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L'UNIVERSITÉ D'ANGERS
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*Mathématiques et Sciences et Technologies
de l'Information et de la Communication*
Spécialité : Mathématiques et leurs interactions

Par

François BERNARD

« **Seminormalisation et fonctions régulières.** »

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Rapporteurs avant soutenance :

Jérémy BLANC Professeur des universités à l'Université de BÂLE
Claus SCHEIDERER Professeur des universités à l'Université de CONSTANCE

Composition du Jury :

Président :	Etienne MANN	Professeur des universités à l'Université d'ANGERS
Examineurs :	Susanna ZIMMERMANN	Professeure des universités à l'Université de PARIS-SACLAY
	Olivier BENOIST	Chargé de recherche à l'ENS ULM
Directeurs de thèse :	Jean-Philippe MONNIER	Maître de conférence à l'Université d'ANGERS.
	Goulwen FICHOU	Maître de conférence à l'Université de RENNES 1.

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NOTATIONS

$\text{Spec}(A)$	le spectre d'un anneau A .
$\text{Max}(A)$	le spectre maximal d'un anneau A .
$\text{R-Spec}(A)$	l'ensemble des idéaux réels d'un anneau A .
$\text{R-Max}(A)$	l'ensemble des idéaux maximaux réels d'un anneau A .
$\text{Cent}(A)$	l'ensemble des idéaux centraux d'un anneau A .
$\text{Indet}(f)$	le lieu d'indétermination d'une fonction rationnelle.
$\text{Dom}(f)$	le domaine d'une fonction rationnelle.
$\mathcal{K}(X)$	l'anneau total des fractions de X .
$\mathcal{K}^0(X(\mathbb{C}))$	l'ensemble des fonctions rationnelles continues sur $X(\mathbb{C})$.
$\mathcal{K}^+(X(\mathbb{R}))$	l'anneau des fonctions sur $X(\mathbb{R})$ devenant polynomiales sur $X^+(\mathbb{R})$.
$\mathcal{K}^{R+}(X(\mathbb{R}))$	l'anneau des fonctions sur $X(\mathbb{R})$ devenant polynomiales sur $X^{R+}(\mathbb{R})$.
$\mathcal{R}^0(X(\mathbb{R}))$	l'anneau des fonctions régulières sur $X(\mathbb{R})$.
X_{sing}	les points singuliers de X .
X_{reg}	les points réguliers de X .
$\text{Norm}(X)$	les points normaux de X .
$\mathcal{SN}(X)$	les points seminormaux de X .
$\text{Rad}(A)$	le radical de Jacobson de A .
$\mathcal{Z}-$	se rapporte à la topologie de Zariski.
Γ_f	le graphe d'une fonction f .
A'_B (resp. A')	la clôture intégrale de A dans B (resp. dans $\mathcal{K}(A)$).
A_B^F	la F-seminormalisation de A dans B .
A_B^+ (resp. A^+)	la seminormalisation de A dans B (resp. dans A').
A_B^{R+} (resp. A^+)	la R-seminormalisation de A dans B (resp. dans A').
A^b	la normalisation birégulière de A .

INTRODUCTION

Cette thèse est consacrée, dans un premier temps, à l'étude de la seminormalisation de variétés algébriques complexes, à son lien avec les fonctions régulières complexes et à l'étude approfondie de ces dernières. Dans un second temps, nous définissons la «R-seminormalisation» d'une variété algébrique réelle, identifions les éléments de son anneau de coordonnées de différentes manières et décrivons la façon dont elle modifie les singularités de la variété en prouvant qu'elle correspond à la normalisation birégulière de la seminormalisation.

Présentation historique :

Le concept de seminormalisation est apparu pour la première fois en 1967 dans un article d'Andreotti et Norguet [4] portant sur l'étude des cycles analytiques d'une variété algébrique complexe. Les auteurs y définissent la notion de «normalisation faible» d'un espace analytique en élargissant le faisceau des fonctions holomorphes à celui des fonctions «c-holomorphes», c'est-à-dire des fonctions holomorphes sur le lieu lisse qui sont continues sur tout l'espace. La normalisation faible d'un espace analytique complexe réduit X est un espace analytique X^* avec une application holomorphe bijective $X^* \rightarrow X$, qui est universelle pour cette propriété. Autrement dit X^* est le plus grand espace analytique homéomorphe à X . En 1969, Andreotti et Bombieri [3] étendent la notion de normalisation faible pour des schémas et un an plus tard, Traverso [38] définit la très proche notion de seminormalisation d'un anneau dans un autre. Pour une extension entière d'anneaux $A \subset B$, ces constructions sont toutes deux obtenues en considérant des «recolllements» d'idéaux de B au-dessus de chaque idéal de A . Concrètement, si A est de caractéristique $p > 0$, alors la normalisation faible de A dans B est donnée par

$$A_B^* := \left\{ b \in B \mid \forall \mathfrak{p} \in \text{Spec}(A), \exists n \in \mathbb{N} \text{ tel que } (b_{\mathfrak{p}})^{p^n} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}}) \right\}$$

où Rad désigne le radical de Jacobson. En caractéristique 0, la normalisation faible coïncide avec la seminormalisation qui est donnée par

$$A_B^+ := \left\{ b \in B \mid \forall \mathfrak{p} \in \text{Spec}(A), b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}}) \right\}$$

La seminormalisation a pour propriété d'être la plus grande extension C de A dans B qui est subintégrale, c'est-à-dire telle que $\text{Spec}(C) \rightarrow \text{Spec}(A)$ est bijective et équirésiduelle (il y a isomorphisme des

corps résiduels). Dans les années suivantes, différentes définitions et caractérisations algébriques de la seminormalisation apparaissent. Celles-ci sont toutes équivalentes à celle de Traverso pour des anneaux de coordonnées sur un corps de caractéristique 0. Pour Hamann [18], un anneau A est seminormal dans B si pour tout $n \in \mathbb{N}^*$, l'anneau A contient tous les éléments $b \in B$ tels que $b^n, b^{n+1} \in A$. La définition de Leahy-Vitulli [28] consiste quant à elle à remplacer $n, n + 1$ par n'importe quel couple d'entiers naturels premiers entre eux. Enfin, pour Swan [37], un anneau est seminormal dans son anneau total des fractions, si pour tout $b, c \in A$ tels que $b^3 = c^2$, alors il existe un élément $a \in A$ tel que $a^2 = b$ et $a^3 = c$. Il est intéressant de noter que cette dernière définition ne fait pas intervenir la normalisation de l'anneau.

En 1980, l'article de Greco et Traverso [16] ainsi que celui de Leahy et Vitulli [28] (corrigé ensuite dans [39]) s'intéressent à la géométrie de la seminormalisation d'une variété algébrique. Pour une variété algébrique X , définie sur un corps algébriquement clos, la seminormalisation X^+ est la plus grande variété intermédiaire entre X et sa normalisation X' dont les points fermés sont en bijection avec ceux de X . Dans [16], Greco et Traverso établissent notamment des liens entre la seminormalisation et le concept de base d'Andreotti et Norguet de normalisation faible d'un espace analytique. Par exemple, une variété algébrique complexe est seminormale si et seulement si son analytifié est faiblement normal au sens d'Andreotti et Norguet. Les auteurs se sont également intéressés aux singularités que peut avoir une variété seminormale, tout comme Leahy et Vitulli dans [29] qui introduisent au même moment la notion de singularités multicroisées («multicross singularities»). Ils prouvent que l'ensemble des singularités d'une variété seminormale qui ne sont pas multicroisées est un espace de codimension au moins égale à 2. On dit qu'un point $x \in X$ est multicroisé s'il est analytiquement isomorphe à un point $x' \in X'$ où X' est l'union de sous-espaces vectoriels se coupant transversalement le long d'un espace vectoriel commun. Le lecteur ou la lectrice pourra trouver plus de précisions sur les points multicroisés au chapitre 1 dans la section 1.3. Plus globalement, nous renvoyons à Vitulli [40] qui a publié un compte-rendu exhaustif sur la seminormalisation en 2011. Récemment, la seminormalisation est apparue dans l'étude des singularités dans le cadre du programme des modèles minimaux, voir Kollár et Kovács [20], Kollár [21] et Ambro [2]. En géométrie réelle, la seminormalisation et la normalisation faible font déjà l'objet de premières recherches en 1975 dans l'article [1] traitant du cas des espaces analytiques réels, ainsi qu'en 1985 dans [34]. Cependant, en 1981, Marinari et Raimondo [31] montrent que la construction de Traverso pour les spectres réels ne possède pas de propriété universelle naturelle. Une alternative est donnée dans deux récents articles de géométrie algébrique réelle [33] et [15] écrits par Fichou, Monnier et Quarez dont nous fournissons une présentation détaillée dans la section 1.4 du chapitre 1. Ils y introduisent le concept de «seminormalisation centrale» qui est pensée pour être plus adaptée aux variétés réelles et qui va notamment vérifier une propriété universelle au niveau des points réels dit «centraux». Ce sont les points se situant dans l'adhérence euclidienne du lieu lisse de la variété. Ces nouvelles études ont été motivées par le lien que la seminormalisation centrale entretient avec la théorie des «fonctions régules». Une théorie qui a été introduite en 2015, en partie par les deux premiers auteurs dans [12] et au même moment par Kollár et Nowak dans [22]. Une fonction est dite «régule» sur une variété réelle X si elle admet une représentation rationnelle, si elle est continue et si sa restriction à toute sous-variété réelle de X admet encore une représentation rationnelle. Sur une variété lisse, toute fonction rationnelle continue est régulue mais ce n'est pas le cas en général (voir l'exemple 1.2.3 issu de [22]). Parmi les nombreux intérêts des fonctions régules, on peut citer par exemple leur capacité à vérifier certains théorèmes de géométrie

algébrique classique, comme le Nullstellensatz, qui ne sont normalement plus valables en géométrie réelle. Ces dernières années, elles ont fait l'objet de nombreuses études comme par exemple [24], [26], [27], [13], [43], [32]. Plus de détails sur les fonctions régulières peuvent être trouvés dans le chapitre 1 à la section 1.2.3 ou bien dans le compte rendu [25] de 2018 écrit par Kucharz et Kurdyka.

Contributions du chapitre 2 :

Probablement inspirés par la construction d'Andreotti et Norguet de la normalisation faible des espaces analytiques complexes qui consistait à étendre le faisceau des fonctions holomorphes au faisceau des fonctions c -holomorphes, Leahy et Vitulli ont voulu montrer dans [28] que la seminormalisation d'une variété algébrique pouvait être obtenue grâce aux fonctions rationnelles qui sont continues pour la topologie de Zariski. Cependant ce résultat est erroné notamment car, si f est une fonction rationnelle vérifiant une relation de la forme $qf - p = 0$ quand $q \neq 0$, alors supposer que f est Zariski continue ne suffit pas en général à étendre la relation à tout l'espace. On peut penser, par exemple, à la fonction $g : \mathbb{A}^1(\mathbb{C}) \rightarrow \mathbb{A}^1(\mathbb{C})$ qui à z associe $1/z$ si $z \neq 0$ et 0 sinon. Cette fonction est Zariski continue mais la relation $zg(z) - 1 = 0$ ne s'étend pas en 0. De manière générale, la continuité de Zariski est une hypothèse faible en dimension 1 où, par exemple, toute fonction bijective est Zariski continue. Par soucis de clarté, nous redémontrons de manière détaillée, dans les sections 2.1 et 2.2, les résultats fondamentaux de la seminormalisation. Dans la section 2.3, nous montrons que, pour une variété affine complexe, le bon ensemble de fonctions à considérer pour construire la seminormalisation est celui des fonctions rationnelles qui sont continues pour la topologie euclidienne. Plus précisément : soit X une variété affine complexe, on définit l'anneau $\mathcal{K}^0(X(\mathbb{C}))$ des fonctions qui sont rationnelles et continues sur $X(\mathbb{C})$ pour la topologie euclidienne induite par $X(\mathbb{C}) \subset \mathbb{C}^n$. Pour un morphisme fini $\pi : Y \rightarrow X$, on s'intéresse au morphisme induit $f \mapsto f \circ \pi_c$ entre les deux anneaux de fonctions rationnelles continues et nous montrons que son image est donnée par

$$\left\{ f \in \mathcal{K}^0(Y(\mathbb{C})) \text{ avec } f \text{ constante sur les fibres de } \pi_c \right\}$$

Ceci nous permet de réinterpréter la notion d'extension subintégrale d'anneaux de coordonnées sur les anneaux de fonctions rationnelles continues. Plus précisément, si $\pi : Y \rightarrow X$ est un morphisme de variétés affines complexes, alors $\mathbb{C}[X] \hookrightarrow \mathbb{C}[Y]$ est subintégrale si et seulement si $\mathcal{K}^0(X(\mathbb{C})) \simeq \mathcal{K}^0(Y(\mathbb{C}))$. Les résultats obtenus sur les extensions subintégrales peuvent être résumés par le théorème suivant :

Théorème (2.2.1 and 2.3.12). *Soit $\pi : Y \rightarrow X$ un morphisme fini entre deux variétés complexes affines. Alors les propriétés suivantes sont équivalentes :*

- 1) *L'extension $\pi^* : \mathbb{C}[X] \hookrightarrow \mathbb{C}[Y]$ est subintégrale.*
- 2) *Le morphisme π_c est bijectif.*
- 3) *Les anneaux $\mathcal{K}^0(Y(\mathbb{C}))$ et $\mathcal{K}^0(X(\mathbb{C}))$ sont isomorphes.*
- 4) *π_c est un homéomorphisme pour la topologie euclidienne.*
- 5) *π_c est un homéomorphisme pour la topologie de Zariski.*
- 6) *π est un homéomorphisme.*

Au début de la sous-section 2.3.1, nous démontrons que les fonctions rationnelles continues sont régulières sur le lieu lisse de la variété. Cela nous permet de voir que, si X est normale, alors la finesse de $X_{\text{sing}}(\mathbb{C})$

implique $\mathcal{K}^0(X(\mathbb{C})) = \mathbb{C}[X]$. En combinant ceci avec le théorème précédent, nous obtenons le résultat principal de cette thèse :

Théorème (2.3.13). *Soit X une variété affine complexe et soit $\pi^+ : X^+ \rightarrow X$ son morphisme de seminormalisation. Alors, le morphisme suivant est un isomorphisme :*

$$\begin{aligned} \varphi : \mathcal{K}^0(X(\mathbb{C})) &\rightarrow \mathbb{C}[X^+] \\ f &\mapsto f \circ \pi_c^+ \end{aligned}$$

Dans la partie 2.5 nous définissons le faisceau des fonctions rationnelles continues $\mathcal{K}_{X(\mathbb{C})}^0$ et généralisons ces deux théorèmes sur des variétés algébriques complexes quelconques. Ainsi, de la même manière que pour obtenir la normalisation faible d'un espace analytique il faut étendre son faisceau de fonctions holomorphes à celui des fonctions c-holomorphes, considérer la seminormalisation d'une variété algébrique complexe revient à remplacer son faisceau structural par $\mathcal{K}_{X(\mathbb{C})}^0$.

Comme nous l'avons souligné dans la présentation historique de cette introduction, les fonctions rationnelles continues et les fonctions régulières forment des classes de fonctions très intéressantes dans le cadre de la géométrie algébrique réelle. Les premiers résultats obtenus dans la thèse nous montrent que les fonctions rationnelles continues complexes possèdent également d'intéressantes propriétés. Il est alors naturel de se demander ce qu'il en est des fonctions régulières complexes. La réponse est donnée dans la section 2.3 : en géométrie complexe, les fonctions régulières et les fonctions rationnelles continues coïncident.

Dans les articles [7] et [8] datant de 2020, les auteurs s'intéressent aux fonctions définies sur un espace analytique complexe qui sont c-holomorphes et dont le graphe est algébrique. Ils présentent celles-ci comme une version complexe des fonctions régulières réelles. Nous démontrons dans la sous-section 2.3.3 que cette classe de fonctions coïncide avec $\mathcal{K}^0(X(\mathbb{C}))$. Cela revient notamment à démontrer une version algébrique du théorème de Whitney ([41], Théorème 4.5Q) que nous obtenons comme un corollaire du résultat principal de cette sous-section :

Théorème (2.3.17). *Soit X une variété affine complexe et $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ une fonction continue pour la topologie euclidienne. Supposons qu'il existe un polynôme $P \in \mathbb{C}[X][t]$ qui est non nul sur chaque composante irréductible de $X(\mathbb{C})$ et tel que $P(x, f(x)) = 0$, pour tout $x \in X(\mathbb{C})$. Alors*

$$f \in \mathcal{K}^0(X(\mathbb{C}))$$

Ce résultat nous permet également de voir $\mathcal{K}^0(X(\mathbb{C}))$ comme la clôture intégrale de $\mathbb{C}[X]$ dans l'anneau $\mathcal{C}^0(X(\mathbb{C}), \mathbb{C})$ des fonctions continues sur $X(\mathbb{C})$ pour la topologie euclidienne.

Nous apportons dans la section 2.3.4 plusieurs exemples non triviaux de fonctions régulières complexes, ces derniers étant quasiment inexistant dans la littérature scientifique. Cela est rendu possible grâce au théorème 2.3.20 qui nous démontrons dans la même section :

Théorème (2.3.20). *Soit X une variété affine complexe. Une fonction $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ appartient à $\mathcal{K}^0(X(\mathbb{C}))$ si et seulement si elle est rationnelle, entière sur $\mathbb{C}[X]$ et si son graphe est Zariski fermé.*

Nous pouvons résumer l'ensemble des résultats obtenus dans la section 2.3 par le graphique suivant qui montre les différentes implications entre les quatre hypothèses «être continue», «être rationnelle», «être entière sur $\mathbb{C}[X]$ » et «avoir un graphe Zariski fermé» que l'on peut apposer à une fonction définie sur $X(\mathbb{C})$.

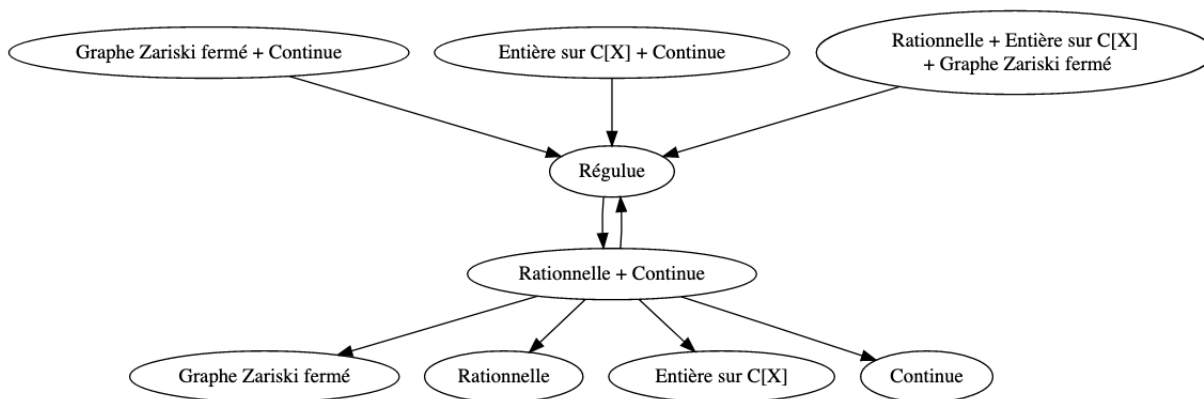


FIGURE 1 – Implications entre les propriétés des fonctions régulières complexes.

Au début de cette introduction, nous avons évoqué l'existence de plusieurs définitions de la seminormalisation utilisées par Leahy-Vitulli, Hamann et Swan. Nous avons également indiqué que toutes ces définitions sont équivalentes à celles de Traverso dans le cadre de la géométrie algébrique complexe. C'est pourquoi, dans la section 2.4, nous les réinterprétons au travers des fonctions rationnelles continues.

Contributions du chapitre 3 :

Dans la construction de Traverso, la seminormalisation d'une variété X est construite en recollant tous les points se trouvant dans une même fibre du morphisme de normalisation π' . Nous avons vu précédemment qu'elle est caractérisée comme étant la plus grande variété intermédiaire entre X et X' dont les points fermés sont en bijection avec ceux de X . On pourrait imaginer qu'en adaptant la construction de Traverso pour une variété réelle, c'est-à-dire en recollant les points réels de X' qui se trouvent au-dessus d'un même point réel de X , alors on obtiendrait la plus grande variété intermédiaire entre X et X' dont les points réels seraient en bijection avec ceux de X . En réalité, cette construction ne possède pas de bonnes propriétés notamment car un morphisme fini n'est pas nécessairement surjectif au niveau des points réels. Par exemple, la normalisation de la variété définie par l'équation $x^2 + y^2 = 0$ est donnée par l'équation $y^2 + 1 = 0$ qui n'a pas de point réel. Plus généralement, l'exemple 5.6 de [15] montre que pour une variété donnée, il n'existe pas nécessairement de plus grande variété qui lui soit liée par un morphisme fini, birationnel et bijectif au niveau des points réels. La seminormalisation centrale de Fichou, Monnier et Quarez fournit une alternative en produisant, pour toute variété réelle, la plus grande variété qui lui soit liée par un morphisme fini, birationnel et bijectif au niveau des points centraux. Dans les sections 3.1 et 3.2 nous définissons une autre alternative à la seminormalisation dans le cadre de la géométrie réelle : la «R-seminormalisation». Pour une variété algébrique réelle affine, celle-ci est obtenue en recollant tous les points complexes de la normalisation vivant au dessus d'un même point réel. La R-seminormalisation

X^{R+} d'une variété X est alors la plus grande variété intermédiaire entre X et X' possédant un unique point complexe au-dessus de chaque point réel de X .

En s'inspirant de ce qui a été fait avec les fonctions régulières complexes dans le chapitre 2, nous cherchons dans la section 3.3 à identifier l'anneau $\mathcal{K}^{R+}(X(\mathbb{R}))$ des fonctions à valeurs sur $X(\mathbb{R})$ qui deviennent polynomiales sur la R-seminormalisation. On pourrait penser que les éléments de $\mathcal{K}^{R+}(X(\mathbb{R}))$ correspondent aux fonctions qui sont rationnelles, entières sur $\mathbb{R}[X]$ et continues pour la topologie euclidienne de $X(\mathbb{C})$, mais il s'avère que ceci n'est vrai que dans le cas des courbes. En général, on obtient la caractérisation suivante :

Proposition (3.3.7). *Soit X une variété réelle affine et soit $f : X(\mathbb{R}) \rightarrow \mathbb{R}$. Alors $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$ si et seulement si elle vérifie les propriétés suivantes :*

- 1) *La fonction f est rationnelle.*
- 2) *La fonction f est entière sur $\mathbb{R}[X]$.*
- 3) *Pour tout $x \in X(\mathbb{R})$, pour tout $(z_n)_n \in \text{Dom}(f) \cup X(\mathbb{R})$ tel que $z_n \rightarrow x$ alors $f(z_n) \rightarrow f(x)$.*

Nous établissons également une caractérisation des éléments de $\mathcal{K}^{R+}(X(\mathbb{R}))$ à l'aide de conditions sur le graphe d'une fonction :

Théorème (3.3.11). *Soit X une variété réelle affine et soit $f : X(\mathbb{R}) \rightarrow \mathbb{R}$. Alors $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$ si et seulement si elle vérifie les propriétés suivantes :*

- 1) *La fonction f est rationnelle.*
- 2) *Il existe un polynôme unitaire $P(t) \in \mathbb{R}[X][t]$ tel que $P(f) = 0$ sur $X(\mathbb{R})$.*
- 3) *Le graphe Γ_f de f est un fermé de Zariski dans $X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$.*
- 4) *Pour tout $x \in X(\mathbb{R})$, on a $\overline{\Gamma_f}^{\mathbb{C}} \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) = \{(x; f(x))\}$*

Ici, $\overline{\Gamma_f}^{\mathbb{C}}$ désigne l'adhérence de Zariski de Γ_f dans $X(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C})$. Grâce à ce résultat nous fournissons plusieurs exemples d'éléments de $\mathcal{K}^{R+}(X(\mathbb{R}))$ et comparons la R-seminormalisation avec la seminormalisation et la seminormalisation centrale.

Dans l'article [14] de Fichou, Monnier et Quarez, les auteurs étudient différentes clôtures intégrales d'anneaux de fonctions, notamment en exploitant le fait que les fonctions polynomiales ne coïncident pas avec les fonctions régulières sur une variété réelle. Ils définissent la «normalisation birégulière» X^b d'une variété X en considérant la clôture intégrale de $\mathbb{R}[X]$ dans l'anneau des fonctions régulières sur $X(\mathbb{R})$. Nous renvoyons à la sous-section 1.4.1 du chapitre 1 pour plus d'informations sur la normalisation birégulière. La dernière contribution de cette thèse est de lier la R-seminormalisation et la normalisation birégulière en établissant, dans la section 3.5, le théorème suivant :

Théorème (3.5.8). *Soit X une variété affine réelle. Alors*

$$X^{R+} \simeq (X^+)^b \simeq (X^b)^+$$

Au niveau des singularités, ce résultat démontre que la R-seminormalisation normalise les points purement complexes de la variété et seminormalise ses points réels.

PRÉREQUIS

Dans ce chapitre, nous introduisons divers outils mathématiques qui ont servi à l'élaboration de cette thèse. Tout d'abord, nous présentons les notions, qui seront omniprésentes, de clôture intégrale d'un anneau dans un autre et de celle d'un anneau dans son anneau total des fractions. Ceci nous amène naturellement vers la présentation de la normalisation d'une variété algébrique, l'objectif global de ce travail de thèse étant d'en étudier différentes variantes. Le chapitre 3 est notamment consacré à l'une de ces variantes s'appliquant spécifiquement dans le cadre de la géométrie réelle. C'est pourquoi nous menons ensuite une discussion sur la notion de variété réelle. De manière générale, les idées des résultats obtenus dans ce manuscrit proviennent d'une part de la théorie des fonctions régulées réelles, auxquelles nous consacrons donc une sous-section, et d'autre part de la seminormalisation que nous présentons à la suite. Le lien entre ces deux objets a été mis en avant dans les articles [15], [33] et [14] qui sont à la base de ce travail de recherche. Nous finissons donc ce chapitre en leur y consacrant une section explicative.

1.1 Résultats généraux d'algèbre commutative.

Nous présentons dans cette section différents résultats d'algèbre commutative concernant les extensions entières d'anneaux et concernant la clôture intégrale d'un anneau dans un autre. Les extensions entières apparaîtront naturellement au cours de notre étude car nous considérerons principalement des sous-extensions de la normalisation. La clôture intégrale sera également omniprésente dans les chapitres suivants, notamment car nous essayerons régulièrement d'identifier certains anneaux comme la clôture intégrale d'un anneau de fonctions dans un autre.

1.1.1 La clôture intégrale dans un anneau.

Les prochains énoncés proviennent du chapitre 5 de [5]. Tous les anneaux sont supposés unitaires et commutatifs.

Définition 1.1.1

Soit $A \hookrightarrow B$ une extension d'anneau. Un élément $b \in B$ est dit entier sur A s'il existe un polynôme unitaire $P \in A[X]$ tel que $P(b) = 0$.

Cela signifie que si b est un élément entier sur A , alors il existe $n \in \mathbb{N}$ tel que $b^n = a_{n-1}b^{n-1} + \dots + a_0$. On en déduit que tous les éléments de $A[b]$ s'expriment en fonction de $\{1, b, \dots, b^{n-1}\}$.

Proposition 1.1.2

Soit $A \hookrightarrow B$ une extension d'anneau. Un élément $b \in B$ est entier sur A si et seulement si l'anneau $A[b]$ est un A -module finiment engendré.

Par récurrence, on obtient le corollaire suivant.

Corollaire 1.1.3

Soit $\{b_i\}_{i \in [1;n]}$ un nombre fini d'éléments de B entiers sur A . Alors $A[b_1, \dots, b_n]$ est un A -module finiment engendré.

En particulier, pour deux éléments b et c entiers sur A , alors $b + c$ et bc appartiennent à $A[b, c]$. Donc le fait d'être entier sur A est stable par addition et multiplication.

Définition 1.1.4

Soit $A \hookrightarrow B$ une extension d'anneau. L'ensemble des éléments de B entiers sur A est un anneau que l'on note A'_B et que l'on appelle la clôture intégrale de A dans B . Si $A = A'_B$, on dit que A est intégralement clos dans B . Si $A'_B = B$, on dit que B est entier sur A ou encore que l'extension $A \hookrightarrow B$ est entière.

La propriété d'être entier sur un anneau est transitive.

Proposition 1.1.5

Soit $A \hookrightarrow B \hookrightarrow C$ des extensions d'anneaux telles que B est entier sur A et C est entier sur B , alors C est entier sur A .

Ainsi on peut voir la clôture intégrale de A'_B comme la plus grande sous-extension entière de $A \hookrightarrow B$.

Une propriété importante des extensions entières est le théorème de «Going-up» qui dit qu'une telle extension respecte la dimension des idéaux. On donne tout d'abord une version plus faible appelée le théorème de «Lying-over».

Théorème 1.1.1 (Lying-over)

Soit $A \hookrightarrow B$ une extension entière et soit \mathfrak{p} un idéal premier de A . Alors il existe un idéal premier \mathfrak{q} de B au dessus de \mathfrak{p} , c'est à dire tel que $\mathfrak{q} \cap A = \mathfrak{p}$.

Théorème 1.1.2 (Going-up)

Soit $A \hookrightarrow B$ une extension entière, soit $\mathfrak{p}_1 \subseteq \dots \subseteq \mathfrak{p}_n$ une chaîne d'idéaux premiers de A et soit $\mathfrak{q}_1 \subseteq \dots \subseteq \mathfrak{q}_m$ une chaîne d'idéaux premiers de B . On suppose que $m < n$ et que $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ pour tout $i \leq m$. Alors la chaîne $\mathfrak{q}_1 \subseteq \dots \subseteq \mathfrak{q}_m$ peut-être étendue en une chaîne $\mathfrak{q}_1 \subseteq \dots \subseteq \mathfrak{q}_n$ telle que $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ pour tout $i \leq n$.

Un point important à souligner est que prendre la clôture intégrale d'un anneau commute avec la localisation.

Proposition 1.1.6

Soit $A \hookrightarrow B$ une extension d'anneaux et soit S une partie multiplicative de A . Alors $S^{-1}(A'_B)$ est la clôture intégrale de $S^{-1}A$ dans $S^{-1}B$.

Si A est intègre, on dit que A est intégralement clos s'il l'est dans son anneau des fractions. Ainsi, par la proposition précédente, on voit que la propriété d'être intégralement clos est une propriété locale.

Corollaire 1.1.7

Soit A un anneau intègre. Alors les énoncés suivants sont équivalents :

1. A est intégralement clos.
2. $A_{\mathfrak{p}}$ est intégralement clos pour tout idéal premier \mathfrak{p} de A .
3. $A_{\mathfrak{m}}$ est intégralement clos pour tout idéal maximal \mathfrak{m} de A .

Ceci nous permettra plus tard de définir la notion de «point normal» d'une variété.

1.1.2 Clôture intégrale dans l'anneau total des fractions.

Dans le cas où A n'est pas intègre, on a une notion qui généralise celle du corps des fractions de A .

Définition 1.1.8

Soit A un anneau et soit $S \subset A$ l'ensemble des éléments qui ne sont pas des diviseurs de zéros. On appelle anneau total des fractions et on note $\mathcal{K}(A)$ le localisé de A par S .

Dans le cas particulier (qui est vérifié pour un anneau de coordonnées) où A est un anneau réduit avec un nombre fini de premiers minimaux $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, on a

$$(0) = \sqrt{(0)} = \bigcap_{\mathfrak{p} \text{ premier}} \mathfrak{p} = \bigcap_{\mathfrak{p} \text{ minimal}} \mathfrak{p} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$$

et donc par le théorème chinois

$$A \hookrightarrow A/\mathfrak{p}_1 \times \dots \times A/\mathfrak{p}_n \hookrightarrow K_1 \times \dots \times K_n =: K$$

où les K_i sont les corps de fractions des A/\mathfrak{p}_i . On a alors $\mathcal{K}(A) = K$.

Proposition 1.1.9

Soit A un anneau réduit avec un nombre fini de premiers minimaux $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Alors

$$A'_{\mathcal{K}(A)} \simeq (A/\mathfrak{p}_1)'_{K_1} \times \dots \times (A/\mathfrak{p}_n)'_{K_n}$$

Démonstration : Soit $f = (f_1, \dots, f_n) \in \mathcal{K}(A)$ et $P \in A[t]$ unitaire et annulant f . Pour tout i on peut considérer un polynôme $P_i \in (A/\mathfrak{p}_i)[t]$ qui correspond au polynôme P dont les coefficients ont été réduit modulo \mathfrak{p}_i . Ainsi P_i est unitaire et $P_i(f_i) = 0$. Donc $f_i \in (A/\mathfrak{p}_i)'_{K_i}$. Réciproquement, si pour tout i , il existe $P_i \in (A/\mathfrak{p}_i)[t]$ unitaire et annulant f_i . Alors on peut considérer le produit P des P_i en choisissant des représentants de leurs coefficients dans A . Ainsi P est unitaire et $P(f) = (\prod P_i(f_1), \dots, \prod P_i(f_n)) = (0, \dots, 0) = 0$. ■

1.2 Résultats généraux de géométrie algébrique.

Soit A un anneau. Nous pouvons considérer l'ensemble $\text{Spec}(A)$ de ses idéaux premiers que nous appellerons le *spectre* de A . On peut munir $\text{Spec}(A)$ de la topologie de Zariski dont les fermés sont engendrés par les ensembles de la forme

$$\mathcal{V}(E) := \{ \mathfrak{p} \in \text{Spec}(A) \mid E \subset \mathfrak{p} \}$$

Ainsi les idéaux maximaux correspondent aux points fermés de $\text{Spec}(A)$. Soit k un corps commutatif. Si A est une k -algèbre de type fini réduite, alors $\text{Spec}(A)$ muni de son faisceau des fonctions régulières $\mathcal{O}_{\text{Spec}(A)}$ est une variété algébrique affine sur k . Soit X une variété algébrique affine sur k , nous noterons $k[X]$ l'anneau de coordonnées de X . Toute k -algèbre de type fini s'écrit sous la forme $k[x_1, \dots, x_n]/I$ où I est un idéal de $k[x_1, \dots, x_n]$. Si k est algébriquement clos et I est radical, alors le Nullstellensatz nous donne un homéomorphisme de Zariski entre l'ensemble algébrique $\mathcal{Z}(I)$ de k^n défini par I et les points fermés $X(k)$ de X . L'anneau de coordonnées $k[X]$ est alors isomorphe à l'anneau des fonctions polynomiales sur $\mathcal{Z}(I)$.

1.2.1 Normalisation d'une variété algébrique.

Nous présentons à présent la notion de normalisation d'une variété algébrique. Si X une variété algébrique affine, alors $k[X]$ est un anneau réduit et noethérien. Ainsi, il possède un nombre fini de premiers minimaux qui correspondent à ses composantes irréductibles. Nous pouvons donc lui appliquer la proposition 1.1.9 et étudier $k[X]_{\mathcal{K}(X)}$ composante par composante.

Définition 1.2.1

Soit X une variété algébrique affine sur k . On note X' la variété affine définie par la k -algèbre de type finie $k[X]_{\mathcal{K}(X)}$. La variété X' est appelée la *normalisation* de X et vient avec un morphisme $\pi' : X' \rightarrow X$ induit par l'extension $k[X] \hookrightarrow k[X']$. On dit que X est *normale* si $X = X'$, autrement dit $k[X]$ est intégralement clos.

Par le corollaire 1.1.7, la normalisation et la localisation commutent. On peut donc étudier la normalisation localement.

Définition 1.2.2

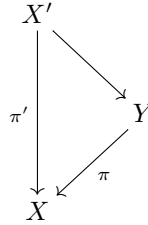
Soit X une variété algébrique sur k . On dit qu'un point $X \in X$ est *normal* si l'anneau $\mathcal{O}_{X,x}$ est intégralement clos. On dit que X est *normale* si tous ces points sont normaux.

Remarque. Si x est un point régulier, alors $\mathcal{O}_{X,x}$ est factoriel par le théorème d'Auslander-Buchsbaum. Un anneau factoriel étant intégralement clos, on obtient que toute variété non-singulière est normale.

La normalisation d'une variété X a pour propriété d'être la plus grande variété qui soit liée à X par un morphisme fini et birationnel.

Proposition 1.2.3 ([36], Chapitre 2 Partie 5.1)

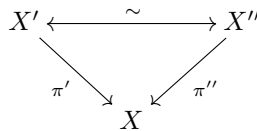
Soit $\pi : Y \rightarrow X$ un morphisme fini et birationnel. Alors il existe un morphisme $X' \rightarrow Y$ qui fasse commuter le diagramme suivant



On a donc unicité de la normalisation à isomorphisme près.

Corollaire 1.2.4 ([36], Chapitre 2 Partie 5.1)

Soit X'' une variété normale et $\pi'' : X'' \rightarrow X$ un morphisme fini et birationnel. Alors il existe un isomorphisme entre X' et X'' qui fasse commuter le diagramme suivant



Les variétés normales ont été introduite par Zariski dans [42] car elles interviennent dans l'étude des singularités, notamment en raison du théorème suivant.

Théorème 1.2.1

Le lieu singulier d'une variété algébrique normale est de codimension au moins 2.

Dans le cas des courbes la normalisation fournit une désingularisation de la variété.

Corollaire 1.2.5

Une courbe algébrique est normale si et seulement si elle ne possède aucun point singulier.

Exemple. Soit $X := \text{Spec}(\mathbb{R}[x, y] / \langle y^2 - x^2(x + 1) \rangle)$. On a $y/x \in K(X)$ qui est entière sur $\mathbb{R}[X]$ car elle est annihilée par le polynôme unitaire $P(t) = t^2 - (x + 1)$. L'anneau de coordonnées de X' est alors donné par $\mathbb{R}[X][y/x] \simeq \mathbb{R}[x, y] / \langle y^2 - x - 1 \rangle$.

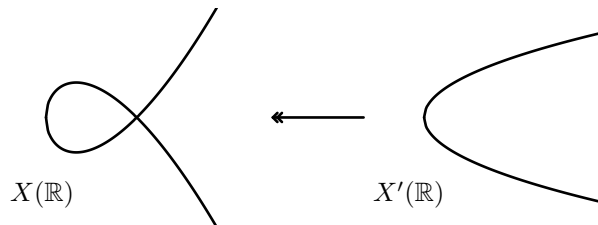


FIGURE 1.1 – Normalisation d'une boucle.

1.2.2 Résultats de géométrie algébrique réelle.

Dans la littérature, il existe différentes définitions de variété algébrique et différentes définitions de variété algébrique *réelle*. Par exemple, on peut penser les variétés réelles comme des \mathbb{R} -schémas de type fini, comme des variétés complexes munies d'une involution ou encore comme les zéros réels de polynômes réels. Les deux derniers points de vue, par exemple, sont ceux adoptés respectivement par Mangolte dans [30] et Bochnak, Coste, Roy dans [9]. Il convient de choisir l'un plutôt que l'autre suivant le type de problème que l'on cherche à résoudre. L'objectif de cette sous-section est de présenter la définition de variété réelle que nous utiliserons au chapitre 3. Pour cela, commençons par donner la définition générale d'une variété sur \mathbb{R} .

Définition 1.2.6

On appelle *variété affine sur \mathbb{R}* un espace localement annelé (X, \mathcal{O}_X) qui est isomorphe à un espace de la forme $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ où A est une \mathbb{R} -algèbre réduite de type fini. On appelle *variété sur \mathbb{R}* un espace localement annelé (X, \mathcal{O}_X) en \mathbb{R} -algèbres qui admet un recouvrement fini par des ouverts X_i tels que chaque (X_i, \mathcal{O}_{X_i}) est une variété affine sur \mathbb{R} .

Avant d'introduire les variétés réelles, il nous faut introduire la notion d'*idéal réel*. Ce type d'idéal apparaît notamment dans la version réelle du Nullstellensatz. Sous sa forme classique, celui-ci s'énonce de la façon suivante.

Théorème 1.2.2 (Nullstellensatz)

Soit K un corps algébriquement clos et I un idéal de $K[x_1, \dots, x_n]$. Alors

$$\mathcal{J}(\mathcal{Z}(I)) = I \text{ si et seulement si } I \text{ est radical.}$$

où $\mathcal{J}(\mathcal{Z}(I))$ désigne l'ensemble des polynômes s'annulant $\mathcal{Z}(I)$.

Cependant, si K n'est plus algébriquement clos, ce théorème n'est plus vérifié. Dans le cas des nombres réels, une alternative est donnée par la notion d'idéal réel.

Définition 1.2.7

Soit A un anneau commutatif. Un idéal I de A est un idéal réel si pour tous $a_1, \dots, a_n \in A$, on a

$$a_1^2 + \dots + a_n^2 \in I \implies \forall i \in \llbracket 1, \dots, n \rrbracket \quad a_i \in I$$

Nous appelons *spectre des premiers réels* de A et notons $\text{R-Spec}(A)$ (resp. $\text{R-Max}(A)$) l'ensemble des idéaux premiers (resp. maximaux) réels de A .

Remarque. Un idéal réel est toujours radical.

Théorème 1.2.3 (Nullstellensatz réel, [9] Théorème 4.1.4)

Soit I un idéal de $\mathbb{R}[x_1, \dots, x_n]$. Alors

$$\mathcal{J}(\mathcal{Z}(I)) = I \text{ si et seulement si } I \text{ est réel.}$$

Définition 1.2.8

Soit X une variété algébrique sur un corps k . On note $X(k)$ l'ensemble des points définis par

$$X(k) := \{x \in X \mid \kappa(x) \simeq k\}$$

Corollaire 1.2.9

Soit X une variété algébrique affine sur \mathbb{R} . Alors

$$X(\mathbb{R}) = \text{R-Max}(\mathbb{R}[X])$$

Corollaire 1.2.10 ([30] Exercice 2.2.6)

Soit I un idéal radical de $\mathbb{R}[x_1, \dots, x_n]$ et X la variété affine associée à I . Alors les conditions suivantes sont équivalentes :

1. L'idéal I est un réel.
2. L'ensemble des points réels $X(\mathbb{R})$ est Zariski dense dans $X(\mathbb{C})$.

Exemple. Soit $X := \text{Spec}(\mathbb{R}[x; y] / \langle x^2 + y^2 \rangle)$. Alors $X(\mathbb{C}) \simeq \{(x; y) \in \mathbb{C}^2 \mid (x - iy)(x + iy) = 0\}$ correspond à l'intersection de deux droites complexes, tandis que $X(\mathbb{R}) \simeq \{(0; 0)\}$. On voit donc, du fait que $\langle x^2 + y^2 \rangle$ n'est pas un idéal réel, que $X(\mathbb{R})$ n'est pas Z-dense dans $X(\mathbb{C})$.

Définition 1.2.11

Soit X une variété algébrique sur \mathbb{R} . Si X est affine, on dit que X est réelle si son anneau de coordonnées est de la forme $\mathbb{R}[x_1, \dots, x_n]/I$ où I est un idéal réel. Si X n'est pas affine, alors on dit qu'elle est réelle s'il existe un recouvrement de X par des ouverts affines réels.

Proposition 1.2.12 ([9], Lemma 4.1.5)

Si X est une variété affine sur \mathbb{R} , alors X est réelle si et seulement si chacune de ses composantes irréductibles est réelle.

Remarque. La définition est cohérente avec les ensembles algébriques réels que l'on retrouve dans Bochnak, Coste, Roy [9] : Soit $E \subset \mathbb{R}[x_1, \dots, x_n]$ et $V := \mathcal{Z}_{\mathbb{R}}(E) \subset \mathbb{R}^n$ l'ensemble algébrique associé. On note $\mathcal{P}(V)$ l'anneau des fonctions polynomiales sur V . Alors la variété $\text{Spec}(\mathcal{P}(V))$ est une variété réelle.

1.2.3 Fonctions régulières.

Une majeure partie de cette thèse étant consacrée à l'étude des fonctions régulières complexes, nous présentons ici un aperçu de la théorie des fonctions régulières réelles qui émerge vers 2015, notamment dans [22] et [12]. Tout comme les fonctions rationnelles continues, qui suscitent déjà l'intérêt des géomètres réels quelques années auparavant (voir [23]), les fonctions régulières naissent des spécificités propres à la géométrie réelle. Les fonctions régulières ou rationnelles ne suffisant pas à capturer à elles seules toute la subtilité du cadre réel, il est alors nécessaire de considérer différentes classes de fonctions moins classiques.

Nous renvoyons vers le compte-rendu [25] de 2018 «From continuous rational to regulous functions» de Kucharz et Kurdyka qui retrace l'émergence des fonctions rationnelles continues et régulières en géométrie réelle.

Nous commençons par donner la définition précise de fonction rationnelle sur une variété.

Definition 1.2.13

Soit X une variété algébrique réelle. Une fonction **rationnelle** est une classe d'équivalence de l'ensemble des couples (f, U) où U est un Z -ouvert Z -dense de $X(\mathbb{R})$ et f est régulière sur U avec la relation d'équivalence suivante :

$$(f, U) \sim (g, V) \text{ ssi } \exists W \subset (U \cap V) \text{ } Z\text{-ouvert } Z\text{-dense tel que } f|_W = g|_W$$

Remarque. Soit (f, U) une fonction rationnelle. On peut ordonner l'ensemble

$$\mathcal{F} = \left\{ (g, V) \mid (f, U) \sim (g, V) \right\}$$

par $(g, V) \leq (h, W)$ lorsque $V \subseteq W$.

Cet ensemble est inductif : pour une chaîne (g_i, V_i) on peut prendre l'union des V_i et définir une fonction qui sera bien régulière. On peut alors appliquer le lemme de Zorn et obtenir un élément maximal (f_0, U_0) . L'ouvert U_0 est appelé *domaine de définition* de f et est noté $\text{dom}(f)$.

Definition 1.2.14

Soit X une variété algébrique réelle et f une fonction rationnelle. On dit que f est régulière si elle s'étend par continuité sur $X(\mathbb{R})$ et si elle est héréditairement rationnelle c'est-à-dire, pour toute sous-variété réelle $V \subset X$, on a $f|_{V(\mathbb{R})}$ rationnelle. On note $\mathcal{R}^0(X(\mathbb{R}))$ l'anneau des fonctions régulières sur $X(\mathbb{R})$.

Exemple. La fonction

$$f : \mathbb{R}^2 \rightarrow \mathbb{R} \\ (x, y) \mapsto \begin{cases} \frac{x^3}{x^2+y^2} & \text{si } y \neq 0 \\ 0 & \text{sinon} \end{cases}$$

est régulière sur \mathbb{R}^n .

Si X est lisse, alors les fonctions régulières coïncident avec les fonctions rationnelles continues sur $X(\mathbb{R})$. Ce n'est plus le cas si la variété n'est pas lisse comme le montre l'exemple suivant.

Exemple. Soit $X(\mathbb{R}) := \{x^3 - (1 + z^2)y^3 = 0\} \subset \mathbb{R}^3$. On pose

$$f : X(\mathbb{R}) \rightarrow \mathbb{R} \\ (x, y, z) \mapsto \begin{cases} x/y & \text{si } y \neq 0 \\ (1 + z^2)^{1/3} & \text{sinon} \end{cases}$$

Bien que f soit rationnelle et continue sur $X(\mathbb{R})$, elle n'est pas rationnelle sur $\{y = 0\}$.

Une fonction régulière ayant la propriété d'être héréditairement rationnelle, sa restriction sur son lieu d'indétermination est encore une fonction rationnelle. On peut de nouveau considérer la restriction de la fonction sur ce nouveau lieu d'indétermination et obtenir à nouveau une fonction rationnelle. En poursuivant cette opération, on va obtenir une stratification de la variété en sous-espaces localement fermés (c.a.d l'intersection d'un ouvert et d'un fermé) sur lesquels f sera régulière.

Théorème 1.2.4 ([12], Théorème 4.1)

Soit $f : X(\mathbb{R}) \rightarrow \mathbb{R}$ une fonction continue sur $X(\mathbb{R})$. Alors f est régulière si et seulement s'il existe une stratification finie de $X(\mathbb{R})$:

$$\mathbb{R}^n = \bigsqcup_{i=1}^m S_i$$

en sous-ensembles localement fermés au sens de Zariski telle que chacune des restrictions $f|_{S_i}$ est régulière.

De la même manière que l'on définit la topologie de Zariski sur \mathbb{R}^n comme l'ensemble des zéros communs de polynômes réels, on peut définir la topologie régulière comme l'ensemble des zéros de fonctions régulières.

Definition 1.2.15

Les ensembles de la forme $Z(E) := \{x \in \mathbb{R}^n \mid \forall f \in E f(x) = 0\}$ avec $E \subseteq \mathcal{R}^0(\mathbb{R}^n)$ sont les fermés d'une topologie appelée la topologie régulière.

La topologie régulière est strictement plus fine que la topologie de Zariski.

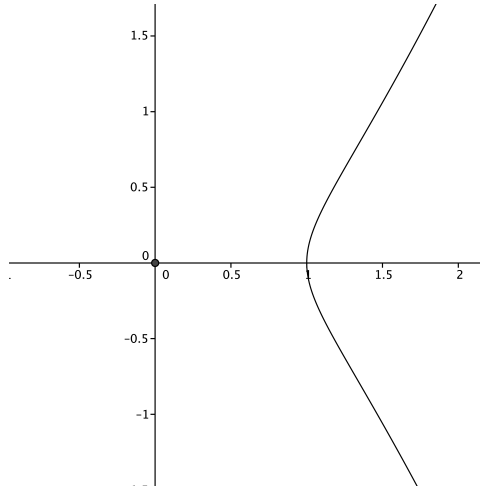


FIGURE 1.2 – La courbe d'équation $y^2 - x^2(x - 1) = 0$

Soit $C(\mathbb{R}) := \{y^2 - x^2(x - 1) = 0\} \subset \mathbb{R}^2$ et

$$f(x, y) = \frac{y^2 - x^2(x - 1)}{y^2 + x^2} = 1 - \frac{x^3}{x^2 + y^2}$$

On voit que f s'étend par continuité en posant $f(0,0) = 1$. Ainsi $Z(f) = C \setminus \{0\}$ est un fermé régulier. Cependant $C \setminus \{0\}$ n'est pas Z-fermé car $C = (C \setminus \{0\}) \cup \{0\}$ et C est irréductible.

Cet exemple illustre un fait plus général : la topologie régulue sur \mathbb{R}^n correspond à la topologie dite «constructible» dont les fermés sont les unions finies de sous-ensembles localement fermés de \mathbb{R}^n qui sont également fermés pour la topologie euclidienne.

Remarque. Signalons au passage que la topologie régulue est une topologie noethérienne alors que l’anneau $\mathcal{R}^0(\mathbb{R}^n)$ ne l’est pas !

Un intérêt des fonctions réguliues vient du fait qu’elles vérifient des théorèmes de géométrie algébrique complexe qui font défaut à la géométrie algébrique réelle. On a, par exemple, un Nullstellensatz régulue.

Théorème 1.2.5 (Nullstellensatz faible, [12] Théorème 5.23)

Soit $n \in \mathbb{N}$ et I un idéal de $\mathcal{R}^0(\mathbb{R}^n)$. Alors

$$\mathcal{Z}(I) = \emptyset \implies I = \mathcal{R}^0(\mathbb{R}^n)$$

Théorème 1.2.6 (Nullstellensatz fort, [12] Théorème 5.24)

Soit $n \in \mathbb{N}$ et I un idéal de $\mathcal{R}^0(\mathbb{R}^n)$. Alors

$$\mathcal{J}(\mathcal{Z}(I)) = \sqrt{I}$$

où $\mathcal{J}(\mathcal{Z}(I))$ est l’ensemble des fonctions réguliues s’annulant sur $\mathcal{Z}(I)$

1.3 Résultats classiques sur la seminormalisation.

Nous présentons dans cette sous-section les résultats classiques de seminormalisation que nous utiliserons ou évoquerons dans cette thèse. Pour ce faire nous utilisons le compte-rendu de Vitulli [40] de 2011 sur la seminormalisation. La définition que nous utiliserons tout au long de la thèse est celle de Traverso introduite dans [38] en 1970. La section 2.1 du chapitre 2 la présente de façon très détaillée, de même que la notion d’extension subintégrale et nous y redémontrons la propriété universelle de la seminormalisation. C’est cela que nous commençons par rappeler brièvement.

Définition 1.3.1 (Traverso)

Soit $A \hookrightarrow B$ une extension entière d’anneaux. L’anneau A_B^+ est défini par

$$A_B^+ := \left\{ b \in B \mid \forall \mathfrak{p} \in \text{Spec}(A), b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}}) \right\}$$

où $\text{Rad}(B_{\mathfrak{p}})$ est le radical de Jacobson de $B_{\mathfrak{p}}$.

Définition 1.3.2

Une extension entière $A \hookrightarrow B$ est dite subintégrale si elle vérifie les deux conditions suivantes :

1. L’application induite $\text{Spec}(B) \rightarrow \text{Spec}(A)$ est bijective.
2. Pour tout $\mathfrak{p} \in \text{Spec}(A)$ et $\mathfrak{q} \in \text{Spec}(B)$ tels que $\mathfrak{q} \cap A = \mathfrak{p}$, alors le morphisme induit sur les corps résiduels $\kappa(\mathfrak{p}) \hookrightarrow \kappa(\mathfrak{q})$ est un isomorphisme.

Proposition 1.3.3 (Propriété universelle de la seminormalisation, [38] Proposition 1.1)

Soit $A \hookrightarrow B$ une extension entière d'anneau. Alors pour toute sous-extension $A \hookrightarrow C \hookrightarrow B$ telle que $A \hookrightarrow C$ est subintégrale, on a $C \hookrightarrow A_B^+$.

Démontrer que la localisation et la seminormalisation commute n'est pas évident à partir de cette définition. Traverso y parvient en introduisant la notion de recollement au-dessus d'un point.

Définition 1.3.4

Soit A un anneau noethérien et $A \hookrightarrow B$ une extension finie. Le recollement de B au dessus d'un point \mathfrak{p} de $\text{Spec}(A)$ est défini par l'anneau

$$A_{\mathfrak{p}}^+ := \left\{ b \in B \mid b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}}) \right\}$$

Théorème 1.3.1 ([38], Théorème 2.1)

Soit A un anneau noethérien et $A \hookrightarrow B$ une extension finie. Si A est seminormal dans B , alors on a une suite finie $B = B_0 \supseteq B_1 \supseteq \dots \supseteq B_n = A$ où B_{i+1} est le recollement de B_i au-dessus d'un point \mathfrak{p} de $\text{Spec}(A)$.

En montrant aisément que la localisation commute aux recollements, Traverso obtient le théorème suivant comme un corollaire du précédent.

Théorème 1.3.2 ([38], Théorème 2.2)

Soit S une partie multiplicative de A . Si A est seminormal dans B , alors

$$S^{-1}A \text{ est seminormal dans } S^{-1}B$$

Corollaire 1.3.5

Soit $A \hookrightarrow B$ une extension entière d'anneaux et S une partie multiplicative de A . Alors

$$S^{-1}A_B^+ = (S^{-1}A)_{S^{-1}B}^+$$

Remarque. Contrairement à ce qui est fait dans Traverso [38], nous énonçons ce résultat sans l'hypothèse que A est noethérien car il a été démontré pour un anneau quelconque par Leahy et Vitulli (voir [28], Corollaire 1.6.).

La commutation avec la localisation nous dit que la seminormalisation peut être vérifiée point par point.

Proposition 1.3.6 ([28], Proposition 1.7)

Un anneau A est seminormal dans B si et seulement si, pour tout $\mathfrak{m} \in \text{Max}(A)$, l'anneau $A_{\mathfrak{m}}$ est seminormal dans $B_{\mathfrak{m}}$.

L'approche de Traverso que nous utilisons ici est pensée pour le cadre géométrique. Cependant différentes définitions, plus adaptées au cadre algébrique abstrait, existent. Celles-ci sont équivalentes à celle de Traverso si l'on considère l'anneau de coordonnées d'une variété algébrique sur un corps de caractéristique zéro.

Proposition 1.3.7 (Critère de Hamann-Vitulli, [28] Proposition 1.4)

Soit $A \hookrightarrow B$ une extension entière d'anneau. La seminormalisation de A dans B peut aussi être définie des façons suivantes :

1. Pour tout couple fixé $(e; f) \in \mathbb{N}^2$ tel que $e \wedge f = 1$, on a

$$A_B^+ = \{b \in B \mid b^e \in A \text{ et } b^f \in A\}$$

2. La seminormalisation de A dans B est donnée par

$$A_B^+ = \{b \in B \mid \exists n \in \mathbb{N}^*, b^n \in A \text{ et } b^{n+1} \in A\}$$

3. La seminormalisation de A dans B est donnée par

$$A_B^+ = \{b \in B \mid \exists n \in \mathbb{N}, \forall m > n, b^m \in A\}$$

On a également un critère de seminormalité dans l'anneau total des fractions qui est donné par Swan.

Définition 1.3.8 (Swan [37])

Soit A un anneau. Alors A est seminormal au sens de Swan si pour tous $b, c \in A$ tels que $b^3 = c^2$, il existe $a \in A$ tel que $a^2 = b$ et $a^3 = c$.

Remarque. Un tel élément a est unique. Un anneau seminormal au sens de Swan est nécessairement réduit. Si l'anneau total des fractions de A est un produit de corps, alors la définition de Swan et celle de Traverso coïncident ([40] Lemme 2.18).

Dans le cas d'une variété algébrique sur un corps algébriquement clos k de caractéristique 0, la seminormalisation est donnée par l'ensemble des fonctions polynomiales sur la normalisation qui prennent la même valeur dans les fibres du morphisme de normalisation. Plus généralement, on a le théorème suivant

Théorème 1.3.3 ([28], Théorème 2.2)

Soit $\pi : Y \rightarrow X$ un morphisme entre deux variétés (affines) sur k . Alors

$$k[X]_{k[Y]}^+ := \left\{ p \in k[Y] \mid \forall y_1, y_2 \in Y(k), \pi(y_1) = \pi(y_2) \implies p(y_1) = p(y_2) \right\}$$

La seminormalisation se définit aussi pour des schémas plus généraux, le cas a été traité par Andriotti et Bombieri [3] puis par Greco et Traverso [16].

Définition 1.3.9

Un schéma X est dit de Mori s'il est réduit et si sa normalisation est finie sur X . On dit que X est seminormal s'il est de Mori et si, pour tout $x \in X$, l'anneau $\mathcal{O}_{X,x}$ est seminormal.

De la même manière que la normalisation intervient dans l'étude des singularités des variétés algébriques, la seminormalisation désingularise légèrement la variété de départ. Plus précisément, elle est liée à la notion de «point multicroisé».

Définition 1.3.10

Soit X une variété algébrique sur un corps algébriquement clos k de caractéristique 0. Un point $x \in X$ est dit multicroisé («multicross» en anglais) s'il existe une collection $\mathcal{C} = \{T_1, \dots, T_r\}$ non vide de sous-ensembles disjoints de $\llbracket 1; n \rrbracket$ telle que l'on a un isomorphisme de k -algèbres

$$\widehat{\mathcal{O}}_{X,x} \simeq k[[z_1, \dots, z_n]] / \langle z_\alpha z_\beta \mid \alpha \in T_i, \beta \in T_j, i \neq j \rangle$$

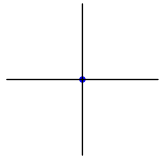
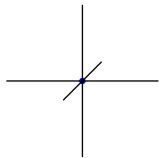
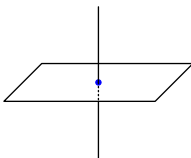
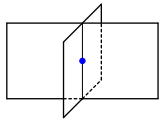
$\mathcal{C} = \{\{x\}, \{y\}\}$	$k[[x, y]] / \langle xy \rangle$	
$\mathcal{C} = \{\{x\}, \{y\}, \{z\}\}$	$k[[x, y, z]] / \langle xy, xz, yz \rangle$	
$\mathcal{C} = \{\{xy\}, \{z\}\}$	$k[[x, y, z]] / \langle xz, yz \rangle$	
$\mathcal{C} = \{\{x\}, \{y\}\}$	$k[[x, y, z]] / \langle xy \rangle$	

FIGURE 1.3 – Multicross dans k^2 et k^3

Les résultats suivant sont un analogue aux théorèmes 1.2.1 et 1.2.5 pour la seminormalisation.

Proposition 1.3.11 ([29], Proposition 3.2)

Soit X une variété algébrique sur k . Si $x \in X$ est un point multicroisé, alors $\mathcal{O}_{X,x}$ est seminormal.

Théorème 1.3.4 ([29], Théorème 3.8)

Soit X une variété algébrique sur k et soit Z l'ensemble de ses points multicroisés. Alors Z est une sous-variété de X telle que

$$\text{codim}(Z) \geq 2$$

Corollaire 1.3.12

Soit X une courbe algébrique sur k dont l'espace tangent de Zariski est de dimension n , alors X est seminormale si et seulement si toutes ses singularités sont des points de multiplicité n avec n tangentes distinctes. Plus précisément, si $x \in X_{\text{sing}}(k)$, alors

$$\widehat{\mathcal{O}}_{X,x} \simeq k[[x_1, \dots, x_n]] / \langle x_i x_j, i \neq j \rangle$$

Exemple. Soit k un corps algébriquement clos et soit $X := \text{Spec}(k[x, y, z] / \langle xyz \rangle)$. On peut montrer que cette variété est seminormale car si l'on se donne un polynôme quelconque sur chaque plan et que ceux-ci coïncident sur les droites d'intersection, alors on peut construire un polynôme défini sur $X(k)$ tel que sa restriction à chaque plan est donnée par l'un des polynômes de départ.

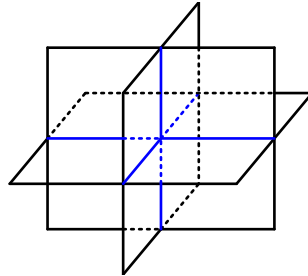


FIGURE 1.4 – Intersection de trois plans dans k^3 .

Soit $x \in X(k)$. Si x n'est dans aucune intersection alors c'est un point lisse, donc il est seminormal. Si x est dans l'intersection de deux plans, alors nous avons vu dans le tableau 1.3 que c'est un point multicroisé et donc qu'il est seminormal. En revanche le point $(0, 0, 0)$ n'est pas multicroisé car il n'apparaît pas dans le tableau 1.3. Au final, l'ensemble des points non multicroisés de $X(k)$ est bien une sous-variété de codimension 2, ce qui est cohérent avec le fait que X est seminormale.

1.4 Clôtures intégrales et seminormalisations en géométrie réelle.

Le sujet de cette thèse s'inscrit dans une étude globale, entamée par Fichou, Monnier et Quarez, de la normalisation et de la seminormalisation en géométrie algébrique réelle. La corrélation entre fonctions régulières réelles et seminormalisation est notamment mise en lumière dans les articles [14], [15] et [33] que nous résumons dans cette section. Les objets présentés dans ces derniers apparaissant de plus dans l'étude de la \mathbb{R} -seminormalisation au chapitre 3, il semblait donc nécessaire d'y consacrer une section explicative.

Dans la première partie, nous survolons l'article [14] qui s'intéresse principalement à la question suivante : Sachant que l'anneau des fonctions régulières $\mathcal{O}(X(\mathbb{R}))$ d'une variété réelle X ne coïncide pas avec l'anneau $\mathbb{R}[X]$ des fonctions polynomiales sur $X(\mathbb{R})$, que peut-on dire des clôtures intégrales $\mathcal{O}(X(\mathbb{R}))'_{\mathcal{K}(X)}$ et $\mathbb{R}[X]'_{\mathcal{O}(X(\mathbb{R}))}$? Après les avoir comparées avec l'anneau $\mathcal{O}(X'(\mathbb{R}))$ des fonctions régulières sur la normalisation, nous présentons la normalisation birégulière X^b dont l'anneau de coordonnées est défini

par $\mathbb{R}[X]'_{\mathcal{O}(X(\mathbb{R}))}$. Nous verrons qu'il s'agit de la plus grande variété biregulièrément isomorphe à X et regarderons la façon dont elle modifie ses singularités.

Dans la seconde partie nous présentons la seminormalisation centrale et la normalisation faible centrale étudiées dans [15] et [33]. Globalement, ces objets sont obtenus en recollant, dans la normalisation d'une variété, les points adhérents au lieu lisse (i.e. les points «centraux») qui vivent au dessus des points centraux de la variété considérée. L'apparition des points centraux dans les définitions est dû au manque de surjectivité des morphismes finis en géométrie réelle. Après avoir donné toutes les définitions nécessaires, nous verrons comment ces variantes de la seminormalisation sont liées à la théorie des fonctions régulières réelles.

1.4.1 La normalisation birégulière.

Nous présentons dans cette sous-section la notion de normalisation birégulière introduite dans [14]. Cet objet intervient dans l'étude de la R-seminormalisation présentée au chapitre 3.

Soit A un anneau et soit $\mathcal{T}(A) := 1 + \sum A^2 = \{1 + a_1^2 + \dots + a_n^2 \mid n \in \mathbb{N} \text{ et } a_i \in A\}$. L'ensemble $\mathcal{T}(A)$ est une partie multiplicative de A et $0 \notin \mathcal{T}(A)$ si et seulement si A est un anneau réel (c'est-à-dire -1 n'est pas somme de carrés). Plus généralement, nous dirons qu'un anneau A vérifie **(mp)** s'il est réduit et s'il possède un nombre fini de premiers minimaux qui sont tous réels. Un tel anneau est en particulier réel. Le localisé de A par $\mathcal{T}(A)$ sera noté $\mathcal{O}(A) := \mathcal{T}(A)^{-1}A$. Remarquons que cet anneau est non nul si et seulement si A est réel. La variété $X = \text{Spec}(A)$ est une variété affine sur \mathbb{R} si A une \mathbb{R} -algèbre réduite de type fini. Si de plus A vérifie **(mp)**, alors X est une variété affine *réelle*. Pour une variété affine réelle X , l'anneau $\mathcal{O}(\mathbb{R}[X])$ coïncide avec l'ensemble $\mathcal{O}(X(\mathbb{R}))$ des fonctions régulières sur $X(\mathbb{R})$. C'est à dire l'ensemble des fonctions $p/q : X(\mathbb{R}) \rightarrow \mathbb{R}$ telles que $p, q \in \mathbb{R}[X]$ avec $\mathcal{Z}(q) = \emptyset$. En effet, dans ce cas, on peut supposer $q > 0$ sur $X(\mathbb{R})$. Ainsi le Positivstellensatz ([9], Théorème 4.4.3) nous dit qu'il existe $h, g \in \sum \mathbb{R}[X]^2$ tels que $gq = 1 + h$. Ainsi $p/q = (gp)/(gq) = (gp)/(1 + h) \in \mathcal{O}(\mathbb{R}[X])$. Contrairement au cas complexe où les fonctions régulières sont polynomiales, on obtient les extensions strictes suivantes :

$$\mathbb{R}[X] \hookrightarrow \mathcal{O}(X(\mathbb{R})) \hookrightarrow \mathcal{K}(X)$$

Le fait que les fonctions régulières ne coïncident pas avec les fonctions polynomiales en géométrie réelle a poussé les auteurs de [14] à s'interroger sur les répercussions que cela pouvait avoir dans le processus de normalisation. Par exemple, quelles différences y a-t-il entre la normalisation de l'anneau des fonctions régulières $\mathcal{O}(X(\mathbb{R}))'_{\mathcal{K}(X)}$ et l'anneau $\mathcal{O}(X'(\mathbb{R}))$ des fonctions régulières sur la normalisation ? Sachant que la normalisation est définie par $\mathbb{R}[X'] := \mathbb{R}[X]'_{\mathcal{K}(X)}$ et que les opérations de localisation et de clôture intégrale commutent, on a

$$\mathcal{O}(X(\mathbb{R}))'_{\mathcal{K}(X)} \hookrightarrow \mathcal{O}(X'(\mathbb{R}))$$

Il s'avère que cette injection est un isomorphisme si et seulement si toutes les fibres réelles du morphisme de normalisation sont réelles ([14], Proposition 3.2).

Exemple. Soit $X := \text{Spec}(\mathbb{R}[x, y]/\langle y^2 - x^2(x - 1) \rangle)$ la cubique avec un point isolé à l'origine. La normalisation X' est donnée par $\mathbb{R}[X][y/x] \simeq \mathbb{R}[X'] = \mathbb{R}[x, t]/\langle t^2 - (x - 1) \rangle$. La fonction $f = 1/(1 + t^2)$ est régulière sur $X'(\mathbb{R})$. En revanche, en utilisant la Proposition 2.12 de [14] on peut voir que $f \notin \mathcal{O}(X(\mathbb{R}))'$.

On peut aussi voir la différence entre ces deux anneaux au niveau des points normaux de la variété.

Proposition 1.4.1 ([14] Proposition 2.15)

Soit X une variété affine réelle. Alors

1. $\mathbb{R}[X]$ est intégralement clos si et seulement si $\text{Norm}(X(\mathbb{C})) = X(\mathbb{C})$
2. $\mathcal{O}(X(\mathbb{R}))$ est intégralement clos si et seulement si $X(\mathbb{R}) \subset \text{Norm}(X(\mathbb{C}))$.

Après avoir étudié la clôture intégrale de $\mathcal{O}(X(\mathbb{R}))$ dans $\mathcal{K}(X)$, les auteurs de [14] ce sont intéressés à la clôture intégrale de $\mathbb{R}[X]$ dans $\mathcal{O}(X(\mathbb{R}))$ et ont introduit la notion de normalisation birégulière.

Définition 1.4.2

Soit A un anneau vérifiant **(mp)**. L'anneau définit par

$$A^b := A'_{\mathcal{O}(A)}$$

est appelé la normalisation birégulière de A . Dans le cas où A est l'anneau de coordonnées d'une variété affine X , on obtient une nouvelle variété X^b dont l'anneau de coordonnées est défini par

$$\mathbb{R}[X^b] := \mathbb{R}[X]^b$$

Remarque. La finitude de l'extension $\mathbb{R}[X] \hookrightarrow \mathbb{R}[X]'_{\mathcal{O}(X(\mathbb{R}))} \hookrightarrow \mathbb{R}[X']$ nous dit que l'on a bien une \mathbb{R} -algèbre de type fini et donc que $\mathbb{R}[X]'_{\mathcal{O}(X(\mathbb{R}))}$ est bien l'anneau de coordonnées d'une variété.

Les éléments de $\mathbb{R}[X^b]$ peuvent également être vus comme des éléments spécifiques de $\mathbb{R}[X']$.

Proposition 1.4.3

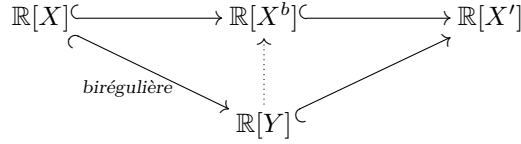
Soit X une variété affine réelle. Alors

$$\mathbb{R}[X^b] \simeq \left\{ p \in \mathbb{R}[X'] \mid \exists f \in \mathcal{O}(X(\mathbb{R})) \text{ telle que } f \circ \pi'_{\mathbb{R}} = p \text{ sur } \pi'^{-1}(X(\mathbb{R})) \right\}$$

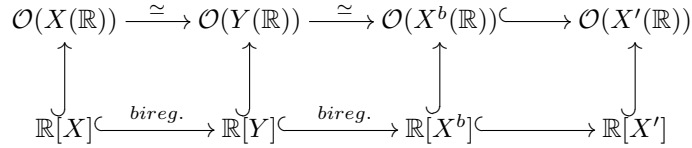
La normalisation birégulière vient avec la propriété universelle suivante : c'est la plus grande variété intermédiaire entre X et X' qui soit birégulièrement isomorphe à X .

Proposition 1.4.4 ([14], Théorème 4.3)

Soit X et Y deux variétés affines réelles telles que $\mathbb{R}[X] \hookrightarrow \mathbb{R}[Y] \hookrightarrow \mathbb{R}[X']$ et $\mathcal{O}(X(\mathbb{R})) \simeq \mathcal{O}(Y(\mathbb{R}))$. Alors $\mathbb{R}[Y] \hookrightarrow \mathbb{R}[X^b]$.



Autrement dit, soit $\pi : Y \rightarrow X$ un morphisme fini et birégulier entre deux variétés affines réelles. Alors on a le diagramme suivant.



Nous allons voir à présent une version plus algébrique de la propriété universelle de normalisation birégulière car nous nous en servons à la fin du chapitre 3. On commence par donner une réinterprétation, sur les spectres réels, du fait d'être birégulier.

Proposition 1.4.5 ([14], Proposition 4.13)

Soit A un anneau vérifiant **(mp)** et soit B tel que $A \hookrightarrow B \hookrightarrow A'$. Alors B satisfait également **(mp)** et les propriétés suivantes sont équivalentes :

1. L'extension $A \hookrightarrow B$ est birégulière.
2. Pour tout $\mathfrak{m} \in \text{R-Max}(A)$, il existe un unique idéal $\mathfrak{m}' \in \text{max}(B)$ tel que $\mathfrak{m}' \cap A = \mathfrak{m}$. De plus le morphisme $A_{\mathfrak{m}} \rightarrow B_{\mathfrak{m}'}$ est un isomorphisme.
3. Pour tout $\mathfrak{p} \in \text{R-Spec}(A)$, il existe un unique idéal $\mathfrak{q} \in \text{Spec}(B)$ tel que $\mathfrak{q} \cap A = \mathfrak{p}$. De plus le morphisme $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ est un isomorphisme.

On retrouve bien que A^b est la plus grande sous-extension birégulière de $A \hookrightarrow A'$.

Théorème 1.4.1 ([14], Théorème 4.14)

Soit A un anneau vérifiant **(mp)** et soit A' sa clôture intégrale. Alors

1. L'extension $A \hookrightarrow A^b$ est birégulière.
2. Pour toute sous-extension $A \hookrightarrow B \hookrightarrow A'$, on a $A \hookrightarrow B$ birégulière si et seulement si $B \subset A^b$.

Enfin, concluons cette présentation de la normalisation birégulière en regardant de quelle manière celle-ci modifie les singularités d'une variété réelle.

Proposition 1.4.6 ([14], Proposition 4.6)

Soit A un anneau vérifiant **(mp)**. Alors

1. Soit $\mathfrak{m} \in \text{Max}(A)$ tel que $\mathfrak{m} \notin \text{R-Spec}(A)$. Alors

$$(A^b)_{\mathfrak{m}} = (A_{\mathfrak{m}})' = (A'_{\mathfrak{m}})$$

2. Soit $\mathfrak{m} \in \text{R-Max}(A)$. Alors

$$(A^b)_{\mathfrak{m}} = A_{\mathfrak{m}}$$

Proposition 1.4.7 ([14], Proposition 4.8)

Soit A un anneau vérifiant **(mp)**. On a alors

$$A^b = \bigcap_{\mathfrak{m} \in \text{R-Max}(A)} A_{\mathfrak{m}} \cap \bigcap_{\mathfrak{m} \in \text{Max}(A) \setminus \text{R-Max}(A)} A'_{\mathfrak{m}}$$

Si X est une variété affine réelle, la proposition précédente nous dit que X^b normalise les points de $X(\mathbb{C})$ sans toucher à $X(\mathbb{R})$. Illustrons ce phénomène sur un exemple.

Exemple. Soit $X := \text{Spec}(\mathbb{R}[x, y] / \langle y^2 - x^3(x^2 + 1)^2 \rangle)$. Alors la fonction $f = y / (x^2 + 1)$ est régulière sur $X(\mathbb{R})$ et est annulée par le polynôme $t^2 - x^3 \in \mathbb{R}[X][t]$. Ainsi $f \in \mathbb{R}[X^b]$ et on a $\mathbb{R}[X][f] \simeq \mathbb{R}[x, y] / \langle y^2 - x^3 \rangle$, ce qui correspond à $\mathbb{R}[X^b]$ puisqu'elle n'a plus de singularités complexes.

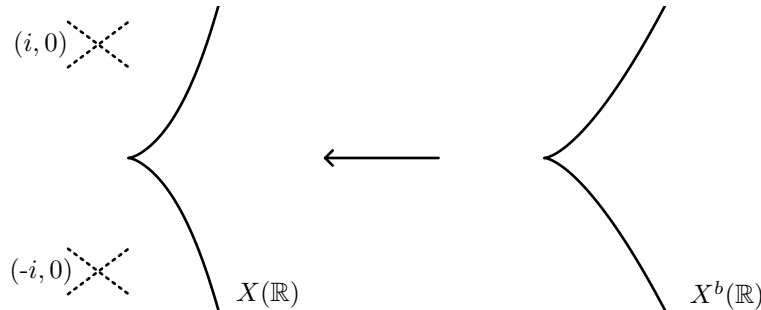


FIGURE 1.5 – La normalisation birégulière de $\{y^2 - x^3(x^2 + 1)^2 = 0\}$.

1.4.2 La seminormalisation centrale.

Si $A \hookrightarrow B$ est une extension entière d'anneaux, la propriété de lying-over habituelle nous dit que le morphisme induit $\text{Spec}(B) \rightarrow \text{Spec}(A)$ est surjectif. Cependant, sa restriction au spectre des premiers réels $\text{R-Spec}(B) \rightarrow \text{R-Spec}(A)$ ne l'est pas nécessairement. Ce fait est problématique si l'on essaye de définir une seminormalisation réelle. Une approche proposée par Fichou, Monnier et Quarez dans [15] est de s'intéresser aux points *centraux* de variétés réelles. Ils définissent donc une seminormalisation *centrale* que Monnier étudie plus en détail dans [33]. Le but de cette sous-section est de présenter cette approche que nous comparerons dans le chapitre 3 avec la R-seminormalisation.

Définition 1.4.8

Soit A un anneau et I un idéal de A . On dit que I est un idéal central si pour tout $a \in A$ et pour tout $b \in \sum \mathcal{K}(A)^2$ on a

$$a^2 + b \in I \implies a \in I$$

On a alors I central $\implies I$ réel.

Exemple. Soit $X := \text{Spec}(\mathbb{R}[x, y, z] / \langle x^3 - z(x^2 + y^2) \rangle)$ le parapluie de Cartan. L'idéal $\mathfrak{p} = (x, y)$ est réel mais n'est pas central. En effet

$$b := x^2 + y^2 - z^2 = x^2 + y^2 - \frac{x^6}{(x^2 + y^2)^2} = \frac{3x^4y^2 + 2x^2y^4 + y^6}{(x^2 + y^2)^2} \in \sum \mathcal{K}(A)^2$$

On a alors $z^2 + b = x^2 + y^2 \in \mathfrak{p}$ mais $z \notin \mathfrak{p}$.

L'ensemble des idéaux centraux de A est noté $\text{C-Spec}(A)$ et l'ensemble des idéaux maximaux centraux est $\text{Cent}(A)$. Concrètement, les maximaux centraux correspondent aux points fermés se trouvant dans l'adhérence euclidienne du lieu lisse de la variété.

Proposition 1.4.9 ([33] Corollaire 3.7)

Soit X une variété affine sur \mathbb{R} . Alors

$$\text{Cent}(X) = \overline{X_{\text{reg}}(\mathbb{R})}^E$$

Une propriété importante du spectre central est que, contrairement au spectre des premiers réels, il vérifie la propriété de Lying-over suivante.

Proposition 1.4.10 ([15] Proposition 2.8)

Soit $A \hookrightarrow B$ une extension d'anneaux entière et birationnelle vérifiant **(mp)**. Alors les applications

$$\begin{array}{ccc} \text{C-Spec}(B) & \rightarrow & \text{C-Spec}(A) \\ \mathfrak{q} & \mapsto & \mathfrak{q} \cap A \end{array}$$

et sa restriction $\text{Cent}(B) \rightarrow \text{Cent}(A)$ sont définies et surjectives.

C'est ici que l'on voit l'intérêt de regarder les idéaux centraux. Par exemple l'extension

$$A = \mathbb{R}[x, y] / \langle y^2 - x^2(x - 1) \rangle \hookrightarrow \mathbb{R}[x, y] / \langle y^2 - (x - 1) \rangle = A'$$

est entière et birationnelle mais il n'y a pas d'idéal réel de A' au dessus de l'idéal $(x, y) \subset A$.

Maintenant que l'on a une propriété de Lying-over pour les idéaux centraux, on peut débiter la construction de la seminormalisation et de la weak-normalisation centrale.

Définition 1.4.11

Soit $A \in B$ une extension entière d'anneaux satisfaisants **(mp)**. L'extension est dite

- s_C -subintégrale si l'application $\text{C-Spec}(B) \rightarrow \text{C-Spec}(A)$ est bijective et equirésiduelle.
- w_C -subintégrale si l'application $\text{Cent}(B) \rightarrow \text{Cent}(A)$ est bijective et equirésiduelle.

Définition 1.4.12

Soit A un anneau, le radical de Jacobson central de A , noté $\text{Rad}^C(A)$, est l'intersection de idéaux centraux maximaux de A .

On définit la w_C -normalisation de A qui consiste à recoller les idéaux centraux de la normalisation au-dessus des idéaux centraux maximaux de A .

Définition 1.4.13

Soit A un anneau et A' sa clôture intégrale. L'anneau

$$A^{w_c} = \left\{ f \in A' \mid \forall \mathfrak{m} \in \text{Cent}(A) \ f_{\mathfrak{m}} \in A_{\mathfrak{m}} + \text{Rad}^C(A'_{\mathfrak{m}}) \right\}$$

est appelé la w_C -normalisation de A .

La w_C -normalisation est la plus grande sous-extension de $A \hookrightarrow A'$ qui soit w_C -subintégrale.

Proposition 1.4.14 ([15], Proposition 2.20)

Soit A un anneau satisfaisant **(mp)** et soit A' sa clôture intégrale. Soit B tel que $A \hookrightarrow B \hookrightarrow A'$, alors les propriétés suivantes sont équivalentes :

1. L'extension $A \hookrightarrow B$ est w_C -subintégrale.
2. $B \subset A^{w_c}$.

On note $\mathcal{K}^0(\text{Cent}(X))$ l'ensemble des fonctions $f : \text{Cent}(X) \rightarrow \mathbb{R}$ qui sont euclidiennement continues sur $\text{Cent}(X)$ et qui admettent une représentation rationnelle. La proposition suivante établit un lien entre les fonctions rationnelles continues et les extensions w_C -subintégrales.

Proposition 1.4.15 ([15], Théorème 3.6)

Soit $\pi : Y \rightarrow X$ un morphisme fini entre deux variétés réelles affines. Alors les propriétés suivantes sont équivalentes :

1. L'extension $\pi^* : \mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ est w_C -subintégrale.
2. Le morphisme $\pi|_{\text{Cent}(Y)} : \text{Cent}(Y) \rightarrow \text{Cent}(X)$ est bijectif.
3. Le morphisme $\pi|_{\text{Cent}(Y)}$ est un homéomorphisme pour la topologie Euclidienne.
4. Le morphisme $\mathcal{K}^0(\text{Cent}(X)) \hookrightarrow \mathcal{K}^0(\text{Cent}(Y)), f \mapsto f \circ \pi|_{\text{Cent}(X)}$ est un isomorphisme.

On définit la s_C -normalisation de A qui consiste à recoller les idéaux centraux de la normalisation au-dessus des idéaux centraux de A .

Définition 1.4.16

Soit A un anneau et A' sa clôture intégrale. L'anneau

$$A^{s_c} = \left\{ f \in A' \mid \forall \mathfrak{p} \in \text{C-Spec}(A) \ f_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}^C(A'_{\mathfrak{p}}) \right\}$$

est appelé la s_C -normalisation de A .

La s_C -normalisation est la plus grande sous-extension de $A \hookrightarrow A'$ qui soit s_C -subintégrale.

Proposition 1.4.17 ([15], Proposition 2.23)

Soit A un anneau satisfaisant **(mp)** et soit A' sa clôture intégrale. Soit B tel que $A \hookrightarrow B \hookrightarrow A'$, alors les propriétés suivantes sont équivalentes :

1. L'extension $A \hookrightarrow B$ est s_C -subintégrale.
2. $B \subset A^{s_c}$.

L'ensemble des fonctions régulières définies sur $\text{Cent}(X)$ est noté $\mathcal{R}^0(\text{Cent}(X))$. La proposition suivante montre comment les fonctions régulières sont liées avec la notion d'extension s_C -subintégrale.

Proposition 1.4.18 ([15], Théorème 3.13)

Soit $\pi : Y \rightarrow X$ un morphisme fini entre deux variétés réelles affines. Alors les propriétés suivantes sont équivalentes :

1. L'extension $\pi^* : \mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ est s_C -subintégrale.
2. Le morphisme $\mathcal{R}^0(\text{Cent}(X)) \hookrightarrow \mathcal{R}^0(\text{Cent}(Y))$, $f \mapsto f \circ \pi_{\text{Cent}(X)}$ est un isomorphisme.

Au final, on obtient

$$A \subset A^+ \subset A^{s_c} \subset A^{w_c} \subset A'$$

Si A est l'anneau de coordonnées d'une variété réelle X , comme les extensions considérées sont finies, on obtient deux nouvelles variétés X^{w_c} et X^{s_c} définies par

$$\mathbb{R}[X^{w_c}] := \mathbb{R}[X]_{\mathbb{R}[X']}^{w_c} \text{ et } \mathbb{R}[X^{s_c}] := \mathbb{R}[X]_{\mathbb{R}[X']}^{s_c}$$

Ainsi, on obtient

$$X' \rightarrow X^{w_c} \rightarrow X^{s_c} \rightarrow X^+ \rightarrow X$$

Terminons cette section en donnant les anneaux de fonctions sur $\text{Cent}(X)$ qui correspondent à $\mathbb{R}[X^{w_c}]$ et $\mathbb{R}[X^{s_c}]$.

Théorème 1.4.2 ([15], Théorèmes 4.11 et 4.16)

Soit X une variété affine réelle. Alors

$$\mathbb{R}[X^{w_c}] := \mathbb{R}[X]_{\mathcal{K}^0(\text{Cent}(X))}' \text{ et } \mathbb{R}[X^{s_c}] := \mathbb{R}[X]_{\mathcal{R}^0(\text{Cent}(X))}'$$

On peut également réécrire les choses en regardant les fonctions polynomiales de X' qui proviennent d'éléments de $\mathbb{R}[X^{s_c}]$ et $\mathbb{R}[X^{w_c}]$.

Proposition 1.4.19 ([15], Lemme 4.4)

Soit X une variété affine réelle et $\pi' : X' \rightarrow X$ le morphisme de normalisation. On a alors

$$\mathbb{R}[X^{wc}] \simeq \left\{ p \in \mathbb{R}[X'] \mid \exists f \in \mathcal{K}^0(\text{Cent}(X)) \text{ telle que } f \circ \pi'_{|\text{Cent}(X')} = p \text{ sur } \text{Cent}(X') \right\}$$

et

$$\mathbb{R}[X^{sc}] \simeq \left\{ p \in \mathbb{R}[X'] \mid \exists f \in \mathcal{R}^0(\text{Cent}(X)) \text{ telle que } f \circ \pi'_{|\text{Cent}(X')} = p \text{ sur } \text{Cent}(X') \right\}$$

Exemple. Reprenons l'exemple 1.2.3 avec la surface de Kollár $X := \text{Spec}(\mathbb{R}[x, y, z] / \langle y^3 - x^3(1 + z^2) \rangle)$ et la fraction y/x qui s'étend par continuité mais qui ne s'étend pas en une fonction régulue. Remarquons d'abord que $\text{Cent}(X) = X$ par la Proposition 1.4.9. De plus, on peut voir que $\mathbb{R}[X'] = \mathbb{R}[X][y/x]$ et donc on obtient $X^{wc} = X'$. Comme y/x ne peut pas être prolongée en une fonction régulue, on a $X^{sc} \neq X'$. En fait, l'anneau de cordonnées $\mathbb{R}[X^{sc}]$ est donné par $\mathbb{R}[X][y^2/x]$ avec y^2/x qui, étendue par 0 en $(0, 0)$, est régulue sur $X(\mathbb{R})$.

Exemple. Soit $X := \text{Spec}(\mathbb{R}[x, y] / \langle y^2 - x^2(x - 1)^3 \rangle)$. On a $\text{Cent}(X) = X(\mathbb{R}) \cap \{x > 0\}$ et on considère $f : \text{Cent}(X) \rightarrow \mathbb{R}$ la fonction rationnelle $y/x(x - 1)$ prolongée continûment par 0 en $(1, 0)$. Comme f est une racine de $t^2 - (x - 1) = 0$ et que f est continue, on a $f \in \mathbb{R}[X]_{\mathcal{K}^0(\text{Cent}(X))}' = \mathbb{R}[X]_{\mathcal{R}^0(\text{Cent}(X))}'$. Il s'avère que $\mathbb{R}[X^{sc}] = \mathbb{R}[X][f] = \mathbb{R}[x, y] / \langle y^2 - (x - 1) \rangle$ et l'on observe que l'on a bien $\text{Cent}(X) \simeq \text{Cent}(X^{sc})$ malgré le fait que $X(\mathbb{R}) \not\simeq X^{sc}(\mathbb{R})$.

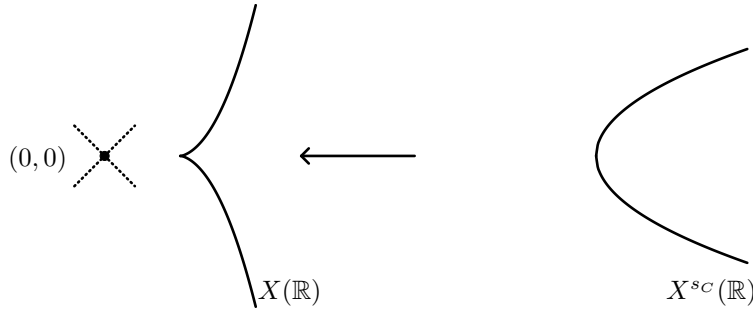


FIGURE 1.6 – La seminormalisation centrale de $\{y^2 - x^2(x - 1)^3 = 0\}$.

SEMINORMALIZATION OF COMPLEX ALGEBRAIC VARIETIES

This chapter is devoted to the study of seminormalization of complex algebraic varieties, to its link with continuous rational functions and to the study of those functions.

Around 1970 Traverso [38] introduced the notion of the seminormalization A_B^+ of a commutative ring A in an integral extension B . The idea is to glue together the prime ideals of B lying over the same prime ideal of A . The seminormalization A_B^+ has the property that it is the biggest extension C of A in B which is subintegral i.e. such that the map $\text{Spec}(C) \rightarrow \text{Spec}(A)$ is bijective and equiresidual (it gives isomorphisms between the residue fields).

In the paper [28], the authors tried to identify the coordinate ring of the seminormalization of a variety as the ring of rational functions which are continuous for the Zariski topology. Unfortunately, the Zariski topology is not strong enough for this to be true. The first aim of this chapter is to show that the correct functions to consider are rational functions which are continuous for the Euclidean topology. The idea of studying the concept of seminormalization with that kind of functions comes from [15] and [33] in the context of real algebraic geometry. Those functions appeared recently in real algebraic geometry (see [12] and [22]) under the name of "regulous functions". They allow to recover some classical theorems of complex algebraic geometry, such as the Nullstellensatz, which normally do not hold anymore in real algebraic geometry. A complex analog of regulous functions has been studied in [8] and [7] in the point of view of complex analytic geometry. The second aim of this chapter is to bring a study of complex regulous functions in the point of view of complex algebraic geometry.

This chapter is organized as follows. In Section 2.1 we rewrite Traverso's construction of the seminormalization of a ring and its universal property regarding to subintegral extensions of rings.

In Section 2.2 we look at the seminormalization of an affine variety over an algebraically closed field of characteristic zero and to its universal property. The seminormalization of an affine variety X can be seen as the biggest birational variety such that its closed points are in bijection with those of X .

In Section 2.3 we introduce continuous rational functions on a complex affine variety X . More precisely

we consider the functions $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ which are rational on a Zariski dense open set of $X(\mathbb{C})$ and which are continuous, for the Euclidean topology, on all $X(\mathbb{C})$. The ring of those functions is denoted by $\mathcal{K}^0(X(\mathbb{C}))$. For $\pi : Y \rightarrow X$ a finite morphism between complex affine varieties and π_c its restriction to $Y(\mathbb{C})$, we look at the induced morphism $f \mapsto f \circ \pi_c$ between rings of continuous rational functions. We show that the image of such a morphism is

$$\{f \in \mathcal{K}^0(Y(\mathbb{C})) \text{ with } f \text{ constant on the fibers of } \pi_c\}$$

It allows us to reinterpret subintegral extensions between coordinate rings of varieties. For X and Y two varieties, one gets that $\mathbb{C}[X] \hookrightarrow \mathbb{C}[Y]$ is subintegral if and only if $\mathcal{K}^0(X(\mathbb{C})) \simeq \mathcal{K}^0(Y(\mathbb{C}))$. The first half of this chapter can be summarized with the following result :

Theorem (2.2.1 and 2.3.12). *Let $\pi : Y \rightarrow X$ be a finite morphism between affine complex varieties. Then the following properties are equivalent :*

- 1) π is subintegral.
- 2) π_c is bijective.
- 3) The rings $\mathcal{K}^0(Y(\mathbb{C}))$ and $\mathcal{K}^0(X(\mathbb{C}))$ are isomorphic.
- 4) π_c is an homeomorphism for the Euclidean topology.
- 5) π_c is an homeomorphism for the Zariski topology.
- 6) π is an homeomorphism.

In the beginning of Subsection 2.3.1, we prove that continuous rational functions are regular on the smooth points of a variety. It allows us to see that, for X a normal variety, the thinness of X_{sing} implies $\mathcal{K}^0(X(\mathbb{C})) = \mathbb{C}[X]$. This fact combined with the previous theorem leads us to the main result of this chapter :

Theorem (2.3.13). *Let X be an affine complex variety and $\pi^+ : X^+ \rightarrow X$ be the seminormalization morphism. We have the following isomorphism*

$$\begin{array}{ccc} \varphi : \mathcal{K}^0(X(\mathbb{C})) & \xrightarrow{\simeq} & \mathbb{C}[X^+] \\ f & \mapsto & f \circ \pi_c^+ \end{array}$$

The results of Subsection 2.3.1 can be summarized with the following diagramm: for every morphism $\pi : Y \rightarrow X$ such that $\mathbb{C}[X] \hookrightarrow \mathbb{C}[Y]$ is subintegral, we get

$$\begin{array}{ccccccc} \mathcal{K}^0(X(\mathbb{C})) & \xrightarrow{\simeq} & \mathcal{K}^0(Y(\mathbb{C})) & \xrightarrow{\simeq} & \mathcal{K}^0(X^+(\mathbb{C})) & \hookrightarrow & \mathcal{K}^0(X'(\mathbb{C})) \\ \uparrow & & \uparrow & & \parallel & & \parallel \\ \mathbb{C}[X] & \xrightarrow{\text{subint.}} & \mathbb{C}[Y] & \xrightarrow{\text{subint.}} & \mathbb{C}[X^+] & \hookrightarrow & \mathbb{C}[X'] \end{array}$$

In subsection 2.3.2 we look at a consequence of theorem 2.3.13 : the restriction of a complex continuous rational function on a subvariety is still a rational function. It is an interesting fact because it says that, unlike the real case, continuous rational functions are regulous functions.

In the remaining of Section 2.3, we are interesting in finding criteria for a continuous function to be rational and then for a rational function to be continuous. In Subsection 2.3.3, we show that a continuous

function on $X(\mathbb{C})$ which is a root for a polynomial with coefficients in $\mathbb{C}[X]$ is necessarily rational. It implies two results. First, we get that $\mathcal{K}^0(X(\mathbb{C}))$ is the integral closure of $\mathbb{C}[X]$ in $\mathcal{C}^0(X(\mathbb{C}), \mathbb{C})$. Secondly, we get an algebraic version of Whitney's theorem 4.5Q in [41], saying that a continuous function on the closed points of an affine variety is rational if and only if its graph is Zariski closed. The second point says that c-holomorphic functions with algebraic graph studied in [8] and [7] correspond, for algebraic varieties, to the continuous rational functions considered in this thesis.

Finally, in Subsection 2.3.4, we give several nontrivial examples of continuous rational functions thanks to the following criterion.

Theorem (2.3.20). *Let X be an affine complex variety. A function $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ is a continuous rational function if and only if it is rational, integral over $\mathbb{C}[X]$ and its graph is Zariski closed.*

In section 2.4 we reinterpret several classical results about seminormalization in terms of rational continuous functions. In the first subsection, we look at criteria for a variety to be seminormal given by Leahy-Vitulli, Hamann and Swan (see the review [40]). To prove that those criteria are sufficient, we show that if f is an element of $\mathcal{K}^0(X(\mathbb{C})) \setminus \mathbb{C}[X]$, then we can always find a function $g \in \mathbb{C}[X][f] \setminus \mathbb{C}[X]$ such that $g^n \in \mathbb{C}[X]$ for all $n \geq 2$. To see that they are necessary, we construct explicit continuous rational functions from the relations appearing in the different criteria. The second subsection is dedicated to see what the commutation between the localization and the seminormalization means for continuous rational functions.

Finally, in Section 2.5, we generalize the results for non necessarily affine complex varieties.

2.1 Universal property of the seminormalization.

We recall in this section the construction of the seminormalization introduced by Traverso [38] for commutative rings. This construction is linked to the notion of subintegrality in the way that the seminormalization of a ring is its biggest subintegral extension.

Let A be a ring, we note $\text{Spec}(A) := \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal of } A\}$ the spectrum of A and $\text{Spm}(A) := \{\mathfrak{m} \subset A \mid \mathfrak{m} \text{ is a maximal ideal of } A\}$ the maximal spectrum of A . Let $\mathfrak{p} \in \text{Spec}(A)$, then $A_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1}A$ is the localization of A at \mathfrak{p} and $\kappa(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ the residue field of \mathfrak{p} .

Since the seminormalization is defined for integral extensions, we recall this notion here.

Definition 2.1.1

Let $A \hookrightarrow B$ be an extension of rings.

1. An element $b \in B$ is integral over A if there exists a monic polynomial $P \in A[X]$ such that $P(b) = 0$.
2. We call integral closure of A in B and we write A'_B the ring defined by

$$A'_B := \{b \in B \mid b \text{ integral on } A\}$$

3. The extension $A \hookrightarrow B$ is integral if $A'_B = B$.

Now we define the seminormalization of a ring in an integral extension. The idea behind this definition is to *glue* the prime ideals of B above those of A . If one thinks of it in terms of algebraic varieties, it consists of gluing points in the fibers together.

Definition 2.1.2

Let $A \hookrightarrow B$ be an integral extension of rings. We define

$$A_B^+ := \left\{ b \in B \mid \forall \mathfrak{p} \in \text{Spec}(A), b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}}) \right\}$$

where $\text{Rad}(B_{\mathfrak{p}}) := \bigcap_{\mathfrak{m} \in \text{Spm}(B_{\mathfrak{p}})} \mathfrak{m}$ is the Jacobson radical of $B_{\mathfrak{p}}$.

We say that A_B^+ is the *seminormalization* of A in B . If $A = A_B^+$, then A is said to be *seminormal* in B .

We introduce now the notion of subintegral extension which is strongly related with that of seminormalization.

Definition 2.1.3

An integral extension of rings $A \hookrightarrow B$ is called *subintegral* if the two following conditions hold :

1. The induced map $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is bijective.
2. For all $\mathfrak{p} \in \text{Spec}(A)$ and $\mathfrak{q} \in \text{Spec}(B)$ with $\mathfrak{q} \cap A = \mathfrak{p}$, the induced map on the residue fields $\kappa(\mathfrak{p}) \hookrightarrow \kappa(\mathfrak{q})$ is an isomorphism.

When the second condition holds, we say that $A \hookrightarrow B$ is *equiresidual*.

The following statement gives the link between the two last definitions. It gives us a universal property of the seminormalization : the seminormalization of a ring in another one as its biggest subintegral subextension.

Proposition 2.1.4

Let $A \hookrightarrow C \hookrightarrow B$ be integral extensions of rings. Then the following statements are equivalent :

- 1) The extension $A \hookrightarrow C$ is subintegral.
- 2) The image of $C \hookrightarrow B$ is a subring of A_B^+ .

Before proving this, we give several lemmas. First, we recall some classical algebraic results. Those are crucial to make things work and they are the reason we consider integral extensions.

Lemma 2.1.5 ([5])

Let $A \hookrightarrow B$ be an integral extension of rings. Then

1. (*Lying-Over property*) The map $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is surjective.
2. Let $\mathfrak{p} \in \text{Spec}(A)$ and $\mathfrak{q} \in \text{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Then $\mathfrak{q} \in \text{Spm}(B)$ if and only if $\mathfrak{p} \in \text{Spm}(A)$.
3. (*Going-up property*) Let $\mathfrak{p} \subseteq \mathfrak{p}' \in \text{Spec}(A)$ and $\mathfrak{q} \in \text{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Then there exists $\mathfrak{q}' \subseteq \mathfrak{q}' \in \text{Spec}(B)$ such that $\mathfrak{q}' \cap A = \mathfrak{p}'$.
4. Let S be any multiplicative subset of A , then $S^{-1}A \hookrightarrow S^{-1}B$ is integral.

Now see that a subextension of a subintegral extension is necessarily subintegral.

Lemma 2.1.6

Let $A \hookrightarrow C \hookrightarrow B$ be integral extensions of rings. Then the following properties are equivalent

- 1) The extension $A \hookrightarrow B$ is subintegral.
- 2) The extensions $A \hookrightarrow C$ and $C \hookrightarrow B$ are subintegral.

Proof: Let us prove that 1) implies 2). We start by showing the bijection between spectra. First, see that the induced maps between $\text{Spec}(C) \rightarrow \text{Spec}(A)$ and $\text{Spec}(B) \rightarrow \text{Spec}(C)$ are surjective because the extensions are integral (Lemma 2.1.5, 2)). Now let $\mathfrak{p} \in \text{Spec}(C)$, suppose that there exists $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{Spec}(B)$ such that $\mathfrak{q}_1 \cap C = \mathfrak{q}_2 \cap C = \mathfrak{p}$. In that case $\mathfrak{q}_1 \cap A = \mathfrak{q}_1 \cap C \cap A = \mathfrak{p} \cap A$ and the same is true for \mathfrak{q}_2 . Since $A \hookrightarrow B$ is subintegral, we have

$$\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A \implies \mathfrak{q}_1 = \mathfrak{q}_2$$

This shows that $\text{Spec}(B) \rightarrow \text{Spec}(C)$ is bijective. We now suppose that there exists $\mathfrak{p} \in \text{Spec}(A)$ and $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{Spec}(C)$ such that $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A = \mathfrak{p}$. By what we've just shown, we can consider some unique $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{Spec}(B)$ such that $\mathfrak{q}_1 \cap C = \mathfrak{p}_1$ and $\mathfrak{q}_2 \cap C = \mathfrak{p}_2$. Then $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A = \mathfrak{p}$, so $\mathfrak{q}_1 = \mathfrak{q}_2$ and finally $\mathfrak{p}_1 = \mathfrak{p}_2$. It shows that $\text{Spec}(C) \rightarrow \text{Spec}(A)$ is bijective.

We now show the isomorphisms on the residue fields. Let's consider the following commutative diagram

$$\begin{array}{ccccc}
 A & \xhookrightarrow{i_1} & C & \xhookrightarrow{i_2} & B \\
 \downarrow \pi & & \downarrow \pi' & & \downarrow \pi'' \\
 A/\mathfrak{p} & \xhookrightarrow{f_1} & C/\mathfrak{p}' & \xhookrightarrow{f_2} & B/\mathfrak{p}'' \\
 \downarrow & & \downarrow & & \downarrow \\
 \kappa(\mathfrak{p}) & \hookrightarrow & \kappa(\mathfrak{p}') & \hookrightarrow & \kappa(\mathfrak{p}'')
 \end{array}$$

The diagram is indeed commutative because the f_i are obtained by the universal property of quotient and their injectivity comes, for example with f_1 , from the following equality :

$$\ker(\pi' \circ i_1) = \{a \in A \text{ such that } i_1(a) \in \mathfrak{p}'\} = \mathfrak{p}' \cap A = \mathfrak{p}$$

By assumption, $A \hookrightarrow B$ is subintegral, so $\kappa(\mathfrak{p}) \rightarrow \kappa(\mathfrak{p}'')$ is an isomorphism. This implies that $\kappa(\mathfrak{p}) \rightarrow \kappa(\mathfrak{p}')$ and $\kappa(\mathfrak{p}') \rightarrow \kappa(\mathfrak{p}'')$ are also isomorphisms. The first implication is proved.

We now show that 2) implies 1). Let's suppose that $A \hookrightarrow C$ and $C \hookrightarrow B$ are subintegral. Let $\mathfrak{p} \in \text{Spec}(A)$, then there exists a unique element $\mathfrak{p}' \in \text{Spec}(C)$ above \mathfrak{p} . In the same way, there exists a unique element $\mathfrak{p}'' \in \text{Spec}(B)$ above \mathfrak{p}' , so \mathfrak{p}'' is the unique element of $\text{Spec}(B)$ above \mathfrak{p} . The fact that we have $\kappa(\mathfrak{p}') \simeq \kappa(\mathfrak{p})$ and $\kappa(\mathfrak{p}'') \simeq \kappa(\mathfrak{p}')$ gives us the equiresiduality. We have shown that $A \hookrightarrow B$ is subintegral. ■

The three following classical results brings a better understanding of the Jacobson radical in the definition of the seminormalization.

Lemma 2.1.7

Let $A \hookrightarrow B$ be an extension of rings, $\mathfrak{p} \in \text{Spec}(A)$ and $\mathfrak{q} \in \text{Spec}(B)$. Then

$$\mathfrak{q} \cap A = \mathfrak{p} \iff \mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$$

Proof: \Leftarrow : Let $a \in \mathfrak{p}$, then $\frac{a}{1} \in \mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}}$. So $\frac{a}{1} = \frac{b}{s}$ such that $b \in \mathfrak{q}$ and $s \in A \setminus \mathfrak{p}$. So $as = b$ and since $s \notin \mathfrak{q}$, we obtain $a \in \mathfrak{q}$. So $\mathfrak{p} \subseteq \mathfrak{q} \cap A$. Conversely, if $a \in \mathfrak{q} \cap A$, then $\frac{a}{1} = \frac{b}{s}$ with $b \in \mathfrak{p}$ and $s \in A \setminus \mathfrak{p}$. So $as = b$ and $s \notin \mathfrak{p}$ thus $a \in \mathfrak{p}$. Finally $\mathfrak{p} = \mathfrak{q} \cap A$.

\Rightarrow : Let $\frac{a}{s} \in \mathfrak{p}A_{\mathfrak{p}}$, then $a \in \mathfrak{p} = \mathfrak{q} \cap A$ and so $\frac{a}{s} \in \mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}}$. Thus $\mathfrak{p}A_{\mathfrak{p}} \subseteq \mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}}$. Conversely, if $\frac{a}{s} \in \mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}}$, then $\frac{a}{s} = \frac{b}{s'}$ with $b \in \mathfrak{q}$. So $as' = bs \in \mathfrak{q}$ and $s' \notin \mathfrak{q}$, so $a \in \mathfrak{q} \cap A = \mathfrak{p}$. Finally $\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}}$ ■

Lemma 2.1.8

Let $A \hookrightarrow B$ be an integral extension and $\mathfrak{p} \in \text{Spec}(A)$. Then

$$\text{Spm}(B_{\mathfrak{p}}) = \{\mathfrak{q}B_{\mathfrak{p}} \mid \mathfrak{q} \in \text{Spec}(B) \text{ and } \mathfrak{q} \cap A = \mathfrak{p}\}$$

Proof: By the description of prime ideals in a localization, we have:

$$\begin{aligned} \text{Spec}(B_{\mathfrak{p}}) &= \{\mathfrak{q}B_{\mathfrak{p}} \mid \mathfrak{q} \in \text{Spec}(B) \text{ and } \mathfrak{q} \cap A \setminus \mathfrak{p} = \emptyset\} \\ &= \{\mathfrak{q}B_{\mathfrak{p}} \mid \mathfrak{q} \in \text{Spec}(B) \text{ and } \mathfrak{q} \cap A \subseteq \mathfrak{p}\} \end{aligned}$$

Furthermore if $A \hookrightarrow B$ is integral then $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$ is also integral by 2.1.5, so $\text{Spm}(B_{\mathfrak{p}})$ is the inverse image of $\text{Spm}(A_{\mathfrak{p}})$ by the map $\text{Spec}(B_{\mathfrak{p}}) \rightarrow \text{Spec}(A_{\mathfrak{p}})$. Thus, since $\mathfrak{p}A_{\mathfrak{p}}$ is the unique maximal of $A_{\mathfrak{p}}$, we have

$$\mathfrak{q}B_{\mathfrak{p}} \in \text{Spm}(B_{\mathfrak{p}}) \iff \mathfrak{q}B_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$$

and so, thanks to the previous lemma,

$$\mathfrak{q}B_{\mathfrak{p}} \in \text{Spm}(B_{\mathfrak{p}}) \iff \mathfrak{q} \cap A = \mathfrak{p}$$

Remark. If $A \hookrightarrow B$ is an integral extension and $\mathfrak{p} \in \text{Spec}(A)$, then

$$\text{Rad}(B_{\mathfrak{p}}) = \bigcap_{\mathfrak{m} \in \text{Spm}(B_{\mathfrak{p}})} \mathfrak{m} = \bigcap_{\mathfrak{q} \in \text{Spec}(B), \mathfrak{q} \cap A = \mathfrak{p}} \mathfrak{q}B_{\mathfrak{p}}$$

Lemma 2.1.9

Let $A \hookrightarrow B$ be an integral extension of rings. Then

$$\text{Rad}(A) = \text{Rad}(B) \cap A$$

Proof: Lemma 2.1.5 tells us that the map $\text{Spm}(B) \rightarrow \text{Spm}(A)$ is surjective so

$$\text{Rad}(B) \cap A = \left(\bigcap_{\mathfrak{m} \in \text{Spm}(B)} \mathfrak{m} \right) \cap A = \bigcap_{\mathfrak{m} \in \text{Spm}(B)} (\mathfrak{m} \cap A) = \bigcap_{\mathfrak{m}' \in \text{Spm}(A)} \mathfrak{m}' = \text{Rad}(A)$$

Now, we have all the tools to demonstrate that the seminormalization of a ring in another one is its biggest subintegral subextension. But before that, we must check that the seminormalization gives a subintegral extension.

Proposition 2.1.10

Let $A \hookrightarrow B$ be an integral extension of rings. Then

$$A \hookrightarrow A_B^+ \text{ is subintegral}$$

Proof: We recall that $A_B^+ = \{b \in B \mid \forall \mathfrak{p} \in \text{Spec}(A) \ b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}})\} \subset B$. To clarify the proof, we will use the following notation abuse : $A^+ = A_B^+$ and $A_{\mathfrak{p}}^+ = (A^+)_{\mathfrak{p}}$.

Let's start by showing the bijection between spectra. In fact, we just have to show that $\text{Spec}(A^+) \rightarrow \text{Spec}(A)$ is injective since the extension is integral. Let $\mathfrak{p} \in \text{Spec}(A)$ and $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{Spec}(A^+)$ such that $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A = \mathfrak{p}$. Suppose that there exists $b \in \mathfrak{q}_1 \cap \mathfrak{q}_2^c$ and write $b_{\mathfrak{p}} = \alpha + \beta \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}})$. Then $\alpha = b_{\mathfrak{p}} - \beta \in \mathfrak{q}_1 A_{\mathfrak{p}}^+ \cap A_{\mathfrak{p}}$. But, by Lemma 2.1.7, we have $\mathfrak{q}_1 A_{\mathfrak{p}}^+ \cap A_{\mathfrak{p}} = \mathfrak{p} A_{\mathfrak{p}} = \mathfrak{q}_2 A_{\mathfrak{p}}^+ \cap A_{\mathfrak{p}}$ and thus $\alpha \in \mathfrak{q}_2 A_{\mathfrak{p}}^+$. Moreover $\beta = b_{\mathfrak{p}} - \alpha \in A_{\mathfrak{p}}^+$ and $\beta \in \text{Rad}(B_{\mathfrak{p}})$. So, by Lemma 2.1.9, we get $\beta \in \text{Rad}(B_{\mathfrak{p}}) \cap A_{\mathfrak{p}}^+ = \text{Rad}(A_{\mathfrak{p}}^+) \subset \mathfrak{q}_2 A_{\mathfrak{p}}^+$. It gives $b_{\mathfrak{p}} \in \mathfrak{q}_2 A_{\mathfrak{p}}^+$ which is not possible by assumption on b . Since \mathfrak{q}_1 and \mathfrak{q}_2 can be inverted in the proof, we obtain $\mathfrak{q}_1 = \mathfrak{q}_2$ and so $\text{Spec}(A^+) \rightarrow \text{Spec}(A)$ is bijective.

We show now the equiresiduality. Let $\mathfrak{p} \in \text{Spec}(A)$ and $\mathfrak{q} \in \text{Spec}(A^+)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Let $\mathfrak{q}' \in \text{Spec}(B)$ such that $\mathfrak{q}' \cap A = \mathfrak{p}$. We can see that $\mathfrak{q}' \cap A^+ = \mathfrak{q}$ because $\mathfrak{q}' \cap A^+$ is a prime ideal above \mathfrak{p} and $\text{Spec}(A^+) \rightarrow \text{Spec}(A)$ is bijective. We obtain the following commutative diagram :

$$\begin{array}{ccccc}
 A^{\subset} & \longrightarrow & A^{+\subset} & \longrightarrow & B \\
 \downarrow & & \downarrow & & \downarrow \\
 A_{\mathfrak{p}}^{\subset} & \longrightarrow & A_{\mathfrak{p}}^{+\subset} & \longrightarrow & B_{\mathfrak{p}} \\
 \downarrow \pi & & \downarrow \pi^+ & & \downarrow \pi' \\
 A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}^{\subset} & \xrightarrow{f} & A_{\mathfrak{p}}^+/\mathfrak{q}A_{\mathfrak{p}}^{+\subset} & \longrightarrow & B_{\mathfrak{p}}/\mathfrak{q}'B_{\mathfrak{p}} \\
 \parallel & & \downarrow & & \downarrow \\
 A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}^{\subset} & \longrightarrow & A_{\mathfrak{q}}^+/\mathfrak{q}A_{\mathfrak{q}}^{+\subset} & \longrightarrow & B_{\mathfrak{q}'}/\mathfrak{q}B_{\mathfrak{q}'}
 \end{array}$$

We want to show that f is surjective. Let $\pi^+(\frac{b}{s}) \in A_{\mathfrak{p}}^+/\mathfrak{q}A_{\mathfrak{p}}^{+\subset}$. Then $b \in A^+$ so there exists $\alpha \in A_{\mathfrak{p}}$ and $\beta \in \text{Rad}(B_{\mathfrak{p}})$ such that $b_{\mathfrak{p}} = \alpha + \beta$. The remark under Proposition 2.1.8 tells us that $\text{Rad}(B_{\mathfrak{p}}) \subset \mathfrak{q}'B_{\mathfrak{p}}$ so $\pi'(b_{\mathfrak{p}}) = \pi'(\alpha)$ and since the diagram is commutative, we get $\pi^+(b_{\mathfrak{p}}) = \pi^+(\alpha) = f \circ \pi(\alpha)$. Finally we obtain

$$f\left(\frac{\pi(\alpha)}{\pi(s)}\right) = \frac{\pi^+(b_{\mathfrak{p}})}{\pi^+(s)} = \pi^+\left(\frac{b}{s}\right)$$

which proves the surjectivity of f and gives us $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \simeq A_{\mathfrak{p}}^+/\mathfrak{q}A_{\mathfrak{p}}^+$.

Finally one can see that $A_{\mathfrak{p}}^+/\mathfrak{q}A_{\mathfrak{p}}^+ \simeq A_{\mathfrak{q}}^+/\mathfrak{q}A_{\mathfrak{q}}^+$ because

$$A_{\mathfrak{q}}^+/\mathfrak{q}A_{\mathfrak{q}}^+ \simeq \text{Frac}(A^+/\mathfrak{q}) \text{ and } A^+/\mathfrak{q} \hookrightarrow A_{\mathfrak{p}}^+/\mathfrak{q}A_{\mathfrak{p}}^+ \implies A_{\mathfrak{q}}^+/\mathfrak{q}A_{\mathfrak{q}}^+ \hookrightarrow A_{\mathfrak{p}}^+/\mathfrak{q}A_{\mathfrak{p}}^+$$

This gives us the result

$$\kappa(\mathfrak{p}) \simeq A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \simeq A_{\mathfrak{q}}^+/\mathfrak{q}A_{\mathfrak{q}}^+ \simeq \kappa(\mathfrak{q}) \quad \blacksquare$$

Finally, we get the main result of this section that we have already stated in Proposition 2.1.4.

Proposition (Proposition 2.1.4). *Let $A \hookrightarrow C \hookrightarrow B$ be integral extensions of rings. Then the following statements are equivalent :*

- 1) *The extension $A \hookrightarrow C$ is subintegral.*
- 2) *The image of $C \hookrightarrow B$ is a subring of A_B^+ .*

Proof: Let's prove that 1) implies 2). Suppose that $A \hookrightarrow C$ is subintegral, we want to show :

$$\forall c \in C, \forall \mathfrak{p} \in \text{Spec}(A), c_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}})$$

Let $\mathfrak{p} \in \text{Spec}(A)$. We write $\mathfrak{q} \in \text{Spec}(C)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$ and so we get the following diagram :

$$\begin{array}{ccc}
 A & \hookrightarrow & C \\
 \downarrow & & \downarrow \\
 A_{\mathfrak{p}} & \xrightarrow{i} & C_{\mathfrak{p}} \\
 \downarrow \pi & & \downarrow \iota \\
 & & C_{\mathfrak{q}} \\
 \downarrow & & \downarrow \pi' \\
 \kappa(\mathfrak{p}) & \xrightarrow{f} & \kappa(\mathfrak{q})
 \end{array}$$

ϕ (curved arrow from $C_{\mathfrak{p}}$ to $\kappa(\mathfrak{q})$)

One can see that the diagram is commutative thanks to the following equality

$$\text{Ker}(\phi) = \{c_{\mathfrak{p}} \in C_{\mathfrak{p}} \text{ such that } \pi' \circ \iota(c_{\mathfrak{p}}) = 0\} = \{c_{\mathfrak{p}} \in C_{\mathfrak{p}} \text{ such that } \iota(c_{\mathfrak{p}}) \in \mathfrak{q}C_{\mathfrak{q}}\} = \mathfrak{q}C_{\mathfrak{q}} \cap C_{\mathfrak{p}} = \mathfrak{q}C_{\mathfrak{p}}.$$

Thus, by Lemma 2.1.7, we have

$$\text{ker}(\phi \circ i) = \mathfrak{q}C_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$$

We now consider $c \in C$ and we want to show $c \in A_B^+$. We have $\phi(c_{\mathfrak{p}}) \in \kappa(\mathfrak{q})$, so one can consider $\frac{a}{s} \in A_{\mathfrak{p}}$ such that $f \circ \pi\left(\frac{a}{s}\right) = \phi(c_{\mathfrak{p}})$. Thanks to the diagram's commutativity, we have $f \circ \pi = \phi \circ i$, which gives us

$$\phi(c_{\mathfrak{p}}) = f \circ \pi\left(\frac{a}{s}\right) = \phi\left(i\left(\frac{a}{s}\right)\right)$$

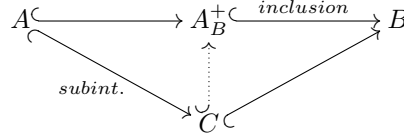
Thus $\phi(c_{\mathfrak{p}} - i\left(\frac{a}{s}\right)) = 0$ and so $c_{\mathfrak{p}} - i\left(\frac{a}{s}\right) \in \mathfrak{q}C_{\mathfrak{p}}$. But, since \mathfrak{q} is the only ideal above \mathfrak{p} , we have $\mathfrak{q}C_{\mathfrak{p}} = \text{Rad}(C_{\mathfrak{p}})$. So, by Lemma 2.1.9, we get $C_{\mathfrak{p}} - i\left(\frac{a}{s}\right) \in \text{Rad}(C_{\mathfrak{p}}) = \text{Rad}(B_{\mathfrak{p}}) \cap C_{\mathfrak{p}} \subset \text{Rad}(B_{\mathfrak{p}})$. And so $c_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}})$, which shows $C \subseteq A_B^+$.

We now prove that 2) implies 1). Suppose we have $A \hookrightarrow C \hookrightarrow A_B^+ \hookrightarrow B$. Those extensions are integral and, by Proposition 2.1.10, the extension $A \hookrightarrow A_B^+$ is subintegral. Then Lemma 2.1.6 tells us that $A \hookrightarrow C$ is subintegral. ■

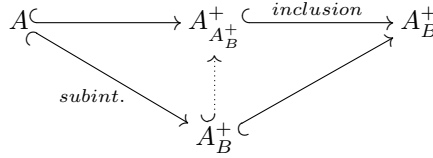
As said previously, Proposition 2.1.4 can be put into the form of a universal property. We rewrite it in the following way :

Theorem 2.1.11 (Universal property of seminormalization)

Let $A \hookrightarrow B$ be an integral extension of rings. For every intermediate extension C of $A \hookrightarrow B$ such that $A \hookrightarrow C$ is subintegral, the image of C by the injection $C \hookrightarrow B$ is contained in A_B^+ .



Remark. Let $A \hookrightarrow B$ be an integral extension. We have $A \hookrightarrow A_B^+$ subintegral by Proposition 2.1.10. So we can apply the universal property in the following way :



Thus $A_B^+ \hookrightarrow A_{A_B^+}^+$. But, by definition, $A_{A_B^+}^+$ is included in A_B^+ . We get the following idempotency property

$$A_B^+ = A_{A_B^+}^+$$

2.2 Universal property of the seminormalization in the geometric case.

Let k be an algebraically closed field of characteristic zero and $X = \text{Spec}(A)$ be an affine algebraic variety with A a reduced k -algebra of finite type. Let $k[X] := A$ denote the coordinate ring of X . We have $k[X] \simeq k[x_1, \dots, x_n]/I$ for a radical ideal I of $k[x_1, \dots, x_n]$. We recall that X is irreducible if and only if $k[X]$ is a domain. A morphism $\pi : Y \rightarrow X$ between two varieties induces the morphism $\pi^* : k[X] \rightarrow k[Y]$ which is injective if and only if π is dominant. We say that π is of finite type (resp. is finite) if π^* makes $k[Y]$ a $k[X]$ -algebra of finite type (resp. a finite $k[X]$ -module).

The space X is equipped with the Zariski topology for which the closed sets are of the form $\mathcal{V}(I) := \{\mathfrak{p} \in \text{Spec}(k[X]) \mid I \subset \mathfrak{p}\}$ where I is an ideal of $k[X]$. We define $X(k) := \{\mathfrak{m} \in \text{Spm}(k[X]) \mid \kappa(\mathfrak{m}) = k\}$. Thus, if we write $k[X] = k[x_1, \dots, x_n]/I$, the elements of $X(k)$ can be seen as elements of $\text{Spm}(k[x_1, \dots, x_n])$ containing I . The Nullstellensatz gives us a Zariski homeomorphism between $X(k)$ and the algebraic set $\mathcal{Z}(I) := \{x \in k^n \mid \forall f \in I f(x) = 0\} \subset k^n$. We will call $\pi_k : Y(k) \rightarrow X(k)$ the restriction of π to $Y(k)$. We will add the prefix «Z-» before a property if it holds for the Zariski topology. If X is irreducible, then we write $\mathcal{K}(X) := \text{Frac}(k[X])$.

If needed, for $x \in X$ (resp. $X(k)$), we will write \mathfrak{p}_x (resp. \mathfrak{m}_x) its associated ideal in $k[X]$.

The goal of this section is to prove the following theorem and to write a universal property of the seminormalization for affine varieties over k .

Theorem 2.2.1

Let $\pi : Y \rightarrow X$ be a finite morphism between affine varieties. Then the following properties are equivalent.

- 1) The morphism $\pi_k : Y(k) \rightarrow X(k)$ is bijective.
- 2) The extension $\pi^* : k[X] \hookrightarrow k[Y]$ is subintegral.
- 3) The morphism $\pi : Y \rightarrow X$ is a Z -homeomorphism.
- 4) The morphism $\pi_k : Y(k) \rightarrow X(k)$ is a Z -homeomorphism.

The following classical result will be useful in the sequel.

Lemma 2.2.2

Let $\pi : Y \rightarrow X$ be a finite morphism between irreducible varieties. Then there exists a non empty Z -open subset U of X such that

$$\forall u \in U(k), \# \pi_k^{-1}(u) = [\mathcal{K}(Y) : \mathcal{K}(X)]$$

Proof: Let U be a non empty Z -open set of X such that U is normal. Since the characteristic of k is zero, the extension $\mathcal{K}(X) \hookrightarrow \mathcal{K}(U)$ is separable. So we can apply Theorem 7 p.117 from [35] on the morphism π restricted to $\pi^{-1}(U)$. ■

The following proposition gives the main part of Theorem 2.2.1. The Nullstellensatz is the key to prove it, which is not surprising for a statement of this kind.

Proposition 2.2.3

Let $\pi : Y \rightarrow X$ be a finite morphism between affine varieties. The following properties are equivalent

- 1) The extension $\pi^* : k[X] \hookrightarrow k[Y]$ is subintegral.
- 2) The morphism $\pi_k : Y(k) \rightarrow X(k)$ is bijective.

Proof: Suppose that $k[X] \hookrightarrow k[Y]$ is subintegral, then $Y \rightarrow X$ is a bijection and as the extension is integral, the inverse image of $X(k)$ is $Y(k)$ (cf. lemma 2.1.5). Therefore

$$\pi_k : Y(k) \rightarrow X(k) \text{ is bijective}$$

Conversely, suppose that $Y(k) \rightarrow X(k)$ is bijective. We start by checking that π is bijective. Let $y_1, y_2 \in Y$ such that $\pi(y_1) = \pi(y_2)$ and write $\mathfrak{p}_{y_1}, \mathfrak{p}_{y_2}$ for the associated prime ideals of $k[Y]$. We want to prove that $y_1 = y_2$. By the Nullstellensatz, it is equivalent to show

$$\mathfrak{p}_{y_1} = \bigcap_{\mathfrak{p}_{y_1} \subseteq \mathfrak{m}_y} \mathfrak{m}_y = \bigcap_{\mathfrak{p}_{y_2} \subseteq \mathfrak{m}_y} \mathfrak{m}_y = \mathfrak{p}_{y_2}$$

Let $y \in Y(k)$ such that $\mathfrak{p}_{y_1} \subset \mathfrak{m}_y$. We then have $\mathfrak{p}_{\pi(y_1)} \subset \mathfrak{m}_{\pi(y)}$, so $\mathfrak{p}_{\pi(y_2)} \subset \mathfrak{m}_{\pi(y)}$. Thus, by the going-up property, we can consider $y' \in Y(k)$ such that $\mathfrak{p}_{y_2} \subseteq \mathfrak{m}_{y'}$ and $\pi(y) = \pi(y')$. As we have supposed that π is injective on $Y(k)$, we get $y = y'$.

Finally

$$\forall y \in Y(k) \quad \mathfrak{p}_{y_1} \subseteq \mathfrak{m}_y \implies \mathfrak{p}_{y_2} \subseteq \mathfrak{m}_y$$

The role of y_1 and y_2 can be inverted in the proof, so we get

$$\mathfrak{p}_{y_1} = \bigcap_{y \in Y(k), \mathfrak{p}_{y_1} \subseteq \mathfrak{m}_y} \mathfrak{m}_y = \bigcap_{y \in Y(k), \mathfrak{p}_{y_2} \subseteq \mathfrak{m}_y} \mathfrak{m}_y = \mathfrak{p}_{y_2} \quad \text{i.e.} \quad y_1 = y_2$$

It remains to check the equiresiduality. Let $x \in X$ and $y \in Y$ such that $\pi(y) = x$. We want to prove $\kappa(\mathfrak{p}_x) \simeq \kappa(\mathfrak{p}_y)$. If we write $V = \text{Spec}(k[X]/\mathfrak{p}_x)$ and $W = \text{Spec}(k[Y]/\mathfrak{p}_y)$, we get the following commutative diagram :

$$\begin{array}{ccc} k[X] & \xleftarrow{\pi^*} & k[Y] \\ \downarrow \pi_X & & \downarrow \pi_Y \\ k[V] & \xleftarrow{(\pi|_W)^*} & k[W] \\ \downarrow & & \downarrow \\ \mathcal{K}(V) & \xleftarrow{\quad} & \mathcal{K}(W) \end{array}$$

As $k[Y]$ is a finite $k[X]$ -module, we have that $k[W]$ is a finite $k[V]$ -module. Thus $\pi|_W$ is a finite morphism between two irreducible varieties. Therefore we can apply Lemma 2.2.2 to consider a non empty Z -open set U of W such that

$$\forall u \in U(k) \quad \#(\pi|_{W(k)})^{-1}(u) = [\mathcal{K}(W) : \mathcal{K}(V)]$$

But we have previously shown that π is bijective so $[\mathcal{K}(W) : \mathcal{K}(V)] = 1$ and so we get $\kappa(\mathfrak{p}_x) \simeq \kappa(\mathfrak{p}_y)$. ■

The following lemma completes the results needed to get Theorem 2.2.1.

Lemma 2.2.4

Let $A \hookrightarrow B$ be an integral extension of rings and $\pi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be the induced map. If the morphism π is bijective, then π is a Z -homeomorphism.

Proof: Since π is Z -continuous, even when the extension is not integral, we just have to show that π is Z -closed. Let $\mathfrak{p} \in \text{Spec}(A)$ and $\mathfrak{q} \in \text{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. We have

$$\pi(\mathcal{V}(\mathfrak{q})) = \{\mathfrak{q}' \cap A \mid \mathfrak{q} \subset \mathfrak{q}' \in \text{Spec}(B)\}$$

so $\pi(\mathcal{V}(\mathfrak{q})) \subset \mathcal{V}(\mathfrak{q} \cap A) = \mathcal{V}(\mathfrak{p})$. If $\mathfrak{p}' \in \mathcal{V}(\mathfrak{p})$, then the going-up property says that there exists $\mathfrak{q}' \in \text{Spec}(B)$ such that $\mathfrak{q}' \cap A = \mathfrak{p}'$ and $\mathfrak{q} \subset \mathfrak{q}'$. Therefore $\mathcal{V}(\mathfrak{p}) \subset \pi(\mathcal{V}(\mathfrak{q}))$ and so $\pi(\mathcal{V}(\mathfrak{q})) = \mathcal{V}(\mathfrak{p})$ which is Z -closed. ■

We can finally prove the main theorem of this section.

Theorem (Theorem 2.2.1). *Let $\pi : Y \rightarrow X$ be a finite morphism between two affine varieties. Then the following properties are equivalent.*

- 1) *The morphism $\pi_k : Y(k) \rightarrow X(k)$ is bijective.*
- 2) *The extension $\pi^* : k[X] \hookrightarrow k[Y]$ is subintegral.*
- 3) *The morphism $\pi : Y \rightarrow X$ is a Z -homeomorphism.*
- 4) *The morphism $\pi_k : Y(k) \rightarrow X(k)$ is a Z -homeomorphism.*

Proof: First of all, Proposition 2.2.3 gives us the first equivalence. Now suppose that $k[X] \hookrightarrow k[Y]$ is subintegral. Then π is bijective and since $k[X] \hookrightarrow k[Y]$ is integral, we can apply Lemma 2.2.4 and get that π is a Z -homeomorphism. Finally, if π is a Z -homeomorphism then π_k is a Z -homeomorphism and thus π_k is bijective. ■

Before going further, we must define the seminormalization of an affine variety. If $\pi : Y \rightarrow X$ is a finite morphism between two affine varieties, then $k[X]_{k[Y]}^+$ is a finite $k[X]$ -module because it is a submodule of $k[Y]$. Thus $k[X]_{k[Y]}^+$ is a k -algebra of finite type because so is $k[X]$.

Definition 2.2.5

Let $\pi : Y \rightarrow X$ be a finite morphism between two affine varieties. The affine variety defined by

$$X_Y^+ = \text{Spec}(k[X]_{k[Y]}^+)$$

is called the seminormalization of X in Y .

As a consequence of the previous theorem, we are going to see that, in order to get the seminormalization of a variety in an other one, it is sufficient to glue together the closed points in the fibers of π_k .

Definition 2.2.6

Let $A \hookrightarrow B$ be an integral extension of rings.

We define

$$A_B^{+max} = \left\{ b \in B \mid \forall \mathfrak{m} \in \text{Spm}(A), b_{\mathfrak{m}} \in A_{\mathfrak{m}} + \text{Rad}(B_{\mathfrak{m}}) \right\}$$

Corollary 2.2.7

Let $\pi : Y \rightarrow X$ be a finite morphism between two affine varieties.

Then

$$k[X]_{k[Y]}^{+max} = k[X]_{k[Y]}^+$$

Proof: We write $A := k[X]$ and $B := k[Y]$. So we have to show $A_B^{+max} = A_B^+$. First of all, note that the inclusion $A_B^+ \subseteq A_B^{+max}$ is obvious.

In order to prove the other inclusion, we have to show that $\text{Spm}(A_B^{+max}) \rightarrow \text{Spm}(A)$ is bijective. If we succeed, we will be able to deduce, thanks to Proposition 2.2.3, that $A \hookrightarrow A_B^{+max}$ is subintegral and the universal property of the seminormalization will give us the result.

Let $\mathfrak{m} \in \text{Spm}(A)$ and $\mathfrak{m}_1, \mathfrak{m}_2 \in \text{Spm}(A_B^{+max})$ such that $\mathfrak{m}_1 \cap A = \mathfrak{m}_2 \cap A = \mathfrak{m}$. Suppose that there exists $b \in \mathfrak{m}_1 \cap \mathfrak{m}_2^c$ and write $b_{\mathfrak{m}} = \alpha + \beta \in A_{\mathfrak{m}} + \text{Rad}(B_{\mathfrak{m}})$. Then $\beta = b_{\mathfrak{m}} - \alpha \in A_{B, \mathfrak{m}}^{+max}$, so $\beta \in \text{Rad}(B_{\mathfrak{m}}) \cap A_{B, \mathfrak{m}}^{+max}$. By Lemma 2.1.9, we get $\beta \in \text{Rad}(A_{B, \mathfrak{m}}^{+max}) \subset \mathfrak{m}_1 A_{B, \mathfrak{m}}^{+max} \cap \mathfrak{m}_2 A_{B, \mathfrak{m}}^{+max}$. In particular, $\beta \in \mathfrak{m}_1 A_{B, \mathfrak{m}}^{+max}$ and by assumption on b , we also have $b_{\mathfrak{m}} \in \mathfrak{m}_1 A_{B, \mathfrak{m}}^{+max}$. We then have $\alpha = b_{\mathfrak{m}} - \beta \in \mathfrak{m}_1 A_{B, \mathfrak{m}}^{+max} \cap A_{\mathfrak{m}}$ and $\mathfrak{m}_1 A_{B, \mathfrak{m}}^{+max} \cap A_{\mathfrak{m}} = \mathfrak{m} A_{\mathfrak{m}} = \mathfrak{m}_2 A_{B, \mathfrak{m}}^{+max} \cap A_{\mathfrak{m}}$, by Lemma 2.1.7. Thus $\alpha \in \mathfrak{m}_2 A_{B, \mathfrak{m}}^{+max}$ and since $\beta \in \mathfrak{m}_2 A_{B, \mathfrak{m}}^{+max}$, we get $b_{\mathfrak{m}} \in \mathfrak{m}_2 A_{B, \mathfrak{m}}^{+max}$ which is not possible by assumption on b . Considering the fact that the role of \mathfrak{m}_1 and \mathfrak{m}_2 can be inverted in the proof, we get $\mathfrak{m}_1 = \mathfrak{m}_2$. ■

We are now able to write a geometric version of the universal property of seminormalization. By just adapting the one we have obtained at the end of the first section with coordinate rings, we get :

$$\begin{array}{ccccc}
 k[X] \subset & \xrightarrow{\quad} & k[X]_{k[Y]}^+ \simeq k[X_Y^+] \subset & \xrightarrow{\quad} & k[Y] \\
 & \searrow \text{subint.} & \uparrow \text{---} & \nearrow & \\
 & & k[Z] \subset & &
 \end{array}$$

This gives us the following statement for varieties :

Theorem 2.2.8 (Universal property of the seminormalization)

Let $Y \rightarrow Z \rightarrow X$ be finite morphisms of affine varieties. Then $\pi_c : Z(k) \rightarrow X(k)$ is bijective if and only if there exists a morphism $\pi_Z^+ : X_Y^+ \rightarrow Z$ such that $\pi \circ \pi_Z^+ = \pi^+$. Moreover π_Z^+ is unique and $\pi_{Z(k)}^+ : X_Y^+(k) \rightarrow Z(k)$ is bijective.

$$\begin{array}{ccccc}
 Y & \xrightarrow{\quad} & X_Y^+ & \xrightarrow{\pi^+} & X \\
 & \searrow & \downarrow \pi_Z^+ & \nearrow \pi_{bij} & \\
 & & Z & &
 \end{array}$$

Proof: We have the following equivalences :

$$\begin{aligned}
 \pi_c : Z(k) \rightarrow X(k) \text{ bijective} & \iff k[X] \hookrightarrow k[Z] \text{ subintegral (by Theorem 2.2.1)} \\
 & \iff k[Z] \subset k[X]_{k[Y]}^+ \simeq k[X_{k[Y]}^+] \text{ (by Proposition 2.1.4)} \\
 & \iff \exists \pi_Z^+ : X_Y^+ \rightarrow Z \text{ dominant, such that } \pi \circ \pi_Z^+ = \pi^+
 \end{aligned}$$

We get the uniqueness of π_Z^+ by injectivity of π . Since $k[X] \hookrightarrow k[X_Y^+]$ is subintegral then Lemma 2.1.6 says that $k[Z] \hookrightarrow k[X_Y^+]$ is also subintegral. So Theorem 2.2.1 tells us that $\pi_{Z(k)}^+ : X_Y^+(k) \rightarrow Z(k)$ is bijective. ■

2.3 Continuous rational functions on complex affine varieties.

This section is dedicated to the introduction and study of continuous rational functions on affine varieties and to its link with the concept of seminormalization. Since we use the Euclidean topology, we restrict ourself to complex varieties. A generalization on algebraic variety over algebraic closed field of characteristic zero is given in [6].

The first subsection is dedicated to the study of the ring of continuous rational functions. Concretely, we show that this ring corresponds to the coordinate ring of the seminormalization of the variety.

In the second subsection, we show that continuous rational functions are always regulous in the case of complex affine varieties. The theory of regulous functions comes from real algebraic geometry and was introduced in [12] and [22]. In this theory, continuous rational functions and regulous functions are not the same on real singular algebraic sets.

In the third subsection we look at continuous functions which are a root of a polynomial in $\mathbb{C}[X][t]$. This leads to identify continuous rational functions with "c-holomorphic" functions with algebraic graphs on algebraic varieties. This kind of functions is studied in [8] and [7] in the point of view of complex

analytic geometry. We show, with algebraic arguments, that they coincide on algebraic varieties. Finally, the fourth subsection presents examples of continuous rational functions.

We begin by recalling classical results on the normalization of an affine variety.

Definition 2.3.1

Let A be a commutative ring. We note \mathcal{K} the localization $S^{-1}A$ where S is the set of non-zero-divisors of A . The ring \mathcal{K} is called the total ring of fractions of A .

If A is reduced with a finite number of minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, then

$$\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n = \bigcap_{p \in \text{Spec}(A)} p = (0)$$

We get the following injections :

$$A \hookrightarrow A/\mathfrak{p}_1 \times \dots \times A/\mathfrak{p}_n \hookrightarrow K_1 \times \dots \times K_n \text{ where } K_i := \text{Frac}(A/\mathfrak{p}_i)$$

Then the total ring of fractions of A corresponds to the product of the fields K_i .

We define A' to be the integral closure of A in \mathcal{K} and we simply call it *the integral closure of A* . In the same spirit, the seminormalization of A in \mathcal{K} is denoted by A^+ and is simply called *the seminormalization of A* . Finally, we say that A is *seminormal* if $A^+ = A$.

For X an affine variety, the total ring of fractions of $\mathbb{C}[X]$ is denoted by $\mathcal{K}(X)$. The ring $\mathcal{K}(X)$ (which is a field when X is irreducible) is also the ring of classes of rational fractions on X and is called the ring of rational functions on X . It means that it represents the set of classes of regular functions f on a \mathbb{Z} -dense \mathbb{Z} -open set U of $X(\mathbb{C})$ with the equivalence relation $(f_1, U_1) \sim (f_2, U_2)$ iff $f_1 = f_2$ on $U_1 \cap U_2$.

We say that a morphism $\varphi : Y \rightarrow X$ is birational if the associated morphism $\mathcal{K}(X) \hookrightarrow \mathcal{K}(Y)$ is an isomorphism.

The integral closure of $\mathbb{C}[X]$ being a finite $\mathbb{C}[X]$ -module (see [11] Thm 4.14), it is also a \mathbb{C} -algebra of finite type. Thus we can define the *normalization* X' of X such that $X' = \text{Spec}(\mathbb{C}[X]')$. We get a finite and birational morphism $\pi' : X' \rightarrow X$. The normalization of X is the biggest affine variety finitely birational to X . It means that for every finite, birational morphism $\varphi : Y \rightarrow X$, there exists $\psi : X' \rightarrow Y$ such that $\pi' = \varphi \circ \psi$.

The seminormalization X^+ of X is the variety $X_{X'}^+$, defined in Definition 2.2.5. The seminormalization X^+ comes with a finite, birational and bijective morphism $\pi^+ : X^+ \rightarrow X$ whose universal property is given by Proposition 2.2.8.

Finally for every affine variety X , we have the following extensions of rings :

$$\mathbb{C}[X] \hookrightarrow \mathbb{C}[X^+] \hookrightarrow \mathbb{C}[X'] \hookrightarrow \mathcal{K}(X).$$

Definition 2.3.2

Let X be an affine variety. We denote by $\mathcal{K}^0(X(\mathbb{C}))$ the set of continuous functions $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ for the Euclidean topology which are rational on $X(\mathbb{C})$.

Example. The most classical example of such a function is the following :

Let $X = \text{Spec}(\mathbb{C}[x, y]/\langle y^2 - x^3 \rangle)$, consider the function f defined on $X(\mathbb{C})$ by

$$f = \begin{cases} y/x & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$$

Remark. Let $f \in \mathcal{K}^0(X(\mathbb{C}))$ and let $(p_1/q_1, U_1)$ be a rational representation of f . If $(p_2/q_2, U_2)$ is such that $(p_1/q_1, U_1) \sim (p_2/q_2, U_2)$, then $f = p_1/q_1 = p_2/q_2$ on $U_1 \cap U_2$. By continuity, we get that $f = p_2/q_2$ on U_2 . Moreover, if a rational function $(p/q, U)$ represents two functions $f, g \in \mathcal{K}^0(X(\mathbb{C}))$, then $f = p/q = g$ on U and so $f = g$ on $X(\mathbb{C})$ by continuity. In conclusion, we have an injection

$$\mathcal{K}^0(X(\mathbb{C})) \hookrightarrow \mathcal{K}(X)$$

We will sometimes make an amalgam between the elements of $\mathcal{K}^0(X(\mathbb{C}))$ and their rational representations.

Now, let us show that we can always assume X to be irreducible.

Lemma 2.3.3

Let E be a topological space and $\{E_i\}_{i \in I}$ be a covering of E . Let A be a subspace of E such that $A \cap E_i$ is dense in E_i for all $i \in I$. Then A is dense in E .

Proof: Let U be a non-empty open set of E . Then there exists $i \in I$ such that $U \cap E_i \neq \emptyset$. Since $A \cap E_i$ is dense in E_i and $U \cap E_i$ is a non-empty open set of E_i , we get $A \cap U \cap E_i \neq \emptyset$. Then $A \cap U \neq \emptyset$. So A is dense in E . ■

Lemma 2.3.4

Let X be an affine variety and $f : X(\mathbb{C}) \rightarrow \mathbb{C}$. We write $X = \bigcup_{i=1}^n X_i$ where the X_i are the irreducible components of X . The following properties are equivalent

- 1) $f \in \mathcal{K}^0(X(\mathbb{C}))$
- 2) $\forall i \in \{1, \dots, n\} \quad f|_{X_i(\mathbb{C})} \in \mathcal{K}^0(X_i(\mathbb{C}))$

Proof: Let $f \in \mathcal{K}^0(X(\mathbb{C}))$, we denote by U the Z-dense Z-open set on which f is regular. For $j \in \{1, \dots, n\}$, the set $X \setminus \bigcup_{i \neq j} X_i$ is a Z-open set contained in X_j , which is not empty because $X_j \not\subseteq \bigcup_{i \neq j} X_i$. Thus, since U is Z-dense, we have $X_j(\mathbb{C}) \cap U \neq \emptyset$. So f is regular on $X_j(\mathbb{C}) \cap U$ which is Z-dense because X_j is irreducible. Then, $f|_{X_j(\mathbb{C})}$ being clearly continuous, we have $f|_{X_j(\mathbb{C})} \in \mathcal{K}^0(X_j(\mathbb{C}))$.

Let $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ such that $f|_{X_i(\mathbb{C})} \in \mathcal{K}^0(X_i(\mathbb{C}))$ for all $i \in \{1, \dots, n\}$. Then f is regular on a Z-dense Z-open set of every component and since, by Lemma 2.3.3, a union of dense open sets of each irreducible component of X is a dense open set of X , we get that f is regular on a Z-dense Z-open set of $X(\mathbb{C})$. Let's show that f is continuous on $X(\mathbb{C})$. First of all, f is clearly continuous on every point of the set

$$X(\mathbb{C}) \setminus \left(\bigcup_{i \neq j} X_i(\mathbb{C}) \cap X_j(\mathbb{C}) \right)$$

which are the points that do not belong to any intersection of components. Now let's take a look at the continuity near the other points. Let $x \in \bigcap_{j \in J} X_j(\mathbb{C})$ where J is a subset of $\{1, \dots, n\}$. Let $\varepsilon > 0$ and $j \in J$. Since f is continuous on $X_j(\mathbb{C})$, we can consider a Euclidean open set U_j containing x such that $\forall y \in X_j(\mathbb{C}) \cap U_j$ we have $|f(x) - f(y)| < \varepsilon$. By doing the same for all $j \in J$, we obtain

$$\forall y \in X(\mathbb{C}) \cap \left(\bigcap_{j \in J} U_j \right) \quad |f(x) - f(y)| < \varepsilon \quad \blacksquare$$

Remark. Let X be an affine variety and $X = \bigcup_{i=1}^n X_i$ its decomposition into irreducible components. Then

$$\mathcal{K}^0(X(\mathbb{C})) \simeq \left\{ (f_1, \dots, f_n) \in \mathcal{K}^0(X_1(\mathbb{C})) \times \dots \times \mathcal{K}^0(X_n(\mathbb{C})) \mid f_{i|X_i(\mathbb{C}) \cap X_j(\mathbb{C})} = f_{j|X_i(\mathbb{C}) \cap X_j(\mathbb{C})} \right\}$$

The next proposition shows that the continuity allows us to be more precise concerning the Z -open set where an element of $\mathcal{K}^0(X(\mathbb{C}))$ is regular. We write X_{reg} (resp. X_{sing}) the set of regular (resp. singular) points of X and $X_{reg}(\mathbb{C})$ (resp. $X_{sing}(\mathbb{C})$) those of $X(\mathbb{C})$.

Proposition 2.3.5

A function belongs to $\mathcal{K}^0(X(\mathbb{C}))$ if and only if it is continuous for the Euclidean topology and it is regular on $X_{reg}(\mathbb{C})$.

Remark. Let $x \in X$, we write $\mathcal{O}_{X,x} := \mathbb{C}[X]_{\mathfrak{p}_x}$ the ring of functions which are regular at x . If X is irreducible and W is a subvariety of X , we write $\mathcal{O}_X(W) := \bigcap_{x \in W} \mathcal{O}_{X,x}$.

Proof: We assume X irreducible thanks to Lemma 2.3.4. Let $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ be regular on the Z -dense Z -open set $U(\mathbb{C})$ and continuous on $X(\mathbb{C})$. Then there exists $q, p \in \mathbb{C}[X]$ such that $pf = q$ on $U(\mathbb{C})$. As a Z -dense Z -open set is dense for the Euclidean topology and $pf - q$ is continuous, we get $pf - q = 0$ on $X(\mathbb{C})$.

Let $x \in X_{reg}(\mathbb{C})$, we have to show $f \in \mathcal{O}_{X,x}$. If p is a unit in $\mathcal{O}_{X,x}$, then $f = q.p^{-1} \in \mathcal{O}_{X,x}$. Otherwise, since $x \in X_{reg}(\mathbb{C})$, the Auslander-Buchsbaum theorem states that $\mathcal{O}_{X,x}$ is a UFD. So, up to multiplying q by some unit elements of $\mathcal{O}_{X,x}$, we can write

$$p = \prod_{i=1}^n p_i^{s_i}$$

with p_i some prime elements of $\mathcal{O}_{X,x}$. We now consider a Z -open neighbourhood $W_1(\mathbb{C})$ of x such that $pf = q$ on $W_1(\mathbb{C})$ and p_1 is a prime element of $\mathcal{O}_X(W_1(\mathbb{C}))$. Thus q vanishes on $\mathcal{Z}(p_1)$ and by the Nullstellensatz we get that $\mathcal{J}(\mathcal{Z}(p_1)) = p_1 \mathcal{O}_X(W_1(\mathbb{C}))$, so $q \in p_1 \mathcal{O}_X(W_1(\mathbb{C}))$. Then there exists $q_1 \in \mathcal{O}_X(W_1(\mathbb{C}))$ such that $q = p_1 q_1$. We obtain

$$\prod_{i=2}^n p_i^{s_i} p_1^{s_1-1} f|_{W_1(\mathbb{C})} = q_1|_{W_1(\mathbb{C})}$$

In the case where $s_1 > 1$ we have $\mathcal{Z}(p_1) \subset \mathcal{Z}(q_1)$ so we can iterate the process and get $q_{s_1} \in \mathcal{O}_X(W_1(\mathbb{C}))$ such that

$$\prod_{i=2}^n p_i^{s_i} f|_{W_1(\mathbb{C})} = q_{s_1}|_{W_1(\mathbb{C})}$$

If we take $W_2(\mathbb{C})$ a Z -open neighbourhood of x on which p_2 is prime, we can repeat the previous argument on $W_1(\mathbb{C}) \cap W_2(\mathbb{C})$. Thus, by doing this n times, we can consider a Z -open neighbourhood $W_n(\mathbb{C})$ of x in $X(\mathbb{C})$ and $q_{\Sigma s_i} \in \mathcal{O}_X(W_n(\mathbb{C}))$ such that $f|_{W_n(\mathbb{C})} = q_{\Sigma s_i}|_{W_n(\mathbb{C})}$. Then we finally conclude that $f \in \mathcal{O}_{X,x}$.



Remark. Note that the Euclidean continuity is essential in the proof to apply the argument of density at the end of the first paragraph. In particular, the Zariski continuity used in [28] wouldn't be enough because it doesn't allow us in general to extend an equality which is true on a Z -dense set. As an example we can consider the function f defined on $\mathbb{A}^1(\mathbb{C})$ by

$$f = \begin{cases} \frac{1}{z} & \text{if } z \neq 0 \\ 0 & \text{else} \end{cases}$$

This function is Z -continuous because it is a bijection and the Z -open sets of $\mathbb{A}^1(\mathbb{C})$ are of the form $\mathbb{A}^1(\mathbb{C}) \setminus \{\text{finite nb of points}\}$. However, although $zf(z) = 1$ on the Z -dense Z -open set $\mathbb{A}^1(\mathbb{C}) \setminus \{0\}$, this equality does not extend on the whole space $\mathbb{A}^1(\mathbb{C})$.

2.3.1 Connection between continuous rational functions and seminormalization.

The goal of this subsection is to study the ring $\mathcal{K}^0(X(\mathbb{C}))$ for X a complex affine variety. The main result being Proposition 2.3.13 which tells us that this ring is in fact the coordinate ring of the seminormalization of X , in other words $\mathcal{K}^0(X(\mathbb{C})) = \mathbb{C}[X^+]$. To do this, we will see how the continuous rational functions behave when they are composed with finite morphisms of affine varieties.

As we have seen previously, the functions in $\mathcal{K}^0(X(\mathbb{C}))$ are regular on the regular points of $X(\mathbb{C})$. Thus, if X is normal, the singular locus is too thin for a continuous rational function not to be polynomial. This allows us to identify the ring $\mathcal{K}^0(X(\mathbb{C}))$ when X is normal.

Proposition 2.3.6

Let X be an affine normal variety. Then $\mathcal{K}^0(X(\mathbb{C})) = \mathcal{O}_X(X(\mathbb{C})) = \mathbb{C}[X]$.

Proof: First $\mathbb{C}[X] \subset \mathcal{K}^0(X(\mathbb{C}))$ is obvious. Conversely, let $f \in \mathcal{K}^0(X(\mathbb{C}))$. By the previous proposition we get $f \in \mathcal{O}_X(X_{reg}(\mathbb{C}))$. But since X is normal we have $\text{codim}(X_{sing}(\mathbb{C})) \geq 2$. Thus, by [19] p.124, there exists a function $\tilde{f} \in \mathcal{O}_X(X(\mathbb{C}))$ which coincides with f on $X_{reg}(\mathbb{C})$. As the function $f - \tilde{f}$ is continuous for the Euclidean topology and vanishes on $X_{reg}(\mathbb{C})$ which is a dense open set of $X(\mathbb{C})$, we get $f = \tilde{f} \in \mathcal{O}_X(X(\mathbb{C})) = \mathbb{C}[X]$. ■

In the following proposition, we improve the result of Proposition 2.3.5. We write $\text{Norm}(X) := \{x \in X \mid \mathcal{O}_{X,x} \text{ is integrally closed}\}$ and $\text{Norm}(X(\mathbb{C})) = \text{Norm}(X) \cap X(\mathbb{C})$.

Proposition 2.3.7

Let X be an affine variety and $f \in \mathcal{K}^0(X(\mathbb{C}))$. Then

$$\forall x \in \text{Norm}(X(\mathbb{C})) \quad f \in \mathcal{O}_{X,x}$$

Proof: Let $\pi' : X' \rightarrow X$ be the normalization morphism of X and $f \in \mathcal{K}^0(X(\mathbb{C}))$. Proposition 2.3.6 tells us that $f \circ \pi' \in \mathbb{C}[X']$. Let $x \in \text{Norm}(X(\mathbb{C}))$, then there exists a unique $x' \in X'$ such that $\pi'(x') = x$ and so $\mathcal{O}_{X,x} \hookrightarrow$

$\mathcal{O}_{X',x'}$. Since normalization commutes with localization (one can see [5] for example), we get $\mathcal{O}'_{X,x} \simeq \mathcal{O}_{X',x'}$. But $x \in \text{Norm}(X(\mathbb{C}))$ implies $\mathcal{O}_{X,x} = \mathcal{O}'_{X,x}$ so

$$\mathcal{O}_{X,x} \simeq \mathcal{O}_{X',x'}$$

Finally, since $f \circ \pi' \in \mathcal{O}_{X',x'}$, we get $f \in \mathcal{O}_{X,x}$. ■

Remark. A UFD being integrally closed, we have $X_{\text{reg}} \subset \text{Norm}(X)$. Therefore Proposition 2.3.7 implies Proposition 2.3.5.

Before continuing the study of continuous rational functions, we have to establish some properties of finite morphisms that we will need later.

Lemma 2.3.8

Let $\pi : Y \rightarrow X$ be a finite morphism of affine varieties. Then

$\pi_c : Y(\mathbb{C}) \rightarrow X(\mathbb{C})$ is surjective and closed for the Euclidean topology.

Proof: Proposition 2.1.5 tells us that the morphism π is surjective and that the inverse image of $X(\mathbb{C})$ is $Y(\mathbb{C})$. Hence $\pi_c : Y(\mathbb{C}) \rightarrow X(\mathbb{C})$ is surjective.

To show that π_c is closed, we are going to prove that it is proper for the Euclidean topology. Let K be a compact subset of $X(\mathbb{C})$. We first have that $\pi_c^{-1}(K)$ is closed because π_c is continuous. Suppose $\mathbb{C}[Y] = \mathbb{C}[y_1, \dots, y_n]/I_Y$ and $Y(\mathbb{C}) \subset \mathbb{C}^n$. We write $Y_i : y \mapsto y_i$ the projection giving the i^{th} coordinate of an element of $Y(\mathbb{C})$. We then have $Y_i \in \mathbb{C}[Y]$ and, since by assumption $\mathbb{C}[Y]$ is a finite $\mathbb{C}[X]$ -module, there exists an identity of the form

$$Y_i^k + a_{k-1} \circ \pi_c \cdot Y_i^{k-1} + \dots + a_0 \circ \pi_c = 0 \text{ with } a_i \in \mathbb{C}[X]$$

Let $y \in \pi_c^{-1}(K)$. We write $\pi_c(y) = x$. If $y_i \neq 0$, then

$$\begin{aligned} Y_i^k(y) + a_{k-1} \circ \pi_c(y) \cdot Y_i^{k-1}(y) + \dots + a_0 \circ \pi_c(y) &= 0 \\ \implies y_i^k + a_{k-1}(x)y_i^{k-1} + \dots + a_0(x) &= 0 \\ \implies 1 + a_{k-1}(x)/y_i + \dots + a_0(x)/y_i^k &= 0 \end{aligned}$$

As K is a compact set, the $a_j(K)$ are bounded. So

$$\forall (y_n)_n \in \pi_c^{-1}(K)^{\mathbb{N}} \quad Y_i(y_n) \rightarrow +\infty$$

This means that $Y_i(\pi_c^{-1}(K))$ is bounded for all i and so that $\pi_c^{-1}(K)$ is a compact set. More generally, we have shown :

$$\pi_c : Y(\mathbb{C}) \rightarrow X(\mathbb{C}) \text{ is proper for the Euclidean topology.}$$

Since a proper continuous map is closed, the lemma is proved. ■

Remark. What we have just shown implies that for every finite morphism $\pi : Y \rightarrow X$ of affine varieties with π_c bijective, the morphism π_c is a Euclidean homeomorphism.

Henceforth, for a given morphism $\pi : Y \rightarrow X$ of affine varieties, we shall write $\varphi : \mathcal{K}^0(X(\mathbb{C})) \rightarrow \mathcal{C}(Y(\mathbb{C}), \mathbb{C})$ the map $f \mapsto f \circ \pi_c$. The purpose of this notation is to distinguish this map from the

morphism $\pi^* : \mathbb{C}[X] \rightarrow \mathbb{C}[Y]$. We will see that if π is a finite morphism we can determine the image of φ . This will be useful for us since the normalization and seminormalization morphisms are finite.

Lemma 2.3.9

Let $\pi : Y \rightarrow X$ be a surjective morphism of affine varieties. Then φ is injective and

$$\text{Im}(\varphi) \subset \left\{ f \in \mathcal{K}^0(Y(\mathbb{C})) \text{ with } f \text{ constant on the fibers of } \pi_{\mathbb{C}} \right\}$$

Proof: If $f \in \mathcal{K}^0(X(\mathbb{C}))$ then we can consider $p, q \in \mathbb{C}[X]$ with q a non-zero-divisor such that $q \cdot f - p = 0$ on $X(\mathbb{C})$. Write $Y = \bigcup_{i=1}^n Y_i$ the decomposition in irreducible component of Y . Then, for all $i \in \{1, \dots, n\}$, we get $q \circ \pi_{|Y_i(\mathbb{C})} \cdot f \circ \pi_{|Y_i(\mathbb{C})} - p \circ \pi_{|Y_i(\mathbb{C})} = 0$ on $Y_i(\mathbb{C})$. Thus $f \circ \pi_{|Y_i(\mathbb{C})} \in \mathcal{K}(Y_i)$ and since $\pi_{\mathbb{C}}$ is continuous, we have $f \circ \pi_{|Y_i(\mathbb{C})} \in \mathcal{K}^0(Y_i(\mathbb{C}))$. By Lemma 2.3.4, we get $f \circ \pi_{\mathbb{C}} \in \mathcal{K}^0(Y(\mathbb{C}))$. Moreover

$$\forall y_1, y_2 \in Y(\mathbb{C}) \quad \pi_{\mathbb{C}}(y_1) = \pi_{\mathbb{C}}(y_2) \implies f \circ \pi_{\mathbb{C}}(y_1) = f \circ \pi_{\mathbb{C}}(y_2)$$

which shows the lemma's inclusion. It remains to prove the injectivity of φ :

Let $f \in \mathcal{K}^0(X(\mathbb{C}))$ be such that $\varphi(f) = f \circ \pi_{\mathbb{C}} = 0$ and take $x \in X(\mathbb{C})$. As π is surjective, there exists $y \in Y(\mathbb{C})$ such that $\pi(y) = x$. So we can write $f(x) = f \circ \pi_{\mathbb{C}}(y) = 0$. Thus we get $f = 0$ which shows that φ is injective. ■

We deduce from Lemma 2.3.9 that continuous rational functions on an affine variety can be seen as polynomial functions on its normalization.

Proposition 2.3.10

Let X be an affine variety and $f \in \mathcal{K}^0(X(\mathbb{C}))$. Then f is integral on $\mathbb{C}[X]$.

Proof: Let $f \in \mathcal{K}^0(X(\mathbb{C}))$ and $\pi : X' \rightarrow X$ be the normalization morphism of X . Since π is a finite morphism, the previous lemma gives us $\mathcal{K}^0(X(\mathbb{C})) \hookrightarrow \mathcal{K}^0(X'(\mathbb{C}))$. But, by Proposition 2.3.6, we get $\mathcal{K}^0(X'(\mathbb{C})) = \mathbb{C}[X']$ so $\mathcal{K}^0(X(\mathbb{C})) \hookrightarrow \mathbb{C}[X']$ and so $f \circ \pi_{\mathbb{C}} \in \mathbb{C}[X']$. By definition of $\mathbb{C}[X']$, we can consider a relation of the form $(f \circ \pi_{\mathbb{C}})^n + (a_{n-1} \circ \pi_{\mathbb{C}})(f \circ \pi_{\mathbb{C}})^{n-1} + \dots + (a_0 \circ \pi_{\mathbb{C}}) = 0$ on $X'(\mathbb{C})$ with $a_0, \dots, a_{n-1} \in \mathbb{C}[X]$. Since $\pi_{\mathbb{C}}$ is surjective, we get that $f^n + a_{n-1}f^{n-1} + \dots + a_0 = 0$ on $X(\mathbb{C})$. Hence f is integral on $\mathbb{C}[X]$. ■

Remark. In general, a function $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ is integral on $\mathbb{C}[X]$ if and only if it is rational and locally bounded on $X(\mathbb{C})$.

We obtain the following commutative diagram which sums up the situation :

$$\begin{array}{ccc} \mathcal{K}^0(X(\mathbb{C})) & \xhookrightarrow{\varphi} & \mathcal{K}^0(X'(\mathbb{C})) \\ \uparrow & & \parallel \\ \mathbb{C}[X] & \xhookrightarrow{\quad} & \mathbb{C}[X'] \end{array}$$

As previously announced, we are going to give a description of the image of φ in the case π is finite.

Proposition 2.3.11

Let $\pi : Y \rightarrow X$ be a finite morphism of affine varieties. Then the image of $\varphi : \mathcal{K}^0(X(\mathbb{C})) \rightarrow \mathcal{K}^0(Y(\mathbb{C}))$ is

$$\text{Im}(\varphi) = \left\{ f \in \mathcal{K}^0(Y(\mathbb{C})) \text{ with } f \text{ constant on the fibers of } \pi_c \right\}$$

Proof: Let $f \in \mathcal{K}^0(Y(\mathbb{C}))$ be such that

$$\forall y_1, y_2 \in Y(\mathbb{C}) \quad \pi_c(y_1) = \pi_c(y_2) \implies f(y_1) = f(y_2)$$

We consider

$$\begin{aligned} g : X(\mathbb{C}) &\longrightarrow \mathbb{C} \\ x &\mapsto f(y_i) \quad \text{with } y_i \in \pi_c^{-1}(\{x\}) \end{aligned}$$

The map is well defined by assumption and because π is surjective. Moreover we have $f = g \circ \pi_c$ and $f \in \mathcal{K}^0(Y(\mathbb{C}), \mathbb{C})$ so $g \circ \pi_c \in \mathcal{C}^0(Y(\mathbb{C}), \mathbb{C})$.

We now show that $g \in \mathcal{C}^0(X(\mathbb{C}), \mathbb{C})$. Let F be a closed subset of \mathbb{C} . Then $(g \circ \pi_c)^{-1}(F) = \pi_c^{-1}(g^{-1}(F))$ is closed because $g \circ \pi_c = f$ is continuous and $\pi_c(\pi_c^{-1}(g^{-1}(F))) = g^{-1}(F)$ because π_c is surjective. Thus, since π_c is closed (see lemma 2.3.9), the set $g^{-1}(F)$ is closed and so g is continuous.

It remains to prove that g is a rational function. We have

$$\begin{aligned} g \circ \pi_c \in \mathcal{K}^0(Y(\mathbb{C})) &\implies g \circ \pi_c \text{ is integral on } \mathbb{C}[Y] && \text{by Proposition 2.3.10} \\ &\implies g \circ \pi_c \text{ is integral on } \mathbb{C}[X] && \text{by [5] Corollary 5.4 p.60} \\ &\implies g \text{ is integral on } \mathbb{C}[X] && \text{because } \pi \text{ is surjective} \\ &\implies g \in \mathcal{K}^0(X(\mathbb{C})) && \text{by Proposition 2.3.18 that we prove further away.} \end{aligned}$$

This shows $f \in \text{Im}(\varphi)$. So we have proved that the set of functions in $\mathcal{K}^0(Y(\mathbb{C}))$ which are constant on the fibers of π_c is included in $\text{Im}(\varphi)$. The reverse inclusion being given by Lemma 2.3.9, we finally get

$$\text{Im}(\varphi) = \{ f \in \mathcal{K}^0(Y(\mathbb{C})) \text{ with } f \text{ constant on the fibers of } \pi_c \}$$

■

Now, considering the last proposition, it is natural to wonder what happens when there is only one element in every fiber of π_c . The answer is given in the following proposition.

Proposition 2.3.12

Let $\pi : Y \rightarrow X$ be a finite morphism of affine varieties. Then the following properties are equivalent :

- 1) The extension $\pi^* : \mathbb{C}[X] \hookrightarrow \mathbb{C}[Y]$ is subintegral.
- 2) The morphism $\varphi : \mathcal{K}^0(X(\mathbb{C})) \rightarrow \mathcal{K}^0(Y(\mathbb{C}))$ is an isomorphism.
- 3) The morphism π_c is a homeomorphism for the Euclidean topology.

Proof: Suppose $\mathbb{C}[X] \hookrightarrow \mathbb{C}[Y]$ is subintegral, by Theorem 2.2.1 it means that $\pi_c : Y(\mathbb{C}) \rightarrow X(\mathbb{C})$ is bijective. So, in this case, every function of $\mathcal{K}^0(Y(\mathbb{C}))$ is clearly constant on the fibers of π_c . So, by Proposition 2.3.11, the map φ is surjective and Lemma 2.3.9 gives us the injectivity.

Conversely, if π_c is not bijective, there exists $y_1 \neq y_2 \in Y(\mathbb{C})$ such that $\pi(y_1) = \pi(y_2)$ and we can find $f \in \mathbb{C}[Y]$ such that $f(y_1) \neq f(y_2)$. Thus $f \in \mathcal{K}^0(Y(\mathbb{C}))$ but $f \notin \text{Im}(\varphi)$.

Finally

$$\begin{aligned} \varphi : \mathcal{K}^0(X(\mathbb{C})) \rightarrow \mathcal{K}^0(Y(\mathbb{C})) \text{ is an isomorphism} &\iff \pi_c : Y(\mathbb{C}) \rightarrow X(\mathbb{C}) \text{ is bijective} \\ &\iff \pi^* : \mathbb{C}[X] \hookrightarrow \mathbb{C}[Y] \text{ subintegral} \end{aligned}$$

The third statement comes from Lemma 2.3.8 ■

Let X be an affine variety, the coordinate ring of X^+ is the largest ring subintegral over $\mathbb{C}[X]$. Thus for every finite morphism between two varieties $\pi : Y \rightarrow X$ with $\mathbb{C}[X] \hookrightarrow \mathbb{C}[Y]$ subintegral, we obtain the following commutative diagram :

$$\begin{array}{ccccccc} \mathcal{K}^0(X(\mathbb{C})) & \xrightarrow{\simeq} & \mathcal{K}^0(Y(\mathbb{C})) & \xrightarrow{\simeq} & \mathcal{K}^0(X^+(\mathbb{C})) & \hookrightarrow & \mathcal{K}^0(X'(\mathbb{C})) \\ \uparrow & & \uparrow & & \uparrow & & \parallel \\ \mathbb{C}[X] & \xrightarrow{\text{subint.}} & \mathbb{C}[Y] & \xrightarrow{\text{subint.}} & \mathbb{C}[X^+] & \hookrightarrow & \mathbb{C}[X'] \end{array}$$

We will complete the diagram by showing $\mathbb{C}[X^+] = \mathcal{K}^0(X^+(\mathbb{C}))$. We prove it in the next theorem by saying that the polynomial functions on the seminormalization are the polynomial functions on the normalization which are constant on the fibers of $\pi'_c : X'(\mathbb{C}) \rightarrow X(\mathbb{C})$.

Theorem 2.3.13

Let X be an affine complex variety and $\pi^+ : X^+ \rightarrow X$ be the seminormalization morphism. We have the following isomorphism

$$\begin{aligned} \varphi : \mathcal{K}^0(X(\mathbb{C})) &\xrightarrow{\simeq} \mathbb{C}[X^+] \\ f &\mapsto f \circ \pi_c^+ \end{aligned}$$

Proof: We have shown in Corollary 2.2.7 that

$$\mathbb{C}[X^+] = \mathbb{C}[X^{+max}] = \{f \in \mathbb{C}[X'] / \forall x \in X(\mathbb{C}) f_x \in \mathcal{O}_{X,x} + \text{Rad}(\mathcal{O}_{X',x})\}$$

Let $\pi' : X' \rightarrow X$ be the normalization morphism of X . We want to show

$$\mathbb{C}[X^{+max}] = \{f \in \mathbb{C}[X'] / f \text{ constant on the fibers of } \pi'_c\}$$

Let $x \in X(\mathbb{C})$ and $f \in \mathbb{C}[X^{+max}]$. We write $\pi_c'^{-1}(\{x\}) = \{x'_1, \dots, x'_n\}$. The goal is to show that $f(x'_i) = f(x'_j)$ for all $i, j \in \{1, \dots, n\}$. First, the ideals of $\mathcal{O}_{X',x}$ above \mathfrak{m}_x are of the form $\mathfrak{m}_{x'_i} \mathcal{O}_{X',x}$:

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \hookrightarrow & \mathcal{O}_{X',x} \\ \mathfrak{m}_x \mathcal{O}_{X,x} & \hookleftarrow & \mathfrak{m}_{x'_1} \mathcal{O}_{X',x} \\ & & \vdots \\ & & \mathfrak{m}_{x'_n} \mathcal{O}_{X',x} \end{array}$$

By definition, we have $f_x \in \mathcal{O}_{X,x} + \text{Rad}(\mathcal{O}_{X',x})$. So we can write $f_x = \alpha + \beta$ with $\alpha \in \mathcal{O}_{X,x} \subset \mathcal{O}_{X',x}$ and $\beta \in \mathfrak{m}_{x'_1} \mathcal{O}_{X',x} \cap \dots \cap \mathfrak{m}_{x'_n} \mathcal{O}_{X',x}$. Thus, for all $i \in \{1, \dots, n\}$

$$f_x(x'_i) = \alpha(\pi'(x'_i)) + \beta(x'_i)$$

But $\beta(x'_i) = 0$ because $\beta \in \mathfrak{m}_{x'_i} \mathcal{O}_{X',x}$ and $\alpha(\pi'(x'_i)) = \alpha(x)$. So $\alpha(x) = f(x'_1) = \dots = f(x'_n)$ and we obtain

$$\mathbb{C}[X^{+max}] \subset \{f \in \mathbb{C}[X'] / f \text{ constant on the fibers of } \pi'_c\}.$$

Conversely, let $f \in \mathbb{C}[X']$ be constant on the fibers of π'_c . Let $x \in X(\mathbb{C})$, then for all $y \in \pi'^{-1}_c(x)$, we have $f(y) = \alpha \in \mathbb{C}$. We get that

$$f_x - \alpha \in \bigcap_{y \in \pi'^{-1}_c(x)} \mathfrak{m}_y \mathcal{O}_{X',x} = \text{Rad}(\mathcal{O}_{X',x})$$

and so $f \in \mathbb{C}[X^{+max}]$. We have proved

$$\mathbb{C}[X^{+max}] = \{f \in \mathbb{C}[X'] / f \text{ constant on the fibers of } \pi'_c\}.$$

But since $\mathcal{K}^0(X'(\mathbb{C})) = \mathbb{C}[X']$ by Proposition 2.3.6 and since

$$\varphi : \mathcal{K}^0(X(\mathbb{C})) \xrightarrow{\sim} \{f \in \mathcal{K}^0(X'(\mathbb{C})) / f \text{ constant on the fibers of } \pi'_c\}$$

by Proposition 2.3.11, we get

$$\varphi : \mathcal{K}^0(X(\mathbb{C})) \xrightarrow{\sim} \mathbb{C}[X^{+max}] = \mathbb{C}[X^+]. \quad \blacksquare$$

We have managed to identify the ring of continuous rational functions of an affine complex variety : it corresponds to the coordinate ring of its seminormalization. We can now complete the previous diagram. For every morphism $\pi : Y \rightarrow X$ of affine varieties such that $\mathbb{C}[X] \hookrightarrow \mathbb{C}[Y]$ is subintegral, we get

$$\begin{array}{ccccccc} \mathcal{K}^0(X(\mathbb{C})) & \xrightarrow{\simeq} & \mathcal{K}^0(Y(\mathbb{C})) & \xrightarrow{\simeq} & \mathcal{K}^0(X^+(\mathbb{C})) & \hookrightarrow & \mathcal{K}^0(X'(\mathbb{C})) \\ \uparrow & & \uparrow & & \parallel & & \parallel \\ \mathbb{C}[X] & \xrightarrow{\text{subint.}} & \mathbb{C}[Y] & \xrightarrow{\text{subint.}} & \mathbb{C}[X^+] & \hookrightarrow & \mathbb{C}[X'] \end{array}$$

2.3.2 Continuous rational functions and regulous functions.

As explained before, continuous rational functions are of particular interest in real algebraic geometry. They are very related to an other kind of functions : the *regulous* functions. Let X be a real algebraic set and let $f : X \rightarrow \mathbb{R}$ be continuous. We say that f is regulous on X if for every algebraic subset $Z \subset X$, the restriction $f|_Z$ has a rational representation. Those two types of functions are not the same in the case of singular real algebraic sets. One can consider the following example (from [22]) of a continuous rational function which is not regulous.

Example. Let $X := \{x^3 - (1 + z^2)y^3\} \subset \mathbb{R}^3$ be a real algebraic set. Consider $f : X \rightarrow \mathbb{R}$ such that $f(x, y, z) = (1 + z^2)^{\frac{1}{3}}$. See that, although f is continuous on X and $f = x/y$ if $y \neq 0$, the function f is not rational on $\{y = 0\}$.

For more details, we refer the reader to [25] section 3 or Chapter 1 Section 1.2.3. We show, in the following proposition, that these two kinds of functions are the same in complex algebraic geometry.

Proposition 2.3.14

Let $f \in \mathcal{K}^0(X(\mathbb{C}))$. Then for every subvariety $V \subset X$, we have

$$f|_{V(\mathbb{C})} \in \mathcal{K}^0(V(\mathbb{C}))$$

Proof: As usual, Lemma 2.3.4 allows us to suppose that V is irreducible. Denote by $\pi^+ : X^+ \rightarrow X$ the seminormalization morphism of X and take $V = \mathcal{V}(\mathfrak{p})$ an irreducible subvariety of X . Like in Proposition 2.2.3, we have the following commutative diagram :

$$\begin{array}{ccc} \mathbb{C}[X] & \xrightarrow{(\pi^+)^*} & \mathbb{C}[X^+] \\ \downarrow & & \downarrow \\ \mathbb{C}[V] & \xrightarrow{(\pi^+|_V)^*} & \mathbb{C}[W] \\ \downarrow & & \downarrow \\ \kappa(V) & \xlongequal{\quad} & \kappa(W), \end{array}$$

with $W = \mathcal{V}(\mathfrak{q})$ where \mathfrak{q} is the unique prime ideal of $\mathbb{C}[X^+]$ above \mathfrak{p} . By the going-up property and the description of prime ideals for quotient rings, one can see that $W \rightarrow V$ is a bijection. Thus Theorem 2.2.1 tells us that $\mathbb{C}[V] \hookrightarrow \mathbb{C}[W]$ is subintegral. Since $\mathbb{C}[W]$ is a finite $\mathbb{C}[V]$ -module, we can apply Proposition 2.3.12 and get that $\mathcal{K}^0(V) \hookrightarrow \mathcal{K}^0(W)$ is an isomorphism. Thus

$$\begin{aligned} f \in \mathcal{K}^0(X(\mathbb{C})) &\implies f \circ \pi_c^+ \in \mathcal{O}_{X^+}(X^+(\mathbb{C})) \implies f \circ \pi_{|W(\mathbb{C})}^+ \in \mathcal{O}_{X^+}(W(\mathbb{C})) \\ &\implies f_{|V(\mathbb{C})} \circ \pi_{|W(\mathbb{C})}^+ = f \circ \pi_{|W(\mathbb{C})}^+ \in \mathcal{K}^0(W(\mathbb{C})) \implies f_{|V(\mathbb{C})} \in \mathcal{K}^0(V(\mathbb{C})) \end{aligned}$$

Remark. In real algebraic geometry, regulous functions can also be defined in the following equivalent way. Let $f : X \rightarrow \mathbb{R}$ be a continuous function on a real algebraic set. We say that f is *regulous* (or sometimes *stratified-regular*) if there exists a finite stratification \mathcal{S} of X , with Zariski locally closed (i.e. the intersection of a closed and an open set) strata such that for all $S \in \mathcal{S}$ the restriction $f|_S$ is regular. It also applies in our case, if $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ is a continuous rational function, then we can write

$$f = \begin{cases} p_1/q_1 & \text{if } q_1 \neq 0 \\ p_2/q_2 & \text{if } q_1 = 0 \text{ and } q_2 \neq 0 \\ p_3/q_3 & \text{if } q_1 = q_2 = 0 \text{ and } q_3 \neq 0 \\ \dots & \end{cases}$$

and for every $n > 1$, we have $\mathcal{Z}(q_n) \subset \text{Sing}(\mathcal{Z}(q_{n-1}))$.

2.3.3 The ring of continuous rational functions seen as an integral closure and algebraic Whitney theorem.

In Whitney's book [41], one can find a chapter dedicated to a certain type of functions : the "c-holomorphic" functions. They are defined as continuous functions on an analytic variety which are holomorphic on the smooth points of the variety. Note that, by Proposition 2.3.5, continuous rational functions are c-holomorphic. A characterization of c-holomorphic functions, given by Whitney, is that a continuous function is c-holomorphic if and only if it has an analytic graph. This theorem naturally leads to wonder if we can have the same characterization for continuous rational functions. In other words, do we have, on an affine algebraic variety, that a continuous function is rational if and only if its graph is algebraically

closed ? The answer is yes and a proof with arguments from analytic geometry can be found in [8]. The goal of this section is to prove a slightly stronger version with arguments from algebraic geometry. More precisely we aim to show that if X is an affine variety then every continuous function from $X(\mathbb{C})$ to \mathbb{C} , for which there exists $P(t) \in \mathbb{C}[X][t]$ such that $P(f) = 0$, is rational. It allows us to deduce the algebraic version of Whitney’s theorem discussed above but also to identify the ring in which $\mathcal{K}^0(X(\mathbb{C}))$ is the integral closure of $\mathbb{C}[X]$.

We start by proving the theorem in the case where X is irreducible and where the polynomial, for which the continuous function is a root, is irreducible in $\mathcal{K}(X)[t]$. With those assumptions we can give a proof similar to the one given in [36] (theorem 8.4 p.176). It is very important for the polynomial to be irreducible in $\mathcal{K}(X)[t]$ otherwise the new variety created from it won’t necessarily be irreducible, whereas the key argument uses the irreducibility of this new variety.

Notation. Let P be a polynomial, we note $\text{disc}(P)$ its discriminant.

Lemma 2.3.15

Let X be an irreducible affine variety and $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ be a continuous function. Suppose there exists an irreducible polynomial $P \in \mathcal{K}(X)[t]$ such that

$$\exists U \text{ a } Z\text{-open set} \quad \forall x \in U(\mathbb{C}) \quad P(x, f(x)) = 0$$

then

$$f \in \mathcal{K}^0(X(\mathbb{C}))$$

Proof: First we consider the affine Z -open set X_1 such that P is a monic polynomial of $\mathbb{C}[X_1][t]$. Then we write $Y_1 = \text{Spec}(\mathbb{C}[X_1][t]/\langle P \rangle)$, which is irreducible because P is irreducible in $\mathcal{K}(X)[t] = \mathcal{K}(X_1)[t]$, and $\pi : Y_1 \rightarrow X_1$ the induced finite morphism. We note X_2 the affine Z -open set where $\text{disc}(P)$ does not vanish. Finally we write $Y_2 = \pi^{-1}(X_2)$. Now X_2 and Y_2 are two irreducible affine varieties with $\pi : Y_2 \rightarrow X_2$ finite and

$$\forall x \in X_2(\mathbb{C}) \quad \#\pi_c^{-1}(x) = [\mathcal{K}(Y_2) : \mathcal{K}(X_2)] = \text{deg}(P).$$

We write $m := \text{deg}(P)$ and we prove by contradiction that $m = 1$. Let’s suppose $m > 1$.

Let $x \in X_2(\mathbb{C})$, we can consider U_x a Euclidean open set such that $X_2(\mathbb{C}) \cap U_x$ is connected and

$$\pi^{-1}(X_2(\mathbb{C}) \cap U_x) = \bigsqcup_{i=1}^m V_x^i \text{ where } V_x^i \subset Y_2(\mathbb{C}) \text{ are two by two disjoint connected open sets}$$

We note

$$\begin{aligned} \varphi : X_2(\mathbb{C}) &\rightarrow Y_2(\mathbb{C}) \\ x &\mapsto (x; f(x)) \end{aligned}$$

which is, by assumption on f , a continuous section of π . Thus $\varphi(X_2(\mathbb{C}) \cap U_x)$ is connected and so it corresponds to one of the V_x^i which we denote $V_x^{i_0}$. We then have that $\varphi(X_2(\mathbb{C}))$ and $Y_2(\mathbb{C}) \setminus \varphi(X_2(\mathbb{C}))$ are open sets because

$$\varphi(X_2(\mathbb{C})) = \bigcup_{x \in X_2(\mathbb{C})} V_x^{i_0}$$

and

$$Y_2(\mathbb{C}) \setminus \varphi(X_2(\mathbb{C})) = \bigcup_{x \in X_2(\mathbb{C})} \bigsqcup_{i \neq i_0} V_x^i$$

Moreover, since $m > 1$, we clearly have $\varphi(X_2(\mathbb{C})) \neq Y_2(\mathbb{C})$. But since Y_2 is irreducible, the set $Y_2(\mathbb{C})$ must be connected, a contradiction. So m must be equal to 1 and then f is rational. ■

In order to prove the desired theorem with Lemma 2.3.15, we need to find an irreducible polynomial of $\mathcal{K}(X)[t]$ for which the continuous function is a root. This is what Lemma 2.3.16 gives us.

Lemma 2.3.16

Let X be an affine irreducible smooth variety and $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ be continuous with respect to the Euclidean topology. Suppose there exists a monic polynomial $P \in \mathbb{C}[X][t]$ such that

$$\forall x \in X(\mathbb{C}) \quad P(x, f(x)) = 0$$

Then f is a root of an irreducible polynomial of $\mathcal{K}(X)[t]$.

Proof: Since X is supposed to be smooth and P to be a monic polynomial, we can apply [41] Lemma 2J Chapter 4 to get that f is holomorphic on $X(\mathbb{C})$. In particular, we get $f \in \mathcal{M}(X(\mathbb{C}))$ which is a field because X is irreducible (see [36] Theorem 7.1). Now consider the morphism

$$\begin{aligned} ev_f : \mathcal{K}(X)[t] &\rightarrow \mathcal{M}(X(\mathbb{C})) \\ Q(t) &\mapsto Q(f) \end{aligned}$$

We have that $\mathcal{K}(X)[t]$ is a principal ideal ring because $\mathcal{K}(X)$ is a field. Since $P(f) = 0$ in $\mathcal{M}(X(\mathbb{C}))$, then $\ker(ev_f) \neq 0$. So there exists $F \neq 0$ such that $\ker(ev_f) = \langle F \rangle$. Since $\mathcal{M}(X(\mathbb{C}))$ is a domain, the polynomial F is irreducible in $\mathcal{K}(X)[t]$. ■

We now have all the arguments we need to demonstrate the main theorem of this section.

Theorem 2.3.17

Let X be an affine variety and $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ be a continuous function for the Euclidean topology. Suppose there exists a polynomial $P \in \mathbb{C}[X][t]$ which is non-zero on each irreducible component of $X(\mathbb{C})$ and such that

$$\forall x \in X(\mathbb{C}) \quad P(x, f(x)) = 0$$

then

$$f \in \mathcal{K}^0(X(\mathbb{C}))$$

Proof: Let $X = \bigcup_{i=1}^n X_i$ be its decomposition into irreducible components. Let $i \in \{1, \dots, n\}$ then $f|_{X_i(\mathbb{C})}$ is a root of the polynomial P with its coefficients restricted to $X_i(\mathbb{C})$. Thus, by Lemma 2.3.4, it is enough to prove the theorem for an irreducible affine variety.

If $X_{sing}(\mathbb{C}) = \mathcal{Z}(\langle q_1, \dots, q_s \rangle)$ and a_n is the leading coefficient of P , we can replace X by $\mathcal{D}(q_1 a_n)$ and then suppose that X is smooth and that P is a monic polynomial. It allows us to use Lemma 2.3.16 and to get an

irreducible polynomial $F \in \mathcal{K}(X)[t]$ such that there exists a Z -open set U where for all $x \in U(\mathbb{C})$, $F(x, f(x)) = 0$. The conclusion is now given by Lemma 2.3.15. ■

Thanks to Theorem 2.3.17 we can now see the ring of continuous rational functions as an integral closure of $\mathbb{C}[X]$.

Corollary 2.3.18

Let X be an affine variety. Then

$$\mathcal{K}^0(X(\mathbb{C})) = \mathbb{C}[X]_{\mathcal{C}^0(X(\mathbb{C}), \mathbb{C})}'$$

Proof: The result follows from Proposition 2.3.10 and Theorem 2.3.17. ■

Remark. By using Corollary 2.3.18, one can give a very short proof of Proposition 2.3.14. Indeed, if V is a subvariety of X and if $f \in \mathcal{K}^0(X(\mathbb{C}))$, then there exists a monic polynomial in $\mathbb{C}[X][t]$ for which f is a root. So $f|_{V(\mathbb{C})}$ is a root of the same polynomial with its coefficients restricted to $V(\mathbb{C})$. Since $f|_{V(\mathbb{C})}$ is continuous, we get $f|_{V(\mathbb{C})} \in \mathbb{C}[V]_{\mathcal{C}^0(V(\mathbb{C}), \mathbb{C})}' = \mathcal{K}^0(V(\mathbb{C}))$.

Let's conclude this section by proving the algebraic version of Whitney's Theorem 4.5Q in [41] introduced at the beginning of this section.

Corollary 2.3.19

Let X be an affine variety and $f : X(\mathbb{C}) \rightarrow \mathbb{C}$ be a continuous function. We denote by Γ_f the graph $\{(x, f(x)) \mid x \in X(\mathbb{C})\} \subset X(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C})$ of f . Then the following properties are equivalent :

- 1) The graph Γ_f is Z -closed.
- 2) $f \in \mathcal{K}^0(X(\mathbb{C}))$

Proof: The implication 1) implies 2) comes from Theorem 2.3.17. Conversely, we suppose $f \in \mathcal{K}^0(X(\mathbb{C}))$ and we note $\pi : X^+ \rightarrow X$ the seminormalization morphism. By Proposition 2.3.13, we have $f \circ \pi \in \mathbb{C}[X^+]$. Thus

$$\Gamma_{f \circ \pi} = \{(y, f \circ \pi(y)) \mid y \in X^+(\mathbb{C})\}$$

is Z -closed. Moreover the map $\pi \times Id$ is Z -closed because, by Theorem 2.2.1, π is a Z -homeomorphism. So $\pi \times Id(\Gamma_{f \circ \pi}) = \{(\pi(y), f \circ \pi(y)); y \in X^+(\mathbb{C})\}$ is Z -closed. Finally, since π is bijective, we get

$$\{(\pi(y), f \circ \pi(y)) \mid y \in X^+(\mathbb{C})\} = \{(x, f(x)) \mid x \in X(\mathbb{C})\} = \Gamma_f$$

Remark. In [7] and [8], the authors consider c -holomorphic functions with an algebraic graph. Corollary 2.3.19 tells us that those functions are the same as the ones considered here when we work on algebraic varieties.

Remark. In real algebraic geometry, the zero sets of regulous functions are the closed sets of a thinner topology than the Zariski topology called the *regulous topology*. In [12], the authors show that we can recover some classical theorems of complex algebraic geometry if we work with the regulous topology

instead of the Zariski topology. In our case, if $f \in \mathcal{K}^0(X(\mathbb{C}))$ then Corollary 4.18 tells us that the set $\{x \in X(\mathbb{C}) \mid f(x) = 0\} = \Gamma_f \cap (X(\mathbb{C}) \times \{0\})$ is a Zariski closed set.

2.3.4 Examples of continuous rational functions.

In general, it is not easy to determine the seminormalization of a variety. We present in this subsection several examples of continuous rational functions and also some explicit seminormalizations of affine varieties. In order to do this we give a useful criterion to identify continuous rational functions.

Theorem 2.3.20

Let X be an affine variety and $f : X(\mathbb{C}) \rightarrow \mathbb{C}$. Then $f \in \mathcal{K}^0(X(\mathbb{C}))$ if and only if it verifies the following properties :

- 1) $f \in \mathcal{K}(X)$
- 2) There exists a monic polynomial $P(t) \in \mathbb{C}[X][t]$ such that $P(f) = 0$ on $X(\mathbb{C})$.
- 3) The graph Γ_f is Zariski closed in $X(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C})$.

Proof: The direct implication is given by Propositions 2.3.10 and 2.3.19. Conversely, suppose that f verifies the three properties above. We consider the map

$$\begin{aligned} \psi : \mathbb{C}[X][t] &\rightarrow \mathcal{K}(X) \\ Q(t) &\mapsto Q(f) \end{aligned}$$

and write $\mathbb{C}[Y] \simeq \mathbb{C}[X][t]/\ker \psi \simeq \mathbb{C}[X][f]$ with $\pi : Y \rightarrow X$ the morphism induced by $\mathbb{C}[X] \hookrightarrow \mathbb{C}[Y]$. We then have

$$\mathbb{C}[X] \hookrightarrow \mathbb{C}[Y] \simeq \mathbb{C}[X][f] \subset \mathcal{K}(X)$$

So $\mathcal{K}(X) \simeq \mathcal{K}(Y)$ and π is birational. Moreover $\mathbb{C}[Y]$ is a finite $\mathbb{C}[X]$ -module because so is $\mathbb{C}[X][t]/\langle P(t) \rangle$ and

$$\mathbb{C}[Y] \simeq \mathbb{C}[X][t]/\ker \psi \simeq (\mathbb{C}[X][t]/\langle P(t) \rangle)/(\ker \psi/\langle P(t) \rangle)$$

Hence $\pi : Y \rightarrow X$ is a finite birational morphism. We want to show that π_c is bijective. By assumption, there exists an ideal $I_f \subset \mathbb{C}[X][t]$ such that $\Gamma_f = \mathcal{Z}(I_f)$. We have $I_f \subset \ker \psi$ because $\forall Q \in I_f, \forall x \in X(\mathbb{C}), Q(x, f(x)) = 0$. So

$$Y(\mathbb{C}) = \mathcal{Z}(\ker \psi) \subset \mathcal{Z}(I_f) = \Gamma_f = \{(x, f(x)) \mid x \in X(\mathbb{C})\}$$

Then

$$\forall x \in X(\mathbb{C}), \pi_c^{-1}(x) = \emptyset \text{ or } \{f(x)\}.$$

Since π is finite, π_c is surjective. So, for all $x \in X(\mathbb{C})$, $\pi_c^{-1}(x)$ is not empty, which means that $\pi_c^{-1}(x) = \{f(x)\}$. We have shown $Y(\mathbb{C}) = \Gamma_f$ thus π_c is bijective with the inverse map $x \mapsto (x, f(x))$. Thus π_c is a finite birational and bijective morphism. From the universal property of the seminormalization, we get

$$\mathbb{C}[X] \hookrightarrow \mathbb{C}[Y] \hookrightarrow \mathbb{C}[X^+]$$

that induces

$$X^+ \xrightarrow{\pi_Y^+} Y \xrightarrow{\pi} X$$

So, if we note $t \in \mathbb{C}[Y]$ such that $t : (x, f(x)) \mapsto f(x)$, Theorem 2.3.13 gives us the existence of $g \in \mathcal{K}^0(X(\mathbb{C}))$ such that $t \circ (\pi_Y^+)_\mathbb{C} = g \circ \pi_c \circ (\pi_Y^+)_\mathbb{C}$ so $t = g \circ \pi_c$. Therefore, since π_c is surjective, we get for all $x \in X(\mathbb{C})$

$$g(x) = g \circ \pi_c(x; f(x)) = t(x; f(x)) = f(x)$$

Thus $f = g \in \mathcal{K}^0(X(\mathbb{C}))$ which concludes the proof. ■

Example. Let $V = \text{Spec}(\mathbb{C}[x, y] / \langle y^2 + (x^2 - 1)x^4 = 0 \rangle)$ and

$$f : V(\mathbb{C}) \rightarrow \mathbb{C} \\ (x; y) \mapsto \begin{cases} y/x & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$$

The function f is a root of the polynomial $P_{(x;y)}(t) = t^2 + x^2(x^2 - 1)$. Since $P_{(0;0)}(t) = t^2$, all the values of f are given by the roots of $xt - y = 0$ on $\{x \neq 0\}$ and else by the root of $P_{(0;0)}(t)$. Thus we have $\Gamma_f = \mathcal{Z}(\langle y^2 + (x^2 - 1)x^4; xt - y; t^2 + x^2(x^2 - 1) \rangle)$ which is a Zariski closed set. So Theorem 2.3.20 tells us that $f \in \mathcal{K}^0(V(\mathbb{C}))$. We have in fact $\Gamma_f = \mathbb{C}[X^+]$.

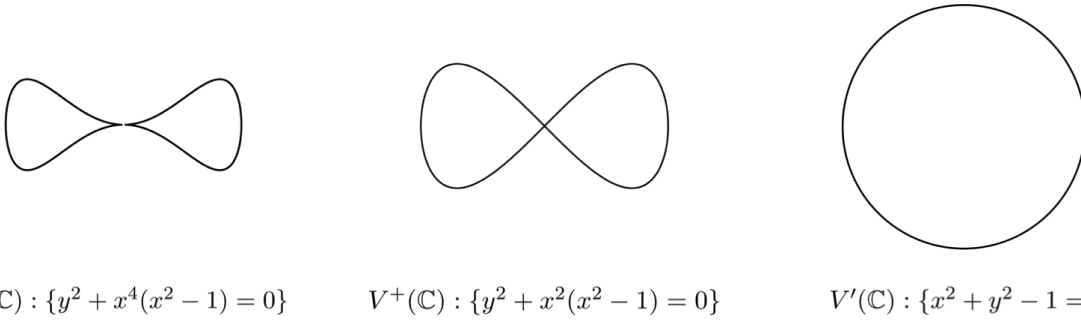


Figure 2.1 – Normalization and seminormalization of a bow tie

Remark. The key thing in the criterion we gave is that f is defined on all $V(\mathbb{C})$. When one add rational functions to the coordinate ring of a variety to get its normalization, the functions are only defined on a Z-open set. Consider again the previous example but with the fraction $\frac{y}{x^2} \in \mathcal{K}(V)$. It is a root of $P_{(x;y)}(t) = t^2 + (x^2 - 1)$ on $\{x \neq 0\}$. But $P_{(0;0)} = t^2 - 1$ as two distinct roots, so $\mathcal{Z}(\langle y^2 + (x^2 - 1)x^4; x^2t - y; t^2 + (x^2 - 1) \rangle)$ cannot be the graph of a map on $V(\mathbb{C})$.

Example. Now we give an example which illustrates the fact that a continuous rational function is a stratified-regular function (see remark after Proposition 2.3.14). Let V be a variety such that the set of its closed points, seen in $\mathbb{A}^4(\mathbb{C})$, is defined by the following equations

$$V(\mathbb{C}) : \begin{cases} x^2 + zyx + ty^2 = 0 & (1) \\ z^2 + z^2t + t^3 + yt = 0 & (2) \\ t^2x^2 + x^2y - y^2z^2 = 0 & (3) \end{cases}$$

Let $f : V(\mathbb{C}) \rightarrow \mathbb{C}$ such that

$$f = \begin{cases} x/y & \text{if } y \neq 0 \\ z/t & \text{if } y = 0 \text{ and } t \neq 0 \\ 0 & \text{else} \end{cases}$$

To show that f is indeed a continuous rational function on $V(\mathbb{C})$, we show that f satisfies the three properties of Theorem 2.3.20. In particular we look at its graph Γ_f and show that it is the following Z-closed set in $\mathbb{A}^5(\mathbb{C})$ defined by

$$\Gamma_f = \begin{cases} x^2 + zyx + ty^2 = 0 & (1) \\ z^2 + z^2t + t^3 + yt = 0 & (2) \\ t^2x^2 + x^2y - y^2z^2 = 0 & (3) \\ yX - x = 0 & (4) \\ X^2 + zX + t = 0 & (5) \\ t^2X^2 + xX - z^2 = 0 & (6) \end{cases}$$

First of all, let's verify that f is indeed a root of the polynomials (4), (5) and (6) on $V(\mathbb{C})$. We start by looking on $\mathcal{D}(y) \subset V(\mathbb{C})$ where $f = x/y$:

$$(4) : y \left(\frac{x}{y} \right) - x = 0$$

$$(5) : \left(\frac{x}{y} \right)^2 + z \left(\frac{x}{y} \right) + t = \frac{x^2 + zyx + ty^2}{y^2} = 0 \text{ by (1)}$$

$$(6) : t^2 \left(\frac{x}{y} \right)^2 + x \left(\frac{x}{y} \right) - z^2 = \frac{t^2x^2 + x^2y - y^2z^2}{y^2} = 0 \text{ by (3)}$$

Now we check that it is still true on $\mathcal{Z}(y) \cap \mathcal{D}(t)$:

$$(5) : \left(\frac{z}{t} \right)^2 + z \left(\frac{z}{t} \right) + t = \frac{z^2 + z^2t + t^3 + yt}{t^2} = 0 \text{ by (2)}$$

$$(6) : t^2 \left(\frac{z}{t} \right)^2 + x \left(\frac{z}{t} \right) - z^2 = \frac{t^2z^2 + xzt - z^2t^2}{t^2} = 0 \text{ since } y = 0 \text{ implies } x = 0 \text{ by (1)}.$$

We get that f is a root of the polynomials (4), (5) and (6). It remains to see if the values of f are completely determined by those polynomials.

If $y \neq 0$, then the equation (4) forces the value of f to be x/y on $\mathcal{D}(y)$. If $y = 0$ and $t \neq 0$, then the system (4), (5), (6) becomes $\begin{cases} X^2 + zX + t = 0 \\ X^2 = z^2/t^2 \end{cases}$ which forces the value of f to be z/t on $\mathcal{Z}(y) \cap \mathcal{D}(t)$.

Finally if $y = t = 0$, then the system becomes $X^2 = 0$.

We have shown that Γ_f is completely described by the system given above. Thus Γ_f is Z-closed, by (5) f is integral on $\mathbb{C}[X]$ and by (4) f is rational on $V(\mathbb{C})$. So Theorem 2.3.20 tells us that $f \in \mathcal{K}^0(V(\mathbb{C}))$.

Remark. The jacobian matrix of the equations defining V is

$$\text{Jac}(V) = \begin{pmatrix} 2x + yz & zx + 2ty & xy & y^2 \\ 0 & t & 2z + 2zt & z^2 + y \\ 2t^2x + 2yx & x^2 - 2yz^2 & -2y^2z & 2tx^2 \end{pmatrix}$$

and, if $y = 0$, it becomes :

$$\text{Jac}(V)|_{\{y=0\}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t & 2z + 2zt & z^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So we have $\{y = 0\} \subset V_{\text{sing}}(\mathbb{C})$ which is coherent with Proposition 2.3.5.

Remark. In the equations defining Γ_f , we could replace (6) by $(x - zt^2)X - (t^3 + z^2)$.

Example. It is shown in [10] that, for plane curves, the seminormality can be read on the geometry of the singularities. A curve in $\mathbb{A}^2(\mathbb{C})$ is seminormal if and only if its singularities are double points whose tangents are linearly independent. We illustrate this by looking at the example of three lines crossing at the origin in $\mathbb{A}^2(\mathbb{C})$.

Let $V = \text{Spec}(\mathbb{C}[x; y] / \langle xy(y - x) \rangle)$. Its closed points correspond to three lines in \mathbb{C}^2 crossing at the origin as represented in the left picture of Figure 2.2. It is clear that V is not seminormal because the lines are not linearly independents. Let $f : V(\mathbb{C}) \rightarrow \mathbb{C}$ be such that

$$f = \begin{cases} \frac{2xy}{x+y} & \text{if } (x; y) \neq (0; 0) \\ 0 & \text{else} \end{cases}$$

We can see that f is a root of the polynomial $P_{(x;y)}(t) = t^2 - xy$ and that Γ_f is equal to the Z -closed set $\mathcal{Z}(xy(y - x); (x + y)t - 2xy; t^2 - xy)$ because 0 is the only root of $P_{(0;0)}$. So, by Theorem 2.3.20, we have $f \in \mathcal{K}^0(V(\mathbb{C}))$. Furthermore, we have $\Gamma_f = V^+(\mathbb{C})$ because the graph corresponds to three linearly independent lines in $\mathbb{A}^3(\mathbb{C})$. Indeed it is the union of three lines crossing at the origin with direction vectors $(1,0,0)$, $(0,1,0)$ and $(1,1,1)$ as drawn in the right picture of Figure 2.2.

Another way to see that f is continuous is that $f|_{x=0} = 0$, $f|_{y=0} = 0$ and $f|_{x=y} = x$. So f is a continuous rational function on each irreducible component of $V(\mathbb{C})$. Thus $f \in \mathcal{K}^0(V(\mathbb{C}))$ by Lemma 2.3.4.

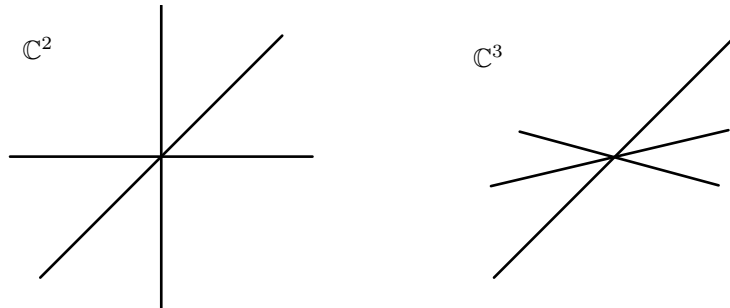


Figure 2.2 – The seminormalization of three lines of \mathbb{C}^2

Let X be an affine variety. Then the extension $\mathbb{C}[X] \hookrightarrow \mathbb{C}[X^+]$ is finite and $\mathbb{C}[X]$ is a noetherian ring. The next proposition shows that the process of adding elements $f_i \in \mathcal{K}^0(X(\mathbb{C}))$ with $f_{i+1} \notin \mathbb{C}[X][f_1, \dots, f_i]$ ends after a finite number of steps.

Proposition 2.3.21

Let $A \hookrightarrow B$ be an integral extension of rings with A noetherian and consider $(C_k)_{k \in \mathbb{N}}$ defined by

$$\begin{cases} C_{k+1} = C_k[f_k] \text{ with } f_k \in B \setminus C_k \\ C_0 = A \end{cases}$$

Then there exists $N \in \mathbb{N}$ such that $C_N = B$.

Proof: Since $A \hookrightarrow B$ is an integral extension, we know that $B = A[b_1, \dots, b_d]$ with all b_i being integral over A . So there exists $n \in \mathbb{N}$ such that for all $b \in B$, we have

$$b = \sum_{\underline{i} \in \{1, \dots, n\}^d} a_{\underline{i}}(b_1 \dots b_d)^{\underline{i}} \text{ with } a_{\underline{i}} \in A$$

Let $\underline{i} \in \{1, \dots, n\}^d$ and $k \in \mathbb{N}$. We consider the ideal

$$I_{\underline{i}}(C_k) := \{a \in A \mid \exists c \in C_k \text{ such that } c = a(b_1 \dots b_d)^{\underline{i}} + \text{terms with smaller exponent}\}$$

and we get $I_{\underline{i}}(C_k) \subseteq I_{\underline{i}}(C_{k+1})$. Now let us show that, for all $k \in \mathbb{N}$, we have $C_k = C_{k+1}$ if and only if $I_{\underline{i}}(C_k) = I_{\underline{i}}(C_{k+1})$, for all $\underline{i} \in \{1, \dots, n\}^d$. One way is trivial, we show the other one by considering an element $c_{k+1} \in C_{k+1} \setminus C_k$ while supposing that $I_{\underline{i}}(C_k) = I_{\underline{i}}(C_{k+1})$, for all $\underline{i} \in \{1, \dots, n\}^d$. Then $c_{k+1} = a_{\underline{i}}(b_1 \dots b_d)^{\underline{i}} + \sum$ smaller terms and there exists $c_k \in C_k$ such that $c_k = a_{\underline{i}}(b_1 \dots b_d)^{\underline{i}} + \sum$ smaller terms. So $\deg(c_{k+1} - c_k) < \underline{i}$ and one can see that $c_{k+1} - c_k \in C_{k+1} \setminus C_k$. By repeating the argument, we obtain $c \in C_{k+1} \setminus C_k$ such that $\deg(c) = 0$, so $c \in A$. This is absurd because $A \subset C_k$.

Since A is noetherian, the increasing sequences of ideals $I_{\underline{i}}(C_k) \subseteq I_{\underline{i}}(C_{k+1})$ become stationary, for all $\underline{i} \in \{1, \dots, n\}^d$. In particular there exists $N \in \mathbb{N}$ such that, for all $\underline{i} \in \{1, \dots, n\}^d$, we have $I_{\underline{i}}(C_N) = I_{\underline{i}}(C_{N+1})$. It means that $C_N = C_{N+1}$ and so $B \setminus C_N = \emptyset$. ■

2.3.5 Nullstellensatz for complex regulous functions.

A very important property of the regulous functions in real algebraic geometry is the regulous version of the Nullstellensatz ([12] Theorem 5.24). We give here a regulous version of the Nullstellensatz for complex affine varieties. One can also find a proof of this result for c-holomorphic functions with algebraic graph in [7].

We consider the same notations as in Theorem 2.3.13. So, if X is an affine variety and $\pi : X^+ \rightarrow X$ is its seminormalization morphism, we consider the isomorphism

$$\begin{aligned} \varphi : \mathcal{K}^0(X(\mathbb{C})) &\xrightarrow{\simeq} \mathbb{C}[X^+] \\ f &\mapsto f \circ \pi_c \end{aligned}$$

Let $I \subset \mathcal{K}^0(X(\mathbb{C}))$. We write

$$\mathcal{Z}^0(I) := \{x \in X(\mathbb{C}) \mid \forall f \in I, f(x) = 0\}$$

Let $E \subset X(\mathbb{C})$. We write

$$\mathcal{J}^0(E) := \{f \in \mathcal{K}^0(X(\mathbb{C})) \mid \forall x \in E, f(x) = 0\}$$

Let I be an ideal of $\mathcal{K}^0(X(\mathbb{C}))$, then see that I is of the form $I = \langle g_1, \dots, g_n \rangle$ by noetherianity of $\mathbb{C}[X^+]$.

So

$$\pi_c^{-1}(\mathcal{Z}^0(I)) = \pi_c^{-1}(\cap \mathcal{Z}^0(g_i)) = \cap \pi_c^{-1}(\mathcal{Z}^0(g_i)) = \cap \mathcal{Z}(g_i \circ \pi_c) = \mathcal{Z}(\varphi(I))$$

Theorem 2.3.22 (Nullstellensatz)

Let X be an affine complex variety and I be an ideal of $\mathcal{K}^0(X(\mathbb{C}))$. Then

$$\mathcal{J}^0(\mathcal{Z}^0(I)) = \sqrt{I}$$

Proof: One of the inclusion is clear. For the other inclusion, we consider $f \in \mathcal{J}^0(\mathcal{Z}^0(I))$. It is equivalent to say that $\mathcal{Z}^0(I) \subset \mathcal{Z}^0(f)$. Then $\mathcal{Z}(\varphi(I)) = \pi_c^{-1}(\mathcal{Z}^0(I)) \subset \pi_c^{-1}(\mathcal{Z}^0(f)) = \mathcal{Z}(f \circ \pi_c)$. Then, by the classical Nullstellensatz on $\mathbb{C}[X^+]$, we can consider $n \in \mathbb{N}$ such that $(f \circ \pi_c)^n \in \varphi(I)$. So we get $f^n = \varphi^{-1}(\varphi(f)^n) \in \varphi^{-1}(\varphi(I)) = I$ and finally $f \in \sqrt{I}$. ■

We also get a version of the Nullstellensatz where we want to study only one element of $\mathcal{K}^0(X(\mathbb{C}))$. We will need this result in Section 2.4.1. One can do the exact same proof as Theorem 2.3.22 by adapting it with the following notations. For $f \in \mathcal{K}^0(X(\mathbb{C}))$, $I \subset \mathbb{C}[X][f]$ and $E \subset X(\mathbb{C})$, consider $\mathcal{Z}^f(I) := \{x \in X(\mathbb{C}) \mid \forall g \in I, g(x) = 0\}$ and $\mathcal{J}^f(E) := \{g \in \mathbb{C}[X][f] \mid \forall x \in E, g(x) = 0\}$. Also, consider $\mathbb{C}[Y] \simeq \mathbb{C}[X][t]/I_f$, $\pi : Y \rightarrow X$ and $\varphi : \mathbb{C}[X][f] \xrightarrow{\sim} \mathbb{C}[Y]$.

Theorem 2.3.23

Let X be an affine complex variety. Let $f \in \mathcal{K}^0(X(\mathbb{C}))$ and I be an ideal of $\mathbb{C}[X][f]$ then

$$\mathcal{J}^f(\mathcal{Z}^f(I)) = \sqrt{I}$$

2.4 Classical results on seminormality with regulous functions.

We revisit several results on seminormality using regulous functions. If $f \in \mathcal{K}^0(X(\mathbb{C}))$, then we have shown in the previous section that Γ_f , the graph of f , is a Z -closed set of $X(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C})$. So there exists an ideal $I_f \subset \mathbb{C}[X][t]$ such that $\Gamma_f = \mathcal{Z}(I_f)$. Moreover, we have $\mathbb{C}[X][t]/I_f \simeq \mathbb{C}[X][f]$ and, since f is integral over $\mathbb{C}[X]$, the ring $\mathbb{C}[X][f]$ is a $\mathbb{C}[X]$ -module of finite type. We note $\text{Cond}(f) := (\mathbb{C}[X] : \mathbb{C}[X][f]) = \{p \in \mathbb{C}[X] \mid p \cdot \mathbb{C}[X][f] \subseteq \mathbb{C}[X]\}$ the conductor of $\mathbb{C}[X]$ in $\mathbb{C}[X][f]$.

2.4.1 Definitions and criteria of seminormality in commutative algebra.

In this thesis, we have used Traverso's definition of the seminormalization [38] where, for an integral extension of rings $A \hookrightarrow B$, the seminormalization of A in B is given by

$$A_B^\dagger = \{b \in B \mid \forall \mathfrak{p} \in \text{Spec}(A), b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}})\}$$

But, as explain in [40], there are several definitions of the seminormalization for commutatives rings. For Hamann a ring A is seminormal in B if, for $n \in \mathbb{N}^*$, A contains all the elements $b \in B$ such that $b^n, b^{n+1} \in A$. The equivalent definition used by Leahy and Vitulli consist in replacing n and $n + 1$ by any positive relatively prime integers. Finally Swan gave an other definition of the seminormalization which is not equivalent to the previous ones for general commutative rings. Our goal in this section is to reinterpret those definitions in term of regulous functions and to see that they are all equivalent for affine rings.

Definition 2.4.1

Let $A \hookrightarrow B$ be an extension of rings and $b \in B$ be such that $b^2, b^3 \in \mathbb{C}[X]$. In this case, we say that $A \hookrightarrow A[b]$ is an elementary subintegral extension.

It is shown in [37] that, if a ring A is not seminormal in an other ring B , then we can always find a proper elementary subintegral subextension of $A \hookrightarrow B$. The following proposition gives a similar result with regulous functions.

Proposition 2.4.2

Let X be a complex affine variety and $f \in \mathcal{K}^0(X(\mathbb{C})) \setminus \mathbb{C}[X]$. Then there exists an element $g \in \mathbb{C}[X][f] \setminus \mathbb{C}[X]$ such that $g^n \in \text{Cond}(f) \subset \mathbb{C}[X]$, for all integer $n \geq 2$.

Proof: We know by Proposition 2.3.14 that f can be written in the following way

$$f = \begin{cases} p_1/q_1 & \text{if } q_1 \neq 0 \\ p_2/q_2 & \text{if } q_1 = 0 \text{ and } q_2 \neq 0 \\ \vdots & \\ p_{n-1}/q_{n-2} & \text{if } q_1 = \dots = q_{n-2} = 0 \text{ and } q_{n-1} \neq 0 \\ p_n & \text{if } q = q_1 = \dots = q_{n-1} = 0 \end{cases}$$

We consider the minimal integer s such that $q_{s+1}f \notin \mathbb{C}[X]$. If s exists, we continue the proof with $q_{s+1}f - p_{s+1} \notin \mathbb{C}[X]$. If s doesn't exists, we take $f - p_n$. So we can suppose that

$$f = \begin{cases} p_1/q_1 & \text{if } q_1 \neq 0 \\ p_2/q_2 & \text{if } q_1 = 0 \text{ and } q_2 \neq 0 \\ \vdots & \\ p_s/q_s & \text{if } q_1 = \dots = q_{s-1} = 0 \text{ and } q_s \neq 0 \\ 0 & \text{if } q = q_1 = \dots = q_s = 0 \end{cases}$$

with $q_i f \in \mathbb{C}[X]$ and so $q_i \in \sqrt{\text{Cond}(f)}$ for all $i \leq s$. Let's consider $I = \langle q_1^{n_1}, \dots, q_s^{n_s} \rangle$ with $n_i \in \mathbb{N}$ such that $q_i^{n_i} \in \text{Cond}(f)$. See that $\mathcal{Z}^f(I) \subset \mathcal{Z}^f(f)$. So, by Theorem 2.3.23, we have $f \in \sqrt{I}$. Since $I \subset \text{Cond}(f)$,

we get $f \in \sqrt{\text{Cond}(f)}$. So we can consider the minimal integer $m \geq 1$ such that $f^m \notin \text{Cond}(f)$ and $f^{m+1} \in \text{Cond}(f)$. The fact that $f^m \notin \text{Cond}(f)$ means that there exists $h = a_0 + a_1.f + \dots + a_d.f^d \in \mathbb{C}[X][f]$, where $d := \deg(f) - 1$, such that $f^m.h = a_0.f^m + a_1.f^{m+1} + \dots + a_{m+d}.f^{m+d} \notin \mathbb{C}[X]$. But since $f^{m+1} \in \text{Cond}(f)$, we get $a_1.f^{m+1} + \dots + a_{m+d}.f^{m+d} \in \mathbb{C}[X]$. It implies that $a_0.f^m \notin \mathbb{C}[X]$ and so $f^m \notin \mathbb{C}[X]$. Finally, we write $g := f^m$ and we have find an element $g \in \mathbb{C}[X][f] \setminus \mathbb{C}[X]$ such that $g^n \in \text{Cond}(f) \subset \mathbb{C}[X]$, for all integer $n \geq 2$. ■

We recover now, with regulous functions, that Traverso, Hamann and Leahy-Vitulli's definitions of the seminormalization are equivalent. In order to do this, we show the following Lemma.

Lemma 2.4.3

Let $f \in K(X)$ such that there exists $n, m \in \mathbb{N}^*$ with $\gcd(n, m) = 1$ and $f^n, f^m \in \mathbb{C}[X]$. Consider $u, v \in \mathbb{Z}$ such that $un + vm = 1$ and assume that $u > 0$ and $v < 0$. Then

$$g = \begin{cases} f & \text{if } f^m \neq 0 \\ 0 & \text{else} \end{cases} \in \mathcal{K}^0(X(\mathbb{C}))$$

Proof: Let X_i be an irreducible component of X . If $f^m|_{X_i(\mathbb{C})} = 0$, then $f|_{X_i(\mathbb{C})} = 0$ so we define $g|_{X_i(\mathbb{C})} = 0$. So, by Lemma 2.3.4, we can suppose that X is irreducible and that $f^m \neq 0$. We have

$$g = \begin{cases} f & \text{if } f^m \neq 0 \\ 0 & \text{else} \end{cases} = \begin{cases} (f^n)^u / (f^m)^{-v} & \text{if } f^m \neq 0 \\ 0 & \text{else} \end{cases}$$

So the graph of g is $\Gamma_g = \mathcal{Z}((f^m)^{-v}t - (f^n)^u; t^m - f^m)$ and, by Theorem 2.3.20, we get $g \in \mathcal{K}^0(X(\mathbb{C}))$. ■

The Lemma tells us that a fraction with one of the property appearing in the following criteria extend into a regulous function. So if X is seminormal it has to contain the elements mentioned in criteria 3),4) and 5). Moreover Proposition 5.2 shows that the seminormalization is the reunion of all of this kind of elements. So it is sufficient for X to contain those elements in order to be seminormal. This is how we obtain the following proposition.

Proposition 2.4.4 (Hamann and Leahy-Vitulli's criteria)

Let X be an affine complex variety. Then the following statements are equivalent :

1. X is seminormal.
2. $\forall f \in \mathbb{C}[X']$ the conductor of $\mathbb{C}[X]$ in $\mathbb{C}[X][f]$ is a radical ideal of $\mathbb{C}[X][f]$.
3. $\forall f \in \mathcal{K}(X)$ $f^2, f^3 \in \mathbb{C}[X] \implies f \in \mathbb{C}[X]$.
4. $\forall f \in \mathcal{K}(X)$ $f^n, f^m \in \mathbb{C}[X] \implies f \in \mathbb{C}[X]$, for some $m, n \in \mathbb{N}$ relatively prime.
5. $\forall f \in \mathcal{K}(X)$ $f^n, f^{n+1} \in \mathbb{C}[X] \implies f \in \mathbb{C}[X]$, for some $n \in \mathbb{N}$.

Proof: 2) \implies 1). If X is not seminormal, then there exists $f \in \mathcal{K}^0(X(\mathbb{C})) \setminus \mathbb{C}[X]$. So Proposition 2.4.2 gives an element $g \in \mathbb{C}[X][f]$ such that g belongs to the radical of $(\mathbb{C}[X] : \mathbb{C}[X][f])$ but not to the conductor itself. The fact that $g \notin \mathbb{C}[X]$ and $g^n \in \mathbb{C}[X]$ for all $n \geq 2$, shows that 3), 4) or 5) \implies 1).

1) \implies 2). Suppose there exists $f \in \mathcal{K}(X)$ and $g \in \sqrt{\text{Cond}(f)} \setminus \text{Cond}(f)$. We can consider $n \in \mathbb{N}^*$ such that $g^{n-1} \notin \text{Cond}(f)$ and $g^n \in \text{Cond}(f)$. So there exists $h \in \mathbb{C}[X][f]$ such that $g^{n-1}h \notin \mathbb{C}[X]$ and $(g^{n-1}h)^2, (g^{n-1}h)^3 \in \mathbb{C}[X]$. Then, by Proposition 2.4.3, we get

$$\psi = \begin{cases} g^{n-1}h & \text{if } (g^{n-1}h)^2 \neq 0 \\ 0 & \text{else} \end{cases} \in \mathcal{K}^0(X(\mathbb{C}))$$

So $\psi \in \mathcal{K}^0(X(\mathbb{C})) \setminus \mathbb{C}[X]$ which means that X is not seminormal.

1) \implies 4). Take $n, m \in \mathbb{N}$ such that $\gcd(n; m) = 1$ and assume X is seminormal. Consider $f \in \mathcal{K}(X)$ such that $f^n, f^m \in \mathbb{C}[X]$. Then, by Proposition 2.4.3, we can extend f to a regulous function. So we get $f \in \mathcal{K}^0(X(\mathbb{C})) = \mathbb{C}[X]$. Since, for all $n \in \mathbb{N}$, we have $\gcd(n; n+1) = 1$, we also get 1) \implies 3) and 5). ■

We recover now that Traverso and Swan's definitions of the seminormalization are equivalent for affine rings by using regulous functions. First we get the following proposition which gives us a way to construct regulous functions from polynomials that respect a certain type of relation.

Proposition 2.4.5

Let $p, q \in \mathbb{C}[X]$ be such that there exists $n \in \mathbb{N}^*$ with $p^n \in \langle q^{n+1} \rangle$. Then

$$f = \begin{cases} p/q & \text{if } q \neq 0 \\ 0 & \text{else} \end{cases} \in \mathcal{K}^0(X(\mathbb{C}))$$

Proof: Consider $n \in \mathbb{N}^*$ such that $p^n \in \langle q^{n+1} \rangle$. Then there exists $h \in \mathbb{C}[X]$ such that $p^n = hq^{n+1}$. So, if X_i is an irreducible component of X such that $q = 0$, we get that $p = 0$ and so we define $f|_{X_i(\mathbb{C})} = 0$. Then, by Lemma 2.3.4, we can suppose X irreducible and $q \neq 0$. In this case, the graph of f is given by $\Gamma_f = \mathcal{Z}(I_X; qt - p; t^n - qh)$ and we can apply Theorem 2.3.20 to conclude. ■

Lemma 2.4.6

Let X be an affine variety and $p, q \in \mathbb{C}[X]$. We write

$$f = \begin{cases} p/q & \text{if } q \neq 0 \\ 0 & \text{else} \end{cases}$$

Then

$$p^2 = q^3 \text{ if and only if } f^2 = q \text{ and } f^3 = p$$

In this case $f \in \mathcal{K}^0(X(\mathbb{C}))$ and

$$\Gamma_f = \mathcal{Z}(I_X; qt - p; t^2 - q) = \mathcal{Z}(I_X; t^2 - q; t^3 - p)$$

Proof: Let X_i be an irreducible component of X such that $q = 0$. Then $f|_{X_i(\mathbb{C})} = 0$ and the lemma becomes trivial. So, by Lemma 2.3.4, we can suppose X irreducible with $q \neq 0$. In this case, if $p^2 = q^3$ then $f^2 = p^2/q^2 = q^3/q^2 = q$

and $f^3 = p^3/q^3 = p^3/p^2 = p$ if $p, q \neq 0$. Moreover, if $q = 0$, then $f^2 = q = f^3 = p = 0$. So $f^2 = q$ and $f^3 = p$ on $X(\mathbb{C})$. Conversely, if $f^2 = q$ and $f^3 = p$ then $p^2 = (f^3)^2 = (f^2)^3 = q^3$. We get that $f \in \mathcal{K}^0(X(\mathbb{C}))$ by Proposition 2.4.5. ■

The Lemma shows that the relations of the form $p^2 = q^3$ produce regulous functions and Proposition 2.4.2 tells us that the seminormalization is the reunion of all of this kind of functions. Hence we obtain Swan's criterion.

Proposition 2.4.7 (*Swan's criterion*)

Let X be an affine complex variety. Then the following statements are equivalent :

1. X is seminormal.
2. For all $p, q \in \mathbb{C}[X]$ such that $p^2 = q^3$ there exists $f \in \mathbb{C}[X]$ with $f^2 = q$ and $f^3 = p$.

Proof: 1) \implies 2). Suppose X is seminormal and let $p, q \in \mathbb{C}[X]$ with $p^2 = q^3$. Then by Lemma 2.4.6 we get an element $f \in \mathcal{K}^0(X(\mathbb{C}))$ such that $f^2 = q$ and $f^3 = p$. Since X is seminormal, we have $f \in \mathbb{C}[X]$.

2) \implies 1). Suppose that X is not seminormal, then Proposition 2.4.2 gives us an element $g \in \mathcal{K}^0(X(\mathbb{C})) \setminus \mathbb{C}[X]$ with $g^2, g^3 \in \mathbb{C}[X]$. So if we write $q := g^2$ and $p := g^3$, Lemma 2.4.6 tells us that $p^2 = q^3$. Thus, if there exists $f \in \mathbb{C}[X]$ with $f^2 = q$ and $f^3 = p$, we get $f = g$ on $\mathcal{D}(q)$. By continuity, we get $f = g$ on $X(\mathbb{C})$ which is impossible because $g \notin \mathbb{C}[X]$. ■

2.4.2 Localization and seminormalization.

It is shown, for general rings, that the operation of localization and seminormalization commute. In Traverso [38], it is proved by considering special subextensions between the seminormalization and the normalization of the ring. In Swan [37], it is proved by considering elementary subintegral extensions of the ring (see Definition 2.4.1). We propose here a proof with regulous functions but only in the case of the localization by a single element because we need $S^{-1}\mathbb{C}[X]$ to be affine.

Proposition 2.4.8 (*Seminormalization and localization by a single element*)

Let X be a complex affine variety and S be a multiplicative set of $\mathbb{C}[X]$ such that $S = \{1, q, q^2, \dots\}$ with $q \in \mathbb{C}[X]$. Then

$$S^{-1}\mathbb{C}[X^+] = (S^{-1}\mathbb{C}[X])^+$$

Proof: First, see that it is equivalent to show

$$S^{-1}\mathcal{K}^0(X(\mathbb{C})) = \mathcal{K}^0(\mathcal{D}(q))$$

The inclusion $S^{-1}\mathcal{K}^0(X(\mathbb{C})) \subset \mathcal{K}^0(\mathcal{D}(q))$ is clear because if $f \in \mathcal{K}^0(X(\mathbb{C}))$, then for all $s \in S$ the function f/s is still rational and continuous on $\mathcal{D}(q)$. To get the other inclusion, we must show

$$\forall g \in \mathcal{K}^0(\mathcal{D}(q)) \quad \exists s \in S \quad sg = \begin{cases} s(x)g(x) & \text{if } x \in \mathcal{D}(q) \\ 0 & \text{else} \end{cases} \in \mathcal{K}^0(X(\mathbb{C}))$$

So let $g \in \mathcal{K}^0(\mathcal{D}(q))$. Then, by Theorem 2.3.20, it verifies the three following properties :

1. $g \in \mathcal{K}(\mathcal{D}(q))$
2. g is the root of a monic polynomial whose coefficients are in $S^{-1}\mathbb{C}[X]$.
3. The graph $\Gamma_g \subset \mathcal{D}(q) \times \mathbb{A}^1(\mathbb{C})$ of g is \mathcal{Z} -closed.

In other words, $g \in \mathcal{K}(\mathcal{D}(q))$ and there exists a system of polynomials

$$(*) : \begin{cases} P(x, t) = t^d + \frac{a_{d-1}}{s_{d-1}}.t^{d-1} + \dots + \frac{a_0}{s_0} = 0 \\ F_1(x, t) = \frac{a_{1,d_1}}{s_{1,d_1}}.t^{d_1} + \frac{a_{1,d_1-1}}{s_{1,d_1-1}}.t^{d_1-1} + \dots + \frac{a_{1,0}}{s_{1,0}} = 0 \\ \vdots \\ F_n(x, t) = \frac{a_{n,d_n}}{s_{n,d_n}}.t^{d_n} + \frac{a_{n,d_n-1}}{s_{n,d_n-1}}.t^{d_n-1} + \dots + \frac{a_{n,0}}{s_{n,0}} = 0 \end{cases}$$

such that, for all $x \in \mathcal{D}(q)$, the element $g(x)$ is its only solution. We have to see if there exists an $s \in S$ such that sg verifies a system of the similar form on $\mathbb{C}[X]$. Lets consider

$$s = \left(\prod_{k=0}^{d-1} s_k \right)^2 \cdot \left(\prod_{i,j \in \{1, \dots, n\} \times \{0, \dots, d_i\}} s_{i,j} \right)^2$$

We show that sg is the only solution of the following system whose coefficients are in $\mathbb{C}[X]$:

$$(**) : \begin{cases} \tilde{P}(x, t) = t^d + \frac{s}{s_{d-1}}a_{d-1}.t^{d-1} + \dots + \frac{s}{s_0}a_0.s^{d-1} = 0 \\ \tilde{F}_1(x, t) = \frac{s}{s_{1,d_1}}a_{1,d_1}.t^{d_1} + \frac{s}{s_{1,d_1-1}}a_{1,d_1-1}.s.t^{d_1-1} + \dots + \frac{s}{s_{1,0}}a_{1,0}.s^{d_1} = 0 \\ \vdots \\ \tilde{F}_n(x, t) = \frac{s}{s_{n,d_n}}a_{n,d_n}.t^{d_n} + \frac{s}{s_{n,d_n-1}}a_{n,d_n-1}.s.t^{d_n-1} + \dots + \frac{s}{s_{n,0}}a_{n,0}.s^{d_n} = 0 \end{cases}$$

Indeed, for all $x \in \mathcal{D}(q)$, we have

$$\begin{cases} \tilde{P}(x, sg(x)) = s^d(x)P(x, g(x)) = 0 \\ \tilde{F}_i(x, sg(x)) = s^{d_i+1}(x)F_i(x, g(x)) = 0 \end{cases}$$

So $sg(x)$ is the only solution of the system (**) for all $x \in \mathcal{D}(q)$. Now see that, in the definition of s , we carefully took squared elements so that if $x \in \mathcal{Z}(s) = \mathcal{Z}(q)$ then all the coefficients in (**) vanish except t^d in $P(x, t)$. Thus, for all $x \notin \mathcal{D}(q)$, we get $t = 0$ and so $sg(x)$ is the only solution of the system (**) for all $x \in X(\mathbb{C})$. ■

The fact that localization and seminormalization commute leads to look at seminormality directly at the points of a variety.

Definition 2.4.9

Let X be an affine variety. We define the set of seminormal points in X by

$$\mathcal{SN}(X) := \{x \in X \mid \mathcal{O}_{X,x} \text{ is seminormal}\}$$

and the seminormal points of $X(\mathbb{C})$ by $\mathcal{SN}(X(\mathbb{C})) = \mathcal{SN}(X) \cap X(\mathbb{C})$.

Now we can improve proposition 2.3.7 and be more precise about the points where regular functions are regular.

Proposition 2.4.10

Let X be an affine variety and $f \in \mathcal{K}^0(X(\mathbb{C}))$. Then $f \in \mathcal{O}_{X,x}$ for all $x \in \mathcal{SN}(X(\mathbb{C}))$.

Proof: We prove 1) \implies 2). Let $f \in \mathcal{K}^0(X(\mathbb{C}))$ and $x \in \mathcal{SN}(X(\mathbb{C}))$, we consider $\pi^+ : X^+ \rightarrow X$ the seminormalization morphism of X and $x^+ \in X^+(\mathbb{C})$ such that $\pi^+(x^+) = x$. By Theorem 2.3.13, we have $f \circ \pi_{\mathbb{C}}^+ \in \mathbb{C}[X^+]$. Since seminormalization and localization commute, we have $f \circ \pi_{\mathbb{C}}^+ \in \mathcal{O}_{X^+,x^+} = \mathcal{O}_{X,x}^+$ and since $x \in \mathcal{SN}(X(\mathbb{C}))$, we have that $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{X,x}^+$ is an isomorphism. So $f \in \mathcal{O}_{X,x}$. ■

Remark. An element in $\bigcap_{x \in \mathcal{SN}(X(\mathbb{C}))} \mathcal{O}_{X,x}$ does not always extends by continuity. One can take the example $X = \text{Spec}(\mathbb{C}[x, y]/\langle y^2 + (x^2 - 1)x^4 \rangle)$ given after Theorem 2.3.20. We have $\mathcal{SN}(X(\mathbb{C})) = \{(0; 0)\} = \{x = 0\}$ but the fraction $\frac{y}{x^2}$ cannot be continuously extended on $X(\mathbb{C})$.

We can deduce from Proposition 2.4.10 a classical result about seminormalization.

Corollary 2.4.11

Let X be an affine variety. Then X is seminormal if and only if $\mathcal{SN}(X(\mathbb{C})) = X(\mathbb{C})$.

Proof: Suppose that X is seminormal, then $\mathbb{C}[X^+] = \mathbb{C}[X]$. So it is clear that $\mathcal{O}_{X,x} = \mathcal{O}_{X^+,x^+}$ for all $x \in X(\mathbb{C})$. Conversely, suppose that $\mathcal{SN}(X(\mathbb{C})) = X(\mathbb{C})$ and let $f \in \mathcal{K}^0(X(\mathbb{C}))$. Then, by Proposition 2.4.10, we have $f \in \bigcap_{x \in X(\mathbb{C})} \mathcal{O}_{X,x} = \mathbb{C}[X]$. So $\mathcal{K}^0(X(\mathbb{C})) = \mathbb{C}[X]$ and X is seminormal. ■

2.5 Generalization on general complex algebraic varieties.

The goal of this section is to generalize the main results of Chapter 2 for non-necessarily affine varieties.

A ring extension $i : A \rightarrow B$ induces a map $\text{Spec}(i) : \text{Spec}(B) \rightarrow \text{Spec}(A)$, given by $\mathfrak{p} \mapsto (\mathfrak{p} \cap A) = i^{-1}(\mathfrak{p})$. For a morphism $\pi : Y \rightarrow X$ between algebraic varieties over k , we have an associated morphism of sheaves of rings $\mathcal{O}_X \rightarrow \pi_*\mathcal{O}_Y$ such that for any open subset $U \subset X$ it gives a ring morphism $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(\pi^{-1}(U))$ which is injective if and only if π is dominant.

A dominant morphism $\pi : Y \rightarrow X$ between algebraic varieties over k is said of finite type (resp. finite, birational, integral) if for any $U \subset X$ the ring extension $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(\pi^{-1}(U))$ is of finite type (resp. finite, birational, integral).

Let X be an algebraic variety over k . The normalization X' of X is an algebraic variety that comes with a finite birational morphism $\pi' : X' \rightarrow X$ called the normalization morphism such that, for any open subset $U \subset X$, we have $\mathcal{O}_{X'}(\pi'^{-1}(U)) = \mathcal{O}_X(U)'$. We say that X is normal if π' is an isomorphism. A point $x \in X$ is said normal if $\mathcal{O}_{X,x}$ is integrally closed.

We start by giving the definition of the seminormalization of an algebraic variety in an other. Let $\pi : Y \rightarrow X$ be a dominant morphism between algebraic varieties over k , then we can define the \mathcal{O}_X -module

$(\mathcal{O}_X)_Y^+$ such that $(\mathcal{O}_X)_Y^+(U) = (\mathcal{O}_X(U))_{\pi_*\mathcal{O}_Y(U)}^+$ for each open set $U \subset X$. It is shown by Andreotti and Bombieri in [3] that $(\mathcal{O}_X)_Y^+$ is quasi-coherent and so, by [17] Proposition 1.3.1, it correspond to the structural sheaf of a variety. This leads to the following definition.

Definition 2.5.1

Let $\pi : Y \rightarrow X$ be a dominant morphism between algebraic varieties over k . The seminormalization of X in Y is an algebraic variety X_Y^+ over k with a finite birational morphism $\pi_Y^+ : X_Y^+ \rightarrow X$ factorising π such that $(\pi_Y^+)_*\mathcal{O}_{X_Y^+} = (\mathcal{O}_X)_Y^+$.

We call X^+ the seminormalization of X in its normalization $Y = X'$. We say that X is seminormal in Y (resp. seminormal) if $X = X_Y^+$ (resp. $X = X^+$).

Definition 2.5.2

Let $\pi : Y \rightarrow X$ be an integral morphism between algebraic varieties over k . We say that π is subintegral if π is bijective and if, for all $y \in Y$ the field extension $\kappa(\pi(y)) \rightarrow \kappa(y)$ is an isomorphism.

Remark. The morphism π is subintegral if and only if, for all open set $U \subset X$, the ring extension $\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_Y(\pi^{-1}(U))$ is subintegral.

Proposition 2.5.3

Let $Y \xrightarrow{\pi_1} Z \xrightarrow{\pi_2} X$ be a sequence of dominant morphisms between algebraic varieties over k . Then $Z \rightarrow X$ is subintegral if and only if $X_Y^+ \rightarrow X$ factorizes through Z .

Proof: For every open set $U \subset X$, we have an extension $\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_Z(\pi_1^{-1}(U))$ and, by Theorem 2.1.11, it is subintegral if and only if we have

$$\mathcal{O}_Z(\pi_1^{-1}(U)) \subset \mathcal{O}_X(U)_{\mathcal{O}_Y(\pi_2^{-1}(U))}^+ = (\pi^+)_*(\mathcal{O}_{X_Y^+})(U)$$

This inclusion inducing a dominant morphism $X_Y^+ \rightarrow Z$, we get the proposition. ■

Let X be an algebraic variety over \mathbb{C} . Consider a Zariski open set $U \subset X$, a finite number of regular functions f_1, \dots, f_m on U and a number $\varepsilon \in \mathbb{R}$. Then the sets of the form

$$V(U; f_1, \dots, f_m, \varepsilon) := \left\{ x \in U(\mathbb{C}) \mid |f_i(x)| < \varepsilon \text{ for } i = 1, \dots, m \right\}$$

form a basis for the open sets of a topology on $X(\mathbb{C})$ called the *strong topology*. If X is affine, then $X(\mathbb{C}) \subset \mathbb{C}^n$, for some $n \in \mathbb{N}$ and the strong topology is induced by the Euclidean topology of \mathbb{C}^n .

Definition 2.5.4

Let X be an algebraic variety over \mathbb{C} and $U(\mathbb{C})$ be a Z -open set of $X(\mathbb{C})$. Then we write $\mathcal{K}_{X(\mathbb{C})}^0(U(\mathbb{C}))$ the set of continuous functions $f : U(\mathbb{C}) \rightarrow \mathbb{C}$ for the strong topology which are regular on a Z -open Z -dense subset of $U(\mathbb{C})$.

Now that we have a local definition for regulous functions, we define the sheaf $\mathcal{K}_{X(\mathbb{C})}^0$.

Proposition 2.5.5

Let X be an algebraic variety over \mathbb{C} . The presheaf defined by

$$\begin{aligned} \mathcal{K}_{X(\mathbb{C})}^0 : \{ \text{Zariski-open sets of } X(\mathbb{C}) \}^{op} &\rightarrow \mathbf{Ring} \\ U(\mathbb{C}) &\mapsto \mathcal{K}_{X(\mathbb{C})}^0(U(\mathbb{C})) \end{aligned}$$

is a sheaf

Proof: It is a presheaf because if $V(\mathbb{C}) \subset U(\mathbb{C})$ are Z-open sets of $X(\mathbb{C})$, we have a restriction morphism

$$\begin{aligned} \mathcal{K}_{X(\mathbb{C})}^0(U(\mathbb{C})) &\rightarrow \mathcal{K}_{X(\mathbb{C})}^0(V(\mathbb{C})) \\ f &\mapsto f|_{V(\mathbb{C})} \end{aligned}$$

In order to prove that it is a sheaf, we consider a Z-open set $U(\mathbb{C})$ and an open cover $\{U_i(\mathbb{C})\}_{i \in I}$ of $U(\mathbb{C})$. Let $\{f_i\}_{i \in I}$ be such that $f_i \in \mathcal{K}_{X(\mathbb{C})}^0(U_i(\mathbb{C}))$ for all $i \in I$ and such that for all $i, j \in I$

$$(f_i)|_{U_i(\mathbb{C}) \cap U_j(\mathbb{C})} = (f_j)|_{U_i(\mathbb{C}) \cap U_j(\mathbb{C})}$$

Then we can define the continuous function

$$\begin{aligned} f : U(\mathbb{C}) &\rightarrow \mathbb{C} \\ x &\mapsto f_i(x) \text{ if } x \in U_i(\mathbb{C}) \end{aligned}$$

Moreover, for all $i \in I$, there is a Z-open set $V_i(\mathbb{C}) \cap U_i(\mathbb{C})$ which is Z-dense for the induced topology on $U_i(\mathbb{C})$, such that $f_i \in \mathcal{O}_{X(\mathbb{C})}(V_i(\mathbb{C}))$. By Lemma 2.3.3, we have that $\bigcup V_i(\mathbb{C})$ is Z-dense in $X(\mathbb{C})$ and since $\mathcal{O}_{X(\mathbb{C})}$ is a sheaf, we get $f \in \mathcal{O}_{X(\mathbb{C})}(\bigcup V_i(\mathbb{C}))$. Hence $f \in \mathcal{K}_{X(\mathbb{C})}^0(X(\mathbb{C}))$. ■

A dominant morphism $\pi : Y \rightarrow X$ between varieties over \mathbb{C} induces an extension $\mathcal{K}_{X(\mathbb{C})}^0 \rightarrow (\pi_c)_* \mathcal{K}_{Y(\mathbb{C})}^0$, hence a morphism $(Y(\mathbb{C}), \mathcal{K}_{Y(\mathbb{C})}^0) \rightarrow (X(\mathbb{C}), \mathcal{K}_{X(\mathbb{C})}^0)$. We now give a generalization of the main results of this chapter, starting with those on subintegrality.

Theorem 2.5.6

Let $\pi : Y \rightarrow X$ be a finite morphism between algebraic varieties over \mathbb{C} . The following properties are equivalent:

- 1) π is subintegral.
- 2) π_c is bijective.
- 4) The ringed spaces $(Y(\mathbb{C}), \mathcal{K}_{Y(\mathbb{C})}^0)$ and $(X(\mathbb{C}), \mathcal{K}_{X(\mathbb{C})}^0)$ are isomorphic.
- 5) π_c is an homeomorphism for the strong topology.
- 6) π_c is an homeomorphism for the Zariski topology.
- 7) π is an homeomorphism.

Proof: The equivalences 1) \iff 2) \iff 6) \iff 7) are true for every affine open subset of $X(\mathbb{C})$ by Theorem 3.1. and the equivalences 1 \iff 4) \iff 5) comes form Proposition 4.12. ■

We now get the generalization of Theorem 2.3.13.

Theorem 2.5.7

Let X be an algebraic variety and $\pi^+ : X^+ \rightarrow X$ be its seminormalization morphism. Then $(\pi_c^+, (\pi_c^+)^*)$ is an isomorphism of ringed spaces between $(X^+(\mathbb{C}), \mathcal{O}_{X^+(\mathbb{C})})$ and $(X(\mathbb{C}), \mathcal{K}_{X(\mathbb{C})}^0)$.

Proof: By Theorem 2.5.6, the morphism π_c^+ is an homeomorphism for the Zariski topology. Moreover, we have shown in Theorem 2.3.13 that, for all affine open set $U(\mathbb{C})$, we have an isomorphism $(\pi_c^+)^* : \mathcal{K}_{X(\mathbb{C})}^0(U(\mathbb{C})) \rightarrow (\pi_c^+)_* \mathcal{O}_{X^+(\mathbb{C})}(U(\mathbb{C}))$. So $(\pi_c^+, (\pi_c^+)^*)$ is an isomorphism of ringed spaces. ■

One can find a generalization of those results for algebraic varieties over any algebraic closed field of characteristic zero in [6].

THE R-SEMINORMALIZATION OF REAL ALGEBRAIC VARIETIES

In this chapter, we introduce the R-seminormalization of real algebraic varieties. The idea behind this object is the following : since the seminormalization of a variety is obtained by gluing together the complex points that have been separated in the normalization, one may want to do the same thing but with real points of real varieties. Unfortunately, gluing the real points of the normalization in the fibers of the real points of a variety can lead to some problems because of the fact that the restriction to the real closed points of a finite morphism of real varieties is not necessarily surjective. For example, the normalization of the real variety $X := \text{Spec}(\mathbb{R}[x, y]/\langle y^2 - x^2(x-1) \rangle)$ is $X' := \text{Spec}(\mathbb{R}[x, y]/\langle y^2 - x + 1 \rangle)$ and we have $(\pi')^{-1}(\{0\}) = \emptyset$ where π' is the normalization morphism. In fact, for a general variety X , one can not find a variety Y that would be maximal for the property of having a morphism $\pi : Y \rightarrow X$ which is finite, birational and such that $\pi_{\mathbb{R}}$ is bijective. A proof that such a variety does not exist in general can be found in [15] Example 5.6. This lack of surjectivity led the authors of [15] to consider the "central" seminormalization where they glue the real central points (i.e. the points in the Euclidean closure of the regular locus) of the normalization above the real central points of the variety. See Chapter 1 Section 1.4.2 for more details about central seminormalization. An other way to counter this lack of surjectivity is to consider all the complex points in the normalization lying over the real points of the variety. This is the idea of the R-seminormalization.

This chapter is organized as follows. In Section 3.1, we give a generalization of Section 2.1. This allows us to define and to provide a universal property of the R-seminormalization. This universal property states that the R-seminormalization of a ring A in an integral extension of A is the biggest subextension such that there exists a unique prime ideal above each real prime ideal of A and their residue fields are isomorphic.

In Section 3.2, we look at the R-seminormalization of real algebraic varieties and we prove a geometrical universal property for this object. We show that the R-seminormalization is the biggest variety between a variety X and its normalization such that there exists a unique complex closed point above each element of $X(\mathbb{R})$. In Section 3.3, we identify by several ways, as we did for the seminormalization in Chapter 2, the ring of functions on $X(\mathbb{R})$ which becomes polynomials on the closed points of the R-seminormalization. On the normalization, those functions correspond to the polynomials which are constant on the complex

fibers over the real closed points of X . They also correspond to the integral rational functions which are (almost) continuous on $X(\mathbb{R})$ for the topology of $X(\mathbb{C})$. More precisely, we get the following version of Theorem 2.3.13 for the R -seminormalization.

Proposition (3.3.7). *Let X be a real affine variety and let $f : X(\mathbb{R}) \rightarrow \mathbb{R}$. Then f becomes polynomial on the R -seminormalization if and only if it verifies the following properties :*

1. *The function f is rational.*
2. *The function f is integral over $\mathbb{R}[X]$.*
3. *For all $x \in X(\mathbb{R})$, for all $(z_n)_n \in \text{Dom}(f) \cup X(\mathbb{R})$ such that $z_n \rightarrow x$ then $f(z_n) \rightarrow f(x)$.*

We give a third characterization of those functions using their graphs in the same spirit as Theorem 2.3.20. In Section 3.4, we look at the functions on $X(\mathbb{R})$ which are the restriction of continuous rational functions on $X(\mathbb{C})$. This ring of functions correspond to the coordinate ring of the seminormalization X^+ . Then we present several examples in order to compare the seminormalization, the R -seminormalization, the central seminormalization and the central weak-normalization. Finally, in Section 3.5, we prove that the R -seminormalization of a real variety X is related to its seminormalization X^+ and to its biregular normalization X^b which has been presented in Chapter 1 Section 1.4.1.

Theorem (3.5.8). *Let X be a real affine variety. Then*

$$\mathbb{R}[X^{R+}] \simeq \mathbb{R}[X^+]^b \simeq \mathbb{R}[X^b]^+$$

This result also tells us how the R -seminormalization modifies the singularities of a real algebraic variety: it normalizes the purely complex points and seminormalizes the real points.

3.1 Generalization of the seminormalization for general rings.

In this section, we present a generalization of the construction given in Section 2.1. It defines and provide a universal property of the R -seminormalization and the R -Max-seminormalization of a ring in an other.

Let A be a ring. We say that an ideal I of A is real if it has the following property : If a sum of square elements $\sum a_i^2 \in \sum A^2$ belongs to I , then every element a_i belongs to I . We denote by $R\text{-Spec}(A)$ (resp. $R\text{-Max}(A)$) the set of real prime (resp. maximal) ideals of A . Let F be one of the functors $\text{Spec}, \text{Max}, R\text{-Max}$ or $R\text{-Spec}$. In particular, $F : \mathbf{Ann} \rightarrow \mathbf{Top}$ is a contravariant subfunctor of Spec . It means that for all $A \in \mathbf{Ann}$ we have $F(A) \subset \text{Spec}(A)$ and for all extension of rings $A \hookrightarrow B$, we get a continuous application $F(B) \rightarrow F(A)$ given by $\mathfrak{p} \rightarrow \mathfrak{p} \cap A$. One may want to take a general subfunctor of Spec but we will need, at some point in this section, the specificity of the four considered subfunctors. See the remark after Definition 3.1.2 for more details.

Definition 3.1.1

Let $A \hookrightarrow B$ be an integral extension of rings. We define

$$A_B^F := \{b \in B \mid \forall \mathfrak{p} \in F(A), b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(B_{\mathfrak{p}})\}$$

Remark. If $F = \text{Spec}$, the ring A_B^F is the seminormalization A_B^+ of A in B .

Definition 3.1.2

Let $A \hookrightarrow B$ be an integral extension of rings. The extension $A \hookrightarrow B$ is called *F-subintegral* if it verifies the following conditions :

1. For all $\mathfrak{p} \in F(A)$, there exists a unique $\mathfrak{q} \in \text{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$.
2. For such \mathfrak{p} and \mathfrak{q} , we have $\mathfrak{q} \in F(B)$ and the induced map $\kappa(\mathfrak{p}) \hookrightarrow \kappa(\mathfrak{q})$ on the residue fields, is an isomorphism.

Remark. It is important to see that, if $\mathfrak{p} \in F(A)$ and $\mathfrak{q} \in \text{Spec}(B)$ are such that $\mathfrak{q} \cap A = \mathfrak{p}$. The condition $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{q})$ and the fact that the extension is integral implies that $\mathfrak{q} \in F(B)$. It is really specific to the fact that an ideal is real if and only if its residual field is a real field.

We give here a first geometric property of F-subintegral extensions.

Proposition 3.1.3

Let $A \hookrightarrow B$ be an F-subintegral extension of rings and $\pi : F(B) \rightarrow F(A)$ be the induced map. Then π is a Z-homeomorphism for the induced topology.

Proof: The morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is Z-continuous, so is its restriction to $F(B)$, thus we just have to show that π is Z-closed. Let $\mathfrak{p} \in F(A)$ and $\mathfrak{q} \in F(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. We have

$$\pi(\mathcal{V}(\mathfrak{q})) = \{\mathfrak{q}' \cap A \mid \mathfrak{q} \subset \mathfrak{q}' \in F(B)\}$$

so $\pi(\mathcal{V}(\mathfrak{q})) \subset \mathcal{V}(\mathfrak{q} \cap A) = \mathcal{V}(\mathfrak{p})$. If $\mathfrak{p}' \in \mathcal{V}(\mathfrak{p}) \cap F(A)$, then the going-up property says that there exists $\mathfrak{q}' \in \text{Spec}(B)$ such that $\mathfrak{q}' \cap A = \mathfrak{p}'$ and $\mathfrak{q} \subset \mathfrak{q}'$. Since the extension is F-subintegral, the ideal \mathfrak{q}' belongs to $F(B)$. Hence $\mathfrak{p}' \in \pi(\mathcal{V}(\mathfrak{q}))$ and finally $\mathcal{V}(\mathfrak{p}) \subset \pi(\mathcal{V}(\mathfrak{q}))$, so $\pi(\mathcal{V}(\mathfrak{q})) = \mathcal{V}(\mathfrak{p})$ which is Z-closed. ■

The goal of this section is to prove the following universal property. It says that the F-seminormalization of A in B is the biggest F-subintegral extension of $A \hookrightarrow B$.

Proposition 3.1.4

Let $A \hookrightarrow C \hookrightarrow B$ be integral extensions of rings. Then the following statements are equivalent :

- 1) The extension $A \hookrightarrow C$ is F-subintegral.
- 2) The image of $C \hookrightarrow B$ is a subring of A_B^F .

We prove the universal property through a series of propositions as in Section 2.1.

Proposition 3.1.5

Let $A \hookrightarrow B$ be an integral extension of rings. Then

$$A \hookrightarrow A_B^F \text{ is F-subintegral}$$

Proof: By doing the exact same proof as in Proposition 2.1.10, one can show that for all $\mathfrak{p} \in F(A)$, there exists a unique $\mathfrak{q} \in \text{Spec}(A_B^F)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$ and $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{q})$. Because of the special nature of F (see remark after Definition 3.1.2), the equiresiduality implies that $\mathfrak{q} \in F(A_B^F)$ and so $A \hookrightarrow A_B^F$ is F-subintegral.

Proposition 3.1.6

Let $A \hookrightarrow C \hookrightarrow B$ be integral extensions of rings. Then the following properties are equivalent

- 1) The extension $A \hookrightarrow B$ is F -subintegral.
- 2) The extensions $A \hookrightarrow C$ and $C \hookrightarrow B$ are F -subintegral.

Proof: We prove 1) implies 2). Let $\mathfrak{p} \in F(C)$ and $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{Spec}(B)$ such that $\mathfrak{q}_1 \cap C = \mathfrak{q}_2 \cap C = \mathfrak{p}$. We have $\mathfrak{q}_1 \cap A = \mathfrak{q}_1 \cap C \cap A = \mathfrak{p} \cap A$ and the same is true for \mathfrak{q}_2 . Since $A \hookrightarrow B$ is F -subintegral and $\mathfrak{p} \cap A \in F(A)$, then

$$\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A \implies \mathfrak{q}_1 = \mathfrak{q}_2$$

Moreover $\kappa(\mathfrak{p} \cap A) \hookrightarrow \kappa(\mathfrak{p}) \hookrightarrow \kappa(\mathfrak{q}_i)$ and $\kappa(\mathfrak{p} \cap A) \simeq \kappa(\mathfrak{q}_i)$. So $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{q}_i)$. Now, let us consider $\mathfrak{p} \in F(A)$ and $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{Spec}(C)$ such that $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A = \mathfrak{p}$. We know that $\text{Spec}(B) \rightarrow \text{Spec}(C)$ is surjective, so we can find $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{Spec}(B)$ such that $\mathfrak{q}_1 \cap C = \mathfrak{p}_1$ and $\mathfrak{q}_2 \cap C = \mathfrak{p}_2$. Then $\mathfrak{q}_1 \cap A = \mathfrak{p}$ and $\mathfrak{q}_2 \cap A = \mathfrak{p}$. Since $A \hookrightarrow B$ is F -subintegral and $\mathfrak{p} \in F(A)$, we get $\mathfrak{q}_1 = \mathfrak{q}_2$ and so $\mathfrak{p}_1 = \mathfrak{p}_2$. Moreover $\kappa(\mathfrak{p}) \hookrightarrow \kappa(\mathfrak{p}_i) \hookrightarrow \kappa(\mathfrak{q}_i)$ and $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{q}_i)$. So $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{p}_i)$.

We now show that 2) implies 1). Let's suppose that $A \hookrightarrow C$ and $C \hookrightarrow B$ are F -subintegral. Let $\mathfrak{p} \in F(A)$, then there exists a unique element $\mathfrak{p}' \in \text{Spec}(C)$ such that $\mathfrak{p}' \cap A = \mathfrak{p}$ and $\mathfrak{p}' \in F(C)$. Moreover $\kappa(\mathfrak{p}') \simeq \kappa(\mathfrak{p})$. Then, since $C \hookrightarrow B$ is F -subintegral, there exists a unique element $\mathfrak{p}'' \in \text{Spec}(B)$ such that $\mathfrak{p}'' \cap C = \mathfrak{p}'$ and $\kappa(\mathfrak{p}'') \simeq \kappa(\mathfrak{p}')$. So \mathfrak{p}'' is the unique element of $\text{Spec}(B)$ such that $\mathfrak{p}'' \cap A = \mathfrak{p}$ and $\mathfrak{p}'' \in F(B)$. Moreover $\kappa(\mathfrak{p}'') \simeq \kappa(\mathfrak{p})$. Hence $A \hookrightarrow B$ is F -subintegral. ■

Proposition (Proposition 3.1.4). *Let $A \hookrightarrow C \hookrightarrow B$ be integral extensions of rings. Then the following statements are equivalent :*

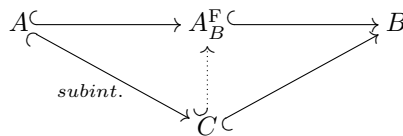
- 1) The extension $A \hookrightarrow C$ is F -subintegral.
- 2) The image of $C \hookrightarrow B$ is a subring of A_B^F .

Proof: One can show 1) implies 2) by doing the exact same proof of Proposition 2.1.4 with A_B^F instead of A_B^+ . The converse is also very similar : suppose that we have $A \hookrightarrow C \hookrightarrow A_B^F \hookrightarrow B$. Those extensions are integral and, by Proposition 3.1.5, the extension $A \hookrightarrow A_B^F$ is F -subintegral. Then Lemma 3.1.6 tells us that $A \hookrightarrow C$ is F -subintegral. ■

Let us conclude this section by rewriting Proposition 3.1.4 in the form of a universal property theorem.

Theorem 3.1.7 (Universal property of the F -seminormalization)

Let $A \hookrightarrow B$ be an integral extension of rings. For every intermediate extension C of $A \hookrightarrow B$ such that $A \hookrightarrow C$ is F -subintegral, the image of C by the injection $C \hookrightarrow B$ is contained in A_B^F .



Remark. Let $A \hookrightarrow B$ be an integral extension. We have that $A \hookrightarrow A_B^F$ is F-subintegral by Proposition 3.1.5. So we can apply the universal property in the following way :

$$\begin{array}{ccccc}
 A \hookrightarrow & & A_{A_B^F}^F \hookrightarrow & & A_B^F \\
 & \searrow & \uparrow & \swarrow & \\
 & & A_B^F & & \\
 & \swarrow & & \searrow & \\
 & & & &
 \end{array}$$

subint.

Thus $A_B^F \hookrightarrow A_{A_B^F}^F$. But, by definition, $A_{A_B^F}^F$ is included in A_B^F . We get the following idempotency property

$$A_B^F = A_{A_B^F}^F$$

3.2 The R-seminormalization for geometric rings.

Let $X = \text{Spec}(A)$ be an affine algebraic variety with A a \mathbb{R} -algebra of finite type. Let $\mathbb{R}[X] := A$ denote the coordinate ring of X . We have $\mathbb{R}[X] \simeq \mathbb{R}[x_1, \dots, x_n]/I$ for an ideal I of $\mathbb{R}[x_1, \dots, x_n]$. We will say that X is a real affine variety if I is a real ideal. A morphism $\pi : Y \rightarrow X$ between two real varieties induces the morphism $\pi^* : \mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ which is injective if and only if π is dominant. We say that π is of finite type (resp. is finite) if π^* makes $\mathbb{R}[Y]$ a $\mathbb{R}[X]$ -algebra of finite type (resp. a finite $\mathbb{R}[X]$ -module). The space X is equipped with the Zariski topology for which the closed sets are of the form $\mathcal{V}(I) := \{\mathfrak{p} \in \text{Spec}(\mathbb{R}[X]) \mid I \subset \mathfrak{p}\}$ where I is an ideal of $\mathbb{R}[X]$. We define $X(\mathbb{R}) := \{\mathfrak{m} \in \text{Max}(\mathbb{R}[X]) \mid \kappa(\mathfrak{m}) \simeq \mathbb{R}\} = \text{R-Max}(\mathbb{R}[X])$. The Real Nullstellensatz gives us a Zariski homeomorphism between $X(\mathbb{R})$ and the algebraic set $\mathcal{Z}_{\mathbb{R}}(I) := \{x \in \mathbb{R}^n \mid \forall f \in I f(x) = 0\} \subset \mathbb{R}^n$. For any real variety, we can look at its complexification whose coordinate ring is given by the change of coordinate $\mathbb{C}[X] := \mathbb{R}[X] \otimes_{\mathbb{R}} \mathbb{C}$. Hence we can consider the set of its closed points $X(\mathbb{C})$. We note $\pi_{\mathbb{R}} : Y(\mathbb{R}) \rightarrow X(\mathbb{R})$ the restriction of π to $Y(\mathbb{R})$ and $\pi_{\mathbb{C}}$ the restriction of π to $Y(\mathbb{C})$. If X is irreducible, then we write $\mathcal{K}(X) := \text{Frac}(\mathbb{R}[X])$.

The goals of this section are to provide a geometric universal property of the R-seminormalization and to see that the R-seminormalization coincide with the R-Max-seminormalization for real algebraic varieties. The following theorem gives a reinterpretation of the R-subintegrality in a geometric point of view.

Theorem 3.2.1

Let $\pi : Y \rightarrow X$ be a finite morphism between real affine varieties. The following properties are equivalent

- 1) *The extension $\pi^* : \mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ is R-subintegral.*
- 2) *The extension $\pi^* : \mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ is R-Max-subintegral.*
- 3) *The restriction $\tilde{\pi} : \pi_{\mathbb{C}}^{-1}(X(\mathbb{R})) \rightarrow X(\mathbb{R})$ of the morphism $\pi_{\mathbb{C}} : Y(\mathbb{C}) \rightarrow X(\mathbb{C})$ is bijective.*
- 4) *The morphism $\pi_{\mathbb{R}} : Y(\mathbb{R}) \rightarrow X(\mathbb{R})$ is bijective and $\pi_{\mathbb{C}}^{-1}(X(\mathbb{R})) = Y(\mathbb{R})$.*

Proof: 1) \implies 2). Suppose that $\mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ is R-subintegral, then for all $\mathfrak{p} \in \text{R-Spec}(\mathbb{R}[X])$, there exists a unique $\mathfrak{q} \in \text{Spec}(\mathbb{R}[Y])$ such that $\mathfrak{q} \cap \mathbb{R}[X] = \mathfrak{p}$. Moreover $\mathfrak{q} \in \text{R-Spec}(\mathbb{R}[Y])$. In particular, it is true if \mathfrak{p} is maximal and so we get Property 2.

2) \implies 1). Suppose that for all $\mathfrak{m} \in \text{R-Max}(\mathbb{R}[X])$, we have a unique element $\mathfrak{m}' \in \text{Spm}(\mathbb{R}[Y])$ such that $\mathfrak{m}' \cap \mathbb{R}[X] = \mathfrak{m}$ and $\mathfrak{m}' \in \text{R-Max}(\mathbb{R}[Y])$. Let $\mathfrak{p} \in \text{R-Spec}(\mathbb{R}[X])$ and $\mathfrak{q} \in \text{Spec}(\mathbb{R}[Y])$ such that $\mathfrak{q} \cap \mathbb{R}[X] = \mathfrak{p}$. We note $V = \mathcal{V}(\mathfrak{p})$ and $W = \mathcal{V}(\mathfrak{q})$. Then $\dim V = \dim W$ since π is finite. Moreover $\dim V(\mathbb{R}) = \dim V(\mathbb{C})$ because \mathfrak{p} is a real ideal and we also have $\dim V(\mathbb{R}) = \dim W(\mathbb{R})$ since $\pi_{\mathbb{R}}$ is bijective. We get

$$\dim W(\mathbb{R}) = \dim V(\mathbb{R}) = \dim V(\mathbb{C}) = \dim V = \dim W = \dim W(\mathbb{C})$$

So $\mathfrak{q} \in \text{R-Spec}(\mathbb{R}[Y])$. Let us show that \mathfrak{q} is unique. Suppose there exists $\mathfrak{q}' \in \text{Spec}(\mathbb{R}[Y])$ such that $\mathfrak{q}' \cap \mathbb{R}[X] = \mathfrak{p}$. By the previous arguments we have $\mathfrak{q}' \in \text{R-Spec}(\mathbb{R}[Y])$. Let $\mathfrak{m} \in \text{R-Max}(\mathbb{R}[Y])$ such that $\mathfrak{q} \subset \mathfrak{m}$ then $\mathfrak{q} \cap \mathbb{R}[X] \subset \mathfrak{m} \cap \mathbb{R}[X]$. Moreover, by the going-up property, we can consider $\mathfrak{m}' \in \text{Spm}(\mathbb{R}[Y])$ such that $\mathfrak{q}' \subset \mathfrak{m}'$ and $\mathfrak{m}' \cap \mathbb{R}[X] = \mathfrak{m} \cap \mathbb{R}[X]$. Then, by assumption, we get $\mathfrak{m} = \mathfrak{m}'$. So $\mathcal{Z}_{\mathbb{R}}(\mathfrak{q}) = \mathcal{Z}_{\mathbb{R}}(\mathfrak{q}')$ and the real Nullstellensatz gives us $\mathfrak{q} = \mathfrak{q}'$.

It remains to see that $\kappa(\mathfrak{p}') \simeq \kappa(\mathfrak{p})$. If we write $V = \text{Spec}(\mathbb{R}[X]/\mathfrak{p})$ and $W = \text{Spec}(\mathbb{R}[Y]/\mathfrak{p}')$, we get the following commutative diagram :

$$\begin{array}{ccc} \mathbb{R}[X] & \xrightarrow{\pi^*} & \mathbb{R}[Y] \\ \downarrow \pi_X & & \downarrow \pi_Y \\ \mathbb{R}[V] & \xrightarrow{(\pi|_W)^*} & \mathbb{R}[W] \\ \downarrow & & \downarrow \\ \mathcal{K}(V) & \xrightarrow{\quad} & \mathcal{K}(W) \end{array}$$

As $\mathbb{R}[Y]$ is a finite $\mathbb{R}[X]$ -module, we have that $\mathbb{R}[W]$ is a finite $\mathbb{R}[V]$ -module. Thus $\pi|_W$ is a finite morphism between two irreducible varieties. Therefore we get $n := [K(W) : K(V)] < +\infty$. Since the characteristic is zero, the extension $K(V) \hookrightarrow K(W)$ is separable and finite. Hence we can consider a primitive element $a \in K(W)$ such that $K(W) = K(V)(a)$. Let F be the minimal polynomial of a . Then $\deg F = n$ and $\text{disc}(F) \neq 0$ where $\text{disc}(F)$ is the discriminant of F . So there exists a \mathbb{Z} -open set of V such that $\text{disc}(F) \neq 0$. Since $\mathfrak{p} \in \text{R-Spec}(\mathbb{R}[X])$, we have that $V(\mathbb{R})$ is \mathbb{Z} -dense in V and so there exists $y \in V(\mathbb{R})$ such that $\text{disc}(F)(y, \cdot) \neq 0$. This means

$$\#\pi_{\mathbb{C}}^{-1}(y) = \#\{\text{complex roots of } F(y, \cdot)\} = \deg F = n$$

By assumption, there is a unique element of $Y(\mathbb{C})$ above every element of $X(\mathbb{R})$. So $\#\pi_{\mathbb{C}}^{-1}(y) = n = 1$ and so $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{p}')$.

Now, let us see that the properties 2, 3 and 4 are equivalent.

2) \implies 3). Property 2 implies that for all $\mathfrak{m} \in \text{R-Max}(\mathbb{R}[X])$, there exists a unique $\mathfrak{m}' \in \text{Spec}(\mathbb{R}[Y])$ such that $\mathfrak{m}' \cap \mathbb{R}[X] = \mathfrak{m}$. By the going-up property, we have $\mathfrak{m}' \in \text{Spm}(\mathbb{R}[Y])$. So, by the Nullstellensatz, we get the third property.

3) \implies 4). Let $x \in X(\mathbb{R})$ and $z \in Y(\mathbb{C})$ such that $\pi_{\mathbb{C}}(z) = x$, then $\pi_{\mathbb{C}}(\bar{z}) = x$. Since z is supposed to be unique, we get $z = \bar{z}$ and so $z \in Y(\mathbb{R})$.

4) \implies 2). Suppose 4 and take $\mathfrak{m} \in \text{R-Max}(\mathbb{R}[X])$. By the going-up property, we know that there is a finite number of prime ideals in $\mathbb{R}[Y]$ lying over \mathfrak{m} and that those ideals are maximal. Since we have $\pi_{\mathbb{C}}^{-1}(X(\mathbb{R})) = Y(\mathbb{R})$, then all the \mathfrak{m}' are real. So, the morphism $\pi_{\mathbb{R}}$ being bijective, we get that there is a unique prime ideal \mathfrak{m}' of $\mathbb{R}[Y]$ lying over \mathfrak{m} . Moreover $\mathfrak{m}' \in \text{R-Max}(\mathbb{R}[Y])$ so $\kappa(\mathfrak{m}) \simeq \kappa(\mathfrak{m}') \simeq \mathbb{R}$ and we get the second property. \blacksquare

Let $A \hookrightarrow B$ be an integral extension of rings, we note

$$A_B^{\text{R-Spec}} = A_B^{R+} \text{ and } A_B^{\text{R-Max}} = A_B^{R+\max}$$

Let us see that, if A is a coordinate ring, then A^{R+} and $A^{R+\max}$ are also the coordinate rings of some real varieties.

Proposition 3.2.2

Let $\pi : Y \rightarrow X$ be a finite morphism between affine varieties over \mathbb{R} . Let A be an intermediate ring between $\mathbb{R}[X]$ and $\mathbb{R}[Y]$. Then there exists a unique affine variety Z such that $A = \mathbb{R}[Z]$. Moreover, if X and Y are real varieties and π is birational, then Z is a real affine variety.

Proof: We have that A is a $\mathbb{R}[X]$ -module because it is a submodule of $\mathbb{R}[Y]$. Thus it is a \mathbb{R} -algebra of finite type because so is $\mathbb{R}[X]$. If π is also birational, then we get $\mathbb{R}[X] \hookrightarrow \mathbb{R}[Z] \hookrightarrow \mathbb{R}[X]'$ and by [14] Lemma 2.8, the ring $\mathbb{R}[Z]$ is real. ■

This lead us to define, for every real variety, a new variety called its \mathbb{R} -seminormalization.

Definition 3.2.3

Let $\pi : Y \rightarrow X$ be a finite morphism between two real affine varieties. The affine variety defined by

$$X_Y^{R+} := \text{Spec}(\mathbb{R}[X]_{\mathbb{R}[Y]}^{R+})$$

is called the \mathbb{R} -seminormalization of X in Y .

The \mathbb{R} -subintegrality being equivalent the \mathbb{R} -Max-subintegrality for affine rings, we naturally get that the \mathbb{R} -Max-seminormalization correspond to the \mathbb{R} -seminormalization. Note that, for the central seminormalization (see 1.4.2), this property is not true and we get two different varieties : the central seminormalization X^{sc} and the central weak-normalization X^{wc} .

Corollary 3.2.4

Let $\pi : Y \rightarrow X$ be a finite morphism between two real affine varieties. Then

$$\mathbb{R}[X]_{\mathbb{R}[Y]}^{R+\max} = \mathbb{R}[X]_{\mathbb{R}[Y]}^{R+}$$

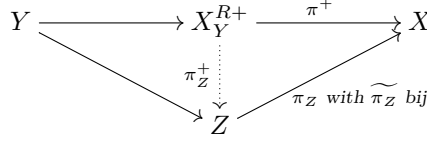
Proof: First, the inclusion $\mathbb{R}[X]_{\mathbb{R}[Y]}^{R+} \subset \mathbb{R}[X]_{\mathbb{R}[Y]}^{R+\max}$ is clear. Now, by Proposition 3.1.5, we know that $\mathbb{R}[X]_{\mathbb{R}[Y]}^{R+\max}$ is \mathbb{R} -Max-subintegral, so by Theorem 3.2.1, it is also \mathbb{R} -subintegral. Then the universal property of $\mathbb{R}[X]_{\mathbb{R}[Y]}^{R+}$ gives us $\mathbb{R}[X]_{\mathbb{R}[Y]}^{R+\max} \subset \mathbb{R}[X]_{\mathbb{R}[Y]}^{R+}$. ■

We can now rewrite the universal property of the \mathbb{R} -seminormalization for the geometric case. It is the biggest variety such that there is a unique complex closed point in the fiber of every real closed point.

Theorem 3.2.5 (Universal property of the \mathbb{R} -seminormalization)

Let $Y \rightarrow Z \rightarrow X$ be finite morphisms between real affine varieties. Then the restriction $\widetilde{\pi}_Z : \pi_{Z(\mathbb{C})}^{-1}(X(\mathbb{R})) \rightarrow X(\mathbb{R})$ of the morphism $\pi_{Z(\mathbb{C})} : Z(\mathbb{C}) \rightarrow X(\mathbb{C})$ is bijective if and only if there exists a morphism $\pi_Z^+ : X_Y^{R+} \rightarrow Z$ such that $\pi_Z \circ \pi_Z^+ = \pi^+$. Moreover π_Z^+ is unique and the restriction

$\widetilde{\pi}_Z^+ : (\pi_Z^+)^{-1}(Z(\mathbb{R})) \rightarrow Z(\mathbb{R})$ of the morphism $\pi_Z^+ : X_Y^{R+}(\mathbb{C}) \rightarrow Z(\mathbb{C})$ is bijective.



Proof: We have the following equivalences :

$$\begin{aligned}
 \text{The morphism } \widetilde{\pi}_Z^+ \text{ is bijective} &\iff \mathbb{R}[X] \hookrightarrow \mathbb{R}[Z] \text{ is } R\text{-subintegral (by Theorem 3.2.1)} \\
 &\iff \mathbb{R}[Z] \hookrightarrow \mathbb{R}[X_{\mathbb{R}[Y]}^{R+}] \text{ (by Proposition 3.1.4)} \\
 &\iff \exists \pi_Z^+ : X_Y^{R+} \rightarrow Z \text{ dominant, such that } \pi_Z \circ \pi_Z^+ = \pi^+
 \end{aligned}$$

We get the uniqueness of π_Z^+ by injectivity of π_Z . Since $\mathbb{R}[X] \hookrightarrow \mathbb{R}[X_Y^{R+}]$ is R -subintegral then Lemma 3.1.6 says that $\mathbb{R}[Z] \hookrightarrow \mathbb{R}[X_Y^{R+}]$ is also R -subintegral. So, by Theorem 3.2.1, the morphism π_Z^+ is bijective. ■

3.3 Functions which become polynomial on the R -seminormalization.

Let X be a real variety. The normalization X' of a real variety is defined by the coordinate ring $\mathbb{R}[X]_{\mathcal{K}(X)}$ and its seminormalization X^+ is defined by the coordinate ring $\mathbb{R}[X]_{\mathbb{R}[X']}$. Moreover, we have $\mathbb{C}[X]' = \mathbb{R}[X'] \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C}[X]^+ = \mathbb{R}[X^+] \otimes_{\mathbb{R}} \mathbb{C}$ by [16] Corollary 5.7.

We recall from Chapter 1 Proposition 1.1.9 that, for a reduced affine variety X with a finite number of irreducible components, the total ring of fractions $\mathcal{K}(X)$ is a product of fields. Moreover, in order to look at the normalization of $\mathbb{R}[X]$ in $\mathcal{K}(X)$, one can look at the normalization of each irreducible component in its field of fractions.

The R -seminormalization X^{R+} of X is the variety $X_{X'}^{R+}$ defined in Definition 3.2.3. The R -seminormalization X^{R+} comes with a finite, birational morphism $\pi^{R+} : X^{R+} \rightarrow X$ such that $\pi_{\mathbb{R}}^{R+}$ is bijective and $(\pi_{\mathbb{C}}^{R+})^{-1}(X(\mathbb{R})) = X^{R+}(\mathbb{R})$. Its universal property is given by Theorem 3.2.5.

We have the following extensions of rings :

$$\mathbb{R}[X] \hookrightarrow \mathbb{R}[X^+] \hookrightarrow \mathbb{R}[X^{R+}] \hookrightarrow \mathbb{R}[X'] \hookrightarrow \mathcal{K}(X).$$

In all this section, we will consider real varieties with real irreducible components. The set of real closed points $X(\mathbb{R})$ can be seen as a subset of \mathbb{R}^n for some n . Hence we will be able to use the Euclidean topology on $X(\mathbb{R})$ induced by \mathbb{R}^n .

In the same spirit as Section 2.3, we want to study the set $\mathcal{K}^{R+}(X(\mathbb{R}))$ of functions defined on $X(\mathbb{R})$ which become polynomials on the R -seminormalization. This can be useful, for example, to construct

the R-seminormalization of a given real variety. We start by identifying the elements of $\mathbb{R}[X']$ which comes from an element of $\mathbb{R}[X^{R+}]$. Then we show that the elements of $\mathcal{K}^{R+}(X(\mathbb{R}))$ are integral rational functions which are almost continuous on $X(\mathbb{R})$ for the topology of $X(\mathbb{C})$. Finally, we provide a last characterization of the elements of $\mathcal{K}^{R+}(X(\mathbb{R}))$ with properties concerning their graph.

Definition 3.3.1

Let X be a real variety and $\pi_{\mathbb{R}}^{R+}$ be its R-seminormalization morphism. We will denote by $\mathcal{K}^{R+}(X(\mathbb{R}))$ the ring of functions $f : X(\mathbb{R}) \rightarrow \mathbb{R}$ such that $f \circ \pi_{\mathbb{R}}^{R+} \in \mathbb{R}[X^{R+}]$.

We start by identifying the elements of $\mathbb{R}[X^{R+}]$ in $\mathbb{R}[X']$. Note that this identification is interesting but does not identify the ring $\mathcal{K}^{R+}(X(\mathbb{R}))$ independently from the variety X^{R+} .

Proposition 3.3.2

Let X be a real affine variety and let $\pi' : X' \rightarrow X$ be the normalization morphism of X . Then

$$\mathbb{R}[X^{R+}] \simeq \{p \in \mathbb{R}[X'] \mid \forall z_1, z_2 \in \pi_c'^{-1}(X(\mathbb{R})), \pi_c'(z_1) = \pi_c'(z_2) \implies p_c(z_1) = p_c(z_2)\}$$

Proof: We will denote by $\pi : X' \rightarrow X^{R+}$ the morphism induced by the extension $\mathbb{R}[X^{R+}] \hookrightarrow \mathbb{R}[X']$. Let $q \in \mathbb{R}[X^{R+}]$ and $x \in X(\mathbb{R})$. We want to show that for all $z_1, z_2 \in \pi_c^{-1}(x)$, we have $q \circ \pi_c(z_1) = q \circ \pi_c(z_2)$. We have

$$\begin{array}{ccc} \mathbb{R}[X]_{\mathfrak{m}_x} & \hookrightarrow & \mathbb{R}[X']_{\mathfrak{m}_x} \\ \mathfrak{m}_x \mathbb{R}[X]_{\mathfrak{m}_x} & \hookleftarrow & \mathfrak{m}_1 \mathbb{R}[X']_{\mathfrak{m}_x} \\ & & \vdots \\ & & \mathfrak{m}_n \mathbb{R}[X']_{\mathfrak{m}_x} \end{array}$$

where the $\mathfrak{m}_i \in \text{Max } \mathbb{R}[X']$ are such that $\mathfrak{m}_i \cap \mathbb{R}[X] = \mathfrak{m}_x$. By definition of $\mathbb{R}[X^{R+}]$, we can write $(q \circ \pi_c)_x = \alpha \circ \pi_{\mathbb{R}}' + \beta$ with $\alpha \in \mathbb{R}[X]_x$ and $\beta \in \text{Rad}(\mathbb{R}[X']_x)$. Then

$$\forall z \in \pi_c'^{-1}(\{x\}) \quad (q \circ \pi_c)_x(z) = \alpha \circ \pi_c'(z) + \beta_c(z) = \alpha_c \circ \pi_c'(z) + \beta_c(z) = \alpha_c(x) + \beta_c(z)$$

Moreover, if we write $\pi_c'^{-1}(\{x\}) = \{x_1, \dots, x_k, z_{k+1}, \overline{z_{k+1}}, \dots, z_n, \overline{z_n}\}$, then we have

$$\beta_c \in \bigcap_{i=1}^n \mathfrak{m}_i \mathbb{C}[X']_{\mathfrak{m}_x} = \bigcap_{i=1}^k \mathfrak{m}_{x_i} \mathbb{C}[X']_{\mathfrak{m}_x} \cap \bigcap_{i=k+1}^n \mathfrak{m}_{z_i} \cap \mathfrak{m}_{\overline{z_i}} \mathbb{C}[X']_{\mathfrak{m}_x}$$

So for all $z \in \pi_c'^{-1}(\{x\})$, we get $\beta_c(z) = 0$ and finally

$$\forall z \in \pi_c'^{-1}(\{x\}) \quad q \circ \pi_c(z) = \alpha(x)$$

Conversely, let $p \in \mathbb{R}[X']$, $x \in X(\mathbb{R})$ and $\pi_c'^{-1}(\{x\}) = \{x_1, \dots, x_k, z_{k+1}, \overline{z_{k+1}}, \dots, z_n, \overline{z_n}\}$ with $x_i \in X'(\mathbb{R})$ and $z_i \in X'(\mathbb{C}) \setminus X'(\mathbb{R})$. Suppose there exists $\alpha \in \mathbb{C}$ such that, for all $z \in \pi_c'^{-1}(\{x\})$, we have $p_c(z) = \alpha$. Since $p(z) = p(\overline{z}) = \overline{p(z)} = \alpha$, we get that $\alpha \in \mathbb{R}$ and so $\alpha_x \in \mathbb{R}[X]_{\mathfrak{m}_x}$. Then we can write

$$p_x = \alpha_x + (p_x - \alpha_x) \in \mathbb{R}[X']_{\mathfrak{m}_x}$$

Moreover, for all $z \in \pi_c^{-1}(x)$, we have $(p - \alpha)_c \in \mathfrak{m}_z \mathbb{C}[X']$. So

$$\begin{aligned}
 (p_x - \alpha_x)_c &\in \mathbb{R}[X']_{\mathfrak{m}_x} \cap \bigcap_{z \in \pi_c'^{-1}(\{x\})} \mathfrak{m}_z \mathbb{C}[X']_{\mathfrak{m}_x} \\
 &= \mathbb{R}[X']_{\mathfrak{m}_x} \cap \bigcap_{i=1}^k \mathfrak{m}_{x_i} \mathbb{C}[X']_{\mathfrak{m}_x} \cap \bigcap_{i=k+1}^n \mathfrak{m}_{z_i} \cap \overline{\mathfrak{m}_{z_i}} \mathbb{C}[X']_{\mathfrak{m}_x} \\
 &= \bigcap_{i=1}^k \mathfrak{m}_{x_i} \mathbb{R}[X']_{\mathfrak{m}_x} \cap \bigcap_{i=k+1}^n (\mathfrak{m}_{z_i} \cap \overline{\mathfrak{m}_{z_i}}) \mathbb{R}[X']_{\mathfrak{m}_x} \\
 &= \bigcap_{\mathfrak{m} \cap \mathbb{R}[X] = \mathfrak{m}_x} \mathfrak{m} \mathbb{R}[X'] \\
 &= \text{Rad}(\mathbb{R}[X']_{\mathfrak{m}_x})
 \end{aligned}$$

So $p_x - \alpha_x \in \text{Rad}(\mathbb{R}[X']_{\mathfrak{m}_x})$. ■

Corollary 3.3.3

Let X be a real affine variety and $\pi' : X' \rightarrow X$ be the normalization morphism. Then

$$\mathcal{K}^{R+}(X(\mathbb{R})) \simeq \{p \in \mathbb{R}[X'] \mid \forall z_1, z_2 \in \pi_c'^{-1}(X(\mathbb{R})), \pi_c'(z_1) = \pi_c'(z_2) \implies p_c(z_1) = p_c(z_2)\}$$

Remark. Note that if $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$ and p is the element of $\mathbb{R}[X']$ above f , then for all $x \in X(\mathbb{R})$ and for all $z \in \pi_c'^{-1}(x)$, we have $p_c(z) = f(x) \in \mathbb{R}$.

In the paper [15], the authors introduced the central seminormalization (resp. weak-normalization) of real algebraic varieties. The idea of those constructions is to glue together the central points of the normalization over the central points of the variety (resp. over the maximal central points). They have shown that, for a real algebraic variety X , the coordinate ring of its central weak-normalization X^{w_c} corresponds to the integral closure of $\mathbb{R}[X]$ in the ring $\mathcal{K}^0(\text{Cent}(X))$ of continuous rational functions on the central points of $X(\mathbb{R})$. Also, the coordinate ring of its central seminormalization X^{s_c} correspond to the integral closure of $\mathbb{R}[X]$ in the ring $\mathcal{R}^0(\text{Cent}(X))$ of regulous functions on the central points of $X(\mathbb{R})$. Those are the continuous rational functions which stay rational by restriction to the real closed points of a real subvariety of X . The elements of $\mathcal{K}^0(X(\mathbb{R}))$ and $\mathcal{R}^0(X(\mathbb{R}))$ have been extensively studied in real algebraic geometry. More details can be found in Chapter 1 Section 1.2.3.

The following lemmas 3.3.4, 3.3.5 and 3.3.6 shows that the elements of $\mathcal{K}^{R+}(X(\mathbb{R}))$ are regulous integral functions and so that they become polynomial on $\mathbb{R}[X^{s_c}]$ and $\mathbb{R}[X^{w_c}]$.

Lemma 3.3.4

Let X be a real affine variety and $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. Then f is rational and there exists a monic polynomial $P(t) \in \mathbb{C}[X][t]$ such that $P(f) = 0$ on $X(\mathbb{R})$.

Proof: Let $f : X(\mathbb{R}) \rightarrow \mathbb{R}$ be such that $f \circ \pi_{\mathbb{R}}^{R+} \in \mathbb{R}[X^{R+}]$. First we have $f \in \mathcal{K}(X)$ because π^{R+} is birational.

Now, since $f \circ \pi_{\mathbb{R}}^{R+} \in \mathbb{R}[X^{R+}] \implies f \circ \pi'_{\mathbb{R}} \in \mathbb{R}[X']$, then there exists a monic polynomial $P(t) \in \mathbb{R}[X][t]$ such that

$$P(f \circ \pi'_{\mathbb{R}}) = (f \circ \pi'_{\mathbb{R}})^n + (a_{n-1} \circ \pi'_{\mathbb{R}})(f \circ \pi'_{\mathbb{R}})^{n-1} + \dots + (a_0 \circ \pi'_{\mathbb{R}}) = 0$$

Since we have an injection $\mathbb{R}[X^{R+}] \hookrightarrow \mathbb{R}[X']$, we get

$$(f \circ \pi_{\mathbb{R}}^{R+})^n + (a_{n-1} \circ \pi_{\mathbb{R}}^{R+})(f \circ \pi_{\mathbb{R}}^{R+})^{n-1} + \dots + (a_0 \circ \pi_{\mathbb{R}}^{R+}) = 0$$

Now, by Proposition 2.1.5, we have $\pi_{\mathbb{R}}^{R+}$ surjective. So

$$f^n + a_{n-1}f^{n-1} + \dots + a_0 = 0$$

So f is integral over $\mathbb{R}[X]$. ■

Lemma 3.3.5

Let X be a real affine variety and $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. Then

f is continuous for the Euclidean topology on $X(\mathbb{R})$.

Proof: Let $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$ and F be an Euclidean closed set of \mathbb{R} . Since $f \circ \pi_{\mathbb{R}}^{R+} \in \mathbb{R}[X^{R+}]$, we have that $(\pi_{\mathbb{R}}^{R+})^{-1}(f^{-1}(F)) = (f \circ \pi_{\mathbb{R}}^{R+})^{-1}(F)$ is closed. By [15] Lemma 3.1, the function $\pi_{\mathbb{R}}^{R+}$ is closed for the Euclidean topology. Then

$$\pi_{\mathbb{R}}^{R+} \left((\pi_{\mathbb{R}}^{R+})^{-1}(f^{-1}(F)) \right) = f^{-1}(F) \text{ is closed.}$$

and so f is continuous. ■

Lemma 3.3.6

Let X be a real affine variety, $V \subset X$ be a real subvariety of X and $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. Then

$f|_{V(\mathbb{R})}$ is rational

Proof: Let V be a subvariety of X . Then there exists $\mathfrak{p} \in \text{R-Spec}(\mathbb{R}[X])$ such that $\mathbb{R}[V] \simeq \mathbb{R}[X]/\mathfrak{p}$. So there is a unique $\mathfrak{q} \in \text{Spec}(\mathbb{R}[X^{R+}])$ such that $\mathfrak{q} \cap \mathbb{R}[X] = \mathfrak{p}$. We note W the subvariety of X^{R+} such that $\mathbb{R}[W] \simeq \mathbb{R}[X^{R+}]/\mathfrak{q}$. Then we get the following commutative diagram.

$$\begin{array}{ccc} \mathbb{R}[X] & \xrightarrow{(\pi_{\mathbb{R}}^{R+})^*} & \mathbb{R}[X^{R+}] \\ \downarrow & & \downarrow \\ \mathbb{R}[V] & \xrightarrow{(\pi_{|W}^{R+})^*} & \mathbb{R}[W] \\ \downarrow & & \downarrow \\ \mathcal{K}(V) & \xlongequal{\quad} & \mathcal{K}(W), \end{array}$$

We have $f \circ \pi_{\mathbb{R}}^{R+} \in \mathbb{R}[X^{R+}]$ so $f|_{V(\mathbb{R})} \circ \pi_{|W(\mathbb{R})}^{R+} = (f \circ \pi_{\mathbb{R}}^{R+})|_{W(\mathbb{R})} \in \mathbb{R}[W]$. Then $\frac{f|_{V(\mathbb{R})} \circ \pi_{|W(\mathbb{R})}^{R+}}{1} \in \mathcal{K}(W)$ and so $f|_{V(\mathbb{R})} \in \mathcal{K}(V)$. ■

Remark. We have shown that the elements of $\mathcal{K}^{R+}(X(\mathbb{R}))$ are regulous functions and that they are integral over $\mathbb{R}[X]$. So

$$\mathcal{K}^{R+}(X(\mathbb{R})) \subset \mathbb{R}[X]'_{\mathcal{R}^0(X(\mathbb{R}))} \subset \mathbb{R}[X]'_{\mathcal{K}^0(X(\mathbb{R}))}$$

In the same spirit as for X^{sc} and X^{wc} , one may wonder if the ring $\mathcal{K}^{R+}(X(\mathbb{R}))$ can also be seen as the integral closure of $\mathbb{R}[X]$ in a ring of functions over $X(\mathbb{R})$. Moreover, we might expect this ring to be the set of functions on $X(\mathbb{R})$ which are continuous for the Euclidean topology of $X(\mathbb{C})$. Indeed, the idea of the seminormalization is to glue together the points of the normalization lying over a same point of X and its coordinate ring corresponds to the ring of rational functions which are continuous on $X(\mathbb{C})$ for the Euclidean topology of $X(\mathbb{C})$. Since the idea behind the \mathbb{R} -seminormalization is to glue together the points of the normalization lying over a same real point of X , the continuity on $X(\mathbb{R})$ for the topology of $X(\mathbb{C})$ seems to be the right notion to consider. There is, in fact, a little subtlety. The elements of $\mathcal{K}^{R+}(X(\mathbb{R}))$ are only defined on $X(\mathbb{R})$ and they can not necessarily be extended on all $X(\mathbb{C})$. This is why the correct type of functions to use are the rational functions which satisfies Condition 3 of the next proposition.

Proposition 3.3.7

Let X be a real affine variety and let $f : X(\mathbb{R}) \rightarrow \mathbb{R}$. Then $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$ if and only if it verifies the following properties :

1. The function f is rational.
2. The function f is integral over $\mathbb{R}[X]$.
3. For all $x \in X(\mathbb{R})$, for all $(z_n)_n \in \text{Dom}(f) \cup X(\mathbb{R})$ such that $z_n \rightarrow x$ then $f(z_n) \rightarrow f(x)$.

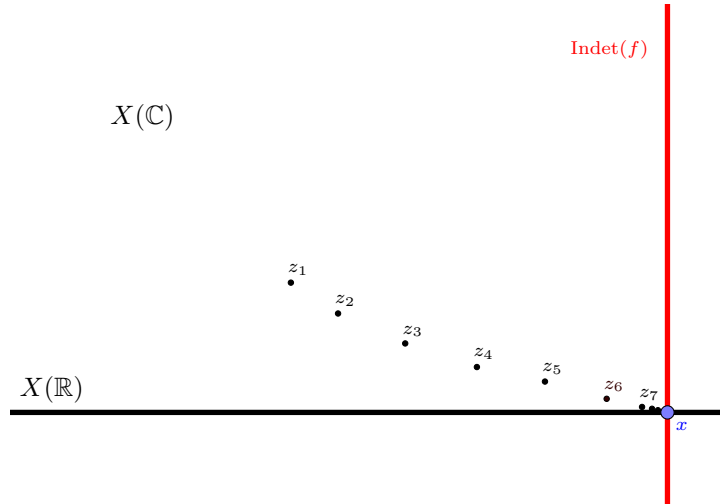


Figure 3.1 – Illustration of Condition 3 in Proposition 3.3.7

Proof: Let $f : X(\mathbb{R}) \rightarrow \mathbb{R}$ be integral over $\mathbb{R}[X]$ and suppose there exists $p, q \in \mathbb{R}[X]$ such that $f = p/q$ on $\mathcal{D}(q)$. Then there exists $g \in \mathbb{R}[X']$ such that $f \circ \pi'_c = g$ on $(\pi'_c)^{-1}(\mathcal{D}(q)) = \mathcal{D}(q \circ \pi'_c)$. Note that f can be extended on $\mathcal{D}(q_c) \cup X(\mathbb{R})$. Let $x \in X(\mathbb{R})$, by Proposition 3.3.2 and Lemma 3.3.4, we just have to show that the following propositions are equivalent

1. For all $z'_1, z'_2 \in (\pi'_c)^{-1}(x)$ we have $g_c(z'_1) = g_c(z'_2) = f(x)$.
2. For all $(z_n)_n \in \mathcal{D}(q_c) \cup X(\mathbb{R})$ such that $z_n \rightarrow x$ then $f(z_n) \rightarrow f(x)$.

We prove 1 \implies 2. Let $(z_n)_n \in \mathcal{D}(q_c) \cup X(\mathbb{R})^{\mathbb{N}}$ be such that $z_n \rightarrow x$ and suppose that $(f(z_n))_n$ is convergent. We can consider an open ball $B(x, \epsilon) \subset X(\mathbb{C})$ and an integer $N \in \mathbb{N}$ such that for all $n > N$ we have $z_n \in B(x, \epsilon)$. In particular, it is also true for $B := \overline{B(x, \epsilon)}$ which is compact. By Lemma 2.3.8, the map π'_c is a proper map and so $(\pi'_c)^{-1}(B)$ is compact. Now, if for all $n > N$ we consider $z'_n \in (\pi'_c)^{-1}(z_n)$, then we obtain a sequence $(z'_n)_{n > N}$ whose elements are contained in the compact set $(\pi'_c)^{-1}(B)$. So there exists a convergent subsequence $(z'_{n_k})_{n_k > N}$ and its limit, that we note l , belongs to $X'(\mathbb{C})$ because this set is closed for the Euclidean topology. Moreover, since π'_c is continuous, we have $z_{n_k} = \pi'_c(z'_{n_k}) \rightarrow \pi'_c(l)$. So $\pi'_c(l) = x$. Then, by continuity of g , we have $f(z_{n_k}) = g(z'_{n_k}) \rightarrow g(l) = f(x)$. This means that the limit of $(f(z_n))_n$ is $f(x)$.

Now suppose that $(f(z_n))_n$ is not necessarily convergent. Since, for all $n > N$, $f \circ \pi'_c(z'_n) = g(z'_n)$, then $\{f(z_n)\}_{n > N} \subset g((\pi'_c)^{-1}(B))$ which is a compact set because g is continuous. Then, by applying the arguments of the preceding paragraph, we can show that every convergent subsequence of $(f(z_n))_{n > N}$ admits the same limit which is $f(x)$. So the sequence $(f(z_n))_{n > N}$ is convergent and we get $f(z_n) \rightarrow f(x)$.

Now we prove 2 \implies 1. Let $z \in (\pi'_c)^{-1}(x)$. Since $\mathcal{D}(q \circ \pi'_c)$ is dense in $X'(\mathbb{C})$, we can consider a sequence $(z_n)_n \in \mathcal{D}(q \circ \pi'_c)^{\mathbb{N}}$ such that $z_n \rightarrow z$. By continuity of π'_c , we have $\pi'_c(z_n) \rightarrow \pi'_c(z) = x$, so $f(\pi'_c(z_n)) \rightarrow f(\pi'_c(z))$, by assumption on f . The function g being continuous, we have $g(z_n) \rightarrow g(z)$ and since for all $n \in \mathbb{N}$, $g(z_n) = f \circ \pi'_c(z_n)$, we get that $g(z) = f(\pi'_c(z)) = f(x)$. ■

In the case of real curves, the poles of f are small enough for f to be well defined in a neighborhood of $X(\mathbb{R})$ in $X(\mathbb{C})$. Thus we get the following corollary.

Corollary 3.3.8

Let X be a real curve. Then

$$\mathcal{K}^{R+}(X(\mathbb{R})) = \left\{ f : X(\mathbb{R}) \rightarrow \mathbb{R} \mid f \in \mathbb{R}[X]' \text{ and } f \text{ is continuous on } X(\mathbb{R}) \text{ in } X(\mathbb{C}) \right\}$$

Proof: Let $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. Since X is a curve, then $\text{Dom}(f_c) = X(\mathbb{C}) \setminus \{\text{finite number of points}\}$. So, for all $x \in X(\mathbb{R})$, there exists $\epsilon > 0$ such that $B_{X(\mathbb{C})}(x, \epsilon) \cap \text{Dom}(f)^c = \emptyset$. By the previous proposition, for all $(z_n)_n \in B_{X(\mathbb{C})}(x, \epsilon)^{\mathbb{N}}$ such that $z_n \rightarrow x$, then $f(z_n) \rightarrow f(x)$. So f is continuous at x in $X(\mathbb{C})$. ■

We want to give a characterization of the elements of $\mathcal{K}^{R+}(X(\mathbb{R}))$ with properties on their graphs as in Theorem 2.3.20. This will allow us, for instance, to provide non trivial examples of elements of $\mathcal{K}^{R+}(X(\mathbb{R}))$. We start by proving two necessary conditions on the graph for a function to be in $\mathcal{K}^{R+}(X(\mathbb{R}))$.

Lemma 3.3.9

Let X be a real affine variety and $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. Then the graph Γ_f is Zariski closed in $X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$.

Proof: Let $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$ and $\pi^{R+} : X^{R+} \rightarrow X$ be the R-seminormalization morphism. By Theorem 3.2.1, we have that $\pi_{\mathbb{R}}^{R+}$ is bijective. So

$$\Gamma_f = \{(x; f(x)) \mid x \in X(\mathbb{R})\} = \{(\pi_{\mathbb{R}}^{R+}(x); f \circ \pi_{\mathbb{R}}^{R+}(x)) \mid x \in X^{R+}(\mathbb{R})\}$$

But since $f \circ \pi_{\mathbb{R}}^{R+}$ is a polynomial and, by Proposition 3.1.3, the morphism $\pi_{\mathbb{R}}^{R+} \times \text{Id}$ is a \mathbb{Z} -homeomorphism, we get that

$$\left\{ \left(\pi_{\mathbb{R}}^{R+}(x); f \circ \pi_{\mathbb{R}}^{R+}(x) \right) \mid x \in X^{R+}(\mathbb{R}) \right\} = \pi_{\mathbb{R}}^{R+} \times \text{Id} \left(\Gamma_{f \circ \pi_{\mathbb{R}}^{R+}} \right)$$

is \mathbb{Z} -closed and so that Γ_f is a \mathbb{Z} -closed subset of $X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$. ■

Notation. We will denote by $\overline{\Gamma_f}^{\mathbb{C}}$ the Zariski closure of Γ_f in the set $X(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C})$.

Lemma 3.3.10

Let X be a real affine variety and $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. Then, for all $x \in X(\mathbb{R})$, we have

$$\overline{\Gamma_f}^{\mathbb{C}} \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) = \{(x; f(x))\}$$

Proof: Let $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. We can consider $p, q \in \mathbb{R}[X]$ such that $f = p/q$ if $q \neq 0$. Let $(x, t) \in \overline{\Gamma_f}^{\mathbb{C}} \cap (\{x\} \times \mathbb{A}^1(\mathbb{C}))$ with $x \in X(\mathbb{R})$ and consider the \mathbb{Z} -dense \mathbb{Z} -open set $\mathcal{D}(q) \subset \overline{\Gamma_f}^{\mathbb{C}}$. Since it is also dense for the Euclidean topology, we can consider a sequence $(z_n, t_n)_n \in \mathcal{D}(q)^{\mathbb{N}}$ such that $(z_n, t_n) \rightarrow (x, t)$. Moreover, we have

$$\overline{\Gamma_f}^{\mathbb{C}} \subset \{qct - p_c\} \subset X(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C})$$

and so we get, for all $n \in \mathbb{N}$, that $(z_n, t_n) = (z_n, p(z_n)/q(z_n))$. So, by Proposition 3.3.7, we have

$$(z_n, t_n) = (z_n, f(z_n)) \rightarrow (x, f(x))$$

This means that $\overline{\Gamma_f}^{\mathbb{C}} \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) = \{(x; f(x))\}$, which conclude the proof. ■

We can now show that if an integral rational function on $X(\mathbb{R})$ verifies Lemma 3.3.9 and 3.3.10, then it is an element of $\mathcal{K}^{R+}(X(\mathbb{R}))$.

Theorem 3.3.11

Let X be a real affine variety and let $f : X(\mathbb{R}) \rightarrow \mathbb{R}$. Then $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$ if and only if it verifies the following properties :

- 1) $f \in \mathcal{K}(X)$
- 2) There exists a monic polynomial $P(t) \in \mathbb{R}[X][t]$ such that $P(f) = 0$ on $X(\mathbb{R})$.
- 3) The graph Γ_f is Zariski closed in $X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$.
- 4) For all $x \in X(\mathbb{R})$, we have $\overline{\Gamma_f}^{\mathbb{C}} \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) = \{(x; f(x))\}$

Proof: Let $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. Then, by Lemma 3.3.4, 3.3.9 and 3.3.10, it verifies the four properties of the proposition. Conversely, suppose that $f : X(\mathbb{R}) \rightarrow \mathbb{R}$ verifies the four properties above. We consider the map

$$\begin{aligned} \psi : \mathbb{R}[X][t] &\rightarrow \mathcal{K}(X) \\ Q(t) &\mapsto Q(f) \end{aligned}$$

and write $\mathbb{R}[Y] \simeq \mathbb{R}[X][t]/\ker \psi \simeq \mathbb{R}[X][f]$ with $\pi : Y \rightarrow X$ the morphism induced by $\mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$. We then

have

$$\mathbb{R}[X] \hookrightarrow \mathbb{R}[Y] \simeq \mathbb{R}[X][f] \subset \mathcal{K}(X)$$

So $\mathcal{K}(X) \simeq \mathcal{K}(Y)$ and π is birational. Moreover $\mathbb{R}[Y]$ is a finite $\mathbb{R}[X]$ -module because so is $\mathbb{R}[X][t]/\langle P(t) \rangle$ and

$$\mathbb{R}[Y] \simeq \mathbb{R}[X][t]/\ker \psi \simeq (\mathbb{R}[X][t]/\langle P(t) \rangle)/(\ker \psi/\langle P(t) \rangle)$$

Hence $\pi : Y \rightarrow X$ is a finite birational morphism. We want to show that the restriction $\tilde{\pi} : \pi_c^{-1}(X(\mathbb{R})) \rightarrow X(\mathbb{R})$ of the morphism π_c is bijective. By assumption, we can consider the real ideal $I_f := \mathcal{I}(\Gamma_f)$ and we have $\Gamma_f = \mathcal{Z}(I_f)$. One can see that $I_f \subset \ker \psi$ because

$$\forall Q \in I_f \quad \forall x \in X(\mathbb{R}) \quad Q(x, f(x)) = 0$$

So

$$Y(\mathbb{C}) = \mathcal{Z}_{\mathbb{C}}(\ker \psi) \subseteq \mathcal{Z}_{\mathbb{C}}(I_f) = \overline{\Gamma_f}^{\mathbb{C}}$$

Let $x \in X(\mathbb{R})$. By the fourth condition, we have

$$\pi_c^{-1}(x) = Y(\mathbb{C}) \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) \subseteq \overline{\Gamma_f}^{\mathbb{C}} \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) = \{(x; f(x))\}$$

and since $\pi_c : Y(\mathbb{C}) \rightarrow X(\mathbb{C})$ is finite, then it is surjective and we get $\pi_c^{-1}(x) = \{(x; f(x))\}$. We have shown that π is a finite birational morphism and $\tilde{\pi}$ is bijective. From the universal property of the \mathbb{R} -seminormalization, we get

$$\mathbb{R}[X] \hookrightarrow \mathbb{R}[Y] \hookrightarrow \mathbb{R}[X^{R+}]$$

So $f \circ \pi_{\mathbb{R}}^{R+} \in \mathbb{R}[X^{R+}]$. ■

Example. Let $X = \text{Spec}(\mathbb{R}[x, y]/\langle y^3 - x^2y^2 + yx^2(x+1) - x^4(x+1) \rangle)$ and consider

$$f = \begin{cases} y/x & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$$

We have that f is a root of the polynomial $P_f(t) = t^3 - xt^2 + t(x+1) - x(x+1)$. Since 0 is the only real root of P_f when $x = 0$, we get that Γ_f is a \mathbb{Z} -closed subset of $X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$ given by

$$\Gamma_f = \begin{cases} y^3 - x^2y^2 + yx^2(x+1) - x^4(x+1) = 0 \\ xt - y = 0 \\ t^3 - xt^2 + t(x+1) - x(x+1) = 0 \end{cases}$$

But it doesn't verify the fourth condition because $\overline{\Gamma_f}^{\mathbb{C}} \cap (\{0\} \times \mathbb{A}^1(\mathbb{C})) = \{(0; 0), (0; \pm i)\}$

3.4 Rational functions extending continuously on the complex points.

In the case of complex varieties, it has been shown in Chapter 2 that the ring of polynomial functions on the seminormalization corresponds to the ring $\mathcal{K}^0(X(\mathbb{C}))$ of rational functions that extend continuously

on $X(\mathbb{C})$. The purpose of this section is to show that, in the case of real varieties, the real valued functions which are the restriction to $X(\mathbb{R})$ of an element of $\mathcal{K}^0(X(\mathbb{C}))$ correspond to the polynomial functions on the seminormalization. Those functions can also be seen as the elements f in $\mathcal{K}^0(X(\mathbb{C}))$ such that $\overline{f(\bar{z})} = f(z)$, for all $z \in X(\mathbb{C})$. We end this section by giving a characterization of those functions with their graphs, as in Theorem 2.3.20 or like Theorem 3.3.11. This will allow us to construct examples of different functions that become polynomials on X^+ , X^{R+} , X^{sc} or X^{wc} .

Definition 3.4.1

Let X be a real affine variety, we define the set

$$\mathcal{K}^+(X(\mathbb{R})) := \{f : X(\mathbb{R}) \rightarrow \mathbb{R} \mid \exists g \in \mathcal{K}^0(X(\mathbb{C})) \text{ such that } g|_{X(\mathbb{R})} = f\}$$

Remark. By Proposition 3.3.7, it is clear that $\mathcal{K}^+(X(\mathbb{R})) \subset \mathcal{K}^{R+}(X(\mathbb{R}))$. Note also that those rings inject in $\mathcal{K}(X)$ and so, even if their elements are defined on all $X(\mathbb{R})$, they are uniquely represented by the equivalent class of a rational representation.

Every real rational function can be extended as a complex rational function. The next lemma says that, for an element of $\mathcal{K}^+(X(\mathbb{R}))$, this extension is a rational representation of an element of $\mathcal{K}^0(X(\mathbb{C}))$.

Lemma 3.4.2

Let $f : X(\mathbb{R}) \rightarrow \mathbb{R}$ and $p, q \in \mathbb{R}[X]$ be such that $f = p/q$ if $q \neq 0$. Then the following statements are equivalent

1. The function f belongs to $\mathcal{K}^+(X(\mathbb{R}))$.
2. The rational function $p_{\mathbb{C}}/q_{\mathbb{C}}$ extends continuously to an element $f_{\mathbb{C}} \in \mathcal{K}^0(X(\mathbb{C}))$.

Proof: Suppose that $f \in \mathcal{K}^+(X(\mathbb{R}))$, then there exists $g = p'/q' \in \mathcal{K}^0(X(\mathbb{C}))$ such that $g|_{X(\mathbb{R})} = f$. Then we have $p'q_{\mathbb{C}} = q'p_{\mathbb{C}}$ on $X(\mathbb{R})$. So $p'q_{\mathbb{C}} = q'p_{\mathbb{C}}$ on $X(\mathbb{C})$ because $X(\mathbb{R})$ is \mathbb{Z} -dense in $X(\mathbb{C})$. So we get $g = f_{\mathbb{C}}$ on $\mathcal{D}(q_{\mathbb{C}}) = \mathcal{D}(q')$ and finally $g = f_{\mathbb{C}}$ on $X(\mathbb{C})$ by continuity. Conversely, it is clear that if $f_{\mathbb{C}} \in \mathcal{K}^0(X(\mathbb{C}))$, then $f = (f_{\mathbb{C}})|_{X(\mathbb{R})} \in \mathcal{K}^+(X(\mathbb{R}))$. ■

By definition, we know that for each element $f \in \mathcal{K}^+(X(\mathbb{R}))$, there is an extension $f_{\mathbb{C}}$ that belongs to $\mathcal{K}^0(X(\mathbb{C}))$. One can ask the reverse question : what are the elements of $\mathcal{K}^0(X(\mathbb{C}))$ which are the extension of an element of $\mathcal{K}^+(X(\mathbb{R}))$? The answer is given by the next proposition.

Notation. Let $E \subset \mathcal{F}(X(\mathbb{C}); \mathbb{C})$ and $f \in E$. We note σf the function defined by $\sigma f(z) = \overline{f(\bar{z})}$, for all $z \in X(\mathbb{C})$. We will say that f is σ -invariant if $\sigma f = f$ and we note σE the elements of E which are σ -invariant.

Proposition 3.4.3

Let X be a real algebraic variety. Then we have the following isomorphism

$$\begin{array}{ccc} \psi : \sigma \mathcal{K}^0(X(\mathbb{C})) & \xrightarrow{\sim} & \mathcal{K}^+(X(\mathbb{R})) \\ & & g \quad \rightarrow \quad g|_{X(\mathbb{R})} \end{array}$$

Proof: First of all, the morphism ψ is well defined because if $g \in {}^\sigma \mathcal{K}^0(X(\mathbb{C}))$, then for all $x \in X(\mathbb{R})$ we have $g(x) = {}^\sigma g(x) = \overline{g(\bar{x})} = \overline{g(x)}$ and so $g(x) \in \mathbb{R}$. The morphism ψ is surjective because, by Lemma 3.4.2, if $f \in \mathcal{K}^+(X(\mathbb{R}))$, then $f_{\mathbb{C}} \in \mathcal{K}^0(X(\mathbb{C}))$ and it is clear that $f_{\mathbb{C}}$ is σ -invariant. Let us see now that it is injective. Let $g_1, g_2 \in \mathcal{K}^0(X(\mathbb{C}))$ be such that $g_1|_{X(\mathbb{R})} = g_2|_{X(\mathbb{R})}$. We can write $g_1 = p_1/q_1$ on $\mathcal{D}(q_1)$ and $g_2 = p_2/q_2$ on $\mathcal{D}(q_2)$ for $p_i, q_i \in \mathbb{C}[X]$. Then $q_1 q_2 g_1 = q_1 q_2 g_2$ on $X(\mathbb{R})$, so $p_1 q_2 = p_2 q_1$ on $X(\mathbb{R})$. Since $X(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})$, the relation extends to $X(\mathbb{C})$ and so $p_1/q_1 = p_2/q_2$ on $\mathcal{D}(q_1 q_2)$. Hence $g_1 = g_2$ by continuity. ■

We obtain a real version of Theorem 2.3.13 saying that the polynomial functions on the seminormalization correspond to the rational functions of X which extend continuously on $X(\mathbb{C})$.

Corollary 3.4.4

Let X be a real algebraic variety. Then we have the following isomorphism

$$\begin{array}{ccc} \mathcal{K}^+(X(\mathbb{R})) & \xrightarrow{\sim} & \mathbb{R}[X^+] \\ f & \mapsto & f \circ \pi_{\mathbb{R}}^+ \end{array}$$

Proof: By Theorem 2.3.13, we have the following isomorphism

$$\begin{array}{ccc} \mathcal{K}^0(X(\mathbb{C})) & \xrightarrow{\sim} & \mathbb{C}[X^+] \\ f & \mapsto & f \circ \pi_{\mathbb{C}}^+ \end{array}$$

Let us show that it induces an isomorphism on the σ -invariant elements

$$\begin{array}{ccc} {}^\sigma \mathcal{K}^0(X(\mathbb{C})) & \xrightarrow{\sim} & {}^\sigma \mathbb{C}[X^+] \\ f & \mapsto & f \circ \pi_{\mathbb{C}}^+ \end{array}$$

Let $f \in {}^\sigma \mathcal{K}^0(X(\mathbb{C}))$ and $z \in X(\mathbb{C})$, since ${}^\sigma \pi_{\mathbb{C}}^+ = \pi_{\mathbb{C}}^+$, we get $\pi_{\mathbb{C}}^+(\bar{z}) = \overline{{}^\sigma \pi_{\mathbb{C}}^+(z)} = \overline{\pi_{\mathbb{C}}^+(z)}$ and so

$$\overline{f \circ \pi_{\mathbb{C}}^+(\bar{z})} = \overline{f(\pi_{\mathbb{C}}^+(\bar{z}))} = \overline{f(\overline{\pi_{\mathbb{C}}^+(z)})} = {}^\sigma f(\pi_{\mathbb{C}}^+(z)) = f(\pi_{\mathbb{C}}^+(z))$$

Since ${}^\sigma \mathbb{C}[X^+] \simeq \mathbb{R}[X^+]$, we deduce from the previous proposition that

$$\begin{array}{ccccccc} \mathcal{K}^+(X(\mathbb{R})) & \xrightarrow{\sim} & {}^\sigma \mathcal{K}^0(X(\mathbb{C})) & \xrightarrow{\sim} & {}^\sigma \mathbb{C}[X^+] & \xrightarrow{\sim} & \mathbb{R}[X^+] \\ f & \mapsto & f_{\mathbb{C}} & \mapsto & f_{\mathbb{C}} \circ \pi_{\mathbb{C}}^+ & \mapsto & (f_{\mathbb{C}} \circ \pi_{\mathbb{C}}^+)|_{X(\mathbb{R})} \end{array}$$

One can see that $(f_{\mathbb{C}} \circ \pi_{\mathbb{C}}^+)|_{X(\mathbb{R})} = f \circ \pi_{\mathbb{R}}^+$ so it concludes the proof. ■

We now give a characterization of the elements of $\mathcal{K}^+(X(\mathbb{R}))$ with their graphs. It can be seen as the real version of Theorem 2.3.20.

Lemma 3.4.5

Let X be a real affine variety and $f \in \mathcal{K}^+(X(\mathbb{R}))$. Let $P \in \mathbb{R}[X][t]$ such that $P(f) = 0$ on $X(\mathbb{R})$, then $P(f_{\mathbb{C}}) = 0$ on $X(\mathbb{C})$.

Proof: Let p and q be such that $f = p/q$ on $\mathcal{D}(q)$ and let $P \in \mathbb{C}[X][t]$ be such that $P(f) = 0$ on $X(\mathbb{R})$. In particular $P(p/q) = 0$ on $\mathcal{D}(q)$. Then $q_{\mathbb{C}}^{\deg(P)} P(p_{\mathbb{C}}/q_{\mathbb{C}}) = 0$ on $X(\mathbb{R})$. Since $X(\mathbb{R})$ is \mathbb{Z} -dense in $X(\mathbb{C})$, we get

$q_{\mathbb{C}}^{\deg(P)} P(p_{\mathbb{C}}/q_{\mathbb{C}}) = 0$ on $X(\mathbb{C})$. Thus $P(f) = P(p_{\mathbb{C}}/q_{\mathbb{C}}) = 0$ on $\mathcal{D}(q_{\mathbb{C}})$ and finally, by continuity, we get $P(f) = 0$ on $X(\mathbb{C})$. ■

Theorem 3.4.6

Let X be a real affine variety and $f : X(\mathbb{R}) \rightarrow \mathbb{R}$, then $f \in \mathcal{K}^+(X(\mathbb{R}))$ if and only if it verifies the following properties :

- 1) $f \in \mathcal{K}(X)$
- 2) There exists a monic polynomial $P \in \mathbb{R}[X][t]$ such that $P(f) = 0$ on $X(\mathbb{R})$.
- 3) The graph Γ_f is Zariski closed in $X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$
- 4) For all $z \in X(\mathbb{C})$, we have $\# \left(\overline{\Gamma_f}^{\mathbb{C}} \cap \{z\} \times \mathbb{A}^1(\mathbb{C}) \right) = 1$

Remark. If $f \in \mathcal{K}^+(X(\mathbb{R}))$ then the last property says that $\overline{\Gamma_f}^{\mathbb{C}} = \Gamma_{f_{\mathbb{C}}}$.

Proof: Let $f \in \mathcal{K}^+(X(\mathbb{R}))$. Then $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$ and by Theorem 3.3.11 the function f verifies condition 1, 2 and 3. By Lemma 3.4.2, we have $f_{\mathbb{C}} \in \mathcal{K}^0(X(\mathbb{C}))$ and by Theorem 2.3.20, we get that $\Gamma_{f_{\mathbb{C}}}$ is Z-closed. Since $\Gamma_f \subset \Gamma_{f_{\mathbb{C}}}$, we have $\overline{\Gamma_f}^{\mathbb{C}} \subset \Gamma_{f_{\mathbb{C}}}$. Moreover, by Lemma 3.4.5, we have that $f_{\mathbb{C}}$ is a complex root of every polynomial defining Γ_f . So if I_f is a real ideal such that $\mathcal{Z}_{\mathbb{R}}(I_f) = \Gamma_f$, then $\Gamma_{f_{\mathbb{C}}} \subset \mathcal{Z}_{\mathbb{C}}(I_f) = \overline{\Gamma_f}^{\mathbb{C}}$. We obtain $\Gamma_{f_{\mathbb{C}}} = \overline{\Gamma_f}^{\mathbb{C}}$ and so f verifies Condition 4).

Conversely, let $f : X(\mathbb{R}) \rightarrow \mathbb{R}$ be a function that verifies all the four conditions. Thanks to Condition 4), we can define a function $g : X(\mathbb{C}) \rightarrow \mathbb{C}$ such that

$$\forall z \in X(\mathbb{C}) \quad \left\{ (z; g(z)) \right\} = \overline{\Gamma_f}^{\mathbb{C}} \cap \{z\} \times \mathbb{A}^1(\mathbb{C})$$

Then $\Gamma_g = \overline{\Gamma_f}^{\mathbb{C}}$ so g is rational, integral over $\mathbb{C}[X]$ and Γ_g is Z-closed. By Theorem 2.3.20, this mean that $g \in \mathcal{K}^0(X(\mathbb{C}))$. Moreover $\Gamma_f \subset \Gamma_g$, so $g|_{X(\mathbb{R})} = f$ and we get $f \in \mathcal{K}^+(X(\mathbb{R}))$. ■

Thanks to Theorems 3.3.11 and 3.4.6, we give several examples of functions of $\mathcal{K}^+(X(\mathbb{R}))$ and $\mathcal{K}^{R+}(X(\mathbb{R}))$. In particular, we show that the following inclusions are strict in general.

$$\mathbb{R}[X] \subset \mathcal{K}^+(X(\mathbb{R})) \subset \mathcal{K}^{R+}(X(\mathbb{R})) \subset \mathbb{R}[X]_{\mathcal{R}^0(\text{Cent}(X))}' \subset \mathbb{R}[X]_{\mathcal{K}^0(\text{Cent}(X))}'$$

Example. We give an example of a function in $\mathcal{K}^+(X(\mathbb{R})) \setminus \mathbb{R}[X]$. Let $X = \text{Spec}(\mathbb{R}[x, y] / \langle y^2 - x^3 \rangle)$ and consider the function

$$f = \begin{cases} y/x & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$$

We have that f is a root of the polynomial $P(t) = t^2 - x$. Since 0 is the only complex root of P when $x = 0$, we get that Γ_f is Z-closed in $X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$ and that $\Gamma_{f_{\mathbb{C}}}$ is Z-closed in $X(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C})$. So by Theorem 3.4.6, we have $f \in \mathcal{K}^+(X(\mathbb{R}))$.

Example. We give an example of a function in $\mathcal{K}^{R+}(X(\mathbb{R})) \setminus \mathcal{K}^+(X(\mathbb{R}))$. Consider $X = \text{Spec}(\mathbb{R}[x, y]/\langle y^2 - x^3(x^2 + 1)^2 \rangle)$ and the function

$$f = \begin{cases} y/x(x^2 + 1) & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$$

We have that f is a root of the polynomial $P(t) = t^2 - x$. Since 0 is the only complex root of P when $x = 0$, we get that Γ_f is Z -closed in $X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$ and that, for all $x \in X(\mathbb{R})$, we have $\overline{\Gamma_f}^{\mathbb{C}} \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) = \{(x; f(x))\}$. So by Theorem 3.3.11, we get $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. However, if $x = \pm i$, then $P(t)$ has two distinct complex roots. So $\overline{\Gamma_f}^{\mathbb{C}}$ is not the graph of $f_{\mathbb{C}}$ extended by continuity.

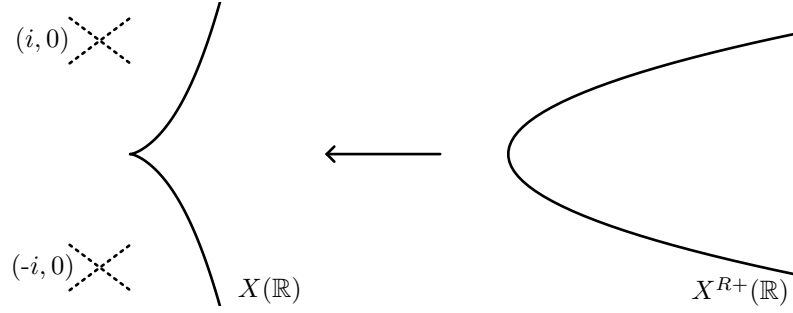


Figure 3.2 – The R -seminormalization of $\{y^2 - x^3(x^2 + 1)^2 = 0\}$.

Example. We give an example of a function in $\mathbb{R}[X]'_{\mathcal{R}^0(\text{Cent}(X))} \setminus \mathcal{K}^{R+}(X(\mathbb{R}))$. Consider the central variety $X = \text{Spec}(\mathbb{R}[x, y]/\langle y^4 - x(x^2 + y^2) \rangle)$ and the function

$$f = \begin{cases} y^2/x & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$$

which is a root of the polynomial $P_{(x,y)}(t) = t^2 - t - x$. Moreover f is continuous so $f \in \mathbb{R}[X]'_{\mathcal{K}^0(X(\mathbb{R}))}$. Since X is a curve, we have $\mathcal{K}^0(X(\mathbb{R})) = \mathcal{R}^0(X(\mathbb{R}))$ and so $f \in \mathbb{R}[X]'_{\mathcal{R}^0(X(\mathbb{R}))}$. However, since P has two roots if $x = 0$, then the graph $\Gamma_f \not\subseteq \mathcal{Z}(\langle y^4 - x(x^2 + y^2); xt - y^2; t^2 - t - x \rangle)$ is not Z -closed. By Lemma 3.3.10, we get that $f \notin \mathcal{K}^{R+}(X(\mathbb{R}))$.

Example. We give an example of a function in $\mathbb{R}[X]'_{\mathcal{K}^0(\text{Cent}(X))} \setminus \mathbb{R}[X]'_{\mathcal{R}^0(\text{Cent}(X))}$. Consider the variety $X = \text{Spec}(\mathbb{R}[x, y]/\langle x^3 - y^3(1 + z^2) \rangle)$ and the function

$$f = \begin{cases} x/y & \text{if } y \neq 0 \\ \sqrt[3]{1 + z^2} & \text{else} \end{cases}$$

We have that f is continuous, rational and is a root of the polynomial $P(t) = t^3 - 1 - z^2$. But it is not in $\mathcal{R}^0(\text{Cent}(X))$.

To conclude, we summarize this section with the following diagram

$$\begin{array}{ccccccccc}
 \mathbb{R}[X] & \hookrightarrow & \mathbb{R}[X^+] & \hookrightarrow & \mathbb{R}[X^{R^+}] & \hookrightarrow & \mathbb{R}[X^{s_c}] & \hookrightarrow & \mathbb{R}[X^{w_c}] & \hookrightarrow & \mathbb{R}[X'] \\
 \simeq \uparrow & & \simeq \uparrow & & \simeq \uparrow & & \simeq \uparrow & & \simeq \uparrow & & \simeq \uparrow \\
 \mathbb{R}[X] & \hookrightarrow & \mathcal{K}^+(X(\mathbb{R})) & \hookrightarrow & \mathcal{K}^{R^+}(X(\mathbb{R})) & \hookrightarrow & \mathbb{R}[X]_{\mathcal{R}^0(\text{Cent}(X))}' & \hookrightarrow & \mathbb{R}[X]_{\mathcal{K}^0(\text{Cent}(X))}' & \hookrightarrow & \mathbb{R}[X]_{\mathcal{K}(X)}'
 \end{array}$$

3.5 Biregular normalization and R -seminormalization.

Let A be a ring, we say that A satisfies the condition **(mp)** if A is reduced and has a finite number of minimal primes which are all real ideals. Note that the ring of polynomial functions on an algebraic set always verifies condition **(mp)**. Moreover, by [14] Lemma 2.8, if A satisfies **(mp)** and B is such that $A \hookrightarrow B \hookrightarrow A'$, then B satisfy **(mp)**. For such a ring, we have an injection

$$A \hookrightarrow A/\mathfrak{p}_1 \times \dots \times A/\mathfrak{p}_n \hookrightarrow \kappa(\mathfrak{p}_1) \times \dots \times \kappa(\mathfrak{p}_n) \simeq \mathcal{K}(A)$$

where the \mathfrak{p}_i are the minimal primes of A . Remark that the A/\mathfrak{p}_i are real rings and so A is also a real ring.

In the paper [14], the authors introduced the biregular normalization A^b of a ring A which is defined in the following way. Let A be a ring satisfying condition **(mp)** and let $\mathcal{T}(A)$ be the multiplicative part $1 + \sum A^2$ of A which does not contain any zero divisors since A is real. Then we can consider $\mathcal{O}(A) := \mathcal{T}(A)^{-1}A$ and the biregular normalization A^b is define as the integral closure of A in $\mathcal{O}(A)$. This ring can also be defined as the biggest biregular subextension of $A \hookrightarrow A'$. We recall here Proposition 4.13 from [14] :

Proposition. *Let A be a ring which verifies **(mp)** and let B such that $A \hookrightarrow B \hookrightarrow A'$. Then the following statements are equivalent :*

1. *The extension $A \hookrightarrow B$ is biregular.*
2. *For all $\mathfrak{m} \in \text{R-Max}(A)$, there exists a unique $\mathfrak{m}' \in \text{max}(B)$ such that $\mathfrak{m}' \cap A = \mathfrak{m}$. Moreover, the morphism $A_{\mathfrak{m}} \rightarrow B_{\mathfrak{m}'}$ is an isomorphism.*
3. *For all $\mathfrak{p} \in \text{R-Spec}(A)$, there exists a unique $\mathfrak{q} \in \text{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Moreover, the morphism $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is an isomorphism.*

For a more detailed presentation of the biregular normalization, one can read Chapter 1 Section 1.4.1.

We will say that a ring A is a real affine ring if it is the coordinate ring of a real affine variety. In this case it satisfy the condition **(mp)** and we have shown in Corollary 3.2.4 that $A^{R^+\text{max}} = A^{R^+}$. Moreover, by Corollary 2.2.7, we also have $A^{+\text{max}} = A^+$.

The goal of this section is to compare the notion of biregular normalization, seminormalization and R -seminormalization. We start by giving two lemmas that will lead us to the first comparison.

Lemma 3.5.1

Let $A \hookrightarrow B$ be an integral extension of rings with A that satisfy **(mp)**. If $A \hookrightarrow B$ is biregular, then $A \hookrightarrow B$ is R -subintegral.

Proof: Let $\mathfrak{p} \in \text{R-Spec}(A)$, then by [14] Proposition 4.13, there exists $\mathfrak{q} \in \text{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$ and $A_{\mathfrak{p}} \simeq B_{\mathfrak{q}}$. So we get

$$\kappa(\mathfrak{p}) \simeq A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \simeq B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}} \simeq \kappa(\mathfrak{q})$$

and finally $A \hookrightarrow B$ is R-subintegral. ■

Remark. As a consequence, we get $A^b \subset A^{R+}$. More precisely, we have the following commutative diagram

$$\begin{array}{ccccc}
 & & A^{+\max} \hookrightarrow & A^{R+\max} \hookrightarrow & A' \\
 & & \uparrow \simeq & \uparrow \simeq & \\
 A \hookrightarrow & A^+ \hookrightarrow & A^{R+} & & \\
 \searrow & & \nearrow & & \\
 & & A^b \hookrightarrow & &
 \end{array}$$

Lemma 3.5.2

Let $A \hookrightarrow B$ be an integral extension of rings and let $A \hookrightarrow C \hookrightarrow A_B^{\text{F}}$ be a subextension of the F-seminormalization of A in B . Then

$$A_B^{\text{F}} = C_B^{\text{F}}$$

Proof: By Proposition 3.1.5, the extension $C \hookrightarrow A_B^{\text{F}}$ is F-subintegral. So, by the universal property of C_B^{F} , we get

$$A \hookrightarrow C \hookrightarrow A_B^{\text{F}} \hookrightarrow C_B^{\text{F}} \hookrightarrow B$$

and the first three extensions are F-subintegral. This implies that $A \hookrightarrow C_B^{\text{F}}$ is F-subintegral and so, by the universal property of C_B^{F} , we obtain $C_B^{\text{F}} \hookrightarrow A_B^{\text{F}}$. ■

By applying this lemma to the subextensions of the previous diagram, we obtain the first comparisons between the considered notions.

Proposition 3.5.3

Let A be a real affine ring. Then

1. $(A^b)^{R+} = (A^{R+})^b = A^{R+}$
2. $(A^+)^{R+} = (A^{R+})^+ = A^{R+}$
3. $(A^b)^+ \subset A^{R+}$
4. $(A^+)^b \subset A^{R+}$

Proof: By applying the previous proposition with $\text{F} = \text{R-Spec}$ to the subextension $A \hookrightarrow A^b \hookrightarrow A^{R+}$, we get $(A^b)^{R+} = A^{R+}$. Moreover $A^{R+} \hookrightarrow (A^{R+})^b \hookrightarrow (A^{R+})^{R+}$, so $(A^{R+})^b = A^{R+}$ and 1 is proved. One can do the exact same thing with the subextension $A \hookrightarrow A^+ \hookrightarrow A^{R+}$ to get 2. The two inclusions 3 and 4 follows because we have

$$A^b \hookrightarrow (A^b)^+ \hookrightarrow (A^b)^{R+} = A^{R+}$$

and

$$A^+ \hookrightarrow (A^+)^b \hookrightarrow (A^+)^{R+} = A^{R+} \quad \blacksquare$$

We now prove that the inclusion 3 of the previous proposition is an equality. It will be the first part of the main theorem of this section.

Proposition 3.5.4

Let A be a real affine ring. Then

$$(A^b)^+ = A^{R+}$$

Proof: By Proposition 3.5.3, we have $(A^b)^+ \hookrightarrow (A^b)^{R+}$. We want to show that the extension $A^b \hookrightarrow (A^b)^{R+}$ is subintegral. Since we deal with affine rings, by Theorem 3.2.1, it is enough to show that it is Max-subintegral.

Let $\mathfrak{m}^b \in \text{Max}(A^b)$ and let us consider $\mathfrak{m} := \mathfrak{m}^b \cap A \in \text{Max}(A)$. If $\mathfrak{m} \in \text{R-Max}(A)$, because $A \hookrightarrow A^b$ is biregular, then \mathfrak{m}^b is the unique prime ideal of A^b above \mathfrak{m} and \mathfrak{m}^b is real. But since $A^b \hookrightarrow (A^b)^{R+}$ is R -subintegral, there exists a unique prime ideal of $(A^b)^{R+}$ above \mathfrak{m}^b .

Now, if $\mathfrak{m} \notin \text{R-Max}(A)$, then we have

$$(A^b)_{\mathfrak{m}} \hookrightarrow ((A^b)^{R+})_{\mathfrak{m}} \hookrightarrow (A^b)'_{\mathfrak{m}} = A'_{\mathfrak{m}}$$

By [14] Proposition 4.6, we have $(A^b)_{\mathfrak{m}} = A'_{\mathfrak{m}}$ so it follows that $(A^b)_{\mathfrak{m}} = ((A^b)^{R+})_{\mathfrak{m}}$. So if we consider $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{Max}(A^{R+})$ such that $\mathfrak{q}_1 \cap A^b = \mathfrak{q}_2 \cap A^b = \mathfrak{m}^b$, then $\mathfrak{q}_1(A^{R+})_{\mathfrak{m}} = \mathfrak{q}_2(A^{R+})_{\mathfrak{m}}$. Suppose there exists $a \in \mathfrak{q}_1 \cap \mathfrak{q}_2^c$. Then $\frac{a}{1} \in \mathfrak{q}_1(A^{R+})_{\mathfrak{m}} = \mathfrak{q}_2(A^{R+})_{\mathfrak{m}}$. So there exists $b \in \mathfrak{q}_2$ and $s \in A \setminus \mathfrak{m}$ such that $\frac{a}{1} = \frac{b}{s}$ and so there exists $u \in A \setminus \mathfrak{m}$ such that $asu = bu$. We have $a \notin \mathfrak{q}_2$ by assumption and also $su \notin \mathfrak{q}_2$ or else su would belong to $A \cap \mathfrak{q}_2 = \mathfrak{m}$. Since $bu \in \mathfrak{q}_2$ and \mathfrak{q}_2 is prime, we obtain a contradiction. This means that a does not exist and so $\mathfrak{q}_1 = \mathfrak{q}_2$. \blacksquare

It is shown in [14] that the biregular normalization of a real variety can be seen as a normalization of its non-real points. We want to understand what the R -seminormalization does locally to the real points and to the non-real points of a variety. This description will also be useful to prove the second part of the main theorem.

Lemma 3.5.5

Let A be a real affine ring and $\mathfrak{m} \in \text{Max}(A)$. Then

1. If $\mathfrak{m} \in \text{R-Max}(A)$, then $(A^{R+})_{\mathfrak{m}} = A_{\mathfrak{m}}^+$.
2. If $\mathfrak{m} \notin \text{R-Max}(A)$, then $(A^{R+})_{\mathfrak{m}} = A'_{\mathfrak{m}}$.

Proof: By Proposition 3.5.4, we have $(A^b)^+ = A^{R+}$. So, for $\mathfrak{m} \in \text{Max}(A)$, we get $((A^b)^+)_{\mathfrak{m}} = (A^{R+})_{\mathfrak{m}}$ and since the seminormalization commutes with the localization (see [38] for example), we get $((A^b)^+)_{\mathfrak{m}} = ((A^b)_{\mathfrak{m}})^+$. By [14] Proposition 4.6, we have $(A^b)_{\mathfrak{m}} = A_{\mathfrak{m}}$ if \mathfrak{m} is real and $(A^b)_{\mathfrak{m}} = A'_{\mathfrak{m}}$ else. It follows that $(A^{R+})_{\mathfrak{m}} = A_{\mathfrak{m}}^+$ if \mathfrak{m} is real and $(A^{R+})_{\mathfrak{m}} = (A'_{\mathfrak{m}})^+ = A'_{\mathfrak{m}}$ else. \blacksquare

Proposition 3.5.6

Let A be a real affine ring. Then

$$A^{R+} = \bigcap_{\mathfrak{m} \in \text{R-Max}(A)} A_{\mathfrak{m}}^+ \cap \bigcap_{\mathfrak{m} \in \text{Max}(A) \setminus \text{R-Max}(A)} A'_{\mathfrak{m}}$$

Proof: We have

$$A^{R+} \subset \bigcap_{\mathfrak{m} \in \text{Max}(A)} (A^{R+})_{\mathfrak{m}} \subset \bigcap_{\mathfrak{m} \in \text{Max}(A^{R+})} (A^{R+})_{\mathfrak{m}}$$

Since A^{R+} is an affine ring, the last term is equal to A^{R+} . So we get

$$A^{R+} = \bigcap_{\mathfrak{m} \in \text{Max}(A^{R+})} (A^{R+})_{\mathfrak{m}}$$

and we conclude with Lemma 3.5.5. ■

Remark. Let X be a real reduced algebraic variety with real irreducible components. From the previous proposition, we see that X^{R+} is the normalization of the non-real locus of X and the seminormalization of its real locus.

Using the previous property, we show the second part of the main theorem of this section.

Proposition 3.5.7

Let A be a real affine ring. Then

$$(A^+)^b = A^{R+}$$

Proof: By [14] Proposition 4.8, we have

$$\begin{aligned} (A^+)^b &= \bigcap_{\mathfrak{m} \in \text{R-Max}(A^+)} (A^+)_{\mathfrak{m}} \cap \bigcap_{\mathfrak{m} \notin \text{R-Max}(A^+)} (A^+)_{\mathfrak{m}}' \\ &= \bigcap_{\mathfrak{m} \in \text{R-Max}(A^+)} A_{\mathfrak{m}}^+ \cap \bigcap_{\mathfrak{m} \notin \text{R-Max}(A^+)} A'_{\mathfrak{m}} \end{aligned}$$

By Proposition 3.5.3, we have $(A^+)^{R+} = A^{R+}$, so

$$\begin{aligned} A^{R+} = (A^+)^{R+} &= \bigcap_{\mathfrak{m} \in \text{R-Max}(A^+)} (A^+)_{\mathfrak{m}}^+ \cap \bigcap_{\mathfrak{m} \notin \text{R-Max}(A^+)} (A^+)_{\mathfrak{m}}' \\ &= \bigcap_{\mathfrak{m} \in \text{R-Max}(A^+)} A_{\mathfrak{m}}^+ \cap \bigcap_{\mathfrak{m} \notin \text{R-Max}(A^+)} A'_{\mathfrak{m}} \end{aligned}$$

Hence $A^{R+} = (A^+)^b$. ■

From Proposition 3.5.7 and Proposition 3.5.4, we get the following theorem.

Theorem 3.5.8

Let X be a real affine variety. Then

$$\mathbb{R}[X^{R+}] \simeq \mathbb{R}[X^+]^b \simeq \mathbb{R}[X^b]^+$$

We illustrate Theorem 3.5.8 with the following example.

Example. Let $X = \text{Spec}(\mathbb{R}[x, y] / \langle y^2 - x^6(x - 1)^3(x - 2)^2(x^2 + 1)^2(x^2 + 4)^3 \rangle)$. This variety has at least seven singularities : The real isolated point $(0, 0)$ is a non-seminormal point, the real point $(1, 0)$ is a non-seminormal point, the point $(2, 0)$ is the crossing of two real lines, the two complex conjugate points $(\pm i, 0)$ are the crossing of two complex lines and the two complex conjugate points $(\pm 2i, 0)$ are non-seminormal points. We represent the closed points of X, X^b, X^+, X^{R+} in the following figure.

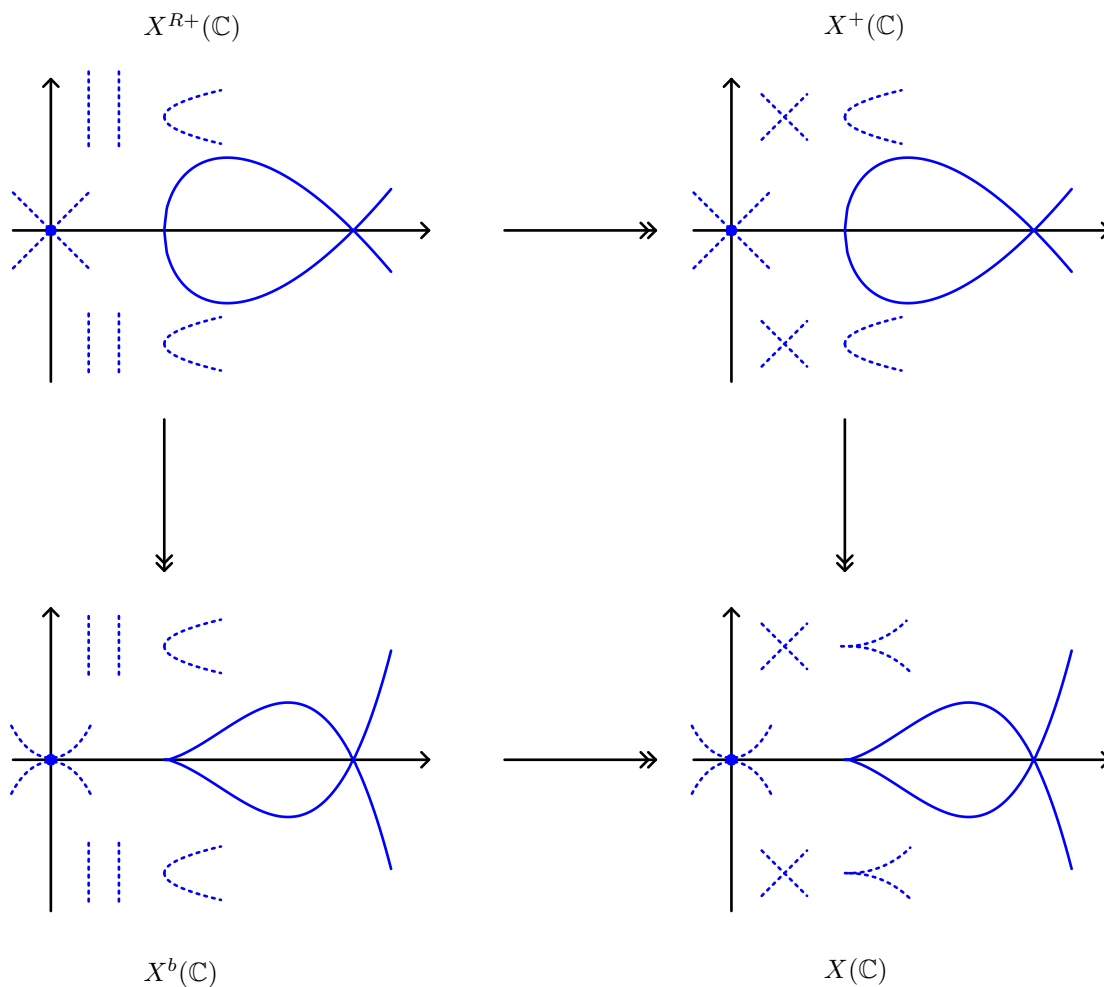


Figure 3.3 – Illustration of Theorem 3.5.8

PUBLICATIONS

Titre : "*Seminormalization and regulous functions on complex affine varieties*".

Résumé : We study seminormalization of affine complex varieties. We show that polynomials on the seminormalization correspond to the rational functions which are continuous for the Euclidean topology. We further study this type of functions which can be seen as complex regulous functions, a class of functions recently introduced in real algebraic geometry, or as the algebraic counterpart of c -holomorphic functions.

Lien ArXiv : <https://arxiv.org/abs/2109.06542v2>.

Titre : "*A notion of seminormalization for real algebraic varieties*".

Résumé : The seminormalization of an algebraic variety X is the biggest variety linked to X by a finite, birational and bijective morphism. In this paper we introduce a variant of the seminormalization, suited for real algebraic varieties, called the R -seminormalization. This object have a universal property of the same kind of the one of the seminormalization but related to the real closed points of the variety. In a previous paper, the author studied the seminormalization of complex algebraic varieties using rational functions that extend continuously to the closed points for the Euclidean topology. We adapt here some of those results to the R -seminormalization and we provide several examples. We also show that the R -seminormalization modifies the singularities of a real variety by normalizing the purely complex points and seminormalizing the real ones.

Lien ArXiv : <https://arxiv.org/abs/2209.03598>.

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Titre : Seminormalisation et fonctions régulières.

Mot clés : Normalisation, Seminormalisation, Fonctions régulières, Fonctions rationnelles continues

Résumé :

Dans le cadre de la géométrie algébrique réelle, Fichou, Monnier et Quarez ont mis en lumière une corrélation entre la seminormalisation, les fonctions rationnelles continues et les fonctions régulières (une classe de fonctions récemment introduite en géométrie réelle). Dans la première partie de cette thèse, nous montrons que cette corrélation existe également en géométrie algébrique complexe et nous approfondissons le concept de fonctions régulières complexes. Nous donnons plusieurs caractérisations de ces fonctions et en fournissons de nombreux exemples. La seconde partie de la thèse est consa-

crée à l'étude de la «R-seminormalisation», une variante de la seminormalisation adaptée au cadre de la géométrie algébrique réelle. Nous montrons que cet objet possède une propriété universelle du même type que celle de la seminormalisation classique. Puis, nous identifions de différentes manières les éléments de son anneau de coordonnées et donnons des exemples de constructions de la R-seminormalisation. Enfin, nous montrons comment cet objet modifie les singularités d'une variété en établissant un lien entre la R-seminormalisation, la seminormalisation et la «normalisation birégulière» introduite récemment en géométrie réelle.

Title: Seminormalization and regulous functions.

Keywords: Normalization, Seminormalization, Regulous functions, Continuous rational functions

Abstract:

In the context of real algebraic geometry, Fichou, Monnier and Quarez highlighted a strong connection between seminormalization, rational continuous functions and regulous functions (a class of functions extensively studied in real algebraic geometry). In the first part of this thesis, we show that the same is true for complex varieties, and we explore the concept of complex regulous functions. We manage to give several characterizations of these functions, and we provide many examples. The second part of this thesis is devoted to the study of the

"R-seminormalization", a variant of the seminormalization for real varieties. We show that it has a universal property of the same kind as the one of the seminormalization. Then we identify, by several different ways, the coordinate ring of the R-seminormalization and we give some examples of its construction. Finally, we explain how the R-seminormalization modifies the singularities of a variety by showing how it is linked with the seminormalization and the "biregular normalization" which has been recently introduced in real algebraic geometry.