Phenomenology of nilmanifolds : Yang-Mills theory on the Heisenberg manifold

Fabio Dogliotti

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I will now write in french for the rest of this section.

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Chapter 0

Introduction

In this thesis, we will discuss a new model for gauge-Higgs unification based on the compactification of a nilmanifold. Nilmanifolds are a class of solvmanifold well studied in the mathematical literature (for harmonic analysis see [1, 2]). They have also been used in the context of string theory [3–12] to investigate new possibilities for compactification. In our context, this new approach gives a new tool to the already rich field of gauge-Higgs unification. The main new tool presented in this thesis rely on the use of a twisted torus. This twist results in a mass for the scalar fields along with a potential at tree-level which circumvent the usual issue of having to rely on quantum corrections to generate such a potential.

In 2012, the Higgs boson was discovered at the Large Hadron Collider (LHC). This discovery solidified the idea put forward by Robert Brout and François Englert [13], Peter Higgs [14], and by Gerald Guralnik, C. R. Hagen, and Tom Kibble [15]. In 1964, they independently proposed a model capable of producing consistent massive gauge bosons, which later made possible the use of Yang-Mills theories to describe weak interactions. This mechanism requires the introduction of a scalar field responsible for the breaking of the gauge symmetry, which results in the presence of massive gauge bosons in the spectrum. This mechanism, now known as the Higgs mechanism, is one of the corner stones of the celebrated Standard Model (SM) of particle physics. In the SM, the Higgs field is in the two-dimensional complex representation of the gauge group $SU(2)$ and charged under $U(1)$. The key ingredient is the self-interacting potential of this scalar field, whose non-trivial minimum reduces the gauge symmetry to $U(1)$ at low energies. The Higgs mechanism is compatible with all the current observations, but an interesting question is raised if we assume the existence of a higher energy scale for the cutoff of the theory. Consider the possibility of a grand unification scale for the fundamental forces of nature. The running of the couplings of the SM seem to approximately converge for energies around $\sim 10^{16}$ GeV. If we assume the existence of a fundamental scalar field responsible for the breaking of $SU(2) \times U(1)$ and it is part of this grand unification, quantum corrections to the mass of this scalar field are quadratically divergent in the

cutoff $\Lambda \sim 10^{16}$ GeV. Observations of the W, Z and Higgs boson masses force the parameter $-\mu^2$ to be of the order $\sim -(100 \text{ GeV})^2$. If one wants to have an effective value for the $-\mu^2$ parameter of the right order of magnitude, one would need to take the bare value for $-\mu^2$ to be of order $-\Lambda^2$ and have the quantum corrections of order Λ^2 produce the desired value. This fine tuning of the mass squared of the Higgs would span 28 orders of magnitude. The question of why the observed Higgs mass should be so small compared to the grand unification scale is referred to as the *gauge hierarchy problem*. It possible that this grand unification scale is somehow related to the unification of gravity with the other forces. The typical scale at which gravity would become relevant for fundamental particles is of the order of the Planck mass $m_P \sim 10^{19}$ GeV. The effects of gravity would surely already be present at a scale of $\sim 10^{18}$ GeV, not so far from the already huge value suggested by the running of the couplings. In this scenario, an explanation might be given by a theory of quantum gravity, though this remains uncertain at this point.

It should be pointed out that the hierarchy problem may not be as much of a problem as it has been suggested. This observation is made for a specific choice of renormalization. How the scenario would change if other methods were picked is not so clear as Mooij and Shapshnikov have argued in [16]. In their paper, they use renormalization schemes that avoid UV divergences. The method is based on the Callan-Symanzik (CS) equation [17, 18]. Taking the simple example of two massive scalar fields, one at with a much larger mass than the other, they recall the standard procedure that leads to corrections to the small mass that are proportional to the large mass squared. They then derive the corrections for the same model using the CS method and show that no infinities appear at any stage in the computation, and that the large mass does not appear in the corrections to the small mass at one-loop order.

Scalar fields also lack any form of guiding principle. Gauge bosons and fermions are well constrained by symmetry and anomaly cancellation. Unfortunatly we don't have anything similar for scalar fields. It may be that scalar fields are not fundamental fields of nature, as suggested for example by composite models where the Higgs is taken to be a fermionic bound state.

In any case, there have been many attempts at solving this seemingly paradoxical observation and in understanding how to constrain scalar fields. Among possible directions, supersymmetry has been a fruitful candidate to explain the cancellations evoked earlier, among many other things. The absence of observation of super-partners at the LHC has raised some doubts on the validity of the Minimal Supersymmetric Standard Model (MSSM), though it has not been entirely excluded so far. Here we will consider another possibility called gauge-Higgs unification.

Gauge-Higgs unification models are a class of models that seek to explain the existence of the Higgs boson as some components of a higher dimensional gauge field. Different scenarios have been put forward. Their common feature is the existence of extra dimensions which give rise to scalar fields in four-dimensions at low energies. Among the proposed models are large extra dimensions and ADD type models [19–23]. Another class of models are based on Kaluza-Klein

compactifications. These types of model are often encountered in the context of string theory, as it requires an explanation for the presence of the extra dimensions compared to the four observed in our universe. In gauge-Higgs theories, the concept can be traced back to Manton [24] and Hosotani [25] who presented models where gauge degrees of freedom in the extra dimensions give effective scalar fields once the compactification is performed. The first idea is that the scalars are identified with the zero modes on the additional manifold. Typically, the manifold is taken to be S^1 or a torus. The scalars are massless at tree level because they descend from the zero modes. One has to rely on quantum corrections to generate the masses and effective potential. Another approach is possible if the manifold for the extra dimensions is not simply connected. One can introduce non-contractible Wilson loops whose phases can be identified with scalar fields. These phases are dynamical quantities which modify the masses of the Kaluza-Klein spectrum. Upon considering quantum corrections, an effective potential is generated for the phases which can result in a non-trivial configuration for the phases, possibly breaking the gauge. This mechanism is often referred to as the Wilson loop mechanism or the Hosotani mechanism.

In all these scenarios, the extra dimensions are small and the manifold is compact, such as to be below current observational limits, but usually bigger than the Planck's length. Another possibility, inspired by string theory, is to have the Standard model of particle constrained to a four-dimensional brane. Gravity is the only force allowed to propagate through the bulk of the theory. An example of such models is the Randall-Sundrum model (RS) where the extra dimension is AdS-like [26].This class of gauge-Higgs models has been extensively studied both in the electroweak sector [27, 28] and in the application to grand-unified theories [29, 30]. Originally, the model was composed of a finite fifth dimension. The SM was located on a brane at one of the boundaries where the natural energy scale is set at the TeV. Another brane at the other boundary is at the Planck scale. The hierarchy is solved in this context by the distance separating the two branes. These models are also constrained by collider observations [31]. Radiative breaking of the electroweak symmetries is also studied in brane models [32].

0.1 The Higgs mechanism

The observation of the W and Z bosons dates back to 1983, but the weak interaction was known to exist before that. Enrico Fermi was the first to propose a model based on contact interaction of four fermions to describe beta decays. In this section we will review the Higgs mechanism, as it will be relevant for the developments of this thesis. We start with the abelian case and then move on the more general setting of non-abelian theories.

Consider the Lagrangian

$$\mathcal{L} = -(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi) \ . \tag{1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and ϕ is a complex scalar field. The squared terms are understood to be contracted with the Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, so with two η factors for the first term, and one in the second. This Lagrangian is symmetric under local $U(1)$ gauge transformations if we assume that the potential only depends on $|\phi|$. A local gauge transformation takes the form

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \quad \text{and} \quad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \ . \tag{2}$$

Let us fix an expression for the potential. The usual choice is

$$V(\phi) = -\mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \ , \tag{3}$$

with $\mu^2 > 0$. The potential has two extrema given by $\langle\phi\rangle = \phi_0 = 0$ and $|\phi| = \sqrt{\frac{\mu^2}{\lambda}}$. These are the solutions to the equation

$$\frac{\delta V}{\delta\phi^*}(\phi) = 0 \ . \tag{4}$$

The complex conjugate of the field obey a similar equation which ensures that $\phi_0^* = 0$ if $\phi_0 = 0$. The point $\phi_0 = 0$ is a local maximum while the other one is a global minimum. The minimum has a $U(1)$ symmetry. This means that there is an infinite number of equivalent vacua given by $\langle\phi\rangle = \sqrt{\frac{\mu^2}{\lambda}}e^{i\theta}$. We can choose the simplest $\langle\phi\rangle = \phi_0 = \sqrt{\frac{\mu^2}{\lambda}}$. We can now expand the potential around the minimum by setting $\phi = \phi_0 + \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. We obtain

$$V(\phi) = -\frac{\mu^4}{2\lambda} + \frac{1}{2}2\mu^2\phi_1^2 + \dots \ . \tag{5}$$

The dots represent higher order terms. So we observe that the field ϕ_1 acquires a mass $\sqrt{2}\mu$ while ϕ_2 remains massless. We now consider how the kinetic term transforms

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \sqrt{2}e\phi_0 A_\mu \ \partial^\mu\phi_2 + e^2\phi_0^2 A_\mu A^\mu + \dots \ . \tag{6}$$

The last term is a mass term for the photon. The mass is given by

$$m_A = \sqrt{2}e\phi_0 \tag{7}$$

By expanding the theory around the minimum of the potential, the theory now describes two scalar fields, one massive, the other massless interacting with a massive photon. The VEV of the scalar field is responsible for the mass of the broken gauge boson. The model is said to exhibit spontaneous symmetry breaking. The photon at the beginning was a massless particle. Massless particle only have two degrees of freedom corresponding to the two helicities. If the photon acquire a mass by this mechanism, an additional degree of freedom

should play the role of the longitudinal polarization of the particle. It is the field ϕ_2 who is responsible for that. To see this, consider the term

$$\sqrt{2}e\phi_0 A_\mu \, \partial^\mu \phi_2 \ . \tag{8}$$

The vertex associated to this term in the quantum theory is

$$i\sqrt{2}e\phi_0(-ik_\mu) = m_A k_\mu \ . \tag{9}$$

If we look at the contributions to the two-point function for the photon, we will have at leading order contributions of the form

$$\begin{aligned} & im_A^2\eta_{\mu\nu} \ + m_A k_\mu \frac{i}{k^2}(-m_A k_\nu) \\ =& im_A^2\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \end{aligned} \tag{10}$$

The first term is simply the mass term considered as a vertex. The second corresponds to a photon transitioning to a scalar with the vertex (9) and then back to a photon. The total contribution is exactly the one needed for the longitudinal polarization of the photon. The field ϕ_2 is called a Goldstone boson in this context.

The Goldstone boson is not a physical particle of the theory. The easiest way to see this is to fix a particular gauge called the *unitary* gauge in the starting Lagrangian. This gauge corresponds to taking a local gauge transformation such that the phase of the scalar field is zero everywhere. In this gauge, the Lagrangian becomes

$$\mathcal{L} = -(F_{\mu\nu})^2 + (\partial_\mu\phi)^2 + e\phi A_\mu A^\mu - V(\phi) \ . \tag{11}$$

We can see directly that if the scalar ϕ acquire a VEV through the potential, the interaction with the photon will result in a mass term. The field ϕ_2 drops completely from the action, making it clear that the associated particle is not physical.

The mechanism presented is the Higgs mechanism. It allows a theory to have massive gauge bosons in a consistent way. The main ingredient was the scalar field along with non-trivial minimum in the potential. The non-trivial minimum is seen as a VEV for the scalar field, which is responsible for the mass term of the gauge boson. This mechanism can be generalized to non-Abelian gauge theories. The generalization is rather straightforward, but we will present it here for completeness.

In order to generalize the mechanism presented above, we must introduce additional scalar fields. These scalar fields will be taken in some representation of the gauge group G such that upon gauge transformation, the scalars obey the transformation rule

$$\phi \to e^{i\theta_a t^a}\phi \ , \tag{12}$$

where the field ϕ is thought of as a vector with components ϕ_1, ϕ_2, ... and the t^a are matrices acting on ϕ as

$$(t^a\phi)_i = t^a_{ij}\phi_j \ , \tag{13}$$

with a sum over j. The θ_a are real parameters. Generally, the gauge group G is taken to be a Lie group because we want to enhance the global symmetry to a local one. The generators t^a then form a representation of the associated Lie algebra. The Lagrangian is typically built by using a bilinear form to write real quantities dependent on the scalars. If we take for example $\phi^\dagger\phi$, then invariance under the group action corresponds to having $(e^{i\theta_a t^a})^\dagger = (e^{i\theta_a t^a})^{-1}$, i.e. it is an element of some unitary group. This implies $(t^a)^\dagger = t^a$ meaning that t^a is an hermitian matrix. We can write equivalently $it^a = T^a$ with T^a anti-hermitian. We will take ϕ to be real as we can always write a complex scalar field as two real fields. The generators are then anti-symmetric, with the T^a being real and the t^a pure imaginary. We now make the global symmetry local by imposing invariance of the action under transformations with $\theta_a(x)$ a function of position. The covariant derivative is written

$$D^\mu\phi = (\partial^\mu - igA^\mu_a t^a)\phi = (\partial^\mu + gA^\mu_a T^a)\phi \ . \tag{14}$$

The kinetic term of the Lagrangian can be expended and takes the form

$$\frac{1}{2}(\partial^\mu\phi_i)^2 + gA^\mu_a(\partial_\mu\phi_i T^a_{ij}\phi_j) + \frac{1}{2}g^2A^\mu_a A^{\mu b}(T^a\phi)_i(T^b\phi)_i \ . \tag{15}$$

Now, without specifying a potential for the scalars, imagine that they acquire some VEV

$$\langle\phi_i\rangle = (\phi_0)_i \ . \tag{16}$$

We can expand the field around this vacuum. The last term of (15) generates the term

$$\Delta\mathcal{L} = \frac{1}{2}g^2A^\mu_a A^{\mu b}(T^a\phi_0)_i(T^b\phi_0)_i \tag{17}$$

and so defines a mass matrix for the gauge bosons

$$m^2_{ab} = g^2(T^a\phi_0)_i(T^b\phi_0)_i \ . \tag{18}$$

This matrix is non-negative since all its diagonal entries $m^2_{aa} = g^2(T^a\phi_0)^2_i$ are positive or zero. The directions a such that the entry is zero are the generators that leave the vacuum invariant

$$T^a\phi_0 = 0 \ . \tag{19}$$

The symmetries of the vacuum correspond then to the residual gauge since for each generator that leave the vacuum invariant, there is an associated massless gauge boson. Just like in the abelian case, the Goldstone bosons will contribute

to the longitudinal polarization of the broken gauge bosons. To see this, consider the term

$$gA_a^\mu(\partial_\mu\phi_i(T^a\phi_0)_j) \tag{20}$$

coming from the second term in (15). These term will contribute to the two point function of A_a^μ in the same way than in the abelian case. For each A_a^μ we will have a first contribution from the mass term, and a second from the transition of the gauge boson to a scalar and then back to a boson. Those contributions take the form of a sum

$$\sum_j (gk_\mu(T^a\phi_0)_j)\,\frac{i}{k^2}\left(-gk_\mu(T^b\phi_0)_j\right)\ . \tag{21}$$

and we can consider all j since the term will be zero if it does not contribute. We obtain again the same form for the correction to the two-point function

$$im_{ab}^2\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right) \tag{22}$$

which accounts for the longitudinal polarization.

We are now ready to detail the Higgs mechanism for the Standad Model. This theory is called the Glashow-Weinberg-Salam theory of weak interactions [33, 34], bearing the name of the authors who developed the model. Here, the gauge group is chosen to be $SU(2)\times U(1)$. We choose a scalar field in the fundamental representation of $SU(2)$ and with a charge $1/2$ under $U(1)$. Its gauge transformation is therefore

$$\phi\to e^{i\alpha^a\tau^a}e^{i\frac{\beta}{2}}\phi\ . \tag{23}$$

where $\tau^a=\sigma^a/2$ for σ^a the Pauli matrices. We choose the following VEV for the scalar field

$$\langle\phi\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix}\ , \tag{24}$$

which is left invariant by the gauge transformation generated by $\beta\tau^3+\beta/2$ for arbitrary β. This is the only symmetry of the vacuum so the theory will have one massless boson , while the other three will acquire a mass.

We write the covariant derivative as

$$D^\mu\phi=(\partial^\mu-igA_a^\mu\tau^a-ig'B^\mu)\phi\ . \tag{25}$$

The gauge bosons will acquire a mass according to equation (18). The resulting term in the Lagrangian after simplification is

$$\Delta\mathcal{L}=\frac{1}{2}\frac{v^2}{4}\left(g^2(A_\mu^1)^2+g^2(A_\mu^2)^2+(-g^2A_\mu^3+g'B_\mu)^2\right)\ . \tag{26}$$

As expected, three bosons acquire a mass. We can change the basis in order to have mass eigenstates. The change in basis is explicitly

$$W_\mu^\pm = \frac{1}{2}(A_\mu^1 \mp iA_\mu^2), \quad Z_\mu \ = \frac{1}{\sqrt{g^2+g'^2}}(g^2 A_\mu^3 - g' B_\mu) \ . \tag{27}$$

The last component of the gauge, orthogonal to Z_μ is given by

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(g^2 A_\mu^3 + g' B_\mu) \ . \tag{28}$$

The masses now read

$$m_W = g\frac{v}{2} \quad , \quad m_Z = \sqrt{g^2+g'^2}\frac{v}{2} \quad \text{and} \quad m_A = 0 \ . \tag{29}$$

For an arbitrary field in a representation T^a of $SU(2)$ and with an hypercharge Y, the covariant derivative is written in the eigenmass basis

$$D_\mu = \partial_\mu \ - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{g}{\cos(\theta)} Z_\mu (T^3 - \sin(\theta_w)^2 Y) - ieA_\mu Q \tag{30}$$

where we have defined the various terms as

$$\begin{aligned} &\cos(\theta_w) = \frac{g}{\sqrt{g^2+g'^2}} \quad , \quad \sin(\theta_w) = \frac{g'}{\sqrt{g^2+g'^2}} \quad , \\ &T^\pm = T^1 \pm iT^2 \quad , \quad Q = T^3 + Y \quad \text{and} \quad e = g\sin(\theta_w) = g'\cos(\theta_w) \end{aligned} \tag{31}$$

The model describes a massless field A_μ which is identified with the electromagnetic potential, along with the three massive bosons $W_\mu^\pm$ and Z_μ. The electric charge is given by Q. The model is defined by three parameters, e, m_W and θ_w. The first one is the electromagnetic coupling constant. The second is the mass of the W boson. The The last parameter is the Weinberg angle, which is responsible for the mixing of the two part of the initial gauge. This can be seen more clearly by noting that the change of basis can be written

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} \cos(\theta_w) & -\sin(\theta_w) \\ \sin(\theta_w) & \cos(\theta_w) \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix} \ . \tag{32}$$

There is more that can be said about this model. The logical continuity would be to study fermions. In our universe, it so happens that only left handed fermions couple to the W_μ boson. The theory is said to be chiral because left handed fermions have different couplings to the gauge than right handed ones. Classically, this wouldn't be a problem. But in the quantum theory, anomalies can arise in chiral theories. Anomalies are the failure of a symmetry to be carried over to the quantum theory. They arise as terms violating the Ward identities. For a chiral theory, the gauge anomaly restricts the possible fermionic content. In the case of the Standard Model, all the anomalies vanish because of the quantum numbers assigned to the fermions. But we will end this section here as this question is not relevant to the development of this thesis.

0.2 Differential geometry and physics

Differential geometry is the mathematical theory that describes smooth spaces. In physics, smoothness is often taken for granted, making differential geometry a natural framework to describe physical processes, at least classically. Throughout this thesis, we will use notions and notations that come from differential geometry. In this part, we will introduce the reader to the basics of differential geometry. The interested reader not familiar with differential geometry can find a more complete introduction to this very large subject in Nakahara's book [35] for example.

The first concept we are going to introduce is the concept of a smooth manifold. A manifold is a space that locally looks like a vector space, here we take this vector space to be $\mathbb{R}^n$. Intuitively, if we think of a sphere, we can imagine zooming on a point of the sphere. As we zoom, the space starts to look more and more like a flat plane around that point. This intuitive idea is precisely what the notion of a manifold formalize. Close enough to any given point of a manifold, the space should look like $\mathbb{R}^n$ for a n-dimensional manifold. Let us be more specific.

A n-dimensional manifold $\mathcal{M}$ is a space equipped with the following structures :

(i) $\mathcal{M}$ is a topological space.

(ii) For each open set U_i of the topology, there is an onto homeomorphism[1] $\phi_i : U_i \to V_i$ for V_i an open set of $\mathbb{R}^n$.

(iii) U_i is a cover of $\mathcal{M}$, meaning $\bigcup_i U_i = \mathcal{M}$

The pair (U_i, ϕ_i) is called a chart and the collection of all the pairs is called an atlas. ϕ_i defines a local coordinate system on the manifold. The U_i are called coordinate neighborhoods while the maps ϕ_i are the coordinate functions, or simply the coordinates. That is because the maps ϕ_i take value in a open set of $\mathbb{R}^n$, so they can be written as a collection of functions $\{x^1(p), \dots x^n(p)\}$ which can be understood as the coordinates in V_i.

The smoothness in the name *smooth* manifold comes from an additional requirement for the maps ϕ_i. Smoothness is a term used to talk about functions that are infinitely differentiable. Here the ϕ_i are maps from a subset of $\mathcal{M}$ to a subset of $\mathbb{R}^n$, so we don't have any well defined notion of differentiability. Instead, what we can do is take two charts (U_i, ϕ_i) and (U_j, ϕ_j) such that $U_i \cap U_j \neq \emptyset$, and ask for the following requirement to be fulfilled:

$$\Phi_{ij} = \phi_i \circ \phi_j^{-1} \colon V_i \to V_j \in C^\infty(\mathbb{R}^n) \ . \tag{33}$$

This amount to ask that the possible coordinates are smoothly related. We can define the differentiability of any map $f \colon \mathcal{M} \to \mathcal{N}$ from a manifold to another

[1] Recall that a homeomorphism is a natural map in topology, it is a continuous function with an inverse which is also continuous, so $\phi_i \circ \phi_i^{-1} = \mathrm{id}_{V_i}$ and $\phi_i^{-1} \circ \phi_i = \mathrm{id}_{U_i}$ with ϕ_i and ϕ_i^{-1} continuous.

manifold. We take two charts (U_i', ψ_i) and (U_j, ϕ_j) on $\mathcal{N}$ and $\mathcal{M}$ respectively. For such a function, the condition takes the form

$$\psi_i \circ f \circ \phi_j^{-1} \colon V_j \to V_i' \in C^\infty \ , \quad V_j \subset \mathbb{R}^m \ , \ V_i' \subset \mathbb{R}^n \ . \tag{34}$$

We can restrict the class of smooth maps further by selecting only homeomorphic maps f, whose inverse are also smooth. Those maps are called diffeomorphisms. Clearly, if there exists a diffeomorphism from $\mathcal{M}$ to $\mathcal{N}$, it implies that $\dim \mathcal{M} = \dim \mathcal{N}$. Diffeomorphisms form a group under composition, noted $Diff(\mathcal{M})$, since for every map $f \in Diff(\mathcal{M})$ there exist a map $g \in Diff(\mathcal{M})$ such that

$$f \circ g = g \circ f = \mathrm{id}_{\mathcal{M}} \ . \tag{35}$$

In a chart (U, ϕ) such that $\phi(p) = x^\mu(p)$, a map f takes a point $p \in \mathcal{M}$ and send it to $f(p)$ such that $\phi(f(p)) = x^\mu(f(p))$ (assuming $f(p) \in U$).

In the same line of reasoning, we can define curves and functions on $\mathcal{M}$. A curve is a smooth map $c : (a, b) \to \mathcal{M}$, it is a list of points on $\mathcal{M}$ which define a smooth trajectory. A function on the other hand is a map of the form $f : \mathcal{M} \to \mathbb{R}$, which a real valued function $f(p)$. With this we can define the map $f \circ c : (a, b) \to R$. This is simply a real valued smooth function f(c(t)). Now, we can take the derivative of this function, which is well defined. Assume the existence of some chart such that $\phi(c(t)) = x^\mu(c(t))$. The derivative at the point $t = 0$[2] can be written

$$\begin{aligned} \frac{d}{dt}\left(f \circ c\right)\Big|_{t=0} &= \frac{d}{dt}\left(f \circ \phi^{-1} \circ \phi \circ c\right)\Big|_{t=0} \\ &= \partial_\mu f \ \frac{dx^\mu(c(t))}{dt}\Big|_{t=0} \ . \end{aligned} \tag{36}$$

The vector $\frac{dx^\mu(c(t))}{dt}$ is defined by the equivalence class of curves giving the same coefficients. Dual to this vector is the differential of f, a function that depends on the point $c(t)$. These two vector-like object can be defined independently. We write

$$\begin{gathered} V = V^\mu \partial_\mu \quad \text{and} \quad \mathrm{d}f = \partial_\mu f dx^\mu \ , \\ \text{with} \quad V^\mu = \frac{dx^\mu(c(t))}{dt}\Big|_{t=0} \quad \text{and} \quad \langle dx^\mu, \partial_\nu \rangle = \langle \partial_\nu, dx^\mu \rangle = \frac{\partial x^\mu}{\partial x^\nu} = \delta^{\mu\nu} \ . \end{gathered} \tag{37}$$

V is called a vector. $\mathrm{d}f$ is the differential of f. The last equation means that the two are dual to one another. We can write the expression for a generic vector and a generic "differential", which in this case is referred to as a one-form

$$X = X^\mu \partial_\mu \quad \text{and} \quad w = w_\mu dx^\mu \ . \tag{38}$$

The scalar product of a vector with a one-form is given by

$$\langle X, w \rangle = X^\mu w_\mu \ . \tag{39}$$

[2] This is a generic point, it could be any value but for simplicity we assume a parametrization such that the point p at which we evaluate the derivative corresponds to $t = 0$.

These are the basics ingredients to construct the structures we are going to develop. It can be easily shown that these objects are independent of the coordinate system.

Notice that here we have two types of diffeomorphisms. The first one is Φ_{ij} which is a change of chart. This is in some sense a *passive* transformation, as it corresponds only to a new choice of coordinates. The second one is $f \in Diff(\mathcal{M})$, which sends a point $p \in \mathcal{M}$ to another point $f(p) \in \mathcal{M}$ (this may come with an additional passive transformation if $f(p)$ is not in the same chart as p). This transformation is *active* in the sense that the point in $\mathcal{M}$ was sent to another point. Both results in a transformation for the vectors and one-forms. Take the first case, we have two charts with the coordinates $\phi_i = x^\mu$ and $\phi_j = y^\mu$. The map $\Phi_{ij} = x^\mu(y)$ can be seen as a list of differentiable function depending on the y^μ coordinates. We have

$$\begin{aligned} V^\mu = \frac{dx^\mu(c(t))}{dt} &= \frac{d\phi_i(c(t))}{dt} \\ &= \frac{d}{dt}\left(\phi_i \circ \phi_j^{-1} \circ \phi_j \circ c(t)\right) = \frac{\partial x^\mu}{\partial y^\nu}\frac{dy^\nu(c(t))}{dt} \ . \end{aligned} \tag{40}$$

In the case of an active transformation, we have

$$\begin{aligned} V^\mu = \frac{dx^\mu(f(c(t)))}{dt} &= \frac{d\phi_i(f(c(t)))}{dt} \\ &= \frac{d}{dt}\left(\phi_i \circ f \circ \phi_i^{-1} \circ \phi_i \circ c(t)\right) = \frac{\partial f^\mu}{\partial x^\nu}\frac{dx^\nu(c(t))}{dt} \ . \end{aligned} \tag{41}$$

Hence it is easy to mix up the two at the level of the coordinate of the vectors since they are both written as the Jacobian of the transformation.

We denote $T_p(\mathcal{M})$ the space of vectors at a point p and $T_p^*(\mathcal{M})$ the space of one-forms dual to those vectors. Once vectors and one-forms are defined, we can define any tensor as a multi-linear map from the tensor product of $T_p(\mathcal{M})$ and $T_p^*(\mathcal{M})$ to $\mathbb{R}$

$$T : T_p(\mathcal{M}) \otimes \ldots T_p(\mathcal{M}) \otimes T_p^*(\mathcal{M}) \otimes \cdots \otimes T_p^*(\mathcal{M}) \to \mathbb{R} \ . \tag{42}$$

A special class of tensors are the differential forms. They correspond to

$$w : T_p^*(\mathcal{M}) \wedge \cdots \wedge T_p^*(\mathcal{M}) \to \mathbb{R} \tag{43}$$

where $\wedge$ is the antisymmetric part of the tensor product, so $a \wedge b = 1/2(a \otimes b - b \otimes a)$. The form is then totally antisymmetric. We denote $\Omega_p^r(\mathcal{M})$ the space of r-forms, so the space is described by elements with r indices

$$w = \frac{1}{r!} w_{\mu_1 \ldots \mu_r} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \ . \tag{44}$$

Now that forms are defined, we can describe an operation similar to the differential on functions. Recall that the differential take a function $f : \mathcal{M} \to \mathbb{R}$

and send it to the one-form $\mathrm{d}f \in \Omega^1_p(\mathcal{M})$. We can extend the definition of this operation in the following way

$$\mathrm{d} : \Omega^r_p(\mathcal{M}) \to \Omega^{r+1}_p(\mathcal{M}) \tag{45}$$

$$w \mapsto \mathrm{d}w \tag{46}$$

where we define

$$\mathrm{d}w = \frac{1}{r!}\partial_\mu w_{\mu_1\ldots\mu_r}\mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \ . \tag{47}$$

One can show with an explicit computation that $\mathrm{d}^2 w = 0$ for *any* form. This operator, called the exterior derivative, can be used to further define an important structure on forms which is the de-Rham cohomology group. To define this group we must first define two types of forms. We call an r-form w exact if $w = \mathrm{d}W$ for some $W \in \Omega^{r-1}_p(\mathcal{M})$, and we call an r-form closed if $\mathrm{d}w = 0$. From the definition, we see that an exact form is necessarily closed, but the converse is not true in general. Understanding by how much the converse is not true is the purpose of the de Rham cohomology group. Note that closed forms can be understood as the kernel of d and the exact forms as the image of d. The r^{th} de-Rham cohomology group is then defined by

$$H^r(\mathcal{M}) = ker(\mathrm{d}_{r+1})/im(\mathrm{d}_r) \ . \tag{48}$$

Let us give a simple physical example for concreteness. If we take $\mathcal{M} = \mathbb{R}^n$, then we know by Poincaré's lemma that all the cohomology groups are trivial (except for $H^0(\mathbb{R}^n) \simeq \mathbb{R}$ since $\mathbb{R}^n$ is contractible. In electromagnetism, half of Maxwell's equations can be formulated as $\mathrm{d}F = 0$, where F is the electromagnetic 2-form, so we know that it is closed. The equation is also immediately obeyed by $F' = F + \mathrm{d}w$ for $w \in \Omega^1_p(\mathbb{R}^n)$. So F is in the cohomology group $H^2(\mathbb{R}^n)$, which is trivial in this case. So we can conclude that F is an exact form globally. This allows us to define the one-form potential A such that $F = dA$. Since any manifold is locally $\mathbb{R}^n$, this can always be done locally for any manifold. The obstruction to make this expression global is given by the second cohomology group $H^2(\mathcal{M})$.

In the case where the space $\mathcal{M}$ is endowed with a metric, the space is called a Riemannian manifold. Additional structure exists for forms on a Riemannian manifold. In particular, we have the Hodge transformation defined on an r-form by

$$*w = \frac{\sqrt{|g|}}{(n-r)!} w_{\mu_1\ldots\mu_r}\epsilon^{\mu_1\ldots\mu_r}{}_{\mu_{r+1}\ldots\mu_n}\mathrm{d}x^{\mu_{r+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \ . \tag{49}$$

We have $*1 = \sqrt{|g|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ which is the volume form, it has the property of being invariant under diffeomorphisms. The Hodge structure allows for the identification of any $w \in \Omega^r(\mathcal{M})$ with a form $*w \in \Omega^{n-r}(\mathcal{M})$, and conversely. The Hodge star is an involution of $\Omega^r(\mathcal{M})$ (up to a sign). We can also define

an inner product on $\Omega^r(\mathcal{M})$ by

$$\langle w, \eta \rangle = \int_{\mathcal{M}} w \wedge *n \ , \text{ for } w, \eta \in \Omega^r(\mathcal{M}) \ , \tag{50}$$

where the measure for this integral is the volume form. This can be used for example to define the Lagrangian of the abelian Yang-Mills theory as

$$\mathcal{L} = -\frac{1}{4} \int_{\mathcal{M}} F \wedge *F = -\frac{1}{4} \int_{\mathcal{M}} F_{\mu\nu} F^{\mu\nu} \ , \tag{51}$$

for F the 2-form electromagnetic tensor. Hodge's duality can also be used to define the adjoint $\mathrm{d}^\dagger$ of the exterior derivative d. It is defined as

$$\mathrm{d}^\dagger = (-1)^k * \mathrm{d} * \ , \tag{52}$$

where k depends on the degree of the form, the dimension of the space, and on whether the space is Riemannian or pseudo-Riemannian. It is a map $\Omega^r(\mathcal{M}) \to \Omega^{r-1}(\mathcal{M})$. With $\mathrm{d}^\dagger$ and d, we can defined one last operation on $\Omega^r(\mathcal{M})$, and that is the Laplacian. It is defined as

$$\Delta = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d} \ . \tag{53}$$

This operator reduces to the usual definition of the Laplacian in the case of a flat space with Euclidian metric. Coming back to the example of the electromagnetic tensor F, the rest of Maxwell's equations can written using $\mathrm{d}^\dagger$. We had $F = \mathrm{d}A$ in some chart, which solved the first set of equation $\mathrm{d}F = 0$. The other equations are

$$\mathrm{d}^\dagger F = \mathrm{d}^\dagger \mathrm{d} A = j \ , \tag{54}$$

where j is the one-form matter current. We can fix a certain gauge A' such that $d^\dagger A' = d^\dagger A + d^\dagger df = 0$, which corresponds to the Lorentz gauge. Maxwell's equations in this gauge take the form

$$\mathrm{d}^2 A' = 0 \quad , \quad \Delta A' = j \ . \tag{55}$$

A central theorem of Riemannian geometry is the Hodge decomposition theorem. It states

$$\forall w \in \Omega^r(\mathcal{M}) \ , \ \exists \quad h \in \mathrm{Harm}^r, b \in \Omega^{r-1}, z \in \Omega^{r+1} \quad | \quad w = h + \mathrm{d}b + \mathrm{d}^\dagger z \ . \tag{56}$$

Harm^r is defined as the space of harmonic forms, i.e. such that $\Delta w = 0$. So any form on a Riemannian manifold can be written as a sum of an harmonic form, an exact form and a co-exact form.

The last ingredient we are going to discuss is the connection. A connection on a manifold is a way to capture how vector fields change for an infinitesimal displacement along a certain direction. A vector field is simply a vector that evolves smoothly along a curve. We write the space of vectors fields $\mathfrak{X}(\mathcal{M})$. A connection is a map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ with the following properties

(i) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(ii) $\nabla_{X+Z} Y = \nabla_X Y + \nabla_Z Y$,

(iii) $\nabla_{fX} Y = f \nabla_X Y$,

(iv) $\nabla_X fY = X[f]Y + f\nabla_X Y$.

In a chart, the connection, also called in this context the covariant derivative, is specified by its action on the basis vectors. So we define

$$\langle \nabla_{\partial_\mu} \partial_\nu, \mathrm{d}x^\rho \rangle = \Gamma_{\mu\nu}{}^{\rho} \ , \tag{57}$$

the connection coefficients. They can be thought of as a list of matrices $\{\Gamma_{1\nu}{}^{\rho}, \ldots, \Gamma_{n\nu}{}^{\rho}\}$, one for each direction. These matrices are the transformations that act on a vector when "moving" infinitesimally along each direction.

A generalization of what we presented is the concept of a fiber bundle. The basic construction is very similar to the one of a manifold, though somewhat more abstract. In fact a smooth manifold is a particular case of a fiber bundle called the tangent bundle. In the language of fiber bundles, the tangent bundle is the following. The space of vectors at a point $p \in \mathcal{M}$ was written $T_p\mathcal{M}$. We can take the union of those spaces for all the points p in a given chart U_i to talk about all the tangent spaces in a chart U_i. So we write

$$TU_i = \bigcup_{p \in U_i} T_p\mathcal{M} \ . \tag{58}$$

This means that for any $p \in U_i$, we have a space $T_p\mathcal{M} \in TU_i$ associated to this point. Since U_i and $T_p\mathcal{M}$ are both homeomorphic to $\mathbb{R}^n$, TU_i can be understood as the direct product $\mathbb{R}^n \times \mathbb{R}^n$ with the identification given by $(p, V) \mapsto (x^\mu(p), V^\mu(p))$. A point $u \in TU_i$ can always be decomposed as a vector V and a point p at which the vector is evaluated. This is formalized by a projection $\pi(u) = p$ for $u \in TU_i$. The fiber at a point p is then defined as $\pi^{-1}(p)$, meaning all the points $u \in TU_i$ such that their base point is p. In the present case, we have for a fixed p, $\pi^{-1}(p) \simeq \mathbb{R}^n$, which is the space of vectors at a point p. We can define a diffeomorphism from the direct product $U_i \times T_p\mathcal{M}$ to the fiber by

$$\varphi_i : \quad U_i \times T_p\mathcal{M} \to \pi^{-1}(U_i) \tag{59}$$

$$(p, V) \mapsto V(p) \quad , \tag{60}$$

which are compatible with the projection : $\pi \circ \varphi_i(p, V) = p$. On $U_i \cap U_j \neq \emptyset$, the maps $\varphi_i(p, V)$ and $\varphi_j(p, V)$ for a fixed point p are diffeomorphism of the fiber $T_p\mathcal{M}$. They are related by

$$\varphi_{i,p}(V) = t_{ij}(p)\varphi_{j,p}(V) \Leftrightarrow V^\mu = \frac{\partial x^\mu}{\partial y^\nu} \tilde{V}^\nu \tag{61}$$

where $t_{ij}(p) \in \mathrm{GL}_n(\mathbb{R})$. $\mathrm{GL}_n(\mathbb{R})$ is called the structure group, it tells us how to stitch the different fiber patches together. Finally, we call the collection of all the charts on $\mathcal{M}$ the tangent bundle, and write it

$$T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M} \ . \tag{62}$$

A vector field is then defined by the map $X : \mathcal{M} \to T\mathcal{M}$. This maps associate for each p a vector in $T_p\mathcal{M}$, so the vector field obeys the relation

$$\pi \circ X(p) = p \ . \tag{63}$$

Any map σ from the base space to the fiber is called a section if it obeys the above condition. In physics, sections correspond to the fields we use, such as the one-form potential which is a section of a principal bundle.

Let us look at the example of a principal bundle more closely. A principal bundle is defined by the fact that the fiber is the structure group itself. For a principal bundle, the local trivializations are $\varphi_i : U_i \times G \to \pi^{-1}(U_i)$ where G is a Lie group. The projection is defined by $\pi : P \to \mathcal{M}$ where P is the total space of the principal bundle, locally diffeomorphic to the tensor space $\mathcal{M} \times G$. The action of the transition functions on the fiber are simply given by the multiplication on the left in the group, so for $p \in U_i \cap U_j \neq \emptyset$,

$$\varphi_i(p, g_i) = \varphi_j(p, t_{ji} g_i) = \varphi_j(p, g_j) \ . \tag{64}$$

But because of the group structure of the fiber, we also have the possibility of acting on the right,

$$\varphi_i(p, g_i)a = \varphi_i(p, g_i a) \tag{65}$$

This definition makes sense because $\varphi_i(p, g)$ is an element of the fiber at p, so multiplication is well defined. This multiplication is independent of the local trivialization since we have for $p \in U_i \cap U_j \neq \emptyset$,

$$\varphi_i(p, g_i)a = \varphi_i(p, g_i a) = \varphi_i(p, t_{ij}(p) g_j a) = \varphi_j(p, g_j a) \ . \tag{66}$$

The right multiplication allows us to define a preferred local trivialization φ_i given a particular section $\sigma_i(p)$. We can always build a local trivialization by multiplying a section $\sigma_i(p)$ by an element $g \in G$ such that $\varphi_i(p, g) = s_i(p)g$. In this local trivialization, the section σ_i is expressed

$$\sigma_i(p) = \varphi_i(p, e) \tag{67}$$

where $e \in G$ is the identity element of the group. This choice of local trivialization is called the canonical local trivialization. Conversely, given a local trivialization $\varphi_i(p, g)$, we can build a canonical section $\sigma_i(p)$ by $\varphi_i(p, g) = \varphi_i(p, e)g = \sigma_i(p)g$.

A connection in this context is a separation of the tangent space T_uP into a *vertical* subspace (V_uP) and an *horizontal* one (H_uP). These are define in the

following way. The right action on the bundle defines a right action on T_uP. This is written for $X \in T_uP$ and A a vector field such that $g = \exp(tA) \in G$

$$\mathcal{A} = R_{g*}X = \frac{\mathrm{d}}{\mathrm{d}t}(f(ue^{tA}))\Big|_{t=0} \tag{68}$$

The vector field $\mathcal{A}$ lies in the vertical subspace of T_uP because if we note $\pi(u) = p$ then $\pi(ue^{tA}) = p$. So the vector field $\mathcal{A}$ lies in the fiber at p, the vertical subspace is generated by moving only along the fiber.

A connection is then a $\mathfrak{g}$-valued one-form that projects T_uP onto V_uP. More precisely, writing this one-form w, we have

(i) $w(\mathcal{A}) = A \in \mathfrak{g}$,

(ii) $R_g^* w = g^{-1} w g$.

The first line means that the w maps V_uP onto the Lie algebra. The second line means that the one-form w transforms in the adjoint representation upon right action of the group. Given the canonical trivialization, the connection is locally the one-form potential, familiar to physicists.

Fiber bundles are the natural language for classical gauge theories. It gives an interesting alternative point of view on physics. In this thesis, we will make use of notions related to differential geometry and fiber bundles, while trying to keep notations closer to the usual physics conventions. The basics of differential geometry and fiber bundles explained above are sufficient to grasp the rest of this manuscript, though much more should be covered for an extensive introduction. We now dive into the main development, the study of nilmanifolds and their use for gauge-Higgs unification.

Our study will focus on compact manifolds and more specifically on nilmanifolds, also known as twisted tori. The twist will be the main new ingredient in our study. Our starting point will be a pure Yang-Mills theory in 7 dimensions. The three extra dimensions form the simplest example of a nilmanifold, namely the Heisenberg manifold (see for example [36]). Our first task will be to review the geometry of this manifold and the solutions to the Laplace equation on this space. Once we know the modes of the Laplacian, we will be able to study the compactification of the model. The effective theory comprises adjoint scalar fields with a mass and a potential at tree level. We then study the potential to produce different breaking patterns. We also find a general approach to the breaking of the gauge for an arbitrary semi-simple gauge group, allowing us to study the renormalization on general grounds. Lastly we conclude our analysis with an explicit renormalized example and some comments on the possibilities and limits of the model.

0.3 Contributions

This manuscript is composed of a review of existing results along with my original work. The preexisting material consists of the study of the geometry of

the Heisenberg manifold and the study of the Laplace equation for the various forms on this space. The scalar case was worked out for an arbitrary metric, while the one-forms were only worked out for the case of the flat metric. I have worked on solving the Dirac equation for an arbitrary metric and also on understanding how the eigenvalues of the constant eigen-1-forms of the Laplacian are modified in the case of a general metric. The sections on the compactification, the study of the potential and the discussion that follows are all part the work I did during my PhD.

Chapter 1

Nilmanifolds

In this section, we are going to explore the subject of nilmanifolds. The first part will focus on definitions and general concepts. Next, we will focus on a particular nilmanifold called the Heisenberg manifold. It will be the focus of the rest of the manuscript so we will spend some time understanding as much as we can about this specific space. We will first build the manifold explicitly by choosing a set of coordinates. Then we will study the moduli space of metrics on the manifold. We will then solve the Laplace equation for the various forms on the space. We will end the section by solving the Dirac equation. We will then have all the tools we need to study the compactification of a Yang-Mills theory on this space.

1.1 Introduction

A nilmanifold is a type of differentiable manifold whose tangent vectors locally look like a nilpotent Lie algebra. A nilpotent Lie algebra is a Lie algebra $\mathfrak{g}$, such that the following series arrives at 0 at some rank n. Define the series by $\mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]$ and $\mathfrak{g}_0 = \mathfrak{g}$, then we have the inclusions:

$$\mathfrak{g} \geq \mathfrak{g}_1 \geq \cdots \geq \mathfrak{g}_n \ . \tag{1.1}$$

If $\mathfrak{g}_n = 0$ (i.e. it is the zero algebra) for some n then the Lie algebra is said to be n-step nilpotent. It was shown that every compact nilmanifolds can be obtained by the following construction. We start with a nilpotent Lie group N and a discrete subgroup $\Gamma \subset N$. If the quotient space N/Γ is compact then the space is called a compact nilmanifold. With compactification in mind, we will only study compact nilmanifolds in the following. Additionally, we interested only in Lie algebras that are in one-to-one correspondence with their Lie group. Stated otherwise the group N considered is connected and nilpotent. As we will see, that is the case for the Heisenberg manifold.

Let us be more specific by introducing a basis for a d-dimensional group manifold. We call V_a, $a \in \{1, \ldots, d\}$ the tangent vectors (left-invariant vector

fields that are globally defined). The Lie brackets between the vectors are defined by

$$[V_a, V_b] = f^c{}_{ab} V_c \tag{1.2}$$

with $f^c{}_{ab} = -f^c{}_{ba}$ the structure constants. Dual to those vectors are the one-forms obeying the Maurer-Cartan equations

$$de^c = -\frac{1}{2} f^c{}_{ab} e^a \wedge e^b \ . \tag{1.3}$$

By endowing the space with the metric δ_{ab}, we can compute the Ricci tensor. For a unimodular Lie algebra (i.e. $|\det(Ad(g))| = 1$ for $g \in N$, or equivalently $\mathrm{Tr}(ad(X)) = 0$ for $X \in \mathfrak{g}$, which is the case for nilpotent algebras), we get

$$\mathcal{R}_{cd} = \frac{1}{2}\left(-f^b{}_{ac} f^a{}_{bd} - \delta^{bg}\delta_{ah} f^h{}_{gc} f^d{}_{ab} + \frac{1}{2}\delta^{ah}\delta^{bj}\delta_{ci}\delta_{bg} f^b{}_{ac} f^i{}_{aj} f^g{}_{hb} \right) \tag{1.4}$$

In the case of a nilpotent Lie algebra, the first term vanishes (it is the Killing form on the space which is zero for nilpotent algebras). Further contracting the Ricci tensor to obtain the Ricci scalar curvature gives

$$\mathcal{R} = -\frac{1}{4}\delta_{ad}\delta^{be}\delta^{cg} f^a{}_{bc} f^f{}_{de} \tag{1.5}$$

which is constant and negative. Therefore nilmanifolds equipped with a flat Euclidian metric are negatively curved spaces. Let us now take the lowest dimensional nilmanifold and study this case in more details.

1.2 The Heisenberg manifold

This section will develop all the basic structure of the Heisenberg manifold. References for the following derivations can be found in [37, 38].

The simplest non-trivial example starts at $d = 3$. There are three possible algebras. Only one of them is nilpotent. The vector algebra reads

$$[V_1, V_2] = -\mathtt{f} V_3 \ , \ [V_1, V_3] = [V_2, V_3] = 0 \ . \tag{1.6}$$

such that $\mathtt{f} = -f^3{}_{12}$. Technically, this is a family of algebras all ismorphic to one another that are labeled by the specific value assigned to $\mathtt{f}$. Equations (1.3) simplify to

$$de^3 = \mathtt{f} e^1 \wedge e^2 \ ; \ de^1 = 0 \ ; \ de^2 = 0 \ . \tag{1.7}$$

This manifold is known as the Heisenberg manifold because the commutation relations form the algebra of the position and momentum operators (along with the identity).

In order to do explicit computations on this manifold, and study the physics on this space we need a system of coordinates such that the above equations are obeyed. A possible choice is

$$e^1 = dX^1 \ ; \quad e^2 = dX^2 \ ; \ e^3 = dX^3 + \mathtt{f} X^1 dX^2 \ . \tag{1.8}$$

This is the most general solution up to redefinition of the coordinates. We can choose to rescale the coordinates such that

$$X^i = r^i x^i \tag{1.9}$$

where the r^i are positive "radii" and the x^i are "angular" coordinates. In those new coordinates, equation (1.8) becomes

$$e^1 = r^1 dx^1 \ ; \quad e^2 = r^2 dx^2 \ ; \ e^3 = r^3 \left(dx^3 + N x^1 dx^2 \right) \ , \tag{1.10}$$

$$\text{where} \quad N = \frac{r^1 r^2}{r^3} \mathtt{f} \ . \tag{1.11}$$

We will see later that consistency requires

$$N \in \mathbb{Z} \ . \tag{1.12}$$

As explained in the beginning of the chapter, we now impose discrete identifications in order to make the manifold compact. Those identifications correspond to the quotient by the discrete lattice. These are explicitly

$$\begin{aligned} &x^1 \sim x^1 + n^1 \ ; \ x^2 \sim x^2 + n^2 \ ; \ x^3 \sim x^3 + n^3 - n^1 N x^2 \ , \\ &n^1, n^2, n^3 \in \{0, 1\} \ , \end{aligned} \tag{1.13}$$

which are just translations by a unit angle, analogous to a full rotation along a coordinate on a torus. This is not the only possible choice for the n^i. Choosing different integers would correspond to choosing a different lattice for the quotient. A non unitary identification would result in a cell with the following parameters

$$r'^i = n^i r^i \ , \ N' = N \frac{n^1 n^2}{n^3} = \mathtt{f} \frac{r'^1 r'^2}{r'^3} \ . \tag{1.14}$$

For the same reason as in the case of $n^1, n^2, n^3 \in \{0, 1\}$ (see 1.2.1 below), N' must be an integer. This implies that

$$N \frac{n^1 n^2}{n^3} \in \mathbb{Z}^* \ . \tag{1.15}$$

Once the unit cell of the lattice is defined by equation (1.13), any combination with $n^i \in \mathbb{Z}$ is also an identification. The identification is therefore consistent throughout the manifold. From now, we take the identifications to be with $n^i \in \{0, 1\}$.

This is a good time to pause and think about the geometry we are describing. The x^1 and x^2 directions form a torus as can be seen from (1.7) and (1.8). The x^3 direction can be interpreted as a $U(1)$ fiber on the base torus. The fibration is trivial in the case $N = 0$, where we recover the usual 3-torus. Here, the case is more general, the fibration can be twisted, the "amount of twisting" is characterized by N. As stated in Section 1, nilmanifolds can always be constructed as a principal torus bundle over a principal torus bundle ... over a torus. This fiber bundle interpretation is particularly useful in understanding the integrality of N. We now turn to this question.

1.2.1 Integrality of N

There are several different ways to prove the integrality of N. The first one is immediate, given some knowledge of fiber bundles. Indeed, viewing the Heisenberg manifold as a principal $U(1)$ bundle over a 2-torus, the first Chern class is integral in cohomology and given by $c_1 = N dx^1 \wedge dx^2$. Let us now show this result explicitly.

We start by parametrising the circle fiber by $g = e^{2\pi i x^3}$ with $x^3 \in [0, 1]$. Next we can read off of (1.8) the vertical displacement on the fiber

$$dx^3 + N x^1 dx^2 = \frac{1}{2\pi i} g^{-1} \mathcal{D} g \ , \ \mathcal{D} = d + A \ , \tag{1.16}$$

with $A = 2\pi i N x^1 dx^2$ the connection on the fiber. We now need to specify the charts on the base torus. Since the torus is the tensor product of two circles $T_2 = S^1_{x^1} \times S^1_{x^2}$, we use the following cover

$$\mathcal{B}_+ = U_+ \times S^1_{x^2} \ ; \ \mathcal{B}_- = U_- \times S^1_{x^2} \tag{1.17}$$

where $U_+ = \{-\epsilon < x^1 < 1 - \epsilon\}$ and $U_- = \{1 - 2\epsilon < x^1 < 1 + \epsilon\}$. The overlap $U_+ \cap U_-$ in the limit $\epsilon \to 0$ is then given by the point $x^1_+ = 0$ or equivalently from the point of view of the chart U_-, $x^1_- = 1$. The intersection of the charts $\mathcal{B}_\pm$ is

$$\mathcal{B}_+ \cap \mathcal{B}_- = \mathcal{Q} \times S^1_{x^2} \ , \tag{1.18}$$

where $\mathcal{Q} = \{x^1_+ = 0\} = \{x^1_- = 1\}$. The transition function that relates the connections on the two charts is a map $t : \mathcal{Q} \times S^1_{x^2} \simeq S^1 \to U(1)$. This is equivalent to a map from S^1 to $S^1 \simeq U(1)$ and is classified by the fundamental group $\pi(S^1) = \mathbb{Z}$. We can then write the map as $t : x^2 \to exp\left(2\pi i m x^2\right)$ with $m \in \mathbb{Z}$. Writing the connection on both charts as $A_\pm = 2\pi i x^1_\pm dx^2_\pm$, we express the relation between the two on the overlap $\mathcal{Q} \times S^1_{x^2}$ by the use of t,

$$A_+ = t^{-1} A_- t + t^{-1} dt \ . \tag{1.19}$$

Applying this relation at the point $\mathcal{Q}$, this equation simplifies to

$$0 = 2\pi i N + 2\pi i m \Leftrightarrow N = -m \ . \tag{1.20}$$

This shows that $N \in \mathbb{Z}$. Finally we can understand how the fibers are patched together. Taking $x^3_\pm$ to be the coordinates of the fiber on the charts $\mathcal{B}_\pm$, a point of the fiber of each chart is written $g_\pm = e^{2\pi i x^3_\pm}$. The two points are related on the overlap by the left action of the transition function,

$$g_+ = t g_- \Leftrightarrow x^3_+ = x^3_- - N x^2 \ . \tag{1.21}$$

This equation describes the twist of the fiber, characterized by the integer N. This relation is invariant under the lattice action $x^2 \to x^2 + 1$ since the fiber coordinate x^3 is defined up to translations by an integer. For a fixed x^2, this equation can be interpreted as the number of turns by which the circle fiber is rotated when going once along the x^1 coordinate. This rotation depends on where we are on the circle $S^1_{x^2}$. As an example, take $N = 1$, for $x^2 = 0$ there is no rotation of the fiber, while on the opposite side, for $x^2 = 1/2$, the fiber circle will rotate by π.

Lastly, let us discuss in a little more details the different possibilities for the lattice. Taking the identifications to be with $n^i \in \mathbb{Z}$, we have the following relations that are true

$$\begin{aligned}(x^1, x^2, x^3) &\sim (x^1 + n^1, x^2, x^3) \sim (x^1, x^2 + n^2, x^3) \\ &\sim (x^1, x^2 + n^2, x^3 + n^3) \sim (x^1 + n^1, x^2, x^3 - n^1 N x^2)\end{aligned} \tag{1.22}$$

Using combinations of these identifications, we can arrive at the following chain of identifications

$$(x^1, x^2, x^3) \sim (x^1, x^2, x^3 + n^3) \sim \cdots \sim (x^1, x^2, x^3 + \tilde{N} n^3) \tag{1.23}$$

for any $\tilde{N} \in \mathbb{Z}$. But we also have

$$\begin{aligned}(x^1, x^2, x^3) &\sim (x^1 + n^1, x^2, x^3 - n^1 N x^2) \\ &\sim (x^1 + n^1, x^2 + n^2, x^3 - n^1 N n^2 - n^1 N x^2) \sim (x^1, x^2, x^3 - n^1 N n^2) \ .\end{aligned} \tag{1.24}$$

For consistency throughout the lattice, we should have for some fixed $\tilde{N} \in \mathbb{Z}$

$$\tilde{N} = \frac{n^1 n^2}{n^3} N \Leftrightarrow N = \frac{n^3}{n^1 n^2} \tilde{N} \in \mathbb{Q}^* \tag{1.25}$$

Remembering that the structure constant is proportional to N, we find the result due to Mal'cev stating that a lattice exist if and only if there exists a basis in which the structure constants are rational.

This concludes our study of nilmanifolds. After having discussed nilmanifolds on rather general grounds in the first part, we picked the simplest example, the 3 dimensional manifold called the Heisenberg manifold. By specifying a coordinate system on the cotangent space, we were able to understand its interpretation as a circle bundle over a torus. We also showed how topological restrictions forced N to be an integer. The degrees of freedom we need to specify to fix the geometry are the radii r^i which have dimension of lengths, and the twist parameter `f`, which has dimension of mass. Those parameters are constrained by equation (1.11). We now proceed with the study of the possible metrics on the Heisenberg manifold.

1.2.2 Metrics on the Heisenberg manifold

So far we have been interested in understanding the geometry of the space. We also have the liberty to endow the space with a metric. Let us take a look at the space of left-invariant metrics. In vielbein coordinates, we can always choose a basis such that the metric is the Euclidian metric globally. That is because of the group structure of the initial nilpotent group.

$$ds^2 = \delta_{ab} E^a E^b \ . \tag{1.26}$$

Because we are interested in left invariant metrics, we can relate the E^a basis to the previous one using a transformation in $GL(3,\mathbb{R})$ such that

$$E^a = (L^{-1})^a{}_b e^b \ . \tag{1.27}$$

In this class of transformations, there exist transformations that are trivial from the point of view of the local nilpotent algebra. More specifically, there are transformations that leave the defining equation (1.3) invariant. We should mod out those transformations from $GL(3,\mathbb{R})$ to be left with only those giving nonequivalent metrics on the space. We write (1.3) in another basis and impose the condition that the equation should remain the same. We denote $(Me)^a$ with $M \in GL(3,\mathbb{R})$ the new basis. We have

$$d(Me)^a = M^a{}_b de^b = -M^a{}_b f^b{}_{cd}\ e^c \wedge e^d = -f^a{}_{bc}\ (Me)^b \wedge (Me)^c \ . \tag{1.28}$$

And so we find the constraint

$$M^a{}_b f^b{}_{cd} = f^a{}_{be} M^b{}_c M^e{}_d \ . \tag{1.29}$$

Applying this constraint to the case of the Heinsenberg manifold, we find that M is of the form

$$M = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ M_{31} & M_{32} & M_{11}M_{22} - M_{12}M_{21} \end{pmatrix} \ , \text{ with } M_{11}M_{22} - M_{12}M_{21} \neq 0 \ . \tag{1.30}$$

These matrices form a subgroup of $GL(3,\mathbb{R})$ denoted *Aut*. If two "L" transformations are related by $L_1^{-1} = L_2^{-1} M$, they are in the same class since the two associated basis will only differ by M, which leave equation (1.3) invariant. It is therefore natural to mod out the group of transformations $GL(3,\mathbb{R})$ by the group *Aut* of automorphism of the Heisenberg algebra. This is done by introducing the equivalence relation $L^{-1} \sim L^{-1} M$ on $GL(3,\mathbb{R})$. Our task is now to find a representative of each class. To do so, we will now denote M_G the matrix in *Aut* such that

$$M_G = \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ G_{31} & G_{32} & G_{11}G_{22} - G_{12}G_{21} \end{pmatrix} \ , \tag{1.31}$$

where G_{ij} are the coefficient of a matrix $G \in GL(3,\mathbb{R})$. Now, introduce an arbitrary matrix $G \in GL(3,\mathbb{R})$. We have the relation

$$GM_{G^{-1}} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix} \ , \tag{1.32}$$

with

$$a = \frac{G_{13}G_{33}}{\det(G)} \ , \ b = \frac{G_{23}G_{33}}{\det(G)} \ , \ c = \frac{(G_{33})^2}{\det(G)} \ . \tag{1.33}$$

Therefore, $\forall G \in GL(3,\mathbb{R})$, we have $G = L\left(M_{G^{-1}}\right)^{-1}$ where L is an element of

$$\mathcal{E} \ = \left\{ L = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix} , \ a,b \in \mathbb{R} \ , \ c \in \mathbb{R}^* \right\} \ . \tag{1.34}$$

This set turns out to be exactly the set of representative of the coset $GL(3,\mathbb{R})/Aut$. Indeed, if we take $L_1 \in \mathcal{E}$, we can easily show $L_1 M \in \mathcal{E} \Leftrightarrow M = \mathbb{I}$. If we now take two different $L_1, L_2 \in \mathcal{E}$, then $\forall M \in Aut\backslash\{\mathbb{I}\}$, $L_2 M \notin \mathcal{E}$, which necessarily implies $L_2 M \neq L_1$. In the case where M is the identity, we still have $L_2 \neq L_1$ by assumption. From this we conclude that $\forall M \in Aut$, $L_2 M \neq L_1$ and so the L matrices belong to different equivalence classes. This shows that $\mathcal{E}$ is indeed a set of representative of the coset $GL(3,\mathbb{R})/Aut$.

We can now apply a transformation with a generic element from the set $\mathcal{E}$ to equation (1.26). This gives the most general metric on the nilmanifold

$$ds^2 = (e^1 + ae^3)^2 + (e^2 + be^3)^2 + c^2(e^3)^2, \quad a,b \in \mathbb{R}, \ c \in \mathbb{R}^* \ . \tag{1.35}$$

We deduce $\sqrt{g} = r^1 r^2 r^3 |c|$, and the volume

$$V = \int dx^3 \sqrt{g} = r^1 r^2 r^3 |c| \ . \tag{1.36}$$

To conclude this section, let us discuss the geometrical meaning of the parameters a, b, c. First of all, the parameter c is redundant. It can be absorbed by redefining the other parameters as follow:

$$r^3 \to \frac{r^3}{|c|}, \quad a \to a|c|, \quad b \to b|c|, \quad \mathtt{f} \to \frac{\mathtt{f}}{|c|} \ . \tag{1.37}$$

So we can set $c = 1$ without loss of generality. The parameters a and b can be interpreted as analogues of a complex structure on a torus. To see that, we fix x^2 to a constant value, along with $b = 0$ and $N = 0$. The resulting structure is an untwisted 2-torus with metric

$$ds^2 = (r^1)^2 \left(\left(dx^1 + a\frac{r^3}{r^1} dx^3 \right)^2 + \left(\frac{r^3}{r^1} dx^3 \right)^2 \right) \ . \tag{1.38}$$

To compare this with the torus, introduce the metric on a 2-torus with area ν and complex structure $\tau = \tau_1 + i\tau_2$:

$$\mathrm{d}s^2 = \frac{\nu}{\tau_2}\left(\left(\mathrm{d}x^1 + \tau_1 \mathrm{d}x^2\right)^2 + (\tau_2 \mathrm{d}x^3)^2\right) \ . \tag{1.39}$$

Comparing the two, we arrive at the relations $\nu = r^1 r^3$, $\tau_1 = ar^3/r^1$ and $\tau_2 = r^3/r^1$. The two sets are therefore two equivalent parameterizations of the torus. In particular, we have $a = \tau_1/\tau_2$ which relates it to the complex structure. A completely similar analysis can be made for b.

1.3 Laplacian on the Heisenberg manifold

Our objective is to study the compactification of Yang-Mills theories on nilmanifolds. A requirement is to understand the eigenforms of the Laplace operator on the space since the compactification will be effectively done by selecting the first lightest modes. The eigenvalues of the Laplacian are of mass dimension 2, hence their interpretation as the masses squared of the modes. We will begin this section by studying the case of the scalar field for an arbitrary metric. Using those results, we will then workout the spectrum of one-forms, but only in the case the Euclidian metric. After that, we will see how the first modes are modified in the case of a general metric. Lastly, we will study the spectrum of the fermions in the general case. We will follow the derivations given in [37–39].

1.3.1 The scalar spectrum

With the metric given in the previous section, we can workout the expression for the Laplace operator on a scalar field. We start with the general expression for the Laplacian acting on a scalar field. From Hodge theory, we know that it can expressed as

$$\Delta\phi = (\mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger)\phi = \mathrm{d}^\dagger \mathrm{d}\phi = -\frac{1}{\sqrt{g}}\partial_m\left(\sqrt{g}g^{mn}\partial_n\phi\right) \tag{1.40}$$

The determinant of the metric is constant here, so it drops out of the equation. Let us start with the simplest case possible, and go from there to study the more general setting.

The simplest case of $a = b = 0$

We start by studying the case where $a = b = 0$. Additionally, we set the radii $r^i = 1$, $c = 1$ and $\mathtt{f} = 1$ (or equivalently $N = 1$). The metric in this case reads

$$\mathrm{d}s^2 = \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3 + x^1\mathrm{d}x^2\right)^2 \ , \tag{1.41}$$

and gives the Laplacian

$$\nabla^2 u = \Delta u = \left(\partial_1^2 + (\partial_2 - x^1\partial_3)^2 + \partial_3^2\right)u \ . \tag{1.42}$$

This setup has the advantage of simplifying the expressions while still retaining the possibility to recover the dependence on the parameters at the end. The objective is to find a basis of function that are invariant under (1.13).

A simple starting point is to observe the following: for functions that don't depend on x^3 we can use the usual Fourier modes on the torus. If we set

$$\tilde{v}_{p,q}(x^1,x^2) = e^{2\pi p i x^1} e^{2\pi q i x^2} \;; \quad p,q \in \mathbb{Z} \;, \tag{1.43}$$

then the action of the Laplacian on those functions gives the eigenvalue equation

$$\big(\nabla^2 + \mu_{p,q}^2\big)\tilde{v}_{p,q} = 0 \;, \tag{1.44}$$

where we define

$$\mu_{p,q}^2 = 4\pi^2(p^2+q^2) \;. \tag{1.45}$$

This set of function is manifestly invariant under the identifications (1.13) and forms the first set of eigenscalar. These are the modes that are limited to the base torus.

Consider now the following set of functions, built from Weil-Brezin-Zak transforms [40],

$$u_{k,l}(x^1,x^2,x^3) = e^{2\pi k i(x^3+x^1x^2)} e^{2\pi l i x^1} \sum_{m\in\mathbb{Z}} e^{2\pi k m i x^1} f(x^2+m) \;; \quad k,l \in \mathbb{Z} \;. \tag{1.46}$$

First of all, one can check that these functions are invariant under the lattice action (1.13). Acting on it with the Laplacian gives

$$\begin{aligned}\nabla^2 u_{k,l} =& e^{2\pi k i(x^3+x^1x^2)} e^{2\pi l i x^1} \\ &\times \sum_{m\in\mathbb{Z}} e^{2\pi k m i x^1} \Big\{\partial_2^2 - 4\pi^2\Big(k^2 + \big(k(x^2+m)+l\big)^2\Big)\Big\} f(x^2+m) \;.\end{aligned} \tag{1.47}$$

We consider here $k \neq 0$, otherwise the function wouldn't depend on x^3. Setting $z_m = x^2 + m + l/k$ and $g(z_m) = f(x^2+m)$, we obtain

$$\nabla^2 u_{k,l} = e^{2\pi k i(x^3+x^1x^2)} e^{2\pi l i x^1} \sum_{m\in\mathbb{Z}} e^{2\pi k m i x^1} \Big\{\partial_{z_m}^2 - (2\pi k)^2(z_m^2+1)\Big\} g(z_m) \;. \tag{1.48}$$

We are now looking for eigen-functions of the differential operator inside the sum. To this end, we recall the definition of the normalized Hermite functions

$$\Phi_n(z) = e^{-\frac{1}{2}z^2} H_n(z) \;; \quad n \in \mathbb{N} \;, \tag{1.49}$$

where H_n are the Hermite polynomials. Recall that the Hermite polynomials are defined as $H_n(y) = (-1)^n e^{y^2}\partial_y^n e^{-y^2}$ and satisfy the differential equation $\partial_y^2 H_n - 2y\partial_y H_n + 2nH_n = 0$. For $\lambda \in \mathbb{R}^*$ we define

$$\Phi_n^\lambda(z) = |\lambda|^{\frac{1}{4}} \Phi_n(|\lambda|^{\frac{1}{2}} z) \;, \tag{1.50}$$

which obeys the differential equation

$$(\partial_z^2 - \lambda^2 z^2)\Phi_n^\lambda(z) = -(2n+1)|\lambda|\Phi_n^\lambda(z) \ . \tag{1.51}$$

Hence by setting $\lambda = 2\pi k$ and $g(z_m) = \Phi_n^{2\pi k}(z_m)$ in (1.48) we obtain:

$$\big(\nabla^2 + M_{k,l,n}^2\big)\tilde{u}_{k,l,n} = 0 \ , \tag{1.52}$$

where the eigenvalues (i.e. the masses) $M_{k,l,n}$ are given by

$$M_{k,l,n}^2 = (2\pi k)^2 \left(1 + \frac{2n+1}{2\pi|k|}\right) \ , \tag{1.53}$$

and we defined

$$\begin{aligned}\tilde{u}_{k,l,n}(x^1,x^2,x^3) = e^{2\pi k \mathrm{i}(x^3 + x^1x^2)} e^{2\pi l \mathrm{i} x^1} \sum_{m\in\mathbb{Z}} e^{2\pi k m \mathrm{i} x^1} \Phi_n^{2\pi k}(x^2 + m + \frac{l}{k}) \ ; \\ l = 0,\dots,|k|-1 \ , \ k \in \mathbb{Z}^* \ , \ n \in \mathbb{N}.\end{aligned} \tag{1.54}$$

Observe that the eigenvalues are degenerate since equation (1.53) doesn't depend on l. The parameter l is defined modulo $|k|$, so the degeneracy is of order $|k|$. This can be seen from the following identity

$$\tilde{u}_{k,l+pk,n}(x^1,x^2,x^3) = \tilde{u}_{k,l,n}(x^1,x^2,x^3) \ , \quad \forall p \in \mathbb{Z} \ , \tag{1.55}$$

which can be shown using the explicit form of $\tilde{u}_{k,l,n}$. We can reintroduce the parameters r^i and $\mathtt{f}$ (along with c). We obtain

$$\begin{aligned} M_{k,l,n}^2 &= k^2 \left(\frac{2\pi}{r^3 c}\right)^2 + (2n+1)|k| \frac{2\pi|\mathtt{f}|}{r^3} \ , \\ \mu_{p,q}^2 &= p^2 \left(\frac{2\pi}{r^1}\right)^2 + q^2 \left(\frac{2\pi}{r^2}\right)^2 \ , \end{aligned} \tag{1.56}$$

The associated modes are

$$\begin{aligned} u_{k,l,n}(x^1,x^2,x^3) &= \sqrt{\frac{r^2}{|N|V}} \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{2\pi \mathrm{i} k(x^3 + N\, x^1 x^2)} e^{2\pi \mathrm{i} l x^1} \sum_{m\in\mathbb{Z}} e^{2\pi \mathrm{i} k m x^1} \Phi_n^\lambda(w_m) \ , \\ v_{p,q}(x^1,x^2) &= \frac{1}{\sqrt{V}} e^{2\pi \mathrm{i} p x^1} e^{2\pi \mathrm{i} q x^2} \ , \end{aligned} \tag{1.57}$$

with $\lambda = k\frac{c}{|c|}\frac{2\pi\mathtt{f}}{r^3}$ and $w_m = r^2\left(x^2 + \frac{m}{N} + \frac{l}{kN}\right)$. Note that the only massless mode is $\tilde{v}_{0,0}$, which is a mode coming from the base torus.

General case

Remember from section 1.2.2, the orthonormal basis E^a can be given in terms of the defining basis e^a of equation (1.3) by $E^a = L^a{}_b e^b$. We can write the relation between the coordinate basis and the basis e^a using the veilbeins e with components $e^a{}_m$ (e can be seen as a matrix acting on the coordinate basis). The relation is

$$e = \begin{pmatrix} r^1 & & \\ & r^2 & \\ & r^3 N x^1 & r^3 \end{pmatrix} \ , \ e^a = e^a{}_m \mathrm{d}x^m \ , \tag{1.58}$$

and we denote $(e^{-1})^m{}_a \equiv e^m{}_a$. The general metric (1.35) can be written as a matrix $g = (Le)^T \delta Le$ for $L \in \mathcal{E}$ (see section 1.2.2) with $\delta = \mathbb{1}_3$. This is simply the Euclidian metric seen from the point of view of the coordinate basis, giving the metric g. It is easy to verify the value of $\sqrt{g}$, already given in (1.36). The inverse metric is then simply $g^{-1} = (Le)^{-1}\delta^{-1}(Le)^{-T}$. The vector $e^{-T}\partial$, of component $(e^{-T})_a{}^m \partial_m = e^m{}_a \partial_m$, is the dual vector or co–frame to the above one-forms; analogously, $(Le)^{-T}\partial = L^{-T}e^{-T}\partial$ is the co–frame to E^a (the L here is the inverse of the one in (1.27)). Using (1.10), those vectors are given by

$$(Le)^{-T}\partial = \begin{pmatrix} \frac{\partial_1}{r^1} \\ \frac{\partial_2}{r^2} - \mathtt{f}\, r^1 x^1 \frac{\partial_3}{r^3} \\ \frac{1}{c}\left(-a\frac{\partial_1}{r^1} - b\frac{\partial_2}{r^2} + (1+\mathtt{f}\, b r^1 x^1)\frac{\partial_3}{r^3}\right) \end{pmatrix} \ . \tag{1.59}$$

Since the determinant of the metric is constant, and because of the easily checked property $\partial_m e^m{}_a = 0$ (without sum), the Laplacian (1.40) is simply given by the square of these vectors

$$\nabla^2 \Phi = \left((Le)^{-T}\partial\right)^2 \Phi \ , \tag{1.60}$$

that results here in

$$\nabla^2 = \left(\frac{\partial_1}{r^1}\right)^2 + \left(\frac{\partial_2}{r^2} - \mathtt{f}\, r^1 x^1 \frac{\partial_3}{r^3}\right)^2 + \frac{1}{c^2}\left(-a\frac{\partial_1}{r^1} - b\frac{\partial_2}{r^2} + (1+\mathtt{f}\, b r^1 x^1)\frac{\partial_3}{r^3}\right)^2 \ . \tag{1.61}$$

Introducing the dimensionful coordinates $X^m = r^m x^m$, the Laplacian reads

$$\nabla^2 = (\partial_{X^1})^2 + \left(\partial_{X^2} - \mathtt{f}\, X^1 \partial_{X^3}\right)^2 + \frac{1}{c^2}\left(\partial_{X^3} - a\partial_{X^1} - b\left(\partial_{X^2} - \mathtt{f}\, X^1 \partial_{X^3}\right)\right)^2 \ . \tag{1.62}$$

In terms of the X^m's the discrete identifications (1.13) take the form

$$\begin{aligned} X^1 \sim X^1 + n^1 r^1 \ ; \quad X^2 \sim X^2 + n^2 r^2 \ ; \\ X^3 \sim X^3 + n^3 r^3 - \mathtt{f}\, n^1 r^1 X^2 \ , n^1, n^2, n^3 \in \{0,1\} \ . \end{aligned} \tag{1.63}$$

The following set of functions is invariant under these identifications,

$$U_{k,l}(X^1, X^2, X^3) = e^{2\pi K \mathrm{i}(X^3 + \mathtt{f}\, X^1 X^2)} e^{2\pi L \mathrm{i} X^1} \sum_{m\in\mathbb{Z}} e^{2\pi K M \mathrm{i} X^1} F\left(X^2 + \frac{M}{\mathtt{f}}\right) \ , \tag{1.64}$$

where

$$K = \frac{k}{r^3} \ , \quad L = \frac{l}{r^1} \ , \quad M = \frac{r^3}{r^1} m \ ; \quad m, k, l \in \mathbb{Z} \ . \tag{1.65}$$

Now we follow a similar procedure as in the simple case before. We consider $K \neq 0$. Plugging (1.64) into the Laplacian (1.62), setting $z_m = X^2 + M/\mathtt{f} + L/(K\mathtt{f})$ and $G(z_m) = F\left(X^2 + \frac{M}{\mathtt{f}}\right)$ we obtain

$$\begin{aligned} \nabla^2 U_{k,l} =& e^{2\pi K \mathrm{i}(X^3 + \mathtt{f} X^1 X^2)} e^{2\pi L \mathrm{i} X^1} \sum_{m \in \mathbb{Z}} e^{2\pi K M \mathrm{i} X^1} \\ & \times \left\{ \partial_{z_m}^2 - (2\pi K \mathtt{f})^2 z_m^2 + \frac{1}{c^2} \left(2\pi K \mathrm{i}(1 - \mathtt{f} a z_m) - b \partial_{z_m}\right)^2 \right\} G(z_m) \ . \end{aligned} \tag{1.66}$$

The above can also be rewritten as

$$\begin{aligned} \nabla^2 U_{k,l} =& e^{2\pi K \mathrm{i}(X^3 + \mathtt{f} X^1 X^2)} e^{2\pi L \mathrm{i} X^1} \sum_{m \in \mathbb{Z}} e^{2\pi K M \mathrm{i} X^1} \exp\left[-\frac{\mathrm{i}\pi K b}{b^2 + c^2} \ z_m (\mathtt{f} a z_m - 2) \right] \\ & \times \left\{ \partial_{z_m}^2 - \frac{4\pi^2 K^2 c^2}{(b^2 + c^2)^2} \left[(1 - \mathtt{f} a z_m)^2 + \mathtt{f}^2 (b^2 + c^2) z_m^2 \right] \right\} H(z_m) \ , \end{aligned} \tag{1.67}$$

where we have defined

$$H(z) = \frac{b^2 + c^2}{c^2} \exp\left[\frac{\mathrm{i}\pi K b}{b^2 + c^2} \ z(\mathtt{f} a z - 2) \right] G(z) \ . \tag{1.68}$$

With a further change of variable

$$w_m = z_m - \frac{a}{\mathtt{f}(a^2 + b^2 + c^2)} \ , \tag{1.69}$$

equation (1.67) becomes (for convenience we do not change notation in the first row: z_m should be thought of as a function of w_m)

$$\begin{aligned} \nabla^2 U_{k,l} =& e^{2\pi K \mathrm{i}(X^3 + \mathtt{f} X^1 X^2)} e^{2\pi L \mathrm{i} X^1} \sum_{m \in \mathbb{Z}} e^{2\pi K M \mathrm{i} X^1} \exp\left[-\frac{\mathrm{i}\pi K b}{b^2 + c^2} \ z_m (\mathtt{f} a z_m - 2) \right] \\ & \times \left\{ \partial_{w_m}^2 - \frac{4\pi^2 K^2 c^2}{(b^2 + c^2)^2} \left[\frac{b^2 + c^2}{a^2 + b^2 + c^2} + \mathtt{f}^2 (a^2 + b^2 + c^2) w_m^2 \right] \right\} T(w_m) \ , \end{aligned} \tag{1.70}$$

where $T(w_m) = H(z_m)$. Finally substituting in (1.70) $T(w_m) = \Phi_n^\lambda(w_m)$, with

$$\lambda = \frac{2\pi K c}{b^2 + c^2} (a^2 + b^2 + c^2)^{\frac{1}{2}} \mathtt{f} \ , \tag{1.71}$$

taking (1.51) into account, we obtain

$$\left(\nabla^2 + M_{k,l,n}^2\right) U_{k,l,n} = 0 \ , \tag{1.72}$$

where the Klein-Gordon masses $M_{k,l,n}$ are given by

$$M^2_{k,l,n} = \frac{4\pi^2 k^2}{(r^3)^2(a^2+b^2+c^2)}\left[1+\frac{(2n+1)r^3|\mathtt{f}|}{2\pi|kc|}(a^2+b^2+c^2)^{\frac{3}{2}}\right] , \qquad (1.73)$$

and we defined

$$U_{k,l,n}(x^1,x^2,x^3) = \sqrt{\frac{r^2}{|N|V}}\frac{1}{\sqrt{2^n n!\sqrt{\pi}}} e^{2\pi K\mathrm{i}(X^3+\mathtt{f}\,X^1X^2)} e^{2\pi L\mathrm{i}X^1}\sum_{m\in\mathbb{Z}} e^{2\pi KM\mathrm{i}X^1}$$
$$\times\exp\left[-\frac{\mathrm{i}\pi Kb}{b^2+c^2}\; z_m(\mathtt{f}az_m-2)\right]\Phi^\lambda_n(w_m)\;; \quad k\in\mathbb{Z}^*\,, n\in\mathbb{N}. \qquad (1.74)$$

Note that the $U_{k,l,n}$ differ from the $U_{k,l}$ by a factor $\frac{b^2+c^2}{c^2}\sqrt{\frac{r^2}{|N|V}}\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}$, V being the volume (1.36). This has of course no influence on the mass, but allows the $U_{k,l,n}$ to be orthonormal as we will verify later on. We rewrite the above as (with λ given in (1.71))[1]

$$U_{k,l,n}(x^1,x^2,x^3) =$$
$$\sqrt{\frac{r^2}{|N|V}}\frac{1}{\sqrt{2^n n!\sqrt{\pi}}} e^{2\pi K\mathrm{i}(X^3+\mathtt{f}\,X^1X^2)} e^{2\pi L\mathrm{i}X^1}\sum_{m\in\mathbb{Z}} e^{2\pi KM\mathrm{i}X^1}$$
$$\times\exp\left[-\frac{\mathrm{i}\pi Kb}{b^2+c^2}\left(X^2+\frac{M}{\mathtt{f}}+\frac{L}{K\mathtt{f}}\right)\left(\mathtt{f}a\left(X^2+\frac{M}{\mathtt{f}}+\frac{L}{K\mathtt{f}}\right)-2\right)\right]$$
$$\times\Phi^\lambda_n\left(X^2+\frac{M}{\mathtt{f}}+\frac{L}{K\mathtt{f}}-\frac{a}{\mathtt{f}(a^2+b^2+c^2)}\right)\;;$$
$$l=0,\dots,|k|-1\;,\; k\in\mathbb{Z}^*\,,n\in\mathbb{N}\,. \qquad (1.75)$$

As in the simple case, the range of l is finite, leading to a finite degeneracy in the masses. This can be seen from the invariance of (1.75) under $L\to L+KP$, where $P=\frac{r^3}{r^1}p$ with $p\in\mathbb{Z}$: taking (1.65) into account it then follows that (1.75) is invariant under $l\to l+kp$. It is also straightforward to recover the results of Section 1.3.1 by setting the parameters to the appropriate values.

The masses obtained in (1.73) only depend on the radius of the fiber, r^3 (or r^3c). This is a way to understand the modes just discussed as coming from the fiber. As in the simple case, there should be other modes coming from the base, independent of the fiber coordinate. The most general decomposition of

[1] As a side remark, we indicate that the product of exponentials can be rewritten as

$$\exp\left[\frac{2\mathrm{i}\pi K}{b^2+c^2}c^2(X^3+\mathtt{f}\,X^1z_m)\right]\exp\left[\frac{2\mathrm{i}\pi K}{b^2+c^2}b\left(b(X^3+\mathtt{f}\,X^1z_m)-\frac{\mathtt{f}a}{2}z_m^2+z_m\right)\right]\,,$$

where each of those two exponentials gets acted on non-trivially by only one of the three terms in the Laplacian (1.62). This might be of interest in generalizing the solution to other nilmanifolds.

such modes, invariant under the identifications (1.63), can be given in terms of a Fourier basis, i.e. in orthonormal form

$$V_{p,q}(x^1,x^2)=\frac{1}{\sqrt{V}}\,e^{2\pi\mathrm{i}PX^1}e^{2\pi\mathrm{i}QX^2}\ ,\quad P=\frac{p}{r^1},\ Q=\frac{q}{r^2},\ p,q\in\mathbb{Z}\ . \tag{1.76}$$

One then gets

$$(\nabla^2+\mu_{p,q}^2)V_{p,q}=0\ ,\quad \mu_{p,q}^2=4\pi^2\left(\frac{p^2}{(r^1)^2}+\frac{q^2}{(r^2)^2}+\frac{1}{c^2}\left(a\frac{p}{r^1}+b\frac{q}{r^2}\right)^2\right)\ . \tag{1.77}$$

As expected, these masses only depend on the base radii. The modes $V_{p,q}$ and $U_{k,l,n}$ form the complete set of eigenmodes of the Laplacian on $\mathcal{M}_3$, as will be verified later on.

We conclude this section with a summary of the results. We studied the spectrum of the Laplace operator acting on scalar fields on the Heisenberg manifold. The study revealed two types of modes. The first ones are limited to the base torus (the $V_{p,q}$'s), while the others are modes on the whole space (the $U_{k,l,n}$'s). As expected, these modes reproduce the results of the simple case $a=b=0$ of 1.3.1. These eigenfunctions of the Laplacian will be the building blocks we will use to obtain the eigenforms and solve the Dirac equation on the space. For convenience in practical applications, we give a summary of the results. The eigenscalars are

$$V_{p,q}(x^1,x^2)=\frac{1}{\sqrt{V}}\,e^{2\pi\mathrm{i}PX^1}e^{2\pi\mathrm{i}QX^2}\ , \tag{1.78}$$

with $P=\frac{p}{r^1},\ Q=\frac{q}{r^2},\ p,q\in\mathbb{Z}$, and

$$\begin{aligned}U_{k,l,n}(x^1,x^2,x^3)=&\sqrt{\frac{r^2}{|N|V}}\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\,e^{2\pi K\mathrm{i}(X^3+\mathtt{f}\,X^1X^2)}e^{2\pi L\mathrm{i}X^1}\sum_{m\in\mathbb{Z}}e^{2\pi KM\mathrm{i}X^1}\\&\times\exp\left[-\frac{\mathrm{i}\pi Kb}{b^2+c^2}\left(X^2+\frac{M}{\mathtt{f}}+\frac{L}{K\mathtt{f}}\right)\left(\mathtt{f}a\left(X^2+\frac{M}{\mathtt{f}}+\frac{L}{K\mathtt{f}}\right)-2\right)\right]\\&\times\Phi_n^\lambda\left(X^2+\frac{M}{\mathtt{f}}+\frac{L}{K\mathtt{f}}-\frac{a}{\mathtt{f}(a^2+b^2+c^2)}\right)\ ,\end{aligned} \tag{1.79}$$

with $l=0,\dots,|k|-1\ ,\ k\in\mathbb{Z}^*\ ,n\in\mathbb{N}$. Their eigenvalues are respectively

$$\mu_{p,q}^2=4\pi^2\left(\frac{p^2}{(r^1)^2}+\frac{q^2}{(r^2)^2}+\frac{1}{c^2}\left(a\frac{p}{r^1}+b\frac{q}{r^2}\right)^2\right)\ , \tag{1.80}$$

$$M_{k,l,n}^2=\frac{4\pi^2k^2}{(r^3)^2(a^2+b^2+c^2)}\left[1+\frac{(2n+1)r^3|\mathtt{f}|}{2\pi|kc|}(a^2+b^2+c^2)^{\frac{3}{2}}\right]\ . \tag{1.81}$$

The eigenscalars and their eigenvalues are related to one another by the two equations

$$(\nabla^2+\mu_{p,q}^2)V_{p,q}=0\ ,\quad \big(\nabla^2+M_{k,l,n}^2\big)U_{k,l,n}=0 \tag{1.82}$$

where the Laplacian reads

$$\nabla^2 = \left(\frac{\partial_1}{r^1}\right)^2 + \left(\frac{\partial_2}{r^2} - \mathtt{f}\, r^1 x^1 \frac{\partial_3}{r^3}\right)^2 + \frac{1}{c^2}\left(-a\frac{\partial_1}{r^1} - b\frac{\partial_2}{r^2} + (1+\mathtt{f}\, b r^1 x^1)\frac{\partial_3}{r^3}\right)^2 . \tag{1.83}$$

Orthonormality and completeness of the basis

We now show that the modes (1.74) are orthonormal, i.e.

$$\int \mathrm{d}^3x \sqrt{g}\, U_{k,l,n}(x^1,x^2,x^3) U^*_{k',l',n'}(x^1,x^2,x^3) = \delta_{k,k'}\delta_{l,l'}\delta_{n,n'} . \tag{1.84}$$

To that end, we compute the left-hand side given by

$$\begin{aligned} I = \frac{r^2}{|N|}\frac{1}{2^n n!\sqrt{\pi}} \times \\ \sum_{m,m'\in\mathbb{Z}} \int_{[0,1]^3} \mathrm{d}^3x\, e^{2\pi \mathrm{i} x^3(k-k')} e^{2\pi \mathrm{i} \mathtt{f}\frac{r^1}{r^3} x^1 (k z_{m,k,l} - k' z_{m',k',l'})} \\ \times e^{-\frac{\mathrm{i}\pi b}{r^3(b^2+c^2)}\left(k z_{m,k,l}(\mathtt{f} a z_{m,k,l} - 2) - k' z_{m',k',l'}(\mathtt{f} a z_{m',k',l'} - 2)\right)} \\ \times \Phi^\lambda_n(w_{m,k,l}) \Phi^\lambda_{n'}(w_{m',k',l'}) , \end{aligned} \tag{1.85}$$

where $z_{m,k,l}$ and $w_{m,k,l}$ correspond to z_m and w_m, and we refer to Section 1.3.1 for the definitions of the various terms. First, the integral over x^3 gives $\delta_{k,k'}$. Then, the integral over x^1 imposes similarly $l - l' = -k(m-m')$. Since $0 \le l \le |k| - 1$ and similarly for l', one deduces $|m - m'| < 1$ i.e. $m = m'$, thus $l = l'$. We deduce $z_{m,k,l} = z_{m',k',l'}$ and similarly for w_m. We are then left with

$$I = \delta_{k,k'}\delta_{l,l'} \frac{r^2}{|N|}\frac{1}{2^n n!\sqrt{\pi}} \sum_{m\in\mathbb{Z}} \int_0^1 \mathrm{d}x^2\, \Phi^\lambda_n(w_{m,k,l}) \Phi^\lambda_{n'}(w_{m,k,l}) . \tag{1.86}$$

We recall that $w_{m,k,l} = \frac{r^2}{N}\left(Nx^2 + m + \omega\right)$ with $\omega = \frac{l}{k} - \frac{aN}{r^2\mathtt{f}(a^2+b^2+c^2)}$. So

$$\begin{aligned}
I &= \delta_{k,k'}\delta_{l,l'}\frac{r^2}{|N|}\frac{1}{2^n n!\sqrt{\pi}}\sum_{m\in\mathbb{Z}}\int_m^{N+m}\frac{\mathrm{d}y}{N}\,\Phi_n^\lambda\left(\frac{r^2}{N}(y+\omega)\right)\Phi_{n'}^\lambda(\dots) \\
&= \delta_{k,k'}\delta_{l,l'}\frac{r^2}{2^n n!\sqrt{\pi}}\sum_{m\in\mathbb{Z}}\int_m^{m+1}\frac{\mathrm{d}y}{|N|}\,\Phi_n^\lambda\left(\frac{r^2}{N}(y+\omega)\right)\Phi_{n'}^\lambda(\dots) \\
&= \delta_{k,k'}\delta_{l,l'}\frac{r^2}{2^n n!\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}y}{|N|}\,\Phi_n^\lambda\left(\frac{r^2}{N}(y+\omega)\right)\Phi_{n'}^\lambda(\dots) \\
&= \delta_{k,k'}\delta_{l,l'}\frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}z\,\Phi_n^\lambda(z)\Phi_{n'}^\lambda(z) \\
&= \delta_{k,k'}\delta_{l,l'}\frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}u\,\Phi_n(u)\Phi_{n'}(u) \\
&= \delta_{k,k'}\delta_{l,l'}\frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}u\,e^{-u^2}H_n(u)H_{n'}(u) \\
&= \delta_{k,k'}\delta_{l,l'}\delta_{n,n'}\ .
\end{aligned} \tag{1.87}$$

This concludes our proof of (1.84).

Lastly, the interested reader can find the proof for the completeness of the set of modes in [37].

1.3.2 One-form spectrum

Following the study of the scalars on the Heisenberg manifold, we now turn to the one-forms. Of course, our knowledge of the eigenscalars will be our main tool in this study. The approach will be the following. We start by laying out the equations for the one-forms, along with introducing the Hodge decomposition for it. We then study each part of the decomposition. We will see that the study of one-forms can be essentially reduced to the study of co-closed one-forms. Once done with solving the Laplace equation for co-closed one-forms, we discuss higher forms using Hodge duality and give a summary of the results.

Hodge decomposition and exact one-forms

We are interested in the one-forms B on M satisfying the eigenvalue equation

$$\Delta B = \Upsilon B\ , \tag{1.88}$$

for a constant Υ. The Laplacian operator is given by $\Delta B = (*\,\mathrm{d}*\,\mathrm{d} + \mathrm{d}*\,\mathrm{d}*)B$ where d is the exterior derivative on M with the Hodge star $*$. The latter is

defined on a p-form in D dimensions as

$$\begin{aligned} * \left(\mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_p}\right) &= \frac{\sqrt{|g|}}{(D-p)!} \epsilon^{m_1 \ldots m_p}{}_{n_{p+1} \ldots n_D} \, \mathrm{d}x^{n_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{n_D} \\ \Leftrightarrow \; * \left(e^{a_1} \wedge \ldots \wedge e^{a_p}\right) &= \frac{1}{(D-p)!} \epsilon^{a_1 \ldots a_p}{}_{a_{p+1} \ldots a_D} \, e^{a_{p+1}} \wedge \ldots \wedge e^{a_D} \; , \end{aligned} \tag{1.89}$$

with $\epsilon_{1 \ldots d} = 1$ and curved or flat indices raised by g_{mn} or δ_{ab} respectively. We also recall for a p-form A_p that $*^2 A_p = s(-1)^{p(D-p)} A_p = s(-1)^{p(D+1)} A_p$ where s is the signature of the D-dimensional space, i.e. $*^2 = 1$ for our M.

The Hodge decomposition of B gives

$$B = \mathrm{d}\varphi + \mathrm{d}^\dagger b_2 + h \; , \tag{1.90}$$

where φ is a globally-defined scalar, b_2 a globally-defined two-form and h a globally-defined harmonic one-form on the nilmanifold. The three terms on the right-hand side above are orthogonal to each other with respect to the canonical pairing of one-forms on M. Let us examine the first one and look for exact one-forms solving the eigenvalue equation. The function φ can be expanded on the basis of eigenfunctions already found. In addition, one verifies that

$$\Delta \mathrm{d}v_{p,q} = -\mu_{p,q}^2 \, \mathrm{d}v_{p,q} \; , \quad \Delta \mathrm{d}u_{k,l,n} = -M_{k,l,n}^2 \, \mathrm{d}u_{k,l,n} \; , \tag{1.91}$$

from which we deduce a basis of exact one-eigenforms. These do not admit a zero-mode, since one should exclude $\mathrm{d}v_{0,0} = 0$. Using the orthonormality of the eigenfunctions and proceeding as in the case of the scalars, one verifies that these exact one-forms are orthonormal, up to a rescaling by $\mu_{p,q}$ or $M_{k,l,n}$.

We now study the other two pieces of (1.90), which are co-closed: this amounts to imposing the condition

$$\mathrm{d}^\dagger B = 0 \; \Leftrightarrow \; \mathrm{d} * B = 0 \; . \tag{1.92}$$

If one considers a theory with a gauge symmetry, (1.92) can also be viewed as a gauge-fixing condition. Indeed, for such a theory, the exact piece of B could be removed by a gauge transformation, without loss of generality.

Co-closed one-forms

We expand B as

$$B = \varphi_a(x) e^a \; , \tag{1.93}$$

with some scalars $\varphi_a(x)$, $a = 1, 2, 3$. The exterior differential is also expressed on this basis as

$$\mathrm{d} = e^a V_a \; , \tag{1.94}$$

where the vectors V_a, introduced in (1.6), are given by

$$V_1 = \frac{\partial_1}{r^1}, \; V_2 = \frac{\partial_2}{r^2} - \mathtt{f} x^1 r^1 \frac{\partial_3}{r^3}, \; V_3 = \frac{\partial_3}{r^3} \; . \tag{1.95}$$

Using (1.89), the co-closed condition (1.92) takes the form

$$\delta^{ab} V_a \varphi_b = 0 \ . \tag{1.96}$$

Moreover using the above equation (1.96), and the fact that $\Delta\varphi = \delta^{ab} V_a V_b \, \varphi$ (see (1.60)), we compute

$$\begin{aligned} \Delta B = &\left(\mathtt{f} V_2\varphi_3 + \mathtt{f} V_3\varphi_2 - \Delta\varphi_1\right) e^1 + \left(-\mathtt{f} V_1\varphi_3 - \mathtt{f} V_3\varphi_1 - \Delta\varphi_2\right) e^2 \\ &+ \left(\mathtt{f} V_1\varphi_2 - \mathtt{f} V_2\varphi_1 - \Delta\varphi_3 + \mathtt{f}^2\varphi_3\right) e^3 \ , \end{aligned} \tag{1.97}$$

where we have used (1.96) and the commutations (1.6) of the V_a. The eigenvalue equation (1.88) then gets decomposed on its various components as

$$\mathtt{f} V_2\varphi_3 + \mathtt{f} V_3\varphi_2 - \Delta\varphi_1 = \Upsilon\varphi_1 \tag{1.98}$$

$$-\mathtt{f} V_1\varphi_3 - \mathtt{f} V_3\varphi_1 - \Delta\varphi_2 = \Upsilon\varphi_2 \tag{1.99}$$

$$\mathtt{f} V_1\varphi_2 - \mathtt{f} V_2\varphi_1 - \Delta\varphi_3 + \mathtt{f}^2\varphi_3 = \Upsilon\varphi_3 \ . \tag{1.100}$$

We now expand our functions φ_a on the basis of Laplacian eigenfunctions previously determined,

$$\varphi_a = \delta_{ab} \sum_{k,l,n} C^b_{k,l,n} \, u_{k,l,n} + \delta_{ab} \sum_{p,q} D^b_{p,q} \, v_{p,q} \ , \tag{1.101}$$

with C and D constant coefficients. The equations to be solved are linear, and the dependence on x^3 in $u_{k,l,n}$ is exponential, so this dependence will not get mixed between the two sums above; this is related to the orthogonality of the eigenfunctions. We then treat these two sums independently.

Forms independent of x^3

We start with the expansion of the φ_a (1.101) on the $v_{p,q}$. The fact that they are x^3-independent simplifies the action of the V_a. The condition (1.96) and the three equations (1.98), (1.99), (1.100), become

$$P D^1_{p,q} + Q D^2_{p,q} = 0 \tag{1.102}$$

$$\mathtt{i}\mathtt{f} Q D^3_{p,q} + (P^2+Q^2) D^1_{p,q} = \Upsilon D^1_{p,q} \tag{1.103}$$

$$-\mathtt{i}\mathtt{f} P D^3_{p,q} + (P^2+Q^2) D^2_{p,q} = \Upsilon D^2_{p,q} \tag{1.104}$$

$$\mathtt{i}\mathtt{f} P D^2_{p,q} - \mathtt{i}\mathtt{f} Q D^1_{p,q} + (P^2+Q^2) D^3_{p,q} + \mathtt{f}^2 D^3_{p,q} = \Upsilon D^3_{p,q} \ , \tag{1.105}$$

where we introduce

$$P = 2\pi \frac{p}{r^1} \ , \ Q = 2\pi \frac{q}{r^2} \ , \ P^2 + Q^2 = \mu^2_{p,q} \ . \tag{1.106}$$

All solutions (except the trivial $D^{1,2,3}_{p,q} = 0$) lead to the following eigenvalues $\Upsilon = Y^{p,q}_{\pm}$

$$Y^{p,q}_{\pm} = P^2 + Q^2 + \frac{\mathtt{f}^2}{2} \pm \sqrt{\left(P^2 + Q^2 + \frac{\mathtt{f}^2}{2}\right)^2 - (P^2+Q^2)^2} \quad \geq 0 \ , \tag{1.107}$$

which solve the equation

$$(P^2+Q^2)\mathtt{f}^2 = (\Upsilon - (P^2+Q^2) - \mathtt{f}^2)(\Upsilon - (P^2+Q^2)) \ . \tag{1.108}$$

The coefficients $D^a_{p,q}$ are fixed by the previous equations, giving the following eigenforms

$$\begin{aligned} &\text{For } p^2+q^2 \neq 0: \quad B^{p,q}_{\pm} = D_{p,q}\, v_{p,q} \left(Qe^1 - Pe^2 + \frac{\Upsilon^{p,q}_{\pm} - (P^2+Q^2)}{\mathtt{if}} e^3 \right) \\ &\text{For } p=q=0: \quad Y^{0,0}_- = 0: \ B^{0,0}_1 = D_{0,0}\, v_{0,0}\, e^1 \ , \ B^{0,0}_2 = D_{0,0}\, v_{0,0}\, e^2 \\ &\qquad\qquad\qquad\quad Y^{0,0}_+ = \mathtt{f}^2: \ B^{0,0}_3 = D_{0,0}\, v_{0,0}\, e^3 \ . \end{aligned} \tag{1.109}$$

Using the orthonormality of the $v_{p,q}$, one verifies non-trivially that the above forms are orthonormal

$$\int B^{p,q}_{\epsilon} \wedge * \overline{B^{p',q'}_{\epsilon'}} = \delta_{p,p'}\delta_{q,q'}\delta_{\epsilon,\epsilon'} \ , \tag{1.110}$$

upon fixing the normalisation constant $D_{p,q}$, using (1.108), to the value

$$D_{p,q} = \frac{1}{\sqrt{\Upsilon^{p,q}_{\epsilon} + P^2 + Q^2}} \text{ for } p \text{ or } q \neq 0 \ , \quad D_{0,0} = 1 \ . \tag{1.111}$$

Forms dependent on x^3

We turn to the expansion of the φ_a (1.101) on the $u_{k,l,n}$, which depend on x^3. We first compute from (1.57), with $y = |\lambda|^{\frac{1}{2}} w_m$, $\lambda = k\frac{2\pi\mathtt{f}}{r^3}$, and the normalisation factor $\mathrm{norm}_n \propto (2^n n!)^{-\frac{1}{2}}$,

$$V_1 u_{k,l,n} = \mathrm{norm}_n |\lambda|^{\frac{1}{2}} e^{2\pi k \mathrm{i}(x^3 + N x^1 x^2)} e^{2\pi l \mathrm{i} x^1} \sum_{m\in\mathbb{Z}} e^{2\pi k m \mathrm{i} x^1} |\lambda|^{\frac{1}{4}} e^{-\frac{y^2}{2}} \mathrm{i}\ \mathrm{sgn}(\lambda) y H_n(y) \ , \tag{1.112}$$

$$\begin{aligned} &V_2 u_{k,l,n} = \\ &\quad \mathrm{norm}_n |\lambda|^{\frac{1}{2}} e^{2\pi k \mathrm{i}(x^3 + N x^1 x^2)} e^{2\pi l \mathrm{i} x^1} \sum_{m\in\mathbb{Z}} e^{2\pi k m \mathrm{i} x^1} |\lambda|^{\frac{1}{4}} e^{-\frac{y^2}{2}} \left(-y H_n(y) + H'_n(y)\right) \ . \end{aligned} \tag{1.113}$$

We use the following properties of the Hermite polynomials $\forall n \in \mathbb{N}$, with $H_{-1} = 0$,

$$\begin{aligned} H'_n(w) &= 2n H_{n-1}(w) \\ 2w H_n(w) &= H_{n+1}(w) + 2n H_{n-1}(w) \ , \end{aligned} \tag{1.114}$$

to reconstruct the various $u_{k,l,n}$. We get $\forall n \in \mathbb{N}$

$$V_1 u_{k,l,n} = |\lambda|^{\frac{1}{2}} \frac{1}{2} \mathrm{i}\ \mathrm{sgn}(\lambda) \left(\sqrt{2(n+1)}\, u_{k,l,n+1} + \sqrt{2n}\, u_{k,l,n-1} \right) \ , \tag{1.115}$$

$$V_2 u_{k,l,n} = |\lambda|^{\frac{1}{2}} \frac{1}{2} \left(-\sqrt{2(n+1)}\, u_{k,l,n+1} + \sqrt{2n}\, u_{k,l,n-1} \right) \ , \tag{1.116}$$

$$V_3 u_{k,l,n} = \frac{|\lambda|}{\mathtt{f}} \mathrm{i}\ \mathrm{sgn}(\lambda) u_{k,l,n} \ , \tag{1.117}$$

with $u_{k,l,-1} = 0$. For convenience, we now change notations with respect to the constant C of (1.101), after which we have

$$\begin{aligned} \varphi_1 &= \sum_{n\in\mathbb{N}} c_n^1 u_{k,l,n} \sqrt{2^n n!} \ , \quad \varphi_2 = \sum_{n\in\mathbb{N}} \frac{c_n^2}{\mathrm{i}\ \mathrm{sgn}(\lambda)} u_{k,l,n} \sqrt{2^n n!} \ , \\ \varphi_3 &= \sum_{n\in\mathbb{N}} \frac{c_n^3}{2} u_{k,l,n} \sqrt{2^n n!} \ , \end{aligned} \tag{1.118}$$

where in the new constants c_n^a we drop for simplicity the indices k, l although they should be understood as present.

This material allows us to reformulate the various constraints. We start with the condition (1.96) that becomes

$$\begin{aligned} &\sum_{n\in\mathbb{N}} |\lambda|^{\frac{1}{2}} \frac{1}{2} \mathrm{i}\ \mathrm{sgn}(\lambda) \left(c_{n-1}^1 + c_{n-1}^2 + 2(n+1)(c_{n+1}^1 - c_{n+1}^2) + c_n^3 \frac{|\lambda|^{\frac{1}{2}}}{\mathtt{f}} \right) u_{k,l,n} \sqrt{2^n n!} \\ &\quad = 0 \ , \end{aligned} \tag{1.119}$$

where we introduced $c_{-1}^{1,2} = 0$. Each term of the sum should vanish, leading to, $\forall n \in \mathbb{N}$,

$$c_n^3 \frac{|\lambda|^{\frac{1}{2}}}{\mathtt{f}} = -(c_{n-1}^1 + c_{n-1}^2) + 2(n+1)(-c_{n+1}^1 + c_{n+1}^2) \ . \tag{1.120}$$

We introduce for future convenience $c_{-2}^{1,2} = 0$, giving with (1.120) $c_{-1}^3 = 0$. We turn to the Laplacian equations: (1.98) and (1.99) lead respectively to, $\forall n \in \mathbb{N}$

$$\begin{aligned} &c_n^1 (M_{k,l,n}^2 - \Upsilon) + c_n^2 |\lambda| + \frac{|\lambda|^{\frac{1}{2}} \mathtt{f}}{4} (2(n+1) c_{n+1}^3 - c_{n-1}^3) = 0 \\ &c_n^2 (M_{k,l,n}^2 - \Upsilon) + c_n^1 |\lambda| + \frac{|\lambda|^{\frac{1}{2}} \mathtt{f}}{4} (2(n+1) c_{n+1}^3 + c_{n-1}^3) = 0 \ , \end{aligned} \tag{1.121}$$

where we used (1.91) for the mass. We add and subtract the above two equations, use (1.120), and obtain $\forall n \in \mathbb{N}$

$$(M_{k,l,n}^2 - \Upsilon + |\lambda| - (n+1)\mathtt{f}^2)(c_n^1 + c_n^2) - 2(n+1)(n+2)\mathtt{f}^2 (c_{n+2}^1 - c_{n+2}^2) = 0 \tag{1.122}$$

$$(M_{k,l,n}^2 - \Upsilon - |\lambda| + n\mathtt{f}^2)(-c_n^1 + c_n^2) - \frac{\mathtt{f}^2}{2} (c_{n-2}^1 + c_{n-2}^2) = 0 \ . \tag{1.123}$$

Finally, using again (1.120), (1.100) becomes $\forall n \in \mathbb{N}$

$$(M^2_{k,l,n}-\Upsilon-|\lambda|+\mathtt{f}^2)(c^1_{n-1}+c^2_{n-1})+2(n+1)(M^2_{k,l,n}-\Upsilon+|\lambda|+\mathtt{f}^2)(c^1_{n+1}-c^2_{n+1})=0\,. \tag{1.124}$$

Introducing

$$\forall n \geq -2,\ c^{\pm}_n = c^1_n \pm c^2_n\ , \tag{1.125}$$

and $c^{1,2}_{-2}=0$, $c^{1,2,3}_{-1}=0$, we summarize the conditions to be solved, (1.120), (1.122), (1.123) and (1.124), as follows $\forall n \in \mathbb{N}$

$$c^3_n \frac{|\lambda|^{\frac{1}{2}}}{\mathtt{f}} = -c^+_{n-1} - 2(n+1)c^-_{n+1} \tag{1.126}$$

$$c^+_n(M^2_{k,l,n} - \Upsilon + |\lambda| - (n+1)\mathtt{f}^2) - 2(n+1)(n+2)\mathtt{f}^2 c^-_{n+2} = 0 \tag{1.127}$$

$$c^-_n(M^2_{k,l,n} - \Upsilon - |\lambda| + n\mathtt{f}^2) + \frac{\mathtt{f}^2}{2}c^+_{n-2} = 0 \tag{1.128}$$

$$c^+_{n-1}(M^2_{k,l,n} - \Upsilon - |\lambda| + \mathtt{f}^2) + 2(n+1)c^-_{n+1}(M^2_{k,l,n} - \Upsilon + |\lambda| + \mathtt{f}^2) = 0\ . \tag{1.129}$$

Equation (1.126) determines the coefficients c^3_n in terms of $c^{1,2}_n$. In their turn the $c^{1,2}_n$ are determined by the system of equations (1.127)-(1.129), which is overdetermined. To see this more clearly it is convenient to perform a shift in the index n, to bring the system to the following form,[2] $\forall n \in \mathbb{N}$,

$$c^+_n(\alpha_n - (n+1)\mathtt{f}^2) - 2(n+1)(n+2)\mathtt{f}^2 c^-_{n+2} = 0 \tag{1.130}$$

$$c^+_n\mathtt{f}^2 + 2c^-_{n+2}(\alpha_n + 2|\lambda| + (n+2)\mathtt{f}^2) = 0 \tag{1.131}$$

$$c^+_n(\alpha_n + \mathtt{f}^2) + 2(n+2)c^-_{n+2}(\alpha_n + 2|\lambda| + \mathtt{f}^2) = 0\ , \tag{1.132}$$

where we have introduced $\alpha_n = M^2_{k,l,n} - \Upsilon + |\lambda|$, and we have taken into account that $M^2_{k,l,n+p} = M^2_{k,l,n} + 2p|\lambda|$.

Equations (1.130)-(1.132) constitute a homogeneous system of three equations for two unknowns. Generically this system can only admit the trivial solution where both c^+_n and c^-_{n+2} vanish identically. The necessary and sufficient condition for the existence of a non-trivial solution is the vanishing of all 2×2 sub-determinants of the 3×2 matrix of coefficients. Remarkably, the three conditions thus obtained turn out to be identical: the system admits non-trivial solutions for c^+_n, c^-_{n+2}, provided α_n obeys the following condition

$$\alpha^2_n + \alpha_n(\mathtt{f}^2 + 2|\lambda|) - 2(n+1)|\lambda|\mathtt{f}^2 = 0\ , \tag{1.133}$$

which ensures that all three equations (1.130)-(1.132) become equivalent. c^+_n and c^-_{n+2} can then be non-zero, and one is given in terms of the other. From

[2] We are looking for a solution to the eigenvalue problem (1.88) with a given Υ, and the latter should therefore be considered as fixed. The solution is given by a set of coefficients $c^{1,2,3}_n$. Equations (1.127)-(1.129) involving these coefficients have been obtained by projecting a sum over n on each $u_{k,l,n}$, but the projection could have been done equivalently on each $u_{k,l,n\pm 1}$, leading to shifted equations. Doing so would have given equations describing the same solution with eigenvalue Υ. So when shifting equations (1.127)-(1.129) as done to reach (1.130)-(1.132), Υ is considered as fixed.

(1.133), α_n and therefore Υ is determined in terms of n; having other coefficients $c_m^\pm$, $m \in \mathbb{N}$, would then lead to different Υ and thus correspond to different eigenmodes. For a given n, the system is then solved by setting $c_m^\pm = 0$ for all m, except for c_n^+ and c_{n+2}^-. This implies that the non-vanishing coefficients of the eigenmode are $c_n^{1,2}$, $c_{n+2}^{1,2}$ and c_{n+1}^3. Moreover, those are determined up to an overall constant, corresponding to the normalization of the one-form.

Explicitly, the eigenforms $B_\pm^{k,l,n}$ and their eigenvalues $\Upsilon = Y_\pm^{k,l,n}$ (we recall $k \in \mathbb{Z}^*$, $n \in \mathbb{N}$, $l = 0, \dots, |k| - 1$) are given by

$$\begin{aligned} B_\pm^{k,l,n} &= \sum_{a=1}^{3} \varphi_a^{k,l,n}\, e^a \\ \text{where}\quad \varphi_1^{k,l,n} &= \frac{1}{2} c \sqrt{2^n n!} \left(u_{k,l,n} - \frac{\sqrt{(n+1)(n+2)}\mathtt{f}^2}{\alpha_n + 2|\lambda| + (n+2)\mathtt{f}^2}\; u_{k,l,n+2} \right) \\ \varphi_2^{k,l,n} &= -\frac{\mathrm{i}}{2} \mathrm{sgn}(\lambda) c \sqrt{2^n n!} \left(u_{k,l,n} + \frac{\sqrt{(n+1)(n+2)}\mathtt{f}^2}{\alpha_n + 2|\lambda| + (n+2)\mathtt{f}^2}\; u_{k,l,n+2} \right) \\ \varphi_3^{k,l,n} &= -\frac{1}{2|\lambda|^{\frac{1}{2}}} c \sqrt{2^n n!}\; \frac{\mathtt{f}\sqrt{2(n+1)}(\alpha_n + 2|\lambda|)}{\alpha_n + 2|\lambda| + (n+2)\mathtt{f}^2}\; u_{k,l,n+1}\ , \end{aligned} \tag{1.134}$$

with c a constant to be determined by the overall normalization, and

$$Y_\pm^{k,l,n} = M_{k,l,n}^2 + 2|\lambda| + \frac{1}{2}\mathtt{f}^2 \pm \sqrt{\left(|\lambda| + \frac{1}{2}\mathtt{f}^2\right)^2 + 2(n+1)|\lambda|\mathtt{f}^2} \quad > 0\ , \tag{1.135}$$

with $M_{k,l,n}^2 = \frac{\lambda^2}{\mathtt{f}^2} + (2n+1)|\lambda|$, or more explicitly

$$\begin{aligned} Y_\pm^{k,l,n} =& k^2 \left(\frac{2\pi}{r^3}\right)^2 + (2n+3)\frac{2\pi}{r^3}|k\mathtt{f}| + \frac{1}{2}\mathtt{f}^2 \\ & \pm \sqrt{\left(\frac{2\pi}{r^3}|k\mathtt{f}| + \frac{1}{2}\mathtt{f}^2\right)^2 + 2(n+1)\frac{2\pi}{r^3}|k\mathtt{f}|\mathtt{f}^2}\ . \end{aligned} \tag{1.136}$$

As the coefficients $\varphi_a^{k,l,n}$ depend on $\alpha_n = M_{k,l,n}^2 - Y_\pm^{k,l,n} + |\lambda|$, each of $Y_\pm$ leads to a different eigenmode $B_\pm$. The orthonormality of the eigenforms is expressed as

$$\int B_\epsilon^{k,l,n} \wedge * \overline{B_{\epsilon'}^{k',l',n'}} = \int \mathrm{d}^3x \sqrt{g}\ \delta^{ab} \varphi_a^{k,l,n,\epsilon} \overline{\varphi_b^{k',l',n',\epsilon'}} = \delta_{k,k'}\delta_{l,l'}\delta_{n,n'}\delta_{\epsilon,\epsilon'}\ , \tag{1.137}$$

for $\epsilon = \pm$, by appropriately choosing the constant c. The orthogonality can be verified using the orthonormality of the $u_{k,l,n}$.[3]

[3] More generally, one can consider two co-closed one-eigenforms B_1, B_2 of eigenvalues Y_1, Y_2 in three dimensions. Using that $A \wedge *B = B \wedge *A$ for forms of same degree, and integration

1.3.3 Higher forms and summary

In a three-dimensional space, the spectrum of the two- and three-forms can be deduced respectively from that of the one-forms and the scalars. Indeed, one can always rewrite a p-form A_p in terms of its Hodge dual as $A_p = *B_{3-p}$, and one verifies that

$$\Delta A_2 = \Upsilon A_2 \Leftrightarrow \Delta B_1 = \Upsilon B_1 \ , \ \Delta A_3 = \Upsilon A_3 \Leftrightarrow \Delta B_0 = \Upsilon B_0 \ . \tag{1.139}$$

The complete spectrum of scalars and one-forms, as well as the basis of eigenmodes, thus provides those of the two- and three-forms by a simple application of the Hodge star. We summarize the former in Table 1.1.[4]

Eigenmodes		Eigenvalues
Scalars	$v_{p,q}$ in (1.57)	$-\mu^2_{p,q}$ in (1.56)
	$u_{k,l,n}$ in (1.57)	$-M^2_{k,l,n}$ in (1.56)
Exact one-forms	$\mathrm{d}v_{p,q}$ with $pq \neq 0$	$-\mu^2_{p,q}$
	$\mathrm{d}u_{k,l,n}$	$-M^2_{k,l,n}$
Co-closed one-forms	$B^{p,q}_{\epsilon}$ in (1.109)	$Y^{p,q}_{\pm}$ in (1.107)
	$B^{k,l,n}_{\epsilon}$ in (1.134)	$Y^{k,l,n}_{\pm}$ in (1.135)

Table 1.1: Scalar and one-form eigenmodes with respective eigenvalues for the Laplacian on the three-dimensional Heisenberg nilmanifold.

Finally, note we can introduce the following real eigenforms

$$B^{k,l,n}_{r\,\epsilon} = \frac{1}{\sqrt{2}}(B^{k,l,n}_{\epsilon} + \overline{B^{k,l,n}_{\epsilon}}) \ , \ B^{p,q}_{r\,\epsilon} = \frac{1}{\sqrt{2}}(B^{p,q}_{\epsilon} + \overline{B^{p,q}_{\epsilon}}) \ . \tag{1.140}$$

One has

$$\int_3 \mathrm{d}^3x\sqrt{g}\ u_{k,l,n}u_{k',l',n'} = \delta_{k,-k'} \times \dots \ , \ \int_3 \mathrm{d}^3x\sqrt{g}\ v_{p,q}v_{p',q'} = \delta_{p,-p'}\delta_{q,-q'} \ . \tag{1.141}$$

This implies that $\{B^{k,l,n}_{r\,\epsilon}\}$ or $\{B^{p,q}_{r\,\epsilon}\}$ form an orthonormal set if one restricts e.g. to $kk' > 0$ or $pp' > 0, qq' > 0$.

Importantly, it should be noted that the case of an arbitrary metric hasn't been discussed. The study could be repeated in a similar fashion to obtain the spectrum with the dependence on the parameters a and b of metric moduli

by parts, one can show

$$Y_2 \int_3 B_1 \wedge *\overline{B_2} = \int_3 *B_1 \wedge \Delta\overline{B_2} = Y_1 \int_3 B_1 \wedge *\overline{B_2} \ . \tag{1.138}$$

This implies that B_1 is orthogonal to B_2 if $Y_1 \neq Y_2$.

[4] Our convention for the Laplacian operator is such that $\Delta = *\mathrm{d}*\mathrm{d} + \mathrm{d}*\mathrm{d}*$ on any p-form, $p \geq 0$. This is the reason for the different signs of the eigenvalues summarized in Table 1.1, where we recall that $Y^{p,q}_{\pm} \geq 0$ and $Y^{k,l,n}_{\pm} > 0$. The more conventional definition, $\Delta = \mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$, would provide a positive sign to all eigenvalues.

space. When we will study the compactification of Yang-Mills theory on the Heisenberg manifold, we will use the Euclidian metric δ since the goal is to develop a basic understanding of these type of spaces. This is still however an important question. If we were to study gravity on the product space of Minkowski times the Heinsenberg manifold, gravitational couplings between the two space could lead to dynamical modifications of the spectrum. This is outside the scope of this thesis, but would certainly be of interest if extensions of the Standard Model can be found through the use of this geometry. For now, we will simply show how the first massive mode of the one-forms is modified in the case of a non-trivial metric.

As mentioned in 1.2.2, once a metric has been chosen for the space, we can always choose a basis in which the metric is the Euclidian one. This gives the basis of one-forms E^a from equation (1.26). The eigen one-forms are linear combinations of the coframe associated with the metric. We write

$$A = \sum_{a=1}^{3} c_a E^a, \tag{1.142}$$

where the c_a's are real constants to be determined in the following and,

$$E^1 = e^1 + ae^3 \; ; \quad E^2 = e^2 + be^3 \; ; \quad E^3 = e^3. \tag{1.143}$$

From the expression of the eigenvalues for the one-forms, we see that in the case of an undeformed metric (a,$b = 0$), the low-lying Laplacian eigen one-forms are $e^{1,2}$, which are harmonic, and e^3, which has eigenvalue $\mathtt{f}^2$ (see eq. (1.102)). To see how this spectrum is modified for a general metric (a,$b \neq 0$), we need to compute the action of the Laplacian Δ on A

$$\Delta A = \sum_{a=1}^{3} c_a (\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d}) E^a, \tag{1.144}$$

where $\mathrm{d}^\dagger \equiv \star \mathrm{d} \star$. The Hodge star is calculated with respect to the metric (1.26), and operates canonically on E^a

$$\star\, E^a = \frac{1}{2} \sum_{b,c=1}^{3} \varepsilon^{abc} E^b \wedge E^c, \tag{1.145}$$

while the action of the exterior derivative on E^a is calculated from (1.7), (1.143). It is then easily verified that the coframe is co-closed, $\mathrm{d}^\dagger E^a = 0$, and,

$$\Delta A = \mathtt{f}^2 (a^2 + b^2 + 1)(ac_1 + bc_2 + c_3)(aE^1 + bE^2 + E^3). \tag{1.146}$$

Setting $c_1 = a$, $c_2 = b$, $c_3 = 1$, it follows that $aE^1 + bE^2 + E^3$ is a Laplacian eigen one-form with eigenvalue $\mathtt{f}^2(a^2 + b^2 + 1)^2$. Moreover, we obtain a two-dimensional space of harmonic one-forms, parameterized by the solutions of $ac_1 + bc_2 + c_3 = 0$. As we will later see, after compactification, the use of a

non-trivial metric on the space doesn't really bring any new parameters, it just modifies the "geometrical" mass. But more on this later.

This concludes our study of forms on the Heisenberg manifold. The last ingredients we need in order to have all the tools for compactification and model building are the eigen-spinors of the Dirac operator. We now address this question.

1.3.4 Dirac operator

In this section we will solve the Dirac equation on the Heisenberg manifold [39]. To do this end, we must first fix some conventions for our spinors. Once this is done, we will lay out the system of equation explicitly, and use the basis of eigen-scalars to try to find a family of solution for the Dirac operator. As in the previous case, we will first study the Dirac operator with a flat metric on the space. This will build our intuition in order to solve the general case. The story goes similarly to the case of the one-forms.

Simple case $a = b = 0$

As a warmup let us consider the Dirac operator eigenvalue problem on $\mathcal{M}_3$ in the simplified case $a = b = 0$. The Dirac eigenvalue equation reads,

$$(D - \lambda)\psi = 0 \; ; \quad D \equiv \gamma^a e^m{}_a(\partial_m + \tfrac{1}{4}\omega_{mbc}\gamma^{bc}), \tag{1.147}$$

where ω_{mbc} is the spin connection. The independent non-vanishing components of the flat spin connection $\omega_{abc} = e^m{}_a\omega_{mbc}$ on $\mathcal{M}_3$ are given by

$$\omega_{123} = \omega_{231} = -\omega_{312} = \frac{\mathtt{f}}{2}. \tag{1.148}$$

The Dirac operator then takes the form

$$D = \gamma^a V_a + i\frac{\mathtt{f}}{4}, \tag{1.149}$$

where the V_a's were given in (1.6). The γ^a's can be taken to be the Pauli matrices, so that (1.147) reduces to the following set of equations,

$$\begin{aligned} V_3\psi_1 + (V_1 - iV_2)\psi_2 - (\lambda - i\frac{\mathtt{f}}{4})\psi_1 = 0 \\ -V_3\psi_2 + (V_1 + iV_2)\psi_1 - (\lambda - i\frac{\mathtt{f}}{4})\psi_2 = 0. \end{aligned} \tag{1.150}$$

Next we expand the two-component Dirac spinor on the complete basis of Laplacian eigenfunctions on $\mathcal{M}_3$,[5]

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \sum_{k,l,n}\begin{pmatrix}C^1_{k,l,n} \\ C^2_{k,l,n}\end{pmatrix} u_{k,l,n} + \sum_{p,q}\begin{pmatrix}D^1_{p,q} \\ D^2_{p,q}\end{pmatrix} v_{p,q}, \tag{1.151}$$

[5] This expansion implicitly assumes that the Dirac spinor on $\mathcal{M}_3$ returns to its original value after a going around each of the three circles parameterized by x^a. This condition can be relaxed, leading to what is known as nontrivial spin structures [41].

where, following [38], we use the notation $u_{k,l,n}$ and $v_{p,q}$, for the $a, b \to 0$ limit of the functions given in (1.75) and (1.76) respectively. Since the two sets of functions are orthogonal to each other, we can treat the two cases independently. The action of the V_a on the v-basis reads,

$$V_1 v_{p,q} = iP v_{p,q} \ ; \quad V_2 v_{p,q} = iQ v_{p,q} \ , \quad V_3 v_{p,q} = 0, \tag{1.152}$$

so that (1.150) reduces to

$$\begin{aligned} i(P - iQ)D^2_{p,q} - (\lambda_{p,q} - i\frac{\mathtt{f}}{4})D^1_{p,q} = 0 \\ i(P + iQ)D^1_{p,q} - (\lambda_{p,q} - i\frac{\mathtt{f}}{4})D^2_{p,q} = 0. \end{aligned} \tag{1.153}$$

From this system we deduce the eigenvalues

$$-i\lambda_{p,q} = \frac{\mathtt{f}}{4} \pm \sqrt{P^2 + Q^2}, \tag{1.154}$$

and the corresponding eigen-spinors

$$\psi_{p,q} = C \begin{pmatrix} v_{p,q} \\ \alpha v_{p,q} \end{pmatrix}; \quad \alpha = \pm \frac{\sqrt{P^2 + Q^2}}{(P - iQ)}, \tag{1.155}$$

up to a normalization constant $C \in \mathbb{C}$.

Let us now turn to the $u_{k,l,n}$ series. Using the action of the $V's$ on the basis,

$$\begin{aligned} \mathrm{i}V_3 u_{k,l,n} &= -\frac{|\sigma|}{\mathtt{f}} sign(\sigma) u_{k,l,n} \\ V_2 u_{k,l,n} &= \frac{|\sigma|^{\frac{1}{2}}}{2} \left(-\sqrt{2(n+1)} u_{k,l,n+1} + \sqrt{2n} u_{k,l,n-1} \right) \\ \mathrm{i}V_1 u_{k,l,n} &= -\frac{|\sigma|^{\frac{1}{2}}}{2} sign(\sigma) \left(\sqrt{2(n+1)} u_{k,l,n+1} + \sqrt{2n} u_{k,l,n-1} \right), \end{aligned} \tag{1.156}$$

with σ defined in (1.71), we obtain

$$\begin{aligned} \left[\mathrm{i}\lambda_{k,l,n} + (\frac{\mathtt{f}}{4} + \frac{\sigma}{\mathtt{f}})\right] C^1_{k,l,n} &= -|\sigma|^{\frac{1}{2}} \left[p^+(\sigma)\sqrt{2n} C^2_{k,l,n-1} + p^-(\sigma)\sqrt{2(n+1)} C^2_{k,l,n+1} \right] \\ \left[\mathrm{i}\lambda_{k,l,n} + (\frac{\mathtt{f}}{4} - \frac{\sigma}{\mathtt{f}})\right] C^2_{k,l,n} &= -|\sigma|^{\frac{1}{2}} \left[p^-(\sigma)\sqrt{2n} C^1_{k,l,n-1} + p^+(\sigma)\sqrt{2(n+1)} C^1_{k,l,n+1} \right], \end{aligned} \tag{1.157}$$

where we have defined $p^{\pm}(\sigma) \equiv \frac{sign(\sigma) \pm 1}{2}$. We thus obtain

$$-\mathrm{i}\lambda_{k,l,n} = \frac{f}{4} \pm \sqrt{\left(\frac{\sigma}{f}\right)^2 + |\sigma|(p^+(\sigma)2n + p^-(\sigma)2(n+1))}. \tag{1.158}$$

Note that the set of eigenvalues is the same for either sign of σ: they are just offset by one increment of n. Moreover there is a degeneracy, since in there is no dependence of the eigenvalues on l. The associated eigen-spinors read

$$\psi_{k,l,n} = C \begin{pmatrix} u_{k,l,n} \\ p^+(\sigma)\alpha u_{k,l,n-1} + p^-(\sigma)\beta u_{k,l,n+1} \end{pmatrix}, \tag{1.159}$$

where $C \in \mathbb{C}$ is a normalization constant and,

$$\alpha = -\frac{\frac{\sigma}{\mathtt{f}} + \sqrt{\left(\frac{\sigma}{f}\right)^2 + |\sigma| 2n}}{|\sigma|^{\frac{1}{2}}\sqrt{2n}} \tag{1.160}$$

$$\beta = -\frac{\frac{\sigma}{\mathtt{f}} + \sqrt{\left(\frac{\sigma}{f}\right)^2 + |\sigma| 2(n+1)}}{|\sigma|^{\frac{1}{2}}\sqrt{2(n+1)}}. \tag{1.161}$$

Non-trivial metric

For the most general metric given in (1.35), the Dirac operator takes the form

$$D = \gamma^a E^m{}_a(\partial_m + \tfrac{1}{4}\omega_{mbc}\gamma^{bc}), \tag{1.162}$$

where $E^m{}_a \equiv (E^{-1})^m{}_a$ is the inverse vielbein associated with (1.35), so that [37],

$$E^{-T}\partial = \begin{pmatrix} V_1 \\ V_2 \\ V_3 - aV_1 - bV_2 \end{pmatrix}, \tag{1.163}$$

with the V_a's given in (1.6). Moreover the independent non-vanishing components of the flat spin connection $\omega_{abc} = E^m{}_a\omega_{mbc}$ read

$$\begin{aligned} &\omega_{112} = -\mathtt{f}a\ ; \quad \omega_{113} = \mathtt{f}ab\ ; \quad \omega_{123} = \tfrac{1}{2}\mathtt{f}(-a^2+b^2+1) \\ &\omega_{212} = -\mathtt{f}b\ ; \quad \omega_{213} = -\tfrac{1}{2}\mathtt{f}(a^2-b^2+1)\ ; \quad \omega_{223} = -\mathtt{f}ab \\ &\omega_{312} = \tfrac{1}{2}\mathtt{f}(a^2+b^2-1)\ ; \quad \omega_{313} = \mathtt{f}b\ ; \quad \omega_{323} = -\mathtt{f}a. \end{aligned} \tag{1.164}$$

Taking (1.163), (1.164) into account, (1.162) reduces to

$$D = \gamma^1 V_1 + \gamma^2 V_2 + \gamma^3\big(V_3 - aV_1 - bV_2\big) + i\frac{\mathtt{f}}{4}(1+a^2+b^2). \tag{1.165}$$

As in the case of the scalars and one-forms, the spectrum falls into two distinct series, depending on whether or not there is non-trivial dependence on the x^3 coordinate. Let us first examine the x^3-independent case. We expand the Dirac spinor as follows

$$\psi_{p,q} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} V_{p,q}, \tag{1.166}$$

for some constants α, $\beta \in \mathbb{C}$. The action of the V_a on the $V_{p,q}$-basis (1.76) reads

$$V_1 V_{p,q} = iPV_{p,q}\ ; \quad V_2 V_{p,q} = iQV_{p,q}\ , \quad V_3 V_{p,q} = 0, \tag{1.167}$$

so that the eigenvalue equation

$$D\psi = \lambda_{p,q}\psi, \tag{1.168}$$

reduces to

$$\begin{aligned} -aP\alpha + (P - iQ)\beta + \left[\mathrm{i}\lambda_{p,q} + \tfrac{1}{4}\mathtt{f}(a^2+b^2+1)\right]\alpha = 0 \\ bQ\beta + (P + iQ)\alpha + \left[\mathrm{i}\lambda_{p,q} + \tfrac{1}{4}\mathtt{f}(a^2+b^2+1)\right]\beta = 0, \end{aligned} \tag{1.169}$$

where we have taken (1.165) into account. From this system we deduce the eigenvalues

$$\boxed{-\mathrm{i}\lambda_{p,q} = \tfrac{1}{4}\mathtt{f}(a^2+b^2+1) + \tfrac{1}{2}(bQ - aP) \pm \sqrt{P^2+Q^2+\tfrac{1}{4}(aP+bQ)^2}.} \tag{1.170}$$

The Dirac eigen-spinor (1.166) is determined, up to an overall normalization constant, by the equation

$$\beta = \frac{\frac{1}{2}(aP+bQ) \pm \sqrt{P^2+Q^2+\frac{1}{4}(aP+bQ)^2}}{P - \mathrm{i}Q}\alpha. \tag{1.171}$$

In the case of nontrivial x^3-dependence the relevant basis is given by the polynomials (1.75). In order not to clutter the notation, we will present the calculation of the spectrum for the case $\sigma > 0$, cf. (1.71). The U-polynomials obey

$$(V_3 - aV_1 - bV_2)U_{k,l,n} = \kappa U_{k,l,n} + \sqrt{\tfrac{\sigma}{2}}\left(w^*\sqrt{(n+1)}\; U_{k,l,n+1} - w\sqrt{n}\; U_{k,l,n-1}\right) \tag{1.172}$$

$$\begin{aligned} &(V_1 + iV_2)U_{k,l,n} = \\ &\kappa z U_{k,l,n} + i\sqrt{\tfrac{\sigma}{2}}\left(\sqrt{(n+1)}(B_- + bw^*)U_{k,l,n+1} + \sqrt{n}(B_+ - bw)U_{k,l,n-1}\right) \end{aligned} \tag{1.173}$$

$$\begin{aligned} &(V_1 - iV_2)U_{k,l,n} = \\ &\kappa z^* U_{k,l,n} + i\sqrt{\tfrac{\sigma}{2}}\left(\sqrt{(n+1)}(B_+ - bw^*)U_{k,l,n+1} + \sqrt{n}(B_- + bw)U_{k,l,n-1}\right), \end{aligned} \tag{1.174}$$

where we have defined

$$z := a+ib\,; \quad \kappa := \frac{2\pi i K}{1+|z|^2}\,; \quad w := b + \frac{ia}{\sqrt{1+|z|^2}}\,; \quad B_\pm := \pm(1+b^2) + \frac{1+b^2}{\sqrt{1+|z|^2}}. \tag{1.175}$$

Let us now come to the Dirac eigenvalue problem. We start with the following ansatz for the Dirac spinor:

$$\psi = \begin{pmatrix} \alpha U_{k,l,n} + \beta U_{k,l,n-1} \\ \gamma U_{k,l,n-1} + \delta U_{k,l,n} \end{pmatrix}. \tag{1.176}$$

Imposing

$$\left[D - i\frac{\mathtt{f}}{4}(1+|z|^2)\right]\psi = \begin{pmatrix} \alpha' U_{k,l,n} + \beta' U_{k,l,n-1} \\ \gamma' U_{k,l,n-1} + \delta' U_{k,l,n} \end{pmatrix}, \tag{1.177}$$

for arbitrary coefficients α', β', γ', δ', taking (1.172) into account, leads to a system of four homogeneous equations for the four coefficients $\alpha, \ldots, \gamma$. Noting the identity

$$(B_- + bw)(B_+ - bw) = w^2, \tag{1.178}$$

this system turns out to be equivalent to the following two conditions

$$\beta = \frac{i(B_- + bw)}{w}\ \gamma\ ; \quad \delta = \frac{i(B_- + bw^*)}{w^*}\ \alpha, \tag{1.179}$$

where it is assumed that a, b are not both zero. Imposing in addition the eigenvalue equation

$$D\psi = \lambda_{k,l,n}\psi\ , \tag{1.180}$$

taking (1.179) into account, leads to a system of four homogeneous equations for α, γ. Clearly this is highly overdetermined. Remarkably, however, the system admits a nontrivial solution, provided

$$\boxed{-i\lambda_{k,l,n} = \tfrac{1}{4}\mathtt{f}(1+|z|^2) \pm \sqrt{\tfrac{4\pi K^2}{1+|z|^2} + 4\pi n K \mathtt{f}\sqrt{1+|z|^2}},} \tag{1.181}$$

where we took (1.71) into account. As was observed in the case of the scalar spectrum, the eigenvalues depend on the a, b parameters only through the norm of z. As in the previous cases, we see that there is a degeneracy, since the eigenvalues are independent of l.

1.4 Summary

In this chapter, we explored the concept of nilmanifolds. After a general introduction, we discussed in great detail the structure of the Heisenberg manifold, the simplest (i.e. the lowest dimensional) example of a nilmanifold. We worked out a geometrical interpretation of this space as a torus with a circle fiber. We then studied fields living on the space, first by solving the Laplace equation for scalars, then using this result to solve the Laplace equation for forms. The various eigenmodes always come in two families, one living on the base 2-torus, the other living on the whole space. One of the attractive feature of this space is that all the Kaluza-Klein spectrum is explicitly known for all the fields, allowing great control over the compactification of the theory. Let us remind the reader of the main properties of the space, they will be relevant for the following chapters. The space is first characterized by its radii r^1, r^2 and r^3, the first two form the base 2-torus, while r^3 is the radius of the circle fiber. Next we have the parameter $\mathtt{f}$, the only non-vanishing structure constant of the local

nilpotent algebra. It is proportional to N which was the parameter controlling the twist of the circle fiber. In our study we found the relation $\mathtt{f} = Nr^3/r^1r^2$ with N an integer, and we also were able to compute the volume $V = r^1r^2r^3$. These two parameters will be the main leftovers of the geometry once our theory will be compactified, along with the field content. We are now finally ready to dive into the physics of this space.

Chapter 2

Compactification

The last chapter was dedicated to the study of nilmanifolds and in particular to the Heisenberg manifold. We solved the Laplace equation for various fields, giving us a basis on which we can expand fields living on the Heisenberg manifold. With these tools in our hands, we can dive into the study of the compactification. Of course, in order to study the physics associated with the compactification on this space, we must specify a theory. A sensible choice is to study the compactification of Yang-Mills theory. This has two main interesting aspects. The first and foremost is the hope to find a model which would relate the gauge group to the scalars of the theory. In quantum field theory, the gauge group is structured by the local symmetries of our theory. The scalar sector on the other hand has no guiding principle. Trying to explain the origin and structure of the observed Higgs field is a major question. Compactification models have this very property of generating scalars as a byproduct of the compactification. These scalars are intimately related to gauge, giving an origin story to such fields. In that framework, scalars are not fundamental objects of nature, but rather the approximate degrees of freedom of the gauge fields living on the extra dimensions. Of course this is not a new idea. The model from Kaluza and Klein [42, 43] initiated this line of reasoning. Their work was concerned with how electromagnetism could be related to gravity in 5 dimensions. This model inspired several lines of developments. The usual story for compactifications in the context of Yang-Mills theories uses tori as the compact space. The problem of such models is that the low lying scalars that are generated are massless, leaving the task of giving them a mass to quantum corrections. This of course sets the scale for the masses of the scalars to be of the same order as the coupling constant. This fails to give high enough masses in practical models. But our story is different. As we will see, the 4-dimensional model keeps some information about the non-trivial structure of our compact space, more specifically of the local algebra, resulting in a mass at tree level for one of the scalars. This is one of the main results of this model. The twisting of the Heisenberg manifold is seen as a mass scale for the scalar of the 4-dimensional theory. This chapter will be composed of two parts based on the work done

in [39] and [44]. The first part will focus on the compactification of the gauge fields, giving rise to an effective field theory composed of scalar fields and the 4-dimensional gauge fields. This compactification will be general in the sense that we won't need to specify the gauge group. We will then move on to the study of the compactification of fermions.

2.1 Reduction of 7-dimensional Yang-Mills theory to 4 dimensions

We start by giving the seven-dimensional (7d) pure Yang-Mills action for a gauge group in the adjoint representation of a general Lie group G:

$$\mathcal{S} = \int_{M_7} \mathrm{Tr}(\mathcal{F} \wedge *_7 \mathcal{F}) = \int_{M_7} \mathrm{Tr}(t_a t_b)\, \mathcal{F}^a \wedge *_7 \mathcal{F}^b \ , \tag{2.1}$$

where $\mathcal{F}$ is the field strength, a two-form that takes values in the Lie algebra of G, the t_a's are a basis of the Lie algebra under consideration (Einstein's convention on the summation of indices is implied), and $*_7$ is the Hodge star in seven dimensions. The integral is taken over M_7, a 7d space formed of the direct product of Minkowski space $\mathcal{M}_4$ and the Heisenberg manifold H_3, i.e.

$$M_7 = \mathcal{M}_4 \times H_3 \ . \tag{2.2}$$

The field strength is expressed in terms of the gauge potential $\mathcal{A}$ as:

$$\mathcal{F} = t_a \left(\mathrm{d}\mathcal{A}^a + \tfrac{1}{2} f^a{}_{bc} \mathcal{A}^b \wedge \mathcal{A}^c \right) \ , \quad \mathcal{F}^a_{MN} = 2\partial_{[M} \mathcal{A}^a_{N]} + f^a{}_{bc} \mathcal{A}^b_M \mathcal{A}^c_N \ , \tag{2.3}$$

where the indices M, N run through the different coordinates on the 7d space and $f^a{}_{bc}$ are the structure constants defined as:

$$[t_b, t_c] = t_a f^a{}_{bc} \ . \tag{2.4}$$

Inserting the expression for the field strength in terms of the gauge potential, allows us to rewrite the action in the following way:

$$\mathcal{S} = \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 \ , \tag{2.5}$$

$$\text{where} \quad \mathcal{S}_2 = \int_7 \mathrm{Tr}(t_a t_b)\, \mathrm{d}\mathcal{A}^a \wedge *_7 \mathrm{d}\mathcal{A}^b \ , \tag{2.6}$$

$$\mathcal{S}_3 = 2 \int_7 \mathrm{Tr}(t_a t_b) \tfrac{1}{2} f^b{}_{cd}\, \mathrm{d}\mathcal{A}^a \wedge *_7 (\mathcal{A}^c \wedge \mathcal{A}^d) \ , \tag{2.7}$$

$$\mathcal{S}_4 = \int_7 \mathrm{Tr}(t_a t_b) \tfrac{1}{4} f^a{}_{cd}\, f^b{}_{ef} \mathcal{A}^c \wedge \mathcal{A}^d \wedge *_7 (\mathcal{A}^e \wedge \mathcal{A}^f) \ . \tag{2.8}$$

Now we are going to use the following ansatz to separate the gauge potential as a sum of 4d-forms and 3d-forms:

$$\mathcal{A}^a = \mathcal{A}^a_M(x^M)\mathrm{d}x^M = \mathcal{A}^a_\mu(x^M)\mathrm{d}x^\mu + \mathcal{A}^a_m(x^M)\mathrm{d}x^m \tag{2.9}$$

$$= \sum_I U_I(x^m) A^{aI}_\mu(x^\mu)\mathrm{d}x^\mu + \phi^{aI}(x^\mu) B_{Im}(x^m)\mathrm{d}x^m \ , \tag{2.10}$$

where $M = 0, \ldots, 6$, $\mu = 0, \ldots, 3$ and $m = 4, 5, 6$. In a shorter notation, we can write the gauge potential as:

$$\mathcal{A}^a = \sum_I A^{aI} U_I + \phi^{aI} B_I \ , \tag{2.11}$$

where U_I and B_I are respectively 3d eigenscalars and 3d eigen-one-forms of the Laplacian on the nilmanifold, while A^{aI} and ϕ^{aI} are a 4d one-form and a 4d scalar respectively. The I is a multi-index that sums over the basis of 3d eigenforms. The analytical expression for these eigenforms was found in the last chapter. By the property of Laplacian eigenforms, we have:

$$*_3 \, \mathrm{d} *_3 \, \mathrm{d} U_I = \lambda_{U_I} U_I \ , \ *_3 \mathrm{d} *_3 \, \mathrm{d} B_I = \lambda_{B_I} B_I \ , \ \lambda_{U_I} \leq 0 \ , \ \lambda_{B_I} \geq 0 \ , \tag{2.12}$$

where the conditions on the signs of the eigenvalues come from the explicit solution of the system, and we have restricted to co-closed one-forms. We refer the reader to the last chapter for a detailed explanation. After certain manipulations,[1] we get the following expressions for the different parts of the action:

$$\begin{aligned} \mathcal{S}_2 = \int_4 \mathrm{vol}_4 \ \mathrm{Tr}(t_a t_b) \Big(& \left(2\partial_{[\mu} A^{aI}_{\nu]} \partial^\mu A^{bJ\,\nu} - \lambda_{U_J} A^{aI}_\mu A^{bJ\,\mu} \right) Y_{U_I U_J} \\ & + \left(\partial_\mu \phi^{aI} \partial^\mu \phi^{aJ} + \lambda_{B_J} \phi^{aI} \phi^{aJ} \right) Y_{B_I B_J} \Big) \ , \end{aligned} \tag{2.13}$$

$$\begin{aligned} \mathcal{S}_3 = 2 \int_4 \mathrm{vol}_4 \ \mathrm{Tr}(t_a t_b) \tfrac{1}{2} f^b{}_{cd} \big(& 2\partial_{[\mu} A^{aI}_{\nu]} A^{cJ\,\mu} A^{dK\,\nu} \ Y_{U_I U_J U_K} \\ & + 2\partial_\mu \phi^{aI} A^{cJ\,\mu} \phi^{dK} \ Y_{U_J B_I B_K} \\ & - 2 A^{aI}_\mu A^{cJ\,\mu} \phi^{dK} \ Y_{U_J d U_I B_K} \\ & + \phi^{aI} \phi^{cJ} \phi^{dK} \ Y_{d B_I B_J B_K} \big) \ , \end{aligned} \tag{2.14}$$

$$\begin{aligned} \mathcal{S}_4 = \int_4 \mathrm{vol}_4 \ \mathrm{Tr}(t_a t_b) \tfrac{1}{4} f^a{}_{cd} f^b{}_{ef} \big(& 2 A^{cJ}_{[\mu} A^{dK}_{\nu]} A^{eL\,\mu} A^{fM\,\nu} \ Y_{U_J U_K U_L U_M} \\ & + 4 A^{cJ}_\mu A^{eL\,\mu} \phi^{dK} \phi^{fM} \ Y_{U_J U_L B_K B_M} \\ & + \phi^{cJ} \phi^{dK} \phi^{eL} \phi^{fM} \ Y_{B_J B_K B_L B_M} \big) \ , \end{aligned} \tag{2.15}$$

where the Y's are given by:

$$Y_{U_I U_J} = \int_3 \mathrm{vol}_3 U_I U_J = \int_3 U_I \wedge *_3 U_J \ , \quad Y_{B_I B_J} = \int_3 B_I \wedge *_3 B_J \ ; \tag{2.16}$$

[1] One needs in particular to decompose the 7d Hodge star into 4d and 3d ones. To that end, the following definitions and properties are useful: we first take $\epsilon_{0123456} = 1$, and have $(*_3)^2 = 1$, $*_3 \mathrm{vol}_3 = 1$, and $(*_4)^2 A_p = (-1)^{p+1}$, $*_4 1 = \mathrm{vol}_4$, $*_4 \mathrm{vol}_4 = -1$ with A_p a 4d form and B_p a 3d form. We also have $*_7 A_p = *_4 A_p \wedge \mathrm{vol}_3$, $*_7 B_p = \mathrm{vol}_4 \wedge *_3 B_p$, and $*_7 (A_1 \wedge B_1) = - *_4 A_1 \wedge *_3 B_1$.

$$Y_{U_I U_J U_K} = \int_3 \mathrm{vol}_3 U_I U_J U_K \ , \tag{2.17}$$

$$Y_{U_J B_I B_K} = \int_3 U_J B_I \wedge *_3 B_K \ , \tag{2.18}$$

$$Y_{U_J dU_I B_K} = \int_3 U_J \mathrm{d}U_I \wedge *_3 B_K \ , \tag{2.19}$$

$$Y_{dB_I B_J B_K} = \int_3 \mathrm{d}B_I \wedge *_3 (B_J \wedge B_K) \ ; \tag{2.20}$$

$$Y_{U_J U_K U_L U_M} = \int_3 \mathrm{vol}_3 U_J U_K U_L U_M \ , \tag{2.21}$$

$$Y_{U_J U_L B_K B_M} = \int_3 U_J U_L B_K \wedge *_3 B_M \ , \tag{2.22}$$

$$Y_{B_J B_K B_L B_M} = \int_3 B_J \wedge B_K \wedge *_3 (B_L \wedge B_M) \ . \tag{2.23}$$

A problem remains: the sums over the indices I, J, K, L, M, are infinite sums over the basis of eigenforms. If we want to make this action manageable, a possibility is to organize these terms according to their masses and select only the light modes.

2.1.1 The truncation

We would like to organize this infinite series of modes according to their masses, which can be read off from the quadratic terms in $\mathcal{S}_2$. To space out the masses as much as possible, we take the following geometrical limit:

$$|\mathtt{f}| \ll \frac{1}{r^i} \ , \ i = 1, 2, 3 \quad \Rightarrow \quad \frac{1}{r^{1,2}} \ll \frac{1}{r^3} \ . \tag{2.24}$$

This limit, known as the small fiber/large base limit [38] can be understood by considering the expression for the masses λ_{U_I}. Remember the scalar modes come in two families:

$$\mu^2_{p,q} = p^2 \left(\frac{2\pi}{r^1}\right)^2 + q^2 \left(\frac{2\pi}{r^2}\right)^2 \tag{2.25}$$

$$M^2_{k,l,n} = k^2 \left(\frac{2\pi}{r^3}\right)^2 + (2n+1) \mid k \mid \frac{2\pi \mid \mathtt{f} \mid}{r^3} \ , \tag{2.26}$$

where $p, q \in \mathbb{Z}$, $k \in \mathbb{Z}^*$, $n \in \mathbb{N}$, $l = 0, 1, \ldots, \mid k \mid$. In the case of one-forms, the expression for the masses λ_{B_I} were:

$$Y_{\pm}^{p,q} = P^2 + Q^2 + \frac{\mathtt{f}^2}{2} \pm \sqrt{\left(P^2 + Q^2 + \frac{\mathtt{f}^2}{2}\right)^2 - (P^2 + Q^2)^2} \ , \tag{2.27}$$

$$Y_{\pm}^{k,l,n} = M_{k,l,n}^2 + 2 \mid k \frac{2\pi\mathtt{f}}{r^3} \mid \tag{2.28}$$

$$+ \frac{1}{2}\mathtt{f}^2 \pm \sqrt{\left(\mid k \frac{2\pi\mathtt{f}}{r^3} \mid + \frac{1}{2}\mathtt{f}^2\right)^2 + 2(n+1) \mid k \frac{2\pi\mathtt{f}}{r^3} \mid \mathtt{f}^2} \ , \tag{2.29}$$

where $P = p2\pi/r^1$, $Q = q2\pi/r^2$. As shown in [38], we then have the following lightest modes (in a basis of real orthonormal eigenmodes):

Scalars:

$$U_{I=1} = \frac{1}{\sqrt{V}} \ ; \quad \lambda_{U_1} = 0 \ , \tag{2.30}$$

where we recall that $V = r^1 r^2 r^3$ is the volume of the nilmanifold. This mode correspond to the eigenvalue $\mu_{p,q}$ for $p = q = 0$q. The following eigenvalues go like $\sim 1/(r^{1,2})^2$. The lowest mass for the $M_{k,l,n}$ is of order $\sim 1/(r^3)^2$ (for $|k| = 1$, $l = 0$, $n = 0$). This will be important when we will compare it to the rest of the masses of the one-forms.

Co-closed one-forms:

$$\begin{aligned} B_{I=1} &= \frac{1}{\sqrt{V}} e^1 \ ; \quad \lambda_{B_1} = 0 \\ B_{I=2} &= \frac{1}{\sqrt{V}} e^2 \ ; \quad \lambda_{B_2} = 0 \\ B_{I=3} &= \frac{1}{\sqrt{V}} e^3 \ ; \quad \lambda_{B_3} = \mathtt{f}^2 \ , \end{aligned} \tag{2.31}$$

where the e^a one-forms satisfy the Maurer Cartan equation (1.8). We observe that the lowest eigenvalue for the $Y_{\pm}^{k,l,n}$ is of order $\sim M_{k,l,n} \sim 1/(r^3)^2$ in our geometrical limit. We can therefore conclude that the next masses in any of the eigenvalue series are much higher than the ones we picked. The eigenvalues define the masses for the selected modes. Almost all of these are massless except for the one-form B_3, which has a mass $\mathtt{f}^2$ directly related to the local algebra of the geometry, cf. Eqs. (1.6), (1.8). The decomposition in Eq. (2.79) of the gauge potential simplifies to:

$$\mathcal{A}^a = A^a U_1 + \sum_{I=1}^{3} \phi^{aI} B_I \ . \tag{2.32}$$

The resulting non-vanishing couplings are as follows,

$$Y_{U_1 U_1} = 1 \ , \quad Y_{B_I B_J} = \delta_{IJ} \ , \tag{2.33}$$

for the quadratic couplings, together with

$$\begin{aligned}
Y_{U_I U_J U_K}: &\quad Y_{U_1 U_1 U_1} = \frac{1}{\sqrt{V}} \\
Y_{U_J B_I B_K}: &\quad Y_{U_1 B_L B_L} = \frac{1}{\sqrt{V}}\ ,\ L = 1,2,3 \\
Y_{dB_I B_J B_K}: &\quad Y_{dB_3 B_2 B_1} = -Y_{dB_3 B_1 B_2} = \frac{\mathtt{f}}{\sqrt{V}} \\
Y_{U_I U_J U_K U_L}: &\quad Y_{U_1 U_1 U_1 U_1} = \frac{1}{V} \\
Y_{U_I U_J B_K B_L}: &\quad Y_{U_1 U_1 B_L B_L} = \frac{1}{V}\ ,\ L = 1,2,3 \\
Y_{B_I B_J B_K B_L}: &\quad Y_{B_1 B_2 B_1 B_2} = Y_{B_1 B_3 B_1 B_3} = Y_{B_2 B_3 B_2 B_3} = \frac{1}{V}\ ,
\end{aligned} \tag{2.34}$$

where in the last line we can also have anti-symmetric permutations of the first two and/or the last two indices. After the truncation to these light modes, we can finally write the action as:

$$S = \int \mathrm{d}x^4 \mathrm{Tr}\Big(\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} + \sum_{I=1}^{3} D_\mu \phi^I D^\mu \phi^I + M^2 (\phi^3)^2 + \mathcal{U}\Big)\ , \tag{2.35}$$

where

$$\mathcal{U} = \mathrm{Tr}\Big(- 2gM[\phi^1, \phi^2]\phi^3 + \tfrac{1}{2} g^2 \sum_{I,J=1}^{3} [\phi^I, \phi^J][\phi^I, \phi^J]\Big)\ , \tag{2.36}$$

with $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} + g[A_\mu, A_\nu]$ and $D_\mu = \partial_\mu + g[A_\mu, \cdot]$. We have relabelled the parameters such that $g = 1/\sqrt{V}$ and $M = \mid \mathtt{f} \mid$.

Lastly, we want to perform a set of transformations in order to retrieve the usual Yang-Mills conventions:

$$\begin{aligned}
t_a &\to i t_a \\
\eta_{\mu\nu} &\to -\eta_{\mu\nu}\ ,
\end{aligned} \tag{2.37}$$

resulting in the final expression:

$$S = \int \mathrm{d}x^4 \mathrm{Tr}\Big(- \tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} + \sum_{I=1}^{3} D_\mu \phi^I D^\mu \phi^I - M^2 (\phi^3)^2 - \mathcal{U}\Big)\ , \tag{2.38}$$

where:

$$\mathcal{U} = \mathrm{Tr}\Big(- 2igM[\phi^1, \phi^2]\phi^3 + \tfrac{1}{2} g^2 \sum_{I,J=1}^{3} [\phi^I, \phi^J][\phi^I, \phi^J]\Big)\ . \tag{2.39}$$

We see that the original seven dimensional pure Yang-Mills theory gives rise to, upon compactification and truncation to the light modes, a low energy effective action consisting of a four-dimensional Yang-Mills coupled to three scalar fields

in the adjoint representation. Moreover, one of these scalars is massive and all three scalars interact via the potential. The next step would be to understand the structure of the potential in order the find a vacuum of the theory, i.e. a local minimum of the potential. This is generally-speaking not a trivial question. The potential has a total number of independent variables that are three times the dimension of the Lie algebra associated to G, meaning that even for low dimensional Lie algebras, the number of real independent variables can be quite large. We will discuss this in great detail in the next chapter. For the moment, let us turn our attention to the compactification of fermions.

2.2 Reduction of gauge-fermion theory

In order to include fermions in our model, we now analyze how fermions living on the 7-dimensional manifold would be compactified in the scenario mentioned in the last section. Our approach is straight forward, we will use the same ansatz for the gauge fields, along with the geometrical limit of a large base, small fiber of (2.62). Let us start by fixing some conventions on the fermions in the various spaces of interest before diving into the actual compactification.

2.2.1 Fermion conventions

In a space of arbitrary dimension and Lorentzian signature, the gamma matrices are taken to satisfy

$$(\Gamma^M)^\dagger = \Gamma^0 \Gamma^M \Gamma^0. \tag{2.40}$$

We define the antisymmetric product of n gamma matrices by

$$\Gamma_{M_1 \dots M_n} := \Gamma_{[M_1} \dots \Gamma_{M_n]}. \tag{2.41}$$

Given a spinor ψ we define

$$\overline{\psi} := \psi^+ \Gamma^0. \tag{2.42}$$

Given a spinor ψ in a space of arbitrary dimension *and arbitrary signature*, we define

$$\tilde{\psi} := \psi^{\mathrm{Tr}} C, \tag{2.43}$$

where C is the charge conjugation matrix. This has the property that for any spinors ψ, χ, the bilinear $\tilde{\psi} \Gamma_{M_1 \dots M_n} \chi$ is an antisymmetric tensor of order n.

4D Minkowski space

The charge conjugation matrix in $1+3$ dimensions satisfies

$$C^{\mathrm{Tr}} = -C; \qquad (C\gamma^\mu)^{\mathrm{Tr}} = -C\gamma^\mu. \tag{2.44}$$

The chirality matrix is defined by

$$\gamma^5 := i\gamma^0 \dots \gamma^3; \qquad (\gamma^5)^2 = 1. \tag{2.45}$$

The fundamental, positive-chirality (Weyl), two-component, spinor representation ψ_+ is complex, meaning that its complex conjugate ψ_- has negative chirality. The complex conjugate ψ_- of ψ_+ is *defined* by

$$\tilde{\psi}_- := \overline{\psi}_+, \tag{2.46}$$

which also implies

$$\tilde{\psi}_+ = -\overline{\psi}_-. \tag{2.47}$$

We stress that these are *not* reality conditions: they simply define ψ_- in terms of ψ_+ or vice-versa. Indeed a reality condition would equate (up to a constant) $\overline{\psi}_+$ and $\tilde{\psi}_+$, which is impossible in four dimensions.

Let $\chi_\pm$, $\psi_\pm$ be arbitrary *anticommuting* Weyl spinors of positive or negative chirality. We have the following useful relations

$$\tilde{\psi}_\pm \chi_\mp = 0 \; ; \quad \tilde{\psi}_\pm \gamma_\mu \chi_\pm = 0. \tag{2.48}$$

The following symmetry relations are valid for Weyl spinors of any chirality

$$\tilde{\psi}\chi = \tilde{\chi}\psi \; ; \quad \tilde{\psi}\gamma_\mu\chi = \tilde{\chi}\gamma_\mu\psi. \tag{2.49}$$

It is also useful to note the following complex conjugation relations

$$(\overline{\psi}_\pm \gamma_\mu \chi_\pm)^* = -\overline{\chi}_\pm \gamma_\mu \psi_\pm \; ; \quad (\overline{\psi}_\pm \chi_\mp)^* = \overline{\chi}_\mp \psi_\pm. \tag{2.50}$$

Let us now consider an arbitrary anticommuting Dirac spinor ψ_D. It can be written in terms of two arbitrary Weyl spinors χ_+ and θ_+

$$\psi_D = \chi_+ + \theta_-, \tag{2.51}$$

where θ_- is the complex conjugate of θ_+, given by (2.46). The "Dirac mass" is given by

$$\overline{\psi}_D \psi_D = \overline{\chi}_+ \theta_- + \overline{\theta}_- \chi_+ = \tilde{\chi}_- \theta_- - \tilde{\theta}_+ \chi_+, \tag{2.52}$$

which is real, as can be verified using (2.50).

A Weyl spinor can be considered as a special case of a Dirac spinor whose component of negative or positive chirality vanishes. Therefore it is sometimes said that the Dirac mass of a Weyl spinor vanishes: indeed setting χ or θ to zero would make the right hand side of (2.52) vanish. Nevertheless a mass term can be defined for a single Weyl spinor: it suffices to set $\theta_\pm = \chi_\pm$ in (2.52). This is sometimes described as defining a Majorana spinor

$$\psi_M = \chi_+ + \chi_-, \tag{2.53}$$

which is nothing other than a Dirac spinor whose negative-chirality component is the complex conjugate of its positive-chirality component.[2] A real mass term for a Weyl spinor χ can then be written in terms of ψ_M,

$$\overline{\psi}_M \psi_M = \overline{\chi}_+ \chi_- + \overline{\chi}_- \chi_+ = \tilde{\chi}_- \chi_- - \tilde{\chi}_+ \chi_+. \tag{2.54}$$

This is sometimes called the Majorana mass.

[2] In our conventions the Majorana spinor satisfies the reality condition: $\overline{(\gamma_5 \psi_M)} = \tilde{\psi}_M$.

3D Riemannian space

In a 3D space of Euclidean signature the gamma matrices can be taken to be the Pauli matrices, while the charge conjugation matrix can be taken as $C = i\sigma_2$. We have

$$C^{\mathrm{Tr}} = -C; \qquad (C\gamma^m)^{\mathrm{Tr}} = C\gamma^m. \tag{2.55}$$

The irreducible spinor representation of $Spin(3)$ has two *complex* components. We thus have the following useful symmetry properties

$$\tilde{\psi}\gamma^{m_1\ldots m_p}\chi = (-1)^{\frac{1}{2}(p-1)(p-2)}\tilde{\chi}\gamma^{m_1\ldots m_p}\psi, \tag{2.56}$$

where χ, ψ are arbitrary *commuting* spinors. We define the complex conjugate ψ_c of ψ via

$$\tilde{\psi}_c = \psi^\dagger, \tag{2.57}$$

so that ψ_c transforms as a spinor. We then have the complex conjugation properties,

$$(\tilde{\psi}_c\gamma^{m_1\ldots m_p}\chi)^* = -(-1)^p\tilde{\psi}\gamma^{m_1\ldots m_p}\chi_c. \tag{2.58}$$

Spinors in 7D Lorentzian space

The irreducible spinor representation of $Spin(1,6)$ has eight complex components. In terms of an $Spin(1,6) \to Spin(1,3) \times Spin(3)$ decomposition, the 7D spinor ψ decomposes as

$$\psi = (\chi_+ + \theta_-) \otimes \xi, \tag{2.59}$$

where χ, θ are irreducible Weyl spinors of $Spin(1,3)$ and ξ is an irreducible spinor of $Spin(3)$. The seven-dimensional gamma matrices Γ^M decompose as

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{I}_2\ ; \quad \Gamma^{m+3} = \gamma^5 \otimes \gamma^m, \tag{2.60}$$

where $\mu = 0, \ldots, 3$ and $m = 1, 2, 3$. We are now ready to proceed with the model and its compactification.

2.2.2 Compactification of fermions

We would now like to examine the 4D effective theory arising as the low-energy limit of a 7D gauge-fermion theory compactified on the Heisenberg nilmanifold $\mathcal{M}_3$. The 7D Lagrangian consists of a Yang-Mills term $\mathcal{L}^{\mathrm{YM}}_{7\mathrm{D}}$ studied in the previous section and a fermion term $\mathcal{L}^{\mathrm{f}}_{7\mathrm{D}}$. The effective theory in four dimensions, $\mathcal{L}^{\mathrm{eff}}_{4D}$, will be given by

$$\mathcal{L}^{\mathrm{eff}}_{4D} = \int \mathrm{d}^3y\ \left(\mathcal{L}^{\mathrm{YM}}_{7\mathrm{D}} + \mathcal{L}^{\mathrm{f}}_{7\mathrm{D}}\right). \tag{2.61}$$

The right-hand side above indicates the KK reduction of the seven-dimensional theory, and involves integrating over the three-dimensional internal space parameterized by the y-coordinates.

Moreover, we will place ourselves in the same limit as in the previous section, i.e. we are taking the small fiber/large base limit,

$$|\mathtt{f}| \ll \frac{1}{r^i}\ ,\ i=1,2,3 \quad \Rightarrow \quad \frac{1}{r^{1,2}} \ll \frac{1}{r^3}. \tag{2.62}$$

In this limit all fields whose masses carry an r^i dependence (i.e. all the KK modes) decouple, leaving in the theory only those fields with masses of either the order of $|\mathtt{f}|$, or zero. The fermionic Lagrangian will be taken to be of the form

$$\mathcal{L}^{\rm f}_{\rm 7D} = \overline{\psi}_i \Gamma^M (\delta^{ij}\nabla_M + i\mathcal{A}^a_M \rho^{ij}_a)\psi_j + \tfrac{1}{6} F_{MNP}\overline{\psi}_i \Gamma^{MNP}\psi_i + M_0 \overline{\psi}_i \psi_i, \tag{2.63}$$

where the hermitian matrices ρ^{ij}_a provide a representation R of the Lie algebra of the gauge group, so that $i,j=1,\ldots,\dim(R)$, and ψ_i transforms in the R representation. We have also allowed for a constant background zero-form flux (a mass term) M_0, and a three-form flux F_{MNP}, which will be assumed to be along the internal manifold $\mathcal{M}_3$ in order not to break the 4D Lorentz invariance.[3] This implies that the only non-vanishing component is given by

$$F_{mnp} = M_1 \varepsilon_{mnp}, \tag{2.64}$$

for some real constant M_1, and the Levi-Civita symbol is a tensor in our conventions.

Our reduction ansatz for the 7D spinors is as in (2.59),

$$\psi_i = (\chi_{i+} + \theta_{i-}) \otimes \xi, \tag{2.65}$$

where χ_i, θ_i are Weyl 4D spinors and ξ is a spinor on $\mathcal{M}_3$. We can already see that the resulting 4D model will necessarily be non-chiral since, the 4D positive and negative chiralities of the spinors both transform in the same representation R.

We expand ξ on a basis of normalized eigen-spinors of the Dirac operator in 3D

$$\sigma^m \nabla_m \xi = \lambda \xi, \tag{2.66}$$

where the eigenvalues of the Dirac operator on $\mathcal{M}_3$ were given in 1.3.4. In the limit (2.62) of decoupling of the KK modes, only the lowest eigenspinor is kept, corresponding to eigenvalue $\lambda = i\frac{\mathtt{f}}{4}(a^2+b^2+1)$, cf. (1.170). Moreover we assume that ξ is normalized: $\int \mathrm{d}^3 y\ \xi^\dagger \xi = 1$. Putting everything together we obtain

$$\int \mathrm{d}^3 y\ \mathcal{L}^{\rm f}_{\rm 7D} = \mathcal{L}_{\rm kinetic} + \mathcal{L}_{\rm Yukawa} + \mathcal{L}_{\rm mass}, \tag{2.67}$$

where

$$\begin{aligned}
\mathcal{L}_{\rm kinetic} &= \overline{\chi}_{i+}\gamma^\mu(\delta^{ij}\partial_\mu + igA^a_\mu \rho^{ij}_a)\chi_{j+} + \overline{\theta}_{i-}\gamma^\mu(\delta^{ij}\partial_\mu + igA^a_\mu\rho^{ij}_a)\theta_{j-} \\
\mathcal{L}_{\rm Yukawa} &= ig(\overline{\theta}_{i-}\chi_{j+} - \overline{\chi}_{i+}\theta_{j-})\rho^{ij}_a \Phi^a \\
\mathcal{L}_{\rm mass} &= M_c\ \overline{\theta}_{i-}\chi_{i+} + M_c^*\ \overline{\chi}_{i+}\theta_{i-},
\end{aligned} \tag{2.68}$$

[3] We may also allow for other types of fluxes, however this will result in a similar mass terms as in (2.72) below, so we do not introduce them independently.

and we have defined a complex "mass" M_c

$$M_c = M_0 + i\big[M_1 + \tfrac{\mathtt{f}}{4}(a^2+b^2+1)\big]. \tag{2.69}$$

The adjoint scalar Φ^a is defined by

$$\Phi^a := \sum_{I=1}^{3} \phi^{aI} c^I, \tag{2.70}$$

where the three constants c^I are given by $c^I := \xi^\dagger \sigma^I \xi$. To make this more explicit, let us give a parameterization for ξ. Up to an unimportant overall phase we may set

$$\xi = \begin{pmatrix} e^{-i\beta}\cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} \end{pmatrix}, \tag{2.71}$$

for some angles α, β. We then find $c^1 = \sin\alpha\cos\beta$, $c^2 = \sin\alpha\sin\beta$, $c^3 = \cos\alpha$. I.e. $\vec{c}$ can be thought of as a unit vector of $\mathbb{R}^3$.

The final expression for the fermionic 4D theory can also be expressed in terms of 4D Dirac spinors $\Psi_i := \mathrm{i}\chi_{i+} + \theta_{i-}$

$$\int \mathrm{d}^3y\ \mathcal{L}^{\mathrm{f}}_{7\mathrm{D}} = \overline{\Psi}_i \Gamma^\mu (\delta^{ij}\partial_\mu + igA^a_\mu \rho^{ij}_a)\Psi_j + g\rho^{ij}_a \Phi^a \overline{\Psi}_i \Psi_j \tag{2.72}$$

$$+ \big[M_1 + \tfrac{\mathtt{f}}{4}(a^2+b^2+1)\big]\overline{\Psi}_i\Psi_i - \mathrm{i}M_0 \overline{\Psi}_i \gamma_5 \Psi_i. \tag{2.73}$$

The 7D parameters $M_{0,1}$ are free (up to flux quantization), and can be thought of as arising from the inclusion of constant background flux on the nilmanifold.

With the compactification of gauge fields and fermions, we now have the possibility to explore realistic model building in this setting. The Yang-Mills theory in 7 dimensions gives rise to a Yang-Mills theory coupled to three scalars in 4 dimensions. The scalars have a potential involving three terms, a quadratic mass term, a trilinear and a quartic term. In the context of a realistic model, the fermions introduced here would not make good candidates for the matter content of the standard model since they are both massive and not chiral. Those problems could be circumvented by introducing localized matter on the nilmanifold. This scenario would make sense if we were to add for example orbifolds to the geometry. This will be discussed after we study the potential for the scalars, which will be done in the following chapter. But before that, I would like to propose an introduction to a point of view on fermions not so popular among physicists. The usual approach to fermions is done through the use of their representations. When we discuss gamma matrices and their actions on fermionic fields, we are really using matrix representations of the $Spin(p,q)$ group and their action on a vector space representing the spinors. A reasonable question one may wonder about is how is the $Spin$ group fundamentally defined, and hence how are the spinors really defined on a given space. This next section is an invitation to the formalism used by mathematicians to talk about spinors. This part sits slightly outside of the main discussion, but is nonetheless useful to the avid learner. The reader interested only in the main development of this thesis may skip this part.

2.2.3 Spinors as Clifford minimal ideals

We are going to use the notion of Clifford algebras to define spinors. The purpose is to propose an alternative way of thinking about spinors. Clifford algebras are central objects in the study of vector spaces equipped with a metric. It is the natural space to describe all the geometrical objects associated to such spaces. It can be used to formulate classical physics in a unified and compact formalism. This algebra can be referred to as the *geometric algebra* [45] in this context to emphasis its central role. This is not a complete introduction to Clifford algebras and their full theory, but rather an invitation to another point of view, less used in physics, but useful in my opinion. A This formalism is powerful as it can be extended to encapsulate We start with the definition of a Clifford algebra.

A (real) Clifford algebra is a (real) vector space, equipped with a metric $g_{\mu\nu}$ such that the following product is defined on the space :

$$\{e_\mu, e_\nu\} = e_\mu e_\nu + e_\nu e_\mu = 2g_{\mu\nu} \tag{2.74}$$

where $\{e_\mu\}$ is an orthonormal basis of the vector space. This product can be decomposed into the two following identities,

$$e_\mu e_\nu = -e_\nu e_\mu \text{ for } \mu \neq \nu \ , \text{ and } \ e_\mu e_\mu = g_{\mu\mu} = \pm 1 \ , \tag{2.75}$$

where the sign depends on the whether the direction is timelike or spacelike. So the elements of the algebra under consideration anticommutes (from the first relation), and each element squares to the identity (or minus the identity). We will denote $Cl(p,q)$ (or $Cl_{p,q}$) the metric space equipped with the Clifford product we just defined, and where the metric has signature $(+,\dots,+,-,\dots,-)$ with p "+" and q "−". So for example, Minkowski space in the mostly "+" signature is associated with the Clifford algebra $Cl(3,1)$, while the mostly "−" signature it is associated with $Cl(1,3)$. The distinction between the two space is important since they are isomorphic as vector spaces but not as real algebras. Elements of this algebra can be decomposed on the standard basis composed of the elements

$$1 \ ; \ e_\mu \ ; \ e_\mu e_\nu \ ; \ \dots \ ; \ e_1 \dots e_n \ , \tag{2.76}$$

where n is the dimension of the vector space. We identify the starting vector space with the grade 1 generated by the elements e_μ. We call vectors elements of this subspace and scalars element of the grade 0 (proportional to the identity).

Let us now study the Clifford algebra associated to our case. Locally, the total space for our theory is given by a seven dimensional space with two possible choice for the metric, giving a different algebra for each choice:

$$Cl_{1,6} \simeq \mathbb{C}(8) \tag{2.77}$$

$$Cl_{6,1} \simeq \mathbb{H}(4) \oplus \mathbb{H}(4) \tag{2.78}$$

where we used the notation $\mathbb{A}(n)$ for the matrix algebra of dimension $n \times n$ with entries in $\mathbb{A}$. $\mathbb{C}$ denotes the complex numbers while $\mathbb{H}$ represent the quarternions.

Note the equality for the dimensions of the spaces: $\dim(\mathbb{C}(8)) = 8*8*2 = 128$ and $\dim(\mathbb{H}(4)\oplus\mathbb{H}(4)) = 4*4*4+4*4*4 = 128$. Of course this also matches the dimension of the Clifford algebra, $\dim(Cl_{p,q}) = 2^{p+q} = 2^7 = 128$, hence the fact that they are isomorphic as vector spaces but not as algebras. Without making a definite choice for the moment, we can use the following two isomorphisms of Clifford algebra to decompose the algebra in a Minkowski part and a nilmanifold part. The isomorphism is

$$\begin{aligned} Cl_{1,6} &\simeq Cl_{1,3}\otimes Cl_3 \\ &\text{or} \\ Cl_{6,1} &\simeq Cl_{3,1}\otimes Cl_{0,3} \end{aligned} \tag{2.79}$$

This is possible here because Minkowski space is even dimensional. We pick for notation $\{e_a\}$, $a \in \{1,2,3\}$ for the nilmanifold and $\{\gamma_\mu\}$, $\mu \in \{0,1,2,3\}$ for Minkowski space. The map is given explicitly by :

$$(x,y) \to x\otimes 1 + \omega \otimes y \tag{2.80}$$

with $x \in Cl_{1,3}$ or $Cl_{3,1}$, $y \in Cl_3$ or $Cl_{0,3}$ and $\omega = \gamma^0\gamma^1\gamma^2\gamma^3$ such that $\omega^2 = -1$. Note the change in signature for Cl_3 and $Cl_{0,3}$ in (2.79) because of the square of ω being -1. Let us look at how the vectors are mapped. Let us take the example of $Cl_{6,1}$. We denote the generators for the vectors in $Cl_{6,1}$ as $\{\Gamma_0,\ldots,\Gamma_3,\Gamma_4,\ldots,\Gamma_6\}$, and we have the relations $\Gamma_0^2 = -1$, and $\Gamma_i^2 = 1$ for all the others. The map gives

$$\Gamma_0 \to \gamma_0\otimes 1 \ , \ \Gamma_1 \to \gamma_1\otimes 1 \ , \Gamma_2 \to \gamma_2\otimes 1 \ , \ \Gamma_3 \to \gamma_3 \otimes 1 \tag{2.81}$$

and

$$\Gamma_4 \to \omega\otimes e_1 \ , \ \Gamma_5 \to \omega\otimes e_2 \ , \Gamma_6 \to \omega\otimes e_3 \ . \tag{2.82}$$

Observe that $(\omega\otimes e_1)^2 = -1\otimes -1 = 1\otimes 1 = \Gamma_4^2$, hence the fact that the signature changes with the decomposition.

The spinor space $\mathcal{S}$ is defined as a left minimal ideal in $Cl_{6,1}$ or $Cl_{1,6}$ depending on the choice. This means that $\mathcal{S}$ is a subalgebra such that when acting on by an element of $Cl_{6,1}$, the product remains in $\mathcal{S}$. Explicitly we write $\mathcal{S}$ as $\mathcal{S} = Cl_{6,1}f$, where f is a primitive idempotent element of $Cl_{6,1}$, meaning that $f^2 = f$ (idempotent) and $Cl_{6,1}f$ cannot be decomposed into a direct sum (primitive). This element acts as a projector on the space. To find such an element, we use a non-scalar elements Γ_T (from the standard basis, it may be a product of vectors) such that $\Gamma_T^2 = 1$. We can then build two orthogonal projectors $e = 1/2(1-\Gamma_T)$ and $f = 1/2(1+\Gamma_T)$ such that $ef = fe = 0$ and $e+f = 1$. We may find another element Γ_{T2} such that it commutes with Γ_T. In that case we project again building the projectors in the same way. More generally, one first find all the elements Γ_{Ti} such that $\Gamma_{Ti}\Gamma_{Tj} = \Gamma_{Tj}\Gamma_{Ti}$ and $\Gamma_{Ti}^2 = 1$. If we find the maximum number k of such elements then f is written as the product of projectors:

$$f = \frac{1}{2^k}(1+\Gamma_{T1})\ldots(1+\Gamma_{Tk}) \tag{2.83}$$

Thankfully, we don't have to try the entire algebra to know the maximum number of such elements, the general theory tells us that for the Clifford algebra $Cl_{p,q}$ there are $k = q - r_{q-p}$ elements Γ_T where the r_i are the Radon-Hurwitz numbers given by:

$$\begin{aligned} r_0 &= 0\ , \quad r_1 = 1\ , \\ r_2 &= 2\ , \quad r_3 = 2\ , \\ r_4 &= 3\ , \quad r_5 = 3\ , \\ r_6 &= 3\ , \quad r_7 = 3\ , \end{aligned} \tag{2.84}$$

along with $r_{i+8} = r_i + 4$ and $r_{-i} = 1 - i + r_{i-2}$. In the case of $Cl_{6,1}$ and $Cl_{1,6}$ we have $k = 3$. Looking at the first case, consider the following elements:

$$\begin{aligned} a &= \Gamma_3 = \gamma_3 \otimes 1, \\ b &= \Gamma_{01} = \gamma_{01} \otimes 1 = \gamma_0\gamma_1 \otimes 1, \\ c &= \Gamma_{0123456} = \Gamma_{0123}\Gamma_{456} = (\omega \otimes 1)(\omega^3 \otimes e_{123}) = 1 \otimes e_{123} \end{aligned} \tag{2.85}$$

where we used the notation $\Gamma_{\mu\nu} = \Gamma_\mu\Gamma_\nu$. We have $a^2 = b^2 = c^2 = 1$ and a, b, c all commute with each other. f can be decomposed using the algebra isomorphism presented above:

$$\begin{aligned} f &= \frac{(1+a)}{2}\frac{(1+b)}{2}\frac{(1+c)}{2} \\ &= \frac{(1\otimes 1 + \gamma_3 \otimes 1)}{2}\frac{(1\otimes 1 + \gamma_{01} \otimes 1)}{2}\frac{(1\otimes 1 + 1 \otimes e_{123})}{2} \\ &= \left(\frac{(1+\gamma_3)\otimes 1}{2}\right)\left(\frac{(1+\gamma_{01})\otimes 1}{2}\right)\left(\frac{1\otimes(1+e_{123})}{2}\right) \\ &= \frac{(1+\gamma_3)}{2}\frac{(1+\gamma_0\gamma_1)}{2} \otimes \frac{(1+e_{123})}{2} \\ &= f_1 \otimes f_2 \end{aligned} \tag{2.86}$$

where in the first line we factorized the projectors. This decomposition is not trivial, and requires that the element f may be factorized on the tensor product space. By acting with the basis of the algebra on f we can generate the spinor spaces which are :

$$Cl_{3,1}f_1 \simeq \mathbb{C}^2 \tag{2.87}$$

$$Cl_{0,3}f_2 \simeq \mathbb{H}\ . \tag{2.88}$$

Therefore we can decompose our 7-dimensional spinors into a tensor product of the usual Weyl spinor and a quaternion. It should be pointed out that for $Cl_{3,1}$ we find $k = 2$ and for $Cl_{0,3}$ we have $k = 1$. f_1 and f_2 are indeed primitive idempotent in there respective space. So the decomposition is pretty straightforward here.

We now proceed in the same way for the opposite metric. This case is more interesting because as we will see, the decomposition isn't as clear as in the last

example. We make the following choice for the element:

$$\begin{aligned} a &= \Gamma_0 = \gamma_0 \otimes 1, \\ b &= -\Gamma_{1234} = -\gamma_{123}\omega \otimes e_3 = \gamma_0 \otimes e_3 \\ c &= -\Gamma_{01234} = -\omega^2 \otimes e_3 = 1 \otimes e_3 \end{aligned} \tag{2.89}$$

where the minus signs where picked for convenience. We have again a, b and c squaring to 1, and they all commute with each other. Note that we have $ab = c$. Now writing the primitive idempotent element f, we find

$$\begin{aligned} f &= \frac{(1+a)}{2}\frac{(1+b)}{2}\frac{(1+c)}{2} = \left(\frac{1}{2}(1+\gamma_0)\otimes\frac{1}{2}(1+e_3)\right)\frac{1}{2}\left(1\otimes 1+\gamma_0\otimes e_3\right) \\ &= \frac{1}{2}\left(\frac{1}{2}(1+\gamma_0)\otimes\frac{1}{2}(1+e_3)+\frac{1}{2}(1+\gamma_0)\gamma_0\otimes\frac{1}{2}(1+e_3)e_3\right) \\ &= \frac{1}{2}(1+\gamma_0)\otimes\frac{1}{2}(1+e_3) = f_1\otimes f_2 \end{aligned} \tag{2.90}$$

This is somewhat surprising. The three projectors found in the Clifford algebra $Cl_{1,6}$ combine into only two projectors on the tensor product. The two projectors match what we would find in the spaces $Cl_{1,3}$ and Cl_3. As we said before, $k = 1$ for both of them. We can take γ_0 for $Cl_{1,3}$ and e_3 for Cl_3. Those give exactly the same result as in (2.90). There spinor spaces are respectively

$$Cl_{1,3}f_1 \simeq \mathbb{C}^4 \ , \tag{2.91}$$

$$Cl_3 f_2 \simeq \mathbb{C}^2 \ . \tag{2.92}$$

The spinor space derived from the 7-dimensional Clifford algebra $Cl_{1,6}$ (so acting with the three projectors) is $2^{7-3} = 16$ dimensional, while the spinor space found from the tensor product is $2^{4-1}\times 2^{3-1} = 32$. So it seems that the isomorphism of Clifford algebra doesn't go through as an isomorphism of spinors, instead, a double copy of the 7d spinors is isomorphic to the tensor product of spinors. We showed that the spinor space $Cl_{1,6}f$ can be decomposed into $Cl_{1,3}f_1\otimes Cl_3 f_2$ on which the Clifford algebra $Cl_{1,3}\otimes Cl_3$ acts naturally by multiplication on the left.

Chapter 3

The potential

In the last chapter we have obtained the effective 4 dimensional theory coming from the a 7 dimensional Yang-Mills theory. Upon compactification and restriction to the first modes, the theory is equivalent to a 4 dimensional Yang-Mills theory coupled to a scalar sector. The three scalars are in the adjoint representation of the gauge group as they stem from the higher dimensional gauge fields, and are coupled via the potential. We recall for clarity the expression of the Lagrangian and the potential:

$$\mathcal{L} = \mathrm{Tr}\Big(-\tfrac{1}{2} F_{\mu\nu}F^{\mu\nu} + \sum_{I=1}^{3} D_\mu \phi^I D^\mu \phi^I - M^2(\phi^3)^2 - \mathcal{U}\Big) \ , \tag{3.1}$$

where:

$$\mathcal{U} = \mathrm{Tr}\Big(-2igM[\phi^1,\phi^2]\phi^3 + \tfrac{1}{2}g^2 \sum_{I,J=1}^{3} [\phi^I,\phi^J][\phi^I,\phi^J]\Big) \ . \tag{3.2}$$

This part is based on the work developed in [39,44,46] and extends it by proposing a general approach to the study of this potential.

3.1 A first approach to the potential

To obtain the masses of the fields we must diagonalize the mass matrix of the quadratic fluctuations around the vacuum configuration. First, let us define the potential we want to study as the sum of the interaction terms and the mass term. Explicitly:

$$\frac{\mathcal{V}}{M^2} = \mathrm{Tr}(\phi^3)^2 + \frac{\mathcal{U}}{M^2} \ . \tag{3.3}$$

A vacuum configuration is a solution of:

$$\frac{\mathcal{V}(\phi+\delta\phi)-\mathcal{V}(\phi)}{M^2} = \\ \mathrm{Tr}\left(2\phi^3\delta\phi^3\right) - \quad 2i\tfrac{g}{M}\mathrm{Tr}\Big([\phi^1,\phi^2]\delta\phi^3+[\phi^3,\phi^1]\delta\phi^2+[\phi^2,\phi^3]\delta\phi^1\Big) \\ +2\tfrac{g^2}{M^2}\mathrm{Tr}\Big(\textstyle\sum_{I,J=1}^{3}[\phi^I,\phi^J][\phi^I,\delta\phi^J]\Big)=0\ . \qquad (3.4)$$

We can see the theory contains a class of vacua of the form ϕ^I = constant, where,

$$\phi^3=0\ ;\quad [\phi^1,\phi^2]=0\ . \qquad (3.5)$$

Once a vacuum $\phi^I=\phi_0^I$ is found, infinite classes of other vacua are generated by conjugation of arbitrary elements $U\in G$, $\phi_0^I\to U^\dagger\phi_0^I U$. Since this conjugation is a symmetry of the potential, it is a map from vacua to vacua. Moreover, if the vacuum ϕ_0^I is not invariant under the conjugation, it will be mapped to a different vacuum with the same energy. This is of course the Goldstone mechanism, implying the existence of various massless 4d scalars in the vacuum, as we will confirm explicitly in the following.

Now that a class of minima has been identified in (3.5), we can develop the potential to second order in the variations of the fields. The variation of the potential reads:

$$\frac{\delta^2\mathcal{V}}{2M^2} = \mathrm{Tr}(\delta\phi^3)^2-\frac{ig}{M}\mathrm{Tr}\left(\left[\delta\phi^1,\phi_0^2\right]\delta\phi^3+\left[\phi_0^1,\delta\phi^2\right]\delta\phi^3\right) \\ +\frac{g^2}{M^2}\mathrm{Tr}\Big(\left[\delta\phi^1,\phi_0^2\right]^2+\left[\delta\phi^2,\phi_0^1\right]^2+\left[\delta\phi^3,\phi_0^1\right]^2+\left[\delta\phi^3,\phi_0^2\right]^2 \\ +2\left[\delta\phi^1,\phi_0^2\right]\left[\phi_0^1,\delta\phi^2\right]\Big)\ , \qquad (3.6)$$

where ϕ_0^1 and ϕ_0^2 are the values of the fields at the minimum. This expression gives a matrix of dimension $(3\times\dim(Lie(G))^2$, where $\dim(Lie(G))$ is the dimension of the Lie algebra associated to the gauge group G. In order to obtain the masses of the fields around this minimum one would have to diagonalize this matrix once a specific gauge group has been chosen. There is actually a way to compute the masses and the potential for an arbitrary gauge group. We leave this approach for later, as we will first build some intuition about the model by fixing some specific gauge groups. The question is, what are sensible options to look first ? It may be obvious to some that the purpose here is to reproduce the Higgs field since it is the only fundamental scalar field observed in nature. The Higgs field before the electroweak symmetry breaking is in the fundamental representation of the weak $SU(2)$ and has an hypercharge equal to 1. In our context, the vacuum we select is going to break the gauge symmetry to some subgroup. We could start with an $SU(2)$ gauge, but remember that the scalar fields are in the adjoint representation. So it won't reproduce the right representation. The next guess would be SU(3). This is an interesting choice since it contains exactly $SU(2)\times U(1)$ along with a fundamental representation

with the right quantum numbers. Bigger gauge group of the $SU(N)$ type will also be considered. Their study will give us the first hint that the masses don't really depend on the gauge structure.

3.1.1 From $SU(3)$ to $SU(2) \times U(1)$

The first case of interest is $SU(3)$: it is a minimal setup that contains $SU(2) \times U(1)$, which is suitable to model the electroweak sector. Also, its Lie algebra is eight dimensional, making it rather manageable. We denote by $\mathfrak{su}(3)$ the Lie algebra associated to $SU(3)$. A matrix $A \in \mathfrak{su}(3)$ is characterized, in our conventions, by $A = A^\dagger$ and $\mathrm{Tr}(A) = 0$. We shall use the Gell-Mann convention for the $SU(3)$ generators [47]. As already remarked, the vacuum solution that we choose may have some residual symmetries. These symmetries will determine the residual gauge symmetry after the vacuum solution has been selected.

Let us start by explaining how to choose the vacuum in order to have an unbroken $SU(2) \times U(1)$ gauge: we want that ϕ_0^1 and ϕ_0^2 commute, in order for them to satisfy condition (3.5) for a minimum. We can thus simply choose them as combinations of the diagonal Gell-Mann matrices, which we know commute: indeed the diagonal Gell-Mann matrices form the Cartan subalgebra of $\mathfrak{su}(3)$, i.e. its maximal Abelian subalgebra. A simple way to parameterize our vacuum is to take $\phi_0^1 = \phi_0^2$ and take ϕ_0 to be a generic element of the Cartan subalgebra:

$$\phi_0^1 = \phi_0^2 = \phi_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} , \tag{3.7}$$

with a, b, c real parameters and $a+b+c = 0$, since $\phi_0 \in \mathfrak{su}(3)$. The factor $1/\sqrt{3}$ was picked for convenience. We want to leave an $\mathfrak{su}(2)$ subalgebra unbroken, in other words, we want that ϕ_0 commutes with the elements that generate an $\mathfrak{su}(2)$. If we take $a = b$, so that the trace condition imposes $c = -2a$, it follows that $\phi_0 = a\lambda_8$. Only one parameter remains free, the normalization of the field. In this case ϕ_0 commutes with the Gell-Mann matrices λ_1, λ_2, λ_3 and λ_8, which generate an $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ subalgebra of $\mathfrak{su}(3)$.

We need to diagonalize the matrix (3.37) of second derivatives of the potential at the point $\phi^3 = 0, \phi^1 = \phi^2 = \phi_0$ (this is a 24×24 matrix). The result is a list of masses, where the multiplicity of each mass indicates the dimension of the representation of the new gauge group (i.e. after symmetry breaking) in which this mass transforms. In our case we obtain the following masses for the scalar fields:

$$12 \text{ massless degrees of freedom (dof)} \tag{3.8}$$

$$M_0^2 = M^2 \quad (4 \text{ dof}) \tag{3.9}$$

$$M_\pm^2 = \frac{1}{2} M^2 \left(1 + 3a^2 \frac{g^2}{M^2} \pm \sqrt{1 + 6a^2 \frac{g^2}{M^2}} \right) \quad (4{+}4 \text{ dof}) , \tag{3.10}$$

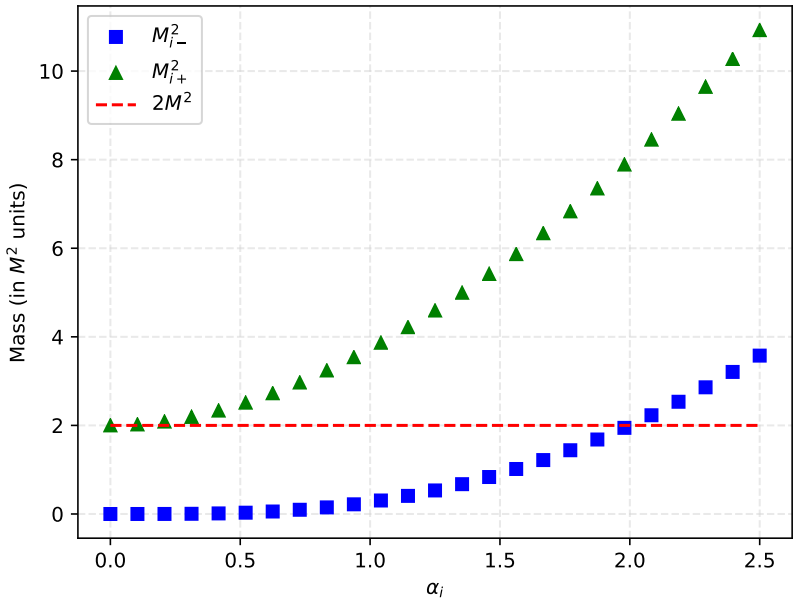


Figure 3.1: Diagram of the three types of scalar masses of the $SU(3)$ model. This diagram is relevant for the other models too since all the masses are of the form $M^2_{i\pm} = 1 + \alpha_i^2 \pm \sqrt{1 + 2\alpha_i^2}$ with α_i being linear combinations of the vacuum parameters. We can see that M^2_{i-} goes to zero for small α_i while the two other masses remain around the M^2 scale.

where in the last line (4+4 dof) means that the subspace associated to the mass M_+^2 is four dimensional, and likewise for the one associated to M_-^2.

Let us see how to associate each mass to a representation. From the eigenvectors (not listed here) one can read that the masses $M_\pm^2$ are in a (2,1) representation of the gauge group $SU(2) \times U(1)$. The mass M_0^2 is in the adjoint representation of $SU(2) \times U(1)$. Lastly, we can deduce the representation of the massless modes by a counting argument. We started with three copies of the adjoint representation of $SU(3)$. We used twice the two- (complex) dimensional representation for the masses $M_\pm^2$. We used one adjoint representation for M_0^2. We are therefore left with two copies of the adjoint representation and one of the two-dimensional representation. The total dimension of these representations is $2 \times 4 + 4 = 12$, just like the multiplicity of the massless modes. We can also note that the massless two-dimensional representation is not really in the spectrum and serves as the longitudinal degree of freedom of the the broken gauge boson.

There is also a more intuitive reasoning that could have let us guess the solution: it can be seen that the masses $M_0^2 = M^2$ are in the adjoint representation, since we chose ϕ_0 to commute with the generators of $\mathfrak{su}(2)$ and λ_8 that generates $U(1)$. We would therefore expect the mass of these states not to be modified by the vacuum we selected. Hence the $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ part of ϕ^3 remains with a mass M^2. The other two massive states depend on the parameter a and are in the only representation that doesn't commute with ϕ_0, namely the two-dimensional one. We will see that this pattern persists for bigger gauge groups.

Finally, we can also calculate the mass of the gauge bosons corresponding to the broken generators. It is simply given by the coupling terms between the scalars and the gauge fields:

$$m_{bosons}^2 = \frac{3}{2} g^2 a^2 \ , \tag{3.11}$$

where on the right-hand side the sum is over the gauge fields associated to the broken generators λ_4, λ_5, λ_6, λ_7.

Now that all the masses have been determined, the effective theory is fully determined. However there is still some freedom in the hierarchy of the masses. Let us now turn to this point: the model has two scalar fields in a (2,1) representation of the gauge, that correspond to the masses M_-^2 and M_+^2. These representations are SU(2) doublets, suitable for the description of the Higgs field. The mass M_-^2 is a difference of two terms. In the limit where $a^2 g^2 / M^2 \to 0$, this mass also goes to zero, while M_+^2 will be of order M^2. If we consider the region $M^2 \gg a^2 g^2 \gg m_{limit}^2$ where m_{limit} is the maximum detectable mass, we can adjust the ratio $a^2 g^2 / M^2$ so that $M_-^2 = M_{Higgs}^2$, while all the other fields can have masses well beyond detection.

3.1.2 The case of $SU(5)$

In this section we study an $SU(5)$ gauge symmetry that breaks spontaneously to $SU(3) \times SU(2) \times U(1)$. We start again by giving the explicit matrix repre-

sentation of $\mathfrak{su}(5)$, where the general element of $\mathfrak{su}(5)$ is of the form:

$$\begin{pmatrix} a_1+c & a_4 & a_5 & h_1 & h_2 \\ a_4^* & a_2+c & a_6 & h_3 & h_4 \\ a_5^* & a_6^* & -(a_1+a_2)+c & h_5 & h_6 \\ h_1^* & h_3^* & h_5^* & b_1-\frac{3}{2}c & b_3 \\ h_2^* & h_4^* & h_6^* & b_3^* & -b_1-\frac{3}{2}c \end{pmatrix} . \tag{3.12}$$

This matrix is traceless and Hermitian (the diagonal elements are taken to be real) and therefore lies in $\mathfrak{su}(5)$. The parameters have been chosen so that the decomposition is made explicit. Indeed, the a's generate an $\mathfrak{su}(3)$ algebra: the b's generate an $\mathfrak{su}(2)$, the c a $\mathfrak{u}(1)$, while the h's are in the (3,2)- (complex) dimensional representation of respectively $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$. The h's are also charged under $\mathfrak{u}(1)$. We see that this situation is very similar to the one we had before for $SU(3)$. Again, we want to find a vacuum $\phi^1 = \phi^2 = \phi_0$ such that this vacuum commutes with the generators of the unbroken gauge group. We can construct ϕ_0 in a similar fashion as before. Let us take ϕ_0 to be:

$$\phi_0 = \begin{pmatrix} a & & & & \\ & a & & & \\ & & a & & \\ & & & -\frac{3}{2}a & \\ & & & & -\frac{3}{2}a \end{pmatrix} . \tag{3.13}$$

The invariance of ϕ_0 can then be seen at the level of the algebra since:

$$\forall\ \lambda = \begin{pmatrix} A & \\ & B \end{pmatrix} \in \mathfrak{su}(5)\ , \quad [\lambda, \phi_0] = 0\ , \tag{3.14}$$

where $A \in \mathfrak{su}(3)$, $B \in \mathfrak{su}(2)$ and of course $Tr(\lambda) = Tr(A) = Tr(B) = 0$. Note that ϕ_0 also commutes with all the generators of the Cartan subalgebra since ϕ_0 itself is in the Cartan subalgebra.

Applying the same procedure as before to the now 72×72 mass matrix, we find the masses to be:

$$\text{Massless} \quad (36 \text{ dof}) \tag{3.15}$$

$$M_0^2 = M^2 \quad (12 \text{ dof}) \tag{3.16}$$

$$M_\pm^2 = \frac{1}{2}M^2 \left(1 + 100a^2\frac{g^2}{M^2} \pm \sqrt{1 + 200a^2\frac{g^2}{M^2}} \right) \quad (12{+}12 \text{ dof})\ . \tag{3.17}$$

These masses are very similar to the masses found for the $SU(3)$ gauge. The masses M_+^2 and M_-^2 fall again into the only representation that does not commute with the gauge, namely the (3,2,1) representation generated by the h's in Eq. (3.12). The mass M_0^2 is again in the adjoint representation of the new gauge. The dimension of this adjoint representation is the sum of the dimensions of each adjoint representation, $8+3+1 = 12$. Again, we can compute the

mass of the broken gauge bosons that we find to be:

$$M_{boson} = \frac{25}{4} g^2 a^2 \ . \tag{3.18}$$

These also transform in the (3,2,1) representation of the gauge group.

The argument about the masses discussed for SU(3) is also valid here, and we can have the field with the mass M_-^2 at a much lower scale than the rest of the masses. This state is also charged under the strong interactions in this case.

3.1.3 The case of $SU(6)$

$SU(6)$ has the advantage of breaking into $SU(3) \times SU(2) \times U(1) \times U(1)$, which is close to the SM gauge, and allows for a (1,2,1,1) representation of the new gauge group (i.e. after symmetry breaking) at an arbitrary low mass. Quite similarly to the $\mathfrak{su}(5)$ case, we can write a general element of $\mathfrak{su}(6)$ as:

$$\begin{pmatrix} a_1+c+d & a_4 & a_5 & h_1 & h_2 & l_1 \\ a_4^* & a_2+c+d & a_6 & h_3 & h_4 & l_2 \\ a_5^* & a_6^* & -(a_1+a_2)+c+d & h_5 & h_6 & l_3 \\ h_1^* & h_3^* & h_5^* & b_1-\frac{3}{2}c+d & b_3 & p_1 \\ h_2^* & h_4^* & h_6^* & b_3^* & -b_1-\frac{3}{2}c+d & p_2 \\ l_1^* & l_2^* & l_3^* & p_1^* & p_2^* & -4d \end{pmatrix} . \tag{3.19}$$

Again the parameterization makes explicit the decomposition. The a's generate an $\mathfrak{su}(3)$, the b's an $\mathfrak{su}(2)$, c and d are two copies of $\mathfrak{u}(1)$, the h's are in the (3,2,1,0), the l's in the (3,1,2,1), the p's in the (1,2,1,1). For each representation the charge under the two $U(1)$'s is calculated simply by using the commutation relations between the generators of the $U(1)$'s and the generators of the representation. This example is interesting because it is the first example that gives different massive representations of the gauge group. We can parameterize the vacuum in the following way:

$$\phi_0 = \begin{pmatrix} a & & & & & & \\ & a & & & & & \\ & & a & & & & \\ & & & b-a & & & \\ & & & & b-a & & \\ & & & & & -(2b+a) \end{pmatrix} . \tag{3.20}$$

This is the first example where the parameterization of ϕ_0 leaves two parameters free. The reason is simply that here we decompose $\mathfrak{su}(6)$ into sufficiently many subalgebras, so that even with the trace condition we still have two free parameters. For this vacuum, the diagonalization of the now 105×105 mass

matrix gives:

$$\text{Massless} \quad (48 \text{ dof}) \tag{3.21}$$

$$M_0^2 = M^2 \ (13 \text{ dof, rep:(8,3,1,1)}) \tag{3.22}$$

$$M_{i\pm}^2 = \frac{1}{2}M^2\left(1 + \frac{g^2}{M^2}\alpha_i^2 \pm \sqrt{1 + 2\frac{g^2}{M^2}\alpha_i^2}\right) \ , \tag{3.23}$$

where

$$\alpha_1^2 = 144b^2 \quad (\text{rep:(1,2,1,1)}) \tag{3.24}$$

$$\alpha_2^2 = 64(a+b)^2 \quad (\text{rep:(3,1,2,1)}) \tag{3.25}$$

$$\alpha_3^2 = 16(2a-b)^2 \quad (\text{rep:(3,2,1,0)}) \ . \tag{3.26}$$

The representations can be associated to the masses simply by using the dimension of each representation and the multiplicity of the masses in the diagonalized mass matrix. We can reach the same conclusion by using the following reasoning. Let us first see why α_1 is associated with the (1,2,1,1) representation. If we take the limit $b \to 0$, ϕ_0 becomes invariant under the group $SU(3) \times SU(3) \times U(1)$. The mass M_1^2 goes to zero, and the masses M_2^2 and M_3^2 become equal. In this case only one non-adjoint representation remains and is a (3,3,1) representation of the gauge $SU(3) \times SU(3) \times U(1)$, generated by the h's and l's in the decomposition (3.19). So we see that the p's generate the mass M_1^2. We can use a similar reasoning in order to associate the correct representations to the masses M_2^2 and M_3^2. We take the limit where $b \to 2a$, so that the mass M_3^2 goes to zero. The masses M_1^2 and M_2^2 become equal. The gauge group in this case is $SU(5) \times U(1)$, and of course the massive states are generated by the l's and p's. We already know that the p's were associated to the mass M_1^2, therefore we know that the l's are associated to the mass M_2^2. Once again, we can calculate the masses of the broken gauge bosons:

$$M_{boson1}^2 = 9g^2b^2 \quad (\text{rep:(1,2,1,1)}), \tag{3.27}$$

$$M_{boson3}^2 = 4g^2(a+b)^2 \quad (\text{rep:(3,1,2,1)}), \tag{3.28}$$

$$M_{boson2}^2 = g^2(b-2a)^2 \quad (\text{rep:(3,2,1,0)}) \ . \tag{3.29}$$

In order to have the (1,2,1,1) representation at a much lower mass than the other massive representations, all we have to do is take $M^2, a^2 \gg b^2$. Indeed, taking $M^2 \gg b^2$ allows for M_{1-}^2 to be very small, while $a^2 \gg b^2$ forces the other masses to be much higher. Just like before, we can then adjust the ratio $9g^2b^2/M^2$ in order to match M_1^2 with the mass of the Higgs boson.

3.1.4 Some insights

Having solved those specific examples, we see a pattern emerge. The masses always have the same generic expressions. The details of the coefficient in front of the vacuum parameter seem to depend on the specific algebra under consideration. This is a clear hint that the expression for the masses may

be found in a more general setting. This will be the main focus of the next part. At this stage of our understanding of the model, we would like to expand the potential around the vacuum and have all the coupling terms involving the scalars around the selected vacuum. We would then be able to compute Feynman diagrams, and understand how the model behaves under one-loop renormalization. This would have to be done using the rotation matrix used to diagonalize the mass matrix. As we saw, the number of scalar degrees of freedom becomes quickly quite significant, and we would have to treat each case separately. Instead we will reserve this step for the next section, when we will have a general approach to the diagonalization problem.

3.2 General treatment

In order to treat the potential on more general grounds, without having to specify the gauge group, we can notice the following point. Remember the expression for the mass matrix (3.37). When putting this expression into matrix form, off-diagonal terms come from the commutators. They mix terms of different directions in the Lie algebra. Notice how all the commutators in the mass matrix are of the form $[\phi_0^i, \ \cdot \ ]$. When studying our examples, we picked the ϕ_0^1 and ϕ_0^2 as the diagonal matrices of the representation of $SU(N)$ to ensure that the two vacua commute, and later restrict to the study of $\phi_0^1 = \phi_0^2$ for simplicity. This can be generalized for any semi-simple Lie algebra by making use of what is called the Weyl-Cartan basis. This basis is characterized by picking the maximal abelian subalgebra, calling it the Cartan subalgebra, and then looking at its action on the rest of the generators, which are referred to as the roots. The action of the Cartan subalgebra on the roots is diagonal, so if we pick the vacuum to be in the Cartan subalgebra the mass matrix will be block diagonal in this basis. From there, the diagonalization follows quite simply, as it only takes place in 3×3 blocks of the matrix. We start by establishing our conventions and reviewing the aforementioned basis. Then we move on to the study of the potential.

3.2.1 Lie Algebra and the Cartan-Weyl basis

We begin by considering a Lie algebra in the Cartan-Weyl basis. Any semi-simple Lie algebra can be put into this form. The algebra decomposes into two parts, the Cartan subalgebra and the complementary space of root vectors. The commutation relations take the form

$$\begin{aligned} [H_i, H_j] = 0 \ , \ [H_i, E_\alpha] = \alpha_i E_\alpha \ , \\ [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \ , \quad [E_\alpha, E_{-\alpha}] = \sum_i \alpha_i H_i \ . \end{aligned} \tag{3.30}$$

In the third equation, $N_{\alpha\beta} = 0$ if $\alpha+\beta$ is not a root. This can be shown by acting on the equation with a generator of the Cartan subalgebra and then using the Jacobi identity. This computation can also be used to prove the last equation,

up to some choice of normalization for the H_i. Additionally, and without loss of generality, we can assume we are working with some matrix representation of the algebra such that

$$\begin{aligned} &([H_i, E_\alpha])^\dagger = (\alpha_i E_\alpha)^\dagger \\ \Leftrightarrow \quad &-[H_i^\dagger, E_\alpha^\dagger] = \alpha_i^* E_\alpha^\dagger \\ \Leftrightarrow \quad &[H_i^\dagger, E_{-\alpha}] = -\alpha_i^* E_{-\alpha} \ . \end{aligned} \tag{3.31}$$

where we chose $E_\alpha^\dagger = E_{-\alpha}$. This in turns imply that if H_i is hermitian then the α's are real, and if H_i is antihermitian then the α's are pure imaginary. We choose the latter for the rest of the proof. This is our reality condition that will ensure that the potential is real. Similarly, we can fix our reality condition for the other commutators to be

$$\begin{aligned} [E_\alpha, E_\beta]^\dagger &= (N_{\alpha\beta}\ E_{\alpha+\beta})^\dagger \\ -[E_{-\alpha}, E_{-\beta}] &= N_{\alpha\beta}^*\ E_{-\alpha-\beta} \\ [E_{-\alpha}, E_{-\beta}] &= -N_{\alpha\beta}^*\ E_{-\alpha-\beta} \end{aligned} \tag{3.32}$$

where the step from the first to the second line is made by using $E_\alpha^\dagger = E_{-\alpha}$. This implies the relation $N_{\alpha\beta}^* = -N_{(-\alpha)(-\beta)}$. Lastly, we assume the existence of a Trace on the representation, matching the Killing form, with the normalization

$$\mathrm{Tr}\,(E_\alpha E_\beta) = \mathcal{C}\delta_{\alpha+\beta,0}\ , \tag{3.33}$$

$$\mathrm{Tr}\,(H_i H_j) = \mathcal{C}\delta_{ij}\ . \tag{3.34}$$

We also have the consistency condition

$$\mathrm{Tr}\,([E_\alpha, E_\beta\ ]E_\gamma) = \mathcal{C}N_{\alpha\beta}\ , \tag{3.35}$$

where we define $\gamma = -(\alpha+\beta)$. Using the fact that the trace is invariant by cyclic permutations it implies the relation

$$N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha}\ . \tag{3.36}$$

3.3 The Mass Matrix

Our starting point is the mass matrix for the potential found in (3.37). In order to limit the factors of i we write it in the antihermitian convention

$$\begin{aligned} \frac{\delta^2\mathcal{V}}{2M^2} = \mathrm{Tr}(\delta\phi^3)^2 - \frac{g}{M}\mathrm{Tr}\left(\left[\delta\phi^1, \phi_0^2\right]\delta\phi^3 + \left[\phi_0^1, \delta\phi^2\right]\delta\phi^3\right) \\ + \frac{g^2}{M^2}\mathrm{Tr}\Big(\left[\delta\phi^1, \phi_0^2\right]^2 + \left[\delta\phi^2, \phi_0^1\right]^2 + \left[\delta\phi^3, \phi_0^1\right]^2 + \left[\delta\phi^3, \phi_0^2\right]^2 \\ + 2\left[\delta\phi^1, \phi_0^2\right]\left[\phi_0^1, \delta\phi^2\right]\Big)\ , \end{aligned} \tag{3.37}$$

Since the potential is real, the fields must also obey some reality conditions:

$$\begin{aligned} \left(\delta\phi^I\right)^\dagger &= -\delta\phi^I \\ \Leftrightarrow \left(\delta\phi^{Ii}\right)^* H_i^\dagger + \left(\delta\phi^{I\alpha}\right)^* E_\alpha^\dagger &= -\delta\phi^I \\ \Leftrightarrow -\left(\delta\phi^{Ii}\right)^* H_i + \left(\delta\phi^{I\alpha}\right)^* E_{-\alpha} &= -\delta\phi^I \end{aligned} \tag{3.38}$$

which implies

$$\left(\delta\phi^{Ii}\right)^* = \delta\phi^{Ii} \text{ and } \left(\delta\phi^{I\alpha}\right)^* = -\delta\phi^{I(-\alpha)} \tag{3.39}$$

We now develop the fields using the basis of 3.2.1. The vacuum condition for the fields $\phi_0^{1,2}$ can be fulfilled if we take them to be in the Cartan subalgebra:

$$\begin{aligned} \delta\phi^I &= \delta\phi^{Ii} H_i + \delta\phi^{I\alpha} E_\alpha \\ \phi_0^{1,2} &= \phi_0^{1,2i} H_i \ . \end{aligned} \tag{3.40}$$

In this basis, the mass matrix reads

$$\begin{aligned} \frac{\delta^2\mathcal{V}}{2M^2} =& \Big(\delta\phi^{3\alpha}\delta\phi^{3\beta} - 2\frac{g}{M}\left(-\alpha_i\phi_0^{2i}\delta\phi^{1\alpha}\delta\phi^{3\beta} + \alpha_i\phi_0^{1i}\delta\phi^{2\alpha}\delta\phi^{3\beta}\right) \\ &+ \frac{g^2}{M^2}\Big(\alpha_i\phi_0^{2i}\beta_j\phi_0^{2j}\delta\phi^{1\alpha}\delta\phi^{1\beta} + \alpha_i\phi_0^{1i}\beta_j\phi_0^{1j}\delta\phi^{2\alpha}\delta\phi^{2\beta} \\ &+ \alpha_i\phi_0^{1i}\beta_j\phi_0^{1j}\delta\phi^{3\alpha}\delta\phi^{3\beta} + \alpha_i\phi_0^{2i}\beta_j\phi_0^{2j}\delta\phi^{3\alpha}\delta\phi^{3\beta} \\ &- 2\alpha_i\phi_0^{1i}\beta_j\phi_0^{2j}\delta\phi^{2\alpha}\delta\phi^{1\beta}\Big)\Big)\mathcal{C}\delta_{\alpha+\beta,0} + \left(\delta\phi^{3i}\delta\phi^{3j}\right)\mathcal{C}\delta_{ij} \end{aligned} \tag{3.41}$$

In order not to clutter the notation, we now write

$$b_I^{(\alpha)} \equiv \frac{g}{M}\alpha_i\phi_0^{Ii} \ . \tag{3.42}$$

We rewrite the mass term as a the scalar product

$$\frac{\delta^2\mathcal{V}}{2M^2} = (\delta\phi)^\dagger \mathcal{M}\delta\phi \tag{3.43}$$

and define $\delta\phi^{tr} = \left(\delta\phi^{1\alpha_1}, \delta\phi^{2\alpha_1}, \ldots, \delta\phi^{3\alpha_n}, \ldots, \delta\phi^{1i}, \ldots\right)$ such that the α_n are organized by pairs of opposite roots. The matrix $\mathcal{M}$ takes the form

$$\mathcal{M} = \begin{pmatrix} \mathcal{C}A_{\alpha_1} & & & & & \\ & \mathcal{C}A_{\alpha_1}^* & & & & \\ & & \ddots & & & \\ & & & \mathcal{C}\mathbb{J}_i & & \\ & & & & \ddots & \end{pmatrix} \tag{3.44}$$

where A_α is a 3×3 matrix. The first dots represent the other roots (with the same structure). The matrix $\mathbb{J}_i$ is also a 3×3 matrix along the H_i direction and takes the form

$$\mathbb{J}_i = E_{33} \tag{3.45}$$

with E_{ij} the canonical basis of 3×3 matrices. Using the fact that $b_I^{(-\alpha)} = -b_I^{(\alpha)}$, we find the expression for the matrix A_α

$$A_\alpha = \begin{pmatrix} -(b_2^\alpha)^2 & b_1^\alpha b_2^\alpha & b_2^\alpha \\ b_1^\alpha b_2^\alpha & -(b_1^\alpha)^2 & -b_1^\alpha \\ -b_2^\alpha & b_1^\alpha & 1-(b_1^\alpha)^2-(b_2^\alpha)^2 \end{pmatrix} . \tag{3.46}$$

As expected, $\mathcal{M}$ is hermitian (since $b_I^{(\alpha)}$ is pure imaginary). This matrix can be diagonalized with eigenvalues

$$0\ ,\ \frac{1}{2}\left(1-2\big((b_1^\alpha)^2+(b_2^\alpha)^2\big) \pm \sqrt{1-4\big((b_1^\alpha)^2+(b_2^\alpha)^2\big)}\right) . \tag{3.47}$$

Writing explicitly the imaginary unit in the roots α, we recover the expressions for the masses derived in section 3.1:

$$0\ ,\ (m_\alpha^\pm)^2 = \frac{1}{2}\left(1+2\big((b_1^\alpha)^2+(b_2^\alpha)^2\big) \pm \sqrt{1+4\big((b_1^\alpha)^2+(b_2^\alpha)^2\big)}\right) . \tag{3.48}$$

where this expression is to be understood with $\alpha_i \in \mathbb{R}$, which we will continue to use for the rest of this section.

The diagonalization of A_α can be done explicitly. We define $\mathcal{P}$ such that

$$A_\alpha = \mathcal{P}_\alpha^\dagger D_\alpha \mathcal{P}_\alpha \quad \text{and } P_\alpha^\dagger P_\alpha = P_\alpha P_\alpha^\dagger = \mathbb{I}, \tag{3.49}$$

with

$$\mathcal{D}_\alpha = \begin{pmatrix} (m_\alpha^-)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (m_\alpha^+)^2 \end{pmatrix} , \tag{3.50}$$

$$\mathcal{P}_\alpha = \begin{pmatrix} -i\cos(\theta)\cos(\nu) & i\cos(\theta)\sin(\nu) & \sin(\theta) \\ i\sin(\nu) & i\cos(\nu) & 0 \\ i\sin(\theta)\cos(\nu) & -i\sin(\theta)\sin(\nu) & \cos(\theta) \end{pmatrix} . \tag{3.51}$$

The sines and cosines are expressed in terms of the vacuum parameters as

$$\cos(\nu) = \frac{1}{\sqrt{\left|\frac{b_1}{b_2}\right|^2+1}} , \tag{3.52}$$

$$\cos(\theta) = \sqrt{\frac{1+2b_1^2+2b_2^2+\sqrt{1+4b_1^2+4b_2^2}}{1+4b_1^2+4b_2^2+\sqrt{1+4b_1^2+4b_2^2}}} . \tag{3.53}$$

We drop the α index for clarity, but it should be emphasized that the angles depend explicitly on the root under consideration.

Lastly, we will make use in the following of two simple relation for the matrices A_α and $\mathcal{P}_\alpha$

$$A_{-\alpha} = A_\alpha^* \text{ and } \mathcal{P}_{-\alpha} = (\mathcal{P}_\alpha)^* . \tag{3.54}$$

Our analysis shows that the diagonalization of the mass matrix can be done in general given the appropriate basis for the Lie algebra. The resulting masses are given in (3.48) for a given root vector. Along this direction, two fields become massive with masses $m_{\pm}^2$, while one remain massless. This massless degree of freedom is actually the one used for the longitudinal polarization of the associated broken gauge boson. To see this, consider the mass term coming from the coupling of the scalars to the gauge, and write it the Cartan-Weyl basis:

$$\begin{aligned} g^2 \mathrm{Tr}\left([\phi_0^I, A_\mu]^2\right) =& g^2 \eta^{\mu\nu} \phi_0^{Ii} A_\mu^\alpha \phi_0^{Ij} A_\nu^\beta \mathrm{Tr}\left([H_i, E_\alpha][H_j, E_\beta]\right) \\ =& g^2 \mathcal{C} \eta^{\mu\nu} \alpha_i \phi_0^{Ii} A_\mu^\alpha \beta_j \phi_0^{Ij} A_\nu^\beta \delta_{\alpha+\beta,0} \\ =& M^2 \mathcal{C} \eta^{\mu\nu} b_I^{(\alpha)} b_I^{(\beta)} A_\mu^\alpha A_\nu^\beta \delta_{\alpha+\beta,0} \ . \end{aligned} \tag{3.55}$$

Therefore the masse for the boson in the α direction is

$$(m_{gauge}^\alpha)^2 = M^2 b_I^{(\alpha)} b_I^{(\alpha)} \ . \tag{3.56}$$

If the direction is left untouched by the vacuum, we have $b_I^\alpha = 0$ and the boson remains massless. For non zero b_I^α, the boson becomes massive, and the longitudinal polarization is given by the massless scalar in the same direction, making it effectively drop out of the spectrum.

So for each root α with non-zero b_I^α, we have two massive scalar degrees of freedom, and a massive broken gauge boson. On the other hand, the degrees of freedom untouched by the vacuum, meaning the roots α such that $b_I^\alpha = 0$ and the H_i directions of the Cartan subalgebra, still have each two massless degrees of freedom coming directly from ϕ^1 and ϕ^2, and a massive degree of freedom of mass $M^2 = \mathtt{f}^2$ coming from ϕ^3. The results are summed up in Table 3.1.

	Spin	Mass^2	Representation of $G_{residual}$	Generators in $\mathfrak{g}$
ϕ^1	0	0	Adjoint	H_i and E_α $(b_I^\alpha = 0)$
ϕ^2	0	0	Adjoint	H_i and E_α $(b_I^\alpha = 0)$
ρ	0	$\mathtt{f}^2$	Adjoint	H_i and E_α $(b_I^\alpha = 0)$
H_1	0	$(m_\alpha^-)^2$	Depends on breaking	E_α $(b_I^\alpha \neq 0)$
H_2	0	$(m_\alpha^+)^2$	Depends on breaking	E_α $(b_I^\alpha \neq 0)$
A_μ	1	0	Adjoint (gauge fields)	H_i and E_α $(b_I^\alpha = 0)$
X_μ	1	$(m_{gauge}^\alpha)^2$	Depends on breaking	E_α $(b_I^\alpha \neq 0)$

Table 3.1: List of the different types of fields after symmetry breaking. $G_{residual}$ is the new gauge group after the symmetry breaking, and $\mathfrak{g}$ denotes the Lie algebra associated with the original gauge group G. There may be numerous fields H_1, H_2 and X_μ, the number of degree of freedom in a representation depends on the number of directions with identical $(b_I^{(\alpha)})^2$. So we could have several H_1-type fields, with different masses $(m_{\alpha_1}^-)^2$, $(m_{\alpha_2}^-)^2, \ldots$ and in different representations.

This is our theory after symmetry breaking at the classical level. The generic features, independent of the gauge, are the adjoint scalar fields, which were

expected to appear because of the adjoint scalars they come from. We have the H^- fields whose masses go to zero when the vacuum parameters are taken to be small, the H^+ fields whose masses go to $M^2 = \mathtt{f}^2$ in the same limit, and the broken boson X_μ. A natural question at this point is how is the mass hierarchy modified when loop effects are considered. To answer this, we have to know the potential in the broken phase of the theory, so we now turn our attention to this issue.

3.4 The coupling terms

The study of the different coupling terms will be split into two parts. First we will study the scalar potential, treating first the trilinear term then the quartic one. Once done we will move on to the study of the coupling terms with the broken gauge boson, where we will follow the same approach. Our goal is to organize the terms in such a way that a systematic approach for any gauge group is possible, simplifying expressions in order to make practical computations as straight forward as possible.

3.4.1 The Trilinear term

Having found the mass basis for the system, we can now write the potential in the new basis using the matrix $\mathcal{P}_\alpha$. We start with the trilinear term given in (3.2):

$$\begin{aligned}\mathrm{Tr}([\phi^1,\phi^2]\phi^3) =& \frac{\epsilon_{IJK}}{3!}\mathrm{Tr}\left([\phi^I,\phi^J]\phi^K\right)\\ =& \frac{\epsilon_{IJK}}{3!}\Big(\phi^{I\alpha}\phi^{J\beta}\phi^{K\gamma}\mathrm{Tr}\left([E_\alpha,E_\beta]E_\gamma\right)\\ &+\phi^{I\alpha}\phi^{J\beta}\phi^{Ki}\mathrm{Tr}\left([E_\alpha,E_\beta]H_i\right)\\ &+\phi^{Ii}\phi^{J\beta}\phi^{K\gamma}\mathrm{Tr}\left([H_i,E_\beta]E_\gamma\right)\\ &+\phi^{I\alpha}\phi^{Ji}\phi^{K\gamma}\mathrm{Tr}\left([E_\alpha,H_i]E_\gamma\right)\Big)\end{aligned} \tag{3.57}$$

The last three terms take similar expressions,

$$\begin{aligned}\mathrm{Tr}([\phi^1,\phi^2]\phi^3) =& \; N_{\alpha\beta}\frac{\epsilon_{IJK}}{3!}\phi^{I\alpha}\phi^{J\beta}\phi^{K\gamma}\delta_{\alpha+\beta+\gamma,0}\\ &+3i\alpha_i\frac{\epsilon_{IJK}}{3!}\phi^{Ii}\phi^{J\alpha}\phi^{K\beta}\delta_{\alpha+\beta,0}\end{aligned} \tag{3.58}$$

where $\alpha_i \in \mathbb{R}$ as already mentioned. We now use the fact that

$$\phi^{I\alpha} = (P^\dagger_\alpha)^I{}_{I'}\tilde{\phi}^{I'\alpha}\ , \tag{3.59}$$

and define the tensor

$$\mathcal{T}_{I'\alpha,J'\beta,K'\gamma} \equiv \epsilon_{IJK}(P^\dagger_\alpha)^I{}_{I'}(P^\dagger_\beta)^J{}_{J'}(P^\dagger_\gamma)^K{}_{K'}\ . \tag{3.60}$$

In order to keep expressions compact, we can extend the definition of the above tensor to the Cartan subalgebra by adding the condition $P_i^\dagger = \mathbb{I}$ for i a direction in the Cartan subalgebra (along H_i). So the trilinear term is

$$\begin{aligned}\mathrm{Tr}([\phi^1,\phi^2]\phi^3) = \frac{1}{3!}\, & N_{\alpha\beta}\mathcal{T}_{I\alpha,J\beta,K\gamma}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\beta}\tilde{\phi}^{K\gamma}\delta_{\alpha+\beta+\gamma,0} \\ & + \frac{1}{2}i\alpha_i\mathcal{T}_{I\alpha,J\beta,Ki}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\beta}\tilde{\phi}^{Ki}\delta_{\alpha+\beta,0} \end{aligned} \tag{3.61}$$

The second line can be further simplified. The sum is over all the roots $(\alpha, -\alpha, \dots)$ and β is fixed to be the opposite of α, so we have the terms

$$\alpha_i\mathcal{T}_{I\alpha,J(-\alpha),Ki}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J(-\alpha)} \text{ and } \quad -\alpha_i\mathcal{T}_{I(-\alpha),J\alpha,Ki}\tilde{\phi}^{I(-\alpha)}\tilde{\phi}^{J\alpha}\ . \tag{3.62}$$

Taking the expression for $\mathcal{T}$, we observe

$$\begin{aligned}\mathcal{T}_{I(-\alpha),J\alpha,Ki} &= \frac{1}{3!}\epsilon_{I'J'K}(P_{-\alpha}^\dagger)^{I'}{}_I(P_\alpha^\dagger)^{J'}{}_J \\ &= -\frac{1}{3!}\epsilon_{J'I'K}(P_\alpha^\dagger)^{J'}{}_J(P_{-\alpha}^\dagger)^{I'}{}_I \\ &= -\mathcal{T}_{J\alpha,I(-\alpha),Ki}\ . \end{aligned} \tag{3.63}$$

And so, up to a change in the I, J indices, we see that those two terms are actually the same. The second line takes the simple form

$$-\sum_{\alpha^+} i\alpha_i\mathcal{T}_{I\alpha,J(-\alpha),Ki}\tilde{\phi}^{I\alpha}(\tilde{\phi}^{J\alpha})^*\tilde{\phi}^{Ki} \tag{3.64}$$

where the sum is taken over each positive root. Now we are left with the task of actually computing the tensor $\mathcal{T}$. We first note

$$\begin{aligned}\mathcal{T}_{I\alpha,J(-\alpha),Ki} =& \epsilon_{I'J'K'}(P_\alpha^\dagger)^{I'}{}_I(P_{-\alpha}^\dagger)^{J'}{}_J\mathbb{I}^{K'}_K \\ =& \epsilon_{I'J'K}(P_\alpha^\dagger)^{I'}{}_I(P_\alpha{}^*)^{\dagger J'}_J \\ =& \epsilon_{I'J'K}(P_\alpha^\dagger)^{I'}{}_I(P_\alpha^T)^{J'}{}_J \\ \equiv& \mathcal{T}^{\alpha i}_{IJK} \end{aligned} \tag{3.65}$$

where the last line is just to make the notation more evident. The non-zero entries for the tensor $\mathcal{T}$ are

$$\begin{aligned}
\mathcal{T}^{\alpha i}_{323} &= \sin(\theta) \\
\mathcal{T}^{\alpha i}_{233} &= \sin(\theta) \\
\mathcal{T}^{\alpha i}_{213} &= -i\cos(\theta) \\
\mathcal{T}^{\alpha i}_{123} &= i\cos(\theta) \\
\mathcal{T}^{\alpha i}_{111} &= \sin(2\theta)\sin(\nu) \\
\mathcal{T}^{\alpha i}_{212} &= \sin(\theta)\sin(\nu) \\
\mathcal{T}^{\alpha i}_{122} &= \sin(\theta)\sin(\nu) \\
\mathcal{T}^{\alpha i}_{321} &= -i\cos(\theta)\cos(\nu) \\
\mathcal{T}^{\alpha i}_{231} &= i\cos(\theta)\cos(\nu) \\
\mathcal{T}^{\alpha i}_{312} &= i\cos(2\theta)\cos(\nu) \\
\mathcal{T}^{\alpha i}_{132} &= -i\cos(2\theta)\cos(\nu) \\
\mathcal{T}^{\alpha i}_{211} &= -\cos(\nu)\sin(\theta) \\
\mathcal{T}^{\alpha i}_{311} &= i\cos(2\theta)\sin(\nu) \\
\mathcal{T}^{\alpha i}_{121} &= -\cos(\nu)\sin(\theta) \\
\mathcal{T}^{\alpha i}_{131} &= -i\cos(2\theta)\sin(\nu) \\
\mathcal{T}^{\alpha i}_{331} &= -2\cos(\theta)\sin(\theta)\sin(\nu) \\
\mathcal{T}^{\alpha i}_{112} &= \cos(\nu)\sin(2\theta) \\
\mathcal{T}^{\alpha i}_{322} &= i\cos(\theta)\sin(\nu) \\
\mathcal{T}^{\alpha i}_{232} &= -i\cos(\theta)\sin(\nu) \\
\mathcal{T}^{\alpha i}_{332} &= -2\cos(\theta)\cos(\nu)\sin(\theta)\ .
\end{aligned} \tag{3.66}$$

Of course, the dependence on α is through the angles θ and ν as defined in (3.52). We now turn to terms coupling the roots with each other. We start by studying the trilinear term

$$\frac{1}{3!}\ N_{\alpha\beta}\mathcal{T}_{I\alpha,J\beta,K\gamma}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\beta}\tilde{\phi}^{K\gamma} \tag{3.67}$$

Let us fix three roots α, β and γ such that

$$\gamma = -(\alpha+\beta) \tag{3.68}$$

First there is also a term

$$\frac{1}{3!}\ N_{\beta\alpha}\mathcal{T}_{I\beta,J\alpha,K\gamma}\tilde{\phi}^{I\beta}\tilde{\phi}^{J\alpha}\tilde{\phi}^{K\gamma} \tag{3.69}$$

Using the properties $N_{\alpha\beta} = -N_{\beta\alpha}$ and $\mathcal{T}_{I\beta,J\alpha,K\gamma} = -\mathcal{T}_{J\alpha,I\beta,K\gamma}$, we see that those two terms are the same (upon renaming the dummy indices $I \leftrightarrow J$). So

we can further simplify the expression by adding a factor of 2 and counting pairs of roots only once.

In the sum there is an additional term involving the opposite roots.

$$\begin{aligned}&\frac{1}{3}\, N_{(-\alpha)(-\beta)}\mathcal{T}_{I(-\alpha),J(-\beta),K-(\gamma)}\tilde{\phi}^{I(-\alpha)}\tilde{\phi}^{J(-\beta)}\tilde{\phi}^{K(-\gamma)}\\ &\quad=\frac{1}{3}\, N^*_{\alpha\beta}\mathcal{T}^*_{I\alpha,J\beta,K\gamma}(\tilde{\phi}^{I\alpha})^*(\tilde{\phi}^{J\beta})^*(\tilde{\phi}^{K\gamma})^*\end{aligned} \tag{3.70}$$

which is the complex conjugate of the first term. Additionally we can invert the relation between the roots to obtain additional terms involving those three roots (and their opposite)

$$\beta=-(\alpha+\gamma) \tag{3.71}$$

$$\alpha=-(\beta+\gamma) \tag{3.72}$$

All in all, for three fixed roots, the related terms are

$$\begin{aligned}&\frac{1}{3}\, N_{\alpha\beta}\mathcal{T}_{I\alpha,J\beta,K\gamma}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\beta}\tilde{\phi}^{K\gamma}+\text{c.c.}\\ +&\frac{1}{3}\, N_{\gamma\alpha}\mathcal{T}_{I\gamma,J\alpha,K\beta}\tilde{\phi}^{I\gamma}\tilde{\phi}^{J\alpha}\tilde{\phi}^{K\beta}+\text{c.c.}\\ +&\frac{1}{3}\, N_{\beta\gamma}\mathcal{T}_{I\beta,J\gamma,K\alpha}\tilde{\phi}^{I\gamma}\tilde{\phi}^{J\beta}\tilde{\phi}^{K\alpha}+\text{c.c.}\end{aligned} \tag{3.73}$$

Now we can use the fact that $N_{\alpha\beta}=N_{\gamma\alpha}=N_{\beta\gamma}$ from section 3.2.1. Relabeling the I,J,K indices and using the fact that $\mathcal{T}$ is invariant under cyclic permutations, we arrive at the simple term

$$N_{\alpha\beta}\mathcal{T}_{I\alpha,J\beta,K\gamma}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\beta}\tilde{\phi}^{K\gamma}+c.c. \tag{3.74}$$

where we have to consider one triplet for the roots only once. The trilinear terms are then simply written

$$\begin{aligned}\mathrm{Tr}([\phi^1,\phi^2]\phi^3)=&-\sum_{\alpha^+} i\alpha_i\mathcal{T}_{I\alpha,J(-\alpha),Ki}\tilde{\phi}^{I\alpha}(\tilde{\phi}^{J\alpha})^*\tilde{\phi}^{Ki}\\ &+\sum_{\substack{\text{pairs }\alpha,\beta,\\ \gamma=-(\alpha+\beta)}} N_{\alpha\beta}\mathcal{T}_{I\alpha,J\beta,K\gamma}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\beta}\tilde{\phi}^{K\gamma}+c.c.\end{aligned} \tag{3.75}$$

3.4.2 Quartic terms

Proceeding similarly, we can compute the quartic term of the scalar potential:

$$\begin{aligned}\mathrm{Tr}([\phi^I,\phi^J]^2) &= \phi^{I\alpha}\phi^{J\beta}\phi^{I\gamma}\phi^{J\delta}\mathrm{Tr}([E_\alpha,E_\beta][E_\gamma,E_\delta]) \\ &\quad + 2\phi^{Ii}\phi^{J\beta}\phi^{I\gamma}\phi^{J\delta}\mathrm{Tr}([H_i,E_\beta][E_\gamma,E_\delta]) \\ &\quad + 2\phi^{I\alpha}\phi^{Ji}\phi^{I\gamma}\phi^{J\delta}\mathrm{Tr}([E_\alpha,H_i][E_\gamma,E_\delta]) \\ &\quad + 2\phi^{Ii}\phi^{J\beta}\phi^{Ij}\phi^{J\delta}\mathrm{Tr}([H_i,E_\beta][H_j,E_\delta]) \\ &\quad + 2\phi^{I\alpha}\phi^{Ji}\phi^{Ij}\phi^{J\delta}\mathrm{Tr}([E_\alpha,H_i][H_j,E_\delta]) \\ &= \; N_{\alpha\beta}N_{\gamma\delta}\phi^{I\alpha}\phi^{J\beta}\phi^{I\gamma}\phi^{J\delta}\delta(\alpha+\beta+\gamma+\delta) \\ &\quad + 2i^2\alpha_i\delta_j\left(\phi^{Ii}\phi^{J\alpha}-\phi^{I\alpha}\phi^{Ji}\right)\phi^{Ij}\phi^{J\delta}\delta_{\alpha+\delta,0}\ . \end{aligned} \tag{3.76}$$

We use the same trick as before to further simplify the second term so that the sum is only on the positive roots.

$$\sum_{\alpha^+} 4\alpha_i\alpha_j\phi^{Ii}\phi^{Jj}\phi^{I\alpha}(\phi^{J\alpha})^* - 4\alpha_i\alpha_j\phi^{Ii}\phi^{Ij}\phi^{J\alpha}(\phi^{J\alpha})^* \tag{3.77}$$

We now transform the scalars in the eigen-mass basis.

$$\begin{aligned}&\sum_{\alpha^+} 4\alpha_i\alpha_j(P^\dagger_\alpha)_{II'}(P^\dagger_\alpha)^*_{JJ'}\tilde\phi^{Ii}\tilde\phi^{Jj}\tilde\phi^{I'\alpha}(\tilde\phi^{J'\alpha})^* \\ &\qquad\qquad - 4\alpha_i\alpha_j(P^\dagger_\alpha)^J{}_{J'}(P^\dagger_\alpha)^{*J}{}_K\tilde\phi^{Ii}\tilde\phi^{Ij}\tilde\phi^{J'\alpha}(\tilde\phi^{K\alpha})^*\end{aligned} \tag{3.78}$$

Notice the first term uses the product $P^\dagger_\alpha(P^\dagger_\alpha)^{*T}$, which is of course the identity by unicity of P_α. Hence the term simplifies

$$\sum_{\alpha^+} 4\alpha_i\alpha_j\mathcal{Q}_{IJI'J'}\tilde\phi^{Ii}\tilde\phi^{Jj}\tilde\phi^{I'\alpha}(\tilde\phi^{J'\alpha})^* - 4\alpha_i\alpha_j\tilde\phi^{Ii}\tilde\phi^{Ij}\tilde\phi^{J\alpha}(\tilde\phi^{J\alpha})^* \tag{3.79}$$

where we define

$$\mathcal{Q}_{IJI'J'} \equiv (P^\dagger_\alpha)_{II'}(P^\dagger_\alpha)^*_{JJ'} \tag{3.80}$$

The entries relevant for mass renormalization are

$$\begin{aligned}\mathcal{Q}_{3322} &= 0 \\ \mathcal{Q}_{3311} &= \sin^2(\theta) \\ \mathcal{Q}_{3333} &= \cos^2(\theta) \\ \mathcal{Q}_{1122} &= \sin^2(\nu) \\ \mathcal{Q}_{2222} &= \cos^2(\nu) \\ \mathcal{Q}_{2233} &= \sin^2(\theta)\sin^2(\nu) \\ \mathcal{Q}_{1111} &= \cos^2(\theta)\cos^2(\nu) \\ \mathcal{Q}_{1133} &= \cos^2(\nu)\sin^2(\theta) \\ \mathcal{Q}_{2211} &= \cos^2(\theta)\sin^2(\nu)\end{aligned} \tag{3.81}$$

Let us now study the term involving only roots

$$N_{\alpha\beta}N_{\gamma\delta}\phi^{I\alpha}\phi^{J\beta}\phi^{I\gamma}\phi^{J\delta} \tag{3.82}$$

We fix the roots such that

$$\alpha+\beta=-(\gamma+\delta) \tag{3.83}$$

and $\alpha+\beta$ is a root (and therefore $\gamma+\delta$ too). Let us start with the term involving the same two roots for both commutator. This means we are taking $\gamma=\alpha$ and $\delta=\beta$ or $\gamma=\beta$ and $\gamma=\alpha$. Listing the terms give the following sum:

$$2N_{\alpha\beta}N_{-\alpha-\beta}\phi^{I\alpha}\phi^{J\beta}(\phi^{I\alpha})^*(\phi^{J\beta})^*+2N_{\alpha\beta}N_{-\beta-\alpha}\phi^{I\alpha}\phi^{J\beta}(\phi^{I\beta})^*(\phi^{J\alpha})^* \tag{3.84}$$

where the factor of 2 comes from taking $\alpha\leftrightarrow-\alpha$, $\beta\leftrightarrow-\beta$ in the first term and $\alpha\leftrightarrow-\beta$, $\beta\leftrightarrow-\alpha$ in the second one. An additional factor of 2 comes from taking the symmetry $\alpha\leftrightarrow\beta$ in both terms and relabeling the dummy indices I,J. We have

$$-4|N_{\alpha\beta}|^2\phi^{I\alpha}\phi^{J\beta}(\phi^{I\alpha})^*(\phi^{J\beta})^*+4|N_{\alpha\beta}|^2\phi^{I\alpha}\phi^{J\beta}(\phi^{I\beta})^*(\phi^{J\alpha})^*\ . \tag{3.85}$$

Transforming the fields in the mass basis gives

$$-4|N_{\alpha\beta}|^2\Big(\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\beta}(\tilde{\phi}^{I\alpha})^*(\tilde{\phi}^{J\beta})^*-\mathcal{R}_{I'\alpha,K\beta,J'\beta,L\alpha}\tilde{\phi}^{I'\alpha}(\tilde{\phi}^{K\beta})^*\tilde{\phi}^{J'\beta}(\tilde{\phi}^{L\alpha})^*\Big)\ . \tag{3.86}$$

and we define

$$\begin{aligned}\mathcal{R}_{I'\alpha,K\beta,J'\beta,L\alpha}=&(P^\dagger_\alpha)^I{}_{I'}(P^{\dagger *}_\beta)^I{}_K(P^\dagger_\beta)^J{}_{J'}(P^{\dagger *}_\alpha)^J{}_L\\ =&(P_\beta P^\dagger_\alpha)_{KI'}(P_\alpha P^\dagger_\beta)_{LJ'}\end{aligned} \tag{3.87}$$

Now we can consider how the terms simplify in the case where γ and δ are different roots whose sum give the opposite of $\alpha+\beta$. We can use symmetries again to regroup similar terms.

$$\begin{aligned}&N_{\alpha\beta}N_{\gamma\delta}\phi^{I\alpha}\phi^{J\beta}\phi^{I\gamma}\phi^{J\delta}+N_{\alpha\beta}N_{\delta\gamma}\phi^{I\alpha}\phi^{J\beta}\phi^{I\delta}\phi^{J\gamma}\\ &N_{\beta\alpha}N_{\gamma\delta}\phi^{I\beta}\phi^{J\alpha}\phi^{I\gamma}\phi^{J\delta}+N_{\beta\alpha}N_{\delta\gamma}\phi^{I\beta}\phi^{J\alpha}\phi^{I\delta}\phi^{J\gamma}\ .\end{aligned} \tag{3.88}$$

Using the symmetries and relabeling the indices we obtain

$$2N_{\alpha\beta}N_{\gamma\delta}\phi^{I\alpha}\phi^{J\beta}\phi^{I\gamma}\phi^{J\delta}-2N_{\alpha\beta}N_{\gamma\delta}\phi^{I\alpha}\phi^{J\beta}\phi^{J\gamma}\phi^{I\delta} \tag{3.89}$$

An additional factor of 2 comes from exchanging the two pairs in each term:

$$4N_{\alpha\beta}N_{\gamma\delta}\phi^{I\alpha}\phi^{J\beta}\phi^{I\gamma}\phi^{J\delta}-4N_{\alpha\beta}N_{\gamma\delta}\phi^{I\alpha}\phi^{J\beta}\phi^{J\gamma}\phi^{I\delta}+\text{c.c}\ . \tag{3.90}$$

The sum is over the pairs of roots such that their sum is opposite. Rotating the basis to have mass eigenstates, we obtain

$$\begin{aligned}&4N_{\alpha\beta}N_{\gamma\delta}\Big(\mathcal{S}_{I\alpha,J\gamma,K\beta,L\delta}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\gamma}\tilde{\phi}^{K\beta}\tilde{\phi}^{L\delta}\\ &-\mathcal{S}_{I\alpha,J\delta,K\beta,L\gamma}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\delta}\tilde{\phi}^{K\beta}\tilde{\phi}^{L\gamma}\Big)+\text{c.c}\ .\end{aligned} \tag{3.91}$$

defining the tensor $\mathcal{S}$ as

$$\mathcal{S}_{I\alpha,J\gamma,K\beta,L\delta} = (P^\dagger_\alpha)^{I'}{}_I (P^\dagger_\gamma)^{I'}{}_J (P^\dagger_\beta)^{J'}{}_K (P^\dagger_\delta)^{J'}{}_L \tag{3.92}$$

Putting everything together, we obtain the quartic term

$$\begin{aligned}
&\mathrm{Tr}([\phi^I,\phi^J]^2) = \\
&-4\sum_{\alpha^+}\Big(\alpha_i\alpha_j\tilde{\phi}^{Ii}\tilde{\phi}^{Ij}\tilde{\phi}^{J\alpha}(\tilde{\phi}^{J\alpha})^* - \alpha_i\alpha_j\mathcal{Q}_{IJI'J'}\tilde{\phi}^{Ii}\tilde{\phi}^{Jj}\tilde{\phi}^{I'\alpha}(\tilde{\phi}^{J'\alpha})^*\Big) \\
&-4|N_{\alpha\beta}|^2\Big(\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\beta}(\tilde{\phi}^{I\alpha})^*(\tilde{\phi}^{J\beta})^* - \mathcal{R}_{I'\alpha,K\beta,J'\beta,L\alpha}\tilde{\phi}^{I'\alpha}(\tilde{\phi}^{K\beta})^*\tilde{\phi}^{J'\beta}(\tilde{\phi}^{L\alpha})^*\Big) \\
&+4N_{\alpha\beta}N_{\gamma\delta}\Big(\mathcal{S}_{I\alpha,J\gamma,K\beta,L\delta}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\gamma}\tilde{\phi}^{K\beta}\tilde{\phi}^{L\delta} - \mathcal{S}_{I\alpha,J\delta,K\beta,L\gamma}\tilde{\phi}^{I\alpha}\tilde{\phi}^{J\delta}\tilde{\phi}^{K\beta}\tilde{\phi}^{L\gamma}\Big) + \text{c.c}\ .
\end{aligned} \tag{3.93}$$

The sum in the second line is over pairs of roots whose sum is a root, i.e. $\alpha+\beta \in$ Roots, each pair being counted once only. In the last line, the sum is over pairs of roots such that $\alpha+\beta = -(\gamma+\delta)$ and $\alpha+\beta$ is a root, each pair of pairs being counted once.

The expressions obtained for the scalar potential are fairly simple and allow the study of any breaking pattern. The tensors $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ are all defined in terms of the rotation matrix $P^\dagger_\alpha$, making their computation fairly straightforward. Since we are interested in the renormalization of the masses, not all the coefficient will be needed as we will see. We now move on to the terms coupling the gauge to the scalars.

3.4.3 Gauge bosons

The gauge couplings are more numerous and the terms are less simple because of the fewer symmetries making their expressions a little less compact. The computation is very similar to what we have done so far. We start with the terms involving the vacuum. We first consider the trilinear term that include the vacuum. The philosophy of the computation hasn't changed. We find:

$$\begin{aligned}
2g\mathrm{Tr}\left(\partial_\mu\phi^I[A^\mu,\phi^I_0]\right) =& 2g\partial_\mu\phi^{I\alpha}A^{\mu\beta}\phi_0^{Ii}\mathrm{Tr}\left(E_\alpha[E_\beta,H_i]\right) \\
=& -2g\mathcal{C}\partial_\mu\phi^{I\alpha}A^{\mu\beta}\beta_i\phi_0^{Ii}\delta_{\alpha+\beta,0} \\
=& 2g\mathcal{C}\alpha_i\phi_0^{Ii}\partial_\mu\phi^{I\alpha}A^{\mu(-\alpha)} \\
=& 2M\mathcal{C}b_I^{(\alpha)}\partial_\mu\phi^{I\alpha}A^{\mu(-\alpha)}\ .
\end{aligned} \tag{3.94}$$

Rotating the scalars in the mass basis, we obtain:

$$\boxed{2g\mathrm{Tr}\left(\partial_\mu\phi^I[A^\mu,\phi^I_0]\right) = 2M\mathcal{C}b_I^{(\alpha)}(P^\dagger_\alpha)^I{}_{I'}\partial_\mu\tilde{\phi}^{I'\alpha}A^{\mu(-\alpha)}\ .} \tag{3.95}$$

Next we have the quartic term involving the vacuum:

$$\begin{aligned}\frac{1}{\mathcal{C}}g^2\mathrm{Tr}\left([\phi_0^I,A_\mu][\phi^I,A^\mu]\right) &= g^2\delta^{\mu\nu}\phi_0^{Ii}A_\mu^\alpha\phi^{I\beta}A_\nu^\gamma\mathrm{Tr}\left([H_i,E_\alpha][E_\beta,E_\gamma]\right)\\ &\quad+g^2\delta^{\mu\nu}\phi_0^{Ii}A_\mu^\alpha\phi^{Ij}A_\nu^\gamma\mathrm{Tr}\left([H_i,E_\alpha][H_j,E_\gamma]\right)\\ &\quad+g^2\delta^{\mu\nu}\phi_0^{Ii}A_\mu^\alpha\phi^{I\beta}A_\nu^j\mathrm{Tr}\left([H_i,E_\alpha][E_\beta,H_j]\right)\\ &= \ g^2i\alpha_iN_{\beta\gamma}\delta^{\mu\nu}\phi_0^{Ii}A_\mu^\alpha\phi^{I\beta}A_\nu^\gamma\delta_{\beta+\gamma,\alpha}\\ &\quad+g^2i\alpha_ii\gamma_j\delta^{\mu\nu}\phi_0^{Ii}A_\mu^\alpha\phi^{Ij}A_\nu^\gamma\delta_{\alpha+\gamma,0}\\ &\quad-g^2i\alpha_ii\beta_j\delta^{\mu\nu}\phi_0^{Ii}A_\mu^\alpha\phi^{I\beta}A_\nu^j\delta_{\alpha+\beta,0}\\ &= g^2i\alpha_iN_{\beta\gamma}\delta^{\mu\nu}\phi_0^{Ii}A_\mu^\alpha\phi^{I\beta}A_\nu^\gamma\delta_{\beta+\gamma,\alpha}\\ &\quad+g^2\alpha_i\alpha_j\delta^{\mu\nu}\phi_0^{Ii}A_\mu^\alpha\phi^{Ij}A_\nu^{-\alpha}\\ &\quad-g^2\alpha_i\alpha_j\delta^{\mu\nu}\phi_0^{Ii}A_\mu^\alpha\phi^{I(-\alpha)}A_\nu^j\ .\end{aligned}\tag{3.96}$$

Taking into account the various related terms in each line, and rotating to the mass basis, we obtain

$$\boxed{\begin{aligned}\frac{1}{\mathcal{C}}g^2\mathrm{Tr}\left([\phi_0^I,A_\mu][\phi^I,A^\mu]\right) &= igMN_{\beta\gamma}\delta^{\mu\nu}b_I^{(\alpha)}(\mathcal{P}_\alpha^\dagger)^I{}_{I'}A_\mu^\alpha\tilde{\phi}^{I'\beta}A_\nu^\gamma+\text{c.c.}\\ &\quad+\text{cyclic permutations in }\alpha,\,\beta,\,\gamma\\ &\quad+gM\alpha_j\delta^{\mu\nu}b_I^{(\alpha)}(\mathcal{P}_\alpha^\dagger)^I{}_{I'}A_\mu^\alpha\tilde{\phi}^{I'j}A_\nu^{-\alpha}+\text{c.c.}\\ &\quad-gM\alpha_j\delta^{\mu\nu}b_I^{(\alpha)}(\mathcal{P}_\alpha^\dagger)^I{}_{I'}A_\mu^\alpha\tilde{\phi}^{I'(-\alpha)}A_\nu^j+\text{c.c.}\end{aligned}}\tag{3.97}$$

Now moving on to the trilinear and quartic terms involving the scalars around the vacuum:

$$\begin{aligned}2g\mathrm{Tr}\left(\partial_\mu\phi^I[A^\mu,\phi^I]\right) &= 2g\mathcal{C}\partial_\mu\phi^{I\alpha}A^{\mu\beta}\phi^{I\gamma}\mathrm{Tr}\left(E_\alpha[E_\beta,E_\gamma]\right)\\ &\quad+2g\mathcal{C}\partial_\mu\phi^{I\alpha}A^{\mu\beta}\phi^{Ii}\mathrm{Tr}\left(E_\alpha[E_\beta,H_i]\right)\\ &\quad+2g\mathcal{C}\partial_\mu\phi^{I\beta}A^{\mu i}\phi^{I\alpha}\mathrm{Tr}\left(E_\beta[H_i,E_\alpha]\right)\\ &\quad+2g\mathcal{C}\partial_\mu\phi^{Ii}A^{\mu\beta}\phi^{I\alpha}\mathrm{Tr}\left(H_i[E_\beta,E_\alpha]\right)\\ &= 2g\mathcal{C}N_{\beta\gamma}\partial_\mu\phi^{I\alpha}A^{\mu\beta}\phi^{I\gamma}\delta_{\alpha+\beta+\gamma}\\ &\quad+2ig\mathcal{C}\alpha_iA^{\mu(-\alpha)}\left(\partial_\mu\phi^{I\alpha}\phi^{Ii}-\phi^{I\alpha}\partial_\mu\phi^{Ii}\right)\\ &\quad+2ig\mathcal{C}\alpha_iA^{\mu i}\partial_\mu\phi^{I(-\alpha)}\phi^{I\alpha}\end{aligned}\tag{3.98}$$

The first line can be further simplified. Let us fix three roots such that $\alpha=-(\beta+\gamma)$. In the sum over the roots, we have the terms

$$\begin{aligned}&2g\mathcal{C}N_{\beta\gamma}\partial_\mu\phi^{I\alpha}A^{\mu\beta}\phi^{I\gamma}+(\alpha\leftrightarrow\gamma)\\ &=2g\mathcal{C}N_{\beta\gamma}\partial_\mu\phi^{I\alpha}A^{\mu\beta}\phi^{I\gamma}+2g\mathcal{C}N_{\beta\alpha}\partial_\mu\phi^{I\gamma}A^{\mu\beta}\phi^{I\alpha}\\ &=2g\mathcal{C}N_{\beta\gamma}A^{\mu\beta}\left(\partial_\mu\phi^{I\alpha}\phi^{I\gamma}-\phi^{I\alpha}\partial_\mu\phi^{I\gamma}\right)\end{aligned}\tag{3.99}$$

where in the last line we used $N_{\beta\alpha} = -N_{\beta\gamma}$. On top of those terms, there are also the cyclic permutations. We also perform the change of basis resulting in the expression

$$\boxed{\begin{aligned}
&2g\mathrm{Tr}\left(\partial_\mu\phi^I[A^\mu,\phi^I]\right)\\
&\quad= 2g\mathcal{C}N_{\beta\gamma}(\mathcal{P}^\dagger_\alpha)^I{}_{I'}(\mathcal{P}^\dagger_\gamma)^I{}_{J'}A^{\mu\beta}\left(\partial_\mu\tilde{\phi}^{I'\alpha}\tilde{\phi}^{J'\gamma}-\phi^{I'\alpha}\partial_\mu\phi^{J'\gamma}\right)+\text{c.c.}\\
&\qquad+\text{cyclic permutations}\\
&\qquad+2ig\mathcal{C}(\mathcal{P}^\dagger_\alpha)^I{}_{I'}\alpha_i A^{\mu(-\alpha)}\left(\partial_\mu\tilde{\phi}^{I'\alpha}\phi^{Ii}-\tilde{\phi}^{I'\alpha}\partial_\mu\phi^{Ii}\right)\\
&\qquad+2ig\mathcal{C}\alpha_i A^{\mu i}\partial_\mu\tilde{\phi}^{I(-\alpha)}\tilde{\phi}^{I\alpha}
\end{aligned}} \tag{3.100}$$

Lastly, we have the quartic term

$$\begin{aligned}
\frac{1}{\mathcal{C}}g^2\mathrm{Tr}\left([\phi^I,A_\mu][\phi^I,A^\mu]\right) &= g^2\eta^{\mu\nu}\phi^{I\alpha}A^\beta_\mu\phi^{I\gamma}A^\lambda_\nu\mathrm{Tr}\left([E_\alpha,E_\beta][E_\gamma,E_\lambda]\right)\\
&\quad+2g^2\eta^{\mu\nu}\phi^{Ii}A^\alpha_\mu\phi^{I\beta}A^\gamma_\nu\mathrm{Tr}\left([H_i,E_\alpha][E_\beta,E_\gamma]\right)\\
&\quad+2g^2\eta^{\mu\nu}\phi^{I\alpha}A^i_\mu\phi^{I\beta}A^\gamma_\nu\mathrm{Tr}\left([E_\alpha,H_i][E_\beta,E_\gamma]\right)\\
&\quad+g^2\eta^{\mu\nu}\phi^{Ii}A^\alpha_\mu\phi^{Ij}A^\gamma_\nu\mathrm{Tr}\left([H_i,E_\alpha][H_j,E_\gamma]\right)\\
&\quad+g^2\eta^{\mu\nu}\phi^{I\alpha}A^i_\mu\phi^{I\gamma}A^j_\nu\mathrm{Tr}\left([E_\alpha,H_i][E_\gamma,H_j]\right)\\
&\quad+2g^2\eta^{\mu\nu}\phi^{Ii}A^\alpha_\mu\phi^{I\beta}A^j_\nu\mathrm{Tr}\left([H_i,E_\alpha][E_\beta,H_j]\right)\\
&= g^2\eta^{\mu\nu}\phi^{I\alpha}A^\beta_\mu\phi^{I\gamma}A^\lambda_\nu N_{\alpha\beta}N_{\gamma\lambda}\delta(\alpha+\beta+\gamma+\lambda)\\
&\quad+2g^2\eta^{\mu\nu}\alpha_i\phi^{Ii}A^\alpha_\mu\phi^{I\beta}A^\gamma_\nu\delta_{\alpha,\beta+\gamma}\\
&\quad-2g^2\eta^{\mu\nu}\alpha_i\phi^{I\alpha}A^i_\mu\phi^{I\beta}A^\gamma_\nu\delta_{\alpha,\beta+\gamma}\\
&\quad+g^2\eta^{\mu\nu}\alpha_i\gamma_j\phi^{Ii}A^\alpha_\mu\phi^{Ij}A^\gamma_\nu\delta_{\alpha+\gamma,0}\\
&\quad+g^2\eta^{\mu\nu}\alpha_i\gamma_j\phi^{I\alpha}A^i_\mu\phi^{I\gamma}A^j_\nu\delta_{\alpha+\gamma,0}\\
&\quad-2g^2\eta^{\mu\nu}\alpha_i\beta_j\phi^{Ii}A^\alpha_\mu\phi^{I\beta}A^j_\nu\delta_{\alpha+\beta,0}
\end{aligned} \tag{3.101}$$

And putting the expression in the mass basis we have

$$\boxed{\begin{aligned}
&\frac{1}{\mathcal{C}}g^2\mathrm{Tr}\left([\phi^I,A_\mu][\phi^I,A^\mu]\right)\\
&\quad= g^2\eta^{\mu\nu}N_{\alpha\beta}N_{\gamma\lambda}(\mathcal{P}^\dagger_\alpha)^I{}_{I'}(\mathcal{P}^\dagger_\gamma)^I{}_{J'}\tilde{\phi}^{I'\alpha}A^\beta_\mu\tilde{\phi}^{J'\gamma}A^\lambda_\nu\delta(\alpha+\beta+\gamma+\lambda)\\
&\qquad+2g^2\eta^{\mu\nu}\alpha_i(\mathcal{P}^\dagger_\beta)^I{}_{I'}\phi^{Ii}A^\alpha_\mu\tilde{\phi}^{I'\beta}A^\gamma_\nu\delta_{\alpha,\beta+\gamma}\\
&\qquad-2g^2\eta^{\mu\nu}\alpha_i(\mathcal{P}^\dagger_\alpha)^I{}_{I'}(\mathcal{P}^\dagger_\beta)^I{}_{J'}\tilde{\phi}^{I'\alpha}A^i_\mu\tilde{\phi}^{J'\beta}A^\gamma_\nu\delta_{\alpha,\beta+\gamma}\\
&\qquad+g^2\eta^{\mu\nu}\alpha_i\gamma_j\phi^{Ii}A^\alpha_\mu\phi^{Ij}A^\gamma_\nu\delta_{\alpha+\gamma,0}\\
&\qquad+g^2\eta^{\mu\nu}\alpha_i\gamma_j\tilde{\phi}^{I\alpha}A^i_\mu\tilde{\phi}^{I\gamma}A^j_\nu\delta_{\alpha+\gamma,0}\\
&\qquad-2g^2\eta^{\mu\nu}\alpha_i\beta_j(\mathcal{P}^\dagger_\beta)^I{}_{I'}\phi^{Ii}A^\alpha_\mu\tilde{\phi}^{I'\beta}A^j_\nu\delta_{\alpha+\beta,0}
\end{aligned}} \tag{3.102}$$

Notice that here, we did not go through the usual simplifications. That is because we do not really need to. Only a few of these terms actually contribute

the renormalization of the masses, it will be easier to only consider the relevant terms and simplify them later.

Having now all the formulas needed for mass renormalization, we can move on to the actual renormalization of the masses at one-loop.

3.5 Mass renormalization

We are now finally ready to renormalize the masses of the model. Not all the masses present in the model are of interest. Recall Table 3.1. For a small vacuum parameter, the masses separate into essentially two scales. The first one is a the geometrical scale $M^2 = \mathtt{f}^2$, the second is given by the masses $(m_\alpha^-)^2$, $(m^\alpha_{gauge})^2$ along with the massless degrees of freedom. Since these would be the observable modes comprising the Higgs boson, we will only renormalize those. The rest will remain at the geometrical scale anyway. In our new notations, these fields correspond respectively to the components $\tilde{\phi}^{1\alpha}$, A^α_μ, $\tilde{\phi}^{1i}$ and $\tilde{\phi}^{2i}$. The gauge symmetry ensures that every component falling into one representation will have the same renormalized mass, allowing us to limit our study to the components just mentioned.

Before diving into the study of the renormalization of the masses, we are going to fix some notation. When computing the various contributions, we will encounter the usual divergent integrals, such as

$$\int dq^4 \frac{1}{((p+q)^2 - m_1^2)(q^2 - m_2^2)} \ . \tag{3.103}$$

These integrals have to be regularized in order to extract the finite part which actually contribute to the mass. The finite part of the regularized integrals will be denoted in the following way

$$\mathrm{FC}\left[\int dq^4 \frac{1}{((p+q)^2 - m_1^2)(q^2 - m_2^2)}\right] \tag{3.104}$$

where FC stands for finite contribution. We will slightly abuse notation in order to keep things as clear as possible. We will use the propagators or products of propagators to mean the finite part of the regularized integral. Explicitly

$$\mathrm{FC}\left[\int dq^4 \frac{i}{((q)^2 - m_{I\alpha}^2)}\right] \equiv \langle \tilde{\phi}^{I\alpha} (\tilde{\phi}^{I\alpha})^* \rangle \ , \tag{3.105}$$

$$\mathrm{FC}\left[\int dq^4 \frac{i^2}{((p+q)^2 - m_{I\alpha}^2)(q^2 - m_{J\beta}^2)}\right] \equiv \langle \tilde{\phi}^{I\alpha} (\tilde{\phi}^{I\alpha})^* \rangle \langle \tilde{\phi}^{J\beta} (\tilde{\phi}^{J\beta})^* \rangle \ . \tag{3.106}$$

We choose dimensional regularization as our method to extract the finite part. All the integrals we encounter are standard, and their regularization can be found in *Peskin & Schroeder "An introduction to quantum field theory"* for example. Finally, the renormalization is done in the Feynman gauge ($\xi = 1$).

3.5.1 Renormalization of the light field $\tilde{\phi}^{1\alpha}$

We are interested in the renormalization of the component $\tilde{\phi}^{1\alpha}$. We first fix a root α such that $b_I^{(\alpha)} \neq 0$, meaning that the vacuum we picked is not orthogonal to this root. This component corresponds to the H^- fields of Table 3.1. The terms involving $\tilde{\phi}^{2\alpha}$ are not taken into account since this degree of freedom is used for the associated broken boson A_μ^α.

Scalar potential terms

The relevant terms of the scalar potential for the renormalization of $\phi^{1\alpha}$ are

$$\begin{aligned}
&-i\alpha_i \mathcal{T}_{I\alpha,J(-\alpha),Ki} \tilde{\phi}^{I\alpha} (\tilde{\phi}^{J\alpha})^* \tilde{\phi}^{Ki} \\
&+ \sum_\beta N_{\alpha\beta} \mathcal{T}_{I\alpha,J\beta,K(\alpha+\beta)} \tilde{\phi}^{I\alpha} \tilde{\phi}^{J\beta} (\tilde{\phi}^{K(\alpha+\beta)})^* + c.c. \\
&- 4 \left(\alpha_i \alpha_j \tilde{\phi}^{Ii} \tilde{\phi}^{Ij} \tilde{\phi}^{J\alpha} (\tilde{\phi}^{J\alpha})^* - \alpha_i \alpha_j \mathcal{Q}_{IJI'J'} \tilde{\phi}^{Ii} \tilde{\phi}^{Jj} \tilde{\phi}^{I'\alpha} (\tilde{\phi}^{J'\alpha})^* \right) \\
&-4 \sum_\beta |N_{\alpha\beta}|^2 \tilde{\phi}^{I\alpha} \tilde{\phi}^{J\beta} (\tilde{\phi}^{I\alpha})^* (\tilde{\phi}^{J\beta})^* \ .
\end{aligned} \tag{3.107}$$

where in the last line, the terms involving the tensor $\mathcal{R}$ don't contribute to the renormalization. Note also that terms with $I = J$ vanish, which can be seen directly from (3.93). We recall also that the tensors $\mathcal{T}$ and $\mathcal{Q}$ don't really depend on the specific algebra under consideration. This dependence is only through the parameters α, β, ... which fixes the value of the angles for the sines and cosines inside $\mathcal{P}_\alpha$. Let us start with the first line. The terms are explicitly

$$\begin{aligned}
\alpha_i \mathcal{T}_{I\alpha,J(-\alpha),Ki} \tilde{\phi}^{I\alpha} (\tilde{\phi}^{J\alpha})^* \tilde{\phi}^{Ki} = &\ i \cos(2\theta) \sin(\nu) \alpha_i \tilde{\phi}^{1i} \left(\tilde{\phi}^{3\alpha} (\tilde{\phi}^{1\alpha})^* - (\tilde{\phi}^{3\alpha})^* \tilde{\phi}^{1\alpha} \right) \\
&+ i \cos(2\theta) \cos(\nu) \alpha_i \tilde{\phi}^{2i} \left(\tilde{\phi}^{3\alpha} (\tilde{\phi}^{1\alpha})^* - (\tilde{\phi}^{3\alpha})^* \tilde{\phi}^{1\alpha} \right) \\
&+ \sin(2\theta) \sin(\nu) \alpha_i \tilde{\phi}^{1i} \tilde{\phi}^{1\alpha} (\tilde{\phi}^{1\alpha})^* \\
&+ \sin(2\theta) \cos(\nu) \alpha_i \tilde{\phi}^{2i} \tilde{\phi}^{1\alpha} (\tilde{\phi}^{1\alpha})^*
\end{aligned} \tag{3.108}$$

where the dependence on the root for the angles $\theta(b_I^\alpha)$ and $\nu(b_I^\alpha)$ is not written explicitly. The first two terms give the following integral when considering their contribution to the mass. We denote g_{eff} the effective coupling constant of the vertex and obtain

$$\begin{aligned}
&\frac{i^2 g_{eff}^2}{2} \tilde{\phi}^{1i} \tilde{\phi}^{1i} i^2 \left(\tilde{\phi}^{3\alpha} (\tilde{\phi}^{1\alpha})^* - (\tilde{\phi}^{3\alpha})^* \tilde{\phi}^{1\alpha} \right)^2 \\
&= \cdots + 2 \frac{i^2 g_{eff}^2}{2} \tilde{\phi}^{1i} \tilde{\phi}^{1i} \tilde{\phi}^{3\alpha} (\tilde{\phi}^{1\alpha})^* (\tilde{\phi}^{3\alpha})^* \tilde{\phi}^{1\alpha} + \ldots
\end{aligned} \tag{3.109}$$

Upon contractions we have

$$i^2 g_{eff}^2 \langle \tilde{\phi}^{1i} \tilde{\phi}^{1i} \rangle \langle \tilde{\phi}^{3\alpha} (\tilde{\phi}^{3\alpha})^* \rangle \ . \tag{3.110}$$

The correlators give the usual two propagator integral. Similarly, the last two lines give similar integrals

$$i^2 g_{eff}^2 \langle \tilde{\phi}^{1i} \tilde{\phi}^{1i} \rangle \langle \tilde{\phi}^{1\alpha} (\tilde{\phi}^{1\alpha})^* \rangle \ . \tag{3.111}$$

The second line doesn't really simplify much in the general case and is better left in this form, giving a contribution of the form

$$i^2 \ |N_{\alpha\beta}|^2 |\mathcal{T}_{1\alpha,J\beta,K\gamma}|^2 \langle \tilde{\phi}^{J\beta} (\tilde{\phi}^{J\beta})^* \rangle \langle \tilde{\phi}^{K\gamma} (\tilde{\phi}^{K\gamma})^* \rangle \ . \tag{3.112}$$

The third line is composed of quartic terms. They are explicitly given by

$$\begin{aligned} &- i4\alpha_i \alpha_j \left(1 - \sin(\theta)^2\right) \phi^{3i} \phi^{3j} \langle \phi^{1\alpha} (\phi^{1\alpha})^* \rangle \\ &- i4\alpha_i \alpha_j \left(1 - \cos(\theta)^2 \cos(\nu)^2\right) \phi^{1i} \phi^{1j} \langle \phi^{1\alpha} (\phi^{1\alpha})^* \rangle \\ &- i4\alpha_i \alpha_j \left(1 - \cos(\theta)^2 \sin(\nu)^2\right) \phi^{2i} \phi^{2j} \langle \phi^{1\alpha} (\phi^{1\alpha})^* \rangle \end{aligned} \tag{3.113}$$

All those terms are tadpole like integrals with one propagator (taking $i = j$ for the contractions to be non zero). Finally we have the fourth line which gives contributions

$$-i4 |N_{\alpha\beta}|^2 \langle \phi^{J\beta} (\phi^{J\beta})^* \rangle \ . \tag{3.114}$$

Of course, when computing the mass, we will have to add the constants in front of the trilinear and quartic terms, which are respectively $(2gM)^2$ and $g^2/2$ (see the expression for the potential in (3.2)).

Gauge coupling terms

The terms coupling $\phi^{1\alpha}$ to the gauge that are relevant to the renormalization are

$$\begin{aligned} &2g\mathcal{C} N_{\beta\gamma} (\mathcal{P}_\alpha^\dagger)^I{}_{I'} (\mathcal{P}_\gamma^\dagger)^I{}_{J'} A^{\mu\beta} \left(\partial_\mu \tilde{\phi}^{I'\alpha} \tilde{\phi}^{J'\gamma} - \phi^{I'\alpha} \partial_\mu \phi^{J'\gamma} \right) + \text{c.c.} \\ &+ \text{cyclic permutations} \\ &+ 2ig\mathcal{C} (\mathcal{P}_\alpha^\dagger)^I{}_{I'} \alpha_i A^{\mu(-\alpha)} \left(\partial_\mu \tilde{\phi}^{I'\alpha} \phi^{Ii} - \tilde{\phi}^{I'\alpha} \partial_\mu \phi^{Ii} \right) \\ &+ 2ig\mathcal{C} \alpha_i A^{\mu i} \partial_\mu \tilde{\phi}^{I(-\alpha)} \tilde{\phi}^{I\alpha} \end{aligned} \tag{3.115}$$

The last two terms can be computed in all generality. We start with the third line

$$2ig\mathcal{C} (\mathcal{P}_\alpha^\dagger)^I{}_{I'} \alpha_i A^{\mu(-\alpha)} \left(\partial_\mu \tilde{\phi}^{I'\alpha} \phi^{Ii} - \tilde{\phi}^{I'\alpha} \partial_\mu \phi^{Ii} \right) + (\alpha \to -\alpha) = \tag{3.116}$$

$$2ig\mathcal{C} \cos(\theta) \cos(\nu) \alpha_i A^{\mu(-\alpha)} \left(\partial_\mu \tilde{\phi}^{1\alpha} \phi^{1i} - \tilde{\phi}^{1\alpha} \partial_\mu \phi^{1i} \right) + \text{c.c.} \tag{3.117}$$

$$- 2ig\mathcal{C} \cos(\theta) \sin(\nu) \alpha_i A^{\mu(-\alpha)} \left(\partial_\mu \tilde{\phi}^{1\alpha} \phi^{2i} - \tilde{\phi}^{1\alpha} \partial_\mu \phi^{2i} \right) + \text{c.c.} \tag{3.118}$$

$$+ 2ig\mathcal{C} (-i \sin(\theta)) \alpha_i A^{\mu(-\alpha)} \left(\partial_\mu \tilde{\phi}^{1\alpha} \phi^{3i} - \tilde{\phi}^{1\alpha} \partial_\mu \phi^{3i} \right) + \text{c.c.} \tag{3.119}$$

The contributions will be similar. Let us treat the first term explicitly. We first write the term in momentum space :

$$ig_{eff}(A^{\mu\alpha})^*\left(\partial_\mu\tilde{\phi}^{1\alpha}\phi^{1i}-\tilde{\phi}^{1\alpha}\partial_\mu\phi^{1i}\right)\to ig_{eff}\left(ip_\mu-iq_\mu\right)(A^{\mu\alpha}(k))^*\tilde{\phi}^{1\alpha}(p)\phi^{1i}(q) \tag{3.120}$$

The contribution takes the form

$$\begin{aligned}-i^2\frac{g_{eff}}{2}i^2 2\eta_{\mu\nu}\eta_{\rho\delta}\left(ip_1^\mu-iq_1^\mu\right)\left(ip_2^\rho-iq_2^\rho\right)\\ \times(A^{\nu\alpha}(k_1))^*A^{\delta\alpha}(k_2)\tilde{\phi}^{1\alpha}(p_1)(\tilde{\phi}^{1\alpha}(p_2))^*\phi^{1i}(q_1)\phi^{1i}(q_2)\end{aligned} \tag{3.121}$$

Upon taking contractions and using conservation of momentum at each vertex we obtain the integral

$$g_{eff}^2\mathcal{I}_2(p^2;m_{gauge}^\alpha,m_i)=g_{eff}^2\ \mathrm{FC}\left[\int\frac{dq^4}{(2\pi)^4}\frac{(p-q)^\mu(p-q)_\mu}{\left((p+q)^2-m_{gauge}^2\right)\left(q^2-m_i^2\right)}\right] \tag{3.122}$$

The fourth line is similar, only with one type of scalar

$$2g\alpha_i\mathcal{C}iA^{\mu i}\left(\partial_\mu\tilde{\phi}^{I(-\alpha)}\tilde{\phi}^{I\alpha}-\tilde{\phi}^{I(-\alpha)}\partial_\mu\tilde{\phi}^{I\alpha}\right) \tag{3.123}$$

The contribution in momentum space is

$$\begin{aligned}&-\frac{i^2}{2}(2g)^2\mathcal{C}^2\alpha_i^2 2i^2A_\mu^i(k_1)A_\nu^i(k_2)ip_1^\mu(\tilde{\phi}^{I\alpha}(p_1))^*\tilde{\phi}^{I\alpha}(q_1)(\tilde{\phi}^{I\alpha}(q_2))^*ip_2^\nu\tilde{\phi}^{I\alpha}(p_2)\\ &=i^2(2g)^2\mathcal{C}^2\alpha_i^2ip_1^\mu ip_2^\nu A_\mu^i(k_1)A_\nu^i(k_2)(\tilde{\phi}^{1\alpha}(p_1))^*\tilde{\phi}^{1\alpha}(q_1)(\tilde{\phi}^{1\alpha}(q_2))^*\tilde{\phi}^{1\alpha}(p_2)\end{aligned} \tag{3.124}$$

There are two possible contractions

$$i^2(2g)^2\mathcal{C}^2\alpha_i^2ip_1^\mu ip_2^\nu\langle A_\mu^i(k_1)A_\nu^i(k_2)\rangle\langle(\tilde{\phi}^{1\alpha}(p_1))^*\tilde{\phi}^{1\alpha}(p_2)\rangle \tag{3.125}$$

and

$$i^2(2g)^2\mathcal{C}^2\alpha_i^2ip_1^\mu ip_2^\nu\langle A_\mu^i(k_1)A_\nu^i(k_2)\rangle\langle(\tilde{\phi}^{1\alpha}(q_2))^*\tilde{\phi}^{1\alpha}(q_1)\rangle\ , \tag{3.126}$$

which can be written as one integral after a change of variable:

$$\mathcal{I}_1(p;m_{i,gauge},m_{1\alpha})=\mathrm{FC}\left[\int\frac{dq^4}{(2\pi)^4}\frac{p_\mu p^\mu+q_\mu q^\mu}{\left((p+q)^2-m_{i,gauge}^2\right)\left(q^2-m_{1\alpha}^2\right)}\right] \tag{3.127}$$

The final terms to consider are the quartic couplings with the gauge. The ones relevant to us are

$$\begin{aligned}&-g^2\mathcal{C}\eta^{\mu\nu}|N_{\alpha\beta}|^2A_\mu^\beta(A_\nu^\beta)^*\tilde{\phi}^{I\alpha}(\tilde{\phi}^{I\alpha})^*\\ &+g^2\mathcal{C}\eta^{\mu\nu}\alpha_i^2A_\mu^iA_\nu^i\tilde{\phi}^{I\alpha}\tilde{\phi}^{I(-\alpha)}\ ,\end{aligned} \tag{3.128}$$

which form tadpole-like integrals.

We have now all the ingredients to renormalize the mass of $\tilde{\phi}^{1\alpha}$. We shall end this part by giving the final expression.

Correction to the mass of $\tilde{\phi}^{1\alpha}$

Finally the expression for the correction to the mass of $\tilde{\phi}^{1\alpha}$ is

$$\begin{aligned}
\frac{1}{\mathcal{C}^2}\delta m^2_{\tilde{\phi}^{1\alpha}} =&(2gM)^2\ (N_{\alpha\beta})^2(\mathcal{T}_{1\alpha,J\beta,K\gamma})^2\langle\tilde{\phi}^{J\beta}(\tilde{\phi}^{J\beta})^*\rangle\langle\tilde{\phi}^{K\gamma}(\tilde{\phi}^{K\gamma})^*\rangle \\
&+(2gM)^2\alpha_i^2(\cos(2\theta)\sin(\nu))^2\langle\tilde{\phi}^{1i}\tilde{\phi}^{1i}\rangle\langle\tilde{\phi}^{3\alpha}(\tilde{\phi}^{3\alpha})^*\rangle \\
&+(2gM)^2\alpha_i^2(\cos(2\theta)\cos(\nu))^2\langle\tilde{\phi}^{2i}\tilde{\phi}^{2i}\rangle\langle\tilde{\phi}^{3\alpha}(\tilde{\phi}^{3\alpha})^*\rangle \\
&+(2gM)^2\alpha_i^2(\sin(2\theta)\sin(\nu))^2\langle\tilde{\phi}^{1i}\tilde{\phi}^{1i}\rangle\langle\tilde{\phi}^{1\alpha}(\tilde{\phi}^{1\alpha})^*\rangle \\
&+(2gM)^2\alpha_i^2(\sin(2\theta)\cos(\nu))^2\langle\tilde{\phi}^{2i}\tilde{\phi}^{2i}\rangle\langle\tilde{\phi}^{1\alpha}(\tilde{\phi}^{1\alpha})^*\rangle \\
&\\
&+\frac{g^2}{2}4|N_{\alpha\beta}|^2\langle\phi^{J\beta}(\phi^{J\beta})^*\rangle \\
&+\frac{g^2}{2}4\alpha_i^2\left(1-\sin(\theta)^2\right)\langle\tilde{\phi}^{3i}\tilde{\phi}^{3i}\rangle \\
&+\frac{g^2}{2}4\alpha_i^2\left(1-\cos(\theta)^2\cos(\nu)^2\right)\langle\tilde{\phi}^{1i}\tilde{\phi}^{1i}\rangle \\
&+\frac{g^2}{2}4\alpha_i^2\left(1-\cos(\theta)^2\sin(\nu)^2\right)\langle\tilde{\phi}^{2i}\tilde{\phi}^{2i}\rangle \\
&\\
&+(2g)^2|N_{\beta\gamma}(\mathcal{P}^\dagger_\alpha)^I{}_{I'}(\mathcal{P}^\dagger_\gamma)^J{}_{J'}|^2\mathcal{I}_2(m^2_{1\alpha};m_{\beta,gauge},m_{J'\gamma}) \\
&+(2g)^2|N_{\gamma\alpha}(\mathcal{P}^\dagger_\alpha)^I{}_{I'}(\mathcal{P}^\dagger_\beta)^J{}_{J'}|^2\mathcal{I}_2(m^2_{1\alpha};m_{\gamma,gauge},m_{J'\beta}) \\
&+\alpha_i^2(2g)^2\mathcal{I}_1(m^2_{1\alpha};m_{i,gauge},m_{1\alpha}) \\
&+\alpha_i^2(2g\cos(\theta)\cos(\nu))^2\mathcal{I}_2(m^2_{1\alpha};m_{\alpha,gauge},m_{1i}) \\
&+\alpha_i^2(2g\cos(\theta)\sin(\nu))^2\mathcal{I}_2(m^2_{1\alpha};m_{\alpha,gauge},m_{2i}) \\
&+\alpha_i^2(2g\sin(\theta))^2\mathcal{I}_2(m^2_{1\alpha};m_{\alpha,gauge},m_{3i}) \\
&\\
&+g^2\mathcal{C}\eta^{\mu\nu}|N_{\alpha\beta}|^2\langle A^\beta_\mu(A^\beta_\nu)^*\rangle \\
&+g^2\mathcal{C}\eta^{\mu\nu}\alpha_i^2\langle A^i_\mu A^i_\nu\rangle
\end{aligned} \tag{3.129}$$

where we recall the expression for the integrals

$$\begin{aligned}
\mathcal{I}_1(p^2;m^\alpha_{gauge},m_{I\alpha}) &= \int\frac{dq^4}{(2\pi)^4}\frac{p_\mu p^\mu+q_\mu q^\mu}{\left((p+q)^2-m^2_{gauge}\right)\left(q^2-m^2_{I\alpha}\right)} \\
\mathcal{I}_2(p^2;m^\alpha_{gauge},m_{I\alpha}) &= \int\frac{dq^4}{(2\pi)^4}\frac{(p-q)^\mu(p-q)_\mu}{\left((p+q)^2-m^2_{gauge}\right)\left(q^2-m^2_{I\alpha}\right)}\ .
\end{aligned} \tag{3.130}$$

A few words are in order about this result. First, notice that this result is in fact independent of the angle ν. That is because ϕ_{1i} and ϕ_{2i} are both massless.

For example, if we take the second and third line

$$\begin{aligned} &+ (2gM)^2\alpha_i^2(\cos(2\theta)\sin(\nu))^2\langle\tilde{\phi}^{1i}\tilde{\phi}^{1i}\rangle\langle\tilde{\phi}^{3\alpha}(\tilde{\phi}^{3\alpha})^*\rangle \\ &+ (2gM)^2\alpha_i^2(\cos(2\theta)\cos(\nu))^2\langle\tilde{\phi}^{2i}\tilde{\phi}^{2i}\rangle\langle\tilde{\phi}^{3\alpha}(\tilde{\phi}^{3\alpha})^*\rangle \ , \end{aligned} \tag{3.131}$$

we see that the correlators are the same in both line, leading to the simplification of the $\cos(\nu)$ and $\sin(\nu)$. This happens in the same way everywhere the ν angle appears. Therefore the mass correction is independent of the angle. Next, tadpole terms involving a massless degree of freedom vanish when explicitly computed, leading to additional simplifications. Lastly, terms involving only roots are the more intricate. It is not really enlightening to develop them more since they mix sines and cosines with different angles, leading to very few simplifications in general. We will see that for the breaking $SU(3) \to SU(2)\times U(1)$, they simplify greatly. Bigger gauge groups would give rise to complex combinatorics between the different roots. It would be of course possible to automatize the computation for any gauge group since the only data that depends on the algebra is though the roots α ... and the coefficients $N_{\alpha\beta}$. Given this data, the computation can be readily automated.

3.5.2 Renormalization of ϕ^{1i} and ϕ^{2i}

The renormalization of these terms is actually even easier than the previous case. We start by writing the relevant terms from the potential

$$\begin{aligned} &-2gM\sum_{\alpha^+} i\alpha_i\mathcal{T}_{I\alpha,J(-\alpha),Ki}\tilde{\phi}^{I\alpha}(\tilde{\phi}^{J\alpha})^*\tilde{\phi}^{Ki} \\ &-\frac{g^2}{2}\left(\sum_{\alpha^+} 4\alpha_i\alpha_j\phi^{Ii}\phi^{Ij}\phi^{J\alpha}(\phi^{J\alpha})^* - 4\alpha_i\alpha_j\mathcal{Q}_{IJI'J'}\phi^{Ii}\phi^{Jj}\phi^{I'\alpha}(\phi^{J'\alpha})^*\right) \\ \\ &+2ig\mathcal{C}(\mathcal{P}_\alpha^\dagger)^I{}_{I'}\alpha_i A^{\mu(-\alpha)}\left(\partial_\mu\tilde{\phi}^{I'\alpha}\phi^{Ii} - \tilde{\phi}^{I'\alpha}\partial_\mu\phi^{Ii}\right) \\ &+g^2\eta^{\mu\nu}\alpha_i\gamma_j\phi^{Ii}A_\mu^\alpha\phi^{Ij}A_\nu^\gamma\delta_{\alpha+\gamma,0} \end{aligned} \tag{3.132}$$

Next we are going to select the terms involving ϕ^{1i}.

$$\begin{aligned} &-2gM\sum_{\alpha^+} i\alpha_i\mathcal{T}_{I\alpha,J(-\alpha),1i}\tilde{\phi}^{I\alpha}(\tilde{\phi}^{J\alpha})^*\tilde{\phi}^{1i} \\ &-\frac{g^2}{2}\left(\sum_{\alpha^+} 4\alpha_i^2(1-\mathcal{Q}_{11I'I'})\phi^{1i}\phi^{1i}\phi^{I'\alpha}(\phi^{I'\alpha})^*\right) \\ &+2ig(\mathcal{P}_\alpha^\dagger)^1{}_{I'}\alpha_i A^{\mu(-\alpha)}\left(\partial_\mu\tilde{\phi}^{I'\alpha}\phi^{1i} - \tilde{\phi}^{I'\alpha}\partial_\mu\phi^{1i}\right) \\ &+g^2\eta^{\mu\nu}\alpha_i^2\phi^{1i}\phi^{1i}A_\mu^\alpha(A_\nu^\alpha)^* \end{aligned} \tag{3.133}$$

The case of ϕ^{2i} is exactly the same. Already we can give the expression for the masses.

$$\boxed{\begin{aligned}\frac{1}{\mathcal{C}^2}\delta m^2_{\phi^{1i}} =&(2gM)^2\alpha_i^2|\mathcal{T}_{I\alpha,J(-\alpha),1i}|^2\langle\tilde{\phi}^{I\alpha}(\tilde{\phi}^{I\alpha})^*\rangle\langle\tilde{\phi}^{J\alpha}(\tilde{\phi}^{J\alpha})^*\rangle\\ &+\frac{g^2}{2}4\alpha_i^2(1-\mathcal{Q}_{11I'I'})\langle\phi^{I'\alpha}(\phi^{I'\alpha})^*\rangle\\ &+(2g)^2\alpha_i^2(\mathcal{P}^\dagger_\alpha)^1{}_{I'}\mathcal{I}_2(m_i^2;m^\alpha_{gauge},m_{I'\alpha})\\ &+g^2\eta^{\mu\nu}\alpha_i^2\langle A^\alpha_\mu(A^\alpha_\nu)^*\rangle\end{aligned}} \tag{3.134}$$

3.5.3 Renormalization of the broken boson A^α_μ

The relevant terms are

$$\begin{aligned}&2g\mathcal{C}N_{\alpha\gamma}(\mathcal{P}^\dagger_\beta)^I{}_{I'}(\mathcal{P}^\dagger_\gamma)^I{}_{J'}A^{\mu\alpha}\left(\partial_\mu\tilde{\phi}^{I'\beta}\tilde{\phi}^{J'\gamma}-\phi^{I'\beta}\partial_\mu\phi^{J'\gamma}\right)+\text{c.c.}\\ &+2ig\mathcal{C}(\mathcal{P}^\dagger_\alpha)^I{}_{I'}\alpha_iA^{\mu(-\alpha)}\left(\partial_\mu\tilde{\phi}^{I'\alpha}\phi^{Ii}-\tilde{\phi}^{I'\alpha}\partial_\mu\phi^{Ii}\right)+\text{c.c.}\ ,\end{aligned} \tag{3.135}$$

along with the quartic terms

$$\begin{aligned}&+g^2|N_{\alpha\beta}|^2\eta^{\mu\nu}\tilde{\phi}^{I\beta}(\tilde{\phi}^{I\beta})^*A^\alpha_\mu(A^\alpha_\nu)^*\\ &+g^2\eta^{\mu\nu}\alpha_i^2\tilde{\phi}^{Ii}\tilde{\phi}^{Ii}A^\alpha_\mu(A^\alpha_\nu)^*\ .\end{aligned} \tag{3.136}$$

We start with the contribution from the first term. We can recycle the computation done for the renormalization of $\tilde{\phi}^\alpha$ in (3.121) when considering the coupling to the gauge boson. We then apply the appropriate delta functions coming from the contractions of the scalars to find

$$g^2_{eff}\mathcal{I}_3(p^2;m_\beta,m_\gamma)=g^2_{eff}\int\frac{dq^4}{(2\pi)^4}\frac{(p+2q)^\mu(p+2q)_\mu}{\left((p+q)^2-m^2_\beta\right)\left(q^2-m^2_\gamma\right)}\ . \tag{3.137}$$

Here g_{eff} refers to $2g\mathcal{C}N_{\alpha\gamma}(\mathcal{P}^\dagger_\beta)^I{}_{I'}(\mathcal{P}^\dagger_\gamma)^I{}_{J'}$. The following line give the same integral with the appropriate masses. The quartic terms give the usual tadpole we already encountered. Putting all this together, we obtain the contribution to the mass of the gauge boson:

$$\boxed{\begin{aligned}(\delta m^\alpha_{gauge})^2 =&(2g)^2\eta^{\mu\nu}|N_{\alpha\gamma}(\mathcal{P}^\dagger_\beta)^I{}_{I'}(\mathcal{P}^\dagger_\gamma)^I{}_{J'}|^2\mathcal{I}_3(p^2;m_{I'\beta},m_{J'\gamma})\\ &+(2g)^2\eta^{\mu\nu}|N_{\alpha\gamma}(\mathcal{P}^\dagger_\beta)^I{}_{I'}\mathcal{I}_3(p^2;m_{I'\alpha},m_{Ii})|^2\\ &+g^2\eta^{\mu\nu}|N_{\alpha\beta}|^2\langle\tilde{\phi}^{I\beta}(\tilde{\phi}^{I\beta})^*\rangle\\ &+g^2\eta^{\mu\nu}\alpha_i^2\langle\tilde{\phi}^{Ii}\tilde{\phi}^{Ii}\rangle\ .\end{aligned}} \tag{3.138}$$

We are finally done computing the corrections to the masses at one-loop. In the following and final chapter, we will discuss specific examples, trying to understand general features of this type of models, and their limitations.

Chapter 4

Discussion

In this final chapter, we are going to use the machinery developed in the previous chapters to study the renormalization of a gauged $SU(3)$ model. We will see how the mass hierarchy organizes and what can be said for future work for this class of model. We will then discuss qualitative properties of other gauge groups, and how we can restrict the choice of the group in which we embed $SU(2) \times U(1)$ to reproduce a scalar field similar to the Higgs field before electroweak symmetry breaking.

4.1 $SU(3) \to SU(2) \times U(1)$, masses at one-loop

In order to compute the correction to the masses for a given gauge group, all we need to do is specify the gauge by setting the values for the roots α, ... and the $N_{\alpha\beta}$ coefficients. We are interested in the gauge group $SU(3)$, as it provides the simplest non-trivial example with $SU(2) \times U(1)$ embedded, along with a fundamental representation of $SU(2) \times U(1)$ to act as a Higgs field. We start by giving the roots for $SU(3)$:

$$\alpha = (-1/2, \sqrt{3}/2) \tag{4.1}$$

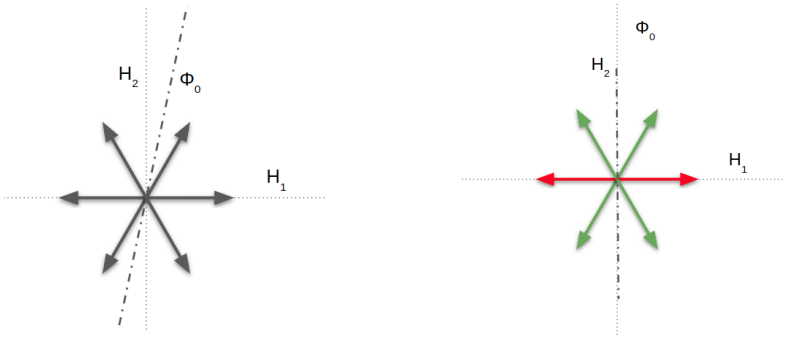
$$\beta = (1, 0) \tag{4.2}$$

$$\gamma = (1/2, \sqrt{3}/2) \tag{4.3}$$

along with their opposites. We also have the relation $\alpha + \beta = \gamma$. The coefficients $N_{\alpha\beta}$ are all the same and given by

$$N_{\alpha\beta} = \frac{i}{\sqrt{2}} \, . \tag{4.4}$$

There is a real advantage of working in the Cartan-Weyl basis like we did, we can easily understand the breaking of the gauge through the use of root diagrams (at least for groups of rank two or three, for larger groups we can use Dynkins diagrams, but it may not be as intuitive). Consider the diagram in



(a) $SU(3) \to U(1) \times U(1)$ (b) $SU(3) \to SU(2) \times U(1)$

Figure 4.1: (a) No root is orthogonal to the vacuum, the gauge after breaking is generated by H_1 and H_2 which commute with one another, resulting in the $U(1)\times U(1)$ gauge. (b) The horizontal root (in red) is orthogonal to the vacuum. Those two roots along with H_1 form an $\mathfrak{su}(2)$ subalgebra, H_2 generates the $U(1)$ part. The other roots (in green) form a fundamental representation of $SU(2)$ and charged under the $U(1)$ generated by H_2.

Figure 4.1. Since our vacuum belongs to the Cartan subalgebra, we can think of it as a direction in root space. As we saw, the various masses for a given direction E_α in the algebra depend on the parameter $b_I^\alpha = g/M\alpha_i\phi_0^{Ii}$, that is, on the scalar product of the root α with the vacuum. This interpretation makes breaking patterns evident when looking at the root diagram. Of course, since our vacuum belongs to the Cartan subalgebra, the generators of this subalgebra cannot be broken by this mechanism since they all commute with each others.

The electroweak symmetry breaking takes an $SU(2)\times U(1)$ gauge and breaks it to $U(1)$. Here, we cannot reproduce this breaking pattern because we have no way of breaking a $U(1)$ factor (seen as a direction in the Cartan subalgebra). Let us study the breaking pattern $SU(3) \to SU(2) \times U(1)$ in more detail, we will discuss possible solutions to break residual $U(1)$ factors later on. To this end, we take the vacuum perpendicular to the root β (the horizontal root). The field content after the breaking is given in Table 3.1 with exactly one H^-, one H^+ and one broken boson X_μ which are all in the fundamental representation of $SU(2)$ and charged under $U(1)$. Let us give the Lagrangian explicitly for clarity. In the eigenmass basis, the dynamical terms read

$$\begin{aligned}\mathcal{L}_{dyn} = &-\frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu}) - D_\mu X_\nu^\dagger \cdot D^\mu X^\nu + M_X^2 X^2 \\ &+ D_\mu (H^\pm)^\dagger \cdot D^\mu H^\pm - (m_\pm)^2 (H^\pm)^\dagger . H^\pm \\ &+ Tr(D_\mu \phi_i D^\mu \phi_i) + Tr(D_\mu \rho D^\mu \rho) - M_\rho^2 Tr(\rho^2) \ , \end{aligned} \tag{4.5}$$

where $i = 1, 2$; $F_{\mu\nu}$ is the field strength tensor for the $SU(2)\times U(1)$ gauge; X_μ is the broken boson (gauge-fixed so the dynamical term is no longer antisymmetric in μ, ν); the H's, the ϕ_i's and ρ are scalars. We recall that the masses take the

following form,

$$M_1 = Mf_-(x) \qquad M_2 = Mf_+(x) \tag{4.6}$$
$$M_\rho = M \qquad M_X = Mx \ , \tag{4.7}$$

where,

$$x = \frac{\sqrt{b_1^2 + b_2^2}}{2} \ , \tag{4.8}$$

and we define the function $f_\pm(x)$ as,

$$f_\pm(x) = \sqrt{(1 + x^2 \pm \sqrt{1 + 2x^2})} \ . \tag{4.9}$$

As already mentioned, all fields of this model transform in either of two $SU(2) \times U(1)$ representations: the adjoint (ϕ_i, ρ) and the fundamental ($H^\pm$, X_μ). The adjoint representation is to be understood as the span of ($\frac{\sqrt{3}}{2}\mathbb{I}$, $\frac{\sigma_1}{2}$, $\frac{\sigma_2}{2}$, $\frac{\sigma_3}{2}$). The coupling constants are,

$$g_{SU(2)} = \frac{g}{2} \quad \text{and} \quad g_{U(1)} = \frac{\sqrt{3}}{2} g \ . \tag{4.10}$$

The cubic terms of the scalar potential around the selected vacuum are given by,

$$\begin{aligned} -2gMiTr([\phi_1,\phi_2]\phi_3) = -2gM\Big(& -2\cos(2\theta)\cos(\nu)\mathrm{Im}\,(H_1^\dagger \cdot \phi_2 \cdot H_2) \\ & -2\cos(2\theta)\sin(\nu)\mathrm{Im}\,(H_1^\dagger \cdot \phi_1 \cdot H_2) \\ & +\sin(2\theta)\cos(\nu) H_1^\dagger \cdot \phi_2 \cdot H_1 \\ & +\sin(2\theta)\sin(\nu) H_1^\dagger \cdot \phi_1 \cdot H_1 \\ & -\sin(2\theta)\cos(\nu) H_2^\dagger \cdot \phi_2 \cdot H_2 \\ & -\sin(2\theta)\sin(\nu) H_2^\dagger \cdot \phi_1 \cdot H_2 + 2iTr(\phi_2 \cdot [\rho, \phi_1])\Big) \ , \end{aligned} \tag{4.11}$$

as well as the cubic terms coupling the broken boson X_μ to the scalars,

$$\begin{aligned} & -2g\sin(\theta)\mathrm{Re}\left(\partial_\mu H_1^\dagger \cdot \rho \cdot X^\mu - H_1^\dagger \cdot \partial_\mu \rho \cdot X^\mu\right) \\ & +2g\cos(\theta)\mathrm{Im}\left(X^{\mu\dagger} \cdot \rho \cdot \partial_\mu H_2 - X^{\mu\dagger} \cdot \partial_\mu \rho \cdot H_2\right) \\ & +2g\cos(\theta)\cos(\nu)\mathrm{Im}\left(X^{\mu\dagger} \cdot \phi_1 \cdot \partial_\mu H_1 - X^{\mu\dagger} \cdot \partial_\mu \phi_1 \cdot H_1\right) \\ & -2g\sin(\theta)\cos(\nu)\mathrm{Re}\left(\partial_\mu H_2^\dagger \cdot \phi_1 \cdot X_\mu - H_2^\dagger \cdot \partial_\mu \phi_1 \cdot X_\mu\right) \\ & +2g\cos(\theta)\sin(\nu)\mathrm{Im}\left(X^{\mu\dagger} \cdot \phi_2 \cdot \partial_\mu H_1 - X^{\mu\dagger} \cdot \partial_\mu \phi_1 \cdot H_1\right) \\ & -2g\sin(\theta)\sin(\nu)\mathrm{Re}\left(\partial_\mu H_2^\dagger \cdot \phi_2 \cdot X_\mu - H_2^\dagger \cdot \partial_\mu \phi_1 \cdot X_\mu\right) . \end{aligned} \tag{4.12}$$

In addition there is a large number of quartic terms listed below:

$$
\begin{aligned}
&g^2 Tr([\phi_1,\rho]^2) + g^2 Tr([\phi_2,\rho]^2) + g^2(1-\sin^2(\theta)\cos^2(\nu))H_2^\dagger\cdot\phi_1^2\cdot H_2\\
&+ g^2(1-\sin(\theta)^2\sin^2(\nu))H_2^\dagger\cdot\phi_2^2\cdot H_2\\
&+ g^2(1-\cos^2(\theta)\cos^2(\nu))H_1^\dagger\cdot\phi_1^2\cdot H_1\\
&+ g^2(1-\cos^2(\theta)\sin^2(\nu))H_1^\dagger\cdot\phi_2^2\cdot H_1\\
&+ g^2\cos^2(\theta)H_1^\dagger\cdot\rho^2\cdot H_1\\
&+ g^2\sin^2(\theta)H_2^\dagger\cdot\rho^2\cdot H_2\\
&+ g^2 X_\mu^\dagger\cdot\rho^2\cdot X^\mu + g^2\left((H_2^\dagger\cdot H_2)(X_\mu^\dagger\cdot X_\mu) - (H_2^\dagger\cdot\sigma_3\cdot H_2)(X_\mu^\dagger\cdot\sigma_3\cdot X_\mu)\right)\\
&+ g^2\left((H_1^\dagger\cdot H_1)(X_\mu^\dagger\cdot X_\mu) - (H_1^\dagger\cdot\sigma_3\cdot H_1)(X_\mu^\dagger\cdot\sigma_3\cdot X_\mu)\right)\\
&+ g^2\left((H_1^\dagger\cdot H_1)(H_2^\dagger\cdot H_2) - (H_1^\dagger\cdot\sigma_3\cdot H_1)(H_2^\dagger\cdot\sigma_3\cdot H_2)\right)\ .
\end{aligned}
\tag{4.13}
$$

The masses after renormalization are given in figure 4.2. All the scalars become massive under renormalization. The massless scalars (denoted by ϕ in figure 4.2) acquire a mass of the same order of magnitude as H^-. Another important point is that the fields $\phi_{1,2}^{U(1)}$ do not couple to the gauge at all. These are the fields corresponding to the H_2 direction of figure 4.1, meaning that they commute with all the generators of the new gauge. This model does not include fermions. If we were to couple fermions to the scalars, fermionic loops would lower the masses, making this mass hierarchy possibly different in a more realistic model.

The $SU(3)$ model offers an interesting setup for Gauge-Higgs unification as it allows for three mass scales given by the low mass scalars, the geometrical mass scale $\mathtt{f}^2$ and the Kaluza-Klein modes whose masses go like one over the radii. The model has few free parameters, namely the geometrical mass M (or twist), the Yang-Mills coupling constant g, and the vacuum parameters. One would expect the vacuum parameters to be fixed to some value that depends on the geometrical parameters of the model g and $\mathtt{f}$ if we were to proceed with a complete renormalization of the potential. But this is outside of the scope of the present thesis. We can wonder if there are other interesting choices for the gauge group. Let us turn our attention to this question.

4.2 Rank-2 semi-simple Lie algebras

We are interested in gauge groups that contain $SU(2)\times U(1)$ as a subgroup. Because we cannot break the Cartan subalgebra, we are considering essentially rank 2 algebras, as they are the only ones capable of giving exactly a gauge $SU(2)\times U(1)$ in the present set-up. Using the classification of semi-simple algebras, they are four possibilities for rank 2 algebras: A_2, B_2, D_2 and G_2 (C_2 is not counted here since it is isomorphic to B_2). Their root diagrams are listed in Figure 4.3. A_2 corresponds to $SU(3)$, so we already explored

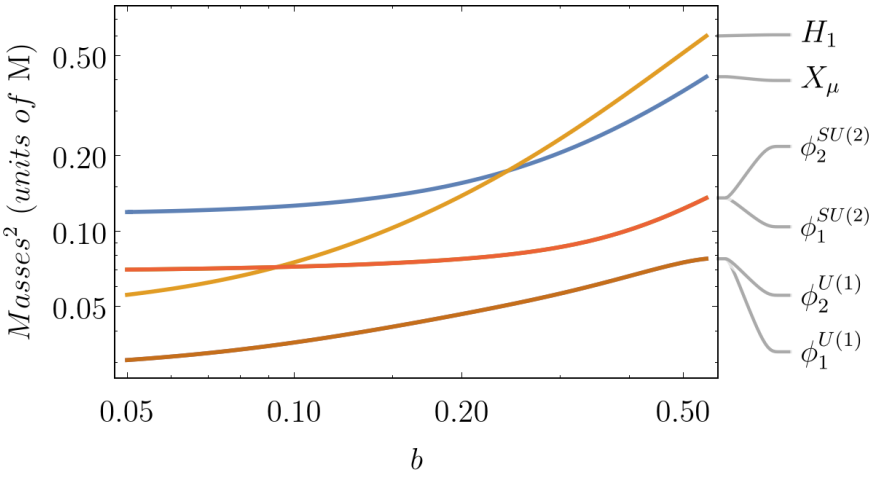
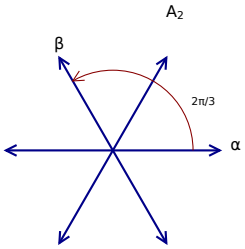
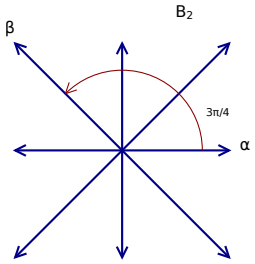


Figure 4.2: One-loop renormalized mass spectrum of the theory as a function of the dimensionless vacuum parameter $b = b_1^\alpha = b_2^\alpha$. The masses are in units of M, the geometrical mass. The value of g sets the renormalization scale. This plot was computed for a coupling constant $g = 10^{-1}$, a UV cutoff $\mu = M$ and the angle $\nu = \pi/4$. The impact of the angle ν is simply to separate the masses of the states ϕ_1 and ϕ_2 by changing the value of some couplings. The massless scalars are of the same order of magnitude as the Higgs-like field, but couplings to fermions could change the mass hierarchy.

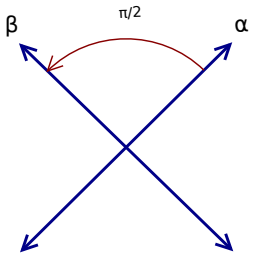
this possibility. Next we have B_2 which has a fundamental representation of $SU(2) \times U(1)$, similarly to A_2. B_2 also has an extra complex scalar field upon breaking to $SU(2) \times U(1)$ which would add unobserved scalar content to the model. This is not a problem as such, but it doesn't really add anything to our model. The Weinberg angle for B_2 is $\pi/4$. D_2 is isomorphic to $SU(2) \times SU(2)$, and upon breaking, would only generate a complex scalar field charged under $U(1)$, and therefore does not contain the appropriate representation for the Higgs field. The last candidate is G_2. G_2 can be seen roughly speaking as two copies of $SU(3)$ coupled in a certain way. If we consider the breaking pattern $G_2 \to SU(2) \times U(1)$, we would obtain two fundamental representation of $SU(2)$ with different hypercharges, along with a charged complex scalar field, singlet of $SU(2)$ (along with the adjoint scalar fields like in the $SU(3)$ model). The additional scalar content would also raise the masses of the scalars. The only advantage of working with G_2 might be the value of the Weinberg angle. When $SU(2) \times U(1)$ is embedded in this algebra, one of the two fundamental representation would have a (tree level) Weinberg angle of $\pi/6$, close to the measured value of $\sim 29°$. By contrast, the $SU(3)$ model has a tree-level Weinberg angle of $\pi/3$, which is significantly larger than the observed value. Of course, one would have to study the running of the angle under renormalization to make any precise statement. Depending on the details of the model, the value could



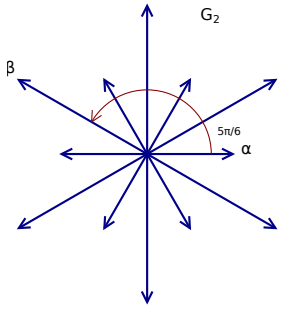
(a) A_2 root system (i.e. $SU(3)$).



(b) B_2 root system. Isomorphic to C_2.



(c) D_2 root system. Isomorphic to $A_1 \times A_1$.



(d) G_2 root system.

Figure 4.3: The four rank 2 semi-simple Lie algebras. The arrows represent the roots of the algebra. C_2 is isomorphic to B_2 since its diagram is a $\pi/2$ rotation of B_2's diagram. D_2 is isomorphic to $A_1 \times A_1$, whose diagram is also a $\pi/2$ rotation of D_2's diagram.

be significantly modified at one-loop.

4.3 Orbifolding and $U(1)$ factors

A limiting aspect of this model is its inability to have a breaking pattern changing the rank of the algebra. As we saw, the Cartan subalgebra of the gauge will always remain unbroken, leaving a product of $U(1)$ factors as residual gauge, one for each direction in the Cartan subalgebra. By contrast, the current model for electroweak symmetry breaking uses $SU(2) \times U(1)$ and breaks the gauge to $U(1)$, lowering the rank by one. A possibility to circumvent this issue would be the use of orbifolds on the geometry of the extra dimensions. In the presence of orbifolds, another mechanism is possible to break the gauge, independent of the scalar potential. This mechanism is known as the Hosotani mechanism or Wilson loop mechanism [25]. The idea is that Wilson lines around a non-contractible loop become physical degrees of freedom. On an orbifold, not all the

Wilson lines survive the orbifolding procedure. Once we have fixed the boundary conditions, so we have fixed the holonomy of the gauge. The phase they acquire going around those loops cannot be gauged away once the boundary conditions are fixed. At the classical level, the field strength is zero, and so this looks like a degeneracy in the vacuum of the theory. But at the quantum level, the effective potential for the Wilson lines develop a non trivial potential whose true vacuum minimizes this potential. If the vacuum is at a non-trivial configuration, the non zero expectation value for the Wilson line gives a mass to some of the 4 dimensional gauge bosons. Possible orbifolds in the case of the Heisenberg manifold were worked out in [37]. In this paper, two possible orbifolds are identified. The first takes place only on the base torus, while the second one takes place only on the fiber. Denoting by y the coordinates on the manifold, these are defined by

$$T : y^{1,2} \to -y^{1,2} \ , \ y^3 \to y^3 \text{ and } \ P : y^{1,2} \to y^{1,2}, y^3 \to y^3 - N y^1 y^2 \ . \quad (4.14)$$

Let us focus on the first transformation since the mechanism can be directly applied in this case. In the case of the T orbifold, we must impose $a = b = 0$, the rest of the parameters are not constrained. The resulting orbifold is $T^2/\mathbb{Z}_2$, which is a square. The space is parametrized by $y^{1,2} \in [0, 1/2]$. There are four fixed fiber circles stemming from the points $(0,0)$, $(1/2,0)$, $(0,1/2)$, $(1/2,1/2)$. We can organize the modes in eigenstates of the orbifold involution. We have first the linear combinations of the modes on the base torus

$$\frac{1}{\sqrt{2}} \left(v_{l,n} \pm v_{-l,-n} \right) \quad (4.15)$$

which are even, odd respectively. For the modes propagating on the fiber, we have the combinations

$$\frac{1}{\sqrt{2}} \left(u_{k,l,n} \pm (-1)^n u_{k,|k|-l,-n} \right) \quad (4.16)$$

that are even, odd respectively. The modes v and u are defined in section 1.3.1. Following the derivation in [48] for the orbifold $S^1/\mathbb{Z}_2$, we will sketch the Hosotani mechanism for this case. First, when a gauge field is going around a non-contractible loop, the field has to go back to its original value only up to a global gauge transformation. So here in our case, it means

$$A_m(x, y^1, y^2) = U_1 A_m(x, y^1 + 1, y^2) U_1^\dagger \quad (4.17)$$

$$A_m(x, y^1, y^2) = U_2 A_m(x, y^1, y^2 + 1) U_2^\dagger \quad (4.18)$$

For simplicity, we will consider $U_2 = \mathbb{I}$, so only loops around the first circle are non-periodic.

Assume we turn on a Wilson line around this first circle such that $\langle A_m \rangle = \frac{1}{g} \Lambda_m$ with constant Λ_m. We can undo the Wilson line with a local gauge transformation $V_m = \exp(i y^1 \Lambda_m)$. We then have

$$\langle A_m^{'} \rangle = V_m \langle A_m \rangle V_m^\dagger \ - \frac{i}{g} V_m \partial_{y^1} V_m^\dagger = 0 \quad (4.19)$$

with $\frac{i}{g}V_m\partial_{y^1}V_m^\dagger = \langle A_m\rangle$. The boundary conditions are modified such that

$$A_m(y^1+1) = A_m(y^1)\ ; \tag{4.20}$$

$$A_m^{'}(y^1+1) = V(y^1+1)A_m(y^1)V^\dagger(y^1+1) - \langle A_m\rangle\ . \tag{4.21}$$

Using the expression for V, we find

$$V(y^1+1) = \exp\left(i(y^1\Lambda+\Lambda)\right) = e^{i\Lambda}V(y^1)\ . \tag{4.22}$$

And so we have

$$A_m^{'}(y^1+1) = e^{i\Lambda}A_m^{'}(y^1)e^{-i\Lambda}\ . \tag{4.23}$$

So the situation with Wilson line and periodic boundary conditions is equivalent to the situation without Wilson lines but with non-periodic boundary conditions. This breaks the gauge symmetry to the subgroup that commutes with Λ_m.

We now impose the orbifold conditions from [25]

$$A_\mu(x,-y^1,y^2) = PA_\mu(x,y^1,y^2)P^\dagger\ ;\ A_m(x,-y^1,y^2) = -PA_m(x,y^1,y^2)P^\dagger\ . \tag{4.24}$$

The first condition is trivial since those fields do not acquire a vacuum expectation values (VEV). In our model, the vacuum solution was of the form $\phi^3 = 0$ and $[\phi^1,\phi^2] = 0$. This resulted in ϕ^1 and ϕ^2 acquiring a VEV. If we take the example of $SU(3)$, we can set

$$P = e^{i\theta\lambda_1} \quad \text{and} \quad \langle A\rangle = \frac{1}{g}\phi_0\lambda_3\ . \tag{4.25}$$

where λ_a is the Gell-Mann basis of $SU(3)$. The condition takes the form

$$\begin{aligned} P\langle\lambda_3\rangle P^\dagger =& e^{i\theta[\lambda_1,\ \cdot\ ]}\lambda_3 \\ =& \lambda_3 + i\theta[\lambda_1,\lambda_3] + \frac{1}{2}[\lambda_1,[\lambda_1,\lambda_3]] + \dots \\ =& \lambda_3 + i\theta(-2i)\lambda_2\ + \frac{1}{2}(i\theta)^2(-2i)2i\lambda_3 + \dots \\ =& \cos(2\theta)\lambda_3 + \sin(2\theta)\lambda_2\ . \end{aligned} \tag{4.26}$$

If we set $\theta = \pi/2$ the condition is fulfilled. This corresponds to the case where the vacuum is aligned with a root. In that case the gauge symmetry is first broken by the VEV to $U(1)\times U(1)$ with the masses for the other roots being all the same. The orbifold breaks the $U(1)$ factor along λ_3 and preserves the factor along λ_8, reducing further the gauge symmetry to $U(1)$.

Another advantage of the orbifolding method is that it allows for chiral fermions. The fixed points of the orbifold, which are really in this case fixed circles because of the fiber at each point, can be used to localize fermions. Fermions at different fixed points can be coupled as we wish, so long as the theory remains anomaly free.

4.4 Higher dimensional nilmanifolds

In this thesis we have focused our attention on the simplest nilmanifold. One could wonder what would be different if we had considered a higher dimensional nilmanifold. Of course, a complete study would require solving the Laplacian on the nilmanifold, which is not a simple task in general. Here we will just try to justify why we would expect the first modes to be dependent only on the structure constants of the tangent algebra. To see this, take an orthonormal frame e_α and its associated co-frame θ^α, define the coefficients of the connection as $\omega^\alpha{}_{\gamma\beta} = \langle \nabla_{e_\gamma} e_\beta, \theta^\alpha \rangle$. The torsion-less condition is $\omega^\alpha{}_{\gamma\beta} - \omega^\alpha{}_{\beta\gamma} = f^\alpha{}_{\gamma\beta}$, where $f^\alpha{}_{\gamma\beta}$ are the structure constants of the nilpotent algebra. The Maurer-Cartan equations take the form

$$d\theta^\alpha + \frac{1}{2} f^\alpha{}_{\gamma\beta} \theta^\gamma \wedge \theta^\beta = 0 \tag{4.27}$$

Now, consider the action of the Laplacian on a one-form of the co-frame. By direct computation, we find

$$\Delta\theta^\alpha = \mathrm{d}^\dagger \mathrm{d}\theta^\alpha = -\frac{1}{2} f^\alpha{}_{\gamma\beta} (-1)^{3n+1} * \mathrm{d} * \left(\theta^\gamma \wedge \theta^\beta\right)) = \frac{1}{2} \sum_{\gamma\beta} f^\alpha{}_{\gamma\beta}\ f^\epsilon{}_{\gamma\beta}\ \delta_{\epsilon\delta}\ \theta^\delta \tag{4.28}$$

The matrix $\sum_{\gamma\beta} f^\alpha{}_{\gamma\beta}\ f^\epsilon{}_{\gamma\beta}$ is real, constant and symmetric, so we know the constant eigen-one-forms will be some linear combination of the co-frame. The eigenvalues will be constant linear combination of the structure constants. It is reasonable to assume that modes with non constant coefficients will have eigenvalues with the typical $1/(r^i)^2$ dependence, where r^i is the radii of the circles of the torus. So the model will be composed at tree level of massive and massless scalar fields in the adjoint representation. Since these low-lying scalars are given by the constant one-forms, the total number of scalar degrees of freedom will be equal to the dimension of the nilmanifold (i.e. the number different θ^α). The potential will have trilinear and quartic couplings, with the various couplings between the scalars given by the structure constants in the appropriate basis. The main consequence would be the addition of several mass scales given by the eigenvalues of the constant modes. The vacuum condition would most likely be more complex. This could also be applied to more general compact group manifolds.

This is the last ingredient we needed for the framework developped in this thesis. We now have everything needed for realistic model building.

Chapter 5

Conclusion

In this thesis, we discussed a new mechanism for spontaneous symmetry breaking based on nilmanifolds. We started with a pure Yang-Mills theory in seven dimensions, the three extra dimensions being the lowest dimensional example of a compact nilmanifold. This space can be understood as a 2-torus with a twisted circle fiber. By studying the spectrum of the Laplace and Dirac operators, we found a infinite discrete basis on which we could expand any form or spinor with periodic boundary conditions. This allowed us to rewrite the seven-dimensional Lagrangian as an effective four dimensional theory by integrating the degrees of freedom on the nilmanifold. We then truncated the Lagrangian, keeping only the lightest modes according to their masses. Unlike the usual case of the torus, we found that the "zero" modes for the one forms contain a non-zero eigenvalue given by the square of the twist `f` of the geometry. The Kaluza-Klein tower retained its typical $1/r^2$ behavior, where r is the typical radius of the circles. The effective theory was composed of three scalar fields in the adjoint representation of the gauge group, one of them being massive, the other two being massless. The scalars are coupled together via a potential present at tree level. We used this potential to induce spontaneous symmetry breaking of the gauge by finding a class of vacua, i.e. minimums, of the theory and developing the scalars around that point. We found a general way to compute the masses of the scalars around any minimum of the class for any gauge group, this allowed us to compute explicitly the potential. With the explicit form of the potential, we were able to renormalize the masses at one-loop. We graphed the renormalized mass spectrum of a model with $SU(3)$ gauge symmetry and computed the model explicitly for the case where the gauge is broken to $SU(2)\times U(1)$. We discussed bigger gauge groups qualitatively and did not found sensible alternatives for the model, at least if taken as it is. We also discussed the use of higher-dimensional nilmanifolds. The constant structures of the nilmanifold are responsible for the masses of the "zero" modes. The separation of the scale of the Kaluza-Klein modes and the zero modes is still possible in this case.

The model should be thought of as another tool for gauge-Higgs unification.

It offers a rich framework for model building of the scalar sector. The pure Yang-Mills theory is rigid, with few free parameters, yet offers a range of possible breaking patterns and mass hierarchies. The model typically predicts three mass scales. The first one is given by the Kaluza-Klein tower and is of order $m_{KK}^2 \sim 1/r^2$. It is at this scale that the geometry becomes apparent. At an arbitrary lower scale sits the scale of the twist of the geometry at $m_{twist}^2 \sim \mathtt{f}^2$. The final scale is the renormalization scale, the masses are roughly of order $m_r^2 \sim N(\mathtt{f}g/4\pi)^2$ where N is the typical number of heavy contributions. We can approximate the number of contributions as $N \sim d^2$ where d is the dimension of the Lie algebra. We can compare this estimation with what we found for the case of $SU(3)$. The graph was plotted for $g = 10^{-1}$, taking $d^2 \sim 10^2$ and $(4\pi)^2 \sim 10^2$, we find $m_r^2 \sim 10^{-2}M^2$, which is in the ballpark of what we found.

The Hosotani mechanism in conjunction with the use of orbifolds could provide additional flexibility to the model. As we have already discussed, topologies with non-contractible loops can have dynamical Wilson loop phases. These phases along with the orbifold condition can generate a spontaneous symmetry breaking. Orbifolds also offer a way to have chiral fermions located at the different fix points (fix circles in our case) of the orbifold.

With all these tools at hand, we have all the ingredients needed for realistic model building, which should be the focus of future work. A more complete study of the breaking patterns compatible with the orbifold is also a necessity. The possible breaking patterns by the orbifold are constrained by our vacuum solution as the vacuum is given by a direction in the Cartan subalgebra of the gauge algebra. A better understanding of the possibilities in this context would be a key point to reproduce the electroweak symmetry breaking.

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