\infty} \frac{n-i'}{n^{1/3}} < \infty.$$

The sum in (3.4.4) is over the trajectories of a simple random walk with k steps conditioned on connecting i to i'. Let us give a heuristic argument as to why the normalized sum converges to an integral with respect to the law of the Brownian bridge connecting x to y. The summand of (3.4.4) can be rewritten as

$$\exp\Big(\frac{1}{2}\sum_{l=1}^{k}\log\Big(1-\frac{n-i_l\wedge i_{l-1}}{n}\Big)+\sum_{l=1}^{k}\log\Big(1+\frac{\xi(i_l\wedge i_{l-1})}{\sqrt{i_l\wedge i_{l-1}}}\Big)\Big). \tag{3.4.6}$$

For terms with $\limsup_{n\to\infty}\frac{n-i^*}{n^{1/3}}<\infty$, we use the approximation $\log(1+z)\approx z$. Here, z is of the order $n^{-2/3}$, and there are order $n^{2/3}$ summands. For the second sum, consider the Taylor expansion

$$\log(1+\frac{\xi(i_l\wedge i_{l-1})}{\sqrt{l_l\wedge i_{l-1}}})=\frac{\xi(i_l\wedge i_{l-1})}{\sqrt{l_l\wedge i_{l-1}}}-\frac{1}{2}\frac{\xi(i_l\wedge i_{l-1})^2}{i_l\wedge i_{l-1}}+\cdots,$$

where the second term is of order n^{-1} in expectation. Since there are order $n^{2/3}$ summands, only the first term can contribute to the $n\to\infty$ limit. In that limit, we can therefore replace (3.4.6) by

$$\exp\Big(-\frac{1}{2n}\sum_{l=1}^{k}(n-i_l\wedge i_{l+1})+\sum_{l=1}^{k}\frac{\xi(i_l\wedge i_{l-1})}{\sqrt{i_l\wedge i_{l-1}}}\Big). \tag{3.4.7}$$

In order to obtain local times at the limit $n\to\infty$, interpret the second term of (3.4.7) in terms of occupation times:

$$\sum_{l=1}^{k}\frac{\xi(i_l\wedge i_{l-1})}{\sqrt{i_l\wedge i_{l-1}}}=\sum_{h=i^*}^{n}\frac{\xi(h)}{\sqrt{h}}\big|\,\{l:i_l\wedge i_{l+1}=h\}\,\big|. \tag{3.4.8}$$

A simple random walk bridge with k steps connecting i to i' typically visits an order of $k^{1/2}$ sites, and the corresponding occupation times $\big|\,\{l:i_l\wedge i_{l-1}=h\}\,\big|$ are of the order $k^{1/2}$. For such a trajectory, the right hand side of (3.4.8) is a sum of independent random variables with means of orders $o(h^{-2/3}k^{1/2})=o(n^{-1/3})$ and variances of orders $O(h^{-1}k)=O(n^{-1/3})$. Since there are an order of $n^{1/3}$ summands, the limit of the sum is given by the Central Limit Theorem.
More specifically, the random walk bridge converges in the limit $n\to\infty$ to a standard Brownian bridge on $[0,T]$ connecting x to y and its occupation times normalized by $n^{1/3}$ converge to the local times of the Brownian bridge. Therefore, the variance of the limiting centered Gaussian random variable comes out to $s_\xi^2\int_0^\infty \mathsf{L}_z(B^{x,y})^2\mathrm{d}z$.
With the Brownian motion $s_\xi W_\xi(z)=\lim\limits_{n\to\infty}n^{-1/6}\sum_{h=n-\lfloor n^{1/3}z\rfloor}^{n}\xi(h)$, this random variable can be written

$$s_\xi\int_0^\infty \mathsf{L}_z(B^{x,y})\mathrm{d}W_\xi(z).$$

An application of Stirling's formula shows that the number of random walk bridges of length $k=\big\lfloor Tn^{2/3}\big\rfloor$ connecting i to i' behaves asymptotically as $2^k n^{-1/3}\sqrt{\frac{2}{\pi T}}e^{-(z-y)^2/(2T)}$. Viewing (3.4.4) as a multiple of the expectation of a

functional with respect to the law of such a random walk bridge, its asymptotic behavior is the same multiple of the corresponding functional of the Brownian bridge $B^{x,y}$, i.e.:

$$2^k n^{-1/3}\sqrt{\frac{2}{\pi T}}e^{-(x-y)^2/(2T)}$$
$$\cdot\,\mathbb{E}_{B^{x,y}}\Big(1_{\{\forall t: B^{x,y}(t)\geqslant 0\}}\exp\big(-\frac{1}{2}\int_0^T B^{x,y}(t)\mathrm{d}t + s_\xi\int_0^\infty \mathsf{L}_a(B^{x,y})\mathrm{d}W(a)\big)\Big).$$

Consider now a sequence of (3.4.3) with k horizontal segments, assuming first $k=2p$ even. This means we take a sequence of length $k-2p$ with no horizontal segments and insert $2p$ horizontal segments at arbitrary spots. If $p=1$ and we normalize by 2^k, the corresponding part in the sum (3.4.3) becomes

$$\frac{1}{2^{k-2}}\sum_{\substack{1\leqslant i_0,i_1,\ldots,i_{k-2}\leqslant n\\ |i_j-i_{j-1}|=1 \text{ for all } j\\ i_0=i,\, i_{k-2}=i'}}\prod_{l=1}^{k-2}\frac{\sqrt{i_l\wedge i_{l-1}}}{\sqrt{n}}\left(1+\frac{\xi\,(i_l\wedge i_{l-1})}{\sqrt{i_l\wedge i_{l-1}}}\right)\cdot\left(\frac{1}{(2\sqrt{n})^2}\sum_{0\leqslant j\leqslant l\leqslant k-2}\mathfrak{a}\,(i_j)\,\mathfrak{a}\,(i_l)\right).$$

The last factor can be written as the sum of the terms $\frac{1}{2}\frac{1}{(2\sqrt{n})^2}\big(\sum_{j=0}^{k-2}\mathfrak{a}(i_j)\big)^2$ and $\frac{1}{2}\frac{1}{(2\sqrt{n})^2}\sum_{j=0}^{k-2}\mathfrak{a}(i_j)^2$. As before, we can show that the first term tends to $\frac{1}{2}$ times the square of a Gaussian random variable with mean 0 and variance $\frac{s_\mathfrak{a}^2}{4}\int_0^\infty \mathsf{L}_z(B^{x,y})^2\mathrm{d}z$, which we write as

$$\frac{s_\mathfrak{a}}{2}\int_0^\infty \mathsf{L}_z(B^{x,y})\mathrm{d}W_\mathfrak{a}(z),$$

with the Brownian motion $s_\mathfrak{a}W_\mathfrak{a}(z)=\lim_{n\to\infty}n^{-1/6}\sum_{h-n-\lfloor n^{1/3}z\rfloor}^{n}\mathfrak{a}(h)$. The second term is of order $O(n^{-1}k)=O(n^{-1/3})$ in expectation, and so negligible in the limit $n\to\infty$. Similarly, for any number $2p$ of horizontal segments, their leading order contribution is a factor of

$$\frac{1}{(2p)!(2\sqrt{n})^{2p}}\Big(\sum_{j=0}^{k-2p}a(i_j)\Big)^{2p}\underset{n\to\infty}{\sim}\frac{1}{(2p)!}\Big(\frac{s_\mathfrak{a}}{2}\int_0^\infty \mathsf{L}_a(B^{x,y})\mathrm{d}W_a(a)\Big)^{2p}.$$

Adding $(M_n)^{k-1}\,[i,i']$, we get the total leading order contribution exp $\big(\frac{s_\mathfrak{a}}{2}\int_0^\infty \mathsf{L}_z(B^{x,y})\,\mathrm{d}W_\mathfrak{a}(z)\big)$. Gathering everything, we obtain the asymptotics

$$\frac{1}{2}\Big(\big(\frac{M_n}{2}\big)^k+\big(\frac{M_n}{2}\big)^{k-1}\Big)[i,i']\underset{n\to\infty}{\sim}n^{-1/3}\sqrt{\frac{1}{2\pi T}}e^{-\frac{(x-y)^2}{2}}$$
$$\cdot\,\mathbb{E}_{B^{x,y}}\left(1_{\{\forall t, B^{x,y}(t)\geqslant 0\}}\exp\Big(-\frac{1}{2}\int_0^T B^{x,y}(t)\mathrm{d}t+\int_0^\infty \mathsf{L}_z(B^{x,y})(s_\xi\mathrm{d}W_\xi(z)+\frac{s_\mathfrak{a}}{2}\mathrm{d}W_\mathfrak{a}(z))\Big)\right).$$

Recall that $s_\xi^2+\frac{s_\mathfrak{a}^2}{4}=\frac{1}{\beta}$ and set $W=\sqrt{\beta}(s_\xi W_\xi+\frac{s_s}{2}W_\mathfrak{a})$. Theorem 3.4.1 follows by summing over the relevant indices i,i' and replacing the sums by the integrals that they approximate.

3.5 Operator level hard-to-soft transition

For the β-Laguerre ensemble $\mathbb{L}^{\beta}_{n,m}$, the *soft* or *hard* nature of the left edge depends on the compared growth of m and n. Dumaz, Li and Valkó [23] proved that taking $a \to +\infty$ in the parameter regime $m = n + a$ induces a transition between soft edge universality and hard edge universality.
For $\beta > 0$ and $a > -1$, recall m_a from (3.1.12) and consider the 'stretching' transformation defined via $(\theta_a f)(x) = f(a^{2/3}x)$. Define the following transform for the stochastic Bessel operator of parameter $2a$:

$$\mathrm{G}^{\beta}_{2a} = \theta_a^{-1}\left(m_{2a}^{1/2}\mathfrak{G}^{\beta}_{2a}m_{2a}^{-1/2}\right)\theta_a. \tag{3.5.1}$$

Then G^{β}_{2a} is a self-adjoint operator with the same spectrum as $\mathfrak{G}^{\beta}_{2a}$, and the operators $(\mathcal{A}^{\beta})^{-1}$ and $\left(\mathrm{G}^{\beta}_{2a} - a^2\right)^{-1}$ are Hilbert-Schmidt integral operators acting on the same space of $L^2(\mathbb{R}_+)$ functions (see [23, Section 2]). An operator level *hard-to-soft transition* occurs.

Theorem 3.5.1 (Operator level hard-to-soft transition [23, Theorem 3])**.**
Let B' be white noise on $\mathbb{R}_+$ and consider the Brownian motion $B(x) := \int_0^x B'(y)dy$. For $a > 0$, set $B_{2a}(x) = a^{-1/3}B\left(a^{2/3}x\right)$. Define $\mathcal{A}_{\beta}$ with the white noise B' and G^{β}_{2a} with the Brownian motion B_{2a} via (3.5.1).
Then we have the convergence:

$$a^{4/3}\left(\mathrm{G}^{\beta}_{2a} - a^2\right)^{-1} \underset{a\to+\infty}{\longrightarrow} (\mathcal{A}^{\beta})^{-1},$$

almost surely in Hilbert-Schmidt norm.

High temperature limit of β-ensembles

Chapter 4

The Riccati transform

We showed in Chapter 3 that, for any $\beta > 0$, the rescaled lowest eigenvalues of the β-Hermite ensemble and the β-Laguerre ensemble converge respectively to the lowest eigenvalues of the stochastic Airy operator and the stochastic Bessel operator. Our goal in the remaining of this thesis is to study the behavior of those eigenvalues as the parameter β tends to 0. To that purpose, this chapter introduces the *Riccati transform*, a classical tool in the study of the spectrum of one-dimensional random Schrödinger operators, first used by Halperin [38].
The Riccati transform takes a linear second order operator into one of first order, at the price of introducing a quadratic nonlinearity. Applying it to the stochastic Airy operator and the stochastic Bessel operator provides us with characterizations of the eigenvalues of $\mathcal{A}^\beta$ and $\mathfrak{G}_a^\beta$ in terms of explosion times of two families of *diffusions*, which will be useful in our $\beta \to 0$ analysis in Chapters 5 and 6.
Sections 4.1 and 4.2 apply the Riccati transform to $\mathfrak{G}_a^\beta$ and $\mathcal{A}^\beta$ respectively and Sections 4.3 and 4.4 show how the obtained characterizations of the eigenvalues can be used to derive properties of the eigenvalues processes.

4.1 Diffusion at the hard edge

4.1.1 The Riccati transform

Set $a > -1$ and $\beta > 0$. Recall the eigenvalues problem of the stochastic Bessel Operator from (3.1.14) and its differential form from Proposition 3.1.5:

$$\mathrm{d}\psi'(t) = \frac{2}{\sqrt{\beta}}\psi'(t)\mathrm{d}B(t) + \left((a+\frac{2}{\beta})\psi'(t) - \lambda e^{-t}\psi(t)\right)\mathrm{d}t,\ \mathrm{d}\psi(t) = \psi'(t)\mathrm{d}t. \tag{4.1.1}$$

Riccati's map (or transform) is the change of variables

$$p_\lambda(t) = \psi'(t)/\psi(t), \tag{4.1.2}$$

valid away from the zeros of ψ. We derive from (4.1.1), in the Itô sense

$$\mathrm{d}p_\lambda^{\mathrm{hard}}(t) = \frac{2}{\sqrt{\beta}}p_\lambda^{\mathrm{hard}}(t)\mathrm{d}B(t) + \left((a+\frac{2}{\beta})p_\lambda^{\mathrm{hard}}(t) - \left(p_\lambda^{\mathrm{hard}}(t)\right)^2 - \lambda e^{-t}\right)\mathrm{d}t. \tag{4.1.3}$$

First, see that replacing the $-\lambda e^{-t}$ term in the drift of p_λ^{hard} with any negative constant produces a homogeneous motion begun at $+\infty$ and which hits $-\infty$ with probability one. Construction of such processes is treated in [43]. Successive dominations of the inhomogeneous p_λ^{hard} by homogeneous versions over all short times leads to the existence of uniqueness of p_λ^{hard}.
Solutions of (4.1.3) can blow up to $-\infty$ in finite time, whenever ψ vanishes, in which case p_λ^{hard} is immediately restarted at $+\infty$. We can think of p_λ^{hard} as taking values in the disjoint union of countable copies of the reals, $\mathbb{R}_0$, $\mathbb{R}_1, \ldots$ In this space, points (n, t) are ordered lexicographically but we will refer to them by their second coordinate t. A natural topology on this space is provided by the two-point compactification of each copy of the reals, glued together at the endpoints so as to respect the lexicographic ordering (see [64, Section 3]).

Proposition 4.1.1 (Family of diffusions at the hard edge [63, Section 3])**.** *The solution p_λ^{hard} of (4.1.3) is unique and continuous for the topology discussed above, and decreasing in λ for each t. It defines a family of Markov processes, or* family of diffusions at the hard edge*, parametrized by λ.*

Recall that $0 < \Lambda_0(\mathfrak{G}_a^\beta) < \ldots < \Lambda_{k-1}(\mathfrak{G}_a^\beta)$ denote the k lowest eigenvalues of $\mathfrak{G}_a^\beta$. Our goal is to prove the following connection between the family of diffusions at the hard edge and the eigenvalues of the stochastic Bessel operator.

Theorem 4.1.1 (Law of explosions at the hard edge [63, Theorem 2])**.** *Let $\mathbb{P}_{x,t}$ denote the law of $p_\lambda^{\text{hard}}(\cdot)$ starting from position x at time t. Let also $\nu_t(\mathrm{d}z) = \mathbb{P}_{+\infty,t}(\mathfrak{r} \in \mathrm{d}z)$ where $\mathfrak{r}$ is the explosion time of p_λ^{hard} to $-\infty$. Then,*

$$\mathbb{P}\left(\Lambda_0(\mathfrak{G}_a^\beta) > \lambda\right) = \nu_0(\{\infty\}) \text{ and, more generally,}$$
$$\mathbb{P}\left(\Lambda_k(\mathfrak{G}_a^\beta) < \lambda\right) = \int_{\mathbb{R}^{k+1}} \nu_0(\mathrm{d}t_1)\, \nu_{t_1}(\mathrm{d}t_2) \cdots \nu_{t_k}(\mathrm{d}t_{k+1}).$$

Remark 4.1.1 (Equality of events at the hard edge)*. The proof of Theorem 4.1.1 will show an equality of events, stronger than the stated equality in law:*

$$\forall k \in \mathbb{N},\ \{p_\lambda^{\text{hard}} \text{ explodes at most } k \text{ times on } (0, \infty[\} = \{\Lambda_k(\mathfrak{G}_a^\beta) > \lambda\}. \quad (4.1.4)$$

4.1.2 Proof of Theorem 4.1.1

We follow the proof of Ramírez and Rider [63, Section 3]. For $L > 0$, consider first the restriction of $\mathfrak{G}_a^\beta$ to $[0, L]$ with Dirichlet conditions in 0 and L, writing $\mathfrak{G}_a^{\beta,L}$ this truncated operator.

Lemma 4.1.1. *[63, Lemma 3.2] Fix λ and consider the unique diffusion $p_\lambda^{\text{hard}}(t)$ started at $+\infty$ at $t = 0$, and restarted at $+\infty$ immediately after any passage to $-\infty$. The number of eigenvalues of $\mathfrak{G}_a^{\beta,L}$ less than λ is equal in law to the number of explosions of p before $t = L$.*

Proof. Take $\psi_0(t, \lambda)$ the solution of (4.1.1) with $\psi_0(0, \lambda) = 0$ and $\psi_0'(0, \lambda) = 1$. Then Λ is an eigenvalue of $\mathfrak{G}_a^{\beta,L}$ if and only if $\psi_0(L, \Lambda) = 0$. Let $\Lambda_0(\mathfrak{G}_a^{\beta,L})$ be the smallest eigenvalue of $\mathfrak{G}_a^{\beta,L}$. Suppose that

$$\forall\ 0 < t \leqslant L,\ \psi_0(t, \lambda) > 0.$$

The monotonicity of $p_\lambda^{\mathrm{hard}}$ in λ implies that

$$\forall \lambda' \leqslant \lambda,\ \forall\ 0 < t \leqslant L,\ \psi_0(t, \lambda') > 0.$$

Therefore, we have the equality of events:

$$\{t \mapsto \psi_0(t, \lambda) \text{ has no roots before } t = L\} = \{\Lambda_0(\mathfrak{G}_a^{\beta,L}) > \lambda\}.$$

The same monotonicity argument shows that additional zeros of the almost surely continuous function $\lambda \mapsto \psi_0(L, \lambda)$ only occur by increasing λ, whereupon all other roots (in the t variable) move to the left, yielding the equality of events for general $k \in \mathbb{N}$:

$$\{t \mapsto \psi_0(t, \lambda) \text{ has at most } k \text{ roots on } (0, L)\} = \{\Lambda_k(\mathfrak{G}_a^{\beta,L}) > \lambda\}. \tag{4.1.5}$$

Consider now $p(t, \lambda)$ formed from $\psi_0(t, \lambda)$ and its derivative. By uniqueness of the solutions to (4.1.1), ψ_0 and ψ_0' cannot vanish simultaneously. In particular, the zeros of ψ_0 are isolated. At any root $\mathfrak{r}$ of $t \mapsto \psi_0(t, \lambda)$, including $\mathfrak{r} = 0$, $\psi_0(\mathfrak{r}+\varepsilon, \lambda) > 0$ and $\psi_0'(\mathfrak{r}+\varepsilon, \lambda) > 0$ for ε small enough, so $\lim_{\varepsilon \downarrow 0} p(\mathfrak{r}+\varepsilon, \lambda) = +\infty$. Likewise, when $\mathfrak{r} > 0$, $\psi_0(\mathfrak{r}+\varepsilon, \lambda) > 0$ and $\psi_0'(\mathfrak{r}+\varepsilon, \lambda) < 0$ for ε small enough, so $\lim_{\varepsilon \downarrow 0} p(\mathfrak{r}-\varepsilon, \lambda) = -\infty$. Therefore, to count roots of $\psi_0(\cdot, \lambda)$ is to count passages of the corresponding $p(\cdot, \lambda)$ to $-\infty$, after subsequent restarts at $+\infty$. □

Lemma 4.1.2. *[63, Lemma 3.3] For $k \in \mathbb{N}^*$, as $L \to \infty$, the top k eigenvalues of $\mathfrak{G}_a^{\beta,L}$ converge to the top k eigenvalues of $\mathfrak{G}_a^{\beta}$ with probability one.*

Proof. $\left(\mathfrak{G}_a^{\beta,L}\right)^{-1}$ acts on $L^2([0, L], m_a)$ via

$$\left(\left(\mathfrak{G}_{\beta,a}^{L}\right)^{-1} f\right)(x) = \int_0^\infty s_L(x, y) f(y) m_a(\mathrm{d}y),$$

where

$$s_L(x, y) = \left(\int_0^{x \wedge y} s(dz)\right) \times \left(\frac{\int_{x \vee y}^L s(\mathrm{d}z)}{\int_0^L s(\mathrm{d}z)}\right) 1_{\{x,y \in [0,L]\}}.$$

We have $s_L(x, y) \leqslant \int_0^{x \wedge y} s(dz)$ and $\lim_{L \to \infty} s_L(x, y) = \int_0^{x \wedge y} s(dz)$ pointwise in x and y, almost surely. By dominated convergence, almost surely,

$$\int_0^\infty \int_0^\infty f(x) s_L(x, y) g(y) m_a(\mathrm{d}x) m_a(\mathrm{d}y) \underset{L \to \infty}{\longrightarrow} \int_0^\infty \int_0^\infty f(x) \left(\int_0^{x \wedge y} s(dz)\right) g(y) m_a(\mathrm{d}x) m_a(\mathrm{d}y)$$

for all $f, g \in L^2(\mathbb{R}_+, m)$, and

$$\mathrm{tr}\left((\mathfrak{G}_a^{\beta,L})^{-1}\right) = \int_0^L s_L(x, x) m_a(\mathrm{d}x) \underset{L \to \infty}{\longrightarrow} \int_0^\infty \int_0^x s(\mathrm{d}y) m_a(\mathrm{d}x) = \mathrm{tr}\left((\mathfrak{G}_a^{\beta})^{-1}\right).$$

This implies the convergence of $\mathfrak{G}_a^{\beta,L}$ to $\mathfrak{G}_a^{\beta}$ in trace norm (see [69], Theorem 2.20), from which we can derive the convergence of the eigenvalues using the same arguments as in the proof of Theorem 3.3.1. □

Taking $L \to \infty$ in Lemma 4.1.1 and using Lemma 4.1.2 proves Theorem 4.1.1 and the equality of events 4.1.4.

4.1.3 Simulations

Fix $a = 2$ and $\beta = 2$. Our numerical simulations are set to start at 10 at $t = 0$ and to restart at 10 when they hit -10. This approximation is made for computational convenience and does not change the number of explosions. Figure 7 shows a realization of p_{50}^{hard}, exploding 4 times. As t increases, it becomes harder for the diffusion to explode to $-\infty$.

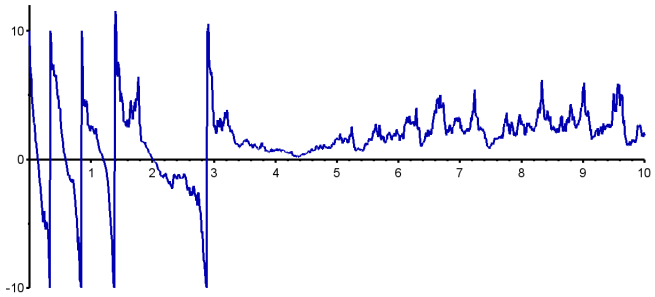

Figure 7: **Diffusion p_{50}^{hard}.**

Figure 8 shows a realization of p_1^{hard}, p_5^{hard} and p_{10}^{hard} using the same Brownian motion $B(\omega)$, exploding respectively 0, 1 and 2 times. From the equality (4.1.4), on this event ω:

$$1 < \Lambda_0(\mathfrak{G}_2^2) \leqslant 5 < \Lambda_1(\mathfrak{G}_2^2) \leqslant 10 < \Lambda_2(\mathfrak{G}_2^2).$$

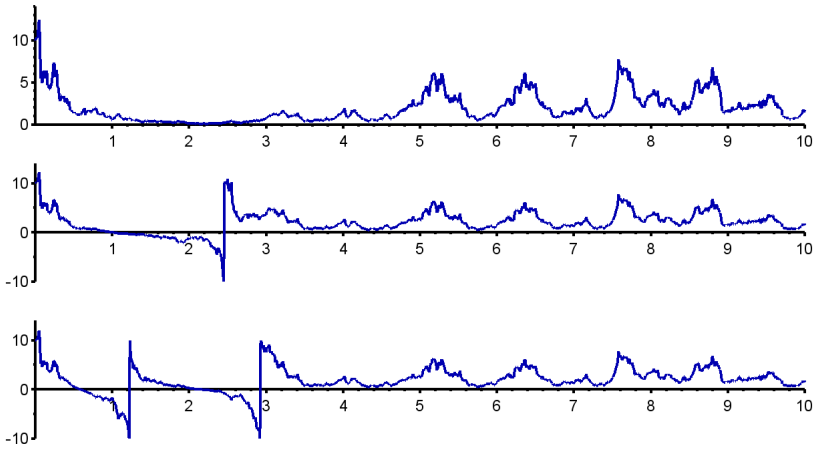

Figure 8: **Diffusions p_1^{hard} (top), p_5^{hard} (middle) and p_{10}^{hard} (bottom).**

4.2 Diffusion at the soft edge

We follow the arguments of Ramírez, Rider and Virág [64, Section 3] to skecth the proof of Theorem 4.2.1, the analog of Theorem 4.1.1 for the soft edge.

4.2.1 The Ricatti transform

Recall the eigenvalue problem (3.1.7) for the stochastic Airy operator from Proposition 3.1.3:

$$\psi''(t) = \frac{2}{\sqrt{\beta}}\psi(t)B'(t) + (t-\lambda)\psi(t). \tag{4.2.1}$$

The Riccati transform $p_\lambda^{\mathrm{soft}}(t) = \psi'(t)/\psi(t)$ turns (4.2.1) into a first-order differential equation, understood in the Itô sense:

$$\mathrm{d}p_\lambda^{\mathrm{soft}}(t) = \left(t - \lambda - \left(p_\lambda^{\mathrm{soft}}(t)\right)^2\right)\mathrm{d}t - \frac{2}{\sqrt{\beta}}\mathrm{d}B(t). \tag{4.2.2}$$

Proposition 4.2.1 (Family of diffusions at the soft edge [64, Section 3])**.** *The solution $p_\lambda^{\mathrm{soft}}$ of (4.2.2) is unique, continuous for the topology discussed in Subsection 4.1.1, and decreasing in λ for each t. It defines a family of Markov processes, or* family of diffusions at the soft edge*, parametrized by λ.*

Arguments similar to those of Section 4.1 prove the following results, where $\mathcal{A}^{\beta,L}$ is the truncation of $\mathcal{A}^\beta$, defined on the finite interval $[0, L]$ with Dirichlet conditions at both endpoints.

Lemma 4.2.1. *[64, Lemma 3.2] For fixed λ, the number of blowups of $p_\lambda^{\mathrm{soft}}(t)$ to $-\infty$ on $[0, L]$ equals the number of eigenvalues of $\mathcal{A}^{\beta,L}$ at most λ.*

Lemma 4.2.2. *[64, Lemma 3.3] As $L \to \infty$ the first k eigenvalues of $\mathcal{A}^{\beta,L}$ converge to the first k eigenvalues of $\mathcal{A}^\beta$.*

Recall that $0 < \Lambda_0(\mathcal{A}^\beta) < \ldots < \Lambda_{k-1}(\mathcal{A}^\beta)$ denote the k lowest eigenvalues of $\mathcal{A}^\beta$. Taking the $L \to \infty$ limit in Lemma 4.2.1 and using Lemma 4.2.2 yields the following results.

Proposition 4.2.2 (Equality of events at the soft edge [64, Proposition 3.4])**.** *For almost all λ,*

$$\forall k \in \mathbb{N},\ \{p_\lambda^{\mathrm{soft}} \text{ explodes at most } k \text{ times on } (0, \infty[\} = \{\Lambda_k(\mathcal{A}^\beta) > \lambda\}. \tag{4.2.3}$$

Finally, using the translation equivariance of $p_\lambda^{\mathrm{soft}}$ with regards to λ, Theorem 4.2.1 characterizes the spectrum of $\mathcal{A}^\beta$ in terms of the explosion probability of the one-dimensional diffusion $t \mapsto p_0(t)$.

Theorem 4.2.1 (Law of explosions at the hard edge [64, Theorem 1.2])**.** *Let $\kappa(t, \cdot)$ be the distribution of the first time passage to $-\infty$ of the diffusion $p_0^{\mathrm{soft}}(t)$ when started from $+\infty$ at time t. Then*

$$\mathbb{P}(\Lambda_0(\mathcal{A}^\beta) > \lambda) = \kappa(-\lambda, \{\infty\}), \qquad \text{and, for } k \geqslant 0,$$
$$\mathbb{P}(\Lambda_k(\mathcal{A}^\beta) < \lambda) = \int_{\mathbb{R}^{k+1}} \kappa(-\lambda, \mathrm{d}t_1)\,\kappa(t_1, \mathrm{d}t_2)\ldots\kappa(t_k, \mathrm{d}t_{k+1}).$$

4.2.2 Simulations

Fix $a = 2$ and $\beta = 2$. Once again, our numerical simulations are set to start at 10 at $t = 0$ and to restart at 10 when they hit -10. This approximation is made for computational convenience and does not change the number of explosions. Figure 7 shows a realization p_{10}^{soft}, exploding 7 times. The more t increases, the harder it becomes for the diffusion to explode to $-\infty$.

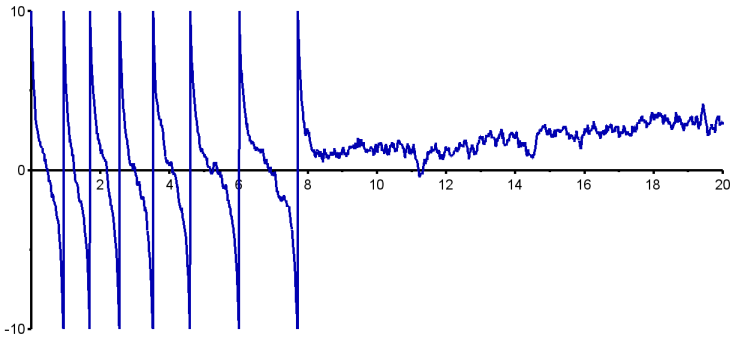

Figure 9: **Diffusion p_{10}^{soft}.**

Figure 10 shows a realization of p_1^{soft}, p_2^{soft} and p_5^{soft} using the same Brownian motion $B(\omega)$, exploding respectively 0, 1 and 3 times. From the equality (4.2.2), on this event ω:

$$1 < \Lambda_0(\mathcal{A}_2^2) < 2 < \Lambda_1(\mathcal{A}_2^2) < \Lambda_2(\mathcal{A}_2^2) \leqslant 5 < \Lambda_3(\mathcal{A}_2^2).$$

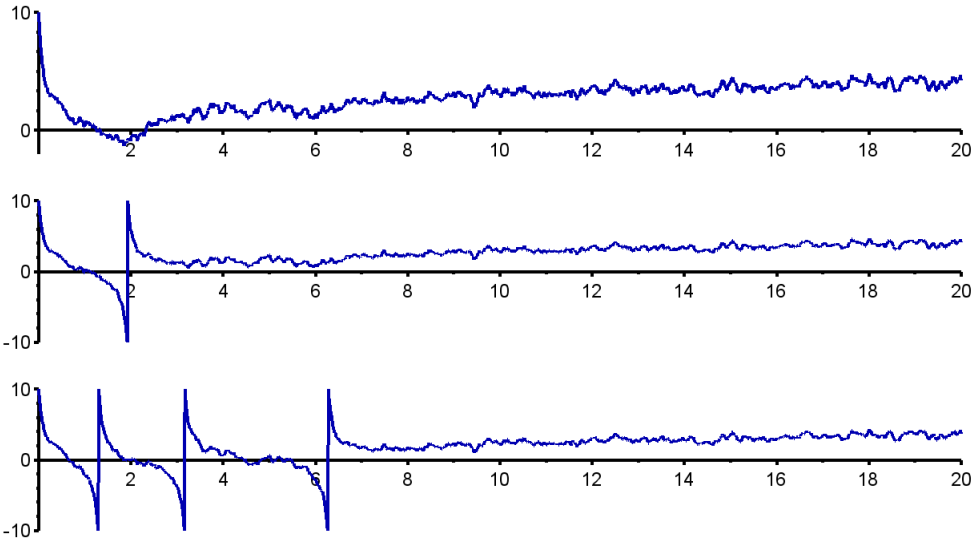

Figure 10: **Diffusions p_1^{soft} (top), p_2^{soft} (middle) and p_5^{soft} (bottom).**

4.3 Tracy-Widom tail bounds

The characterization of the spectrum of $\mathcal{A}^\beta$ by the explosions of p_λ^{soft} as in (4.2.2) can be used to prove tail estimates of the Tracy-Widom distribution.

Theorem 4.3.1 (Tracy-Widom tail bounds [64, Theorem 1.3])**.**
With $TW_\beta = -\Lambda_0(\mathcal{A}^\beta)$, we have:

$$\mathbb{P}\left(TW_\beta > a\right) = \exp\left(-\frac{2}{3}\beta a^{3/2}(1 + \underset{a\to+\infty}{o}(1))\right),$$
$$\mathbb{P}\left(TW_\beta < -a\right) = \exp\left(-\frac{1}{24}\beta a^{3}(1 + \underset{a\to+\infty}{o}(1))\right).$$

The characterization (4.2.3) is used in the proof of the lower bound of the left tail and of the upper bound of the right tail. We explain the former here and refer to [64, Section 4] for the proof of the latter.
With Theroem 4.2.1, we have:

$$\begin{aligned}\mathbb{P}\left(TW_\beta < -a\right) &= \mathbb{P}(\Lambda_0(\mathcal{A}^\beta) > a)\\ &= \mathbb{P}_{(\infty,-a)}\big(p_0^{\text{soft}} \text{ never explodes}\big),\end{aligned}$$

where the subscript of $\mathbb{P}$ indicates the space-time starting point of p_0.
By monotonicity and strong Markov property,

$$\begin{aligned}&\mathbb{P}_{(\infty,-a)}\big(p_0^{\text{soft}} \text{ never explodes}\big) \geqslant \mathbb{P}_{(1,-a)}\big(p_0^{\text{soft}} \text{ never explodes}\big)\\ &\qquad\geqslant \mathbb{P}_{(1,-a)}\big(\forall\ t\in[-a,0],\ p_0^{\text{soft}}(t)\in[0,2]\big)\ \mathbb{P}_{(0,0)}\big(p_0^{\text{soft}} \text{ never explodes}\big).\end{aligned}$$

The second factor is a positive term not depending on a. To bound the first probability, we use the Cameron-Martin-Girsanov formula:

$$\begin{aligned}&\mathbb{P}_{(1,-a)}\big(\forall\ t\in[-a,0],\ p_0^{\text{soft}}(t)\in[0,2]\big)\\ &= \mathbb{E}_{(1,-a)}\Big(\exp\big(-\frac{\beta}{4}\int_{-a}^{0}\left(t-B(t)^2\right)\mathrm{d}B(t) - \frac{\beta}{8}\int_{-a}^{0}\left(t-B(t)^2\right)^2\mathrm{d}t\big); B(t)\in[0,2]\ \forall\ t\leqslant 0\Big),\end{aligned}$$

where B_t is a Brownian motion with diffusion coefficient $2/\sqrt{\beta}$. On the event in question,

$$\frac{\beta}{8}\int_{-a}^{0}\left(t-B_t^2\right)^2\mathrm{d}t = \frac{\beta}{24}a^3 + \underset{a\to+\infty}{O}\left(a^2\right),$$

and

$$\int_{-a}^{0}\left(t-B(t)^2\right)\mathrm{d}B(t) = aB(-a) + \frac{1}{3}\left(B(-a)^3 - B(0)^3\right) + \left(\frac{4}{\beta}-1\right)\int_{-a}^{0}B(t)\mathrm{d}t = \underset{a\to+\infty}{O}(a).$$

To finish, note that $\mathbb{P}_{(-a,0)}\big(\forall\ t\leqslant 0,\ B(t)\in[0,2]\big) \geqslant e^{-ca}$ does not interfere with the leading order. We have proved

$$\mathbb{P}(TW_\beta < -a) \geqslant \exp\Big(-\frac{1}{24}\beta a^3(1 + \underset{a\to+\infty}{o}(1))\Big).$$

To illustrate Theorem 4.3.1 numerically, we approximate TW_1 by the empirical distribution of the largest eigenvalue of a size 10^3 Gaussian Orthogonal Matrix, computed over 10^4 samples. Figure 11 compares this empirical distribution with the theoretical tail estimates.

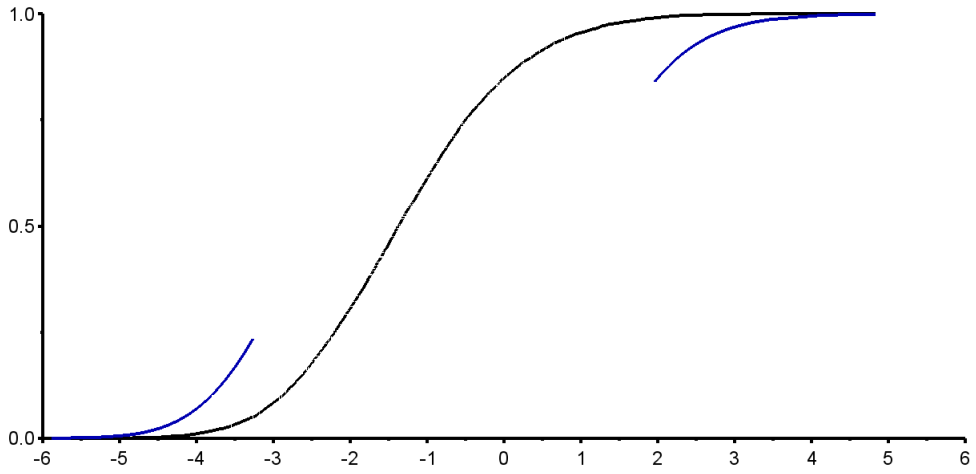

Figure 11: TW_1 **(black) and left and right tail estimates (blue).**

4.4 The hard-to-soft transition

Borodin and Forrester [14] proved a transition between the hard and soft edge distributions at $\beta = 1, 2$, and 4. In the unitary case $\beta = 2$, we can use the determinantal structure from Chapter 2, Section 2.2 to write these distributions in terms of *Fredholm determinants*:

$$P\left(\Lambda_0(\mathfrak{G}_a^2) > \lambda\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_0^\lambda \mathrm{d}x_1 \dots \int_0^\lambda \mathrm{d}x_k \det\left[K^{\text{Bessel}}(x_i, x_j)\right]_{i,j=1,\dots,k},$$

and

$$P\left(TW_2 < \lambda\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_\lambda^\infty \mathrm{d}x_1 \dots \int_\lambda^\infty \mathrm{d}x_k \det\left[K^{\text{Airy}}(x_i, x_j)\right]_{i,j=1,\dots,k},$$

where the kernels K^{Airy} and K^{Bessel} are as in Section 2.2. Using the the Riccati transform and characterization (4.1.4), Ramírez and Rider showed the same transition holds at all $\beta > 0$.

Theorem 4.4.1 (Hard to soft transition [63, Theorem 3])**.**
For $\beta > 0$,

$$\frac{\eta - \Lambda_0(\mathfrak{G}^\beta_{2\sqrt{\eta}-2/\beta})}{\eta^{2/3}} \xrightarrow[\eta\to\infty]{\mathrm{d}} TW_\beta.$$

Chapter 5

The stochastic Airy operator

The inverse temperature parameter β shapes the global and local behavior of the β-ensembles. The last two chapters of this thesis study respectively the β-Hermite ensemble from (1.1.2) and the β-Laguerre ensemble from (1.1.4) in the double limit $n \to \infty,\ \beta \to 0$, which we call *high-temperature regime*.

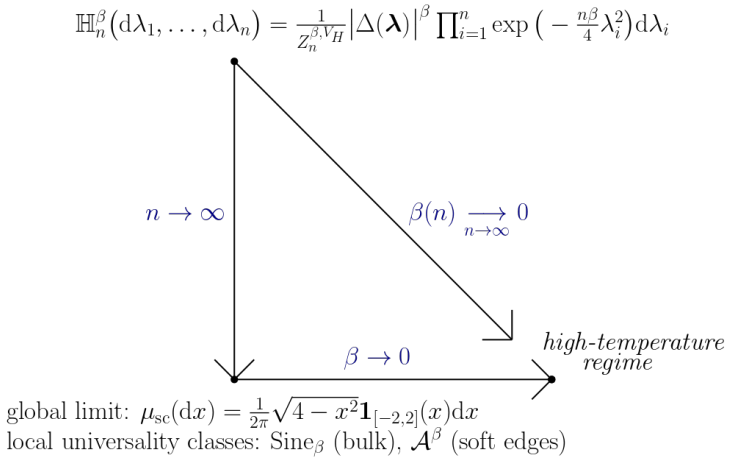

Figure 12: **High-temperature regime for $\mathbb{H}_n^\beta$.**

The high-temperature regime can be obtained in different ways. Taking $\beta = \beta(n)$ allows to achieve simultaneously $n \to \infty$ and $\beta \to 0$ and yields Poissonian limit local statistics for the eigenvalues in the bulk (see [61], [27], [11]) and at the edges (see [5], [8] and [62]), for specific $\beta(n)$ regimes.
In this thesis, we achieve the high-temperature regime by taking first $n \to \infty$ and then $\beta \to 0$. The limit local statistics of the β-Hermite ensemble are determined by the behavior, when $\beta \to 0$, of the Sine_β operator from Theorem 3.1.1 (in the bulk) and of the stochastic Airy operator $\mathcal{A}^\beta$ from Definition 3.1.5 (at the edges). For the bulk statistics, Allez and Dumaz [6] proved the convergence of the Sine_β point process to a Poisson point process on $\mathbb{R}$.
We focus here on the (soft) edges statistics. Dumaz and Labbé [22] proved the convergence of the rescaled eigenvalues of $\mathcal{A}_\beta$ to a Poisson point process on $\mathbb{R}$. This chapter is dedicated to the proof of this result, stated in Theorem 5.1.1,

with techniques that we use again in Chapter 6 in our study of the stochastic Bessel operator.

5.1 Statement of results

Recall that $\Lambda_0(\mathcal{A}^\beta) < \Lambda_1(\mathcal{A}^\beta) < \dots$ denote the smallest eigenvalues of $\mathcal{A}_\beta$.

Theorem 5.1.1 ($\mathcal{A}^\beta$ at high temperature [22, Theorem 1])**.**
Setting $c_\beta = \left(\frac{3}{2\beta}\ln\frac{1}{\pi\beta}\right)^{2/3}$*, we have:*

$$\left(\beta\sqrt{c_\beta}(\Lambda_k(\mathcal{A}^\beta) + c_\beta)\right)_{k\geqslant 0} \xrightarrow[\beta\to 0]{\mathrm{d}} (\Lambda_k)_{k\geqslant 0},$$

where $(\Lambda_k)_{k\geqslant 0}$ *are the atoms of a Poisson point process on* $\mathbb{R}$ *with intensity* $e^x\mathrm{d}x$*.*

To prove Theorem 5.1.1, we follow the arguments of Allez and Dumaz [7] and of Dumaz and Labbé [22], who use the fruitful connection between the eigenvalues of $\mathcal{A}^\beta$ and the diffusions obtained through its Riccati map (see Chapter 4, Section 4.2). To this end, we introduce the stochastic linear operator $\mathcal{L}^\beta$, for $x > 0$:

$$\mathcal{L}^\beta = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\beta}{4}x + B'(x).$$

The operator $\mathcal{L}^\beta$ proves more convenient to study than $\mathcal{A}^\beta$ in the high temperature regime because it removes the $2/\sqrt{\beta}$ term, which diverges as $\beta \to 0$. Denoting by $\Lambda_0(\mathcal{L}^\beta) < \Lambda_1(\mathcal{L}^\beta) < \dots$ its eigenvalues, the change of function $\tilde{\psi}(x) = c\psi(x/c)$ with $c = (4/\beta)^{1/3}$ in the differential equation (3.1.7) satisfied by the eigenfunctions ψ of $\mathcal{A}^\beta$ shows that:

$$\left\{\Lambda_k(\mathcal{L}^\beta), k \in \mathbb{N}\right\} \overset{\mathrm{d}}{=} \left(\frac{\beta}{4}\right)^{2/3}\left\{\Lambda_k(\mathcal{A}^\beta), k \in \mathbb{N}\right\}. \tag{5.1.1}$$

Implementing the Riccati transform from Chapter 4 on $\mathcal{L}^\beta$, we obtain a new family of diffusions on $\mathbb{R}_+$ (see [22, Section 2]).

Proposition 5.1.1 (A second family of diffusions [22, Theorem 1])**.**
Let $\beta > 0$ *and* B *a Brownian motion. Introduce* $(Z_a)_{a\in\mathbb{R}}$*, the family of diffusions on* $\mathbb{R}_+$*:*

$$\mathrm{d}Z_a(t) = (a + \frac{\beta}{4}t - Z_a(t)^2)\mathrm{d}t + \mathrm{d}B(t),\ Z_a(0) = +\infty,\ a \in \mathbb{R}. \tag{5.1.2}$$

When exploding to $-\infty$*,* Z_a *is immediately restarted from* $+\infty$*.*
We have the key equality of events:

$$\{\Lambda_k(\mathcal{L}^\beta) \leqslant a\} = \{Z_{-a} \text{ blows up to } -\infty \text{ a least } k+1 \text{ times in } \mathbb{R}_+\}. \tag{5.1.3}$$

The proof of Theorem 5.1.1 has three steps. In Section 5.2, we study the explosions process of a stationary version of the diffusion Z_a. In Section 5.3, we prove the convergence of the explosions process of Z_a using stationary approximations. In Section 5.4, we derive the convergence of the eigenvalues process of $\mathcal{A}^\beta$.

5.2 A family of stationary diffusions

5.2.1 Statement of results

Definition 5.2.1 (A family of stationary diffusions)**.**
For B a Brownian motion, $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R} \cup \{+\infty\}$, introduce $(Y_a^{(t_0,x_0)})_{a\in\mathbb{R}}$ the family of stationary diffusions on $\mathbb{R}_+$:

$$\mathrm{d}Y_a^{(t_0,x_0)}(t) = \left(a - Y_a^{(t_0,x_0)}(t)^2\right)\mathrm{d}t + \mathrm{d}B(t),\ Y_a^{(t_0,x_0)}(t_0) = x_0,\ a \in \mathbb{R}. \quad (5.2.1)$$

Each time $Y_a^{(t_0,x_0)}$ hits $-\infty$, it is restarted immediately from $+\infty$.

The diffusion $Y_a^{(t_0,x_0)}$ evolves in a potential $V^Y(z) = -az + z^3/3$, which does not depend on time t. Consequently, Y_a can be seen as *a stationary version of Z_a in the case $\beta = 0$*, and understanding its explosions process is the first step to prove Theorem 5.1.1. To this end, we will use the following key property.

Proposition 5.2.1 (Monotonicity property of $Y_a^{(t_0,x_0)}$ [22, Section 4.1])**.**
Almost surely, for all $a \leqslant a'$, all (t_0, x_0), (t_0', x_0') and all $s \in [t_0 \vee t_0']$,

$$\text{if } Y_a^{(t_0,x_0)}(s) \leqslant Y_a^{(t_0',x_0')}(s) \text{ then } Y_a^{(t_0,x_0)}(s+\cdot) \leqslant Y_a^{(t_0',x_0')}(s+\cdot), \quad (5.2.2)$$

up to the next explosion time of $Y_a^{(t_0,x_0)}$.

Definition 5.2.2 (Explosion times point process of $Y_a^{(0,+\infty[}$)**.**
For $a \in \mathbb{R}$, introduce the first explosion time *of $Y_a^{(t_0,x_0)}$:*

$$\zeta^{(t_0,x_0)}(a) = \inf\{t \geqslant 0 : Y_a^{(t_0,x_0)}(t) = -\infty\}.$$

The diffusion $Y_a^{(0,+\infty[}$ blows up to $-\infty$ in a finite time almost surely (later proved in (5.2.4)) and is then reset at $+\infty$, so we can define the successive explosions times:

$$\zeta_1(a) = \zeta^{(0,+\infty[}(a) < \zeta_2(a) = \zeta^{(\zeta_1,+\infty[}(a) < \zeta_3(a) = \zeta^{(\zeta_2,+\infty[}(a)\ldots$$

The explosion times point process of $Y_a^{(0,+\infty[}$ *is the empirical measure μ_a of the rescaled explosion times:*

$$\mu_a(B) = \sum_{k=1}^{+\infty} \delta_{\zeta_k(a)/m(a)}(B),\ \textit{for } B \textit{ Borel set of } \mathbb{R}_+,$$

where $m(a)$ is later defined in (5.2.5). In particular, the random variable $\mu_a([0,t])$ is the number of explosions of the diffusion $Y_a^{(0,+\infty[}$ in the interval $[0, m(a)t]$.

We can now state the main result of this section.

Theorem 5.2.1 (Vague convergence of μ_a [7, Theorem 3.6])**.**
When $a \to +\infty$, the explosion times point process μ_a of the diffusion $Y_a^{(0,+\infty[}$ converges weakly (in the space of Radon measures on $\mathbb{R}_+$ equipped with the topology of vague convergence [47])to a Poisson point process with intensity 1 on $\mathbb{R}_+$.

5.2.2 Proof of Theorem 5.2.1

First, we study the law of the first explosion time. The potential V^Y presents a local minimum in $z = \sqrt{a}$ and a local maximum in $z = -\sqrt{a}$. The potential barrier $\Delta V^Y = \frac{4}{3}a^{3/2}$ gets very large when $a \to +\infty$, while the noise remains constant. A typical sample path of the diffusion spends most of its time near the bottom of the well, and from time to time manages to reach the unstable equilibrium point $-\sqrt{a}$, from where it either explodes to $-\infty$ or comes back to the bottom of the well within a short time. From Kramer's theory [39], we expect $\zeta^{(t_0,x_0)}(a)$ to be distributed according to an exponential law with parameter $\exp(-2\Delta V^Y)$. To prove this, consider the Laplace transform:

$$g_\alpha(x) = \mathbb{E}\left[e^{-\alpha\zeta^{(0,x)}(a)}\right]. \tag{5.2.3}$$

Using classical theory on infinitesimal generators of diffusion processes (see [43]), we get the following characterization.

Proposition 5.2.2 (Laplace transform of $\zeta^{(t_0,x_0)}(a)$ [7, Proposition 3.1])**.** *Let $\alpha > 0$. The function g_α defined in (5.2.3) is the unique bounded and twice continuously differentiable solution of the boundary value problem*

$$\frac{1}{2}g''_\alpha - \left(x^2 - a\right)g'_\alpha = \alpha g_\alpha,\ g_\alpha(x) \to 1 \text{ when } x \to -\infty.$$

In addition, it satisfies the fixed point equation

$$g_\alpha(x) = 1 - 2\alpha\int_{-\infty}^{x}\mathrm{d}t\int_{t}^{+\infty}\mathrm{d}s\exp\left(2a(s-t) + \frac{2}{3}\left(t^3 - s^3\right)\right)g_\alpha(u).$$

The probability that $Y_a^{(0,x)}$ explodes in a finite time is given by taking $\alpha = 0$:

$$g_0(x) := \lim_{\alpha\downarrow 0}\mathbb{E}(e^{-\alpha\zeta^{(0,x)}(a)}) = \mathbb{P}(\zeta^{(0,x)}(a) < +\infty[= 1. \tag{5.2.4}$$

The mean exit time $m^{(0,x)}(a)$ of the diffusion starting at time $t = 0$ from position x is obtained by differentiating $g_\alpha(x)$ with respect to α and taking $\alpha = 0$.

$$m^{(0,x)}(a) = 2\int_{-\infty}^{x}\mathrm{d}t\int_{t}^{+\infty}\mathrm{d}u\exp\left(2a(u-t) + \frac{2}{3}\left(t^3 - u^3\right)\right).$$

Set $m(a) = m^{(0,+\infty[}(a)$. Its asymptotic when $a \to +\infty$ can be determined using the saddle point method (see [7], Appendix C):

$$m(a) = \frac{\pi}{\sqrt{a}}\exp\left(\frac{8}{3}a^{3/2}\right)\left(1 + \frac{5}{48}\frac{1}{a^{3/2}} + \underset{a\to+\infty}{o}\left(\frac{1}{a^3}\right)\right). \tag{5.2.5}$$

Theorem 5.2.2 shows that, when $a \to +\infty$, the explosion time $\zeta^{(t_0,x_0)}(a)$ rescaled by its mean $m(a)$ converges to the exponential distribution with parameter 1, faithfully to Kramer's theory. This result is valid for a large range of starting points, mainly all the points which are in the potential well (i.e. above the local maximum of the potential).

Theorem 5.2.2 (Convergence in law of $\zeta^{(t_0,x_0)}(a)$ [7, Theorem 3.3])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ such that $a^{1/4}\left(f(a) + a^{1/2}\right) \underset{a\to+\infty}{\to} +\infty$. Then,*

$$\sup_{x \geqslant f(a)} \left| g_{\alpha/m(a)}(x) - \frac{1}{1+\alpha} \right| \underset{a\to+\infty}{\longrightarrow} 0.$$

In particular, for any $x \geqslant f(a)$, the first rescaled explosion time $\zeta^{(0,x)}(a)/m(a)$ converges in law when $a \to +\infty$ to an exponential law with parameter 1.

We can now prove the convergence of the explosion times point process to a Poisson point process with intensity 1 on $\mathbb{R}_+$.

Proof of Theorem 5.2.1. Using the criterion from Kallenberg [47], it is sufficient to prove that, for any finite union I of disjoint and bounded intervals, $a \to +\infty$,

$$\mathbb{E}\big(\mu_a(I)\big) \underset{a\to+\infty}{\longrightarrow} |I|, \tag{5.2.6}$$

$$\mathbb{P}\big(\mu_a(I) = 0\big) \underset{a\to+\infty}{\longrightarrow} \exp\big(-|I|\big), \tag{5.2.7}$$

where $|I|$ denotes the length of the set I. For (5.2.6), by linearity, we just need to prove that $\mathbb{E}(\mu_a[0,t]) \underset{a\to+\infty}{\longrightarrow} t$. With Theorem 5.2.2, we have

$$\mathbb{E}\big(\mu_a[0,t]\big) = \sum_{k=0}^{+\infty} \mathbb{P}\big(\mu_a[0,t] \geqslant k\big) = 1 + \sum_{k=1}^{+\infty} \mathbb{P}\Big(\frac{\zeta_k(a)}{m(a)} \leqslant t\Big).$$

For each k, the strong Markov property tells us that the random variable $\zeta_{k+1} - \zeta_k$ is independent of $(\zeta_1, \zeta_2 - \zeta_1, \ldots, \zeta_k - \zeta_{k-1})$ and has the same distribution as ζ_1. Therefore, $\zeta_k(a)/m(a)$ converges in law when $a \to +\infty$ to the Gamma distribution $\Gamma(k,1)$. Using Cramer's large deviation principle, one can prove the existence of a constant $C > 0$ independent of k such that $\mathbb{P}\left[\zeta_k(a)/m(a) \leqslant t\right] \leqslant C/k^2$. The bounded convergence theorem applies and:

$$\mathbb{E}\big(\mu_a[0,t]\big) \underset{a\to+\infty}{\longrightarrow} 1 + \sum_{k=1}^{+\infty} \mathbb{P}\big(\Gamma(k,1) \leqslant t\big) = t.$$

We now turn to the second equality (5.2.7). For any $k \geqslant 1$, the strong Markov property for the diffusion $Y_a^{(0,+\infty[}$ shows that the first k explosions times converge jointly in law to the first k occurrence times $\xi_1 < \xi_2 < \ldots < \xi_k$ of a Poisson point process with intensity 1:

$$\frac{1}{m(a)}\big(\zeta_1(a), \zeta_2(a), \ldots, \zeta_k(a)\big) \xrightarrow[a\to+\infty]{\mathrm{d}} (\xi_1, \xi_2, \ldots, \xi_k). \tag{5.2.8}$$

Indeed, the increments are independent and each of them converges to an exponential distribution thanks to Theorem 5.2.2. A large deviation argument can again bound the probability of having more than k explosion times before time $m(a)t$. Using the convergence (5.2.8) on the overwhelming event $\{\zeta_k(a) > m(a)t\}$, we get the convergence (5.2.7). □

5.3 Convergence of the explosions times process

5.3.1 A first convergence result

We now study the explosions times point process of the non-stationary diffusion Z_a using approximations by the stationary diffusion Y_a. If β tends to 0 while a is fixed, the diffusion Z_a converges in law to the stationary diffusion Y_a, which explodes at least k times with probability one, for any fixed $k \in \mathbb{N}$. Therefore, if we look for a non trivial limit in law for the eigenvalue $\Lambda_k(\mathcal{L}^\beta)$ when $\beta \to 0$, the parameter a should increase to $+\infty$. Consider the rescaling $a_{L^\beta} = m^{-1}(L^\beta)$, with $L^\beta = \beta^{-1}(\frac{3}{8}\ln 1/\beta)^{-1/3}$. Dumaz and Labbé [22, Section 4.1] proved that

$$a_{L^\beta} \underset{\beta\to 0}{=} \left(\frac{3}{8}\ln\frac{1}{\beta}\right)^{2/3}\left(1 - \frac{2}{3}\frac{\ln\pi}{\ln(1/\beta)} + \underset{\beta\to 0}{o}\Big(\frac{1}{\ln(1/\beta)}\Big)\right). \tag{5.3.1}$$

To account for the fact that Z_a feels the evolution in time due to the linear term $\frac{\beta}{4}$ in the drift, introduce $a^\beta(r) = a_{L^\beta} - \frac{r}{4\sqrt{a_{L^\beta}}}$, $r \in \mathbb{R}$. The asymptotic in (5.3.1) shows us that

$$a^\beta(r) \underset{\beta\to 0}{=} a_{L^\beta} - \frac{1}{2}\frac{1}{3^{1/3}}\ln\big(\frac{1}{\beta}\big)^{-1/3} r + \underset{\beta\to 0}{o}\Big(\frac{1}{(\ln\frac{1}{\beta})^{4/3}}\Big).$$

This definition is motivated by the fact that, between times 0 and $L^\beta\, t$, the first part of the drift of $Z_{a_{L^\beta}}$ evolves from a_{L^β} to

$$a_{L^\beta} + \frac{\beta}{4}L^\beta\, t \underset{\beta\to 0}{\sim} a^\beta(t).$$

Fix $r \in \mathbb{R}$ and introduce $(\zeta_k)_{k\in\mathbb{N}^*}$ the family of explosion times of $Z_{a^\beta(r)}$ (with no risks of confusion with the explosions of Y^a from Section 5.2, which we no longer use), where we omit the dependency on β and r to alleviate notations.

Definition 5.3.1 (Explosion times point process of $Z_{a^\beta(r)}$)**.**
The explosion times point process of $Z_{a^\beta(r)}$ *is the empirical measure* ν_β*,*

$$\nu_\beta(B) = \sum_{k=1}^{+\infty}\delta_{\zeta_k/L^\beta}(B) = \sum_{k=1}^{+\infty}\delta_{\zeta_k/m(a_{L^\beta})}(B),\ \textit{for } B \textit{ Borel set of } \mathbb{R}_+.$$

For almost all r, the number $\nu_\beta(\mathbb{R}_+)$ of explosions in $\mathbb{R}_+$ is equal to the number of eigenvalues smaller than $-a^\beta(r)$.

Theorem 5.3.1 (Vague convergence of ν_β [7, Theorem 4.1])**.**
When $\beta \to 0$, the explosion times point process ν_β converges weakly (in the space of Radon measures equipped with the topology of vague convergence [47]) to a Poisson point process with inhomogeneous intensity $e^r \times e^{-t}\mathrm{d}t$. It readily implies the following convergence.
For any $t < t', k \in \mathbb{N}$,

$$\mathbb{P}\Big(\nu_\beta([t;t']) = k\Big) \underset{\beta\to 0}{\longrightarrow} \exp\left(-e^r\int_t^{t'} e^{-x}\mathrm{d}x\right)\frac{\Big(e^r\int_t^{t'} e^{-x}\mathrm{d}x\Big)^k}{k!}.$$

Proof of Theorem 5.3.1. For simplification purposes, we notate here Z the diffusion $Z_{a^\beta(r)}$. Again, from Kallenberg's criterion [28], it is sufficient to prove that, for any finite union I of disjoint and bounded intervals,

$$\mathbb{E}(\nu_\beta(I)) \underset{\beta\to 0}{\longrightarrow} e^r \int_I e^{-t}\mathrm{d}t \tag{5.3.2}$$

$$\mathbb{P}(\nu_\beta(I)=0) \underset{\beta\to 0}{\longrightarrow} \exp\left(-e^r \int_I e^{-t}\mathrm{d}t\right). \tag{5.3.3}$$

Write $I = J \cup [t_1;t_2]$ and let $t_0 < t_1$ be the supremum of J. Set $s_i = L^\beta\ t_i$ for $i = 0,1,2$. By linearity of the expectation, it is enough to prove (5.3.2) for intervals I of the form $I=[0,t]$. For (5.3.3), the simple Markov property yields

$$\mathbb{P}(\nu_\beta(I)=0) = \mathbb{P}\Big(\mathbf{1}_{\nu_\beta(J)=0}\mathbb{P}_{Z(s_0),s_0}\Big(\nu_\beta(\,[t_1;t_2]\,)=0\Big)\Big),$$

where $\mathbb{P}_{Z(s_0),s_0}$ is the law of the diffusion Z started at time s_0 with value $Z(s_0)$. Therefore, showing (5.3.3) reduces to see that with probability going to 1 as $\beta\to 0$,

$$\mathbb{P}_{Z(s_0),s_0}\Big(\nu_\beta(\,[t_1;t_2]\,)=0\Big) \underset{\beta\to 0}{\longrightarrow} \exp\left(-e^r\int_{t_1}^{t_2} e^{-t}\mathrm{d}t\right).$$

The idea is to decompose the interval $[t_1;t_2]$ (with $t_1=0$ for (5.3.2)) into a finite number of small intervals and to approximate the number of explosions of Z on each small interval of the subdivision by those of stationary diffusions. Define a sequence $(\tau_k)_{k\in\mathbb{N}}$ of i.i.d. random variables with uniform law in $[0,1]$, independent of the diffusion Z. Let δ small enough such that $0<\delta<s_1-s_0$. Then, construct iteratively the sequence of random times $(S_k)_k \geqslant 0$ such that $S_0 = s_1 - \delta\tau_0$, $S_k = S_{k-1} + \delta\tau_k$ for $k\geqslant 1$. On each interval $[S_k, S_{k+1}]$ of the random subdivision, define two diffusions m_k and M_k (independent of the times τ_k) driven by the same Brownian motion as Z:

$$m_k(S_k) = Z(S_k) \text{ and } \mathrm{d}m_k(s) = \left(a^\beta(r) + \frac{\beta}{4}S_k - m_k^2(s)\right)\mathrm{d}s + \mathrm{d}B_s \text{ for } s\in[S_k,S_{k+1}],$$

$$M_k(S_k) = Z(S_k) \text{ and } \mathrm{d}M_k(s) = \left(a^\beta(r) + \frac{\beta}{4}S_{k+1} - M_k^2(s)\right)\mathrm{d}s + \mathrm{d}B_s \text{ for } s\in[S_k,S_{k+1}].$$

The drifts of the diffusion $m_k(s), Z(s)$ and $M_k(s)$ are in increasing order for $s\in[S_k,S_{k+1}]$: $a^\beta(r)+\frac{\beta}{4}S_k \leqslant a^\beta(r)+\frac{\beta}{4}s \leqslant a^\beta(r)+\frac{\beta}{4}S_{k+1}$. The monotonicity property of the stationary diffusions implies that the number of explosions $\nu_\beta(\,[S_k,S_{k+1}]\,)$ of the diffusion Z is stochastically dominated from above (resp. from below) by the number of explosions of the diffusion m_k (resp. M_k).

It remains to check that the starting points of m_k and M_k fall within the hypotheses of Theorem 5.2.2, with probability one as β tends to 0. For those technical estimates, we refer the reader to [7, Section 4.2]. Theorem 5.2.1 adapted to m_k and M_k then shows that their respective explosions point processes converge weakly in the space of Radon measure, when the time scale is renormalized respectively by $m\left(a^\beta(r)+\beta S_k/4\right)$ and $m\left(a^\beta(r)+\beta S_{k+1}/4\right)$, to Poisson point processes with intensity 1, independently of the exact location of $Z(S_k)$. This proves (5.3.2) and (5.3.3) and concludes the proof of the theorem. □

5.3.2 A stronger convergence result

A shortcoming of Theorem 5.3.1 is that the topology of vague convergence of Radon measures does not allow to control the mass at infinity, while this is required to study the eigenvalues of the operator $\mathcal{L}^\beta$. To prove the convergence of the eigenvalues process in the next section, we will need to consider the topology of weak convergence of finite measures. This was done by Dumaz and Labbé [22], who controlled the behavior of the diffusion Z at infinity to improve Theorem 5.3.1 and obtained the following convergence.

Theorem 5.3.2 (Weak convergence of ν_β [22, Theorem 4])**.**
Fix $r \in \mathbb{R}$. As $\beta \to 0$, the random measure ν_β converges in law for the topology of weak convergence of finite measures to a Poisson point process on $\mathbb{R}_+$ with intensity $e^r e^{-t}\mathrm{d}t$.

As a straightforward application of Theorem 5.3.1, we get the progressive deformation of TW_β into a *Gumbel law* when $\beta \to 0$.

Theorem 5.3.3 (From Tracy Widom to Gumbel [7, Corollary 3.5])**.**
When properly rescaled and centered, the Tracy-Widom law TW_β converges weakly to the Gumbel law. More precisely, the following convergence holds:

$$2\cdot 3^{1/3}\cdot\left(\ln\frac{1}{\beta}\right)^{1/3}\left(\left(\frac{\beta}{4}\right)^{2/3}TW_\beta-\left(\frac{3}{8}\right)^{2/3}\left(\ln\frac{1}{\beta\pi}\right)^{2/3}\right)\underset{\beta\to 0}{\longrightarrow} e^{-x}\exp\left(-e^{-x}\right)\mathrm{d}x.$$

The Gumbel law governs the fluctuations of the maximum of independent Gaussian variables, so this convergence in consistent with the independence of eigenvalues showed by the limit Poissonian statistics of Theorem 5.3.1.

5.4 Convergence of the eigenvalues process

We now sketch the proof of Theorem 5.1.1, deriving from Theorem 5.3.2 the convergence of the marginal distributions of the bottom eigenvalues of the linear stochastic operator $\mathcal{L}^\beta$ when $\beta \to 0$. We introduce a discretization scheme to approximate the diffusion Z_a. First, let $0 = t_0^n < t_1^n < \ldots < t_{2^n}^n = +\infty$ be the points that satisfy:

$$\int_{t_j^n}^{t_{j+1}^n} e^{-s}\mathrm{d}s = \frac{1}{2^n}, \quad \forall j \in \{0,\ldots,2^n-1\}.$$

Using Theorem 5.3.2, as $\beta \to 0$, the number of explosions of the diffusion $Z_{a^\beta(r)}$ in the time interval $[t_j^n L^\beta, t_{j+1}^n L^\beta]$ is given by a Poisson random variable of intensity $2^{-n}e^r$ and the first eigenvalues of $\mathcal{L}^\beta$ typically deviate from a_{L^β} like $1/\sqrt{a_{L^\beta}}$. Therefore, we set $\varepsilon > 0$ and introduce the grid around a_{L^β}:

$$\mathcal{M}_\varepsilon^\beta = \left\{ a_{L^\beta} + p\frac{\varepsilon}{4\sqrt{a_{L^\beta}}} : p \in \mathbb{Z} \cap \left[-1/\varepsilon^2, 1/\varepsilon^2\right] \right\}.$$

For every $j \in \{0, \dots, 2^n - 1\}$ and every $a \in \mathcal{M}_\varepsilon^\beta$, we use the diffusion $Z_a^{\left(t_j^n L^\beta, +\infty\right)}$, notated Z_a^j, to approximate the diffusion Z_a on the interval $[t_j^n L^\beta, t_{j+1}^n L^\beta]$. The justification behind this approximation is provided by the following lemma, we refer to [22] for its technical proof.

Lemma 5.4.1 (Approximation of Z_a by Z_a^j [22, Lemma 5.1])**.**
With a probability going to 1 *as* β *tends to* 0 *and then* n *goes to* ∞*, the following holds. For all* $a \in \mathcal{M}_\varepsilon^\beta$ *and all* $j \in \{0, \dots, 2^n - 1\}$*:*

- Z_a *explodes at most one time on* $(t_j^n L^\beta, t_{j+1}^n L^\beta]$*,*
- Z_a *explodes on* $(t_j^n L^\beta, t_{j+1}^n L^\beta]$ *if and only if* Z_a^j *explodes on* $(t_j^n L^\beta, t_{j+1}^n L^\beta]$*.*

Denote by $(q_i)_{i=1\dots m}$ the elements of $\mathcal{M}_\varepsilon^\beta$ in decreasing order $q_1 > \dots > q_m$ and let r_i be such that

$$q_i = a_L - \frac{r_i}{4\sqrt{a}_L}, \quad i = 1, \dots, m.$$

For every $j \in \{0, \dots, 2^n - 1\}$ set $V_j(i) = 1$ if the diffusion $Z_{q_i}^j$ explodes on $(t_j^n L^\beta, t_{j+1}^n L^\beta]$, and $V_j(i) = 0$ otherwise. Set also $q_0 = +\infty$, $r_0 = -\infty$ and $V_j(0) = 0$. Finally, define

$$Q_n^\beta(i) = \sum_{j=0}^{2^n-1} \left(V_j(i) - V_j(i-1)\right), \quad i = 1 \dots m.$$

For every i, the r.v. $Q_n^\beta(i)$ counts the number of intervals $(t_j^n L^\beta, t_{j+1}^n L^\beta]$ where the diffusion $Z_{q_i}^j$ explodes but the diffusion $Z_{q_{i-1}}^j$ does not. By Lemma 5.4.1, this is a good approximation of the total number of explosions of Z_{q_i} minus the total number of explosions of $Z_{q_{i-1}}$ in the $\beta \to 0$ and $n \to +\infty$ limit.

Lemma 5.4.2 (Convergence of $\left(Q_n^\beta(i)\right)_{i=1,\dots,m}$ [22, Lemma 5.2])**.**
The vector $\left(Q_n^\beta(i)\right)_{i=1,\dots,m}$ *converges in distribution as* $\beta \to 0$ *and* $n \to +\infty$ *to a vector of independent Poisson random variables with parameters* $p_i = \int_{r_{i-1}}^{r_i} e^s \, \mathrm{d}s$*.*

Proof. By the monotonicity property of the stationary diffusions, for any given $j \in \{0, \dots, 2^n - 1\}$, the diffusions $(Z_{q_i}^j, i = 1, \dots, m)$ on the interval $[t_j^n L^\beta, \infty[$ are ordered up to their first explosion times, so $V_j(1) \leqslant V_j(2) \leqslant \dots \leqslant V_j(m)$. Since these r.v. are $\{0, 1\}$-valued, we get the identities:

$$\begin{aligned}
\mathbb{P}\left(V_j(1) = 0, \dots, V_j(i-1) = 0, V_j(i) = 1, \dots, V_j(m) = 1\right) &= \mathbb{P}\left(V_j(i) = 1\right) - \mathbb{P}\left(V_j(i-1) = 1\right) \\
\mathbb{P}\left(V_j(1) = 0, \dots, V_j(m) = 0\right) &= \mathbb{P}\left(V_j(m) = 0\right) \\
\mathbb{P}\left(V_j(1) = 1, \dots, V_j(m) = 1\right) &= \mathbb{P}\left(V_j(1) = 1\right).
\end{aligned}$$

Theorem 5.3.2 yields:

$$\mathbb{P}\big(V_j(i) = 1\big) \underset{\beta \to 0}{\longrightarrow} 1 - \exp\left(-2^{-n} e^{r_i}\right). \tag{5.4.1}$$

Noting that the random vectors $\big(V_j(1), V_j(2), \dots, V_j(m)\big)_{j=0,\dots,2^n-1}$ are independent, we now compute the law of $\big(Q_n^\beta(i)\big)_{i=1,\dots,m}$. For any given integers $\ell_1, \dots, \ell_m$, set $\ell = \sum_i \ell_i$. Then $\mathbb{P}\big(Q_n^\beta = (\ell_1, \dots, \ell_m)\big)$ equals

$$\sum_{\substack{S_1,\dots,S_m \subset \{0,\dots,2^n-1\} \\ S_i \cap S_{i'} = \emptyset \\ \#S_i = \ell_i}} \prod_{j \in S_i} \Big(\mathbb{P}(V_j(i) = 1) - \mathbb{P}(V_j(i-1) = 1) \Big) \prod_{j \notin S_1 \cup \dots \cup S_m} \mathbb{P}(V_j(m) = 0).$$

Using (5.4.1), we deduce that the $\beta \to 0$ limit of the last expression equals

$$\binom{2^n}{\ell_1, \dots, \ell_m, 2^n - \ell} \prod_{i=1}^m \Big(\exp\big(-2^{-n} e^{r_{i-1}}\big) - \exp\big(-2^{-n} e^{r_i}\big) \Big)^{\ell_i} \Big(\exp\big(-2^{-n} e^{r_m}\big) \Big)^{2^n - \ell}.$$

As $n \to \infty$, one can show that this converges to the required quantity $\prod_{i=1}^m \frac{p_i^{\ell_i}}{\ell_i!} e^{-p_i}$. □

To conclude the proof of Theorem 5.1.1, we define

$$\mathcal{Q}^\beta = \sum_{k \geqslant 1} \delta_{4\sqrt{a_{L^\beta}}(\lambda_k + a_{L^\beta})}$$

as a random variable in the space of measures on $(-\infty, \infty[$ which are finite on all intervals bounded to the right. We endow this space with the weak topology towards $-\infty$ and the vague topology towards $+\infty$, so as to control the increasing sequence of atom locations of $\mathcal{Q}^\beta$ from its first point.
For any $\varepsilon > 0$, on the event on which Lemma 5.4.1 holds true, we have, for every $i \in \{1, \dots, m\}$,

$$\mathcal{Q}^\beta\big((r_{i-1}, r_i] \big) = Q_n^\beta(i).$$

Thus, by Lemma 5.4.2, the random vector $\mathcal{Q}^\beta\big((r_{i-1}, r_i] \big), i = 1, \dots, m$, converges in distribution as $\beta \to 0$ to a vector of independent Poisson random variables of intensity $e^{r_i} - e^{r_i - 1}$. This provides the required control on the mass given by $\mathcal{Q}_L$ to $(-\infty, r]$ for any given r to obtain the tightness of $\Big(\mathcal{Q}^\beta\Big)_\beta$. Furthermore, the marginals of any limiting point are uniquely identified thanks to this convergence. Therefore, $(\mathcal{Q}^\beta)_\beta$ converges in law to a Poisson point process of intensity $e^x \mathrm{d}x$, and then standard arguments ensure that the increasing sequence of its atom locations converges in law for the product topology to the increasing sequence of atom locations of this Poisson point process.

Chapter 6

The stochastic Bessel operator

6.1 Introduction

This chapter presents our results on the high temperature limit of the eigenvalues of the stochastic Bessel operator introduced by Ramírez and Rider [63].

Recall that the Stochastic Bessel operator (SBO for short) is a random differential operator depending on two parameters $\beta > 0$ and $a > -1$ which writes:

$$\mathfrak{G}_a^\beta = -\exp\Big((a+1)x + \frac{2}{\sqrt{\beta}}B(x)\Big)\cdot\frac{d}{dx}\Big(\exp\Big(-ax - \frac{2}{\sqrt{\beta}}B(x)\Big)\frac{d}{dx}\Big)\,.$$

In this paper, for technical reasons, we will restrict to $a > 0$ that we will fix for the rest of the paper.

SBO's eigenvalues that we denote $\Lambda_{\beta,a}(0) < \Lambda_{\beta,a}(1) < \dots$ can be seen as interacting particles on $\mathbb{R}_+$ at temperature $1/\beta$, confined in $\mathbb{R}_+$ and whose interaction with the hard edge 0 depends on the parameter a. Indeed they correspond to the scaling limit of the smallest points of the (β, a)-Laguerre particles whose density writes

$$\frac{1}{Z_{n,\beta,a}}\prod_{i<j}|\lambda_i^{(n)} - \lambda_j^{(n)}|^\beta \times \prod_{k=0}^{n-1}(\lambda_k^{(n)})^{\frac{\beta}{2}(a+1)-1}e^{-\frac{\beta}{2}\lambda_k^{(n)}}1_{\lambda_k^{(n)}>0}\,. \tag{6.1.1}$$

They can be described via a family of coupled Riccati diffusions $(p_\lambda^\beta,\ \lambda \in \mathbb{R}_+^*)$ with initial condition $p_\lambda^\beta(0) = +\infty$:

$$\mathrm{d}p_\lambda^\beta(t) = \frac{2}{\sqrt{\beta}}p_\lambda^\beta(t)\mathrm{d}b(t) + \big((a+\frac{2}{\beta})p_\lambda^\beta(t) - p_\lambda^\beta(t)^2 - \lambda e^{-t}\big)\mathrm{d}t\,.$$

The diffusion p_λ^β may explode to $-\infty$, in this case it immediately restarts from $+\infty$.
It is crucial to note that the *same* Brownian motion B drives the whole family of SDE. It implies important properties such as for example the monotonicity of the number of explosions of p_λ^β (which turns out to be finite). In fact, the number of explosions of p_λ^β over $\mathbb{R}_+$ corresponds to N_λ^β, the counting function of the eigenvalues of the SBO.

When β tends to 0, the smallest eigenvalues get close to the hard edge at 0 at an exponential rate. In order to get a non trivial limit, we therefore consider the rescaled eigenvalues $\mu_{\beta,a}(i) := \beta \ln(1/\Lambda_{\beta,a}(i))$ for $i \geqslant 0$ (note that this reverses the ordering of the eigenvalues).

Our main result is the following theorem:

Theorem 6.1.1 (Convergence of the eigenvalues)**.** *When $\beta \to 0$, the rescaled eigenvalue point process of SBO $(\mu_{\beta,a}(i),\ i \geqslant 0)$ converges in law towards a random simple point process on $\mathbb{R}_+$ which can be described using coupled SDEs.*

The convergence holds for a well chosen topology of Radon measures on $\mathbb{R}_+$, corresponding to a left-vague/right-weak topology (see below for more details). We will also give a characterization (similar to the one for SBO eigenvalues) of the limiting point process through coupled diffusions. This characterization enables one to compute various statistics on the limiting point process.

Let us now make a few comments on this result. Usually, one expects that when the temperature is large, the limiting point process is no longer repulsive and corresponds to a Poisson point process as the noise becomes dominant. Here we get a different result. It comes from the competition with the strong interaction with the hard edge (which is attractive when β is small, see (6.1.1)). Because of this interaction, the repulsive factor does not disappear at the limit.

6.2 Strategy of proof and limiting point process

6.2.1 Rescaled diffusions

We will study the small beta limit of the family (p^β_λ) when λ is properly rescaled with β, i.e. when λ is such that $\beta \ln(1/\lambda)$ is of order 1. More precisely, we aim at understanding the number of explosion times of p^β_λ on $\mathbb{R}_+$, as it corresponds to the number of eigenvalues below λ.
Notice that when p^β_λ reaches 0, the term in front of the noise vanishes and the drift is negative. It implies that p^β_λ never reaches 0 from below. It is easy to check that the hitting times of 0 form a discrete point process.
Let us fix $\mu > 0$. Using this property, we define the diffusion $q^\beta_\mu(t)$, which equals

$$q^+_\mu(t) := \beta \ln\big(p^\beta_{\Lambda^\beta}(t/(4\beta))\big) \text{ when } p^\beta_{\Lambda^\beta}(t/(4\beta)) > 0,$$
$$q^-_\mu(t) := -\beta \ln\big(-p^\beta_{\Lambda^\beta}(t/(4\beta))\big) - \mu - t/4 \text{ when } p^\beta_{\Lambda^\beta}(t/(4\beta)) < 0,$$

where $\Lambda^\beta := \exp(-\mu/\beta)$.

The diffusions $q^+_\mu(t)$ and $q^-_\mu(t)$ follow the SDEs:

$$\mathrm{d}q^+_\mu = \mathrm{d}W(t) + \frac{1}{4}\Big(a - \exp(q^+_\mu(t)/\beta) - \exp(-(q^+_\mu(t) + t/4 + \mu)/\beta)\Big)\mathrm{d}t \tag{6.2.1}$$

$$\mathrm{d}q^-_\mu = \mathrm{d}W(t) + \frac{1}{4}\Big(-(a+1) - \exp(q^-_\mu(t)/\beta) - \exp(-(q^-_\mu(t) + t/4 + \mu)/\beta)\Big)\mathrm{d}t \tag{6.2.2}$$

where W is a Brownian motion (corresponding to (different) rescaling of the initial Brownian motion B). The diffusions $q_\mu^\pm$ may explode to $-\infty$ in a finite time. By definition, the diffusion q_μ^β alternates between q_μ^+ and q_μ^-: it starts to follow q_μ^+ and each time $q_\mu^\beta = q_\mu^+$ (resp. q_μ^-) reaches $-\infty$, q_μ^β immediatly restarts from $+\infty$ and follow q_μ^- (resp. q_μ^+) .

Let us define the critical line:

$$c_\mu(t) := -\mu - t/4\,. \tag{6.2.3}$$

Figure 13 shows an example of trajectory of the diffusion q_μ^β. On this event, the diffusion q_μ^β explodes one time as q_μ^+ (blue), then one time as q_μ^- (red), and then stays above the critical line $c_\mu(t)$ as q_μ^+ (blue) and does not explode anymore.

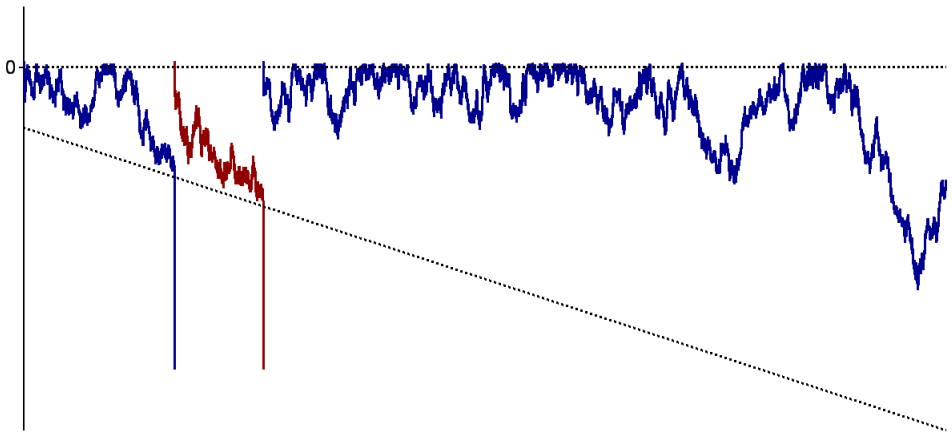

Figure 13: **Sample trajectory of the diffusion q_μ^β.**

Roughly, the diffusion q_μ^+ behaves as follows after each explosion time. It first quickly goes down to values around 0. Then, it spends some time between the line $c_\mu(t)$ and 0, where it behaves as a reflected (downwards) Brownian motion with drift $a/4$. If it reaches the line $t \mapsto c_\mu(t)$ in a finite time then it quickly explodes to $-\infty$ after this hitting time.
The behavior of the diffusion q_μ^- is similar except that in the interval $[c_\mu(t), 0]$, it behaves as a reflected (downwards) Brownian motion with drift $-(a+1)/4$. Therefore, it almost surely hits $c_\mu(t)$ when $a > 0$.
There are two types of explosions for q_μ^β: either q_μ^β explodes at a time ξ such that $q_\mu^\beta(\xi^+) = q_\mu^+(\xi^+)$. It corresponds to the (rescaled) hitting times of 0 by the initial diffusion $p_{\Lambda^\beta}^\beta$. Or q_μ^β explodes at time ξ such that $q_\mu^\beta(\xi^-) = q_\mu^-(\xi^-)$. In this case, we get the (rescaled) explosion times by the initial diffusion $p_{\Lambda^\beta}^\beta$.

In the following, we denote by $\xi_\beta^+(0) < \xi_\beta^-(0) < \xi_\beta^+(1) < \xi_\beta^-(1) < \dots$ the explosion times of the diffusion q_μ^β and by

$$\nu_\mu^\beta := \sum_{i \geqslant 0} \delta_{\xi_\beta^-(i)} \,. \tag{6.2.4}$$

the measure corresponding to the (rescaled) explosions of $p_{\Lambda^\beta}^\beta$.
We will prove that, for a well-chosen topology, the trajectory of the diffusion q_μ^β converges in law when $\beta \to 0$ towards a non-trivial limit, that we describe in the following paragraph.

6.2.2 Description of the limiting point process

Let us define now the limiting diffusion r_μ, which will characterize the limiting point process. Its definition involves Brownian motions with drift reflected downwards at 0.
By definition, a Brownian motion with drift q reflected downwards at 0 is a diffusion Markov process with infinitesimal operator $G \ : \ f \in \mathcal{D} \mapsto \frac{1}{2} f'' + q f'$ acting on the domain

$$\mathcal{D} := \left\{ f \in C_b[0, +\infty[, Gf \in C_b([0, +\infty[), \ \lim_{x \downarrow 0} f'(x) = 0 \right\},$$

where $C_b([0, +\infty[)$ denotes the continuous and bounded functions on $[0, +\infty[$. Thanks to Skorohod problem, we can write this diffusion as

$$(W(t) + qt) - \sup_{s \leqslant t} (W(s) + qs) \vee 0 \,,$$

where W is a Brownian motion starting at 0 or any negative point.

Let us now define the diffusion r_μ. It starts at 0 at time 0 i.e. $r_\mu(0) := 0$. It then follows a Brownian motion with drift $a/4$ reflected downwards at 0 that we denote r^+, until its first hitting time of the critical line (6.2.3). If it reaches the critical line in a finite time, then it immediately restarts at 0 at this time and follows a reflected downwards at 0 Brownian motion with another drift $-(a+1)/4$ denoted r^-, and so on, alternating between r^+ and r^- each time it hits $t \mapsto c_\mu(t)$ and restarts at 0. Note that the probability that r^+ reaches the critical line decreases with time. On the other hand, r^- almost surely hits the critical line in a finite time.

Let us denote by $\xi_0^+(0) < \xi_0^-(0) < \xi_0^+(1) < \xi_0^-(1) < \dots$ the hitting times of the critical line by the diffusion r_μ and define the random measure associated to the point process $(\xi_0^-(k), \ k \geqslant 0)$ only:

$$\nu_\mu^0 := \sum_{i \geqslant 0} \delta_{\xi_0^-(i)} \,.$$

We will then use the coupled random measures $(\nu_\mu^0, \ \mu > 0)$ to define a discrete point process on $\mathbb{R}_+$. As $\mu \in \mathbb{R}_+ \mapsto \nu_\mu(\mathbb{R}_+)$ is decreasing from $+\infty$ to 0 almost surely, it is easy to prove the following proposition:

Proposition 6.2.1 (Definition of the limiting point process)**.** *There is a unique random variable M_0 valued on the Borel measures on $]0,+\infty[$ such that, for all fixed $\mu_1 < \cdots < \mu_k$,*

$$M_0[\mu_i, +\infty[= \nu^0_{\mu_i}(\mathbb{R}_+)\,.$$

Almost surely, the measure M_0 is discrete, bounded from above and has an accumulation point at 0.

6.2.3 Convergence towards the limiting measures

We can now state the desired convergence result:

Proposition 6.2.2 (Convergence of the explosion times)**.**
When $\beta \to 0$, the measure ν^β_μ converges in law to the measure ν^0_μ for the topology of weak convergence.

It is immediate to extend this proposition for the joint law of $\nu^\beta_{\mu_i}$ when $\mu_1, \ldots, \mu_k$ are fixed positive numbers. It directly implies the following result on the finite dimensional laws of the point process $\{\mu_{\beta,a}(i),\ i \geqslant 0\}$. Let us denote by M_β the measure associated to this point process, i.e. $M_\beta := \sum_{i\geqslant 0} \delta_{\mu_{\beta,a}(i)}$. Then, we have the following result:

Proposition 6.2.3 (Convergence of the finite-marginals of the eigenvalue process)**.**
Fix $\mu_1 < \ldots < \mu_k$. When $\beta \to 0$, the random vector $(M_\beta[\mu_1, +\infty[, \ldots, M_\beta[\mu_k, +\infty[)$ converges in law to the random vector $(M_0[\mu_1, +\infty[, \ldots, M_0[\mu_k, +\infty[)$.

Consider the space of measures on $(0, \infty[$ with the topology that makes continuous the maps $m \mapsto \langle f, m\rangle$ for any continuous and bounded function f with support bounded to the left: in other words, this is the vague topology towards 0 and the weak topology towards $+\infty$.
The previous proposition shows that the family $(M_\beta)_{\beta>0}$ is tight: indeed the above convergence provides the required control on the mass at $+\infty$.
As the finite-marginals of any limiting point is identified, we deduce the convergence of the left-vague/right-weak topology of the eigenvalue point process stated in Theorem 6.1.1.

Let us now give an outline of the rest of the paper which is devoted to the proof of Proposition 6.2.2. We will first control the first explosion time of the diffusion q^β_μ in the next section 6.3 and deduce the weak convergence of the k first explosions times. The section 6.4 focuses on the tightness of the family $(\nu^\beta_\mu)_{\beta>0}$.

6.2.4 Useful results

We will use the following estimates for any Brownian motion W:

$$\forall x \geqslant 0,\ \mathbb{P}\Big(\sup_{s\in[0,1]} W(s) > x\Big) \leqslant \mathbb{P}\Big(\sup_{s\in[0,1]} |W(s)| > x\Big) \leqslant 4e^{-x^2/2}, \tag{6.2.5}$$

$$\forall x \geqslant 0,\ \mathbb{P}\Big(\sup_{s\in[0,1]} W(s) > x\Big) = \mathbb{P}\Big(|W(1)| > x\Big) \geqslant 1 - \sqrt{\frac{2}{\pi}}x. \tag{6.2.6}$$

Consider a diffusion y started from 0 and its counterpart $\overline{y}$ reflected downwards at the origin:

$$\overline{y}(t) = y(t) - \sup_{s \leqslant t} y(s).$$

For all $\delta > 0$, $\{\overline{y}(t) < -\delta\} = \{\exists s < t,\ y(t) - y(s) < -\delta\}$, therefore:

$$\sup_{s\in[0,t]} |y(s)| < \delta/2 \Rightarrow \inf_{s\in[0,t]} \overline{y}(s) > -\delta. \tag{6.2.7}$$

6.3 Control of the explosion times

In this section, we fix $\mu > 0$. Recall the definition of $c = c_\mu$ in (6.2.3). Consider the first two explosion times $\xi^+ := \xi^+_\beta(0)$ and $\xi^- := \xi^-_\beta(0)$ of the diffusion q^β_μ. Until the first explosion time ξ^+, by definition $q^\beta_\mu(0) = +\infty$ and $q^\beta_\mu(t) = q^+(t)$ follows the SDE (6.2.1).
Set $\delta = \beta^{1/8}$ and introduce the first hitting times by the diffusion q^+:

$$\tau_0 := \inf\{t \geqslant 0,\ q^+(t) \leqslant 0\} \text{ and } \tau_c := \inf\{t \geqslant 0,\ q^+(t) \leqslant c(t) + \delta\}. \tag{6.3.1}$$

We decompose the trajectory of q^+ into three parts: first it reaches the axis $x = 0$ in a short time (descent from $+\infty$). Then it spends some time of order $O(1)$ in the region $[c(t)+\delta, 0]$ and behaves like a reflected Brownian motion with a constant drift $a/4$. If it approaches the critical line $t \mapsto c(t)$ closer than δ, then it explodes with high probability within a short time (explosion to $-\infty$).
Recall the first hitting times $\xi^+_0(0) < \xi^-_0(0)$ of the line c by the diffusion r.

Proposition 6.3.1 (Limit behavior of the diffusion q^+ and q^-)**.**
Fix $T > 0$ (independent of β). There exist a deterministic $\eta \to 0$ and an event $\mathcal{E}_0$ such that $\mathbb{P}(\mathcal{E}_0) \to 1$, on which, for β small enough:

(a) $\tau_0 < \beta$,

(b) $\sup_{[\tau_0, \tau_c \wedge T]} |q^+(t) - r^+(t)| < \delta$,

(c) $\tau_c < T \Rightarrow |\xi^+ - \tau_c| < \eta$ *and* $|\xi^+_0(0) - \tau_c| < \eta$.

Besides, these properties also hold for the diffusions q^- from (6.2.2) *and r^- and their corresponding hitting and explosion times. As a consequence, on the event $\mathcal{E}_0$, for β small enough,*

$$\xi^- < T \Rightarrow |\xi^- - \xi^-_0(0)| < 2\eta. \tag{6.3.2}$$

The control (6.3.2) readily extends to any $\xi^-_\beta(k)$ and $\xi^-_0(k)$ for $k \in \mathbb{N}$ and ensures that, for any $T > 0$, $\mathbb{P}(\xi^-_\beta(k) \leqslant T) \to \mathbb{P}(\xi^-_0(k) \leqslant T)$, thus identifying the measure ν^0_μ as the unique possible limit for ν^β_μ.

The rest of this section is dedicated to the proof of Proposition 6.3.1. Recall that the diffusion q^- differs from its counterpart q^+ only by its constant drift component $-(a+1)/4$ (instead of $+a/4$ for q^+), which makes q^- decrease faster

than q^+. We prove the results for the diffusion q^+, they can be extended to the diffusion q^- with the same arguments.

We introduce the stationary diffusion $\overline{q}$ on $\mathbb{R}_+$, which we use to approximate q^+ in the region when the drift component $\exp\big[-\frac{1}{\beta}(c(t)+q^+(t))\big]$ becomes negligible as β tends to 0:

$$\mathrm{d}\overline{q}(t) = \mathrm{d}W(t) + \frac{1}{4}(a - e^{\overline{q}(t)/\beta})\mathrm{d}t. \tag{6.3.3}$$

6.3.1 Descent from $+\infty$: proof of (a)

It suffices to prove property (a) for the diffusion $\overline{q}$, which bounds the diffusion q^+ from above. Set the level $l_1 = \beta^{3/4}$, so that $\beta = o(l_1)$. As β tends to 0, when the diffusion $\overline{q}$ is above the level l_1, the term of leading order in the right-hand side of (6.3.3) is $e^{\overline{q}(t)/\beta}$.

Consider the ordinary differential equation on $\mathbb{R}_+$:

$$\mathrm{d}y(t) = \frac{1}{4}(a - \frac{1}{2}e^{y/\beta})\mathrm{d}t,\ y(0) = +\infty,$$

which has for solution $y(t) = -\beta \ln\big(\frac{1}{2a}(1 - e^{-at/4\beta})\big)$. The time t_1 at which the diffusion y reaches the level $l_1/2$ has the asymptotics

$$t_1 = 8\beta e^{-l_1/(2\beta)} + o(\beta e^{-l_1/\beta}).$$

Introduce the diffusion $\overline{q}_1(t) = \overline{q}(t) - W(t)$. Its evolution writes:

$$\mathrm{d}\overline{q}_1(t) = \frac{1}{4}\Big(a - e^{\big(\overline{q}_1(t)+W(t)\big)/\beta}\Big)\mathrm{d}t.$$

Let $\mathcal{E}_1' = \{\sup_{[0,t_1]}|W(t)| \leqslant \beta^2\}$. By the Brownian tail bound (6.2.5), $\mathbb{P}(\mathcal{E}_1') \longrightarrow 1$.

On the event $\mathcal{E}_1'$, $e^{(\overline{q}_1(t)+W(t))/\beta} \geqslant e^{-1/\beta}e^{\overline{q}_1(t)/\beta}$, so the diffusion $\overline{q}_1$ is bounded from above by the diffusion y for β small enough and hits the level $l_1/2$ before time t_1. Since $|\overline{q}_1(t) - \overline{q}(t)| \leqslant \beta^2$ before time t_1, for β small enough, the diffusion $\overline{q}$ hits the level l_1 before time t_1 .

After the level l_1 is reached, we use the Brownian motion $W(t_1 + \cdot) - W(t_1)$ to reach $x = 0$ in a short additional time. Set the event

$$\mathcal{E}_1'' = \Big\{\inf_{[0,\beta/2]}\Big\{W(t_1+t) - W(t_1) + \frac{a}{4}t\Big\} < -l_1\Big\},$$

on which $\tau_0 < t_1 + \beta/2$. Since $\mathbb{P}(\mathcal{E}_1'') \geqslant \mathbb{P}\Big(\inf_{[0,\beta/2]} W(t_1+t) - W(t) < -l_1 - \frac{a}{8}\beta\Big)$, the lower bound (6.2.6) and the asymptotics $\beta = o(l_1)$ and $l_1 = o(\sqrt{\beta})$ imply that $\mathbb{P}(\mathcal{E}_0') \longrightarrow 1$, thus proving the property (a) on the event $\mathcal{E}_1 = \mathcal{E}_1' \cap \mathcal{E}_1''$.

6.3.2 Convergence to r^+: proof of (b)

The bound (6.2.5) shows that the probability of the following event tends to 1 as β tends to 0:

$$\mathcal{E}_2' = \{ \sup_{t\in[0,\beta]} |W(t)| \leqslant \beta^{1/4}\}.$$

Note that $\beta^{1/4} = o(\delta)$ and that the diffusion $r^+(\tau_0 + t) - r^+(\tau_0)$ is equal in law to r^+ by the strong Markov property. Thus, to prove property (b), it suffices to show that, with overwhelming probability as β tends to 0, for β small enough,

$$\sup_{[0,\tau_c'\wedge T]} \left|q_0^+(t) - r^+(t)\right| < \delta/2, \tag{6.3.4}$$

where q_0^+ denotes the diffusion q^+ started from $x = 0$ at time $t = 0$ and τ_c' its first hitting time of $c(t) + \delta$.

We denote by $\overline{q}_0$ the stationary diffusion $\overline{q}$ from (6.3.3) started from $x = 0$ at time $t = 0$. For $t \in [0, \tau_c' \wedge T]$, $q_0^+(t) \geqslant c(t) + \delta$, so we get the bounds:

$$\forall t \in [0, \tau_c' \wedge T],\ \overline{q}_0(t) - e^{-\delta/\beta}T \leqslant q_0^+(t) \leqslant \overline{q}_0(t).$$

Since $e^{-\delta/\beta} = o(\delta)$, to prove property (6.3.4), it is enough to show that, on an event $\mathcal{E}_2'$ of probability going to 1 as β tends to 0, for β small enough,

$$\sup_{[0,\tau_c'\wedge T]} \left|\overline{q}_0(t) - r^+(t)\right| < \delta/4, \tag{6.3.5}$$

To that end, we bound the diffusion $\overline{q}_0(t)$ from below and above by two reflected diffusions r_1^+ and r_2^+ that converge to r^+ as β tends to 0.

We set a level $l_2 = \beta^{1/6}$, so that $l_2 = o(\delta)$ and $\delta = o(\sqrt{l_2})$.

Lower bound

For any stochastic process Z, we denote by $L_t^x(Z)$ its local time at position x and time t. Since the element of drift $-e^{\overline{q}_0(t)/\beta}$ is decreasing on $]-\infty, -l_2]$, we have the lower bound:

$$\forall t \in [0, \tau_c' \wedge T],\ \overline{q}_0(t) \geqslant r_1^+(t), \tag{6.3.6}$$

where $r_1^+(t)$ is the following diffusion (reflected downwards at the barrier $-l_2$):

$$r_1^+(t) = -l_2 + W(t) + at/4 - e^{-\frac{\delta}{\beta}}T/4 - L_t^0\Big(W(t) + at/4 - e^{-\frac{\delta}{\beta}}T/4\Big),$$

It is straightforward that

$$\forall t \in [0, \tau_c' \wedge T],\ r^+(t) - e^{-\frac{\delta}{\beta}}T/4 - l_2 \leqslant r_1^+(t) \leqslant r^+(t) - l_2.$$

Since $e^{-\frac{\delta}{\beta}} = o(l_2)$, for β small enough,

$$\sup_{[0,\tau_c'\wedge T]} \left|r^+(t) - r_1^+(t)\right| < 2l_2. \tag{6.3.7}$$

Upper bound

We wish to bound the diffusion $\overline{q}_0(t)$ from above by the diffusion $r^+(t) + l_2$. To prove that this upper bound holds with high probability as β tends to 0, we use the following result, that shows how unlikely it becomes for the diffusion $\overline{q}_0(t)$ to hit the level l_2 before any negative level.

Lemma 6.3.1 (Levels hitting times for the diffusion $\overline{q}_0(t)$)**.**
For any $\gamma < 0$,

$$\mathbb{P}\Big(\inf\{t \geqslant 0,\ \overline{q}_0(t) = l_2\} < \inf\{t \geqslant 0,\ \overline{q}_0(t) = \gamma\}\Big) \longrightarrow 0.$$

Lemma 6.3.1 proved in the Appendices using standard tools of diffusion analysis.

The choice of level $\gamma = -\mu(T)$ in Lemma 6.3.1 provides the existence of an event $\mathcal{E}_2'$ of probability going to 1 as β tends to 0 on which the diffusion $\overline{q}_0(t)$ hits the barrier $c(t)$ before the level l_2, and thus:

$$\forall t\ \in [0, \tau_c' \wedge T],\ \overline{q}_0(t) \leqslant r^+(t) + l_2. \tag{6.3.8}$$

Conclusion

Gathering (6.3.6) and (6.3.8), we get that, on $\mathcal{E}_2'$, for β small enough,

$$\forall t\ \in [0, \tau_c' \wedge T], r_1^+(t) \leqslant \overline{q}_0(t) \leqslant r^+(t) + l_2,$$

which implies $\sup_{[0,\tau_c' \wedge T]} |\overline{q}_0(t) - r^+(t)| < 2l_2$. This in turn implies (6.3.5) and thus proves property (b).

6.3.3 Explosion to $-\infty$: proof of (c)

We denote by q_δ^+ (resp. $q_{-\delta}^+$) the diffusion q^+ started at time $t = 0$ from position $-\mu + \delta$ (resp. $-\mu - \delta$). We introduce the first hitting time τ_δ of the level $c(t) - \delta$ by the diffusion $q_{+\delta}^+$, and the explosion time $\tau_{-\infty}$ of the diffusion $q_{-\delta}^+$ to $-\infty$.
To prove property (c), we choose $\eta = 2l_2$, with $l_2 = \beta^{1/6}$, and show that there exists an event of probability going to 1 as β tends to 0 on which, for β small enough, $\tau_\delta < \eta/2$ and $\tau_{-\infty} < \eta/2$.

Control of τ_δ

Recall that $\delta = \beta^{1/8}$, so that $l_2 = o(\delta)$ and $\delta = o(\sqrt{l_2})$.
We use the variations of the Brownian motion W to cross the critical line $c(t)$. The upper bound $q_\delta^+(t) \leqslant -\mu + \delta + W(t) + \frac{a}{4}t$ implies that $\tau_\delta < l_2$ on the event

$$\mathcal{E}_3 = \left\{\inf_{[0,l_2]}\Big\{W(s) + \frac{a}{4}s + \frac{1}{4}s + 2\delta\Big\} < 0\right\}.$$

The bound (6.2.6) ensures that $\mathbb{P}(\mathcal{E}_3) \to 1$.

Note that the inclusion of events $\mathcal{E}_3' \subset \left\{ \inf\{t \geqslant 0,\ r^+(t) + t/4 \leqslant -2\delta\} < l_2 \right\}$ ensures that, on a subevent of $\mathcal{E}_2$ (where $|q^+(\tau_c) - r^+(\tau_c)| < \delta$ if $\tau_c < +\infty$) of probability going to $\mathbb{P}(\mathcal{E}_2)$ as β tends to 0, the diffusion r^+ hits the critical line $c(t)$ while the diffusion q^+ crosses this line, between times τ_c and $\tau_c + \eta$.

Control of $\tau_{-\infty}$

On each Brownian trajectory, the diffusion $q^+_{-\delta}$ is bounded from above by the diffusion z, with

$$\mathrm{d}z(t) = \mathrm{d}W(t) + \frac{1}{4}(a - e^{-\frac{1}{\beta}\left(t/4+\mu_\beta+z(t)\right)})\mathrm{d}t,\ z(0) = -\mu - \delta.$$

Define the diffusion $z_1(t) = z(t) + \mu - W(t) - at/4$, with evolution

$$\mathrm{d}z_1(t) = -\frac{1}{4}e^{-\frac{1}{\beta}\left(z_1(t)+t/4+W(t)+at/4\right)}\mathrm{d}t,\ z_1(0) = -\delta.$$

Consider the event $\mathcal{E}_3^{'} = \{\sup_{[0,\beta]} |W(t)| \leqslant \delta/4\}$, with $\mathbb{P}(\mathcal{E}_3^{'}) \to 1$.
On $\mathcal{E}_3^{'}$, for β small enough:

$$\forall t \leqslant \beta,\ W(t) + \frac{a+1}{4}t \leqslant \delta/2,$$

so the diffusion z_1 is bounded from above by the diffusion z_2 with evolution

$$\mathrm{d}z_2(t) = -\frac{1}{4}e^{-\frac{1}{\beta}\left(z_3(t)+\delta/2\right)}\mathrm{d}t,\ z_2(0) = -\delta.$$

The diffusion z_2 has the solution

$$z_2(t) = -\delta/2 + \beta \ln\left(e^{-\delta/(2\beta)} - \frac{t}{4\beta}\right),$$

which explodes to $-\infty$ in a time $4\beta e^{-\delta/(2\beta)}$, smaller than β for β small enough. This remains true for z_1 and thus for z, since $|z_1 - z| \leqslant \mu + a\beta/4 + \beta$ while $t \leqslant \beta$. Since $\beta = o(\eta)$, this proves the desired control on $\tau_{-\infty}$.

6.4 Tightness of the explosion times measures

In this section, we fix $\mu > 0$. Recall from (6.2.4) the measure of the explosion times ν_μ^β. We prove in this section that there are $\beta_0 > 0$ and $\alpha > 0$, such that, for all $\epsilon > 0$, there exist a finite time T_ϵ and a finite number of explosions N_ϵ so that:

$$\inf_{\beta \leqslant \beta_0} \mathbb{P}\left(\left\{ \nu_\mu^\beta([0, \alpha T_\epsilon]) \leqslant N_\epsilon \right\} \bigcap \left\{ \nu_\mu^\beta([\alpha T_\epsilon, +\infty[) = 0 \right\} \right) > 1 - \epsilon. \tag{6.4.1}$$

Introduce $\mathcal{L}_\mu^\beta$, the law of the random measure ν_μ^β. The bound (6.4.1) readily implies the tightness condition:

$$\exists \beta_0,\ \forall \epsilon > 0,\ \exists K_\epsilon \text{ compact},\ \sup_{\beta < \beta_0} \mathcal{L}_\beta(K_\epsilon) \geqslant 1 - \epsilon.$$

Indeed, the closure K_ϵ of $\{\mu \in \mathcal{P},\ \mu([0,\alpha T_\epsilon]) \leqslant N_\epsilon$ and $\mu([\alpha T_\epsilon, +\infty[) = 0\}$, where $\mathcal{P}$ is the space of locally finite measures on $\mathbb{R}_+$, satisfies the Kallenberg criterion from [47] for weak relative compacity:

$$\sup_{\mu\in K_\epsilon} \mu(\mathbb{R}_+) < +\infty,$$
$$\inf_{t>0} \sup_{\mu\in K_\epsilon} \mu([t,+\infty[) = 0.$$

The rest of this section is dedicated to the proof of (6.4.1), which provides the relative compacity of the family $(\mathcal{L}_\beta)_{\beta<\beta_0}$ in $\mathcal{M}^1(\mathcal{P})$ by the Prokhorov theorem and thus concludes the proof of Proposition 6.2.2.

We show a preliminary result that will be helpful to control the number of explosions. Recall that $\xi^+ = \xi^+_\beta(0)$ is the first explosion time of the diffusion q^+ from (6.2.1), started from $+\infty$ at time 0.

Lemma 6.4.1 (Lower bound on the explosion time of q^+)**.**
For β small enough,

$$\forall t > 0,\ \mathbb{P}(\xi^+ > t) \geqslant 1 - 4e^{-\mu^2/(32t)}.$$

Proof. We fix a deterministic δ_0 such that $0 < \delta_0 < \mu/4$. Recall the definition of the critical line c_μ in (6.2.3). When the diffusion q^+ is in the region between $-c_\mu(t) + \delta_0$ and $-\delta_0$, we have the lower bound, for β small enough:

$$a - e^{q^+(t)/\beta} - e^{-\frac{1}{\beta}(\mu(t)+q^+(t))} \geqslant a - 2e^{-\delta_0/\beta} \geqslant a/2. \tag{6.4.2}$$

Introduce the diffusion $\widehat{q}$ on $\mathbb{R}_+$, started from $-\delta_0$ at time 0 and reflected downwards at $-\delta_0$:

$$\widehat{q}(t) = -\epsilon + W(t) + at/8 - L^0_t(W(t) + at/8).$$

The bound (6.4.2) shows that, for β small enough, the diffusion q^+ is bounded from below by the diffusion $\widehat{q}$ on each Brownian trajectory, up until the first hitting time of $c_\mu(t) + \delta_0$ by the diffusion q^+.

Set $t > 0$ and introduce the event $\mathcal{E} = \{\sup_{s\in[0,t]} |W(t)| < \mu/4\}$. By the Brownian tail bound (6.2.5), we have $\mathbb{P}(\mathcal{E}) \geqslant 1 - 4e^{-\mu^2/(32t)}$. Besides, on the event $\mathcal{E}$, using (6.2.7),

$$\forall s \in [0,t],\ \widehat{q}(s) > -\delta_0 - \mu/2.$$

This means that the diffusion $\widehat{q}$ stays above $-\mu$ until time t. Thus, for β small enough so that (6.4.2) holds, on the event $\mathcal{E}$, we have $\xi^+ > t$. □

We now turn to the proof of (6.4.1). Fix $\epsilon > 0$. We will control the diffusion q^β_μ with two diffusions. The first diffusion Q_1 starts at time T_ϵ at position -1 and is reflected below the horizontal line -1 with drift $a/8$. The second diffusion Q_2 starts at time $2T$ at position $c_\mu(T_\epsilon)$, has a drift $a/8$ as well and is also reflected below -1.

We can choose T_ϵ large enough such that the diffusions Q_1 and Q_2 do not hit $c_\mu(t)+1$ with probability greater than $1-\epsilon/10$. Indeed, the sublinearity of the Brownian motion $W(T_\epsilon+t)-W(T_\epsilon)$ (implied for instance by the law of iterated logarithm) is such that there exists a time T_0 after which $|W(T_\epsilon+t)-W(t)|< t/16$ with probability greater than $1-\epsilon/40$. Using (6.2.7), this implies that Q_1 stays above $-1-t/8$ after time T_0, and thus above the critical line $c_\mu(t)$. We choose T_ϵ large enough so that $|W(T_\epsilon+t)-W(t)|< c_\mu(T)/4$ with probability greater than $1-\epsilon/40$ until time T_0, as a result Q_1 stays above $c_\mu(t)$ before time T_0 as well. Therefore, on an event of probability greater than $1-\epsilon/20$, the diffusion Q_1 never hits $c_\mu(t)$. Similar arguments can be used for the second diffusion Q_2.

The term $-(a+1)/4$ in the drift of the diffusion q^- implies the existence of $\alpha>0$ such that, almost surely, when started before time $2T_\epsilon+1$, the diffusion q^- explodes before time αT_ϵ. If, at time T_ϵ, the diffusion q_μ^β evolves as the diffusion q^-, then it almost surely explodes before time αT_ϵ, after which it evolves as q^+ and stays above Q_1 (for β small enough such that $2e^{-1/\beta}<a/8$) and does not explode.

If, at time T, the diffusion q_μ^β evolves as the diffusion q^+, then we distinguish between three cases:
First, if the diffusion q_μ^β hits -1 between times T_ϵ and $2T_\epsilon$, then q_β stays above Q_1 and therefore does not explode.
Else, following the proof of property (c) from Proposition 6.3.1 in Section 6.3, we can choose a deterministic level $\delta_1>0$ so that, if q_μ^β reaches $t\mapsto c_\mu(t)+\delta_1$ between time T_ϵ and $2T_\epsilon$, then it explodes before time $2T_\epsilon+1$ with probability greater than $1-\varepsilon/10$. After that, q_μ^β behaves as q^- and almost surely explodes one last time before time αT_ϵ, as previously.
Finally, if the diffusion q_μ^β starts above $c_\mu(T_\epsilon)+\delta_1$ at time T_ϵ and stays in the interval $[c_\mu(t)+\delta_1,-1]$ for all $t\in[T_\epsilon,2T_\epsilon]$, then it is bounded from below by a Brownian motion with a positive drift $a/8$, and therefore it will be above $c_\mu(T)$ at time $2T_\epsilon$ with probability greater than $1-\epsilon/10$. In this event, q_μ^β stays above the diffusion Q_2 after time $2T_\epsilon$ and thus does not explode.

Gathering the different cases, we obtain the existence of an event of probability greater than $1-\epsilon/2$ on which q_μ^β explodes at most once after time T_ϵ and does not explode after time αT_ϵ.

To conclude the proof of the tightness criterion (6.4.1), we apply Lemma 6.4.1 to get the existence of a finite N_ϵ and of an event of probability greater than $1-\epsilon/2$ on which the diffusion q_μ^β explodes at most $N_\epsilon-1$ times before time T_ϵ.

Appendices

Lemmas for Theorem 1.3.5

Proof of Lemma 1.3.1

For the first property, if $(m_n)_{n\geqslant 1}$ is an integer sequence with $\lim_{n\to\infty} m_n = +\infty$ and $\lim_{n\to\infty} \frac{m_n}{n} = 0$, then

$$\mathbb{P}\left(i_n > m_n\right) = \frac{n - m_n}{n} = 1 - \frac{m_n}{n} \underset{n\to\infty}{\longrightarrow} 1.$$

The second property follows from the fact that $n + 1 - i_n$ and i_n are equally distributed. For the third property, we write, for all $t \in [0, 1]$,

$$\mathbb{P}\left(i_n \leqslant nt\right) = \mathbb{P}\left(i_n \leqslant \lfloor nt \rfloor\right) = \frac{\lfloor nt \rfloor}{n} \underset{n\to\infty}{\longrightarrow} t.$$

For the last property, we can simply take $U \sim \text{Uniform}\left([0, 1]\right)$ on an arbitrary probability space and define $i_n = 1 + \lfloor nU \rfloor$ for all $n \geqslant 1$.

Proof of Lemma 1.3.2

Since

$$\max_{1\leqslant i\leqslant n} |x_i| \leqslant |x| = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2},$$

we get, for all $\varepsilon \in (0, +\infty[$,

$$\max_{1\leqslant i\leqslant d} \mathbb{P}\left(\left|X_i^{(n)} - X_i\right| \geqslant \varepsilon\right) \leqslant \mathbb{P}\left(\left|X^{(n)} - X\right| \geqslant \varepsilon\right) \leqslant \sum_{i=1}^{d} \mathbb{P}\left(\left|X_i^{(n)} - X_i\right| \geqslant \frac{\varepsilon}{d}\right),$$

by the union bound for the last inequality.

Lemmas for Theorem 3.3.1

Proof of Lemma 3.3.2

For χ_r a chi random variable of index $r > 0$, for any $p > -r$, we use the estimates (see [52]):

$$\mathbf{E}(\chi_r^p) = 2^p \Gamma(\frac{r+p}{2})(\Gamma(\frac{r}{2}))^{-1}, \tag{6.4.3}$$

$$\mathbf{E}(\log \chi_r) = \frac{1}{2}\log r - \frac{3}{2r} + \underset{r\to+\infty}{O}(1/r^2),\ \mathrm{Var}(\log \chi_r) = \frac{1}{2r} + \underset{r\to+\infty}{O}(1/r^2), \tag{6.4.4}$$

$$\forall m \in \mathbb{N}^*,\ \mathbf{E}\big((\log \chi_r - \mathbf{E}\log \chi_r)^{2m}\big) = \underset{r\to+\infty}{O}(1/r^m) \tag{6.4.5}$$

Start with the convergence (3.3.3). For any fixed $x \in (0,1]$, it is an instance of the law of large numbers. With (6.4.3), one can obtain the increment bound

$$\mathbf{E}\Big(\sqrt{\frac{(r+1)\beta}{\chi^2_{(r+a+1)\beta}}} - \sqrt{\frac{r\beta}{\chi^2_{(r+a)\beta}}}\Big)^2 = \underset{r\to+\infty}{O}(1/r^2),$$

which gives the tightness required to ensure process level convergence.
Turning to the proof of (3.3.4), by the first estimate of (6.4.4), we have:

$$\lim_{n\to\infty} \sum_{k=\lfloor ny\rfloor}^{\lfloor nx\rfloor} \big(\mathbf{E}(\log \bar{\chi}_{k\beta}) - \mathbf{E}(\log \chi_{(k+a)\beta})\big) = \frac{a}{2}\log(y/x),$$

uniformly for $y < x$ restricted to compact sets of $(0,1]$. Thus, for (3.3.4), it is enough to demonstrate the weak convergence

$$\sum_{k=\lfloor nx\rfloor}^{n} \big(\log \chi_{(k+c)\beta} - \mathbf{E}\log \chi_{(k+c)\beta}\big) \xrightarrow[n\to\infty]{\mathrm{d}} \int_x^1 \frac{1}{\sqrt{2\beta z}}\mathrm{d}B(z), \tag{6.4.6}$$

where c is any fixed number, and apply it to the two independent sums of the kernel. To that purpose, we use the following result.

Proposition 6.4.1. *[32, Chapter 7] Let $y_{n,k}$ be a sequence of mean-zero processes starting at 0 with independent increments $\Delta y_{n,k}$. Assume that, when $n, k \to \infty$,*

$$n\mathbf{E}(\Delta y_{n,k})^2 = f(k/n) + o(1), \quad n\mathbf{E}(\Delta y_{n,k})^4 = o(1) \tag{6.4.7}$$

uniformly for k/n in compact sets of $[0,T)$ with a continuous $f \in L^1_{\mathrm{loc}}([0,T))$. Then, in the Skorohod topology,

$$y_n(t) := y_{\mathrm{n},\lfloor \mathrm{nt}\rfloor} \xrightarrow[n\to\infty]{\mathrm{d}} \int_0^t f^{1/2}(s)\mathrm{d}B(s)$$

with B a standard Brownian motion.

Take $t = 1 - x$ in the proposition to apply it to (6.4.6). The second estimate of (6.4.4) yields the first assumption of (6.4.7) with $f(t) = 1/(2\beta t)$. The estimate (6.4.5) with $m = 2$ produces the second half of (6.4.7), as x is always positive. This proves (3.3.4).

Proof of Lemma 3.3.4

First, for $C \in \mathbb{R}$ and $a > -1$, the integral operator on $L^2([0,1])$ with kernel

$$k_C(x,y) = C \exp\Big(C\big(\log(1/x)\big)^{3/4} + C\big(\log(1/y)\big)^{3/4} \Big) \frac{y^{a/2}}{x^{(a+1)/2}} 1_{y<x}$$

is Hilbert-Schmidt. Indeed, the change of variables $x = e^{-s}$ and $y = e^{-t}$ produces

$$\int_0^1 \int_0^1 |k_C(x,y)|^2 \mathrm{d}x\mathrm{d}y = C^2 \int_0^\infty e^{2Cs^{3/4}+as} \int_s^\infty e^{2Ct^{3/4}-(a+1)t} \mathrm{d}t\mathrm{d}s,$$

and the latter is finite if and only if $a > -1$.
Using Lemma 3.3.2, there is a subsequence (still noted (n) for simplification purposes) over which we have the joint convergence in law:

$$\frac{\sqrt{n\beta}}{\chi_{(\lfloor nx \rfloor + a)\beta}} \xrightarrow[n\to\infty]{\mathrm{d}} (\frac{1}{\sqrt{x}}, 0 < x \leqslant 1),$$

$$\sum_{k=\lfloor ny \rfloor}^{\lfloor nx \rfloor} (\log \tilde{\chi}_{k\beta} - \log \chi_{(k+a)\beta}) \xrightarrow[n\to\infty]{\mathrm{d}} ((a/2)\log(y/x) + \int_y^x \frac{db_z}{\sqrt{\beta z}}, 0 < y \leqslant x < 1),$$

$$\kappa_n, \kappa_n' \xrightarrow[n\to\infty]{\mathrm{d}} \kappa, \kappa'.$$

By Skorohod's representation theorem ([32, Theorem 1.8]), there is a probability space on which each of the above occurs with probability one. On this new space

$$\mathbb{P}\big(\lim_{n\to\infty} k_n^{\beta,a}(x,y)(\omega) = k_{\beta,a}(x,y)(\omega) \text{ for almost every } x,y \in [0,1]^2 \big) = 1.$$

To apply Lebesgue's dominated convergence theorem, we need to find an almost surely finite constant $C(\omega)$ such that, for almost all $x, y \in [0,1]$ and ω:

$$\sup_{n>0} k_n^{\beta,a}(x,y)(\omega) \leqslant k_{C(\omega)}(xy) \text{ and } k_{\beta,a}(x,y)(\omega) \leqslant k_{C(\omega)}(x,y). \tag{6.4.8}$$

We use Lemma 3.3.3. For each n, for $x \in [x_i, x_{i+1}), y \in [y_j, y_{j+1})$,

$$k_n^{\beta,a}(x,y)(w) \leqslant \kappa_n(\omega) x_i^{-(a+1)/2} y_j^{a/2} \exp\Big(\kappa_n'(\omega)\big(1 + T^{3/4}(x_i) + T^{3/4}(y_j)\big)\Big). \tag{6.4.9}$$

Since κ_n and κ_n' converge, they are bounded almost surely. Using the continuity of T, we can fit the right hand side of (6.4.9) under a fixed k_C independent of n. For the limit kernel $k_\infty^{\beta,a}(x,y)(\omega)$, its exponent can be expressed as equal in law to

$$\int_y^x \frac{dB(z)}{\sqrt{\beta z}} = \frac{1}{\sqrt{\beta}}\Big(\tilde{B}\big(\log(1/x)\big) - \tilde{B}\big(\log(1/y)\big)\Big),$$

with $\tilde{B}$ a different Brownian motion living on the same probability space. By the law of the iterated logarithm, $\tilde{B}(a) \leqslant c(\omega)\Big(1 + \big(a \log\log(1+a)\big)^{1/2}\Big)$ for a random $c(\omega)$ and all $a > 0$, and $\big(a\log\log(1+a)\big)^{1/2} \leqslant c' a^{3/4}$ with a (non-random) c' and all a large enough. Thus, the second half of (6.4.8) holds with $C(\omega) \leqslant \beta^{-1/2} c' c(\omega)$. This concludes the proof, with

$$\int_0^1 \int_0^1 \Big| k_n^{\beta,a}(x,y)(\omega) - k_\infty^{\beta,a}(x,y)(\omega) \Big|^2 \mathrm{d}x\mathrm{d}y \underset{n\to\infty}{\longrightarrow} 0 \text{ a.s.}$$

Proof of Lemma 6.3.1

Recall that $l_1 = \beta^{3/4}$ and $l_2 = \beta^{1/6}$. In this proof we denote by q the diffusion $\overline{q}_0^+$. Set $\gamma < 0$, $\tau = \inf\{t \geqslant 0,\ q_0^+(t) = l_2\}$ and $\tau' = \inf\{t \geqslant 0,\ q_0^+(t) = \gamma\}$. Let $f_\beta(x) = \frac{1}{4}(a - \exp(x/\beta))$. To compute the hitting times of q, introduce the scale functions $\mathfrak{s}_\beta$ and $\mathfrak{s}$:

$$\mathfrak{s}_\beta(x) = \int_{-1}^{x} \exp\Big(-2\int_0^u f_\beta(v)\mathrm{d}v\Big)\mathrm{d}u,$$
$$\mathfrak{s}(x) = \int_{-1}^{x} \exp\Big(-2\int_0^u \frac{a}{4}\mathrm{d}v\Big)\mathrm{d}u = \frac{2}{a}(e^{a/2} - e^{ax/2}).$$

The following lemma explicits the asymptotic behavior of $\mathfrak{s}_\beta$.

Lemma 6.4.2 (Convergence of the scale functions)**.**
For any $x_0 < 0$,

$$\mathfrak{s}'_\beta \longrightarrow \mathfrak{s}' \text{ and } \mathfrak{s}_\beta \longrightarrow \mathfrak{s} \text{ uniformly on } [x_0, 0].$$

Furthermore, $\mathfrak{s}_\beta(l_2) \longrightarrow +\infty$.

Since $\mathfrak{s}_\beta(q(\cdot))$ is a local martingale, $\mathfrak{s}_\beta(q(\cdot \wedge \tau(q) \wedge \tau'(q))$ is a martingale. By the stopping theorem, we get:

$$\mathbb{P}(\tau < \tau') = \frac{\mathfrak{s}_\beta(\gamma) - \mathfrak{s}_\beta(0)}{\mathfrak{s}_\beta(\gamma) - \mathfrak{s}_\beta(l_2)}.$$

Lemma 6.4.2 readily implies that this probability tends to 0 as β tends to 0.

Proof of Lemma 6.4.2. We have, for all $x \leqslant 0$,

$$\Big|\int_0^x f_\beta(v) - \frac{a}{4}\mathrm{d}v\Big| \leqslant \int_x^0 e^{v/\beta}\mathrm{d}v \leqslant \beta \longrightarrow 0,$$

which means that $\ln \mathfrak{s}'_\beta$ converges uniformly to $\ln \mathfrak{s}'$ on $]-\infty, 0]$. Besides, the functions $\ln \mathfrak{s}'_\beta$ and $\ln \mathfrak{s}'$ are bounded on $[x_0, 0]$, and $x \mapsto e^x$ is uniformly continuous on $[x_0, 0]$ so $\mathfrak{s}'_\beta$ converges uniformly to $\mathfrak{s}'$ on $[x_0, 0]$. Therefore, $\mathfrak{s}_\beta$ converges uniformly to $\mathfrak{s}$ on $[x_0, 0]$.
Now turning to $\mathfrak{s}_\beta(l_2)$. We can compute explicitly:

$$\ln \mathfrak{s}'_\beta(x) = -2\int_0^x f_\beta(v)\mathrm{d}v = -2ax + 2\int_0^x e^{v/\beta}\mathrm{d}v = -2ax + 2\beta(e^{x/\beta} - 1).$$

A study of the variations of $\ln \mathfrak{s}'_\beta$ shows that $\mathfrak{s}'_\beta$ decreases until $\beta \ln(2a)$ and increases afterwards. For β small enough, $\ln \mathfrak{s}'_\beta(l_1) \geqslant \beta e^{l_1/\beta}$, which tends to $+\infty$ as β tends to 0, and $\mathfrak{s}'_\beta$ increases on $[l_1, l_2]$, so that:

$$\mathfrak{s}_\beta(l_2) \geqslant \int_{l_1}^{l_2} \mathfrak{s}'_\beta(v)\mathrm{d}v \geqslant (l_2 - l_1)\mathfrak{s}'_\beta(l_1),$$

therefore $\mathfrak{s}_\beta(l_2) \longrightarrow +\infty$. □

Bibliography

[1] M. Abért, A. B. Thom, and B. Virág. Benjamini-schramm convergence and pointwise convergence of the spectral measure. 2014. URL: `https://tu-dresden.de/mn/math/geometrie/thom/ressourcen/dateien/forschung/preprint_app?lang=en`.

[2] G. Akemann, J. Baik, and P. Di Francesco, editors. *The Oxford handbook of random matrix theory*. Oxford: Oxford University Press, 2011.

[3] D. Aldous and R. Lyons. Processes on unimodular random networks. *Electronic Journal of Probability*, 12:1454–1508, 2007. `doi:10.1214/EJP.v12-463`.

[4] D. Aldous and J. Steele. The objective method: Probabilistic combinatorial optimization and local weak convergence. *Probability on Discrete Structures*, 110, 2003. `doi:10.1007/978-3-662-09444-0_1`.

[5] R. Allez, J. P. Bouchaud, and A. Guionnet. Invariant beta ensembles and the gauss-wigner crossover. *Physical Review Letters*, 109(9), 2012. `doi:10.1103/physrevlett.109.094102`.

[6] R. Allez and L. Dumaz. From sine kernel to Poisson statistics. *Electron. J. Probab.*, 19:25, 2014. Id/No 114. `doi:10.1214/EJP.v19-3742`.

[7] R. Allez and L. Dumaz. Tracy–widom at high temperature. *Journal of Statistical Physics*, 156(6), 2014. `doi:10.1007/s10955-014-1058-z`.

[8] R. Allez and A. Guionnet. A diffusive matrix model for invariant β-ensembles. *Electronic Journal of Probability*, 18, 2013. `doi:10.1214/ejp.v18-2073`.

[9] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*, volume 118 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2010.

[10] G. Ben Arous and A. Guionnet. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Relat. Fields*, 108(4):517–542, 1997. `doi:10.1007/s004400050119`.

[11] F. Benaych-Georges and S. Péché. Poisson statistics for matrix ensembles at large temperature. *J. Stat. Phys.*, 161(3):633–656, 2015. `doi:10.1007/s10955-015-1340-8`.

[12] I. Benjamini and O. Schramm. Recurrence of distributional limits of finite planar graphs. *Electronic Journal of Probability*, 6:1–13, 2001. doi:10.1214/EJP.v6-96.

[13] C. Bordenave, P. Caputo, D. Chafaï, and J. Salez. Local weak convergence of random graphs. book in preparation, 2022.

[14] A. Borodin and P. Forrester. Increasing subsequences and the hard-to-soft edge transition in matrix ensembles. *Journal of Physics A: Mathematical and General*, 36(12):2963–2981, 2003. doi:10.1088/0305-4470/36/12/307.

[15] G. Borot and A. Guionnet. Asymptotic expansion of beta matrix models in the one-cut regime. *Communications in Mathematical Physics*, 317(2):447–483, 2012. doi:10.1007/s00220-012-1619-4.

[16] P. Bourgade, L. Erdős, and H. T. Yau. Bulk universality of general β-ensembles with non-convex potential. *Journal of Mathematical Physics*, 53(9), 2012. doi:10.1063/1.4751478.

[17] P. Bourgade, L. Erdős, and H. T. Yau. Universality of general β-ensembles. *Duke Mathematical Journal*, 163(6), 2014. doi:10.1215/00127094-2649752.

[18] P. Deift. *Orthogonal polynomials and random matrices: A Riemann-Hilbert approach*. New York University, 1999.

[19] P. Deift and D. Gioev. Universality at the edge of the spectrum for unitary, orthogonal, and symplectic ensembles of random matrices. *Commun. Pure Appl. Math.*, 60(6):867–910, 2007. doi:10.1002/cpa.20164.

[20] P. Deift and D. Gioev. Universality in random matrix theory for orthogonal and symplectic ensembles. *IMRP, Int. Math. Res. Pap.*, 2007:116, 2007. Id/No rpm004. doi:10.1093/imrp/rpm004.

[21] P. Deift, D. Gioev, T. Kriecherbauer, and M. Vanlessen. Universality for orthogonal and symplectic Laguerre-type ensembles. *J. Stat. Phys.*, 129(5-6):949–1053, 2007. doi:10.1007/s10955-007-9325-x.

[22] L. Dumaz and C. Labbé. The stochastic airy operator at large temperature. 2020. doi:10.48550/ARXIV.1908.11273.

[23] L. Dumaz, Y. Li, and B. Valkó. Operator level hard-to-soft transition for β-ensembles. *Electron. J. Probab.*, 26:28, 2021. Id/No 39. doi:10.1214/21-EJP602.

[24] I. Dumitriu and A. Edelman. Matrix models for beta ensembles. *J. Math. Phys.*, 43(11):5830–5847, 2002. doi:10.1063/1.1507823.

[25] I. Dumitriu and A. Edelman. Global spectrum fluctuations for the β-hermite and β-laguerre ensembles via matrix models. *Journal of Mathematical Physics*, 47(6), 2006. doi:10.1063/1.2200144.

[26] T. K. Duy. On spectral measures of random Jacobi matrices. *Osaka J. Math.*, 55(4):595–617, 2018.

[27] T. K. Duy and T. Shirai. The mean spectral measures of random Jacobi matrices related to Gaussian beta ensembles. *Electron. Commun. Probab.*, 20:13, 2015. Id/No 68. doi:10.1214/ECP.v20-4252.

[28] F. J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.*, 3:1191–1198, 1962. doi:10.1063/1.1703862.

[29] A. Edelman and N. R. Rao. Random matrix theory. *Acta Numerica*, 14:233–297, 2005. doi:10.1017/S0962492904000236.

[30] A. Edelman and B. Sutton. From random matrices to stochastic operators. *Journal of Statistical Physics*, 127(6):1121–1165, 2007. doi:10.1007/s10955-006-9226-4.

[31] L. Erdős and H. T. Yau. Gap universality of generalized Wigner and β-ensembles. *J. Eur. Math. Soc. (JEMS)*, 17(8):1927–2036, 2015. doi:10.4171/JEMS/548.

[32] S. Ethier and T. Kurtz. *Markov processes: characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., 1986.

[33] P. J. Forrester. Log-gases, random matrices and the Fisher-Hartwig conjecture. *J. Phys. A, Math. Gen.*, 26(5):1179–1191, 1993. doi:10.1088/0305-4470/26/5/035.

[34] J. Franchi and Y. Le Jan. *Hyperbolic dynamics and Brownian motion. An introduction*. Oxford Math. Monogr. Oxford: Oxford University Press, 2012.

[35] P. Y. Gaudreau Lamarre. On the convergence of random tridiagonal matrices to stochastic semigroups. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 56(4):2686–2731, 2020. doi:10.1214/20-AIHP1055.

[36] H. H. Goldstine and J. Von Neumann. *Planning and Coding of Problems for an Electronic Computing Instrument*. Annals of Mathematics, 1947.

[37] V. Gorin and M. Shkolnikov. Stochastic Airy semigroup through tridiagonal matrices. *Ann. Probab.*, 46(4):2287–2344, 2018. doi:10.1214/17-AOP1229.

[38] B. I. Halperin. Green's functions for a particle in a one-dimensional random potential. *Phys. Rev.*, 139:A104–A117, 1965. doi:10.1103/PhysRev.139.A104.

[39] P. Hänggi, P. Talkner, and M. Borkovec. Reaction-rate theory: fifty years after kramers. *Rev. Mod. Phys.*, 62:251–341, 1990. doi:10.1103/RevModPhys.62.251.

[40] F. Hiai and D. Petz. *The semicircle law, free random variables and entropy*, volume 77 of *Math. Surv. Monogr.* Providence, RI: American Mathematical Society (AMS), 2000.

[41] F. Hiai and D. Petz. Large deviations for functions of two random projection matrices. *Acta Sci. Math.*, 72(3-4):581–609, 2006.

[42] D. Holcomb and G. R. Moreno Flores. Edge scaling of the β-Jacobi ensemble. *J. Stat. Phys.*, 149(6):1136–1160, 2012. doi:10.1007/s10955-012-0634-3.

[43] K. Itô and H. P. McKean. *Diffusion Processes and Their Sample Paths*. Springer-Verlag, 1974.

[44] S. Jacquot and B. Valkó. Bulk scaling limit of the laguerre ensemble. *Electronic Journal of Probability*, 16(none):314–346, 2011. doi:10.1214/EJP.v16-854.

[45] K. Johansson. On fluctuations of eigenvalues of random hermitian matrices. *Duke Mathematical Journal*, 91(1):151–204, 1998. doi:10.1215/S0012-7094-98-09108-6.

[46] I. M. Johnstone. Multivariate analysis and Jacobi ensembles: largest eigenvalue, Tracy-Widom limits and rates of convergence. *Ann. Stat.*, 36(6):2638–2716, 2008. doi:10.1214/08-AOS605.

[47] O. Kallenberg. *Random Measures, Theory and Applications*, volume 77. 2017. doi:10.1007/978-3-319-41598-7.

[48] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics (113) (Book 113). Springer New York, 1991.

[49] H. Kesten. Symmetric random walks on groups. *Trans. Am. Math. Soc.*, 92:336–354, 1959. doi:10.2307/1993160.

[50] R. Killip and I. Nenciu. Matrix models for circular ensembles. *Int. Math. Res. Not.*, 2004(50):2665–2701, 2004. doi:10.1155/S1073792804141597.

[51] W. König. Orthogonal polynomial ensembles in probability theory. *Probab. Surv.*, 2:385–447, 2005. URL: citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.234.5003, doi:10.1214/154957805100000177.

[52] K. Krishnamoorthy. *Handbook of statistical distributions with applications*. Boca Raton, FL: CRC Press, 2nd edition edition, 2016. doi:10.1201/b19191.

[53] M. Krishnapur, B. Rider, and B. Virág. Universality of the stochastic airy operator. *Communications on Pure and Applied Mathematics*, 69(1):145–199, 2015. doi:10.1002/cpa.21573.

[54] H-H. Kuo. *White noise distribution theory*. Probability and Stochastics Series, 1996.

[55] N. N. Lebedev. Special functions and their applications. *SIAM Rev.*, 7(4):577–580, 1965. doi:10.1137/1007133.

[56] V. A. Marchenko and L. A. Pastur. Distribution of eigenvalues for some sets of random matrices. *Math. USSR, Sb.*, 1:457–483, 1968. doi:10.1070/SM1967v001n04ABEH001994.

[57] B. D. McKay. The expected eigenvalue distribution of a large regular graph. *Linear Algebra Appl.*, 40:203–216, 1981. doi:10.1016/0024-3795(81)90150-6.

[58] M. L. Mehta. Random matrices and the statistical theory of energy levels. 1967.

[59] M. L. Mehta. *Random Matrices*. 3rd edition, 2004.

[60] C. Min and Y. Chen. Global and local scaling limits for linear eigenvalue statistics of Jacobi beta-ensembles, 2021. doi:10.48550/ARXIV.2111.05661.

[61] F. Nakano and T. K. Duy. Gaussian beta ensembles at high temperature: eigenvalue fluctuations and bulk statistics. *J. Stat. Phys.*, 173(2):295–321, 2018. doi:10.1007/s10955-018-2131-9.

[62] C. Pakzad. Poisson statistics at the edge of Gaussian β-ensemble at high temperature. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 16(1):871–897, 2019. URL: alea.impa.br/articles/v16/16-32.pdf.

[63] J. Ramírez and B. Rider. Diffusion at the random matrix hard edge. *Communications in Mathematical Physics*, 288(3):887–906, 2009. doi:10.1007/s00220-008-0712-1.

[64] J. A. Ramírez, B. Rider, and B. Virág. Beta ensembles, stochastic Airy spectrum, and a diffusion. *J. Am. Math. Soc.*, 24(4):919–944, 2011. doi:10.1090/S0894-0347-2011-00703-0.

[65] B. Rider and P. Waters. Universality of the stochastic Bessel operator. *Probab. Theory Relat. Fields*, 175(1-2):97–140, 2019. doi:10.1007/s00440-018-0888-z.

[66] W. Rudin. *Functional Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1991.

[67] M. Shcherbina. Orthogonal and symplectic matrix models: Universality and other properties. *Communications in Mathematical Physics*, 307(3):761–790, Sep 2011. doi:10.1007/s00220-011-1351-5.

[68] J. W. Silverstein. The smallest eigenvalue of a large dimensional Wishart matrix. *Ann. Probab.*, 13:1364–1368, 1985. doi:10.1214/aop/1176992819.

[69] B. Simon. *Trace Ideals and Their Applications*. Mathematical surveys and monographs. American Mathematical Society, 1074.

[70] B. Simon. *Functional Integration and Quantum Physics*. Academic Press, 1979.

[71] P. Sosoe and P. Wong. Local semicircle law in the bulk for gaussian β-ensemble. *Journal of Statistical Physics*, 148(2):204–232, 2012. doi:10.1007/s10955-012-0536-4.

[72] B. Sutton. *The stochastic operator approach to random matrix theory*. PhD thesis, Virginia Polytechnic Institute and State University, 2005.

[73] G. Szego. *Orthogonal Polynomials*. American Math. Soc: Colloquium publ. American Mathematical Society, 1975.

[74] C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Commun. Math. Phys.*, 159(1):151–174, 1994. doi:10.1007/BF02100489.

[75] C. A. Tracy and H. Widom. On orthogonal and symplectic matrix ensembles. *Communications in Mathematical Physics*, 177(3):727–754, 1996. doi:10.1007/bf02099545.

[76] H. F. Trotter. Eigenvalue distributions of large Hermitian matrices; Wigner's semi- circle law and a theorem of Kac, Murdock, and Szegö. *Adv. Math.*, 54:67–82, 1984. doi:10.1016/0001-8708(84)90037-9.

[77] B. Valkò and B. Virág. Continuum limits of random matrices and the brownian carousel. *Inventiones mathematicae*, 177(3):463–508, 2009. doi:10.1007/s00222-009-0180-z.

[78] B. Valkó and B. Virág. The sine β operator. *Inventiones mathematicae*, 209(1):275–327, 2016. doi:10.1007/s00222-016-0709-x.

[79] P. Waters. Feynman-kac formula for the stochastic bessel operator. 2017. doi:10.48550/ARXIV.1711.00908.

[80] J. Weidmann. *Spectral theory of ordinary differential operators*, volume 1258 of *Lect. Notes Math.* Springer, Cham, 1987.

[81] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Ann. Math. (2)*, 62:548–564, 1955. doi:10.2307/1970079.

[82] E. P. Wigner. On the distribution of the roots of certain symmetric matrices. *Ann. Math. (2)*, 67:325–327, 1958. doi:10.2307/1970008.

[83] J. Wishart. The generalised product moment distribution in samples from a normal multivariate population. *Biometrika*, 20A(1/2):32–52, 1928.

[84] P. Wong. Local semicircle law at the spectral edge for gaussian β-ensembles. *Communications in Mathematical Physics*, 312(1):251–263, 2012. doi:10.1007/s00220-012-1456-5.

RÉSUMÉ

Cette thèse porte sur les comportements global et local des β-ensembles lorsque leur dimension tend vers l'infini, d'abord à $\beta > 0$ fixé puis dans la limite haute température, lorsque β tend vers 0. Dans une première partie, nous étudions les ensembles β-*Hermite*, β-*Laguerre* et β-*Jacobi* lorsque $\beta > 0$. Le chapitre 1 introduit des représentations matricielles de ces ensembles et explique la convergence de leurs mesures spectrales vers des mesures d'équilibre. Les chapitres 2 et 3 présentent des classes d'universalité pour les comportements locaux des β-ensembles lorsque leur dimension tend vers l'infini, définies par les processus ponctuels des valeurs propres de certains opérateurs stochastiques. Dans une seconde partie, les chapitres 4, 5 et 6 s'intéressent au comportement des valeurs propres de ces opérateurs stochastiques dans la limite haute température. Le chapitre 5 présente les résultats précédents de Laure Dumaz et Cyril Labbé sur l'*opérateur stochastique d'Airy*. Le Chapitre 6 contient notre contribution de recherche, avec l'étude des plus petites valeurs propres l'*opérateur stochastique de Bessel* dans la limite haute température.

MOTS CLÉS

Beta-ensembles - Gas de Coulomb - Matrices aléatoires - Opérateurs stochastiques - Diffusions - Physique statistique

ABSTRACT

This thesis focuses on the global and local behavior of β-ensembles when their size tends to infinity, first at set $\beta > 0$, then in the high temperature limit, when β tends to 0. In a first part, we study the β-*Hermite*, the β-*Laguerre* and the β-*Jacobi* ensembles when $\beta > 0$. Chapter 1 introduces matrix models for these ensembles and explains the convergence of their spectral measures to equilibrium measures. Chapters 2 and 3 present universality classes for the local behaviors of β-ensembles in the large dimension limit, defined using the eigenvalues point processes of stochastic operators. In a second part, we analyze the eigenvalues of these stochastic operators in the high temperature limit. Chapter 5 presents the previous results from Laure Dumaz and Cyril Labbé on the *stochastic Airy operator*. Chapter 6 contains our research contribution, with the study of the smallest eigenvalues *stochastic Bessel operator* in the high temperature limit.

KEYWORDS

Beta-ensembles - Coulomb gases - Random matrices - Stochastic operators - Diffusions - Statistical Physics