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THÈSE DE DOCTORAT

pour l'obtention de grade de
Docteur de l'Université Gustave Eiffel

Spécialité Mathématiques

au titre de École doctorale MSTIC, n°532
Laboratoire LAMA, UMR 8050

Théorèmes limites pour les champs aléatoires et applications

Présentée par Han-Mai LIN

23/11/2022

**Sous la direction de Florence Merlevède
et Dalibor Volný**

Après les rapports de

Clément Dombry et Cristina Tone

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Abstract

Cette thèse est consacrée au théorème central limite (TCL) et à sa version fonctionnelle pour les champs aléatoires strictement stationnaires.

Dans un premier temps, nous traiterons des champs aléatoires strictement stationnaires non nécessairement adaptés à la filtration naturelle, et lorsque la variance asymptotique ne croit pas nécessairement linéairement. Dans cette partie, nous donnons le TCL et sa version fonctionnelle pour les champs aléatoires strictement stationnaires sous la normalisation $(s_{\underline{n}})_{\underline{n} \in \mathbb{Z}^d}$, où $(s_{\underline{n}})_{\underline{n} \in \mathbb{Z}^d}$ est une suite de réels positifs qui tend vers infini quand \underline{n} tend vers infini.

Dans un deuxième temps, nous nous intéresserons à l'extension du TCL de Gordin [25, 1973] sous une condition \mathbb{L}^1 projective. On verra que dans le cadre des champs aléatoires indexés par \mathbb{Z}^d avec $d \geq 2$, une condition supplémentaire par rapport au cadre des suites ($d = 1$) est nécessaire pour obtenir le TCL.

Dans le dernier chapitre, nous nous intéresserons au comportement asymptotique des sommes partielles associées à des champs aléatoires à valeurs dans des espaces de Banach. En particulier nous établirons la version fonctionnelle du TCL pour les champs de différences d'ortho-martingales ergodiques et à valeurs dans un espace de Banach soit 2-lisse ou de cotype 2. Puis à l'aide d'une approximation ortho-martingale, nous obtiendrons une version fonctionnelle du TCL pour des champs aléatoires strictement stationnaires à valeurs dans un espace \mathbb{L}^p , $p \in [1, 2]$, sous des critères projectifs.

Mots clés :

Champs aléatoires; Processus stationnaires; Ortho-martingales; Théorème limite central; Principe d'invariance faible; Processus de Weiner; Approximation par ortho-martingales; Décomposition en cobord; Critères projectifs; Espaces de Banach; Distribution empirique; Distance \mathbb{L}^p ; Distance de Wasserstein.

Classifications AMS : *60F17, 60G60, 60F25,*

Abstract

This thesis is devoted to the central limit theorem (CLT) and its functional version for strictly stationary random fields.

In the first part, we deal with strictly stationary random fields that are not necessarily adapted to the natural filtration, and when the asymptotic variance does not necessarily grow linearly. In this part, we give the CLT and its functional version for strictly stationary random fields under the normalization $(s_{\underline{n}})_{\underline{n} \in \mathbb{Z}^d}$, where $(s_{\underline{n}})_{\underline{n} \in \mathbb{Z}^d}$ is a sequence of positive reals which tends to infinity when \underline{n} tends to infinity.

In the second part, we are interested in the extension of Gordin's CLT [25, 1973] under a projective \mathbb{L}^1 condition. We will see that for random fields indexed by \mathbb{Z}^d with $d \geq 2$, an additional condition with respect to sequences ($d = 1$) is necessary in order to obtain the CLT.

In the last part, we are interested in the asymptotic behavior of partial sums associated to random fields with values in Banach spaces. In Particular, we obtained the functional version of the CLT for fields of ortho-martingales differences that are ergodic and take values in a 2-smooth or cotype 2 Banach space. Then, using an ortho-martingale approximation, we derive a functional version of the CLT for strictly stationary random fields with values in a \mathbb{L}^p space, $p \in [1, 2]$, under projective criteria.

Keywords:

Random fields; Stationary processes; Ortho-martingales; Central limit theorem; Weak invariance principle; Weiner process; Ortho-martingales approximation; Coboundary decomposition; Projective criteria; Banach spaces; Empirical distribution; \mathbb{L}^p distance; Wasserstein distance.

AMS classification: *60F17, 60G60, 60F25,*

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Chapter 1

Introduction

Dans cette thèse nous nous intéressons à des théorèmes centraux limites pour des champs aléatoires non nécessairement indépendants. Nous travaillerons tout au long de cette thèse avec des champs aléatoires strictement stationnaires. Ce sont des variables aléatoires indexées par \mathbb{Z}^d telles que les lois fini-dimensionnelles sont invariantes par translation du temps. Pour définir les champs aléatoires strictement stationnaires, on introduit naturellement les \mathbb{Z}^d -actions dans un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$. On note les éléments de \mathbb{Z}^d par $\underline{i} := (i_1, \dots, i_d) \in \mathbb{Z}^d$. On définit maintenant le groupe des \mathbb{Z}^d -actions $(T_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ dans un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$. Ces \mathbb{Z}^d -actions sont générées par $\{T_{e_1}, \dots, T_{e_d}\}$ qui sont des transformations bimesurables préservant la mesure \mathbb{P} , où pour tout $1 \leq i \leq d$, e_i est un vecteur de \mathbb{Z}^d avec 1 à i -ième place et 0 ailleurs. Soit alors $(X \circ T_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ un champ aléatoire strictement stationnaire définie sur $(\Omega, \mathcal{A}, \mathbb{P})$. On notera $X \circ T_{\underline{k}}$ par $X_{\underline{k}}$. Une des problématiques de cette thèse sera d'établir des conditions suffisantes pour obtenir la convergence asymptotique suivante :

$$\frac{1}{\sqrt{|\underline{n}|}} S_{\underline{n}}(X) \Rightarrow \sigma \mathcal{N}(0, 1),$$

où $S_{\underline{n}}(X) := \sum_{k_1=1}^{n_1} \dots \sum_{k_d=1}^{n_d} X \circ T_{\underline{k}}$ et $|\underline{n}| = n_1 \times \dots \times n_d$. Ici \Rightarrow désigne la convergence en loi, et $\mathcal{N}(0, 1)$ la loi normale centrée réduite. Dans le cas $d = 1$, et si les X_i sont indépendantes, et identiquement distribuées (i.i.d.) et dans \mathbb{L}^2 , le théorème central limite (TCL) assure que $n^{-1/2} S_n(X)$ converge en loi vers une loi

normale. On dira également que le TCL a lieu pour la suite $(X \circ T_i)_{i \in \mathbb{Z}}$. Dans le cadre indépendant, le théorème peut se généraliser aisément aux champs aléatoires avec $d \geq 2$. On peut également obtenir la version fonctionnelle de ce résultat, aussi appelé principe d'invariance faible (WIP). Plus précisément, on cherche à établir le comportement asymptotique suivant pour le processus des sommes partielles :

$$\left\{ \frac{1}{\sqrt{n}} S_{[nt]}(X) \right\}_{t \in [0,1]} \Rightarrow \{\sigma \mathcal{B}_t\}_{t \in [0,1]} ,$$

où $(\mathcal{B}_t)_{t \in [0,1]}$ est un mouvement brownien. Dans le cadre i.i.d, ce résultat est dû à Donsker [4, 1951]. Pour de nombreux modèles, l'hypothèse d'indépendance n'est pas satisfaite et étendre ces résultats dans le cadre des suites ou des champs dépendants a donc de multiples applications. Dans la suite, nous commencerons par rappeler quelques résultats concernant ces problématiques.

Théorèmes limites pour les suites de variables aléatoires dépendantes

Soit $(\Omega, \mathcal{A}, \mathbb{P})$ un espace probabilisé. Comme dans la section précédente, on définit les suites strictement stationnaires à l'aide d'une variable aléatoire dans $(\Omega, \mathcal{A}, \mathbb{P})$ et d'une transformation T bijective, bimesurable qui préserve la mesure \mathbb{P} . Ainsi on considère la suite $(X_k)_{k \in \mathbb{Z}}$ définie par $X_k = X \circ T^k$ où X est une variable aléatoire dans $(\Omega, \mathcal{A}, \mathbb{P})$. On considèrera aussi une filtration $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ stationnaire définie par $\mathcal{F}_k = T^{-k}(\mathcal{F}_0)$ où \mathcal{F}_0 est une sous tribu de \mathcal{A} telle que $\mathcal{F}_0 \subset T^{-1}(\mathcal{F}_0)$. Les suites de différences de martingale étant un exemple de suites dépendantes, commençons par en rappeler la définition.

Définition (A1). Soit $d \in \mathbb{L}^1(\mathbb{P})$, on définit la suite strictement stationnaire $(d_k := d \circ T^k)_{k \in \mathbb{Z}}$. On dit que $(d_k)_{k \in \mathbb{Z}}$ est une suite de différences de martingale pour la filtration $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ si d est \mathcal{F}_0 -mesurable et

$$\mathbb{E}(d | \mathcal{F}_{-1}) = 0 \text{ p.s. .}$$

D'autre part, $M_n := \sum_{k=1}^n d_k$ est une martingale pour la filtration $(\mathcal{F}_k)_{k \in \mathbb{Z}}$.

Nous sommes ainsi en mesure d'énoncer le théorème central limite pour les martingales de carré intégrable à accroissements strictement stationnaires. Ce théorème est dû à Ibragimov [49, 1963] et Billingsley [48, 1961] (le résultat a été obtenu de façon indépendante).

Théorème (B1). *Soit $(d_k := d \circ T^k)_{k \in \mathbb{Z}}$ une suite strictement stationnaire de différences de martingale pour la filtration $(\mathcal{F}_k)_{k \in \mathbb{Z}}$. Si d est de carré intégrable, et T est ergodique alors*

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n d \circ T^k \Rightarrow \mathcal{N}(0, \sigma^2),$$

où $\sigma^2 = \|d_0\|_2^2$.

Dans [3, 1969], Gordin fut l'un des premiers à exploiter ce théorème dans un contexte de dépendance au sens large, en utilisant notamment la décomposition en cobord. Cette décomposition consiste à écrire :

$$X = d + \theta - \theta \circ T, \tag{1}$$

où $(d \circ T^n)_{n \in \mathbb{Z}}$ est une suite de différences de martingale dans \mathbb{L}^1 , et $([\theta - \theta \circ T] \circ T^n)_{n \in \mathbb{Z}}$ est le cobord. On en déduit alors que

$$S_n = \sum_{k=1}^n d_k + (\theta - \theta_n). \tag{2}$$

On remarque que si la suite $(d_n)_{n \in \mathbb{Z}}$ est ergodique dans $\mathbb{L}^2(\mathbb{P})$ alors elle satisfait le TCL pour les suites des différences de martingale. Ainsi, si le cobord renormalisé est négligeable pour la convergence en loi, alors le TCL pour $(X_n)_{n \in \mathbb{Z}}$ se déduira de celui de $(d_n)_{n \in \mathbb{Z}}$. Gordin a ensuite proposé le TCL suivant à l'aide de cette décomposition en cobord.

Théorème (B2). *Soit $(X_k)_{k \geq 0}$ une suite strictement stationnaire et ergodique dans*

$\mathbb{L}^2(\mathbb{P})$, adaptée à la filtration $(\mathcal{F}_i)_{i \geq 0}$. Suppose que

$$\sum_{k \geq 1} \|\mathbb{E}(X_k | \mathcal{F}_0)\|_2 < \infty. \quad (3)$$

Alors $n^{-1/2}S_n(X)$ converge en loi vers une loi normale centrée de variance $\sigma^2 = \|d_0\|_2^2 := \sum_{k \in \mathbb{Z}} \text{Cov}(X_0, X_k)$.

Heyde [5, 1975] a ensuite montré que le WIP est satisfait dès que l'on a

$$\sum_{k \geq 1} \mathbb{E}(X_k | \mathcal{F}_0) \text{ converge dans } \mathbb{L}^2. \quad (4)$$

D'autre part, Volný [30, 1993] a montré que (4) est une condition nécessaire et suffisante pour obtenir la décomposition (1), avec d et θ dans \mathbb{L}^2 . Une question naturelle est donc de savoir si le TCL reste vrai, si on relaxe (3) demandant seulement la convergence \mathbb{L}^1 . Dans [25, 1973] Gordin a donné une réponse positive à cette question et a établi le TCL suivant :

Théorème (B3) (Gordin 1973). *Soit $(X_k)_{k \in \mathbb{Z}}$ une suite strictement stationnaire et ergodique dans $\mathbb{L}^1(\mathbb{P})$ adapté à la filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$. Supposons que*

$$\sum_{i \geq 0} \mathbb{E}(X_i | \mathcal{F}_0) \text{ converge dans } \mathbb{L}^1(\mathbb{P}) \quad (5)$$

et

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|S_n(X)|)}{\sqrt{n}} < \infty. \quad (6)$$

Alors $n^{-1/2}S_n(X)$ converge en loi vers une loi normale centrée.

C.G. Esseen et S. Janson [23, 1985] ont ensuite donné une démonstration détaillée de ce Théorème en utilisant l'inégalité de Burkholder suivante

Lemme (C1). *Soit $(d_k)_{k \in \mathbb{Z}}$ une suite de différences de martingale dans $\mathbb{L}^1(\mathbb{P})$. Il*

existe une constante universelle C tel que pour tout $n \geq 1$ et $\lambda > 0$, on a

$$\mathbb{P} \left(\left[\sum_{k=1}^n d_k^2 \right]^{1/2} > \lambda \right) \leq \frac{C}{\lambda} \mathbb{E} (|S_n(d)|) .$$

En effet, la condition (5) entraîne la décomposition en cobord de $(X_k)_{k \geq 0}$ dans $\mathbb{L}^1(\mathbb{P})$ (voir [30, 1993]). Il existe donc $(d_k)_{k \geq 0}$ une suite de différences de martingale dans $\mathbb{L}^1(\mathbb{P})$ telle que $S_n(X) = \sum_{k=1}^n d_k - Z_n$ avec $\sup_{n \geq 1} \|Z_n\|_1 < \infty$. Ainsi, si $(d_k)_{k \geq 0}$ est dans $\mathbb{L}^2(\mathbb{P})$, elle satisfait le TCL pour les différences de martingale, et entrainera le TCL pour la suite $(X_k)_{k \geq 0}$. Supposons par simplicité que $n^{-1/2} \mathbb{E} (|S_n(X)|) \leq A < \infty$. Comme $\sup_{n \geq 1} \|Z_n\|_1 < \infty$, on en déduit que $n^{-1/2} \mathbb{E} (|S_n(d)|) \leq B < \infty$. Par le lemme (C1), pour λ assez grand, on a :

$$\mathbb{P} \left(\left[\frac{1}{n} \sum_{k=1}^n d_k^2 \right]^{1/2} \geq \lambda \right) \leq \frac{C}{\lambda} \mathbb{E} \left(\frac{|S_n(d)|}{\sqrt{n}} \right) \leq \frac{CB}{\lambda} < 1. \quad (7)$$

Or, par le théorème ergodique

$$\frac{1}{n} \sum_{k=1}^n d_k^2 \longrightarrow \mathbb{E} (d_0^2) \text{ presque sûrement quand } n \rightarrow \infty .$$

Si $\mathbb{E} (d_0^2) = \infty$, alors $n^{-1} \sum_{k=1}^n d_k^2 \longrightarrow \infty$ presque sûrement, et la premier partie de l'inégalité (7) converge vers 1 quand n converge vers infini, d'où la contradiction. Cependant cette démonstration ne peut pas être généralisée pour les champs aléatoires, car l'inégalité de Burkholder n'a pas d'équivalent pour les ortho-martingales qui peuvent être vues comme la version multi-index des martingales. Le chapitre 2 de cette thèse est consacré à l'étude de l'extension du Théorème (B3) aux champs aléatoires. Afin d'exposer les résultats obtenus, introduisons quelques notations.

Notation pour des champs aléatoires dans \mathbb{Z}^2 . En complément des notations précédentes, on introduit S une transformation bijective, bimesurable qui préserve la mesure \mathbb{P} . On définit ainsi un champ aléatoire strictement stationnaire de la

façon suivante : Soit $(X_{k,\ell})_{k,\ell \in \mathbb{Z}}$ le champ aléatoire strictement stationnaire défini sur $(\Omega, \mathcal{A}, \mathbb{P})$ par $X_{k,\ell} := X \circ T^k \circ S^\ell$, avec X une v.a. définie sur $(\Omega, \mathcal{A}, \mathbb{P})$. Soit $\mathcal{F}_{0,0}$ une sous tribu de \mathcal{A} . Pour tout $i, j \in \mathbb{Z}$, on définit $\mathcal{F}_{i,j} = T^{-i} S^{-j} \mathcal{F}_{0,0}$. On dit que $(\mathcal{F}_{i,j})_{i,j \in \mathbb{Z}}$ est une filtration, si pour tout $j \in \mathbb{Z}$ fixé, on a $\mathcal{F}_{i,j} \subset \mathcal{F}_{k,j}$ pour tout $i \leq k$ et de même pour indice j . Introduisons également la notion de filtration commutante :

Définition (A2). *On dit que la filtration $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$ est complètement commutante si $\forall i, j, h, k \in \mathbb{Z}$, et pour tout v.a. X , on a*

$$\begin{aligned} \mathbb{E} [\mathbb{E} [X | \mathcal{F}_{i,j}] | \mathcal{F}_{h,k}] &= \mathbb{E} [\mathbb{E} [X | \mathcal{F}_{h,k}] | \mathcal{F}_{i,j}] \\ &= \mathbb{E} [X | \mathcal{F}_{i \wedge h, j \wedge k}] \text{ p.s. ,} \end{aligned}$$

avec $i \wedge j = \min(i, j)$.

Nous pouvons maintenant donner la définition des ortho-martingales.

Définition (A3). *Soit $(d_{i,j})_{(i,j) \in \mathbb{Z}^2}$ un champ aléatoire intégrable adapté à une filtration complètement commutante $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$. On dit que $(d_{i,j})_{(i,j) \in \mathbb{Z}^2}$ est un champ de différences d'ortho-martingale pour la filtration $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$, si pour tout $i, j, k, \ell \in \mathbb{Z}$, tel que $k < i$, ou $\ell < j$ on a :*

$$\mathbb{E} (d_{i,j} | \mathcal{F}_{k,\ell}) = 0 \text{ p.s.}$$

De plus, $M_{n_1, n_2} := \sum_{k=1}^{n_1} \sum_{\ell=1}^{n_2} d_{k,\ell}$ est une ortho-martingale pour la filtration $(\mathcal{F}_{k,\ell})_{k,\ell \in \mathbb{Z}}$.

Soit $(X_{i,j})_{(i,j) \in \mathbb{Z}^2}$ un champ aléatoire strictement stationnaire adapté à la filtration complètement commutante $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$ définie sur $(\Omega, \mathcal{A}, \mathbb{P})$. On peut établir une décomposition ortho-martingale-cobord. Elle est due à Gordin [8, 2009] (voir aussi [22, 2016] pour une application au TCL). Pour définir correctement la décomposition, introduisons les notations supplémentaires :

$$\mathcal{F}_{i, -\infty} = T^{-i} \sigma \left\{ \bigcap_{j \in \mathbb{Z}} S^{-j} \mathcal{F}_{0,0} \right\} \text{ et } \mathcal{F}_{-\infty, j} = S^{-j} \sigma \left\{ \bigcap_{i \in \mathbb{Z}} T^{-i} \mathcal{F}_{0,0} \right\} .$$

La décomposition ortho-martingale-cobord doit satisfaire :

$$X = d + (g_1 - g_1 \circ T) + (g_2 - g_2 \circ S) + (g_3 - g_3 \circ T - g_3 \circ S + g_3 \circ T \circ S), \quad (8)$$

où $(d_{i,j} := d \circ T^i \circ S^j)_{(i,j) \in \mathbb{Z}^2}$ est une différence d'ortho-martingales pour la filtration $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$. Pour tout $j \in \mathbb{Z}$ fixé, $([g_2 - g_2 \circ S] \circ T^i \circ S^j)_{i \in \mathbb{Z}}$ est une différence de martingale pour la filtration $(\mathcal{F}_{i,\infty})_{i \in \mathbb{Z}}$, et pour tout $i \in \mathbb{Z}$ fixé, $([g_2 - g_2 \circ S] \circ T^i \circ S^j)_{j \in \mathbb{Z}}$ renormalisé est négligeable pour la convergence en loi. Pour tout $i \in \mathbb{Z}$ fixé, $([g_1 - g_1 \circ T] \circ T^i \circ S_j)_{j \in \mathbb{Z}}$ est une martingale pour la filtration $(\mathcal{F}_{\infty,j})_{j \in \mathbb{Z}}$ et pour tout $j \in \mathbb{Z}$ fixé, $([g_1 - g_1 \circ T] \circ T^i \circ S_j)_{i \in \mathbb{Z}}$ renormalisé est négligeable pour la convergence en loi. Ces deux champs sont appelés les cobord mixtes. Le champ $([g_3 - g_3 \circ T - g_3 \circ S + g_3 \circ T \circ S] \circ T^i \circ S^j)_{(i,j) \in \mathbb{Z}^2}$ est le cobord qui, renormalisé, est négligeable pour la convergence en loi.

Tout comme dans le cadre des suites, il existe une condition nécessaire et suffisante pour obtenir la décomposition (8). En particulier Volný [29, 2018] a montré que

$$\sum_{i,j \geq 0} \mathbb{E}(X_{i,j} | \mathcal{F}_{0,0}) \text{ converge in } \mathbb{L}^1(\mathbb{P}) \quad (9)$$

est une condition suffisante pour obtenir la décomposition (8) (elle est également nécessaire dans le cadre régulier (voir définition 2.1.2)). Dans le chapitre 2, nous établirons le résultat suivant :

Théorème (B4). *Soit $(X_{k,\ell})_{k,\ell \geq 0}$ un champ aléatoire centré et ergodique dans $\mathbb{L}^1(\mathbb{P})$, et adapté à la filtration complètement commutante $(\mathcal{F}_{i,j})_{i,j \geq 0}$. On pose $S_{n_1, n_2}(X) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} X_{i,j}$. Supposons (9) et que les conditions suivantes sont satisfaites :*

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\| \sum_{i=0}^{n-1} X_{i,0} \right\|_1 < \infty, \quad \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{j=0}^{N-1} X_{0,j} \right\|_1 < \infty \quad (10)$$

et

$$\liminf_{n_1 \rightarrow \infty} \liminf_{n_2 \rightarrow \infty} \frac{\mathbb{E}(|S_{n_1, n_2}(X)|)}{\sqrt{n_1 n_2}} < \infty, \quad \liminf_{n_2 \rightarrow \infty} \liminf_{n_1 \rightarrow \infty} \frac{\mathbb{E}(|S_{n_1, n_2}(X)|)}{\sqrt{n_1 n_2}} < \infty. \quad (11)$$

Alors, les v.a. d, g_1, g_2 et g_3 définies dans (8) sont dans $\mathbb{L}^2(\mathbb{P})$.

Cependant contrairement au cadre des suites ($d = 1$), le résultat précédent ne garantit pas le TCL. Cela vient du fait que les cobord mixtes renormalisés ne sont pas forcément négligeables dans les deux directions (voir le contre-exemple 3.1.4). Ainsi une condition supplémentaire est nécessaire pour que les cobords mixtes renormalisés soient négligeables. C'est l'objet du résultat suivant :

Théorème (B5). *Sous les conditions de théorème (B4), et si*

$$\lim_{\min(n_1, n_2) \rightarrow \infty} \frac{\mathbb{E}(|S_{n_1, n_2}(X)|)}{\sqrt{n_1 n_2}} \text{ existe,} \quad (12)$$

alors $(n_1 n_2)^{-1/2} S_{n_1, n_2}(f)$ converge en loi vers une variable gaussienne centrée (qui peut être dégénérée).

D'autres types de conditions projectives peuvent également être considérés et induisent une approximation par une ortho-martingale. Ces conditions n'entraînent pas une décomposition en cobord mais assurent la négligeabilité des "restes renormalisés". En particulier, dans le cadre régulier (voir définition 2.1.2), la condition de Hannan [6, 1973] :

$$\sum_{k \geq 1} \|P_0(X_k)\|_2 < \infty \quad (13)$$

où $P_i(X_k) := \mathbb{E}(X_k | \mathcal{F}_i) - \mathbb{E}(X_k | \mathcal{F}_{i-1})$, est suffisante pour assurer que la suite $(X_n)_{n \in \mathbb{Z}}$ satisfait le WIP (Voir Dedecker et al. [11, 2007]). Mentionnons également le critère projectif dû à Maxwell-Woodroffe. Pour ce faire, commençons par rappeler le résultat dû à Merlevède et al. [47, 2012] :

$$\|S_n(X) - S_n(d)\|_2 \leq n^{1/2} \sum_{k \geq n} \frac{\|\mathbb{E}(S_k | \mathcal{F}_0)\|_2}{n^{3/2}}.$$

On a donc :

Théorème (B6). *Soit $(X_k)_{k \geq 0}$ une suite strictement stationnaire et ergodique dans*

$\mathbb{L}^2(\mathbb{P})$, et $(\mathcal{F}_k)_{k \geq 0}$ la filtration engendré par $(X_k)_{k \geq 0}$. Si on a

$$\sum_{k \geq 1} \frac{\|\mathbb{E}(S_k | \mathcal{F}_0)\|_2}{n^{3/2}} < \infty, \quad (14)$$

alors $n^{-1/2}S_n(X)$ converge en loi vers une loi normale centrée de variance $\sigma^2 = \|X_0\|_2^2$.

Tous les résultats susmentionnés sont sous la normalisation \sqrt{n} . Mais il existe des processus où la normalisation \sqrt{n} n'est pas suffisante pour obtenir le TCL ou le WIP. Par exemple, les processus linéaires. Considérons f une fonction C^1 lipschitzienne, et $(\xi_i)_{i \in \mathbb{Z}}$ une suite de v.a. i.i.d. dans \mathbb{L}^2 . Définissons le processus $(X_k)_{k \in \mathbb{Z}}$ avec

$$X_k = f \left(\sum_{i \geq 0} \frac{\xi_{k-i}}{i+1} \right) - \mathbb{E} \left[f \left(\sum_{i \geq 0} \frac{\xi_{k-i}}{i+1} \right) \right].$$

Dans [11, 2007], Dedecker et al. ont montré que le TCL (et sa version fonctionnelle) a lieu pour la suite $(X_k)_{k \in \mathbb{Z}}$ sous la normalisation $\sqrt{n \log(n)}$. Ce résultat est une conséquence de résultats plus généraux que nous décrivons. Pour une suite des réels positifs $(s_n)_{n \in \mathbb{Z}}$ qui tend vers infini quand n tend vers infini, Dedecker et al. ont établi une approximation martingale pour $(X_k)_{k \in \mathbb{Z}}$, une suite strictement stationnaire non nécessairement adaptée à la filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$.

Théorème (B7). *Soit d une variable aléatoire définie sur $(\Omega, \mathcal{A}, \mathbb{P})$ dans $\mathbb{L}^2(\mathbb{P})$ telle que d est \mathcal{F}_0 -mesurable, et $\mathbb{E}(d | \mathcal{F}_{-1}) = 0$. Les conditions suivantes sont équivalentes*

$$\mathbf{C}_0(s_n): \quad \lim_{n \rightarrow \infty} \left\| \frac{S_n(X)}{s_n} - \frac{1}{\sqrt{n}} \sum_{i=1}^n d \circ T^i \right\|_2 = 0.$$

$$\mathbf{C}_1(s_n): \quad \begin{cases} (a) & \|\mathbb{E}[S_n(X) | \mathcal{F}_0]\|_2 = o(s_n), \\ (b) & \|S_n(X) - \mathbb{E}[S_n(X) | \mathcal{F}_n]\|_2 = o(s_n), \\ (c) & \lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left\| \frac{\sqrt{n}}{s_n} \sum_{k=1-i}^{n-i} P_0(X_k) - m \right\|_2^2 = 0. \end{cases}$$

Si une des conditions est satisfaite et que la transformation T est ergodique, alors $s_n^{-1}S_n(X)$ converge en loi vers $\sigma\mathcal{N}(0, 1)$, où $\mathcal{N}(0, 1)$ est la loi normale centrée réduite, et $\sigma^2 = \mathbb{E}(d^2)$.

Ensuite, en utilisant cette approximation martingale, Dedecker et al. [11, 2007] ont montré le WIP pour les suites strictement stationnaires non nécessairement adaptées à la filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$.

Théorème (B8). *Supposons que $\mathbf{C}_1(s_n)(c)$ est satisfaite, et que pour tout $t \in [0, 1]$ $s_{[nt]}/s_n$ est bornée. Supposons T ergodique et que la condition suivante est satisfaite :*

$$\mathbf{C}_2(s_n): \left\{ \begin{array}{l} (a) \quad \|\max_{1 \leq k \leq n} |\mathbb{E}[S_k(X)|\mathcal{F}_0]|\|_2 = o(s_n), \\ (b) \quad \|\max_{1 \leq k \leq n} |S_n(X) - \mathbb{E}[S_n(X)|\mathcal{F}_n]|\|_2 = o(s_n), \\ (c) \quad \text{pour certaine suite positive } (u_i)_{i \in \mathbb{Z}} \text{ tel que} \\ \quad \frac{\sqrt{n}}{s_n} \sum_{i=-n}^n u_i \text{ est bornée,} \\ \quad \lim_{\lambda \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\sqrt{n}}{s_n} \sum_{i=-n}^n \mathbb{E} \left[\frac{P_0^2(X_i)}{u_i} \mathbf{1}_{P_0^2(X_i) > \lambda u_i^2} \right] = 0, \end{array} \right.$$

alors $\{s_n^{-1}S_{[nt]}, t \in [0, 1]\}$ converge en loi dans $D([0, 1])$ vers $\{\sigma\mathcal{B}_t\}_{t \in [0, 1]}$, avec $\sigma = \sqrt{\mathbb{E}(d^2)}$.

Dans le chapitre 1, nous étendons ces deux résultats aux champs aléatoires strictement stationnaires non nécessairement adaptés à une filtration sous-jacente complètement commutante. Notons que le TCL pour les ortho-martingales a été établi par Volný [28, 2015]. Rappelons son énoncé dans le cadre des champs aléatoire indexés par \mathbb{Z}^2 .

Théorème (B9) (D.Volný 2015). *Soit $X \in \mathbb{L}^2(\mathbb{P})$ tel que $(U^k V^\ell X)_{k, \ell \in \mathbb{Z}}$ est un champ aléatoire de différences d'ortho-martingale pour la filtration complètement commutante $(\mathcal{F}_{i,j})_{i,j \in \mathbb{Z}}$. Si T ou S est ergodique alors $(n_1 n_2)^{-1/2} S_{n_1, n_2}(X)$ converge en loi vers une loi normale centrée de variance $\|X\|_2^2$.*

Il faut aussi noter que la convergence a toujours lieu même si aucune des transformation n'est ergodique. Cependant la somme partielle convergera vers un mélange des lois normales (voir [29, 2018]).

Dans le chapitre 3, nous nous intéresserons à établir des TCL et leurs version fonctionnelles pour des champs aléatoires à valeurs dans des espaces de Banach. Mentionnons quelques résultats antérieurs sur ce sujet : Tout d'abord dans $\mathbb{L}^1(S, \mu)$ l'espace des fonctions réelles μ -intégrables avec μ σ -finie (on supposera $\mathbb{L}^1(S, \mu)$ séparable), Dédé [9, 2009] a montré que pour une suite $(d_i)_{i \in \mathbb{Z}}$ strictement stationnaire et ergodique de différences de martingales à valeurs dans $\mathbb{L}^1(S, \mu)$ telle que,

$$\int_S \|d_0(t)\|_2 \mu(dt) < \infty, \quad (15)$$

$n^{-1/2} \sum_{k=1}^n d_k$ satisfait le TCL dans $\mathbb{L}^1(S, \mu)$. Puis en utilisant une approximation martingale, elle a montré que si $(X_k)_{k \in \mathbb{Z}}$ était une suite strictement stationnaire, ergodique à valeurs dans $\mathbb{L}^1(S, \mu)$ telle que

$$\int_S \|X_0(t)\|_2 \mu(dt) < \infty \text{ et } \sum_{i \in \mathbb{Z}} \int_S \|P_i(X_0(t))\|_2 \mu(dt) < \infty, \quad (16)$$

alors $n^{-1/2} \sum_{k=1}^n X_k$ convergeait en loi dans $\mathbb{L}^1(S, \mu)$ vers une variable gaussienne. Notons que (16) est une condition de type Hannan. Plus récemment, Cuny [35, 2017] a établi des TCL et leurs version fonctionnelles pour des suites de v.a. à valeurs dans des espaces de Banach sous des conditions projectives. Il a d'abord montré le WIP pour les suites de différences de martingale ergodiques à valeurs dans un espace de Banach séparable 2-lisse ou de cotype 2 (les espace \mathbb{L}^1 sont de cotype 2). Puis, il a obtenu le WIP pour les suites strictement stationnaires et ergodiques à valeurs dans un espace de Banach de cotype 2 sous une condition de type Maxwell-Woodrooffe généralisée. Dans $\mathbb{L}^1(S, \mu)$, cette condition s'écrit :

$$\sum_{k \geq 1} \int_S \frac{\|\mathbb{E}(S_k(t) | \mathcal{F}_0)\|_2}{n^{3/2}} \mu(dt) < \infty.$$

Enfin, Dedecker et Merlevède [36, 2015] ont obtenu le WIP dans l'espace $\mathbb{L}^1(S, \mu)$ sous un critère projectif de type \mathbb{L}^1 . Dans le chapitre 3, nous nous intéresserons à l'extension des résultats de Dédé [9] dans le cadre des champs aléatoires indexés par \mathbb{Z}^d et à l'obtention de la version fonctionnelle. On commencera par étendre aux ortho-martingales à valeurs dans des espaces de Banach 2-lisse ou de cotype 2, le WIP obtenu par Cuny [35] dans le cadre des martingales. Puis à l'aide d'une approximation ortho-martingale, nous en déduirons le WIP pour des champs strictement stationnaires et ergodiques à valeurs dans $\mathbb{L}^p(S, \mu)$, $1 \leq p \leq 2$, sous une condition projective de type Hannan.

Les trois chapitres suivants de cette thèse détaillent les résultats énoncés dans l'introduction et font l'objet des trois articles suivants :

1. H. M. Lin, On the weak invariance principle for non-adapted stationary random fields under projective criteria, *Stoch Dyn*, 22 - 05 - 2250013, 2022. <https://doi.org/10.1142/S0219493722500137>.
2. H. M. Lin, F. Merlevède, D. Volný, On the central limit theorem for stationary random fields under \mathbb{L}^1 -projective condition. *Electron. Commun. Probab.* 271 - 12, 2022. <https://doi.org/10.1214/22-ECP486>.
3. H. M. Lin, F. Merlevède, On the weak invariance principle for ortho-martingale in Banach spaces. Application to stationary random fields. *Stoch. Process. Their Appl.* 153 - 198-220, 2022. <https://doi.org/10.1016/j.spa.2022.08.003>.

Chapter 2

On the weak invariance principle for non-adapted stationary random fields under projective criteria

In this chapter, we study the central limit theorem (CLT) and its weak invariance principle (WIP) for sums of stationary random fields non necessarily adapted, under different normalizations. To do so, we first state sufficient conditions for the validity of a suitable ortho-martingale approximation. Then, with the help of this approximation, we derive projective criteria under which the CLT as well as the WIP holds. These projective criteria are in the spirit of the Hannan's condition and are well adapted to linear random fields with ortho-martingale innovations and which exhibit long memory.

2.1 Introduction

Let $(X_i)_{i \in \mathbb{Z}^d}$, $d \geq 1$, be a stationary random field with zero mean and finite variance. Let also $S_{\underline{n}}$ be its associated partial sum with $\underline{n} = (n_1 \cdots n_d) \in \mathbb{N}^d$, defined as

$$S_{\underline{n}} = \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} X_{\underline{i}}.$$

In this paper, we are interested in the central limit theorem and the weak invariance principle. In particular the WIP addresses the question of finding a positive sequence $s_{\underline{n}}$ such that $s_{\underline{n}} \rightarrow \infty$ and

$$\left\{ \frac{S_{[\underline{n} \cdot t]}}{s_{\underline{n}}} \right\}_{t \in [0,1]^d} \Rightarrow \{\sigma W_d(t)\}_{t \in [0,1]^d}, \quad (2.1.1)$$

in $D([0,1]^d)$ equipped with the uniform topology. Here $[\underline{n} \cdot t]$ means $([n_1 t_1], \dots, [n_d t_d])$ and the notation " \Rightarrow " means the convergence in distribution. When the X_k are functions of an i.i.d random field, under the so-called Hannan's condition [6] generalized to \mathbb{Z}^d , Wang and Volný [31] have established that (2.1.1) holds with $s_{\underline{n}}^2 = n_1 \dots n_d$. When $d = 2$, their condition reads as

$$\sum_{(i,j) \in \mathbb{Z}^2} \|P_{0,0}(X_{i,j})\|_2 < \infty, \quad (2.1.2)$$

where $P_{0,0}$ is the projection operator defined by (2.1.4). We also refer to [20] for a quenched version of the CLT under (2.1.2). The result stated in [31] does not allow to consider models for which the normalization in the WIP does not have a linear growth with respect to \underline{n} . Our paper is a step in this direction and aims at relaxing the condition (2.1.2) to still get the WIP. Our results can be viewed as the random field counterparts of those established in Dedecker et al. [11]. To do so, we start with a new ortho-martingale approximation using the notion of commuting filtration (note that there is no natural ordering of future and past in higher dimension).

Before giving a flavor of the results obtained in this paper by considering a simple

example of linear random field with long memory, let us first mention additional earlier results to [11], involving projective type criteria and using the notion of ortho-martingale. First, in [32], Wang and Woodroffe proved the WIP for stationary random fields indexed by rectangular sets with the help of an m -dependent random field approximation. Their condition is in term of conditional expectation and is in the spirit of the Maxwell-Woodroffe's condition in dimension 1 (see [14]). Then, as an application, they proved the WIP for ortho-martingales when they are functions of an i.i.d random field. In [39], using the same method (m -dependent random field approximation) and the physical dependence measure, El Machkouri et al. proved a more general result in the sense that their result apply to a wider class of stationary random fields.

Concerning stationary ortho-martingales, a CLT has been obtained by Volný [28]. More precisely, let us consider the case $d = 2$ and assume that $(T_{i,j})_{(i,j) \in \mathbb{Z}^2}$ is a group of commuting probability preserving transformation of $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose in addition that h is a random variable in \mathbb{L}^2 and that $(h \circ T_{i,j})_{(i,j) \in \mathbb{Z}^2}$ is an ortho-martingale (see definition 4.2.2) with respect to a commuting filtration $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$ (see definition 2.1.1). In this situation, it has been proved in [28] that, as $n_1 \wedge n_2 \rightarrow \infty$, $(n_1 n_2)^{-1/2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} h \circ T_{i,j}$ converges in distribution to a centered gaussian random variable with variance $\|h\|_2^2$ provided that one of the transformation $T_{0,1}$ or $T_{1,0}$ is ergodic (here and along the paper the symbol $a \wedge b$ stands for the minimum between a and b). Our strategy of proof is then to derive a new ortho-martingale approximation for $S_{\underline{n}}/s_{\underline{n}}$ which combined with the CLT in [28] will lead us to new projective criteria ensuring the convergence in law of $S_{\underline{n}}/s_{\underline{n}}$. To derive the corresponding WIP, we then state a new criterion ensuring the uniform integrability of $s_{\underline{n}}^{-2} \max_{k \leq \underline{n}} S_k^2$.

To illustrate our results, let us consider the following linear fields in dimension 2 with long memory. Let $X_{k,\ell} = \sum_{i,j \in \mathbb{N}^2} \frac{1}{(i+1)(j+1)} \xi_{0,0} \circ T_{k-i,\ell-j}$, with T the shift transformation defined as in subsection 1.1 and $\xi_{0,0}$ a centered random variable in \mathbb{L}^2 and such that

$$\mathbb{E}(\xi_{0,0} | \mathcal{F}_{-1,0}) = \mathbb{E}(\xi_{0,0} | \mathcal{F}_{0,-1}) = 0 \text{ a.s. ,}$$

where $\mathcal{F}_{i,j} = \sigma\{\xi_{h,k}, h \leq i, k \leq j\}$. As a consequence of our result, we get that for

this process (2.1.1) holds with the normalization $s_{n_1, n_2} = \sqrt{n_1 n_2} \log(n_1) \log(n_2)$ (see our Corollary 2.4.1). This kind of limiting behavior seems to be new in the context of linear random fields. Even if all our results hold for any $d \geq 1$, for the sake of clarity due to complicated notations, we shall rather state and prove them in case $d = 2$. However, some indications concerning the statements of the results in case $d > 2$ are given in Section 5.

Our paper is organized as follows. In Sections 2 and 3, we state our main results in case of random fields with dimension 2. In Section 4, we apply our results to linear fields with ortho-martingale innovations. Then, in Section 5, we extend our results to higher dimension. Section 6 is devoted to the proofs of the main results. We end this section by giving some notations and definitions used all along the paper.

Notations and Definitions. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. To define properly a stationary filtration, as in [20], it is convenient to start with an auxiliary stationary process $(\xi_{i,j})_{i,j \in \mathbb{Z}^2}$ and then to set

$$\mathcal{F}_{i,j} = \sigma\{\xi_{h,k}, h \leq i, k \leq j\}. \quad (2.1.3)$$

For all i, j , we also define $\mathcal{F}_{\infty, \infty} = \bigvee_{(i,j) \in \mathbb{Z}^2} \mathcal{F}_{i,j}$, $\mathcal{F}_{i, -\infty} = \bigwedge_{j \in \mathbb{Z}} \mathcal{F}_{i,j}$, $\mathcal{F}_{-\infty, j} = \bigwedge_{i \in \mathbb{Z}} \mathcal{F}_{i,j}$, and $\mathcal{F}_{-\infty, -\infty} = \bigwedge_{(i,j) \in \mathbb{Z}^2} \mathcal{F}_{i,j}$.

Définition 2.1.1. *The filtration $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$ defined as above is said to be commuting if $\forall i, j, h, k \in \mathbb{Z}$,*

$$\begin{aligned} \mathbb{E}[\mathbb{E}[X | \mathcal{F}_{i,j}] | \mathcal{F}_{h,k}] &= \mathbb{E}[\mathbb{E}[X | \mathcal{F}_{h,k}] | \mathcal{F}_{i,j}] \\ &= \mathbb{E}[X | \mathcal{F}_{i \wedge h, j \wedge k}] \text{ a.s.}, \end{aligned}$$

provided all the above conditional expectations are well defined.

Note that a filtration defined by an independent and identically distributed (i.i.d) random field is commuting. This kind of filtrations can also be constructed using stationary random fields with independent rows or columns (see [39]). *From now on, we assume that the filtration defined in (2.1.3) is commuting.*

Next we introduce the projection operators defined by: for all $i, j \in \mathbb{Z}$ and $X \in \mathbb{L}^1(\Omega)$,

$$\begin{aligned} P_{i,\tilde{j}}(X) &= \mathbb{E}[X|\mathcal{F}_{i,j}] - \mathbb{E}[X|\mathcal{F}_{i,j-1}] \\ P_{\tilde{i},j}(X) &= \mathbb{E}[X|\mathcal{F}_{i,j}] - \mathbb{E}[X|\mathcal{F}_{i-1,j}]. \end{aligned}$$

Note that, if the filtration is commuting then

$$P_{i,j}(X) = P_{\tilde{i},j} \circ P_{i,\tilde{j}}(X) = P_{i,\tilde{j}} \circ P_{\tilde{i},j}(X).$$

Therefore

$$P_{i,j}(X) = \mathbb{E}[X|\mathcal{F}_{i,j}] - \mathbb{E}[X|\mathcal{F}_{i,j-1}] - \mathbb{E}[X|\mathcal{F}_{i-1,j}] + \mathbb{E}[X|\mathcal{F}_{i-1,j-1}]. \quad (2.1.4)$$

Let us now introduce the shift operators as follows: on $\mathbb{R}^{\mathbb{Z}^2}$, let

$$\begin{aligned} T_{(1)}((x_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}) &= (x_{k+1,\ell})_{(k,\ell) \in \mathbb{Z}^2}, \\ T_{(2)}((x_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}) &= (x_{k,\ell+1})_{(k,\ell) \in \mathbb{Z}^2}. \end{aligned} \quad (2.1.5)$$

We will denote $T_{(1)}^i \circ T_{(2)}^j((x_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2})$ by $T_{i,j}((x_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2})$. Note that the filtration defined above can also be rewritten $\mathcal{F}_{i,j} = T_{-i,-j}(\mathcal{F}_{0,0})$. Now we introduce a stationary random field in the following way. For a real-valued measurable function f on $\mathbb{R}^{\mathbb{Z}^2}$, we define

$$X_{0,0} = f\left((\xi_{i,j})_{(i,j) \in \mathbb{Z}^2}\right) \text{ and } X_{k,\ell} = f\left(T_{k,\ell}(\xi_{i,j})_{(i,j) \in \mathbb{Z}^2}\right) = X_{0,0} \circ T_{k,\ell}. \quad (2.1.6)$$

Définition 2.1.2. *The stationary random field $(X_{i,j})_{(i,j) \in \mathbb{Z}^2}$ will be said to be regular if $X_{0,0}$ is $\mathcal{F}_{\infty,\infty}$ -measurable, and*

$$\mathbb{E}(X_{0,0}|\mathcal{F}_{-\infty,\infty}) = \mathbb{E}(X_{0,0}|\mathcal{F}_{\infty,-\infty}) = \mathbb{E}(X_{0,0}|\mathcal{F}_{-\infty,-\infty}) = 0 \text{ a.s.}$$

We will sometimes use the notation $\sum_{i,j=1}^{n_1,n_2}$ to mean the double sum $\sum_{i=1}^{n_1} \sum_{j=1}^{n_2}$.

2.2 Ortho-Martingale Approximation

Let $(X_{i,j})_{(i,j) \in \mathbb{Z}^2}$ be a stationary random field defined by (2.1.6) and let

$$S_{n_1, n_2} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} X_{i,j} \quad \text{and} \quad \sigma_{n_1, n_2}^2 = \|S_{n_1, n_2}\|_2^2.$$

Définition 2.2.1. *Let m be an integrable $\mathcal{F}_{0,0}$ -measurable function. We say that $(m \circ T_{i,j})_{(i,j) \in \mathbb{Z}^2}$ is a field of ortho-martingale differences with respect to a commuting filtration $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$, if for all $i, j, k, \ell \in \mathbb{Z}$, such that either $k < i$, or $\ell < j$ then*

$$\mathbb{E}(m \circ T_{i,j} | \mathcal{F}_{k,\ell}) = 0 \text{ a.s. .}$$

In addition, $M_{n_1, n_2} := \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m \circ T_{i,j}$ is said to be an ortho-martingale.

Let $(s_{n_1, n_2})_{n_1, n_2 \geq 1}$ be a double indexed sequence of positive numbers such that $s_{n_1, n_2} \rightarrow \infty$ as $n_1 \wedge n_2 \rightarrow \infty$. Below we give an ortho-martingale approximation for the normalized partial sum $S_{n_1, n_2} / s_{n_1, n_2}$.

Théorème 2.2.2. *Let m be a square integrable $\mathcal{F}_{0,0}$ -measurable function such that $\mathbb{E}(m | \mathcal{F}_{-1,0}) = \mathbb{E}(m | \mathcal{F}_{0,-1}) = 0$ a.s. and let $\mathbf{R}_{n_1, n_2} = \|\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, 0}]\|_2 + \|\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{0, n_2}]\|_2 + \|\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{0,0}]\|_2$. Then the following conditions are equivalent*

$$\mathbf{C}_0(s_{n_1, n_2}): \quad \lim_{n_1 \wedge n_2 \rightarrow \infty} \left\| \frac{S_{n_1, n_2}}{s_{n_1, n_2}} - \frac{1}{\sqrt{n_1 n_2}} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m \circ T_{i,j} \right\|_2 = 0.$$

$$\mathbf{C}_1(s_{n_1, n_2}): \quad \left\{ \begin{array}{l} (a) \quad \mathbf{R}_{n_1, n_2} = o(s_{n_1, n_2}), \\ (b) \quad \|S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}]\|_2 = o(s_{n_1, n_2}), \\ (c) \quad \lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{1}{n_1 n_2} \sum_{i,j=1}^{n_1, n_2} \left\| \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h,k=1}^{n_1, n_2} P_{0,0}(X_{h-i, k-j}) - m \right\|_2^2 = 0. \end{array} \right.$$

If one of the conditions is satisfied and $T_{(1)}$ or $T_{(2)}$ is ergodic, then $s_{n_1, n_2}^{-1} S_{n_1, n_2}$ converges in distribution to $\sigma \mathcal{N}$, where \mathcal{N} is a standard Gaussian random variable, and

$\sigma^2 = \mathbb{E}(m^2)$.

Remark 2.2.3. Note that if $\mathbb{E}[m^2] < \infty$ then the convergence in law of $\frac{M_{n_1, n_2}}{\sqrt{n_1 n_2}}$ always takes place, but to a mixture of normal laws (see [19]). However, the additional condition that at least one of the transformation is ergodic ($T_{(1)}$ or $T_{(2)}$) guarantees the convergence towards a normal law. For more detailed results see [28].

Proposition 2.2.4. If $\mathbf{C}_1(\sigma_{\mathbf{n}_1, \mathbf{n}_2})(a)$ and (b) hold then $\frac{\sigma_{n_1, n_2}}{\sqrt{n_1 n_2}}$ is a two-parameter slowly varying function (2p-svf) in the following sense : for any non negative integers k and ℓ

$$\lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{\sigma_{kn_1, \ell n_2}^2}{\sigma_{n_1, n_2}^2} = k \cdot \ell. \quad (2.2.1)$$

In addition, if $\sigma_{n_1, n_2} \rightarrow \infty$ as $n_1 \wedge n_2$ goes to infinity and if

$$\lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{\sigma_{1, n_2}}{\sigma_{n_1, n_2}} = 0, \quad \lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{\sigma_{n_1, 1}}{\sigma_{n_1, n_2}} = 0, \quad (2.2.2)$$

then,

$$\lim_{x \wedge y \rightarrow \infty} \frac{\sigma_{[kx], [\ell y]}^2}{\sigma_{[x], [y]}^2} = k \cdot \ell. \quad (2.2.3)$$

Next we give a similar remark carried from its one dimensional version (see Remark 3 in [11]).

Remark 2.2.5. If $\mathbf{C}_0(\mathbf{s}_{\mathbf{n}_1, \mathbf{n}_2})$ holds then $s_{n_1, n_2}^{-2} \sigma_{n_1, n_2}^2$ converges to $\mathbb{E}(m^2)$. Consequently, if $\mathbb{E}(m^2) > 0$ then $\mathbf{C}_0(\sigma_{\mathbf{n}_1, \mathbf{n}_2})$ holds with $m' = m / \|m\|_2$. Therefore $\mathbf{C}_1(\sigma_{\mathbf{n}_1, \mathbf{n}_2})(a)$, and (b) holds, which implies that $\sigma_{n_1, n_2} / \sqrt{n_1 n_2}$ is a 2p-svf (by using Proposition 2.2.4) and the same is true for $s_{n_1, n_2} / \sqrt{n_1 n_2}$.

Below we give sufficient conditions for $\mathbf{C}_1(s_{n_1, n_2})(c)$ to hold.

Proposition 2.2.6. Condition $\mathbf{C}_1(s_{n_1, n_2})(c)$ is satisfied as soon as the following con-

ditions hold: as $n_1 \wedge n_2 \rightarrow \infty$,

$$\frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=-n_1}^{n_1} \sum_{k=-n_2}^{n_2} P_{0,0}(X_{h,k}) \longrightarrow m \text{ in } \mathbb{L}^2, \quad (2.2.4)$$

$$n_1 \sum_{j=1}^{n_2} \left\{ \left\| \sum_{h=-n_1}^{n_1} \sum_{k=j}^{n_2} P_{0,0}(X_{h,k}) \right\|_2^2 + \left\| \sum_{h=-n_1}^{n_1} \sum_{k=-n_2}^{-j} P_{0,0}(X_{h,k}) \right\|_2^2 \right\} = o(s_{n_1, n_2}^2), \quad (2.2.5)$$

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left\{ \left\| \sum_{h=i}^{n_1} \sum_{k=1-j}^{n_2-j} P_{0,0}(X_{h,k}) \right\|_2^2 + \left\| \sum_{h=-n_1}^{-i} \sum_{k=1-j}^{n_2-j} P_{0,0}(X_{h,k}) \right\|_2^2 \right\} = o(s_{n_1, n_2}^2). \quad (2.2.6)$$

In particular if $X_{0,0}$ is $\mathcal{F}_{0,0}$ -measurable and $\frac{s_{n_1, n_2}}{\sqrt{n_1 n_2}} = h_1(n_1)h_2(n_2)$, with h_1 and h_2 two one-parameter slowly varying functions, then $\mathbf{C}_0(s_{n_1, n_2})$ is satisfied as soon as:

$$\mathbf{R}_{n_1, n_2} = o(s_{n_1, n_2}) \quad \text{and} \quad \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=0}^{n_1} \sum_{k=0}^{n_2} P_{0,0}(X_{h,k}) \rightarrow m \text{ in } \mathbb{L}^2. \quad (2.2.7)$$

As a consequence of the previous proposition, we obtain the following corollary:

Corollary 2.2.7. *Consider the following conditions:*

$$\sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} P_{0,0}(X_{i,j}) \longrightarrow m \text{ in } \mathbb{L}^2, \quad \text{and} \quad \frac{\|S_{n_1, n_2}\|_2}{\sqrt{n_1 n_2}} \longrightarrow \|m\|_2, \quad \text{as } n_1 \wedge n_2 \rightarrow \infty, \quad (2.2.8)$$

$$X_{0,0} \text{ is regular and } \sum_{(i,j) \in \mathbb{Z}^2} \|P_{0,0}(X_{i,j})\|_2 < \infty. \quad (2.2.9)$$

We have the implications $(2.2.9) \Rightarrow (2.2.8) \Rightarrow \mathbf{C}_1(\sqrt{\mathbf{n}_1 \mathbf{n}_2})$.

Note that in dimension one, (2.2.9) is the so-called Hannan's condition. The condition (2.2.8) can be viewed as a non-adapted version of Theorem 5 of [17].

2.3 Weak invariance principle

We start this section by giving sufficient conditions for the sequence $\left(s_{n_1, n_2}^{-2} \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} S_{k, l}^2 \right)_{(n_1, n_2) \in \mathbb{N}^2}$ to be uniformly integrable. With this aim, let us introduce the following notation:

$$R_{n_1, n_2}(S_{k, l}) = \mathbb{E}(S_{k, l} | \mathcal{F}_{n_1, 0}) + \mathbb{E}(S_{k, l} | \mathcal{F}_{0, n_2}) - \mathbb{E}(S_{k, l} | \mathcal{F}_{0, 0}).$$

Proposition 2.3.1. *The sequence $\left(s_{n_1, n_2}^{-2} \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} S_{k, l}^2 \right)_{(n_1, n_2) \in \mathbb{N}^2}$ is uniformly integrable if the following conditions are satisfied:*

$$\mathbf{C}_2(s_{n_1, n_2}): \left\{ \begin{array}{l} (a) \quad \left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} |R_{n_1, n_2}(S_{k, l})| \right\|_2 = o(s_{n_1, n_2}), \\ (b) \quad \left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} |S_{k, l} - \mathbb{E}[S_{k, l} | \mathcal{F}_{n_1, n_2}]| \right\|_2 = o(s_{n_1, n_2}), \\ (c) \quad \text{for some positive sequence } (u_{i, j})_{i, j \in \mathbb{Z}^2} \text{ such that} \\ \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} u_{i, j} \text{ is bounded,} \\ \lim_{\lambda \rightarrow \infty} \limsup_{n_1 \wedge n_2 \rightarrow \infty} \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} \mathbb{E}[U_{0, 0}^2(i, j, \lambda)] = 0, \end{array} \right.$$

where $U_{0, 0}^2(i, j, \lambda) := \frac{P_{0, 0}^2(X_{i, j})}{u_{i, j}} \mathbf{1}_{P_{0, 0}^2(X_{i, j}) > \lambda u_{i, j}^2}$.

Remark 2.3.2. *Note that if the first part of $\mathbf{C}_2(s_{n_1, n_2})(c)$ holds then its second part does as soon as $\left(\frac{P_{0, 0}(X_{i, j})}{u_{i, j}} \right)_{(i, j) \in \mathbb{Z}^2}$ is a square uniformly integrable family.*

We give now sufficient conditions for $\mathbf{C}_2(s_{n_1, n_2})(a)$ and (b) to hold.

Proposition 2.3.3. *$\mathbf{C}_2(s_{n_1, n_2})(a)$ and (b) hold as soon as $X_{0, 0}$ is regular and*

$$\mathbf{C}_3(s_{n_1, n_2}) : \begin{cases} (a) & \sum_{u=1}^{n_1} \sum_{v=1}^{n_2} \sqrt{\sum_{|i| \geq u} \sum_{|j| \geq v} \|P_{0,0}(X_{i,j})\|_2^2} = o(s_{n_1, n_2}), \\ (b) & \sqrt{n_1} \sum_{i=1}^{n_1-1} \sum_{v=1}^{n_2} \sqrt{\sum_{|j| \geq v} \|P_{0,0}(X_{i,j})\|_2^2} = o(s_{n_1, n_2}), \\ (c) & \sqrt{n_2} \sum_{u=1}^{n_1} \sum_{j=1}^{n_2-1} \sqrt{\sum_{|i| \geq u} \|P_{0,0}(X_{i,j})\|_2^2} = o(s_{n_1, n_2}). \end{cases}$$

In addition, in the adapted case, $\mathbf{C}_3(s_{n_1, n_2})$ holds provided

$$\overline{\mathbf{C}_3(s_{n_1, n_2})} : \begin{cases} (a) & \sqrt{n_1} \sum_{u=1}^{n_1} \sum_{v=1}^{n_2} \frac{1}{\sqrt{u}} \|\mathbb{E}(X_{u,v} | \mathcal{F}_{0,0})\|_2 = o(s_{n_1, n_2}), \\ (b) & \sqrt{n_2} \sum_{u=1}^{n_1} \sum_{v=1}^{n_2} \frac{1}{\sqrt{v}} \|\mathbb{E}(X_{u,v} | \mathcal{F}_{0,0})\|_2 = o(s_{n_1, n_2}). \end{cases}$$

Remark 2.3.4. If s_{n_1, n_2}^2 is regular enough, so for instance if there exist α and β in $]0, 1[$ such that $\frac{n_1^{1-\alpha} n_2^{1-\beta} (n_1 + n_2)}{s_{n_1, n_2}^2} \rightarrow \infty$ as $n_1 \wedge n_2 \rightarrow \infty$, then Hölder's inequality combined with Kronecker's Lemma for double indexed sequences implies that $\overline{\mathbf{C}_3(s_{n_1, n_2})}$ is satisfied as soon as $\sum_{u,v \geq 1} \frac{uv}{s_{u,v}^2} \|\mathbb{E}(X_{u,v} | \mathcal{F}_{0,0})\|_2^2 < \infty$.

According to the lemma stated in page 88 in [1], the uniform integrability of $\left(s_{n_1, n_2}^{-2} \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} S_{k,l}^2 \right)_{(n_1, n_2) \in \mathbb{N}^2}$ implies the tightness of $\{s_{n_1, n_2}^{-1} S_{[n_1 t_1], [n_2 t_2]}, (t_1, t_2) \in [0, 1]^2\}$ in $D([0, 1]^2)$. Hence, Theorem 2.2.2 together with Proposition 2.3.1 give the following weak invariance principle.

Théorème 2.3.5. Suppose that $s_{[n_1 t_1], [n_2 t_2]} / s_{n_1, n_2}$ is bounded for any $t \in [0, 1]^2$. If $\mathbf{C}_1(s_{n_1, n_2})(c)$ and $\mathbf{C}_2(s_{n_1, n_2})$ hold and $T_{(1)}$ or $T_{(2)}$ is ergodic, then $\{s_{n_1, n_2}^{-1} S_{[n_1 t_1], [n_2 t_2]}, t \in [0, 1]^2\}$ converges in distribution in $D([0, 1]^2)$ to σW_2 where W_2 is a 2-parameter Brownian sheet and $\sigma = \sqrt{\mathbb{E}(m^2)}$.

Remark 2.3.6. The proof reveals that if we assume that $\mathbf{C}_1(s_{n_1, n_2})(c)$ holds with m such that $\mathbb{E}(m^2) > 0$, then we do not need to assume the boundedness of $(s_{[n_1 t_1], [n_2 t_2]} / s_{n_1, n_2})_{t \in [0, 1]^2}$ in the statement of the previous theorem.

2.4 Application to linear fields of ortho-martingales

Define $X_{0,0} = \sum_{i,j \in \mathbb{Z}^2} a_{i,j} \xi_{0,0} \circ T_{-i,-j}$ where $(a_{i,j})_{(i,j) \in \mathbb{Z}^2}$ a double-indexed sequence of real numbers in ℓ^2 , and $\xi_{0,0}$ is a regular $\mathcal{F}_{0,0}$ -measurable function in \mathbb{L}^2 such that

$$\mathbb{E}(\xi_{0,0} | \mathcal{F}_{-1,0}) = \mathbb{E}(\xi_{0,0} | \mathcal{F}_{0,-1}) = 0 \text{ a.s. .}$$

Let $\xi_{k,\ell} = \xi_{0,0} \circ T_{k,\ell}$ and

$$S_{n_1, n_2} = \sum_{k=1}^{n_1} \sum_{\ell=1}^{n_2} X_{k,\ell} = \sum_{k=1}^{n_1} \sum_{\ell=1}^{n_2} \sum_{(i,j) \in \mathbb{Z}^2} a_{i,j} \xi_{k-i, \ell-j}.$$

Corollary 2.4.1. *Let $\xi_{0,0}$, $(a_{k,\ell})_{k,\ell \in \mathbb{Z}^2}$ and $X_{0,0}$ be defined as above. We then define the double-indexed sequence $s_{n_1, n_2} = \sqrt{n_1 n_2} \left| \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} a_{i,j} \right|$. Assume the following conditions:*

$$\limsup_{n_1 \wedge n_2 \rightarrow \infty} \frac{\sum_{k=-n_1}^{n_1} \sum_{\ell=-n_2}^{n_2} |a_{k,\ell}|}{\left| \sum_{k=-n_1}^{n_1} \sum_{\ell=-n_2}^{n_2} a_{k,\ell} \right|} < \infty, \quad (2.4.1)$$

$$\sum_{u=1}^{n_1} \sum_{v=1}^{n_2} \sqrt{\sum_{|i| \geq u} \sum_{|j| \geq v} a_{i,j}^2} = o(s_{n_1, n_2}), \quad (2.4.2)$$

$$\sqrt{n_1} \sum_{i=1-n_1}^{n_1-1} \sum_{v=1}^{n_2} \sqrt{\sum_{|j| \geq v} a_{i,j}^2} = o(s_{n_1, n_2}), \quad (2.4.3)$$

and

$$\sqrt{n_2} \sum_{u=1}^{n_1} \sum_{j=1-n_2}^{n_2-1} \sqrt{\sum_{|i| \geq u} a_{i,j}^2} = o(s_{n_1, n_2}). \quad (2.4.4)$$

Then $\{s_{n_1, n_2}^{-1} S_{[n_1 t_1][n_2 t_2]}, (t_1, t_2) \in [0, 1]^2\}$ converges in distribution in $D([0, 1]^2)$ to $\sqrt{\mathbb{E}(\xi_{0,0}^2)} W_2$, where W_2 is a 2-parameter Brownian sheet.

Remark 2.4.2. *If Condition (2.4.1) is satisfied and $\sum_{i,j \in \mathbb{Z}} |a_{i,j}| < \infty$ instead of assuming (2.4.2)-(2.4.4), then the conclusion of Corollary 2.4.1 follows from Theorem 5.1 in [31] since, according to Condition (2.4.1), as $n_1, n_2 \rightarrow \infty$, $s_{n_1, n_2} / \sqrt{n_1 n_2} \rightarrow \left| \sum_{i,j \in \mathbb{Z}} a_{i,j} \right| > 0$.*

Comment 2.4.3. Condition (2.4.1) of the above corollary does not allow the following possibility: $\sum_{k=-n_1}^{n_1} \sum_{\ell=-n_2}^{n_2} |a_{k,\ell}|$ diverges but $\sum_{k=-n_1}^{n_1} \sum_{\ell=-n_2}^{n_2} a_{k,\ell}$ converges. For instance let us consider the following sequence: $a_{i,j} = (-1)^{i+j}(i+j)^{-2}$ for $i, j \geq 1$ and 0 otherwise, then Corollary 2.4.1 does not apply. More generally, this is also the case for double alternating series $\sum_{i,j} a_{i,j}$ (in the sense that each row and column is an alternating simple series) that are in addition monotonic which means that $|a_{i,j}| \leq |a_{n,m}|$ for $i \geq n$ and $j \geq m$. Indeed such series are not absolutely convergent if $\sum_{i,j} |a_{i,j}| = \infty$ but convergent as soon as $|a_{i,j} + a_{i+1,j}| \geq |a_{i,j+1} + a_{i+1,j+1}|$ for all i, j (see Meyer [16] for more details) and then, in this situation, condition (2.4.1) fails. However, by simple algebra, we infer that, for such series, condition (2.2.7) holds with $s_{n_1, n_2} \sim \sqrt{n_1 n_2} \left| \sum_{i,j \geq 0} a_{i,j} \right|$ (and then $s_{n,n}^{-1} S_{n,n}$ converges in distribution to a Gaussian random variable) as soon as

$$\sum_{u,v \geq 0} \left(\sum_{k=1}^{n_1} \sum_{\ell=1}^{n_2} a_{k+u, \ell+v} \right)^2 + \sum_{u=1}^{n_1} \left(\sum_{k=1}^{n_1} \sum_{\ell=1}^{n_2} a_{k+u, \ell+v} \right)^2 = o(n_1 n_2)$$

and

$$\sum_{u=1}^{n_1} \sum_{v \geq 0} \left(\sum_{k=u}^{n_1} \sum_{\ell=1}^{n_2} a_{k+u, \ell+v} \right)^2 + \sum_{v=1}^{n_2} \sum_{u \geq 0} \left(\sum_{k=1}^{n_1} \sum_{\ell=v}^{n_2} a_{k+u, \ell+v} \right)^2 = o(n_1 n_2).$$

Clearly, both of the above conditions hold if $\sum_{u,v \geq 0} \left(\sum_{k \geq u} \sum_{\ell \geq v} a_{k,\ell} \right)^2 < \infty$ which is satisfied as soon as $\sum_{i,j \geq 0} a_{i,j}^2 < \infty$ (to see this, use an Abel transformation and the monotonicity conditions on the sequence $(a_{i,j})$).

2.4.1 Examples

Example 2.4.4. As quoted in the introduction, the double indexed sequence $(a_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}$ defined by $a_{k,\ell} = \frac{1}{k+1} \frac{1}{\ell+1}$ for $k \geq 0$ and $\ell \geq 0$, and 0 otherwise, satisfies the condition of Corollary 2.4.1 with $s_{n_1, n_2} \sim \sqrt{n_1 n_2} \log(n_1) \log(n_2)$.

Example 2.4.5. Let us consider now another example in the same spirit but for which the normalizing sequence s_{n_1, n_2} is not a product of a function of n_1 times a function

of n_2 . The double indexed sequence $(a_{k,\ell})_{k,\ell \in \mathbb{Z}^2}$ is this time defined by $a_{k,\ell} = \frac{1}{(k+\ell)^2}$ for k and $\ell > 0$, and 0 otherwise. In this case the conditions of Corollary 2.4.1 are satisfied with $s_{n_1, n_2} \sim \sqrt{n_1 n_2} \log \left(\frac{n_1 n_2}{n_1 + n_2} \right)$.

Comment 2.4.6. It is also possible to apply our Theorem 2.3.5 to non linear random fields. For instance, let us consider the following Volterra process

$$X_{k,\ell} = \sum_{i,j \geq 0} \xi_{k-i,\ell-j} (\alpha_{i,j} \xi_{k-i-1,\ell-j} + \beta_{i,j} \xi_{k-i,\ell-j-1}),$$

where $(\xi_{k,\ell})_{k,\ell}$ is a sequence of ortho-martingale differences. In this case, setting $s_{n_1, n_2} = \sqrt{n_1 n_2} \left| \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} (\alpha_{i,j} + \beta_{i,j}) \right|$ and assuming that there exist reals α and β such that

$$\lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \alpha_{i,j}}{\left| \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} (\alpha_{i,j} + \beta_{i,j}) \right|} = \alpha \text{ and } \lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \beta_{i,j}}{\left| \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} (\alpha_{i,j} + \beta_{i,j}) \right|} = \beta, \quad (2.4.5)$$

one sees that condition (2.2.4) holds with $m = \xi_{0,0}(\alpha \xi_{-1,0} + \beta \xi_{0,-1})$. Moreover, following the lines of the proof of Corollary 2.4.1, we infer that, if in addition to (2.4.5), we assume that

$$\lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} (|\alpha_{i,j}| + |\beta_{i,j}|)}{\left| \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} (\alpha_{i,j} + \beta_{i,j}) \right|} < \infty$$

and conditions (2.4.2)-(2.4.4) are satisfied with $\alpha_{i,j}^2 + \beta_{i,j}^2$ replacing $a_{i,j}^2$ in the numerators, then $\{s_{n_1, n_2}^{-1} S_{[n_1 t_1][n_2 t_2]}, (t_1, t_2) \in [0, 1]^2\}$ converges in distribution in $D([0, 1]^2)$ to $c_{\alpha, \beta} W_2$, where W_2 is a 2-parameter Brownian sheet and $c_{\alpha, \beta}^2 = \mathbb{E}(m^2)$.

2.5 Extension to higher dimension $d > 2$

In this section, we extend our results to dimension $d > 2$. To do so, we first introduce some notations in dimension d . Let $\underline{i}, \underline{j}$ be two elements of \mathbb{Z}^d . For $\underline{i} := (i_1, \dots, i_d)$,

and $\underline{j} := (j_1, \dots, j_d)$, we set

- $\underline{i} + \underline{j} := (i_1 + j_1, \dots, i_d + j_d)$
- $\underline{i} \leq \underline{j}$ mean that $i_k \leq j_k$ for all $1 \leq k \leq d$
- $\underline{i} \wedge \underline{j} := (i_1 \wedge j_1, \dots, i_d \wedge j_d)$
- $|\underline{n}| := (n_1 \times \dots \times n_d)$

Note also that the integers of \mathbb{Z}^d will be in bold, for example $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^d$.

Next for $\underline{n} = (n_1, \dots, n_d)$, and for every $1 \leq k \leq d$, we set $\underline{n}^k = (0, \dots, n_k, \dots, 0)$, $\underline{n}^{k_1 k_2} = (0, \dots, 0, n_{k_1}, 0, \dots, 0, n_{k_2}, 0, \dots, 0)$ and so on. Also $\underline{n}^{-k} = (n_1, \dots, n_{k-1}, 0, n_{k+1}, \dots, n_d)$.

As in dimension 2, let $\mathcal{F}_{\underline{i}} = \sigma(\xi_{\underline{j}}, \underline{j} \leq \underline{i})$ where $(\xi_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ is an auxiliary stationary process. We assume that the filtration is commuting. The shift transformation $(T_{(1)}, \dots, T_{(d)})$ is defined as follows: on $\mathbb{R}^{\mathbb{Z}^d}$, for $1 \leq j \leq d$,

$$T_{(j)}((x_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d}) = (x_{k_1, \dots, k_{j-1}, k_j+1, k_{j+1}, \dots, k_d})_{\underline{k} \in \mathbb{Z}^d}. \quad (2.5.1)$$

For $\underline{i} = (i_1, \dots, i_d)$, we set $T_{(1)}^{i_1} \circ \dots \circ T_{(d)}^{i_d}((w_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d})$ by $T_{\underline{i}}((w_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d})$. Next, for a real valued measurable function f on $\mathbb{R}^{\mathbb{Z}^d}$, we define

$$X_{\mathbf{0}} = f((\xi_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}), \text{ and } X_{\underline{k}} = f(T_{\underline{k}}(\xi_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}) = X_{\mathbf{0}} \circ T_{\underline{k}}.$$

Let $S_{\underline{n}}$ be its associated partial sum defined as $S_{\underline{n}} = \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} X_{\underline{i}}$.

We now clarify the rest $R_{\underline{n}}(S_{\underline{n}})$ in the decomposition of $S_{\underline{n}}$ with the help of the projective operators. We have

$$S_{\underline{n}} = \sum_{k_1=1}^{n_1} \dots \sum_{k_d=1}^{n_d} P_{\underline{k}}(S_{\underline{n}}) + R_{\underline{n}}(S_{\underline{n}}),$$

where

$$\begin{aligned}
R_{\underline{n}}(S_i) &= (-1)^{d-1} \mathbb{E}[S_i | \mathcal{F}_0] + (-1)^{d-2} \sum_{k=1}^d \mathbb{E}[S_i | \mathcal{F}_{\underline{n}^k}] + (-1)^{d-3} \sum_{k_1=1}^{d-1} \sum_{k_2 > k_1}^d \mathbb{E}[S_i | \mathcal{F}_{\underline{n}^{k_1 k_2}}] \\
&\quad + (-1)^{d-4} \sum_{k_1=1}^{d-2} \sum_{k_2 > k_1}^{d-1} \sum_{k_3 > k_2}^d \mathbb{E}[S_i | \mathcal{F}_{\underline{n}^{k_1 k_2 k_3}}] + \cdots \\
&\quad + (-1)^{d-d} \sum_{k_1=1}^{d-(d-1)+1} \sum_{k_2 > k_1}^3 \cdots \sum_{k_{d-2} > k_{d-3}}^{d-1} \sum_{k_{d-1} > k_{d-2}}^d \mathbb{E}[S_i | \mathcal{F}_{\underline{n}^{k_1 k_2 \dots k_{d-1}}}] .
\end{aligned}$$

To simplify its expression, let us introduce some specific notations. We define the set D of integers from 1 to d , ($D := (1, \dots, d)$). Next for any $1 \leq k \leq d$, Q_k will designate a set of the type $\{i_1, \dots, i_k\}$ with $i_u \neq i_v$ for $u \neq v$ and $i_u \in D$ for any $1 \leq u \leq k$.

Notice that for k fixed, there exist different Q_k since there are $\binom{d}{k}$ ways to chose a set of k different integers among D . Thus we will numerate each combination of Q_k by Q_k^ℓ for $1 \leq \ell \leq \binom{d}{k}$, and these combinations will be numerated in the natural increasing way. For example for $k = 1$, $Q_1^1 = \{1\}$, $Q_1^2 = \{2\} \dots Q_1^d = \{d\}$ and for $k = 2$, $Q_2^1 = \{1, 2\}$, $Q_2^2 = \{1, 3\} \dots Q_2^{d-1} = \{1, d\}$, $Q_2^{(d-1)+1} = \{2, 3\} \dots Q_2^{(d-1)+(d-2)+1} = \{3, 4\}$. In addition $(Q_k^\ell)^C$ will denote the complement of Q_k^ℓ in D .

For $Q_k^\ell := \{e_1 \dots e_k\}$, we denote $\mathcal{F}_{Q_k^\ell}$ by $\mathcal{F}_{\underline{n}^{e_1 \dots e_k}}$. Hence, setting $u_{k,d} = \binom{d}{k}$, we have

$$R_{\underline{n}}(S_i) = (-1)^{d-1} \mathbb{E}[S_i | \mathcal{F}_0] + \sum_{k=1}^{d-1} (-1)^{d-1-k} \sum_{\ell=1}^{u_{k,d}} \mathbb{E}(S_i | \mathcal{F}_{Q_k^\ell}). \quad (2.5.2)$$

2.5.1 Ortho-Martingale approximation and CLT in dimension $d \geq 2$

Théorème 2.5.1. *Let m be a \mathcal{F}_0 -measurable function such that, $\mathbb{E}(m | \mathcal{F}_{-1,0,\dots,0}) = \mathbb{E}(m | \mathcal{F}_{0,-1,0,\dots,0}) = \cdots = \mathbb{E}(m | \mathcal{F}_{0,\dots,0,-1}) = 0$. Then the two following conditions are equivalent*

$$\begin{aligned}
\mathbf{C}_0^d(s_{\underline{n}}): \quad & \lim_{\underline{n} \rightarrow \infty} \left\| \frac{S_{\underline{n}}}{s_{\underline{n}}} - \frac{1}{\sqrt{\underline{n}}} \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} m \circ T_{i_i} \right\|_2 = 0. \\
\mathbf{C}_1^d(s_{\underline{n}}): \quad & \left\{ \begin{array}{l}
(a) \quad \text{for every } 0 \leq k \leq d-1, \text{ and } 1 \leq \ell \leq \binom{d}{k} \\
\left\| \mathbb{E} \left(S_{\underline{n}} | \mathcal{F}_{Q_k}^\ell \right) \right\|_2 = o(\sigma_{\underline{n}}), \\
(b) \quad \|S_{\underline{n}} - \mathbb{E}[S_{\underline{n}} | \mathcal{F}_{\underline{n}}]\|_2 = o(s_{\underline{n}}), \\
(c) \quad \lim_{\underline{n} \rightarrow \infty} \frac{1}{\underline{n}} \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} \left\| \frac{\sqrt{\underline{n}}}{s_{\underline{n}}} \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} P_\theta(X_{\underline{k}-i}) - m \right\|_2^2 = 0.
\end{array} \right.
\end{aligned}$$

If one of the conditions is satisfied and one of $T_{(1)}, \dots, T_{(d)}$ is ergodic, then $s_{\underline{n}}^{-1} S_{\underline{n}}$ converges in distribution to $\sigma \mathcal{N}$, where \mathcal{N} is a standard gaussian random variable, and $\sigma^2 = \mathbb{E}(m^2)$.

Remark 2.2.4 writes in dimension $d > 2$ as follows:

Proposition 2.5.2. *If $\mathbf{C}_1^d(s_{\underline{n}})(a)$ and (b) hold, then*

$$\lim_{\underline{n} \rightarrow \infty} \frac{\sigma_{k_1 n_1, \dots, k_d n_d}^2}{\sigma_{n_1, \dots, n_d}^2} = \lim_{\underline{n} \rightarrow \infty} \frac{\mathbb{E} \left(S_{k_1 n_1, \dots, k_d n_d}^2 \right)}{\mathbb{E} \left(S_{n_1, \dots, n_d}^2 \right)} = k_1 \cdots k_d. \quad (2.5.3)$$

In addition, if $\sigma_{n_1, \dots, n_d} \rightarrow \infty$ as $n_1 \wedge \dots \wedge n_d \rightarrow \infty$ and if for every $1 \leq k \leq d-1$, and $1 \leq \ell \leq u_{k,d}$, we have

$$\lim_{n_1 \wedge \dots \wedge n_d \rightarrow \infty} \frac{\sigma_{Q_k}^\ell}{\sigma_{n_1, \dots, n_d}} = 0, \quad (2.5.4)$$

where for example $\sigma_{Q_1} = \sigma_{1, n_2, \dots, n_d}$. Then,

$$\lim_{x_1 \wedge \dots \wedge x_d \rightarrow \infty} \frac{\sigma_{[k_1 x_1], \dots, [k_d x_d]}^2}{\sigma_{[x_1], \dots, [x_d]}^2} = k_1 \cdots k_d. \quad (2.5.5)$$

As in dimension 2, we now give a sufficient condition to prove $\mathbf{C}_1^d(s_{\underline{n}})(c)$.

Proposition 2.5.3. *Condition $\mathbf{C}_1^d(s_n)(c)$ holds as soon as*

$$\frac{\sqrt{|n|}}{s_n} \sum_{k_1=-n_1}^{n_1} \cdots \sum_{k_d=-n_d}^{n_d} P_{\mathbf{o}}(X_{\underline{k}}) \longrightarrow m \text{ in } \mathbb{L}^2, \quad (2.5.6)$$

and

$$\sum_{k=1}^d \sum_{u_1=1}^{n_1} \cdots \sum_{u_d=1}^{n_d} \left\| \sum_{\substack{i_q=-n_q \\ q \in Q_{k-1}^1}}^{n_q} \left(\sum_{i_k=u_k}^{n_k} + \sum_{i_k=-n_k}^{-u_k} \right) \sum_{\substack{i_p=1-u_p \\ p \in (Q_k^1)^C}}^{n_p-u_p} P_{\mathbf{o}}(X_{\underline{i}}) \right\|_2^2 = o(s_n^2), \quad (2.5.7)$$

with $Q_k^1 = \{1, 2, 3, \dots, k\}$ and $Q_0^1 = (Q_d^1)^C = \emptyset$. In particular if $X_{\mathbf{o}}$ is $\mathcal{F}_{\mathbf{o}}$ -measurable and $\frac{s_n}{\sqrt{|n|}} = h_1(n_1) \dots h_d(n_d)$ with h_k a one-parameter slowly varying function for $k = 1, \dots, d$, then $\mathbf{C}_0^d(s_n)$ is satisfied as soon as:

$$\frac{\sqrt{|n|}}{s_n} \sum_{k_1=0}^{n_1} \cdots \sum_{k_d=0}^{n_d} P_{\mathbf{o}}(X_{\underline{k}}) \longrightarrow m \text{ in } \mathbb{L}^2, \quad (2.5.8)$$

and $\mathbf{C}_0^d(s_n)(a)$ is satisfied.

As a consequence, we get a generalized version of Corollary 2.2.7.

Corollary 2.5.4. *Considering the following conditions*

$$\sum_{\underline{i} \in \mathbb{Z}^d} P_{\mathbf{o}}(X_{\underline{i}}) \longrightarrow_{\mathbb{L}^2} m, \text{ and } \frac{\|S_n\|_2}{\sqrt{|n|}} \longrightarrow \|m\|_2, \quad (2.5.9)$$

$$X_{\mathbf{o}} \text{ is regular and } \sum_{\underline{i} \in \mathbb{Z}^d} \|P_{\mathbf{o}}(X_{\underline{i}})\|_2 < \infty. \quad (2.5.10)$$

We have the implications $(2.5.10) \Rightarrow (2.5.9) \Rightarrow \mathbf{C}_1^d(s_n)$.

One can see this as an non-adapted case of Theorem 7 in [17]

2.5.2 Weak invariance principle in dimension $d \geq 2$

Proposition 2.5.5. *The sequence $(s_n^{-2} \max_{\mathbf{1} \leq i \leq n} S_{\underline{i}}^2)_{n \in \mathbb{N}^d}$ is uniformly integrable if and only if the following conditions are satisfied :*

$$\mathbf{C}_2^d(s_n): \left\{ \begin{array}{l} (a) \quad \left\| \sup_{\mathbf{1} \leq i \leq n} |R_n(S_{\underline{i}})| \right\|_2 = o(s_n), \\ (b) \quad \left\| \sup_{\mathbf{1} \leq i \leq n} |S_{\underline{i}} - \mathbb{E}[S_{\underline{i}} | \mathcal{F}_n]| \right\|_2 = o(s_n). \\ (c) \quad \text{for some positive sequence } (u_i)_{i \in \mathbb{Z}^d} \text{ such that } \frac{\sqrt{|n|}}{s_n} \sum_{i=-n}^n u_i \text{ is bounded,} \\ \lim_{\lambda \rightarrow \infty} \limsup_{n_1 \wedge \dots \wedge n_d \rightarrow \infty} \frac{\sqrt{|n|}}{s_n} \sum_{i=-n}^n \mathbb{E} \left[\frac{P_{\mathbf{o}}^2(X_{\underline{i}})}{u_i} \mathbf{1}_{P_{\mathbf{o}}^2(X_{\underline{i}}) > \lambda u_i^2} \right] = 0. \end{array} \right.$$

Remark 2.3.3 can also be adapted to dimension d .

Remark 2.5.6. *Condition $\mathbf{C}_2^d(s_n)(a)$ and (b) hold as soon as, $X_{\mathbf{o}}$ is regular and*

$$\sum_{u_1=1}^{n_1} \cdots \sum_{u_d=1}^{n_d} \sqrt{\sum_{|i_1| \geq u_1} \cdots \sum_{|i_d| \geq u_d} \|P_{\mathbf{o}}(X_{\underline{i}})\|_2^2} = o(s_n), \quad (2.5.11)$$

and for every $1 \leq k \leq d-1$, and $1 \leq \ell \leq \binom{d}{k}$,

$$\prod_{q \in Q_k^\ell} \sqrt{n_q} \sum_{\substack{i_q=1-n_q \\ q \in Q_k^\ell}}^{n_q-1} \sum_{\substack{i_p=1 \\ p \in (Q_k^\ell)^C}}^{n_p} \sqrt{\sum_{\substack{i_p=u_p \\ p \in (Q_k^\ell)^C}}^{+\infty} \|P_{\mathbf{o}}(X_{\underline{i}})\|_2^2} = o(s_n), \quad (2.5.12)$$

$$\prod_{q \in Q_k^\ell} \sqrt{n_q} \sum_{\substack{i_q=1-n_q \\ q \in Q_k^\ell}}^{n_q-1} \sum_{\substack{i_p=1 \\ p \in (Q_k^\ell)^C}}^{n_p} \sqrt{\sum_{\substack{i_p=-\infty \\ p \in (Q_k^\ell)^C}}^{-u_p} \|P_{\mathbf{o}}(X_{\underline{i}})\|_2^2} = o(s_n), \quad (2.5.13)$$

Next, Theorem 2.5.1 together with Proposition 2.5.5 give the following weak invariance principle

Théorème 2.5.7. *Assume $\mathbf{C}_2^d(s_n)$ holds and that one of $T_{(1)}, \dots, T_{(d)}$ is ergodic. In addition suppose that $s_{[n,\underline{t}]} / s_n$ is bounded for any $\underline{t} \in [0, 1]^d$ and $\mathbb{E}(m^2) < \infty$. If $\mathbf{C}_1^d(s_n)(c)$ holds, then $\{s_n^{-1} S_{[n,\underline{t}]}, \underline{t} \in [0, 1]^d\}$ converges in distribution in $D([0, 1]^d)$ to σW_d where W_d is a d -parameter Brownian sheet and $\sigma = \sqrt{\mathbb{E}(m^2)}$.*

The proofs of the results stated in this section are of the same flavor as those developed in Section 6 when $d = 2$ but with more tedious computations due to the higher dimension. They are therefore omitted and left to the reader.

2.6 Proofs

2.6.1 Proof of Theorem 2.2.2

Note first that the following decomposition holds :

$$S_{n_1, n_2} = S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}] + R_{n_1, n_2}(S_{n_1, n_2}) + \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}] - R_{n_1, n_2}(S_{n_1, n_2}), \quad (2.6.1)$$

with

$$R_{n_1, n_2}(S_{n_1, n_2}) = \mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{n_1, 0}) + \mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{0, n_2}) - \mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{0, 0}).$$

Then we denote $\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m \circ T_{i,j}$ by M_{n_1, n_2} . Note first that with the definition of m , we have that $\mathbb{E}(M_{n_1, n_2} | \mathcal{F}_{0, 0}) = \mathbb{E}(M_{n_1, n_2} | \mathcal{F}_{n_1, 0}) = \mathbb{E}(M_{n_1, n_2} | \mathcal{F}_{0, n_2}) = 0$. Therefore recalling that

$$\mathbf{R}_{n_1, n_2} = \|\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, 0}]\|_2 + \|\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{0, n_2}]\|_2 + \|\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{0, 0}]\|_2,$$

we derive that

$$\mathbf{R}_{n_1, n_2} \leq 3 \|S_{n_1, n_2} - M_{n_1, n_2}\|_2.$$

Hence $\mathbf{C}_0(s_{n_1, n_2})$ entails $\mathbf{C}_1(s_{n_1, n_2})(a)$. Next, using (2.6.1) and orthogonality, we have :

$$\begin{aligned} \left\| \frac{S_{n_1, n_2}}{s_{n_1, n_2}} - \frac{M_{n_1, n_2}}{\sqrt{n_1 n_2}} \right\|_2^2 &= \left\| \frac{S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}]}{s_{n_1, n_2}} \right\|_2^2 + \left\| \frac{R_{n_1, n_2}(S_{n_1, n_2})}{s_{n_1, n_2}} \right\|_2^2 \\ &\quad + \left\| \frac{\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}] - R_{n_1, n_2}(S_{n_1, n_2})}{s_{n_1, n_2}} - \frac{M_{n_1, n_2}}{\sqrt{n_1 n_2}} \right\|_2^2 \\ &:= A^2 + B^2 + C^2. \end{aligned} \quad (2.6.2)$$

Therefore $\mathbf{C}_0(s_{n_1, n_2})$ implies $\mathbf{C}_1(s_{n_1, n_2})(b)$ and $C \rightarrow 0$. Note now that the following decomposition is valid:

$$\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}] - R_{n_1, n_2}(S_{n_1, n_2}) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} P_{i, j}(S_{n_1, n_2}). \quad (2.6.3)$$

For more details concerning the decomposition we refer to [20]. Using (2.6.3), it follows that

$$C = \left\| \frac{1}{\sqrt{n_1 n_2}} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left(\frac{\sqrt{n_1 n_2} \sum_{h=1}^{n_1} \sum_{k=1}^{n_2} P_{i, j}(X_{h, k})}{s_{n_1, n_2}} - m \circ T_{i, j} \right) \right\|_2.$$

Hence, by orthogonality, we get

$$C^2 = \frac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left\| \frac{\sqrt{n_1 n_2} \sum_{h=1}^{n_1} \sum_{k=1}^{n_2} P_{i, j}(X_{h, k})}{s_{n_1, n_2}} - m \circ T_{i, j} \right\|_2^2.$$

Using stationarity, we then derive

$$C^2 = \frac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left\| \frac{\sqrt{n_1 n_2} \sum_{h=1}^{n_1} \sum_{k=1}^{n_2} P_{0, 0}(X_{h-i, k-j})}{s_{n_1, n_2}} - m \right\|_2^2. \quad (2.6.4)$$

Therefore, $C \rightarrow 0$ is equivalent to $\mathbf{C}_1(s_{n_1, n_2})(c)$. This ends the proof of $\mathbf{C}_0(s_{n_1, n_2}) \Rightarrow \mathbf{C}_1(s_{n_1, n_2})$. For the converse, we use (2.6.2), (2.6.4) and the fact that $\|R_{n_1, n_2}(S_{n_1, n_2})\|_2 \leq R_{n_1, n_2}$. Now if either $\mathbf{C}_0(s_{n_1, n_2})$ or $\mathbf{C}_1(s_{n_1, n_2})$ holds then, by Remark 2.2.3, we derive

that $s_{n_1, n_2}^{-1} S_{n_1, n_2}$ converges in distribution to $\sqrt{\mathbb{E}(m^2)} \mathcal{N}$, provided that one of the transformations $T_{(1)}$ or $T_{(2)}$ is ergodic. \square

2.6.2 Proof of Proposition 2.2.4

Assume that the conditions $\mathbf{C}_1(\sigma_{n_1, n_2})(a)$ and $\mathbf{C}_1(\sigma_{n_1, n_2})(b)$ hold. To prove σ_{n_1, n_2} is a 2p-svf, it suffices to prove (2.2.1). Note first that the following decomposition holds: for every non-negative integers k and ℓ ,

$$\sigma_{kn_1, \ell n_2}^2 = \|S_{n_1, \ell n_2}\|_2^2 + \left\| \sum_{i=2}^k (S_{in_1, \ell n_2} - S_{(i-1)n_1, \ell n_2}) \right\|_2^2 + 2B, \quad (2.6.5)$$

where $B = \mathbb{E} \left[S_{n_1, \ell n_2} \left(\sum_{i=2}^k (S_{in_1, \ell n_2} - S_{(i-1)n_1, \ell n_2}) \right) \right]$. Hence by stationarity,

$$\begin{aligned} \sigma_{kn_1, \ell n_2}^2 &= k \|S_{n_1, \ell n_2}\|_2^2 + 2 \sum_{i=2}^{k-1} \sum_{j=i+1}^k \mathbb{E} [S_{n_1, \ell n_2} (S_{(j-i+1)n_1, \ell n_2} - S_{(j-i)n_1, \ell n_2})] + 2B \\ &:= k \|S_{n_1, \ell n_2}\|_2^2 + 2A + 2B. \end{aligned}$$

We shall then focus on $\|S_{n_1, \ell n_2}\|_2^2$. Proceeding as before, we have

$$\begin{aligned} \|S_{n_1, \ell n_2}\|_2^2 &= \ell \|S_{n_1, n_2}\|_2^2 + 2 \sum_{i=2}^{\ell-1} \sum_{j=i+1}^{\ell} \mathbb{E} [S_{n_1, n_2} (S_{n_1, (j-i+1)n_2} - S_{n_1, (j-i)n_2})] \\ &\quad + 2\mathbb{E} \left[S_{n_1, n_2} \left(\sum_{i=2}^{\ell} (S_{n_1, in_2} - S_{n_1, (i-1)n_2}) \right) \right] \\ &:= \ell \|S_{n_1, n_2}\|_2^2 + 2C + 2D. \end{aligned}$$

Therefore,

$$\sigma_{kn_1, \ell n_2}^2 = k \cdot \ell \|S_{n_1, n_2}\|_2^2 + 2A + 2B + 2C + 2D.$$

Hence to prove (2.2.1), it is sufficient to show that A , B , C and D are $o(\sigma_{n_1, n_2}^2)$. We

first handle D and write

$$\begin{aligned}
|D| &\leq \left| \mathbb{E} \left[(S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}]) \left(\sum_{i=2}^{\ell} (S_{n_1, in_2} - S_{n_1, (i-1)n_2}) \right) \right] \right| \\
&\quad + \left| \mathbb{E} \left[\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}] \left(\sum_{i=2}^{\ell} (S_{n_1, in_2} - S_{n_1, (i-1)n_2}) \right) \right] \right| \\
&:= D_1 + D_2.
\end{aligned}$$

By stationarity and using $\mathbf{C}_1(\sigma_{n_1, n_2})(b)$ we infer that

$$\begin{aligned}
D_1 &\leq \|S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}]\|_2 \left\| \sum_{i=2}^{\ell} (S_{n_1, in_2} - S_{n_1, (i-1)n_2}) \right\|_2 \\
&\leq (\ell - 1) \|S_{n_1, n_2}\|_2 \|S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}]\|_2 = o(\sigma_{n_1, n_2}^2).
\end{aligned}$$

On another hand, using stationarity and $\mathbf{C}_1(\sigma_{n_1, n_2})(a)$ we get

$$\begin{aligned}
D_2 &\leq \|S_{n_1, n_2}\|_2 \left\| \mathbb{E} \left(\sum_{i=2}^{\ell} (S_{n_1, in_2} - S_{n_1, (i-1)n_2}) | \mathcal{F}_{n_1, n_2} \right) \right\|_2 \\
&\leq (\ell - 1) \|S_{n_1, n_2}\|_2 \|\mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{n_1, 0})\|_2 = o(\sigma_{n_1, n_2}^2).
\end{aligned}$$

Thus $D = o(\sigma_{n_1, n_2}^2)$. We handle now the quantity C and write

$$\begin{aligned}
|C| &\leq \left| \sum_{i=2}^{\ell-1} \sum_{j=i+1}^{\ell} \mathbb{E} \left[(S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}]) (S_{n_1, (j-i+1)n_2} - S_{n_1, (j-i)n_2}) \right] \right| \\
&\quad + \left| \sum_{i=2}^{\ell-1} \sum_{j=i+1}^{\ell} \mathbb{E} \left[\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}] (S_{n_1, (j-i+1)n_2} - S_{n_1, (j-i)n_2}) \right] \right| \\
&:= C_1 + C_2.
\end{aligned}$$

Using stationarity $\mathbf{C}_1(\sigma_{n_1, n_2})(b)$, we get

$$\begin{aligned} C_1 &\leq \sum_{i=2}^{\ell-1} \sum_{j=i+1}^{\ell} \|S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}]\|_2 \|S_{n_1, (j-i+1)n_2} - S_{n_1, (j-i)n_2}\|_2 \\ &\leq \ell^2 \|S_{n_1, n_2}\|_2 \|S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}]\|_2 = o(\sigma_{n_1, n_2}^2). \end{aligned}$$

Next, by stationarity and $\mathbf{C}_1(\sigma_{n_1, n_2})(a)$ we infer that,

$$\begin{aligned} C_2 &\leq \sum_{i=2}^{\ell-1} \sum_{j=i+1}^{\ell} \|S_{n_1, n_2}\|_2 \left| \mathbb{E}[S_{n_1, (j-i+1)n_2} - S_{n_1, (j-i)n_2} | \mathcal{F}_{n_1, n_2}] \right| \\ &\leq \ell^2 \|S_{n_1, n_2}\|_2 \|\mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{n_1, 0})\|_2 = o(\sigma_{n_1, n_2}^2). \end{aligned}$$

Recall that, $\|S_{n_1, \ell n_2}\|_2^2 = \ell \|S_{n_1, n_2}\|_2^2 + 2C + 2D$. Thereby,

$$\lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{\|S_{n_1, \ell n_2}\|_2^2}{\|S_{n_1, n_2}\|_2^2} = \ell. \quad (2.6.6)$$

We handle now the quantity B . We infer that

$$\begin{aligned} |B| &\leq \left| \mathbb{E} \left[(S_{n_1, \ell n_2} - \mathbb{E}[S_{n_1, \ell n_2} | \mathcal{F}_{n_1, \ell n_2}]) \left(\sum_{i=2}^k (S_{in_1, \ell n_2} - S_{(i-1)n_1, \ell n_2}) \right) \right] \right| \\ &+ \left| \mathbb{E} \left[\mathbb{E}[S_{n_1, \ell n_2} | \mathcal{F}_{n_1, \ell n_2}] \left(\sum_{i=2}^k (S_{in_1, \ell n_2} - S_{(i-1)n_1, \ell n_2}) \right) \right] \right| \\ &:= B_1 + B_2. \end{aligned}$$

Using stationarity and $\mathbf{C}_1(\sigma_{n_1, n_2})(b)$ and (2.6.6), we get

$$\begin{aligned} B_1 &\leq \|S_{n_1, \ell n_2} - \mathbb{E}[S_{in_1, \ell n_2} | \mathcal{F}_{in_1, \ell n_2}]\|_2 \left\| \sum_{i=2}^k (S_{in_1, \ell n_2} - S_{(i-1)n_1, \ell n_2}) \right\|_2 \\ &\leq (k-1) \|S_{n_1, \ell n_2} - \mathbb{E}[S_{in_1, \ell n_2} | \mathcal{F}_{in_1, \ell n_2}]\|_2 \|S_{n_1, \ell n_2}\|_2 = o(\sigma_{n_1, n_2}^2). \end{aligned}$$

Using stationarity and $\mathbf{C}_1(\sigma_{n_1, n_2})(a)$ and (2.6.6), we derive

$$\begin{aligned} B_2 &\leq \|S_{n_1, \ell n_2}\|_2 \left\| \mathbb{E} \left(\sum_{i=2}^k (S_{in_1, \ell n_2} - S_{(i-1)n_1, \ell n_2}) \mid \mathcal{F}_{n_1, \ell n_2} \right) \right\|_2 \\ &\leq (k-1) \|S_{n_1, \ell n_2}\|_2 \|\mathbb{E}(S_{n_1, \ell n_2} \mid \mathcal{F}_{0, \ell n_2})\|_2 = o(\sigma_{n_1, n_2}^2). \end{aligned}$$

Therefore $B = o(\sigma_{n_1, n_2}^2)$. Now we handle the quantity A and write

$$\begin{aligned} |A| &\leq \left| \sum_{i=2}^{k-1} \sum_{j=i+1}^k \mathbb{E} \left[(S_{n_1, \ell n_2} - \mathbb{E}[S_{n_1, \ell n_2} \mid \mathcal{F}_{n_1, \ell n_2}]) (S_{(j-i+1)n_1, \ell n_2} - S_{(j-i)n_1, \ell n_2}) \right] \right| \\ &\quad + \left| \sum_{i=2}^{k-1} \sum_{j=i+1}^k \mathbb{E} \left[(\mathbb{E}[S_{n_1, \ell n_2} \mid \mathcal{F}_{n_1, \ell n_2}]) (S_{(j-i+1)n_1, \ell n_2} - S_{(j-i)n_1, \ell n_2}) \right] \right| \\ &:= A_1 + A_2. \end{aligned}$$

Thereby, with similar arguments as above,

$$\begin{aligned} A_1 &\leq k^2 \|S_{n_1, \ell n_2}\|_2 \|S_{n_1, \ell n_2} - \mathbb{E}(S_{n_1, \ell n_2} \mid \mathcal{F}_{0, \ell n_2})\|_2 = o(\sigma_{n_1, n_2}^2), \\ A_2 &\leq k^2 \|S_{n_1, \ell n_2}\|_2 \|\mathbb{E}(S_{n_1, \ell n_2} \mid \mathcal{F}_{0, \ell n_2})\|_2 = o(\sigma_{n_1, n_2}^2). \end{aligned}$$

Hence $A = o(\sigma_{n_1, n_2}^2)$. This ends the proof of (2.2.1). Now we prove (2.2.3). Let introduce the two following conditions :

$$\lim_{x \wedge y \rightarrow \infty} \frac{\left\| \sum_{i=1}^{k[x]+k+1} \sum_{j=1}^{\ell[y]+\ell+1} X_{i,j} - \sum_{i=1}^{[kx]} \sum_{j=1}^{[\ell y]} X_{i,j} \right\|_2}{\sigma_{[x],[y]}} = 0 \quad (2.6.7)$$

and

$$\lim_{x \wedge y \rightarrow \infty} \frac{\left\| \sum_{i=1}^{k[x]+k+1} \sum_{j=1}^{\ell[y]+\ell+1} X_{i,j} - \sum_{i=1}^{k[x]} \sum_{j=1}^{\ell[y]} X_{i,j} \right\|_2}{\sigma_{[x],[y]}} = 0. \quad (2.6.8)$$

We infer that if (2.6.7) and (2.6.8) hold then (2.2.1) can be extended to (2.2.3). Next, it is easy to see that (2.6.7) and (2.6.8) hold as soon as $\sigma_{[x],[y]} \rightarrow \infty$ as $[x] \wedge [y] \rightarrow \infty$, and (2.2.2) is satisfied. \square

2.6.3 Proof of Proposition 2.2.6

The first part of the proof is quite direct. Indeed, let us denote the double sums in the norm of the conditions (2.2.5) and (2.2.6) by a, b, c, d , and the quantities appearing in the left hand side of the conditions (2.2.5) and (2.2.6) by A, B, C , and D . For example:

$$a = \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=-n_1}^{n_1} \sum_{k=j}^{n_2} P_{0,0}(X_{h,k}) \text{ and } A = n_1 \sum_{j=1}^{n_2} \left\| \sum_{h=-n_1}^{n_1} \sum_{k=j}^{n_2} P_{0,0}(X_{h,k}) \right\|_2^2.$$

We have

$$\begin{aligned} & \frac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left\| \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=1-i}^{n_1-i} \sum_{k=1-j}^{n_2-j} P_{0,0}(X_{h,k}) - m \right\|_2^2 \\ &= \frac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left\| \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=-n_1}^{n_1} \sum_{k=-n_2}^{n_2} P_{0,0}(X_{h,k}) - m - a - b - c - d \right\|_2^2 \\ &\leq \left\| \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=-n_1}^{n_1} \sum_{k=-n_2}^{n_2} P_{0,0}(X_{h,k}) - m \right\|_2^2 + \frac{A + B + C + D}{s_{n_1, n_2}^2}. \end{aligned}$$

Hence if (2.2.4)-(2.2.6) are satisfied then $\mathbf{C}_1(s_{n_1, n_2})(c)$ holds.

Now assume that $X_{0,0}$ is $\mathcal{F}_{0,0}$ -measurable and $\frac{s_{n_1, n_2}}{\sqrt{n_1 n_2}}$ is a composition of two one-parameter slowly varying functions. In this case, conditions (2.2.4)-(2.2.6) reduce to

$$\frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=0}^{n_1} \sum_{k=0}^{n_2} P_{0,0}(X_{h,k}) \longrightarrow m \text{ in } \mathbb{L}^2, \quad (2.6.9)$$

$$n_1 \sum_{j=1}^{n_2} \left\| \sum_{h=0}^{n_1} \sum_{k=n_2-j+1}^{n_2} P_{0,0}(X_{h,k}) \right\|_2^2 = o(s_{n_1, n_2}^2), \quad (2.6.10)$$

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left\| \sum_{h=n_1-i+1}^{n_1} \sum_{k=0}^{n_2-j} P_{0,0}(X_{h,k}) \right\|_2^2 = o(s_{n_1, n_2}^2). \quad (2.6.11)$$

Note that the following decomposition holds: for every $n_1 \geq 1$, $n_2 > j > 0$,

$$\begin{aligned} \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=0}^{n_1} \sum_{k=n_2-j+1}^{n_2} P_{0,0}(X_{h,k}) &= \\ &= \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=0}^{n_1} \sum_{k=0}^{n_2} P_{0,0}(X_{h,k}) - \frac{\sqrt{n_1(n_2-j)}}{s_{n_1, n_2-j}} \sum_{h=0}^{n_1} \sum_{k=0}^{n_2-j} P_{0,0}(X_{h,k}) \\ &\quad + \frac{\sqrt{n_1(n_2-j)}}{s_{n_1, n_2-j}} \sum_{h=0}^{n_1} \sum_{k=0}^{n_2-j} P_{0,0}(X_{h,k}) \left(1 - \frac{\sqrt{n_1 n_2}}{\sqrt{n_1(n_2-j)}} \frac{s_{n_1, n_2-j}}{s_{n_1, n_2}} \right). \end{aligned}$$

Since $s_{n_1, n_2} = \sqrt{n_1 n_2} h_1(n_1) h_2(n_2)$ with h_1 and h_2 two slowly varying functions, we get

$$\begin{aligned} \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=0}^{n_1} \sum_{k=n_2-j+1}^{n_2} P_{0,0}(X_{h,k}) &= \\ &= \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{h=0}^{n_1} \sum_{k=0}^{n_2} P_{0,0}(X_{h,k}) - \frac{\sqrt{n_1(n_2-j)}}{s_{n_1, n_2-j}} \sum_{h=0}^{n_1} \sum_{k=0}^{n_2-j} P_{0,0}(X_{h,k}) \\ &\quad + \frac{\sqrt{n_1(n_2-j)}}{s_{n_1, n_2-j}} \sum_{h=0}^{n_1} \sum_{k=0}^{n_2-j} P_{0,0}(X_{h,k}) \left(1 - \frac{h_2(n_2-j)}{h_2(n_2)} \right). \end{aligned}$$

It follows that condition (2.6.9) implies condition (2.6.10) provided

$$\frac{1}{n_2} \sum_{j=1}^{n_2} \left(1 - \frac{h_2(j-1)}{h_2(n_2)} \right)^2 \longrightarrow 0.$$

This holds since h_2 is a one-parameter slowly varying function. With similar arguments and using that h_1 is a one-parameter slowly varying function, we get that (2.6.9) implies (2.6.11). \square

2.6.4 Proof of Corollary 2.2.7

This proof is quite similar to its one dimensional version (see Corollary 1in [11]). We give it for completeness.

Let first prove (2.2.8) $\Rightarrow \mathbf{C}_1(\sqrt{n_1 n_2})$. Taking $s_{n_1, n_2} = \sqrt{n_1 n_2}$ and $m = \sum_{i, j \in \mathbb{Z}^2} P_{0,0}(X_{i,j})$, it follows that if the first part of (2.2.8) holds then the conditions (2.2.4)-(2.2.6) of Proposition 2.2.6 are clearly satisfied. Therefore $\mathbf{C}_1(\sqrt{n_1 n_2})(c)$ holds. Now starting from the decomposition (2.6.1) and using orthogonality, we derive

$$\begin{aligned} \frac{\|S_{n_1, n_2}\|_2^2}{n_1 n_2} &= \frac{1}{n_1 n_2} \|S_{n_1, n_2} - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}]\|_2^2 + \frac{1}{n_1 n_2} \|R_{n_1, n_2}(S_{n_1, n_2})\|_2^2 \\ &\quad + \frac{1}{n_1 n_2} \|\mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}] - R_{n_1, n_2}(S_{n_1, n_2})\|_2^2 \\ &:= A^2 + B^2 + C^2, \end{aligned} \tag{2.6.12}$$

where $R_{n_1, n_2}(S_{n_1, n_2}) = \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{n_1, 0}] + \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{0, n_2}] - \mathbb{E}[S_{n_1, n_2} | \mathcal{F}_{0, 0}]$.

According to the proof of Theorem 2.2.2, since $\mathbf{C}_1(\sqrt{n_1 n_2})(c)$ holds, C^2 converges to $\|m\|_2^2$. Since by assumption, $(n_1 n_2)^{-1} \|S_{n_1, n_2}\|_2^2$ converges to $\|m\|_2^2$, we get overall that A and B converge to zero. Note that $A \rightarrow 0$ is exactly $\mathbf{C}_1(\sqrt{n_1 n_2})(b)$. It remains to prove that $\mathbf{C}_1(\sqrt{n_1 n_2})(a)$ is satisfied. With this aim, note that since $M_{n_1, n_2} := \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m \circ T_{i,j}$ is an ortho-martingale with respect to filtration $(\mathcal{F}_{i,j})_{i,j \in \mathbb{Z}}$, we have

$$\begin{aligned} &\mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{n_1, 0}) \\ &= \mathbb{E}[\mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}) - R_{n_1, n_2}(S_{n_1, n_2}) + R_{n_1, n_2}(S_{n_1, n_2}) - M_{n_1, n_2} | \mathcal{F}_{n_1, 0}]. \end{aligned}$$

Therefore

$$\frac{\|\mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{n_1, 0})\|_2}{\sqrt{n_1 n_2}} \leq B + \frac{1}{\sqrt{n_1 n_2}} \|\mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{n_1, n_2}) - R_{n_1, n_2}(S_{n_1, n_2}) - M_{n_1, n_2}\|_2,$$

which converges to zero by $\mathbf{C}_1(\sqrt{n_1 n_2})(c)$ and the fact that $B \rightarrow 0$. Similarly, we get that $(n_1 n_2)^{-1/2} \|\mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{0, n_2})\|_2 \rightarrow 0$ and $(n_1 n_2)^{-1/2} \|\mathbb{E}(S_{n_1, n_2} | \mathcal{F}_{0, 0})\|_2 \rightarrow 0$. Hence

$\mathbf{C}_1(\sqrt{n_1 n_2})(a)$ holds.

It remains to prove that (2.2.9) \Rightarrow (2.2.8). Clearly if (2.2.9) holds so does the first part of (2.2.8). Next we shall prove that

$$\frac{1}{n_1 n_2} \mathbb{E} [S_{n_1, n_2}^2] \rightarrow \sum_{(k_1, k_2) \in \mathbb{Z}^2} \mathbb{E} [X_{0,0} X_{k_1, k_2}] \text{ a.s. and } \mathbb{E} [m^2] = \sum_{(k_1, k_2) \in \mathbb{Z}^2} \mathbb{E} [X_{0,0} X_{k_1, k_2}], \quad (2.6.13)$$

which implies the second part of (2.2.8). Since $X_{0,0}$ is regular, the following decomposition holds

$$\mathbb{E} [X_{0,0} X_{k_1, k_2}] = \sum_{(i_1, i_2) \in \mathbb{Z}^2} \sum_{(j_1, j_2) \in \mathbb{Z}^2} \mathbb{E} [P_{i_1, i_2}(X_{0,0}) P_{j_1, j_2}(X_{k_1, k_2})].$$

By orthogonality (which comes from the fact that the filtration $(\mathcal{F}_{i,j})$ is commuting) and stationarity, we derive

$$\mathbb{E} [X_{0,0} X_{k_1, k_2}] = \sum_{(i_1, i_2) \in \mathbb{Z}^2} \mathbb{E} [P_{0,0}(X_{i_1, i_2}) P_{0,0}(X_{k_1+i_1, k_2+i_2})]. \quad (2.6.14)$$

Hence

$$|\mathbb{E} [X_{0,0} X_{k_1, k_2}]| \leq \sum_{(i_1, i_2) \in \mathbb{Z}^2} \|P_{0,0}(X_{i_1, i_2})\|_2 \|P_{0,0}(X_{k_1+i_1, k_2+i_2})\|_2,$$

so that

$$\sum_{(k_1, k_2) \in \mathbb{Z}^2} |\mathbb{E} [X_{0,0} X_{k_1, k_2}]| \leq \left(\sum_{(i_1, i_2) \in \mathbb{Z}^2} \|P_{0,0}(X_{i_1, i_2})\|_2 \right)^2,$$

which is finite under condition (2.2.9). Therefore, the series $\sum_{(k_1, k_2) \in \mathbb{Z}^2} |\mathbb{E} [X_{0,0} X_{k_1, k_2}]|$ converges. In addition, note that with (2.6.14), we have

$$\sum_{(k_1, k_2) \in \mathbb{Z}^2} \mathbb{E} [X_{0,0} X_{k_1, k_2}] = \sum_{(i_1, i_2) \in \mathbb{Z}^2} \sum_{(j_1, j_2) \in \mathbb{Z}^2} \mathbb{E} [P_{0,0}(X_{i_1, i_2}) P_{0,0}(X_{j_1, j_2})].$$

In addition,

$$\mathbb{E}(m^2) = \sum_{(i_1, i_2) \in \mathbb{Z}^2} \sum_{(j_1, j_2) \in \mathbb{Z}^2} \mathbb{E}[P_{0,0}(X_{i_1, i_2}) P_{0,0}(X_{j_1, j_2})].$$

So,

$$\mathbb{E}(m^2) = \sum_{(k_1, k_2) \in \mathbb{Z}^2} \mathbb{E}[X_{0,0} X_{k_1, k_2}].$$

On another hand,

$$\frac{1}{n_1 n_2} \mathbb{E}(S_{n_1, n_2}^2) = \sum_{i_1 = -n_1 + 1}^{n_1 - 1} \sum_{i_2 = -n_2 + 1}^{n_2 - 1} \left(1 - \frac{|i_1|}{n_1}\right) \left(1 - \frac{|i_2|}{n_2}\right) \mathbb{E}(X_{0,0} X_{i_1, i_2}),$$

which then converges to $\sum_{(k_1, k_2) \in \mathbb{Z}^2} \mathbb{E}[X_{0,0} X_{k_1, k_2}] = \mathbb{E}(m^2)$. This ends the proof of (2.2.8). \square

2.6.5 Proof of Proposition 2.3.1

By simplicity in the next proof, we denote $\max_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}}$ by $\max_{k,l}$, and $\max_{1 \leq k \leq n_1}$ by \max_k respectively for l . All along the proof, C will be a universal positive constant (depending on the dimension) which may vary from line to line. Our main aim is to prove that

$$\lim_{\lambda \rightarrow \infty} \limsup_{n_1 \wedge n_2 \rightarrow \infty} \frac{1}{s_{n_1, n_2}^2} \mathbb{E} \left[\max_{k,l} |S_{k,l}^2| \mathbb{1}_{\max_{k,l} |S_{k,l}| > \lambda s_{n_1, n_2}} \right] = 0.$$

Note that the following decomposition holds: for every $1 \leq k \leq n_1$, $1 \leq l \leq n_2$,

$$S_{k,l} = S_{k,l} - \mathbb{E}[S_{k,l} | \mathcal{F}_{n_1, n_2}] + R_{n_1, n_2}(S_{k,l}) + \mathbb{E}[S_{k,l} | \mathcal{F}_{n_1, n_2}] - R_{n_1, n_2}(S_{k,l}), \quad (2.6.15)$$

where $R_{n_1, n_2}(S_{k,l}) = \mathbb{E}[S_{k,l} | \mathcal{F}_{n_1, 0}] + \mathbb{E}[S_{k,l} | \mathcal{F}_{0, n_2}] - \mathbb{E}[S_{k,l} | \mathcal{F}_{0, 0}]$. Starting from (2.6.15) and using the fact that for any $A \geq 0$, $x^2 \mathbb{1}_{|x| > A} \leq 4 \left(|x| - \frac{A}{2} \right)_+^2$, where $(x)_+ =$

$x\mathbb{1}_{x \geq 0}$, we get by convexity that, for any $\lambda \geq 0$,

$$\begin{aligned} \mathbb{E} \left[\max_{k,l} |S_{k,l}^2| \mathbb{1}_{\max_{k,l} |S_{k,l}| > 2\lambda s_{n_1, n_2}} \right] &\leq 4\mathbb{E} \left[\left(\max_{k,l} |S_{k,l}| - \lambda s_{n_1, n_2} \right)_+^2 \right] \\ &\leq 12\mathbb{E} \left[\left(\max_{k,l} |S_{k,l} - \mathbb{E}[S_{k,l} | \mathcal{F}_{n_1, n_2}]| \right)^2 \right] + 12\mathbb{E} \left[\left(\max_{k,l} |R_{n_1, n_2}(S_{k,l})| \right)^2 \right] \\ &\quad + 12\mathbb{E} \left[\left(\max_{k,l} |\mathbb{E}[S_{k,l} | \mathcal{F}_{n_1, n_2}] - R_{n_1, n_2}(S_{k,l})| - \lambda s_{n_1, n_2} \right)_+^2 \right]. \end{aligned}$$

Therefore by the condition $\mathbf{C}_2(s_{n_1, n_2})(a)$ and (b), it is sufficient to show that,

$$\lim_{\lambda \rightarrow \infty} \lim_{n_1 \wedge n_2 \rightarrow \infty} \frac{1}{s_{n_1, n_2}^2} \mathbb{E} \left[\left(\max_{k,l} |\mathbb{E}[S_{k,l} | \mathcal{F}_{n_1, n_2}] - R_{n_1, n_2}(S_{k,l})| - \lambda s_{n_1, n_2} \right)_+^2 \right] = 0. \quad (2.6.16)$$

For the sake of simplicity, we set $\tilde{S}_{k,l} := \mathbb{E}[S_{k,l} | \mathcal{F}_{n_1, n_2}] - R_{n_1, n_2}(S_{k,l})$, $\tilde{S}_{n_1, n_2}^+ := \max(0, \tilde{S}_{1,1}, \dots, \tilde{S}_{n_1, n_2})$, and $\tilde{S}_{n_1, n_2}^- := \max(0, -\tilde{S}_{1,1}, \dots, -\tilde{S}_{n_1, n_2})$.

Therefore if we assume that $\mathbf{C}_2(s_{n_1, n_2})(a)$ and $\mathbf{C}_2(s_{n_1, n_2})(b)$ hold, then $\left(\max_{k,l} \frac{S_{k,l}^2}{s_{n_1, n_2}^2} \right)_{(n_1, n_2) \in (\mathbb{N}^*)^2}$ will be uniformly integrable as soon as

$$\lim_{\lambda \rightarrow \infty} \limsup_{n_1 \wedge n_2 \rightarrow \infty} \frac{1}{s_{n_1, n_2}^2} \mathbb{E} \left[\left(\tilde{S}_{n_1, n_2}^+ - \lambda s_{n_1, n_2} \right)_+^2 \right] = 0 \quad (2.6.17)$$

and

$$\lim_{\lambda \rightarrow \infty} \limsup_{n_1 \wedge n_2 \rightarrow \infty} \frac{1}{s_{n_1, n_2}^2} \mathbb{E} \left[\left(\tilde{S}_{n_1, n_2}^- - \lambda s_{n_1, n_2} \right)_+^2 \right] = 0. \quad (2.6.18)$$

Note that using the projective operators, we have

$$\begin{aligned}\tilde{S}_{k,l} &= \sum_{u=1}^k \sum_{v=1}^l \sum_{i=u-n_1}^{u-1} \sum_{j=v-n_2}^{v-1} P_{u-i,v-j}(X_{u,v}) \\ &= \sum_{i=1-n_1}^{k-1} \sum_{j=1-n_2}^{l-1} \sum_{u=1\vee(i+1)}^{k\wedge(n_1+i)} \sum_{v=1\vee(j+1)}^{l\wedge(n_2+j)} P_{u-i,v-j}(X_{u,v}).\end{aligned}$$

For any fixed positive integer i and j , we introduce the double indexed sequence $(Y_{i,j,k,l})_{k,l \geq 1}$ defined by

$$Y_{i,j,k,l} = \sum_{u=1\vee(i+1)}^{k\wedge(n_1+i)} \sum_{v=1\vee(j+1)}^{l\wedge(n_2+j)} P_{u-i,v-j}(X_{u,v}).$$

Notice that, $(Y_{i,j,k,l})_{k,l \geq 1}$ is an ortho-martingale w.r.t the filtration $(\mathcal{F}_{i,j})_{i,j \in \mathbb{Z}}$.

With these notations, $\tilde{S}_{k,l} = \sum_{i=1-n_1}^{k-1} \sum_{j=1-n_2}^{l-1} Y_{i,j,k,l}$. Setting $b_{i,j} = u_{i,j} (\sum_{k=-n_1}^{n_1} \sum_{\ell=-n_2}^{n_2} u_{k,\ell})^{-1}$, with $(u_{i,j})_{i,j \in \mathbb{Z}}$ a positive sequence such that

$$\frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{i=1-n_1}^{n_1-1} \sum_{j=1-n_2}^{n_2-1} u_{i,j} < \infty,$$

we get

$$\left(\tilde{S}_{k,l} - \lambda s_{n_1, n_2} \right)_+ \leq \sum_{i=1-n_1}^{k-1} \sum_{j=1-n_2}^{l-1} (Y_{i,j,k,l} - \lambda b_{i,j} s_{n_1, n_2})_+. \quad (2.6.19)$$

By Hölder's inequality and taking the maximum over (k, l) on both sides, we derive

that

$$\begin{aligned} & \left(\tilde{S}_{n_1, n_2}^+ - \lambda s_{n_1, n_2} \right)_+^2 \\ & \leq \left(\sum_{i=1-n_1}^{n_1-1} \sum_{j=1-n_2}^{n_2-1} u_{i,j} \right) \sum_{i=1-n_1}^{n_1-1} \sum_{j=1-n_2}^{n_2-1} \frac{1}{u_{i,j}} \left(Y_{i,j,n_1,n_2}^+ - \lambda b_{i,j} s_{n_1,n_2} \right)_+^2, \end{aligned}$$

where $Y_{i,j,n_1,n_2}^+ = \max_{k,l} (\max(0, Y_{i,j,k,l}))$. Hence to prove (2.6.17), it is sufficient to show that,

$$\lim_{\lambda \rightarrow \infty} \limsup_{n_1 \wedge n_2 \rightarrow \infty} \frac{1}{\sqrt{n_1 n_2} s_{n_1, n_2}} \sum_{i=1-n_1}^{n_1-1} \sum_{j=1-n_2}^{n_2-1} \frac{1}{u_{i,j}} \mathbb{E} \left[\left(Y_{i,j,n_1,n_2}^+ - \lambda b_{i,j} s_{n_1,n_2} \right)_+^2 \right] = 0. \quad (2.6.20)$$

To prove this, we will need the two following lemmas, whose proofs are postponed to Section 7.

Définition 2.6.1. *We say that a random field $(M_{n_1, n_2})_{(n_1, n_2) \in \mathbb{N}^2}$ is an ortho-submartingale w.r.t the commuting filtration $(\mathcal{F}_{n_1, n_2})_{(n_1, n_2) \in \mathbb{Z}^2}$ if M_{n_1, n_2} is integrable, \mathcal{F}_{n_1, n_2} -measurable, and for all integers i, j, k, ℓ , we have*

$$\mathbb{E} (M_{i,j} | \mathcal{F}_{k,\ell}) \geq M_{\min(i,k), \min(j,\ell)}.$$

Lemma 2.6.2. *Let $M_{n_1, n_2} \equiv \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} d_{i,j}$ be an ortho-submartingale. Then for all $p \in]1, +\infty[$, and $\lambda > 0$*

$$\mathbb{E} \left[\max_{k,l} |M_{k,l}|^p \mathbf{1}_{\max_{k,l} |M_{k,l}| > \lambda} \right] \leq 2^{2p} \left(\frac{p}{p-1} \right)^2 \mathbb{E} \left[|M_{n_1, n_2}|^p \mathbf{1}_{|M_{n_1, n_2}| > \frac{\lambda}{4}} \right].$$

Lemma 2.6.3. *Let $M_{n_1, n_2} \equiv \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} d_{i,j}$ be an ortho-martingale in \mathbb{L}^2 . Then for all $\lambda > 0$*

$$\mathbb{E} \left[(M_{n_1, n_2} - \lambda)_+^2 \right] \leq \sum_{i=1}^{n_1} \mathbb{E} \left[\left(\sum_{j=1}^{n_2} d_{i,j} \right)^2 \mathbf{1}_{\max_k |M_{k,n_2}| > \lambda} \right], \quad (2.6.21)$$

and

$$\mathbb{E} \left[|M_{n_1, n_2}|^2 \mathbb{1}_{|M_{n_1, n_2}| > \lambda} \right] \leq 8 \sum_{i=1}^{n_1} \mathbb{E} \left[\left(\sum_{j=1}^{n_2} d_{i,j} \right)^2 \mathbb{1}_{\max_k |M_{k, n_2}| > \frac{\lambda}{2}} \right]. \quad (2.6.22)$$

In addition, for any fixed i (the same goes for j), we have

$$\mathbb{E} \left[\left(\sum_{j=1}^{n_2} d_{i,j} - \lambda \right)_+^2 \right] \leq \sum_{j=1}^{n_2} \mathbb{E} \left[d_{i,j}^2 \mathbb{1}_{\max_{l \leq n_2} |\sum_{j=1}^l d_{i,j}| > \lambda} \right]. \quad (2.6.23)$$

To reduce the complexity of the notation, we write $\sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2-1}$ for $\sum_{i,j}$, and $\sum_{u=1 \vee (i+1)}^{k \wedge (n_1+i)}$ by \sum_u^k , the same goes for \sum_v^l . Now we denote

$$U(n_1, n_2, \lambda) := \frac{1}{\sqrt{n_1 n_2 s_{n_1, n_2}}} \sum_{i,j} \frac{1}{u_{i,j}} \mathbb{E} \left[(Y_{i,j, n_1, n_2}^+ - \lambda b_{i,j} s_{n_1, n_2})_+^2 \right].$$

Since $(|x| - \lambda)_+^2 \leq x^2 \mathbb{1}_{|x| > \lambda}$, we have

$$U(n_1, n_2, \lambda) \leq \frac{1}{\sqrt{n_1 n_2 s_{n_1, n_2}}} \sum_{i,j} \frac{1}{u_{i,j}} \mathbb{E} \left[(Y_{i,j, n_1, n_2}^+)^2 \mathbb{1}_{Y_{i,j, n_1, n_2}^+ > \lambda b_{i,j} s_{n_1, n_2}} \right].$$

Next applying Lemma 2.6.2, we get

$$U(n_1, n_2, \lambda) \leq \frac{C}{\sqrt{n_1 n_2 s_{n_1, n_2}}} \sum_{i,j} \frac{1}{u_{i,j}} \mathbb{E} \left[Y_{i,j, n_1, n_2}^2 \mathbb{1}_{|Y_{i,j, n_1, n_2}| > \frac{\lambda}{4} b_{i,j} s_{n_1, n_2}} \right].$$

Then using (2.6.22) of Lemma 2.6.3, we derive

$$\begin{aligned} & U(n_1, n_2, \lambda) \\ & \leq \frac{C}{\sqrt{n_1 n_2 s_{n_1, n_2}}} \sum_{i,j} \frac{1}{u_{i,j}} \sum_u^{n_1} \mathbb{E} \left[\left(\sum_v^{n_2} P_{u-i, v-j}(X_{u,v}) \right)^2 \mathbb{1}_{\max_u |Y_{i,j, u, n_2}| > \frac{\lambda}{8} b_{i,j} s_{n_1, n_2}} \right]. \end{aligned}$$

Let $A > 0$ and set $A_{i,j,n_2} = Au_{i,j}^2 n_2$, we have that $U(n_1, n_2, \lambda)$ is less than or equal to

$$\begin{aligned} & \frac{C}{\sqrt{n_1 n_2} s_{n_1, n_2}} \sum_{i,j} \frac{1}{u_{i,j}} \sum_u^{n_1} \mathbb{E} \left[\left(\sum_v^{n_2} P_{u-i, v-j}(X_{u,v}) \right)^2 \mathbb{1}_{\left(\sum_v^{n_2} P_{u-i, v-j}(X_{u,v}) \right)^2 > A_{i,j,n_2}} \right] \\ & + \frac{C}{\sqrt{n_1 n_2} s_{n_1, n_2}} \sum_{i,j} \frac{1}{u_{i,j}} n_1 A_{i,j,n_2} \mathbb{P} \left(\max_u |Y_{i,j,u,n_2}| > \frac{\lambda}{8} b_{i,j} s_{n_1, n_2} \right) \\ & := C(I + II). \end{aligned}$$

Using (2.6.23) of Lemma 2.6.3, we get

$$I \leq \frac{C_1}{\sqrt{n_1 n_2} s_{n_1, n_2}} \sum_{i,j} \frac{1}{u_{i,j}} \sum_u^{n_1} \sum_v^{n_2} \mathbb{E} \left[P_{u-i, v-j}^2(X_{u,v}) \mathbb{1}_{\max_l \left| \sum_v^l P_{u-i, v-j}(X_{u,v}) \right| > \frac{\sqrt{A_{i,j,n_2}}}{2}} \right].$$

It follows that for any $B > 0$

$$\begin{aligned} I & \leq \frac{C_1}{\sqrt{n_1 n_2} s_{n_1, n_2}} \sum_{i,j} \frac{1}{u_{i,j}} \sum_u^{n_1} \sum_v^{n_2} \mathbb{E} \left[P_{u-i, v-j}^2(X_{u,v}) \mathbb{1}_{P_{u-i, v-j}^2(X_{u,v}) > Bu_{i,j}^2} \right] \\ & + \frac{C_1}{\sqrt{n_1 n_2} s_{n_1, n_2}} \sum_{i,j} \frac{1}{u_{i,j}} \sum_u^{n_1} \sum_v^{n_2} Bu_{i,j}^2 \mathbb{P} \left(\max_l \left| \sum_v^l P_{u-i, v-j}(X_{u,v}) \right| > \frac{\sqrt{A_{i,j,n_2}}}{2} \right). \end{aligned}$$

By stationarity, it follows that,

$$\begin{aligned} I & \leq C_1 \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{i,j} \frac{1}{u_{i,j}} \mathbb{E} \left[P_{0,0}^2(X_{i,j}) \mathbb{1}_{P_{0,0}^2(X_{i,j}) > Bu_{i,j}^2} \right] \\ & + \frac{C_1}{\sqrt{n_1 n_2} s_{n_1, n_2}} \sum_{i,j} \frac{1}{u_{i,j}} \sum_u^{n_1} \sum_v^{n_2} Bu_{i,j}^2 \mathbb{P} \left(\max_l \left| \sum_v^l P_{u-i, v-j}(X_{u,v}) \right| > \frac{\sqrt{A_{i,j,n_2}}}{2} \right). \\ & := C_1(I_1 + I_2). \end{aligned}$$

The quantity I_1 converges to zero by $\mathbf{C}_2(s_{n_1, n_2})(c)$, if we first let n_1, n_2 tend to infinity

and then B to infinity. To handle I_2 , we use Doob's inequality that leads to

$$I_2 \leq \frac{4}{\sqrt{n_1 n_2} s_{n_1, n_2}} \sum_{i,j} \frac{1}{u_{i,j}} \sum_u^{n_1} \sum_v^{n_2} B u_{i,j}^2 \sum_{v=1 \vee (j+1)}^{n_2 \wedge (n_2+j)} \frac{\mathbb{E} [P_{u-i, v-j}(X_{u,v})]^2}{A_{i,j, n_2}}.$$

Using stationarity again, and recalling that $A_{i,j, n_2} = A u_{i,j}^2 n_2$, we derive

$$\begin{aligned} I_2 &\leq \frac{4}{\sqrt{n_1 n_2} s_{n_1, n_2}} \sum_{i,j} \frac{1}{u_{i,j}} \sum_u^{n_1} \sum_v^{n_2} B u_{i,j}^2 n_2 \frac{\mathbb{E} [P_{0,0}^2(X_{i,j})]}{A u_{i,j}^2 n_2} \\ &\leq 4 \frac{B \sqrt{n_1 n_2}}{A s_{n_1, n_2}} \sum_{i,j} \mathbb{E} \left[\frac{P_{0,0}^2(X_{i,j})}{u_{i,j}} \right]. \end{aligned}$$

Now by $\mathbf{C}_2(s_{n_1, n_2})(c)$, we have

$$\sup_{\substack{n_1 \geq 1 \\ n_2 \geq 1}} \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{i=1-n_1}^{n_1-1} \sum_{j=1-n_2}^{n_2-1} \mathbb{E} \left[\frac{P_{0,0}^2(X_{i,j})}{u_{i,j}} \right] < \infty. \quad (2.6.24)$$

Hence I_2 converges to zero if we first let A goes to infinity and then B to infinity. So overall I converges to zero by letting first $n_1 \wedge n_2$ tend to infinity and then A . Now to deal with II , we proceed as for I_2 . Indeed, Doob's maximal inequality leads to

$$\begin{aligned} II &= A \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{i,j} u_{i,j} \mathbb{P} \left(\max_u |Y_{i,j, u, n_2}| > \frac{\lambda}{8} b_{i,j} s_{n_1, n_2} \right), \\ &\leq C_2 A \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{i,j} u_{i,j} \frac{\mathbb{E} [Y_{i,j, n_1, n_2}^2]}{\lambda^2 b_{i,j}^2 s_{n_1, n_2}^2}. \end{aligned}$$

Hence, by stationarity,

$$II \leq C_2 \frac{A}{\lambda^2} \frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{i,j} \mathbb{E} \left[\frac{P_{0,0}^2(X_{i,j})}{u_{i,j}} \right] \left(\frac{\sqrt{n_1 n_2}}{s_{n_1, n_2}} \sum_{i,j} u_{i,j} \right)^2,$$

which converges to zero by letting first $n_1 \wedge n_2$ tend to infinity and then λ . This ends the proof of (2.6.17). The proof of (2.6.18) can be done similarly. \square

2.6.6 Proof of Proposition 2.3.3

Notice first that the condition $\mathbf{C}_2(s_{n_1, n_2})(a)$ holds as soon as,

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} |\mathbb{E}[S_{k,l} | \mathcal{F}_{n_1,0}] - \mathbb{E}[S_{k,l} | \mathcal{F}_{0,0}]| \right\|_2 = o(s_{n_1, n_2}), \quad (2.6.25)$$

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} |\mathbb{E}[S_{k,l} | \mathcal{F}_{0, n_2}] - \mathbb{E}[S_{k,l} | \mathcal{F}_{0,0}]| \right\|_2 = o(s_{n_1, n_2}), \quad (2.6.26)$$

and

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} |\mathbb{E}[S_{k,l} | \mathcal{F}_{0,0}]| \right\|_2 = o(s_{n_1, n_2}). \quad (2.6.27)$$

Note that

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} |\mathbb{E}[S_{k,l} | \mathcal{F}_{0,0}]| \right\|_2 \leq \sum_{k=1}^{n_1} \sum_{\ell=1}^{n_2} \|\mathbb{E}(X_{k,\ell} | \mathcal{F}_{0,0})\|_2,$$

and since $X_{k,\ell} = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} P_{0,0}(X_{k,\ell})$,

$$\mathbb{E}(X_{k,\ell} | \mathcal{F}_{0,0}) = \sum_{i=-\infty}^0 \sum_{j=-\infty}^0 P_{i,j}(X_{k,\ell}).$$

Therefore, the condition (2.6.27) is implied by $\mathbf{C}_3(s_{n_1, n_2})(a)$. Now, we focus on (2.6.25). Notice first that

$$\mathbb{E}[S_{k,l} | \mathcal{F}_{n_1,0}] - \mathbb{E}[S_{k,l} | \mathcal{F}_{0,0}] = \sum_{u=1}^k \sum_{v=1}^{\ell} \sum_{i=1}^{n_1} \sum_{j=-\infty}^0 P_{i,j}(X_{u,v}).$$

Thus (2.6.25) is equivalent to

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} \left| \sum_{u=1}^k \sum_{v=1}^{\ell} \sum_{i=1}^{n_1} \sum_{j=-\infty}^0 P_{i,j}(X_{u,v}) \right| \right\|_2 = o(s_{n_1, n_2}). \quad (2.6.28)$$

But

$$\begin{aligned}
& \left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} \left\| \sum_{u=1}^k \sum_{v=1}^{\ell} \sum_{i=1}^{n_1} \sum_{j=-\infty}^0 P_{i,j}(X_{u,v}) \right\| \right\|_2 \\
& \leq \sum_{v=1}^{n_2} \left\| \sup_{1 \leq k \leq n_1} \left\| \sum_{u=1}^k \sum_{i=1}^{n_1} \sum_{j=-\infty}^0 P_{i,j}(X_{u,v}) \right\| \right\|_2 \\
& \leq \sum_{v=1}^{n_2} \left\| \sup_{1 \leq k \leq n_1} \left\| \sum_{u=1}^k \sum_{i=u-n_1}^{u-1} \sum_{j=-\infty}^0 P_{u-i,j}(X_{u,v}) \right\| \right\|_2 \\
& \leq \sum_{v=1}^{n_2} \left\| \sup_{1 \leq k \leq n_1} \left\| \sum_{i=1-n_1}^{k-1} \sum_{u=(i+1) \vee 1}^{k \wedge (i+n_1)} \sum_{j=-\infty}^0 P_{u-i,j}(X_{u,v}) \right\| \right\|_2.
\end{aligned}$$

We denote $\sum_{j=-\infty}^0 P_{u-i,j}(X_{u,v})$ by $D_{i,u}(v)$, and notice that $D_{i,u}(v)$ is orthogonal with respect to u . Next denote $\sum_{u=(i+1) \vee 1}^{k \wedge (i+n_1)} D_{i,u}(v)$ by $Y_{i,k}(v)$. Let $(U_i(v))_{i \geq 1-n_1}$ be a sequence of positive real numbers that we will define later. By Cauchy-Schwarz's inequality,

$$\begin{aligned}
\left| \sum_{i=1-n_1}^{k-1} Y_{i,k}(v) \right|^2 &= \left| \sum_{i=1-n_1}^{k-1} \sqrt{U_i(v)} \frac{1}{\sqrt{U_i(v)}} Y_{i,k}(v) \right|^2 \\
&\leq \left(\sum_{i=1-n_1}^{k-1} U_i(v) \right) \sum_{i=1-n_1}^{k-1} \frac{1}{U_i(v)} (Y_{i,k}(v))^2.
\end{aligned}$$

Therefore,

$$\left\| \sup_{1 \leq k \leq n_1} \left\| \sum_{i=1-n_1}^{k-1} Y_{i,k}(v) \right\| \right\|_2^2 \leq \left(\sum_{i=1-n_1}^{n_1-1} U_i(v) \right) \sum_{i=1-n_1}^{n_1-1} \frac{1}{U_i(v)} \left\| \sup_{1 \leq k \leq n_1} |Y_{i,k}(v)| \right\|_2^2.$$

Note now that $(Y_{i,k}(v))_{k \geq 1}$ is a martingale. Using Doob's inequality and stationarity, it follows that

$$\begin{aligned} \left\| \sup_{1 \leq k \leq n_1} |Y_{i,k}(v)| \right\|_2^2 &\leq 4 \sum_{u=(i+1) \vee 1}^{n_1 \wedge i + n_1} \|D_{i,u}(v)\|_2^2 \\ &\leq 4 \sum_{u=(i+1) \vee 1}^{n_1 \wedge i + n_1} \sum_{j=-\infty}^0 \|P_{u-i,j}(X_{u,v})\|_2^2 \\ &\leq 4n_1 \sum_{j=v}^{+\infty} \|P_{0,0}(X_{i,j})\|_2^2. \end{aligned}$$

So overall taking $U_i(v) = \sqrt{\sum_{j=v}^{+\infty} \|P_{0,0}(X_{i,j})\|_2^2}$, we have

$$\left\| \sup_{1 \leq k \leq n_1} \left| \sum_{i=1-n_1}^{k-1} Y_{i,k}(v) \right| \right\|_2^2 \leq 4n_1 \left(\sum_{i=1-n_1}^{n_1-1} \sqrt{\sum_{j=v}^{+\infty} \|P_{0,0}(X_{i,j})\|_2^2} \right)^2.$$

Therefore

$$\sum_{v=1}^{n_2} \left\| \sup_{1 \leq k \leq n_1} \left| \sum_{i=1-n_1}^{k-1} Y_{i,k}(v) \right| \right\|_2 \leq 2\sqrt{n_1} \sum_{i=1-n_1}^{n_1-1} \sum_{v=1}^{n_2} \sqrt{\sum_{j=v}^{+\infty} \|P_{0,0}(X_{i,j})\|_2^2}.$$

Thus (2.6.25) holds as soon as $\mathbf{C}_3(s_{n_1, n_2})(b)$ holds. Since the filtration is commuting, arguments applied to (2.6.25) can still be applied to show that (2.6.26) holds. Thus (2.6.26) holds as soon as $\mathbf{C}_3(s_{n_1, n_2})(c)$ does.

Now to give sufficient condition for $\mathbf{C}_2(s_{n_1, n_2})(b)$ to hold, note first that,

$$\begin{aligned} S_{k,\ell} - \mathbb{E}(S_{k,\ell} | \mathcal{F}_{n_1, n_2}) = \\ \sum_{u=1}^k \sum_{v=1}^{\ell} \left(\sum_{i=-\infty}^{n_1} \sum_{j=n_2+1}^{+\infty} P_{i,j}(X_{u,v}) + \sum_{i=n_1+1}^{+\infty} \sum_{j=-\infty}^{n_2} P_{i,j}(X_{u,v}) + \sum_{i=n_1+1}^{+\infty} \sum_{j=n_2+1}^{+\infty} P_{i,j}(X_{u,v}) \right). \end{aligned}$$

Hence $\mathbf{C}_2(s_{n_1, n_2})(b)$ holds as soon as,

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} \left| \sum_{u=1}^k \sum_{v=1}^l \sum_{i=-\infty}^{n_1} \sum_{j=n_2+1}^{+\infty} P_{i,j}(X_{u,v}) \right| \right\|_2 = o(s_{n_1, n_2}), \quad (2.6.29)$$

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} \left| \sum_{u=1}^k \sum_{v=1}^l \sum_{i=n_1+1}^{+\infty} \sum_{j=-\infty}^{n_2} P_{i,j}(X_{u,v}) \right| \right\|_2 = o(s_{n_1, n_2}), \quad (2.6.30)$$

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} \left| \sum_{u=1}^k \sum_{v=1}^l \sum_{i=n_1+1}^{+\infty} \sum_{j=n_2+1}^{+\infty} P_{i,j}(X_{u,v}) \right| \right\|_2 = o(s_{n_1, n_2}). \quad (2.6.31)$$

Clearly (2.6.31) holds as soon as $\mathbf{C}_3(s_{n_1, n_2})(a)$ holds. For (2.6.29), notice now

$$\sum_{i=-\infty}^{n_1} \sum_{j=n_2+1}^{+\infty} P_{i,j}(X_{u,v}) = \sum_{i=1}^{n_1} \sum_{j=n_2+1}^{+\infty} P_{i,j}(X_{u,v}) + \sum_{i=-\infty}^0 \sum_{j=n_2+1}^{+\infty} P_{i,j}(X_{u,v}).$$

Therefore, (2.6.29) holds as soon as

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} \left| \sum_{u=1}^k \sum_{v=1}^l \sum_{i=1}^{n_1} \sum_{j=n_2+1}^{+\infty} P_{i,j}(X_{u,v}) \right| \right\|_2 = o(s_{n_1, n_2}), \quad (2.6.32)$$

and

$$\left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} \left| \sum_{u=1}^k \sum_{v=1}^l \sum_{i=-\infty}^0 \sum_{j=n_2+1}^{+\infty} P_{i,j}(X_{u,v}) \right| \right\|_2 = o(s_{n_1, n_2}). \quad (2.6.33)$$

Clearly (2.6.33) holds as soon as $\mathbf{C}_3(s_{n_1, n_2})(a)$ is satisfied. With similar argument as before, we infer that (2.6.32) holds as soon as $\mathbf{C}_3(s_{n_1, n_2})(b)$ does. The proof of (2.6.30) and (2.6.31) can be done arguing as before.

We now prove that in adapted case $\mathbf{C}_3(s_{n_1, n_2})$ holds as soon as $\overline{\mathbf{C}_3(s_{n_1, n_2})}$ does.

In adapted case $\mathbf{C}_3(s_{n_1, n_2})(a)$ reads as

$$\sum_{u=1}^{n_1} \sum_{v=1}^{n_2} \|\mathbb{E}(X_{u,v} | \mathcal{F}_{0,0})\|_2 = o(s_{n_1, n_2}),$$

which is implied by $\overline{\mathbf{C}_3(s_{n_1, n_2})}(a)$. Next, note that $\mathbf{C}_3(s_{n_1, n_2})(b)$ can be rewritten as

$$\sqrt{n_1} \sum_{i=1}^{n_1} \sum_{v=1}^{n_2} \|P_{-\tilde{i}, -v}(X_{0,0})\|_2 = o(s_{n_1, n_2}),$$

where $P_{i,j}(X) = \mathbb{E}(X | \mathcal{F}_{i,j}) - \mathbb{E}(X | \mathcal{F}_{i-1,j})$. We then apply Lemma 6.1 in [12] to the partial sum with $b_i = 1$ and $u_i = \|P_{-\tilde{i}, -v}(X_{0,0})\|_2$. Hence we get that

$$\sum_{i=1}^{n_1} \sum_{v=1}^{n_2} \|P_{-\tilde{i}, -v}(X_{0,0})\|_2 \leq C \sum_{i=1}^{n_1} \frac{1}{\sqrt{i}} \sum_{v=1}^{n_2} \left(\sum_{k=i}^{n_1} \|P_{-\tilde{k}, -v}(X_{0,0})\|_2^2 \right)^{1/2},$$

Therefore, we deduce that

$$\sum_{i=1}^{n_1} \sum_{v=1}^{n_2} \|P_{-\tilde{i}, -v}(X_{0,0})\|_2 \leq C \sum_{i=1}^{n_1} \sum_{v=1}^{n_2} \frac{1}{\sqrt{i}} \|\mathbb{E}(X_{i,v} | \mathcal{F}_{0,0})\|_2.$$

Hence, $\mathbf{C}_3(s_{n_1, n_2})(b)$ is implied by $\overline{\mathbf{C}_3(s_{n_1, n_2})}(a)$. Similar arguments are applied to $\mathbf{C}_3(s_{n_1, n_2})(c)$ (implied by $\overline{\mathbf{C}_3(s_{n_1, n_2})}(b)$). \square

2.6.7 Proof of Theorem 2.3.5

Since $\mathbf{C}_2(s_{n_1, n_2})$ holds, the sequence $\left(s_{n_1, n_2}^{-2} \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq l \leq n_2}} S_{k,l}^2 \right)_{(n_1, n_2) \in \mathbb{N}^2}$ is uniformly integrable, and the process $\{s_{n_1, n_2}^{-1} S_{[n_1 t_1], [n_2 t_2]}\}_{(t_1, t_2) \in [0, 1]^2}$ is tight. It remains to prove that the finite-dimensional distributions converge, that is, for any $t^{(1)} = (t_1^1, t_2^1), \dots, t^{(k)} = (t_1^k, t_2^k) \in [0, 1]^2$,

$$\frac{1}{s_{n_1, n_2}} \left(\sum_{k=1}^{[n_1 t_1^{(1)}]} \sum_{l=1}^{[n_2 t_2^{(1)}]} X_{k,l}, \dots, \sum_{k=1}^{[n_1 t_1^{(k)}]} \sum_{l=1}^{[n_2 t_2^{(k)}]} X_{k,l} \right) \Rightarrow \sigma(W_2(t^{(1)}), \dots, W_2(t^{(k)})).$$

This follows from the invariance principle for fields of stationary ortho-martingale differences (see Theorem 1 in [18]), provided that for any $(t_1, t_2) \in [0, 1]^2$,

$$\lim_{n_1 \wedge n_2 \rightarrow \infty} \left\| \frac{S_{[n_1 t_1], [n_2 t_2]}}{s_{n_1, n_2}} - \frac{1}{\sqrt{n_1 n_2}} \sum_{k_1=1}^{[n_1 t_1]} \sum_{k_2=1}^{[n_2 t_2]} m \circ T_{k_1, k_2} \right\|_2 = 0. \quad (2.6.34)$$

Note that Theorem 1 in [18] is stated for reversed ortho-martingale differences, but it is also obviously true in case of ortho-martingales in the usual sense. Now in order to prove (2.6.34), since $\mathbf{C}_2(s_{n_1, n_2})$ and $\mathbf{C}_1(s_{n_1, n_2})(c)$ hold, then $\mathbf{C}_0(s_{n_1, n_2})$ holds. Two cases arise: either $m = 0$ almost surely then (2.6.34) follows directly from $\mathbf{C}_0(s_{n_1, n_2})$ and the fact that $s_{[n_1 t_1], [n_2 t_2]} / s_{n_1, n_2}$ is bounded; or $\mathbb{E}(m^2) > 0$ and, for this case, since by Remark 2.2.5, $s_{n_1, n_2} / \sqrt{n_1 n_2}$ is a slowly varying function, then (2.6.34) is equivalent $\mathbf{C}_0(s_{n_1, n_2})$. \square

2.6.8 Proof of Corollary 2.4.1

We shall apply Theorem 2.3.5, and then we only need to show that conditions $\mathbf{C}_1(s_{n_1, n_2})(c)$ and $\mathbf{C}_2(s_{n_1, n_2})$ hold. We first prove $\mathbf{C}_2(s_{n_1, n_2})(a)$, and $\mathbf{C}_2(s_{n_1, n_2})(b)$. Using Remark 2.3.3, they hold as soon as $\mathbf{C}_3(s_{n_1, n_2})$ is satisfied. Note that $P_{0,0}(X_{i,j}) = a_{i,j} \xi_{0,0}$. Therefore $\mathbf{C}_3(s_{n_1, n_2})$ is implied by (2.4.2), (2.4.3), and (2.4.4).

We prove now that $\mathbf{C}_2(s_{n_1, n_2})(c)$ holds with $u_{i,j} = |a_{i,j}|$. Since $P_{0,0}(X_{i,j}) = a_{i,j} \xi_{0,0}$ and $s_{n_1, n_2} = \sqrt{n_1 n_2} \left| \sum_{i=-n_1}^{n_1} \sum_{j=-n_2}^{n_2} a_{i,j} \right|$, this follows directly from Remark 2.3.2 provided that (2.4.1) holds. It remains to prove $\mathbf{C}_1(s_{n_1, n_2})(c)$. By Proposition 2.2.6, this condition holds as soon as (2.2.4)-(2.2.6) are satisfied. This follows from easy computations by taking into account conditions (2.4.1)-(2.4.4). \square

2.7 Appendix

2.7.1 Proof of Lemma 2.6.2

By Proposition 2.2.1 in [26], we know that $(\max_{1 \leq k \leq n_1} M_{k, n_2})_{n_2 \in \mathbb{N}}$ and $(\max_{1 \leq l \leq n_2} M_{n_1, l})_{n_1 \in \mathbb{N}}$ are both one parameter submartingales. Therefore, for any fixed pair (n_1, n_2) apply-

ing Corollary 2.10 in [15] to the submartingale $(\max_{1 \leq k \leq n_1} M_{k,n_2})_{n_2 \in \mathbb{N}}$ we have

$$\begin{aligned} \mathbb{E} \left[\max_{1 \leq l \leq n_2} \max_{1 \leq k \leq n_1} |M_{k,l}|^p \mathbf{1}_{\max_{1 \leq l \leq n_2} \max_{1 \leq k \leq n_1} |M_{k,l}| > \lambda} \right] \\ \leq 2^p \frac{p}{p-1} \mathbb{E} \left[\max_{1 \leq k \leq n_1} |M_{k,n_2}|^p \mathbf{1}_{\max_{1 \leq k \leq n_1} |M_{k,n_2}| > \frac{\lambda}{2}} \right]. \end{aligned}$$

The desired result follows by applying Corollary 2.10 in [15] again to $(M_{k,n_2})_{k \in \mathbb{N}}$. \square

2.7.2 Proof of Lemma 2.6.3

The inequality (2.6.21) comes from inequality (3.6) in [10] with $X_i = \sum_{j=1}^{n_2} d_{i,j}$ by taking the expectation and by noticing that $(X_i)_{i \geq 1}$ is a sequence of martingale differences. Next (2.6.22) comes from (2.6.21) by taking account that $x^2 \mathbf{1}_{|x| > \lambda} \leq 4 \left(|x| - \frac{\lambda}{2} \right)_+^2$. Finally (2.6.23) is again an application of inequality (3.6) in [10]. \square

Chapter 3

On the central limit theorem for stationary random fields under \mathbb{L}^1 -projective condition

The first aim of this chapter is to wonder to what extent we can generalize the central limit theorem of Gordin [25] under the so-called \mathbb{L}^1 -projective criteria to ergodic stationary random fields when completely commuting filtrations are considered. Surprisingly it appears that this result cannot be extended to its full generality and that an additional condition is needed.

3.1 Introduction and main results

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be an *ergodic* bijective bimeasurable transformation preserving the probability \mathbb{P} . Let \mathcal{F}_0 be a sub- σ -algebra of \mathcal{A} satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$ and f be a $\mathbb{L}^1(\mathbb{P})$ real-valued centered random variable adapted to \mathcal{F}_0 . By U we denote the operator $U : f \mapsto f \circ T$. The notation I will denote the identity operator. Define then the stationary sequence $(f_i)_{i \in \mathbb{Z}}$ by $f_i = f \circ T^i = U^i f$, its associated stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ by $\mathcal{F}_i = \mathcal{F}_0 \circ T^{-i}$ and let $S_n(f) = \sum_{i=0}^{n-1} U^i f$.

The following theorem is essentially due to Gordin [25] and gives sufficient conditions for $(U^i f)_{i \in \mathbb{Z}}$ to satisfy the central limit theorem.

Theorem 3.1.1 (Gordin). *Assume that the series*

$$\sum_{i \geq 0} \mathbb{E}(U^i f | \mathcal{F}_0) \text{ converges in } \mathbb{L}^1(\mathbb{P}) \quad (3.1.1)$$

and

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|S_n(f)|)}{\sqrt{n}} < \infty. \quad (3.1.2)$$

Then $n^{-1/2} S_n(f)$ converges in distribution to a centered normal variable (that can be degenerate).

The proof of this result is based on the following coboundary martingale decomposition (see [30] for more details concerning necessary and sufficient conditions for the existence of such a decomposition): Under (3.1.1),

$$f = m + (I - U)g \quad (3.1.3)$$

where m and g are in $\mathbb{L}^1(\mathbb{P})$ and $(U^i m)_{i \geq 0}$ is a stationary sequence of martingale differences, and on the following theorem whose complete proof can be found in Esseen and Janson [23].

Theorem 3.1.2 (Esseen-Janson). *If $(U^i m)_{i \geq 0}$ is a stationary and ergodic sequence of martingale differences in $\mathbb{L}^1(\mathbb{P})$ satisfying*

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|\sum_{i=0}^{n-1} U^i m|)}{\sqrt{n}} < \infty, \quad (3.1.4)$$

then $m \in \mathbb{L}^2(\mathbb{P})$.

Clearly, using the coboundary martingale decomposition (3.1.3), (3.1.2) implies (3.1.4).

The aim of this paper is to prove that Theorem 3.1.1 can be extended to random fields when the underlying filtrations are completely commuting (see [26, Chap. 1])

for a definition of this notion). To fix the idea, let us first state the result in case of multidimensional index of dimension $d = 2$ (the general case will be stated in Section 3.4). Then, in complement to the previous notation, let S be an ergodic bimeasurable and measure preserving bijection of Ω . By V we denote the operator $V : f \mapsto f \circ S$.

In what follows we shall assume that the ergodic transformations T and S are commuting. Note that $T_{i,j} = T^i S^j$ is an ergodic \mathbb{Z}^2 action on $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\mathcal{F}_{0,0}$ be a sub-sigma field of \mathcal{A} and for all $(i, j) \in \mathbb{Z}^2$ define $\mathcal{F}_{i,j} = T^{-i} S^{-j}(\mathcal{F}_{0,0})$. Suppose that the filtration $(\mathcal{F}_{i,j})_{(i,j) \in \mathbb{Z}^2}$ is increasing in i for every j fixed and increasing in j for every i fixed, and is *completely commuting* in the sense that, for any integrable f ,

$$\mathbb{E}(\mathbb{E}(f|\mathcal{F}_{i,j})|\mathcal{F}_{u,v}) = \mathbb{E}(f|\mathcal{F}_{i \wedge u, j \wedge v}).$$

To extend Theorem 3.1.1 to random fields indexed by the lattice \mathbb{Z}^2 , the first tool is a suitable coboundary orthomartingale decomposition. In what follows, f is a $\mathcal{F}_{0,0}$ -measurable centered $\mathbb{L}^1(\mathbb{P})$ function. According to Volný [29], the condition:

$$\text{the series } \sum_{i,j \geq 0} \mathbb{E}(U^i V^j f | \mathcal{F}_{0,0}) \text{ converges in } \mathbb{L}^1(\mathbb{P}) \quad (3.1.5)$$

implies the existence of the following decomposition:

$$f = m + (I - U)g_1 + (I - V)g_2 + (I - U)(I - V)g_3, \quad (3.1.6)$$

where $m, g_1, g_2, g_3 \in \mathbb{L}^1(\mathbb{P})$, $(U^i V^j m)$ is a stationary field of orthomartingale differences, $(V^j g_1)_j$ is a stationary martingale differences sequence with respect to the filtration $(\mathcal{F}_{\infty, j})_j$, and $(U^i g_2)_i$ is a stationary martingale differences sequence with respect to the filtration $(\mathcal{F}_{i, \infty})_i$. To fix the ideas, setting $\mathbb{E}_{a,b}(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_{a,b})$, we have

$$m = \sum_{i,j \geq 0} (\mathbb{E}_{0,0}(U^i V^j f) - \mathbb{E}_{-1,0}(U^i V^j f) - \mathbb{E}_{0,-1}(U^i V^j f) + \mathbb{E}_{-1,-1}(U^i V^j f)),$$

$$g_1 = \sum_{i,j \geq 0} (\mathbb{E}_{-1,0}(U^i V^j f) - \mathbb{E}_{-1,-1}(U^i V^j f)), \quad g_2 = \sum_{i,j \geq 0} (\mathbb{E}_{0,-1}(U^i V^j f) - \mathbb{E}_{-1,-1}(U^i V^j f)),$$

and $g_3 = \sum_{i,j \geq 0} \mathbb{E}_{-1,-1}(U^i V^j f)$. Recall also that $(U^i V^j m)$ is said to be a orthomartingale differences field w.r.t. $(\mathcal{F}_{i,j})$ if

$$\mathbb{E}_{i-1,j}(U^i V^j m) = \mathbb{E}_{i,j-1}(U^i V^j m) = \mathbb{E}_{i-1,j-1}(U^i V^j m) = 0 \text{ a.s.}$$

Note that if f is additionally assumed to be regular in the sense that f is $\mathcal{F}_{\infty,\infty}$ -measurable and $\mathbb{E}(f|\mathcal{F}_{0,-\infty}) = \mathbb{E}(f|\mathcal{F}_{-\infty,0}) = 0$ then, it is proved in Volný [29] that the converse is true, meaning that if f satisfies the decomposition (3.1.6) then (3.1.5) holds. We also refer to [22] where the existence of the decomposition (3.1.6) is proved under a reinforcement of (3.1.5) (they assume that the series of the \mathbb{L}^1 -norm is convergent). We also mention [24, Theorem 2.2] where a necessary and sufficient condition for an orthomartingale-coboundary decomposition is established when all the underlying random elements are square integrable.

Our first result is the following:

Theorem 3.1.3. *Let f be a $\mathcal{F}_{0,0}$ -measurable centered $\mathbb{L}^1(\mathbb{P})$ random variable. Let $S_{n_1, n_2}(f) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} U^i V^j f$. Assume that condition (3.1.5) is satisfied and that*

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\| \sum_{i=0}^{n-1} U^i f \right\|_1 < \infty, \quad \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{j=0}^{N-1} V^j f \right\|_1 < \infty \quad (3.1.7)$$

and

$$\liminf_{n_1 \rightarrow \infty} \liminf_{n_2 \rightarrow \infty} \frac{\mathbb{E}(|S_{n_1, n_2}(f)|)}{\sqrt{n_1 n_2}} < \infty, \quad \liminf_{n_2 \rightarrow \infty} \liminf_{n_1 \rightarrow \infty} \frac{\mathbb{E}(|S_{n_1, n_2}(f)|)}{\sqrt{n_1 n_2}} < \infty. \quad (3.1.8)$$

Then the random variables m , $(I-U)g_1$ and $(I-V)g_2$ defined in (3.1.6) are in $\mathbb{L}^2(\mathbb{P})$.

Compared to the case of random sequences a natural question is then to wonder if condition (3.1.5) together with conditions (3.1.7) and (3.1.8) are sufficient to ensure that, when $\min(n_1, n_2) \rightarrow \infty$, the limiting distribution behavior of $(n_1 n_2)^{-1/2} S_{n_1, n_2}(f)$ is the same as that of the orthomartingale part $(n_1 n_2)^{-1/2} S_{n_1, n_2}(m)$. In other terms

one can wonder if assuming the conditions of Theorem 3.1.3 is enough to ensure that the coboundaries' behavior, i.e. $(n_1 n_2)^{-1/2}(S_{n_1, n_2}(f) - S_{n_1, n_2}(m))$ is negligible for the convergence in distribution. Surprisingly the answer to this question is negative as shown by the next counterexample.

Theorem 3.1.4. *There exist a probability space $(\Omega, \mathcal{A}, \mu)$, a function $g \in \mathbb{L}^1(\mu)$, measurable with respect to a σ -algebra $\mathcal{F}_{0,0} \subset \mathcal{A}$ and bijective bimeasurable ergodic transformations T and S such that $f = (I - U)g$ is in $\mathbb{L}^2(\mu)$, satisfies the conditions (3.1.5), (3.1.7) and (3.1.8) but such that $(n_1 n_2)^{-1/2} S_{n_1, n_2}(f)$ does not converge in distribution to zero as $\min(n_1, n_2) \rightarrow \infty$.*

This result proves a drastically different behavior for the case of random fields with dimension $d \geq 2$ compared to the case of random sequences ($d = 1$) for which the coboundary is negligible for the convergence in distribution as soon as (3.1.1) is assumed. Let us also indicate that even if (3.1.1) is reinforced to a convergence in $\mathbb{L}^p(\mu)$ for some $p \in [1, 2)$ this is still not enough for $(n_1 n_2)^{-1/2} S_{n_1, n_2}(f)$ and $(n_1 n_2)^{-1/2} S_{n_1, n_2}(m)$ to have the same limiting behavior (to see this it suffices to take $n_k = \lceil 2^{k/2} \rceil$ and $m_k \sim (n_k/k)^{p/(2-p)}$ in the construction of the counterexample of Theorem 3.1.4).

However, reinforcing the conditions of Theorem 3.1.3, we can prove the following CLT.

Theorem 3.1.5. *In addition to the conditions of Theorem 3.1.3, assume that*

$$\lim_{\min(n_1, n_2) \rightarrow \infty} \frac{\mathbb{E}(|S_{n_1, n_2}(f)|)}{\sqrt{n_1 n_2}} \text{ exists.} \quad (3.1.9)$$

Then, as $\min(n_1, n_2) \rightarrow \infty$, $(n_1 n_2)^{-1/2} S_{n_1, n_2}(f)$ converges in distribution to a centered normal variable (that can be degenerate).

Remark 3.1.6. *Under the conditions of Theorem 3.1.3, condition (3.1.9) is equiva-*

lent to

$$\begin{aligned} \limsup_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \frac{1}{\sqrt{nk}} \|S_{n,k}(f)\|_1 &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \|S_{n,n}(f)\|_1 \text{ and} \\ \limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{nk}} \|S_{n,k}(f)\|_1 &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \|S_{n,n}(f)\|_1. \end{aligned}$$

This is consequence of the proof of Theorem 3.1.5. The existence of $\lim_{n \rightarrow \infty} n^{-1} \|S_{n,n}(f)\|_1$ has been mentioned there.

It is noteworthy to indicate that f does not need to be in \mathbb{L}^2 but only in \mathbb{L}^1 to apply Theorem 3.1.5 (see Example 5.1.10 given below). Theorem 3.1.5 then gives alternative projective conditions compared to those required in [31, Th. 5.1] or in [27, Th. 1] for the central limit theorem under the normalization $\sqrt{n_1 n_2}$ to hold. Note that the proofs of the two above mentioned results are also based on an orthomartingale approximation. We refer also to [32] where the notion of orthomartingales and completely commuting filtrations have been previously used in the particular case of functions of iid random fields. Let us also indicate that when filtrations in the lexicographic order rather than completely commuting filtrations are considered, [21, Th. 1] provides a projective type condition in the spirit of the \mathbb{L}^1 -projective condition (3.1.5) (but still requiring f to be in \mathbb{L}^2) for the normalized partial sums associated with a stationary random field to satisfy the central limit theorem. His proof is based on the so-called Lindeberg method.

3.2 Examples

3.2.1 An example when f is in \mathbb{L}^1 but not in \mathbb{L}^2

For $k \in \mathbb{N}^*$ and $i, j \in \mathbb{Z}$, let $e_{k,i,j}$ be mutually independent zero mean random variables with $Ue_{k,i,j} = e_{k,i+1,j}$, $Ve_{k,i,j} = e_{k,i,j+1}$. Let $\mathcal{F}_{a,b} = \sigma(e_{k,i,j}, k \in \mathbb{Z}, i \leq a, j \leq b)$. We denote $e_k = e_{k,0,0}$. Let $(v_k)_{k \geq 1}$ be a sequence of nonnegative reals and $(p_k)_{k \geq 1}$ a sequence of reals in $[0, 1]$. Assume that for any $i, j \in \mathbb{Z}^2$, $\mathcal{L}(e_{k,i,j}) = \mathcal{L}(e_k)$ and that e_k takes value v_k with probability p_k , $-v_k$ with probability p_k and 0 with probability

$1 - 2p_k$. It follows that $\|e_k\|_1 = 2v_k p_k$ and $\|e_k\|_2^2 = 2v_k^2 p_k$. We use the following selection of $(v_k)_{k \geq 1}$ and $(p_k)_{k \geq 1}$: $v_k = k^2(\log(k+1))^2$ and $p_k = \frac{1}{2k^2(\log(k+1))^4}$. For any $k \geq 1$ and $i \geq 0$, let $a_{k,i} = (k+i)^{-2}$, and define

$$g = \sum_{k \geq 1} \sum_{i \geq 0} a_{k,i} U^{-k-i} e_k, \quad h = \sum_{k \geq 1} \frac{1}{k} U^{-1} V^{-1} e_k, \quad m = \sum_{k \geq 1} \frac{1}{k^2} e_k,$$

and $f = m + (I - U)g + (I - U)(I - V)h$. By simple computations we have $\|g\|_1 < \infty$ and $\|h\|_1 < \infty$ but $\|g\|_2 = \infty$ and $\|h\|_2 = \infty$. In addition m is in $\mathbb{L}^2(\mathbb{P})$. On another hand, f is in $\mathbb{L}^1(\mathbb{P})$ but not in $\mathbb{L}^2(\mathbb{P})$, and one can verify that $\sum_{i,j \geq 0} \|\mathbb{E}(U^i V^j f | \mathcal{F}_{0,0})\|_1 < \infty$. Moreover, for any positive integer ℓ , by independence of the r.v.'s $e_{k,i,j}$, we infer that

$$\|(I - U^\ell)g\|_2^2 = \sum_{k \geq 1} \sum_{i=0}^{\ell-1} a_{k,i}^2 \|e_k\|_2^2 + \sum_{k \geq 1} \sum_{i \geq 0} (a_{k,i} - a_{k,i+\ell})^2 \|e_k\|_2^2.$$

Hence, by simple algebra, there exists a positive constant C such that

$$\|(I - U^\ell)g\|_2^2 \leq C \log(\ell + 1) \text{ for any positive integer } \ell. \quad (3.2.1)$$

In particular, we get $\|(I - U)g\|_2 < \infty$. In addition, by (3.2.1),

$$\frac{\|\sum_{i=0}^{n-1} \sum_{j=0}^{N-1} U^i V^j (I - U)g\|_2^2}{nN} = \frac{\|(I - U^n)g\|_2^2}{n} \xrightarrow{n \rightarrow \infty} 0,$$

which combined with $\frac{\|\sum_{i=0}^{n-1} \sum_{j=0}^{N-1} U^i V^j (I - U)(I - V)h\|_1}{\sqrt{nN}} \rightarrow 0$, as $\max(n, N) \rightarrow \infty$, implies that $\frac{\|S_{n,N}(f - m)\|_1}{\sqrt{nN}} \rightarrow 0$, as $\min(n, N) \rightarrow \infty$. Next, since, $\lim_{\min(n,N) \rightarrow \infty} \frac{\|\sum_{i=0}^{n-1} \sum_{j=0}^{N-1} U^i V^j m\|_1}{\sqrt{nN}}$ exists (it is equal to $\sqrt{\frac{2}{\pi}} \|m\|_2$), we can deduce that $\lim_{\min(n,N) \rightarrow \infty} \frac{\|\sum_{i=0}^{n-1} \sum_{j=0}^{N-1} U^i V^j f\|_1}{\sqrt{nN}}$ also exists. Hence all the conditions of Theorem 3.1.5 are satisfied. So, as $\min(n_1, n_2) \rightarrow \infty$, $(n_1 n_2)^{-1/2} S_{n_1, n_2}(f)$ converges in distribution to a centered normal variable.

3.2.2 An example where f does not satisfy Hannan's \mathbb{L}^2 -condition

We exhibit an example where f satisfies all the conditions of Theorem 3.1.5 but not the Hannan's \mathbb{L}^2 -condition required in [31, Th. 5.1]. We consider the random field $(e_{k,i,j})_{k,i,j}$ of mutually independent random variables as in Example 5.1.10 with the following choices of $(v_k)_{k \geq 1}$ and $(p_k)_{k \geq 1}$. Let $\alpha > 4$. Then for any $k \geq 1$, define

$$v_k = k^\alpha \text{ and } p_k = \frac{1}{2k^5 \log(k+1)^2}.$$

For any $k \geq 1$ and $i, j \geq 0$, let $a_{k,i,j} = (k+i+j)^{-\alpha}$. Then, define

$$f = \sum_{k \geq 1} \sum_{u,v \geq 0} a_{k,u,v} U^{-u} V^{-v} e_k.$$

$(U^i V^j f)_{i,j}$ is usually called a super linear random field. Let $\mathcal{F}_{a,b} = \sigma(e_{k,i,j}, k \in \mathbb{Z}, i \leq a, j \leq b)$. Note that f is a $\mathcal{F}_{0,0}$ -measurable random variable, centered and in $\mathbb{L}^2(\mathbb{P})$. In addition, one can check that condition (3.1.5) is satisfied implying that the orthomartingale coboundary decomposition (3.1.6) holds. Moreover, by simple algebra, one can verify that $m \in \mathbb{L}^2(\mathbb{P})$ and that there exists a positive constant K (depending on α) such that $\|(I - U^\ell)g_1\|_2^2 + \|(I - U^\ell)g_2\|_2^2 \leq K\ell(\log(\ell+1))^{-1}$. Proceeding as in Example 5.1.10, one can verify that conditions (3.1.7), (3.1.8) and (3.1.9) are satisfied. Hence, Theorem 3.1.5 applies to $(n_1 n_2)^{-1/2} S_{n_1, n_2}(f)$.

On another hand, defining $P_{0,0}(\cdot) = \mathbb{E}_{0,0}(\cdot) - \mathbb{E}_{-1,0}(\cdot) - \mathbb{E}_{0,-1}(\cdot) + \mathbb{E}_{-1,-1}(\cdot)$, we get, for any $i, j \geq 0$,

$$\|P_{0,0}(U^i V^j f)\|_2^2 = \left\| \sum_{k \geq 1} a_{k,i,j} e_k \right\|_2^2 = \sum_{k \geq 1} a_{k,i,j}^2 \|e_k\|_2^2 \geq \sum_{k \geq i+j+1} \frac{k^{2\alpha-5}}{(k+i+j)^{2\alpha} (\log(k+1))^2},$$

implying that

$$\|P_{0,0}(U^i V^j f)\|_2^2 \geq \frac{C}{(i+j+1)^4 (\log(i+j+2))^2}.$$

Hence $\sum_{i,j} \|P_{0,0}(U^i V^j f)\|_2$ diverges. So the Hannan's \mathbb{L}^2 -condition in the random

fields setting does not hold, and [31, Th. 5.1] does not apply. Note also that for this example, [21, Th. 1] that involves filtrations in the lexicographic order, cannot be applied.

3.3 Proofs

3.3.1 Proof of Theorem 3.1.3

Recall the decomposition (3.1.6) and let

$$m' = m + (I - V)g_2. \quad (3.3.1)$$

It follows that $(U^i m')_i$ is a stationary sequence of $\mathbb{L}^1(\mathbb{P})$ martingale differences with respect to $(\mathcal{F}_{i,\infty})_i$. Since T is ergodic, according to Theorem 3.1.2, if

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|\sum_{i=0}^{n-1} U^i m'|)}{\sqrt{n}} < \infty, \quad (3.3.2)$$

then $m' \in \mathbb{L}^2(\mathbb{P})$. By (3.1.6), $\left\| \sum_{i=0}^{n-1} U^i m' \right\|_1 \leq \left\| \sum_{i=0}^{n-1} U^i f \right\|_1 + 2\|g_1\|_1 + 4\|g_3\|_1$. Hence, since g_1 and g_3 are in $\mathbb{L}^1(\mathbb{P})$, under the first part of (3.1.7)

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|\sum_{i=0}^{n-1} U^i m'|)}{\sqrt{n}} \leq \liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|\sum_{i=0}^{n-1} U^i f|)}{\sqrt{n}} < \infty,$$

Therefore (3.3.2) holds and $m' \in \mathbb{L}^2(\mathbb{P})$. Next recall that $m' = m + (I - V)g_2$ and recall that $(V^j m)_j$ is a stationary sequence of $\mathbb{L}^1(\mathbb{P})$ martingale differences with respect to $(\mathcal{F}_{\infty,j})_j$. Since S is ergodic, according again to Theorem 3.1.2, to prove that $m \in \mathbb{L}^2(\mathbb{P})$, it suffices to prove that

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|\sum_{j=0}^{n-1} V^j m|)}{\sqrt{n}} < \infty. \quad (3.3.3)$$

But since $m' = m + (I - V)g_2$ and g_2 is in $\mathbb{L}^1(\mathbb{P})$, proving (3.3.3) is reduced to show that

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|\sum_{j=0}^{n-1} V^j m'|)}{\sqrt{n}} < \infty. \quad (3.3.4)$$

With this aim, recall first that $m' \in \mathbb{L}^2(\mathbb{P})$ and note that

$$\frac{1}{\sqrt{n}} \left\| \sum_{j=0}^{n-1} V^j m' \right\|_1 \leq \frac{1}{\sqrt{n}} \left\| \sum_{j=0}^{n-1} V^j m' \right\|_2 := \sigma_n. \quad (3.3.5)$$

For any fixed positive integer n , let $d := n^{-1/2} \sum_{j=0}^{n-1} V^j m'$. Since $(U^i m')_i$ is a stationary and ergodic sequence of martingale differences in $\mathbb{L}^2(\mathbb{P})$, so is $(U^i d)_i$. By the CLT for stationary and ergodic martingales in $\mathbb{L}^2(\mathbb{P})$, as $N \rightarrow \infty$, $N^{-1/2} \sum_{i=0}^{N-1} U^i d$ converges in distribution to a centered Gaussian random variable G_n with variance σ_n^2 . Hence, by [1, Th. 3.4] and noticing that $\mathbb{E}|G_n| = \sigma_n \sqrt{2/\pi}$, for any fixed positive integer n , we get

$$\sigma_n \leq \sqrt{\frac{\pi}{2}} \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{i=0}^{N-1} U^i d \right\|_1 = \sqrt{\frac{\pi}{2}} \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{nN}} \left\| \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} U^i V^j m' \right\|_1. \quad (3.3.6)$$

But, recalling (3.1.6) and that $m' = m + (I - V)g_2$, we derive

$$\frac{1}{\sqrt{N}} \left\| \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} U^i V^j m' \right\|_1 \leq \frac{1}{\sqrt{N}} \left\| \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} U^i V^j f \right\|_1 + \frac{2n}{\sqrt{N}} \|g_1\|_1 + \frac{4}{\sqrt{N}} \|g_3\|_1.$$

Since g_1 and g_3 are in $\mathbb{L}^1(\mathbb{P})$, the two last terms of the right-hand side are converging to zero as N tends to infinity. Hence, taking into account (3.3.5) and (3.3.6), we get

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|\sum_{j=0}^{n-1} V^j m'|)}{\sqrt{n}} \leq \sqrt{\frac{\pi}{2}} \liminf_{n \rightarrow \infty} \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{nN}} \left\| \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} U^i V^j f \right\|_1.$$

which is finite by the second part of condition (3.1.8). This ends the proof of (3.3.4) (and then of (3.3.3)) and leads to the fact that m is in $\mathbb{L}^2(\mathbb{P})$. Next recall that we have proved that m' defined in (3.3.1) is in $\mathbb{L}^2(\mathbb{P})$ which combined with the fact

that m is in $\mathbb{L}^2(\mathbb{P})$ implies that $(I - V)g_2$ is in $\mathbb{L}^2(\mathbb{P})$. On another hand setting $m'' = m + (I - U)g_1$, we can use previous arguments to infer that m'' is in $\mathbb{L}^2(\mathbb{P})$. Hence taking into account that $m \in \mathbb{L}^2(\mathbb{P})$, we get that $(I - U)g_1$ is in $\mathbb{L}^2(\mathbb{P})$. This ends the proof of the theorem.

3.3.2 Proof of Theorem 3.1.4

Let $(X_{i,j})_{(i,j) \in \mathbb{Z}^2}$ be an iid random field on $(\Omega, \mathcal{B}, \mu)$. Let $\mathcal{A} = \sigma\{X_{i,j}, (i,j) \in \mathbb{Z}^2\}$. Then, there exist transformations T and S such that $X_{i,j} \circ T = X_{i+1,j}$ and $X_{i,j} \circ S = X_{i,j+1}$. These transformations on $(\Omega, \mathcal{A}, \mu)$ are bijective, commuting, probability preserving and ergodic. Let $Uf = f \circ T$ and $Vf = f \circ S$. Denote $e = X_{0,0}$ and $\mathcal{C} = \sigma\{U^i e : i \in \mathbb{Z}\}$. (\mathcal{C}, T) is thus a Bernoulli dynamical system and the sigma algebras $S^j \mathcal{C}$, $j \in \mathbb{Z}$, are mutually independent. Let us recall the so-called Rokhlin lemma.

Lemma 3.3.1 (Rokhlin lemma). *Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic dynamical system, n a positive integer, and $\epsilon > 0$. Then there exists a set $F \in \mathcal{A}$ such that $F, T^{-1}F, \dots, T^{-n+1}F$ are disjoint and $\mu(\cup_{i=0}^{n-1} T^{-i}F) > 1 - \epsilon$.*

For any integer $k \geq 1$, we set $n_k = \lfloor 2^{k/2} \rfloor$ and $m_k = 2^k$. Let $\epsilon > 0$. By Lemma 3.3.1 there exists a Rokhlin tower $F, T^{-1}F, \dots, T^{-N_k+1}F$ with $N_k = m_k n_k \sim 2^{3k/2}$. Note that $1/N_k \geq \mu(T^{-i}F) > (1 - \epsilon)/N_k$, for any $i = 0, \dots, N_k - 1$. We define

$$g_k(\omega) = \begin{cases} (j+1)\sqrt{m_k/n_k} & \text{on } T^{-j}F, \quad j = 0 \dots, n_k - 1, \\ (2n_k - j - 1)\sqrt{m_k/n_k} & \text{on } T^{-j}F, \quad j = n_k, \dots, 2n_k - 1, \\ 0 & \text{on the rest of } \Omega. \end{cases}$$

We can notice that $g_k - Ug_k$ is equal to $\sqrt{m_k/n_k}$ on $T^{-j}F$, $j = 0 \dots, n_k - 1$, to $-\sqrt{m_k/n_k}$ on $T^{-j}F$, $j = n_k \dots, 2n_k - 1$, and to 0 on the rest of Ω . In addition, we

have

$$\|g_k - Ug_k\|_2^2 \leq \frac{2}{n_k m_k} \sim \frac{2}{2^{k/2}}, \quad \|g_k\|_1 \leq 2\sqrt{\frac{n_k}{m_k}} \leq 2^{1-k/4} \text{ and}$$

$$\left\| \sum_{i=0}^{2n_k-1} U^i(g_k - Ug_k) \right\|_2 = \|g_k - U^{2n_k}g_k\|_2 \leq 2\sqrt{n_k}.$$

Notice that all sums of $U^i(g_k - Ug_k)$ are \mathcal{C} -measurable hence the random variables $(V^j(g_k - U^{2n_k}g_k))_j$ are iid. In addition, the support of $g_k - U^{2n_k}g_k$ is included in the union B_k of $F, \dots, T^{-4n_k+1}F$ hence is of measure $\leq 4/m_k = 2^{-k+2}$. Next, for any $k \geq 1$, let $A_k = \{|g_k - U^{2n_k}g_k| \geq (1/2)\sqrt{m_k n_k}\}$. The set A_k is included in B_k and is of measure $2/m_k = 2^{-k+1}$. Because the sigma algebras $S^i\mathcal{C}$, $i \in \mathbb{Z}$, are mutually independent, the sets $S^{-j}A_k$, $j = 0, \dots, m_k - 1$, are independent. For $h = 1_{A_k}$ using that $e^{2\ln(1-x)/x} \geq e^{-4}$ for any $x \in]0, 1/2]$ and that $m_k\mu(A_k) = 2$, we thus have, for any $k \geq 2$,

$$\mu\left(\sum_{j=0}^{m_k-1} V^j h = 1\right) = m_k\mu(A_k)(1 - \mu(A_k))^{m_k-1} \geq 2/e^4.$$

We then conclude that

$$\mu\left(\frac{1}{\sqrt{n_k}}\frac{1}{\sqrt{m_k}}\left|\sum_{i=0}^{2n_k-1}\sum_{j=0}^{m_k-1}U^iV^j(g_k - Ug_k)\right|\geq 1/2\right) \geq 2/e^4. \quad (3.3.7)$$

By recursion we shall define a strictly increasing sequence $k_\ell \nearrow \infty$ and then set

$$g = \sum_{\ell=1}^{\infty} g_{k_\ell} \text{ and } f = g - Ug.$$

For $\ell = 1$ we put $k_\ell = 1$. Suppose now that for $1 \leq \ell' < \ell$ the $k_{\ell'}$ have been defined. All the functions $g_{k_{\ell'}}$ are bounded ($0 \leq g_{k_{\ell'}} \leq \sqrt{n_{k_{\ell'}} m_{k_{\ell'}}}$) hence the sums

$$\sum_{i=0}^{n-1} \sum_{\ell'=1}^{\ell-1} U^i(g_{k_{\ell'}} - Ug_{k_{\ell'}}) = \sum_{\ell'=1}^{\ell-1} (g_{k_{\ell'}} - U^n g_{k_{\ell'}}),$$

$n \geq 1$, are uniformly bounded. If k_ℓ is sufficiently large we thus get

$$\frac{1}{\sqrt{n_{k_\ell}}} \sum_{\ell'=1}^{\ell-1} \left| \sum_{i=0}^{2n_{k_\ell}-1} U^i(g_{k_{\ell'}} - U g_{k_{\ell'}}) \right| < 1/2^\ell.$$

Next note that $\left(V^j \left(\frac{1}{\sqrt{n_{k_\ell}}} \sum_{\ell'=1}^{\ell-1} \sum_{i=0}^{2n_{k_\ell}-1} U^i(g_{k_{\ell'}} - U g_{k_{\ell'}}) \right) \right)_{j \geq 0}$ are martingale differences. Hence

$$\left\| \frac{1}{\sqrt{n_{k_\ell}}} \frac{1}{\sqrt{m_{k_\ell}}} \sum_{i=0}^{2n_{k_\ell}-1} \sum_{j=0}^{m_{k_\ell}-1} U^i V^j \sum_{\ell'=1}^{\ell-1} (g_{k_{\ell'}} - U g_{k_{\ell'}}) \right\|_2 \leq \frac{1}{2^\ell}. \quad (3.3.8)$$

On another hand, recall that $\|g_k - U g_k\|_2 \leq \sqrt{2}/2^{k/4}$. Hence choosing k_ℓ sufficiently large we get $\left\| \sum_{i=0}^{2n_{k_{\ell'}}-1} U^i(g_{k_{\ell'}} - U g_{k_{\ell'}}) \right\|_2 \leq 4^{-\ell}$, for all $1 \leq \ell' < \ell$. Having constructed the sequence of k_ℓ this way we thus have $\left\| \frac{1}{\sqrt{n_{k_\ell}}} \sum_{\ell'=\ell+1}^{\infty} \sum_{i=0}^{2n_{k_{\ell'}}-1} U^i(g_{k_{\ell'}} - U g_{k_{\ell'}}) \right\|_2 < 2^{-\ell}$. Since $\left(V^j \left(\frac{1}{\sqrt{n_{k_\ell}}} \sum_{\ell'=\ell+1}^{\infty} \sum_{i=0}^{2n_{k_{\ell'}}-1} U^i(g_{k_{\ell'}} - U g_{k_{\ell'}}) \right) \right)_{j \geq 0}$ are martingale differences, we get

$$\left\| \frac{1}{\sqrt{n_{k_\ell}}} \frac{1}{\sqrt{m_{k_\ell}}} \sum_{i=0}^{2n_{k_\ell}-1} \sum_{j=0}^{m_{k_\ell}-1} U^i V^j \sum_{\ell'=\ell+1}^{\infty} (g_{k_{\ell'}} - U g_{k_{\ell'}}) \right\|_2 \leq \frac{1}{2^\ell}. \quad (3.3.9)$$

Then, the upper bounds (3.3.8) and (3.3.9) entail that

$$\left\| \frac{1}{\sqrt{n_{k_\ell}}} \frac{1}{\sqrt{m_{k_\ell}}} \sum_{i=0}^{2n_{k_\ell}-1} \sum_{j=0}^{m_{k_\ell}-1} U^i V^j (I - U)(g - g_{k_\ell}) \right\|_2 \leq \frac{2}{2^\ell}. \quad (3.3.10)$$

Hence taking into account (3.3.7) and (3.3.10), it follows that, for $f = g - U g$, the sequence $(n_1 n_2)^{-1/2} \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} U^i V^j f$ cannot converge in distribution to zero.

On another hand, for any p and q fixed, by independence, $\sum_{i=0}^p \sum_{j=0}^q \mathbb{E}(U^i V^j f | \mathcal{F}_{0,0}) = g - \mathbb{E}(U^{p+1} g | \mathcal{F}_{0,0})$. But, by construction, $\lim_{p \rightarrow \infty} \|U^{p+1} g\|_1 = 0$. Hence $\|\mathbb{E}(U^{p+1} g | \mathcal{F}_{0,0})\|_1$ is going to zero as $p \rightarrow \infty$. Therefore condition (3.1.5) is satisfied.

It remains to prove that the conditions (3.1.7) and (3.1.8) are satisfied with $f = g - U g$. With this aim, note first that $n^{-1/2} \sum_{i=0}^{n-1} U^i f = n^{-1/2} (g - U^n g) \rightarrow_{n \rightarrow \infty} 0$

in \mathbb{L}^1 (recall that $g \in \mathbb{L}^1$). Next, since the random variables $(V^j f)_{j \geq 0}$ are independent and square integrable, $m^{-1/2} \left\| \sum_{j=0}^{m-1} V^j f \right\|_2 = \|g - Ug\|_2 < \infty$. Hence both conditions in (3.1.7) are satisfied. On another hand, for every m fixed,

$$\frac{1}{\sqrt{n}} \frac{1}{\sqrt{m}} \left\| \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} U^i V^j (g - Ug) \right\|_1 \leq \frac{2\sqrt{m}}{\sqrt{n}} \|g\|_1 \xrightarrow{n \rightarrow \infty} 0,$$

proving the second part of condition (3.1.8). It remains to prove its first part. Here we use particular properties of g constructed above. We have found a sequence of n_{k_ℓ} for which there exists a positive constant $c > 0$ such that $(n_{k_\ell})^{-1/2} \left\| \sum_{i=0}^{2n_{k_\ell}-1} U^i (g - Ug) \right\|_2 \leq c(2^{-\ell} + 1)$. Since $(V^j \sum_{i=0}^{2n_{k_\ell}-1} U^i (g - Ug))_{j \geq 0}$ is a stationary sequence of martingale differences in \mathbb{L}^2 , it follows that

$$\frac{1}{\sqrt{n_{k_\ell}}} \frac{1}{\sqrt{m}} \left\| \sum_{i=0}^{2n_{k_\ell}-1} \sum_{j=0}^{m-1} U^i V^j (g - Ug) \right\|_2 \leq c(2^{-\ell} + 1) \leq 2c.$$

Since the upper bound is uniform for all n_{k_ℓ} , the first part of condition (3.1.8) holds true.

3.3.3 Proof of Theorem 3.1.5

We shall use the coboundary decomposition (3.1.6). Note first that $(U^i V^j m)_{i,j}$ is an ergodic and stationary $\mathbb{L}^2(\mathbb{P})$ orthomartingale field. Then, according to the CLT for ergodic fields of martingale differences as obtained in [28], as $\min(n_1, n_2) \rightarrow \infty$, the sequence $(n_1 n_2)^{-1/2} \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} U^i V^j m$ converges in distribution to a centered Gaussian random variable with variance $\|m\|_2^2$. Theorem 3.1.5 then follows if one can prove that under the conditions of Theorem 3.1.3 and if condition (3.1.9) is satisfied, then

$$\lim_{\min(n_1, n_2) \rightarrow \infty} \frac{\|S_{n_1, n_2}(f) - S_{n_1, n_2}(m)\|_1}{\sqrt{n_1 n_2}} = 0. \quad (3.3.11)$$

Clearly, since $\left\| \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} U^i V^j (I-U)(I-V)g_3 \right\|_1 \leq 4\|g_3\|_1$, the convergence (3.3.11) will follow if one can prove that, as $\min(n_1, n_2) \rightarrow \infty$,

$$(n_1 n_2)^{-1/2} \left(\left\| \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} U^i V^j (I-U)g_1 \right\|_1 + \left\| \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_2-1} U^i V^j (I-V)g_2 \right\|_1 \right) \rightarrow 0. \quad (3.3.12)$$

Since $(V^j(I-U)g_1)_{j \geq 0}$ and $(U^i(I-V)g_2)_{j \geq 0}$ are sequences of martingale differences in $\mathbb{L}^2(\mathbb{P})$, we shall rather prove (3.3.12) in $\mathbb{L}^2(\mathbb{P})$ and show that

$$\lim_{n \rightarrow \infty} \frac{\|(I-U^n)g_1\|_2}{\sqrt{n}} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\|(I-V^n)g_2\|_2}{\sqrt{n}} = 0. \quad (3.3.13)$$

With this aim, we start by noticing that, for any n fixed, $d_{1,n} := \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} U^i (m + (I-U)g_1)$ is such that $(V^j d_{1,n})_{j \geq 0}$ is an ergodic and stationary sequence of $\mathbb{L}^2(\mathbb{P})$ martingale differences with respect to the filtration $(\mathcal{F}_{\infty,j})_j$. Hence, by the CLT for ergodic and stationary martingales, as $N \rightarrow \infty$, $N^{-1/2} \sum_{j=1}^N V^j d_{1,n} \rightarrow^{\mathcal{D}} G_{1,n}$ where $G_{1,n}$ is a centered random Gaussian with standard deviation $C_n = \|d_{1,n}\|_2$. Since $(N^{-1/2} |\sum_{j=1}^N V^j d_{1,n}|)_{N \geq 1}$ is uniformly integrable, by the convergence of moments theorem (see [1, Th. 3.5]) we have in particular that $\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{j=1}^N V^j d_{1,n} \right\|_1 = \|G_{1,n}\|_1 = \sqrt{\frac{2}{\pi}} C_n$. But $\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{j=0}^{N-1} V^j d_{1,n} \right\|_1 = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{nN}} \|S_{n,N}(f)\|_1$. So, overall, for any n fixed, $\lim_{N \rightarrow \infty} \frac{1}{\sqrt{nN}} \|S_{n,N}(f)\|_1 = \sqrt{\frac{2}{\pi}} C_n$, implying by standard arguments that there exists an increasing subsequence (n_k) tending to infinity such that

$$\lim_{k \rightarrow \infty} \left| \frac{1}{\sqrt{n_k k}} \|S_{n_k, k}(f)\|_1 - \sqrt{\frac{2}{\pi}} C_{n_k} \right| = 0. \quad (3.3.14)$$

Next, note that $C_n^2 = \|m\|_2^2 + \frac{2}{n} \mathbb{E} \left((I-U^n)g_1 \sum_{i=0}^{n-1} U^i m \right) + \frac{\|(I-U^n)g_1\|_2^2}{n}$. But, since $(V^j(I-U^n)g_1)_{j \geq 0}$ is an ergodic and stationary sequence of $\mathbb{L}^2(\mathbb{P})$ martingale differences with respect to the filtration $(\mathcal{F}_{\infty,j})_{j \geq 0}$, we have, by using [1, Th. 3.4] and arguments used to get (3.3.6), $\|(I-U^n)g_1\|_2 \leq \sqrt{\frac{\pi}{2}} \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{i=0}^{n-1} \sum_{j=0}^{N-1} U^i V^j (I-U)g_1 \right\|_1$. Moreover, according to the coboundary decomposition (3.1.6), for any n

fixed,

$$\liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{i=0}^{n-1} \sum_{j=0}^{N-1} U^i V^j (I - U) g_1 \right\|_1 = \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{i=0}^{n-1} \sum_{j=0}^{N-1} U^i V^j (f - m) \right\|_1.$$

In addition, $\left\| \sum_{i=0}^{n-1} \sum_{j=0}^{N-1} U^i V^j (f - m) \right\|_1 \leq \left\| \sum_{i=0}^{n-1} \sum_{j=0}^{N-1} U^i V^j f \right\|_1 + \sqrt{nN} \|m\|_2$. So, overall, taking into account condition (3.1.9), we get

$$\kappa := \limsup_{n \rightarrow \infty} \frac{\|(I - U^n)g_1\|_2}{\sqrt{n}} \leq \sqrt{\frac{\pi}{2}} \left(\|m\|_2 + \lim_{n, N \rightarrow \infty} \frac{1}{\sqrt{nN}} \|S_{n, N}(f)\|_1 \right) < \infty. \quad (3.3.15)$$

Now, for any positive real A , write

$$\begin{aligned} \left| C_n^2 - \|m\|_2^2 - \frac{\|(I - U^n)g_1\|_2^2}{n} \right| &\leq \frac{2A}{\sqrt{n}} \|(I - U^n)g_1\|_1 \\ &+ 2 \frac{\|(I - U^n)g_1\|_2}{\sqrt{n}} \times \left(\frac{1}{n} \mathbb{E} \left(\left| \sum_{i=1}^n U^i m \right|^2 \mathbf{1}_{\{|\sum_{i=1}^n U^i m| > A\sqrt{n}\}} \right) \right)^{1/2}. \end{aligned} \quad (3.3.16)$$

Hence, using that $n^{-1/2} \|(I - U^n)g_1\|_1 \rightarrow_{n \rightarrow \infty} 0$ and taking into account (3.3.15) and the fact that $(n^{-1} (\sum_{i=0}^{n-1} U^i m)^2)_{n \geq 1}$ is uniformly integrable, we derive that the terms in the right-hand side of (3.3.16) tend to zero by letting first n goes to infinity and after A . Therefore $\lim_{n \rightarrow \infty} \left| C_n^2 - \|m\|_2^2 - \frac{\|(I - U^n)g_1\|_2^2}{n} \right| = 0$. Assume now that

$$\kappa = \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|(I - U^n)g_1\|_2 > 0. \quad (3.3.17)$$

Then, there exists an increasing subsequence $(n'_\ell)_{\ell \geq 1}$ tending to infinity such that

$$\lim_{\ell \rightarrow \infty} \frac{1}{\sqrt{n'_\ell}} \|(I - U^{n'_\ell})g_1\|_2 = \kappa \text{ and then } \lim_{\ell \rightarrow \infty} C_{n'_\ell}^2 = \|m\|_2^2 + \kappa^2. \quad (3.3.18)$$

According to (3.3.14) and (3.3.18), we then infer that if (3.3.17) holds true then there exist two increasing subsequences $(n_\ell'')_{\ell \geq 1}$ and $(k_\ell'')_{\ell \geq 1}$ tending to infinity such that

$$\lim_{\ell \rightarrow \infty} \frac{1}{\sqrt{n_\ell'' k_\ell''}} \left\| S_{n_\ell'', k_\ell''}(f) \right\|_1 > \sqrt{\frac{2}{\pi}} \|m\|_2. \quad (3.3.19)$$

But, using once again the coboundary decomposition (3.1.6), note that

$$\frac{1}{n} S_{n,n}(f) = \frac{1}{n} S_{n,n}(m) + \frac{1}{n} (I - U^n) \sum_{j=1}^n V^j g_1 + \frac{1}{n} (I - V^n) \sum_{i=1}^n U^i g_2 + \frac{1}{n} (I - V^n)(I - V^n) g_3.$$

Birkhoff's theorem in $\mathbb{L}^1(\mathbb{P})$ implies that $\lim_{n \rightarrow \infty} \frac{1}{n} \|(I - U^n) \sum_{j=1}^n V^j g_1\|_1 = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \|(I - V^n) \sum_{i=1}^n U^i g_2\|_1 = 0$. Using in addition that g_3 is in $\mathbb{L}^1(\mathbb{P})$, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|S_{n,n}(f)\|_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \|S_{n,n}(m)\|_1. \quad (3.3.20)$$

But since $n^{-1} S_{n,n}(m)$ converges in distribution to a centered Gaussian random variable with variance $\|m\|_2^2$ and $(\frac{1}{n} |S_{n,n}(m)|)_{n \geq 1}$ is uniformly integrable, we derive, by the convergence of moments theorem, that $\lim_{n \rightarrow \infty} \frac{1}{n} \|S_{n,n}(m)\|_1 = \sqrt{\frac{2}{\pi}} \|m\|_2$. This result together with (3.3.20) imply that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \|S_{n,n}(f)\|_1 = \sqrt{\frac{2}{\pi}} \|m\|_2. \quad (3.3.21)$$

Clearly, under condition (3.1.9), if (3.3.17) is supposed to be true, (3.3.19) and (3.3.21) are not compatible. This proves that (3.3.17) cannot be true and then that the first part of (3.3.13) is satisfied. With similar arguments, one can prove that, provided the additional condition (3.1.9) is assumed, the second part of (3.3.13) is also satisfied. This ends the proof of the theorem.

3.4 Extension to multidimensional index of higher dimension

To state the extension of Theorems 3.1.3 and 3.1.5 to higher dimensions, some additional notations are needed. Let $d \geq 1$ and $(T_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ be \mathbb{Z}^d actions on $(\Omega, \mathcal{A}, \mathbb{P})$ generated by commuting invertible and measure-preserving transformations T_{ε_q} , $1 \leq q \leq d$. Here ε_q is the vector of \mathbb{Z}^d which has 1 at the q -th place and 0 elsewhere. By $U_{\underline{i}}$ we denote the operator in \mathbb{L}^p ($1 \leq p \leq \infty$) defined by $U_{\underline{i}}f = f \circ T_{\underline{i}}$, $\underline{i} \in \mathbb{Z}^d$. By $\underline{i} \leq \underline{j}$, we understand $i_k \leq j_k$ for all $1 \leq k \leq d$. Let $\langle d \rangle := \{1, \dots, d\}$. For any subset J of $\langle d \rangle$, let $U_J = \prod_{\ell=1}^{|J|} U_{\varepsilon_\ell}$ where $|J|$ is the cardinal of J , and define $U_J^{\underline{s}} = \prod_{\ell=1}^{|J|} U_{\varepsilon_\ell}^{s_\ell}$ for any $\underline{s} = (s_1, \dots, s_{|J|})$ in J .

We suppose that there is a completely commuting filtration $(\mathcal{F}_{\underline{j}})_{\underline{j} \in \mathbb{Z}^d}$, i.e. there is a σ -algebra $\mathcal{F}_{\underline{0}}$ such that $\mathcal{F}_{\underline{i}} = T_{-\underline{i}}\mathcal{F}_{\underline{0}}$, for $\underline{i} \leq \underline{j}$ we have $\mathcal{F}_{\underline{i}} \subset \mathcal{F}_{\underline{j}}$ and for an integrable f ,

$$\mathbb{E}(\mathbb{E}(f|\mathcal{F}_{i_1, \dots, i_d})|\mathcal{F}_{j_1, \dots, j_d}) = \mathbb{E}(f|\mathcal{F}_{i_1 \wedge j_1, \dots, i_d \wedge j_d}).$$

By $\mathcal{F}_\ell^{(k)}$ we denote the σ -algebra generated by all $\mathcal{F}_{\underline{i}}$ with $\underline{i} = (i_1, \dots, i_d)$ with $i_k \leq \ell$ and $i_j \in \mathbb{Z}$ for $1 \leq j \leq d$, $j \neq \ell$. For σ -algebras $\mathcal{G} \subset \mathcal{F} \subset \mathcal{A}$, by $\mathbb{L}^p(\mathcal{F}) \ominus \mathbb{L}^p(\mathcal{G})$ we denote the space of $f \in \mathbb{L}^p(\mathcal{F})$ for which $\mathbb{E}(f|\mathcal{G}) = 0$ a.s.

For f a $\mathcal{F}_{\underline{0}}$ -measurable centered $\mathbb{L}^1(\mathbb{P})$ random variable, it has been proved in Volný [29, Th. 4] that the condition

$$\text{the series } \sum_{i_1, \dots, i_d \geq 0} \mathbb{E}(U_{i_1, \dots, i_d} f | \mathcal{F}_{\underline{0}}) \text{ converges in } \mathbb{L}^1(\mathbb{P}) \quad (3.4.1)$$

ensures the existence of the following orthomartingale-coboundary decomposition:

$$f = m + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_{\varepsilon_s}) m_J + \prod_{s=1}^d (I - U_{\varepsilon_s}) g \quad (3.4.2)$$

where m , g and m_J belong to $\mathbb{L}^1(\mathcal{F}_{\underline{0}}, \mathbb{P})$, $\mathbb{L}^1(\prod_{s=1}^d T_{\varepsilon_s} \mathcal{F}_{\underline{0}}, \mathbb{P})$ and $\mathbb{L}^1(\prod_{s \in J} T_{\varepsilon_s} \mathcal{F}_{\underline{0}}, \mathbb{P})$ respectively and $(U_{\langle d \rangle}^{\underline{i}} m)_{\underline{i} \in \mathbb{Z}^d}$ and $(U_{J^c}^{\underline{i}} m_J)_{\underline{i} \in \mathbb{Z}^d - |J|}$ are orthomartingale differences

random fields for $\emptyset \subsetneq J \subsetneq \langle d \rangle$. Above $J^c = \langle d \rangle \setminus J$.

For any positive integer k less than d , define $\mathcal{S}_{k,d}$ the set of all the injections from $\{1, \dots, k\}$ to $\{1, \dots, d\}$. We state now the extension of Theorems 3.1.3 and 3.1.5 .

Theorem 3.4.1. *Let $d \geq 1$ and f a \mathcal{F}_0 -measurable centered $\mathbb{L}^1(\mathbb{P})$ random variable. Let $\underline{n}_d = (n_1, \dots, n_d)$ and $S_{\underline{n}_d}(f) = \sum_{i_1=0}^{n_1-1} \dots \sum_{i_d=0}^{n_d-1} U_{i_1, \dots, i_d} f$. Assume that each of the transformations T_{ε_q} , $1 \leq q \leq d$, is ergodic. Suppose also that condition (3.4.1) holds and that for any integer $k \in \{1, \dots, d\}$ and any σ in $\mathcal{S}_{k,d}$,*

$$\liminf_{n_{\sigma(1)} \rightarrow \infty} \dots \liminf_{n_{\sigma(k)} \rightarrow \infty} \frac{\mathbb{E}(|\sum_{i_1=0}^{n_{\sigma(1)}-1} \dots \sum_{i_k=0}^{n_{\sigma(k)}-1} U_{i_1, \dots, i_k} f|)}{(\prod_{i=1}^k n_{\sigma(i)})^{1/2}} < \infty. \quad (3.4.3)$$

Then $m \in \mathbb{L}^2(\mathbb{P})$ and for any set J such that $\emptyset \subsetneq J \subsetneq \langle d \rangle$, $\prod_{s \in J} (I - U_{\varepsilon_s}) m_J \in \mathbb{L}^2(\mathbb{P})$ (m and m_J are defined in (3.4.2)). If in addition,

$$\lim_{\min(n_1, n_2, \dots, n_d) \rightarrow \infty} \frac{\mathbb{E}(|S_{\underline{n}_d}(f)|)}{(\prod_{i=1}^d n_i)^{1/2}} \text{ exists,} \quad (3.4.4)$$

then $(\prod_{i=1}^d n_i)^{-1/2} S_{\underline{n}_d}(f)$ converges in distribution to a centered normal variable (that can be degenerate) as $\min(n_1, n_2, \dots, n_d) \rightarrow \infty$.

Proof of Theorem 3.4.1. The result will follow by recurrence. Note that it holds for $d = 1$ and also for $d = 2$ as shown in the previous section. Assume that it holds for $d - 1$ and let us prove it for d . Recall the decomposition (3.4.2) and let

$$m' = m + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_{\varepsilon_s}) m_J, \quad (3.4.5)$$

where $\langle d \rangle_1 = \langle d \rangle \setminus \{1\} = \{2, \dots, d\}$. Note that $(U_{\varepsilon_1}^i m')_{i \in \mathbb{Z}}$ is a stationary sequence of $\mathbb{L}^1(\mathbb{P})$ martingale differences w.r.t. $(\mathcal{F}_{i,0,\dots,0})_{i \in \mathbb{Z}}$. Since T_{ε_1} is ergodic, by Theorem 3.1.2, if $\liminf_{n \rightarrow \infty} n^{-1/2} \|\sum_{i=0}^{n-1} U_{\varepsilon_1}^i m'\|_1 < \infty$, then $m' \in \mathbb{L}^2(\mathbb{P})$. This follows from (3.4.2) and the fact that, by condition (3.4.3), $\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(|\sum_{i=0}^{n-1} U_{\varepsilon_1}^i f|)}{\sqrt{n}} < \infty$. Next, starting from (3.4.5), and taking into account the induction hypothesis, namely: Theorem 3.4.1 holds for $d - 1$, we infer that if for any integer k in $[2, d]$ and any

injection σ from $\{2, \dots, k\}$ to $\langle d \rangle_1$,

$$\liminf_{n_{\sigma(2)} \rightarrow \infty} \dots \liminf_{n_{\sigma(k)} \rightarrow \infty} \frac{\mathbb{E}(|\sum_{i_2=0}^{n_{\sigma(2)}-1} \dots \sum_{i_k=0}^{n_{\sigma(k)}-1} U_{0,i_2,\dots,i_k} m'|)}{(\prod_{i=2}^k n_{\sigma(i)})^{1/2}} < \infty. \quad (3.4.6)$$

then $m \in \mathbb{L}^2(\mathbb{P})$ and, for any set J such that $\emptyset \subsetneq J \subsetneq \langle d \rangle_1$, $\prod_{s \in J} (I - U_{\varepsilon_s}) m_J \in \mathbb{L}^2(\mathbb{P})$. By using similar arguments as those developed in the proof of Theorem 3.1.3, we infer that (3.4.6) is satisfied under condition (3.4.3). Hence $m \in \mathbb{L}^2(\mathbb{P})$. Then, using in addition that $m' \in \mathbb{L}^2(\mathbb{P})$, we conclude that, for any set J such that $\emptyset \subsetneq J \subseteq \langle d \rangle_1$, $\prod_{s \in J} (I - U_{\varepsilon_s}) m_J \in \mathbb{L}^2(\mathbb{P})$. The first part of Theorem 3.4.1 follows by using $d - 1$ times the same arguments and replacing $\langle d \rangle_1$ by $\langle d \rangle_i$ for $i = 2, \dots, d$. The second part of the theorem follows by applying the CLT for ergodic orthomartingales as proved in Volný [28] for $(\prod_{i=1}^d n_i)^{-1/2} S_{n_d}(m)$ and by using similar arguments as those developed in the proof of Theorem 3.1.5.

Chapter 4

On the weak invariance principle for ortho-martingale in Banach spaces. Application to stationary random fields

In this chapter, we study the weak invariance principle for stationary ortho-martingales with values in 2-smooth or cotype 2 Banach spaces. Then, with the help of a suitable maximal ortho-martingale approximation, we derive the weak invariance principle for stationary random fields in L^p , $1 \leq p \leq 2$, under a condition in the spirit of Hannan. As an application, we get an asymptotic result for the L^p -distances ($1 \leq p \leq 2$) between the common distribution function and the corresponding empirical distribution function of stationary random fields.

4.1 Introduction

Let $(X_i)_{i \in \mathbb{Z}}$ be a stationary random sequence. If one can represent $(X_i)_{i \in \mathbb{Z}}$ in the form

$$X_i = d_i + \zeta_i, \tag{4.1.1}$$

where $(d_i)_{i \in \mathbb{Z}}$ is a stationary sequence of martingale differences, and $(\zeta_i)_{i \in \mathbb{Z}}$ is a coboundary, which means that it can be written as $\zeta_i = \theta_i - \theta_{i-1}$ with $(\theta_i)_{i \in \mathbb{Z}}$ a stationary random sequence, then one may derive the CLT as well as other limit theorems from the corresponding ones for the martingale differences $(d_i)_{i \in \mathbb{Z}}$. This method is usually called the martingale approximation, but is also known as Gordin's method (see [3]). Note that, as a variation of this method, it is not necessary that the sequence $(\zeta_i)_{i \in \mathbb{Z}}$ is a coboundary, but only is negligible, for instance in probability, under a suitable normalisation. This kind of approach has been developed in many papers. We refer to Chapter 4 in Merlevède et al. [15] for a survey concerning Gaussian approximation via martingale methods.

In this paper, we will use an adaptation of the martingale approximation method to prove the CLT and its functional form for the partial sums associated with multi-indexed sequences (also called random fields) with values in some Banach spaces \mathcal{X} . Let us first recall some recent results concerning the limiting behavior of the partial sums associated with random sequences in Banach spaces.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, T be an invertible bi-measurable measure preserving transformation on Ω . Let $\mathcal{F} \subset \mathcal{A}$ be a σ -algebra such that $T^{-1}\mathcal{F} \subset \mathcal{F}$, and define a non-decreasing filtration $\mathcal{F}_i = T^{-i}(\mathcal{F})$. We also define the stationary sequence of random elements $X := (X_i)_{i \in \mathbb{Z}}$ by $X_i = X_0 \circ T^i$. Below, let us consider the Banach space $\mathcal{X} = L^1(\mathbb{R}, \mu)$, where μ is a σ -finite measure on the real line, and $L^\infty(\mathbb{R}, \mu)$ be its dual space. When $(X_i)_{i \in \mathbb{Z}}$ is a L^1 -valued stationary ergodic sequence of martingale differences (m.d.s.), Dede [9] proved the following CLT: If

$$\int_{\mathbb{R}} \|X_0(t)\|_2 \mu(dt) < \infty, \quad (4.1.2)$$

then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n X \circ T^i \Rightarrow_{n \rightarrow \infty} G \text{ in } L^1. \quad (4.1.3)$$

Here and along the paper, \Rightarrow stands for the convergence in distribution and G is a centered Gaussian random variable with covariance operator \mathcal{K}_X (see definition 4.3.1). Then, with the help of a martingale approximation, she extended the CLT to

stationary and ergodic sequences of regular random elements with values in $L^1(\mathbb{R}, \mu)$ provided that (4.1.2) holds and

$$\sum_{i \in \mathbb{Z}} \int_{\mathbb{R}} \|P_0(X_i(t))\|_2 \mu(dt) < \infty, \quad (4.1.4)$$

where $P_0(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_0) - \mathbb{E}(\cdot | \mathcal{F}_{-1})$. Her condition is in the spirit of Hannan [6].

Later, Cuny [35] has proved the CLT and its functional form for stationary and ergodic sequences of m.d.s. with values in more general Banach spaces than L^1 , namely cotype 2 or 2-smooth Banach spaces. Next, again with the help of a martingale approximation, he derived sufficient conditions in the spirit of Maxwell-Woodroffe [14] ensuring the weak invariance principle for a stationary and ergodic sequence with values in either 2-smooth or cotype 2 Banach spaces. Note that when applied to the L^1 -space, his condition and (4.1.4) have a different range of applications.

The first aim of this paper is to generalize the functional CLT for martingale differences sequences with values in some Banach spaces as stated in [35, Prop 3.2] to higher dimension in the sense of multi-indexed sequences. Recall that there is no natural ordering in higher dimension and, in this paper, we shall use the notion of completely commuting filtration (see Section 4.2 for a definition of this notion). It follows that the generalization of [35, Prop 3.2] will mean extending the functional CLT to ortho-martingale sequences with values in a 2-smooth or cotype 2 Banach spaces (see Section 4.3). Then, in Section 4.4, with the help of a suitable ortho-martingale approximation, we derive a functional CLT for L^p -valued stationary random fields, where $p \in [1, 2]$. As in [9], our conditions are in the spirit of Hannan [6]. Hence, our theorem 4.4.1 can be viewed as a generalization of the results stated in [9] in several directions. First, it proves the functional form of [9, Th 2.3]. Secondly, it extends it to multi-indexed random sequences. Finally, it considers the case of variables with values in a broader class of Banach spaces. As an application, in Section 4.5, we give asymptotic results for the L^p -distances, $1 \leq p \leq 2$, between the common distribution function and the corresponding empirical distribution function for stationary random fields. Section 4.6 is devoted to a suitable maximal ortho-martingale approximation in 2-smooth Banach spaces and has interest in itself. All the proofs are postponed

to Section 4.7.

4.2 Notations and definitions

We will use the same notations as in [18] and [44]. We shall consider Banach-valued random fields. Hence, in all the paper, $(\mathcal{X}, |\cdot|_{\mathcal{X}})$ will be a real separable Banach space. We denote by \mathcal{X}^* the topological dual of \mathcal{X} . We shall also denote by $L^0(\mathcal{X})$ the space of functions from $\Omega \rightarrow \mathcal{X}$ that are limits \mathbb{P} -a.s. of simple functions. In addition, for every $p \geq 1$, we define the usual Bochner spaces L^p as follows

$$L^p(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{X}) = L^p(\mathcal{X}) = \{Z \in L^0(\mathcal{X}) : \mathbb{E}(|Z|_{\mathcal{X}}^p) < \infty\}.$$

For every Z in $L^p(\mathcal{X})$, write $\|Z\|_{p, \mathcal{X}} = (\mathbb{E}(|Z|_{\mathcal{X}}^p))^{1/p} = \| |Z|_{\mathcal{X}} \|_p$. To define now random fields, we start by introducing \mathbb{Z}^d -actions. With this aim, we denote elements of \mathbb{Z}^d by $\underline{i} := (i_1, \dots, i_d)$. Let $(T_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ be a \mathbb{Z}^d -action on $(\Omega, \mathcal{A}, \mathbb{P})$ generated by commuting invertible and measure-preserving transformations $\{T_{e_1}, \dots, T_{e_d}\}$, where, for $1 \leq i \leq d$, e_i is a \mathbb{Z}^d vector with 1 at i -th place and 0 elsewhere. We denote those transformations by $\{T_{(1)}, \dots, T_{(d)}\}$. By $U_{\underline{i}}$ we denote the operator in $L^p(\mathcal{X})$ ($1 \leq p < \infty$) defined by $U_{\underline{i}}f = f \circ T_{\underline{i}}$. We assume that the filtration $(\mathcal{F}_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d}$ is completely commuting, i.e. there exists a σ -algebra \mathcal{F} such that $\mathcal{F}_{\underline{i}} = T_{-\underline{i}}\mathcal{F}$, for $\underline{i} \leq \underline{j}$ we have $\mathcal{F}_{\underline{i}} \subset \mathcal{F}_{\underline{j}}$ ($\underline{i} \leq \underline{j}$ means $i_q \leq j_q$ for all $1 \leq q \leq d$) and for an integrable X , we have

$$\mathbb{E}(\mathbb{E}(X | \mathcal{F}_{i_1, \dots, i_d}) | \mathcal{F}_{j_1, \dots, j_d}) = \mathbb{E}(X | \mathcal{F}_{i_1 \wedge j_1, \dots, i_d \wedge j_d}),$$

where $i \wedge j = \min\{i, j\}$. Note that the natural filtration associated with an independent and identically distributed (i.i.d.) random field $(\xi_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ and defined by $\mathcal{F}_{\underline{i}} = \sigma(\xi_{\underline{k}}, \underline{k} \leq \underline{i})$, is completely commuting. Completely commuting filtrations can also be constructed using stationary random fields with independent rows or columns (see [39]).

For a fixed $q \in \{1, \dots, d\}$ and a fixed $\ell \in \mathbb{Z}$, we denote by $\mathcal{F}_{\ell}^{(q)}$ the σ -algebra generated by all $\mathcal{F}_{\underline{i}}$ with \underline{i} such that $i_q \leq \ell$. For every $1 \leq q \leq d$, define also $\mathcal{F}_{-\infty}^{(q)} = \bigcap_{i \in \mathbb{Z}} \mathcal{F}_i^{(q)}$, and $\mathcal{F}_{\infty} = \bigvee_{\underline{i} \in \mathbb{Z}^d} \mathcal{F}_{\underline{i}}$.

For σ -algebras $\mathcal{G} \subset \mathcal{F} \subset \mathcal{A}$ and $1 \leq p < \infty$, we denote by $L^p(\mathcal{F}, \mathcal{X}) \ominus L^p(\mathcal{G}, \mathcal{X})$ the space of $X \in L^p(\mathcal{F}, \mathcal{X})$ for which $\mathbb{E}(X|\mathcal{G}) = 0$. We can define projection operators $P_\ell^{(q)}$ onto $L^p(\mathcal{F}_\ell^{(q)}, \mathcal{X}) \ominus L^p(\mathcal{F}_{\ell-1}^{(q)}, \mathcal{X})$ by $P_\ell^{(q)}(X) = \mathbb{E}(X|\mathcal{F}_\ell^{(q)}) - \mathbb{E}(X|\mathcal{F}_{\ell-1}^{(q)})$. Those operators commute and for $\ell \neq k$, $P_\ell^{(q)}P_k^{(q)} = 0$ (see [31, Lemma 2.4]). We now define the projections $P_{\underline{i}} = P_{i_1}^{(1)} \cdots P_{i_d}^{(d)}$ onto $\bigcap_{1 \leq q \leq d} L^p(\mathcal{F}_{i_q}^{(q)}, \mathcal{X}) \ominus L^p(\mathcal{F}_{i_q-1}^{(q)}, \mathcal{X})$.

Définition 4.2.1. *Let $X \in L^1(\mathcal{X})$. The stationary random fields $(U_{\underline{i}}X)_{\underline{i} \in \mathbb{Z}^d}$ is said to be regular if X is \mathcal{F}_∞ -measurable, and for every $1 \leq q \leq d$, $\mathbb{E}(X|\mathcal{F}_{-\infty}^{(q)}) = 0$ a.s.*

Next we give the definition of ortho-martingales.

Définition 4.2.2. *Let $D \in L^1(\mathcal{X})$. We say that $(U_{\underline{i}}D)_{\underline{i} \in \mathbb{Z}^d}$ is a field of ortho-martingale differences with respect to a completely commuting filtration $(\mathcal{F}_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$, if D is \mathcal{F}_0 -measurable and for all $\underline{i}, \underline{j} \in \mathbb{Z}^d$, such that $j_q < i_q$ for some $q \in \{1, \dots, d\}$, we have*

$$\mathbb{E}\left(D \circ T_{\underline{i}}|_{\mathcal{F}_{\underline{j}}}\right) = 0.$$

In addition, $M_{\underline{n}} := \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} D \circ T_{\underline{i}}$ is said to be an ortho-martingale.

Since our results are stated for random fields in Banach spaces that are 2-smooth or of cotype 2, let us recall their definitions (see [43] for more details). We say that \mathcal{X} is 2-smooth, if there exists $L \geq 1$, such that

$$|x + y|_{\mathcal{X}}^2 + |x - y|_{\mathcal{X}}^2 = 2(|x|_{\mathcal{X}}^2 + L|y|_{\mathcal{X}}^2) \quad \forall x, y \in \mathcal{X}. \quad (4.2.1)$$

For such a L , we say that \mathcal{X} is $(2, L)$ -smooth. We shall also recall the following inequality for m.d.s. in 2-smooth Banach space ([33, Prop 1]): Assume that \mathcal{X} is $(2, L)$ -smooth, then for every m.d.s. $(D_i)_{i \in \mathbb{Z}}$, we have

$$\mathbb{E}\left(|D_1 + \cdots + D_N|_{\mathcal{X}}^2\right) \leq 2L^2 \sum_{i=1}^N \mathbb{E}\left(|D_i|_{\mathcal{X}}^2\right) \quad \text{for all } N \in \mathbb{N}. \quad (4.2.2)$$

For example, for $p \geq 2$ the spaces $L^p(\mathbb{R})$ are $(2, \sqrt{p-1})$ -smooth. As a counterpart, we recall that a separable Banach space \mathcal{X} is said of cotype 2 if there exists $L > 0$ such

that for every independent random variables $D_1, \dots, D_N \in L^2(\mathcal{X})$, with $\mathbb{E}(D_1) = \dots = \mathbb{E}(D_N) = 0$, (4.2.2) holds in the reverse direction. As an example, note that for $1 \leq p \leq 2$, $L^p(\mathbb{R})$ is of cotype 2.

4.3 Functional CLT for ortho-martingales in Banach spaces

Let X be a random element from Ω to \mathcal{X} . Define its associated partial sum $S_{\underline{n}}(X)$ by

$$S_{\underline{n}}(X) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} U_{\underline{i}} X.$$

In addition, for $\underline{t} = (t_1, \dots, t_d) \in [0, 1]^d$ and $[\underline{nt}] = ([n_1 t_1], \dots, [n_d t_d])$, we set

$$S_{\underline{n}, \underline{t}}(X) = \sum_{0 \leq k \leq [\underline{nt}]} \prod_{i=1}^d (k_i \wedge (n_i t_i - 1) - k_i + 1) U_{\underline{k}} X$$

and

$$T_{\underline{nt}}(X) := \frac{1}{\sqrt{n_1 \cdots n_d}} S_{\underline{n}, \underline{t}}(X). \quad (4.3.1)$$

Before stating the main result of this section, as in [35], we need to recall the definitions of Gaussian and pregaussian random elements.

Définition 4.3.1. *Let \mathcal{K} be a bounded symmetric bilinear operator from $\mathcal{X}^* \times \mathcal{X}^*$ to \mathbb{R} . We say that $\mathcal{K} = \mathcal{K}_X$ is the covariance operator associated with X if*

$$\forall x^*, y^* \in \mathcal{X}^* \quad \mathcal{K}(x^*, y^*) = \mathbb{E}[x^*(X)y^*(X)].$$

Définition 4.3.2. *The random variable $W \in L^0(\mathcal{X})$ is said to be Gaussian if, for every $x^* \in \mathcal{X}^*$, $x^*(W)$ has a normal distribution. We say that a random variable $X \in L^0(\mathcal{X})$ such that for every $x^* \in \mathcal{X}^*$, $\mathbb{E}(x^*(X)^2) < \infty$ and $\mathbb{E}(x^*(X)) = 0$, is pregaussian, if there exists a Gaussian variable $W \in L^0(\mathcal{X})$ with the same covari-*

ance operator. We denote by $G(X)$ a Gaussian variable having the same covariance operator as X .

Définition 4.3.3. Let $\mathbb{G}(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{X}) = \mathbb{G}(\mathcal{X})$ be the set of pregaussian random elements that are in $L^2(\mathcal{X})$. For every $X \in \mathbb{G}(\mathcal{X})$, denote $\|X\|_{\mathbb{G}(\mathcal{X})} = \|X\|_{2, \mathcal{X}} + \|G(X)\|_{2, \mathcal{X}}$.

We are now in position to state the functional form of the CLT for sequences of ortho-martingale differences with values in some Banach spaces as well as a L^2 -maximal inequality for the corresponding partial sums. Below and all along the paper, $\underline{n} \rightarrow \infty$ means $\min(n_1, \dots, n_d) \rightarrow \infty$. Next theorem is the extension to the random fields setting of [35, Prop. 3.2].

Théorème 4.3.4. Let \mathcal{X} be a real separable Banach space that is either 2-smooth or of cotype 2. Let $D \in L^2(\mathcal{X})$ be such that $(U_{\underline{i}}D)_{\underline{i} \in \mathbb{Z}^d}$ is a field of ortho-martingale differences w.r.t. a completely commuting filtration $(\mathcal{F}_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$. Assume in addition that $D \in \mathbb{G}(\mathcal{F}_0, \mathcal{X})$ and that at least one of the $T_{(i)}$ for $1 \leq i \leq d$ is ergodic. Then, as $\underline{n} \rightarrow \infty$, $\{T_{\underline{nt}}(D)\}_{\underline{t} \in [0, 1]^d}$ converges in distribution in $C([0, 1]^d, \mathcal{X})$ (equipped with the uniform topology) to a Brownian motion $(W_{\underline{t}})_{\underline{t} \in [0, 1]^d}$ with covariance \mathcal{K}_D associated with D . In addition, there exists $C \geq 0$, such that

$$\left\| \max_{1 \leq k \leq n} |S_{\underline{k}}(D)|_{\mathcal{X}} \right\|_2 \leq C(n_1 \cdots n_d)^{1/2} \|D\|_{\mathbb{G}(\mathcal{X})}. \quad (4.3.2)$$

4.4 Application to stationary random fields in L^p ($1 \leq p \leq 2$)

Let $1 \leq p \leq 2$. In this section we consider a σ -finite measure space (S, \mathcal{S}, μ) such that $\mathcal{X} := L^p(S, \mathcal{S}, \mu)$ is separable. Recall that, since $1 \leq p \leq 2$, $L^p(S, \mu)$ is a Banach space of cotype 2. Let $L^q(S, \mu)$, for $q = p/(p-1)$, be its dual space.

In all this section, X is a random variable in $L^2(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{X})$ and $(\mathcal{F}_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ with $\mathcal{F}_{\underline{i}} = T_{-\underline{i}}\mathcal{F}$ is a completely commuting filtration. Let also $X_{\underline{i}} = U_{\underline{i}}X$.

Theorem 4.3.4 combined with a suitable ortho-martingale approximation (see Proposition 4.6.1 stated in Section 4.6) leads to the following result.

Théorème 4.4.1. *Assume that X is regular and that*

$$\int_S \|X(s)\|_2^p \mu(ds) < \infty. \quad (4.4.1)$$

Assume also that

$$\int_S \left(\sum_{\underline{i} \in \mathbb{Z}^d} \|P_{\underline{0}}(X_{\underline{i}}(s))\|_2 \right)^p \mu(ds) < \infty. \quad (4.4.2)$$

Then there exists a stationary random field of ortho-martingale differences $(U_{\underline{i}}D)_{\underline{i} \in \mathbb{Z}^d}$ w.r.t $(\mathcal{F}_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ such that $D \in \mathbb{G}(\mathcal{F}_{\underline{0}}, L^p(S, \mu))$ and

$$\lim_{\underline{n} \rightarrow \infty} \int_S \left\| \max_{1 \leq k \leq \underline{n}} \left| \frac{S_k(X(s))}{\sqrt{n_1 \cdots n_d}} - \frac{S_k(D(s))}{\sqrt{n_1 \cdots n_d}} \right| \right\|_2^p \mu(ds) = 0. \quad (4.4.3)$$

Suppose, in addition, that at least one of the transformations $T_{(i)}$ for $1 \leq i \leq d$ is ergodic, then

$$\{T_{\underline{nt}}(X), \underline{t} \in [0, 1]^d\} \Longrightarrow_{\underline{n} \rightarrow \infty} W \text{ in } C([0, 1]^d, \mathcal{X}), \quad (4.4.4)$$

where W is a \mathcal{X} -valued centered Brownian motion with covariance operator \mathcal{K}_D that can be defined as follows: for all f and g in $L^q(S, \mu)$ (where q is the conjugate of p),

$$\begin{aligned} \mathcal{K}_D(f, g) &= \mathbb{E} \left(\int_S \int_S f(t)g(s)D(t)D(s)\mu(dt)\mu(ds) \right) \\ &= \sum_{\underline{k} \in \mathbb{Z}^d} \int_S \int_S f(t)g(s) \text{Cov}(f(X_{\underline{0}}(s)), g(X_{\underline{k}}(t)))\mu(dt)\mu(ds). \end{aligned} \quad (4.4.5)$$

where $D(s) = \sum_{\underline{i} \in \mathbb{Z}^d} P_{\underline{0}}(X_{\underline{i}}(s))$.

Remark 4.4.2. Note that by [35, Lemma 2.4], condition (4.4.1) implies that $X \in \mathbb{G}(L^1(S, \mu))$. Note also that when X is regular and in $L^1(\Omega, \mathcal{A}, \mathbb{P}, L^1(S, \mu))$ then the (reverse-)martingale convergence theorem in the Banach-valued case (see e.g. [43]) ensures that (4.4.2) implies (4.4.1).

Next result gives sufficient conditions for (4.4.2) to hold.

Corollary 4.4.3. *Let $p \in [1, 2]$. Let $X := (X_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d}$ be a regular centered random variable in $L^0(\Omega, \mathcal{A}, \mathbb{P}, L^p(S, \mu))$ such that $\int_S \|X_{\underline{0}}(s)\|_2^p \mu(ds) < \infty$. If the following conditions hold:*

$$\int_S \left(\sum_{i_1=1}^{\infty} \cdots \sum_{i_d=1}^{\infty} \frac{1}{\sqrt{i_1 \cdots i_d}} \|\mathbb{E}(X_{\underline{i}}(s) | \mathcal{F}_{\underline{0}})\|_2 \mu(ds) \right)^p < \infty \quad (4.4.6)$$

and

$$\int_S \left(\sum_{i_1=1}^{\infty} \cdots \sum_{i_d=1}^{\infty} \frac{1}{\sqrt{i_1 \cdots i_d}} \|X_{-\underline{i}}(s) - \mathbb{E}(X_{-\underline{i}}(s) | \mathcal{F}_{\underline{0}})\|_2 \right)^p \mu(ds) < \infty, \quad (4.4.7)$$

then (4.4.2) is satisfied.

Remark 4.4.4. *Condition (4.4.6) together with (4.4.7) imply that X is regular.*

Comment 4.4.5. *Using Proposition 4.6.1, a similar result as Theorem 4.4.1 can be obtained when $X \in L^0(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{X})$ where $\mathcal{X} = L^p(S, \mu)$ with $p > 2$. More precisely if $\|X_{\underline{0}}\|_{2, \mathcal{X}} < \infty$ and*

$$\sum_{\underline{i} \in \mathbb{Z}^d} \left\| \left(\int_S |P_{\underline{0}}(X_{\underline{i}}(s))|^p \mu(ds) \right)^{1/p} \right\|_2 < \infty, \quad (4.4.8)$$

then

$$\lim_{n \rightarrow \infty} \prod_{i=1}^d n_i^{-1/2} \left\| \sup_{1 \leq \underline{k} \leq \underline{n}} |S_{\underline{k}} - M_{\underline{k}}|_{\mathcal{X}} \right\|_2 = 0,$$

and the convergence (4.4.4) holds.

Note that if $X_{\underline{0}}(s) = \sum_{\underline{i} \in \mathbb{Z}^d} a_{\underline{i}} Y_{-\underline{i}}(s)$ where $(Y_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ is an ortho-martingale differences, (4.4.8) holds as soon as

$$\|Y_{\underline{0}}\|_{2, \mathcal{X}} < \infty \text{ and } \sum_{\underline{i} \in \mathbb{Z}^d} |a_{\underline{i}}| < \infty. \quad (4.4.9)$$

When $d = 1$, our condition (4.4.8) and condition (3) in [35] have a different range of applications. For instance (4.4.9) is not enough for [35, Condition (3)] to hold.

4.5 Application to empirical processes

Let Y be a real random variable which is \mathcal{F}_0 -measurable. For every $\underline{i} \in \mathbb{Z}^d$, we define $Y_{\underline{i}} := U_{\underline{i}}Y$ and for every $s \in \mathbb{R}$, $X_{\underline{i}}(s) := \mathbb{1}_{Y_{\underline{i}} \leq s} - F(s)$, where $F(s) = \mathbb{P}(Y \leq s)$. Let $S_{\underline{n}}(X)(s) := \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} X_{\underline{i}}(s)$. We also denote by $F_{\underline{n}}$ the empirical distribution function of $(Y_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$:

$$\forall s \in \mathbb{R} \quad F_{\underline{n}}(s) := \frac{1}{n_1 \cdots n_d} \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} \mathbb{1}_{Y_{\underline{i}} \leq s}.$$

For $1 \leq p \leq 2$, we are interested in deriving the asymptotic behavior of

$$\int_{\mathbb{R}} \left| \frac{1}{n_1 \cdots n_d} S_{\underline{n}}(X)(s) \right|^p ds = \int_{\mathbb{R}} |F_{\underline{n}}(s) - F(s)|^p ds. \quad (4.5.1)$$

When $p = 1$, (4.5.1) is the L^1 -Wasserstein distance between F the distribution function of Y , and $F_{\underline{n}}$ the corresponding empirical distribution function.

The following result is a direct application of Theorem 4.4.1 (with $S := \mathbb{R}$ and μ the Lebesgue measure) combined with Corollary 4.4.3.

Corollary 4.5.1. *Let p in $[1, 2]$. Assume that Y is regular,*

$$\int_{\mathbb{R}} (F(s)(1 - F(s)))^{p/2} ds < \infty \quad (4.5.2)$$

and

$$\int_{\mathbb{R}} \left(\sum_{i_1=1}^{\infty} \cdots \sum_{i_d=1}^{\infty} \frac{1}{\sqrt{i_1 \cdots i_d}} \|\mathbb{P}(Y_{\underline{i}} \leq s | \mathcal{F}_0) - F(s)\|_2 \right)^p ds < \infty. \quad (4.5.3)$$

Suppose, in addition, that at least one of the transformations $T_{(i)}$ for $1 \leq i \leq d$ is ergodic. Then the sequence of processes $\{\sqrt{n_1 \cdots n_d} (F_{\underline{n}}(s) - F(s)), s \in \mathbb{R}\}$ converges in L^p to a centered Gaussian random variable G , with covariance operator \mathcal{K}_μ defined

by: for every $f, g \in L^q(\mathbb{R}, \lambda)$ (where q is the conjugate of p):

$$\mathcal{K}_\mu(f, g) = \sum_{\underline{i} \in \mathbb{Z}^d} \mathbb{E} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} f(s)g(t) (\mathbf{1}_{Y_{\underline{0}} \leq s} - F(s)) (\mathbf{1}_{Y_{\underline{i}} \leq t} - F(t)) ds dt \right).$$

In particular, as $\underline{n} \rightarrow \infty$,

$$(n_1 \cdots n_d)^{p/2} \int_{\mathbb{R}} |F_{\underline{n}}(s) - F(s)|^p ds \Rightarrow \int_{\mathbb{R}} |G(s)|^p ds.$$

Note that by Remark 4.4.2, condition (4.5.2) implies that $X \in \mathbb{G}(\mathcal{F}_{\underline{0}}, L^p(\mathbb{R}, \lambda))$. Next we give a sufficient condition for (4.5.3) to hold, in terms of dependence coefficients. With this aim, in the spirit of [38], define

$$\phi_Y(\underline{i}) = \sup_{s \in \mathbb{R}} \|\mathbb{P}(Y_{\underline{i}} \leq s | \mathcal{F}_{\underline{0}}) - \mathbb{P}(Y_{\underline{i}} \leq s)\|_\infty. \quad (4.5.4)$$

Since

$$\|\mathbb{P}(Y_{\underline{i}} \leq s | \mathcal{F}_{\underline{0}}) - F(s)\|_2^2 \leq 2F(t)(1 - F(t))\phi_Y(\underline{i}),$$

we get the following proposition:

Proposition 4.5.2. *Condition (4.5.3) holds as soon as (4.5.2) holds and*

$$\sum_{i_1=1}^{\infty} \cdots \sum_{i_d=1}^{\infty} \frac{1}{\sqrt{i_1 \cdots i_d}} \sqrt{\phi_Y(\underline{i})} < \infty. \quad (4.5.5)$$

Remark 4.5.3. *Note that the filtration defining the coefficients $\phi_Y(\underline{i})$ is commuting, so Proposition 4.5.2 is not comparable to other results for mixing random fields such as those obtained in Bolthausen [34] or Dedecker [10] where the underlying filtrations are not the same.*

Comment 4.5.4. *Note that by Item 3 in [38, Lemma 2], we get that if the distribution function F of $Y_{\underline{0}}$ is continuous then, for any $\underline{i} \geq \underline{0}$,*

$$\phi_Y(\underline{i}) \leq K \|F(Y_{\underline{i}}) - F(Y_{\underline{i}}^*)\|_\infty, \quad (4.5.6)$$

where $Y_{\underline{i}}^*$ is a random element distributed as $Y_{\underline{i}}$ and independent of \mathcal{F}_0 .

To give an example, let us consider $d = 2$. Let $(\varepsilon_{i,j})_{(i,j) \in \mathbb{Z}^2}$ be a iid random field such that $\mathbb{P}(\varepsilon_{0,0} = 0) = \mathbb{P}(\varepsilon_{0,0} = 1) = 1/2$, and define $\mathcal{F}_{i,j} = \sigma(\varepsilon_{u,v}, u \leq i, v \leq j)$. Let $f(0,0) = 0$ and $f(i,j) = (i+j)(i+j+1)/2 + i$ for any $(i,j) \in \mathbb{N}^2 \setminus \{0,0\}$. Then f is a one-to-one onto map from \mathbb{N}^2 to \mathbb{N} . Let now $U_{k,\ell} = \sum_{i \geq 0, j \geq 0} 2^{-f(i,j)-1} \varepsilon_{k-i, \ell-j}$. Clearly each of the $U_{k,\ell}$'s is distributed as $\sum_{i \geq 0, j \geq 0} 2^{-f(i,j)-1} \varepsilon_{-f(i,j), 0}$ and then as $\sum_{u \geq 0} 2^{-u-1} \varepsilon_{-u, 0}$ which is uniformly distributed over $[0, 1]$. Next, setting $E_{k,\ell} = \{i \geq k, j \geq \ell\}$ and $S_{k,\ell} = \mathbb{N}^2 \setminus E_{k,\ell}$, and considering $(\varepsilon_{i,j}^*)_{(i,j) \in \mathbb{Z}^2}$ an independent copy of $(\varepsilon_{i,j})_{(i,j) \in \mathbb{Z}^2}$, we define

$$U_{k,\ell}^* = \sum_{(i,j) \in S_{k,\ell}} 2^{-f(i,j)-1} \varepsilon_{k-i, \ell-j} + \sum_{(i,j) \in E_{k,\ell}} 2^{-f(i,j)-1} \varepsilon_{k-i, \ell-j}^*.$$

Clearly $U_{k,\ell}^*$ is distributed as $U_{k,\ell}$ and is independent of $\mathcal{F}_{0,0}$. Moreover

$$\|U_{k,\ell} - U_{k,\ell}^*\|_\infty \leq \sum_{(i,j) \in E_{k,\ell}} 2^{-f(i,j)-1} \|\varepsilon_{k-i, \ell-j} - \varepsilon_{k-i, \ell-j}^*\|_\infty \leq c 2^{-k} 2^{-\ell},$$

implying by (4.5.6) that $\phi_U((k, l)) \leq C 2^{-k} 2^{-\ell}$. Let now $Y_{k,\ell} = h(U_{k,\ell})$ with $h(x) = x^{-a}$ for $a > 0$. Clearly $\phi_Y((k, l)) = \phi_U((k, l))$. Then, according to Proposition 4.5.2, Condition (4.5.3) holds provided (4.5.2) is satisfied which requires in our case that $a < p/2$.

4.6 Ortho-martingale approximation in Banach spaces

The following result allows to derive a useful ortho-martingale approximation in 2-smooth Banach spaces. It is an extension to multidimensional index sets and to 2-smooth Banach spaces of the estimate (3.3) in [37]. It can also be viewed as an extension in several directions of [45, Th. 1 (ii)] and gives a more precise estimate than [37, Prop. 3.1]. Below $(\mathcal{F}_{\underline{i}})_{\underline{i} \in \mathbb{Z}^d}$ defined by $\mathcal{F}_{\underline{i}} = U_{\underline{i}} \mathcal{F}$ is assumed to be a completely commuting filtration, and we denote by $X_{\underline{i}} = U_{\underline{i}} X$.

Proposition 4.6.1. *Let \mathcal{X} be a $(2, L)$ -smooth Banach space. Let $p \geq 2$ and $X_{\underline{0}}$ be a regular r.v. with values in \mathcal{X} such that $\|X_{\underline{0}}|_{\mathcal{X}}\|_p < \infty$. Assume that*

$$\sum_{\underline{i} \in \mathbb{Z}^d} \|P_{\underline{0}}(X_{\underline{i}})|_{\mathcal{X}}\|_p < \infty. \quad (4.6.1)$$

Then setting $D_{\underline{k}} = \sum_{\underline{i} \in \mathbb{Z}^d} P_{\underline{k}}(X_{\underline{i}})$, $(D_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d}$ are stationary $\mathbb{L}^p(\mathcal{X})$ ortho-martingale differences with respect to $(\mathcal{F}_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d}$ and the corresponding ortho-martingale $M_{\underline{n}} = \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} D_{\underline{k}}$ satisfies the following maximal inequality: there exists a positive constant C only depending on (L, p, d) such that for any positive integers n_1, \dots, n_d, m and for any $\lambda \geq 1$,

$$\begin{aligned} \prod_{i=1}^d n_i^{-1/2} \left\| \sup_{1 \leq k \leq n} |S_{\underline{k}} - M_{\underline{k}}|_{\mathcal{X}} \right\|_p &\leq C \sum_{i=1}^d \sum_{|k_i| > m} \sum_{\underline{k} \setminus \{k_i\} \in \mathbb{Z}^{d-1}} \|P_{\underline{k}}(X_{\underline{0}})|_{\mathcal{X}}\|_p \\ &+ Cm^{2d-1} \left(\frac{\lambda^2}{\min_{1 \leq i \leq d} n_i^{1/2}} + \frac{\lambda^{-1}}{\min_{1 \leq i \leq d} n_i^{(p-2)/(2p)}} \right) (\|X_{\underline{0}}|_{\mathcal{X}}\|_p + \|D_{\underline{0}}|_{\mathcal{X}}\|_p) \\ &+ Cm^{2d-1} \frac{1}{\min_{1 \leq i \leq d} n_i^{(p-2)/(2p)}} \left(\|X_{\underline{0}}|_{\mathcal{X}} \mathbf{1}_{\{|X_{\underline{0}}|_{\mathcal{X}}| > \lambda \|X_{\underline{0}}|_{\mathcal{X}}\|_p}\|_p + \|D_{\underline{0}}|_{\mathcal{X}} \mathbf{1}_{\{|D_{\underline{0}}|_{\mathcal{X}}| > \lambda \|D_{\underline{0}}|_{\mathcal{X}}\|_p}\|_p \right). \end{aligned} \quad (4.6.2)$$

The proof of this result is postponed to Section 4.7.4.

Remark 4.6.2. *Similarly to Remark 4.4.2, note that when X is regular and in $L^1(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{X})$ then (4.6.1) implies that $\|X_{\underline{0}}|_{\mathcal{X}}\|_p < \infty$.*

Remark 4.6.3. *A careful analysis of the proof of Proposition 4.6.1 reveals that when \mathcal{X} is of cotype 2, then a similar maximal inequality to that stated in (4.6.2) can be derived but only for $p = 2$. Indeed it suffices to replace in the proof the martingale inequality (4.2.2) by the inequality (4.3.2) valid for martingales with values in Banach spaces of cotype 2. Hence, when $X_{\underline{0}}$ is a regular r.v. with values in $\mathbb{G}(\mathcal{X})$ where \mathcal{X} is of cotype 2 and if $\sum_{\underline{i} \in \mathbb{Z}^d} \|P_{\underline{0}}(X_{\underline{i}})\|_{G(\mathcal{X})} < \infty$ then, when $p = 2$, Inequality (4.6.2) remains valid provided we replace in the right-hand side the norm $\|\cdot\|_p$ by the pre-Gaussian norm $\|\cdot\|_{G(\mathcal{X})}$.*

4.7 Proofs

4.7.1 Proof of Theorem 4.3.4

The idea of this proof is essentially the same as the proof of [35, Prop 3.2], except for the convergence of finite dimensional laws.

We first prove (4.3.2). By the Cairoli's strong (p, p) inequality for ortho-submartingales [26, Th. 2.3.1], we have

$$\left\| \max_{1 \leq k \leq \underline{n}} |S_{\underline{k}}(D)|_{\mathcal{X}} \right\|_2^2 \leq 2^{2d} \left\| |S_{\underline{n}}(D)|_{\mathcal{X}} \right\|_2^2.$$

Assume first that \mathcal{X} is $(2, L)$ -smooth, then applying (4.2.2) to each index, we derive

$$\left\| |S_{\underline{n}}(D)|_{\mathcal{X}} \right\|_2^2 \leq 2^d L^{2d} \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} \mathbb{E} (|D_{i_d}|_{\mathcal{X}}^2).$$

Recall that any 2-smooth Banach space is of type 2. Therefore (4.3.2) holds by stationarity and the fact that, in Banach spaces of type 2, the norms $\|\cdot\|_{\mathbb{G}(\mathcal{X})}$ and $\|\cdot\|_{2, \mathcal{X}}$ are equivalent (see [41, Prop 9.24]).

Suppose now that \mathcal{X} is of cotype 2. Since D is assumed to be pregaussian, so is $S_{\underline{n}}(D)$. In addition, by the orthogonality of ortho-martingale increments, we get $G(S_{\underline{n}}(D)/\sqrt{n_1 \cdots n_d}) = G(D)$. Hence by [41, Prop 9.25], we deduce that

$$\|S_{\underline{n}}(D)\|_{2, \mathcal{X}} \leq C \|G(S_{\underline{n}}(D))\|_{2, \mathcal{X}} = C \sqrt{n_1 \cdots n_d} \|G(D)\|_{2, \mathcal{X}} \leq C \sqrt{n_1 \cdots n_d} \|D\|_{\mathbb{G}(\mathcal{X})}.$$

This ends the proof of (4.3.2).

Now we prove the functional central limit theorem by first proving the tightness and then the convergence of finite dimensional laws. For the tightness, the idea in [35, Proof of Prop 3.2] also applies in higher dimension. For the reader's convenience, let us give the details.

Since \mathcal{X} is separable, then $\sigma(D)$ is countably generated. Thus there exists an

increasing filtration $(\mathcal{G}_m)_{m \in \mathbb{N}^*}$, such that \mathcal{G}_m is finite for every $m \geq 1$ and $\sigma(D) = \bigcup_{m \geq 1} \mathcal{G}_m$. We set $D_m := \mathbb{E}(D | \mathcal{G}_m)$. Since \mathcal{G}_m is finite, there exist $A_1^{(m)}, \dots, A_{k_m}^{(m)} \in \mathcal{G}_m$ and $x_1^{(m)}, \dots, x_{k_m}^{(m)} \in \mathcal{X}$ such that $D_m = \sum_{k=1}^{k_m} x_k^{(m)} \mathbf{1}_{A_k^{(m)}}$. Using [35, Lemma 2.3], we have

$$\|D_m - D\|_{\mathbb{G}(\mathcal{X})} \longrightarrow 0 \text{ as } m \rightarrow \infty.$$

Now, setting

$$\tilde{D}_m := P_{\underline{0}}(D_m).$$

One can see that $(U_{\underline{i}} \tilde{D}_m)_{\underline{i} \in \mathbb{Z}^d}$ is a sequence of ortho-martingale differences with respect to the completely commuting filtration $(\mathcal{F}_{\underline{i}})_{\underline{i} \in \mathbb{Z}}$. Then using [35, Lemma 2.2] and the fact that $\mathbb{E}(D | \mathcal{F}_{-e_1}) = \dots = \mathbb{E}(D | \mathcal{F}_{-e_d}) = 0$, we deduce that $(U_{\underline{i}} \tilde{D}_m)_{m \geq 1}$ converges in $\mathbb{G}(\mathcal{X})$ to D .

By [18, Theorem 1], for every $m \geq 1$, $\left\{ (n_1 \dots n_d)^{-1/2} S_{\underline{n}, \underline{t}}(\tilde{D}_m) \right\}_{\underline{t} \in [0,1]^d}$ is tight in $C([0,1]^d, \mathcal{X})$. Indeed \tilde{D}_m takes only a finite number of values and therefore we can work on the finite dimensional vector space generated by these values. Note that Theorem 1 in [18] is stated for reversed ortho-martingale differences, but it is also obviously true in case of ortho-martingales in the usual sense. Now, by (4.3.2), we deduce that

$$\frac{1}{\sqrt{n_1 \dots n_d}} \left\| \max_{1 \leq \underline{k} \leq \underline{n}} \left| \left(S_{\underline{k}}(\tilde{D}_m) - S_{\underline{k}}(D) \right) \right|_{\mathcal{X}} \right\|_2 \leq C \| \tilde{D}_m - D \|_{\mathbb{G}(\mathcal{X})} \longrightarrow 0 \text{ as } m \rightarrow \infty.$$

Hence the tightness of $\left\{ (n_1 \dots n_d)^{-1/2} S_{\underline{n}, \underline{t}}(D) \right\}_{\underline{t} \in [0,1]^d}$ follows (see [35, Lemma B.1]).

We turn now to the proof of the convergence of the finite dimensional laws. Recall the notation (4.3.1) for $T_{\underline{nt}}(D)$. When no confusion is possible, we will denote $T_{\underline{nt}}(D)$ by $T_{\underline{nt}}$. Let $\underline{t}^{(k)} = (t_1^{(k)}, \dots, t_d^{(k)})$. Our aim is to prove that for any $m \geq 1$ and any $(0, 0, \dots, 0) < \underline{t}^{(1)} < \dots < \underline{t}^{(m)} \leq (1, 1, \dots, 1)$,

$$(T_{\underline{nt}^{(1)}}, \dots, T_{\underline{nt}^{(m)}}) \Rightarrow (W_{\underline{t}^{(1)}}, \dots, W_{\underline{t}^{(m)}}), \quad (4.7.1)$$

where $(W_{\underline{t}})_{\underline{t} \in [0,1]^d}$ is a Brownian sheet with covariance operator \mathcal{K}_D . For the reader's convenience, we shall give the complete proof in case $d = 2$, noticing that the general

case can be proved by induction. Since either $T_{(1)}$ or $T_{(2)}$ is ergodic, let us assume from now that $T_{(2)}$ is (recall that $T_{(2)}$ is equal to $T_{(0,1)}$). Using the Cramer-Wold device, it is sufficient to prove that for any $m \geq 1$, any $(0, 0) < \underline{t}^{(1)} < \dots < \underline{t}^{(m)} \leq (1, 1)$ and any $x_1^*, \dots, x_m^* \in \mathcal{X}^*$,

$$\sum_{i=1}^m x_i^* (T_{\underline{nt}^{(i)}}) \Rightarrow \sum_{i=1}^m x_i^* (W_{\underline{t}^{(i)}}). \quad (4.7.2)$$

With this aim we shall use similar arguments as those developed in [18, Section 3.2] and, in a sake of clarity, we shall give most of the details. Notice first that $\sum_{i=1}^m x_i^* (T_{\underline{nt}^{(i)}})$ can be written as a weighted sum over disjoint and adjacent rectangles. Hence proving (4.7.2) is equivalent to show that for any positive integer m , any $0 = t_0 < t_1 < \dots < t_m \leq 1$, $0 = s_0 < s_1 < \dots < s_m \leq 1$ and $(x_{i,j}^*)_{1 \leq i, j \leq m} \in \mathcal{X}^*$

$$\begin{aligned} V_{n_1 n_2} &:= \frac{1}{\sqrt{n_1 n_2}} \sum_{i=1}^m \sum_{j=1}^m x_{i,j}^* \left(\sum_{k=[n_1 t_{i-1}]+1}^{[n_1 t_i]} \sum_{\ell=[n_2 s_{j-1}]+1}^{[n_2 s_j]} D \circ T_{k,\ell} \right) \\ &\Rightarrow \sum_{i=1}^m \sum_{j=1}^m x_{i,j}^* (W_{t_i, s_j} + W_{t_{i-1}, s_{j-1}} - W_{t_i, s_{j-1}} - W_{t_{i-1}, s_j}), \end{aligned} \quad (4.7.3)$$

as $n_1 \wedge n_2 \rightarrow \infty$. Notice that the random variable on the right-hand side is distributed according to $\mathcal{N}(0, \sigma^2)$ with

$$\sigma^2 = \sum_{i=1}^m \sum_{j=1}^m (t_i - t_{i-1}) (s_j - s_{j-1}) \mathbb{E} (x_{i,j}^* (D)^2).$$

Clearly, it suffices to prove the convergence (4.7.3) when $n_1, n_2 \rightarrow \infty$ along any sequence $(n_r, N_r)_{r \geq 1}$. Hence, let us fix a sequence $(n_r, N_r)_{r \geq 1}$ such that $n_r, N_r \rightarrow \infty$ as $r \rightarrow \infty$. It remains to prove that

$$V_r = \frac{1}{\sqrt{n_r N_r}} \sum_{i=1}^m \sum_{j=1}^m x_{i,j}^* \left(\sum_{k=[n_r t_{i-1}]+1}^{[n_r t_i]} \sum_{\ell=[N_r s_{j-1}]+1}^{[N_r s_j]} D \circ T_{k,\ell} \right) \Rightarrow \mathcal{N}(0, \sigma^2). \quad (4.7.4)$$

As in [18], the proof of (4.7.4) is based on the usual central limit theorem for trian-

gular arrays of martingale differences due McLeish (see [40, Theorem 3.2] for an easy reference). The rest of the proof then follows the lines of [18, Section 3.2] by noticing that $\sup_{1 \leq i, j \leq m} \|x_{i,j}^*(D)\|_2 < \infty$ and the following modification of [18, Lemma 4].

Lemma 4.7.1. *Let $\Delta_i = \sum_{j=1}^m (s_j - s_{j-1}) \mathbb{E} (x_{i,j}^*(D)^2)$ and $\varepsilon > 0$. If $T_{0,1}$ is ergodic, there exist integers $v > 0$ (large enough) and $p(v)$ (large enough), such that for every $n \geq p(v)$*

$$\left\| \frac{1}{v} \sum_{k=1}^v \left(\sum_{j=1}^m \frac{1}{\sqrt{n}} \sum_{\ell=[ns_{j-1}]+1}^{[ns_j]} x_{i,j}^*(D) \circ T_{k,\ell} \right)^2 - \Delta_i \right\|_1 < \varepsilon. \quad (4.7.5)$$

Proof of Lemma 4.7.1. Note first that for any fixed $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have that $(x_{i,j}^*(D) \circ T_{k,\ell})_{\ell \in \mathbb{Z}}$ is a sequence of martingale differences with respect to the filtration $(\mathcal{F}_\ell^{(2)})_{\ell \in \mathbb{Z}}$. Let $d_{k,\ell}(i, j) = x_{i,j}^*(D) \circ T_{k,\ell}$ and define

$$D_\ell(i, j) = (d_{1,\ell}(i, j), \dots, d_{v,\ell}(i, j))^t.$$

Note that $(D_\ell(i, j))_{\ell \geq 1}$ is a stationary and ergodic (since $T_{(2)}$ is ergodic) sequence of \mathbb{R}^v -valued martingale differences. Therefore from the functional form of the central limit theorem for \mathbb{R}^v -valued stationary and ergodic \mathbb{L}^2 martingale differences, we get that for any positive integer m , and any $0 \leq s_1 < \dots < s_m \leq 1$

$$\left(\frac{1}{\sqrt{n}} \sum_{\ell=1}^{[ns_1]} D_\ell(i, j), \dots, \frac{1}{\sqrt{n}} \sum_{\ell=[ns_{m-1}]+1}^{[ns_m]} D_\ell(i, j) \right)$$

converges in distribution to (G_1, \dots, G_m) , where $(G_u)_{1 \leq u \leq m}$ are independent and centered Gaussian random variables with respective covariance matrix

$$\mathbb{E} (G_\ell G_\ell^t) = (s_\ell - s_{\ell-1}) \mathbb{E} (D_1(i, j) D_1(i, j)^t) = (s_\ell - s_{\ell-1}) \|x_{i,j}^*(d)\|_2^2 Id,$$

and for any i, j, k , $(n^{-1} \max_{u \leq n} (\sum_{\ell=1}^u d_{k,\ell}(i, j))^2)_{n \geq 1}$ is uniformly integrable. In

particular, we have

$$\left(\frac{1}{\sqrt{n}} \sum_{j=1}^m \sum_{\ell=[ns_{j-1}]+1}^{[ns_j]} x_{i,j}^*(D) \circ T_{k,\ell} \right)_{1 \leq k \leq v} \quad (4.7.6)$$

converges in distribution to $(N_k)_{1 \leq k \leq v}$ where N_k are i.i.d random variables with common distribution $\mathcal{N}\left(0, \sum_{j=1}^m (s_j - s_{j-1}) \mathbb{E}(x_{i,j}^*(d)^2)\right)$. Now using the notations

$$F_{i,k,n} = \sum_{j=1}^m \frac{1}{\sqrt{n}} \sum_{\ell=[ns_{j-1}]+1}^{[ns_j]} x_{i,j}^*(d) \circ T_{k,\ell} \text{ and } V_{i,k,n} = F_{i,k,n}^2 - \Delta_i,$$

note that $\Delta_i = \mathbb{E}(F_{i,k,n}^2) = \mathbb{E}(N_k^2)$. To soothe the notation, we will drop the index i in the rest of the proof. Let $\varepsilon > 0$, $M > 0$ and define

$$A = \mathbb{E} \left| \frac{1}{v} \sum_{k=1}^v V_{k,n} \mathbf{1} \left(\left| \frac{1}{v} \sum_{k=1}^v V_{k,n} \right| \leq \varepsilon \right) \right|, \quad B = \mathbb{E} \left| \frac{1}{v} \sum_{k=1}^v V_{k,n} \mathbf{1} \left(\varepsilon < \left| \frac{1}{v} \sum_{k=1}^v V_{k,n} \right| \leq M \right) \right|$$

and

$$C = \mathbb{E} \left| \frac{1}{v} \sum_{k=1}^v V_{k,n} \mathbf{1} \left(\left| \frac{1}{v} \sum_{k=1}^v V_{k,n} \right| > M \right) \right|.$$

We have

$$\mathbb{E} \left| \frac{1}{v} \sum_{k=1}^v V_{k,n} \right| \leq A + B + C.$$

Clearly $A \leq \varepsilon$. Next,

$$B \leq M \mathbb{P} \left(\left| \frac{1}{v} \sum_{k=1}^v V_{k,n} \right| > \varepsilon \right)$$

Since $(F_{i,k,n})_{1 \leq k \leq v} \Rightarrow (N_k)_{1 \leq k \leq v}$, we get that, for any $\varepsilon > 0$,

$$\mathbb{P} \left(\left| \frac{1}{v} \sum_{k=1}^v V_{k,n} \right| > \varepsilon \right) \rightarrow_{n \rightarrow \infty} \mathbb{P} \left(\left| \frac{1}{v} \sum_{k=1}^v N_k^2 - \Delta_i \right| > \varepsilon \right)$$

which converges to zero as $v \rightarrow \infty$ by the law of large numbers.

Let us now deal with C . Letting $h_M(x) = \left(|x| - \frac{M}{2}\right)_+$, where $x_+ = x\mathbb{1}_{x>0}$, and noticing that $|x|\mathbb{1}_{|x|>M} \leq 2h_M(x)$, we get

$$C \leq 2\mathbb{E} \left(h_M \left(\frac{1}{v} \sum_{k=1}^v V_{k,n} \right) \right).$$

Since h_M is a convex function, we deduce that

$$C \leq 2\frac{1}{v} \sum_{k=1}^v \mathbb{E} (h_M (V_{k,n})).$$

But since for each i and k , $(F_{i,k,n}^2)_{n \geq 1}$ is a uniformly integrable family,

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} (h_M (V_{k,n})) = 0.$$

So overall the lemma follows by finally letting $\varepsilon \rightarrow 0$. □

4.7.2 Proof of Theorem 4.4.1

By applying Proposition 4.6.1 to $X_0(s)$ and $\mathcal{X} = \mathbb{R}$, it follows that there exists a positive constant C only depending on p and d such that for any positive integers n_1, \dots, n_d, m and for any real $\lambda \geq 1$,

$$\begin{aligned}
& (n_1 \cdots n_d)^{-1/2} \left(\int_S \left\| \max_{1 \leq k \leq n} |S_{\underline{k}}(X(s)) - S_{\underline{k}}(D(s))| \right\|_2^p \mu(ds) \right)^{1/p} \\
& \leq C \left(\int_S \left(\sum_{i=1}^d \sum_{|k_i| > m} \sum_{\underline{k} \setminus \{k_i\} \in \mathbb{Z}^{d-1}} \|P_{\underline{k}}(X_{\underline{0}}(s))\|_2 \right)^p d\mu(s) \right)^{1/p} \\
& + Cm^{2d-1} \left(\frac{\lambda^2}{\min_{1 \leq i \leq d} n_i^{1/2}} + \frac{1}{\lambda} \right) \left(\int_S (\|X_{\underline{0}}(s)\|_2^p + \|D_{\underline{0}}(s)\|_2^p) d\mu(s) \right)^{1/p} \\
& + Cm^{2d-1} \left(\int_S \left(\|X_{\underline{0}}(s) \mathbf{1}_{\{|X_{\underline{0}}(s)| > \lambda \|X_{\underline{0}}(s)\|_2}\|_2^p + \|D_{\underline{0}}(s) \mathbf{1}_{\{|D_{\underline{0}}(s)| > \lambda \|D_{\underline{0}}(s)\|_2}\|_2^p \right) \mu(ds) \right)^{1/p} \\
& := I_1 + I_2 + I_3.
\end{aligned}$$

Letting first $\underline{n} \rightarrow \infty$, then $\lambda \rightarrow \infty$ and after $m \rightarrow \infty$, the R.H.S. is tending to zero. Indeed, I_1 is tending to zero as $m \rightarrow \infty$ by using the dominated convergence theorem and by taking into account condition (4.4.2). I_2 is tending to zero by letting first $\underline{n} \rightarrow \infty$, then $\lambda \rightarrow \infty$ and by taking into account that (4.4.2) implies that $\int_S \|D_{\underline{0}}(s)\|_2^p \mu(ds) < \infty$ (and then, by [35, Lemma 2.4], that $D_{\underline{0}}(\cdot)$ is pre-Gaussian). Finally, I_3 is tending to zero by letting $\lambda \rightarrow \infty$ and using the dominated convergence theorem and conditions (4.4.1) and (4.4.2). Hence (4.4.3) holds. Note now that, since $1 \leq p \leq 2$, for $\mathcal{X} = L^p(S, \mu)$,

$$\begin{aligned}
\|X\|_{2, \mathcal{X}} &= \left\| \left(\int_S |X(s)|^p d\mu(s) \right)^{1/p} \right\|_2 = \left\| \int_S |X(s)|^p d\mu(s) \right\|_{2/p}^{1/p} \\
&\leq \left(\int_S \| |X(s)|^p \|_{2/p} d\mu(s) \right)^{1/p} = \left(\int_S \|X(s)\|_2^p d\mu(s) \right)^{1/p}.
\end{aligned}$$

Hence, (4.4.3) implies that

$$\lim_{\underline{n} \rightarrow \infty} \left\| \max_{1 \leq k \leq n} \left| \frac{S_{\underline{k}}(X)}{\sqrt{n_1 \cdots n_d}} - \frac{S_{\underline{k}}(D)}{\sqrt{n_1 \cdots n_d}} \right| \right\|_{2, \mathcal{X}} = 0. \quad (4.7.7)$$

Then the convergence in distribution (4.4.4) follows by using (4.7.7) together with Theorem 4.3.4. Next to prove (4.4.5), as in the proof of [18, Th 8], we use [18, Lemma 7]. Hence it suffices to prove that for any $f \in L^q(S, \mu)$, where q is the con-

jugate of p , we have $\sum_{i \in \mathbb{Z}^d} \|P_0(f(X_i))\|_2 < \infty$, which is implied by (4.4.2) by direct computations. Note that [18, Lemma 7] is stated for reversed ortho-martingales, but it holds also for ortho-martingales in the usual sense. \square

4.7.3 Proof of Corollary 4.4.3

For the sake of clarity, we shall give the complete proof in the case $d = 2$ (the case $d > 2$ can be proved with similar arguments). We shall need the following lemma, whose proof follows by applying two times [12, Lemma A.1].

Lemma 4.7.2. *For any double indexed sequence $(u_{i,j})_{i,j \in \mathbb{N}}$ of non-negative numbers, there exists a positive constant C such that*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u_{i,j} \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{1}{ij} \sum_{k=i}^{\infty} \sum_{\ell=j}^{\infty} u_{k,\ell} \right)^{1/2}.$$

From Lemma 4.7.2 with $u_{i,j} = \|P_{-i,-j}(X_{0,0}(s))\|_2$, we get:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|P_{-i,-j}(X_{0,0}(s))\|_2 \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{1}{ij} \sum_{k=i}^{\infty} \sum_{\ell=j}^{\infty} \|P_{-k,-\ell}(X_{0,0}(s))\|_2^2 \right)^{1/2}.$$

But by orthogonality and regularity

$$\sum_{k=i}^{\infty} \sum_{\ell=j}^{\infty} \|P_{-k,-\ell}(X_{0,0}(s))\|_2^2 = \left\| \sum_{k=i}^{\infty} \sum_{\ell=j}^{\infty} P_{-k,-\ell}(X_{0,0}(s)) \right\|_2^2 = \|\mathbb{E}(X_{i,j}(s) | \mathcal{F}_{0,0})\|_2^2.$$

Hence

$$\int_S \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \|P_{-i,-j}(X_{0,0}(s))\|_2 \right)^p \mu(ds) \leq C \left(\int_S \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{\sqrt{ij}} \|\mathbb{E}(X_{i,j}(s) | \mathcal{F}_{0,0})\|_2 \right)^p \mu(ds),$$

which is finite by condition (4.4.6) in case $d = 2$. It remains to prove that:

$$\int_S \left(\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \|P_{0,0}(X_{i,-j}(s))\|_2 \right)^p \mu(ds) < \infty, \quad \int_S \left(\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \|P_{0,0}(X_{-i,j}(s))\|_2 \right)^p \mu(ds) < \infty, \quad (4.7.8)$$

and

$$\int_S \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|P_{0,0}(X_{-i,-j}(s))\|_2 \right)^p \mu(ds) < \infty. \quad (4.7.9)$$

By Lemma 4.7.2 again,

$$\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \|P_{0,0}(X_{i,-j}(s))\|_2 \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{1}{ij} \sum_{k=i+1}^{\infty} \sum_{\ell=j}^{\infty} \|P_{k,-\ell}(X_{0,0}(s))\|_2^2 \right)^{1/2}. \quad (4.7.10)$$

But, for any $j \geq 0$, by stationarity,

$$\sum_{k=i+1}^{\infty} \sum_{\ell=j}^{\infty} \|P_{k,-\ell}(X_{0,0}(s))\|_2^2 \leq \sum_{k=i+1}^{\infty} \sum_{\ell=-j}^{\infty} \|P_{k,-\ell}(X_{0,0}(s))\|_2^2 \leq \sum_{k=1}^{\infty} \sum_{\ell=-\infty}^0 \|P_{k,\ell}(X_{-i,-j}(s))\|_2^2.$$

Next, by orthogonality,

$$\sum_{k=1}^{\infty} \sum_{\ell=-\infty}^0 \|P_{k,\ell}(X_{-i,-j}(s))\|_2^2 = \left\| \sum_{k=1}^{\infty} \sum_{\ell=-\infty}^0 P_{k,\ell}(X_{-i,-j}(s)) \right\|_2^2.$$

But, note that,

$$\begin{aligned} X_{-i,-j}(s) - \mathbb{E}(X_{-i,-j}(s) | \mathcal{F}_{0,0}) &= \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} P_{k,\ell}(X_{-i,-j}(s)) \\ &\quad + \sum_{k=-\infty}^0 \sum_{\ell=1}^{\infty} P_{k,\ell}(X_{-i,-j}(s)) + \sum_{k=1}^{\infty} \sum_{\ell=-\infty}^0 P_{k,\ell}(X_{-i,-j}(s)) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Hence, by orthogonality, it follows that

$$\|X_{-i,-j}(s) - \mathbb{E}(X_{-i,-j}(s)|\mathcal{F}_{0,0})\|_2^2 = \|I_1\|_2^2 + \|I_2\|_2^2 + \|I_3\|_2^2.$$

So, overall,

$$\sum_{k=i+1}^{\infty} \sum_{\ell=j}^{\infty} \|P_{k,-\ell}(X_{0,0}(s))\|_2^2 \leq \|I_3\|_2^2 \leq \|X_{-i,-j}(s) - \mathbb{E}(X_{-i,-j}(s)|\mathcal{F}_{0,0})\|_2^2. \quad (4.7.11)$$

Therefore, starting from (4.7.10) and taking into account (4.7.11), it follows that the first part of (4.7.8) holds provided condition (4.4.7) (in case $d = 2$) is assumed. Using similar arguments, we derive that the second part of (4.7.8) as well as (4.7.9) are satisfied provided (4.4.7) holds. This ends the proof of the corollary in case $d = 2$. \square

4.7.4 Proof of Proposition 4.6.1

In the proof the notation $a \ll b$ means that there exists a universal constant C (here only depending on (L, p, d)) such that $a \leq Cb$.

For the reader's convenience, let us consider the case $d = 2$ (the case $d = 1$ is even simpler and for $d > 2$ some indications are given at the end of the proof). When $d = 2$, (4.6.1) reads as

$$\sum_{i,j \in \mathbb{Z}} \|P_{0,0}(X_{i,j})\|_{\mathcal{X}} < \infty. \quad (4.7.12)$$

Then $D_{k,\ell} = \sum_{i,j \in \mathbb{Z}} P_{k,\ell}(X_{i,j})$. Moreover $(D_{k,\ell})_{k,\ell}$ is a stationary field of $L^p(\mathcal{X})$ ortho-martingale differences with respect to the filtration $(\mathcal{F}_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}$. In addition, the corresponding ortho-martingale is $M_{n_1, n_2} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} D_{i,j}$. Next (4.6.2) reads as: there exists a positive constant C only depending on (L, p) such that for any positive integers n_1, n_2, m and for any real $\lambda \geq 1$,

$$\begin{aligned}
& (n_1 n_2)^{-1/2} \left\| \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} |S_{k,\ell} - M_{k,\ell}|_{\mathcal{X}} \right\|_p \\
& \leq C \sum_{|u| > m, v \in \mathbb{Z}} \|P_{u,v}(X_{0,0})|_{\mathcal{X}}\|_p + C \sum_{u \in \mathbb{Z}, |v| > m} \|P_{u,v}(X_{0,0})|_{\mathcal{X}}\|_p \\
& \quad + C m^3 \lambda^2 \left(\frac{1}{\sqrt{n_1 \wedge n_2}} + \frac{\lambda^{-3}}{(n_1 \wedge n_2)^{(p-2)/(2p)}} \right) (\|X_{0,0}|_{\mathcal{X}}\|_p + \|D_{0,0}|_{\mathcal{X}}\|_p) \\
& + C \frac{m^3}{(n_1 \wedge n_2)^{(p-2)/(2p)}} \left(\|X_{0,0}|_{\mathcal{X}} \mathbf{1}_{\{|X_{0,0}|_{\mathcal{X}}| > \lambda \|X_{0,0}|_{\mathcal{X}}\|_p}\} \|_p + \|D_{0,0}|_{\mathcal{X}} \mathbf{1}_{\{|D_{0,0}|_{\mathcal{X}}| > \lambda \|D_{0,0}|_{\mathcal{X}}\|_p}\} \|_p \right).
\end{aligned} \tag{4.7.13}$$

To prove (4.7.13), the idea is to extend the decomposition (29) in [12] stated in the case $d = 1$, to higher dimension. The idea of this decomposition is then as follows. Fix a positive integer m , and define $M_{k,\ell}^{(m)} = \sum_{i=1}^k \sum_{j=1}^{\ell} D_{i,j} \circ T_{m,m}$. Then we have

$$\sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} |S_{k,\ell} - M_{k,\ell}|_{\mathcal{X}} \leq \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} |S_{k,\ell} - M_{k,\ell}^{(m)}|_{\mathcal{X}} + \sup_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} |M_{k,\ell}^{(m)} - M_{k,\ell}|_{\mathcal{X}}.$$

To take care of the norm of the last term we use among others Doob's maximal inequality. Next, to take care of the norm of the first term in the right-hand side, we shall use a decomposition of $X_{0,0} - D_{0,0} \circ T_{m,m}$ with coboundaries and mixed coboundaries plus a rest. We refer to the forthcoming decomposition (4.7.14) for a more precise statement. In this decomposition, the term $Y_{0,0}^{(m)} - Z_{0,0}^{(m)}$ will be considered as the rest and, once summed over a rectangle, will have at the end a negligible contribution because of the assumption (4.7.12). The first term I in (4.7.14) will be the coboundary and the others involving $g^{(m)}$ and $h^{(m)}$ will be the mixed coboundaries in the sense that they are coboundaries in one direction and martingales in the other. As usual the coboundary structure will allow to simplify the treatment of the norm of those terms since they will behave as a telescoping sum, and next a usual truncation argument can be used. The mixed coboundaries same arguments are used in addition to Doob's maximal inequality.

We now give the details. Since $X_{0,0}$ is regular, we can write $X_{0,0} = \sum_{i,j \in \mathbb{Z}} P_{i,j}(X_{0,0})$ a.s. Let m be a fixed positive integer, and define

$$\theta_{0,0}^{(m)} = \sum_{k=0}^{2m-2} \sum_{\ell=0}^{2m-2} \sum_{i=k-m+1}^{m-1} \sum_{j=\ell-m+1}^{m-1} P_{i,j}(X_{k,\ell}) \text{ and } \theta_{u,v}^{(m)} = \theta_{0,0}^{(m)} \circ T_{u,v}.$$

By simple algebra we have

$$\begin{aligned} I &:= \theta_{0,0}^{(m)} - \theta_{1,0}^{(m)} - \theta_{0,1}^{(m)} + \theta_{1,1}^{(m)} \\ &= \sum_{k=-m+1}^{m-1} \sum_{\ell=-m+1}^{m-1} P_{k,\ell}(X_{0,0}) - \sum_{i=-m+1}^{m-1} \sum_{v=1}^{2m-1} P_{i,m}(X_{0,v}) \\ &\quad - \sum_{u=1}^{2m-1} \sum_{j=-m+1}^{m-1} P_{m,j}(X_{u,0}) + \sum_{u=1}^{2m-1} \sum_{v=1}^{2m-1} P_{m,m}(X_{u,v}). \end{aligned}$$

On another hand setting

$$g_{0,0}^{(m)} = \sum_{u=1}^{2m-1} \sum_{b=0}^{2m-2} \sum_{\ell=b-m+1}^{m-1} P_{m,\ell}(X_{u,b}) \text{ and } g_{i,j}^{(m)} = g_{0,0}^{(m)} \circ T_{i,j},$$

and

$$h_{0,0}^{(m)} = \sum_{v=1}^{2m-1} \sum_{a=0}^{2m-2} \sum_{k=a-m+1}^{m-1} P_{k,m}(X_{a,v}) \text{ and } h_{i,j}^{(m)} = h_{0,0}^{(m)} \circ T_{i,j},$$

we have

$$\sum_{u=1}^{2m-1} \left(\sum_{j=-m+1}^{m-1} P_{m,j}(X_{u,0}) - \sum_{v=1}^{2m-1} P_{m,m}(X_{u,v}) \right) = g_{0,0}^{(m)} - g_{0,1}^{(m)}$$

and

$$\sum_{v=1}^{2m-1} \left(\sum_{i=-m+1}^{m-1} P_{i,m}(X_{0,v}) - \sum_{k=1}^{2m-1} P_{m,m}(X_{k,v}) \right) = h_{0,0}^{(m)} - h_{1,0}^{(m)}.$$

So, overall, for any positive integer m , the following decomposition is valid:

$$\begin{aligned} X_{0,0} - D_{0,0} \circ T_{m,m} &:= I + (g_{0,0}^{(m)} - g_{0,1}^{(m)}) + (h_{0,0}^{(m)} - h_{1,0}^{(m)}) \\ &\quad + Y_{0,0}^{(m)} - Z_{0,0}^{(m)}, \end{aligned} \quad (4.7.14)$$

where we used the following notation:

$$Y_{0,0}^{(m)} := \sum_{(k,\ell) \in \mathbb{Z}^2 \setminus [-m+1, m-1]^2} P_{k,\ell}(X_{0,0}) \text{ and } Z_{0,0}^{(m)} := \sum_{(u,v) \in \mathbb{Z}^2 \setminus [1, 2m-1]^2} P_{m,m}(X_{u,v}).$$

Let $R_{k,\ell} = S_{k,\ell} - M_{k,\ell}$. We then derive that

$$\begin{aligned} R_{k,\ell} &= - \sum_{i=1}^k \sum_{j=1}^{\ell} (D_{i,j} - D_{i,j} \circ T_{m,m}) + (\theta_{1,1}^{(m)} - \theta_{k+1,1}^{(m)} - \theta_{1,\ell+1}^{(m)} + \theta_{k+1,\ell+1}^{(m)}) \\ &\quad + \sum_{i=1}^k (g_{i,1}^{(m)} - g_{i,\ell+1}^{(m)}) + \sum_{j=1}^{\ell} (h_{1,j}^{(m)} - h_{k+1,j}^{(m)}) \\ &\quad + \sum_{i=1}^k \sum_{j=1}^{\ell} Y_{0,0}^{(m)} \circ T_{i,j} - \sum_{i=1}^k \sum_{j=1}^{\ell} Z_{0,0}^{(m)} \circ T_{i,j}. \end{aligned}$$

Therefore

$$\begin{aligned} \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} |R_{k,\ell}|_{\mathcal{X}} \right\|_p &\leq \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} (D_{i,j} - D_{i,j} \circ T_{m,m}) \right|_{\mathcal{X}} \right\|_p + 4 \left\| \max_{\substack{1 \leq k \leq n_1+1 \\ 1 \leq \ell \leq n_2+1}} |\theta_{k,\ell}^{(m)}|_{\mathcal{X}} \right\|_p \\ &\quad + 2 \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2+1}} \left| \sum_{i=1}^k g_{i,\ell}^{(m)} \right|_{\mathcal{X}} \right\|_p + 2 \left\| \max_{\substack{1 \leq k \leq n_1+1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{j=1}^{\ell} h_{k,j}^{(m)} \right|_{\mathcal{X}} \right\|_p \\ &\quad + \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} Y_{0,0}^{(m)} \circ T_{i,j} \right|_{\mathcal{X}} \right\|_p + \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} Z_{0,0}^{(m)} \circ T_{i,j} \right|_{\mathcal{X}} \right\|_p. \end{aligned} \quad (4.7.15)$$

Let $Y_{i,j}^{(m)} = Y_{0,0}^{(m)} \circ T_{i,j}$. Since $X_{0,0}$ is regular,

$$Y_{i,j}^{(m)} = \sum_{u,v \in \mathbb{Z}} P_{u-i,v-j}(Y_{i,j}^{(m)}).$$

Hence

$$\left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} Y_{i,j}^{(m)} \right|_{\mathcal{X}} \right\|_p \leq \sum_{u,v \in \mathbb{Z}} \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} P_{u-i,v-j}(Y_{i,j}^{(m)}) \right|_{\mathcal{X}} \right\|_p.$$

Denoting by $U_{k,\ell} = \left| \sum_{i=1}^k \sum_{j=1}^{\ell} P_{u-i,v-j}(Y_{i,j}^{(m)}) \right|_{\mathcal{X}}$, note that $(U_{k,\ell})_{k,\ell}$ is an ortho-submartingale w.r.t. the completely commuting filtration $(\mathcal{F}_{k,\ell})_{k,\ell}$. By [26, Prop. 2.2.1], it follows that $(\max_{1 \leq k \leq n_1} U_{k,\ell})_{\ell \geq 1}$ and $(\max_{1 \leq \ell \leq n_2} U_{k,\ell})_{k \geq 1}$ are both one parameter submartingales. Therefore, for any fixed pair (n_1, n_2) of natural numbers, applying twice Doob's maximal inequality, we get

$$\left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} Y_{i,j}^{(m)} \right|_{\mathcal{X}} \right\|_p \leq \left(\frac{p}{p-1} \right)^2 \sum_{u,v \in \mathbb{Z}} \left\| \left| \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} P_{u-i,v-j}(Y_{i,j}^{(m)}) \right|_{\mathcal{X}} \right\|_p.$$

Next, since \mathcal{X} is $(2, L)$ -smooth, using twice [42, Th. 2.6] (see also [36, Th. 2.2]), we get

$$\begin{aligned} \left\| \left| \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} P_{u-i,v-j}(Y_{i,j}^{(m)}) \right|_{\mathcal{X}} \right\|_p^2 &\ll \sum_{i=1}^{n_1} \left\| \left| \sum_{j=1}^{n_2} P_{u-i,v-j}(Y_{i,j}^{(m)}) \right|_{\mathcal{X}} \right\|_p^2 \\ &\ll \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left\| \left| P_{u-i,v-j}(Y_{i,j}^{(m)}) \right|_{\mathcal{X}} \right\|_p^2. \end{aligned}$$

Hence, by stationarity,

$$\left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} Y_{i,j}^{(m)} \right|_{\mathcal{X}} \right\|_p \ll \sqrt{n_1 n_2} \sum_{u,v \in \mathbb{Z}} \left\| \left| P_{u,v}(Y_{0,0}^{(m)}) \right|_{\mathcal{X}} \right\|_p.$$

Since $P_{u,v}P_{k,\ell}(\cdot) = 0$ for $(u, v) \neq (i, j)$, it follows that

$$\begin{aligned} & \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} Y_{i,j}^{(m)} \right|_{\mathcal{X}} \right\|_p \\ & \ll \sqrt{n_1 n_2} \left(\sum_{|u|>m} \sum_{v \in \mathbb{Z}} \|P_{u,v}(X_{0,0})|_{\mathcal{X}}\|_p + \sum_{u \in \mathbb{Z}} \sum_{|v|>m} \|P_{u,v}(X_{0,0})|_{\mathcal{X}}\|_p \right). \end{aligned} \quad (4.7.16)$$

Similarly, we get

$$\left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} Z_{i,j}^{(m)} \right|_{\mathcal{X}} \right\|_p \ll \sqrt{n_1 n_2} \sum_{(u,v) \in \mathbb{Z}^2 \setminus [1, 2m-1]^2} \|P_{m,m}(X_{u,v})|_{\mathcal{X}}\|_p,$$

implying that

$$\begin{aligned} & \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} Z_{i,j}^{(m)} \right|_{\mathcal{X}} \right\|_p \\ & \ll \sqrt{n_1 n_2} \left(\sum_{|u|>m} \sum_{v \in \mathbb{Z}} \|P_{u,v}(X_{0,0})|_{\mathcal{X}}\|_p + \sum_{u \in \mathbb{Z}} \sum_{|v|>m} \|P_{u,v}(X_{0,0})|_{\mathcal{X}}\|_p \right). \end{aligned} \quad (4.7.17)$$

We handle now the first term in the R.H.S. of decomposition (4.7.15). Note first that

$$\begin{aligned} \sum_{i=1}^k \sum_{j=1}^{\ell} (D_{i,j} - D_{i,j} \circ T_{m,m}) &= \sum_{i=1}^m \sum_{j=1}^{\ell} D_{i,j} - \sum_{i=k+1}^{k+m} \sum_{j=m+1}^{\ell+m} D_{i,j} \\ &+ \sum_{i=m+1}^k \sum_{j=1}^m D_{i,j} - \sum_{i=m+1}^k \sum_{j=\ell+1}^{\ell+m} D_{i,j}. \end{aligned} \quad (4.7.18)$$

Since for j fixed $(D_{i,j})_j$ is a martingale differences sequence, $(|\sum_{j=1}^{\ell} D_{i,j}|_{\mathcal{X}})_{\ell}$ is a submartingale. Then, by stationarity and Doob's maximal inequality,

$$\left\| \sum_{i=1}^m \max_{1 \leq \ell \leq n_2} \left| \sum_{j=1}^{\ell} D_{i,j} \right|_{\mathcal{X}} \right\|_p \leq m \left(\frac{p}{p-1} \right) \left\| \sum_{j=1}^{n_2} D_{0,j} \right\|_p.$$

Next, using that \mathcal{X} is $(2, L)$ -smooth and [42, Th. 2.6] , we derive

$$\left\| \sum_{i=1}^m \max_{1 \leq \ell \leq n_2} \left| \sum_{j=1}^{\ell} D_{i,j} \right|_{\mathcal{X}} \right\|_p^2 \ll m^2 \sum_{j=1}^{n_2} \|D_{0,j}|_{\mathcal{X}}\|_p^2 \ll m^2 n_2 \|D_{0,0}|_{\mathcal{X}}\|_p^2. \quad (4.7.19)$$

Similarly, we obtain

$$\left\| \sum_{j=1}^m \max_{m+1 \leq k \leq n_1} \left| \sum_{i=m+1}^k D_{i,j} \right|_{\mathcal{X}} \right\|_p^2 \ll m^2 n_1 \|D_{0,0}|_{\mathcal{X}}\|_p^2. \quad (4.7.20)$$

To deal with the quantity coming from the second term in the R.H.S. of (4.7.18), we first note that for any positive real A ,

$$\begin{aligned} & \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=k+1}^{k+m} \sum_{j=m+1}^{\ell+m} D_{i,j} \right|_{\mathcal{X}} \right\|_p \\ & \leq \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \sum_{i=k+1}^{k+m} \left| \sum_{j=m+1}^{\ell+m} D_{i,j} \right|_{\mathcal{X}} \mathbf{1}_{\{\max_{1 \leq \ell \leq n_2} |\sum_{j=m+1}^{\ell+m} D_{i,j}|_{\mathcal{X}}| \leq 4A\sqrt{n_2}\}} \right\|_p \\ & \quad + \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \sum_{i=k+1}^{k+m} \left| \sum_{j=m+1}^{\ell+m} D_{i,j} \right|_{\mathcal{X}} \mathbf{1}_{\{\max_{1 \leq \ell \leq n_2} |\sum_{j=m+1}^{\ell+m} D_{i,j}|_{\mathcal{X}}| > 4A\sqrt{n_2}\}} \right\|_p. \end{aligned}$$

The first term in the R.H.S. is less than $4mA\sqrt{n_2}$ whereas to deal with the second one we note that, by stationarity,

$$\begin{aligned} & \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \sum_{i=k+1}^{k+m} \left| \sum_{j=m+1}^{\ell+m} D_{i,j} \right|_{\mathcal{X}} \mathbf{1}_{\{\max_{1 \leq \ell \leq n_2} |\sum_{j=m+1}^{\ell+m} D_{i,j}|_{\mathcal{X}}| > 4A\sqrt{n_2}\}} \right\|_p^p \\ & \leq n_1 m^p \left\| \max_{1 \leq \ell \leq n_2} \left| \sum_{j=m+1}^{\ell+m} D_{0,j} \right|_{\mathcal{X}} \mathbf{1}_{\{\max_{1 \leq \ell \leq n_2} |\sum_{j=m+1}^{\ell+m} D_{0,j}|_{\mathcal{X}}| > 4A\sqrt{n_2}\}} \right\|_p^p. \end{aligned}$$

But, by [15, Corollary 2.10] and stationarity,

$$\begin{aligned} & \left\| \max_{1 \leq \ell \leq n_2} \left\| \sum_{j=m+1}^{\ell+m} D_{0,j} \right\|_{\mathcal{X}} \mathbf{1}_{\{\max_{1 \leq \ell \leq n_2} |\sum_{j=m+1}^{\ell+m} D_{0,j}|_{\mathcal{X}} > 4A\sqrt{n_2}\}} \right\|_p^p \\ & \leq 2^p \left(\frac{p}{p-1} \right)^p \left\| \left\| \sum_{j=1}^{n_2} D_{0,j} \right\|_{\mathcal{X}} \mathbf{1}_{\{|\sum_{j=1}^{n_2} D_{0,j}|_{\mathcal{X}} > 2A\sqrt{n_2}\}} \right\|_p^p. \end{aligned}$$

Let B be a positive real and define

$$d'_j = D_{0,j} \mathbf{1}_{\{|D_{0,j}|_{\mathcal{X}} \leq B\}} - \mathbb{E}(D_{0,j} \mathbf{1}_{\{|D_{0,j}|_{\mathcal{X}} \leq B\}} | \mathcal{F}_{0,j-1}), \quad M'_{n_2} = \sum_{j=1}^{n_2} d'_j$$

and

$$d''_j = D_{0,j} \mathbf{1}_{\{|D_{0,j}|_{\mathcal{X}} > B\}} - \mathbb{E}(D_{0,j} \mathbf{1}_{\{|D_{0,j}|_{\mathcal{X}} > B\}} | \mathcal{F}_{0,j-1}), \quad M''_{n_2} = \sum_{j=1}^{n_2} d''_j.$$

Observe that $M_{n_2} = \sum_{j=1}^{n_2} D_{0,j} = M'_{n_2} + M''_{n_2}$ and that for any nonnegative reals a , b and ε , $(a+b)^p \mathbf{1}_{\{a+b \geq 2\varepsilon\}} \leq 2^p a^p \mathbf{1}_{\{a \geq \varepsilon\}} + 2^p b^p \mathbf{1}_{\{b \geq \varepsilon\}}$. Therefore

$$\begin{aligned} & \left\| \left\| \sum_{j=1}^{n_2} D_{0,j} \right\|_{\mathcal{X}} \mathbf{1}_{\{|\sum_{j=1}^{n_2} D_{0,j}|_{\mathcal{X}} > 2A\sqrt{n_2}\}} \right\|_p^p \\ & \leq 2^p \left\| \left\| M'_{n_2} \right\|_{\mathcal{X}} \mathbf{1}_{\{|M'_{n_2}|_{\mathcal{X}} > A\sqrt{n_2}\}} \right\|_p^p + 2^p \left\| \left\| M''_{n_2} \right\|_{\mathcal{X}} \mathbf{1}_{\{|M''_{n_2}|_{\mathcal{X}} > A\sqrt{n_2}\}} \right\|_p^p. \end{aligned}$$

Using that \mathcal{X} is $(2, L)$ -smooth, [42, Th. 2.6] and stationarity, we get

$$\left\| \left\| M''_{n_2} \right\|_{\mathcal{X}} \right\|_p^2 \ll n_2 \left\| \left\| D_{0,0} \right\|_{\mathcal{X}} \mathbf{1}_{\{|D_{0,0}|_{\mathcal{X}} > B\}} \right\|_p^2.$$

Next, using again that \mathcal{X} is $(2, L)$ -smooth and [42, Th. 2.6], we derive

$$\begin{aligned} \left\| \left\| M'_{n_2} \right\|_{\mathcal{X}} \mathbf{1}_{\{|M'_{n_2}|_{\mathcal{X}} > A\sqrt{n_2}\}} \right\|_p^p & \leq \frac{1}{A^p (n_2)^{p/2}} \mathbb{E}(\|M'_{n_2}\|_{\mathcal{X}}^{2p}) \ll \frac{1}{A^p (n_2)^{p/2}} \left(\sum_{j=1}^{n_2} \left\| \left\| d'_j \right\|_{\mathcal{X}} \right\|_{2p}^2 \right)^p \\ & \ll \frac{(n_2)^{p/2}}{A^p} \left\| \left\| D_{0,0} \right\|_{\mathcal{X}} \mathbf{1}_{\{|D_{0,0}|_{\mathcal{X}} \leq B\}} \right\|_{2p}^{2p} \ll \frac{(n_2)^{p/2} B^p}{A^p} \left\| \left\| D_{0,0} \right\|_{\mathcal{X}} \right\|_p^p. \end{aligned}$$

So, overall, for any positive reals A and B ,

$$\begin{aligned} & \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=k+1}^{k+m} \sum_{j=m+1}^{\ell+m} D_{i,j} \right|_{\mathcal{X}} \right\|_p \\ & \ll mA\sqrt{n_2} + mn_1^{1/p}\sqrt{n_2} \|D_{0,0}|_{\mathcal{X}} \mathbf{1}_{\{|D_{0,0}|_{\mathcal{X}} > B\}}\|_p + m \frac{n_1^{1/p}\sqrt{n_2}B}{A} \|D_{0,0}|_{\mathcal{X}}\|_p. \end{aligned} \quad (4.7.21)$$

Similarly, we get

$$\begin{aligned} & \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=m+1}^k \sum_{j=\ell+1}^{\ell+m} D_{i,j} \right|_{\mathcal{X}} \right\|_p \\ & \ll mA\sqrt{n_1} + mn_2^{1/p}\sqrt{n_1} \|D_{0,0}|_{\mathcal{X}} \mathbf{1}_{\{|D_{0,0}|_{\mathcal{X}} > B\}}\|_p + m \frac{n_2^{1/p}\sqrt{n_1}B}{A} \|D_{0,0}|_{\mathcal{X}}\|_p. \end{aligned} \quad (4.7.22)$$

Let $\lambda > 0$. Starting from (4.7.18), taking into account (4.7.19)-(4.7.22) and selecting $A = \lambda^2 \|D_{0,0}|_{\mathcal{X}}\|_p$ and $B = A/\lambda$, we derive

$$\begin{aligned} & \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k \sum_{j=1}^{\ell} (D_{i,j} - D_{i,j} \circ T_{m,m}) \right|_{\mathcal{X}} \right\|_p \\ & \ll m(\lambda^2 + 1)(\sqrt{n_1} + \sqrt{n_2}) \|D_{0,0}|_{\mathcal{X}}\|_p \\ & \quad + m(n_1^{1/p}n_2^{1/2} + n_1^{1/2}n_2^{1/p}) \|D_{0,0}|_{\mathcal{X}} \mathbf{1}_{\{|D_{0,0}|_{\mathcal{X}} > \lambda \|D_{0,0}|_{\mathcal{X}}\|_p}\|_p \\ & \quad + \frac{m(n_1^{1/p}n_2^{1/2} + n_1^{1/2}n_2^{1/p})}{\lambda} \|D_{0,0}|_{\mathcal{X}}\|_p. \end{aligned} \quad (4.7.23)$$

Next note that, for any positive real M ,

$$\begin{aligned} & \left\| \max_{\substack{1 \leq u \leq n_1 \\ 1 \leq v \leq n_2}} |\theta_{u,v}^{(m)}|_{\mathcal{X}} \right\|_p \leq 4(2m-1)^4 M \\ & \quad + (n_1 n_2)^{1/p} \sum_{k=0}^{2m-2} \sum_{\ell=0}^{2m-2} \sum_{i=-m+1}^{m-1} \sum_{j=-m+1}^{m-1} \|P_{i,j}(X_{k,\ell} \mathbf{1}_{\{|X_{k,\ell}|_{\mathcal{X}} > M\}})|_{\mathcal{X}}\|_p. \end{aligned}$$

Hence, by stationarity, setting $M = \lambda \|X_{0,0}|_{\mathcal{X}}\|_p$, we get

$$\left\| \max_{\substack{1 \leq u \leq n_1 \\ 1 \leq v \leq n_2}} |\theta_{u,v}^{(m)}|_{\mathcal{X}} \right\|_p \ll \lambda m^4 \|X_{0,0}|_{\mathcal{X}}\|_p + m^4 (n_1 n_2)^{1/p} \|X_{0,0}|_{\mathcal{X}} \mathbf{1}_{\{|X_{0,0}|_{\mathcal{X}} > \lambda \|X_{0,0}|_{\mathcal{X}}\|_p}\} \|_p. \quad (4.7.24)$$

We deal now with the third and fourth term in the R.H.S of decomposition (4.7.15).

For any positive real A , using stationarity, we infer that

$$\begin{aligned} \left\| \max_{\substack{0 \leq k \leq n_1 \\ 0 \leq \ell \leq n_2}} \left| \sum_{i=1}^k g_{i,\ell+1}^{(m)} \right|_{\mathcal{X}} \right\|_p &\leq \sum_{u=1}^{2m-1} \sum_{b=0}^{2m-2} \sum_{v=b-m+1}^{m-1} \left\| \max_{\substack{0 \leq k \leq n_1 \\ 0 \leq \ell \leq n_2}} \left| \sum_{i=1}^k P_{m+i,v}(X_{u+i,b}) \circ T_{0,\ell+1} \right|_{\mathcal{X}} \right\|_p \\ &\leq 4(2m-1)^3 A \sqrt{n_1} + (2m-1)(n_2+1)^{1/p} \sum_{u=1-m}^{m-1} \sum_{v=-m+1}^{m-1} A(n_1, u, v), \end{aligned}$$

where

$$A(n_1, u, v) := \left\| \max_{1 \leq k \leq n_1} \left| \sum_{i=1}^k P_{i,v}(X_{u+i,0}) \right|_{\mathcal{X}} \mathbf{1}_{\{|\sum_{i=1}^k P_{i,v}(X_{u+i,0})|_{\mathcal{X}} > 4A\sqrt{n_1}\}} \right\|_p.$$

Note that $(|\sum_{i=1}^k P_{i,v}(X_{u+i,0})|_{\mathcal{X}})_{k \geq 1}$ is a submartingale. Therefore, by [15, Corollary 2.10] and stationarity,

$$A(n_1, u, v) \leq \frac{2p}{p-1} \left\| \left| \sum_{i=1}^{n_1} P_{i,v}(X_{u+i,0}) \right|_{\mathcal{X}} \mathbf{1}_{\{|\sum_{i=1}^{n_1} P_{i,v}(X_{u+i,0})|_{\mathcal{X}} > 2A\sqrt{n_1}\}} \right\|_p.$$

Next noticing that for any real B ,

$$P_{i,v}(X_{u+i,0}) = P_{i,v}(X_{u+i,0} \mathbf{1}_{\{|X_{u+i,0}|_{\mathcal{X}} \leq B\}}) + P_{i,v}(X_{u+i,0} \mathbf{1}_{\{|X_{u+i,0}|_{\mathcal{X}} > B\}})$$

and, proceeding as to get (4.7.21) and selecting $A = \lambda^2 \|X_{0,0}|_{\mathcal{X}}\|_p$ and $B = A/\lambda$, we

infer that, for any $\lambda > 0$,

$$\begin{aligned} & \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{i=1}^k g_{i,\ell+1}^{(m)} \right|_{\mathcal{X}} \right\|_p \ll m^3 \lambda^2 \sqrt{n_1} \|X_{0,0}|_{\mathcal{X}}\|_p \\ & + m^3 (n_1)^{1/2} (n_2)^{1/p} \|X_{0,0}|_{\mathcal{X}} \mathbf{1}_{\{|X_{0,0}|_{\mathcal{X}}| > \lambda \|X_{0,0}|_{\mathcal{X}}\|_p\}}\|_p + \frac{m^3 (n_1)^{1/2} (n_2)^{1/p}}{\lambda} \|X_{0,0}|_{\mathcal{X}}\|_p. \end{aligned} \quad (4.7.25)$$

With similar arguments, we obtain

$$\begin{aligned} & \left\| \max_{\substack{1 \leq k \leq n_1 \\ 1 \leq \ell \leq n_2}} \left| \sum_{j=1}^{\ell} h_{k+1,j}^{(m)} \right|_{\mathcal{X}} \right\|_p \ll m^3 \lambda^2 \sqrt{n_2} \|X_{0,0}|_{\mathcal{X}}\|_p \\ & + m^3 (n_1)^{1/p} (n_2)^{1/2} \|X_{0,0}|_{\mathcal{X}} \mathbf{1}_{\{|X_{0,0}|_{\mathcal{X}}| > \lambda \|X_{0,0}|_{\mathcal{X}}\|_p\}}\|_p + \frac{m^3 (n_1)^{1/p} (n_2)^{1/2}}{\lambda} \|X_{0,0}|_{\mathcal{X}}\|_p. \end{aligned} \quad (4.7.26)$$

Starting from decomposition (4.7.15) and considering the upper bounds (4.7.16), (4.7.17), (4.7.23), (4.7.24), (4.7.25) and (4.7.26), the inequality (4.7.12) follows.

Let us now indicate the main argument to extend (4.7.13) to dimension $d > 2$ and then get (4.6.2). By simple induction we infer that the following extension of (4.7.14) holds:

$$X_{\underline{0}} - D_{\underline{0}} \circ T_{\underline{m}} = \sum_{\emptyset \subsetneq J \subseteq \langle d \rangle} \prod_{s \in J^c} (I - U_s) g_J^{(m)} + Y_{\underline{0}}^{(m)} - Z_{\underline{0}}^{(m)}, \quad (4.7.27)$$

where

$$Y_{\underline{0}}^{(m)} = \sum_{\underline{k} \in \mathbb{Z}^d \setminus [-m+1, m-1]^d} P_{\underline{k}}(X_{\underline{0}}) \text{ and } Z_{\underline{0}}^{(m)} := \sum_{\underline{u} \in \mathbb{Z}^d \setminus [1, 2m-1]^d} P_{\underline{m}}(X_{\underline{u}}),$$

and $g_J^{(m)} \in L^p(\mathcal{X})$. Even if it is not an easy task to give a rigorous formulation for

g_J^m let us describe them for some given sets J : for $J = J_1 = \{1\}$ we have

$$g_{J_1}^{(m)} = \sum_{u_1=1}^{2m-1} \sum_{a_2=0}^{2m-2} \cdots \sum_{a_d=0}^{2m-2} \sum_{b_2=a_2-m+1}^{m-1} \cdots \sum_{b_d=a_d-m+1}^{m-1} P_{m,b_2,\dots,b_d}(X_{u_1,a_2,\dots,a_d}),$$

for $J = J_2 = \{1, 2\}$ we have

$$g_{J_2}^{(m)} = \sum_{u_1=1}^{2m-1} \sum_{u_2=1}^{2m-1} \sum_{a_3=0}^{2m-2} \cdots \sum_{a_d=0}^{2m-2} \sum_{b_3=a_3-m+1}^{m-1} \cdots \sum_{b_d=a_d-m+1}^{m-1} P_{m,m,b_3,\dots,b_d}(X_{u_1,u_2,a_3,\dots,a_d}),$$

for $J = J_3 = \{1, 3\}$ we have

$$g_{J_3}^{(m)} = \sum_{u_1=1}^{2m-1} \sum_{u_3=1}^{2m-1} \sum_{a_2=0}^{2m-2} \sum_{a_4=0}^{2m-2} \cdots \sum_{a_d=0}^{2m-2} \sum_{b_2=a_2-m+1}^{m-1} \sum_{b_4=a_4-m+1}^{m-1} \cdots \sum_{b_d=a_d-m+1}^{m-1} P_{m,b_2,m,b_4,\dots,b_d}(X_{u_1,a_2,u_3,a_4,\dots,a_d}),$$

and so on. Inequality (4.6.2) then follows using the decomposition (4.7.27) and the arguments used to prove (4.7.13). \square

Chapter 5

Appendix

In this appendix, we give some related results between chapter 2 and 3. Namely the TCL under \mathbb{L}^1 condition in $\mathbb{L}^1(S, \mathcal{S}, \mu)$.

5.1 TCL under \mathbb{L}^1 condition in Banach space

We first recall some notations. Let (S, \mathcal{S}, μ) be a σ -finite measure space such that $\mathcal{X} := \mathbb{L}^1(S, \mathcal{S}, \mu)$ is separable. Recall that $\mathbb{L}^1(S, \mu)$ is a Banach space of cotype 2. Let $\mathbb{L}^\infty(S, \mu)$ be its dual space.

Let T and S be commuting invertible and measure-preserving transformations. By U (resp V) we denote the operator in $\mathbb{L}^p(\mathcal{X})$ defined by $Uf = f \circ U$ (resp $Vf = f \circ S$) for any random variable f . Now for a stationary random variable $f \in \mathbb{L}^1(\mathcal{X})$ if the following condition holds for every μ -almost every t

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \mathbb{E}_{0,0} (U^k V^\ell f(t)) \text{ converges in } \mathbb{L}^1(\mathcal{X}). \quad (5.1.1)$$

Then the coboundary decomposition holds,

$$f(t) = m(t) + (I - U)g_1(t) + (I - V)g_2(t) + (I - U)(I - V)g_3(t), \quad (5.1.2)$$

where m, g_1, g_2 and g_3 are in $\mathbb{L}^1(\mathcal{X})$, $(U^i V^j m(t))$ is a stationary field of orthomartingale differences, $(V^j g_1(t))_j$ is a stationary martingale differences sequence with respect to the filtration $(\mathcal{F}_{\infty, j})_j$, and $(U^i g_2(t))_i$ is a stationary martingale differences sequence with respect to the filtration $(\mathcal{F}_{i, \infty})_i$. Now setting

$$C_1(t) = \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\| \sum_{i=1}^n U^i f(t) \right\|_1, \quad (5.1.3)$$

$$C_2(t) = \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{i=1}^N V^i f(t) \right\|_1, \quad (5.1.4)$$

$$C_3(t) = \liminf_{n_1 \rightarrow \infty} \liminf_{n_2 \rightarrow \infty} \frac{1}{\sqrt{n_1 n_2}} \left\| S_{n_1, n_2}(f(t)) \right\|_1, \quad (5.1.5)$$

and

$$C_4(t) = \liminf_{n_2 \rightarrow \infty} \liminf_{n_1 \rightarrow \infty} \frac{1}{\sqrt{n_1 n_2}} \left\| S_{n_1, n_2}(f(t)) \right\|_1. \quad (5.1.6)$$

We are now in position to give the following result.

Théorème 5.1.1. *Let f be a measurable centered $\mathbb{L}^1(\mathcal{X})$ random variable. Let $S_{n_1, n_2}(f) := \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} U^i V^j f$. Assume that for μ -almost every t condition (5.1.1) holds*

$$C_i(t) < \infty \text{ and } \int C_i(t) \mu(dt) < \infty \text{ for every } i \in \{1, 2, 3, 4\}. \quad (5.1.7)$$

In addition, if

$$\lim_{\min(n_1, n_2) \rightarrow \infty} \int \frac{\mathbb{E}(|S_{n_1, n_2}(f(t))|)}{\sqrt{n_1 n_2}} \mu(dt) \text{ exists.} \quad (5.1.8)$$

Then, the random variables m , $(I - U)g_1$ and $(I - V)g_2$ defined in (5.1.2) are in $\mathbb{L}^2(\mathcal{X})$, and as $\min(n_1, n_2) \rightarrow \infty$, $(n_1 n_2)^{-1/2} S_{n_1, n_2}(f)$ converges in distribution to a centered normal variable.

As an immediate consequence of the Theorem 5.1.1, we get the following Corollary.

Corollary 5.1.2. *Assume that (5.1.8) and (5.1.1) hold and*

$$\int \sqrt{\sum_{k \geq 0} \sum_{\ell \geq 0} \|\max\{1, |f(t)|\} \|\mathbb{E}_{0,0}(U^k V^\ell f(t))\|_1} \mu(dt) < \infty. \quad (5.1.9)$$

Then the conclusion of Theorem 5.1.1 holds.

5.1.1 Empirical distribution

In this section, $S = \mathbb{R}$ and μ is a σ -finite measure on \mathbb{R} equipped with the Borel σ -field. Let $\mathbb{X} = (X_k)_{k \in \mathbb{Z}}$ be a stationary and ergodic sequence of real-valued random variables. Let \mathbb{X}_ℓ , $\ell \in \mathbb{Z}$ be independent copies of \mathbb{X} .

As an example, let us consider $(\xi_{k,\ell})_{k,\ell \in \mathbb{Z}}$ be an i.i.d random field, let h be a measurable function and $(X_{0,\ell})_{\ell \in \mathbb{Z}}$ a sequence of i.i.d. random variables such that $(X_{0,\ell})_{\ell \in \mathbb{Z}}$ is independent of $(\xi_{k,\ell})_{k \geq 1}$. Then for any $\ell \in \mathbb{Z}$ and any $k \geq 1$ we define

$$X_{k,\ell} = h(X_{k-1,\ell}, \xi_{k,\ell}).$$

For each ℓ fixed, $(X_{k,\ell})_{k \geq 1}$ forms a Markov chain and assume that it has a stationary distribution. Next, from the stationary random field $(X_{k,\ell})$ we define $Y_{k,\ell}(t) = \mathbb{1}_{X_{k,\ell} \leq t} - F(t) \forall t \in \mathbb{R}$, where F is the cumulative distribution function of $X_{0,0}$. Let $\mathcal{F}_{i,j} = \sigma\{X_{k,\ell}, k \leq i, j \leq \ell\}$. In this setting, for non negative integers k and ℓ ,

$$\mathbb{E}(Y_{k,\ell}(t) | \mathcal{F}_{0,0}) = \begin{cases} \mathbb{E}(Y_{k,0}(t) | \mathcal{F}_{0,0}) & \text{if } \ell = 0, \\ = 0 & \text{otherwise.} \end{cases} \quad (5.1.10)$$

Then conditions (5.1.1) and (5.1.9) hold as soon as

$$\sum_{k \geq 0} \int \|\mathbb{E}(Y_{k,0}(t) | \mathcal{F}_{0,0})\|_1 \mu(dt) < \infty, \quad (5.1.11)$$

and

$$\int \sqrt{\sum_{k \geq 0} \|\mathbb{E}(Y_{k,0}(t)|\mathcal{F}_{0,0})\|_1} \mu(dt) < \infty. \quad (5.1.12)$$

To apply Corollary 5.1.2, it remains to prove (5.1.8). To do so, notice first that due to the independence in rows, [21, Th 1] applies, and [21, condition (2.3)] reads as

$$\sum_{k > 0} \left\| Y_{0,0}(t) |\mathbb{E}(Y_{k,0}(t)|\mathcal{F}_{0,0}) \right\|_1 < \infty. \quad (5.1.13)$$

Hence by (5.1.11), for μ -almost every t ,

$$\frac{1}{\sqrt{n_1 n_2}} S_{n_1, n_2}(Y(t)) \Rightarrow G(t),$$

where $G(t)$ is a Gaussian-random variable with variance $\sigma^2(t) = \sum_{k \geq 1} \sum_{\ell \in \mathbb{Z}} \mathbb{E}(Y_{0,0}(t)Y_{k,\ell}(t))$. Moreover

$$\lim_{\min(n_1, n_2) \rightarrow \infty} \left\| \frac{1}{\sqrt{n_1 n_2}} S_{n_1, n_2}(Y(t)) \right\|_2^2 = \sigma^2(t).$$

Hence by the convergence of moments Theorem, $\left(\frac{1}{\sqrt{n_1 n_2}} S_{n_1, n_2}(Y(t)) \right)^2$ is uniformly integrable. By the convergence of moments Theorem again, it follows that for μ -almost every t ,

$$\lim_{\min(n_1, n_2) \rightarrow \infty} \left\| \frac{1}{\sqrt{n_1 n_2}} S_{n_1, n_2}(Y(t)) \right\|_1 = \|G(t)\|_1 = \sqrt{\frac{2}{\pi}} \sigma(t). \quad (5.1.14)$$

To end the proof of (5.1.8), note that by stationarity,

$$\int \sigma(t) \mu(dt) \leq \int 2 \sqrt{\sum_{k \geq 0} \|\mathbb{E}(Y_{k,0}(t)|\mathcal{F}_{0,0})\|_1} \mu(dt),$$

which is finite by (5.1.12).

In addition, we define

$$\alpha_k = \sup_{t \in \mathbb{R}} \|\mathbb{E}(\mathbf{1}_{X_k \leq t} | \mathcal{F}_{0,0}) - F(t)\|_1$$

As a direct consequence we get the following corollary

Corollary 5.1.3. *Let $B(t) = F(t)(1 - F(t))$. Assume that*

$$\int \sqrt{\sum_{k \geq 1} \min(\alpha_k, B(t))} \mu(dt) < \infty. \quad (5.1.15)$$

Then the conclusion of theorem 5.1.1 holds.

5.2 Proof

5.2.1 Proof of Theorem 5.1.1

First, we will prove that $m \in \mathbb{L}^2(\mathcal{X})$. Now notice that by (5.1.1) the following decomposition holds for μ -almost every t

$$f(t) = m(t) + (I - U)g_1 + (I - V)g_2 + (I - U)(I - V)g_3 \quad (5.2.1)$$

that for μ -almost every t , all the condition in Theorem 3.1.3 are satisfied. Therefore, we deduce that

$$\mathbb{E}(m(t)^2) + \mathbb{E}(((I - U)g_1(t))^2) + \mathbb{E}(((I - V)g_2(t))^2) < \infty. \quad (5.2.2)$$

For μ -almost every t , $(U^i V^j m(t))_{i,j}$ is a stationary and ergodic fields of orthomartingale differences, so $(V^j m(t))_j$ is a stationary and ergodic sequence of martingale differences with finite variance. By the CLT for stationary and ergodic martingales with finite variance, as $N \rightarrow \infty$, $N^{-1/2} \sum_{i=1}^N U^i m(t)$ converges in distribution to a centered Gaussian random variable $G(t)$ with variance $\|m(t)\|_2^2$. Hence, by [1,

Th. 3.4] and noticing that $\mathbb{E}|G(t)| = \|m(t)\|_2 \sqrt{2/\pi}$, we get

$$\|m(t)\|_2 \leq \sqrt{\frac{\pi}{2}} \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{j=1}^N V^j m(t) \right\|_1. \quad (5.2.3)$$

Now notice that, for any N fixed, $(U^i \frac{1}{\sqrt{N}} \sum_{j=1}^N V^j m(t))_i$ is also a stationary and ergodic sequence of martingale differences. Therefore applying TCL for martingale and [1, Th. 3.4] again, we get

$$\liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{j=1}^N V^j m(t) \right\|_2 \leq \sqrt{\frac{\pi}{2}} \liminf_{n \rightarrow \infty} \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{nN}} \left\| \sum_{i=1}^n \sum_{j=1}^N U^i V^j m(t) \right\|_1. \quad (5.2.4)$$

Following the prof of Theorem 3.1.5, we deduce that

$$\liminf_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left\| \sum_{j=1}^N V^j m(t) \right\|_2 \leq \sqrt{\frac{\pi}{2}} \liminf_{n \rightarrow \infty} \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{nN}} \left\| \sum_{i=1}^n \sum_{j=1}^N U^i V^j f(t) \right\|_1. \quad (5.2.5)$$

Since $\|m(t)\|_2 < \infty$, combining with (5.2.3), (5.2.4) and (5.2.5) we get that

$$\|m(t)\|_2 \leq \sqrt{\frac{\pi}{2}} \liminf_{n \rightarrow \infty} \liminf_{N \rightarrow \infty} \frac{1}{\sqrt{nN}} \left\| \sum_{i=1}^n \sum_{j=1}^N U^i V^j f(t) \right\|_1. \quad (5.2.6)$$

Hence by (5.1.7)

$$\int \|m(t)\|_2 \mu(dt) \leq \int C_3(t) \mu(dt) < \infty. \quad (5.2.7)$$

Therefore, $m \in \mathbb{L}^2(\mathcal{X})$. Next, we will prove that $(I - V)g_2 \in \mathbb{L}^2(\mathcal{X})$ for μ -almost every t let

$$m'(t) = m(t) + (I - V)g_2(t). \quad (5.2.8)$$

One can see that $(U^i m'(t))_i$ is a stationary and ergodic sequence of martingale differences. By (5.2.2) we are in position to apply TLC for martingale and [1, Th. 3.4]

again, we get

$$\|m'(t)\|_2 \leq \sqrt{\frac{\pi}{2}} \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\| \sum_{i=1}^n U^i m'(t) \right\|_1.$$

Since $\|g_1(t)\| < \infty$, by decomposition (5.2.2), we deduce that

$$\|m'(t)\|_2 \leq \sqrt{\frac{\pi}{2}} \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left\| \sum_{i=1}^n U^i f(t) \right\|_1. \quad (5.2.9)$$

Therefore, by (5.1.7)

$$\int m'(t) \mu(dt) \leq \int C_1(t) \mu(dt) < \infty. \quad (5.2.10)$$

Then, by (5.2.8), (5.2.7), and (5.2.10) we deduce that

$$\int (I - V)g_2(t) \mu(dt) < \infty. \quad (5.2.11)$$

Repeating the above argument with

$$m''(t) = m(t) + (I - U)g_1(t),$$

we get that

$$\int (I - V)g_1(t) \mu(dt) < \infty. \quad (5.2.12)$$

Therefore, m , $(I - V)g_2$, $(I - U)g_1$ are in $\mathbb{L}^2(\mathcal{X})$, hence the desired result follows by applying theorem 4.3.4 to $(U^i V^j m)_{i,j}$, if we can prove the martingale approximation, namely

$$\lim_{\min(n_1, n_2) \rightarrow \infty} \int \frac{\|S_{n_1, n_2}(f(t)) - S_{n_1, n_2}(m(t))\|_1}{\sqrt{n_1 n_2}} \mu(dt) = 0. \quad (5.2.13)$$

Now following the arguments used in the proof of Theorem 3.1.5, we get that for μ -almost every t

$$\kappa := \limsup_{n \rightarrow \infty} \frac{\|(I - U^n)g_1(t)\|_2}{\sqrt{n}} \leq \sqrt{\frac{\pi}{2}} \left(\|m(t)\|_2 + \lim_{n, N \rightarrow \infty} \frac{1}{\sqrt{nN}} \|S_{n,N}(f(t))\|_1 \right). \quad (5.2.14)$$

Therefore, using (5.1.8), (5.2.7), (5.2.13) follows from (3.3.15) to (3.3.21) in the proof of Theorem 3.1.5. \square

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