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# Topological dynamics of groupoids for the study of randomizations

Jorge Muñoz Carvajal

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## **Dynamique topologique de groupoïdes pour l'étude des aléatorisations**

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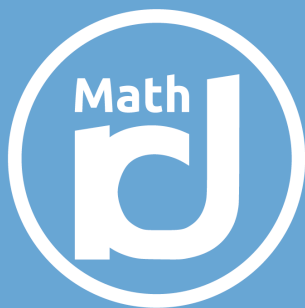
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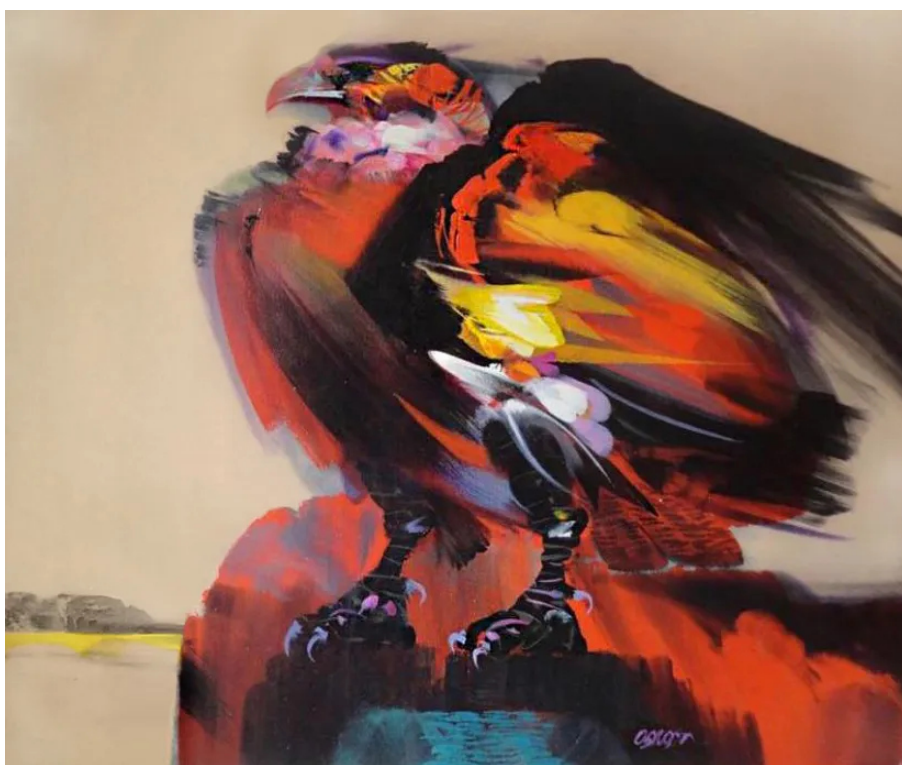




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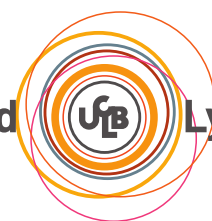
# Dynamique topologique de groupoïdes pour l'étude des aléatorisations



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# Introduction en français

Les aléatorisations ont été définies par H.J. Keisler dans [Kei99] comme des structures classiques<sup>1</sup> à trois sortes, une sorte pour des éléments aléatoires, une autre pour des événements et la troisième pour des scalaires. La dernière sorte interagissait avec celle des événements via la fonction de mesure. Les aléatorisations ont été réintroduites en utilisant le formalisme de la logique continue dans [BYK09]. Ici, la sorte des scalaires est omise et la mesure est considérée comme un prédicat unaire. Cela fournit un autre avantage, la classe d'algèbres de probabilité n'est pas élémentaire en logique classique ordre alors que dans le formalisme continue leur théorie est stable.

D'après [BYK09], étant donné un langage classique  $\mathcal{L}$  et une  $\mathcal{L}$ -structure  $\mathcal{M}$ , une aléatorisation de  $\mathcal{M}$  est une structure continue à deux sortes  $(\mathcal{K}, \mathcal{B})$ , où  $\mathcal{B}$  est l'algèbre d'événements d'un espace de probabilité sans atomes  $(\Omega, \mathcal{B}, \mu)$  et  $\mathcal{K}$  est un ensemble de fonctions sur  $\Omega$  à valeurs dans  $\mathcal{M}$ . Chaque  $\mathcal{L}$ -formule  $\varphi(x)$  définit une fonction dans le langage de l'aléatorisation,  $\llbracket \varphi(\cdot) \rrbracket : \mathcal{K}^{|x|} \rightarrow \mathcal{B}$  donnée par

$$\llbracket \varphi(a) \rrbracket = \{\omega \in \Omega ; \mathcal{M} \models \varphi(a_\omega)\}$$

pour  $a \in \mathcal{K}^{|x|}$ . Il est aussi nécessaire que les formules existentielles aient des témoins et que l'algèbre  $\mathcal{B}$  soit égale à l'ensemble de tous les événements de la forme  $\llbracket \varphi(a) \rrbracket$ .

Étant donnée une théorie classique  $T$ , on définit la théorie de l'aléatorisation,  $T^R$ , comme la théorie commune à toutes les aléatorisations de modèles de  $T$ . Cette théorie admet une axiomatisation simple ainsi que l'élimination des quantificateurs [BYK09, Theorem 2.9]. Elle préserve aussi plusieurs propriétés de la théorie originale, telles que la complétude, la  $\omega$ -catégoricité et la stabilité [BYK09, Theorems 2.1, 3.2 et 5.14]. Ici on remarque que pour la préservation de la stabilité, les auteurs ont montré que si  $\varphi(x, y)$  est une formule stable alors les  $\mu\llbracket \varphi \rrbracket$ -types sont définissables, obtenant ainsi que  $\mu\llbracket \varphi(x, y) \rrbracket$  est aussi stable.

Dans ce contexte, des travaux postérieurs, notamment d'U. Andrews, I. Goldbring et H. J. Keisler, ont étudié certaines propriétés modèle-théoriques des aléatorisations, comme dans [AGK15, AK15, AGK19]. La première mention de l'aléatorisation d'une théorie continue apparaît dans [BY09]. Ici, l'aléatorisation d'une théorie continue est définie comme la théorie continue ayant pour espace de types en  $n$  variables,  $S_n(T^R)$ , l'espace de mesures de Borel de probabilité sur  $S_n(T)$ , muni de la topologie faible\*. Dans cet article, l'auteur montre que si une théorie est NIP alors son aléatorisation  $T^R$  l'est aussi. L'idée de la preuve est d'étendre la correspondance entre NIP et VC-dimension au cadre continu et montrer que l'aléatorisation se comporte bien par rapport à cette notion. La démonstration est néanmoins très technique et des différentes

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<sup>1</sup>Quand on dit qu'une structure, langage, etc. est classique, on veut dire dans la logique du premier ordre à valeurs dans  $\{0, 1\}$ .

tentatives ont été faites pour trouver une preuve alternative.

Ultérieurement, dans [BY13], I. Ben Yaacov donne une présentation beaucoup plus gentille de l'aléatorisation d'une théorie continue. Ici, la sorte d'événements est remplacée par une sorte de variables aléatoires à valeurs dans  $[0, 1]$ . Le langage de l'aléatorisation,  $\mathcal{L}^R$ , est aussi simplifié. Les symboles de fonction du langage original  $\mathcal{L}$  sont préservés, la fonction étant définie point par point sur des uplets d'éléments aléatoires, et chaque symbole de prédicat  $n$ -aire  $P$  de  $\mathcal{L}$  définit un symbole de fonction  $\llbracket P(\cdot) \rrbracket$  dans  $\mathcal{L}^R$ , qui associe à chaque  $n$ -uplet d'éléments aléatoires une variable aléatoire définie par la valeur point par point du prédicat. Dans ce contexte, chaque  $\mathcal{L}$ -formule  $\varphi(x)$  donne lieu à une fonction définissable  $\llbracket \varphi(\cdot) \rrbracket$  définie sur l'ensemble de  $|x|$ -uplets d'éléments aléatoires et à valeurs dans la sorte de variables aléatoires.

L'exemple le plus simple d'une aléatorisation est l'aléatorisation de Borel d'une structure  $\mathcal{M}$ , dénotée  $\mathcal{M}^R$ . On prend un espace de probabilité standard sans atomes  $(\Omega, \mathcal{B}, \mu)$ , on considère toutes les fonctions mesurables de  $\Omega$  dans  $M$  dans la sorte d'éléments aléatoires et  $L_1((\Omega, \mathcal{B}, \mu), [0, 1])$  dans la sorte de variables aléatoires. Dans [BY13], les résultats de préservation de [BYK09] sont montrés pour l'aléatorisation de structures continues. En particulier, la préservation de la stabilité est prouvée en montrant que pour toute  $\mathcal{L}$ -formule stable  $\varphi(x, y)$ , les types dans l'aléatorisation au-dessus de modèles sont  $\mathbb{E}\llbracket \varphi(x, y) \rrbracket$ -définissables.

## Connexité des aléatorisations de groupes

La première partie de ce manuscrit s'occupe de la notion de connexité des aléatorisations de groupes. On dit qu'un groupe définissable est définissablement connexe s'il n'a pas de sous-groupes type-définissables, propres, d'indice borné, quand on l'analyse dans un modèle montre. Avec A. Berenstein dans [BM21], l'auteur de cette thèse avait montré que si  $G$  est un groupe NIP, classique, définissablement moyennable alors  $G^R$ , l'ensemble de fonctions aléatoires à valeurs dans  $G$ , est définissablement connexe. La preuve a été menée à travers un étude de types génériques. Par exemple, lorsque la théorie est  $\omega$ -stable, il y a un nombre fini de génériques dans le groupe de départ, disons  $n$ , le type générique dans l'aléatorisation sera donc l'élément aléatoire prenant chaque valeur générique sur un ensemble de mesure  $1/n$ .

Ici, on généralise ce résultat et on se débarrasse des hypothèses modèle-théoriques. On montre que si  $G$  est un groupe définissable dans une théorie continue, alors  $G^R$  est définissablement connexe. On montre même que  $G^R$  n'a pas de sous-groupes propres, invariants, d'indice borné. C'est à dire que pour tout petit ensemble de paramètres  $A$ ,  $(G^R)_A^\infty = G^R$ . Il y a deux ingrédients principaux pour cette preuve. D'abord, la construction de l'espace homogène. Deuxièmement, le fait que dans les aléatorisations tous les types au-dessus des paramètres provenant des sortes principales sont Lascar fort [BY13, Theorem 5.8]. On rappelle qu'avoir le même type de Lascar est la plus fine relation d'équivalence bornée, invariante.

On prend la présentation de l'espace homogène donnée dans [GN08] et on l'étend au cas continu. L'idée est de prendre un groupe définissable  $G$  dans une structure suffisamment saturée  $\mathcal{C}$ , ajouter une copie affine de  $G$  comme une sorte et ajouter au langage l'action de  $G$  sur cette copie. On dénote  $\tilde{\mathcal{C}}$  cette nouvelle structure. D'après [GN08], on a  $\text{Aut}(\tilde{\mathcal{C}}) = G \rtimes \text{Aut}(\mathcal{C})$ . De plus, si  $\text{Autf}_L(\tilde{\mathcal{C}}/A)$  dénote le groupe d'automorphismes Lascar fort au-dessus de  $A$ , c'est à dire le groupe d'automorphismes fixant tous les types de Lascar au-dessus de  $A$ , alors pour tout petit ensemble de paramètres  $A$ ,  $G_A^\infty = G \cap \text{Autf}_L(\tilde{\mathcal{C}}/A)$ . Cependant, dans l'aléatorisation tous les automorphismes sont Lascar fort. Par conséquent, quand on applique ces résultats à  $G^R$ , on

déduit que dans l'aléatorisation  $(G^R)_A^\infty$  est égal à  $G^R$ .

## Dynamique topologique et l'aléatorisation

On discute à présent les interprétations et les groupes d'automorphismes. Soient  $\mathcal{M}$  et  $\mathcal{N}$  deux structures continues. On suppose que l'on a une interprétation de  $\mathcal{N}$  dans  $\mathcal{M}$ . Un automorphisme de  $\mathcal{M}$  induit un automorphisme de  $\mathcal{N}$ . L'application qui en résulte de  $\text{Aut}(\mathcal{M})$  dans  $\text{Aut}(\mathcal{N})$  est un homomorphisme de groupes topologiques. Une bi-interprétation entre  $\mathcal{M}$  et  $\mathcal{N}$  consiste d'une paire d'interprétations, d'une structure dans l'autre, telle que lorsque l'on compose ces interprétations le modèle est définissablement isomorphe à sa copie interprétée. Si les deux structures sont  $\omega$ -catégoriques alors elles sont bi-interprétables si et seulement si leurs groupes d'automorphismes sont isomorphes comme groupes topologiques. Ce résultat, attribué à T. Coquand, a été publié pour la première fois dans [AZ86] et ensuite généralisé aux structures continues dans [BYK16]. Ainsi, pour des structures  $\omega$ -catégoriques, plusieurs propriétés modèlles-théoriques, au moins celles préservées par bi-interprétation, peuvent être obtenues à partir de propriétés dynamiques du groupe d'automorphismes.

Pour des structures  $\omega$ -catégoriques, I. Ben Yaacov et T. Tsankov ont étudié dans [BYT16] l'équivalent dynamique de la stabilité. On sait qu'une formule est stable dans une structure  $\mathcal{M}$  si elle n'a pas la propriété de l'ordre, suivant un argument de diagonalisation, comme dans [Gro52, Remarque 2], cela équivaut à une propriété d'échange de limites doubles. À savoir,  $\varphi(x, y)$  est stable dans  $\mathcal{M}$  si pour n'importe quelles suites  $(a_n)_{n \in \mathbb{N}}$  et  $(b_n)_{n \in \mathbb{M}}$  dans  $M$  on a

$$\lim_{n \in \mathbb{N}} \lim_{m \in \mathbb{N}} \varphi(a_n, b_m) = \lim_{m \in \mathbb{N}} \lim_{n \in \mathbb{N}} \varphi(a_n, b_m)$$

dès que les deux limites existent. Ici on peut voir le critère de la double limite de Grothendieck pour la compacité faible [Gro52, Théorème 6]. Cela a permis à I. Ben Yaacov et à T. Tsankov d'exprimer la stabilité de  $\varphi$  en termes de la presque-périodicité faible, que l'on procède à définir.

Soit  $G$  un groupe topologique et  $X$  un  $G$ -espace, une fonction  $f \in C_b(X)$  est dite faiblement presque-périodique, ou wap par son nom en anglais, si  $\{gf ; g \in G\}$  est relativement faiblement compact dans  $C_b(X)$ . M. Megrelishvili avec E. Glasner ont remarqué que en représentant un système dynamique sur un espace de Banach, les propriétés géométriques de cet espace nous donnent des informations sur les propriétés dynamiques du système. En fait, ils ont établi une hiérarchie selon laquelle la représentabilité d'une action dans une classe particulière d'espaces de Banach a son équivalent topologique dans le domaine de systèmes dynamiques [GM14]. D'après [Meg03, Théorème 4.6] on obtient que les fonctions wap sont précisément celles qui proviennent de représentations de l'action sur d'espaces de Banach réflexifs.

Ce résultat a été exploité par T. Ibarlucía dans [Iba17] pour donner une preuve alternative de la préservation de la stabilité par l'aléatorisation dans le cas  $\omega$ -catégorique<sup>2</sup>. D'abord, il a obtenu  $\text{Aut}(\mathcal{M}^R)$ , le groupe d'automorphismes de l'aléatorisation de Borel de  $\mathcal{M}$ , en termes de  $\text{Aut}(\mathcal{M})$  pour toute structure séparable  $\mathcal{M}$ . Ensuite, il a montré que tout ensemble de  $\mathcal{L}$ -formules  $\Delta(x, y)$  engendre une famille de  $\mathcal{L}^R$ -formules  $\Delta^R$ , contenant en particulier les formules  $\mathbb{E}[\varphi(x, y)]$  avec  $\varphi(x, y) \in \Delta$ , pour laquelle l'espace de types  $S_{\Delta^R}(M^R)$  peut être facilement décrit à partir de  $S_{\Delta}(M)$ . Finalement, il a montré que si  $S_{\Delta}(M)$  admet une représentation fidèle sur

<sup>2</sup>Voir le commentaire après le Corollaire 3.19. Dans le même travail, il a présenté une autre preuve de la préservation de la stabilité en utilisant la compactification wap d'un groupe, une méthode que l'on ne explore pas ici.

un espace réflexif alors  $S_{\Delta^R}(M^R)$  aussi. Il en découle que si  $\varphi(x, y)$  est stable dans une structure  $\omega$ -catégorique  $\mathcal{M}$  alors  $\mathbb{E}[\varphi(x, y)]$  l'est aussi dans  $\mathcal{M}^R$ .

D'un autre côté, T. Ibarlucía a établi dans [Iba16] les équivalents modèle-théoriques de la hiérarchie précédemment mentionnée. Il a remarqué que NIP correspond à docilité (*tameness* en anglais) laquelle, par [GM12, Theorems 6.7 et 6.9], correspond à la représentabilité dans la classe d'espaces de Rosenthal, des espaces ne contenant pas une copie isomorphe de  $l_1$ . En suivant une procédé similaire, il a montré la préservation de NIP par l'aléatorisation dans le cadre  $\omega$ -catégorique.

La motivation derrière ce projet était de trouver une preuve dynamique de la préservation de NIP par l'aléatorisation dans un cadre plus général que  $\omega$ -catégoricité. En tenant en compte que les résultats impliqués dans la préservation de la stabilité par des méthodes dynamiques sont plus simples, on a décidé de étudier ce problème d'abord, en espérant passer après au cas NIP. Néanmoins, les méthodes utilisées ont été beaucoup plus compliquées que l'on ne le espérait.

D'abord, on a besoin d'un objet, analogue au groupe d'automorphismes d'une structure  $\omega$ -catégorique, qui nous permette de considérer tous les modèles séparables d'une théorie. Cela a été fait par I. Ben Yaacov dans [BY22]. Ici, il a défini la notion de sorte de Skolem universelle. Il a montré que l'on peut associer à toute théorie complète, dénombrable, admettant une telle sorte, un groupoïde Polonais au-dessus  $2^{\mathbb{N}}$  qui détermine la théorie, à bi-interprétation près. Parmi les théories ayant une sorte de Skolem universelle on peut compter les théories classiques et les théories  $\omega$ -catégoriques. On sait cependant que pas toutes les théories admettent une telle sorte [BY22, Example 4.15].

Une sorte  $D$  est dite Skolem pour une autre sorte  $E$  si toute formule en  $D \times E$  admet des fonctions de Skolem approximatives définissables.  $D$  est universelle pour  $E$  si pour tout  $\epsilon > 0$  on peut trouver une application définissable de  $D$  dans  $E$  telle que l'image de toute  $\epsilon$ -boule dans  $D$  est  $\epsilon$ -dense dans  $E$ . Une sorte  $D$  est Skolem universelle si les deux conditions sont vérifiées pour toute sorte  $E$ . De cette façon, une sorte de Skolem nous permet d'obtenir des témoins approximatifs pour les quantificateurs existentiels, tandis que l'universalité nous laisse récupérer tous ces témoins, voir [BY22, Definition 4.6 et Lemma 4.8]. Il en découle que chaque élément d'une sorte de Skolem universelle est interdéfinissable avec un modèle séparable. Le groupoïde associé à la théorie est l'ensemble de types des couples de la sorte de Skolem universelle encodant le même modèle.

Afin d'utiliser cet outil pour montrer les résultats de préservation souhaités, on doit montrer que si une théorie complète, dénombrable  $T$  admet une sorte de Skolem universelle alors son aléatorisation aussi. Cela est précisément fait dans la Section 2.2. On explique brièvement sa construction. Dans l'aléatorisation il y a deux types de sortes, les sortes principales et la sorte auxiliaire provenant de ARV, la théorie de variables aléatoires sans atomes, laquelle comme théorie  $\omega$ -catégorique admet une sorte de Skolem universelle. Il est donc prévisible que dans la sorte de Skolem universelle de l'aléatorisation les deux types de sortes jouent un rôle. En effet, la sorte cherchée est un produit fibré entre l'aléatorisation de la sorte de Skolem universelle de la théorie et la sorte de Skolem universelle de ARV.

Le pas suivant est la description du groupoïde associé à l'aléatorisation en termes du groupoïde associé à la théorie de départ. On rappelle de [Iba17] que pour toute structure séparable  $\mathcal{M}$ ,  $\text{Aut}(\mathcal{M}^R) = \text{Aut}(\mathcal{M})^{\Omega} \rtimes \text{Aut}(\Omega)$ . Ici,  $\Omega$  est un espace de probabilité standard sans atomes,

$\text{Aut}(\mathcal{M})^\Omega$  est la famille de fonctions mesurables de  $\Omega$  dans  $\text{Aut}(\mathcal{M})$  et  $\text{Aut}(\Omega)$  est le groupe d'applications inversibles sur  $\Omega$  qui préservent la mesure. Le groupoïde que l'on obtient est similaire. On montre dans la [Section 2.4](#) que si  $\mathbf{G}(T)$  est le groupoïde associé à  $T$  alors  $\mathbf{G}(T^R) = (\mathbf{G}(T)^\Omega \rtimes \text{Aut}(\Omega))_\pi$ , le changement de base de  $\mathbf{G}(T)^\Omega \rtimes \text{Aut}(\Omega)$  à  $2^\mathbb{N}$  donné par la fonction  $\pi : 2^\mathbb{N} \rightarrow (2^\mathbb{N})^\Omega$ , obtenue à partir de la fonction utilisé pour définir le produit fibré dans la sorte de Skolem universelle de  $T^R$ .

Maintenant on doit décrire la stabilité dans des termes dynamiques, c'est à dire, en utilisant l'action du groupoïde associé à la théorie sur un espace de types. On suppose que  $T$  est complète, dénombrable, admettant une sorte de Skolem universelle,  $D_T$ , et soit  $F$  une sorte quelconque de  $T$ . Dans le cas  $\omega$ -catégorique, pour déterminer si une formule  $\varphi(x, y)$  est stable, il suffit d'étudier l'action  $G \curvearrowright S_x(M)$ , où  $\mathcal{M}$  est le modèle séparable de la théorie. Dans notre cas, puisque la sorte de Skolem universelle nous permet de considérer tous les modèles séparables de  $T$  à la fois, on étudie l'action  $\mathbf{G}(T) \curvearrowright S_{D_T \times F}(T)$ , ainsi que l'action du groupoïde sur l'espace de type partiels  $S_\Delta(T)$  pour  $\Delta$  un ensemble de formules en  $D_T \times F$ . De plus, dans le [Chapitre 4](#), si  $D_{T^R}$  dénote la sorte de Skolem universelle de  $T^R$ , on définit un ensemble de formules  $\Delta^R$  en  $D_{T^R} \times F^R$ , contenant  $\mathbb{E}[\varphi]$  pour tout  $\varphi \in \Delta$ , pour lequel on peut construire  $S_{\Delta^R}(T^R)$  à partir de  $S_\Delta(T)$ .

La traduction de la stabilité en termes de l'action  $\mathbf{G}(T) \curvearrowright S_{D_T \times F}(T)$  a été notre source d'inspiration pour la définition de fonction faiblement presque-périodique dans le contexte d'actions de groupoïdes, voir [Section 3.6](#). Maintenant, on doit trouver une structure similaire aux espaces de Banach, ayant une notion de dual, et sur laquelle on peut représenter les actions de groupoïdes. C'est ainsi que l'on utilise les champs d'espaces de Banach. Pour chaque élément de la base du groupoïde on a un espace de Banach et on les colle ensemble en utilisant une famille de sections. Celle-ci n'est pas une nouvelle idée, en fait plusieurs de nos résultats et définitions proviennent de [\[DD63\]](#).

La dynamique topologique des actions de groupoïdes est un sujet très récent, avec une première présentation détaillée sortie en 2020 dans [\[FM20\]](#) suivie de [\[FM21\]](#). D'un autre côté, une notion de semigroupoïde d'Ellis pour des groupoïdes localement compacts a été proposé dans [\[AB21\]](#). Il y a eu quelques études précédents sur les représentations de groupoïdes, par exemple dans [\[Bos11\]](#), où un analogue du théorème de Peter-Weyl est prouvé pour des groupoïdes propres. Cependant, on n'a pas trouvé l'analogue de la hiérarchie dynamique pour les actions de groupoïdes. Celle-ci est la motivation derrière le [Chapitre 3](#).

On étudie d'abord le champ dual muni de la topologie faible\*. Ensuite, on étudie le champ de fonctions continues. On s'intéresse particulièrement à la topologie de convergence simple, pour laquelle on déduit un critère de compacité relative, à la façon de [\[Gro52\]](#). Après, on définit la notion d'action de groupoïde séparément continue et de dualité. Finalement, on propose une définition de représentation de une action de groupoïde sur un champ de Banach. On montre que les fonctions wap sont précisément celles qui proviennent des représentation sur des champs d'espaces réflexifs.

Pour finir, on doit montrer comment construire une représentation de l'aléatorisation d'une action de groupoïde à partir de d'une représentation de l'action originale. Inspirés par [\[Iba17, Section 3.3\]](#), on définit le champ  $L_2(\Omega, \mathbb{V})_\pi$ , où  $(\mathbb{V}, \Gamma)$  est un champ de Banach et  $\pi : 2^\mathbb{N} \rightarrow (2^\mathbb{N})^\Omega$  est l'application mentionné ci-dessus.  $L_2(\Omega, \mathbb{V})_\pi$  est un champ au dessus  $2^\mathbb{N}$ . On peut penser aux éléments dans la fibre au-dessus  $\alpha \in 2^\mathbb{N}$  comme des fonctions mesurables  $\hat{v} : \Omega \rightarrow \mathbb{V}$  satisfaisant

$\hat{v}(\omega) \in V_{\pi(\alpha)(\omega)}$  pour tout  $\omega \in \Omega$  et telles que leur  $L_2$ -norme,  $\|\hat{v}\|_2 = (\int_{\Omega} \|\hat{v}(\omega)\|^2 d\mu(\omega))^{1/2}$  est finie. Cependant, il nous reste encore de montrer que chaque fibre est réflexive, qui est certainement le cas quand un seul espace de Banach est considéré. Le problème est la difficulté au moment de caractériser son dual à cause d'une absence de résultats à la Radon-Nikodym.

Cette thèse pose une autre question, la caractérisation de la fonction  $\pi$ . Puisque cette application est utilisée pour la définition de la sorte de Skolem universelle, sa présence dans toutes les constructions qui en découlent n'est pas une surprise. On a présenté une condition suffisante pour que le produit fibré défini par  $\pi$  soit définissable, on ne sait pas si elle est aussi nécessaire. De plus, si  $\pi'$  est une autre application vérifiant cette condition, par l'unicité de la sorte de Skolem universelle, il existe une bijection définissable entre la sorte définie par  $\pi$  et celle définie par  $\pi'$ . Cela induit un isomorphisme entre les deux groupoïdes. On se demande si est-il possible de trouver explicitement l'isomorphisme dans des termes topologiques.

# Introduction

## Background

The first attempts to define a  $[0, 1]$ -valued logic can be traced back to C. C. Chang in the 1960's. His results were later generalized in a joint work with H. J. Keisler in [CK66]. In this monograph, any compact, Hausdorff space  $X$  was admitted as the set of truth values, connectors were continuous functions from  $X^n$  into  $X$  and quantifiers were set continuous functions on  $X$ . In this framework, classical model theoretical theorems, such as compactness and Löwenheim-Skolem, were generalized. This approach however was too general, in fact the authors described it as a “theory of model theories”, and it failed to gain much traction.

Later, taking as a starting point the construction of ultraproducts of Banach spaces, given for example in [BDCK66], C. W. Henson provided in [Hen75] a first approximation of a model theory for normed spaces. Subsequently, C.W. Henson and J. Iovino developed in [HI02] the framework of positive bounded formulas with approximate satisfaction in order to study the model theory of normed spaces. In this context a distinguished sort is reserved for the real numbers, a tool that was also used when randomizations were introduced. We can already see here a defining trait of continuous logic, there is no negation. Furthermore, the implication only appears as part of a bounded universal quantifier.

Another approach to a positive model theory was proposed by I. Ben Yaacov while developing a model-theoretical framework for hyperimaginaries in simple theories, see [BY05]. This setting, known as compact abstract theories, or cats for short, encompasses both classical logic<sup>3</sup> and positive bounded formulas with approximate satisfaction for Banach spaces. All these different approaches decanted in the definition of continuous logic, as presented in [BYU10]. The simplicity of its presentation together with its expressive power made it the suitable candidate for the study of the model theory of metric spaces, as witnessed by the abundance of work developed therein, the present thesis being just an example. We refer to reader to [BYBHU08] for a detailed presentation on continuous logic.

Randomizations of classical structures were originally introduced by H. J. Keisler in [Kei99] as three-sorted classical structures, one sort for random elements, one for events and one for scalars. The latter interacted with the sort of events via the measure map. The continuous framework, in which the distance is part of the language and the predicates are uniformly continuous  $[0, 1]$ -valued functions, made superfluous the introduction of the sort of real numbers. This provides another advantage, the class of probability algebras is not elementary in classical first order logic while in the continuous formalism their theory is stable. Keisler's randomization was thus reintroduced as a continuous structure in [BYK09].

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<sup>3</sup>By classical logic we mean first order  $\{0, 1\}$ -valued logic. Likewise, whenever we say that a structure, language etc. is classical, we mean that it is considered in this logic.

Following [BYK09], given a classical language  $\mathcal{L}$  and an  $\mathcal{L}$ -structure  $\mathcal{M}$ , a randomization of  $\mathcal{M}$  is a two sorted continuous structure  $(\mathcal{K}, \mathcal{B})$  where  $\mathcal{B}$  is the algebra of events of an atomless probability space  $(\Omega, \mathcal{B}, \mu)$  and  $\mathcal{K}$  is a set of functions from  $\Omega$  into  $M$ . Each  $\mathcal{L}$ -formula  $\varphi(x)$  defines a function in the randomization language  $\llbracket \varphi(\cdot) \rrbracket : \mathcal{K}^{|x|} \rightarrow \mathcal{B}$  given by

$$\llbracket \varphi(a) \rrbracket = \{\omega \in \Omega ; \mathcal{M} \models \varphi(a_\omega)\}$$

for  $a \in \mathcal{K}^{|x|}$ . It is also required that the existential formulas have witnesses and that the algebra  $\mathcal{B}$  is equal to the set of all events of the form  $\llbracket \varphi(a) \rrbracket$ .

Given a classical theory  $T$  we define the randomization theory  $T^R$  as the common theory of randomizations of models of  $T$ . This theory can be nicely axiomatized and it admits quantifier elimination [BYK09, Theorem 2.9]. It preserves many of the properties of the original theory, such as completeness,  $\omega$ -categoricity and stability [BYK09, Theorems 2.1, 3.2 and 5.14]. Here we remark that for the preservation of stability, the authors showed that if  $\varphi(x, y)$  is a stable formula of a theory  $T$  then the  $\mu\llbracket \varphi(x, y) \rrbracket$ -types are definable, obtaining in this manner that  $\mu\llbracket \varphi(x, y) \rrbracket$  is a stable formula of  $T^R$ .

In this setting, subsequent works, notably of U. Andrews, I. Goldbring and H. J. Keisler, studied several model-theoretical properties of randomizations, as in [AGK15, AK15, AGK19]. The first mention of the randomization of a continuous theory appears in [BY09], therein the randomization of a continuous theory is defined as the continuous theory whose space of types on  $n$  variables,  $S_n(T^R)$ , is the space of Borel probability measures on  $S_n(T)$ , endowed with the weak\* topology. In this article the author shows that if a theory  $T$  is NIP then its randomization  $T^R$  is also NIP. The idea of the proof is to extend the correspondence between NIP and VC-dimension to the continuous setting and to show that the randomization is well-behaved with respect to this notion. The demonstration is however highly technical and different attempts has been made to find an alternative one.

Afterwards, in [BY13] I. Ben Yaacov gives a much more gentle presentation of the randomization of a continuous theory. Here, the sort of events is replaced by a sort of  $[0, 1]$ -valued random variables. The randomization language,  $\mathcal{L}^R$ , is also simplified. The function symbols of the original language  $\mathcal{L}$  are preserved in the randomization, the function being defined pointwise on tuples of random elements, and each  $n$ -ary predicate symbol  $P$  of  $\mathcal{L}$  defines a function symbol  $\llbracket P(\cdot) \rrbracket$  in  $\mathcal{L}^R$ , that to each  $n$ -tuple of random elements assigns a random variable defined by the pointwise value of the predicate. In this setting, each  $\mathcal{L}$ -formula  $\varphi(x)$  gives rise to a definable function  $\llbracket \varphi(\cdot) \rrbracket$  from the set of  $|x|$ -tuples of random elements into the sort of random variables.

The simplest example of a randomization is the Borel randomization of a structure  $\mathcal{M}$ . Pick an atomless standard probability space  $(\Omega, \mathcal{B}, \mu)$ , consider all measurable functions from  $\Omega$  into  $M$  in the sort of random elements and  $L_1((\Omega, \mathcal{B}, \mu), [0, 1])$  in the sort of random variables. In the same article the preservation results from [BYK09] are proven for the randomization of continuous theories. In particular, the preservation of stability is proven by showing that for any stable  $\mathcal{L}$ -formula  $\varphi(x, y)$ , types in the randomization over models are  $\mathbb{E}\llbracket \varphi(x, y) \rrbracket$ -definable.

## Connectedness of randomizations of groups

The first part of this manuscript deals with the notion of connectedness for the randomization of groups. We say that a definable group is definably connected if it has no proper type-definable

subgroups of bounded index, when looked inside a monster model. The author of the thesis together with A. Berenstein had shown in [BM21] that if  $G$  is a classical, definably amenable, NIP group then  $G^R$ , the set of random functions taking values in the group  $G$ , is definably connected. The proof was carried out via a study of generic types. For example, when the theory is  $\omega$ -stable, there are finitely many generics in the original group, say  $n$ , and the generic type of the randomization will be a random element taking each generic value on a set of measure  $1/n$ .

Here we generalize this result and we get rid of the model-theoretical hypothesis. We show that if  $G$  is a definable group in a continuous theory then  $G^R$  is definably connected. We even show that  $G^R$  does not admit proper, invariant subgroups of bounded index, this is, for any small set of parameters  $A$ ,  $(G^R)_A^\infty = G^R$ . There are two main ingredients for this proof. The first one is the construction of the homogeneous space. The second one is the fact that in randomizations, all types over parameters coming from the home sorts are Lascar strong [BY13, Theorem 5.8]. Recall that having the same Lascar type is the finest bounded, invariant equivalence relation on a sort.

We take the presentation of the homogeneous space given in [GN08] and we extend it to the continuous case. The idea is to take a definable group  $G$  on a sufficiently saturated structure  $\mathfrak{C}$ , add a sort consisting of an affine copy of  $G$  and add to the language the action of  $G$  on this copy. Let us denote by  $\tilde{\mathfrak{C}}$  this new structure. From [GN08] we deduce that  $\text{Aut}(\tilde{\mathfrak{C}}) = G \rtimes \text{Aut}(\mathfrak{C})$ . Moreover, if  $\text{Aut}_L(\tilde{\mathfrak{C}}/A)$  denotes the group of Lascar strong automorphisms over  $A$ , that is the group of automorphisms fixing all Lascar types over  $A$ , then for any small set of parameters  $A$ ,  $G_A^\infty = G \cap \text{Aut}_L(\tilde{\mathfrak{C}}/A)$ . However, in randomizations all automorphisms are Lascar strong. Hence, when we apply these results to  $G^R$  we conclude in Section 1.4 that in randomizations  $(G^R)_A^\infty$  is equal to  $G^R$ .

## Topological dynamics and randomizations

Let us now discuss interpretations and the groups of automorphisms. Let  $\mathcal{M}$  and  $\mathcal{N}$  be two continuous structures. Suppose that we have an interpretation of  $\mathcal{N}$  in  $\mathcal{M}$ . An automorphism of  $\mathcal{M}$  induces a uniquely defined automorphism of  $\mathcal{N}$ . The resulting map from  $\text{Aut}(\mathcal{M})$  into  $\text{Aut}(\mathcal{N})$  is a homomorphism of topological groups. A bi-interpretation between  $\mathcal{M}$  and  $\mathcal{N}$  consists of an interpretation of each structure in the other such that when these interpretations are composed the model is definably isomorphic to its interpreted copy. If both structures are  $\omega$ -categorical then they are bi-interpretable if and only if their automorphisms groups are isomorphic as topological groups. This result, attributed to T. Coquand, was first published in the discrete setting in [AZ86] and later generalized to continuous structures in [BYK16]. In this manner, for  $\omega$ -categorical structures, several model-theoretical properties, at least those preserved by bi-interpretation, can be obtained from dynamical properties of the automorphism group.

For  $\omega$ -categorical structures, I. Ben Yaacov and T. Tsankov studied in [BYT16] the dynamical counterpart of the notion of stability. We know that a formula is stable in a structure  $\mathcal{M}$  if it does not have the order property, following a diagonalization argument as illustrated for example in [Gro52, Remarque 2], this is equivalent to a property of interchangeability of double limits. Namely,  $\varphi(x, y)$  is stable in  $\mathcal{M}$  if for every couple of sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(b_m)_{m \in \mathbb{N}}$  in  $M$  we have

$$\lim_{n \in \mathbb{N}} \lim_{m \in \mathbb{N}} \varphi(a_n, b_m) = \lim_{m \in \mathbb{N}} \lim_{n \in \mathbb{N}} \varphi(a_n, b_m)$$

whenever both limits exist. Here we recognize Grothendieck's double limit criterion for weak compactness [Gro52, Théorème 6], which let I. Ben Yaacov and T. Tsankov express the stability of  $\varphi$  in terms of weakly almost periodicity, which we now proceed to define.

Let  $G$  be a topological group and  $X$  a  $G$ -space, a function  $f \in C_b(X)$  is said to be weakly almost periodic, or wap for short, if  $\{gf ; g \in G\}$  is relatively weakly compact in  $C_b(X)$ . M. Megrelishvili together with E. Glassner realized that by representing a dynamical system on a Banach space, the geometrical characteristics of said space give us some insight on the topological properties of the system. They established in fact a hierarchy according to which the representability of the action in a particular class of Banach spaces has its topological counterpart in the realm of dynamical systems [GM14]. From [Meg03, Theorem 4.6] we obtain that wap functions are precisely those that come from representations of the action  $G \curvearrowright X$  on reflexive Banach spaces.

This fact was exploited by T. Ibarlucía in [Iba17] to give an alternative proof of the preservation of stability by the randomization in the  $\omega$ -categorical case<sup>4</sup>. First he obtained  $\text{Aut}(\mathcal{M}^R)$  in terms of  $\text{Aut}(\mathcal{M})$  for any separable structure  $\mathcal{M}$ . Then he showed that any set of  $\mathcal{L}$ -formulas  $\Delta(x, y)$  gives rise to a set of  $\mathcal{L}^R$ -formulas  $\Delta^R$ , containing in particular the formulas  $\mathbb{E}[\varphi(x, y)]$  with  $\varphi(x, y) \in \Delta$ , for which the space of types  $S_{\Delta^R}(M^R)$  can be nicely described from  $S_{\Delta}(M)$ . Finally, he showed that if  $S_{\Delta}(M)$  admits a faithful representation on a reflexive space then so does  $S_{\Delta^R}(M^R)$ . Thus concluding that if  $\varphi(x, y)$  is stable, in a  $\omega$ -categorical structure  $\mathcal{M}$ , then so it is  $\mathbb{E}[\varphi(x, y)]$  in  $\mathcal{M}^R$ .

T. Ibarlucía in [Iba16] established the model-theoretic counterparts of the hierarchy previously mentioned. He remarked that NIP corresponds to tameness, which by [GM12, Theorems 6.7 and 6.9] corresponds in the dynamical hierarchy to representability in the class of Rosenthal spaces, Banach spaces not containing an isomorphic copy of  $l_1$ . Following an analogous procedure to the one just described, he showed the preservation of NIP by the randomization in the  $\omega$ -categorical setting.

The motivation behind this project was to find a dynamical proof of the preservation of stability and NIP by the randomization in a broader setting than  $\omega$ -categoricity. Having in mind that the results involved in the preservation of stability by dynamical methods are much simpler, we decided to tackle this problem first, hoping to then study the NIP case. However, the methods involved were more complicated than expected as we will see.

First, we need an object, analogous to the automorphism group of an  $\omega$ -categorical structure, that allows us to consider all separable models of a theory. This was done by I. Ben Yaacov in [BY22]. Therein, he defined the notion of a universal Skolem sort. He showed that to any countable, complete theory  $T$  admitting such a sort, we can associate a Polish groupoid over  $2^{\mathbb{N}}$  that determines the theory up to bi-interpretability. Among the theories having a universal Skolem sort we can count the classical theories and the  $\omega$ -categorical theories. It is known however that not all theories admit such a sort [BY22, Example 4.15].

A sort  $D$  is Skolem for another sort  $E$  if every formula in  $D \times E$  admits definable approximate Skolem maps.  $D$  is universal for  $E$  if for every  $\epsilon > 0$  we can find a definable map from  $D$  into  $E$  such that the image of every  $\epsilon$ -ball in  $D$  is  $\epsilon$ -dense in  $E$ . A sort  $D$  is universal Skolem if both conditions hold for any sort  $E$ . In this way, a sort being Skolem allow us to obtain approxi-

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<sup>4</sup>See the comment after Corollary 3.19. In the same article he presented another proof of the preservation of stability using the wap compactification of a group, a path that we do not pursue here

mate witnesses of existential quantifiers, while universality let us recover all such witnesses, see [BY22, Definition 4.6 and Lemma 4.8]. We have that every element of a universal Skolem sort is interdefinable with a separable model. The groupoid associated to the theory is the set of types of all couples in the universal Skolem sort coding the same model.

In order to use this tool to show the desired preservation results, we must show that if a complete, countable theory  $T$  admits a universal Skolem sort then so it does its randomization. This is precisely done in Section 2.2. Let us briefly explain its construction. In a randomization there are two kind of sorts, the home sorts coming from the theory and the auxiliary sort coming from ARV, the theory of atomless random variables, which as an  $\omega$ -categorical theory admits a universal Skolem sort. It is therefore expected that in the universal Skolem sort of the randomization the two type of sorts play a role. Indeed, the wanted sort is a fiber product between the randomization of the universal Skolem sort of the theory and the universe Skolem sort of ARV.

The next step was to describe the groupoid associated to the randomization in terms of the groupoid associated to the original theory. We recall from [Iba17, Theorem 2.8] that for any separable structure  $\mathcal{M}$ ,  $\text{Aut}(\mathcal{M}^R) = \text{Aut}(\mathcal{M})^\Omega \rtimes \text{Aut}(\Omega)$ . Here,  $\Omega$  is an atomless standard probability space,  $\text{Aut}(\mathcal{M})^\Omega$  is the family of measurable maps from  $\Omega$  into  $\text{Aut}(\mathcal{M})$  and  $\text{Aut}(\Omega)$  is the group of invertible measure-preserving maps of  $\Omega$ . The groupoid we obtain is quite similar. We show in Section 2.4 that if  $\mathbf{G}(T)$  is the groupoid associated to  $T$  then  $\mathbf{G}(T^R) = (\mathbf{G}(T)^\Omega \rtimes \text{Aut}(\Omega))_\pi$ , that is the base change of  $\mathbf{G}(T)^\Omega \rtimes \text{Aut}(\Omega)$  to  $2^\mathbb{N}$  given by a map  $\pi : 2^\mathbb{N} \rightarrow (2^\mathbb{N})^\Omega$ , obtained from the function used to define the fiber product in the universal Skolem sort of  $T^R$ .

Now we must describe stability dynamically, by which we mean in terms of the action of the groupoid associated to the theory on a space of types. Suppose that  $T$  is a complete, countable theory admitting a universal Skolem sort,  $D_T$ , and let  $F$  be any sort of  $T$ . In the  $\omega$ -categorical case, to determine if a formula  $\varphi(x, y)$  is stable it suffices to study the action  $G \curvearrowright S_x(M)$  where  $\mathcal{M}$  is the separable model of the theory. In our case, since the universal Skolem sort allows us to consider all separable models of  $T$  at once, we study the action  $\mathbf{G}(T) \curvearrowright S_{D_T \times F}(T)$ , as well as the action of the groupoid on the space of partial types  $S_\Delta(T)$  for  $\Delta$  a set of formulas in  $D_T \times F$ . Moreover, in Chapter 4, if  $D_{T^R}$  denotes the universal Skolem sort of the randomization, we exhibit a set of formulas  $\Delta^R$  in  $D_{T^R} \times F^R$ , containing  $\mathbb{E}[\varphi]$  for every  $\varphi \in \Delta$ , for which we can construct  $S_{\Delta^R}(T^R)$  from  $S_\Delta(T)$ .

The translation of stability in the terms of the action  $\mathbf{G}(T) \curvearrowright S_{D_T \times F}(T)$  is what inspired our definition of weakly almost periodic function in the setting of groupoid actions, see Definition 3.6.3. Now it is necessary to find an appropriate notion of representation of a groupoid action in a Banach-like structure for which there is a reasonably well-behaved dual. Towards this end we use the notion of a field of Banach spaces. For each element in the base of the groupoid we have a Banach space and we glue them together via a family of sections. This is not by any means a new idea, in fact many of our results and definitions come from [DD63].

The topological dynamics of groupoid actions is quite a recent subject, with the first systematic exposition given in 2020 in [FM20] followed by [FM21]. Additionally, a notion of Ellis semi-groupoid for locally compact groupoids was proposed in [AB21]. There has been some prior developments on the representation of groupoids, for example in [Bos11] where an analog of the Peter-Weyl theorem is proven for proper groupoids. However, we could not find an analog of the dynamical hierarchy for groupoid actions. This is the motivation behind Chapter 3.

First, we study the dual field endowed with the weak\* topology. Afterwards, we study the field of continuous functions, giving a particular attention to the topology of pointwise convergence for which we deduce a criterion for relative compactness in the style of [Gro52]. Subsequently, we define the notion of a separately continuous groupoid action and that of duality. Finally, we propose a definition for the representation of a groupoid action on a Banach field. We show that wap functions are precisely those coming from representations on fields of reflexive spaces.

At last, we need to show how to build a representation of the randomization of a groupoid action starting from a representation of the original action. Inspired by [Iba17, Section 3.3] we define the field  $L_2(\Omega, \mathbb{V})_\pi$ , where  $(\mathbb{V}, \Gamma)$  is a field of Banach spaces and  $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^\Omega$  is the map discussed above.  $L_2(\Omega, \mathbb{V})_\pi$  is a field over  $2^{\mathbb{N}}$ . We can think of the elements in the fiber over  $\alpha \in 2^{\mathbb{N}}$  as measurable maps  $\hat{v} : \Omega \rightarrow \mathbb{V}$  satisfying  $\hat{v}(\omega) \in V_{\pi(\alpha)(\omega)}$  for every  $\omega \in \Omega$  and such that the  $L_2$ -norm  $\|\hat{v}\|_2 = (\int_\Omega \|\hat{v}(\omega)\|^2 d\mu(\omega))^{1/2}$  is finite. Nevertheless, it still remains to show that each fiber is reflexive, which is certainly the case when a single Banach space is considered. The problem being the difficulty of characterizing its dual space due to a lack of Radon-Nikodym like results.

This thesis rises another question, the characterization of the map  $\pi$ . Since this map is used for the definition of the universal Skolem sort of the randomization, its presence is no surprise in all of the constructions that ensued. We stated a sufficient condition for the fiber product given by  $\pi$  to define a universal Skolem sort, however we do not know whether it is also sufficient. Moreover, if  $\pi'$  is another map verifying this condition, by the uniqueness of the universal Skolem sort, there is a definable bijection between the sort defined by  $\pi$  and the one defined by  $\pi'$ . This induces an isomorphism between the respective groupoids. We ask whether it is possible to find the explicit isomorphism in topological terms.

## Notation

Suppose  $\mathfrak{A}$  is a Boolean algebra and  $a \in \mathfrak{A}$  then  $a^0$  and  $a^1$  will denote respectively  $a$  and  $\neg a$ . In case  $\nu \in 2^m$  and  $(a_i)$  is a sequence of elements of  $\mathfrak{A}$  of length at least  $m$  then  $a_\nu$  stands for  $a_0^{\nu(0)} \wedge \dots \wedge a_{m-1}^{\nu(m-1)}$ . For our purposes,  $\mathfrak{A}$  will either be an algebra of formulas or a measure algebra. For  $A$  a family of sets in a  $\sigma$ -algebra we write  $\sigma(A)$  for its generated  $\sigma$ -algebra.

If  $(X, d)$  is a metric space, then for  $x \in X$  and  $r > 0$ ,  $B_d(x, r)$  and  $\bar{B}_d(x, r)$  will denote, respectively, the open and closed  $d$ -balls with radius  $r$  centered at  $x$ . If the metric is clear from the context we may drop the subindex. All compact spaces are assumed to be Hausdorff. We denote by  $C_b(X)$  the space of continuous, bounded, real-valued functions defined on  $X$ . In case  $X$  is compact we will drop the subindex. Whenever a space of functions is considered under the pointwise convergence topology we will add  $p$  as a subindex. For example,  $C_p(X)$  stands for the space  $C_b(X)$  endowed with the pointwise convergence topology.

For a vector space  $V$  and  $A \subseteq V$ ,  $\text{Span}(A)$  will denote the vector subspace of  $V$  generated by  $A$ .  $\text{co}(A)$  will denote the convex hull of  $A$ . If  $V$  is a normed space then  $B(V)$  will stand for the close unit ball of  $V$ .  $V^*$  will denote the topological dual space.

Whenever  $f$  is a tuple of functions defined on a set  $\Omega$  and  $\psi(x)$  is a classical formula then  $\llbracket \psi(f) \rrbracket$  will denote the set  $\{\omega \in \Omega ; \psi(f(\omega)) \text{ is true}\}$ . Whenever  $X$  is a random variable,  $\mathbb{E}(X)$  will denote the expectation of  $X$ . If  $A_0, \dots, A_{n-1}$  is a partition of  $\Omega$  and  $a_0, \dots, a_{n-1}$  are functions defined on  $\Omega$  then  $\langle A_0, \dots, A_{n-1}, a_0, \dots, a_{n-1} \rangle$  and  $\langle (A_i)_{i < n}, (a_i)_{i < n} \rangle$  stand for the map that agrees with  $a_i$  on  $A_i$  for  $i < n$ .

Whenever we have a condition  $\psi(x) = 0$  and we have not quantified over the variable  $x$ , we will assume that there is an implicit universal quantifier. So, for instance,  $\inf_y \varphi(x, y) = 0$  will stand for  $\sup_x \inf_y \varphi(x, y) = 0$ .

$G$  will denote a group while boldface  $\mathbf{G}$  will stand for a groupoid. For  $(X, \rho)$  a space over  $\mathbf{B}$ , its elements will be denoted by plain characters  $x, \dots$  while boldface characters  $\mathbf{x}, \dots$  will be reserved for sections.

For  $n \in \mathbb{N}$  and  $\nu \in 2^n$  let us denote  $[\nu] = \{\alpha \in 2^{\mathbb{N}} ; \alpha \upharpoonright_n = \nu\}$ .



# Chapter 1

## Randomizations

### Introduction

Given a structure  $\mathfrak{M}$ , continuous or classical, a randomization of  $\mathfrak{M}$  is a new structure consisting of random functions from a probability space into  $M$ . For classical structures, they were first introduced by H.J. Keisler on [Kei99] and were then formalized in the continuous setting by I. Ben Yaacov and H.J. Keisler on [BYK09]. They are defined as two-sorted continuous structures, consisting of a sort of random elements and a sort of events.

Moreover, starting from a theory  $T$  we can obtain a continuous theory  $T^R$ , known as the randomization of  $T$ , defined as the common theory of randomizations of models  $T$ . In [BYK09], an axiomatization of  $T^R$  is given and several preservation results are proven, such as completeness,  $\omega$ -categoricity and stability. This framework proved to be the adequate one for the study of randomizations, and subsequent works of U. Andrews, I. Goldbring and H.J. Keisler were developed in this context, see [AGK15], [AGK19], [AK15].

Finally, I. Ben Yaacov in [BY13] generalized the randomization construction to continuous structures. In this slightly different set-up, although equivalent in the classical case, the sort of events is replaced by a sort of  $[0, 1]$ -valued random variables and the language is simplified. As before, for a continuous theory  $T$  we can construct  $T^R$  and the same preservation results will hold.

We begin this chapter with a brief description of the theory of atomless probability spaces,  $\text{APr}$ , and that of atomless random variables,  $\text{ARV}$ . We then give the definition of randomizations, as presented in [BY13]. We continue with some results on definability on the randomizations, generalizing to the continuous case some results previously stated in [BM21] for the randomization of classical structures. We end the chapter by showing that the randomization of a definable group is definably connected, meaning that it does not have proper, invariant subgroups of bounded index, when looked inside a monster model.

### 1.1 The theories $\text{APr}$ and $\text{ARV}$

The language of probability algebras is defined as  $\mathcal{L}_{\text{Pr}} = \{\top, \perp, \neg, \cup, \cap, \mu\}$ . If  $(\Omega, \mathcal{B}, \mu)$  is a probability space then its measure algebra,  $(\bar{\mathcal{B}}, \mu)$ , admits an obvious interpretation as an  $\mathcal{L}_{\text{Pr}}$ -structure. We will denote by  $\text{Pr}$  the common theory of all measure algebras coming from a probability space. This theory can be axiomatized using the axioms of Boolean algebras, conditions stating that  $\mu$  is a finitely additive probability measure and that the distance corresponds to the measure of the symmetric difference. If we add to  $\text{Pr}$  the condition saying that the algebra

is atomless we get the theory of atomless probability algebras, which we denote  $\text{APr}$ . Every atomless probability algebra can be regarded as a model of  $\text{APr}$ . Conversely, if  $\mathcal{M}$  is a model of  $\text{APr}$  then there is an atomless probability space whose measure algebra is isomorphic to  $\mathcal{M}$ . The theory  $\text{APr}$  is complete,  $\omega$ -categorical and eliminates quantifiers [BY13, Fact 2.10].

Now we pass to the theory of atomless random variables. Let us define  $\mathcal{L}_{\text{RV}} = \{0, \neg, \frac{1}{2}, \dot{\div}\}$ , where  $\frac{1}{2}$  represents a unary function symbol. For  $(\Omega, \mathcal{B}, \mu)$  a probability space, the space of random variables,  $L^1((\Omega, \mathcal{B}, \mu), [0, 1])$  together with the  $L^1$ -distance, can be seen as an  $\mathcal{L}_{\text{RV}}$ -structure by interpreting 0 as the null map and, by setting  $(\frac{1}{2}x)(\omega) = \frac{1}{2}x(\omega)$ ,  $(\neg x)(\omega) = 1 - x(\omega)$  and  $(x \dot{\div} y)(\omega) = x(\omega) \dot{\div} y(\omega) = \max\{0, x(\omega) - y(\omega)\}$  for  $x, y \in L^1((\Omega, \mathcal{B}, \mu), [0, 1])$  and  $\omega \in \Omega$ . The common theory of such structures will be denoted by  $\text{RV}$ . Every model of  $\text{RV}$  is isomorphic to a model of the form  $L^1((\Omega, \mathcal{B}, \mu), [0, 1])$  for some probability space  $(\Omega, \mathcal{B}, \mu)$ . If we add the condition stating that the space must be atomless we get the theory  $\text{ARV}$  of atomless random variables. As before,  $\text{ARV}$  is complete,  $\omega$ -categorical and eliminates quantifiers [BY13, Theorem 2.17].

The theories  $\text{ARV}$  and  $\text{APr}$  are bi-interpretable [BY13, Proposition 2.12]. Therefore, for any  $\mathcal{M}$  and  $\mathcal{N}$  separable models of  $\text{APr}$  and  $\text{ARV}$  respectively,  $\text{Aut}(\mathcal{M})$  and  $\text{Aut}(\mathcal{N})$  are isomorphic as topological groups. Here we give a presentation of such group. Let us begin with the measure algebra on the unit interval,  $(\bar{\mathcal{F}}, \lambda)$ , or with any separable model of  $\text{APr}$  for that matter. Let  $\Omega$  be its Stone space, a compact, Hausdorff topological space. Then, there is an atomless probability measure  $\mu$  on  $\mathcal{B}$ , the measurable subsets of  $\Omega$ , such that  $(\bar{\mathcal{F}}, \lambda)$  is isomorphic to  $(\bar{\mathcal{B}}, \mu)$  [Fre02, Paragraph 321.J]. The ideal of  $\mu$ -negligible subsets of  $\Omega$  corresponds to the ideal of meager subsets. It follows that  $\mu$  is complete. Moreover, a subset of  $\Omega$  is measurable if and only if it differs from a clopen set in a meager set. The group of automorphisms of the measure algebra  $(\bar{\mathcal{B}}, \mu)$  will be denoted by  $\text{Aut}(\Omega)$  and it consists of the group of invertible measure-preserving transformations of  $\Omega$ . The action of  $\text{Aut}(\Omega)$  on  $L^1((\Omega, \mathcal{B}, \mu), [0, 1])$  is given by  $(\tau \cdot f)(\omega) = f(\tau^{-1}(\omega))$  for  $\tau \in \text{Aut}(\Omega)$ ,  $f \in L^1((\Omega, \mathcal{B}, \mu), [0, 1])$  and  $\omega \in \Omega$ .  $\text{Aut}(\Omega)$  endowed with the topology of pointwise convergence is a Polish group (see [Kec10, Chapter I]). A compatible left-invariant metric can be constructed as follows: pick  $\Xi = (\xi_n)_{n \in \mathbb{N}}$  a sequence of events such that the generated subalgebra is dense in  $\bar{\mathcal{B}}$ , and for  $\tau, \rho \in \text{Aut}(\Omega)$  let

$$d_L(\tau, \rho) = \sup_{n \in \mathbb{N}} 2^{-n} \wedge d(\tau(\xi_n), \rho(\xi_n)).$$

## 1.2 The randomization construction

In order to define the notion of randomization, we introduce the concept of integration space.

**Definition 1.2.1.** An *integration space* is a triple  $(\Omega, \mathcal{A}, \mathbb{E})$  such that  $\Omega$  is any set,  $\mathcal{A} \subseteq [0, 1]^\Omega$  with  $0 \in \mathcal{A}$  and closed under the maps  $\neg, \frac{1}{2}, \dot{\div}$ , as defined before, and  $\mathbb{E} : \mathcal{A} \rightarrow [0, 1]$  satisfies  $\mathbb{E}(f + g) = \mathbb{E}(f) + \mathbb{E}(g)$  for  $f, g, f + g \in \mathcal{A}$ .

If we denote by  $\mathbb{E}$  the expected value, namely  $\mathbb{E}(f) = \int f d\mu$ , then  $(\Omega, L^1((\Omega, \mathcal{B}, \mu), [0, 1]), \mathbb{E})$  is an integration space. Moreover, by identifying an event with its characteristic function, we get that  $(\Omega, \mathcal{B}, \mu)$  is also an integration space.

**Definition 1.2.2.** Let  $\mathcal{L}$  be a language, let  $\Omega$  be a non-empty set and let  $\{\mathcal{M}_\omega ; \omega \in \Omega\}$  be a family of  $\mathcal{L}$ -structures. A *randomization based on*  $\{\mathcal{M}_\omega ; \omega \in \Omega\}$  is a triple  $(\mathcal{M}, \mathcal{A}, \mathbb{E})$  where  $M \subseteq \prod_{\omega \in \Omega} M_\omega$  is non-empty,  $\mathcal{A} \subseteq [0, 1]^\Omega$  and  $\mathbb{E} : \mathcal{A} \rightarrow [0, 1]$  such that:

1.  $(\Omega, \mathcal{A}, \mathbb{E})$  is an integration space.

2. For every constant symbol  $c$  in  $\mathcal{L}$ , the map  $\omega \mapsto c^{\mathcal{M}_\omega}$  belongs to  $M$ .
3. For every  $n$ -ary function symbol  $F$  in  $\mathcal{L}$  and every  $a_0, \dots, a_{n-1} \in M$ , the map

$$\omega \mapsto F^{\mathcal{M}_\omega}(a_0(\omega), \dots, a_{n-1}(\omega))$$

belongs to  $M$ .

4. For every  $n$ -ary predicate symbol  $P$  in  $\mathcal{L}$  and every  $a_0, \dots, a_{n-1} \in M$ , the map

$$\omega \mapsto P^{\mathcal{M}_\omega}(a_0(\omega), \dots, a_{n-1}(\omega))$$

belongs to  $\mathcal{A}$ .

In case  $\mathbb{E}$  can be deduced from the context we will omit it and we may simply write  $(\mathcal{M}, \mathcal{A})$ . Whenever  $\mathcal{M}_\omega$  is equal to a single structure  $\mathfrak{M}$  for every  $\omega \in \Omega$ , we say that  $(\mathcal{M}, \mathcal{A})$  is a randomization of  $\mathfrak{M}$ .

**Definition 1.2.3.** Given a language  $\mathcal{L}$ , the *randomization language*,  $\mathcal{L}^R$ , is a continuous language consisting of the same sorts as  $\mathcal{L}$ , the home sorts, together with an additional auxiliary sort satisfying:

1. For each constant symbol in  $\mathcal{L}$  a constant symbol in  $\mathcal{L}^R$  belonging to the same sort.
2. For each function symbol in  $\mathcal{L}$  a function symbol in  $\mathcal{L}^R$  between the corresponding main sorts and with the same modulus of uniform continuity as the original symbol.
3. For each predicate symbol  $P$  in  $\mathcal{L}$  a function symbol  $\llbracket P() \rrbracket$  from the corresponding main sorts into the auxiliary sort and with the same modulus of uniform continuity.
4. If  $\mathcal{L}$  is classical, the auxiliary sort will be equipped with the language  $\mathcal{L}_{\text{Pr}}$ . Otherwise, it will be equipped with the language  $\mathcal{L}_{\text{RV}}$ .

For simplicity, we will assume that  $\mathcal{L}$  consists of a single sort, however the same arguments can be carried out in the many-sorted case with some additional bookkeeping. It follows from the previous two definitions that if  $\mathbf{M} = (\mathcal{M}, \mathcal{A}, \mathbb{E})$  is a randomization based on a family of  $\mathcal{L}$ -structures  $\{\mathcal{M}_\omega ; \omega \in \Omega\}$ , then it admits a natural interpretation as an  $\mathcal{L}^R$ -prestructure, where the distance on the home sort is given by  $d(a, b) = \mathbb{E}\llbracket d(a, b) \rrbracket$  for  $a, b \in \mathcal{M}$ .

The classical example of a randomization is the *Borel randomization* of a structure. Let  $\mathcal{M}$  be a separable  $\mathcal{L}$ -structure and let  $(\Omega, \mathcal{B}, \mu)$  be the standard Borel probability space constructed earlier. Define  $\mathcal{M}^\Omega$  as the set of measurable functions from  $\Omega$  into  $\mathcal{M}$  and take  $(\Omega, \mathcal{A}, \mathbb{E})$  equal to  $(\Omega, \mathcal{B}, \mu)$  or  $(\Omega, L^1((\Omega, \mathcal{B}, \mu), [0, 1]), E)$ , depending on whether  $\mathcal{L}$  is classical or not. Then  $\mathcal{M}^R = (\mathcal{M}^\Omega, \mathcal{A})$  is a randomization based on  $\mathcal{M}$ .

**Definition 1.2.4.** Suppose  $T$  is an  $\mathcal{L}$ -theory. Let  $T^{Ra}$  be the common theory of the randomizations based on models of  $T$ . The *randomization theory*, denoted  $T^R$ , is defined as  $T^{Ra}$  together with the condition stating that the underlying integration space is atomless.

As expected, every model of  $T^{Ra}$  is isomorphic to a randomization based on models of  $T$ . In fact, to every  $\mathbf{M} = (\mathcal{M}, \mathcal{A}, \mathbb{E})$  model of  $T^{Ra}$ , we can canonically associate a randomization based on models of  $T$  whose underlying probability space corresponds to the Stone space of  $\mathcal{A}$ . This randomization will be known as the *canonical representation* and we will identify a model of  $T^{Ra}$  with it, see [BY13, Definition 3.9] and the construction thereafter. In particular, whenever

$\mathbf{M}$  is a separable model of  $T^R$ , we can assume that its underlying probability space is the Stone space  $(\Omega, \mathcal{B}, \mu)$  described above.

Given a randomization  $\mathbf{M} = (\mathcal{M}, \mathcal{A}, \mathbb{E})$  based on a family of  $\mathcal{L}$ -structures  $\{\mathcal{M}_\omega ; \omega \in \Omega\}$ , for every  $\mathcal{L}$ -formula  $\varphi(x)$  and every tuple  $a \in \mathcal{M}$  we can define a map  $\llbracket \varphi(a) \rrbracket : \Omega \rightarrow [0, 1]$  by  $\llbracket \varphi(a) \rrbracket(\omega) = \varphi^{\mathcal{M}_\omega}(a(\omega))$  for  $\omega \in \Omega$ . It turns out that this map belongs to  $\mathcal{A}$  and the resulting function  $\llbracket \varphi(\cdot) \rrbracket : \mathcal{M}^n \rightarrow \mathcal{A}$  is uniformly definable on randomizations based on  $\mathcal{L}$ -structures, see [BY13, Lemma 3.13 and Theorem 3.14]. Moreover, for every  $\mathcal{L}$ -formula  $\varphi(x, y)$  we have that  $\llbracket \inf_y \varphi(x, y) \rrbracket = \inf_y \llbracket \varphi(x, y) \rrbracket$ . In particular, if  $\varphi$  is an  $\mathcal{L}$ -sentence, then  $\llbracket \varphi \rrbracket \in \mathcal{A}$ . Therefore,

$$T \vdash \varphi = 0 \text{ implies } T^{Ra} \vdash \llbracket \varphi \rrbracket = 0.$$

This fact is known as the *Transfer Axiom* and will be used constantly, sometimes without explicit mention. The randomization theory  $T^R$  eliminates quantifiers down to formulas of the form  $\mathbb{E} \llbracket \varphi(x) \rrbracket$  [BY13, Theorem 3.32]. Moreover, an explicit axiomatization of  $T^R$  is given in [BY13]. Randomizations also preserve many of the properties of the original theory such as, completeness,  $\omega$ -categoricity, stability and NIP [BY13, Sections 3 and 4]. Notably, if  $T$  is  $\omega$ -categorical and  $\mathcal{M}$  is a separable model of  $T$  then  $\mathcal{M}^R$ , its Borel randomization, is the unique separable model of  $T^R$ , modulo isomorphism. The following fact is an easy consequence of [BY13, Lemma 3.10].

**Fact 1.2.5.** *Let  $(\mathcal{M}, \mathcal{A}, \mathbb{E}) \models T^R$ . Let  $A_0, \dots, A_{n-1}$  be a partition of the underlying probability space and let  $a_0, \dots, a_{n-1} \in \mathcal{M}$ . Then, there is a unique  $c \in \mathcal{M}$  such that  $c$  is equal to  $a_i$  on  $A_i$  for  $i < n$ .*

The element  $c$  in the previous lemma will be denoted by  $\langle A_0, \dots, A_{n-1}, a_0, \dots, a_{n-1} \rangle$ , or equivalently, by  $\langle (A_i)_{i < n}, (a_i)_{i < n} \rangle$ .

### 1.3 Definability on randomizations

Let us fix a language  $\mathcal{L}$ . Let  $\mathcal{M}$  be an  $\mathcal{L}$ -structure and  $A \subseteq M$ , a predicate  $P : M^n \rightarrow [0, 1]$  is *definable in  $\mathcal{M}$  over  $A$*  if there is a sequence of  $\mathcal{L}(A)$ -formulas that converges uniformly on  $M^n$  to  $P$ . A closed set  $U \subseteq M^n$  is said to be *definable in  $\mathcal{M}$  over  $A$*  if  $d(x, U)$  is a definable predicate in  $\mathcal{M}$  over  $A$ . Finally, a map  $f : M^n \rightarrow M$  is *definable in  $\mathcal{M}$  over  $A$*  if the predicate  $d(f(x), y)$  on  $M^{n+1}$  is definable in  $\mathcal{M}$  over  $A$ . As expected these notions are well-behaved with respect to elementary extensions and substructures [BYBHU08, Section 9]. Whenever we say that a predicate, subset or function is definable we always mean without parameters. We now state some technical results. The first one, characterizing definability of a set, will be used on [Chapter 2](#) when discussing definable closures in randomizations.

**Fact 1.3.1.** *Let  $\mathcal{M}$  be a structure,  $U$  be a closed subset of  $M^n$  and  $A$  be any subset of  $M$ . Then the following assertions are equivalent:*

1.  $U$  is definable in  $\mathcal{M}$  over  $A$ .
2.  $U$  is the set of zeros of an  $A$ -definable predicate  $\psi(x)$  and  $d(x, U) \leq \psi(x)$  on  $M^n$ .
3.  $U$  is the set of zeros of an  $A$ -definable predicate  $\psi(x)$  and, for every  $\epsilon > 0$  there is  $\delta > 0$  such that for every  $x \in M^n$ ,  $\psi(x) \leq \delta$  implies  $d(x, U) \leq \epsilon$ .
4. For every  $\epsilon > 0$  there is a formula with parameters in  $A$ ,  $\psi(x)$ , such that

$$U \subseteq \{a \in M^n ; \psi(a) = 0\} \subseteq \{a \in M^n ; \psi(a) < 1\} \subseteq \{a \in M^n ; d(a, U) < \epsilon\}.$$

5. For every  $A$ -definable predicate  $\varphi(x, y)$ , the predicate  $\psi(y) = \inf_{x \in U} \varphi(x, y)$  is  $A$ -definable.

*Proof.* The equivalence of (1), (2), (4) and (5) is due to [BY10, Fact 1.7]. The equivalence between (1) and (3) corresponds to [BYBHU08, Remark 9.20].  $\square$

**Fact 1.3.2** (Theorem 9.12 in [BYBHU08]). *Let  $\mathcal{M}$  be an  $\mathcal{L}$ -structure. Given a definable predicate  $P : M^n \rightarrow [0, 1]$ , we define the two conditions*

$$E_1 : \sup_x \inf_y P(y) \vee |P(x) - d(x, y)| = 0$$

$$E_2 : \sup_x |P(x) - \inf_y (P(y) + d(x, y))| = 0.$$

Then,  $P(x)$  is equal to the distance to the zerset of  $P$  if and only if  $\mathcal{M}$  satisfies  $E_1$  and  $E_2$ .

We will now show that the randomization of definable sets and functions are definable. The particular case where  $\mathcal{L}$  is a classical language was considered in [BM21, Lemma 3.13].

**Proposition 1.3.3.** *Let  $T$  be an  $\mathcal{L}$ -theory and  $D$  a definable subset of  $T$ . Then, for any  $\mathbf{M} \models T^R$ ,*

$$D^R(\mathbf{M}) = \{a \in M^n ; \mu[a \in D] = 1\}.$$

*is definable in  $\mathbf{M}$ . This definition is moreover uniform across all models of  $T^R$ .*

*Proof.* Suppose that  $d(x, D) = P(x)$  for some definable predicate  $P$ . Then  $D^R$  corresponds to the zerset of  $\mathbb{E}[P(x)]$ . By the transfer axiom applied to  $E_1$ ,  $\mathbf{M} \models \sup_x \inf_y (\mathbb{E}[P(y)] \vee \mathbb{E}[|P(x) - d(x, y)|]) = 0$ . This implies that,  $\sup_x \inf_y (\mathbb{E}[P(y)] \vee \mathbb{E}[|P(x) - d(x, y)|]) = 0$ . Hence,  $\sup_x \inf_y (\mathbb{E}[P(y)] \vee |\mathbb{E}[P(x)] - \mathbb{E}[d(x, y)]|) = 0$ . This means that  $\mathbf{M}$  satisfies  $E_1$  for the definable predicate  $\mathbb{E}[P(x)]$ . Analogously we obtain that  $\sup_x |\mathbb{E}[P(x)] - \inf_y (\mathbb{E}[P(y)] + \mathbb{E}[d(x, y)])| = 0$ , i.e.  $\mathbf{M}$  satisfies  $E_2$  for  $\mathbb{E}[P(x)]$ . The result follows from the previous fact.  $\square$

For the proof of the next lemma we will use the following fact, which although not explicitly stated, is the heart of the proof of [BYBHU08, Theorem 9.17].

**Fact 1.3.4.** *Let  $\mathcal{M}$  be an  $\mathcal{L}$ -structure. If  $D \subseteq M^n$  is definable and  $P : M^m \times M^n \rightarrow [0, 1]$  is a definable predicate, then there is an increasing, continuous function  $\alpha : [0, 1] \rightarrow [0, 1]$  such that for any  $x \in M^m$ :*

- $|P(x, y) - P(x, z)| \leq \alpha(d(y, z))$  for every  $y, z \in M^n$ .
- $\inf_{y \in D} P(x, y) = \inf_y (P(x, y) + \alpha(d(y, D)))$ .

**Lemma 1.3.5.** *Let  $T$  be an  $\mathcal{L}$ -theory,  $D$  a definable subset of  $n$ -tuples and  $P$  be an  $(m+n)$ -ary definable predicate. Then,*

$$\mathbb{E}[\inf_{y \in D} P(x, y)] = \inf_{y \in D^R} \mathbb{E}[P(x, y)].$$

*Proof.* Take  $\mathbf{M} \models T^R$ . Fix  $x \in M^m$ , then for any  $y_0 \in D^R(\mathbf{M})$ ,  $\mathbb{E}[P(x, y_0)] \geq \mathbb{E}[\inf_{y \in D} P(x, y)]$ . By taking the infimum over  $y_0 \in D^R$  we conclude that  $\inf_{y \in D^R} \mathbb{E}[P(x, y)] \geq \mathbb{E}[\inf_{y \in D} P(x, y)]$ . For the other inequality, take  $\alpha : [0, 1] \rightarrow [0, 1]$  as in the previous fact. Given  $\epsilon > 0$ , take  $y_0 \in M^n$  satisfying

$$\llbracket P(x, y_0) + \alpha(d(y_0, D)) \rrbracket \leq \llbracket \inf_{y \in D} P(x, y) \rrbracket + \epsilon.$$

Pick  $\delta > 0$  such that for every  $a, b \in [0, 1]$ ,  $|a - b| \leq \delta$  implies  $|\alpha(a) - \alpha(b)| \leq \epsilon$ . Let  $z_0 \in D^R(\mathbf{M})$  be such that  $\llbracket d(y_0, z_0) \rrbracket \leq \llbracket d(y_0, D) \rrbracket + \delta$ . Then,

$$\begin{aligned} \inf_{y \in D^R} \llbracket P(x, y) \rrbracket &\leq \llbracket P(x, z_0) \rrbracket \\ &\leq \llbracket P(x, y_0) + \alpha(d(y_0, z_0)) \rrbracket \\ &\leq \llbracket P(x, y_0) + \alpha(d(y_0, D) + \delta) \rrbracket \\ &\leq \llbracket P(x, y_0) + \alpha(d(y_0, D)) \rrbracket + \epsilon \\ &\leq \llbracket \inf_{y \in D} P(x, y) \rrbracket + 2\epsilon. \end{aligned}$$

Since  $\epsilon$  was arbitrary, we get the desired inequality.  $\square$

**Proposition 1.3.6.** *Let  $T$  be an  $\mathcal{L}$ -theory and  $f$  an  $n$ -ary definable map in  $T$ . Then, for every  $\mathbf{M} \models T^R$  the map  $f^R: \mathcal{M}^n \rightarrow \mathcal{M}$  given by  $f^R(a)(\omega) = f^{\mathcal{M}_\omega}(a(\omega))$  for  $a \in \mathcal{M}^n$  and  $\omega \in \Omega$  is a definable map in  $\mathbf{M}$ . This definition is uniform across all models of  $T^R$ .*

*Proof.* Follows directly from the fact that if  $P(x, y)$  is the definable  $(n+1)$ -ary predicate defining  $d(f(x), y)$ , then for  $x \in \mathcal{M}^n$  and  $y \in \mathcal{M}$ ,  $d(f^R(x), y) = \mathbb{E}\llbracket d(f(x), y) \rrbracket = \mathbb{E}\llbracket P(x, y) \rrbracket$ .  $\square$

## 1.4 Connectedness of randomizations of groups

In this section we show that the randomization of a group definable in a structure is definably connected. Let  $\mathfrak{M}$  be an  $\mathcal{L}$ -structure and  $G$  a definable group in  $\mathfrak{M}$ , we have that  $G^R$ , defined as the set of variables whose values belong to the group, is a definable group in the randomization. The author together with A. Berenstein proved in [BM21], via a study of generic types, that if  $G$  is a classical definably amenable NIP group then  $G^R$  is definably connected, i.e. it has no proper type-definable subgroups of bounded index, when looked inside a monster model. In this section, we show that in general, without any model-theoretical assumptions on the model,  $G^R$  is definably connected. Moreover, we prove that it has no proper invariant subgroups of bounded index. For the proof, we use the construction of a homogeneous space. We use the presentation given in [GN08], that allows us to embed the group  $G^R$  in the group of automorphisms of a certain structure and then use the fact that in randomizations all types over parameters coming from the home sorts are Lascar strong.

### 1.4.1 The homogeneous space construction

Suppose  $G$  is a definable group in a structure  $\mathfrak{M}$ . The homogeneous space structure consists of  $\mathfrak{M}$  together with a new sort  $X$ , corresponding to an affine copy of  $G$ , and a symbol for the action  $\cdot: G \times X \rightarrow X$ . This is a classical construction and has been considered several times. For instance, in [Zie02, Section 7] it is used to prove an unpublished result of E. Bouscaren, D. Lascar and A. Pillay concerning compact Lie groups. It was later used in [GN08] to describe groups of automorphisms of the homogeneous space in terms of groups of automorphisms of the original structure and groups related to  $G$ . This section is based on the latter, we recreate in the continuous setting the construction given in Section 3 therein.

As previously stated, we let  $X$  be an affine copy of  $G$ . We also add  $G$  itself as a sort, so the group operation can be represented by a symbol in the language and we can avoid dealing with partially defined functions. However, this step is not strictly necessary and the same results can be obtained by just adding the affine copy. Let us denote this new structure by  $\tilde{\mathfrak{M}}$ . As mentioned before, the main goal of the construction is to describe some subgroups of  $\text{Aut}(\tilde{\mathfrak{M}})$  in terms of subgroups of  $\text{Aut}(\mathfrak{M})$  and subgroups of  $G$ . Towards this end, we must embed  $G$  as a normal subgroup of  $\text{Aut}(\tilde{\mathfrak{M}})$ . To make this task easier, we will suppose that the distance on  $G$  is bi-invariant, which is always true in the classical context. Nevertheless, this does not impose us a limitation, since the group always admits a definable bi-invariant metric, see [Fact 1.4.13](#) below.

Let  $\mathfrak{M}$  be an  $\mathcal{L}$ -structure. Suppose  $G$  is a definable group whose metric is bi-invariant. We will denote by  $S$  the sort in which  $G$  is defined. Let  $\tilde{\mathcal{L}}$  be the many-sorted language obtained by adjoining two sorts  $S_1, S_2$  and the following function symbols to  $\mathcal{L}$ :

- A unary function symbol  $\iota : S_1 \rightarrow S$ .
- A binary function symbol  $\cdot_G : S_1 \times S_1 \rightarrow S_1$
- A binary function symbol  $\cdot : S_1 \times S_2 \rightarrow S_2$ .

We require the new unary function symbol to be 1-Lipschitz. To both binary function symbols we assign the same modulus of uniform continuity of the group multiplication. We define  $\tilde{\mathfrak{M}} = (M, G(M), X(M))$  as the  $\tilde{\mathcal{L}}$ -structure where

1. The sort  $M$  has the original structure of  $\mathfrak{M}$ .
2.  $\iota : G(M) \hookrightarrow M$  is the inclusion map.
3.  $\cdot_G : G(M) \times G(M) \rightarrow G(M)$  is the group operation.
4.  $\cdot : G(M) \times X(M) \rightarrow X(M)$  is a regular (free and transitive) action of  $G(M)$  on  $X(M)$ .

We endow  $G(M)$  and  $X(M)$  with the same metric, the one they inherit from  $M$ . So,  $G(M)$  acts by isometries on  $X(M)$ , and for all  $g, h \in G(M)$  and  $x \in X(M)$ ,  $d_X(g \cdot x, h \cdot x) = d_G(g, h)$ . We will simply write  $\cdot$  instead of  $\cdot_G$ , the distinction with the action being clear from the context.

Fix  $\tilde{\mathfrak{C}} = (\mathfrak{C}, G, X)$  a monster model extending  $\tilde{\mathfrak{M}}$ . In particular,  $\mathfrak{C}$  is a monster model of  $\text{Th}(\mathfrak{M})$ . A set is said to be *bounded*, or *small*, if its cardinality is less than the saturation of our monster. As usual, sets of parameters are supposed to be small.

*Remark.* The structure  $\tilde{\mathfrak{C}}$  is interpretable in  $\mathfrak{C}$  (for more details on interpretation in continuous logic see [\[BYK16\]](#)). Indeed, fix  $x_0 \in X$ , the map that is equal to the identity on the sort  $\mathfrak{C}$ , the inclusion on the sort  $G$  and on the sort  $X$  is given by  $h \cdot x_0 \mapsto h$  defines an interpretation of  $\tilde{\mathfrak{C}}$  on  $\mathfrak{C}$ .

In other words, we add two interpretable sorts, namely two copies of  $G$  interacting with each other, and in one of them we forget the inner group structure. We now characterize the elementary substructures of  $\tilde{\mathfrak{C}}$ .

**Proposition 1.4.1.** *Every elementary substructure of  $\tilde{\mathfrak{C}}$  is of the form  $(\mathfrak{N}, G(\mathfrak{N}), G(\mathfrak{N}) \cdot x_0)$  for some  $\mathfrak{N} \leq \mathfrak{C}$  and  $x_0 \in X$ . Conversely, if  $\mathfrak{N} \leq \mathfrak{C}$  then for every  $x \in X$ ,  $(\mathfrak{N}, G(\mathfrak{N}), G(\mathfrak{N}) \cdot x)$  is an elementary substructure of  $\tilde{\mathfrak{C}}$ .*

*Proof.* Let  $\tilde{\mathfrak{N}} = (\mathfrak{N}, G_0, X_0)$  be an elementary substructure of  $\tilde{\mathfrak{C}}$ . It follows that  $\mathfrak{N} \leq \mathfrak{C}$ , which implies  $G_0 = G(\mathfrak{N})$ . Pick any  $x_0 \in X_0$ , since the action  $G_0 \curvearrowright X_0$  is regular, we have  $X_0 = G_0 \cdot x_0 = G(\mathfrak{N}) \cdot x_0$ . We conclude that  $\tilde{\mathfrak{N}} = (\mathfrak{N}, G(\mathfrak{N}), G(\mathfrak{N}) \cdot x_0)$ . For the converse, suppose  $\mathfrak{N} \leq \mathfrak{C}$ , pick  $x \in X$  and define  $\tilde{\mathfrak{N}} = (\mathfrak{N}, G(\mathfrak{N}), G(\mathfrak{N}) \cdot x)$ . Since  $\tilde{\mathfrak{C}}$  is interpretable in  $\mathfrak{C}$ , for every  $\tilde{\mathcal{L}}$ -formula  $\varphi(u, y, z)$ , where  $u, y$  and  $z$  are tuple of variables in the sorts  $S, S_1$  and  $S_2$ , respectively, there is a definable predicate  $P_\varphi(u, v, w)$  in  $\mathcal{L}$  such that for every  $a \in \mathfrak{C}, g, h \in G$

$$\varphi^{\tilde{\mathfrak{C}}}(a, g, h \cdot x) = P_\varphi^{\mathfrak{C}}(a, g, h).$$

Restricting to  $\tilde{\mathfrak{N}}$ , we obtain an interpretation of  $\tilde{\mathfrak{N}}$  in  $\mathfrak{N}$  and an analogous equality for every  $a \in \mathfrak{N}, g, h \in G(\mathfrak{N})$ . It follows that

$$\varphi^{\tilde{\mathfrak{C}}}(a, g, h \cdot x) = P_\varphi^{\mathfrak{C}}(a, g, h) = P_\varphi^{\mathfrak{N}}(a, g, h) = \varphi^{\tilde{\mathfrak{N}}}(a, g, h \cdot x)$$

This completes the proof.  $\square$

Now, take any  $x_0 \in X$ , then  $X = G \cdot x_0$ . For  $g \in G$  we define  $\bar{g} : \tilde{\mathfrak{C}} \rightarrow \tilde{\mathfrak{C}}$  by

$$\bar{g} \upharpoonright_{\mathfrak{C}} = \text{id}_{\mathfrak{C}}, \quad \bar{g} \upharpoonright_G = \text{id}_G, \quad \bar{g}(h \cdot x_0) = (hg^{-1}) \cdot x_0.$$

**Claim:** For any  $g \in G$ ,  $\bar{g}$  is an automorphism of  $\tilde{\mathfrak{C}}$ .

*Proof.* It is immediate that  $\bar{g}$  is a bijection that respects the structure of  $\mathfrak{C}$ , the group operation and that commutes with  $\iota$ . Now, for  $k, h \in G$

$$\bar{g}(k \cdot (h \cdot x_0)) = \bar{g}(kh \cdot x_0) = (khg^{-1}) \cdot x_0 = \bar{g}(k) \cdot \bar{g}(h \cdot x_0),$$

and

$$d_X(\bar{g}(h \cdot x_0), \bar{g}(k \cdot x_0)) = d_X(hg^{-1} \cdot x_0, kg^{-1} \cdot x_0) = d_G(h, k) = d_X(h \cdot x_0, k \cdot x_0). \quad \square$$

On the other hand, suppose  $f \in \text{Aut}(\mathfrak{C})$ , since  $G$  is definable,  $f(G) = G$ . So we can define  $\bar{f} : \tilde{\mathfrak{C}} \rightarrow \tilde{\mathfrak{C}}$  by

$$\bar{f} \upharpoonright_{\mathfrak{C}} = f, \quad \bar{f} \upharpoonright_G = f \upharpoonright_G, \quad \bar{f}(h \cdot x_0) = f(h) \cdot x_0.$$

**Claim:** For any  $f \in \text{Aut}(\mathfrak{C})$ ,  $\bar{f}$  is an automorphism of  $\tilde{\mathfrak{C}}$ .

*Proof.* By definition  $\bar{f}$  is a bijection that respects the structure of  $\mathfrak{C}$ , the group operation and commutes with  $\iota$ . Now, for  $k, h \in G$

$$\bar{f}(k \cdot (h \cdot x_0)) = \bar{f}(kh \cdot x_0) = f(kh) \cdot x_0 = f(k)f(h) \cdot x_0 = \bar{f}(k) \cdot \bar{f}(h \cdot x_0).$$

And

$$d_X(\bar{f}(h \cdot x_0), \bar{f}(k \cdot x_0)) = d_X(f(h) \cdot x_0, f(k) \cdot x_0) = d_G(f(h), f(k)) = d_G(h, k) = d_X(h \cdot x_0, k \cdot x_0). \quad \square$$

In this way, we have embeddings  $\cdot : \text{Aut}(\mathfrak{C}) \hookrightarrow \text{Aut}(\tilde{\mathfrak{C}})$  and  $\cdot : G \hookrightarrow \text{Aut}(\tilde{\mathfrak{C}})$ . Note that for  $f \in \text{Aut}(\tilde{\mathfrak{C}})$  and  $g \in G$ ,  $f \circ \bar{g} \circ f^{-1} = \bar{k}^{-1} f(g) \bar{k}$  where  $k \in G$  satisfies  $f(x_0) = k \cdot x_0$ . Thus,  $G$  is a normal subgroup of  $\text{Aut}(\tilde{\mathfrak{C}})$ . We now define the notion of Lascar types.

**Definition 1.4.2.** Let  $A$  be a set of parameters and,  $a$  and  $b$  be tuples (possibly infinite) of the same length in  $\mathfrak{C}$ . We say that  $d_A(a, b) \leq 1$  if there exists  $\mathfrak{N} \leq \mathfrak{C}$  containing  $A$  such that  $\text{tp}(a/N) = \text{tp}(b/N)$ . We say that  $a$  and  $b$  have the same Lascar type over  $A$ , and we write  $\text{Lstp}(a/A) = \text{Lstp}(b/A)$ , if there is a finite sequence of tuples of the same length  $a_0 = a, a_1, \dots, a_n = b$  such that  $d_A(a_i, a_{i+1}) \leq 1$ .

An equivalence relation is *bounded* if the set of equivalence classes is bounded. It is a well-known fact that having the same Lascar type over  $A$  is the smallest bounded  $A$ -invariant equivalence relation on tuples of the same length, see for example [BY13, Fact 5.2] or [Sim15, Lemma 5.3]. The group of *Lascar strong automorphisms over  $A$*  is the group generated by all  $\text{Aut}(\mathfrak{C}/M)$  where  $M$  ranges over all models containing  $A$ , and it is denoted by  $\text{Autf}_L(\mathfrak{C}/A)$ . It follows from the definition that every automorphism in  $\text{Autf}_L(\mathfrak{C}/A)$  preserves Lascar strong types. Moreover, if  $\text{Lstp}(a/A) = \text{Lstp}(b/A)$  then there is  $f \in \text{Autf}_L(\mathfrak{C}/A)$ , such that  $f(a) = b$ .

**Definition 1.4.3.** Given  $A \subseteq \mathfrak{C}$ , we define  $G_A^\infty$  to be the smallest  $A$ -invariant subgroup of  $G$  of bounded index.

**Proposition 1.4.4.** For  $A \subseteq \mathfrak{C}$ ,  $G_A^\infty = \langle g^{-1}h ; g, h \in G \text{ and } \text{Lstp}(g/A) = \text{Lstp}(h/A) \rangle$ .

*Proof.* Denote  $H = \langle g^{-1}h ; g, h \in G \text{ and } \text{Lstp}(g/A) = \text{Lstp}(h/A) \rangle$ . The relation  $E(g, h)$  given by  $g^{-1}h \in G_A^\infty$  is  $A$ -invariant and bounded. Therefore, if  $g, h \in G$  and  $\text{Lstp}(g/A) = \text{Lstp}(h/A)$  then  $g^{-1}h \in G_A^\infty$ . Hence,  $H \subseteq G_A^\infty$ . For the other inclusion, clearly  $H$  is  $A$ -invariant. Let  $\lambda$  be the index of  $H$  in  $G$  and suppose that  $\{g_\alpha H ; \alpha \in \lambda\}$  is an enumeration of the cosets of  $H$  in  $G$ . Take  $\alpha, \beta \in \lambda$  with  $\alpha \neq \beta$ , then  $g_\alpha^{-1}g_\beta \notin H$ . This implies that  $\text{Lstp}(g/A) \neq \text{Lstp}(h/A)$ . Since there are boundedly many Lascar types over  $A$ , then the index of  $H$  in  $G$  is bounded. Therefore,  $G_A^\infty \subseteq H$  and the equality follows.  $\square$

The following proposition corresponds to [GN08, Propositions 3.3 and 3.4]. Even though they are proved in the classical setting, the proofs readily adapt to continuous structures.

**Proposition 1.4.5.** 1.  $\text{Aut}(\tilde{\mathfrak{C}}) = G \rtimes \text{Aut}(\mathfrak{C})$ . More precisely, for  $F \in \text{Aut}(\tilde{\mathfrak{C}})$ ,  $F = \bar{g} \circ \bar{f}$  where  $\bar{f} = F \upharpoonright_{\mathfrak{C}}$  and  $F(x_0) = g^{-1} \cdot x_0$ .

2. If  $\tilde{\mathfrak{M}}_0 \leq \tilde{\mathfrak{C}}$  with  $\tilde{\mathfrak{M}}_0 = (\mathfrak{M}_0, G_0, X_0)$  and  $X_0 = G_0 \cdot (h_0 \cdot x_0)$  for some  $h_0 \cdot x_0 \in X_0$ . Then  $F \in \text{Aut}(\tilde{\mathfrak{C}}/\tilde{\mathfrak{M}}_0)$  if and only if there is  $f \in \text{Aut}(\mathfrak{C}/\mathfrak{M}_0)$  such that  $F = \bar{f}^{h_0}$ .

3. For any  $A \subseteq \mathfrak{C}$ ,  $\text{Autf}_L(\tilde{\mathfrak{C}}/A) = G_A^\infty \rtimes \text{Autf}_L(\mathfrak{C}/A)$ .

4.  $G_A^\infty = G \cap \text{Autf}_L(\tilde{\mathfrak{C}}/A)$ .

## 1.4.2 Randomization of the homogeneous space

Fix a continuous language  $\mathcal{L}$ ,  $\mathfrak{M}$  an  $\mathcal{L}$ -structure and  $G$  an  $\emptyset$ -definable group. Let  $(\mathcal{K}, \mathcal{A})$  be a monster model of  $\text{Th}(\mathfrak{M})^R$ , which we identify with its canonical representation. By taking  $\mathfrak{C}$  to be a sufficiently saturated elementary extension of  $\mathfrak{M}$ , we can assume that  $(\mathcal{K}, \mathcal{A})$  corresponds to a randomization of  $\mathfrak{C}$  and we will denote it  $\mathfrak{C}^R$ . The randomization of  $G$ ,  $G^R$ , will correspond to the set of functions in  $\mathcal{K}$  whose image is contained in  $G$ . We will show that this is in fact a definable group in  $\mathfrak{C}^R$ .

**Proposition 1.4.6.**  $G^R$  is a definable group in  $\mathfrak{C}^R$  with the group operation defined pointwise, i.e. for  $f_1, f_2 \in G^R$ ,  $(f_1 \cdot f_2)(\omega) = f_1(\omega)f_2(\omega)$  for every  $\omega \in \Omega$ .

*Proof.* By [Proposition 1.3.3](#),  $G^R$  is a definable subset of  $\mathfrak{C}^R$ . Let  $P : \mathfrak{C}^3 \rightarrow [0, 1]$  be the  $\emptyset$ -definable predicate such that  $P(x, y, z) = d(x \cdot y, z)$  when restricted to  $G^2 \times \mathfrak{C}$ . We have that  $\mathfrak{C} \models \sup_{x \in G} \sup_{y \in G} \inf_{z \in G} P(x, y, z) = 0$ . By transference and [Lemma 1.3.5](#),

$$\mathfrak{C}^R \models \sup_{x \in G^R} \sup_{y \in G^R} \inf_{z \in G^R} \llbracket P(x, y, z) \rrbracket = 0.$$

This implies that for every  $x, y \in G^R$  there is  $z \in G^R$  such that  $x \cdot y = z$ . So,  $G^R$  is closed under pointwise multiplication. From [Proposition 1.3.6](#) we deduce that the group operation is definable. It remains to show that  $G^R$  is in fact a group. It is immediate that the operation is associative. Again by transference and [Lemma 1.3.5](#),  $\mathfrak{C}^R \models \inf_{x \in G^R} \sup_{y \in G^R} \llbracket P(x, y, y) \vee P(y, x, y) \rrbracket = 0$  and  $\mathfrak{C}^R \models \sup_{x \in G^R} \inf_{y \in G^R} \sup_{z \in G^R} \llbracket P(x \cdot y, z, z) \vee P(y \cdot x, z, z) \rrbracket = 0$ . So,  $G^R$  has a neutral element and inverses.  $\square$

Assume that the metric in  $G^R$  is bi-invariant. Then, we can construct the  $(\widetilde{\mathcal{L}^R})$ -structure  $\widetilde{\mathfrak{C}^R} = (\mathcal{K}, \mathcal{A}, G^R, X^R)$ . By reordering the sorts of  $(\widetilde{\mathcal{L}^R})$ , we see that  $(\widetilde{\mathcal{L}^R}) \subset (\widetilde{\mathcal{L}})^R$ . The inclusion being strict due to the fact that the mappings  $\llbracket d(-, -) \rrbracket$  from the sorts of  $G^R$  and  $X^R$  do not belong to  $(\widetilde{\mathcal{L}^R})$ . However, if  $x, y \in G^R$  (or  $X^R$ ) then they can be identified with elements in  $\mathcal{K}$ , so we can define these maps in the obvious way, namely:

$$\llbracket d(x, y) \rrbracket(\omega) = d(x(\omega), y(\omega)) \text{ for every } \omega \in \Omega.$$

Hence,  $(\mathcal{K}, G^R, X^R, \mathcal{A})$  admits a natural interpretation as an  $(\widetilde{\mathcal{L}})^R$ -structure. Moreover, it is easy to check that with the maps  $\llbracket d(-, -) \rrbracket$  defined as above,  $\widetilde{\mathfrak{C}^R}$  is a randomization of  $\widetilde{\mathfrak{C}}$ . In this way, when we consider  $(\mathcal{K}, G^R, X^R, \mathcal{A})$  as an  $(\widetilde{\mathcal{L}})^R$ -structure we will write  $\widetilde{\mathfrak{C}^R}$ .

**Proposition 1.4.7.** *The maps  $\llbracket d(-, -) \rrbracket : (G^R)^2 \rightarrow \mathcal{A}$  and  $\llbracket d(-, -) \rrbracket : (X^R)^2 \rightarrow \mathcal{A}$  are definable in  $\widetilde{\mathfrak{C}^R}$ .*

*Proof.* For any  $g, h \in G^R$  we have that  $\llbracket d(g, h) \rrbracket = \llbracket d(i(g), i(h)) \rrbracket$ , a definable map. For the other map, first note that  $e \in G^R$  is definable, the distance predicate being given by  $d_{\mathcal{K}}(x, e) = d_{\mathcal{K}}(x^2, x)$ . Now, given  $x, y \in X^R$  define  $\chi(x, y)$  to be the unique  $g \in G^R$  such that  $g \cdot x = y$ . The map  $\chi : (X^R)^2 \rightarrow G^R$  is definable. Indeed, given  $h \in G^R$ ,  $d_{G^R}(h, \chi(x, y)) = d_{X^R}(h \cdot x, y)$ . Hence,  $\llbracket d(x, y) \rrbracket = \llbracket d(\chi(x, y), e) \rrbracket$  is a definable map.  $\square$

Immediately from the previous proposition we deduce the following result.

**Corollary 1.4.8.** *Given  $A \subseteq \mathcal{K}$ ,  $\text{Aut}(\widetilde{\mathfrak{C}^R}/A) = \text{Aut}(\widetilde{\mathfrak{C}}^R/A)$ .*

The next result states that in randomizations all types are Lascar strong.

**Theorem 1.4.9.** *[BY13, Theorem 5.8] Let  $T$  be any theory,  $(\mathcal{K}, \mathcal{B}) \models T^R$ ,  $a, b \in \mathcal{K}$  and  $A \subseteq \mathcal{K}$  any set of parameters. Then the following are equivalent:*

1.  $\text{Lstp}(a/A) = \text{Lstp}(b/A)$ .
2.  $\text{tp}(a/A) = \text{tp}(b/A)$ .

**Corollary 1.4.10.** *Given  $A \subseteq \mathcal{K}$ ,  $\text{Autf}_L(\widetilde{\mathfrak{C}^R}/A) = \text{Aut}(\widetilde{\mathfrak{C}^R}/A)$*

**Proposition 1.4.11.** *Given  $A \subseteq \mathcal{K}$ ,  $\text{Autf}_L(\widetilde{\mathfrak{C}^R}/A) = \text{Autf}_L(\widetilde{\mathfrak{C}}^R/A)$ .*

*Proof.*  $\supseteq$ ) It follows directly from [Corollary 1.4.8](#) and [Corollary 1.4.10](#).

$\supseteq$ ) Take  $\mathfrak{N} \leq \tilde{\mathfrak{C}}^R$  with  $A \subseteq \mathfrak{N}$  and  $f \in \text{Aut}(\tilde{\mathfrak{C}}^R/\mathfrak{N})$ . Then  $\mathfrak{N}_0$ , the reduct of  $\mathfrak{N}$  to a  $(\mathcal{L}^R)$ -structure, is an elementary substructure of  $\tilde{\mathfrak{C}}^R$ . Hence,  $f \in \text{Aut}(\tilde{\mathfrak{C}}^R/\mathfrak{N}_0)$ . So  $\text{Aut}(\tilde{\mathfrak{C}}^R/\mathfrak{N}) \subseteq \text{Autf}_L(\tilde{\mathfrak{C}}^R/A)$ . Since  $\mathfrak{N}$  was an arbitrary model containing  $A$ , the result follows.  $\square$

**Corollary 1.4.12.** *For any  $A \subseteq \mathcal{K}$ ,  $(G^R)_A^\infty = G^R$ . Thus,  $G^R$  is definably connected.*

*Proof.* By 4. of [Proposition 1.4.5](#),  $(G^R)_A^\infty = G^R \cap \text{Autf}_L(\tilde{\mathfrak{C}}^R/A)$ . Using [Proposition 1.4.11](#) we get  $(G^R)_A^\infty = G^R \cap \text{Autf}_L(\tilde{\mathfrak{C}}^R/A)$ . Now, by [Corollary 1.4.10](#),  $(G^R)_A^\infty = G^R \cap \text{Aut}(\tilde{\mathfrak{C}}^R/A)$ . Finally, since  $G^R \subseteq \text{Aut}(\tilde{\mathfrak{C}}^R/A)$ , then  $(G^R)_A^\infty = G^R$ .  $\square$

In order to prove this corollary we assumed that the metric on the group is bi-invariant. However, this requirement can be removed, as we now show.

**Fact 1.4.13.** [[BY10](#), [Proposition 3.13](#)] *Let  $\mathcal{M}$  be an  $\mathcal{L}$ -structure and let  $G$  be a definable group. Then there is a definable metric on  $\mathcal{M}$  which is bi-invariant on  $G$ .*

**Theorem 1.4.14.** *Let  $G$  be a definable group in a continuous theory. Then, for any  $A \subseteq \mathcal{K}$ ,  $(G^R)_A^\infty = G^R$ .*

*Proof.* The case where the metric in  $G^R$  is bi-invariant corresponds to [Corollary 1.4.12](#). Suppose now that the metric in  $G^R$  is not bi-invariant. Let  $d_1$  be a definable metric on  $\mathcal{K}$  which is bi-invariant on  $G^R$ , as per the previous fact. We define  $\mathfrak{C}_1^R$  as the structure having the same universe that  $\mathfrak{C}^R$  but whose distance predicate on  $\mathcal{K}$  is given by  $d_1$  and the original metric will correspond to a new predicate  $d_2$ . So,

$$d_{\mathcal{K}}^{\mathfrak{C}_1^R} = d_1^{\mathfrak{C}^R} \quad \text{and} \quad d_2^{\mathfrak{C}_1^R} = d_{\mathcal{K}}^{\mathfrak{C}^R}.$$

Since  $\mathfrak{C}^R$  and  $\mathfrak{C}_1^R$  have the same definable predicates, then for any  $A \subseteq \mathcal{K}$ ,  $\text{Aut}(\tilde{\mathfrak{C}}^R/A) = \text{Aut}(\tilde{\mathfrak{C}}_1^R/A)$ . Furthermore, if  $\mathfrak{N} \leq \tilde{\mathfrak{C}}$  then the structure  $\mathfrak{N}_1$ , defined in the same way, is an elementary substructure of  $\tilde{\mathfrak{C}}_1$ . Hence, for any  $A \subseteq \mathcal{K}$ ,  $\text{Autf}_L(\tilde{\mathfrak{C}}^R/A) = \text{Autf}_L(\tilde{\mathfrak{C}}_1^R/A)$ . The metric on  $G^R$ , viewed now as a definable group on  $\tilde{\mathfrak{C}}_1^R$ , is bi-invariant. By [Corollary 1.4.12](#),  $G^R$  does not have non-trivial  $A$ -invariant subgroups of bounded index in the new language. As we have the same definable predicates, the same is true for  $G^R$  in the original structure.  $\square$



## Chapter 2

# Universal Skolem sorts of Randomizations

### Introduction

A classical result, attributed to T. Coquand but first published by G. Ahlbrandt and M. Ziegler in [AZ86], states that two classical, complete,  $\omega$ -categorical theories in countable languages are determined, modulo bi-interpretation, by their automorphism groups. This result was later generalized by I. Ben Yaacov and A. Kaïchouh in [BYK16] to the continuous framework. Namely, two separable  $\omega$ -categorical continuous structures are bi-interpretable if and only if their automorphism groups are isomorphic as topological groups. They went even further and showed a reconstruction procedure, a way to define a structure from the automorphism group which is bi-interpretable with the original structure. Towards this end, starting from the automorphism group  $G$ , they showed that the structure  $\hat{G}$ , consisting of the left-completion of the group,  $\hat{G}_L$ , together with a distance predicate to each orbit closure of the diagonal action of  $G$  on  $(\hat{G}_L)^n$ , is bi-interpretable with the original structure.

In order to generalize this result and obtain a reconstruction procedure that applies beyond the context of  $\omega$ -categoricity, I. Ben Yaacov in [BY22] defined the notion of *universal Skolem sort*. To every complete continuous theory  $T$  in a countable language admitting such a sort we can associate a topological groupoid  $\mathbf{G}(T)$  over  $2^{\mathbb{N}}$  which determines  $T$  up to bi-interpretability, and from which we can define a theory bi-interpretable with  $T$ . We must remark that not every theory has a universal Skolem sort, those that admit it include classical and  $\omega$ -categorical theories.

Randomizations are basically structures consisting of random functions taking values in structures on a given language. They were first defined by H.J. Keisler in [Kei99] and then reformulated using the framework of continuous logic in [BYK09], and again in [BY13]. Formally, a randomization is a continuous two-sorted structure, one sort for the random functions in the structures and a second sort for  $[0, 1]$ -valued random variables. The two sorts are related via maps  $\llbracket \varphi(\cdot) \rrbracket$ , where  $\varphi(x)$  is a formula in the original language, that to each tuple of random elements  $a$  assigns a random variable  $\llbracket \varphi(a) \rrbracket$ . In case the given language is classical we can restrict our attention to  $\{0, 1\}$ -valued random variables, which we identify with events.

Randomizations preserve many desirable model-theoretical properties such as completeness,  $\omega$ -categoricity, stability and NIP. This results can be shown using analytic methods as in [BY13, BY09] or, in the  $\omega$ -categorical case, dynamically as in [Iba17]. In this context, the correspondence between stable (resp. NIP) formulas and weakly almost periodic (resp. tame)

functions, as shown in [Iba16], allows us to describe these model-theoretic properties in terms of representations of the action of the group of automorphisms on the space of types in a particular class of Banach spaces. Thus, by describing the group of automorphisms of the randomization in terms of the group of automorphisms of the original structure and by constructing a representation of the randomized action from the original representation, T. Ibarlucía showed these preservation results.

Using the notion of universal Skolem sort, and that of associated groupoid, we expect to extend T. Ibarlucía's methods to a broader context. The present chapter deals with the first of these issues, obtaining the groupoid associated to the randomization in terms of the original groupoid. Our first objective is to show that if a theory  $T$  admits a universal Skolem sort, then so it does its randomization  $T^R$ . The universal Skolem sort of  $T^R$  will be a fiber product between the randomization of the universal Skolem sort of  $T$  and the universal Skolem sort of the theory of random variables which, as an  $\omega$ -categorical theory, admits such a sort. We then give an explicit description of the groupoid associated to  $T^R$  and we show a purely topological construction of this groupoid starting from the groupoid associated to the original theory.

This chapter is organized as follows, in Section 2.1, we present the notions of universal Skolem sort and topological groupoid, and we show how to associate to a theory admitting a universal Skolem sort the topological groupoid that determines it up to bi-interpretation. In Section 2.2 we exhibit the universal Skolem sort of the randomization theory, assuming the original theory admits such a sort. In Section 2.3, we study the groupoid  $\mathbf{G}^\Omega$  consisting of measurable functions from a probability space  $(\Omega, \mathcal{B}, \mu)$  into a Polish open topological groupoid  $\mathbf{G}$ . We also study  $\mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$ , where  $\text{Aut}(\Omega)$  is the group of invertible measure preserving transformations of  $(\Omega, \mathcal{B}, \mu)$ . Finally, in Section 2.4 we describe the groupoid associated to the randomization and we show that it corresponds to  $(\mathbf{G}(T)^\Omega \rtimes \text{Aut}(\Omega))_\pi$ , the base change of  $\mathbf{G}(T)^\Omega \rtimes \text{Aut}(\Omega)$  to  $2^\mathbb{N}$  given by the map used to define the fiber product mentioned above.

*Notation.* Suppose  $\mathfrak{A}$  is a Boolean algebra and  $a \in \mathfrak{A}$  then  $a^0$  and  $a^1$  will denote respectively  $a$  and  $\neg a$ . In case  $\nu \in 2^m$  and  $(a_i)$  is a sequence of elements of  $\mathfrak{A}$  of length at least  $m$  then  $a_\nu$  stands for  $a_0^{\nu(0)} \wedge \dots \wedge a_{m-1}^{\nu(m-1)}$ . For our purposes,  $\mathfrak{A}$  will either be an algebra of formulas or a measure algebra. For  $A$  a family of sets in a  $\sigma$ -algebra we write  $\sigma(A)$  for its generated  $\sigma$ -algebra. If  $(X, d)$  is a metric space, then for  $x \in X$  and  $r > 0$ ,  $B_d(x, r)$  and  $\bar{B}_d(x, r)$  will denote, respectively, the open and closed  $d$ -balls with radius  $r$  centered at  $x$ . If the metric is clear from the context we may drop the subindex. Whenever  $f$  is a tuple of functions defined on a set  $\Omega$  and  $\psi(x)$  is a classical formula then  $\llbracket \psi(f) \rrbracket$  will denote the set  $\{\omega \in \Omega ; \psi(f(\omega)) \text{ is true}\}$ . Whenever  $X$  is a random variable,  $\mathbb{E}(X)$  will denote the expectation of  $X$ .

## 2.1 Universal Skolem sorts

Most of the results and definitions given in this section come from [BY22]. We fix a theory  $T$  in a countable language.

**Definition 2.1.1.** The *metric sorts* of  $T$  is the smallest collection of objects containing the home sorts of  $T$  and that is closed under the following three operations:

- Infinite product: If  $\{F_n ; n \in \mathbb{N}\}$  is a countable family of sorts of  $T$  then  $F = \prod_{n \in \mathbb{N}} F_n$  is a sort with the metric given by  $d(x, y) = \sup_{n \in \mathbb{N}} 2^{-n} \wedge d_{F_n}(x_n, y_n)$ .
- Definable subset: If  $F$  is a sort of  $T$  and  $E$  is a definable subset of  $F$  then  $E$  together with the metric it inherits from  $F$  is a sort of  $T$ .

- Metric quotient: If  $F$  is a sort of  $T$  and  $\rho$  is a definable pseudo-metric on  $F$  then  $E$ , the completion of the quotient of  $F$  induced by  $\rho$ , is a sort of  $T$  whose metric is  $\rho$ .

A metric sort of a theory is thus a definable subset of an imaginary sort. The formulas on an infinite product are the uniform limits of formulas on a finite subproduct. On the other hand, the formulas on a definable subset are just the restriction of formulas to the subset. Finally, the formulas on a metric quotient of  $F$  are the formulas on  $F$  that are uniformly continuous with respect to the pseudo-metric.

*Remark.*

The next results presents some natural sorts of the randomizations and characterizes their space of types.

**Proposition 2.1.2.** *Let  $F$  be a sort of  $T$ . Then  $F^R$ , defined as the set of random functions on  $F$  together with the metric given by  $d_{FR}(x, y) = \mathbb{E}[\|d_F(x, y)\|]$ , is a sort of  $T^R$ .*

*Proof.* If  $F$  is a home sort, then by definition  $F^R$  with the metric described above is a sort of  $T^R$ . Let  $\{F_n ; n \in \mathbb{N}\}$  be a countable family of sorts of  $T$  such that each  $F_n^R$  is a sort of  $T^R$  and let  $F = \prod_{n \in \mathbb{N}} F_n$ . It follows that  $D = \prod_{n \in \mathbb{N}} F_n^R$  with the metric given by  $d_D(x, y) = \sup_{n \in \mathbb{N}} 2^{-n} \wedge d_{FR}(x_n, y_n)$  is a sort of  $T^R$ . Note that  $D$  can be naturally identified with  $F^R$ . However, the metric  $d_D$  does not agree with  $d_{FR}$ . Nevertheless, it can be easily seen that  $d_{FR}$  is definable. It follows that the two metrics are uniformly equivalent and thus we may take  $d_{FR}$  as the canonical metric on  $D = F^R$ . By [Proposition 1.3.3](#), if  $F$  is a sort of  $T$  and  $E$  is a definable subset of  $F$  then  $E^R$  is a definable subset of  $F^R$ . Hence,  $E^R$  is a sort of  $T^R$ . Finally, suppose that  $F$  is a sort of  $T$ , that  $\rho$  is a definable pseudo-metric on  $F$  and that  $E$  is the quotient sort. We have that  $\rho^R(x, y) = \mathbb{E}[\|\rho(x, y)\|]$  is a definable pseudo-metric on  $F^R$  and that the quotient of  $F^R$  induced by  $\rho^R$  corresponds to  $E^R$ .  $\square$

Given a compact Hausdorff space  $K$ , we denote by  $\mathfrak{R}(K)$  the space of Borel probability measures on  $K$ . Note that  $\mathfrak{R}(K)$  is a subset of the unit ball of  $C(K)^*$  and as such it is naturally endowed with the weak\* topology. Since  $\mathfrak{R}(K)$  is closed, it is a compact space. The proof of the following result is completely analogous to that of [\[BY13, Theorem 3.32\]](#) but considering elements in the sort  $F$  rather than tuples of elements.

**Fact 2.1.3.** *Let  $F$  be a sort of  $T$ . Then  $S_{FR}(T^R)$  is homeomorphic to  $\mathfrak{R}(S_F(T))$ .*

We will now define the notions of universal and Skolem sorts.

**Definition 2.1.4.** Let  $D$  and  $F$  be sorts of  $T$ , we say that  $D$  is a *Skolem sort* for  $F$  if for every formula  $\varphi(x, y)$  in  $D \times F$  and every  $\epsilon > 0$  there is a definable map  $\sigma : D \rightarrow F$  such that for every  $x \in D$ ,  $\varphi(x, \sigma(x)) \leq \inf_y \varphi(x, y) + \epsilon$ . We say that  $D$  is a *universal sort* for  $F$  if for every  $\epsilon > 0$  there is a definable map  $\sigma : D \rightarrow F$  such that the image of every  $\epsilon$ -ball in  $D$  is  $\epsilon$ -dense in  $F$ .

A sort  $D$  is said to be *Skolem* if it is a Skolem sort for every sort  $F$ . Similarly, it is *universal* if it is a universal sort for any sort  $F$ . Notice that by considering  $\varphi(x, y) - \inf_z \varphi(x, z)$  in the previous definition, we may restrict our attention to the formulas satisfying  $\sup_x \inf_y \varphi(x, y) = 0$ . We can further restrict ourselves to a dense subset of such formulas.

*Notation.* Whenever we have a condition  $\psi(x) = 0$  and we have not quantified over the variable  $x$ , we will assume that there is an implicit universal quantifier. So, for instance,  $\inf_y \varphi(x, y) = 0$  will stand for  $\sup_x \inf_y \varphi(x, y) = 0$ .

The next result shows that in order to prove that a sort is universal Skolem it suffices to verify it for a small set of sorts.

**Proposition 2.1.5.** [BY22, Proposition 4.9] *A sort  $D$  is universal Skolem if and only if it is Skolem for every home sort and universal for every finite product of home sorts.*

Between every two universal Skolem sorts there is a definable bijection [BY22, Theorem 4.10]. The next result is a kind of converse of this result.

**Lemma 2.1.6.** *If  $D$  is a universal Skolem sort and  $\eta : D \rightarrow E$  is a definable bijection then  $E$  is a universal Skolem sort.*

*Proof.* Take  $F$  any sort, first we will prove that  $E$  is Skolem for  $F$ . Let  $\varphi(x, y)$  be a formula in  $E \times F$  such that  $\inf_y \varphi(x, y) = 0$  and let  $\epsilon > 0$ . Let  $z$  be a variable of sort  $D$  and define  $\psi(z, y) = \varphi(\eta(z), y)$ . Then,  $\inf_y \psi(z, y) = 0$  and, since  $D$  is a Skolem sort for  $F$ , there is a definable map  $\sigma : D \rightarrow F$  such that  $\psi(z, \sigma(z)) < \epsilon$ . It follows that  $\tilde{\sigma} = \sigma \circ \eta^{-1}$  is an  $\epsilon$ -Skolem map for  $\varphi$ . To show that  $E$  is weakly universal for  $F$ , take  $\epsilon > 0$  and choose  $0 < \delta < \epsilon$  such that for all  $z_1, z_2 \in D$ ,  $d(z_1, z_2) < \delta$  implies  $d(\eta(z_1), \eta(z_2)) < \epsilon$ . We know that there is a definable map  $\sigma : D \rightarrow F$  such that the image of any  $\delta$ -ball in  $D$  is  $\delta$ -dense in  $F$  and let us define  $\tilde{\sigma} = \sigma \circ \eta^{-1}$ . Take  $x \in E$  and  $y \in F$  arbitrary, and let  $z = \eta^{-1}(x)$ . We know that there is some  $z_0 \in D$  such that  $d(z_0, z) < \delta$  and  $d(\sigma(z_0), \sigma(z)) < \delta$ . If  $x_0 = \eta(z_0)$  then  $d(x_0, x) < \epsilon$  and  $d(\tilde{\sigma}(x_0), \tilde{\sigma}(x)) < \epsilon$ , which completes the proof.  $\square$

Not every theory admits a universal Skolem sort [BY22, Example 4.15]. However, all classical and all  $\omega$ -categorical theories admit such a sort.

**Fact 2.1.7.** [BY22, Proposition 4.17] *Suppose  $T$  is an  $\omega$ -categorical theory and take  $\mathcal{M}$  a separable model of  $T$ . Pick a dense enumeration of  $\mathcal{M}$ , then  $D_0 = \text{tp}(a)$  is a sort of  $T$  and  $D = D_0 \times 2^{\mathbb{N}}$  is a universal Skolem sort*

We will now exhibit another universal Skolem sort for an  $\omega$ -categorical theory.

**Lemma 2.1.8.** *Assume that  $T$  is  $\omega$ -categorical. If  $b$  enumerates a sequence whose definable closure is a model  $\mathcal{M} \models T$  and  $D_0$  is the type of  $b$  then  $D = D_0 \times 2^{\mathbb{N}}$  is a universal Skolem sort.*

*Proof.* Let  $a$  be an enumeration of a dense subset of  $M$  and let  $E_0$  be the sort defined by the type of  $a$ , we denote by  $S$  the sort of  $M$ . We will build a definable bijection between  $E_0$  and  $D_0$ . Since  $a$  is definable over  $b$ , there is a formula  $\varphi(x, y)$  in  $D_0 \times S^{\mathbb{N}}$  such that  $\varphi(b, y) = d(y, a)$ . Then, for any  $b' \in D_0$ ,  $\varphi(b', y)$  is a distance predicate on  $S^{\mathbb{N}}$ . Furthermore, if  $\varphi(b', a') = 0$  then  $a' \in E_0$ . Thus, we obtain a definable map  $f : D_0 \rightarrow E_0$  where  $d(f(x), y) = \varphi(x, y)$ . On the other hand, since  $a$  is dense in  $M$ , for every  $n \in \mathbb{N}$ , there is a subsequence  $(a_{n,k})_{k \in \mathbb{N}}$  of  $a$  such that  $\lim_{k \rightarrow \infty} a_{n,k} = b_n$ . Then,  $g(y) = (\lim_{k \rightarrow \infty} y_{n,k})_{n \in \mathbb{N}}$  is a definable map from  $E_0$  into  $D_0$ . It can be easily verified that  $f \circ g = \text{Id}_{E_0}$  and that  $g \circ f = \text{Id}_{D_0}$ . Therefore,  $f \times \text{Id}_{2^{\mathbb{N}}}$  is a definable bijection between  $D$  and  $E_0 \times 2^{\mathbb{N}}$ , the result follows from Lemma 2.1.6.  $\square$

In the case  $T = \text{APr}$ , we take  $\Xi$  to be a sequence of independent subsets of  $\Omega$  of measure  $1/2$  whose generated algebra is dense in  $(\bar{\mathcal{B}}, \mu)$ . It follows that if  $D_{\text{APr},0} = \text{tp}(\Xi)$ , then  $D_{\text{APr}} = D_{\text{APr},0} \times 2^{\mathbb{N}}$  is a universal Skolem sort for  $\text{APr}$ , as well as for  $\text{RV}$  by bi-interpretability.

### 2.1.1 Topological groupoids

We will now give some results on topological groupoids. Groupoids are usually defined as categories in which every morphism is invertible. What we call the base is just the set of objects of the category. However, for our purposes, it is more convenient to take an algebraic approach and see them as groups with a partially defined operation, as in [ADR00].

**Definition 2.1.9.** A *groupoid* is a non-empty set,  $\mathbf{G}$ , together with a partially defined binary operation and an inverse map such that:

- Whenever  $f, g, h \in \mathbf{G}$  and,  $fg$  and  $gh$  are defined, then  $f(gh)$  and  $(fg)h$  are both defined and they agree.
- For every  $g \in \mathbf{G}$ ,  $gg^{-1}$  and  $g^{-1}g$  are both defined.
- For every  $f, g \in \mathbf{G}$  if  $fg$  is defined then  $fgg^{-1} = f$  and  $f^{-1}fg = g$ .

For  $g \in \mathbf{G}$  we define the *target* and the *source* of  $g$  as  $t_g = gg^{-1}$  and  $s_g = g^{-1}g$  respectively. An element  $e \in \mathbf{G}$  is *neutral* if  $e^2 = e$ , the set of neutral elements of  $\mathbf{G}$  is known as the *base* of  $\mathbf{G}$  and it will be denoted by  $\mathbf{B}$ . It is easy to verify that  $(g^{-1})^{-1} = g$ ,  $t_g$  and  $s_g$  are neutral,  $s_{g^{-1}} = t_g$  and  $t_{g^{-1}} = s_g$ , and  $fg$  is defined if and only if  $s_f = t_g$ . We remark that the target and the source define maps from  $\mathbf{G}$  onto  $\mathbf{B}$ .

**Definition 2.1.10.** A *topological groupoid* is a groupoid  $\mathbf{G}$  together with a Hausdorff topology on  $\mathbf{G}$  such that the inverse and the product, where defined, are continuous.

A topological groupoid is said to be *open* if the source  $s : \mathbf{G} \rightarrow \mathbf{B}$ , or equivalently the target  $t : \mathbf{G} \rightarrow \mathbf{B}$ , is an open map. Now we introduce the notion of a groupoid action.

**Definition 2.1.11.** Let  $\mathbf{B}$  be a topological space, a *space over  $\mathbf{B}$*  is a couple  $(X, \rho)$  where  $X$  is a topological space and  $\rho : X \rightarrow \mathbf{B}$  is a continuous, surjective map.

Note that a topological groupoid is a space over its base, via the target or the source map.

**Definition 2.1.12.** Let  $\mathbf{G}$  be a topological groupoid of base  $\mathbf{B}$  and let  $(X, \rho)$  be a space over  $\mathbf{B}$ . The *fiber product between  $\mathbf{G}$  and  $X$*  is defined as

$$\mathbf{G} \times_{\mathbf{B}} X = \{(g, x) \in \mathbf{G} \times X ; s_g = \rho(x)\}.$$

A *continuous action of  $\mathbf{G}$  on  $X$*  is a continuous map  $\cdot : \mathbf{G} \times_{\mathbf{B}} X \rightarrow X$  such that whenever  $fg$  and  $g \cdot x$  are defined then  $f \cdot (g \cdot x) = (fg) \cdot x$ , meaning both sides are defined and they agree.

In this case we say that  $(X, \rho)$  is a  *$\mathbf{G}$ -space over  $\mathbf{B}$*  and we write  $\mathbf{G} \curvearrowright X$ . The dual notion of fiber product between  $X$  and  $\mathbf{G}$ ,  $X \times_{\mathbf{B}} \mathbf{G}$ , is defined analogously but with the target map instead of the source, from which the notions of right action and right  $\mathbf{G}$ -space can be easily defined.

Considering  $\mathbf{G}$  as a space over  $\mathbf{B}$ , we have that the domain of the groupoid operation is the fiber product  $\mathbf{G} \times_{\mathbf{B}} \mathbf{G} = \{(f, g) \in \mathbf{G}^2 ; s_f = t_g\}$ . Furthermore, the groupoid operation  $\mathbf{G} \times_{\mathbf{B}} \mathbf{G} \rightarrow \mathbf{G}$  is simultaneously a left and a right action of  $\mathbf{G}$  on itself. Finally, seeing  $(\mathbf{B}, \text{id}_{\mathbf{B}})$  as a space over itself, we have a natural continuous action of  $\mathbf{G}$  on  $\mathbf{B}$  given by  $(g, s_g) \mapsto t_g$  (do not mistake with the groupoid operation). The next fact characterizes open groupoids in terms of actions.

**Fact 2.1.13.** [BY22, Fact 1.5] *Let  $\mathbf{G}$  be a topological groupoid of base  $\mathbf{B}$ . Then, the following are equivalent:*

1.  $\mathbf{G}$  is an open groupoid.
2. For any  $(X, \rho)$  space over  $\mathbf{B}$ , the projection  $\mathbf{G} \times_{\mathbf{B}} X \rightarrow X$  is open.
3. For any  $\mathbf{G}$ -space  $(X, \rho)$ , the action law  $\mathbf{G} \times_{\mathbf{B}} X \rightarrow X$  is open.
4. The groupoid operation  $\mathbf{G} \times_{\mathbf{B}} \mathbf{G} \rightarrow \mathbf{G}$  is open.

Finally we introduce the notion of a minimal action.

**Definition 2.1.14.** Let  $\mathbf{G}$  be a topological groupoid over  $\mathbf{B}$  and let  $(X, \rho)$  be a  $\mathbf{G}$ -space over  $\mathbf{B}$ . We say that the action  $\mathbf{G} \curvearrowright X$  is *minimal* if every orbit is dense. In particular, when  $\mathbf{G} \curvearrowright \mathbf{B}$  is minimal we say that  $\mathbf{G}$  itself is *minimal*.

In other words,  $\mathbf{G}$  is minimal if and only if for every  $e \in \mathbf{B}$  and  $W$  open subset of  $\mathbf{B}$  there is  $g \in \mathbf{G}$  with  $t_g = e$  and  $s_g \in W$ , or equivalently, there is some  $h \in \mathbf{G}$  with  $s_h = e$  and  $t_h \in W$ .

## 2.1.2 Groupoid associated to a theory

Now we will show how to define the groupoid associated to a theory admitting a universal Skolem sort following [BY22, Section 5]. Given a sort  $D$ , by  $S_{nD}(T)$  we mean the space of types on  $D^n$ . Suppose  $D_T$  is a universal Skolem sort of  $T$ . The *groupoid associated to  $T$* , denoted  $\mathbf{G}(T)$ , is defined as

$$\mathbf{G}(T) = \{\text{tp}(a, b) \in S_{2D_T}(T) ; \text{dcl}(a) = \text{dcl}(b)\}.$$

Its base,  $\mathbf{B}(T)$ , is the set of types of the form  $\text{tp}(a, a)$  with  $a \in D_T$ , which we identify with  $\text{tp}(a)$ , and thus  $\mathbf{B}(T) = S_{D_T}(T)$ , which we know to be homeomorphic to  $2^{\mathbb{N}}$  [BY22, Lemma 4.11]. If  $p = \text{tp}(a, b)$  then its inverse is given by  $p^{-1} = \text{tp}(b, a)$ . If  $q = \text{tp}(b', c')$  and  $\text{tp}(b) = \text{tp}(b')$ , then there is a unique  $c \in D_T$  such that  $\text{tp}(b, c) = q$ , and we define  $pq = \text{tp}(a, c)$ .

**Theorem 2.1.15.** [BY22, Proposition 5.3 and Theorem 5.14] *Let  $T$  be a complete theory admitting a universal Skolem sort. Then,  $\mathbf{G}(T)$ , endowed with the topology it inherits as a subset of  $S_{2D_T}(T)$ , is a Polish, open, minimal, topological groupoid whose base  $\mathbf{B}(T)$  is homeomorphic to  $2^{\mathbb{N}}$ . Furthermore, if  $T'$  is another complete theory admitting a universal Skolem sort then  $\mathbf{G}(T)$  is isomorphic to  $\mathbf{G}(T')$ , as topological groupoids, if and only if  $T$  and  $T'$  are bi-interpretable.*

Assume that  $T$  is  $\omega$ -categorical and let  $\mathcal{M}$  be a separable model of  $T$ . Recall from Lemma 2.1.8 that if  $a$  enumerates a sequence satisfying  $\text{dcl}(a) = \mathcal{M}$  and  $D_0 = \text{tp}(a)$ , then  $D = D_0 \times 2^{\mathbb{N}}$  is a universal Skolem sort. It follows that the groupoid associated to  $T$  is  $\mathbf{G}(T) = 2^{\mathbb{N}} \times \text{Aut}(\mathcal{M}) \times 2^{\mathbb{N}}$ . In particular, when  $T = \text{APr}$  the associated groupoid is  $\mathbf{G}(\text{APr}) = 2^{\mathbb{N}} \times \text{Aut}(\Omega) \times 2^{\mathbb{N}}$ .

## 2.2 Universal Skolem sort of the randomization

From now on, we assume that  $T$  is a complete continuous theory in a countable language that admits a universal Skolem sort  $D_T$ . In this section we will exhibit a universal Skolem sort for  $T^R$ . Assuming  $T$  has only one home sort, the randomization theory is two-sorted, the home sort and the sort of random variables. Therefore, it is natural to think that a universal Skolem sort for  $T^R$  must involve  $D_T^R$ , the randomization of the universal Skolem sort of  $T$ , and  $D_{\text{APr}}$ , the universal Skolem sort of RV, as defined after Lemma 2.1.8. The home sort and the random variable sort are related through the maps  $\llbracket \varphi(\cdot) \rrbracket$ , where  $\varphi(x)$  is an  $\mathcal{L}$ -formula. In order to capture this relation we will define a fiber product between  $D_T^R$  and  $D_{\text{APr}}$ .

From now on  $(\Omega, \mathcal{B}, \mu)$  will refer to the probability space obtained from the Stone space of the usual Borel algebra on the interval  $[0, 1]$ , as described in Section 1.1.  $E$  and  $E_{\text{RV}}$  will stand respectively for the sort of events and the sort of random variables in ARV.  $E(\Omega)$  and  $E_{\text{RV}}(\Omega)$  will refer to the sort of events and random variables in  $(\Omega, \mathcal{B}, \mu)$ . We will identify a member of  $E(\Omega)$  with the corresponding clopen subset of  $\Omega$ .  $x$  will denote a variable of sort  $D_T$ ,  $\bar{x}$  a variable of sort  $D_{\text{APr}, 0}$ , and  $\dot{x}$  a variable of sort  $2^{\mathbb{N}}$ .

We know that  $S_{D_T}(T)$  is homeomorphic to  $2^{\mathbb{N}}$ . Hence, by considering  $2^{\mathbb{N}}$  as a sort of  $T$ , the type function  $\text{tp} : D_T \rightarrow 2^{\mathbb{N}}$  is definable. By [Proposition 1.3.6](#), the corresponding map on the randomization  $\text{tp}^R : D_T^R \rightarrow (2^{\mathbb{N}})^R$  is also definable. On the other hand, the function  $\iota : 2^R \rightarrow E$ , that assigns to every  $f \in 2^R$  the event whose characteristic function is  $f$ , establishes a definable isometric bijection. Following the argument given in the proof of [Proposition 2.1.2](#),  $(2^R)^{\mathbb{N}}$  can be identified with  $(2^{\mathbb{N}})^R$ . Hence, we have a definable bijection  $\psi : (2^{\mathbb{N}})^R \rightarrow E^{\mathbb{N}}$  given by  $\psi(f)_n = \llbracket f_n = 0 \rrbracket$  for  $f \in (2^{\mathbb{N}})^R$  and  $n \in \mathbb{N}$ . Recall from [Proposition 2.1.2](#) that the metric on  $(2^{\mathbb{N}})^R$  is given by the expectation of the product metric on  $2^{\mathbb{N}}$ . It follows that the two metrics on  $(2^{\mathbb{N}})^R$  given by

$$d(f, g) = \mathbb{E}[\llbracket d(f, g) \rrbracket] \quad \text{and} \quad d'(f, g) = \sup_{n \in \mathbb{N}} 2^{-n} \wedge \mu(\llbracket f_n = 0 \rrbracket \Delta \llbracket g_n = 0 \rrbracket),$$

are uniformly equivalent. So, when convenient, we can consider  $\text{tp}^R$  as a map from  $D_T^R$  into  $E^{\mathbb{N}}$ . Moreover, in the separable case, the previous identification commutes with the action of  $\text{Aut}(\Omega)$  i.e. for any  $\tau \in \text{Aut}(\Omega)$  and any  $f \in (2^{\mathbb{N}})^R$ ,  $\psi(\tau \cdot f) = \tau(\psi(f))$ .

**Lemma 2.2.1.** *For every  $\epsilon > 0$  and every  $k \in \mathbb{N}$  there is  $\delta > 0$  such that for every  $x \in D_T^R$  and every  $A \in E^{\mathbb{N}}$  satisfying  $d(\text{tp}^R(x), A) < \delta$ , there exists  $x' \in D_T^R$  such that  $d(x, x') < \epsilon$  and  $\text{tp}_n^R(x') = A_n$  for  $n < k$ .*

*Proof.* First note that, given  $\eta \in \{0, 1\}^k$ , there is  $a_\eta \in D_T^R$  such that  $\text{tp}_\eta^R(a_\eta) = \top \in E$ . Take  $\delta < \min\{2^{-k}, \epsilon/k\}$  and suppose that  $d(\text{tp}^R(x), A) < \delta$ . For  $n < k$  define  $D_n = \text{tp}_n^R(x) \Delta A_n$ , and denote by  $\mathcal{A}$  the algebra generated by  $\{A_n; n < k\} \cup \{\text{tp}_n^R(x); n < k\}$ . Let  $B_0, \dots, B_m$  be the atoms of  $\mathcal{A}$  contained in  $D_0 \cup \dots \cup D_{k-1}$ . We know that for each  $i < m$  there is  $\eta_i \in \{0, 1\}^k$  such that  $B_i \subseteq A_{\eta_i}$ . Define  $x' \in D_T^R$  such that  $x'$  agrees with  $a_{\eta_i}$  on  $B_i$  for  $i < m$ , and agrees with  $x$  on  $\neg(D_0 \cup \dots \cup D_{k-1})$ . Then  $\text{tp}_n^R(x') = A_n$  for  $n < k$  and

$$d(x, x') \leq \mu(D_0 \cup \dots \cup D_{k-1}) \leq \sum_{n < k} \mu(D_n) < k\delta < \epsilon. \quad \square$$

We want a fiber product between  $D_T^R$  and  $D_{\text{APr}}$  over  $E^{\mathbb{N}}$  to be definable. We already have the map  $\text{tp}^R : D_T^R \rightarrow E^{\mathbb{N}}$  and we want to find an appropriate map  $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ . In order to prove definability we will use numeral  $\beta$ . of [Fact 1.3.1](#) with the predicate  $d(\text{tp}^R(x), \pi(\bar{x}, \dot{x}))$ . Assume  $d(\text{tp}^R(x), \pi(\bar{x}, \dot{x}))$  is small, the previous lemma shows that we can take an  $x'$  close to  $x$  so that the first few elements of the sequences  $\text{tp}(x')$  and  $\pi(\bar{x}, \dot{x})$  coincide. The following definition will ensure that we can find  $(\bar{x}', \dot{x}')$  close to  $(\bar{x}, \dot{x})$  so that  $\text{tp}^R(x') = \pi(\bar{x}', \dot{x}')$ . Surprisingly, at least for us, this property will be enough to show that the fiber product is in fact a universal Skolem sort of  $T^R$ .

**Definition 2.2.2.** A map  $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  is said to be *weakly open* if it is definable and for any given  $\epsilon > 0$  and  $N \in \mathbb{N}$ , there is some  $k \in \mathbb{N}$  such that for every  $(\bar{x}, \dot{x}) \in D_{\text{APr}}$ , every separable  $\mathcal{A} \models \text{APr}$  containing  $\bar{x}$  and every  $A \in E^{\mathbb{N}}(\mathcal{A})$  with  $\pi_n(\bar{x}, \dot{x}) = A_n$  for every  $n < k$ , there exists  $(\bar{x}', \dot{x}') \in D_{\text{APr}}(\mathcal{A})$  satisfying  $d(\bar{x}, \bar{x}') < \epsilon$ ,  $\dot{x} \upharpoonright_N = \dot{x}' \upharpoonright_N$ ,  $\pi(\bar{x}', \dot{x}') = A$ . If under the same conditions  $(\bar{x}', \dot{x}')$  can be chosen so that  $\sigma(\bar{x}') = \mathcal{A}$  we say that  $\pi$  is *pseudo-open*.

**Lemma 2.2.3.** *Let  $\pi^0 : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  be weakly open. Define  $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  by  $\pi_n(\bar{x}, \dot{x}) = \pi_{2n}^0(\bar{x}, \dot{x})$  for  $n \in \mathbb{N}$  and  $(\bar{x}, \dot{x}) \in D_{\text{APr}}$ . Then,  $\pi$  is pseudo-open.*

*Proof.* Given  $\epsilon > 0$  and  $N \in \mathbb{N}$ , take  $k \in \mathbb{N}$  as in the definition of weakly open for  $\pi^0$ . We can assume that  $k$  is pair, so  $k = 2m$  for some  $m \in \mathbb{N}$ . Take  $(\bar{x}, \dot{x}) \in D_{\text{APr}}$ ,  $\mathcal{A} \models \text{APr}$  separable with  $\bar{x} \subseteq \mathcal{A}$  and  $A \in E^{\mathbb{N}}$  satisfying  $\pi_n(\bar{x}, \dot{x}) = A_n$  for every  $n < m$ . Let  $(a_l)_{l \in \mathbb{N}}$  be an enumeration of a dense subset of  $\mathcal{A}$ . For  $n \in \mathbb{N}$  we set

$$B_n = \begin{cases} A_l & \text{if } n = 2l \text{ with } l \in \mathbb{N} \\ \pi_n^0(\bar{x}, \dot{x}) & \text{if } n = 2l + 1, \text{ with } l < m \\ a_l & \text{if } n = k + 2l + 1 \text{ with } l \in \mathbb{N}. \end{cases}$$

It follows that  $\pi_n^0(\bar{x}, \dot{x}) = B_n$  for  $n < k$ . Therefore, there is  $(\bar{x}', \dot{x}') \in D_{\text{APr}}(\mathcal{A})$  satisfying  $d(\bar{x}, \bar{x}') < \epsilon$ ,  $\dot{x} \upharpoonright_N = \dot{x}' \upharpoonright_N$  and  $\pi^0(\bar{x}', \dot{x}') = B$ . Hence,  $\pi(\bar{x}', \dot{x}') = A$  and  $\mathcal{A} = \sigma((a_l)_{l \in \mathbb{N}}) \subseteq \sigma(B) \subseteq \sigma(\bar{x}')$ . We conclude that  $\pi$  is pseudo-open.  $\square$

**Lemma 2.2.4.** *Suppose that  $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  is pseudo-open. Then, the fiber product*

$$D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}} = \{(x, \bar{x}, \dot{x}) \in D_T^R \times D_{\text{APr}}; \text{tp}^R(x) = \pi(\bar{x}, \dot{x})\}.$$

*is a definable subset of  $D_T^R \times D_{\text{APr}}$ .*

*Proof.* We will show that for every  $\epsilon > 0$  there exists  $\delta > 0$  such that for every  $(x, \bar{x}, \dot{x}) \in D_T^R \times D_{\text{APr}}$  if  $d(\text{tp}^R(x), \pi(\bar{x}, \dot{x})) < \delta$  then  $d((x, \bar{x}, \dot{x}), D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}) < \epsilon$ . Fix  $\epsilon > 0$  and take  $N \in \mathbb{N}$  with  $2^{-N} < \epsilon$ . For this  $\epsilon$  and this  $N$  let  $k \in \mathbb{N}$  be as in the definition of pseudo-open. Now, for such  $\epsilon$  and  $k$  take  $\delta > 0$  as given by [Lemma 2.2.1](#). Suppose that  $(x, \bar{x}, \dot{x}) \in D_T^R \times D_{\text{APr}}$  and that  $d(\text{tp}^R(x), \pi(\bar{x}, \dot{x})) < \delta$ . Using [Lemma 2.2.1](#), we find  $x' \in D_T^R$  such that  $d(x, x') < \epsilon$  and  $\text{tp}_n^R(x') = \pi_n(\bar{x}, \dot{x})$  for  $n < k$ . As  $\pi$  is pseudo-open, there is  $(\bar{x}', \dot{x}') \in D_{\text{APr}}$  with  $\pi(\bar{x}', \dot{x}') = \text{tp}^R(x')$  such that  $d(\bar{x}, \bar{x}') < \epsilon$  and  $\dot{x} \upharpoonright_N = \dot{x}' \upharpoonright_N$ . Thus,  $(x', \bar{x}', \dot{x}') \in D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$  and  $d((x, \bar{x}, \dot{x}), (x', \bar{x}', \dot{x}')) < \epsilon$ .  $\square$

Assuming that a pseudo-open map  $\pi$  exists, as it will be the case, the previous lemma shows that  $D_{T^R} = D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$  is a sort of  $T^R$ . Now we want to show that it is in fact a universal Skolem sort. By [Proposition 2.1.5](#), it suffices to show that for any sort  $F$  of  $T$ ,  $D_{T^R}$  is universal Skolem for  $F^R$ . So, from now on, we will assume that a pseudo-open map  $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  exists and that  $F$  is a sort of  $T$ . We let  $z$  be a variable of sort  $F$  and  $\hat{x} = (x, \bar{x}, \dot{x})$  be a variable of sort  $D_{T^R}$ . First we will prove that  $D_{T^R}$  is universal for  $F^R$ . We will denote by  $d_\infty$  the supremum metric.

**Lemma 2.2.5.** *For any given  $\epsilon > 0$  there is  $\epsilon' > 0$  such that for every  $\hat{x} \in D_{T^R}$  and every  $y \in D_T^R$ , if  $d_\infty(x, y) < \epsilon'$  then there is  $(\bar{y}, \dot{y}) \in D_{\text{APr}}$  such that  $\hat{y} = (y, \bar{y}, \dot{y}) \in D_{T^R}$  and  $d(\hat{x}, \hat{y}) < \epsilon$ .*

*Proof.* Fix  $\epsilon > 0$ , let  $N \in \mathbb{N}$  verify  $2^{-N} < \epsilon$ , and take  $k \in \mathbb{N}$  as in the definition of being pseudo-open. We know that there is  $\delta > 0$  such that for every  $x_0, x_1 \in D_T$ ,  $d(x_0, x_1) < \delta$  implies  $\text{tp}_i(x_0) = \text{tp}_i(x_1)$  for every  $i < k$ . Let  $\epsilon' = \min\{\epsilon, \delta\}$ , take any  $\hat{x} \in D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$  and take  $y \in D_T^R$  with  $d_\infty(x, y) < \epsilon'$ . So,  $\text{tp}_i^R(y) = \text{tp}_i^R(x) = \pi_i(\bar{x}, \dot{x})$  for  $i < k$ . Since  $\pi$  is pseudo-open, there is some  $(\bar{y}, \dot{y}) \in D_{\text{APr}}$  such that  $d((\bar{x}, \dot{x}), (\bar{y}, \dot{y})) < \epsilon$  and  $\pi(\bar{y}, \dot{y}) = \text{tp}^R(y)$ , completing the proof.  $\square$

**Lemma 2.2.6.** *Let  $F$  be a sort of  $T$ , then the sort  $D_{T^R}$  is universal for  $F^R$ .*

*Proof.* Fix  $\epsilon > 0$ , and take  $0 < \epsilon' < \epsilon$  as in the previous lemma. Since  $D_T$  is universal for  $F$ , there exists a definable map  $\sigma : D_T \rightarrow F$  such that the image of any  $\epsilon'/2$ -ball in  $D_T$  is  $\epsilon'/2$ -dense in  $F$ . Let  $\sigma^R : D_T^R \rightarrow F^R$  be the corresponding map in the randomization. Define the map  $\tau : D_{T^R} \rightarrow F^R$  by  $\tau(\hat{x}) = \sigma^R(x)$ . Pick any  $\hat{x} \in D_{T^R}$  and  $z \in F^R$ . We have that  $T^R \models \sup_x \sup_z \inf_y [d(x, y) \vee d(z, \sigma^R(y))] \leq \epsilon'/2$ . By [[BY13](#), Lemma 3.13], there is some  $y \in D_T^R$  satisfying  $d_\infty(x, y) < \epsilon'$  and  $d_\infty(\sigma^R(y), z) < \epsilon$ . By the previous lemma, there is  $(\bar{y}, \dot{y}) \in D_{\text{APr}}$  such that  $\hat{y} = (y, \bar{y}, \dot{y}) \in D_{T^R}$  and  $d(\hat{x}, \hat{y}) < \epsilon$ . Thus,  $d(\tau(\hat{y}), z) < \epsilon$ , showing that the image via  $\tau$  of  $B(\hat{x}, \epsilon)$  is  $\epsilon$ -dense.  $\square$

The next lemma says that as a result of having defined  $D_{T^R}$  as a fiber product over  $E^{\mathbb{N}}$  with the type function, the map  $\llbracket \varphi(x) \rrbracket$  can be obtained in a definable manner from  $(\bar{x}, \dot{x})$ .

**Lemma 2.2.7.** *Suppose that  $\varphi(x)$  is an  $\mathcal{L}$ -formula in  $D_T$ . Then there is a definable map  $\sigma : D_{\text{APr}} \rightarrow E_{\text{RV}}$  such that for every  $\hat{x} \in D_{TR}$ ,  $\llbracket \varphi(x) \rrbracket = \sigma(\bar{x}, \dot{x})$ .*

*Proof.* We know that there exists a continuous function  $f : 2^{\mathbb{N}} \rightarrow [0, 1]$  such that  $\varphi(x) = f(\text{tp}(x))$  for every  $x \in D_T$ . Identifying  $(2^{\mathbb{N}})^R$  with  $E^{\mathbb{N}}$ , the map  $f^R : E^{\mathbb{N}} \rightarrow E_{\text{RV}}$  is definable. Since for every  $x \in D_T^R$ ,  $\llbracket \varphi(x) \rrbracket = f^R(\text{tp}^R(x))$ , we conclude that for every  $\hat{x} \in D_{TR}$ ,  $\llbracket \varphi(x) \rrbracket = f^R(\pi(\bar{x}, \dot{x}))$ , which is a definable map on  $(\bar{x}, \dot{x}) \in D_{\text{APr}}$ .  $\square$

Assume that  $T$  is a classical theory. Suppose that for  $i < m$ ,  $\sigma_i : D_T \rightarrow F$  and  $\eta_i : D_{\text{APr}} \rightarrow E$  are definable. Furthermore, assume that for every  $(\bar{x}, \dot{x}) \in D_{\text{APr}}$ ,  $\{\eta_0(\bar{x}, \dot{x}), \dots, \eta_{m-1}(\bar{x}, \dot{x})\}$  is a partition of the probability space. Then, the map  $\rho : D_{TR} \rightarrow F^R$  given by

$$\rho(\hat{x}) = \langle \eta_0(\bar{x}, \dot{x}), \dots, \eta_{m-1}(\bar{x}, \dot{x}), \sigma_0^R(x), \dots, \sigma_{m-1}^R(x) \rangle$$

is definable, as

$$d(\rho(\hat{x}), y) = \sum_{i < m} \mu(\llbracket y \neq \sigma_i^R(x) \rrbracket \cap \eta_i(\bar{x}, \dot{x})).$$

**Lemma 2.2.8.** *Suppose that  $T$  is a classical theory and that  $F$  is a sort of  $T$ . Then,  $D_{TR}$  is a Skolem sort for  $F^R$ .*

*Proof.* Fix  $\varphi(\hat{x}, y)$  a formula in  $D_{TR} \times F^R$  with  $\inf_y \varphi(\hat{x}, y) = 0$  and let  $\epsilon > 0$ . We can assume that there is a  $\mathcal{L}_{\text{Pr}}$ -formula  $\psi(z, \bar{x}, \dot{x})$  in  $E^m \times D_{\text{APr}}$  such that

$$\varphi(\hat{x}, y) = \psi(\llbracket \varphi_0(x, y) \rrbracket, \dots, \llbracket \varphi_{m-1}(x, y) \rrbracket, \bar{x}, \dot{x})$$

where  $\varphi_i(x, y)$  is a  $\mathcal{L}$ -formula in  $D_T \times F$  for  $i < m$ . Take  $\delta < \epsilon$  such that for every  $A, B \in E^m$  and every  $(\bar{x}, \dot{x}) \in D_{\text{APr}}$ ,  $d(A, B) < \delta$  implies  $|\psi(A, \bar{x}, \dot{x}) - \psi(B, \bar{x}, \dot{x})| < \epsilon/2$ . Since the sort  $D_T$  is Skolem for  $F$ , for each  $\nu \in 2^m$  there exists a definable map  $\sigma_\nu : D_T \rightarrow F$  such that  $\llbracket \exists u \varphi_\nu(x, u) \rrbracket = \llbracket \varphi_\nu(x, \sigma_\nu(x)) \rrbracket$ . In addition, by [Lemma 2.2.7](#), for  $\nu \in 2^m$  there are  $\eta_\nu, \lambda_\nu : D_{\text{APr}} \rightarrow E$  such that for every  $\hat{x} \in D_{TR}$ ,  $\llbracket \exists u \varphi_\nu(x, u) \rrbracket = \eta_\nu(\bar{x}, \dot{x})$  and  $\llbracket \forall u \varphi_\nu(x, u) \rrbracket = \lambda_\nu(\bar{x}, \dot{x})$ . We define the  $\mathcal{L}_{\text{Pr}}$ -formula

$$\chi(\bar{x}, \dot{x}, z) = \psi(z, \bar{x}, \dot{x}) \vee \left( \sum_{\nu \in 2^m} \mu(z_\nu \setminus \eta_\nu(\bar{x}, \dot{x})) \vee \mu(\lambda_\nu(\bar{x}, \dot{x}) \setminus z_\nu) \right).$$

Note that the condition  $\chi(\bar{x}, \dot{x}, z) = 0$  essentially says that  $\psi(z, \bar{x}, \dot{x}) = 0$  and  $z$  is a tuple of the form  $(\llbracket \varphi_i(x, y) \rrbracket)_{i < m}$ . It follows that,  $\inf_z \chi(\bar{x}, \dot{x}, z) = 0$ . Therefore, there is a definable map  $\xi : D_{\text{APr}} \rightarrow E^m$  such that  $\chi(\bar{x}, \dot{x}, \xi(\bar{x}, \dot{x})) < \delta/2^m$ . Let  $\rho : D_{TR} \rightarrow F^R$  be the definable application given by

$$\rho(\hat{x}) = \langle (\xi_\nu(\bar{x}, \dot{x}) ; \nu \in 2^m), (\sigma_\nu^R(x) ; \nu \in 2^m) \rangle.$$

We will show that  $\varphi(\hat{x}, \rho(\hat{x})) < \epsilon$ . First, note that for  $i < m$

$$\begin{aligned} d(\llbracket \varphi_i(x, \rho(\hat{x})) \rrbracket, \xi_i(\bar{x}, \dot{x})) &= d(\bigcup \{ \xi_\nu(\bar{x}, \dot{x}) \cap \llbracket \exists u \varphi_\nu(x, u) \rrbracket ; \nu \in 2^m, \nu(i) = 0 \}, \xi_i(\bar{x}, \dot{x})) \\ &= \sum_{\nu(i)=0} \mu(\xi_\nu(\bar{x}, \dot{x}) \setminus \llbracket \exists u \varphi_\nu(x, u) \rrbracket) \\ &< \sum_{\nu(i)=0} \delta/2^m \\ &< \delta. \end{aligned}$$

Thus,

$$\begin{aligned}
\phi(\hat{x}, \rho(\hat{x})) &= \psi(\llbracket \varphi_0(x, \rho(\hat{x})) \rrbracket, \dots, \llbracket \varphi_{m-1}(x, \rho(\hat{x})) \rrbracket, \bar{x}, \dot{x}) \\
&< \psi(\xi(\bar{x}, \dot{x}), \bar{x}, \dot{x}) + \epsilon/2 \\
&\leq \chi(\bar{x}, \dot{x}, \xi(\bar{x}, \dot{x})) + \epsilon/2 \\
&< \epsilon.
\end{aligned}$$

□

We will now study the continuous case, which will also cover the classical one. This means that we will no longer work with events but with random variables instead. This poses two main difficulties. Firstly, it is not so easy to say in terms of  $(\bar{x}, \dot{x})$  that  $z$  is a tuple of the form  $(\llbracket \varphi_i(x, y) \rrbracket)_{i < m}$ , as we did in the formula  $\chi(\bar{x}, \dot{x}, z)$ . Secondly, the map  $\xi$  will give us a tuple of random variables that we must use to define the approximate Skolem map. In this case, instead of a partition of the probability space in measurable sets we do a partition of the constant map  $\mathbf{1}$  in random variables.

Fix  $n \in \mathbb{N}$ , for  $i < 2^n$  we will define continuous functions (in each case we truncate the functions at 1)  $\lambda_{n,i} : [0, 1] \rightarrow [0, 1]$  by:

$$\begin{aligned}
\lambda_{n,0}(x) &= 2^{2^n}(1/2^n \dot{-} x), \\
\lambda_{n,i}(x) &= 2^{2^n}[(x \dot{-} (i/2^n - 1/2^{2^n})) \wedge ((i+1)/2^n \dot{-} x)] \text{ for } 0 < i < 2^n - 1, \\
\lambda_{n,2^n-1}(x) &= 2^{2^n}(x \dot{-} ((2^n - 1)/2^n - 1/2^{2^n})).
\end{aligned}$$

These mappings give rise to definable functions  $\lambda_{n,i} : E_{\text{RV}} \rightarrow E_{\text{RV}}$  defined by  $\lambda_{n,i}(z) = \lambda_{n,i} \circ z$ . Since for every  $x \in [0, 1]$ ,  $\sum_{i < 2^n} \lambda_{n,i}(x) = 1$ , then for every  $z \in E_{\text{RV}}$ ,  $\{\lambda_{n,i}(z) ; i < 2^n\}$  is a partition of the constant map  $\mathbf{1}$ . We also remark that  $\lambda_{n,i}(z) = 1$  exactly when  $i/2^n \leq z \leq (i+1)/2^n - 1/2^{2^n}$  and,  $\lambda_{n,i}(z) = 0$  when  $z \leq i/2^n - 1/2^{2^n}$  or  $(i+1)/2^n \leq z$ . Now, for  $m \geq 1$  and  $\bar{i} \in (2^n)^m$ , we define  $\lambda_{n,\bar{i}}^m : [0, 1]^m \rightarrow [0, 1]$  by  $\lambda_{n,\bar{i}}^m(u) = \prod_{k < m} \lambda_{n,i_k}(u_k)$ .

The following two lemmas solve the first problem mentioned above.

**Lemma 2.2.9.** *Let  $F$  be a sort of  $T$  and let  $\varphi(x, y)$  be a formula in  $D_T \times F$ . For  $x \in D_T^R$ , let  $A(x) = \{\llbracket \varphi(x, y) \rrbracket ; y \in F^R\}$ . Then there is a definable predicate  $P : E_{\text{RV}} \times D_{\text{APr}} \rightarrow [0, 1]$  such that for every  $\hat{x} \in D_{TR}$  and every  $z \in E_{\text{RV}}$ ,  $d(z, A(x)) = P(z, \bar{x}, \dot{x})$ .*

*Proof.* Take  $n \in \mathbb{N}$ , for  $i < 2^n$  let  $\varphi_{n,i}(x, y) = |i/2^n - \varphi(x, y)|$ . Now, let us define

$$\tau_n(z, x) = \sum_{i < 2^n} \lambda_{n,i}(z) \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket, \quad P_n(z, x) = \mathbb{E}[\tau_n(z, x)].$$

We have that  $d(z, A(x)) = \inf_y d(z, \llbracket \varphi(x, y) \rrbracket) = \inf_y \mathbb{E}[|z - \llbracket \varphi(x, y) \rrbracket|] = \mathbb{E}[\inf_y |z - \llbracket \varphi(x, y) \rrbracket|]$ , and let us denote by  $\tau(z, x)$  this last function. Note that  $|\tau(z, x) - \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket| \leq 1/2^n$  on  $\llbracket i/2^n \leq z \leq (i+1)/2^n \rrbracket$ . Thus, on this set,

$$\begin{aligned}
|\tau(z, x) - \tau_n(z, x)| &= \left| \tau(z, x) - \lambda_{n,i}(z) \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket - \lambda_{n,i+1}(z) \llbracket \inf_y \varphi_{n,i+1}(x, y) \rrbracket \right| \\
&\leq \left| \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket - \lambda_{n,i}(z) \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket - \lambda_{n,i+1}(z) \llbracket \inf_y \varphi_{n,i+1}(x, y) \rrbracket \right| + 1/2^n \\
&\leq \left| \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket - \llbracket \inf_y \varphi_{n,i+1}(x, y) \rrbracket \right| + 1/2^n \\
&\leq 1/2^{n-1}.
\end{aligned}$$

Hence,  $d(\tau(z, x), \tau_n(z, x)) \leq 1/2^{n-1}$ . In this way,  $d(z, A(x)) = \lim_{n \rightarrow \infty} P_n(z, x)$ . By [Lemma 2.2.7](#), for each  $n \in \mathbb{N}$  and each  $i < 2^n$ , there is a definable map  $\sigma_{n,i} : D_{\text{APr}} \rightarrow E_{\text{RV}}$  such that for every  $\hat{x} \in D_{\text{TR}}$ ,  $\llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket = \sigma_{n,i}(\bar{x}, \hat{x})$ . Therefore, if  $P(z, \bar{x}, \hat{x}) = \lim_{n \rightarrow \infty} \mathbb{E} \left[ \sum_{i < 2^n} \lambda_{n,i}(z) \sigma_{n,i}(\bar{x}, \hat{x}) \right]$  then  $d(z, A(x)) = P(z, \bar{x}, \hat{x})$ .  $\square$

If we generalize the proof of the previous lemma and use the random variables  $\{\lambda_{n,i}^m; \bar{i} \in (2^n)^m\}$  in order to consider multiple formulas simultaneously we get the following result.

**Lemma 2.2.10.** *Let  $F$  be a sort of  $T$  and let  $\varphi_0(x, y), \dots, \varphi_{m-1}(x, y)$  be formulas in  $D_T \times F$ . For  $x \in D_T^R$ , let*

$$A(x) = \{(\llbracket \varphi_0(x, y) \rrbracket), \dots, (\llbracket \varphi_{m-1}(x, y) \rrbracket)\}; y \in F^R\}.$$

*Then, there is a definable predicate  $P : E_{\text{RV}}^m \times D_{\text{APr}} \rightarrow [0, 1]$  such that for every  $\hat{x} \in D_{\text{TR}}$  and every  $z \in E_{\text{RV}}^m$ ,  $d(z, A(x)) = P(z, \bar{x}, \hat{x})$ .*

As we will use the maps  $\{\lambda_{n,i}; i < 2^n\}$  to define our approximate Skolem functions, we want to ensure that our definable map  $\xi : D_{\text{APr}} \rightarrow E_{\text{RV}}$  gives us witnesses where the overlap between  $\lambda_{n,i}(\xi(z))$  and  $\lambda_{n,i+1}(\xi(z))$  occurs on a set of small measure. The next technical lemmas will guarantee precisely that.

**Lemma 2.2.11.** *Let  $a_0 < b_0 < \dots < a_{n-1} < b_{n-1}$  be real numbers in  $[0, 1]$  and let  $\delta = \max_{i < n} \{b_i - a_i\}$ . Then, for any  $r > 0$ , the set  $\{z \in E_{\text{RV}}; \sum_{i < n} \mu[a_i < z < b_i] \leq r\}$  is  $\delta$ -dense.*

*Proof.* Fix  $r > 0$  and take  $z \in E_{\text{RV}}$ . For each  $i < n$ , pick  $c_i, d_i \in \mathbb{R}$  such that  $a_i < c_i < d_i < b_i$  and  $\mu[c_i < z < d_i] \leq r/n$ . Also, let  $g_i$  be the linear function passing through  $(c_i, a_i)$  and  $(d_i, b_i)$ . Define the function  $f : [0, 1] \rightarrow [0, 1]$  by (we let  $b_{-1} = 0$ )

$$f(x) = \begin{cases} \min\{x, a_i\} & \text{if } b_{i-1} \leq x < c_i, i < n \\ g_i(x) & \text{if } c_i \leq x < d_i, i < n \\ b_i & \text{if } d_i \leq x < b_i, i < n \\ x & \text{if } x \geq b_{n-1}. \end{cases}$$

We have that for every  $x \in [0, 1]$ ,  $|f(x) - x| < \delta$ . Thus, if  $z_0 = f \circ z$  then  $d(z, z_0) < \delta$ . Furthermore, for every  $i < n$ ,  $\mu[a_i < z_0 < b_i] = \mu[c_i < z < d_i] \leq r/n$ . Hence,  $\sum_{i < n} \mu[a_i < z_0 < b_i] \leq r$ .  $\square$

Take  $z, z_0 \in E_{\text{RV}}$ , from [\[BY13, Lemma 2.15\]](#), we know that  $\text{tp}(z) = \text{tp}(z_0)$  if and only if they have the same distribution as random variables. Hence, for any  $a, b \in [0, 1]$  with  $a < b$ , the real-valued map that to each type  $p \in S_{E_{\text{RV}}}(\text{ARV})$  assigns  $\mu[a < z < b]$ , where  $z$  is any realization of  $p$ , is well-defined and we will simply denote it by  $\mu[a < p < b]$ .

**Lemma 2.2.12.** *For any  $a, b \in [0, 1]$  with  $a < b$ , the real-valued map defined on  $S_{E_{\text{RV}}}(\text{ARV})$  by  $p \mapsto \mu[a < p < b]$  is lower semi-continuous.*

*Proof.* Take  $r \in \mathbb{R}$  and define  $F = \{p \in S_{E_{\text{RV}}}(\text{ARV}); \mu[a < p < b] \leq r\}$ , we must prove that  $F$  is closed. Suppose  $q \in S_{E_{\text{RV}}}(\text{ARV}) \setminus F$ , so  $\mu[a < q < b] > r$ . Then, there is  $\delta > 0$  such that  $r_0 = \mu[a + \delta \leq q \leq b - \delta] > r$ . Take  $\epsilon = \delta(r_0 - r)$ , we will show that  $B(q, \epsilon) \cap F = \emptyset$ . Let  $p \in F$  and pick any  $z \models q$  and  $z_0 \models p$ . If  $A = [a + \delta \leq z \leq b - \delta] \setminus [a < z_0 < b]$  then  $\mu(A) \geq r_0 - r$  and for  $\omega \in A$ ,  $|z(\omega) - z_0(\omega)| \geq \delta$ . Hence,  $d(z, z_0) \geq \int_A |z - z_0| \geq \epsilon$ . Since  $z$  and  $z_0$  are arbitrary realizations of  $q$  and  $p$ , we conclude that  $d(q, p) \geq \epsilon$ .  $\square$

**Corollary 2.2.13.** *Let  $a_0 < b_0 < \dots < a_{n-1} < b_{n-1}$  and  $r$  be real numbers in  $[0, 1]$ . The set*

$$\mathcal{C} = \left\{ z \in E_{\text{RV}}; \sum_{i < n} \mu \llbracket a_i < z < b_i \rrbracket \leq r \right\}$$

*is definable.*

*Proof.* Since  $ARV$  is  $\omega$ -categorical, it suffices to show that  $\{p \in S_{E_{\text{RV}}}(ARV); \sum_{i < n} \mu \llbracket a_i < p < b_i \rrbracket \leq r\}$  is type-definable, i.e. closed. This follows immediately from the previous lemma and the fact that the sum of lower semi-continuous functions is again lower semi-continuous.  $\square$

In order to prove that  $D_{TR}$  is a Skolem sort we will find approximations of the desired map. Towards this end we will use the notion of a Katětov function.

**Definition 2.2.14.** Let  $(X, d)$  be a metric space. A map  $\psi : X \rightarrow [0, +\infty]$  is said to be *Katětov* if for all  $x, y \in X$ ,  $\psi(x) \leq d(x, y) + \psi(y)$  and  $d(x, y) \leq \psi(x) + \psi(y)$ .

**Lemma 2.2.15.** *Let  $F$  be a sort of  $T$ , then for every formula  $\varphi(\hat{x}, y)$  in  $D_{TR} \times F^R$  and every  $\epsilon > 0$  there is a formula  $\psi_0(\hat{x}, y)$  such that for every  $\hat{x} \in D_{TR}$*

- $\psi_0(\hat{x}, y)$  is a Katětov function on  $y$ .
- There is  $y \in F^R$  such that  $\varphi(\hat{x}, y) + \psi_0(\hat{x}, y) < \inf_y \varphi(\hat{x}, y) + \epsilon$ .

*Proof.* First, we remark that it is enough to show the lemma for a dense set of formulas. So, we can assume that there is an  $\mathcal{L}_{\text{Pr}}$ -formula  $\psi(z, \bar{x}, \dot{x})$  in  $(E_{\text{RV}})^n \times D_{\text{APr}}$  such that

$$\varphi(\hat{x}, y) = \psi(\llbracket \theta_1(x, y) \rrbracket, \dots, \llbracket \theta_m(x, y) \rrbracket, \bar{x}, \dot{x})$$

where  $\theta_i(x, y)$  is an  $\mathcal{L}$ -formula in  $D_T \times F$  with  $\inf_y \theta_i(\hat{x}, y) = 0$ , for  $i \leq m$ . For simplicity we show the case  $m = 1$ , i.e.  $\varphi(\hat{x}, y) = \psi(\llbracket \theta(x, y) \rrbracket, \bar{x}, \dot{x})$ , but the proof generalizes using [Lemma 2.2.10](#) and the functions  $\{\lambda_{n,i}^m; \bar{i} \in (2^n)^m\}$ . Pick  $\delta_1, \delta_0 < \epsilon$  such that for every  $z_0, z_1 \in E_{\text{RV}}$ ,  $d(z_0, z_1) < \delta_0$  implies  $|\psi(z_0, \bar{x}, \dot{x}) - \psi(z_1, \bar{x}, \dot{x})| < \epsilon/3$  and  $d(z_0, z_1) < \delta_1$  implies  $|\psi(z_0, \bar{x}, \dot{x}) - \psi(z_1, \bar{x}, \dot{x})| < \delta_0/5$ . Also, let  $P(z, \bar{x}, \dot{x})$  be the predicate as per [Lemma 2.2.9](#) for the formula  $\theta(x, y)$ . Now, take  $n \in \mathbb{N}$  such that  $2^{-n+2} < \min\{\delta_1, \delta_0/5\}$ . For  $i < 2^n$ , define  $\theta_i(x, y) = |i/2^n - \theta(x, y)|$ . Since  $D_T$  is a Skolem sort for  $F$ , for each  $i < 2^n$ , there exist a definable function  $\sigma_i : D_T \rightarrow F$  such that  $\theta_i(x, \sigma_i(x)) \leq \inf_y \theta_i(x, y) + \delta_0/5$ . Additionally, by [Lemma 2.2.11](#) and [Corollary 2.2.13](#), the set

$$\mathcal{C} = \left\{ z \in E_{\text{RV}}; \sum_{i < 2^n} \mu \left[ \frac{i+1}{2^n} - \frac{1}{2^{2n}} < z < \frac{i+1}{2^n} \right] \leq \delta_0/5 \right\}$$

is a sort of  $T^R$  which is furthermore  $2^{-2n}$ -dense in  $E_{\text{RV}}$ . Next, define  $\chi(\bar{x}, \dot{x}, z) = \psi(z, \bar{x}, \dot{x}) \vee P(z, \bar{x}, \dot{x})$  as a formula in  $D_{\text{APr}} \times \mathcal{C}$ . We claim that  $\inf_{z \in \mathcal{C}} \chi(\bar{x}, \dot{x}, z) \leq \delta_0/5$ . Indeed, take  $r > 0$  and let  $y \in F^R$  be such that  $\psi(\llbracket \theta(x, y) \rrbracket, \bar{x}, \dot{x}) < r$ . Pick  $z \in \mathcal{C}$  such that  $d(\llbracket \theta(x, y) \rrbracket, z) < 2^{-2n}$ . So,  $P(z, \bar{x}, \dot{x}) < 2^{-2n}$  and  $\psi(z, \bar{x}, \dot{x}) < r + \delta_0/5$ , implying that  $\chi(\bar{x}, \dot{x}, z) < r + \delta_0/5$ . Since  $r$  is arbitrary, the result follows. We also know that  $D_{\text{APr}}$  is a Skolem sort for  $\mathcal{C}$ , so there is a definable map  $\xi : D_{\text{APr}} \rightarrow \mathcal{C}$  such that  $\chi(\bar{x}, \dot{x}, \xi(\bar{x}, \dot{x})) < 2\delta_0/5$ . Now, let us define

$$\psi_0(\hat{x}, y) = \mathbb{E} \left[ \sum_{i < 2^n} \lambda_{n,i}(\xi(\bar{x}, \dot{x})) \llbracket d(\sigma_i(x), y) \rrbracket \right].$$

It is a routine verification that  $\psi_0(\hat{x}, y)$  is Katětov in  $y$ . To prove the second part, fix  $\hat{x} \in D_{TR}$  and take  $y \in F^R$  that agrees with  $\sigma_i(x)$  on  $A_i = \llbracket i/2^n \leq \xi(\bar{x}, \dot{x}) \leq (i+1)/2^n - 1/2^{2n} \rrbracket$  for  $i < 2^n$ . Given  $i < 2^n$ , let us denote  $B_i = \llbracket (i+1)/2^n - 1/2^{2n} < \xi(\bar{x}, \dot{x}) < (i+1)/2^n \rrbracket$ . We have that,

$$\begin{aligned}
\psi_0(\hat{x}, y) &= \sum_{i < 2^n} \left[ \int_{B_i} (\lambda_{n,i}(\xi(\bar{x}, \hat{x})) \llbracket d(\sigma_i(x), y) \rrbracket + \lambda_{n,i+1}(\xi(\bar{x}, \hat{x})) \llbracket d(\sigma_{i+1}(x), y) \rrbracket) d\mu \right] \\
&\leq \sum_{i < 2^n} \mu(A_i) \\
&\leq \delta_0/5.
\end{aligned}$$

We also remark that

$$\begin{aligned}
d(\llbracket \theta(x, y) \rrbracket, \xi(\bar{x}, \hat{x})) &\leq \sum_{i < 2^n} \left[ \int_{A_i} |\llbracket \theta(x, \sigma_i(x)) \rrbracket - \xi(\bar{x}, \hat{x})| d\mu + \int_{B_i} d\mu \right] \\
&\leq \sum_{i < 2^n} \left[ \int_{A_i} \llbracket \theta_i(x, \sigma_i(x)) \rrbracket + |i/2^n - \xi(\bar{x}, \hat{x})| d\mu + \mu(B_i) \right] \\
&\leq \sum_{i < 2^n} \left[ \int_{A_i} \llbracket \inf_y \theta_i(x, y) \rrbracket \right] + \delta_0/5 + 1/2^n + \delta_0/5 \\
&\leq \mathbb{E} \left[ \sum_{i < 2^n} \lambda_{n,i}(\xi(\bar{x}, \hat{x})) \llbracket \inf_y \theta_i(x, y) \rrbracket \right] + 1/2^n + 2\delta_0/5 \\
&< P(\xi(\bar{x}, \hat{x}), \bar{x}, \hat{x}) + 1/2^{n-2} + 2\delta_0/5 \\
&< \delta_0.
\end{aligned}$$

Therefore,  $\psi(\llbracket \theta(x, y) \rrbracket, \bar{x}, \hat{x}) < \psi(\xi(\bar{x}, \hat{x}), \bar{x}, \hat{x}) + \epsilon/3 \leq 2\delta_0/5 + \epsilon/3$ . Thus,  $\varphi(\hat{x}, y) + \psi_0(\hat{x}, y) < 3\delta_0/5 + \epsilon/3 < \epsilon$ .  $\square$

**Theorem 2.2.16.** *Suppose that  $T$  is a continuous theory that admits a Skolem sort,  $D_T$ , and that  $F$  is a sort of  $T$ . Then  $D_{TR}$  is a Skolem sort for  $F^R$ .*

*Proof.* Let  $\varphi(\hat{x}, y)$  be a formula in  $D_{TR} \times F^R$  such that  $\inf_y \varphi(\hat{x}, y) = 0$  and let  $\epsilon > 0$ . Take  $0 < \delta < \epsilon$  such that  $d(y, y') \leq \delta$  implies  $|\varphi(\hat{x}, y) - \varphi(\hat{x}, y')| \leq \epsilon/2$ . Now define  $\delta_n = 2^{-n-3}\delta$ . By the previous lemma, there is a formula  $\varphi_0(x, y)$ , which is Katětov in  $y$ , and for every  $\hat{x} \in D_{TR}$  there is  $y_0 \in F^R$  such that  $\varphi(\hat{x}, y_0) + \varphi_0(\hat{x}, y_0) < \delta_0$ . Iterating this procedure, we obtain a sequence of formulas  $(\varphi_n(\hat{x}, y))_{n \in \mathbb{N}}$ , which is Katětov in  $y$ , and for each  $\hat{x} \in D_{TR}$ , a sequence  $(y_n)_{n \in \mathbb{N}}$  such that

$$\varphi_n(\hat{x}, y_{n+1}) + \varphi_{n+1}(\hat{x}, y_{n+1}) < \inf_y \varphi_n(\hat{x}, y) + \delta_{n+1} < \delta_n + \delta_{n+1}.$$

Fix  $\hat{x} \in D_{TR}$ , then  $d(y_n, y_{n+1}) \leq \varphi_n(\hat{x}, y_n) + \varphi_n(\hat{x}, y_{n+1}) < 2\delta_n + \delta_{n+1}$ . Thus,  $(y_n)_{n \in \mathbb{N}}$  is a convergent sequence, say to  $\sigma(\hat{x}) \in F^R$ . Note that  $\varphi_n(\hat{x}, \sigma(\hat{x})) \leq \varphi_n(\hat{x}, y_n) + d(y_n, \sigma(\hat{x})) < \delta_n + d(y_n, \sigma(\hat{x}))$ . So,  $(\varphi_n(\hat{x}, \sigma(\hat{x})))_{n \in \mathbb{N}}$  converges to 0. Hence, for any  $y \in F^R$ , we have that  $|d(y, \sigma(\hat{x})) - \varphi_n(\hat{x}, y)| \leq \varphi_n(\hat{x}, \sigma(\hat{x}))$  converges to 0. Thus,  $(\varphi_n(\hat{x}, y))_{n \in \mathbb{N}}$  converges uniformly to  $d(y, \sigma(\hat{x}))$ , and the map  $\sigma : D_{TR} \rightarrow F^R$  is definable. Since  $d(y_0, \sigma(\hat{x})) \leq \delta$ , we obtain that

$$\varphi(\hat{x}, \sigma(\hat{x})) \leq \varphi(\hat{x}, y_0) + \epsilon/2 < \delta_0 + \epsilon/2 < \epsilon.$$

Therefore,  $\sigma$  is an  $\epsilon$ -Skolem map for  $\varphi$ .  $\square$

The previous theorem together with [Lemma 2.2.6](#) give us the following result.

**Corollary 2.2.17.** *Let  $T$  be a complete continuous theory admitting a universal Skolem sort,  $D_T$ . Then  $D_{TR} = D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$  is a universal Skolem sort of  $T^R$ .*

Suppose now that  $T$  is an  $\omega$ -categorical theory and let  $\mathcal{M}$  be a separable model of  $T$ . We know from [Fact 2.1.7](#) that  $D_T = D_0 \times 2^{\mathbb{N}}$ , where  $D_0 = \text{tp}(a)$  is the type of a dense enumeration of  $M$ , is a universal Skolem sort for  $T$ . Let  $c_a \in D_0^R(M)$  be the map taking constant value  $a$  and take  $\Xi \in D_{\text{APr},0}(\Omega)$  as before. It is easy to see, using for example [Lemma 2.4.6](#) below, that  $\text{dcl}(c_a, \Xi) = \mathcal{M}^R$ . Thus, by [Lemma 2.1.8](#),  $D_0^R \times D_{\text{APr},0} \times 2^{\mathbb{N}} = D_0^R \times D_{\text{APr}}$  is a universal Skolem sort for  $T^R$ . On the other hand,  $D_{T^R} = (D_0 \times 2^{\mathbb{N}})^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$  is also a universal Skolem sort for  $T^R$ . Since  $(D_0 \times 2^{\mathbb{N}})^R = D_0^R \times E^{\mathbb{N}}$  and  $\text{tp} : D_0 \times 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$  coincides with the natural projection, we have that

$$D_{T^R} = (D_0^R \times E^{\mathbb{N}}) \times_{E^{\mathbb{N}}} D_{\text{APr}} \cong D_0^R \times D_{\text{APr}}.$$

Thus, modulo a definable bijection, the sort  $D_T^R$  coincides with the sort we knew beforehand for  $\omega$ -categorical structures. We finish this section by exhibiting a pseudo-open map  $\pi^0 : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ .

For  $i, j \in \mathbb{N}$  with  $i \leq j$  let  $n_{i,j} = \frac{j(j+1)}{2} + i$ . So,  $n_{\cdot, \cdot}$  is a bijection between  $\{(i, j) \in \mathbb{N}^2; i \leq j\}$  and  $\mathbb{N}$ .  $\pi^0$  will be defined recursively, and for  $i \in \mathbb{N}$ , the steps of the form  $n_{i,j}$  will be used to define  $\pi_i^0$ . Now, for  $\nu \in \{0, 1\}^{\mathbb{N}}$  let  $\bar{\nu} = \sum_{i=0}^{\nu-1} \nu_i 2^i$ . The integers  $\bar{\nu}$  will be used to go through the indices of the elements in the sort  $2^{\mathbb{N}}$ .

**Definition 2.2.18.** We define the map  $\pi^0 : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  in a recursive manner. Fix  $(\bar{a}, \alpha) \in D_{\text{APr}}$ , let  $\mathcal{A}_0 = \{\perp, \top\}$ , and for  $n > 0$  let  $\mathcal{A}_n$  be the algebra generated by  $\{a_0, \dots, a_{n-1}\}$ . Suppose we have already carried out our construction up to the step  $n-1$  and we have already used  $\alpha_0, \dots, \alpha_{2^n-2}$ . We know that there exist  $i, j \in \mathbb{N}$  with  $i \leq j$  such that  $n = n_{i,j}$ . If the construction, up to the step  $n-1$ , neither implies  $\pi_i(\bar{a}, \alpha) \cap a_\nu = \emptyset$  nor  $a_\nu \subseteq \pi_i(\bar{a}, \alpha)$ , then:

- in case  $\alpha_{2^{n-1}+\bar{\nu}} = 0$ , we impose the condition  $\pi_i(\bar{a}, \alpha) \cap a_\nu \subseteq a_\nu \cap a_n$ ,
- in case  $\alpha_{2^{n-1}+\bar{\nu}} = 1$ , we impose the condition  $a_\nu \cap a_n \subseteq \pi_i(\bar{a}, \alpha) \cap a_\nu$ .

Since the sequence  $\bar{a}$  is independent, the measure of the atoms of  $\mathcal{A}_n$  tend to zero. Thus, the previous conditions define a unique sequence  $A \in E^{\mathbb{N}}$ . Furthermore, each element in the sequence is a uniform limit of  $\mathcal{L}_{\text{APr}}$ -terms in  $(\bar{a}, \alpha)$ . It follows that the map  $\pi^0$  is definable.

**Lemma 2.2.19.** *The map  $\pi^0 : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  is weakly open.*

*Proof.* Given  $\epsilon > 0$  and  $N \in \mathbb{N}$ , take  $k \in \mathbb{N}$  such that  $2^{n_{k,k}} < \epsilon$  and  $2^{-n_{k,k}} - 1 \geq N$ . We will work inside a fixed separable  $\mathcal{A} \models \text{APr}$ . Suppose that  $(\bar{a}, \alpha) \in D_{\text{APr}}$  and  $A \in E^{\mathbb{N}}$  satisfy  $\pi_n^0(\bar{a}, \alpha) = A_n$  for every  $n < k$ . For  $i < n_{k,k}$  take  $b_i = a_i$  and for  $l < 2^{n_{k,k}} - 1$  let  $\beta_l = \alpha_l$ . So, independent of the choice of the remaining  $b_i$  and  $\beta_l$ ,  $d(\bar{a}, \bar{b}) < \epsilon$  and  $\alpha \upharpoonright_N = \beta \upharpoonright_N$ . Take  $n \geq n_{k,k}$  and suppose that we have already defined  $b_0, \dots, b_{n-1}$  and  $\beta_0, \dots, \beta_{2^n-2}$ . Assume that  $n = n_{i,j}$  for some  $i, j \in \mathbb{N}$ . For  $\nu \in \{0, 1\}^{\mathbb{N}}$ , if  $\mu(b_\nu \cap A_i) \leq \mu(b_\nu)/2$  put  $\beta_{2^{n-1}+\bar{\nu}} = 0$  and take  $D_\nu \in \mathcal{A}$  with  $b_\nu \cap A_i \subseteq D_\nu \subseteq b_\nu$  and  $\mu(D_\nu) = \mu(b_\nu)/2$ . In case  $\mu(b_\nu \cap A_i) > \mu(b_\nu)/2$ , put  $\beta_{2^{n-1}+\bar{\nu}} = 1$  and pick  $D_\nu \in \mathcal{A}$  with  $D_\nu \subseteq b_\nu \cap A_i$  and  $\mu(D_\nu) = \mu(b_\nu)/2$ . Finally, define  $b_n = \bigcup_{\nu \in \{0,1\}^{\mathbb{N}}} D_\nu$ .

By the previous process we obtain an element  $(\bar{b}, \beta) \in D_{\text{APr}}$ . By construction, for every  $i \in \mathbb{N}$ , every  $j \geq i$  and every  $\nu \in 2^{n_{i,j}}$ ,  $\mu(b_\nu \cap A_i) \leq \mu(b_\nu)/2$  if and only if  $\mu(b_\nu \cap \pi_i^0(\bar{b}, \beta)) \leq \mu(b_\nu)/2$ . Hence,  $\pi^0(\bar{b}, \beta) = A$ .  $\square$

[Lemma 2.2.3](#) yields the following result.

**Corollary 2.2.20.** *The map  $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  defined by  $\pi_n(\bar{x}, \dot{x}) = \pi_{2^n}^0(\bar{x}, \dot{x})$  for  $n \in \mathbb{N}$  and  $(\bar{x}, \dot{x}) \in D_{\text{APr}}$  is pseudo-open.*

## 2.3 Random functions on groupoids

In this section we will define the groupoid  $\mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$  for  $\mathbf{G}$  a Polish groupoid. We will show that several properties of  $\mathbf{G}$  are preserved when passing to  $\mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$  such as being open or minimal. Finally, from the map  $\pi : D_{\text{APr}} \rightarrow E^\mathbb{N}$  we will define a base change of  $\mathbf{G} \wr \Omega$  to  $2^\mathbb{N}$ . In [Section 2.4](#), we will see how this construction relates to the groupoid associated to the randomization. We begin this section with some results on the space  $X^\Omega$  for a Polish space  $X$ .

### 2.3.1 The topology on the space of random functions

Let  $X$  be a Polish space and let  $d$  be a compatible metric on  $X$  bounded by 1. Recall from [Section 2.2](#) that  $(\Omega, \mathcal{B}, \mu)$  denotes the Stone space of the usual measure algebra of  $[0, 1]$ . The space  $X^\Omega = L^0((\Omega, \mathcal{B}, \mu), X)$  admits a natural metric, the expectation of the metric on  $X$ . Namely, for  $f, g \in X^\Omega$  let

$$d^\Omega(f, g) = \int d(f(\omega), g(\omega)) d\mu(\omega).$$

On the other hand, convergence in measure induces a topology on  $X^\Omega$  which is moreover metrizable.

**Fact 2.3.1.** [[Fan44](#)] *Let  $X$  be a Polish space and let  $d$  be a compatible metric on  $X$  bounded by 1. Then, the topology of convergence in measure on  $X^\Omega$  is induced by the distance  $d_0$  defined as*

$$d_0(f, g) = \inf\{\epsilon > 0 ; \mu[\![d(f, g) \geq \epsilon]\!] < \epsilon\}$$
 for  $f, g \in X^\Omega$ .

Given  $A \subseteq \Omega$ ,  $U \subseteq X$  and  $\epsilon > 0$  let

$$\mathcal{O}(A, U, \epsilon) = \{f \in X^\Omega ; \mu(\{\omega \in A ; f(\omega) \notin U\}) < \epsilon\}.$$

We will denote by  $\tau$  the topology on  $X^\Omega$  having as a subbase the sets of the form  $\mathcal{O}(A, U, \epsilon)$  where  $A \in \mathcal{B}$ ,  $U \subseteq X$  open and  $\epsilon > 0$ . For  $A_0, \dots, A_{n-1} \subseteq \Omega$ ,  $U_0, \dots, U_{n-1} \subseteq X$  and  $\epsilon > 0$  we set

$$\mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon) = \bigcap_{i < n} \mathcal{O}(A_i, U_i, \epsilon).$$

Let us also denote  $\mathcal{O}(A, U) = \{f \in X^\Omega ; \text{for all } \omega \in A, f(\omega) \in U\}$ .  $\mathcal{O}((A_i)_{i < n}, (U_i)_{i < n})$  is defined analogously. The next lemma, whose proof is an easy but tedious exercise, shows that by only considering the sets defined by partitions of  $\Omega$  we obtain a basis of  $\tau$ .

**Lemma 2.3.2.** *Let  $X$  be a Polish space, then the family of sets of the form  $\mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon)$  where  $n \in \mathbb{N}$ ,  $A_0, \dots, A_{n-1}$  is a partition of  $\Omega$ ,  $U_0, \dots, U_{n-1}$  are open subsets of  $X$  and  $\epsilon > 0$ , is a base for the topology  $\tau$  on  $X^\Omega$ .*

Since every measurable set in  $\Omega$  differs from a clopen set in a set of null measure, we may only consider clopen subsets of  $\Omega$ . The three topologies defined thus far agree, as will be shown below. However, we find it convenient to exhibit a base of this topology that does not explicitly mention the metric on  $X$ .

**Lemma 2.3.3.** *Let  $X$  be a Polish space and let  $d$  be a compatible metric on  $X$  bounded by 1. Then the following topologies on  $X^\Omega$  agree:*

1.  $\tau$ .
2. The topology induced by  $d^\Omega$ .

3. *The topology of convergence in measure.*

Moreover, a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $X^\Omega$  converges to  $f$  if and only if for every subsequence of  $(f_n)_{n \in \mathbb{N}}$  there is a further subsequence that converges to  $f$  almost everywhere. Finally,  $X^\Omega$ , endowed with this topology, is a Polish space.

*Proof.* The equivalence between (2) and (3), and the moreover part correspond to [Moo76, Proposition 6]. It is a well known fact that  $(X^\Omega, d^\Omega)$  is a Polish space, see for example [Moo76, Proposition 7]. It remains to show the equivalence between  $\tau$  and the other topologies.

3.  $\subseteq$  1. Pick any  $f \in X^\Omega$  and any  $\epsilon > 0$ . Let  $\{W_i ; i \in \mathbb{N}\}$  be a countable base of the topology of  $X$  consisting of open sets of diameter less than  $\epsilon$ . For  $i \in \mathbb{N}$  we set  $A_i = f^{-1}(W_i)$ , then there is some  $n \in \mathbb{N}$  such that  $\mu(\bigcup_{i < n} A_i) \geq 1 - \epsilon/2$ . We have that  $f \in \mathcal{O}((A_i)_{i < n}, (W_i)_{i < n}, \epsilon/2n) \subseteq B_{d_0}(f, \epsilon)$ .

1.  $\subseteq$  3. Let  $A \subseteq \Omega$  be measurable,  $U \subseteq X$  be open and  $\epsilon > 0$ . Suppose that  $f \in \mathcal{O}(A, U, \epsilon)$ , let us denote  $A_0 = \{\omega \in A ; f(\omega) \in U\}$  and take  $r > 0$  satisfying  $\mu(A \setminus A_0) + r < \epsilon$ . By Lusin's theorem there is  $A_1 \subseteq A_0$  compact such that  $f \upharpoonright_{A_1}$  is continuous and  $\mu(A_0 \setminus A_1) < r/2$ . Since  $f(A_1) \subseteq U$ , by continuity, there is  $\delta < r/2$  such that  $B_d(f(A_1), \delta) \subseteq U$ . It follows that  $f \in B_{d_0}(f, \delta) \subseteq \mathcal{O}(A, U, \epsilon)$ .  $\square$

For  $\tau \in \text{Aut}(\Omega)$  and  $f \in X^\Omega$  define  $(\tau \cdot f)(\omega) = f(\tau^{-1}(\omega))$  for  $\omega \in \Omega$ . It is easy to see that  $\tau \cdot f \in X^\Omega$  and a straightforward verification shows that this operation defines a continuous action by  $d^\Omega$ -isometries (as well as  $d_0$ -isometries) of  $\text{Aut}(\Omega)$  on  $X^\Omega$ .

### 2.3.2 The groupoid of random functions

Let  $\mathbf{G}$  be a Polish open topological groupoid of base  $\mathbf{B}$ . Our first goal is to show that  $\mathbf{G}^\Omega$  is also a Polish open topological groupoid.

**Proposition 2.3.4.** *Let  $\mathbf{G}$  be a Polish topological groupoid. Then,  $\mathbf{G}^\Omega$ , with the topology of convergence in measure and operations defined pointwise, is a Polish topological groupoid.*

*Proof.* Take  $g, h \in \mathbf{G}^\Omega$  such that  $g(\omega)h(\omega)$  is defined for every  $\omega \in \Omega$ . Take  $U \subseteq \mathbf{G}$  open and let  $V \subseteq \mathbf{G}^2$  be the inverse image of  $U$  under the product map. So,  $V$  is the intersection of a closed and an open set, hence a measurable subset of  $\mathbf{G}^2$ . On the other hand, the map  $f : \Omega \rightarrow \mathbf{G}^2$  defined by  $f(\omega) = (g(\omega), h(\omega))$  is measurable. Therefore,  $(gh)^{-1}(U) = f^{-1}(V)$  is measurable and  $gh \in \mathbf{G}^\Omega$ , i.e.  $\mathbf{G}^\Omega$  is closed under pointwise multiplication. Clearly,  $\mathbf{G}^\Omega$  is also closed under pointwise inverses and we conclude that it is in fact a groupoid.

Take  $g, h \in \mathbf{G}^\Omega$  such that  $gh$  is defined. Suppose moreover that we also have sequences  $(g_n)_{n \in \mathbb{N}}$ ,  $(h_n)_{n \in \mathbb{N}}$  converging to  $g$  and  $h$ , respectively such that  $g_n h_n$  is defined for every  $n \in \mathbb{N}$ . Let  $(n_k)_{k \in \mathbb{N}}$  be any subsequence, take  $(m_i)_{i \in \mathbb{N}}$  a further subsequence such that  $g_{m_i}(\omega)$  converges to  $g(\omega)$ , when  $i \rightarrow \infty$ , for almost every  $\omega \in \Omega$ . Then, there is an even further subsequence  $(m_{i_j})_{j \in \mathbb{N}}$  such that  $h_{m_{i_j}}(\omega)$  converges to  $h(\omega)$ , when  $j \rightarrow \infty$ , for almost every  $\omega \in \Omega$ . Therefore,  $\lim_{j \rightarrow \infty} g_{m_{i_j}} h_{m_{i_j}}(\omega) = gh(\omega)$  for almost every  $\omega \in \Omega$ . It follows, that  $(g_n h_n)_{n \in \mathbb{N}}$  converges to  $gh$  and that the product in  $\mathbf{G}^\Omega$  is continuous. A similar argument shows that the inverse is also continuous. Hence,  $\mathbf{G}^\Omega$  is a topological groupoid and, by Lemma 2.3.3, it is Polish.  $\square$

Note that the base of  $\mathbf{G}^\Omega$  is  $\mathbf{B}^\Omega$ . The source and target maps will be denoted  $s^\Omega : \mathbf{G}^\Omega \rightarrow \mathbf{B}^\Omega$  and  $t^\Omega : \mathbf{G}^\Omega \rightarrow \mathbf{B}^\Omega$  respectively, and they are given by  $s^\Omega(g)(\omega) = s_{g(\omega)}$  and  $t^\Omega(g)(\omega) = t_{g(\omega)}$  for  $\omega \in \Omega$  and  $g \in \mathbf{G}^\Omega$ . To show that if  $\mathbf{G}$  is open then  $\mathbf{G}^\Omega$  is also open, we need a few technical lemmas. We assume that every topological space is endowed with the Borel  $\sigma$ -algebra.

**Theorem 2.3.5.** [*Kec95*, Theorem 15.1] *Let  $X, Y$  be Polish spaces and  $f : X \rightarrow Y$  be continuous. If  $A \subseteq X$  is Borel and  $f \upharpoonright A$  is injective, then  $f(A)$  is Borel.*

**Theorem 2.3.6.** [*Kec95*, Theorem 12.16] *Let  $X$  be a Polish space and  $E$  an equivalence relation such that every equivalence class is closed. Suppose that either the saturation of any open set is Borel, or the saturation of any closed set is Borel. Then  $E$  admits a Borel transversal.*

**Lemma 2.3.7.** *Let  $X$  and  $Y$  be Polish spaces and let  $\pi : X \rightarrow Y$  be a continuous, open, surjective map. Then there is  $f : Y \rightarrow X$  measurable such that  $\pi \circ f = \text{id}_Y$ .*

*Proof.* Define the equivalence relation  $E$  on  $X$  by  $xEx'$  if and only if  $\pi(x) = \pi(x')$ . For  $x \in X$ ,  $[x]_E = \pi^{-1}(\pi(x))$ , a closed set. For  $U \subseteq X$  open, since  $\pi$  is open and continuous, its saturation  $[U]_E = \pi^{-1}(\pi(U))$  is open. By **Theorem 2.3.6**, there is  $T \subseteq X$  Borel that meets every equivalence class in exactly one point. Now we define  $f : Y \rightarrow X$ , for  $y \in Y$  we set  $f(y)$  as the only element in  $T \cap \pi^{-1}(\{y\})$ . Take  $U \subseteq X$  Borel, note that  $f^{-1}(U) = \pi(U \cap T)$  and  $\pi \upharpoonright (U \cap T)$  is injective. Using **Theorem 2.3.5**, we obtain that  $f^{-1}(U)$  is Borel. Therefore,  $f$  is measurable.  $\square$

**Corollary 2.3.8.** *Let  $X, Y$  be Polish spaces and let  $Z$  be a topological space. Suppose that  $\pi : X \rightarrow Y$  is a continuous, open, surjective map and that  $f : Z \rightarrow Y$  is a measurable map. Then, there is a measurable application  $g : Z \rightarrow X$  such that  $f = \pi \circ g$ .*

*Proof.* By the previous lemma, there is  $h : Y \rightarrow X$  measurable with  $\pi \circ h = \text{id}_Y$ , then  $g = h \circ f$  is the desired function.  $\square$

**Corollary 2.3.9.** *Let  $X, Y$  be Polish spaces and let  $Z$  be a topological space. Suppose that  $\pi : X \rightarrow Y$  is a continuous, open map and that  $f : Z \rightarrow Y$  is a measurable map such that  $f(Z) \subseteq \pi(U)$  for some  $U$ , open subset of  $X$ . Then, there is a measurable application  $g : Z \rightarrow U$  such that  $f = \pi \circ g$ .*

*Proof.* Use the previous corollary replacing  $X$  with  $U$  and  $Y$  with  $\pi(U)$ .  $\square$

We borrow the following notation from randomizations, for  $A_0, \dots, A_{n-1}$  a partition of  $\Omega$  and  $f_0, \dots, f_{n-1} \in X^\Omega$ ,  $\langle (A_i)_{i < n}, (f_i)_{i < n} \rangle$  will denote the element in  $X^\Omega$  that agrees with  $f_i$  on  $A_i$  for  $i < n$ .

**Lemma 2.3.10.** *Let  $\mathbf{G}$  be a Polish open topological groupoid. Let  $A_0, \dots, A_{n-1}$  be a partition of  $\Omega$  and let  $U_0, \dots, U_{n-1}$  be open subsets of  $\mathbf{G}$ . Suppose that  $e \in \mathcal{O}((A_i)_{i < n}, (t(U_i))_{i < n})$ , then there is  $g \in \mathcal{O}((A_i)_{i < n}, (U_i)_{i < n})$  such that  $t^\Omega(g) = e$ .*

*Proof.* Fix  $i \leq n$ , take  $a_i \in t(U_i)$  and let  $e_i = \langle A_i, \Omega \setminus A_i, e, a_i \rangle$ . Then,  $e_i : \Omega \rightarrow \mathbf{B}$  is a measurable map whose image is contained in  $t(U_i)$ . By the previous lemma, there is a measurable map  $g_i : \Omega \rightarrow \mathbf{G}$  whose image is contained in  $U_i$  and such that  $t^\Omega(g_i) = t \circ g_i = e_i$ . Then  $g = \langle (A_i)_{i < n}, (g_i)_{i < n} \rangle$  is the desired map.  $\square$

**Theorem 2.3.11.** *Let  $\mathbf{G}$  be a Polish open topological groupoid, then  $\mathbf{G}^\Omega$  is also a Polish open topological groupoid.*

*Proof.* By **Proposition 2.3.4**, it remains to show that it is in fact an open groupoid. We claim that for any  $A_1, \dots, A_n$  partition of  $\Omega$ ,  $U_1, \dots, U_n$  open subsets of  $\mathbf{G}$  and  $\epsilon > 0$ ,  $t^\Omega(\mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon)) = \mathcal{O}((A_i)_{i < n}, (t(U_i))_{i < n}, \epsilon)$ . Pick  $e = t^\Omega(g)$  with  $g \in \mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon)$ . Notice that for  $i < n$ , if  $\omega \in A_i$  and  $g(\omega) \in U_i$  then  $e(\omega) \in t(U_i)$ . Thus, for  $i < n$ ,

$$\mu(\{\omega \in A_i ; e(\omega) \notin t(U_i)\}) \leq \mu(\{\omega \in A_i ; g(\omega) \notin U_i\}) < \epsilon$$

and  $e \in \mathcal{O}((A_i)_{i < n}, (t(U_i))_{i < n}, \epsilon)$ . Now take  $e \in \mathcal{O}((A_i)_{i < n}, (t(U_i))_{i < n}, \epsilon)$ , for  $i < n$  define  $B_i = \{\omega \in A_i ; e(\omega) \notin t(U_i)\}$ . We have that  $e \in \mathcal{O}(A_1 \setminus B_1, B_1 \dots, A_n \setminus B_n, B_n, t(U_1), \mathbf{B}, \dots, t(U_n), \mathbf{B})$ , so, by the previous lemma, there is  $g \in \mathcal{O}(A_1 \setminus B_1, B_1 \dots, A_n \setminus B_n, B_n, U_1, \mathbf{G}, \dots, U_n, \mathbf{G})$  such that  $t^\Omega(g) = e$ . It is true that for  $i < n$ ,  $\{\omega \in \Omega ; g(\omega) \notin U_i\} \subseteq B_i$ . Therefore,  $g \in \mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon)$ , which completes the proof.  $\square$

Let  $\varphi : X \rightarrow Y$  be a continuous map between Polish spaces. Define the application  $\varphi^\Omega : X^\Omega \rightarrow Y^\Omega$  by  $\varphi^\Omega(x) = \varphi \circ x$  for  $x \in X^\Omega$ . This map is continuous since for any  $A \subseteq \Omega$ ,  $V$  open subset of  $Y$  and  $\epsilon > 0$

$$(\varphi^\Omega)^{-1}(\mathcal{O}(A, V, \epsilon)) = \mathcal{O}(A, \varphi^{-1}(V), \epsilon)$$

It follows that if  $(X, \rho)$  is a space over  $\mathbf{B}$  then  $(X^\Omega, \rho^\Omega)$  is a space over  $\mathbf{B}^\Omega$ .

**Lemma 2.3.12.** *Let  $\mathbf{G}$  be a Polish groupoid and let  $(X, \rho)$  be a Polish  $\mathbf{G}$ -space over  $\mathbf{B}$ . Then,  $(X^\Omega, \rho^\Omega)$  is a Polish  $\mathbf{G}^\Omega$ -space over  $\mathbf{B}^\Omega$ .*

*Proof.* The action  $\mathbf{G}^\Omega \curvearrowright X^\Omega$  is defined by  $(g \cdot x)(\omega) = g(\omega) \cdot x(\omega)$  whenever  $t^\Omega(g) = \rho^\Omega(x)$ . The fact that  $g \cdot x$  belongs to  $X^\Omega$  and that the action is continuous is showed in the same way as in the case of Polish groups, see [Iba17, Definition 2.1] and the discussion thereafter.  $\square$

**Lemma 2.3.13.** *Let  $\mathbf{G}$  be a Polish open groupoid acting minimally on  $(X, \rho)$ , space over  $\mathbf{B}$ . Let  $V \subseteq X$  be open. Then, for any  $x \in X^\Omega$ , there is  $g \in \mathbf{G}^\Omega$  such that  $g \cdot x \in \mathcal{O}(\Omega, V)$ .*

*Proof.* Let us fix  $x \in X^\Omega$ . By **Fact 2.1.13**, the projection  $\eta : \mathbf{G} \times_{\mathbf{B}} X \rightarrow X$  is open. Let  $W$  be the inverse image of  $V$  under the action law, so  $W$  is an open subset of  $\mathbf{G} \times_{\mathbf{B}} X$ . Since  $\mathbf{G} \curvearrowright X$  is minimal, for every  $\omega \in \Omega$  there is some  $h \in \mathbf{G}$  such that  $h \cdot x(\omega) \in V$ . Thus,  $(g, x(\omega)) \in W$ , which implies  $x(\omega) \in \eta(W)$ . Since this is true for every  $\omega \in \Omega$ , we obtain that  $x(\Omega) \subseteq \eta(W)$ . Hence, by **Corollary 2.3.9**, there is a measurable map  $\tilde{g} : \Omega \rightarrow W$  such that  $x = \eta \circ \tilde{g}$ . Then, the projection of  $\tilde{g}$  onto the first coordinate is the desired map.  $\square$

**Corollary 2.3.14.** *Suppose  $\mathbf{G}$  is a Polish open groupoid acting on  $(X, \rho)$ , Polish space over  $\mathbf{B}$ . Then,  $\mathbf{G} \curvearrowright X$  is minimal if and only if  $\mathbf{G}^\Omega \curvearrowright X^\Omega$  is minimal. In particular,  $\mathbf{G}$  is minimal if and only if  $\mathbf{G}^\Omega$  is minimal.*

*Proof.* Assume first that  $\mathbf{G}^\Omega \curvearrowright X^\Omega$  is minimal. Fix  $x \in X$  and  $V \subseteq X$  open. Let  $c_x \in X^\Omega$  be the constant map of value  $x$  and consider the open set  $\mathcal{O}(\Omega, V, 1/2)$ . Using minimality, we get  $g \in \mathbf{G}^\Omega$  such that  $g \cdot c_x \in \mathcal{O}(\Omega, V, 1/2)$ . Therefore, for some  $\omega \in \Omega$ ,  $g(\omega) \cdot x \in V$ , which shows minimality. Suppose now that  $\mathbf{G} \curvearrowright X$  is minimal. Let  $A_0, \dots, A_{n-1}$  be a partition of  $\Omega$ ,  $V_0, \dots, V_{n-1}$  be open subsets of  $X$  and  $\epsilon > 0$ . Fix  $x \in X^\Omega$ , by the previous lemma, for every  $i < n$ , there is  $g_i \in \mathbf{G}^\Omega$  satisfying  $g_i \cdot x \in \mathcal{O}(\Omega, V_i)$ . Define  $g = \langle (A_i)_{i < n}, (g_i)_{i < n} \rangle$ , it follows that  $g \cdot x \in \mathcal{O}((A_i)_{i < n}, (V_i)_{i < n}, \epsilon)$ , finishing the proof.  $\square$

We will now study the semidirect product between a groupoid and a group.

**Definition 2.3.15.** Let  $\mathbf{G}$  be a groupoid and  $\Gamma$  a group. An *action of  $\Gamma$  on  $\mathbf{G}$*  is a map  $\cdot : \Gamma \times \mathbf{G} \rightarrow \mathbf{G}$  such that for every  $\gamma, \delta \in \Gamma$  and  $g, h \in \mathbf{G}$

$$1_\Gamma \cdot g = g, \quad \gamma \cdot (\delta \cdot g) = (\gamma\delta) \cdot g, \quad \gamma \cdot (gh) = (\gamma \cdot g)(\gamma \cdot h).$$

The last equality meaning that whenever  $gh$  is defined then  $(\gamma \cdot g)(\gamma \cdot h)$  is also defined and both sides agree. Thus, for any  $e \in \mathbf{B}$  and  $\gamma \in \Gamma$ , we must have  $\gamma e \in \mathbf{B}$ . From this, we deduce that for every  $\gamma \in \Gamma$  and every  $g \in \mathbf{G}$ ,  $(\gamma \cdot g)^{-1} = \gamma \cdot g^{-1}$ ,  $s_{\gamma \cdot g} = \gamma \cdot s_g$  and  $t_{\gamma \cdot g} = \gamma \cdot t_g$ .

**Definition 2.3.16.** Let  $\Gamma$  be a group acting on a groupoid  $\mathbf{G}$ . We define the *semidirect product*  $\mathbf{G} \rtimes \Gamma$  as the groupoid whose underlying set is  $\mathbf{G} \times \Gamma$  and whose operations are given by

$$(g, \gamma)(h, \delta) = (g(\gamma \cdot h), \gamma\delta), \quad (g, \gamma)^{-1} = (\gamma^{-1} \cdot g^{-1}, \gamma^{-1}).$$

In this case the source and target maps are given by

$$t(g, \gamma) = (t_g, 1_\Gamma), \quad s(g, \gamma) = (\gamma^{-1} s_g, 1_\Gamma).$$

The base of  $\mathbf{G} \rtimes \Gamma$  is  $\mathbf{B} \times \{1_\Gamma\} \cong \mathbf{B}$ .

**Lemma 2.3.17.** *Let  $\mathbf{G}$  be a topological groupoid and let  $\Gamma$  be a topological group acting continuously on  $\mathbf{G}$ . Then,  $\mathbf{G} \rtimes \Gamma$ , endowed with the product topology, is a topological groupoid. If  $\mathbf{G}$  is open then  $\mathbf{G} \rtimes \Gamma$  is also open. Finally, if  $\mathbf{G}$  and  $\Gamma$  are Polish then so is  $\mathbf{G} \rtimes \Gamma$ .*

*Proof.* Since the action and the inversion are continuous maps, the application  $\eta : \mathbf{G} \times \Gamma \rightarrow \mathbf{G}$  defined by  $\eta(g, \gamma) = \gamma^{-1} \cdot g^{-1}$  is also continuous. Let  $U \subseteq \mathbf{G}$  and  $V \subseteq \Gamma$  be open, we claim that  $(U \times V)^{-1} = \eta^{-1}(U) \cap (\mathbf{G} \times V^{-1})$ . Indeed,  $(g, \gamma) \in (U \times V)^{-1}$  if and only if  $(\gamma^{-1} \cdot g^{-1}, \gamma^{-1}) \in U \times V$ , which occurs exactly when  $\eta(g, \gamma) \in U$  and  $\gamma \in V^{-1}$ , i.e.  $(g, \gamma) \in \eta^{-1}(U) \cap (\mathbf{G} \times V^{-1})$ .

To show that the groupoid operation is continuous, take  $(g, \gamma), (h, \delta) \in \mathbf{G} \rtimes \Gamma$  whose product is defined and take  $U \subseteq \mathbf{G}$ ,  $V \subseteq \Gamma$  open sets with  $(g(\gamma \cdot h), \gamma\delta) \in U \times V$ . By the continuity of the group operation, there are open sets  $V_0, V_1 \subseteq \Gamma$  with  $\gamma \in V_0$ ,  $\delta \in V_1$  and  $V_0 V_1 \subseteq V$ . Using the continuity of the groupoid operation, we find  $U_0$  and  $U_1$  open subsets of  $\mathbf{G}$ , with  $g \in U_0$ ,  $\gamma \cdot h \in U_1$  and  $U_0 U_1 \subseteq U$ . Finally, since the action is continuous, there are  $U_2 \subseteq \mathbf{G}$  and  $V_2 \subseteq V_0$  open sets with  $h \in U_2$ ,  $\gamma \in V_2$  and  $V_2 \cdot U_2 \subseteq U_1$ . In this way,  $(g, \gamma) \in U_0 \times V_2$ ,  $(h, \delta) \in U_2 \times V_1$  and  $(U_0 \times V_2)(U_2 \times V_1) \subseteq U \times V$ , showing that the groupoid operation is continuous.

Now assume that  $\mathbf{G}$  is open. Let  $U \subseteq \mathbf{G}$  and  $V \subseteq \Gamma$  be open, we have that  $t(U \times V) = t(U)$ , which is an open subset of  $\mathbf{B}$ . Finally, if  $\mathbf{G}$  and  $\Gamma$  are Polish, it is a well-known fact that their product is also a Polish space.  $\square$

Suppose that  $(X, \rho)$  is a  $\mathbf{G}$ -space and that  $\Gamma$  is a group together with continuous actions  $\Gamma \curvearrowright \mathbf{G}$  and  $\Gamma \curvearrowright X$  such that  $\gamma \cdot (g \cdot x) = (\gamma \cdot g) \cdot (\gamma \cdot x)$  for every  $\gamma \in \Gamma$ ,  $g \in \mathbf{G}$  and  $x \in X$  for which  $g \cdot x$  is defined. In particular, this implies that  $\rho(\gamma \cdot x) = \gamma \cdot \rho(x)$ . It is a straightforward verification that  $(g, \gamma) \cdot x = g \cdot (\gamma \cdot x)$ , whenever  $s_g = \rho(\gamma \cdot x)$ , defines a continuous action of  $\mathbf{G} \rtimes \Gamma$  on  $X$ . Furthermore, if  $\mathbf{G} \curvearrowright X$  is minimal then  $(\mathbf{G} \rtimes \Gamma) \curvearrowright X$  is also minimal. Indeed, if  $x \in X$  and  $V \subseteq X$  is open, then there is  $g \in \mathbf{G}$  such that  $g \cdot x \in V$ . It follows that  $(g, 1_\Gamma) \cdot x \in V$ . Note that the action of  $\text{Aut}(\Omega)$  on  $X^\Omega$  satisfies  $\tau \cdot (g \cdot x) = (\tau \cdot g) \cdot (\tau \cdot x)$  for every  $\tau \in \text{Aut}(\Omega)$ ,  $g \in \mathbf{G}^\Omega$  and  $x \in X^\Omega$  for which  $g \cdot x$  is defined. This fact, together with [Corollary 2.3.14](#) and [Lemma 2.3.17](#), implies the following result.

**Corollary 2.3.18.** *If  $\mathbf{G}$  is a Polish open topological groupoid over  $\mathbf{B}$  then  $\mathbf{G} \wr \Omega = \mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$  is a Polish open topological groupoid over  $\mathbf{B}^\Omega$ . If furthermore  $\mathbf{G}$  is minimal then  $\mathbf{G} \wr \Omega$  is minimal.*

We will now discuss the notion of base change.

**Definition 2.3.19.** Suppose  $\mathbf{G}$  is a groupoid of base  $\mathbf{B}$ ,  $\mathbf{C}$  is a set and  $\theta : \mathbf{C} \rightarrow \mathbf{B}$  is any map. The *base change* of  $\mathbf{G}$  to  $\mathbf{C}$  is the groupoid whose underlying set is

$$\mathbf{G}_\theta = \{(b, g, a) \in \mathbf{C} \times \mathbf{G} \times \mathbf{C} ; \theta(b) = t_g \text{ and } \theta(a) = s_g\},$$

and whose operations are given by

$$(c, h, b)(b, g, a) = (c, hg, a), \quad (b, g, a)^{-1} = (a, g^{-1}, b).$$

**Proposition 2.3.20.** *Suppose  $\mathbf{G}$  is a topological groupoid of base  $\mathbf{B}$ ,  $\mathbf{C}$  is a topological space and  $\theta : \mathbf{C} \rightarrow \mathbf{B}$  is a continuous map. Then  $\mathbf{G}_\theta$ , endowed with the product topology, is a topological groupoid whose base is homeomorphic to  $\mathbf{C}$  via  $c \mapsto (c, \theta(c), c)$ . If moreover  $\mathbf{G}$  and  $\mathbf{C}$  are Polish then  $\mathbf{G}_\theta$  is also Polish*

*Proof.* The inverse map is continuous since for  $U_0, U_1 \subseteq \mathbf{C}$  and  $V \subseteq \mathbf{G}$  open sets,

$$((U_0 \times V \times U_1) \cap \mathbf{G}_\theta)^{-1} = (U_1 \times V^{-1} \times U_0) \cap \mathbf{G}_\theta.$$

To see that the product is continuous, pick  $(c, h, b), (b, g, a) \in \mathbf{G}_\theta$  and suppose that  $(c, hg, a) \in U_0 \times V \times U_1$  for some open sets  $U_0, U_1 \subseteq \mathbf{C}$  and  $V \subseteq \mathbf{G}$ . Let  $W_0$  and  $W_1$  be open neighborhoods of  $h$  and  $g$ , respectively, satisfying  $W_0 W_1 \subseteq V$ . It follows that

$$((U_0 \times W_0 \times \mathbf{C}) \cap \mathbf{G}_\theta)((\mathbf{C} \times W_1 \times U_1) \cap \mathbf{G}_\theta) \subseteq U_0 \times V \times U_1.$$

By the definition of  $\mathbf{G}_\theta$ , the neutral elements are those of the form  $(c, \theta(c), c)$  for  $c \in \mathbf{C}$ . The map  $c \mapsto (c, \theta(c), c)$  is clearly a homeomorphism. The moreover part follows from the fact that  $\mathbf{G}_\theta$  is a closed subspace of the product  $\mathbf{C} \times \mathbf{G} \times \mathbf{C}$ .  $\square$

Having in mind that we want to build the groupoid associated to the randomization theory, we must obtain a groupoid whose base is the Cantor space. However, if  $\mathbf{G}$  denotes the groupoid of  $T$ , its base is  $2^{\mathbb{N}}$  and thus the base of  $\mathbf{G} \wr \Omega$  is  $(2^{\mathbb{N}})^{\Omega}$ . So, we will use the map  $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  to define a base change of  $\mathbf{G} \wr \Omega$  to  $2^{\mathbb{N}}$ . If the base change map  $\theta : 2^{\mathbb{N}} \rightarrow \mathbf{B}^{\Omega}$  is continuous and open then the base change of  $\mathbf{G} \wr \Omega$  to  $2^{\mathbb{N}}$  is an open groupoid. However, in our case, the function  $\theta$  need not be open. So, we will show another sufficient condition for the base change to  $2^{\mathbb{N}}$  to be open and minimal.

Recall from [Section 2.2](#) the identification  $E^{\mathbb{N}} \cong (2^{\mathbb{N}})^R$ . In the separable case we have that  $(2^{\mathbb{N}})^R$  agrees with  $(2^{\mathbb{N}})^{\Omega}$ . Hence, the pseudo-open map  $\pi$ , obtained from [Corollary 2.2.20](#), when considered inside the separable model  $(\mathcal{B}, \mu) \models \text{APr}$ , can be written as  $\pi : D_{\text{APr}}(\Omega) \rightarrow (2^{\mathbb{N}})^{\Omega}$ . Also recall that  $\Xi$  is the sequence of independent subsets of measure 1/2 used to define the universal Skolem sort of APr, as after [Lemma 2.1.8](#).  $\Xi$  defines moreover a left-invariant metric  $d_L$  on  $\text{Aut}(\Omega)$ , as in [Section 1.1](#). Given  $\alpha \in 2^{\mathbb{N}}$  let us write  $\pi(\alpha)$  for  $\pi(\Xi, \alpha)$ , this defines a continuous map  $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^{\Omega}$ . The intended use of the map  $\pi$ , as a function defined on  $D_{\text{APr}}$  or on  $2^{\mathbb{N}}$ , will be clear from the context.

Pick  $A, B \in E^{\mathbb{N}}(\Omega)$  and suppose they are identified with  $f_A, f_B \in (2^{\mathbb{N}})^{\Omega}$ . Then, given  $k \in \mathbb{N}$ ,  $A_n = B_n$  for every  $n < k$  if and only if  $d_{\infty}(f_A, f_B) < 2^{-k}$ . Now suppose that  $\bar{a} \in D_{\text{APr}_0}(\Omega)$  generates  $\bar{\mathcal{B}}$ . Then, as  $\Xi$  also generates  $\bar{\mathcal{B}}$ , the map  $\Xi \mapsto \bar{a}$  defines an automorphism of  $(\bar{\mathcal{B}}, \mu)$  which corresponds to some  $\tau \in \text{Aut}(\Omega)$ . It follows that  $d(\Xi, \bar{a}) = d(\Xi, \tau(\Xi)) = d_L(\text{id}_{\Omega}, \tau)$ . For  $n \in \mathbb{N}$  and  $\nu \in 2^n$  let us denote  $[\nu] = \{\alpha \in 2^{\mathbb{N}} ; \alpha \upharpoonright_n = \nu\}$ . Therefore, the map  $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  being pseudo-open translates to the following property.

**Definition 2.3.21.** A map  $\theta : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^{\Omega}$  is said to be *almost uniformly open* if for every  $\epsilon > 0$  and  $N \in \mathbb{N}$  there is  $\delta > 0$  such that for every  $\alpha \in 2^{\mathbb{N}}$

$$B_{d_{\infty}}(\theta(\alpha), \delta) \subseteq B_{d_L}(\text{id}_{\Omega}, \epsilon) \cdot \theta([\alpha \upharpoonright_N]).$$

Since the action of  $\text{Aut}(\Omega)$  on  $(2^{\mathbb{N}})^{\Omega}$  is by  $d_{\infty}$ -isometries and the metric  $d_L$  is invariant on the left, being almost uniformly open implies that for every  $\epsilon > 0$  and  $N \in \mathbb{N}$  there is  $\delta > 0$  such that for every  $\alpha \in 2^{\mathbb{N}}$  and  $\tau \in \text{Aut}(\Omega)$ ,

$$B_{d_\infty}(\tau \cdot \theta(\alpha), \delta) \subseteq B_{d_L}(\tau, \epsilon) \cdot \theta([\alpha \upharpoonright_N]).$$

The name almost uniformly open comes from the fact that the map  $\theta$  itself is not necessarily uniformly open, when considering the metric  $d_\infty$  on  $(2^\mathbb{N})^\Omega$ . However, once we enlarge the image of  $\theta$  by the action of  $\text{Aut}(\Omega)$  we get uniform openness. We will now state a technical lemma in order to show that if the original groupoid is open and minimal, and the base change map is almost uniformly open, then the base change to  $2^\mathbb{N}$  is also open and minimal.

**Lemma 2.3.22.** *Let  $\mathbf{G}$  be a Polish open minimal groupoid over  $2^\mathbb{N}$ . Take  $g \in \mathbf{G}^\Omega$ ,  $e \in (2^\mathbb{N})^\Omega$  and  $\epsilon > 0$ . Then, there exists  $h \in \mathbf{G}^\Omega$  such that  $t^\Omega(h) = t^\Omega(g)$ ,  $\llbracket g \neq h \rrbracket = \llbracket d(e, s^\Omega(g)) \geq \epsilon \rrbracket$  and  $d_\infty(s^\Omega(h), e) < \epsilon$ .*

*Proof.* Let  $\{V_1, \dots, V_k\}$  be a cover of  $2^\mathbb{N}$  in disjoint clopen sets of diameter less than  $\epsilon$ . For  $i \leq k$ , we define  $A_i = e^{-1}(V_i)$ . So,  $\{A_1, \dots, A_k\}$  is a partition of  $\Omega$ , by [Lemma 2.3.13](#), there is  $f \in \mathbf{G}^\Omega$  such that  $t^\Omega(f) = t^\Omega(g)$  and  $s^\Omega(f) \in \mathcal{O}(A_1, \dots, A_k, V_1, \dots, V_k)$ . Let  $B = \llbracket d(e, s^\Omega(g)) < \epsilon \rrbracket$  and define  $h = \langle B, \Omega \setminus B, g, f \rangle$ . Clearly,  $t^\Omega(h) = t^\Omega(g)$  and  $\llbracket g \neq h \rrbracket = \llbracket d(e, s^\Omega(g)) \geq \epsilon \rrbracket$ . Now take  $\omega \in \Omega$ , if  $\omega \in B$  then  $d(s^\Omega(h)_\omega, e_\omega) = d(s^\Omega(g)_\omega, e_\omega) < \epsilon$ . In case  $\omega \in \Omega \setminus B$ , take  $i \leq k$  such that  $\omega \in A_i \cap B$ . We have that  $s^\Omega(h)_\omega = s^\Omega(f)_\omega \in V_i$ , since the diameter of  $V_i$  is less than  $\epsilon$  and  $e_\omega \in V_i$ , we get that  $d(s^\Omega(h)_\omega, e_\omega) < \epsilon$ . We conclude that  $d_\infty(s^\Omega(h), e) < \epsilon$ .  $\square$

**Theorem 2.3.23.** *Let  $\mathbf{G}$  be a Polish open minimal groupoid over  $2^\mathbb{N}$  and let  $\theta : 2^\mathbb{N} \rightarrow (2^\mathbb{N})^\Omega$  be a continuous almost uniformly open map. Then,  $(\mathbf{G} \wr \Omega)_\theta$ , the base change of  $\mathbf{G} \wr \Omega$  to  $2^\mathbb{N}$ , is a Polish open minimal groupoid.*

*Proof.* We already know that  $(\mathbf{G} \wr \Omega)_\theta$  is a Polish groupoid. It remains to show that it is open and minimal. First we will show that for any  $(\alpha, g, \tau, \beta) \in (\mathbf{G} \wr \Omega)_\theta$  and  $\mathcal{U}$  open neighborhood of  $(\alpha, g, \tau, \beta)$  in  $(\mathbf{G} \wr \Omega)_\theta$ ,  $t(\mathcal{U})$  is a neighborhood of  $\alpha$ . We can assume that there are some  $\epsilon > 0$ ,  $N \in \mathbb{N}$  and  $W \subseteq G^\Omega$  open neighborhood of  $g$  such that

$$\mathcal{U} = ([\alpha \upharpoonright_N] \times W \times B_{d_L}(\tau, \epsilon) \times [\beta \upharpoonright_N]) \cap (\mathbf{G} \wr \Omega)_\theta.$$

For such  $\epsilon$  and  $N$  take  $\delta > 0$  as in the definition of almost uniformly open. Now, take  $W_0 \subseteq W$  open set containing  $g$  such that for every  $f \in W_0$ ,  $d_0(s^\Omega(f), s^\Omega(g)) < \delta$ . We may further assume that if  $f \in W_0$  then  $B_{d_0}(f, \delta) \subseteq W$ . Finally, take  $K \geq N$  such that  $[\alpha \upharpoonright_K] \subseteq \theta^{-1}(t^\Omega(W_0))$ . We claim that  $[\alpha \upharpoonright_K] \subseteq t(\mathcal{U})$ . Indeed, take  $\alpha' \in [\alpha \upharpoonright_K]$ , then there is  $f \in W_0$  such that  $t^\Omega(f) = \theta(\alpha')$ . In particular, we have that  $d_0(s^\Omega(g), s^\Omega(f)) < \delta$ . Using the previous lemma, we obtain  $h \in \mathbf{G}^\Omega$  such that  $t^\Omega(h) = t^\Omega(f)$ ,  $\llbracket h \neq f \rrbracket = \llbracket d(s^\Omega(f), s^\Omega(g)) \geq \delta \rrbracket$  and  $d_\infty(s^\Omega(h), s^\Omega(g)) < \delta$ . This implies that  $d_0(f, h) < \delta$  and so  $h \in W$ . Since  $s^\Omega(g) = \tau \cdot \theta(\beta)$  and  $\theta$  is almost uniformly open, we obtain that  $s^\Omega(h) \in B_{d_L}(\tau, \epsilon) \cdot \theta([\beta \upharpoonright_N])$ . Hence, there is  $\rho \in B_{d_L}(\tau, \epsilon)$  and there is  $\beta' \in [\beta \upharpoonright_N]$  such that  $s^\Omega(h) = \rho \cdot \theta(\beta')$ . Therefore,  $(\alpha', h, \rho, \beta') \in \mathcal{U}$  and  $t(\alpha', h, \rho, \beta') = \alpha'$ .

It remains to show that the action  $(\mathbf{G} \wr \Omega)_\theta \curvearrowright 2^\mathbb{N}$  is minimal. Take  $\alpha \in 2^\mathbb{N}$  and  $\nu \in 2^\mathbb{N}$  with  $N \in \mathbb{N}$ . For this  $N$  and  $\epsilon = 1$  take  $\delta > 0$  as in the definition of almost uniformly open. Take  $\{V_1, \dots, V_k\}$  a cover of  $2^\mathbb{N}$  consisting of disjoint clopen sets of diameter less than  $\delta$ . Take any  $\beta \in [\nu]$  and for  $i \leq k$  define  $A_i = \theta(\beta)^{-1}(V_i)$ . In this way,  $\{A_1, \dots, A_k\}$  is a partition of  $\Omega$ , and by [Lemma 2.3.13](#), there is  $g \in \mathbf{G}^\Omega$  such that  $t^\Omega(g) = \theta(\alpha)$  and  $s^\Omega(g) \in \mathcal{O}(A_1, \dots, A_k, V_1, \dots, V_k)$ . This implies that  $d_\infty(s^\Omega(g), \theta(\beta)) < \delta$ . So, there are  $\tau \in \text{Aut}(\Omega)$  and  $\beta' \in [\nu]$  such that  $s^\Omega(g) = \tau \cdot \theta(\beta')$ . Thus,  $(\alpha, g, \tau, \beta') \in (\mathbf{G} \wr \Omega)_\theta$ ,  $t(\alpha, g, \tau, \beta') = \alpha$  and  $s(\alpha, g, \tau, \beta') \in [\nu]$ .  $\square$

## 2.4 The groupoid of the randomization

The following result describes the automorphism group of the Borel randomization.

**Theorem 2.4.1.** *[Iba17, Theorem 2.18] Let  $\mathcal{M}$  be a separable structure and let  $\mathcal{M}^R$  be its Borel randomization. Then,*

$$\text{Aut}(\mathcal{M}^R) = \text{Aut}(\mathcal{M}) \wr \Omega.$$

It follows that if  $T$  is an  $\omega$ -categorical theory and  $\mathcal{M}$  is its separable model, then  $\mathbf{G}(T^R) = 2^{\mathbb{N}} \times (\text{Aut}(\mathcal{M}) \wr \Omega) \times 2^{\mathbb{N}}$ . In order to generalize this result to the context of theories admitting universal Skolem sorts we do the construction of the previous section. We will show that the groupoid associated to the randomization of a theory admitting a universal Skolem sort is of the form  $(\mathbf{G} \wr \Omega)_\pi$ , where  $\mathbf{G}$  is the groupoid associated to the original theory and  $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^\Omega$  is the map discussed above. Furthermore, we show in the  $\omega$ -categorical case that the two presentations of the groupoid are indeed equivalent. Since an element of the groupoid is the type of a couple where the definable closure of its entries agree, we do a brief study of the definable closure in randomizations.

### 2.4.1 Definable closure in randomizations

An extensive treatment of this topic for classical theories can be found in [AGK15]. In this section we generalize some of the results therein to the case where the original theory is continuous. During this subsection  $D$  and  $F$  will denote sorts of  $T$ .  $x$  and  $y$  will denote variables in  $D$  and  $F$  respectively. As usual, for  $a \in M$  we say  $a \in \text{dcl}(A)$  if  $\{a\}$  is definable in  $\mathcal{M}$  over  $A$ . For  $a \in F$  we let  $\text{dcl}_D(a) = \text{dcl}(a) \cap D$ . We will denote by  $(\Upsilon, \mathcal{F}, \mu)$  the underlying probability space of the randomizations we consider in this subsection. It is also worth recalling that  $E$  denotes the sort of events.

**Lemma 2.4.2.** *Let  $D$  and  $F$  be sorts of  $T$ . If  $a \in F^R$  and  $b \in D^R$  then*

$$\llbracket b \in \text{dcl}(a) \rrbracket = \{v \in \Upsilon ; b(v) \in \text{dcl}(a(v))\}$$

*is measurable.*

*Proof.* We know that for any formula  $\psi(x)$  and any  $c$  in the randomization, the set  $\llbracket \psi(c) = 0 \rrbracket$  is measurable. On the other hand, from **Fact 1.3.1**, we have that  $v \in \llbracket a \in \text{dcl}(b) \rrbracket$  if and only if for every  $n > 0$  there is a formula  $\varphi(x, y)$  such that  $\varphi(b(v), a(v)) = 0$  and the diameter of  $\{c \in D ; \varphi(c, a(v)) < 1\}$  is less or equal than  $1/n$ . Furthermore, we can assume that the formulas belong to a fixed countable dense set of formulas, which we will call  $\Sigma$ . Given a formula  $\varphi(x, y)$  we define

$$\tilde{\varphi}_n(y) = \sup_{z_1, z_2} [(1 - \varphi(z_1, y) \vee \varphi(z_2, y)) \wedge d(z_1, z_2) \div 1/n].$$

Then,  $\tilde{\varphi}_n(d) = 0$  precisely when the diameter of  $\{c \in D ; \varphi(c, d) < 1\}$  is less or equal than  $1/n$ . Now, we set

$$B_n = \bigcup_{\varphi \in \Sigma} \llbracket \varphi(b, a) \vee \tilde{\varphi}_n(a) = 0 \rrbracket$$

We thus obtain that  $\llbracket b \in \text{dcl}(a) \rrbracket = \bigcap_{n \in \mathbb{N}} B_n$  is measurable. □

**Lemma 2.4.3.** *Let  $D$  and  $F$  be sorts of  $T$ . Take  $a \in F^R$  and  $b \in D^R$ . If  $b$  is definable over  $a$  then  $\mu[b \in \text{dcl}(a)] = 1$ .*

*Proof.* Suppose on the contrary that  $\mu[b \in \text{dcl}(a)] = 1 - r < 1$ . We will show that there exists some element different from  $b$  that realizes  $\text{tp}(b/a)$ . For  $\varphi(x, y)$  a formula and  $n > 0$  let

$$A_{\varphi, n} = \llbracket \varphi(b, a) > 0 \rrbracket \cup \left( \llbracket \varphi(b, a) = 0 \rrbracket \cap \bigcup_{m > 0} \llbracket \inf_z [(\varphi(z, a) \dot{-} (1 - 1/m)) \wedge (1/n \dot{-} d(z, b))] = 0 \rrbracket \right).$$

In this way,  $v \in A_{\varphi, n}$  when  $\varphi(b(v), a(v)) = 0$  implies that there exists  $z$  with  $\varphi(z, a(v)) < 1$  and  $d(b(v), z) \geq 1/n$ . Let us fix a countable dense set of formulas  $\Sigma$  and define  $A_n = \bigcap_{\varphi \in \Sigma} A_{\varphi, n}$ , then,  $\neg \llbracket b \in \text{dcl}(a) \rrbracket = \bigcup_{n > 0} A_n$ . So, there must be some  $N \in \mathbb{N}$  such that  $\mu(A_N) \geq r/2$ . Now, take any formula  $\varphi(x, y)$ , any  $\epsilon > 0$  and pick  $n \in \mathbb{N}$  with  $1/n < \epsilon/2$ . Given  $i < n$ , define  $B_i = A_i \cap \llbracket i/n \leq \varphi(b, a) < (i+1)/n \rrbracket$  and  $\varphi_i(x, y) = |i/n + 1/2n - \varphi(x, y)| \dot{-} 1/2n$ . Let  $c_i \in D^R$  be such that

$$\llbracket \inf_z [(\varphi_i(z, a) \dot{-} \epsilon/2) \wedge (1/N - d(b, z))] \rrbracket = \llbracket (\varphi_i(c_i, a) \dot{-} \epsilon/2) \wedge (1/N - d(b, c_i)) \rrbracket.$$

Note that for  $v \in B_i$ ,  $\varphi_i(b(v), a(v)) = 0$ . Hence,  $\varphi_i(c_i(v), a(v)) \leq \epsilon/2$  and  $d(b(v), c_i(v)) \geq 1/N$ . Take  $c = \langle B_0, \dots, B_{n-1}, \Upsilon \setminus A_N, c_0, \dots, c_{n-1}, b \rangle$ , then  $d(b, c) = \int_{A_N} d(b(v), c(v)) d\mu(v) \geq \mu(A_N)/N = r/2N$  and  $|\llbracket \varphi(b, a) \rrbracket - \llbracket \varphi(c, a) \rrbracket| \leq (1/n + \epsilon/2) < \epsilon$ . So,  $|\mathbb{E}[\llbracket \varphi(b, a) \rrbracket] - \mathbb{E}[\llbracket \varphi(c, a) \rrbracket]| < \epsilon$ . The same argument generalizes when considering several formulas simultaneously. The result will follow by compactness.  $\square$

Given  $A \subseteq E_{RV}$  we denote by  $\sigma(A)$  the minimal  $\sigma$ -sub-algebra of  $E$  such that every  $f \in A$  is  $\sigma(A)$ -measurable. So,  $\sigma(A)$  is the  $\sigma$ -algebra generated by  $\{\llbracket f < r \rrbracket ; f \in A, r > 0\}$ . In particular, when  $A$  consists of  $\{0, 1\}$ -valued random variables,  $\sigma(A)$  is the generated  $\sigma$ -algebra. In case  $a \in D^R$ , by  $\sigma(a)$  we mean  $\sigma(\{\llbracket \varphi(a) \rrbracket ; \varphi(x) \text{ formula}\})$ .

**Fact 2.4.4.** *Let  $A \subseteq E_{RV}$ , then  $\text{dcl}_E(A) = \sigma(A)$ .*

*Proof.* Assume we are working inside a sufficiently saturated model, by [BY13, Theorem 2.17], for  $U, V \in E$ ,  $\text{tp}(U/A) = \text{tp}(V/A)$  if and only if  $\text{tp}(U/\sigma(A)) = \text{tp}(V/\sigma(A))$ . This implies that,  $\text{dcl}_E(A) = \text{dcl}_E(\sigma(A)) = \sigma(A)$ .  $\square$

**Lemma 2.4.5.** *Let  $D$  be a sort of  $T$  and let  $a \in D^R$ , then  $\text{dcl}_E(a) = \sigma(a)$ .*

*Proof.* Assume we are working inside a sufficiently saturated model and let  $A = \{\llbracket \varphi(a) \rrbracket ; \varphi(x) \text{ formula}\}$ . By quantifier elimination, for any  $U, V \in E$ ,  $\text{tp}(U/a) = \text{tp}(V/a)$  if and only if  $\text{tp}(U/A) = \text{tp}(V/A)$ . Hence,  $\text{dcl}_E(a) = \text{dcl}_E(A) = \sigma(A) = \sigma(a)$ .  $\square$

**Lemma 2.4.6.** *Let  $E_0$  be a sort of  $\text{APr}$  and let  $D$  and  $F$  be sorts of  $T$ . Take  $(a, \bar{a}) \in F^R \times E_0$  and  $b \in D^R$ , then  $b$  is definable over  $(a, \bar{a})$  if and only if:*

1.  $\mu[b \in \text{dcl}(a)] = 1$ .
2.  $\sigma(b, a, \bar{a}) = \sigma(a, \bar{a})$ .

*Proof.*  $\Rightarrow$ ) The first item corresponds to Lemma 2.4.3. The second item follows from the fact that  $\sigma(a, \bar{a}) \subseteq \sigma(b, a, \bar{a}) = \text{dcl}_{E_{RV}}(b, a, \bar{a}) \subseteq \text{dcl}_{E_{RV}}(a, \bar{a}) = \sigma(a, \bar{a})$ .

$\Leftarrow$ ) We can assume that we work inside a sufficiently saturated model. Let  $c \models \text{tp}(b/(a, \bar{a}))$ , we will show that  $c = b$ . Take  $\varphi(x, y)$  any formula in  $D \times F$ , we have that  $\llbracket \varphi(b, a) \rrbracket \in \sigma(b, a) \subseteq \sigma(a, \bar{a})$ .

Hence, there is a predicate  $\psi(u, a, \bar{a})$  such that  $d(u, \llbracket \varphi(b, a) \rrbracket) = \psi(u, a, \bar{a})$ . Since,  $c \models \text{tp}(b/(a, \bar{a}))$  then  $\psi(\llbracket \varphi(c, a) \rrbracket) = 0$ . Therefore,  $\llbracket \varphi(b, a) \rrbracket = \llbracket \varphi(c, a) \rrbracket$  for any formula  $\varphi(x, y)$ . Now fix any  $v \in \llbracket b \in \text{dcl}(a) \rrbracket$  and take  $\epsilon > 0$ , then there is a formula  $\varphi(x, y)$  such that  $\varphi(b(v), a(v)) = 0$  and  $\text{diam}(\{d \in D_2 ; \varphi(d, a(v)) < 1\}) < \epsilon$ . Due to the fact that  $\llbracket \varphi(b, a) \rrbracket = \llbracket \varphi(c, a) \rrbracket$ , we get that  $\varphi(c(v), a(v)) = 0$ . This implies that  $d(b(v), c(v)) < \epsilon$ . Since  $\epsilon$  was arbitrary,  $b(v) = c(v)$ . It follows that  $b(v) = c(v)$  for almost every  $v \in \Upsilon$  and we conclude that  $b = c$ .  $\square$

By taking  $E_0 = E$  and  $\bar{a} = \perp$  we obtain the following result.

**Corollary 2.4.7.** *Let  $D$  and  $F$  be sorts of  $T$ . Let  $a \in F^R$  and  $b \in D^R$ , then  $b$  is definable over  $a$  if and only if:*

1.  $\mu \llbracket b \in \text{dcl}(a) \rrbracket = 1$ .
2.  $\sigma(a, b) = \sigma(a)$ .

Note that if  $\hat{a} \in D_{TR}$  then, by [Lemma 2.2.7](#),  $\text{dcl}_E(\hat{a}) \subseteq \sigma(\bar{a})$ . This fact together with the previous lemma allow us to understand interdefinability in the universal sort of  $T^R$ .

**Corollary 2.4.8.** *Assume that  $T$  admits a universal Skolem sort,  $D_T$ . Let  $\hat{a}, \hat{b} \in D_{TR}$ , then  $\hat{b}$  is definable over  $\hat{a}$  if and only if:*

1.  $\mu \llbracket b \in \text{dcl}(a) \rrbracket = 1$ .
2.  $\sigma(a, b, \bar{b}) \subseteq \sigma(\bar{a})$ .

*Proof.*  $\Rightarrow$  Assume  $\hat{b} \in \text{dcl}(\hat{a})$ , in particular,  $b \in \text{dcl}(a, \bar{a})$ . By [Lemma 2.4.6](#),  $\mu \llbracket b \in \text{dcl}(a) \rrbracket = 1$  and  $\sigma(a, b, \bar{b}) \subseteq \sigma(a, \bar{a}, b, \bar{b}) = \sigma(a, \bar{a}) = \sigma(\bar{a})$ .

$\Leftarrow$  Let  $\hat{b} = (b, \bar{b}, \beta)$ , clearly  $\beta \in \text{dcl}(\hat{a})$ . By (2),  $\bar{b} \subseteq \sigma(\bar{a}) \subseteq \text{dcl}_E(\hat{a})$ , which implies  $\bar{b} \in \text{dcl}(\hat{a})$ . The fact that  $b \in \text{dcl}(\hat{a})$  follows from [Lemma 2.4.6](#).  $\square$

**Corollary 2.4.9.** *Assume that  $T$  admits a universal Skolem sort,  $D_T$ . Let  $\hat{a}, \hat{b} \in D_{TR}$ , then  $\text{dcl}(\hat{b}) = \text{dcl}(\hat{a})$  if and only if:*

1.  $\mu \llbracket \text{dcl}(b) = \text{dcl}(a) \rrbracket = 1$ .
2.  $\sigma(a, b, \bar{b}, \bar{a}) = \sigma(\bar{a}) = \sigma(\bar{b})$ .

Let  $D$  be any sort of  $T$  and let  $\mathbf{M}$  be a separable model of  $T^R$ . An element  $a \in D^R(\mathbf{M})$  gives rise to a measurable function  $g : \Omega \rightarrow S_D(T)$  defined by  $g(\omega) = \text{tp}(a(\omega))$  for  $\omega \in \Omega$ . The next result gives us a converse of this result. From now on we may identify a formula  $\varphi(u)$  in  $D$  with the continuous map it induces on the space of types  $\varphi : S_D(T) \rightarrow [0, 1]$ . It follows, that for  $g \in S_D(T)^\Omega$ ,  $\varphi \circ g \in L^1((\Omega, \mathcal{B}, \mu), [0, 1])$ .

**Lemma 2.4.10.** *Let  $D$  be a sort of  $T$ . Then,  $S_{DR}(T^R)$  is homeomorphic to  $S_D(T)^\Omega / \text{Aut}(\Omega)$ . Furthermore, for any  $f \in S_D(T)^\Omega$ , there is  $a \in D^R$  such that  $f(\omega) = \text{tp}(a(\omega))$  for almost every  $\omega \in \Omega$ .*

*Proof.* Recall from [Fact 2.1.3](#) that we have an identification between  $S_{DR}(T^R)$  and  $\mathfrak{R}(S_D(T))$ . By [\[Iba17, Lemma 3.5\]](#), the latter is homeomorphic to  $S_D(T)^\Omega / \text{Aut}(\Omega)$ . For the last part, take  $f \in S_D(T)^\Omega$ , then, by the previous identification,  $[f]_{\text{Aut}(\Omega)}$  corresponds to a type  $\mathbf{p} \in S_{DR}(T^R)$ . Let  $b \models \mathbf{p}$ , it follows that for any set of formulas  $\varphi_1(u), \dots, \varphi_n(u)$  in  $D$ ,

$$\text{tp}(\varphi_1 \circ f, \dots, \varphi_n \circ f) = \text{tp}(\llbracket \varphi_1(b) \rrbracket, \dots, \llbracket \varphi_n(b) \rrbracket).$$

This implies, that for any  $\epsilon > 0$  there is  $\tau \in \text{Aut}(\Omega)$  such that  $d(\llbracket \varphi_i(\tau \cdot b) \rrbracket, \varphi_i \circ f) < \epsilon$ . for every  $i \leq n$ . This means that the set of conditions

$$\Sigma(u) = \{d(\llbracket \varphi(u) \rrbracket, \varphi \circ f) = 0 ; \varphi \text{ formula in } D\}$$

is approximately finitely satisfiable. Hence,  $a$  in a separable model satisfying  $\Sigma(u)$  is the desired element.  $\square$

Let  $\Xi \in D_{\text{APr},0}(\Omega)$  denote the sequence of independent sets of measure 1/2 used to define  $D_{\text{APr}}$ . Take  $p \in \mathbf{G}(T^R)$  and let  $(\hat{a}, \hat{b}) \models p$  in some separable model  $\mathbf{M}$  of  $T^R$ . We may assume that  $\text{dcl}(\hat{a}) = \text{dcl}(\hat{b}) = M$  and even that  $\bar{a} = \Xi$ . In this way,  $p = \text{tp}((a, \Xi, \alpha), (b, \bar{b}, \beta))$ . By [Corollary 2.4.9](#),  $\sigma(\Xi) = \sigma(\bar{b})$ . This implies that  $\bar{b} = \tau(\Xi)$  for some  $\tau \in \text{Aut}(\Omega)$ . Thus, every type  $p \in \mathbf{G}(T^R)$  is of the form  $\text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$  for some  $a, b \in D_T^R(\mathbf{M})$ , where  $\mathbf{M}$  is a separable randomization, and  $\tau \in \text{Aut}(\Omega)$ . Furthermore, since  $\text{tp}(a, b) \in S_{2D_T^R}(T^R)$ , we obtain a function  $g \in S_{2D_T}(T)^\Omega$  defined by  $g(\omega) = \text{tp}(a(\omega), b(\omega))$  for  $\omega \in \Omega$ . By [Corollary 2.4.9](#), we have that  $g \in \mathbf{G}(T)^\Omega$ .

**Theorem 2.4.11.** *Let  $T$  be a complete, countable theory admitting a universal Skolem sort,  $D_T$ . Then  $\mathbf{G}(T^R)$  is homeomorphic to*

$$\mathbf{H} = \{(g, \alpha, \tau, \beta) \in \mathbf{G}(T)^\Omega \times \mathbf{G}(\text{APr}) ; t^\Omega(g) = \pi(\alpha), s^\Omega(g) = \tau \cdot \pi(\beta)\},$$

where  $\mathbf{H}$  is endowed with the subspace topology, as a subset of the product space  $\mathbf{G}(T)^\Omega \times \mathbf{G}(\text{APr})$ .

*Proof.* We will define a map  $F : \mathbf{G}(T^R) \longrightarrow \mathbf{H}$  in the following manner. Given  $p \in \mathbf{G}(T^R)$ , we know that it is of the form  $\text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$  for some  $a, b \in D_T^R$ ,  $\tau \in \text{Aut}(\Omega)$  and  $\alpha, \beta \in 2^\Omega$ . We define  $g \in \mathbf{G}(T)^\Omega$  by  $g(\omega) = \text{tp}(a(\omega), b(\omega))$ , for  $\omega \in \Omega$  and we set  $F(p) = (g, \alpha, \tau, \beta)$ . Note that  $t^\Omega(g) = \text{tp}^R(a) = \pi(\Xi, \alpha) = \pi(\alpha)$  and similarly  $s^\Omega(g) = \text{tp}^R(b) = \pi(\tau(\Xi), \beta) = \tau \cdot \pi(\beta)$ , so the map is well-defined. To show that the map  $F : \mathbf{G}(T^R) \longrightarrow \mathbf{H}$  is surjective, take  $(g, \alpha, \tau, \beta) \in \mathbf{G}(T)^\Omega \times \mathbf{G}(\text{APr})$  with  $t^\Omega(g) = \pi(\alpha)$  and  $s^\Omega(g) = \tau \cdot \pi(\beta)$ . By [Lemma 2.4.10](#), there is a randomization  $\mathbf{M}$  based on  $\Omega$  and there are  $a, b \in D_T^R(\mathbf{M})$  such that  $g(\omega) = \text{tp}(a(\omega), b(\omega))$  for every  $\omega \in \Omega$ . We have that  $\text{tp}^R(a) = t^\Omega(g) = \pi(\Xi, \alpha)$  and that  $\text{tp}^R(b) = s^\Omega(g) = \pi(\tau(\Xi), \beta)$ . So,  $(a, \Xi, \alpha)$  and  $(b, \tau(\Xi), \beta)$  belong to  $D_{T^R}$ . Since,  $\text{tp}(a(\omega), b(\omega))$  belongs to  $\mathbf{G}(T)$  for almost every  $\omega \in \Omega$ , we get  $\mu[\text{dcl}(b) = \text{dcl}(a)] = 1$ . Furthermore,  $\sigma(a, b, \tau(\Xi), \Xi) = \sigma(\Xi) = \sigma(\tau(\Xi))$ . Hence, by [Corollary 2.4.9](#),  $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta)) \in \mathbf{G}(T^R)$  and we obtain that  $(g, \alpha, \tau, \beta) = F(p)$ . To see that it is injective, take  $p, q \in \mathbf{G}(T^R)$  and suppose that  $F(p) = F(q) = (g, \alpha, \tau, \beta)$ . It follows that  $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$  and  $q = \text{tp}((c, \Xi, \alpha), (d, \tau(\Xi), \beta))$  for some  $a, b, c, d \in D_T^R$ . Let  $\varphi(x, y)$  be any formula in  $D_T^2$ , then  $\llbracket \varphi(a, b) \rrbracket = \varphi \circ g = \llbracket \varphi(c, d) \rrbracket$ . This implies that  $p = q$ .

Next we will show that  $F$  is continuous. Here we will use the maps  $\{\lambda_{n,i} : [0, 1] \longrightarrow [0, 1] ; i < 2^n\}$  defined just before [Lemma 2.2.9](#). First, note that if  $V \subseteq \mathbf{G}(\text{APr})$  is open then  $F^{-1}(\mathbf{G}(T)^\Omega \times V)$  is clearly open. So, it suffices to show that for any  $A \subseteq \Omega$  clopen,  $U \subseteq \mathbf{G}(T)$  open and  $\epsilon > 0$ ,  $W = F^{-1}(\mathcal{O}(A, U, \epsilon))$  is open. We can assume, that  $U = [\varphi(x, y) < 1]$  for some formula  $\varphi(x, y)$ . Take  $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta)) \in W$ , so  $\delta = \mu(A \cap \llbracket \varphi(a, b) = 1 \rrbracket) < \epsilon$ . Now choose any  $r > 0$  with  $\delta + r < \epsilon$ . We know that  $A \in \text{dcl}(\Xi)$ , then there is an  $\mathcal{L}_{\text{APr}}$ -term  $\chi(\bar{x})$  satisfying  $d(A, \chi(\Xi)) < r/2$ . In particular,  $\mu(\chi(\Xi) \cap \llbracket \varphi(a, b) = 1 \rrbracket) < \delta + r/2$ . Moreover, since  $\llbracket \varphi(a, b) = 1 \rrbracket = \lim_{n \rightarrow \infty} \llbracket \lambda_{n, 2^n-1}(\varphi(a, b)) \rrbracket$ , there is  $N \in \mathbb{N}$  such that  $\mathbb{E}(\chi(\Xi) \wedge \llbracket \lambda_{N, 2^N-1}(\varphi(a, b)) \rrbracket) < \delta + r/2$ . We define  $\psi(\hat{x}, \hat{y}) = \mathbb{E}(\chi(\hat{x}) \wedge \llbracket \lambda_{N, 2^N-1}(\varphi(x, y)) \rrbracket)$ . By definition,  $p \in [\psi(\hat{x}, \hat{y}) < \delta + r/2]$ , we claim that  $[\psi(\hat{x}, \hat{y}) < \delta + r/2] \subseteq W$ . Let  $q = \text{tp}((c, \Xi, \alpha), (d, \tau(\Xi), \beta)) \in [\psi(\hat{x}, \hat{y}) < \delta + r/2]$ , this means that  $\mathbb{E}(\chi(\Xi) \wedge \llbracket \lambda_{N, 2^N-1}(\varphi(c, d)) \rrbracket) < \delta + r/2$ . Therefore,

$$\begin{aligned}
\mu(A \cap \llbracket \varphi(c, d) = 1 \rrbracket) &< \mu(\chi(\Xi) \cap \llbracket \varphi(c, d) = 1 \rrbracket) + r/2 \\
&\leq \mathbb{E}(\chi(\Xi) \wedge \llbracket \lambda_{N, 2^{N-1}}(\varphi(c, d)) \rrbracket) + r/2 \\
&< \delta + r \\
&< \epsilon.
\end{aligned}$$

This implies that  $q \in W$ , which shows that  $F$  is continuous. We will now see that  $F$  is also open. Take  $\psi(\hat{x}, \hat{y})$  a formula in  $D_{TR}^2$ , we can assume that it is of the form

$$\psi(\hat{x}, \hat{y}) = \psi^*(\llbracket \varphi_1(x, y) \rrbracket, \dots, \llbracket \varphi_n(x, y) \rrbracket, \bar{x}, \hat{x}, \bar{y}, \hat{y})$$

where  $\psi^*(z, \bar{x}, \hat{x}, \bar{y}, \hat{y})$  is a formula in  $E_{RV}^n \times D_{APr}^2$ . Let  $p \in [\psi(\hat{x}, \hat{y}) < 1]$ , let  $(g, \alpha, \tau, \beta) = F(p)$  and pick  $r > 0$  with  $\psi(\hat{x}, \hat{y})^p + r < 1$ . Pick  $\epsilon > 0$  such that  $d((z, \bar{x}, \hat{x}, \bar{y}, \hat{y}), (z', \bar{x}', \hat{x}', \bar{y}', \hat{y}')) < \epsilon$  implies  $|\psi^*(z, \bar{x}, \hat{x}, \bar{y}, \hat{y}) - \psi^*(z', \bar{x}', \hat{x}', \bar{y}', \hat{y}')| < r$ . Note that for  $i \leq n$ , the map  $\hat{\varphi}_i : \mathbf{G}(T)^\Omega \rightarrow L^1((\Omega, \mathcal{B}, \mu), [0, 1])$  given by  $\hat{\varphi}_i(h) = \varphi_i \circ h$  is continuous. Thus, there is  $U \subseteq \mathbf{G}(T)^\Omega$  open with  $g \in U$  such that for any  $h \in U$ ,  $d(\varphi_i \circ g, \varphi_i \circ h) < \epsilon$  for  $i \leq n$ . Also, there is  $V \subseteq \mathbf{G}(APr)$  open with  $(\alpha, \tau, \beta) \in V$  such that for any  $(\alpha', \tau', \beta') \in V$ ,  $d(\alpha, \alpha') \vee d(\beta, \beta') \vee d(\tau(\Xi), \tau'(\Xi)) < \epsilon$ . We thus get that  $(g, \alpha, \tau, \beta) \in (U \times V) \cap \mathbf{H} \subseteq F([\psi(\hat{x}, \hat{y}) < 1])$ , which completes the proof.  $\square$

The next step is to determine how the groupoid operations are defined on  $\mathbf{H}$ . Take  $(a, \Xi, \alpha) \in D_{TR}$ , then  $\text{tp}^R(a) = \pi(\alpha)$  and  $\text{tp}(a, \Xi, \alpha)$  corresponds to  $(\pi(\alpha), \alpha, \text{id}_\Omega, \alpha)$ . We thus have an identification between  $2^{\mathbb{N}}$  and  $\mathbf{B}(T^R)$  via

$$\alpha \in 2^{\mathbb{N}} \mapsto (\pi(\alpha), \alpha, \text{id}_\Omega, \alpha) \in \mathbf{B}(T^R).$$

Now take  $(g, \alpha, \tau, \beta) \in \mathbf{H}$ , which we identify with  $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$ . Let  $\mathbf{M} = \text{dcl}(a, \Xi, \alpha)$ , which as usual we suppose is based on  $(\Omega, \mathcal{B}, \mu)$ . Given  $\omega \in \Omega$  let  $\mathcal{A}_\omega = \mathcal{M}_{\tau(\omega)}$ . We define a new randomization  $\mathbf{A}$ , based on  $\{\mathcal{A}_\omega ; \omega \in \Omega\}$ , where  $x \in \prod_{\omega \in \Omega} \mathcal{A}_\omega$  belongs to  $\mathbf{A}$  if and only if there is  $a \in \mathbf{M}$  such that  $x(\omega) = a(\tau(\omega))$  for every  $\omega \in \Omega$ . It follows that the function  $f_{\tau^{-1}} : \mathbf{M} \rightarrow \mathbf{A}$ , that assigns to each  $a \in \mathbf{M}$  the map  $\omega \mapsto a(\tau(\omega))$ , is an isomorphism. Moreover, on the sort of events,  $f_{\tau^{-1}} = \tau^{-1}$ . Therefore,  $p^{-1} = \text{tp}((b, \tau(\Xi), \beta), (a, \Xi, \alpha)) = \text{tp}(f_{\tau^{-1}}(b, \tau(\Xi), \beta), f_{\tau^{-1}}(a, \Xi, \alpha)) = \text{tp}((f_{\tau^{-1}}(b), \Xi, \beta), (f_{\tau^{-1}}(a), \tau^{-1}(\Xi), \alpha))$ . It is true that for every  $\omega \in \Omega$ ,

$$\text{tp}((f_{\tau^{-1}}(b)(\omega), (f_{\tau^{-1}}(a)(\omega))) = \text{tp}(b(\tau(\omega)), a(\tau(\omega))) = g(\tau(\omega))^{-1} = (\tau^{-1} \cdot g^{-1})(\omega).$$

Consequently,

$$(g, \alpha, \tau, \beta)^{-1} = (\tau^{-1} \cdot g^{-1}, \beta, \tau^{-1}, \alpha).$$

Let  $(g, \alpha, \tau, \beta), (h, \beta, \rho, \gamma) \in \mathbf{H}$  correspond to  $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$  and  $q = \text{tp}((c, \Xi, \beta), (d, \rho(\Xi), \gamma))$  respectively. Define  $\mathbf{A} = \text{dcl}(a, \Xi, \alpha) = \text{dcl}(b, \tau(\Xi), \beta)$  and  $\mathbf{M} = \text{dcl}(c, \Xi, \beta) = \text{dcl}(d, \rho(\Xi), \gamma)$ . We remark that for almost every  $\omega \in \Omega$ ,  $\mathcal{A}_\omega = \text{dcl}(a(\omega)) = \text{dcl}(b(\omega))$  and  $\mathcal{M}_\omega = \text{dcl}(c(\omega)) = \text{dcl}(d(\omega))$ . Furthermore,

$$\text{tp}(c(\omega)) = t(h_\omega) = \pi(\beta)(\omega) = \tau \cdot \pi(\beta)(\tau(\omega)) = s(g_{\tau(\omega)}) = \text{tp}(b(\tau(\omega))).$$

Hence, the map sending  $c(\omega)$  to  $b(\tau(\omega))$  defines an isomorphism  $f_\omega : M_\omega \rightarrow \mathcal{A}_{\tau(\omega)}$ . This implies that the map

$$f : \mathbf{M} \longrightarrow \mathbf{A}$$

$$u \longmapsto (\omega \mapsto f_{\tau^{-1}(\omega)}(u(\tau^{-1}(\omega))))$$

is an isomorphism sending  $c$  to  $b$  and agreeing with  $\tau$  on the sort of events. It follows that  $q = \text{tp}((b, \tau(\Xi), \beta), (f(d), \tau\rho(\Xi), \gamma))$  and we get that  $pq = \text{tp}((a, \Xi, \alpha), (f(d), \tau\rho(\Xi), \gamma))$ . Now, for  $\omega \in \Omega$ ,

$$\begin{aligned} \text{tp}(a(\omega), f(d)(\omega)) &= \text{tp}(a(\omega), f_{\tau^{-1}(\omega)}(d(\tau^{-1}(\omega)))) \\ &= \text{tp}(a(\omega), b(\omega)) \cdot \text{tp}(b(\omega), f_{\tau^{-1}(\omega)}(d(\tau^{-1}(\omega)))) \\ &= \text{tp}(a(\omega), b(\omega)) \cdot \text{tp}(c(\tau^{-1}(\omega)), d(\tau^{-1}(\omega))) \\ &= g(\omega)(\tau \cdot h)(\omega). \end{aligned}$$

We thus conclude that

$$(g, \alpha, \tau, \beta)(h, \beta, \rho, \gamma) = (g(\tau \cdot h), \alpha, \tau\rho, \gamma).$$

It follows that the target and source maps are given by,

$$t(g, \alpha, \tau, \beta) = (\pi(\alpha), \alpha, \text{id}_\Omega, \alpha) \cong \alpha, \quad s(g, \alpha, \tau, \beta) = (\pi(\beta), \beta, \text{id}_\Omega, \beta) \cong \beta.$$

The next result shows that this identification is independent of the choice of  $\Xi \in D_{\text{APr},0}(\Omega)$  as long as it generates the whole algebra.

**Lemma 2.4.12.** *Fix  $\Delta \in D_{\text{APr},0}(\Omega)$  generating  $\bar{\mathcal{B}}$  and define  $\theta : 2^{\mathbb{N}} \longrightarrow (2^{\mathbb{N}})^\Omega$  by  $\theta(\alpha) = \pi(\Delta, \alpha)$ . Then,*

$$\mathbf{H}_\Delta = \{(g, \alpha, \tau, \beta) \in \mathbf{G}(T)^\Omega \times \mathbf{G}(\text{APr}) ; t^\Omega(g) = \theta(\alpha), s^\Omega(g) = \tau \cdot \theta(\beta)\}$$

is homeomorphic to  $\mathbf{H}$  via a map that respects the identification with  $\mathbf{G}(T^R)$ . This means that there is map  $\Lambda : \mathbf{H} \longrightarrow \mathbf{H}_\Delta$  such that the following diagram commute.

$$\begin{array}{ccc} \mathbf{G}(T^R) & \xrightarrow{F} & \mathbf{H} \\ & \searrow F_\Delta & \downarrow \Lambda \\ & & \mathbf{H}_\Delta \end{array}$$

*Proof.* Let  $\rho \in \text{Aut}(\Omega)$  be such that  $\rho(\Xi) = \Delta$ . It follows that  $\theta(\alpha) = \pi(\rho(\Xi), \alpha) = \rho \cdot \pi(\alpha)$  for every  $\alpha \in 2^{\mathbb{N}}$ . Define

$$\begin{aligned} \Lambda : \mathbf{H} &\longrightarrow \mathbf{H}_\Delta \\ (g, \alpha, \tau, \beta) &\longmapsto (\rho \cdot g, \alpha, \rho\tau\rho^{-1}, \beta). \end{aligned}$$

Note that  $s^\Omega(\rho \cdot g) = \rho \cdot s^\Omega(g) = \rho\tau \cdot \pi(\beta) = \rho\tau\rho^{-1} \cdot \theta(\beta)$ . Analogously,  $t^\Omega(\rho \cdot g) = \theta(\alpha)$ , so the map is well-defined. It is a routine verification that  $\Lambda$  is a bijection. Let  $A_1, \dots, A_n \subseteq \Omega$ , let  $U_1, \dots, U_n$  be open subsets of  $\mathbf{G}(T)$  and let  $\epsilon > 0$ , then

$$\rho^{-1} \cdot \mathcal{O}(A_1, \dots, A_n, U_1, \dots, U_n, \epsilon) = \mathcal{O}(\rho^{-1}(A_1), \dots, \rho^{-1}(A_n), U_1, \dots, U_n, \epsilon).$$

Furthermore, if  $V \subseteq \mathbf{G}(\text{APr})$  is open then  $\rho^{-1}V\rho = \{(\alpha, \rho^{-1}\tau\rho, \beta) \in \mathbf{G}(\text{APr}) ; (\alpha, \tau, \beta) \in V\}$  is also open. Since,

$$\Lambda^{-1}(\mathcal{O}(A_1, \dots, A_n, U_1, \dots, U_n, \epsilon) \times V) = (\rho^{-1}\mathcal{O}(A_1, \dots, A_n, U_1, \dots, U_n, \epsilon) \times \rho^{-1}V\rho) \cap \mathbf{H},$$

we obtain that  $\Lambda$  is continuous. By a similar argument, the map is also open and we conclude that it is in fact a homeomorphism. Now, take  $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta)) \in \mathbf{G}(T^R)$  and let  $(g, \alpha, \tau, \beta) = F(p)$ . We set  $\mathbf{M} = \text{dcl}(a, \Xi, \alpha)$ , and for each  $\omega \in \Omega$  let  $\mathcal{N}_\omega = \mathcal{M}_{\rho^{-1}(\omega)}$ . As before, say that  $x \in \mathbf{A}$  if and only if for some  $a \in \mathbf{M}$ ,  $x(\omega) = a(\rho^{-1}(\omega))$  for every  $\omega \in \Omega$ . Again, we have an isomorphism  $f_\rho : \mathbf{M} \rightarrow \mathbf{A}$  that agrees with  $\rho$  on the sort of events. We then have that,  $p = \text{tp}((f_\rho(a), \Delta, \alpha), (f_\rho(b), \rho\tau\rho^{-1}(\Delta), \beta))$  and for  $\omega \in \Omega$ ,  $\text{tp}(f_\rho(a)(\omega), f_\rho(b)(\omega)) = \text{tp}(a(\rho^{-1}(\omega)), b(\rho^{-1}(\omega))) = (\rho \cdot g)(\omega)$ . This implies that  $F_\Delta(p) = (\rho \cdot g, \alpha, \rho\tau\rho^{-1}, \beta) = \Lambda(g, \alpha, \tau, \beta)$ , which completes the proof.  $\square$

**Theorem 2.4.13.** *Let  $T$  be a complete theory admitting a universal Skolem sort. Let  $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^\Omega$  be the map defined above. Then,  $\mathbf{G}(T^R)$  is isomorphic, as a topological groupoid, to  $(\mathbf{G}(T) \wr \Omega)_\pi$ , the base change of  $\mathbf{G}(T) \wr \Omega$  to  $2^{\mathbb{N}}$  given by  $\pi$ .*

*Proof.* We have that

$$(\mathbf{G}(T) \wr \Omega)_\pi = \{(\alpha, g, \tau, \beta) \in 2^{\mathbb{N}} \times \mathbf{G}(T)^\Omega \times \text{Aut}(\Omega) \times 2^{\mathbb{N}} ; t^\Omega(g) = \pi(\alpha), \tau^{-1} \cdot s^\Omega(g) = \pi(\beta)\}$$

endowed with the topology induced from the product, which is equal to  $\mathbf{H}$  up to a reordering of variables.  $\square$

Suppose  $T$  is  $\omega$ -categorical and  $\mathcal{M}$  is its separable model, we know that  $\mathbf{G}(T) = 2^{\mathbb{N}} \times \text{Aut}(\mathcal{M}) \times 2^{\mathbb{N}}$ . Hence, an element  $g \in \mathbf{G}(T)^\Omega$  can be identified with  $(t^\Omega(g), \tilde{g}, s^\Omega(g))$  for some  $\tilde{g} \in \text{Aut}(\mathcal{M})^\Omega$ . Thus, if  $(\alpha, g, \tau, \beta) \in \mathbf{G}(T^R)$  then

$$(\alpha, g, \tau, \beta) = (\alpha, t^\Omega(g), \tilde{g}, s^\Omega(g), \tau, \beta) = (\alpha, \pi(\alpha), \tilde{g}, \tau \cdot \pi(\beta), \tau, \beta) \cong (\alpha, \tilde{g}, \tau, \beta),$$

which gives the correspondence between  $\mathbf{G}(T^R)$ , as constructed here, and, as previously known, i.e.  $2^{\mathbb{N}} \times (\text{Aut}(\mathcal{M}) \wr \Omega) \times 2^{\mathbb{N}}$ .

## 2.5 Further questions

The reader may have noticed that the map  $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$  is not canonical. Given any two pseudo-open maps the fiber products they define are universal Skolem sorts. Hence, there is a definable bijection between them which induces an isomorphism between the corresponding groupoids. We wonder whether is possible to explicitly build such an isomorphism and show directly the independence of the choice of the pseudo-open map. Furthermore, the property of being pseudo-open is sufficient for the fiber product to be definable. However, we do not know if it is also necessary and in case it is not, what would be an appropriate characterization of the maps whose fiber product define a universal Skolem sort.

## Chapter 3

# Fields of Banach spaces and groupoid actions

### Introduction

Take  $X$  a compact space and  $G$  a topological group acting on  $X$ . It is a standard fact that the action  $G \curvearrowright X$  admits a faithful representation on  $C(X)$ . However, it may happen that the action admits a representation on a Banach space with a much nicer geometry, for example reflexivity. Were this to be the case, one can ask if the geometric properties of the Banach space has any implications on the dynamical properties of the action. This led M. Megreleshvili in [Meg03], and then together with E. Glasner in [GM12], to establish a dynamical hierarchy, where the representability on a particular class of Banach spaces has its dynamical counterpart. We refer the reader to the survey [GM14].

Now suppose that  $T$  is an  $\omega$ -categorical theory,  $\mathcal{M}$  is its separable model and  $G = \text{Aut}(\mathcal{M})$ . Since  $G$  determines  $T$  up to bi-interpretability, see [AZ86, BYK16], it is expected that model-theoretical properties of  $T$  correspond to dynamical properties of  $G$ . In particular, for the notion of stability this was explored in [BYT16], and then retaken in [Iba16]. It turns out that stable formulas correspond to weakly almost periodic functions. This is precisely the first level of the hierarchy mentioned above, which corresponds to representability in the class of reflexive spaces. In fact, [Iba16] establishes the model-theoretical equivalent of the dynamical hierarchy in the  $\omega$ -categorical setting. We note that NIP corresponds to tameness, and so to Rosenthal representability.

The introduction of universal Skolem sorts, and thus of the associated groupoid in [BY22], let us expect that such methods might work on a broader context than  $\omega$ -categoricity. Indeed, this groupoid will act on a space of types, as we will see in the next chapter, and dynamical properties of this action will have model-theoretical implications. In this chapter we explore the first level of the dynamical hierarchy for groupoid actions, that of weakly almost periodic functions. Instead of representing on a Banach space, for each element of the base we will have a Banach space. This lead us to representing groupoid actions on fields of Banach spaces.

A field of Banach spaces essentially consists of a topological space serving as the base, a Banach space over each element of the base and a family of sections. The field is continuous if the norm of sections are continuous. It was defined for the first time for Hilbert spaces over a locally compact space in [God51], while studying unitary representations of  $*$ -algebras. Let us explain the motivation behind it. Suppose that  $\mathbf{A}$  is a  $*$ -algebra, each bounded positive linear form  $f$  on

$\mathbf{A}$  gives rise to a Hilbert space  $H_f$  and a unitary representation of  $\mathbf{A}$  on said space. If  $\mathbf{B}$  is a set of bounded positive linear forms, endowed with the weak\* topology, the notion of field of Banach spaces was defined in such a way that for each  $x \in \mathbf{A}$ , the map  $f \in \mathbf{B} \mapsto x_f \in H_f$  defines a section.

This construction was later generalized to arbitrary Banach spaces in [DD63], which is the presentation we use. The article was mainly concerned with triviality results (a field over  $\mathbf{B}$  is said to be trivial if it is of the form  $V \times \mathbf{B}$ ) and with fields of  $C^*$ -algebras. Since then, multiple works studying the representations of  $C^*$ -algebras and continuous fields of Banach spaces has been carried out. An extensive development of the subject can be found in [FD88a, FD88b].

We begin this chapter by giving the basic definitions and by studying the dual field, which we endow with the weak\*-topology. We showed the analogue of the Banach-Alaoglu theorem for continuous fields. We also prove that if the field is separable then its dual admits sections and it is moreover a lower semi-continuous field. We continue with the study of  $\mathbb{C}_b(X)$ , the field of continuous, bounded, real-valued functions on a normal, Hausdorff space  $(X, \rho)$  over  $\mathbf{B}$ . When  $(X, \rho)$  admits sections we define the topology of pointwise convergence and under suitable hypothesis for  $(X, \rho)$  we deduce a double limit criterion for pointwise compactness in the style of [Gro52].

Afterwards, we define the notions of separately continuous groupoid action and of duality. In particular, we study the continuity of the induced action  $\mathbf{G} \curvearrowright \mathbb{C}_b(X)$ . Subsequently, we study the representations of groupoid actions. A representation will consist of an action by linear isometries on a field of Banach spaces together with an equivariant, weak\* continuous map from the topological space into the unit ball of the dual field.

Finally, we define the notion of a weakly almost periodical function, or wap for short, in the context of groupoid actions. Generalizing the usual notion, a function will be wap if the set in the field of continuous functions obtained after the action of the groupoid is fiberwise relatively weakly compact. We show that wap functions are those precisely coming from representations on fields of reflexive spaces. All of our results hold if the base space is assumed to be the space consisting of a single point, in which case, the groupoid is just a group and we can recover the classical results for group actions.

### 3.1 Fields of Banach spaces

The definitions of this section are based on [DD63, Bos11]. Throughout this section we assume that  $\mathbf{B}$  is a compact (Hausdorff) space. As mentioned before, whenever  $\mathbf{B} = \{*\}$ , the space consisting of a single point, we will rediscover some familiar results from functional analysis.

**Definition 3.1.1.** A *field of Banach spaces over  $\mathbf{B}$*  is a couple  $(\{V_e ; e \in \mathbf{B}\}, \Gamma)$  where each  $V_e$  is a Banach space and  $\Gamma \subseteq \prod_{e \in \mathbf{B}} V_e$  satisfies:

1.  $\Gamma$  is a vector subspace of  $\prod_{e \in \mathbf{B}} V_e$ .
2. For every  $e \in \mathbf{B}$ ,  $\{\mathbf{v}(e) ; \mathbf{v} \in \Gamma\}$  is dense in  $V_e$ .
3. For every  $\mathbf{v} \in \Gamma$  the map  $e \mapsto \|\mathbf{v}(e)\|$  is bounded.

$\Gamma$  is thus a family of sections of the space. From 1. we deduce that  $\Gamma$  contains the zero section, which we will denote by  $\mathbf{0}$ . In case the norm of each section is continuous a field generates what will be known as a *continuous field*.

**Proposition 3.1.2.** [DD63, Proposition 3] Let  $(\{V_e ; e \in \mathbf{B}\}, \Lambda)$  be a field of Banach spaces over  $\mathbf{B}$ . Suppose that for every  $\mathbf{v} \in \Lambda$  the map  $e \mapsto \|\mathbf{v}(e)\|$  is continuous. Then there is a unique  $\Gamma \subseteq \prod_{e \in \mathbf{B}} V_e$  containing  $\Lambda$  and satisfying:

1.  $\Gamma$  is a sub- $C(\mathbf{B})$ -module of  $\prod_{e \in \mathbf{B}} V_e$ .
2. For every  $e \in \mathbf{B}$  and every  $v \in V_e$  there is  $\mathbf{v} \in \Gamma$  such that  $\mathbf{v}(e) = v$ .
3. For every  $\mathbf{v} \in \Gamma$  the map  $e \mapsto \|\mathbf{v}(e)\|$  is continuous.
4.  $\Gamma$  is locally uniformly closed, i.e. if  $\mathbf{w} \in \prod_{e \in \mathbf{B}} V_e$  satisfies that for any  $e \in \mathbf{B}$  and any  $\epsilon > 0$  there are  $\mathbf{v} \in \Gamma$  and  $U \subseteq \mathbf{B}$  open with  $e \in U$  such that  $\|\mathbf{w}(e') - \mathbf{v}(e')\| < \epsilon$  for every  $e' \in U$ , then  $\mathbf{w}$  belongs to  $\Gamma$ .

**Definition 3.1.3.** A continuous field of Banach spaces over  $\mathbf{B}$  is a family of Banach spaces  $\{V_e ; e \in \mathbf{B}\}$  together with a set of sections  $\Gamma \subseteq \prod_{e \in \mathbf{B}} V_e$  such that  $(\{V_e ; e \in \mathbf{B}\}, \Gamma)$  satisfies the four conditions stated in the previous proposition.

If we weaken the third condition in the previous definition and we merely require the maps  $e \mapsto \|\mathbf{v}(e)\|$  to be lower (resp. upper) semi-continuous and bounded we say that  $(\{V_e ; e \in \mathbf{B}\}, \Gamma)$  is a lower (resp. upper) semi-continuous field. Note that

$$\|\mathbf{v}\| = \sup_{e \in \mathbf{B}} \|\mathbf{v}(e)\|$$

defines a norm on  $\Gamma$ , and by 4. this norm is complete.

Let  $(\{V_e ; e \in \mathbf{B}\}, \Gamma)$  be a field of Banach spaces over  $\mathbf{B}$ . Define  $\mathbb{V} = \sqcup_{e \in \mathbf{B}} V_e$  and let  $\text{pr} : \mathbb{V} \rightarrow \mathbf{B}$  be the natural projection. We may write  $(e, v) \in \mathbb{V}$  to indicate that  $v \in V_e$ . From now on,  $(\mathbb{V}, \Gamma)$  will also refer to the field  $(\{V_e ; e \in \mathbf{B}\}, \Gamma)$ . For  $\mathbf{v} \in \Gamma$ ,  $U \subseteq \mathbf{B}$  open and  $\epsilon > 0$  define

$$\mathcal{O}(\mathbf{v}, \epsilon, U) = \{(e, v) \in \mathbb{V} ; e \in U \text{ and } \|v - \mathbf{v}(e)\| < \epsilon\}.$$

The next result shows that for a continuous field of Banach spaces over  $\mathbf{B}$ ,  $\mathbb{V}$  defines a space over  $\mathbf{B}$  for which  $\Gamma$  corresponds to the space of sections. Recall that a space over  $\mathbf{B}$  is a topological space  $X$  together with a continuous, surjective function  $\rho : X \rightarrow \mathbf{B}$ . A section of  $X$  is continuous map  $\mathbf{x} : \mathbf{B} \rightarrow X$  such that  $\rho \circ \mathbf{x} = \text{id}_{\mathbf{B}}$ . The proof of the following fact is given throughout [DD63, Section I.2].

**Fact 3.1.4.** Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces over  $\mathbf{B}$ . Then, the family of sets  $\{\mathcal{O}(\mathbf{v}, \epsilon, U) ; \mathbf{v} \in \Gamma, U \subseteq \mathbf{B} \text{ open and } \epsilon > 0\}$  is a base for a Hausdorff topology on  $\mathbb{V}$ . Under this topology:

1.  $\text{pr}$  is continuous and open.
2. For every  $e \in \mathbf{B}$ , the induced topology on  $V_e$  is the norm topology.
3. If  $\mathbf{v} \in \Gamma$  then  $\mathbf{v}$  is a continuous map from  $\mathbf{B}$  into  $\mathbb{V}$ . Conversely, if  $\mathbf{v} : \mathbf{B} \rightarrow \mathbb{V}$  is continuous and  $\text{pr} \circ \mathbf{v} = \text{id}_{\mathbf{B}}$  then  $\mathbf{v} \in \Gamma$ .
4. The sum from  $\mathbb{V} \times_{\mathbf{B}} \mathbb{V}$  into  $\mathbb{V}$  and the scalar multiplication from  $\mathbb{R} \times \mathbb{V}$  into  $\mathbb{V}$  are continuous maps.

*Remark.* In case the field is just upper-semicontinuous, the family  $\{\mathcal{O}(\mathbf{v}, \epsilon, U) ; \mathbf{v} \in \Gamma, U \subseteq \mathbf{B} \text{ open and } \epsilon > 0\}$  still defines a base for a topology on  $\mathbb{V}$ . However, lower semi-continuity is used in order to show that the topology is Hausdorff.

The next fact characterizes the family sections of a continuous field of Banach spaces.

**Fact 3.1.5.** [DD63, Proposition 1] *Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces over  $\mathbf{B}$ . Then, for any  $\mathbf{v} \in \prod_{e \in \mathbf{B}} V_e$  the following conditions are equivalent:*

1.  $\mathbf{v} \in \Gamma$ .
2. For every  $\mathbf{w} \in \Gamma$ , the map  $e \mapsto \|\mathbf{v}(e) - \mathbf{w}(e)\|$  is continuous.
3. For every  $e \in \mathbf{B}$  and every  $\epsilon > 0$  there are  $\mathbf{w} \in \Gamma$  and  $U \subseteq \mathbf{B}$  open set containing  $e$  such that  $\|\mathbf{v}(e') - \mathbf{w}(e')\| < \epsilon$  for every  $e' \in U$ .

As the following results will show, the sets  $\mathcal{O}(\mathbf{v}, \epsilon, U)$  behave in pretty much the same way as open balls in Banach spaces. In particular the proof that  $\mathbb{V}$  is a Baire space is an adaptation of the proof of the analogous result for Banach spaces. Given  $\mathbf{v} \in \Gamma$ ,  $\epsilon > 0$  and  $V \subseteq \mathbf{B}$  we define

$$\overline{\mathcal{O}}(\mathbf{v}, \epsilon, V) = \{(e, v) \in \mathbb{V} ; e \in V \text{ and } \|v - \mathbf{v}(e)\| \leq \epsilon\}.$$

**Lemma 3.1.6.** *Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces over  $\mathbf{B}$ . Then, for  $\mathbf{v} \in \Gamma$ ,  $\epsilon > 0$  and  $U \subseteq \mathbf{B}$ , the closure of  $\mathcal{O}(\mathbf{v}, \epsilon, U)$  is equal to  $\overline{\mathcal{O}}(\mathbf{v}, \epsilon, \bar{U})$ . In particular, if  $V \subseteq \mathbf{B}$  is closed then  $\overline{\mathcal{O}}(\mathbf{v}, \epsilon, V)$  is closed.*

*Proof.* Let  $A$  denote the closure of  $\mathcal{O}(\mathbf{v}, \epsilon, U)$ . Take  $(e, w) \in A$ , for any open neighborhood  $W$  of  $e$ ,  $\text{pr}^{-1}(W)$  is an open set containing  $(e, w)$ . Then there is  $(e', w') \in \mathcal{O}(\mathbf{v}, \epsilon, U) \cap \text{pr}^{-1}(W)$  which implies  $e' \in W \cap U$ . This implies that  $e \in \bar{U}$ . Let  $r = \|\mathbf{v}(e) - w\|$  and suppose that  $r > \epsilon$ . Fix  $\mathbf{w} \in \Gamma$  passing through  $w$ . We know that there is an open set  $V$  containing  $e$  such that  $\|\mathbf{w} - \mathbf{v}\| > \epsilon$  on  $V$ . It follows that  $\mathcal{O}(\mathbf{w}, r - \epsilon, V)$  is an open set containing  $w$  that does not intersect  $\mathcal{O}(\mathbf{v}, \epsilon, U)$ , which is absurd. Thus,  $r \leq \epsilon$  and  $(e, w) \in \overline{\mathcal{O}}(\mathbf{v}, \epsilon, \bar{U})$ . For the other direction, take  $(e, w) \in \overline{\mathcal{O}}(\mathbf{v}, \epsilon, \bar{U})$ . Pick  $\mathbf{w} \in \Gamma$  passing through  $(e, w)$ ,  $0 < \delta < \epsilon$  and  $V$  open neighborhood of  $e$ . If  $\|\mathbf{v}(e) - \mathbf{w}(e)\| \leq \delta/2$  then there is  $V_0 \subseteq V$  open with  $e \in V_0$  such that  $\|\mathbf{v} - \mathbf{w}\| < \epsilon$  on  $V_0$ . Since  $e \in \bar{U}$ , we can find  $e_0 \in U \cap V_0$  and we have that  $\mathbf{w}(e_0) \in \mathcal{O}(\mathbf{w}, \delta, V) \cap \mathcal{O}(\mathbf{v}, \epsilon, U)$ . Now assume that  $\|\mathbf{v}(e) - \mathbf{w}(e)\| > \delta/2$ . Let  $V_1 \subseteq V$  be open with  $e \in V_1$  satisfying  $\delta/2 < \|\mathbf{v} - \mathbf{w}\| < \epsilon + \delta/2$  on  $V_1$ . Pick  $e_1 \in U \cap V_1$  and define  $w' = \mathbf{w}(e_1) + \frac{\delta}{2\|\mathbf{v}(e_1) - \mathbf{w}(e_1)\|}(\mathbf{v}(e_1) - \mathbf{w}(e_1))$ . We have that  $\|\mathbf{w}(e_1) - w'\| = \delta/2$ ,  $\|\mathbf{v}(e_1) - w'\| = \|\mathbf{v}(e_1) - \mathbf{w}(e_1)\| - \delta/2 < \epsilon$ . Therefore,  $(e_1, w') \in \mathcal{O}(\mathbf{w}, \delta, V) \cap \mathcal{O}(\mathbf{v}, \epsilon, U)$ .  $\square$

**Theorem 3.1.7.** *Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces over  $\mathbf{B}$ . Then,  $\mathbb{V}$  is a Baire space.*

*Proof.* Let  $\{U_n ; n \in \mathbb{N}\}$  be a collection of open dense subsets of  $\mathbb{V}$ . Note in particular that each  $\text{pr}(U_n)$  is a dense open subset of  $\mathbf{B}$ . Let  $\mathcal{V}$  be an open subset of  $\mathbb{V}$ . Since  $\mathcal{V} \cap U_1 \neq \emptyset$ , there is  $\mathbf{v}_0 \in \Gamma$ ,  $0 < \epsilon_0 < 1$  and  $W_0 \subseteq \mathbf{B}$  open such that  $\overline{\mathcal{O}}(\mathbf{v}_0, \epsilon_0, \bar{W}_0) \subseteq \mathcal{V} \cap U_0$ . Suppose  $\mathbf{v}_0, \dots, \mathbf{v}_{n-1}$ ,  $\epsilon_0, \dots, \epsilon_{n-1}$  and  $W_0, \dots, W_{n-1}$  are already defined. Due to the fact that  $U_n$  is dense, there is  $\mathbf{v}_n \in \Gamma$ ,  $0 < \epsilon_n < 1/(n+1)$  and  $W_n \subseteq \mathbf{B}$  open such that  $\overline{\mathcal{O}}(\mathbf{v}_n, \epsilon_n, \bar{W}_n) \subseteq \mathcal{O}(\mathbf{v}_{n-1}, \epsilon_{n-1}, W_{n-1}) \cap U_n$ . We have that  $(\bar{W}_n)_{n \in \mathbb{N}}$  is a decreasing sequence of closed subsets of  $\mathbf{B}$ . Thus, we can find some  $e \in \bigcap_{n \in \mathbb{N}} \bar{W}_n$ . We claim that  $(\mathbf{v}_n(e))_{n \in \mathbb{N}}$  is a Cauchy sequence. Indeed, for  $m < n$  we have  $\overline{\mathcal{O}}(\mathbf{v}_n, \epsilon_n, \bar{W}_n) \subseteq \mathcal{O}(\mathbf{v}_m, \epsilon_m, W_m)$ , implying  $\|\mathbf{v}_n(e) - \mathbf{v}_m(e)\| < \epsilon_m < 1/(m+1)$ . Therefore,  $(\mathbf{v}_n(e))_{n \in \mathbb{N}}$  converges, to say  $(e, v)$ , and  $(e, v) \in \overline{\mathcal{O}}(\mathbf{v}_m, \epsilon_m, \bar{W}_m) \subseteq U_m$  for every  $m \in \mathbb{N}$ . Hence,  $(e, v) \in \mathcal{V} \cap \bigcap_{n \in \mathbb{N}} U_n$ .  $\square$

Let  $(\mathbb{V}, \Gamma)$  be a field of Banach spaces over  $\mathbf{B}$ , a *subfield* is a field  $(\mathbb{W}, \Lambda)$  such that  $\Lambda \subseteq \Gamma$  and for every  $e \in \mathbf{B}$ ,  $W_e$  is a closed subspace of  $V_e$ . Given  $\Lambda \subseteq \Gamma$ , for  $e \in \mathbf{B}$  define  $W_e$  as the closed subspace generated by  $\{\mathbf{v}(e) ; \mathbf{v} \in \Lambda\}$ . Let  $\bar{\Lambda}$  be the vector subspace of  $\Gamma$  generated by  $\Lambda$ , then

$(\mathbb{W}, \bar{\Lambda})$  is a subfield of  $(\mathbb{V}, \Gamma)$  and it is known as the *field generated by  $\Lambda$* . In case  $(\mathbb{V}, \Gamma)$  is a continuous field, define instead  $\bar{\Lambda}$  as the family of sections of  $\Gamma$  that are contained in  $\mathbb{W}$ . Then,  $(\mathbb{W}, \bar{\Lambda})$  is a continuous subfield of  $(\mathbb{V}, \Gamma)$ , see [DD63, Section I.6], and we will refer to it as the *continuous field generated by  $\Lambda$* .

### 3.1.1 The dual field

Let  $(\mathbb{V}, \Gamma)$  be a field of Banach spaces over  $\mathbf{B}$ . For each  $e \in \mathbf{B}$ , let  $B(V_e)$  denote the close unit ball in  $V_e$  and let  $B(\mathbb{V}) = \bigsqcup_{e \in \mathbf{B}} B(V_e)$ . Whenever the field is continuous we will endow  $B(\mathbb{V})$  with the subspace topology. We will now define the dual of a continuous field. The underlying space is given by

$$\mathbb{V}^* = \bigsqcup_{e \in \mathbf{B}} V_e^*.$$

As usual, we will write  $(e, \varphi) \in \mathbb{V}^*$  to indicate  $\varphi \in V_e^*$ . For  $\mathbf{v} \in \Gamma$  we will write  $\langle \mathbf{v}, \varphi \rangle$  instead of  $\langle \mathbf{v}(e), \varphi \rangle$ . Given  $\mathbf{v}_0, \dots, \mathbf{v}_{n-1} \in \Gamma$ ,  $I_0, \dots, I_{n-1} \subseteq \mathbb{R}$  and  $U \subseteq \mathbf{B}$ , define

$$\mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U) = \{(e, \varphi) \in \mathbb{V}^* ; e \in U \text{ and } \langle \mathbf{v}_j, \varphi \rangle \in I_j \text{ for every } j < n\}.$$

**Proposition 3.1.8.** *Let  $(\mathbb{V}, \Gamma)$  be a field of Banach spaces over  $\mathbf{B}$ . Then, the family of sets  $\{\mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U) ; \mathbf{v}_0, \dots, \mathbf{v}_{n-1} \in \Gamma, I_0, \dots, I_{n-1} \subseteq \mathbb{R}, \text{ open and } U \subseteq \mathbf{B} \text{ open}\}$  is a base for a Hausdorff topology on  $\mathbb{V}^*$ . Under this topology:*

1. *The natural projection  $\text{pr}^* : \mathbb{V}^* \rightarrow \mathbf{B}$  is continuous.*
2. *For every  $\mathbf{v} \in \Gamma$  the map  $\langle \mathbf{v}, \cdot \rangle : \mathbb{V}^* \rightarrow \mathbb{R}$  given by  $(e, \varphi) \mapsto \langle \mathbf{v}, \varphi \rangle$  is continuous.*
3. *The sum from  $\mathbb{V}^* \times_{\mathbf{B}} \mathbb{V}^*$  into  $\mathbb{V}^*$  and the scalar multiplication from  $\mathbb{R} \times \mathbb{V}^*$  into  $\mathbb{V}^*$  are continuous maps.*

*Proof.* First we will show that this family of sets is indeed a base for a topology. Clearly they cover  $\mathbb{V}^*$  and  $\mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U) \cap \mathcal{O}^*((\mathbf{w}_i)_{i < m}, (J_i)_{i < m}, V)$  is just  $\mathcal{O}^*((\mathbf{v}_j)_{j < n+m}, (I_j)_{j < n+m}, U \cap V)$  where  $\mathbf{v}_{n+i} = \mathbf{w}_i$  and  $I_{n+i} = J_i$  for  $i < m$ . Continuity of  $\text{pr}^*$  follows from the fact that for any  $U \subseteq \mathbf{B}$  open,  $(\text{pr}^*)^{-1}(U) = \mathcal{O}^*(\mathbf{0}, \mathbb{R}, U)$  where  $\mathbf{0}$  is the section taking the value 0 at each point.

In order to show that the topology is Hausdorff, take  $(e, \varphi), (e', \psi) \in \mathbb{V}^*$  with  $(e, \varphi) \neq (e', \psi)$ . In case  $e \neq e'$ , take  $U, V \subseteq \mathbf{B}$  open sets satisfying  $e \in U$ ,  $e' \in V$  and  $U \cap V = \emptyset$ . Then,  $(\text{pr}^*)^{-1}(U)$  and  $(\text{pr}^*)^{-1}(V)$  are open sets separating  $(e, \varphi)$  and  $(e', \psi)$ . Assume now that  $e = e'$ , then there is  $\mathbf{v} \in \Gamma$  such that  $\langle \mathbf{v}, \varphi \rangle \neq \langle \mathbf{v}, \psi \rangle$ . Let  $I, J \subseteq \mathbb{R}$  be disjoint open sets containing  $\langle \mathbf{v}, \varphi \rangle$  and  $\langle \mathbf{v}, \psi \rangle$  respectively. We then have that  $(e, \varphi) \in \mathcal{O}^*(\mathbf{v}, I, \mathbf{B})$ ,  $(e, \psi) \in \mathcal{O}^*(\mathbf{v}, J, \mathbf{B})$  and  $\mathcal{O}^*(\mathbf{v}, I, \mathbf{B}) \cap \mathcal{O}^*(\mathbf{v}, J, \mathbf{B}) = \emptyset$ .

Assertion 2. follows immediately from the definition of the topology. In order to prove 3., take  $\varphi, \psi \in V_e^*$  and suppose that  $(e, \varphi + \psi) \in \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U)$ . For  $j < n$  take  $K_j, L_j \subseteq \mathbb{R}$  open sets satisfying  $\langle \mathbf{v}_j, \varphi \rangle \in K_j$ ,  $\langle \mathbf{v}_j, \psi \rangle \in L_j$  and  $K_j + L_j \subseteq I_j$ . It follows that  $(e, \varphi) \in \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (K_j)_{j < n}, U)$ ,  $(e, \psi) \in \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (L_j)_{j < n}, U)$  and

$$\mathcal{O}^*((\mathbf{v}_j)_{j < n}, (K_j)_{j < n}, U) + \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (L_j)_{j < n}, U) \subseteq \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U).$$

This shows that the sum is continuous, scalar multiplication is treated in a similar fashion.  $\square$

If  $\{\mathbf{v}(e) ; \mathbf{v} \in \Gamma\} = V_e$ , as is the case for example in a semi-continuous field, then the induced topology on  $V_e^*$  is just the weak\* topology. For this reason, this topology will be known as the *weak\* topology* on  $\mathbb{V}^*$ . It is easy to verify that this topology agrees with the initial topology given by  $\text{pr}^*$  and the maps  $\langle \mathbf{v}, \cdot \rangle$  for  $\mathbf{v} \in \Gamma$ . Let us define  $B(\mathbb{V}^*) = \{(e, \varphi) \in \mathbb{V}^* ; \|\varphi\| \leq 1\}$ . The next result is the analogue of the Banach-Alaoglu theorem for fields of Banach spaces.

**Proposition 3.1.9.** *Let  $(\mathbb{V}, \Gamma)$  be an upper semi-continuous field of Banach spaces over  $\mathbf{B}$ . Then,  $B(\mathbb{V}^*)$  is compact in the weak\* topology.*

*Proof.* Given  $\mathbf{v} \in \Gamma$  let  $D_{\mathbf{v}} = [-\|\mathbf{v}\|, \|\mathbf{v}\|] \subseteq \mathbb{R}$ , and define

$$P = \mathbf{B} \times \prod_{\mathbf{v} \in \Gamma} D_{\mathbf{v}}.$$

By Tychonoff's theorem,  $P$  is compact. Define the map

$$\begin{aligned} \iota : B(\mathbb{V}^*) &\longrightarrow P \\ (e, \varphi) &\longmapsto (e, (\langle \mathbf{v}, \varphi \rangle)_{\mathbf{v} \in \Gamma}). \end{aligned}$$

Note that for every  $\mathbf{v} \in \Gamma$ ,  $|\langle \mathbf{v}, \varphi \rangle| \leq \|\mathbf{v}(e)\| \leq \|\mathbf{v}\|$ . This shows that  $\iota$  is well-defined. It is immediate that  $\iota$  is also one-to-one. We will show that it is indeed a homeomorphism onto its image. Take  $(e, \varphi) \in B(\mathbb{V}^*)$ , and let  $\mathcal{W}$  be the open set in  $P$ ,

$$\mathcal{W} = U \times \{f \in \prod_{\mathbf{v} \in \Gamma} D_{\mathbf{v}} ; |f(\mathbf{v}_j) - \langle \mathbf{v}_j, \varphi \rangle| < \epsilon \text{ for } j < n\}$$

where  $U \subseteq \mathbf{B}$  is an open set with  $e \in U$ ,  $\mathbf{v}_0, \dots, \mathbf{v}_{n-1} \in \Gamma$  and  $\epsilon > 0$ . For  $j < n$ , set  $I_j = (\langle \mathbf{v}_j, \varphi \rangle - \epsilon, \langle \mathbf{v}_j, \varphi \rangle + \epsilon)$ , then  $\iota(\mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U)) = \mathcal{W} \cap \iota(B(\mathbb{V}^*))$ . If we let  $U$  range over the family of open neighborhoods of  $e$ ,  $n$  over  $\mathbb{N}$ ,  $\mathbf{v}_0, \dots, \mathbf{v}_{n-1}$  over  $\Gamma$  and  $\epsilon$  over the set of positive reals, the sets  $\mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U)$  are a local basis at  $(e, \varphi)$  and the sets  $\mathcal{W}$  are a local basis at  $\iota(e, \varphi)$ . This implies that  $\iota$  is a homeomorphism onto its image.

To finish the proof it suffices to show that  $\iota(B(\mathbb{V}^*))$  is closed in  $P$ . Take  $(e, f) \in P$  belonging to the closure of  $\iota(B(\mathbb{V}^*))$  and define  $\varphi : V_e \longrightarrow \mathbb{R}$  by  $\langle v, \varphi \rangle = f(\mathbf{v})$  where  $\mathbf{v}$  is any section passing through  $(e, v)$ . We will prove that  $\varphi \in B_e^*$ , so  $\iota(\varphi) = (e, f)$  and the result will follow. First, let us show that  $\varphi$  is well defined. Take  $v \in V_e$  and suppose  $\mathbf{v}_1(e) = \mathbf{v}_2(e) = v$  for some  $\mathbf{v}_1, \mathbf{v}_2 \in \Gamma$ . Let  $\epsilon > 0$ , by upper semi-continuity of the norm, there is  $U$  open neighborhood of  $e$  such that  $\|\mathbf{v}_1 - \mathbf{v}_2\| < \epsilon$  on  $U$ . Consider the open set

$$\mathcal{W}_0 = U \times \{f' \in \prod_{\mathbf{v} \in \Gamma} D_{\mathbf{v}} ; |f(\mathbf{v}_1) - f'(\mathbf{v}_1)| < \epsilon \text{ and } |f(\mathbf{v}_2) - f'(\mathbf{v}_2)| < \epsilon\}.$$

Then, there is  $(e', \psi) \in B(\mathbb{V}^*)$  such that  $\iota(e', \psi) \in \mathcal{W}_0$ . Thus, for  $i = 1, 2$

$$\begin{aligned} |f(\mathbf{v}_1) - f(\mathbf{v}_2)| &\leq |f(\mathbf{v}_1) - \iota(e', \psi)(\mathbf{v}_1)| + |\iota(e', \psi)(\mathbf{v}_1) - \iota(e', \psi)(\mathbf{v}_2)| + |\iota(e', \psi)(\mathbf{v}_2) - f(\mathbf{v}_2)| \\ &< 2\epsilon + |\langle \mathbf{v}_1 - \mathbf{v}_2, \psi \rangle| \\ &< 3\epsilon. \end{aligned}$$

Since  $\epsilon$  is arbitrary we obtain that  $f(\mathbf{v}_1) = f(\mathbf{v}_2)$ . Now take  $v_1, v_2 \in V_e$ ,  $\alpha, \beta \in \mathbb{R}$  and  $\epsilon > 0$ . Pick  $\mathbf{v}_1, \mathbf{v}_2 \in \Gamma$  with  $\mathbf{v}_i(e) = (e, v_i)$  for  $i = 1, 2$ . Set

$$\mathcal{W}_1 = \mathbf{B} \times \{f' \in \prod_{\mathbf{v} \in \Gamma} D_{\mathbf{v}} ; |f(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) - f'(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2)| < \epsilon, |f(\mathbf{v}_i) - f'(\mathbf{v}_i)| < \epsilon \text{ for } i = 1, 2\}.$$

It follows that there is  $(e', \psi) \in B(\mathbb{V}^*)$  with  $\iota(e', \psi) \in \mathcal{W}_1$ . Therefore,

$$\begin{aligned} |f(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) - \alpha f(\mathbf{v}_1) - \beta f(\mathbf{v}_2)| &\leq |f(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) - \iota(e', \psi)(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2)| \\ &\quad + |\iota(e', \psi)(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) - \alpha f(\mathbf{v}_1) - \beta f(\mathbf{v}_2)| \\ &< \epsilon + |\alpha| |\iota(e', \psi)(\mathbf{v}_1) - f(\mathbf{v}_1)| + |\beta| |\iota(e', \psi)(\mathbf{v}_2) - f(\mathbf{v}_2)| \\ &< (1 + |\alpha| + |\beta|)\epsilon. \end{aligned}$$

Since this is true for any  $\epsilon > 0$  we get that  $f(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = \alpha f(\mathbf{v}_1) + \beta f(\mathbf{v}_2)$ . This means that  $\varphi$  is linear. It remains to show that  $\varphi$  is continuous and its norm is  $\leq 1$ . Towards this end, as usual, take  $\mathbf{v} \in \Gamma$  and  $\epsilon > 0$ . Let  $U$  be a neighborhood of  $e$  such that  $\|\mathbf{v}(e')\| < \|\mathbf{v}(e)\| + \epsilon$  for every  $e' \in U$ . Define,

$$\mathcal{W}_2 = U \times \{f' \in \prod_{\mathbf{v} \in \Gamma} D_{\mathbf{v}} ; |f(\mathbf{v}) - f'(\mathbf{v})| < \epsilon\}.$$

Let  $(e', \psi) \in B(\mathbb{V}^*)$  satisfy  $\iota(e', \psi) \in \mathcal{W}_2$ . We then have that

$$|f(\mathbf{v})| \leq |\iota(e', \psi)(\mathbf{v})| + \epsilon \leq \|\mathbf{v}(e')\| + \epsilon < \|\mathbf{v}(e)\| + 2\epsilon.$$

This implies that  $|f(\mathbf{v})| \leq \|\mathbf{v}(e)\|$ . Thus,  $\varphi$  is continuous and  $\|\varphi\| \leq 1$ .  $\square$

Our aim is now to show that the projection map  $\text{pr}^* : B(\mathbb{V}^*) \rightarrow \mathbf{B}$  is open. The next proposition is a generalization of the remark in [DD63, Section I.1].

**Proposition 3.1.10.** *Let  $(\mathbb{V}, \Gamma)$  be a lower semi-continuous field of Banach spaces over  $\mathbf{B}$ . Let  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \Gamma$  and let  $e \in \mathbf{B}$  be such that  $\mathbf{v}_1(e), \dots, \mathbf{v}_n(e)$  is a set of linearly independent vectors. Then, there is a neighborhood  $U \subseteq \mathbf{B}$  of  $e$  such that  $\mathbf{v}_1(e'), \dots, \mathbf{v}_n(e')$  are linearly independent for every  $e' \in U$ .*

*Proof.* Take  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \Gamma$  and  $e \in \mathbf{B}$  as in the statement. Define  $\alpha : \mathbf{B} \times \mathbb{R}^n \rightarrow \mathbb{R}$  by  $\alpha(e', \lambda_1, \dots, \lambda_n) = \|\sum_{i \leq n} \lambda_i \mathbf{v}_i(e')\|$ . Let  $\beta : \mathbf{B} \rightarrow \mathbb{R}$  be the map given by

$$\beta(e') = \inf\{\alpha(e', \lambda_1, \dots, \lambda_n) ; \sum_{i \leq n} |\lambda_i| = 1\}.$$

We will show that  $\beta$  is lower semi-continuous. Fix  $e_0 \in \mathbf{B}$  and suppose, towards a contradiction, that  $\beta$  is not lower semi-continuous at  $e_0$ . Then, there is  $\epsilon > 0$  such that for every  $W \subseteq \mathbf{B}$  open neighborhood of  $e_0$  there is  $e_W \in W$  with  $\beta(e_W) < \beta(e_0) - \epsilon$ . This means that for every  $W \subseteq \mathbf{B}$  open set containing  $e_0$  we can find  $(\lambda_1^W, \dots, \lambda_n^W) \in \mathbb{R}^n$  with  $\sum_{i \leq n} |\lambda_i^W| = 1$  and  $\alpha(e_W, \lambda_1^W, \dots, \lambda_n^W) < \beta(e_0) - \epsilon$ . By compactness, the net  $\{(\lambda_1^W, \dots, \lambda_n^W) ; W \subseteq \mathbf{B} \text{ open with } e_0 \in W\}$  admits an accumulation point, say  $(\lambda_1, \dots, \lambda_n)$ . Since  $\alpha$  is a lower semi-continuous map, there are  $W \subseteq \mathbf{B}$  and  $U \subseteq \mathbb{R}^n$  open with  $(e_0, \lambda_1, \dots, \lambda_n) \in W \times U$  such that for every  $(e', \lambda'_1, \dots, \lambda'_n) \in W \times U$ ,  $\alpha(e', \lambda'_1, \dots, \lambda'_n) \geq \alpha(e_0, \lambda_1, \dots, \lambda_n) - \epsilon$ . By changing  $W$  to a smaller neighborhood we can assume that  $(\lambda_1^W, \dots, \lambda_n^W) \in U$ . It follows that  $\alpha(e_W, \lambda_1^W, \dots, \lambda_n^W) \geq \alpha(e_0, \lambda_1, \dots, \lambda_n) - \epsilon \geq \beta(e_0) - \epsilon$ , which is absurd. Since  $\mathbf{v}_1(e), \dots, \mathbf{v}_n(e)$  are linearly independent vectors, then  $\beta(e) > 0$ . Therefore, there is  $U \subseteq \mathbf{B}$  open neighborhood of  $e$  such that  $\beta(e') > 0$  for every  $e' \in U$ . Hence,  $\mathbf{v}_1(e'), \dots, \mathbf{v}_n(e')$  are linearly independent for every  $e' \in U$ .  $\square$

Suppose now that  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \Gamma$  and that for some  $e \in \mathbf{B}$ ,  $\mathbf{v}_1(e), \dots, \mathbf{v}_n(e)$  are linearly independent. We define a norm  $\rho_e$  on  $\text{Span}(\mathbf{v}_1(e), \dots, \mathbf{v}_n(e))$  by

$$\rho_e \left( \sum_{i \leq n} \lambda_i \mathbf{v}_i(e) \right) = \sum_{i \leq n} |\lambda_i|.$$

Given  $e \in \mathbf{B}$  we will write  $\|\cdot\|_e$  for the norm of  $V_e$ .

**Lemma 3.1.11.** *Let  $(\mathbb{V}, \Gamma)$  be a lower semi-continuous field of Banach spaces over  $\mathbf{B}$ . Let  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \Gamma$  and  $e \in \mathbf{B}$  be such that  $\mathbf{v}_1(e), \dots, \mathbf{v}_n(e)$  is a set of linearly independent vectors. Suppose furthermore that on  $\text{Span}(\mathbf{v}_1(e), \dots, \mathbf{v}_n(e))$  we have  $\rho_e(\cdot) \leq C\|\cdot\|_e$ . Then, for every  $D \in \mathbb{R}^+$  with  $C < D$  there is a neighborhood  $U \subseteq \mathbf{B}$  of  $e$  such that on  $\text{Span}(\mathbf{v}_1(e'), \dots, \mathbf{v}_n(e'))$ ,  $\rho_{e'}(\cdot) \leq D\|\cdot\|_{e'}$  for every  $e' \in U$ .*

*Proof.* Take  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \Gamma$ ,  $e \in \mathbf{B}$ ,  $C, D \in \mathbb{R}^+$  as in the statement of the lemma. Define  $\alpha : \mathbf{B} \times \mathbb{R}^n \rightarrow \mathbb{R}$  by  $\alpha(e', \lambda_1, \dots, \lambda_n) = D\|\sum_{i \leq n} \lambda_i \mathbf{v}_i(e')\| - \sum_{i \leq n} |\lambda_i|$  and  $\beta : \mathbf{B} \rightarrow \mathbb{R}$  by  $\beta(e') = \inf\{\alpha(e', \lambda_1, \dots, \lambda_n) ; \sum_{i \leq n} |\lambda_i| = 1\}$ . Since  $\alpha$  is lower semi-continuous, as in the proof of the preceding lemma,  $\beta$  is also lower semi-continuous. Hence, there is  $U \subseteq \mathbf{B}$  open set containing  $e$  such that  $\mathbf{v}_1(e'), \dots, \mathbf{v}_n(e')$  are linearly independent and  $f(e') > 0$  for every  $e' \in U_0$ . It follows that  $\rho_{e'}(\cdot) \leq D\|\cdot\|_{e'}$  on  $\text{Span}(\mathbf{v}_1(e'), \dots, \mathbf{v}_n(e'))$  for every  $e' \in U$ .  $\square$

**Lemma 3.1.12.** *Let  $(\mathbb{V}, \Gamma)$  be a lower semi-continuous field of Banach spaces over  $\mathbf{B}$ . Fix  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \Gamma$  and  $r_1, \dots, r_n \in \mathbb{R}$ . Given  $e \in \mathbf{B}$  let  $A(e) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n ; \|\sum_{i \leq n} \lambda_i \mathbf{v}_i(e)\| \leq 1\}$  and define the application  $f : \mathbf{B} \rightarrow \mathbb{R} \cup \{+\infty\}$  by*

$$f(e) = \sup\{|\lambda_1 r_1 + \dots + \lambda_n r_n| ; (\lambda_1, \dots, \lambda_n) \in A(e)\}.$$

*For any  $e \in \mathbf{B}$ , if  $\mathbf{v}_1(e), \dots, \mathbf{v}_n(e)$  are linearly independent then  $f(e) < +\infty$  and  $f$  is upper semi-continuous at  $e$ .*

*Proof.* Fix  $e \in \mathbf{B}$  with  $\mathbf{v}_1(e), \dots, \mathbf{v}_n(e)$  linearly independent. Take  $C > 0$  such that  $\rho_e(\cdot) \leq C\|\cdot\|_e$ . It follows that if  $(\lambda_1, \dots, \lambda_n) \in A(e)$  then  $\sum_{i \leq n} |\lambda_i| = \rho_e(\sum_{i \leq n} \lambda_i \mathbf{v}_i(e)) \leq C\|\sum_{i \leq n} \lambda_i \mathbf{v}_i(e)\| \leq C$ . Hence,  $A(e)$  is bounded, which implies that  $f(e)$  is finite. Furthermore, by lower semi-continuity of the map  $(e', \mu_1, \dots, \mu_n) \mapsto \|\sum_{i \leq n} \mu_i \mathbf{v}_i(e')\|$ ,  $A(e)$  is closed and hence compact. Using **Proposition 3.1.10** and **Lemma 3.1.11** for some  $D > C$ , we can find an open neighborhood  $U$  of  $e$  such that for every  $e' \in U$ ,  $\mathbf{v}_1(e'), \dots, \mathbf{v}_n(e')$  are linearly independent and  $\rho_{e'}(\cdot) \leq D\|\cdot\|_{e'}$ . In particular, for every  $e' \in U$ ,  $A(e')$  is contained in the closed  $l_1$ -ball of radius  $D$ .

Suppose, on the contrary, that  $f$  is not upper semi-continuous at  $e$ . This means that there is  $\epsilon > 0$  such that for every  $W \subseteq U$  open with  $e \in W$  there are  $e_W \in W$  and  $(\lambda_{W,1}, \dots, \lambda_{W,n}) \in A(e_W)$  satisfying  $|\sum_{i \leq n} \lambda_{W,i} r_i| > f(e) + \epsilon$ . By compactness of the closed  $l_1$ -ball of radius  $D$ , the net  $\{(\lambda_{W,1}, \dots, \lambda_{W,n}) ; W \subseteq \mathbf{B} \text{ open with } e \in W\}$  admits an accumulation point  $(\lambda_1, \dots, \lambda_n)$ . Since the map  $(e', \mu_1, \dots, \mu_n) \mapsto \|\sum_{i \leq n} \mu_i \mathbf{v}_i(e')\|$  is lower semi-continuous, we have that  $(\lambda_1, \dots, \lambda_n) \in A(e)$ . Therefore,  $f(e) \geq |\sum_{i \leq n} \lambda_i r_i| \geq f(e) + \epsilon$ , a contradiction. Thus,  $f$  is upper semi-continuous at  $e$ .  $\square$

**Proposition 3.1.13.** *Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces over  $\mathbf{B}$ . Then,  $\text{pr}^* : B(\mathbb{V}^*) \rightarrow \mathbf{B}$  is open.*

*Proof.* Take  $e \in \text{pr}^*(\mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U) \cap B(\mathbb{V}^*))$ , so,  $e \in U$  and there is  $\varphi \in V_e^*$  with  $\|\varphi\| < 1$  and  $\langle \mathbf{v}_j, \varphi \rangle \in I_j$  for  $j < n$ . Throughout this proof, for  $e' \in \mathbf{B}$ , let us denote  $E(e') = \text{Span}(\mathbf{v}_0(e'), \dots, \mathbf{v}_{n-1}(e'))$ . Set  $k = \dim(E(e))$ , we can assume that  $\{\mathbf{v}_0(e), \dots, \mathbf{v}_{k-1}(e)\}$  is a base for  $E(e)$ . For  $k \leq j < n$  write  $\mathbf{v}_j(e) = \sum_{i < k} \lambda_{j,i} \mathbf{v}_i(e)$  and let  $\mathbf{v}'_j = \sum_{i < k} \lambda_{j,i} \mathbf{v}_i$ . In case  $j < k$  set  $\mathbf{v}'_j = \mathbf{v}_j$ . Now take  $r > 0$  such that  $(\langle \mathbf{v}_j, \varphi \rangle - r, \langle \mathbf{v}_j, \varphi \rangle + r) \subseteq I_j$  for every  $j < n$ . Let  $U' \subseteq U$  be open with  $e \in U'$  such that, by virtue of **Proposition 3.1.10**, for every  $e' \in U'$ ,  $\mathbf{v}_0(e'), \dots, \mathbf{v}_{k-1}(e')$  are linearly independent and  $\|\mathbf{v}'_j(e') - \mathbf{v}_j(e')\| < r/2$  for  $k \leq j < n$ . Finally, for  $j < n$ , define  $I'_j = (\langle \mathbf{v}_j, \varphi \rangle - r/2, \langle \mathbf{v}_j, \varphi \rangle + r/2)$ . Pick  $(e', \psi) \in \mathcal{O}^*((\mathbf{v}'_j)_{j < n}, (I'_j)_{j < n}, U') \cap B(\mathbb{V}^*)$  then for  $j < n$

$$\begin{aligned}
|\langle \mathbf{v}_j, \psi \rangle - \langle \mathbf{v}_j, \varphi \rangle| &\leq |\langle \mathbf{v}_j - \mathbf{v}'_j, \psi \rangle| + |\langle \mathbf{v}'_j, \psi \rangle - \langle \mathbf{v}_j, \varphi \rangle| \\
&< \|\mathbf{v}_j(e') - \mathbf{v}'_j(e')\| + r/2 \\
&< r.
\end{aligned}$$

It follows that  $(e, \varphi) \in \mathcal{O}^*((\mathbf{v}'_j)_{j < n}, (I'_j)_{j < n}, U') \cap B(\mathbb{V}^*) \subseteq \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U) \cap B(\mathbb{V}^*)$ . Thus, replacing everything with its primed counterpart, we can assume that  $\mathbf{v}_0, \dots, \mathbf{v}_{k-1}$  are linearly independent on  $U$  and that they span  $\mathbf{v}_k, \dots, \mathbf{v}_{n-1}$ .

For  $j < k$  let  $r_j = \langle \mathbf{v}_j, \varphi \rangle$  and define  $f : \mathbf{B} \rightarrow \mathbb{R} \cup \{+\infty\}$  as in the previous lemma. Namely,

$$f(e') = \sup\{|\lambda_0 r_0 + \dots + \lambda_{k-1} r_{k-1}| ; \|\sum_{i < k} \lambda_i \mathbf{v}_i(e')\| \leq 1\}.$$

Let  $\varphi_0 = \varphi \upharpoonright_{E(e)}$ , remark that  $f(e) = \|\varphi_0\| < 1$ . Therefore, there is  $W \subseteq U$  open set containing  $e$  such that for every  $e' \in W$ ,  $f(e') \leq 1$ . Take  $e' \in W$  and define  $\theta_0 : E(e') \rightarrow \mathbb{R}$  by  $\theta_0(\sum_{i < n} \lambda_i \mathbf{v}_i(e')) = \sum_{i < n} \lambda_i r_i$ . We have that  $\|\theta_0\| = f(e') \leq 1$ . Using the Hahn-Banach theorem we get a linear form  $\theta : V_{e'} \rightarrow \mathbb{R}$  extending  $\theta_0$  with  $\|\theta\| \leq 1$ . Hence,  $(e', \theta) \in \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U) \cap B(\mathbb{V}^*)$  and so  $e' \in \text{pr}^*(\mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U) \cap B(\mathbb{V}^*))$  which completes the proof.  $\square$

We will say that a space over  $\mathbf{B}$ ,  $(X, \rho)$ , admits sections if through every point of  $X$  passes a section. We know by [Proposition 3.1.8](#) that  $(\mathbb{V}^*, \text{pr}^*)$  is a space over  $\mathbf{B}$ . Our next objective is to show that it admits sections. We will restrict our study to the case of a separable field, which we define below.

**Definition 3.1.14.** A field of Banach spaces over  $\mathbf{B}$ ,  $(\mathbb{V}, \Gamma)$ , is said to be *separable* if there is a countable  $\Lambda \subseteq \Gamma$  such that for every  $e \in \mathbf{B}$ ,  $\{\mathbf{v}(e) ; \mathbf{v} \in \Lambda\}$  is dense in  $V_e$ .

It follows immediately that in a separable field each fiber is separable. However, the converse is not true in general, even if the base is separable, see the remark after [\[DD63, Proposition 13\]](#). We note that in a separable field there is a countable  $\Lambda_0 \subseteq \prod_{e \in \mathbf{B}} B_e$  such that  $\{\mathbf{v}(e) ; \mathbf{v} \in \Lambda_0\}$  is dense in  $B_e$  for every  $e \in \mathbf{B}$ .

**Proposition 3.1.15.** *Suppose  $\mathbf{B}$  is a compact, metrizable space. A continuous field of Banach spaces over  $\mathbf{B}$ ,  $(\mathbb{V}, \Gamma)$ , is separable if and only if  $\Gamma$ , as a Banach space, is separable.*

*Proof.*  $\Leftarrow$ ) Take  $\Lambda \subseteq \Gamma$  be a countable dense subset. Fix  $e \in \mathbf{B}$ ,  $v \in V_e$  and  $\epsilon > 0$ , now pick  $\mathbf{v} \in \Gamma$  such that  $\mathbf{v}(e) = v$ . By density, there is  $\mathbf{w} \in \Lambda$  satisfying  $\|\mathbf{v} - \mathbf{w}\| < \epsilon$ . Then,  $\|v - \mathbf{w}(e)\| \leq \|\mathbf{v} - \mathbf{w}\| < \epsilon$ . Thus,  $\{\mathbf{w}(e) ; \mathbf{w} \in \Lambda\}$  is dense in  $V_e$ . Since this is true for every  $e \in \mathbf{B}$ , we conclude that  $(\mathbb{V}, \Gamma)$  is separable.

$\Rightarrow$ ) Let  $\Lambda$  be a countable subset of  $\Gamma$  such that  $\{\mathbf{v}(e) ; \mathbf{v} \in \Lambda\}$  is dense in  $V_e$  for every  $e \in \mathbf{B}$ . Let  $\mathcal{A}$  be a uniformly dense countable subset of  $C(\mathbf{B})$ . Define

$$\Lambda' = \{f_1 \mathbf{v}_1 + \dots + f_n \mathbf{v}_n ; n \in \mathbb{N}, f_1, \dots, f_n \in \mathcal{A} \text{ and } \mathbf{v}_1, \dots, \mathbf{v}_n \in \Lambda\},$$

we will show that  $\Lambda'$  is dense in  $\Gamma$ . Fix  $\mathbf{w} \in \Gamma$  and  $\epsilon > 0$ , then, by the definition of separability and continuity of the norm, for each  $e \in \mathbf{B}$  there is  $U_e$  open neighborhood of  $e$  and there is  $\mathbf{v}_e \in \Lambda$  such that  $\|\mathbf{v}_e(e') - \mathbf{w}(e')\| < \epsilon/2$  for every  $e' \in U_e$ . By compactness, the cover  $\{U_e ; e \in \mathbf{B}\}$  admits a finite subcover, say  $\{U_{e_1}, \dots, U_{e_n}\}$ . Let  $g_1, \dots, g_n \in C(\mathbf{B})$  be a partition of unity subordinated to  $\{U_{e_1}, \dots, U_{e_n}\}$  and assume that  $\text{supp}(g_i) \subseteq U_{e_i}$  for  $i \leq n$ . It follows that  $\|g_1 \mathbf{v}_{e_1} + \dots + g_n \mathbf{v}_{e_n} - \mathbf{w}\| \leq$

$\epsilon/2$ . Now, let  $M = \max\{\|\mathbf{v}_{e_i}\| ; i \leq n\}$  and for each  $i \leq n$  take  $f_i \in \mathcal{A}$  with  $\|f_i - g_i\| < \epsilon/(2nM)$ . Therefore,

$$\begin{aligned} \|f_1 \mathbf{v}_{e_1} + \cdots + f_n \mathbf{v}_{e_n} - \mathbf{w}\| &\leq \|(f_1 - g_1) \mathbf{v}_{e_1} + \cdots + (f_n - g_n) \mathbf{v}_{e_n} - \mathbf{w}\| + \|g_1 \mathbf{v}_{e_1} + \cdots + g_n \mathbf{v}_{e_n} - \mathbf{w}\| \\ &< \|f_1 - g_1\| \|\mathbf{v}_{e_1}\| + \cdots + \|f_n - g_n\| \|\mathbf{v}_{e_n}\| + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

This completes the proof.  $\square$

In order to prove that  $\mathbb{V}^*$  admits sections we will show that  $B(\mathbb{V}^*)$  is completely metrizable and use the following fact, due to Itai Ben Yaacov, whose proof we reproduce here.

**Fact 3.1.16.** *Let  $X$  be a completely metrizable topological space,  $\mathbf{B}$  a compact totally disconnected space and  $\rho : X \rightarrow \mathbf{B}$  a continuous, open surjection. Then  $(X, \rho)$  admits sections.*

*Proof.* Let  $d$  be a complete metric on  $X$  of diameter smaller than 1. Fix  $x \in X$ . We will build recursively a tree  $T$  such that for every  $\sigma \in T$  of level  $k$ , we choose  $n_\sigma \in \mathbb{N}$ ,  $x_\sigma \in X$  and  $U_\sigma$  clopen subset of  $\mathbf{B}$  satisfying:

- $\rho(x_\sigma) \in U_\sigma \subseteq \rho(B(x_\sigma, 2^{-k}))$ ,
- the successors of  $\sigma$  are  $\sigma \frown i$  for  $i < n_\sigma$ ,
- $\{U_{\sigma \frown i} ; i < n_\sigma\}$  is a partition of  $U_\sigma$ .

For the level 0 we start with  $\emptyset \in T$ ,  $U_\emptyset = \mathbf{B}$  and  $x_\emptyset = x$ . Suppose  $\sigma \in T$  of level  $k$  has already been defined. Let  $X_\sigma = \rho^{-1}(U_\sigma) \cap B(x_\sigma, 2^{-k})$ , so  $\rho(X_\sigma) = U_\sigma$ . Since  $\rho$  is open, the collection  $\{\rho(B(z, 2^{-k-1})) ; z \in X_\sigma\}$  is an open cover of  $U_\sigma$ . By compactness, it admits a finite subcover  $\{\rho(B(x_{\sigma \frown i}, 2^{-k-1})) ; i < n_\sigma\}$  with  $x_{\sigma \frown 0} = x_\sigma$ . Take  $\{U_{\sigma \frown i} ; i < n_\sigma\}$ , a partition of  $U_\sigma$  in clopen sets, such that  $\rho(x_{\sigma \frown i}) \in U_{\sigma \frown i} \subseteq \rho(B(x_{\sigma \frown i}, 2^{-k-1}))$ .

Let us define  $\mathbf{x} = \overline{\{x_\sigma ; \sigma \in T\}}$ . Note that for every  $\sigma \in T$  of level  $k$ ,  $d(x_\sigma, x_{\sigma \frown i}) < 2^{-k}$  for every  $i < n_\sigma$ . This implies that  $\mathbf{x}$  is totally bounded, hence compact. Moreover, for every  $\eta \in \partial T$ , the set of branches of  $T$ , the sequence  $(x_{\eta \upharpoonright n})_{n \in \mathbb{N}}$  is Cauchy. By completeness, it admits a limit, say  $x_\eta$ . Furthermore,  $\bigcap_{n \in \mathbb{N}} U_{\eta \upharpoonright n} = \{\rho(x_\eta)\}$ . We claim that  $\mathbf{x} = \{x_\eta ; \eta \in \partial T\}$ . The inclusion  $\{x_\eta ; \eta \in \partial T\} \subseteq \mathbf{x}$  is clear. For the converse, take  $x \in \mathbf{x}$  and choose  $\eta \in \partial T$  such that  $\rho(x) = \rho(x_\eta)$ . For each  $n \in \mathbb{N}$ , we have that  $\rho^{-1}(U_{\eta \upharpoonright n}) \cap B(x, 2^{-n}) \cap \{x_\sigma ; \sigma \in T\} \neq \emptyset$ . By construction, we can find  $m_n \in \mathbb{N}$  such that  $x_{\eta \upharpoonright m_n} \in \rho^{-1}(U_{\eta \upharpoonright n}) \cap B(x, 2^{-n})$ , by adding zeros at the end if necessary, we can assume that  $m_n \geq n$ . Therefore,  $x = \lim_n x_{\eta \upharpoonright m_n} = \lim_n x_{\eta \upharpoonright n} = x_\eta$ , which shows the other inclusion. We conclude that  $\rho \upharpoonright_{\mathbf{x}} : \mathbf{x} \rightarrow \mathbf{B}$  is a bijection, so  $\mathbf{x}$  defines a section.  $\square$

**Theorem 3.1.17.** *Let  $(\mathbb{V}, \Gamma)$  be a separable continuous field of Banach spaces over  $\mathbf{B}$ . Suppose moreover that  $\mathbf{B}$  is completely metrizable. Then, the weak\* topology on  $B(\mathbb{V}^*)$  is completely metrizable.*

*Proof.* Let  $\Lambda_0 = \{\mathbf{v}_n ; n \in \mathbb{N}\}$  be a countable subset of  $\Gamma$  such that  $\{\mathbf{v}_n(e) ; n \in \mathbb{N}\}$  is a dense subset of  $B_e$  for every  $e \in \mathbf{B}$ . Let  $d_{\mathbf{B}}$  be a complete metric for  $\mathbf{B}$  of diameter  $\leq 1$ . For  $(e, \varphi), (e', \psi) \in B(\mathbb{V}^*)$  define

$$d((e, \varphi), (e', \psi)) = \left( \sum_{n \in \mathbb{N}} 2^{-n-1} |\langle \mathbf{v}_n, \varphi \rangle - \langle \mathbf{v}_n, \psi \rangle| \right) + d_{\mathbf{B}}(e, e').$$

We will prove that  $d$  is a complete metric that induces the weak\* topology on  $B(\mathbb{V}^*)$ . Clearly, if  $(e, \varphi) = (e', \psi)$  then  $d((e, \varphi), (e', \psi)) = 0$ . Conversely, if  $d((e, \varphi), (e', \psi)) = 0$  then  $e = e'$  and  $\varphi$  and  $\psi$  agree on a dense subset of  $B_e$  which implies  $\varphi = \psi$ . By definition,  $d$  is symmetric and it can be readily shown that the triangle inequality also holds. Thus,  $d$  defines a metric on  $B(\mathbb{V}^*)$ . We will show that the weak\* topology is finer than the metric topology. Let us fix  $(e, \varphi) \in B(\mathbb{V}^*)$  and  $\epsilon > 0$ . We set  $U = B_{d_{\mathbf{B}}}(e, \epsilon/2)$ . Let  $N \in \mathbb{N}$  be such that  $\sum_{n \geq N} 2^{-n-1} < \epsilon/8$ . For  $n < N$  set  $I_n = (\langle \mathbf{v}_n, \varphi \rangle - \epsilon/4, \langle \mathbf{v}_n, \varphi \rangle + \epsilon/4)$ . Clearly  $(e, \varphi) \in \mathcal{O}^*((\mathbf{v}_n)_{n < N}, (I_n)_{n < N}, U)$ , we claim that the latter is contained in  $B_d((e, \varphi), \epsilon)$ . Indeed, take  $(e', \psi) \in \mathcal{O}^*((\mathbf{v}_n)_{n < N}, (I_n)_{n < N}, U)$ , we then have that

$$\begin{aligned} d((e, \varphi), (e', \psi)) &= \left( \sum_{n < N} 2^{-n-1} |\langle \mathbf{v}_n, \varphi \rangle - \langle \mathbf{v}_n, \psi \rangle| \right) + \left( \sum_{n \geq N} 2^{-n-1} |\langle \mathbf{v}_n, \varphi \rangle - \langle \mathbf{v}_n, \psi \rangle| \right) + d_{\mathbf{B}}(e, e') \\ &\leq \frac{\epsilon}{4} \sum_{n < N} 2^{-n-1} + 2 \sum_{n \geq N} 2^{-n-1} + d_{\mathbf{B}}(e, e') \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

i.e.  $(e', \psi) \in B_d((e, \varphi), \epsilon)$ . In order to show that the metric topology is finer than the weak\* topology, take  $\mathbf{w}_0, \dots, \mathbf{w}_{m-1} \in \Gamma$ ,  $I_0, \dots, I_{m-1}$  open subsets of  $\mathbb{R}$  and  $U \subseteq \mathbf{B}$  open, and suppose that  $(e, \varphi) \in \mathcal{O}^*((\mathbf{w}_j)_{j < m}, (I_j)_{j < m}, U)$ . Pick  $r > 0$  satisfying  $(\langle \mathbf{w}_j, \varphi \rangle - r, \langle \mathbf{w}_j, \varphi \rangle + r) \subseteq I_j$  for  $j < m$ . Since  $\{\mathbf{v}_n(e) ; n \in \mathbb{N}\}$  is a dense subset of  $B_e$ , there are  $n_0, \dots, n_{m-1} \in \mathbb{N}$  with  $\|\mathbf{w}_j(e) - \mathbf{v}_{n_j}(e)\| < r/3$  for  $j < m$ . Take  $\epsilon_0 > 0$  such that  $B_{d_{\mathbf{B}}}(e, \epsilon_0) \subseteq U$  and  $\|\mathbf{w}_j(e') - \mathbf{v}_{n_j}(e')\| < r/3$  for every  $e' \in B_{d_{\mathbf{B}}}(e, \epsilon_0)$ . Set  $N = \max\{n_0, \dots, n_{m-1}\}$  and let  $\epsilon = \min\{2^{-N-1}r/3, \epsilon_0\}$ , we will see that  $B_d((e, \varphi), \epsilon) \subseteq \mathcal{O}^*((\mathbf{w}_j)_{j < m}, (I_j)_{j < m}, U)$ . Take  $(e', \psi) \in B_d((e, \varphi), \epsilon)$ , then  $e' \in B_{d_{\mathbf{B}}}(e, \epsilon_0)$  and

$$\sum_{n \in \mathbb{N}} 2^{-n-1} |\langle \mathbf{v}_n, \varphi \rangle - \langle \mathbf{v}_n, \psi \rangle| < 2^{-N-1}r/3.$$

Therefore, for every  $n \leq N$ ,  $|\langle \mathbf{v}_n, \varphi \rangle - \langle \mathbf{v}_n, \psi \rangle| < r/3$ . Hence, for every  $j < m$ ,

$$\begin{aligned} |\langle \mathbf{w}_j, \varphi \rangle - \langle \mathbf{w}_j, \psi \rangle| &\leq |\langle \mathbf{w}_j, \varphi \rangle - \langle \mathbf{v}_{n_j}, \varphi \rangle| + |\langle \mathbf{v}_{n_j}, \varphi \rangle - \langle \mathbf{v}_{n_j}, \psi \rangle| + |\langle \mathbf{v}_{n_j}, \psi \rangle - \langle \mathbf{w}_j, \psi \rangle| \\ &< \|\mathbf{w}_j(e) - \mathbf{v}_{n_j}(e)\| + \frac{r}{3} + \|\mathbf{w}_j(e') - \mathbf{v}_{n_j}(e')\| \\ &< r. \end{aligned}$$

We thus conclude that the two topologies agree. Now take  $(e_n, \varphi_n)_{n \in \mathbb{N}}$  a Cauchy sequence in  $B(\mathbb{V}^*)$ . Since  $d_{\mathbf{B}}(e_n, e_m) \leq d(\varphi_n, \varphi_m)$  for every  $m, n \in \mathbb{N}$ , then  $(e_n)_{n \in \mathbb{N}}$  is a  $d_{\mathbf{B}}$ -Cauchy sequence in  $\mathbf{B}$ . Thus, it converges to say  $e \in \mathbf{B}$ . Fix  $j \in \mathbb{N}$ , note that  $|\langle \mathbf{v}_j, \varphi_n \rangle - \langle \mathbf{v}_j, \varphi_m \rangle| \leq 2^{j+1}d(\varphi_n, \varphi_m)$ . It follows that  $(\langle \mathbf{v}_j, \varphi_n \rangle)_{n \in \mathbb{N}}$  is a convergent sequence of reals, let  $r_j \in \mathbb{R}$  be its limit. Let  $W_e = \text{Span}\{\mathbf{v}_n(e) ; n \in \mathbb{N}\}$ , then  $W_e$  is a dense subspace of  $V_e$ . Define  $\psi : W_e \rightarrow \mathbb{R}$  by  $\langle \mathbf{v}_j(e), \psi \rangle = r_j$  for  $j \in \mathbb{N}$ , and extend by linearity. We will show that  $\psi$  is a well-defined continuous linear form on  $W_e$ . Suppose  $\sum_{j < k} \lambda_j \mathbf{v}_j(e) = 0$  for some  $\lambda_0, \dots, \lambda_{k-1} \in \mathbb{R}$ . Then,  $\lim_{n \rightarrow \infty} \|\sum_{j < k} \lambda_j \mathbf{v}_j(e_n)\| = 0$ , and since  $|\langle \sum_{j < k} \lambda_j \mathbf{v}_j(e_n), \varphi_n \rangle| \leq \|\sum_{j < k} \lambda_j \mathbf{v}_j(e_n)\|$ , we get that  $\lim_{n \rightarrow \infty} \langle \sum_{j < k} \lambda_j \mathbf{v}_j(e_n), \varphi_n \rangle = 0$ . This implies that  $\sum_{j < k} \lambda_j r_j = 0$ , showing that  $\psi$  is well-defined. By definition,  $\psi$  is linear. To prove that it is continuous, let  $\lambda_0, \dots, \lambda_{k-1} \in \mathbb{R}$ , then

$$\left| \left\langle \sum_{j < k} \lambda_j \mathbf{v}_j(e), \psi \right\rangle \right| = \lim_{n \rightarrow \infty} \left| \left\langle \sum_{j < k} \lambda_j \mathbf{v}_j(e_n), \varphi_n \right\rangle \right| \leq \lim_{n \rightarrow \infty} \left\| \sum_{j < k} \lambda_j \mathbf{v}_j(e_n) \right\| = \left\| \sum_{j < k} \lambda_j \mathbf{v}_j(e) \right\|.$$

Therefore,  $\psi$  is continuous and  $\|\psi\| \leq 1$ . By density,  $\psi$  extends uniquely to a continuous linear form  $\varphi$  on  $V_e$  with  $\|\varphi\| \leq 1$ . To complete the proof we will show that  $(e_n, \varphi_n)_{n \in \mathbb{N}}$  converges to

$(e, \varphi)$ . Fix  $\epsilon > 0$  and pick  $N_0 \in \mathbb{N}$  such that  $\sum_{n \geq N_0} 2^{-n-1} < \epsilon/8$  and  $d_{\mathbf{B}}(e_n, e) < \epsilon/2$  for every  $n \geq N_0$ . Now take  $N_1 > N_0$  such that for every  $n < N_0$  and every  $k \geq N_1$ ,  $|\langle \mathbf{v}_n, \varphi_k \rangle - \langle \mathbf{v}_n, \varphi \rangle| < \epsilon/4$ . In this way, for  $k \geq N_1$ ,

$$\begin{aligned} d((e_k, \varphi_k), (e, \varphi)) &= \left( \sum_{n < N_0} 2^{-n-1} |\langle \mathbf{v}_n, \varphi_k \rangle - \langle \mathbf{v}_n, \varphi \rangle| \right) + \left( \sum_{n \geq N_0} 2^{-n-1} |\langle \mathbf{v}_n, \varphi_k \rangle - \langle \mathbf{v}_n, \varphi \rangle| \right) + d_{\mathbf{B}}(e_k, e) \\ &< \frac{\epsilon}{4} \sum_{n < N_0} 2^{-n-1} + 2 \sum_{n \geq N_0} 2^{-n-1} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

This completes the proof.  $\square$

**Corollary 3.1.18.** *Let  $(\mathbb{V}, \Gamma)$  be a separable continuous field of Banach spaces over  $\mathbf{B}$ . Then,  $(\mathbb{V}^*, \text{pr}^*)$  admits sections.*

*Proof.* By the previous theorem and **Fact 3.1.16** we conclude that  $(B(\mathbb{V}^*), \text{pr}^*)$  admits sections. Let us call  $\Gamma_0^*$  this family of sections. Then,  $\Gamma^* = \{\lambda\nu ; \lambda \in \mathbb{R}, \nu \in \Gamma_0^*\}$  is a family of sections for  $(\mathbb{V}^*, \text{pr}^*)$ .  $\square$

From now on  $\Gamma_0^*$  and  $\Gamma^*$  will always refer to the sections on  $B(\mathbb{V}^*)$  and  $\mathbb{V}^*$  respectively. From the previous corollary we deduce that  $\nu \in \prod_{e \in \mathbf{B}} V_e^*$  belongs to  $\Gamma^*$  if and only if  $e \mapsto \|\nu(e)\|$  is bounded, say by  $M > 0$ , and  $\frac{1}{M}\nu \in \Gamma_0^*$ . The next proposition characterizes the elements of  $\Gamma_0^*$ .

**Proposition 3.1.19.** *Let  $(\mathbb{V}, \Gamma)$  be a separable continuous field of Banach spaces over  $\mathbf{B}$ . For  $\nu \in \prod_{e \in \mathbf{B}} B(V_e^*)$  the following assertions are equivalent:*

1.  $\nu \in \Gamma_0^*$ .
2. For every  $e \in \mathbf{B}$  and every  $\epsilon > 0$  there exist  $\nu' \in \Gamma_0^*$  and  $U \subseteq B$  open neighborhood of  $e$  such that  $\|\nu(e') - \nu'(e')\| < \epsilon$  for every  $e' \in U$ .
3. For every  $e \in B$  and every  $\epsilon > 0$  there exist  $\nu' \in \Gamma_0^*$  and  $U \subseteq B$  open neighborhood of  $e$  such that  $d(\nu(e'), \nu'(e')) < \epsilon$  for every  $e' \in U$ .
4. For every  $\nu' \in \Gamma_0^*$ , the function  $e \mapsto d(\nu(e), \nu'(e))$  is continuous.
5. For every  $\mathbf{v} \in \Gamma$ , the mapping  $e \mapsto \langle \mathbf{v}(e), \nu(e) \rangle$  is continuous.
6. The map  $\mathbb{V} \rightarrow \mathbb{R}$  given by  $(e, v) \mapsto \langle v, \nu(e) \rangle$  is continuous.

*Proof.* 1.  $\Rightarrow$  2. Immediate.

2.  $\Rightarrow$  3. Follows from the fact that  $d(\nu(e'), \nu'(e')) = \sum_{n \in \mathbb{N}} 2^{-n-1} |\langle \mathbf{v}_n(e), \nu(e') - \nu'(e') \rangle| \leq \|\nu(e') - \nu'(e')\|$ .

3.  $\Rightarrow$  4. Fix  $\nu' \in \Gamma_0^*$ , let  $e \in \mathbf{B}$  and let  $\epsilon > 0$ . Then, there exists  $\nu_0 \in \Gamma_0^*$  and  $U \subseteq B$  open set containing  $e$  satisfying  $d(\nu(e'), \nu_0(e')) < \epsilon/3$  for every  $e' \in U$ . Furthermore, since  $\nu', \nu_0 \in \Gamma_0^*$ , there is  $U_0 \subseteq U$  open with  $e \in U_0$  such that  $|d(\nu_0(e'), \nu'(e')) - d(\nu_0(e), \nu'(e))| < \epsilon/3$ . Therefore, for every  $e' \in U_0$

$$\begin{aligned} |d(\nu(e'), \nu'(e')) - d(\nu(e), \nu'(e))| &< \frac{\epsilon}{3} + |d(\nu_0(e'), \nu'(e')) - d(\nu_0(e), \nu'(e))| \\ &< \frac{2\epsilon}{3} + |d(\nu_0(e'), \nu'(e')) - d(\nu_0(e), \nu'(e))| \\ &< \epsilon. \end{aligned}$$

4.  $\Rightarrow$  5. Fix  $\mathbf{v} \in \Gamma$ , let  $e \in \mathbf{B}$  and let  $\epsilon > 0$ . Take  $\nu' \in \Gamma_0^*$  passing through  $\nu(e)$  and pick  $N \in \mathbb{N}$  with  $\|\mathbf{v}(e) - \mathbf{v}_N(e)\| < \epsilon/5$ . Let  $U \subseteq \mathbf{B}$  be open such that  $e \in U$  and,  $\|\mathbf{v}(e') - \mathbf{v}_N(e')\| < \epsilon/5$ ,  $d(\nu'(e'), \nu'(e)) < 2^{-N-1}(\epsilon/5)$  and  $d(\nu'(e'), \nu(e')) < 2^{-N-1}(\epsilon/5)$  for every  $e' \in U$ . It follows that for every  $e' \in U$ ,

$$\begin{aligned} |\langle \mathbf{v}(e'), \nu(e') \rangle - \langle \mathbf{v}(e), \nu(e) \rangle| &\leq |\langle \mathbf{v}(e') - \mathbf{v}_N(e'), \nu(e') \rangle| + |\langle \mathbf{v}_N(e'), \nu(e') - \nu'(e') \rangle| \\ &\quad + |\langle \mathbf{v}_N(e'), \nu'(e') \rangle - \langle \mathbf{v}_N(e), \nu'(e) \rangle| + |\langle \mathbf{v}_N(e), \nu'(e) - \nu(e) \rangle| \\ &\quad + |\langle \mathbf{v}_N(e) - \mathbf{v}(e), \nu(e) \rangle| \\ &\leq \|\mathbf{v}(e') - \mathbf{v}_N(e')\| + 2^{N+1}d(\nu(e'), \nu'(e')) + 2^{N+1}d(\nu'(e'), \nu'(e)) \\ &\quad + 2^{N+1}d(\nu'(e), \nu(e)) + \|\mathbf{v}(e) - \mathbf{v}_N(e)\| \\ &< \epsilon. \end{aligned}$$

5.  $\Rightarrow$  6. Take  $(e, \nu) \in \mathbb{V}$ . Fix  $\epsilon > 0$  and pick  $\mathbf{v} \in \Gamma$  passing through  $(e, \nu)$ . Let  $U$  be an open neighborhood of  $e$  such that  $|\langle \mathbf{v}(e'), \nu(e') \rangle - \langle \mathbf{v}(e), \nu(e) \rangle| < \epsilon/2$  for every  $e' \in U$ . Let  $(e', \nu') \in \mathcal{O}(\mathbf{v}, \epsilon/2, U)$  then

$$\begin{aligned} |\langle \nu', \nu(e') \rangle - \langle \nu, \nu(e) \rangle| &\leq |\langle \nu' - \nu(e'), \nu(e') \rangle| + |\langle \mathbf{v}(e'), \nu(e') \rangle - \langle \mathbf{v}(e), \nu(e) \rangle| \\ &< \|\nu' - \nu(e')\| + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

6.  $\Rightarrow$  1. Fix  $e \in \mathbf{B}$  and suppose that  $\nu(e) \in \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U)$  for some  $\mathbf{v}_0, \dots, \mathbf{v}_{n-1} \in \Gamma$ ,  $I_0, \dots, I_{n-1} \subseteq \mathbb{R}$  open and  $U \subseteq \mathbf{B}$  open. Then, for each  $j < n$ , there is  $\epsilon_j > 0$  and  $U_j$  open neighborhood of  $e$  such that for every  $(e', \nu') \in \mathcal{O}(\mathbf{v}_j, \epsilon_j, U_j)$  we have  $\langle \nu', \nu(e') \rangle \in I_j$ . Define  $U' = U \cap U_0 \cap \dots \cap U_{n-1}$ , it follows that for every  $e' \in U'$ ,  $\langle \mathbf{v}_j(e'), \nu(e') \rangle \in I_j$  for every  $j < n$ . Hence,  $\nu(e') \in \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U)$  for every  $e' \in U'$ , which completes the proof.  $\square$

The next result follows immediately from the previous proposition and the definition of  $\Gamma^*$ .

**Corollary 3.1.20.** *Let  $(\mathbb{V}, \Gamma)$  be a separable, continuous field of Banach spaces over  $\mathbf{B}$  and let  $\nu \in \prod_{e \in \mathbf{B}} V_e^*$ . Then,  $\nu \in \Gamma^*$  if and only if  $e \mapsto \|\nu(e)\|$  is bounded and for every  $\mathbf{v} \in \Gamma$  the map  $e \mapsto \langle \mathbf{v}(e), \nu(e) \rangle$  is continuous.*

**Corollary 3.1.21.** *Let  $(\mathbb{V}, \Gamma)$  be a separable, continuous field of Banach spaces over  $\mathbf{B}$ . Then, for every  $\nu \in \Gamma_0^*$  the function  $e \mapsto \|\nu(e)\|$  is lower semi-continuous.*

*Proof.* Let  $\Lambda_0 = \{\mathbf{v}_n ; n \in \mathbb{N}\}$  be a countable subset of  $\Gamma$  such that  $\{\mathbf{v}_n(e) ; n \in \mathbb{N}\}$  is a dense subset of  $B(V_e)$  for every  $e \in \mathbf{B}$ . We have that for every  $e \in \mathbf{B}$ ,

$$\|\nu(e)\| = \sup\{|\langle \mathbf{v}_n(e), \nu(e) \rangle| ; n \in \mathbb{N}\}.$$

Therefore,  $e \mapsto \|\nu(e)\|$  is the supremum of a family of continuous functions, and as such is lower semi-continuous.  $\square$

As we stated before,  $\Gamma^* = \{\lambda\nu ; \lambda \in \mathbb{R}, \nu \in \Gamma_0^*\}$  is a family of sections for  $(\mathbb{V}^*, \text{pr}^*)$ . Since  $\Gamma_0^*$  is a  $C(\mathbf{B})$ -module then so it is  $\Gamma^*$ . By 2. of the previous proposition,  $\Gamma^*$  is locally uniformly closed and by the previous corollary, the norm of a section is a lower semi-continuous application. We thus have proved the following result.

**Corollary 3.1.22.** *Suppose  $(\mathbb{V}, \Gamma)$  is a separable continuous field of Banach spaces over  $\mathbf{B}$ . Then,  $(\mathbb{V}^*, \Gamma^*)$  is a lower semi-continuous field of Banach spaces over  $\mathbf{B}$ .*

*Remark.* If  $\mathbb{V}$  is trivial, i.e. equal to  $V \times \mathbf{B}$ , then  $\Gamma_0^*$  can be identified with the set of weakly\* continuous maps from  $\mathbf{B}$  into  $B(V^*)$ . Indeed, take  $\nu \in \Gamma_0^*$ , if  $\nu(e) \in \{\varphi \in B^* ; \text{ for every } j \leq n, \langle v_j, \varphi \rangle \in I_j\}$ , take  $U$  open neighborhood of  $e$  such that for every  $e' \in U$ ,  $(e', \nu(e')) \in \mathcal{O}^*((v_j)_{j < n}, (I_j)_{j < n}, \mathbf{B})$ . Now suppose  $\nu : \mathbf{B} \rightarrow B(V^*)$  is weakly\* continuous and assume  $(e, \nu(e)) \in \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U)$ . Take  $r > 0$  such that  $B(\langle \mathbf{v}_j(e), \nu(e) \rangle, r) \subseteq I_j$  for every  $j < n$ . Take  $U_0 \subseteq U$  open set containing  $e$  such that  $\|\mathbf{v}_j(e) - \mathbf{v}_j(e')\| < r/2$  and  $|\langle \mathbf{v}_j(e), \nu(e') \rangle - \langle \mathbf{v}_j(e), \nu(e) \rangle| < r/2$  for every  $e' \in U_0$  and every  $j < n$ . It follows that for every  $e' \in U_0$ ,  $(e', \nu(e')) \in \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U)$ .

In general, this is the best result we can hope for, compare for example with [Bos11, Lemma 3.11]. We ask however if there is a condition on the field that guarantees continuity of the dual field.

### 3.1.2 The weak topology

In this subsection we assume that for any  $(\mathbb{V}, \Gamma)$ , continuous field of Banach spaces over  $\mathbf{B}$ ,  $\mathbb{V}^*$ , endowed with the weak\* topology, admits sections, say  $\Gamma^*$ . By 6. of Proposition 3.1.19, each  $\nu \in \Gamma^*$  defines a continuous mapping  $\mathbb{V} \rightarrow \mathbb{R}$ . The initial topology on  $\mathbb{V}$  given by this family of maps together with the projection on  $\mathbf{B}$  will be known as the *weak topology*. We will write  $\mathbb{V}_w$  whenever we consider the weak topology on  $\mathbb{V}$ . It follows from the definition that the sets

$$\mathcal{O}_w((\nu_j)_{j < n}, (I_j)_{j < n}, U) = \{(e, v) \in \mathbb{V} ; e \in U \text{ and } \langle v, \nu \rangle \in I_j \text{ for every } j < n\},$$

where  $\nu_0, \dots, \nu_{n-1} \in \Gamma^*$ ,  $I_0, \dots, I_{n-1} \subseteq \mathbb{R}$  and  $U \subseteq \mathbf{B}$  are open, form a base for this topology. Moreover, on each fiber this topology coincides with the weak topology on the Banach space. We also remark that each section in  $\Gamma$  is a weakly continuous map. Indeed, let  $\mathbf{v} \in \Gamma$ ,  $e_0 \in \mathbf{B}$  and take  $\mathcal{O}_w((\nu_j)_{j < n}, (I_j)_{j < n}, U)$  a basic weakly open set containing  $\mathbf{v}(e_0)$ . For each  $j < n$ , since the map  $e \mapsto \langle \mathbf{v}(e), \nu_j(e) \rangle$  is continuous, there is an open neighborhood  $U_j$  of  $e_0$  such that  $\langle \mathbf{v}(e), \nu_j(e) \rangle \in I_j$  for every  $e \in U_j$ . Thus, for every  $e \in U \cap U_0 \cap \dots \cap U_{n-1}$ , we have  $\mathbf{v}(e) \in \mathcal{O}_w((\nu_j)_{j < n}, (I_j)_{j < n}, U)$ .

**Definition 3.1.23.** Let  $(\mathbb{V}, \Gamma)$  and  $(\mathbb{W}, \Lambda)$  be fields of Banach spaces over  $\mathbf{B}$ . A map  $T : \mathbb{V} \rightarrow \mathbb{W}$  is said to be *linear* if for every  $e \in \mathbf{B}$ ,  $T_e = T \upharpoonright_{V_e}$  is a linear application from  $V_e$  into  $W_e$ . If both fields are continuous, it is said to be a *homomorphism* if  $T$  is moreover continuous.

We remark that any map  $T : \mathbb{V} \rightarrow \mathbb{W}$  over  $\mathbf{B}$  induces an application  $T : \prod_{e \in \mathbf{B}} V_e \rightarrow \prod_{e \in \mathbf{B}} W_e$  by  $T(\mathbf{v})(e) = T(\mathbf{v}(e))$  for  $\mathbf{v} \in \prod_{e \in \mathbf{B}} V_e$  and  $e \in \mathbf{B}$ . In particular if  $T$  is continuous and  $\mathbf{v} \in \Gamma$  then  $T(\mathbf{v}) \in \Lambda$ . On the other hand, given  $(e, \varphi) \in \mathbb{W}^*$  define  $T'(\varphi) : V_e \rightarrow \mathbb{R}$  by  $\langle v, T'(\varphi) \rangle = \langle T(v), \varphi \rangle$ . In case  $T$  is linear (resp. continuous) then  $T'(\varphi)$  is also a linear (resp. continuous) map. Likewise,  $T$  induces a function  $T' : \prod_{e \in \mathbf{B}} W_e^* \rightarrow \prod_{e \in \mathbf{B}} V_e^*$ .

**Fact 3.1.24.** [DD63, Proposition 5] *Let  $T : (\mathbb{V}, \Gamma) \rightarrow (\mathbb{W}, \Lambda)$  be a linear map between continuous fields of Banach spaces over  $\mathbf{B}$ . Then,  $T$  is a homomorphism if and only if for every  $\mathbf{v} \in \Gamma$ ,  $T(\mathbf{v}) \in \Lambda$  and the map  $e \rightarrow \|T_e\|$  is bounded.*

**Lemma 3.1.25.** *Let  $T : (\mathbb{V}, \Gamma) \rightarrow (\mathbb{W}, \Lambda)$  be a linear map between separable, continuous fields of Banach spaces over  $\mathbf{B}$ . Then, each of the following affirmations implies the ones below:*

1.  $T$  is a homomorphism.
2.  $T'(\Lambda^*) \subseteq \Gamma^*$ .

3.  $T$  is weakly continuous and the map  $e \longrightarrow \|T_e\|$  is bounded.

If every weak section of  $\mathbb{W}$  is also a norm section, then the three affirmations are equivalent.

*Proof.* 1.  $\Rightarrow$  2. Let  $\nu \in \Lambda^*$  and take  $\mathbf{v} \in \Gamma$ . By hypothesis  $T(\mathbf{v}) \in \Lambda$ , this implies that the map  $e \longmapsto \langle T(\mathbf{v})(e), \nu(e) \rangle = \langle \mathbf{v}(e), T'(\nu)(e) \rangle$  is continuous. Moreover, for every  $e \in \mathbf{B}$ ,  $\|T'(\nu)(e)\| \leq \|T_e\| \|\nu(e)\|$ . From [Fact 3.1.24](#) we get that  $e \longmapsto \|T'(\nu)(e)\|$  is a bounded map. Thanks to [Corollary 3.1.20](#), we obtain that  $T'(\nu) \in \Gamma^*$ .

2.  $\Rightarrow$  3. Suppose that  $(e_i, v_i)_{i \in I}$  is a net in  $\mathbb{V}$  converging weakly to  $(e, v)$ . In particular  $(e_i)_{i \in I}$  converges to  $e$ . Take any  $\nu \in \Lambda^*$ , by our assumption  $T'(\nu) \in \Gamma^*$ . Therefore,  $\langle T(v_i), \nu(e_i) \rangle = \langle v_i, T'(\nu)(e_i) \rangle \longrightarrow \langle v, T'(\nu)(e) \rangle = \langle T(v), \nu(e) \rangle$ . Thus,  $(T(e_i, v_i))_{i \in I}$  converges weakly to  $T(e, v)$ . In particular, for every  $e \in \mathbf{B}$ ,  $T'_e(W_e^*) \subseteq V_e^*$ . This implies that  $T_e$  is bounded and so  $\|T_e\| < +\infty$ . Let us denote by  $\Gamma_0^*$  the family of sections on  $B(\mathbb{W}^*)$ . Let  $d$  be the complete metric on  $B(\mathbb{W}^*)$  as per [Theorem 3.1.17](#). This distance induces a uniform metric,  $d_u$ , on  $\Gamma_0^*$ , which by [Proposition 3.1.19](#) is complete. Take  $\nu \in \Lambda_0^*$ , then for every  $(e, v) \in B(\mathbb{V})$ ,  $|\langle T(v), \nu(e) \rangle| = |\langle v, T'(\nu)(e) \rangle| \leq \|T'(\nu)(e)\| \leq \|T'(\nu)\|$ . Therefore,  $\sup\{|\langle T(v), \nu(e) \rangle| ; (e, v) \in B(\mathbb{V})\} < +\infty$ . For  $n \in \mathbb{N}$  define  $X_n = \{\nu \in \Lambda_0^* ; \sup\{|\langle T(v), \nu(e) \rangle| ; (e, v) \in B(\mathbb{V})\} \leq n\}$ . It follows that  $\bigcup_{n \in \mathbb{N}} X_n = \Lambda_0^*$ . By Baire category theorem, there is  $m \in \mathbb{N}$  such that  $X_m$  has non-empty interior. Choose  $\nu \in \Lambda_0^*$  and  $\epsilon > 0$  such that  $\bar{B}_{d_u}(\nu, \epsilon) \subseteq X_m$ . Let  $(e, v) \in B(\mathbb{V})$ , pick  $\varphi \in B(V_e^*)$  such that  $\|T(v)\| = \langle T(v), \varphi \rangle$ , and let  $\eta \in \Lambda_0^*$  passing through  $(e, \varphi)$ . Then,

$$\|T(v)\| = \langle T(v), \varphi \rangle = \frac{1}{\epsilon} (\langle T(v), \epsilon \eta(e) + \nu(e) \rangle - \langle T(v), \nu(e) \rangle) \leq \frac{2m}{\epsilon}.$$

Since this is true for any  $(e, v) \in B(\mathbb{V})$ , we conclude that  $\|T_e\| < \frac{2m}{\epsilon}$  for every  $e \in \mathbf{B}$ .

Suppose now that every weak section of  $\mathbb{W}$  is also a norm section and that 3. holds. Let  $\mathbf{v} \in \Gamma$ , since  $T$  is weakly continuous, we obtain that  $T(\mathbf{v})$  is a weak section, and so  $T(\mathbf{v}) \in \Lambda$ . The result follows from [Fact 3.1.24](#).  $\square$

Notice that the proof of the implication 2.  $\rightarrow$  3. is just an adaptation of the proof of the uniform boundedness principle, which is used on the proof the analogous result for Banach spaces, see for example [[Con90](#), Theorem VI.1.1].

### 3.1.3 The bidual field

Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces over  $\mathbf{B}$  such that  $\mathbb{V}^*$ , endowed with the weak\* topology, admits sections, say  $\Gamma^*$ . Let

$$\mathbb{V}^{**} = \bigsqcup_{e \in \mathbf{B}} V_e^{**}$$

and  $B(\mathbb{V}^{**}) = \{(e, \lambda) \in \mathbb{V}^{**} ; \|\lambda\| \leq 1\}$ . For  $\nu_0, \dots, \nu_{n-1} \in \Gamma^*$ ,  $I_0, \dots, I_{n-1} \subseteq \mathbb{R}$  and  $U \subseteq \mathbf{B}$  let

$$\mathcal{O}^{**}((\nu_j)_{j < n}, (I_j)_{j < n}, U) = \{(e, \lambda) \in \mathbb{V}^{**} ; e \in U \text{ and } \langle \lambda, \nu_j(e) \rangle \in I_j \text{ for every } j < n\}.$$

The following theorem is the analogue of [Proposition 3.1.8](#) for the bidual field and as such the proofs readily translates.

**Proposition 3.1.26.** *Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces over  $\mathbf{B}$  such that  $\mathbb{V}^*$ , endowed with the weak\* topology, admits a family of sections  $\Gamma^*$ . Then, the family of sets  $\{\mathcal{O}^{**}((\nu_j)_{j \leq n}, (I_j)_{j \leq n}, U) ; \nu_1, \dots, \nu_n \in \Gamma^*, I_1, \dots, I_n \subseteq \mathbb{R}, \text{ open and } U \subseteq \mathbf{B} \text{ open}\}$  is a base for a Hausdorff topology on  $\mathbb{V}^{**}$ . Under this topology:*

1.  $\text{pr}^{**} : \mathbb{V}^{**} \longrightarrow \mathbf{B}$  is continuous.
2. For every  $\nu \in \Gamma^*$  the map  $\langle \cdot, \nu \rangle : \mathbb{V}^{**} \longrightarrow \mathbb{R}$  given by  $(e, \lambda) \longmapsto \langle \lambda, \nu \rangle$  is continuous.
3. The sum from  $\mathbb{V}^{**} \times_{\mathbf{B}} \mathbb{V}^{**}$  into  $\mathbb{V}^{**}$  and the scalar multiplication from  $\mathbb{R} \times \mathbb{V}^{**}$  into  $\mathbb{V}^{**}$  are continuous maps.

For every  $e \in \mathbf{B}$ , the induced topology on  $V_e^{**}$  is the weak-\* topology. We will see that the projection from  $\mathbb{V}^{**}$  onto  $\mathbf{B}$  is open. For this, we will need the next result.

**Fact 3.1.27.** [*Yos68*, Corollary of Theorem IV.8.3] *Let  $V$  be a Banach space. Then, the image of  $B$  under the canonical embedding of  $V$  into  $V^{**}$  is weak\* dense in  $B(V^{**})$ .*

For  $V$  a Banach space and  $v \in V$  we will denote its image under the canonical embedding into  $V^{**}$  by  $\hat{v}$ .

**Proposition 3.1.28.** *Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces over  $\mathbf{B}$  such that  $\mathbb{V}^*$  admits sections. Then, the natural projection  $\text{pr}^{**} : B(\mathbb{V}^{**}) \longrightarrow \mathbf{B}$  is open.*

*Proof.* Pick  $e \in \text{pr}^{**}(\mathcal{O}^{**}((\nu_j)_{j \leq n}, (I_j)_{j \leq n}, U) \cap B(\mathbb{V}^{**}))$  where  $\nu_1, \dots, \nu_n \in \Gamma^*$ ,  $I_1, \dots, I_n \subseteq \mathbb{R}$  and  $U \subseteq \mathbf{B}$  are open. This means, that we can find some  $(e, \lambda) \in \mathcal{O}^{**}((\nu_j)_{j \leq n}, (I_j)_{j \leq n}, U) \cap B(\mathbb{V}^{**})$ . Pick  $r > 0$  such that  $(\langle \lambda, \nu_j(e) \rangle - 2r, \langle \lambda, \nu_j(e), \lambda \rangle + 2r) \subseteq I_j$  for every  $j \leq n$ . Consider the following weak\* open neighborhood of  $\lambda$  in  $B_e^{**}$

$$\mathcal{W} = \{ \tau \in B_e^{**} ; |\langle \tau - \lambda, \nu_j(e) \rangle| < r \text{ for every } j \leq n \}.$$

By the previous fact, there is  $v \in B_e$  such that  $\hat{v} \in \mathcal{W}$ . Take  $\mathbf{v} \in \Gamma$  passing through  $(e, v)$ . Since the mappings  $e' \longmapsto \langle \mathbf{v}(e'), \nu_j(e') \rangle$  are continuous, there is  $U_0 \subseteq U$  open set containing  $e$  such that  $|\langle \mathbf{v}(e'), \nu_j(e') \rangle - \langle \mathbf{v}(e), \nu_j(e) \rangle| < r$  for every  $j \leq n$  and every  $e' \in U_0$ . Fix  $e' \in U_0$ , then for every  $j \leq n$ ,

$$|\langle \widehat{\mathbf{v}(e')}, \nu_j(e') \rangle - \langle \lambda, \nu_j(e) \rangle| \leq |\langle \mathbf{v}(e'), \nu_j(e') \rangle - \langle \mathbf{v}(e), \nu_j(e) \rangle| + |\langle \mathbf{v}(e), \nu_j(e) \rangle - \langle \lambda, \nu_j(e) \rangle| < 2r.$$

Therefore,  $\widehat{\mathbf{v}(e')} \in \mathcal{O}^{**}((\nu_j)_{j \leq n}, (I_j)_{j \leq n}, U) \cap B(\mathbb{V}^{**})$  and so,  $e' \in \text{pr}^{**}(\mathcal{O}^*((\nu_j)_{j \leq n}, (I_j)_{j \leq n}, U) \cap B(\mathbb{V}^{**}))$ , showing that  $\text{pr}^{**}$  is open.  $\square$

In general, we do not know whether there is a family of sections  $\Gamma^{**}$  on  $\mathbb{V}^{**}$  so that  $(\mathbb{V}^{**}, \Gamma^{**})$  is a field of Banach spaces. However, this is evidently true if the original field consists of reflexive spaces.

## 3.2 The field $\mathbb{C}_b(X)$

For a topological space  $X$ ,  $C_b(X)$  denotes the Banach space of continuous, real-valued, bounded functions defined on  $X$ . Let  $(X, \rho)$  be a normal Hausdorff space over  $\mathbf{B}$ . For  $e \in \mathbf{B}$  let us set  $X_e = \rho^{-1}(\{e\})$  and for  $\mathbf{f} \in C_b(X)$  let  $\mathbf{f}_e = \mathbf{f} \upharpoonright_{X_e}$ . We define  $\mathbb{C}_b(X) = \bigsqcup_{e \in \mathbf{B}} C_b(X_e)$ . Each  $\mathbf{f} \in C_b(X)$  can be seen as a section of  $\mathbb{C}_b(X)$  via  $e \longmapsto \mathbf{f}_e$ .

**Proposition 3.2.1.** *Let  $(X, \rho)$  be a normal Hausdorff space over  $\mathbf{B}$ . Then, if  $\rho$  is closed,  $(\mathbb{C}_b(X), C_b(X))$  is an upper semi-continuous field of Banach spaces over  $\mathbf{B}$ . In case  $\rho$  is open then the field is lower semi-continuous.*

*Proof.* We will verify that  $(\mathbb{C}_b(X), C_b(X))$  satisfies the four conditions given in **Proposition 3.1.2**. It is a routine verification that  $C_b(X)$  is a  $C(\mathbf{B})$ -module. Pick any  $e \in \mathbf{B}$  and any  $f \in C_b(X_e)$ . Since  $X_e$  is closed, by Tietze extension theorem, there is  $\mathbf{f} \in C_b(X)$  extending  $f$ . Thus,  $\mathbf{f}$  is a section passing through  $f$ . We will first prove that  $C_b(X)$  is locally uniformly closed. Fix  $\mathbf{f} \in \prod_{e \in \mathbf{B}} C_b(X_e)$  satisfying condition 4. of **Definition 3.1.3**, we must show that  $\mathbf{f}$  is continuous and bounded. Pick  $x \in X$  and  $\epsilon > 0$ , take  $U \subseteq B$  open neighborhood of  $e = \rho(x)$  and  $\mathbf{g} \in C_b(X)$  satisfying  $\|\mathbf{f}_{e'} - \mathbf{g}_{e'}\| < \epsilon/3$  for every  $e' \in U$ . We set  $W_0 = \rho^{-1}(U)$ , note that for every  $x' \in W_0$ ,  $|\mathbf{f}(x') - \mathbf{g}(x')| \leq \|\mathbf{f}_{\rho(x')} - \mathbf{g}_{\rho(x')}\| < \epsilon/3$ . Take  $W_1$  open subset of  $X$  containing  $x$  such that  $|\mathbf{g}(x) - \mathbf{g}(x')| < \epsilon/3$  for every  $x' \in W_1$ . Now set  $W = W_0 \cap W_1$ , if  $x' \in W$  then

$$|\mathbf{f}(x') - \mathbf{f}(x)| \leq |\mathbf{f}(x') - \mathbf{g}(x')| + |\mathbf{g}(x') - \mathbf{g}(x)| + |\mathbf{g}(x) - \mathbf{f}(x)| < \epsilon,$$

showing that  $\mathbf{f}$  is continuous. Again by condition 4. and the compactness of  $\mathbf{B}$ , there is  $\{U_1, \dots, U_n\}$  open cover of  $\mathbf{B}$  and  $\mathbf{g}_1, \dots, \mathbf{g}_n \in C_b(X)$  such that  $\|\mathbf{f} - \mathbf{g}_i\| \leq 1$  on  $U_i$ , for  $i = 1, \dots, n$ . It follows that  $\|\mathbf{f}\| \leq \max_{i \leq n} \|\mathbf{g}_i\| + 1$  and so  $\mathbf{f}$  is bounded.

Now fix  $\mathbf{f} \in C_b(X)$  and suppose that  $\rho$  is closed. We will now show that the map  $e \mapsto \|\mathbf{f}_e\|$  is upper semi-continuous. Take  $e \in \mathbf{B}$  and  $\epsilon > 0$ . For every  $x \in X_e$  there exists an open set  $U_x \subseteq X$  with  $x \in U_x$  such that for every  $x' \in U_x$ ,  $|\mathbf{f}(x) - \mathbf{f}(x')| < \epsilon$ . In this way,  $U = \bigcup \{U_x ; x \in X_e\}$  is an open set containing  $X_e$ . Since  $\rho$  is a closed map,  $V = B \setminus \rho(W)$  is an open neighborhood of  $e$ . Furthermore, for every  $e' \in V$ ,  $X_{e'} \subseteq U$ . Take  $e' \in V$  and  $x \in X_{e'}$ , pick  $x_0 \in X_e$  with  $x \in U_{x_0}$ , then  $|\mathbf{f}(x)| < |\mathbf{f}(x_0)| + \epsilon \leq \|\mathbf{f}_e\| + \epsilon$ . It follows that  $\|\mathbf{f}_{e'}\| \leq \|\mathbf{f}_e\| + \epsilon$ .

Suppose now that  $\rho$  is open, take  $\mathbf{f} \in C_b(X)$ ,  $e \in \mathbf{B}$  and  $\epsilon > 0$ . Let  $x_0 \in X_e$  such that  $|\mathbf{f}(x_0)| > \|\mathbf{f}_e\| - \epsilon$ . Let  $U \subseteq X$  be an open set containing  $x_0$  such that  $|\mathbf{f}(x)| > \|\mathbf{f}_e\| - \epsilon$  for every  $x \in U$ . We have that  $V_0 = \rho(U)$  is an open neighborhood of  $e$ . Fix any  $e' \in V_0$ , then we can find some  $x' \in X_{e'} \cap U$ . We then have that  $\|\mathbf{f}_{e'}\| \geq |\mathbf{f}(x')| > \|\mathbf{f}_e\| - \epsilon$ , completing the proof.  $\square$

The family of sections on  $X$  will be denoted by  $\Gamma(X)$ . If  $\mathbf{x} \in \Gamma(X)$  and  $(e, f) \in \mathbb{C}_b(X)$  we set  $f(\mathbf{x}) = f(\mathbf{x}(e))$ .

**Definition 3.2.2.** Let  $(X, \rho)$  be a normal Hausdorff space over  $\mathbf{B}$  admitting sections. Let  $(e_i, f_i)_{i \in I}$  be a net in  $\mathbb{C}_b(X)$ , we say that  $(e_i, f_i)_{i \in I}$  converges pointwise to  $(e, f) \in \mathbb{C}_b(X)$  if the following two conditions hold:

1.  $(e_i)_{i \in I}$  converges to  $e$ .
2. For every  $\mathbf{x} \in \Gamma(X)$ ,  $(f_i(\mathbf{x}))_{i \in I}$  converges to  $f(\mathbf{x})$ .

By  $\mathbb{C}_p(X)$  we mean the field  $\mathbb{C}_b(X)$  endowed with the pointwise convergence topology. It follows from the definition that this space is Hausdorff and that the induced topology on each fiber corresponds to the pointwise convergence topology on  $C_b(X_e)$ . Note that every  $\mathbf{x} \in \Gamma(X)$ , through the evaluation map, defines a section of the dual field  $\mathbb{C}_b(X)^*$ . It follows that, whenever it is defined, the weak topology on  $\mathbb{C}_b(X)$  is finer than the pointwise convergence topology. The next result is an immediate consequence of the definition of the pointwise convergence topology.

**Proposition 3.2.3.** Let  $(X, \rho)$  be a normal Hausdorff space over  $\mathbf{B}$  admitting sections. Then,  $\mathbb{C}_p(X)$  embeds into  $\mathbf{B} \times \prod_{\mathbf{x} \in \Gamma(X)} \mathbb{R}$  via  $(e, f) \mapsto (e, (f(\mathbf{x}))_{\mathbf{x} \in \Gamma(X)})$ .

It follows that a basis of the pointwise convergence topology are the sets

$$\mathcal{O}_p((\mathbf{x}_j)_{j < n}, (I_j)_{j < n}, U) = \{(e, f) \in \mathbb{C}_b(X) ; e \in U \text{ and } f(\mathbf{x}_j) \in I_j \text{ for every } j < n\}.$$

where  $\mathbf{x}_0, \dots, \mathbf{x}_{n-1} \in \Gamma(X)$ ,  $I_0, \dots, I_{n-1}$  are open subsets of  $\mathbb{R}$  and  $U \subseteq \mathbf{B}$  is open. Suppose now that  $\mathbb{C}(X)$  is separable, which is the case if  $X$  is for example compact and metrizable. Pick  $\mathbf{x} \in \Gamma(X)$ , then for every  $\mathbf{f} \in C(X)$ , the function  $e \mapsto \mathbf{f}(\mathbf{x}(e))$  is continuous. By **Proposition 3.1.19**,  $\mathbf{x}$  defines a section on the dual space  $\mathbb{C}_b(X)^*$ . Therefore, every pointwise open subset described above is also weakly open. Consequently, the weak topology is finer than the pointwise convergence topology.

Since the norm topology is finer than both the pointwise convergence topology and the weak topology, every continuous function  $\mathbf{f} \in C_b(X)$  defines a section in these topologies. Our aim now is to show that under suitable hypothesis for  $X$  and  $\mathbf{B}$ , there are no other sections on  $\mathbb{C}_p(X)$ .

**Definition 3.2.4.** A space  $(X, \rho)$  over  $\mathbf{B}$  is said to have the *section extension property* if for every  $F \subseteq \mathbf{B}$  closed and every continuous section over  $F$ ,  $\mathbf{x}_0 : F \rightarrow X$ , there is a continuous section  $\mathbf{x} : \mathbf{B} \rightarrow X$  extending  $\mathbf{x}_0$ .

Notice that a space with the section extension property admits sections.

**Lemma 3.2.5.** *Let  $(X, \rho)$  be a space over  $\mathbf{B}$ . Assume that  $X$  is sequential and that it has the section extension property. Then, a map  $\mathbf{f} : X \rightarrow Y$  is continuous if and only if the following two conditions hold*

1. *For every  $e \in \mathbf{B}$ , the map  $\mathbf{f}_e : X_e \rightarrow Y$  is continuous.*
2. *For every  $\mathbf{x} \in \Gamma(X)$ , the map  $\mathbf{f} \circ \mathbf{x} : \mathbf{B} \rightarrow Y$  is continuous.*

*Proof.* The implication left to right is clear. Suppose now that  $\mathbf{f} : X \rightarrow Y$  satisfies the two conditions stated. Assume, for sake of contradiction, that  $\mathbf{f}$  is not continuous. Then there is a sequence  $(x_n)_{n \in \mathbb{N}} \subseteq X$  converging to  $x \in X$  such that  $(\mathbf{f}(x_n))_{n \in \mathbb{N}}$  does not converge to  $\mathbf{f}(x)$ . If the set  $\{n \in \mathbb{N} ; \rho(x_n) \neq \rho(x)\}$  is finite, then the sequence  $(x_n)_{n \in \mathbb{N}}$  is eventually contained in  $X_{\rho(x)}$ , and this contradicts 1. Suppose now that the aforementioned set is infinite, by passing to a subsequence we can assume that  $\rho(x_n) \neq \rho(x)$  and that  $\rho(x_n) \neq \rho(x_m)$  for  $m \neq n$ . Let us define  $F = \{\rho(x_n) ; n \in \mathbb{N}\} \cup \{\rho(x)\}$ . The map  $\mathbf{x}_0 : F \rightarrow X$  defined by  $\mathbf{x}_0(\rho(x_n)) = x_n$  for  $n \in \mathbb{N}$ , and  $\mathbf{x}_0(\rho(x)) = x$ , is a section over  $F$ . By the section extension property there is  $\mathbf{x} \in \Gamma(X)$  with  $\mathbf{x}_0 \subseteq \mathbf{x}$ . In this way, the map  $\mathbf{f} \circ \mathbf{x}$  is not continuous, contradicting 2. This completes the proof.  $\square$

If we consider each  $X_e$  as a topological space on its own and we denote by  $\iota_e : X_e \rightarrow X$  the natural injection, then, the previous lemma states that the topology on  $X$  is the final topology given by the family of maps  $\{\iota_e ; e \in \mathbf{B}\} \cup \Gamma(X)$ . The next result is a generalization of **Fact 3.1.16** when  $X$  is Polish.

**Lemma 3.2.6.** *Let  $X$  be a completely metrizable space,  $\mathbf{B}$  a compact totally disconnected space, and  $\rho : X \rightarrow \mathbf{B}$  a continuous, open surjection. Then,  $(X, \rho)$  has the section extension property.*

*Proof.* Take  $F \subseteq \mathbf{B}$  closed and  $\mathbf{x}_0 : F \rightarrow X$  a section over  $F$ . Let  $\{U_n ; n \in \mathbb{N}\}$  be a decreasing neighborhood system for  $\mathbf{x}_0$  with  $U_0 = X$ . For each  $n \in \mathbb{N}$ , since  $\rho$  is open,  $\rho(U_n)$  is an open set containing  $F$  and so, there is a clopen set  $O_n$  with  $F \subseteq O_n \subseteq \rho(U_n)$ . By replacing  $U_n$  with  $U_n \cap \rho^{-1}(O_n)$ , we can assume that  $\rho(U_n) = O_n$ . For  $n \in \mathbb{N}$  let  $V_n = O_n \setminus O_{n+1}$ . In this way,  $\{V_n ; n \in \mathbb{N}\}$  is a sequence of disjoint clopen sets whose union is  $\mathbf{B} \setminus F$ . Given  $n \in \mathbb{N}$  let  $W_n = U_n \cup \rho^{-1}(\mathbf{B} \setminus O_n)$ . As an open subset of  $X$ ,  $W_n$  is completely metrizable, by **Fact 3.1.16**, there is a continuous section  $\mathbf{y}_n : \mathbf{B} \rightarrow W_n$ . Define  $\mathbf{x} = \mathbf{x}_0 \cup \bigcup_{n \in \mathbb{N}} \mathbf{y}_n \upharpoonright_{V_n}$ . Clearly,  $\mathbf{x}$  is a map defined on  $\mathbf{B}$  extending  $\mathbf{x}_0$  and satisfying  $\rho \circ \mathbf{x} = \text{id}_{\mathbf{B}}$ , we claim that it is also continuous. If

$e \in \mathbf{B} \setminus F$ , then  $e$  belongs to some  $V_n$ , so  $\mathbf{x}$  agrees with  $\mathbf{y}_n$  on a neighborhood of  $e$  and continuity at  $e$  readily follows. Pick  $e \in F$  and  $W \subseteq X$  open with  $\mathbf{x}(e) \in W$ . Since  $\mathbf{x}_0$  is continuous and  $\rho$  is open, there is  $V' \subseteq \mathbf{B}$  clopen containing  $e$  such that  $\mathbf{x}_0(V' \cap F) \subseteq W$  and  $V' \subseteq \rho(W)$ . Let  $W' = W \cup \rho^{-1}(B \setminus V')$ . We have that  $W'$  is an open set containing  $\mathbf{x}_0$ . Then, there is  $n \in \mathbb{N}$  such that  $U_n \subseteq W'$ . We set  $V'' = V' \cap O_n$  and take  $e' \in V''$ . In case  $e' \in F$  we get  $e' \in V' \cap F$  and thus  $\mathbf{x}(e') = \mathbf{x}_0(e') \in W$ . Suppose now that  $e' \in V'' \setminus F$ , then there exists  $m \in \mathbb{N}$  such that  $e' \in V_m = O_m \setminus O_{m+1}$ . Due to the fact that  $e' \in O_n$ , we obtain that  $m \geq n$  and so  $\mathbf{x}(e') = \mathbf{y}_m(e') \in W_m$ . Since  $e' \in O_m$ , we conclude that  $\mathbf{x}(e') \in U_m \subseteq U_n \subseteq W'$  which implies  $\mathbf{x}(e') \in W$ , as  $e' \in V'$ . This completes the proof.  $\square$

**Proposition 3.2.7.** *Suppose that  $(X, \rho)$  is a completely metrizable space over a compact, totally disconnected space  $\mathbf{B}$ . Let  $\mathbf{f} : \mathbf{B} \rightarrow \mathbb{C}_p(X)$  be a section, then the map defined by  $\mathbf{f}(x) = \mathbf{f}(\rho(x))(x)$  for  $x \in X$ , is continuous.*

*Proof.* By Lemma 3.2.6,  $X$  has the section extension property. On one hand, for every  $e \in \mathbf{B}$ ,  $\mathbf{f}(e)$  is a continuous map on  $X_e$ . On the other hand, by the definition of the pointwise convergence topology on  $\mathbb{C}(X)$ , for every  $\mathbf{x} \in \Gamma(X)$ , the application  $e \rightarrow \mathbf{f}(e)(\mathbf{x})$  is continuous. Since  $X$  is sequential, the result follows from Lemma 3.2.5.  $\square$

Since weak continuity implies pointwise continuity, the next result is immediate.

**Corollary 3.2.8.** *Suppose that  $(X, \rho)$  is a completely metrizable space over a compact, totally disconnected space  $\mathbf{B}$ . Let  $\mathbf{f} : \mathbf{B} \rightarrow \mathbb{C}_w(X)$  be a section, then the map defined by  $\mathbf{f}(x) = \mathbf{f}(\rho(x))(x)$  for  $x \in X$ , is continuous.*

### 3.3 Compactness in $\mathbb{C}_p(X)$

We assume that  $(X, \rho)$  is a compact, metrizable space over a totally disconnected compact space  $\mathbf{B}$ . In this section, inspired by Grothendieck in [Gro52], we will prove a double limit criterion for compactness on  $\mathbb{C}_p(X)$ . We know from Proposition 3.2.3 that  $\mathbb{C}_p(X)$  embeds in  $\mathbf{B} \times \prod_{\mathbf{x} \in \Gamma(X)} \mathbb{R}$ .

The following lemma is a first characterization of compactness in terms of this embedding.

**Lemma 3.3.1.** *Let  $A \subseteq \mathbb{C}(X)$ . Then,  $A$  is relatively pointwise compact if and only if the following conditions are satisfied:*

1. For every  $\mathbf{x} \in \Gamma(X)$ ,  $\{f(\mathbf{x}) ; f \in A\}$  is bounded.
2. The pointwise closure of  $A$  in  $\mathbf{B} \times \prod_{\mathbf{x} \in \Gamma(X)} \mathbb{R}$  is the same as in  $\mathbb{C}_p(X)$ .

*Proof.* Let  $\text{cl}_p(A)$  be the closure of  $A$  in  $\mathbb{C}_p(X)$  and let  $\bar{A}$  denote the closure of  $A$  in  $\mathbf{B} \times \prod_{\mathbf{x} \in \Gamma(X)} \mathbb{R}$ . First suppose that the two conditions are satisfied. For each  $\mathbf{x} \in \Gamma(X)$  take  $M_{\mathbf{x}} \in \mathbb{R}$  such that  $\{f(\mathbf{x}) ; f \in A\} \subseteq [-M_{\mathbf{x}}, M_{\mathbf{x}}]$ . It follows that  $A$  is contained into the compact set  $\mathbf{B} \times \prod_{\mathbf{x} \in \Gamma(X)} [-M_{\mathbf{x}}, M_{\mathbf{x}}]$ . Therefore,  $\bar{A}$  is also contained in this set, and hence it is compact. Since  $\bar{A} = \text{cl}_p(A)$ , we conclude that  $A$  is relatively pointwise compact. For the converse, suppose that  $A$  is relatively pointwise compact, that is  $\text{cl}_p(A)$  is compact in  $\mathbb{C}_p(X)$ . This implies that  $\text{cl}_p(A)$  is also compact, and thus closed, in  $\mathbf{B} \times \prod_{\mathbf{x} \in \Gamma(X)} \mathbb{R}$ . Hence,  $\bar{A} = \text{cl}_p(A)$  and 2. holds. Now fix  $\mathbf{x} \in \Gamma(X)$ , the projection onto the  $\mathbf{x}$ -th component is continuous. It follows that  $\{f(\mathbf{x}) ; f \in \text{cl}_p(A)\}$  is compact which implies that  $\{f(\mathbf{x}) ; f \in A\}$  is bounded.  $\square$

First, we will find a criterion for compactness using doubly adherent points. Then, we will use the diagonalization of double sequences in order to deduce a double limit criterion.

**Definition 3.3.2.** Let  $(x_{m,n})_{m,n \in \mathbb{N}}$  be a double sequence in a topological space  $F$ . We say that  $x \in F$  is *adherent by columns* to  $(x_{m,n})_{m,n \in \mathbb{N}}$  if for every  $V \subseteq F$  open neighborhood of  $x$  there is infinitely many  $n \in \mathbb{N}$  such that  $\{m \in \mathbb{N} ; x_{m,n} \in V\}$  is infinite. If we exchange the roles of  $m$  and  $n$  we get the notion of *adherent by rows*. We say that  $x$  is *doubly adherent* if it is adherent by rows and by columns.

The next result will allow us to obtain a double limit criterion for compactness in  $\mathbb{C}_p(X)$ , as in [Gro52, Théorème 2].  $C(\mathbf{B}, X)$ , as a function space between topological spaces, is naturally endowed with the pointwise convergence topology. Furthermore, since  $X$  is metrizable,  $C(\mathbf{B}, X)$  also admits the uniform convergence topology.  $\Gamma(X)$ , as a subset of  $C(\mathbf{B}, X)$ , inherits these two topologies.

**Theorem 3.3.3.** *Let  $A$  be a subset of  $\mathbb{C}_p(X)$ . The following assertions are equivalent:*

1.  *$A$  is relatively compact.*
2.  *$A$  is relatively countably compact.*
3. *For every sequence  $(e_n, f_n)_{n \in \mathbb{N}}$  in  $A$  and every relatively pointwise compact sequence  $(\mathbf{x}_m)_{m \in \mathbb{N}}$  in  $\Gamma(X)$  there is  $r \in \mathbb{R}$  doubly adherent to the double sequence  $(f_n(\mathbf{x}_m))_{m,n \in \mathbb{N}}$ .*
4. *Same as 3. but with the sequence  $(\mathbf{x}_m)_{m \in \mathbb{N}}$  being relatively uniformly compact in  $\Gamma(X)$ .*

*Proof.* 1.  $\Rightarrow$  2. It is immediate.

2.  $\Rightarrow$  3. Let  $(e_n, f_n)_{n \in \mathbb{N}}$  be an arbitrary sequence in  $A$  and let  $(\mathbf{x}_m)_{m \in \mathbb{N}}$  be a relatively pointwise compact sequence in  $\Gamma(X)$ . By hypothesis the sequence  $(e_n, f_n)_{n \in \mathbb{N}}$  admits an adherence point, say  $(e, f) \in \mathbb{C}_p(X)$ . In particular we have that  $e$  is adherent to  $(e_n)_{n \in \mathbb{N}}$ . By compactness,  $(\mathbf{x}_m)_{m \in \mathbb{N}}$  admits a pointwise adherence point  $\mathbf{x} \in \Gamma$ . We claim that  $f(\mathbf{x})$  is doubly adherent to  $(f_n(\mathbf{x}_m))_{m,n \in \mathbb{N}}$ . Suppose on the contrary that it is not adherent by columns. Then there is an open neighborhood  $V$  of  $f(\mathbf{x})$  and  $N \in \mathbb{N}$  such that for every  $n \geq N$ ,  $\{f_n(\mathbf{x}_m) ; m \in \mathbb{N}\} \cap V$  is finite. This means that for every  $n \geq N$ , there is  $m_0(n) \in \mathbb{N}$  such that  $f_n(\mathbf{x}_m) \notin V$  for  $m \geq m_0(n)$ . Note that for every  $n \in \mathbb{N}$  we have a pointwise continuous function from  $\Gamma(X)$  into  $\mathbb{R}$  by  $\mathbf{x}' \rightarrow f_n(\mathbf{x}')$ . This implies that, for every  $n \geq N$ ,  $f_n(\mathbf{x}) \notin V$ . Hence,  $f(\mathbf{x}) \notin V$ , which is absurd. Now suppose that it is not adherent by rows. This means that there is an open neighborhood  $V$  of  $f(\mathbf{x})$  and  $M \in \mathbb{N}$  such that for every  $m \geq M$ ,  $\{f_n(\mathbf{x}_m) ; n \in \mathbb{N}\} \cap V$  is finite. Then, for every  $m \geq M$ , there is  $n_0(m) \in \mathbb{N}$  such that  $f_n(\mathbf{x}_m) \notin V$  for  $n \geq n_0(m)$ . Therefore, for every  $m \geq M$ ,  $f(\mathbf{x}_m) \notin V$ . Hence,  $f(\mathbf{x}) \notin V$  which is absurd.

3.  $\Rightarrow$  4. Immediate.

4.  $\Rightarrow$  1. If  $\mathbf{x} \in \Gamma(X)$  and  $(e_n, f_n)_{n \in \mathbb{N}}$  is a sequence in  $A$  then, by taking  $\mathbf{x}_m = \mathbf{x}$  for every  $m \in \mathbb{N}$ , we get that  $(f_n(\mathbf{x}))_{n \in \mathbb{N}}$  admits an adherence point. This implies that the set  $\{f(\mathbf{x}) ; f \in A\}$  must be bounded. Hence, the first condition of Lemma 3.3.1 is satisfied. We will show that the second condition also holds. Let  $\bar{A}$  be the closure of  $A$  in  $\mathbf{B} \times \prod_{\mathbf{x} \in \Gamma(X)} \mathbb{R}$  and take  $(e, f) \in \bar{A}$ , we must show that  $f \in C(X_e)$ . This means that  $f$  defines a function on  $X_e$  and this function is moreover continuous. Suppose this is not the case, then there is  $\mathbf{x}_0 \in \Gamma(X)$  and  $\epsilon > 0$  such that for every neighborhood  $V$  of  $\mathbf{x}_0(e)$  in  $X_e$  there is  $\mathbf{x}_V \in \Gamma(X)$  such that  $\mathbf{x}_V(e) \in V$  and  $|f(\mathbf{x}_V) - f(\mathbf{x}_0)| \geq \epsilon$ . Let  $(W_m)_{m \in \mathbb{N}}$  be a decreasing countable neighborhood basis around  $\mathbf{x}_0$ . We will construct, in a recursive manner, two sequences  $(e_n, f_n)_{n \geq 1}$  in  $A$  and  $(\mathbf{x}_m)_{m \geq 1}$  in  $\Gamma(X)$  such that for every  $n \in \mathbb{N}$

a)  $|f_n(\mathbf{x}_m) - f(\mathbf{x}_m)| \leq 1/n$  for  $0 \leq m \leq n - 1$ .

b)  $|f_i(\mathbf{x}_n) - f_i(\mathbf{x}_0)| \leq 1/n$  for  $0 < i \leq n$ .

- c)  $|f(\mathbf{x}_n) - f(\mathbf{x}_0)| \geq \epsilon$ .  
d)  $\mathbf{x}_n \in W_n$ .

Suppose  $(e_1, f_1), \dots, (e_{n-1}, f_{n-1})$  and  $\mathbf{x}_1, \dots, \mathbf{x}_{n-1}$  are already defined. Since  $(e, f)$  is adherent to  $A$ , there is  $(e_n, f_n) \in A$  such that  $|f_n(\mathbf{x}_m) - f(\mathbf{x}_m)| \leq 1/n$  for every  $0 \leq m \leq n-1$ . We know that there exists  $V_n$  open subset of  $X_e$ , which we can safely assume is contained in  $W_n \cap X_e$ , such that for every  $i \leq n$  and for every  $z \in V_n$ , we have  $|f_i(z) - f_i(\mathbf{x}_0)| \leq 1/n$ . We remark that  $W_n$ , being an open subset of  $X$  projecting onto  $\mathbf{B}$ , admits sections itself. Therefore, there is  $\mathbf{x}_n \in \Gamma(W_n)$  satisfying  $\mathbf{x}_n(e) \in V_n$  and  $|f(\mathbf{x}_n) - f(\mathbf{x}_0)| \geq \epsilon$ . By construction, the sequence  $(\mathbf{x}_m)_{m \in \mathbb{N}}$  converges uniformly to  $\mathbf{x}_0$ . Thus, from our hypothesis, we obtain  $r \in \mathbb{R}$  doubly adherent to the double sequence  $(f_n(\mathbf{x}_m))_{m, n \in \mathbb{N}}$ . We fix  $m \in \mathbb{N}$ , by a),  $(f_n(\mathbf{x}_m))_{n \in \mathbb{N}}$  converges to  $f(\mathbf{x}_m)$ . If we fix  $n \in \mathbb{N}$ , by b),  $(f_n(\mathbf{x}_m))_{m \in \mathbb{N}}$  converges to  $f_n(\mathbf{x}_0)$ . Therefore,  $r$  is adherent to the sequences  $(f(\mathbf{x}_m))_{m \in \mathbb{N}}$  and  $(f_n(\mathbf{x}_0))_{n \in \mathbb{N}}$ . Using a) once again, we obtain that  $(f_n(\mathbf{x}_0))_{n \in \mathbb{N}}$  converges to  $f(\mathbf{x}_0)$ . It follows that  $r = f(\mathbf{x}_0)$ . Nevertheless, by c),  $|f(\mathbf{x}_m) - r| \geq \epsilon$ , a contradiction.  $\square$

The condition of relative compactness on the sequence  $(\mathbf{x}_m)_{m \in \mathbb{N}}$  in  $\mathcal{I}$  seems rather artificial. However, what we are essentially requiring is the existence of a doubly adherent point to  $(\mathbf{x}_m(e_n))_{m, n \in \mathbb{N}}$ , in order to obtain a doubly adherent point to  $(f_n(\mathbf{x}_m))_{m, n \in \mathbb{N}}$ .

**Fact 3.3.4** (Diagonalization of double sequences). [*Gro52, Remarque 2*] *Let  $(x_{m,n})_{m, n \in \mathbb{N}}$  be a relatively compact double sequence in a metric space  $F$ . The non-existence of a doubly adherent point to  $(x_{m,n})_{m, n \in \mathbb{N}}$  implies that there is a double subsequence  $(y_{i,j})_{i, j \in \mathbb{N}} = (x_{m_i, n_j})_{i, j \in \mathbb{N}}$  such that  $\lim_i \lim_j y_{i,j}$  and  $\lim_j \lim_i y_{i,j}$  both exist but they are different. On the other hand, if  $\lim_m \lim_n x_{m,n}$  and  $\lim_n \lim_m x_{m,n}$  both exist and they are different then  $(x_{m,n})_{m, n \in \mathbb{N}}$  has no doubly adherent point.*

This fact allows us to obtain the double limit criterion for compactness in  $\mathbb{C}_p(X)$ .

**Corollary 3.3.5.** *Let  $A \subseteq \mathbb{C}_p(X)$  such that  $\{f(x) ; (e, f) \in A \text{ and } x \in X_e\}$  is bounded.  $A$  is relatively compact in  $\mathbb{C}_p(X)$  if and only if for every sequence  $(e_n, f_n)_{n \in \mathbb{N}}$  in  $A$  and every relatively uniformly compact sequence  $(\mathbf{x}_m)_{m \in \mathbb{N}}$  in  $\Gamma(X)$ ,*

$$\lim_n \lim_m (f_n(\mathbf{x}_m)) = \lim_m \lim_n (f_n(\mathbf{x}_m))$$

as soon as both limits exist.

*Proof.*  $\Rightarrow$ ) Let  $(e_n, f_n)_{n \in \mathbb{N}} \subseteq A$  and let  $(\mathbf{x}_m)_{m \in \mathbb{N}} \subseteq \Gamma(X)$  be relatively pointwise compact, such that  $\lim_n \lim_m (f_n(\mathbf{x}_m(e_n)))$  and  $\lim_m \lim_n (f_n(\mathbf{x}_m(e_n)))$  exist. By **Theorem 3.3.3**, there is  $r \in \mathbb{R}$  double adherent to  $(f_n(\mathbf{x}_m(e_n)))_{m, n \in \mathbb{N}}$ . Therefore,  $\lim_n \lim_m (f_n(\mathbf{x}_m(e_n))) = r = \lim_m \lim_n (f_n(\mathbf{x}_m(e_n)))$ .

$\Leftarrow$ ) Suppose that  $A$  is not relatively compact in  $\mathbb{C}_p(X)$ . Then, there are sequences  $(e_n, f_n)_{n \in \mathbb{N}}$  in  $A$  and  $(\mathbf{x}_m)_{m \in \mathbb{N}}$  relatively uniformly compact in  $\Gamma(X)$  such that  $(f_n(\mathbf{x}_m(e_n)))_{m, n \in \mathbb{N}}$  does not admit a doubly adherent point in  $\mathbb{R}$ . Since  $(f_n(\mathbf{x}_m(e_n)))_{m, n \in \mathbb{N}}$  is bounded, by the previous fact, we can pass to a double subsequence, such that  $\lim_m \lim_n f_n(\mathbf{x}_m(e_n))$  and  $\lim_n \lim_m f_n(\mathbf{x}_m(e_n))$  both exist but they are different.  $\square$

### 3.4 Groupoid actions

For  $(X, \rho)$  a space over  $\mathbf{B}$  and  $\mathbf{G}$  a groupoid of base  $\mathbf{B}$  an action of  $\mathbf{G}$  on  $X$  is a map  $\cdot : \mathbf{G} \times_{\mathbf{B}} X \rightarrow X$  such that whenever  $g, h \in \mathbf{G}$  and  $x \in X$  satisfy  $s_h = t_g$  and  $s_g = \rho(x)$  then  $h \cdot (g \cdot x) = (hg) \cdot x$ . We begin by studying separately continuous actions.

### 3.4.1 Separately continuous actions

Let us work first in the setting of group actions. Let  $X$  be a topological space and let  $G$  be a topological group acting (not necessarily continuously) on  $X$ . The action is said to be *left continuous* if for every  $g \in G$  the map  $x \mapsto g \cdot x$  is continuous. It is said to be *right continuous* if for every  $x \in X$  the map  $g \mapsto g \cdot x$  is continuous. We say that the action is separately continuous if it is both left and right continuous.

Let  $\text{Homeo}(X)$  denote the group of homomorphisms from  $X$  to itself. A left continuous action  $G \curvearrowright X$  is just a group homomorphism from  $G$  to  $\text{Homeo}(X)$ . The group  $\text{Homeo}(X)$  is usually endowed with the compact-open topology, under which is a topological group in case  $X$  is compact. However, we will consider the pointwise convergence topology<sup>1</sup> on  $\text{Homeo}(X)$ . Suppose that the action  $G \curvearrowright X$  is moreover right continuous. Let  $(g_i)_{i \in I}$  be a net in  $G$  converging to  $g \in G$ , then for any  $x \in X$ ,  $(g_i \cdot x)_{i \in I}$  converges to  $g \cdot x$ . This means that the orbit maps  $(\tilde{g}_i)_{i \in I}$ , considered as elements of  $\text{Homeo}_p(X)$ , converge pointwise to  $\tilde{g}$ . This means that the homomorphism  $G \rightarrow \text{Homeo}_p(X)$  is continuous. In conclusion, a separately continuous action of a topological group  $G$  on a topological space  $X$  is a continuous homomorphism of  $G$  into  $\text{Homeo}_p(X)$ , the latter endowed with the pointwise convergence topology. This is precisely the notion we will use in order to define separately continuous groupoid actions.

Suppose that  $X$  and  $Y$  are topological spaces, the pointwise convergence topology on a space of functions from  $X$  to  $Y$  is just the topology induced from  $Y^X$ . The space  $\mathcal{S}(X, Y)$ , defined below, is the analogue of the product when considering spaces over  $\mathbf{B}$ .

**Definition 3.4.1.** Let  $(X, \rho)$  and  $(Y, \eta)$  be spaces over  $\mathbf{B}$ . We set

$$\mathcal{S}(X, Y) = \bigsqcup_{e, e' \in \mathbf{B}} (Y_{e'})^{X_e}.$$

We will write  $(e', f, e) \in \mathcal{S}(X, Y)$  to indicate an element  $f \in (Y_{e'})^{X_e}$ .

From now on we will assume that  $(X, \rho)$  admits sections. We will endow  $\mathcal{S}(X, Y)$  with a topology in the following manner. A net  $(e'_i, f_i, e_i)_{i \in I}$  in  $\mathcal{S}(X, Y)$  converges pointwise to  $(e', f, e)$  if the following conditions hold:

1.  $(e_i)_{i \in I}$  converges to  $e$ .
2.  $(e'_i)_{i \in I}$  converges to  $e'$ .
3. For every  $\mathbf{x} \in \Gamma(X)$ ,  $(f_i(\mathbf{x}))_{i \in I}$  converges to  $f(\mathbf{x})$ .

**Proposition 3.4.2.** Let  $(X, \rho)$  and  $(Y, \eta)$  be spaces over  $\mathbf{B}$  with  $X$  admitting sections. Then,

1. The sets of the form

$$\mathcal{O}((\mathbf{x}_i)_{i < n}, (W_i)_{i < n}, U', U) = \{(e', f, e) \in \mathcal{S}(X, Y) ; e' \in U', e \in U, f(\mathbf{x}_i) \in W_i \text{ for } i < n\},$$

where  $\mathbf{x}_0, \dots, \mathbf{x}_{n-1} \in \Gamma(X)$ ,  $W_0, \dots, W_{n-1} \subseteq Y$  are open and  $U, U' \subseteq \mathbf{B}$  are open, form a basis for the pointwise convergence topology on  $\mathcal{S}(X, Y)$ .

2. The maps  $s, t : \mathcal{S}(X, Y) \rightarrow \mathbf{B}$  given by  $s(e', f, e) = e$  and  $t(e', f, e) = e'$  are continuous and open.

<sup>1</sup>Recall that we add the subindex  $p$  to a function space in order to explicit the fact that we are endowing it with the topology of pointwise convergence.

*Proof.* It is an easy exercise to see that pointwise convergence is equivalent to convergence on the topology having these sets as a basis. 2. follows immediately from the definition of pointwise convergence.  $\square$

In case  $X = Y$  we will simply write  $\mathcal{S}(X)$ , which corresponds to [AB21, Definition 5]. This space admits naturally a semigroupoid structure by

$$(e'', g, e') \cdot (e', f, e) = (e'', g \circ f, e).$$

The neutral elements are precisely  $(e, \text{id}_{X_e}, e)$  for  $e \in \mathbf{B}$ . The next result corresponds to [AB21, Theorem 2]. Therein the authors assume that  $X = Y$  and that each fiber of  $X$  is compact. Nevertheless, the proof readily adapts to our context, and the compactness hypothesis is only used to see that  $(X_{e'})^{X_e}$  is compact; which we do not need.

**Fact 3.4.3.** *Let  $(X, \rho)$  and  $(Y, \eta)$  be spaces over  $\mathbf{B}$  with  $X$  admitting sections. Then,*

1. *The pointwise convergence topology on  $\mathcal{S}(X, Y)$  is Hausdorff.*
2. *The induced topology on  $(Y_{e'})^{X_e}$  agrees with the product topology.*
3. *Multiplication on the right on  $\mathcal{S}(X)$  is continuous.*

**Definition 3.4.4.** For  $(X, \rho)$  and  $(Y, \eta)$  spaces over  $\mathbf{B}$  we set

$$\begin{aligned} \mathcal{C}_p(X, Y) &= \{(e', f, e) \in \mathcal{S}(X, Y) ; f \text{ is continuous}\} \\ \mathcal{H}_p(X) &= \{(e', f, e) \in \mathcal{S}(X) ; f \text{ is a homeomorphism}\}, \end{aligned}$$

both spaces endowed with the topology of pointwise convergence induced by  $\mathcal{S}(X, Y)$  and  $\mathcal{S}(X)$  respectively. As usual, we will write  $\mathcal{C}_p(X)$  instead of  $\mathcal{C}_p(X, X)$ . Note that  $\mathcal{H}_p(X)$  is closed under the semi-groupoid operation. Moreover,  $\mathcal{H}_p(X)$  is closed under inverse. It follows that  $\mathcal{H}_p(X)$  is a groupoid and its base is in correspondence with  $\mathbf{B}$ . Since multiplication on the right is continuous, we obtain the following result.

**Lemma 3.4.5.**  *$\mathcal{H}_p(X)$  is a right semi-topological groupoid over  $\mathbf{B}$ .*

**Definition 3.4.6.** Let  $\mathbf{G}$  be a groupoid over  $\mathbf{B}$  and let  $(X, \rho)$  be a space over  $\mathbf{B}$ . An action  $\mathbf{G} \curvearrowright X$  is said to be *separately continuous* if for every  $g \in \mathbf{G}$  the orbit map  $\tilde{g} : X_{s_g} \rightarrow X_{t_g}$  is continuous and the map  $g \mapsto \tilde{g}$  is a continuous homomorphism from  $\mathbf{G}$  into  $\mathcal{H}_p(X)$ .

Fix  $\mathbf{x} \in \Gamma(X)$ , then the function  $\tilde{\mathbf{x}} : \mathbf{G} \rightarrow X$  given by  $\tilde{\mathbf{x}}(g) = g \cdot \mathbf{x}(s_g)$  is continuous. Indeed, if the net  $(g_i)_{i \in I}$  converges to  $g$  then  $(t_{g_i}, \tilde{g}_i, s_i)_{i \in I}$  converges pointwise to  $(t_g, \tilde{g}, s_g)$ , which implies  $g_i \cdot \mathbf{x}(s_{g_i}) \rightarrow g \cdot \mathbf{x}(s_g)$ . Furthermore, if  $\mathbf{g} \in \Gamma_s(\mathbf{G})$ , the space of  $s$ -sections of  $\mathbf{G}$ , then  $e \mapsto \mathbf{g}(e) \cdot \mathbf{x}(e)$  is continuous. To see this, take  $(e_i)_{i \in I}$  converging to  $e$ . It follows that  $(t_{\mathbf{g}(e_i)}, \mathbf{g}(e_i), e_i)_{i \in I}$  converges to  $(t_{\mathbf{g}(e)}, \mathbf{g}(e), e)$  and so  $\mathbf{g}(e_i) \cdot \mathbf{x}(e_i) = \mathbf{g}(e_i)(\mathbf{x}) \rightarrow \mathbf{g}(e)(\mathbf{x}) = \mathbf{g}(e) \cdot \mathbf{x}(e)$ . By a completely analogous argument, if  $\mathbf{g} \in \Gamma_t(\mathbf{G})$ , the family of  $t$ -sections of  $\mathbf{G}$ , then the map  $e \rightarrow \mathbf{g}(e) \cdot \mathbf{x}(s_{\mathbf{g}(e)})$  is continuous and it defines a section in  $\Gamma(X)$ . We may write  $g \cdot \mathbf{x}$  to denote  $g \cdot \mathbf{x}(s_g)$ .

Given  $(e, f) \in \mathcal{C}_b(X)$  and  $g \in \mathbf{G}e$  define  $gf : X_{t_g} \rightarrow \mathbb{R}$  by  $gf(x) = f(g^{-1} \cdot x)$  for  $x \in X_{t_g}$ .  $gf$  is continuous and bounded, as  $gf = f \circ \tilde{g}^{-1}$ . Therefore, we have an induced action  $\mathbf{G} \curvearrowright \mathcal{C}_b(X)$  given by  $g \cdot (e, f) = (t_g, gf)$  whenever  $s_g = e$ . Following the notation introduced just above, for  $f \in \mathcal{C}_b(X)$  and  $g \in \mathbf{G}$ ,  $gf$  will stand for the map  $gf_{s_g}$ .

**Lemma 3.4.7.** *Let  $(X, \rho)$  be a completely metrizable space over a compact, totally disconnected space  $\mathbf{B}$  and let  $\mathbf{G} \curvearrowright X$  be a separately continuous action. Then, the action  $\mathbf{G} \curvearrowright \mathbb{C}_p(X)$  is separately continuous.*

*Proof.* Pick  $g \in \mathbf{G}$  and a net  $(f_i)_{i \in I}$  in  $C(X_{s_g})$  converging pointwise to  $f \in C_b(X_{s_g})$ . This implies that for any  $x \in X_{t_g}$ ,  $gf_i(x) = f_i(g^{-1} \cdot x) \rightarrow f(g^{-1} \cdot x) = gf(x)$ . Thus,  $\tilde{g}: C_b(X_{s_g}) \rightarrow C_b(X_{t_g})$  is continuous. Now take a net  $(g_i)_{i \in I} \subseteq \mathbf{G}$  converging to  $g \in \mathbf{G}$ , we must show that  $(s_{g_i}, \tilde{g}_i, s_{g_i})_{i \in I}$  converges in  $\mathcal{H}_p(\mathbb{C}_p(X))$  to  $(s_g, \tilde{g}, s_g)$ . This occurs precisely when for every section  $\mathbf{f}$  of  $\mathbb{C}_p(X)$ ,  $(\tilde{g}_i(\mathbf{f}))_{i \in I}$  converges to  $\tilde{g}(\mathbf{f})$ . By [Proposition 3.2.7](#),  $\mathbf{f}$  is a continuous map, i.e.  $\mathbf{f} \in C_b(X)$ . Thus, we must see that  $(g_i \mathbf{f})_{i \in I}$  converges pointwise, that is in  $\mathbb{C}_p(X)$ , to  $g \mathbf{f}$ . Fix  $\mathbf{x} \in \Gamma(X)$ , since the action  $\mathbf{G} \curvearrowright X$  is separately continuous, we obtain  $g_i^{-1} \cdot \mathbf{x}(t_{g_i}) \rightarrow g \cdot \mathbf{x}(t_g)$ . The continuity of  $\mathbf{f}$  allows us to conclude that  $g_i \mathbf{f}(\mathbf{x}) = \mathbf{f}(g_i^{-1} \cdot \mathbf{x}(t_{g_i})) \rightarrow \mathbf{f}(g^{-1} \cdot \mathbf{x}(t_g)) = g \mathbf{f}(\mathbf{x})$ . This completes the proof.  $\square$

### 3.4.2 Continuous actions

Let  $(X, \rho)$  be a space over  $\mathbf{B}$  and  $\mathbf{G}$  be a groupoid acting on  $X$ . If the action map  $\cdot: \mathbf{G} \times_{\mathbf{B}} X \rightarrow X$  is continuous, we say that  $\mathbf{G}$  acts continuously on  $X$ , or that  $X$  is a  $\mathbf{G}$ -space. As expected, a continuous action is separately continuous.

We know that a separately continuous action  $\mathbf{G} \curvearrowright X$  induces an action  $\mathbf{G} \curvearrowright \mathbb{C}_b(X)$ . For this action to be separately continuous it is necessary that for every  $\mathbf{f} \in C_b(X)$  the orbit map  $\tilde{\mathbf{f}}: \mathbf{G} \rightarrow \mathbb{C}_b(X)$  is norm continuous. In the framework of group actions this means precisely that the function is right uniformly continuous. We extend this notion to the case of groupoid actions.

**Definition 3.4.8.** A map  $\mathbf{f} \in C_b(X)$  is said to be *right uniformly continuous* if the orbit map  $\tilde{\mathbf{f}}: \mathbf{G} \rightarrow \mathbb{C}_b(X)$  is norm continuous. We will write  $\mathbf{f} \in \text{RUC}_{\mathbf{G}}(X)$ .

This means precisely, that for every  $g \in \mathbf{G}$ , every  $\epsilon > 0$  and every (equivalently some)  $\mathbf{f}_0 \in C_b(X)$  passing through  $g \mathbf{f}$ , there is  $W \subseteq \mathbf{G}$  open set containing  $g$  satisfying  $|\mathbf{f}(h^{-1} \cdot x) - \mathbf{f}_0(x)| < \epsilon$  for every  $h \in W$  and every  $x \in X_{t_h}$ .

**Lemma 3.4.9.** *Suppose  $\mathbf{f} \in C_b(X)$ . If  $\mathbf{f}$  is right uniformly continuous then for every  $\epsilon > 0$  there is  $W \subseteq \mathbf{G}$  open neighborhood of  $\mathbf{B}$  such that for every  $g \in W$  and  $x \in X_{t_g}$ ,  $|\mathbf{f}(g^{-1} \cdot x) - \mathbf{f}(x)| < \epsilon$  for every  $x \in X$ . If  $\mathbf{G}$  admits sections then the converse also holds.*

*Proof.* Fix  $\epsilon > 0$ , for each  $e \in \mathbf{B}$  pick  $W_e$  open neighborhood in  $\mathbf{G}$  of  $e$  such that  $|\mathbf{f}(h^{-1} \cdot x) - \mathbf{f}(x)| < \epsilon$  for every  $h \in W_e$  and every  $x \in X_{t_h}$ . Then  $W = \bigcup_{e \in \mathbf{B}} W_e$  is the desired open set. Now suppose that  $\mathbf{G}$  admits sections and that  $\mathbf{f}$  satisfies the property stated on the right hand side. Pick  $g \in \mathbf{G}$  and  $\epsilon > 0$ , and let  $\mathbf{g} \in \Gamma_t(\mathbf{G})$  pass through  $g$ . It follows that  $\mathbf{g} \mathbf{f}$  is a continuous function passing through  $g \mathbf{f}$ . For this section and said  $\epsilon$ , take  $W_0$  open neighborhood of  $\mathbf{B}$  as in the statement. We have that  $W = \mathbf{g}W_0$  is an open set containing  $g$ . Take  $h \in W$ , since  $\mathbf{g}(t_h)^{-1}h \in W_0$  we get  $|\mathbf{f}(h^{-1}\mathbf{g}(t_h) \cdot y) - \mathbf{f}(y)| < \epsilon$  for every  $y \in X_{s_{\mathbf{g}(t_h)}}$ . Take  $x \in X_{t_h}$ , we have that  $y = \mathbf{g}(t_h)^{-1} \cdot x \in X_{s_{\mathbf{g}(t_h)}}$ , and so

$$|h \mathbf{f}(x) - \mathbf{g} \mathbf{f}(x)| = |\mathbf{f}(h^{-1} \cdot x) - \mathbf{f}(\mathbf{g}(t_h)^{-1} \cdot x)| = |\mathbf{f}(h^{-1} \mathbf{g}(t_h) \cdot y) - \mathbf{f}(y)| < \epsilon.$$

This completes the proof.  $\square$

**Lemma 3.4.10.** *Let  $(X, \rho)$  be a normal space over  $\mathbf{B}$  and let  $\mathbf{G}$  be a groupoid acting on  $X$ . If  $C_b(X) = \text{RUC}_{\mathbf{G}}(X)$  then the actions  $\mathbf{G} \curvearrowright X$  and  $\mathbf{G} \curvearrowright \mathbb{C}_b(X)$  are continuous.*

*Proof.* Assume that  $C_b(X) = \text{RUC}_{\mathbf{G}}(X)$ . Take  $(g, x) \in \mathbf{G} \times_{\mathbf{B}} X$  and let  $W$  be an open subset of  $X$  containing  $g \cdot x$ . Take  $\mathbf{f} : X \rightarrow [0, 1]$  a continuous function such that  $\mathbf{f}(g \cdot x) = 0$  and  $\mathbf{f} \upharpoonright_{X \setminus W} = 1$ . Take  $\mathbf{f}' \in C_b(X)$  extending  $g^{-1}\mathbf{f}$ . We then have that  $g^{-1}\mathbf{f} \in \mathcal{O}(\mathbf{f}', 1/2, \mathbf{B})$ . Since  $\mathbf{f}$  is right uniformly continuous, there is  $U$  open neighborhood of  $g$  such that for any  $h \in U$ ,  $h^{-1}\mathbf{f} \in \mathcal{O}(\mathbf{f}', 1/2, \mathbf{B})$ . Since  $\mathbf{f}'(x) = \mathbf{f}(g \cdot x) = 0$ , there is  $V \subseteq X$  open set containing  $x$  such that  $|\mathbf{f}'(x')| < 1/2$  for every  $x' \in V$ . It follows that for  $(h, x') \in U \times_{\mathbf{B}} V$ ,

$$\mathbf{f}(h \cdot x') \leq |\mathbf{f}(h \cdot x') - \mathbf{f}'(x')| + |\mathbf{f}'(x')| \leq \|h^{-1}\mathbf{f} - \mathbf{f}'_{s_h}\| + |\mathbf{f}'(x')| < 1.$$

Therefore,  $h \cdot x' \in W$  which shows that the action  $\mathbf{G} \curvearrowright X$  is continuous. Now fix  $g \in \mathbf{G}$  and  $(s_g, f) \in \mathbb{C}_b(X)$ . Take  $\mathbf{f}_0 \in C_b(X)$  extending  $gf$  and suppose that  $(t_g, gf) \in \mathcal{O}(\mathbf{f}_0, \epsilon, U)$ . Let  $\mathbf{f} \in C_b(X)$  extend  $f$ , since  $\mathbf{f}$  is right uniformly continuous there is  $W$  neighborhood of  $g$  such that for every  $h \in W$ ,  $(t_h, h\mathbf{f}) \in \mathcal{O}(\mathbf{f}_0, \epsilon/2, U)$ . Therefore, if  $h \in W$  and  $(s_h, f') \in \mathcal{O}(\mathbf{f}, \epsilon/2, \mathbf{B})$  an easy computation shows that  $(t_h, hf') \in \mathcal{O}(\mathbf{f}_0, \epsilon, U)$ . This concludes the proof.  $\square$

From this lemma we conclude that the action  $\mathbf{G} \curvearrowright \mathbb{C}_b(X)$  is separately continuous if and only if it is continuous. In any case, it is necessary that the action  $\mathbf{G} \curvearrowright X$  is continuous. When  $X$  is compact we will see that this condition is also sufficient.

**Lemma 3.4.11.** *If  $(X, \rho)$  is a compact space over  $\mathbf{B}$  and  $\mathbf{G}$  is a groupoid acting continuously on  $X$  then  $C(X) = \text{RUC}_{\mathbf{G}}(X)$ .*

*Proof.* Fix  $\mathbf{f} \in C(X)$ ,  $g \in \mathbf{G}$ ,  $\epsilon > 0$  and  $\mathbf{f}_0 \in C(X)$  extending  $gf$ . For each  $x \in X$  there is  $U_x \subseteq X$  open with  $x \in U_x$  such that  $|\mathbf{f}(x') - \mathbf{f}(x)| < \epsilon/2$  and  $|\mathbf{f}_0(x') - \mathbf{f}_0(x)| < \epsilon/2$  for every  $x' \in U_x$ . By the continuity of the action, for every  $x \in X_{t_g}$  there are  $W_x \subseteq \mathbf{G}$  and  $V_x \subseteq X$  open neighborhoods of  $g^{-1}$  and  $x$  respectively, such that  $W_x \cdot V_x \subseteq U_{g^{-1} \cdot x}$ . We thus get that  $\{V_x \cap U_x ; x \in X_{t_g}\}$  is an open cover of  $X_{t_g}$ . By compactness, there are  $x_0, \dots, x_{n-1} \in X_{t_g}$  and there is  $A \subseteq \mathbf{B}$  open neighborhood of  $t_g$  such that

$$X_{t_g} \subseteq \rho^{-1}(A) \subseteq (V_{x_0} \cap U_{x_0}) \cup \dots \cup (V_{x_{n-1}} \cap U_{x_{n-1}}).$$

We set  $W = t^{-1}(A) \cap (W_{x_0}^0 \cap \dots \cap W_{x_{n-1}}^0)^{-1}$ , which is an open neighborhood of  $g$ . Pick  $h \in W$  and  $x \in X_{t_h}$ . It follows that  $\rho(x) = t_h \in A$  and so there exists  $i < n$  with  $x \in V_{x_i} \cap U_{x_i}$ . Since  $h^{-1} \in W_{x_i}$ , we get  $h^{-1} \cdot x \in U_{g^{-1} \cdot x_i}$ . Thus,

$$|\mathbf{f}(h^{-1} \cdot x) - \mathbf{f}_0(x)| \leq |\mathbf{f}(h^{-1} \cdot x) - \mathbf{f}(g^{-1} \cdot x_i)| + |\mathbf{f}_0(x_i) - \mathbf{f}_0(x)| < \epsilon. \quad \square$$

**Corollary 3.4.12.** *Let  $(X, \rho)$  be a compact space over  $\mathbf{B}$  and let  $\mathbf{G}$  a groupoid acting on  $X$ . Then,  $\mathbf{G} \curvearrowright X$  is continuous if and only if  $\mathbb{C}(X) = \text{RUC}_{\mathbf{G}}(X)$ . Any of these two conditions imply that  $\mathbf{G} \curvearrowright \mathbb{C}(X)$  is continuous.*

### 3.4.3 Dualities

Throughout this section we will assume that  $(X, \rho)$  and  $(Y, \eta)$  are spaces over  $\mathbf{B}$  admitting sections.

**Definition 3.4.13.** A function  $\langle \cdot, \cdot \rangle : Y \times_{\mathbf{B}} X \rightarrow \mathbb{R}$  is said to be *left continuous* if for every  $y \in Y$ , the map  $q_Y(y) : X_{\eta(y)} \rightarrow \mathbb{R}$  given by  $x \mapsto \langle y, x \rangle$  belongs to  $C_b(X_{\rho(x)})$ , and defines a continuous map  $q_Y : Y \rightarrow \mathbb{C}_p(X)$ . *Right continuity* is defined in a similar fashion. The map is said to be a *duality* if it is both left and right continuous. If furthermore  $q_Y : Y \rightarrow \mathbb{C}_p(X)$  is a topological embedding we say that the duality is *left strict*. *Right strict* is defined in an analogous manner. The duality is *strict* if it is both left and right strict. In case  $\mathbf{G}$  is a groupoid acting on  $X$  and  $Y$ , we say that  $\langle \cdot, \cdot \rangle$  is a  $\mathbf{G}$ -*duality* if  $\langle y, g \cdot x \rangle = \langle g^{-1} \cdot y, x \rangle$  for every  $x \in X$ ,  $y \in Y$  and  $g \in \eta(y)\mathbf{G}\rho(x)$ .

Whenever  $\mathbf{x} \in \Gamma(X)$  and  $y \in Y$ , we let  $\langle y, \mathbf{x} \rangle = \langle y, \mathbf{x}(\eta(y)) \rangle$ . In a similar fashion, for  $\mathbf{y} \in \Gamma(Y)$  and  $x \in X$ ,  $\langle \mathbf{y}, x \rangle = \langle \mathbf{y}(\rho(x)), x \rangle$ .

**Lemma 3.4.14.** *Let  $(X, \rho)$  and  $(Y, \eta)$  be spaces over  $\mathbf{B}$  admitting sections on which  $\mathbf{G}$  acts (not necessarily continuously). Let  $\langle \cdot, \cdot \rangle : Y \times_{\mathbf{B}} X \rightarrow \mathbb{R}$  be a duality. Then,*

1. *The duality is left strict if and only if a net  $(y_i)_{i \in I}$  converges to  $y \in Y$  precisely when  $(\eta(y_i))_{i \in I}$  converges to  $\eta(y)$ , and  $(\langle y_i, \mathbf{x} \rangle)_{i \in I}$  converges to  $\langle y, \mathbf{x} \rangle$  for every  $\mathbf{x} \in \Gamma(X)$ .*
2. *The duality is right strict if and only if a net  $(x_i)_{i \in I}$  converges to  $x \in X$  precisely when  $(\rho(x_i))_{i \in I}$  converges to  $\rho(x)$ , and  $(\langle \mathbf{y}, x_i \rangle)_{i \in I}$  converges to  $\langle \mathbf{y}, x \rangle$  for every  $\mathbf{y} \in \Gamma(Y)$ .*
3.  *$\langle \cdot, \cdot \rangle$  is a  $\mathbf{G}$ -duality if and only if  $q_X$  is  $\mathbf{G}$ -equivariant, which in its turn is equivalent to  $q_Y$  being  $\mathbf{G}$ -equivariant.*
4. *If  $\langle \cdot, \cdot \rangle : Y \times_{\mathbf{B}} X \rightarrow \mathbb{R}$  is a right (resp. left) strict  $\mathbf{G}$ -duality and the action  $\mathbf{G} \curvearrowright Y$  (resp.  $\mathbf{G} \curvearrowright X$ ) is separately continuous then  $\mathbf{G} \curvearrowright X$  (resp.  $\mathbf{G} \curvearrowright Y$ ) is separately continuous.*
5. *If  $\langle \cdot, \cdot \rangle : Y \times_{\mathbf{B}} X \rightarrow \mathbb{R}$  is a strict  $\mathbf{G}$ -duality then, the action  $\mathbf{G} \curvearrowright Y$  is separately continuous if and only if  $\mathbf{G} \curvearrowright X$  (resp.  $\mathbf{G} \curvearrowright Y$ ) is separately continuous.*

*Proof.* 1. The duality is left strict if and only if a net  $(y_i)_{i \in I}$  converges to  $y \in Y$  precisely when  $(q_Y(y_i))_{i \in I}$  converges in the pointwise topology to  $q_Y(y)$ . By definition, the latter occurs when  $(\eta(y_i))_{i \in I}$  converges to  $\eta(y)$ , and  $(\langle y_i, \mathbf{x} \rangle)_{i \in I}$  converges to  $\langle y, \mathbf{x} \rangle$  for every  $\mathbf{x} \in \Gamma(X)$ .

2. Analogous to 1.

3. Suppose that we have a  $\mathbf{G}$ -duality, then for every  $x \in X$ ,  $y \in Y$  and  $g \in \eta(y)\mathbf{G}\rho(x)$  we have

$$q_X(g \cdot x)(y) = \langle y, g \cdot x \rangle = \langle g^{-1} \cdot y, x \rangle = q_X(x)(g^{-1} \cdot y) = gq_X(x)(y).$$

Thus,  $q_X$  is  $\mathbf{G}$ -equivariant. A similar argument shows that the converse also holds. The equivalence involving  $q_Y$  is proved in the same manner.

4. We deal with the case of a right strict  $\mathbf{G}$ -duality, the other one being analogous. Let us denote  $\tilde{\cdot} : \mathbf{G} \rightarrow \mathcal{H}_p(Y)$  the corresponding groupoid homomorphism. Fix  $g \in \mathbf{G}$  and let  $\hat{g} : X_{s_g} \rightarrow X_{t_g}$  be the orbit map. Take a net  $(x_i)_{i \in I}$  in  $X_{s_g}$  converging to  $x \in X_{s_g}$ . By right continuity of the duality, for every  $y \in Y_{s_g}$ ,  $\langle y, x_i \rangle \rightarrow \langle y, x \rangle$ . Then, for every  $y \in Y_{t_g}$ ,  $\langle g^{-1} \cdot y, x_i \rangle \rightarrow \langle g^{-1} \cdot y, x \rangle$ . Using the fact that we have a  $\mathbf{G}$ -duality we conclude that  $(\langle y, g \cdot x_i \rangle)_{i \in I}$  converges to  $\langle y, g \cdot x \rangle$  for every  $y \in Y_{t_g}$ . This means precisely that  $(q_X(g \cdot x_i))_{i \in I}$  converges pointwise to  $q_X(g \cdot x)$ . Since the duality is right strict, we conclude that  $(g \cdot x_i)_{i \in I}$  converges to  $g \cdot x$ . This shows that  $\hat{g}$  is continuous and thus that  $\hat{g} \in \mathcal{H}_p(X)$ . Let  $\hat{\cdot} : \mathbf{G} \rightarrow \mathcal{H}_p(X)$  be the groupoid homomorphism. Take  $(g_i)_{i \in I} \subseteq \mathbf{G}$  converging to  $g$ . In particular we have that  $s_{g_i} \rightarrow s_g$  and  $t_{g_i} \rightarrow t_g$ . Take  $\mathbf{x} \in \Gamma(X)$  and  $\mathbf{y} \in \Gamma(Y)$ , since  $\widetilde{g_i^{-1}} \rightarrow \widetilde{g^{-1}}$ , then  $(\widetilde{g_i^{-1}}(\mathbf{y}))_{i \in I}$  converges in  $Y$  to  $\widetilde{g^{-1}}(\mathbf{y})$ . By left continuity of the duality,  $q_Y(\widetilde{g_i^{-1}}(\mathbf{y}))$  converges pointwise to  $q_Y(\widetilde{g^{-1}}(\mathbf{y}))$ . Therefore,

$$\langle \mathbf{y}, \hat{g}_i(\mathbf{x}) \rangle = \langle \widetilde{g_i^{-1}}(\mathbf{y}), \mathbf{x} \rangle = q_Y(\widetilde{g_i^{-1}}(\mathbf{y}))(\mathbf{x}) \rightarrow q_Y(\widetilde{g^{-1}}(\mathbf{y}))(\mathbf{x}) = \langle \widetilde{g^{-1}}(\mathbf{y}), \mathbf{x} \rangle = \langle \mathbf{y}, \hat{g}(\mathbf{x}) \rangle.$$

By 2. we get that  $(\hat{g}_i(\mathbf{x}))_{i \in I}$  converges to  $\hat{g}(\mathbf{x})$ . This shows that  $\hat{\cdot} : \mathbf{G} \rightarrow \mathcal{H}_p(X)$  is continuous, which concludes the proof.

5. Follows directly from 4. □

- Example 3.4.15.** 1. Suppose that  $(X, \rho)$  is a compact, metrizable space over a compact, totally disconnected space  $\mathbf{B}$ . Define  $\langle \cdot, \cdot \rangle : \mathbb{C}_p(X) \times_{\mathbf{B}} X \rightarrow \mathbb{R}$  by  $\langle f, x \rangle = f(x)$ . It is easy to see that this map is a left strict duality. Recall from [Proposition 3.2.7](#) that  $\Gamma(\mathbb{C}_p(X)) = C(X)$  and so  $q_X : X \rightarrow \mathbb{C}_p(\mathbb{C}_p(X))$  is continuous. Moreover, due to the fact that continuous functions separate points in  $X$ , we get that  $q_X$  is injective and thus an embedding. Hence,  $\langle \cdot, \cdot \rangle : \mathbb{C}_p(X) \times_{\mathbf{B}} X \rightarrow \mathbb{R}$  is a strict duality. Furthermore, in case  $\mathbf{G}$  acts in a separately continuous manner on  $X$  we have in fact a strict  $\mathbf{G}$ -duality.
2. Suppose that  $\rho : X \rightarrow \mathbf{B}$  is closed and let us consider the norm topology on  $\mathbb{C}_b(X)$ . In this case we also have that  $\Gamma(\mathbb{C}_b(X)) = C_b(X)$  and that continuous functions separate points. Hence, the duality  $\langle \cdot, \cdot \rangle : \mathbb{C}_b(X) \times_{\mathbf{B}} X \rightarrow \mathbb{R}$  is right strict. If  $\mathbf{G}$  is a topological groupoid and there is a separately continuous action  $\mathbf{G} \curvearrowright X$  then we have in fact a  $\mathbf{G}$ -duality.
3. Let  $(\mathbb{V}, \Gamma)$  be a separable, continuous field of Banach spaces over  $\mathbf{B}$ . It is immediate from the definitions that the natural map  $\langle \cdot, \cdot \rangle : \mathbb{V} \times_{\mathbf{B}} \mathbb{V}^* \rightarrow \mathbb{R}$  defines a right strict duality, which in general is not left strict.
4. Consider at last a separable, continuous field of Banach spaces over  $\mathbf{B}$ ,  $(\mathbb{V}, \Gamma)$ , endowed with the weak topology. It follows directly from the definition of the weak topology that the natural map  $\langle \cdot, \cdot \rangle : \mathbb{V}_w \times_{\mathbf{B}} \mathbb{V}^* \rightarrow \mathbb{R}$  is left continuous. In fact, the map  $q_{\mathbb{V}_w}$  is an embedding. However, if  $\mathbf{u} \in \Gamma(\mathbb{V}_w)$ , there is no reason for the map  $(e, \varphi) \mapsto \langle \mathbf{u}, \varphi \rangle$  to be continuous. This means that the map  $q_{\mathbb{V}^*} : \mathbb{V}^* \rightarrow \mathbb{C}_p(\mathbb{V}_w)$  is not necessarily continuous. Now, we know that  $\Gamma \subseteq \Gamma(\mathbb{V}_w)$ . If  $\Gamma = \Gamma(\mathbb{V}_w)$  we then have that  $q_{\mathbb{V}^*} : \mathbb{V}^* \rightarrow \mathbb{C}_p(\mathbb{V}_w)$  is continuous, and even more, it is an embedding. We thus get a strict duality.

In the case of group actions, bounded strict  $G$ -dualities on compact spaces are restrictions of the canonical reflexive duality, this is  $rB(V)_w \times B(V^*) \rightarrow \mathbb{R}$ , where  $r > 0$ ,  $V$  is a reflexive space on which  $G$ -acts by isometries [[Meg03](#), Theorem 4.5]. In the case of  $\mathbf{G}$ -dualities the picture is more complicated as the previous example shows. This is further illustrated with the following result. It essentially says that under suitable hypothesis left strict  $\mathbf{G}$ -dualities can be reduced to strict  $\mathbf{G}$ -dualities, compare with numeral (4) of [[Meg03](#), Lemma 4.2].

**Lemma 3.4.16.** *Let  $(X, \rho)$  and  $(Y, \eta)$  be spaces over  $\mathbf{B}$  admitting sections. Assume moreover that  $Y$  is completely metrizable,  $\mathbf{B}$  is totally disconnected and  $\mathbf{G}$  is a groupoid acting on  $X$  and  $Y$ . Let  $\langle \cdot, \cdot \rangle : Y \times_{\mathbf{B}} X \rightarrow \mathbb{R}$  be a left strict  $\mathbf{G}$ -duality. Then it can be reduced to a strict  $\mathbf{G}$ -duality  $\langle \cdot, \cdot \rangle_q : Y \times_{\mathbf{B}} X_q \rightarrow \mathbb{R}$  where  $X_q = q_X(X) \subseteq \mathbb{C}_p(y)$ . If  $\mathbf{G} \curvearrowright Y$  is separately continuous then  $\mathbf{G} \curvearrowright X_q$  is also separately continuous.*

*Proof.*  $X_q$  as a subspace of  $\mathbb{C}_p(Y)$  is a space over  $\mathbf{B}$ . Furthermore, each section of  $\mathbf{x}$  defines a corresponding section in  $X_q$ . So,  $X_q$  admits sections. We define  $\langle \cdot, \cdot \rangle_q : Y \times_{\mathbf{B}} X_q \rightarrow \mathbb{R}$  by  $\langle y, z \rangle_q = z(y) = \langle y, x \rangle$  where  $x \in X$  is any element satisfying  $q_X(x) = z$ . In this way, the map  $q_{X_q} : X_q \rightarrow \mathbb{C}_p(Y)$  is just the inclusion map which is a topological embedding. For  $y \in Y$ , the map  $q'_Y(y) : z \mapsto \langle y, z \rangle_q$  is just the evaluation map on  $y$ , which belongs to  $\mathbb{C}_p(X_q)$ . By 1. in [Lemma 3.4.14](#), in order to see that the duality  $\langle \cdot, \cdot \rangle_q$  is left strict, we must show that a net  $(y_i)_{i \in I}$  in  $Y$  converges to  $y$  if and only if  $(\eta(y_i))_{i \in I}$  converges to  $\eta(y)$  and for every  $\mathbf{f} \in \Gamma(X_q)$ ,  $(\mathbf{f}(y_i))_{i \in I}$  converges to  $\mathbf{f}(y)$ . From left to right, it is clear that  $(\eta(y_i))_{i \in I}$  converges to  $\eta(y)$ . Fix  $\mathbf{f} \in \Gamma(X_q)$ , then  $\mathbf{f} \in \Gamma(\mathbb{C}_p(Y))$ . This implies, using [Proposition 3.2.7](#), that  $\mathbf{f} \in C(Y)$ . Thus,  $(\mathbf{f}(y_i))_{i \in I}$  converges to  $\mathbf{f}(y)$ . For the converse, take  $\mathbf{x} \in \Gamma(X)$ , by the right continuity of the original duality, the map  $y' \mapsto \langle y', \mathbf{x} \rangle$  is continuous, and it is moreover a section of  $X_q$ . Hence, by hypothesis,  $\langle y_i, \mathbf{x} \rangle \rightarrow \langle y, \mathbf{x} \rangle$ . Again 1. in [Lemma 3.4.14](#) allows us to conclude that  $(y_i)_{i \in I}$  converges to  $y$ . Therefore,  $\langle \cdot, \cdot \rangle_q$  is strict. An easy verification shows that it is a  $\mathbf{G}$ -duality. The last part follows directly from 4. in [Lemma 3.4.14](#).  $\square$

### 3.5 Representations of groupoid actions

Given two Banach spaces over the same field,  $V$  and  $W$ , we denote by  $\text{Iso}(V, W)$  the space of linear surjective isometries from  $V$  onto  $W$ .  $\text{Iso}(V, W)$  admits several natural topologies, but we will consider it with the topology of pointwise convergence.

Given a field of Banach spaces over  $\mathbf{B}$ ,  $(\mathbb{V}, \Gamma)$ , let

$$\text{Iso}(\mathbb{V}) = \bigsqcup_{e, e' \in \mathbf{B}} \text{Iso}(V_e, V_{e'}).$$

We will write  $(e', T, e) \in \text{Iso}(\mathbb{V})$  to indicate  $T \in \text{Iso}(V_e, V_{e'})$ .  $\text{Iso}(\mathbb{V})$  admits a natural groupoid structure over  $\mathbf{B}$ . Indeed, for  $(e'', S, e')$ ,  $(e', T, e) \in \text{Iso}(\mathbb{V})$  set

$$(e'', S, e')(e', T, e) = (e'', ST, e) \text{ and } (e', T, e)^{-1} = (e, T^{-1}, e').$$

For  $\mathbf{v}, \mathbf{w} \in \Gamma$ ,  $U, U' \subseteq \mathbf{B}$  and  $\epsilon > 0$  define

$$\mathcal{O}(\mathbf{v}, \mathbf{w}, U', U, \epsilon) = \{(e', T, e) \in \text{Iso}(\mathbb{V}) ; e \in U, e' \in U', \text{ and } \|T(\mathbf{v}(e)) - \mathbf{w}(e')\| < \epsilon\}.$$

We will endow  $\text{Iso}(\mathbb{V})$  with the topology having the family of sets of this form, where  $U$  and  $U'$  are open, as a subbase.

**Proposition 3.5.1.** *Let  $(\mathbb{V}, \Gamma)$  be a field of Banach spaces over  $\mathbf{B}$ . Then, the groupoid operations on  $\text{Iso}(\mathbb{V})$  are continuous. If the field is lower semi-continuous then the topology is Hausdorff.*

*Proof.* First we verify that the groupoid operations are continuous. Pick  $(e', T, e)$  and  $(e'', S, e')$  in  $\text{Iso}(\mathbb{V})$ , and suppose that  $(e'', ST, e) \in \mathcal{O}(\mathbf{v}, \mathbf{w}, U'', U, \epsilon)$ . Pick  $r, s > 0$  satisfying  $\|ST(\mathbf{v}(e)) - \mathbf{w}(e'')\| < r < r + 2s < \epsilon$ . Take  $\mathbf{u} \in \Gamma$  with  $\|\mathbf{u}(e') - T\mathbf{v}(e)\| < s$ . Define  $\mathcal{U}_1 = \mathcal{O}(\mathbf{v}, \mathbf{u}, \mathbf{B}, U, s)$  and  $\mathcal{U}_2 = \mathcal{O}(\mathbf{u}, \mathbf{w}, U'', \mathbf{B}, r + s)$ . Clearly  $(e', T, e) \in \mathcal{U}_1$ . We also have

$$\begin{aligned} \|ST\mathbf{v}(e) - \mathbf{w}(e'')\| &\leq \|ST\mathbf{v}(e) - S\mathbf{u}(e')\| + \|S\mathbf{u}(e') - \mathbf{w}(e'')\| \\ &< \|T\mathbf{v}(e) - \mathbf{u}(e')\| + r \\ &< r + s. \end{aligned}$$

So,  $(e'', S, e') \in \mathcal{U}_2$ . Now let  $(e'_0, T_0, e_0) \in \mathcal{U}_1$  and  $(e''_0, S_0, e'_0) \in \mathcal{U}_2$ . Then,  $e_0 \in U$ ,  $e''_0 \in U''$  and by repeating the previous computation we get  $\|S_0T_0\mathbf{v}(e_0) - \mathbf{w}(e''_0)\| < r + 2s < \epsilon$ . This shows that the product is continuous. Continuity of the inverse map follows from the fact that  $\mathcal{O}(\mathbf{v}, \mathbf{w}, U', U, \epsilon)^{-1} = \mathcal{O}(\mathbf{w}, \mathbf{v}, U, U', \epsilon)$ .

We will now assume that the field is lower semi-continuous and we will see that the topology defined above is Hausdorff. Take  $(e', T, e)$  and  $(e'_0, S, e_0)$  two different elements in  $\text{Iso}(\mathbb{V})$ . If  $e \neq e_0$  then there are  $U$  and  $U_0$  open subsets of  $\mathbf{B}$  separating  $e$  and  $e_0$ . It follows that  $\mathcal{O}(\mathbf{0}, \mathbf{0}, \mathbf{B}, U, 1)$  and  $\mathcal{O}(\mathbf{0}, \mathbf{0}, \mathbf{B}, U_0, 1)$  separate  $(e', T, e)$  and  $(e'_0, S, e_0)$ . The case  $e' \neq e'_0$  is treated similarly. Now assume that  $e = e_0$  and  $e' = e'_0$  and take  $v \in V_e$  such that  $T(v) \neq S(v)$ . Let  $r = \|T(v) - S(v)\|$  and pick  $\mathbf{v}, \mathbf{w}, \mathbf{w}_0 \in \Gamma$  passing through  $v$ ,  $T(v)$  and  $S(v)$  respectively. By lower semi-continuity, there is  $U'$  open neighborhood of  $e'$  such that  $\|\mathbf{w} - \mathbf{w}_0\| > r/2$  on  $U'$ . Then,  $\mathcal{O}(\mathbf{v}, \mathbf{w}, U', \mathbf{B}, r/4)$  and  $\mathcal{O}(\mathbf{v}, \mathbf{w}_0, U', \mathbf{B}, r/4)$  separate  $(e', T, e)$  and  $(e'_0, S, e_0)$ .  $\square$

In case  $(\mathbb{V}, \Gamma)$  is a continuous field,  $\text{Iso}(\mathbb{V})$  is a subspace of  $\mathcal{H}_p(\mathbb{V})$ . Indeed, let  $\mathbf{v}, \mathbf{w} \in \Gamma$ ,  $U, U' \subseteq \mathbf{B}$  open,  $\epsilon > 0$  and consider the subbasic open  $\mathcal{O}(\mathbf{v}, \mathbf{w}, U', U, \epsilon)$ . Define  $W = \mathcal{O}(\mathbf{w}, \epsilon, U')$ , an open subset of  $\mathbb{V}$ . We then have that  $\mathcal{O}(\mathbf{v}, \mathbf{w}, U', U, \epsilon) = \mathcal{O}(\mathbf{v}, W, U', U) \cap \text{Iso}(\mathbb{V})$ , where the open set

in the right hand side is defined as per [Proposition 3.4.2](#). Therefore, the topology we have just defined on  $\text{Iso}(\mathbb{V})$  agrees with the induced topology from  $\mathcal{H}_p(\mathbb{V})$ .

Let  $\mathbf{G}$  be a topological groupoid over  $\mathbf{B}$  and  $(\mathbb{V}, \Gamma)$  a field over  $\mathbf{B}$ . An action by linear isometries of  $\mathbf{G}$  on  $\mathbb{V}$  is a homomorphism of  $\mathbf{G}$  into  $\text{Iso}(\mathbb{V})$ . If said homomorphism is continuous we say that the action is separately continuous. In the case of a continuous field, by the previous discussion, this notion agrees with the one given in [Definition 3.4.6](#). The next proposition shows that for continuous fields separate continuity is equivalent to continuity.

**Proposition 3.5.2.** *Let  $\mathbf{G}$  be a topological groupoid acting by linear isometries on  $(\mathbb{V}, \Gamma)$ , a continuous field of Banach spaces over  $\mathbf{B}$ . Then, the groupoid homomorphism  $\tilde{\cdot} : \mathbf{G} \rightarrow \text{Iso}(\mathbb{V})$  is continuous if and only if the corresponding action  $\mathbf{G} \curvearrowright \mathbb{V}$  is continuous.*

*Proof.* Suppose first that  $\tilde{\cdot} : \mathbf{G} \rightarrow \text{Iso}(\mathbb{V})$  is continuous. Fix  $g \in \mathbf{G}$ ,  $v \in V_{s_g}$  and let  $g \cdot v \in \mathcal{O}(\mathbf{w}, \epsilon, U)$  where  $\mathbf{w} \in \Gamma$  passes through  $g \cdot v$ ,  $\epsilon > 0$  and  $U \subseteq \mathbf{B}$  is an open neighborhood of  $t_g$ . Pick  $\mathbf{v} \in \Gamma$  passing through  $v$ , then  $\tilde{g} \in \mathcal{O}(\mathbf{v}, \mathbf{w}, \mathbf{B}, U, \epsilon/2)$ . By the continuity of the homomorphism, there is  $W \subseteq \mathbf{G}$  with  $g \in W$  such that  $\tilde{W} \subseteq \mathcal{O}(\mathbf{v}, \mathbf{w}, U, \mathbf{B}, \epsilon/2)$ . We claim that  $W \cdot \mathcal{O}(\mathbf{w}, \epsilon/2, \mathbf{B}) \subseteq \mathcal{O}(\mathbf{w}, \epsilon, U)$ . Indeed, for  $h \in W$  and  $u \in \mathcal{O}(\mathbf{v}, \mathbf{B}, \epsilon/2)$  verifying  $u \in V_{s_h}$ , we have  $h \cdot u \in V_{t_h}$  with  $t_h \in U$ , and

$$\|\mathbf{w}(t_h) - h \cdot u\| \leq \|\mathbf{w}(t_h) - h \cdot \mathbf{v}(s_h)\| + \|h \cdot \mathbf{v}(s_h) - h \cdot u\| < \epsilon/2 + \|\mathbf{v}(s_h) - u\| < \epsilon.$$

Now suppose that the action  $\mathbf{G} \curvearrowright \mathbb{V}$  is continuous. This implies in particular that the action is separately continuous. By definition, this means that  $\tilde{\cdot} : \mathbf{G} \rightarrow \text{Iso}(\mathbb{V})$  is continuous.  $\square$

Recall from [Subsection 3.4.1](#) that a separately continuous action  $\mathbf{G} \curvearrowright X$  induces an action  $\mathbf{G} \curvearrowright \mathbb{C}_b(X)$ . This action is given by linear isometries. Indeed, for  $(e, f) \in \mathbb{C}_b(X)$  and  $g \in \mathbf{G}e$ ,

$$\|gf\| = \sup_{x \in X_{t_g}} |gf(x)| = \sup_{x \in X_{t_g}} |f(g^{-1} \cdot x)| = \sup_{x \in X_e} |f(x)| = \|f\|.$$

In this way, when we apply the previous lemma on  $\mathbb{C}_b(X)$  we recover [Lemma 3.4.10](#).

Let  $\mathbf{G}$  be a groupoid acting by linear isometries on a continuous field,  $(\mathbb{V}, \Gamma)$ , over  $\mathbf{B}$ . This action induces a dual action on  $\mathbb{V}^*$ . Indeed, for  $(e, \varphi) \in \mathbb{V}^*$  and  $g \in \mathbf{G}e$  define  $g\varphi : V_{t_g} \rightarrow \mathbb{R}$  by  $\langle v, g\varphi \rangle = \langle g^{-1} \cdot v, \varphi \rangle$  for  $v \in V_{t_g}$ . It is easy to verify that  $g\varphi \in V_{t_g}^*$  and that  $\|g\varphi\| = \|\varphi\|$ . In particular, the dual action restricts to the unit ball, i.e.  $\mathbf{G} \curvearrowright B(\mathbb{V}^*)$ .

**Lemma 3.5.3.** *Let  $\mathbf{G}$  be a topological groupoid of base  $\mathbf{B}$  and let  $(\mathbb{V}, \Gamma)$  be a field of Banach spaces over  $\mathbf{B}$ . If an action by linear isometries  $\mathbf{G} \curvearrowright \mathbb{V}$  is separately continuous then the dual action  $\mathbf{G} \curvearrowright B(\mathbb{V}^*)$  is weak\* continuous.*

*Proof.* Pick  $(e, \varphi) \in B(\mathbb{V}^*)$  and  $g \in \mathbf{G}e$  with  $(t_g, g\varphi) \in \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U)$  where each  $\mathbf{v}_j \in \Gamma$ ,  $I_j \subseteq \mathbb{R}$  is open and  $U \subseteq \mathbf{B}$  is also open. We can assume that there is  $r > 0$  such that  $I_j = B(\langle \mathbf{v}_j, g\varphi \rangle, r)$  for every  $j < n$ . For each  $j < n$  take  $\mathbf{w}_j \in \Gamma$  such that  $\|g^{-1} \cdot \mathbf{v}_j(t_g) - \mathbf{w}_j(s_g)\| < r/2$ . Since the action  $\mathbf{G} \curvearrowright \mathbb{V}$  is separately continuous, for each  $j < n$  there is an open neighborhood  $W_j$  of  $g$  such that for every  $h \in W_j$  we have  $\tilde{h}^{-1} \in \mathcal{O}(\mathbf{v}_j, \mathbf{w}_j, \mathbf{B}, U, r/2)$ . Let  $W = W_1 \cap \dots \cap W_n$ , for  $j < n$  define  $I'_j = B(\langle \mathbf{w}_j, \varphi \rangle, r/2)$ . We assert that  $W \cdot \mathcal{O}^*((\mathbf{w}_j)_{j < n}, (I'_j)_{j < n}, \mathbf{B}) \subseteq \mathcal{O}^*((\mathbf{v}_j)_{j < n}, (I_j)_{j < n}, U)$ . Indeed, take  $(e', \psi) \in \mathcal{O}^*((\mathbf{w}_j)_{j < n}, (I'_j)_{j < n}, \mathbf{B})$  and  $h \in We'$ . Then,  $t_h \in U$  and for  $j < n$  we have  $h \in W_j$  and

$$\begin{aligned}
|\langle \mathbf{v}_j, h\psi \rangle - \langle \mathbf{v}_j, g\varphi \rangle| &\leq |\langle h^{-1} \cdot \mathbf{v}_j, \psi \rangle - \langle \mathbf{w}_j, \psi \rangle| + |\langle \mathbf{w}_j, \psi \rangle - \langle \mathbf{v}_j, g \cdot \varphi \rangle| \\
&< \|h^{-1}\mathbf{v}_j - \mathbf{w}_j(s_h)\| + |\langle \mathbf{w}_j, \psi \rangle - \langle \mathbf{w}_j, \varphi \rangle| \\
&< r.
\end{aligned}$$

This concludes the proof.  $\square$

So far we have only considered the norm topology on  $\mathbb{V}$ . We will now suppose that  $(\mathbb{V}, \Gamma)$  is separable, so  $\mathbb{V}^*$  admits sections, and we will consider the weak topology on  $\mathbb{V}$ . The next result shows that separate weak continuity of the action implies separate weak\* continuity of the dual action.

**Proposition 3.5.4.** *Let  $\mathbf{G}$  be a topological groupoid acting by linear isometries on a separable continuous field of Banach spaces  $(\mathbb{V}, \Gamma)$  over  $\mathbf{B}$ . If  $\mathbf{G} \curvearrowright \mathbb{V}$  is separately weakly continuous then  $\mathbf{G} \curvearrowright \mathbb{V}^*$  is separately weakly\* continuous. If  $\Gamma = \Gamma(\mathbb{V}_w)$  then the converse also holds.*

*Proof.* Suppose that  $\mathbf{G} \curvearrowright \mathbb{V}$  is separately weakly continuous and let us denote  $\tilde{\cdot} : \mathbf{G} \rightarrow \mathcal{H}_p(\mathbb{V}_w)$  the corresponding groupoid homomorphism. Fix  $g \in \mathbf{G}$ , we denote by  $\hat{g} : V_{s_g}^* \rightarrow V_{t_g}^*$  the orbit map. Since  $\hat{g}$  is the dual map of the linear isometry  $\widetilde{g^{-1}}$ , it is continuous. So,  $\hat{\cdot} : \mathbf{G} \rightarrow \mathcal{H}_p(\mathbb{V}^*)$  defines a groupoid homomorphism. Let  $(g_i)_{i \in I} \subseteq \mathbf{G}$  converge to  $g$ . Then,  $s_{g_i} \rightarrow s_g$  and  $t_{g_i} \rightarrow t_g$ . Take  $\nu \in \Gamma^*$  and  $\mathbf{v} \in \Gamma$ , since  $(\widetilde{g_i^{-1}})_{i \in I}$  converges in  $\mathcal{H}_p(\mathbb{V}_w)$  to  $\widetilde{g^{-1}}$ , then  $(\widetilde{g_i^{-1}}(\mathbf{v}))_{i \in I}$  converges weakly to  $\widetilde{g^{-1}}(\mathbf{v})$ . Hence,

$$\langle \mathbf{v}, \hat{g}_i(\nu) \rangle = \langle \widetilde{g_i^{-1}}(\mathbf{v}), \nu \rangle \rightarrow \langle \widetilde{g^{-1}}(\mathbf{v}), \nu \rangle = \langle \mathbf{v}, \hat{g}(\nu) \rangle.$$

Thus,  $(\hat{g}_i(\nu))_{i \in I}$  converges weakly\* to  $\hat{g}(\nu)$ , which shows that  $\mathbf{G} \curvearrowright \mathbb{V}^*$  is separately weakly\* continuous. Assume now that  $\Gamma = \Gamma(\mathbb{V}_w)$  and that  $\mathbf{G} \curvearrowright \mathbb{V}^*$  is separately weakly\* continuous. By 4. of [Example 3.4.15](#),  $\langle \cdot, \cdot \rangle : \mathbb{V}_w \times_{\mathbf{B}} \mathbb{V}^* \rightarrow \mathbb{R}$  is a strict duality. The result follows from 5. in [Lemma 3.4.14](#).  $\square$

**Proposition 3.5.5.** *Let  $(X, \rho)$  be a completely metrizable space over a compact, totally disconnected space  $\mathbf{B}$ . Let  $\mathbf{G}$  be a Polish open groupoid of base  $\mathbf{B}$  acting separately continuously on  $\mathbf{X}$ . Then,  $\mathbf{G} \curvearrowright \mathbb{C}(X)$  separately weakly continuous and  $\mathbf{G} \curvearrowright \mathbb{C}(X)^*$  is separately weakly\* continuous.*

*Proof.* Throughout this proof we will consider  $\mathbf{G}$  as a space over  $\mathbf{B}$  via the source map. By [Fact 3.1.16](#),  $\mathbf{G}$  admits  $s$ -sections. Pick  $g \in \mathbf{G}$ , since  $\widetilde{g^{-1}} : C_b(X_{t_g}) \rightarrow C_b(X_{s_g})$  is a linear surjective isometry, we get an induced weak\* homeomorphism  $\widetilde{g^*} : C_b(X_{s_g})^* \rightarrow C_b(X_{t_g})^*$ . Now fix  $e \in \mathbf{G}$  and take a sequence  $(g_n)_{n \in \mathbb{N}} \subseteq \mathbf{G}e$  converging to  $g \in \mathbf{G}e$ . Let  $\nu \in \Gamma(\mathbb{C}(X)^*)$  and let  $\mathbf{f} \in C_b(X)$ . By the continuity of the action,  $\widetilde{g_n^{-1}}\mathbf{f}$  converges pointwise in  $C_b(X_e)$  to  $\widetilde{g^{-1}}\mathbf{f}$ , and since  $|\widetilde{g_n^{-1}}\mathbf{f}(x)| \leq \|\mathbf{f}\|$  for every  $x \in X_e$ , from Lebesgue dominated convergence theorem we get that  $\langle \widetilde{g_n^{-1}}\mathbf{f}, \nu(e) \rangle \rightarrow \langle \widetilde{g^{-1}}\mathbf{f}, \nu(e) \rangle$ . This means that the map  $\tilde{\cdot} : \mathbf{G} \rightarrow \mathcal{H}_p(\mathbb{C}_b(X)^*)$  is continuous on each fiber. Take  $\mathbf{g} \in \Gamma_s(\mathbf{G})$ , and let  $\nu \in \Gamma(\mathbb{C}_b(X)^*)$  and  $\mathbf{f} \in C_b(X)$  be arbitrary. It follows that  $\mathbf{g}^{-1}$  is a  $t$ -section and so  $\mathbf{g}^{-1}\mathbf{f} \in C_b(X)$ . From [Proposition 3.1.19](#), we deduce that the function  $e \mapsto \langle (\mathbf{g}^{-1}\mathbf{f})_e, \nu(e) \rangle = \langle \mathbf{f}_{t_{\mathbf{g}(e)}}, \mathbf{g}(e)\nu(e) \rangle$  is continuous. Since  $\mathbf{f}$  was arbitrary, then the mapping  $e \mapsto \mathbf{g}(e)\nu(e)$  is weakly\* continuous. As this is true for any  $\nu \in \Gamma(\mathbb{C}_b(X)^*)$ , we conclude that  $e \mapsto \widetilde{\mathbf{g}(e)^*}$  is continuous. From [Lemma 3.2.5](#) we deduce that  $\tilde{\cdot} : \mathbf{G} \rightarrow \mathcal{H}_p(\mathbb{C}_b(X)^*)$  is continuous. The separate weak continuity of the action  $\mathbf{G} \curvearrowright \mathbb{C}_b(X)$  is a direct consequence of [Corollary 3.2.8](#) and [Proposition 3.5.4](#).  $\square$

If  $(X, \rho)$  and  $(Y, \eta)$  are spaces over  $\mathbf{B}$ ,  $f : X \rightarrow Y$  is said to be a *function over  $\mathbf{B}$*  if it is continuous and  $\eta \circ f = \rho$ .

**Definition 3.5.6.** Suppose  $\mathbf{G}$  is a topological groupoid of base  $\mathbf{B}$  and  $(\mathbb{V}, \Gamma)$  is a field of Banach spaces over  $\mathbf{B}$ . If  $\mathbf{G}$  acts on  $(X, \rho)$ , a space over  $\mathbf{B}$ , a *representation of  $\mathbf{G} \curvearrowright X$*  on  $(\mathbb{V}, \Gamma)$  is an action by linear isometries  $\mathbf{G} \curvearrowright \mathbb{V}$  together with a weak\* continuous  $\mathbf{G}$ -equivariant map over  $\mathbf{B}$ ,  $\alpha : X \rightarrow B(\mathbb{V}^*)$ .

This means that for every  $x \in X$ ,  $(e, v) \in \mathbb{V}$  and  $g \in e\mathbf{G}\rho(x)$ ,

$$\langle v, \alpha(g \cdot x) \rangle = \langle g^{-1} \cdot v, \alpha(x) \rangle.$$

The representation is said to be *faithful* if  $\alpha$  is a topological embedding. The representation is said to be *strongly continuous* if the action  $\mathbf{G} \curvearrowright \mathbb{V}$  is continuous.

**Proposition 3.5.7.** *Suppose that  $(X, \rho)$  is a compact  $\mathbf{G}$ -space over  $\mathbf{B}$ . Then there is a strongly continuous, faithful representation of the action  $\mathbf{G} \curvearrowright X$  on  $\mathbb{C}(X)$ .*

*Proof.* By [Corollary 3.4.12](#), we have a continuous action by linear isometries  $\mathbf{G} \curvearrowright \mathbb{C}(X)$ . Given  $x \in X$  let  $e = \rho(x)$ , and define  $\alpha(x) \in C(X_e)^*$  by  $\alpha(x) = \text{ev}_x$  i.e.  $\langle f, \alpha(x) \rangle = f(x)$  for  $f \in C(X_e)$ . We thus have a map over  $\mathbf{B}$ ,  $\alpha : X \rightarrow B(\mathbb{V}^*)$ . It is easy to see that  $\alpha$  is  $\mathbf{G}$ -equivariant. We claim that this map is a weak\* embedding. Indeed, take  $\mathbf{f}_1, \dots, \mathbf{f}_n \in C(X)$ ,  $I_1, \dots, I_n \subseteq \mathbb{R}$  open and  $U \subseteq \mathbf{B}$  open, and suppose that  $\alpha(x) \in \mathcal{O}^*((\mathbf{f}_j)_{j \leq n}, (I_j)_{j \leq n}, U)$ . Pick  $W \subseteq X$  open with  $x \in W$  such that for every  $x' \in W$ ,  $\mathbf{f}_j(x) \in I_j$  for every  $j \leq n$ . It follows that if  $x' \in W \cap \rho^{-1}(U)$  then  $\alpha(x') \in \mathcal{O}^*((\mathbf{f}_j)_{j \leq n}, (I_j)_{j \leq n}, U)$ , which shows continuity. Since  $C(X)$  separates points, then  $\alpha$  is one-to-one, hence a topological embedding.  $\square$

### 3.6 wap functions and reflexive representations

We start by giving the definition of a wap function in the context of group actions. Let  $G$  be a topological group and let  $X$  be a  $G$ -space, we say that  $f \in C_b(X)$  is *weakly almost periodic*, or wap for short, if  $\{gf ; g \in G\}$  is relatively weakly compact in  $C_b(X)$ . In order to determine whether a function is wap we have Grothendieck's double limit criterion.

**Fact 3.6.1.** [[Gro52](#), Théorème 6] *Let  $X$  be a topological space and let  $X_0$  be a dense subset of  $X$ . Then, for any  $A \subseteq C_b(X)$  the following assertions are equivalent:*

1.  *$A$  is relatively weakly compact in  $C_b(X)$ .*
2.  *$A$  is bounded and for every pair of sequences  $(f_n)_{n \in \mathbb{N}} \subseteq A$  and  $(x_m)_{m \in \mathbb{N}} \subseteq X_0$  we have*

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m)$$

*as soon as both limits exist.*

Suppose now that  $(X, \rho)$  is a  $\mathbf{G}$ -space over  $\mathbf{B}$ . A way to avoid functions that are just defined on some fiber would be to act by  $t$ -sections on  $C_b(X)$  rather than by elements of the groupoid. This is, if  $\mathbf{g} \in \Gamma_t(\mathbf{G})$  and  $\mathbf{f} \in C_b(X)$ , define  $(\mathbf{gf})(x) = \mathbf{f}(\mathbf{g}(\rho(x))^{-1} \cdot x)$  for  $x \in X$ . So, a first attempt to extend the definition of wap functions to groupoid actions would be to say that  $\mathbf{f} \in C_b(X)$  is wap if  $\{\mathbf{gf} ; \mathbf{g} \in \Gamma_t(\mathbf{G})\}$  is relatively weakly compact in  $C_b(X)$ . However, this does not seem to be the correct definition for our purposes. Mainly because it is too strong as the next example shows.

**Example 3.6.2.** Let  $X = \mathbb{R}/\mathbb{Z}$  and let  $G = \mathbb{R}$  act on  $X$  by translations, as usual. We will always assume that  $x \in X$  is represented by a real in  $[0, 1)$ . Define  $f : X \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 - x & \text{otherwise.} \end{cases}$$

It is easy to check, using compactness and Grothendieck's double limit criterion, that  $f$  is wap. Let  $\mathcal{X} = [0, 1] \times X$  and define  $\mathbf{f} \in C_b(\mathcal{X})$  by  $\mathbf{f}(r, x) = f(x)$ . The corresponding groupoid is  $\mathbf{G} = [0, 1] \times G \times [0, 1]$ . For each  $n \in \mathbb{N}$  define  $\mathbf{g}_n \in \Gamma_t(\mathbf{G})$  by  $\mathbf{g}_n(a) = (a, \min\{na, 1/2\}, a)$  for  $a \in [0, 1]$ . For each  $m \in \mathbb{N}$  let  $(r_m, x_m) = (1/m, 1/2) \in \mathcal{X}$ . On one hand,

$$\lim_n \lim_m (\mathbf{g}_n \mathbf{f})(r_m, x_m) = \lim_n \lim_m f\left(\frac{1}{2} - \min\left\{\frac{n}{m}, \frac{1}{2}\right\}\right) = f\left(\frac{1}{2}\right) = \frac{1}{2}.$$

On the other hand,

$$\lim_m \lim_n (\mathbf{g}_n \mathbf{f})(r_m, x_m) = \lim_m \lim_n f\left(\frac{1}{2} - \min\left\{\frac{n}{m}, \frac{1}{2}\right\}\right) = f(0) = 0.$$

By Grothendieck's double limit criterion  $\{\mathbf{g}\mathbf{f} ; \mathbf{g} \in \Gamma_t(\mathbf{G})\}$  is not relatively weakly compact in  $C_b(\mathcal{X})$ . Even though it would be desirable that  $\mathbf{f}$  is wap.

Having this example in mind, we propose the following definition of a wap function in a  $\mathbf{G}$ -space. We must deal with functions restricted to fibers. Nevertheless, under this notion, if a function  $f$  on a  $G$ -space  $X$  is wap then its natural extension  $\mathbf{f}$  to  $\mathbf{B} \times X$ , defined as in the previous example, is also wap. Recall from [Subsection 3.4.1](#) that for any  $g \in \mathbf{G}$  and  $\mathbf{f} \in C_b(X)$ ,  $g\mathbf{f} \in C_b(X_{t_g})$  is the map defined by  $x \rightarrow \mathbf{f}(g^{-1} \cdot x)$ .

**Definition 3.6.3.** Let  $(X, \rho)$  be a  $\mathbf{G}$ -space over  $\mathbf{B}$ . A function  $\mathbf{f} \in C_b(X)$  is said to be *weakly almost periodic*, or wap for short, if for every  $e \in \mathbf{B}$ , the set  $\{g\mathbf{f} ; g \in e\mathbf{G}\}$  is relatively weakly compact in  $C_b(X_e)$ . The set of wap functions on  $X$  will be denoted by  $\text{WAP}_{\mathbf{G}}(X)$ .

**Lemma 3.6.4.** *If  $\mathbf{f} \in \text{WAP}_{\mathbf{G}}(X)$  and  $\mathbf{g} \in \Gamma_t(\mathbf{G})$  then  $\mathbf{g}\mathbf{f} \in \text{WAP}_{\mathbf{G}}(X)$ .*

*Proof.* It follows directly from the fact that for every  $e \in \mathbf{B}$ ,  $\{h(\mathbf{g}\mathbf{f}) ; h \in e\mathbf{G}\} \subseteq \{k\mathbf{f} ; k \in e\mathbf{G}\}$ , which is a relatively weakly compact set in  $C_b(X_e)$ .  $\square$

**Lemma 3.6.5.**  *$\text{WAP}_{\mathbf{G}}(X)$  is the family of sections of a subfield of  $C_b(X)$ , which we denote  $\text{WAP}_{\mathbf{G}}(X)$ .*

*Proof.* It suffices to show that  $\text{WAP}_{\mathbf{G}}(X)$  satisfies conditions 1. and 4. in [Proposition 3.1.2](#). First we will show that  $\text{WAP}_{\mathbf{G}}(X)$  is a  $C(\mathbf{B})$ -submodule of  $C(X)$ . Let  $\mathbf{f}_1, \mathbf{f}_2 \in \text{WAP}_{\mathbf{G}}(X)$  and take  $e \in \mathbf{B}$ . For  $i = 1, 2$  let  $A_i$  be the weak closure of  $\{g\mathbf{f}_i ; g \in e\mathbf{G}\}$ . We have that  $A_1 + A_2$  is weakly compact and it contains  $\{g(\mathbf{f}_1 + \mathbf{f}_2) ; g \in e\mathbf{G}\} \subseteq A_1 + A_2$ . Therefore,  $\{g(\mathbf{f}_1 + \mathbf{f}_2) ; g \in e\mathbf{G}\}$  is relatively weakly compact. Since this is true for every  $e \in \mathbf{B}$  we conclude that  $\mathbf{f}_1 + \mathbf{f}_2 \in \text{WAP}_{\mathbf{G}}(X)$ . Now take  $\varphi \in C(\mathbf{B})$  and  $\mathbf{f} \in \text{WAP}_{\mathbf{G}}(X)$ . Fix  $e \in \mathbf{B}$  and let  $A = \{g(\varphi\mathbf{f}) ; g \in e\mathbf{G}\}$ . For  $g \in e\mathbf{G}$  we have  $\|g(\varphi\mathbf{f})\| \leq \|\varphi\| \|\mathbf{f}\|$ . This implies that  $A$  is bounded. Let  $(g_n)_{n \in \mathbb{N}} \subseteq e\mathbf{G}$  and  $(x_m)_{m \in \mathbb{N}} \subseteq X_e$  be sequences such that the iterated limits  $\lim_n \lim_m g_n(\varphi\mathbf{f})(x_m)$  and  $\lim_m \lim_n g_n(\varphi\mathbf{f})(x_m)$  exist and are equal to  $L$  and  $K$  respectively. We recall that  $g_n(\varphi\mathbf{f})(x_m) = \varphi(s_{g_n})\mathbf{f}(g_n^{-1} \cdot x)$ . Let us denote  $r_n = \lim_m g_n(\varphi\mathbf{f})(x_m)$  and  $s_m = \lim_n g_n(\varphi\mathbf{f})(x_m)$ . By passing to a subsequence we can assume that  $(s_{g_n})_{n \in \mathbb{N}}$  converges to  $e' \in \mathbf{B}$  and so  $\varphi(s_{g_n}) \rightarrow \varphi(e')$ . Suppose first that  $\varphi(e') = 0$ . Since  $|\mathbf{f}(g_n^{-1} \cdot x_m)| \leq \|\mathbf{f}\|$  for every  $m, n \in \mathbb{N}$ , we obtain that  $|r_n| \leq |\varphi(s_{g_n})| \|\mathbf{f}\|$  and that  $s_m = 0$ . We thus get that  $L = K = 0$ . Now suppose that  $\varphi(e') \neq 0$ . Then, for  $n$  large enough,  $\lim_m g_n\mathbf{f}(x_m) = r_n/\varphi(s_{g_n})$  and we obtain  $\lim_n \lim_m g_n\mathbf{f}(x_m) = L/\varphi(e')$ . Likewise, for  $m \in \mathbb{N}$ ,

$\lim_n g_n \mathbf{f}(x_m) = s_m / \varphi(e')$  and so  $\lim_m \lim_n g_n \mathbf{f}(x_m) = K / \varphi(e')$ . Due to the fact that  $\mathbf{f}$  is weakly almost periodic, we must have  $K = L$ . By Grothendieck's double limit criterion,  $A$  is relatively weakly compact. Therefore,  $\varphi \mathbf{f} \in \text{WAP}_{\mathbf{G}}(X)$ .

It remains to show that  $\text{WAP}_{\mathbf{G}}(X)$  is locally uniformly closed. Take  $\mathbf{f} \in \prod_{e \in \mathbf{B}} C_b(X_e)$  such that for any  $e \in \mathbf{B}$  and any  $\epsilon > 0$  there are  $\mathbf{f}' \in \text{WAP}_{\mathbf{G}}(X)$  and  $U \subseteq \mathbf{B}$  open neighborhood of  $e$  such that  $\|\mathbf{f}_{e'} - \mathbf{f}'_{e'}\| < \epsilon$  for  $e' \in U$ . Towards a contradiction suppose that  $\mathbf{f}$  is not weakly almost periodic. This means that there is  $e \in \mathbf{B}$  for which there exist sequences  $(g_n)_{n \in \mathbb{N}}$  in  $e\mathbf{G}$  and  $(x_m)_{m \in \mathbb{N}}$  in  $X_e$  such that

$$L = \lim_n \lim_m g_n \mathbf{f}(x_m) \neq \lim_m \lim_n g_n \mathbf{f}(x_m) = K.$$

Passing to a subsequence we may assume that  $(s_{g_n})_{n \in \mathbb{N}}$  converges to  $e' \in \mathbf{B}$ . For this  $e'$  and  $\epsilon = |L - K|/3$  take  $\mathbf{f}' \in \text{WAP}_{\mathbf{G}}(X)$  and  $U \subseteq \mathbf{B}$  as stated above. Choose  $N \in \mathbb{N}$  such that  $s_{g_n} \in U$  for every  $n \geq N$ . It follows that for every  $m \in \mathbb{N}$  and every  $n \geq N$ ,  $|\mathbf{f}'(g_n^{-1} \cdot x_m) - \mathbf{f}(g_n^{-1} \cdot x_m)| \leq \|\mathbf{f}'_{s_{g_n}} - \mathbf{f}_{s_{g_n}}\| < \epsilon$ . This implies that the double sequence  $(g_n \mathbf{f}'(x_m))_{m, n \in \mathbb{N}}$  does not admit a doubly adherent point. Therefore, by [Fact 3.3.4](#), there is a double subsequence  $(g_{n_j} \mathbf{f}'(x_{m_i}))_{i, j \in \mathbb{N}}$  such that  $\lim_i \lim_j g_{n_j} \mathbf{f}'(x_{m_i})$  and  $\lim_j \lim_i g_{n_j} \mathbf{f}'(x_{m_i})$  both exist and they are different. This is absurd since  $\mathbf{f}'$  is weakly almost periodic.  $\square$

The example *par excellence* of a wap function is one coming from a reflexive representation, see [\[Meg03, Fact 3.6\]](#). The next proposition is the analogue in the context of groupoid actions.

**Proposition 3.6.6.** *Let  $(X, \rho)$  be a space over  $\mathbf{B}$  and let  $\mathbf{G} \curvearrowright X$  be a separately continuous groupoid action. Let  $(\mathbb{V}, \Gamma)$  be a field of Banach spaces over  $\mathbf{B}$ . Suppose that  $\mathbf{G}$  acts, not necessarily continuously, by linear isometries on  $\mathbb{V}$  and  $\alpha : X \rightarrow B(\mathbb{V}^*)$  is a weakly\* continuous  $\mathbf{G}$ -equivariant map. Then,*

1. *For every  $(e, v) \in \mathbb{V}$  the map  $T(v) : X_e \rightarrow \mathbb{R}$  defined by  $T(v)(x) = \langle v, \alpha(x) \rangle$ , is continuous and  $\|T(v)\| \leq \|v\|$ .*
2.  *$T : \mathbb{V} \rightarrow C_b(X)$  is a  $\mathbf{G}$ -equivariant linear map.*
3. *For every  $\mathbf{v} \in \Gamma$ ,  $T(\mathbf{v}) : X \rightarrow \mathbb{R}$  given by  $T(\mathbf{v})(x) = \langle \mathbf{v}, \alpha(x) \rangle$  belongs to  $C_b(X)$ .*
4. *If  $e \in \mathbf{B}$  and  $V_e$  is reflexive then for any  $\mathbf{v} \in \Gamma$ ,  $\{gT(\mathbf{v}(s_g))\} ; g \in e\mathbf{G}$  is relatively weakly compact in  $C_b(X_e)$ .*
5. *If  $V_e$  is reflexive for every  $e \in \mathbf{B}$  then  $T(\mathbf{v})$  is wap for every  $\mathbf{v} \in \Gamma$ .*

*Proof.* 1. Pick  $(e, v) \in \mathbb{V}$ , we know that the map  $ev_v : V_e^* \rightarrow \mathbb{R}$  is weak\* continuous. It follows that  $T(v) = ev_v \circ \alpha \upharpoonright_{X_e}$  is continuous. Furthermore, for every  $x \in X_e$ ,  $|T(v)(x)| = |\langle v, \alpha(x) \rangle| \leq \|v\|$ . 2. For  $e \in \mathbf{B}$  we write  $T_e = T \upharpoonright_{V_e}$ . It is immediate that  $T_e : V_e \rightarrow C_b(X_e)$  is a linear application which, by 1., satisfies  $\|T_e\| \leq 1$ . To show that it is  $\mathbf{G}$ -equivariant, pick  $e \in \mathbf{B}$ ,  $v \in V_e$ ,  $g \in \mathbf{G}e$  and  $x \in X_{t_g}$ . Then,

$$(gT(v))(x) = T(v)(g^{-1} \cdot x) = \langle v, \alpha(g^{-1} \cdot x) \rangle = \langle v, g^{-1} \cdot \alpha(x) \rangle = \langle g \cdot v, \alpha(x) \rangle = T(g \cdot v)(x).$$

It follows that  $gT(v) = T(g \cdot v)$ .

3. By the definition of the weak\* topology, the map  $ev_{\mathbf{v}} : B(\mathbb{V}^*) \rightarrow \mathbb{R}$  defined by  $(e, \varphi) \mapsto \langle \mathbf{v}, \varphi \rangle$  is continuous. It follows that  $T(\mathbf{v}) = ev_{\mathbf{v}} \circ \alpha$  is continuous. We also have  $\|T(\mathbf{v})\| \leq \|\mathbf{v}\|$ , implying that  $T(\mathbf{v})$  is bounded.

4. Fix  $e \in \mathbf{B}$  with  $V_e$  reflexive and take  $\mathbf{v} \in \Gamma$ . The ball  $\|\mathbf{v}\|B_e$  is weakly compact. Therefore,

$\{g \cdot \mathbf{v}(s_g) ; g \in e\mathbf{G}\} \subseteq \|\mathbf{v}\|B_e$  es relatively weakly compact. Thus, by the continuity of  $T_e$  and the  $\mathbf{G}$ -equivariance of  $T$ , the same is true for  $\{gT(\mathbf{v}(s_g)) ; g \in e\mathbf{G}\} \subseteq C_b(X_e)$ .

5. Follows directly from 4.  $\square$

We remark that under the hypothesis of the previous proposition, if  $(\mathbb{V}, \Gamma)$  and  $\mathbb{C}_b(X)$  are moreover continuous continuous fields then, by [Fact 3.1.24](#), the application  $T$  defines a homomorphism.

Our next objective is to show that every wap function can be obtained in the previous manner. Namely, that if a topological groupoid  $\mathbf{G}$  acts in a separately continuous manner on  $(X, \rho)$  and  $\mathbf{f} \in \text{WAP}_{\mathbf{G}}(X)$  then there is a representation on a field  $(\mathbb{V}, \Gamma)$  such that each fiber is reflexive and  $\mathbf{f} = T(\mathbf{v})$  for some  $\mathbf{v} \in \Gamma$ . We draw our inspiration from [\[Meg03, Theorem 4.5\]](#). First we give some definitions and results on convex sets. For  $A$  subset of a vector space,  $\text{co}(A)$  will denote the convex hull of  $A$ .

Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces over  $\mathbf{B}$  and let  $U \subseteq \Gamma$ . Given  $e \in \mathbf{B}$ , we denote  $U_e = \{\mathbf{v}(e) ; \mathbf{v} \in U\}$ . Note that if  $U$  is either convex, balanced or absorbent, as a subset of the vector space  $\Gamma$ , then the same is true for  $U_e$  for every  $e \in \mathbf{B}$ . Note that  $B(\mathbb{V}) = \bigsqcup_{e \in \mathbf{B}} B(\Gamma)_e$ .

For  $e \in \mathbf{B}$  let  $\rho_e : V_e \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  be the Minkowski functional associated to  $U_e$ . Namely, for  $v \in V_e$ ,  $\rho_e(v) = \inf\{\lambda > 0 ; v \in \lambda U_e\}$  where  $\inf \emptyset = +\infty$ . Let us also define  $\rho : \mathbb{V} \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  by  $\rho(e, v) = \rho_e(v)$ . We will refer to  $\rho$  as the Minkowski functional of  $U$ . A fiberwise analysis yields the following result, see for example [\[Bou07, Proposition II.22\]](#).

**Lemma 3.6.7.** *Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces and let  $U \subseteq \Gamma$ .*

1. *If  $U$  is absorbing then  $\rho$  is real-valued.*
2. *If  $U$  is balanced then  $\rho$  is absolutely homogeneous.*
3. *If  $U$  is convex then  $\rho$  is subadditive.*
4. *If  $U$  is bounded and  $\mathbf{0} \in U$  then  $\rho^{-1}(\{0\}) = \mathbf{0}$ .*

**Lemma 3.6.8.** *Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces. If  $U \subseteq \Gamma$  is a convex and balanced neighborhood of  $\mathbf{0}$ . Then,  $\text{int}(U)$ , as a subset of  $\mathbb{V}$ , is just  $\{(e, v) \in \mathbb{V} ; \rho(e, v) < 1\}$ .*

*Proof.* Let  $(e, v) \in \text{int}(U)$ , then there is  $\delta > 0$   $B_{\|\cdot\|_e}(v, \delta) \subseteq U$ . We set  $r = 1 + \frac{\delta}{(\|v\|+1)}$ , we have that  $\|rv - v\| < \delta$ . Thus,  $rv \in U$  which implies  $\rho(e, v) \leq \frac{1}{r} < 1$ . For the converse, take  $\epsilon > 0$  such that the  $\epsilon$ -ball in  $\Gamma$  centered at  $\mathbf{0}$  is contained in  $U$ . We then have that  $\mathcal{O}(\mathbf{0}, \epsilon, \mathbf{B}) \subseteq U$ . Let  $\mathbf{v} \in U$  and  $0 < r < 1$ , we claim that  $\mathcal{O}(r\mathbf{v}, \frac{1-r}{2}\epsilon, \mathbf{B}) \subseteq U$ , and so  $r\mathbf{v} \subseteq \text{int}(U)$ . Indeed, take  $(w, e) \in \mathcal{O}(r\mathbf{v}, \frac{1-r}{2}\epsilon, \mathbf{B})$ , then  $\|\frac{2r}{1-r}(\frac{1}{r}w - \mathbf{v}(e))\| \leq \frac{2}{1-r}\|w - r\mathbf{v}(e)\| < \epsilon$ . Therefore,  $\frac{2r}{1-r}(\frac{1}{r}w - \mathbf{v}(e)) \in U$  and we get  $\frac{1}{r}w - \mathbf{v}(e) \in \frac{1-r}{2r}U$ . thus,  $\rho(\frac{1}{r}w - \mathbf{v}(e)) \leq \frac{1-r}{2r} < \frac{1-r}{r}$ . It follows that

$$\rho(w) = r\rho\left(\frac{1}{r}w\right) = r\rho\left(\frac{1}{r}w - \mathbf{v}(e) + \mathbf{v}(e)\right) \leq r\left(\rho\left(\frac{1}{r}w - \mathbf{v}(e)\right) + \rho(\mathbf{v}(e))\right) < r\left(\frac{1-r}{r} + 1\right) = 1.$$

This shows that  $w \in U$ . Now pick  $(e, v) \in \mathbb{V}$  with  $\rho(e, v) < 1$ . Then, there exists  $0 < r < 1$  such that  $\frac{1}{r}v \in U$ . Let  $\mathbf{v} \in U$  be a section passing through  $\frac{1}{r}v$ . Hence,  $v = r\mathbf{v}(e) \in r\mathbf{v} \subseteq \text{int}(U)$ .  $\square$

The homogeneity of  $\rho$  gives the following result.

**Corollary 3.6.9.** *If  $U \subseteq \Gamma$  is a convex and balanced neighborhood of  $\mathbf{0}$ . Then,  $\rho$  is upper semi-continuous.*

When  $V$  is a Banach space and  $U$  is a balanced, convex, bounded neighborhood of  $0$  then  $\rho$  defines a norm in  $V$  which is equivalent to the original norm  $\|\cdot\|_V$ . However, in Banach fields, under the same hypothesis for  $U \subseteq \Gamma$ , the Minkowski functional also defines a norm but we cannot guarantee that it is equivalent to the usual norm. The problem lies in the fact that the closure of  $U$  in  $\mathbb{V}$  does not necessarily agree with the union of the fiberwise closures of  $U_e$ . So there could be  $(e, v)$  in the closure of  $U$  for which  $\rho(e, v) > 1$ . We are now ready to prove that wap functions come from reflexive representations.

**Theorem 3.6.10.** *Let  $(X, \rho)$  be a compact space over  $\mathbf{B}$  and let  $\mathbf{G} \curvearrowright X$  be a separately continuous action. Suppose that  $\mathbf{G}$  admits sections and that  $\mathbf{f} \in \text{WAP}_{\mathbf{G}}(X)$ . Then there is  $(\mathbb{V}, \Gamma)$  a field of reflexive Banach spaces over  $\mathbf{B}$ , an action by linear isometries  $\mathbf{G} \curvearrowright \mathbb{V}$ , a weakly\* continuous  $\mathbf{G}$ -equivariant map  $\alpha : X \rightarrow B(\mathbb{V}^*)$  and  $\mathbf{v} \in \Gamma$  such that  $\mathbf{f}(x) = \langle \mathbf{v}, \alpha(x) \rangle$  for every  $x \in X$ .*

*Proof.* Assume that  $\|\mathbf{f}\| \leq 1$ . Let  $Y = \Gamma_t(\mathbf{G})\mathbf{f} = \{\mathbf{g}\mathbf{f} ; \mathbf{g} \in \Gamma_t(\mathbf{G})\} \subseteq C(X)$ . Since  $\mathbf{G}$  admits sections, for every  $e \in \mathbf{B}$  we have  $Y_e = \{\mathbf{g}\mathbf{f} ; \mathbf{g} \in e\mathbf{G}\}$ . We set  $Z = \text{co}(Y \cup -Y)$ , in  $C(X)$ , and we pick  $a > 0$  such that  $\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2} = 1$ . For each  $n \in \mathbb{N}$ , let  $U_n = a^{\frac{n}{2}}Z + a^{-\frac{n}{2}}B(C(X))$  and let  $\rho_n : C(X) \rightarrow \mathbb{R}_+$  be the corresponding Minkowski functional. Since  $Z$  is convex, balanced and bounded then the same is true for  $U_n$ .  $U_n$  is moreover a neighborhood of  $\mathbf{0}$ . Therefore, for each  $e \in \mathbf{B}$ ,  $\rho_{n,e}$  is a norm on  $C(X_e)$  equivalent to its original norm  $\|\cdot\|_e$ . Define  $N : C(X) \rightarrow [0, +\infty]$  by  $N(e, f) = (\sum_{n=1}^{\infty} \rho_n(e, f)^2)^{\frac{1}{2}}$  and let  $\Gamma = \{\mathbf{f}' \in C(X) ; \sup_{e \in \mathbf{B}} N(\mathbf{f}'(e)) < +\infty\}$ . Note that for every  $e \in \mathbf{B}$ ,  $n \in \mathbb{N}$  and  $\mathbf{f}' \in Z$ ,  $\rho_n(\mathbf{f}'(e)) \leq \frac{a^{\frac{n}{2}}}{a^n+1}$  and so  $N(\mathbf{f}'(e)) \leq 1$ . Thus,  $Y \subseteq Z \subseteq \Gamma$ . For each  $e \in \mathbf{B}$  set  $W_e = \{f \in C(X_e) ; N(e, f) < +\infty\}$ , endowed with the norm  $N_e$ , and let  $V_e$  be the closure in  $W_e$  of  $\{\mathbf{f}'(e) ; \mathbf{f}' \in \Gamma\}$ . Defined in this manner,  $(\mathbb{V}, \Gamma)$  is a field of Banach spaces over  $\mathbf{B}$ . We claim that it is the desired field. Since  $\mathbf{f}$  is wap,  $Y_e$  is a relatively weakly compact subset of  $C(X_e)$ . By the Krein-Smulian Theorem,  $Z_e$  is relatively weakly compact as well. It follows from [DFP74, Lemma 1] that  $W_e$  is a reflexive Banach space. Hence, the same holds for  $V_e$  for every  $e \in \mathbf{B}$ .

Since  $Y$  is invariant under the action of  $\mathbf{G}$ , then so it is  $Z$ . Due to the fact that  $\mathbf{G}$  acts on  $C(X)$  by linear isometries,  $B(C(X))$ , and thus  $U_n$ , are also invariant under the action of  $\mathbf{G}$ . This implies that  $N(t_g, gf) = N(e, f)$  for every  $(e, f) \in C(X)$  and every  $g \in \mathbf{G}e$ . Hence, for  $\mathbf{g} \in \Gamma_t(\mathbf{G})$  and  $\mathbf{f}' \in \Gamma$ ,  $\mathbf{g}\mathbf{f}' \in \Gamma$ . Therefore,  $\mathbb{V}$  is  $\mathbf{G}$ -invariant and the action of  $\mathbf{G}$  on  $\mathbb{V}$  is given by  $N$ -isometries. For  $e \in \mathbf{B}$  let  $J_e : (W_e, N_e) \rightarrow C(X_e)$  be the inclusion map, we know from [Meg03, Theorem 4.5] that  $\|J_e\| \leq 1$ . Pick  $x \in X_e$ , define  $\alpha(x) : V_e \rightarrow \mathbb{R}$  by  $\langle f, \alpha(x) \rangle = f(x)$ . The map  $\alpha(x)$  is clearly linear and  $|\langle f, \alpha(x) \rangle| \leq \|f\| \leq N_e(f)$  for every  $(e, f) \in V_e$ . Therefore,  $\|\alpha(x)\| \leq 1$  and we have a function over  $\mathbf{B}$ ,  $\alpha : X \rightarrow B(\mathbb{V}^*)$ . For every  $\mathbf{f}' \in \Gamma$  we have  $\langle \mathbf{f}', \cdot \rangle \circ \alpha = \mathbf{f}'$ . It follows that  $\alpha$  is weak\* continuous. A straightforward verification shows that  $\alpha$  is  $\mathbf{G}$ -equivariant. Finally, note that  $\mathbf{f} \in \Gamma$  and so  $\mathbf{f}(x) = \langle \mathbf{f}, \alpha(x) \rangle$  for every  $x \in X$ . This completes the proof.  $\square$

The field  $(\mathbb{V}, \Gamma)$  is not necessarily continuous. By Corollary 3.6.9, each norm  $\rho_n$  is upper semi-continuous. But this does not imply any continuity condition on the norm  $N$ . The most we can say is that the norm of each section in  $\Gamma$  is a measurable function, a fact that will be used in the next chapter. We do not know whether we can find such a representation on a continuous field nor if we can find a strongly continuous representation.

### 3.7 Further questions

In [Meg03, Theorem 4.5] it is proved that for a semigroup  $S$  bounded strict  $S$ -dualities on compact spaces come as restrictions of  $S$ -dualities of reflexive Banach spaces. The proof passes to the Ellis semigroup in order to get compactness. There is a notion of the Ellis semigroupoid given in [AB21], we would like to know if this notion allows us to extend these results to the context of  $\mathbf{G}$ -dualities. Moreover, in the context of group actions, a compact, metrizable  $G$ -space admits a faithful representation on a reflexive space if and only if the algebra of wap functions is the whole space of continuous functions, we wonder whether a generalization to the setting of groupoid actions is possible.

In order to prove [Fact 3.6.1](#), Grothendieck uses the Eberlein-Šmulian theorem and the Lebesgue's dominated convergence theorem in order to reduce the problem of relative weak compactness to relative pointwise sequential compactness, then he uses the double limit criterion for pointwise compactness, the analogue of our [Theorem 3.3.3](#). We would like to find analogues of these results in the context of Banach fields in order to obtain a double limit criterion to determine whether  $A \subseteq C_b(X)$  is relatively weakly compact.

The works of Glasner and Megrelishvili established a hierarchy of dynamical systems and its representations on Banach spaces, see [GM14]. Here we explored merely one of such correspondences, WAP systems, and notably wap functions, and reflexive representations. In this hierarchy there is a correspondence between tame systems and Rosenthal representations. We would like to establish similar results for groupoid actions. Namely, find the appropriate definition of tame function in the context of groupoid actions and find the corresponding class of Banach fields in which they can be represented.

## Chapter 4

# The space of partial types and groupoid actions

### Introduction

In the  $\omega$ -categorical case, as first noted in [BYT16, Lemma 5.1], stable formulas correspond to weakly almost periodical functions. Indeed, let  $\mathcal{M}$  be an  $\omega$ -categorical structure, from [Iba16, Lemma 2.2] we deduce that a formula  $\varphi(x, y)$  is stable in  $\mathcal{M}$  if and only if for every  $b \in M$ , the induced map  $\varphi^b : S_x(M) \rightarrow [0, 1]$ , given by  $p \mapsto \varphi(x, b)^p$ , is weakly almost periodic. Now suppose that  $\Delta$  is a set of formulas in variables  $x, y$  containing  $\varphi$ . Since the map  $\varphi^b$  factors through the space of  $\Delta$ -types,  $S_\Delta(M)$ , the stability of  $\varphi(x, y)$  can also be derived from the weakly almost periodicity of  $\varphi^b$  in the  $\text{Aut}(\mathcal{M})$ -space  $S_\Delta(M)$ , for  $b \in M$ .

By the dynamical hierarchy established by E. Glasner and M. Megrelishvili, wap functions are those coming from representations on reflexive Banach spaces. In fact, if a compact dynamical system admits a faithful representation on a reflexive Banach space then every continuous function is wap [GM14, Example 5.3 and Theorem 5.4]. If the space is moreover metrizable then the converse also holds [Meg03, Corollary 4.10]. In this case, we say that the system is reflexive-representable.

This fact was used by T. Ibarlucía in [Iba17] to prove the preservation of stability and NIP by the randomization. Let us briefly explain his argument. Fix a set of formulas  $\Delta(x, y)$  and let  $\mathcal{M}^R$  denote the Borel randomization of  $\mathcal{M}$ . We can define a set of formulas  $\Delta^R(x, y)$  in the randomization, containing in particular  $\mathbb{E}[\varphi(x, y)]$  for  $\varphi \in \Delta$ , such that every reflexive representation of  $\text{Aut}(\mathcal{M}) \curvearrowright S_\Delta(\mathcal{M})$  induces a representation of  $\text{Aut}(\mathcal{M}^R) \curvearrowright S_{\Delta^R}(\mathcal{M}^R)$  on another reflexive space. If the original flow is reflexive-representable, which happens to be the case if all the formulas in  $\Delta$  are stable, each continuous function  $S_\Delta(\mathcal{M})$  comes from a reflexive representation, and the family of induced representations of  $\text{Aut}(\mathcal{M}^R) \curvearrowright S_{\Delta^R}(\mathcal{M}^R)$  separate points, which is good enough.

This is the behaviour we try to generalize to the context of theories admitting a universal Skolem sort. For such a theory  $T$ , let  $D_T$  be its universal Skolem sort. Since we do no longer have a single separable model to consider, and the sort  $D_T$  let us consider all separable models, we will replace the space of types  $S_x(M)$  by  $S_{D_T \times F}(T)$ , where  $F$  is an arbitrary sort of  $T$ . The groupoid associated to the theory,  $\mathbf{G}(T)$ , will act naturally on this space on the left. The stability of a formula  $\varphi(x, y)$  in  $D_T \times F$  will correspond to the function defined on the space of types being wap, as in [Definition 3.6.3](#).

For a stable formula  $\varphi(x, y)$ , the work done in [Chapter 3](#) let us build a representation of  $\mathbf{G}(T) \simeq S_{D_T \times F}(T)$  on a field of reflexive spaces from which  $\varphi(x, y)$  comes. This chapter is concerned with the construction of the corresponding representation of the randomized action  $\mathbf{G}(T^R) \simeq S_{D_{T^R} \times F^R}(T)$ . By [Chapter 2](#), we know how to obtain  $\mathbf{G}(T^R)$  from  $\mathbf{G}(T)$ . Following [\[Iba17\]](#), for any set of formulas  $\Delta(y, u)$  in  $D_T \times F$  we define a set of formulas  $\Delta^R(\hat{y}, u)$  in  $D_{T^R} \times F^R$ , containing  $\mathbb{E}[\varphi(x, y)]$  for  $\varphi \in \Delta$ , such that a representation of  $\mathbf{G}(T) \simeq S_\Delta(T)$  on a field of Banach spaces induces a representation of  $\mathbf{G}(T^R) \simeq S_{\Delta^R}(T^R)$ .

We do this construction in a purely topological manner. In [Section 4.1](#), we start with a topological groupoid  $\mathbf{G}$  over  $2^{\mathbb{N}}$  and a compact, metrizable  $\mathbf{G}$ -space  $(X, \rho)$ . We define a space of measures on  $\Omega \times X$ ,  $S(\Omega, X)$ , on which the groupoid  $(\mathbf{G} \wr \Omega)_\pi$ , defined as in [Section 2.3](#), acts. We also study the previous construction when compact, Hausdorff quotients of  $X$  are considered. In [Section 4.2](#), we apply these results when  $X$  is the space of types  $S_{D_T \times F}(T)$ . Finally, in [Section 4.3](#), we show that if  $\mathbf{f} \in C(X)$  comes from a representation of  $\mathbf{G} \wr X$  on a field  $(\mathbb{V}, \Gamma)$  then  $\mathbb{E}[\mathbf{f}] \in C(S(\Omega, X))$  comes from a representation of  $(\mathbf{G} \wr \Omega)_\pi \wr S(\Omega, X)$  on the field  $L_2(\Omega, \mathbb{V})_\pi$ . It remains however the question on whether the reflexivity of the original field implies the reflexivity of the  $L_2(\Omega, \mathbb{V})_\pi$ , we conjecture that this should be the case.

## 4.1 Randomization of a groupoid action

Throughout this chapter  $(\Omega, \mathcal{B}, \mu)$  will refer to the probability space obtained from the Stone space of the usual Borel algebra on the interval  $[0, 1]$ , as described in [Section 1.1](#). As usual, we will identify a randomization with its canonical representation. So we will assume that all separable randomizations have  $(\Omega, \mathcal{B}, \mu)$  as their underlying probability space. Recall that  $E$  will denote the sort of events and  $E_{RV}$  the sort of random variables.  $E(\Omega)$  is the sort of events when interpreted in  $(\mathcal{B}, \mu) \models \text{APr}$ , and it is in correspondence with the algebra of clopen subsets of  $\Omega$ . The following fact is just a restatement of the discussion at the beginning of [Section 2.2](#).

**Fact 4.1.1.** *The map  $\psi : (2^{\mathbb{N}})^\Omega \rightarrow E^{\mathbb{N}}(\Omega)$ , given by  $\psi(f)_n = \llbracket f_n = 0 \rrbracket$  for  $f \in (2^{\mathbb{N}})^R$  and  $n \in \mathbb{N}$ , is a definable bijection. It follows that the two metrics on  $(2^{\mathbb{N}})^\Omega$  given by*

$$d(f, g) = \mathbb{E}[\llbracket d(f, g) \rrbracket] \quad \text{and} \quad d'(f, g) = \sup_{n \in \mathbb{N}} 2^{-n} \wedge \mu(\llbracket f_n = 0 \rrbracket \triangle \llbracket g_n = 0 \rrbracket),$$

are uniformly equivalent.

Following the notation of [Chapter 2](#),  $\Xi \in E^{\mathbb{N}}(\Omega)$  denotes a sequence of independent subsets of measure 1/2 whose generated algebra is dense in  $E(\Omega)$ .

Let  $\mathbf{B}$  be a topological space. A *space over  $\mathbf{B}$*  is a couple  $(X, \rho)$  where  $X$  is a topological space and  $\rho : X \rightarrow \mathbf{B}$  is a continuous, surjective map. If  $\mathbf{G}$  is a topological groupoid acting continuously on  $X$ , we will say that  $(X, \rho)$  is a  *$\mathbf{G}$ -space over  $\mathbf{B}$* . For  $(X, \rho)$  and  $(Y, \eta)$  spaces over  $\mathbf{B}$ , the *fiber product between  $X$  and  $Y$*  is defined as

$$X \times_{\mathbf{B}} Y = \{(x, y) \in X \times Y ; \rho(x) = \eta(y)\}.$$

The fiber product is thus a closed subset of the product  $X \times Y$ . From now on  $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^\Omega$  will be a continuous map,  $\mathbf{G}$  will be a topological groupoid over  $2^{\mathbb{N}}$  and  $(X, \rho)$  will be a compact, metrizable space over  $2^{\mathbb{N}}$ . Let  $(\mathbf{G} \wr \Omega)_\pi$  be the base change of  $\mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$  to  $2^{\mathbb{N}}$  given by  $\pi$ . Namely,

$$(\mathbf{G} \wr \Omega)_\pi = \{(\alpha, g, \tau, \beta) \in 2^{\mathbb{N}} \times \mathbf{G}^\Omega \times \text{Aut}(\Omega) \times 2^{\mathbb{N}} ; t^\Omega(g) = \pi(\alpha) \text{ and } s^\Omega(g) = \tau \cdot \pi(\beta)\}$$

endowed with the product topology and the groupoid operations given by  $(\alpha, g, \tau, \beta)(\beta, h, \eta, \gamma) = (\alpha, g(\tau \cdot h), \tau\eta, \gamma)$  and  $(\alpha, g, \tau, \beta)^{-1} = (\beta, \tau^{-1} \cdot g^{-1}, \tau^{-1}, \alpha)$ . Given  $\alpha \in 2^{\mathbb{N}}$  let

$$\Omega \times_\alpha X = \{(\omega, x) \in \Omega \times X ; \rho(x) = \pi(\alpha)(\omega)\}.$$

Notice that  $\Omega \times_\alpha X$  is a closed, and hence measurable, subset of  $\Omega \times X$ .

**Definition 4.1.2.** Let  $(X, \rho)$  be a compact, metrizable space over  $2^{\mathbb{N}}$ . We define

$$S(\Omega, X) = \{(\alpha, \lambda) \in 2^{\mathbb{N}} \times \mathfrak{A}(\Omega \times X) ; \lambda|_\Omega = \mu \text{ and } \lambda(\Omega \times_\alpha X) = 1\}.$$

We endow this space with the topology it inherits from the product  $2^{\mathbb{N}} \times \mathfrak{A}(\Omega \times X)$ . It follows from the definition that  $S(\Omega, X)$  is closed in  $2^{\mathbb{N}} \times \mathfrak{A}(\Omega \times X)$  and so it is compact. Also, it is naturally a space over  $2^{\mathbb{N}}$  via the projection on the first coordinate.

**Definition 4.1.3.** Let  $V$  be a Banach space and  $1 \leq q \leq \infty$ , we define  $L_q^{w*}(\Omega, V^*)$  as the set of weak\* measurable mappings  $\varphi : \Omega \rightarrow V^*$ , modulo equality almost everywhere, for which there exists  $g \in L^q(\Omega)$  such that  $\|\varphi(\omega)\| \leq g(\omega)$  for almost every  $\omega \in \Omega$ .

This space is normed by  $\|\varphi\|_q = \inf \|g\|_q$ , where the infimum is taken over all such  $g \in L^q(\Omega)$ . This space is precisely the dual of  $L_p(\Omega, V)$ , see [CM97, Theorem 1.5.5], and as such we will endow it with the weak\* topology.

**Lemma 4.1.4.** Let  $(X, \rho)$  be a compact, metrizable space over  $2^{\mathbb{N}}$ , then  $S(\Omega, X)$  is homeomorphic to

$$\{(\alpha, p) \in 2^{\mathbb{N}} \times L_2^{w*}(\Omega, C(X)^*) ; p(\omega) \in \mathfrak{A}(X) \text{ and } p(\omega)(X_{\pi(\alpha)(\omega)}) = 1 \text{ } \mu\text{-a.e.}\}.$$

*Proof.* We know from [Iba17, Section 3.3] that there is a homeomorphism

$$\{\lambda \in \mathfrak{A}(\Omega \times X) ; \lambda|_\Omega = \mu\} \cong \{p \in L_2^{w*}(\Omega, C(X)^*) ; p(\omega) \in \mathfrak{A}(X) \text{ a. e.}\},$$

assigning to a measure  $\lambda$  its disintegration  $p(\omega) = \lambda_\omega$  for almost every  $\omega \in \Omega$ . For any given  $\omega \in \Omega$  we have  $\{x \in X ; (\omega, x) \in \Omega \times_\alpha X\} = \{x \in X ; \rho(x) = \pi(\alpha)(\omega)\} = X_{\pi(\alpha)(\omega)}$ . Since  $\lambda(\Omega \times_\alpha X) = \int_\Omega p(\omega)(X_{\pi(\alpha)(\omega)}) d\mu(\omega)$ , then  $\lambda(\Omega \times_\alpha X) = 1$  if and only if  $p(\omega)(X_{\pi(\alpha)(\omega)}) = 1$  for almost every  $\omega \in \Omega$ . Thus, the map that to  $(\alpha, \lambda) \in S(\Omega, X)$  assigns  $(\alpha, p)$  defines a homeomorphism.  $\square$

Under this identification, for  $(\alpha, \lambda) \in S(\Omega, X)$  and  $F \in C(\Omega \times X)$ ,

$$\langle F, \lambda \rangle = \int_\Omega \left( \int_{X_{\pi(\alpha)(\omega)}} F(\omega, x) dp(\omega)(x) \right) d\mu(\omega).$$

Now suppose that  $(X, \rho)$  is a  $\mathbf{G}$ -space. We will define an action of  $(\mathbf{G} \wr \Omega)_\pi$  on  $S(\Omega, X)$ . Fix  $\alpha, \beta \in 2^{\mathbb{N}}$ , take  $F : \Omega \times_\beta X \rightarrow \mathbb{R}$  and  $\hat{g} = (\alpha, g, \tau, \beta) \in (\mathbf{G} \wr \Omega)_\pi$ , and define  $\hat{g}F : \Omega \times_\alpha X \rightarrow \mathbb{R}$  by

$$\hat{g}F(\omega, x) = F(\tau^{-1}(\omega), g_\omega^{-1} \cdot x) \text{ for } (\omega, x) \in \Omega \times_\alpha X.$$

The map  $\hat{g}F(\omega, x)$  is well-defined. Indeed, if  $(\omega, x) \in \Omega \times_\alpha X$  then  $s_{g_\omega^{-1}}^\Omega(\omega) = \pi(\alpha)(\omega) = \rho(x)$ , so  $g_\omega^{-1} \cdot x$  is defined and  $\rho(g_\omega^{-1} \cdot x) = t_{g_\omega^{-1}}^\Omega = s_{g_\omega}^\Omega = (\tau \cdot \pi(\beta))(\omega) = \pi(\beta)(\tau^{-1}(\omega))$ , which implies that  $(\tau^{-1}(\omega), g_\omega^{-1} \cdot x) \in \Omega \times_\beta X$ . The function is moreover measurable. Indeed, the function  $\hat{g}F$  is obtained as the composition of the following measurable maps,

$$\begin{array}{ccccccc} \Omega \times_{\alpha} X & \longrightarrow & \Omega \times (\mathbf{G} \times_{\mathbf{B}} X) & \longrightarrow & \Omega \times X & \longrightarrow & \mathbb{R} \\ (\omega, x) & \longmapsto & (\omega, g_{\omega}^{-1} \cdot x) & \longmapsto & (\tau^{-1}(\omega), g_{\omega}^{-1} \cdot x) & \longmapsto & F(\tau^{-1}(\omega), g_{\omega}^{-1} \cdot x) \end{array} .$$

Take  $(\beta, \lambda) \in S(\Omega, X)$  and  $\hat{g} = (\alpha, g, \tau, \beta) \in (\mathbf{G} \wr \Omega)_{\pi}$ , we define  $\hat{g}(\beta, \lambda) = (\alpha, \nu)$ , where  $\nu$  is a measure on  $\Omega \times_{\alpha} X$  defined by

$$\langle F, \nu \rangle = \langle \hat{g}^{-1} F, \lambda \rangle \text{ for } F \in C(\Omega \times_{\alpha} X).$$

By definition, for  $V \subseteq \Omega$  Borel,  $\nu|_{\Omega}(V) = \nu(V \times X) = \lambda(\tau^{-1}(V) \times X) = \lambda|_{\Omega}(\tau^{-1}(V)) = \mu(\tau^{-1}(V)) = \mu(V)$ . This implies that  $(\alpha, \nu) \in S(\Omega, X)$ . Now take  $\hat{g} = (\alpha, g, \tau, \beta)$  and  $\hat{h} = (\beta, h, \rho, \gamma) \in (\mathbf{G} \wr \Omega)_{\pi} \simeq S(\Omega, X)$ , and  $F \in C(\Omega \times_{\alpha} X)$ . For  $(\omega, x) \in \Omega \times_{\gamma} X$ , we have

$$\begin{aligned} (\hat{g}(\hat{h}F))(\omega, x) &= (\hat{h}F)(\tau^{-1}(\omega), g_{\omega}^{-1} \cdot x) \\ &= F(\rho^{-1}(\tau^{-1}(\omega)), h_{\tau^{-1}(\omega)}^{-1} g_{\omega}^{-1} \cdot x) \\ &= F((\tau\rho)^{-1}(\omega), (g(\tau \cdot h))_{\omega}^{-1} \cdot x) \\ &= ((\hat{g}\hat{h})F)(\omega, x). \end{aligned}$$

This verifies associativity. We have thus an action  $(\mathbf{G} \wr \Omega)_{\pi} \curvearrowright S(\Omega, X)$ . In order to describe this action using the the homeomorphism given by [Lemma 4.1.4](#) we will need the notion of pushforward measure.

**Definition 4.1.5.** Given two measurable spaces  $(Z_1, \Sigma_1)$  and  $(Z_2, \Sigma_2)$ , a measurable map  $T : Z_1 \rightarrow Z_2$  and a measure  $\mu$  on  $Z_1$ , the *pushforward measure*,  $T_*(\mu)$ , is the measure on  $Z_2$  given by  $T_*(\mu)(A) = \mu(T^{-1}(A))$  for  $A \in \Sigma_2$ .

Take  $(\beta, \lambda) \in S(\Omega, X)$ , let  $(\beta, p) \in 2^{\mathbb{N}} \times L_2^{w*}(\Omega, C(X)^*)$  be the corresponding map and  $\hat{g} = (\alpha, g, \tau, \beta) \in (\mathbf{G} \wr \Omega)_{\pi}$ . For every  $\omega \in \Omega$  we have a homeomorphism  $\tilde{g}_{\omega} : X_{\pi(\beta)(\tau^{-1}(\omega))} \rightarrow X_{\pi(\alpha(\omega))}$ . Then, if  $(\alpha, q) = \hat{g}(\beta, p)$ , a direct verification shows that  $q(\omega) = \tilde{g}_{\omega,*}(p(\tau^{-1}(\omega)))$ .

We will now consider quotients of  $X$  on which  $\mathbf{G}$  acts. Let  $E$  be a closed equivalence relation on  $X$ . Then, the quotient space,  $X/E$ , is a compact space. From now on we assume that  $E$  is such that  $X/E$  is Hausdorff as well. We let  $q_E : X \rightarrow X/E$  be the quotient map and  $[x]_E$  denote the class of  $x \in X$ . We may drop the subindex if the relation is clear from the context. We say that  $E$  is an *equivalence relation over  $\mathbf{B}$*  if for every  $x, y \in X$ ,  $xEy$  implies  $\rho(x) = \rho(y)$ . By the universal property of the quotient map, if  $E$  is an equivalence relation over  $\mathbf{B}$  then there is a unique continuous application  $\rho_E : X/E \rightarrow \mathbf{B}$  such that  $\rho = \rho_E \circ q_E$ . In this way,  $(X/E, \rho_E)$  is a space over  $\mathbf{B}$ . If  $\mathbf{G}$  is a groupoid over  $\mathbf{B}$  acting on  $X$  we say that  $E$  is  *$\mathbf{G}$ -invariant* if  $E$  is an equivalence relation over  $\mathbf{B}$  and  $xEy$  implies  $g \cdot x E g \cdot y$  whenever  $g \cdot x$  (and hence  $g \cdot y$ ) is defined. In this case, we have a well-defined action of  $\mathbf{G}$  on  $X/E$  given by  $g \cdot [x]_E = [g \cdot x]_E$ .

**Lemma 4.1.6.** *Let  $(X, \rho)$  be a compact, metrizable  $\mathbf{G}$ -space over  $\mathbf{B}$  and let  $E$  be a closed,  $\mathbf{G}$ -invariant equivalence relation on  $X$ . Then,  $(X/E, \rho_E)$  is a compact  $\mathbf{G}$ -space over  $\mathbf{B}$ .*

*Proof.* We have already seen that  $(X/E, \rho_E)$  is a compact space over  $\mathbf{B}$  on which  $\mathbf{G}$  acts. It remains to show that the action is continuous. Consider the equivalence relation  $\tilde{E}$  on  $\mathbf{G} \times X$  given by  $(g, x)\tilde{E}(h, y)$  if and only if  $g = h$  and  $xEy$ . It follows that  $\tilde{E}$  is a closed equivalence relation on  $\mathbf{G} \times X$ . We claim that  $(\mathbf{G} \times X)/\tilde{E}$  is homeomorphic to  $\mathbf{G} \times X/E$  via  $[(g, x)]_{\tilde{E}} \mapsto (g, [x]_E)$ . This map is clearly a continuous bijection. Take  $\mathcal{U} \subseteq (\mathbf{G} \times X)/\tilde{E}$  open and let  $\mathcal{U}'$  be its image in  $\mathbf{G} \times X/E$ . Pick  $(g, [x]_E) \in \mathcal{U}'$ , this means that  $[(g, x)]_{\tilde{E}} \in \mathcal{U}$ . Therefore,

$\{g\} \times [x]_E \subseteq q_E^{-1}(\mathcal{U})$ . The set on the right hand side is open, by compactness, there is  $V \subseteq \mathbf{G}$  and  $W \subseteq X$  open sets satisfying  $\{g\} \times [x] \subseteq V \times W \subseteq q_E^{-1}(\mathcal{U})$ . Since  $q_E$  is closed, there is a saturated open  $W_0 \subseteq X$  with  $[x]_E \subseteq W_0 \subseteq W$ . It follows that  $q_E(W_0)$  is an open subset of  $X/E$  and  $(g, [x]_E) \in V \times q_E(W_0) \subseteq \mathcal{U}'$ . Hence,  $\mathcal{U}'$  is open. This shows that the map defined above is indeed a homeomorphism. This application establishes an identification between  $(\mathbf{G} \times_{\mathbf{B}} X)/\tilde{E}$  and  $\mathbf{G} \times_{\mathbf{B}} X/E$ . The map,  $\mathbf{G} \times_{\mathbf{B}} X \rightarrow X/E$ ,  $(g, x) \mapsto [g \cdot x]_E$ , is continuous. By the universal property of the quotient, the function  $\mathbf{G} \times_{\mathbf{B}} X/E \rightarrow X/E$  given by  $(g, [x]_E) \mapsto [g \cdot x]_E$  is continuous. This completes the proof.  $\square$

**Corollary 4.1.7.** *Let  $(X, \rho)$  be a compact  $\mathbf{G}$ -space over  $2^{\mathbb{N}}$  and let  $E$  be a closed,  $\mathbf{G}$ -invariant equivalence relation on  $X$ . Then,  $(\mathbf{G} \wr \Omega)_{\pi}$  acts continuously on  $S(\Omega, X/E)$ .*

We now state a technical lemma concerning pushforward measures.

**Lemma 4.1.8.** *Let  $X$  and  $Y$  be compact, metrizable spaces and let  $T : X \rightarrow Y$  be a continuous map. Then, the pushforward map  $T_* : \mathfrak{R}(X) \rightarrow \mathfrak{R}(Y)$  is continuous. Furthermore, if  $T$  is surjective then  $T_*$  is also surjective.*

*Proof.* Continuity of  $T_*$  follows easily from the fact that for every  $f \in C(Y)$ , the map  $f \circ T \in C(X)$ . Now suppose that  $T$  is surjective. Consider the equivalence relation  $E$  on  $X$  given by  $xEx'$  if and only if  $T(x) = T(x')$ . Since  $T$  is closed, the saturation of any closed set is closed. By [Theorem 2.3.6](#),  $E$  admits a Borel transversal,  $F$ . By [Theorem 2.3.5](#), for any  $W \subseteq F$  Borel,  $T(W)$  is again a Borel set. Now take any  $\mu \in \mathfrak{R}(Y)$  and define  $\lambda \in \mathfrak{R}(X)$  by  $\lambda(A) = \mu(T(A \cap F))$  for  $A \subseteq X$  Borel. Due to the fact that  $T$  is surjective, for any  $B \subseteq Y$ ,  $T(T^{-1}(B) \cap F) = B$ . This implies that  $T_*(\lambda) = \mu$ .  $\square$

For  $E$  a closed equivalence relation on  $X$  define  $p_E : \Omega \times X \rightarrow \Omega \times (X/E)$  by  $p_E(\omega, x) = (\omega, [x]_E)$  for  $(\omega, x) \in \Omega \times X$ . We will define a new equivalence relation  $\tilde{E}$  on  $S(\Omega, X)$ , we say that  $(\alpha, \lambda) \tilde{E} (\beta, \nu)$  if and only if  $\alpha = \beta$  and  $p_{E,*}(\lambda) = p_{E,*}(\nu)$ .

**Lemma 4.1.9.** *Suppose that  $(X, \rho)$  is a compact, metrizable  $\mathbf{G}$ -space over  $2^{\mathbb{N}}$  and that  $E$  is a closed equivalence relation on  $X$  over  $2^{\mathbb{N}}$ , then  $\tilde{E}$  is a closed equivalence relation on  $S(\Omega, X)$  over  $2^{\mathbb{N}}$ . In case  $E$  is  $\mathbf{G}$ -invariant then  $\tilde{E}$  is  $(\mathbf{G} \wr \Omega)_{\pi}$ -invariant.*

*Proof.* The equivalence relation  $\tilde{E}$  is closed because the pushforward map  $p_{E,*}$  is continuous. The fact that it is an equivalence relation over  $2^{\mathbb{N}}$  follows directly from the definition. Suppose  $E$  is  $\mathbf{G}$ -invariant, take  $(\beta, \lambda), (\beta, \lambda_0) \in S(\Omega, X)$  with  $(\beta, \lambda) \tilde{E} (\beta, \lambda_0)$  and  $\hat{g} = (\alpha, g, \tau, \beta) \in (\mathbf{G} \wr \Omega)_{\pi}$ . Let  $(\alpha, \nu) = \hat{g}(\beta, \lambda)$  and  $(\alpha, \nu_0) = \hat{g}(\beta, \lambda_0)$ , and take  $F \in C(\Omega \times_{\alpha} X/E)$ . A straightforward verification shows that  $\hat{g}^{-1}(F \circ p_E) = (\hat{g}^{-1}F) \circ p_E$ . It follows that,

$$\langle F, p_{E,*}(\nu) \rangle = \langle F \circ p_E, \nu \rangle = \langle \hat{g}^{-1}(F \circ p_E), \lambda \rangle = \langle (\hat{g}^{-1}F) \circ p_E, \lambda \rangle = \langle \hat{g}^{-1}F, p_{E,*}(\lambda) \rangle.$$

Due to the fact that  $p_{E,*}(\lambda) = p_{E,*}(\lambda_0)$  we get that  $\langle F, p_{E,*}(\nu) \rangle = \langle F, p_{E,*}(\nu_0) \rangle$ . Hence  $p_{E,*}(\nu) = p_{E,*}(\nu_0)$  and we conclude that  $\hat{g}(\beta, \lambda) \tilde{E} \hat{g}(\beta, \lambda_0)$ .  $\square$

It is easy to see that the quotient space  $S(\Omega, X)/\tilde{E}$  is Hausdorff. Indeed, pick  $[(\alpha, \lambda)]_{\tilde{E}} \neq [(\beta, \nu)]_{\tilde{E}}$ . If  $\alpha \neq \beta$  then there are disjoint open subsets of  $2^{\mathbb{N}}$  separating  $\alpha$  and  $\beta$ . The corresponding open sets in  $S(\Omega, X)$  are disjoint, saturated open sets separating  $(\alpha, \lambda)$  and  $(\beta, \nu)$ . If  $\alpha = \beta$  then there is  $F \in C(\Omega \times X/E)$  such that  $\langle F, p_{E,*}(\lambda) \rangle \neq \langle F, p_{E,*}(\nu) \rangle$ . Use the map  $F \circ p_E \in C(\Omega \times X)$  to define disjoint, saturated open subsets of  $S(\Omega, X)$  separating  $(\alpha, \lambda)$  and  $(\beta, \nu)$ .

In virtue of the previous lemma we have a continuous action  $(\mathbf{G} \wr \Omega)_{\pi} \curvearrowright S(\Omega, X)/\tilde{E}$  which is equivalent to the action  $(\mathbf{G} \wr \Omega)_{\pi} \curvearrowright S(\Omega, X/E)$  as the next theorem shows.

**Theorem 4.1.10.** *Let  $(X, \rho)$  be a compact, metrizable  $\mathbf{G}$ -space over  $\mathbf{B}$ . Suppose that  $E$  is a closed, equivalence relation on  $X$  over  $2^{\mathbb{N}}$ , then  $S(\Omega, X)/\tilde{E}$  is homeomorphic to  $S(\Omega, X/E)$ . If  $E$  is  $\mathbf{G}$ -invariant then said homeomorphism is  $(\mathbf{G} \wr \Omega)_\pi$ -equivariant.*

*Proof.* Define  $\phi : S(\Omega, X)/\tilde{E} \rightarrow S(\Omega, X/E)$  by  $\phi([\alpha, \lambda]_{\tilde{E}}) = (\alpha, p_{E,*}(\lambda))$ . By the definition of  $\tilde{E}$ , the map is well-defined and it is also one-to-one. Now take  $(\alpha, \nu) \in S(\Omega, X/E)$ , by [Lemma 4.1.8](#), there is  $\lambda \in \mathfrak{R}(\Omega \times X)$  such that  $\nu = p_{E,*}(\lambda)$ . For  $A \subseteq \Omega$  measurable, we have

$$\lambda|_{\Omega}(A) = \lambda(A \times X) = p_{E,*}(\lambda)(A \times X/E) = \nu(A \times X/E) = \nu|_{\Omega}(A) = \mu(A).$$

We also have that  $\Omega \times_{\alpha} X = p_E^{-1}(\Omega \times_{\alpha} X/E)$  and so  $\lambda(\Omega \times_{\alpha} X) = \nu(\Omega \times_{\alpha} X/E) = 1$ . It follows that  $(\alpha, \lambda) \in S(\Omega, X)$  and that  $\phi([\alpha, \lambda]_{\tilde{E}}) = (\alpha, \nu)$ . So,  $\phi$  is a bijection. Since  $p_{E,*}$  is continuous then  $\phi$  is also continuous. Due to the fact that  $S(\Omega, X)/\tilde{E}$  is compact and  $S(\Omega, X/E)$  is Hausdorff, we obtain that  $\phi$  is a homeomorphism.

It remains to show that  $\phi$  is  $(\mathbf{G} \wr \Omega)_\pi$ -equivariant. Take  $[\beta, \lambda]_{\tilde{E}} \in S(\Omega, X)/\tilde{E}$  and  $\hat{g} = (\alpha, g, \tau, \beta) \in (\mathbf{G} \wr \Omega)_\pi$ . Define  $(\alpha, \nu) = \hat{g}(\beta, \lambda)$  and  $(\alpha, \nu_0) = \hat{g}(\beta, p_{E,*}(\lambda))$ . For  $F \in C(\Omega \times_{\alpha} X/E)$  we have  $\hat{g}^{-1}(F \circ p_E) = (\hat{g}^{-1}F) \circ p_E$  and so,  $\langle F, \nu_0 \rangle = \langle F, p_{E,*}(\nu) \rangle$ . Thus,

$$\phi(\hat{g}[\beta, \lambda]_{\tilde{E}}) = \phi([\alpha, \nu]_{\tilde{E}}) = (\alpha, p_{E,*}(\nu)) = (\alpha, \nu_0) = \hat{g}(\beta, p_{E,*}(\lambda)) = \hat{g}\phi([\beta, \lambda]_{\tilde{E}}).$$

This completes the proof. □

## 4.2 The space of partial types

Let  $T$  be a complete, countable theory admitting a universal Skolem sort,  $D_T$ , and let  $F$  be any sort of  $T$ . In this section we will study closed equivalence relations on  $S_{D_T \times F}(T)$  that refines equality of the target and are invariant under the action of  $\mathbf{G}(T)$ . We will see that they correspond precisely with equivalence relations defined from a set on formulas on  $D_T \times F$ . The next result shows that in order to study stability, or NIP, it suffices to consider formulas in sorts of the form  $D_T \times F$ .

**Lemma 4.2.1.** *Let  $T$  be a complete, countable theory admitting a universal Skolem sort  $D_T$ . If a formula  $\psi(v, u)$  in the sorts  $E \times F$  is unstable then there is a formula  $\varphi(y, u)$  in  $D_T \times F$  which is unstable.*

*Proof.* Since the sort  $D_T$  is universal, there is a definable surjective map  $\sigma : D_T \rightarrow E$ . Then,  $\varphi(y, u) = \psi(\sigma(y), u)$  also has the order property. □

By essentially the same proof, the analogous result holds for NIP.

**Lemma 4.2.2.** *Let  $T$  be a complete, countable theory admitting a universal Skolem sort  $D_T$ . If a formula  $\psi(v, u)$  in the sorts  $E \times F$  has IP then there is a formula  $\varphi(y, u)$  in  $D_T \times F$  which has IP.*

We now pass to the space of partial types. We should make a disclaimer at this point. We will define the space of partial types as a quotient of the space of types, as it is done in [\[BYU10, Section 6\]](#). First, we will study the  $\omega$ -categorical case in order to gain some intuition. To simplify notation, and contrary to convention, the left tuple of variables will play the role of parameters.

### 4.2.1 The $\omega$ -categorical case

Let  $T$  be an  $\omega$ -categorical theory,  $\mathcal{M}$  a countable model of  $T$ ,  $G = \text{Aut}(\mathcal{M})$  and  $\xi$  an enumeration of a tuple such that  $\text{dcl}(\xi) = M$ . We will denote by  $D_0$  the sort defined by  $\text{tp}(\xi)$ . We know by [BYT16, Section 2.2] that the Roelcke compactification of  $G$ ,  $R(G)$ , corresponds to  $S_{2D_0}(T)$ . We claim that  $R(G)$  can also be identified with  $S_{D_0}(M)$ . Indeed, pick  $\mathcal{M}'$  an  $\omega_1$ -saturated elementary extension of  $\mathcal{M}$ . Take  $q \in R(G)$ , then there is  $a \in D(M')$  such that  $q = \text{tp}(\xi, a)$ , and we identify  $q$  with  $\text{tp}(a/M)$ . Clearly, if  $\text{tp}(\xi, a) = \text{tp}(\xi, a')$  then  $\text{tp}(a/M) = \text{tp}(a'/M)$ , so it is a well-defined map. It is straightforward to verify that it is indeed a homeomorphism. The group  $G$  acts continuously on the left on  $R(G)$  via  $g \text{tp}(\xi, a) = \text{tp}(g^{-1}(\xi), a)$ . Using the previous identification, the action is given by

$$g \text{tp}(a/M) = \{\varphi(g^{-1}(c), u) = \varphi(c, a); \varphi(v, u) \text{ formula}, c \in M^{|v|}\}.$$

In general, for any sort  $F$  we have a continuous left action  $G \curvearrowright S_{D_0 \times F}(T)$ , defined in the same way as before, and a  $G$ -equivariant homeomorphism between  $S_F(M)$  and  $S_{D_0 \times F}(T)$ . We now state a technical result.

**Fact 4.2.3.** [BYU10, Fact 4.7] *Let  $X$  be a compact Hausdorff space and  $\mathcal{A} \subseteq C(X, [0, 1])$ . Define an equivalence relation on  $X$  by  $x \sim y$  if  $f(x) = f(y)$  for every  $f \in \mathcal{A}$ . Then,*

1. *Every  $f \in \mathcal{A}$  factors through the quotient  $\pi : X \rightarrow X/\sim$  as  $f = f_\sim \circ \pi$ .*
2. *The quotient topology on  $X/\sim$  is the initial topology given by the family of such maps  $\{f_\sim; f \in \mathcal{A}\}$ .*
3.  *$X/\sim$  is compact and Hausdorff.*

*Conversely, if  $Y$  is a compact Hausdorff space and  $\pi : X \rightarrow Y$  is a continuous projection, then  $Y$  is homeomorphic to  $X/\sim$  where  $\sim$  is obtained as above using  $\mathcal{A} = \{f \circ \pi; f \in C(Y, [0, 1])\}$ .*

Let  $u$  be a variable in  $F$  and let  $v$  be a tuple of variables in the home sort, infinite perhaps. Let  $\Phi$  be a set of formulas on  $v, u$ , we define the equivalence relation  $\sim_\Phi$  on  $S_F(M)$  by  $p \sim_\Phi q$  if and only if for every  $b \in M^{|v|}$ , and every  $\varphi(v, u) \in \Phi$ ,  $\varphi(b, u)^p = \varphi(b, u)^q$ . By the previous fact, The resulting quotient space, denoted  $S_\Phi(M)$ , is a compact Hausdorff space. The equivalence relation is moreover invariant under the left action of  $G$ . Thus,  $G$  acts continuously on  $S_\Phi(M)$ . We will denote the equivalence class of  $p \in S_F(M)$  by  $p \upharpoonright_\Phi$ .

We recall that for  $\sigma \in \mathbb{N}^{|v|}$  and  $a$  a tuple of length at least  $|v|$ ,  $a_\sigma$  stands for  $(a_{\sigma(0)}, a_{\sigma(1)}, \dots)$ . Since  $\text{dcl}(\xi) = M$ , we may only consider formulas with parameters in  $\xi$ . Hence, for  $p, q \in S_F(M)$ ,  $p \sim_\Phi q$  if and only if for every  $\sigma \in \mathbb{N}^{|v|}$  and every  $\varphi(v, u) \in \Phi$ ,  $\varphi(\xi_\sigma, u)^p = \varphi(\xi_\sigma, u)^q$ . Therefore, identifying  $S_F(M)$  with  $S_{D_0 \times F}(T)$  and letting  $y$  be a variable in  $D_0$ , if  $p, q \in S_{D_0 \times F}(T)$  then  $p \sim_\Phi q$  if and only if for every  $\sigma \in \mathbb{N}^{|v|}$  and every  $\varphi(v, u) \in \Phi$ ,  $\varphi(y_\sigma, u)^p = \varphi(y_\sigma, u)^q$ .

Take  $g \in G$ , then for any  $a \in D_0$  there is a unique  $\mathbf{g}(a) \in D_0$  such that  $\text{tp}(a, \mathbf{g}(a)) = \text{tp}(\xi, g(\xi))$ . This determines a definable map  $\mathbf{g} : D_0 \rightarrow D_0$ . Let  $\Delta$  be a set of formulas in  $D_0 \times F$ . For  $p, q \in S_{D_0 \times F}(T)$  we say that  $p \equiv_\Delta q$  if for every  $\varphi(y, u) \in \Delta$  and every  $g \in G$ ,  $\varphi(\mathbf{g}(y), u)^p = \varphi(\mathbf{g}(y), u)^q$ . The quotient space  $S_{D_0 \times F}(T)/\equiv_\Delta$  will be denoted  $\hat{S}_\Delta(T)$ . For  $p \in S_{D_0 \times F}(T)$  we denote its class in the quotient space as  $\hat{p} \upharpoonright_\Delta$ . Again,  $\hat{S}_\Delta(T)$  is a compact Hausdorff space on which  $G$  acts continuously.

So far we have defined two equivalence relations on  $S_{D_0 \times F}(T)$ . We will now show how to recover  $\sim_\Phi$  from  $\equiv_\Delta$ . Let  $\Phi$  be a set of formulas in  $u, v$ , for  $\varphi(u, v) \in \Phi$  and  $\sigma \in \mathbb{N}^{|v|}$  define

$\varphi_\sigma(y, u) = \varphi(y_\sigma, u)$ . Let us set  $\Delta_0 = \{\varphi_\sigma(y, u) ; \varphi(v, u) \in \Phi \text{ and } \sigma \in \mathbb{N}^{|\nu|}\}$ . It is easy to verify that for every  $p, q \in S_{D_0 \times F}(T)$ ,  $p \sim_\Phi q$  if and only if  $p \equiv_{\Delta_0} q$ . However, using  $\omega$ -categoricity we can find a smaller set of formulas. Since the action  $G \curvearrowright M$  is approximately oligomorphic, the quotient  $M^{|\nu|}/G$  is compact. This means that for every  $n \in \mathbb{N}$  there are  $\sigma_1^n, \dots, \sigma_{k_n}^n \in \mathbb{N}^{|\nu|}$  such that for every  $a \in M^{|\nu|}$  there are  $g \in G$  and  $i \leq k_n$  with  $d(g(\xi_{\sigma_i^n}), a) < 1/n$ . For  $\varphi(v, u) \in \Phi$ ,  $n \in \mathbb{N}$  and  $i \leq k_n$  let  $\varphi_i^n(y, u) = \varphi(y_{\sigma_i^n}, u)$  and define

$$\Delta = \{\varphi_i^n(y, u) ; \varphi(v, u) \in \Phi, n \in \mathbb{N} \text{ and } i \leq k_n\}.$$

Clearly, for every  $p, q \in S_{D_0 \times F}(T)$ ,  $p \equiv_{\Delta_0} q$  implies  $p \equiv_\Delta q$ . Now suppose that  $p \not\equiv_{\Delta_0} q$ , then, by exchanging the roles of  $p$  and  $q$  if necessary, there are  $\varphi(v, u) \in \Phi$ ,  $\sigma \in \mathbb{N}^{|\nu|}$ ,  $g \in G$  and  $\epsilon > 0$  such that  $\varphi_\sigma(\mathbf{g}(y), u)^p + \epsilon < \varphi_\sigma(\mathbf{g}(y), u)^q$ . Taking again  $\delta_\varphi$  as a modulus of uniform continuity for  $\varphi(v, u)$ , pick  $n \in \mathbb{N}$  with  $1/n < \delta_\varphi(\epsilon/2)$ . Let  $i \leq k_n$  and  $h \in G$  be such that  $d(h(\xi_{\sigma_i^n}), g(\xi_\sigma)) < 1/n$ . Then,

$$\varphi_i^n(\mathbf{h}(y), u)^p \leq \varphi(\mathbf{g}(y_\sigma), u)^p + \frac{\epsilon}{2} < \varphi(\mathbf{g}(y_\sigma), u)^q - \frac{\epsilon}{2} \leq \varphi_i^n(\mathbf{h}(y), u)^q.$$

This implies that  $p \not\equiv_\Delta q$ . So, from now on, we only need to consider the equivalence relation  $\equiv_\Delta$ . Moreover, if we close  $\Delta$  under connectors, uniform limits and the action of  $G$ , given by the definable maps  $\mathbf{g} : D_0 \rightarrow D_0$ , we obtain the same equivalence relation. Now assume that  $\cong$  is a closed  $G$ -invariant equivalence relation on  $S_{D_0 \times F}(T)$  such that  $S_{D_0 \times F}(T)/\cong$  is a Hausdorff space. Let  $\pi : S_{D_0 \times F}(T) \rightarrow S_{D_0 \times F}(T)/\cong$  be the quotient map. For any  $f \in C(S_{D_0 \times F}(T)/\cong, [0, 1])$ , the map  $\tilde{f} = f \circ \pi$  is continuous, so, it is obtained from a formula  $\varphi_f(y, u)$ . By [Fact 4.2.3](#),  $\cong$  agrees with the equivalence relation given by equality on the set of formulas

$$\Delta = \{\varphi_f(y, u) ; f \in C(S_{D_0 \times F}(T)/\cong, [0, 1])\},$$

Since  $\cong$  is  $G$ -invariant,  $G$  acts continuously on  $S_{D_0 \times F}(T)/\cong$ . Furthermore, for  $f \in C(S_{D_0 \times F}(T)/\cong, [0, 1])$  and  $g \in G$ ,  $\varphi_f(\mathbf{g}(y), u) = \varphi_{gf}(y, u)$ . Consequently  $\cong$  agrees with  $\equiv_\Delta$ .

Recall from [Subsection 2.1.2](#) that  $D_T = D_0 \times 2^{\mathbb{N}}$  is a universal Skolem sort for  $T$  and that its associated groupoid is  $\mathbf{G}(T) = 2^{\mathbb{N}} \times G \times 2^{\mathbb{N}}$ , where  $(\alpha, g, \beta)$  is identified with  $\text{tp}((\xi, \alpha), (g(\xi), \beta))$ . Our goal now is to define an equivalence relation, invariant under the action of  $\mathbf{G}(T)$ , on the space of types on  $D_T \times F$  from which we can recover the quotients presented thus far. The target map  $t : S_{D_T \times F}(T) \rightarrow 2^{\mathbb{N}}$  given by  $t(\text{tp}(a, \alpha, b)) = \alpha$  renders  $S_{D_T \times F}(T)$  a space over  $2^{\mathbb{N}}$ . In this way, we can define an action of  $\mathbf{G}(T)$  on  $S_{D_T \times F}(T)$  in the following manner, take  $p = \text{tp}(\xi, \beta, b) \in S_{D_T \times F}(T)$  and  $(\alpha, g, \beta) \in \mathbf{G}(T)$  and define

$$(\alpha, g, \beta) \cdot \text{tp}(\xi, \beta, b) = \text{tp}(g^{-1}(\xi), \alpha, b).$$

Let  $\dot{y}$  denote a variable of sort  $2^{\mathbb{N}}$  and let  $\Delta$  be a set of formulas in  $D_T \times F$ . An equivalence relation on  $S_{D_T \times F}(T)$  invariant under the action of  $\mathbf{G}(T)$  should refine equality of the target. Having this in mind, for  $p, q \in S_{D_T \times F}(T)$  we say that  $p =_\Delta q$  if and only if  $t_p = t_q$  and, assuming  $p = \text{tp}(\xi, \beta, b)$  and  $q = \text{tp}(\xi, \beta, c)$ , for every  $\varphi(y, \dot{y}, u) \in \Delta$  and every  $(\alpha, g, \beta) \in \mathbf{G}(T)$ ,  $\varphi(g(\xi), \alpha, b) = \varphi(g(\xi), \alpha, c)$ . The equivalence relation will be closed and invariant under the action of  $\mathbf{G}(T)$ , as will be shown in the general case.

Assume now that the formulas in  $\Delta$  depend only on  $y, u$ . In this case, for  $p, q \in S_{D_T \times F}(T)$ ,  $p =_\Delta q$  if and only if  $t_p = t_q$  and  $p \equiv_\Delta q$ . Thus,  $S_{D_T \times F}(T)/=_\Delta$  is homeomorphic to  $\hat{S}_\Delta(T) \times 2^{\mathbb{N}}$ . Consequently, the  $\mathbf{G}(T)$ -space  $S_{D_T \times F}(T)/=_\Delta$  is just the base change to  $2^{\mathbb{N}}$  of the action  $G \curvearrowright \hat{S}_\Delta(T)$ . Therefore, we will only consider the relation  $=_\Delta$  and extend it to any theory with a universal Skolem sort.

### 4.2.2 The general case

Let  $T$  be a complete, countable theory admitting a universal Skolem sort  $D_T$  and let  $F$  be any sort of  $T$ .  $u$  will denote a variable in  $F$  and  $y$  a variable in  $D_T$ . Take  $a \in D_T$  and  $g \in \mathbf{G}(T)$  with  $\text{tp}(a) = s_g$ , then there is a unique element in  $D_T$ , which we denote  $ga$ , such that  $g = \text{tp}(ga, a)$ . The target map, which by abuse of notation we will also call  $t$ , renders  $S_{D_T \times F}(T)$  a space over  $2^{\mathbb{N}}$ .  $\mathbf{G}(T)$  acts continuously on  $S_{D_T \times F}(T)$  as the following lemma shows.

**Lemma 4.2.4.**  *$\mathbf{G}(T)$  acts continuously on  $S_{D_T \times F}(T)$  in the following manner: for  $\text{tp}(a, b) \in S_{D_T \times F}(T)$  and  $g \in \mathbf{G}(T)$  with  $\text{tp}(a) = s_g$ , let  $g \cdot \text{tp}(a, b) = \text{tp}(ga, b)$ .*

*Proof.* First we will show that it is well-defined. Suppose  $p = \text{tp}(a, b) = \text{tp}(a', b') \in S_{D_T \times F}(T)$  and  $g \in \mathbf{G}(T)$  with  $\text{tp}(a) = s_g$ . Let  $\varphi(y, u)$  be a formula, then there is a formula  $\psi(y, u)$  such that  $\varphi(ga, b) = \psi(a, b) = \psi(a', b') = \varphi(ga', b')$ . This implies that  $\text{tp}(ga, b) = \text{tp}(ga', b')$ . Now we verify that it is indeed an action, take  $\text{tp}(a, b) \in S_{D_T \times F}(T)$  and  $g, h \in \mathbf{G}(T)$  with  $\text{tp}(a) = s_g$  and  $t_g = s_h$ . Then,  $g = \text{tp}(ga, a)$  and  $h = \text{tp}(h(ga), ga)$ . It follows that  $hg = \text{tp}(h(ga), a) = \text{tp}((hg)a, a)$ . Thus,  $h(ga) = (hg)a$  and we obtain that

$$h(g \cdot \text{tp}(a, b)) = h \cdot \text{tp}(ga, b) = \text{tp}(h(ga), b) = \text{tp}((hg)a, b) = hg \cdot \text{tp}(a, b).$$

In order to show that the action is continuous we do a slight modification of the argument given in [BY22, Lemma 5.1]. Take  $g \in \mathbf{G}$  and  $p = \text{tp}(a, b) \in S_{D_T \times F}(T)$  with  $s_g = \text{tp}(a)$ . Let  $\varphi(y, u)$  be a formula with  $\varphi(ga, b) = 0$  and consider the open neighborhood  $[\varphi(y, u) < 1]$ . Since  $ga$  is definable from  $a$  there is a formula  $\psi(u, y)$  such that  $\varphi(ga, u) = \psi(a, u)$ . Let  $z$  denote another variable in  $D_T$  and define  $\chi(y, z) = \sup_u |\varphi(y, u) - \psi(z, u)|$ . It follows that  $g \in [\chi(y, z) < 1/2]$ ,  $p \in [\psi(z, u) < 1/2]$  and  $([\chi(y, z) < 1/2] \cap \mathbf{G}) \cdot [\psi(z, u) < 1/2] \subseteq [\varphi(y, u) < 1]$ .  $\square$

Given a set of formulas  $\Delta$  in  $D_T \times F$  we define an equivalence relation  $=_{\Delta}$  on  $S_{D_T \times F}(T)$ , as follows, if  $p, q \in S_{D_T \times F}(T)$  we say  $p =_{\Delta} q$  if and only if  $t_p = t_q$  and, assuming  $p = \text{tp}(a, b)$  and  $q = \text{tp}(a, c)$ , for every  $g \in \mathbf{G}(T)$  with  $s_g = \text{tp}(a)$  and every  $\varphi(y, u) \in \Delta$ ,  $\varphi(ga, b) = \varphi(ga, c)$ . For  $p \in S_{D_T \times F}(T)$ , let  $p \upharpoonright_{\Delta}$  be its equivalence class.

By [Fact 3.1.16](#),  $\mathbf{G}(T)$  admits  $s$ -sections, as well as  $t$ -sections. We will denote by  $\Gamma_s(\mathbf{G})$  and  $\Gamma_t(\mathbf{G})$  the families of  $s$ -sections and  $t$ -sections respectively. Every  $s$ -section gives rise to a definable map on  $D_T$  in the following way, take  $\mathbf{g} \in \Gamma_s(\mathbf{G}(T))$ , for  $a \in D_T$  let  $\mathbf{g}(a)$  be the unique element in  $D_T$  such that  $\text{tp}(\mathbf{g}(a), a) = \mathbf{g}(\text{tp}(a))$ . The map from  $D_T$  into  $D_T$  will also be denoted  $\mathbf{g}$ . The relation  $=_{\Delta}$  can be described in terms of  $s$ -sections.

**Lemma 4.2.5.** *Let  $\Delta$  be a set of formulas in  $u, y$ . Then, for every  $p, q \in S_{D_T \times F}(T)$ ,  $p =_{\Delta} q$  if and only if  $t_p = t_q$  and  $\varphi(\mathbf{g}(y), u)^p = \varphi(\mathbf{g}(y), u)^q$  for every  $\varphi(y, u) \in \Delta$  and every  $\mathbf{g} \in \Gamma_s(\mathbf{G}(T))$ .*

*Proof.* Let  $p, q \in S_{D_T \times F}(T)$  be such that  $p =_{\Delta} q$ . Assume that  $p = \text{tp}(a, b)$ ,  $q = \text{tp}(a, c)$  and take  $\varphi(y, u) \in \Delta$  and  $\mathbf{g} \in \Gamma_s(\mathbf{G}(T))$ . Let  $g = \mathbf{g}(\text{tp}(a))$  then

$$\varphi(\mathbf{g}(y), u)^p = \varphi(\mathbf{g}(a), b) = \varphi(ga, b) = \varphi(ga, c) = \varphi(\mathbf{g}(a), c) = \varphi(\mathbf{g}(y), u)^q.$$

Conversely, suppose that  $p, q \in S_{D_T \times F}(T)$  satisfy the condition stated in the lemma. Take  $\varphi(y, u) \in \Delta$  and  $g \in \mathbf{G}(T)$  with  $s_g = t_p = t_q$ . Suppose that  $p = \text{tp}(a, b)$ ,  $q = \text{tp}(a, c)$  and  $\mathbf{g}$  is a  $s$ -section passing through  $g$ . Then, as before,  $\varphi(ga, b) = \varphi(\mathbf{g}(b), a)$  and  $\varphi(ga, c) = \varphi(\mathbf{g}(b), c)$ , and the equality follows.  $\square$

**Proposition 4.2.6.**  *$=_{\Delta}$  is an equivalence relation invariant under the action of  $\mathbf{G}(T)$ .  $S_{\Delta}(T) = S_{D_T \times F}(T) / =_{\Delta}$  is a compact, Hausdorff space and  $\mathbf{G}(T)$  acts continuously on  $S_{\Delta}(T)$ .*

*Proof.* Suppose  $p =_{\Delta} q$  and take  $g \in \mathbf{G}(T)$  with  $t_p = t_q = s_g$ . Let  $\mathbf{h} \in \Gamma_s(\mathbf{G}(T))$  be arbitrary and take  $\mathbf{k} \in \Gamma_s(\mathbf{G}(T))$  containing  $\mathbf{h}(t_g)g$ . Suppose that  $p = \text{tp}(a, b)$  and  $q = \text{tp}(a, c)$ , then

$$\text{tp}(\mathbf{k}(a), a) = \mathbf{k}(\text{tp}(a)) = \mathbf{k}(s_g) = \mathbf{h}(t_g)g = \mathbf{h}(\text{tp}(ga))g = \text{tp}(\mathbf{h}(ga), ga)g = \text{tp}(\mathbf{h}(ga), a).$$

This implies that  $\mathbf{k}(a) = \mathbf{h}(ga)$ . Hence, for every  $\varphi(y, u) \in \Delta$ ,

$$\varphi(\mathbf{h}(y), u)^{gp} = \varphi(\mathbf{k}(a), b) = \varphi(\mathbf{k}(y), u)^p = \varphi(\mathbf{k}(y), u)^q = \varphi(\mathbf{k}(a), c) = \varphi(\mathbf{h}(y), u)^{gq}.$$

Thus,  $gp =_{\Delta} gq$ , which shows  $\mathbf{G}(T)$ -invariance. For  $n \in \mathbb{N}$  let  $\eta_n : 2^{\mathbb{N}} \rightarrow \{0, 1\}$  be the projection on the  $n$ -th coordinate. Since for each  $n \in \mathbb{N}$  the map  $p \in S_{D_T \times F}(T) \mapsto \eta_n(t_p)$  belongs to  $C(S_{D_T \times F}(T), [0, 1])$ , then equality on the target can be obtained as equality on a family of continuous functions. In this manner, by [Fact 4.2.3](#),  $S_{\Delta}(T) = S_{D_T \times F}(T) / =_{\Delta}$  is a compact, Hausdorff space. Continuity of the action of  $\mathbf{G}(T)$  on  $S_{\Delta}(T)$  is a consequence of [Lemma 4.1.6](#).  $\square$

As in the  $\omega$ -categorical case, every  $\mathbf{G}(T)$ -invariant equivalence relation on  $S_{D_T \times F}(T)$ , refining equality on the target, for which the quotient is a compact, Hausdorff space is of the form  $=_{\Delta}$  for some set of formulas  $\Delta$ . Suppose that  $\sim$  is an equivalence relation on  $S_{D_T \times F}(T)$  satisfying the previous conditions and let  $\pi : S_{D_T \times F}(T) \rightarrow S_{D_T \times F}(T) / \sim$  be the quotient map. For any given  $\mathbf{f} \in C(S_{D_T \times F}(T) / \sim, [0, 1])$ , the map  $\tilde{\mathbf{f}} = \mathbf{f} \circ \pi$  is continuous, so, it is given by a formula  $\varphi_{\mathbf{f}}(y, u)$ . We define  $\Delta = \{\varphi_{\mathbf{f}}(y, u) ; \mathbf{f} \in C(S_{D_T \times F}(T) / \sim, [0, 1])\}$ . We then have that  $\sim$  agrees with  $=_{\Delta}$ .

Let us fix a set of formulas  $\Delta$  in  $D_T \times F$ . For  $\mathbf{g} \in \Gamma_s(\mathbf{G})$  and  $p \in S_{D_T \times F}(T)$ , we set  $\mathbf{g}p = \mathbf{g}(t_p)p$ . Also, for  $\mathbf{f} \in C(S_{\Delta}(T), [0, 1])$  we have a map  $\mathbf{g}\mathbf{f} \in C(S_{\Delta}(T), [0, 1])$  given by  $\mathbf{g}\mathbf{f}(p \upharpoonright_{\Delta}) = \mathbf{f}(\mathbf{g}p \upharpoonright_{\Delta})$ . Take  $\sim$  to be equal to  $=_{\Delta}$ . The previous procedure gives us a set of formulas  $\Delta'$  such that  $=_{\Delta'}$  agrees with  $=_{\Delta}$ . Given  $\varphi(y, u) \in \Delta$ , let  $\mathbf{f}_{\varphi} : S_{\Delta}(T) \rightarrow [0, 1]$  be the corresponding map on the quotient. Define  $\mathcal{A} = \{\mathbf{f}_{\varphi} ; \varphi(y, u) \in \Delta\} \cup \{\eta_n \circ t ; n \in \mathbb{N}\}$ , where  $t$  denotes the target map on  $S_{\Delta}(T)$  and  $\eta_n$  the projection on the  $n$ -th component on  $2^{\mathbb{N}}$ . Let  $\mathcal{B}_{\Delta}$  be the subalgebra of  $C(S_{\Delta}(T), [0, 1])$  generated by  $\mathcal{A}$  after closing under connectors and left multiplication by  $\Gamma_s(\mathbf{G})$ . Since  $\mathcal{B}_{\Delta}$  separates points in  $S_{\Delta}(T)$ , by the Stone-Weierstrass theorem for lattices, it is uniformly dense in  $C(S_{\Delta}(T), [0, 1])$ . This means that  $\Delta'$  is the closure of  $\Delta \cup \{\eta_n \circ t ; n \in \mathbb{N}\}$  under connectors, left multiplication by  $\Gamma_s(\mathbf{G})$  and uniform limits. We can therefore assume that any set of formulas contains  $\{\eta_n \circ t ; n \in \mathbb{N}\}$  and is closed under these operations.

We will now study the randomization of these quotients. Recall from [Fact 4.1.1](#) that  $E^{\mathbb{N}}(\Omega)$  is in correspondence with  $(2^{\mathbb{N}})^{\Omega}$ , the set of measurable functions from  $\Omega$  into  $2^{\mathbb{N}}$ , by assigning to  $A \in E^{\mathbb{N}}(\Omega)$  the map  $f_A : \Omega \rightarrow 2^{\mathbb{N}}$  defined by  $f_A(\omega)_n = 0$  if and only if  $\omega \in A_n$ , for  $\omega \in \Omega$  and  $n \in \mathbb{N}$ . From [Fact 2.1.3](#) we know that we have a homeomorphism between  $S_{(2^{\mathbb{N}})^R}(T^R)$  and  $\mathfrak{R}(S_{2^{\mathbb{N}}}(T)) = \mathfrak{R}(2^{\mathbb{N}})$ . It follows that to every  $A \in E^{\mathbb{N}}(\Omega)$  we associate a measure  $\mu_A \in \mathfrak{R}(2^{\mathbb{N}})$ , which depends only on  $\text{tp}(A)$ , such that for any formula  $\psi(\dot{x})$  in  $2^{\mathbb{N}}$ ,  $\langle \psi, \mu_A \rangle = \mathbb{E}(\psi(A))$ . Recall that for  $n \in \mathbb{N}$  and  $\nu \in 2^n$ ,  $[\nu] = \{\alpha \in 2^{\mathbb{N}} ; \alpha \upharpoonright_n = \nu\}$ . Hence, for every  $\nu \in 2^{<\omega}$ ,  $\mu_A([\nu]) = \mu(A_{\nu})$ . Thus, if  $f_A$  is the corresponding map in  $(2^{\mathbb{N}})^{\Omega}$  then  $\mu_A$  is just the pushforward measure  $f_{A,*}(\mu)$ . In particular,  $\mu_{\Xi}$  corresponds to the usual coin-toss measure on  $2^{\mathbb{N}}$  and the same is true for any  $A \in D_{\text{APr},0}(\Omega)$ . Likewise, every  $A \in E^{\mathbb{N}}(\Omega)$  defines a measure  $\lambda_A \in \mathfrak{R}(\Omega)$  by  $\lambda_A(\Xi_{\nu}) = \mu(A_{\nu})$  for every  $\nu \in 2^{<\omega}$ . As before, the measure  $\lambda_A$  depends only on the type of  $A$ , and it is related to  $\mu_A$  via  $\langle \psi, \mu_A \rangle = \langle \psi \circ f_{\Xi}, \lambda_A \rangle$  for  $\psi(\dot{x})$  a formula in  $2^{\mathbb{N}}$ . We have thus established a homeomorphism between  $S_{E^{\mathbb{N}}}(T)$  and  $\mathfrak{R}(\Omega)$ . In particular, if  $A \in D_{\text{APr},0}(\Omega)$  then for every  $\nu \in 2^{<\omega}$ ,  $\lambda_A(\Xi_{\nu}) = \mu(A_{\nu}) = \mu(\Xi_{\nu})$  and so  $\lambda_A = \mu$ .

Let us repeat this process while considering an additional sort  $F$  of  $T$ . We let  $u$  denote a variable in the sort  $F$ . We know that  $S_{E^{\mathbb{N}} \times F^{\mathbb{R}}}(T) = S_{(2^{\mathbb{N}} \times F)^{\mathbb{R}}}(T^{\mathbb{R}})$  is homeomorphic to  $\mathfrak{R}(S_{2^{\mathbb{N}} \times F}(T))$ . Since  $S_{2^{\mathbb{N}} \times F}(T)$  is homeomorphic to  $2^{\mathbb{N}} \times S_F(T)$ , we get an identification between  $S_{(2^{\mathbb{N}} \times F)^{\mathbb{R}}}(T^{\mathbb{R}})$  and  $\mathfrak{R}(2^{\mathbb{N}} \times S_F(T))$ , which is homeomorphic to  $\mathfrak{R}(\Omega \times S_F(T))$ . Let  $\mathbf{p} \in S_{E^{\mathbb{N}} \times F^{\mathbb{R}}}(T)$  and let  $\sigma[\bar{y}, (\varphi_i(u))_{i < n}]$  be a formula in  $2^{\mathbb{N}} \times F$ , where  $\sigma$  is an  $L_{\text{RV}}$ -term and  $\varphi_i(u)$  is a formula in  $F$  for  $i < n$ . The measure  $\lambda_{\mathbf{p}} \in \mathfrak{R}(\Omega \times S_F(T))$  is defined by

$$\mathbb{E}(\sigma[\bar{y}, (\varphi_i(u))_{i < n}])^{\mathbf{p}} = \int \sigma[f_{\Xi}(\omega), (\varphi_i(u))_{i < n}]^q d\lambda_{\mathbf{p}}(\omega, q).$$

Note that this is essentially the homeomorphism described in [Iba17, Lemma 3.14] except that we do not consider parameters. In case  $\mathbf{p} \in S_{D_{\text{APr},0} \times F^{\mathbb{R}}}(T)$  then for any  $\nu \in 2^{<\omega}$ ,

$$\lambda_{\mathbf{p}}|_{\Omega}(\Xi_{\nu}) = \lambda_{\mathbf{p}}(\Xi_{\nu} \times S_F(T)) = \mu(\bar{x}_{\nu})^{\mathbf{p}} = \mu(\Xi_{\nu}).$$

Thus,  $\lambda|_{\Omega} = \mu$ . Conversely, if  $\lambda \in \mathfrak{R}(\Omega \times S_F(T))$  and  $\lambda|_{\Omega} = \mu$  then there is a type  $\mathbf{p} = \text{tp}(A, b) \in S_{E^{\mathbb{N}} \times F^{\mathbb{R}}}(T^{\mathbb{R}})$  such that  $\lambda = \lambda_{\mathbf{p}}$ . By a similar argument, for any  $\nu \in 2^{<\omega}$ ,  $\mu(A_{\nu}) = \mu(\bar{x}_{\nu})^{\mathbf{p}} = \lambda|_{\Omega}(\Xi_{\nu}) = \mu(\Xi_{\nu})$ . Therefore,  $A \in D_{\text{APr},0}$  and so  $\mathbf{p} \in S_{D_{\text{APr},0} \times F^{\mathbb{R}}}(T^{\mathbb{R}})$ . This motivates the next definition and proves the proposition that follows.

**Definition 4.2.7.** Let  $X$  be a compact space. We define

$$S_0(\Omega, X) = \{\lambda \in \mathfrak{R}(\Omega \times X) ; \lambda|_{\Omega} = \mu\}.$$

**Proposition 4.2.8.** Let  $F$  be a sort of  $T$ , then  $S_{D_{\text{APr},0} \times F^{\mathbb{R}}}(T^{\mathbb{R}})$  is homeomorphic to  $S_0(\Omega, S_F(T))$ .

Since  $D_{\text{APr}} = D_{\text{APr},0} \times 2^{\mathbb{N}}$ , the next result is immediate.

**Corollary 4.2.9.** Let  $F$  be a sort of  $T$ , then  $S_{D_{\text{APr}} \times F^{\mathbb{R}}}(T^{\mathbb{R}})$  is homeomorphic to  $2^{\mathbb{N}} \times S_0(\Omega, S_F(T))$ .

Suppose  $T$  admits a universal Skolem sort,  $D_T$ , and fix  $F$  any sort of  $T$ . We will now describe  $S_{D_{T^{\mathbb{R}}} \times F^{\mathbb{R}}}(T^{\mathbb{R}})$  in terms of  $S_{D_T \times F}(T)$ . Recall that, under the homeomorphism between  $E^{\mathbb{N}}$  and  $(2^{\mathbb{N}})^{\Omega}$  defined in Section 2.2, the map  $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^{\Omega}$  is defined by  $\pi(\alpha) = \pi(\Xi, \alpha)$  for  $\alpha \in 2^{\mathbb{N}}$ . Also, for any given  $\alpha \in 2^{\mathbb{N}}$ , we defined the fiber product

$$\Omega \times_{\alpha} S_{D_T \times F}(T) = \{(\omega, q) \in \Omega \times S_{D_T \times F}(T) ; \pi(\alpha)(\omega) = t_q\}.$$

Following Definition 4.1.2 we have

$$S(\Omega, S_{D_T \times F}(T)) = \{(\alpha, \lambda) \in 2^{\mathbb{N}} \times S_0(\Omega, S_{D_T \times F}(T)) ; \lambda(\Omega \times_{\alpha} S_{D_T \times F}(T)) = 1\}.$$

On one hand, we know that  $\mathbf{G}(T)$  acts continuously on  $S_{D_T \times F}(T)$ . This induces an action of  $(\mathbf{G}(T) \wr \Omega)_{\pi}$  on  $S(\Omega, S_{D_T \times F}(T))$ . On the other hand,  $(\mathbf{G}(T) \wr \Omega)_{\pi}$  is the groupoid associated to the randomization and as such it acts continuously on  $S_{D_{T^{\mathbb{R}}} \times F^{\mathbb{R}}}(T^{\mathbb{R}})$ . The next result shows that this two actions are isomorphic.

**Theorem 4.2.10.** Let  $T$  be a complete, countable theory admitting a universal Skolem sort,  $D_T$ , and let  $F$  be a sort of  $T$ . Then, there is a  $(\mathbf{G}(T) \wr \Omega)_{\pi}$ -equivariant homeomorphism between  $S_{D_{T^{\mathbb{R}}} \times F^{\mathbb{R}}}(T^{\mathbb{R}})$  and  $S(\Omega, S_{D_T \times F}(T))$ .

*Proof.* By the previous corollary,  $S_{D_T^{\mathbb{R}} \times D_{\text{APr}} \times F^{\mathbb{R}}}(T^{\mathbb{R}})$  is homeomorphic to  $2^{\mathbb{N}} \times S_0(\Omega, S_{D_T \times F}(T))$ . We must show that this map restricts to the desired homeomorphism. If  $\mathbf{p} = \text{tp}(a, \Xi, \alpha, b) \in S_{D_{T^{\mathbb{R}}} \times F^{\mathbb{R}}}(T^{\mathbb{R}})$  then it defines a measure  $\lambda \in S_0(\Omega, S_{D_T \times F}(T))$ . We must show that  $\lambda(\Omega \times_{\alpha} S_{D_T \times F}(T)) = 1$ . For  $n \in \mathbb{N}$  let  $V_n = \{\beta \in 2^{\mathbb{N}} ; \beta_n = 0\}$ ,  $U_n = 2^{\mathbb{N}} \setminus V_n$  and define

$$B_n = (\pi(\alpha)^{-1}(V_n) \times [\text{tp}_n(x) = 0]) \cup (\pi(\alpha)^{-1}(U_n) \times [\text{tp}_n(x) = 1]).$$

Since  $\Omega \times_\alpha S_{D_T \times F}(T) = \bigcap_{n \in \mathbb{N}} B_n$ , it is enough to show that  $\lambda(B_n) = 1$  for every  $n \in \mathbb{N}$ . Take  $n \in \mathbb{N}$ , due to the fact that  $\text{tp}^R(a) = \pi(\alpha)$ , we get

$$\lambda(B_n) = \mu[(\pi(\alpha)_n \cap [\text{tp}_n(a) = 0]) \cup (\neg\pi(\alpha) \cap [\text{tp}_n(a) = 1])] = 1.$$

Conversely, suppose that  $(\alpha, \lambda) \in S(\Omega, S_{D_T \times F}(T))$ . In particular,  $\lambda \in S_0(\Omega, S_{D_T \times F}(T))$ , so, it corresponds to a type  $\mathbf{p}_0 \in S_{D_T^R \times D_{\text{APr}, 0} \times F^R}(T^R)$ . Assume that  $p_0 = \text{tp}(a, \Xi, b)$  and let  $\mathbf{p} = \text{tp}(a, \Xi, \alpha, b)$ . By hypothesis,  $\lambda(B_n) = 1$  for every  $n \in \mathbb{N}$ . This implies that  $\text{tp}_n^R(a) = \pi(\alpha)_n$ . Hence,  $\mathbf{p} \in S_{D_{T^R} \times F}(T^R)$ .

It remains to show that the homeomorphism is  $(\mathbf{G}(T) \wr \Omega)_\pi$ -equivariant. Take  $\hat{g} = (\alpha, g, \tau, \beta) \in (\mathbf{G}(T) \wr \Omega)_\pi$  and  $\mathbf{p} = \text{tp}(a, \tau, \beta, b) \in S_{D_{T^R} \times F^R}(T^R)$ . Then, we can find some  $c \in D_T^R$  such that  $\hat{g} = \text{tp}((c, \tau^{-1}(\Xi), \alpha), (a, \Xi, \beta))$ . It follows,  $\hat{g}\mathbf{p} = \text{tp}(c, \tau^{-1}(\Xi), \alpha, b)$ . In this case, for  $\omega \in \Omega$ ,  $\text{tp}(c_\omega, a_\omega) = g_{\tau(\omega)}$  and so,  $g_{\tau(\omega)} \text{tp}(a_\omega, b_\omega) = \text{tp}(c_\omega, b_\omega)$ . Let  $\psi(\dot{y}, y, u) = \sigma[\dot{y}, (\varphi_i(y, u))_{i < n}]$  be a formula in  $2^{\mathbb{N}} \times D_T \times F$ , where  $\sigma$  is an  $L_{RV}$ -term in  $E_{RV}^{\mathbb{N}} \times E_{RV}^n$  and  $\varphi_i(y, u)$  is a formula in  $D_T \times F$  for  $i < n$ . Let  $(\beta, p)$  be the couple in  $2^{\mathbb{N}} \times L_2^{w*}(\Omega, C(X)^*)$  corresponding to  $(\beta, \lambda_{\mathbf{p}})$  as per [Lemma 4.1.4](#). Then,

$$\begin{aligned} \langle \psi, \lambda_{\hat{g}\mathbf{p}} \rangle &= \mathbb{E}(\sigma[\tau^{-1}(\Xi), (\llbracket \varphi_i(c, b) \rrbracket)_{i < n}]) \\ &= \int_{\Omega} \sigma[f_{\Xi}(\tau(\omega)), (\varphi_i(y, \tau \cdot u))_{i < n}]^{\text{tp}(c_\omega, b_\omega)} d\mu(\omega) \\ &= \int_{\Omega} \sigma[f_{\Xi}(\tau(\omega)), (\varphi_i(y, \tau \cdot u))_{i < n}]^{g_{\tau(\omega)} \text{tp}(a_\omega, b_\omega)} d\mu(\omega) \\ &= \int_{\Omega} \left( \int_{X_{\pi(\beta)(\omega)}} \sigma[f_{\Xi}(\tau(\omega)), (\varphi_i(y, u))_{i < n}]^{g_{\tau(\omega)} \cdot q} dp(\omega)(q) \right) d\mu(\omega) \\ &= \langle \hat{g}^{-1}\psi, \lambda_{\mathbf{p}} \rangle \\ &= \langle \psi, \hat{g}\lambda_{\mathbf{p}} \rangle. \end{aligned}$$

This completes the proof.  $\square$

Fix  $\Delta$  a set of  $\mathcal{L}$ -formulas in the variables  $D_T \times F$ . We define  $\Delta^R$  as the set of formulas of the form  $\mathbb{E}(\sigma[\dot{y}, (\llbracket \varphi_i(y, u) \rrbracket)_{i < n}])$  where  $\sigma$  is a term in  $E_{RV}^{\mathbb{N}} \times E_{RV}^n$  and each  $\varphi_i(y, u)$  is a formula in  $\Delta$ . From [Proposition 4.2.6](#), we know that  $=_{\Delta}$  is a closed equivalence relation on  $S_{D_T \times F}(T)$  invariant under the action of  $\mathbf{G}(T)$ . By [Lemma 4.1.9](#), we get an induced equivalence relation on  $S(\Omega, S_{D_T \times F}(T))$ , this space being homeomorphic to  $S_{D_{T^R} \times F^R}(T^R)$  as we have just seen. In the following theorem we will see that this equivalence relation is just  $=_{\Delta^R}$ . We will denote  $\eta_{\Delta} : S_{D_T \times F}(T) \longrightarrow S_{\Delta}(T)$  and  $\eta_{\Delta^R} : S_{D_{T^R} \times F^R}(T^R) \longrightarrow S_{\Delta^R}(T^R)$  the quotient maps.

**Theorem 4.2.11.** *Let  $\Delta$  be a set of formulas in  $D_T \times F$ , then there is a  $(\mathbf{G} \wr \Omega)_\pi$ -equivariant homeomorphism between  $S_{\Delta^R}(T^R)$  and  $S(\Omega, S_{\Delta}(T))$  such that the following diagram commutes*

$$\begin{array}{ccc} S_{D_{T^R} \times F^R}(T^R) & \longrightarrow & S(\Omega, S_{D_T \times F}(T)) \\ \downarrow \eta_{\Delta^R} & & \downarrow \text{id}_{2^{\mathbb{N}}} \times \eta_{\Delta, * } \\ S_{\Delta^R}(T^R) & \longrightarrow & S(\Omega, S_{\Delta}(T)). \end{array}$$

*Proof.* We can assume that  $\Delta$  consists of the whole sets of formulas on  $S_\Delta(T)$ . Let us denote by  $\tilde{E}_\Delta$  the equivalence relation on  $S(\Omega, S_{D_T \times F}(T))$  induced by  $=_\Delta$  as in [Lemma 4.1.9](#). In virtue of [Theorem 4.1.10](#) and [Theorem 4.2.10](#), we have  $(\mathbf{G}; \Omega)_\pi$ -equivariant homeomorphisms between  $S(\Omega, S_\Delta(T))$  and  $S(\Omega, S_{D_T \times F}(T))/\tilde{E}_\Delta$ , and between  $S(\Omega, S_{D_T \times F}(T))$  and  $S_{D_{T^R} \times F^R}(T^R)$ . Therefore, it suffices to show that under the cited homeomorphism  $=_{\Delta^R}$  agrees with  $\tilde{E}_\Delta$ . Consider the formula  $\psi(y, \dot{y}, u) = \sigma[\dot{y}, (\varphi_i(y, u))_{i < n}]$ , where  $\sigma$  is a term in  $E_{\text{RV}}^{\mathbb{N}} \times E_{\text{RV}}^n$  and  $\varphi_i(y, u) \in \Delta$  for  $i < n$ . Since  $\mathbb{E}[\psi]$  belongs to  $\Delta^R$ , for every  $\mathbf{p} \in S_{D_{T^R} \times F^R}(T^R)$  we have

$$\langle \psi, \eta_{\Delta, *}( \lambda_{\mathbf{p}} ) \rangle = \mathbb{E}[\psi(y, u, \bar{y})]^{\mathbf{p}}.$$

Due to the fact that the family of functions defined by formulas of this form is dense in  $C(\Omega \times S_\Delta(T), [0, 1])$ , we conclude that  $\mathbf{p} =_{\Delta^R} \mathbf{q}$  if and only if  $(t_{\mathbf{p}}, \lambda_{\mathbf{p}}) \tilde{E}_\Delta (t_{\mathbf{q}}, \lambda_{\mathbf{q}})$ .  $\square$

We will now relate stability and the groupoid action on these quotients. Given a formula  $\varphi$  defined on some sort  $D$  we denote by  $\mathbf{f}_\varphi$  the corresponding map on the space of types, i.e.  $\mathbf{f}_\varphi : S_D(T) \rightarrow [0, 1]$  is given by  $\mathbf{f}_\varphi(p) = \varphi^p$ . The next proposition shows that stability of the formula  $\varphi$  is equivalent to the function  $\mathbf{f}_\varphi$  being wap, as defined in [Definition 3.6.3](#).

**Proposition 4.2.12.** *Let  $F$  be a sort of  $T$  and let  $\varphi(y, u)$  be a formula in  $D_T \times F$ . Then,  $\varphi$  is stable if and only if the map  $\mathbf{f}_\varphi : S_{D_T \times F}(T) \rightarrow [0, 1]$  is wap.*

*Proof.* Suppose first that  $\varphi(y, u)$  is stable. Fix  $q \in S_{D_T}(T)$  and let us denote  $S_q(T) = \{p \in S_{D_T \times F}(T) ; t(p) = q\}$ . Let  $(g_n)_{n \in \mathbb{N}} \subseteq q\mathbf{G}(T)$  and  $(p_m)_{m \in \mathbb{N}} \subseteq S_q(T)$  be sequences such that  $\lim_n \lim_m g_n \mathbf{f}_\varphi(p_m)$  and  $\lim_m \lim_n g_n \mathbf{f}_\varphi(p_m)$  both exist. By stability, the double limits must agree. It follows from Grothendieck's criterion that  $\{g \mathbf{f}_\varphi ; g \in q\mathbf{G}(T)\}$  is relatively weakly compact in  $C(S_q(T))$ , which shows that  $\mathbf{f}_\varphi$  is wap. Now suppose that  $\varphi(x, y)$  is not stable. Then there is  $\mathcal{M}$  separable model of  $T$ ,  $(a_n)_{n \in \mathbb{N}} \subseteq D_T(\mathcal{M})$ ,  $(b_m)_{m \in \mathbb{N}} \subseteq F(\mathcal{M})$  and reals  $r < s$  such that  $\varphi(a_n, b_m) \leq r$  for  $n < m$  and  $\varphi(a_n, b_m) \geq s$  for  $n > m$ . By [[BY22](#), Lemma 4.14], the set of elements in  $D_T(\mathcal{M})$  interdefinable with  $\mathcal{M}$  is dense in  $D_T(\mathcal{M})$ . We may therefore assume that each  $a_n$  is interdefinable with  $\mathcal{M}$ . For  $m, n \in \mathbb{N}$  let  $g_n = \text{tp}(a_0, a_n)$  and  $p_m = \text{tp}(a_0, b_m)$ . If we denote  $q = \text{tp}(a_0)$ , we have that  $(g_n)_{n \in \mathbb{N}}$  is a sequence in  $q\mathbf{G}(T)$  and  $(p_m)_{m \in \mathbb{N}}$  is a sequence in  $S_q(T)$  such that  $g_n \mathbf{f}_\varphi(p_m) \leq r$  for  $n < m$  and  $g_n \mathbf{f}_\varphi(p_m) \geq s$  for  $n > m$ . By diagonalization, there is a double subsequence such that the iterated limits exist but they are different. Using Grothendieck's criterion we obtain that  $\{g \mathbf{f}_\varphi ; g \in q\mathbf{G}(T)\}$  is not relatively weakly compact in  $C(S_q(T))$ . We conclude that  $\mathbf{f}_\varphi$  is not wap.  $\square$

Let  $\varphi(y, u)$  be a formula in  $D_T \times F$  and let  $\Delta$  be a set of formulas in  $D_T \times F$  containing  $\varphi$ . By [Fact 4.2.3](#), the map  $\mathbf{f}_\varphi$  factors through the quotient  $S_\Delta(T)$ . If  $\varphi$  is stable, the previous proposition together with [Theorem 3.6.10](#) imply that there is a representation of  $\mathbf{G}(T) \curvearrowright S_\Delta(T)$  on a field of reflexive spaces such that  $\mathbf{f}_\varphi$  agrees with the map defined by a section. Conversely, if we can find such a representation then we get from [Proposition 3.6.6](#) that  $\mathbf{f}_\varphi$  is wap and so  $\varphi$  is stable. We thus have proven the following result.

**Theorem 4.2.13.** *Let  $T$  be a complete, countable theory admitting a universal Skolem sort  $D_T$ . Let  $F$  be any sort of  $T$  and let  $\varphi(y, u)$  be a formula in  $D_T \times F$ . Then,  $\varphi$  is stable if and only if for any set of formulas  $\Delta$  in  $D_T \times F$  containing  $\varphi$  there is a representation  $\mathbf{G}(T) \curvearrowright \mathbb{V}$ ,  $\alpha : S_\Delta(T) \rightarrow B(\mathbb{V}^*)$ , on a field of reflexive Banach spaces  $(\mathbb{V}, \Gamma)$ , and there is  $\mathbf{v} \in \Gamma$  such that  $\varphi(y, u)^p = \langle \mathbf{v}, \alpha(p \upharpoonright \Delta) \rangle$  for every  $p \in S_{D_T \times F}(T)$ .*

### 4.3 Representation of the randomization of an action

Suppose that  $(X, \rho)$  is a compact, metrizable  $\mathbf{G}$ -space over  $2^{\mathbb{N}}$  and that  $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^{\Omega}$  is a continuous map. Recall from [Section 4.1](#) that we have an action of  $(\mathbf{G} \wr \Omega)_{\pi}$  on  $S(\Omega, X)$ . Assume moreover that we have a representation  $\mathbf{G} \curvearrowright \mathbb{V}$ ,  $\alpha : X \rightarrow B(\mathbb{V}^*)$ , of the original action. The aim of this section is to obtain a representation of  $(\mathbf{G} \wr \Omega)_{\pi} \curvearrowright S(\Omega, X)$ . The next definition is just a generalization of the construction of  $L_2(\Omega, V)$  for a Banach space  $V$ , as done for example in [\[DU77, Chapter II\]](#).

**Definition 4.3.1.** Let  $\{V_{\omega} ; \omega \in \Omega\}$  be a family of Banach spaces and let  $\Lambda$  be a vector subspace of  $\prod_{\omega \in \Omega} V_{\omega}$  satisfying:

1. If  $\hat{v}_0, \dots, \hat{v}_{n-1} \in \Lambda$  and  $A_0, \dots, A_{n-1} \in \mathcal{B}$  then  $\sum_{i < n} \chi_{A_i} \hat{v}_i \in \Lambda$ .
2. For every  $\omega \in \Omega$ ,  $\{\hat{v}(\omega) ; \hat{v} \in \Lambda\}$  is dense in  $V_{\omega}$ .
3. For every  $\hat{v} \in \Lambda$ , the map  $\omega \mapsto \|\hat{v}(\omega)\|$  is measurable and

$$\|\hat{v}\|_2 = \left( \int \|\hat{v}(\omega)\|^2 d\mu(\omega) \right)^{\frac{1}{2}} < +\infty.$$

We define  $L_2(\{V_{\omega} ; \omega \in \Omega\}, \Lambda)$  as the family of maps  $\hat{v} \in \prod_{\omega \in \Omega} V_{\omega}$  for which there is a sequence  $(\hat{v}_n)_{n \in \mathbb{N}}$  in  $\Lambda$  satisfying  $\lim_n \|\hat{v}_n - \hat{v}\| = 0$  almost everywhere, and whose  $L_2$ -norm  $\|\hat{v}\|_2$ , defined as above, is finite.

Note that if all  $V_{\omega}$  are equal to a fixed space  $V$  and  $\Lambda$  is the space of simple functions then  $L_2(\{V_{\omega} ; \omega \in \Omega\}, \Lambda)$  agrees with  $L_2(\Omega, V)$ . The proof of the following result is a straightforward adaptation of the analogous result for  $L_2(\Omega, V)$ .

**Lemma 4.3.2.**  $L_2(\{V_{\omega} ; \omega \in \Omega\}, \Lambda)$  together with the norm  $\|\cdot\|_2$  is a Banach space having  $\Lambda$  as a dense subset.

Let  $(\mathbb{V}, \Gamma)$  be a field of Banach spaces over  $2^{\mathbb{N}}$  and let  $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^{\Omega}$  be a continuous map. Fix  $\alpha \in 2^{\mathbb{N}}$ . Given  $A_0, \dots, A_{n-1} \in \mathcal{B}$  and  $\mathbf{v}_0, \dots, \mathbf{v}_{n-1} \in \Gamma$  let us define the map  $\hat{v} \in \prod_{\omega \in \Omega} V_{\pi(\alpha)(\omega)}$  by

$$\hat{v}(\omega) = \sum_{i < n} \chi_{A_i}(\omega) \mathbf{v}_i(\pi(\alpha)(\omega)).$$

Let  $\Lambda_{\alpha}$  denote the family of maps  $\hat{v} \in \prod_{\omega \in \Omega} V_{\pi(\alpha)(\omega)}$  of this form. It is easy to see that  $\Lambda_{\alpha}$  verifies properties 1. and 2. of [Definition 4.3.1](#). Assume that for every  $\mathbf{v} \in \Gamma$  the map  $\beta \rightarrow \|\mathbf{v}(\beta)\|$  is measurable. Then, the maps  $\omega \rightarrow \|\mathbf{v}(\pi(\alpha)(\omega))\|$  are also measurable and they are moreover bounded. Consequently,  $\Lambda_{\alpha}$  also verifies 3. It follows that  $L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha})$  is a Banach space. We set

$$L_2(\Omega, \mathbb{V})_{\pi} = \bigsqcup_{\alpha \in 2^{\mathbb{N}}} L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha}).$$

The construction of sections is quite straightforward. Given  $A_0, \dots, A_{n-1} \in \mathcal{B}$  and  $\mathbf{v}_0, \dots, \mathbf{v}_{n-1} \in \Gamma$  define  $\hat{\mathbf{v}} \in \prod_{\alpha \in 2^{\mathbb{N}}} L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha})$  by  $\hat{\mathbf{v}}(\alpha)(\omega) = \sum_{i < n} \chi_{A_i}(\omega) \mathbf{v}_i(\pi(\alpha)(\omega))$  for  $\alpha \in 2^{\mathbb{N}}$  and  $\omega \in \Omega$ . Let us denote by  $\Lambda$  the family of such sections. Clearly,  $\Lambda$  is closed under linear combinations and  $\Lambda_{\alpha}$  is dense in  $L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha})$  for every  $\alpha \in 2^{\mathbb{N}}$ . It is easy to verify that if  $M > 0$  satisfies  $\|\mathbf{v}_i(\beta)\| \leq M$  for every  $\beta \in 2^{\mathbb{N}}$  and every  $i < n$  then  $\|\hat{\mathbf{v}}(\alpha)\|_2 \leq M$  for every

$\alpha \in 2^{\mathbb{N}}$ . We thus conclude that  $(L_2(\Omega, \mathbb{V})_{\pi}, \Lambda)$  is a field of Banach spaces over  $2^{\mathbb{N}}$ .

When  $(\mathbb{V}, \Gamma)$  is a continuous field then  $\mathbb{V}$  is endowed with a topology as in **Fact 3.1.4**. The next lemma shows that in this case, for a separable field, the elements of  $L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha})$  consists of measurable maps of finite  $L_2$ -norm.

**Lemma 4.3.3.** *Suppose that  $(\mathbb{V}, \Gamma)$  is a separable, continuous field of Banach spaces over  $2^{\mathbb{N}}$ . Then, for  $\alpha \in 2^{\mathbb{N}}$ , a map  $\hat{v} \in \prod_{\omega \in \Omega} V_{\pi(\alpha)(\omega)}$  belongs to  $L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha})$  if and only if  $\hat{v} : \Omega \rightarrow \mathbb{V}$  is measurable and  $\|\hat{v}\|_2 = \left(\int_{\Omega} \|\hat{v}(\omega)\|^2 d\mu(\omega)\right)^{1/2} < +\infty$ .*

*Proof.*  $\Rightarrow$ ) Pick  $\hat{v} \in \Gamma$ , since this map is continuous, the map  $\mathbf{v} \circ \pi(\alpha) : \Omega \rightarrow \mathbb{V}$  is measurable. This implies that every map of  $\Lambda_{\alpha}$  is measurable. Therefore,  $\hat{v}$ , being an almost everywhere pointwise limit of maps of  $\Lambda_{\alpha}$ , is measurable. The  $L_2$ -norm is finite by definition.

$\Leftarrow$ ) Let  $\{\mathbf{v}_k ; k \in \mathbb{N}\}$  be a dense subset of  $\Gamma$  and fix  $n \geq 1$ . For every  $k \in \mathbb{N}$  define  $A_{k,n} = \{\omega \in \Omega ; \hat{v}(\omega) \in \mathcal{O}(\mathbf{v}_k, 1/n, 2^{\mathbb{N}})\}$  and  $B_{k,n} = A_{k,n} \setminus \bigcup_{i < k} A_{i,n}$ . By construction,  $\{B_{k,n} ; k \in \mathbb{N}\}$  is a measurable partition of  $\Omega$ . For every  $n \in \mathbb{N}$  pick  $l_n \in \mathbb{N}$  such that  $\mu(\bigcup_{k < l_n} B_{k,n}) \geq 1 - 1/2^n$ . Define  $\hat{v}_n = \sum_{k < l_n} \chi_{B_{k,n}}(\mathbf{v}_k \circ \pi(\alpha))$ . Let  $E_n = \bigcup_{k < l_n} B_{k,n}$  and  $E = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} E_n$ , we have that  $\mu(E) = 1$ . Take  $\omega \in E$  and let  $\epsilon > 0$  be arbitrary, then there exists  $N \in \mathbb{N}$  such that  $1/N < \epsilon$  and  $\omega \in \bigcap_{n \geq N} E_n$ . It follows that for  $n \geq N$ ,  $\|\hat{v}(\omega) - \hat{v}_n(\omega)\| \leq 1/n < \epsilon$ . This shows that  $\lim_n \|\hat{v} - \hat{v}_n\| = 0$  on  $E$ , which completes the proof.  $\square$

**Proposition 4.3.4.** *Let  $(\mathbb{V}, \Gamma)$  be a continuous field of Banach spaces. Then there is a unique family of sections  $\hat{\Lambda} \supseteq \Lambda$  such that  $(L_2(\Omega, \mathbb{V})_{\pi}, \hat{\Lambda})$  is a continuous field of Banach spaces.*

*Proof.* By **Proposition 3.1.2** it suffices to show that for every  $\hat{\mathbf{v}} \in \Lambda$  the map  $\alpha \rightarrow \|\hat{\mathbf{v}}(\alpha)\|_2$  is continuous. Suppose that  $\hat{\mathbf{v}} = \sum_{i < n} \chi_{A_i}(\mathbf{v}_i \circ \pi(\cdot))$  where  $\{A_0, \dots, A_{n-1}\}$  is a partition of  $\Omega$  and  $\mathbf{v}_0, \dots, \mathbf{v}_{n-1} \in \Gamma$ . Take  $(\alpha_m)_{m \in \mathbb{N}} \subseteq 2^{\mathbb{N}}$  converging to  $\alpha$ . Choose  $r > 0$  with  $r \geq \|\mathbf{v}_i(\beta)\|^2$  for every  $\beta \in 2^{\mathbb{N}}$  and every  $i < n$ . Now fix  $i < n$ . Due to the fact that the map  $\beta \rightarrow \|\mathbf{v}_i(\beta)\|^2$  is uniformly continuous, there is  $\delta_i > 0$  such that for every  $\beta, \beta' \in 2^{\mathbb{N}}$  if  $d(\beta, \beta') \leq \delta_i$  then  $|\|\mathbf{v}_i(\beta)\|^2 - \|\mathbf{v}_i(\beta')\|^2| < \epsilon^2/2$ . For  $m \in \mathbb{N}$ , let  $C_{i,m} = \llbracket d(\pi(\alpha_m), \pi(\alpha)) > \delta_i \rrbracket$ . By the continuity of  $\pi$ , there is  $M_i \in \mathbb{N}$  such that for every  $m \geq M_i$  we have  $\mu(C_{i,m}) < \frac{\epsilon^2}{4r}$ . It follows that for  $m \geq M_i$ ,

$$\int_{A_i} \left| \|\mathbf{v}_i(\pi(\alpha)(\omega))\|^2 - \|\mathbf{v}_i(\pi(\alpha_m)(\omega))\|^2 \right| d\mu(\omega) < 2r\mu(A_i \cap C_{i,m}) + \frac{\epsilon^2}{2}\mu(A_i \setminus C_{i,m}).$$

Define  $M = \max_{i < n} M_i$ , for  $m \geq M$  we have that

$$\left| \|\hat{\mathbf{v}}(\alpha)\|_2^2 - \|\hat{\mathbf{v}}(\alpha_m)\|_2^2 \right| \leq \sum_{i < n} \int_{A_i} \left| \|\mathbf{v}_i(\pi(\alpha)(\omega))\|^2 - \|\mathbf{v}_i(\pi(\alpha_m)(\omega))\|^2 \right| d\mu(\omega) < 2r\mu(C_{i,m}) + \frac{\epsilon^2}{2} < \epsilon^2.$$

We conclude that  $(\|\hat{\mathbf{v}}(\alpha_m)\|_2)_{m \in \mathbb{N}}$  converges to  $\|\hat{\mathbf{v}}(\alpha)\|_2$ .  $\square$

**Lemma 4.3.5.** *If  $(\mathbb{V}, \Gamma)$  is a separable, continuous field of Banach spaces then  $(L_2(\Omega, \mathbb{V})_{\pi}, \hat{\Lambda})$  is separable.*

*Proof.* Let  $\Gamma_0$  countable subset of  $\Gamma$  such that  $\{\mathbf{v}(\beta) ; \mathbf{v} \in \Lambda\}$  is dense in  $V_{\beta}$  for every  $\beta \in 2^{\mathbb{N}}$ . We may suppose that  $\Gamma_0$  is closed under rational linear combinations. Choose  $\mathcal{A}$  a countable, dense subalgebra of  $(\hat{\mathcal{B}}, \mu)$ , the measure algebra of  $(\Omega, \mathcal{B}, \mu)$ . Define  $\Lambda_0$  as the sections of  $\Lambda$  defined from sections in  $\Gamma_0$  and partitions of  $\Omega$  consisting of elements of  $\mathcal{A}$ , then for every  $\alpha \in 2^{\mathbb{N}}$ ,  $\{\hat{\mathbf{v}}(\alpha) ; \hat{\mathbf{v}} \in \Lambda_0\}$  is dense in  $L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha})$ .  $\square$

Suppose that  $\mathbf{G}$  is a topological groupoid over  $2^{\mathbb{N}}$  acting by linear isometries on a Banach field  $(\mathbb{V}, \Gamma)$ . Given  $(\beta, \hat{v}) \in L_2(\Omega, \mathbb{V})_{\pi}$  and  $(\alpha, g, \tau, \beta) \in (\mathbf{G} \wr \Omega)_{\pi}$  define  $(\alpha, g, \tau, \beta) \cdot (\beta, \hat{v})(\omega) = g_{\omega} \cdot \hat{v}(\tau^{-1}(\omega))$  for  $\omega \in \Omega$ . A straightforward computation shows that  $\hat{v}(\tau^{-1}(\omega)) \in V_{s_{g_{\omega}}}$  for every  $\omega \in \Omega$ . Hence,  $(\alpha, g, \tau, \beta) \cdot (\beta, \hat{v})(\omega)$  is well-defined. It is easy to check that it belongs to  $\prod_{\omega \in \Omega} V_{\pi(\alpha)(\omega)}$ . Suppose that the field  $(\mathbb{V}, \Gamma)$  is continuous and separable, and that the action  $\mathbf{G} \curvearrowright \mathbb{V}$  is continuous as well. We claim that  $(\alpha, g, \tau, \beta) \cdot (\beta, \hat{v}) \in L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha})$ . Indeed,  $(\alpha, g, \tau, \beta) \cdot (\beta, \hat{v})$  is the composition of the maps

$$\begin{array}{ccccc} \Omega & \longrightarrow & \mathbf{G} \times_{\mathbf{B}} \mathbb{V} & \longrightarrow & \mathbb{V} \\ \omega & \longmapsto & (g_{\omega}, \hat{v}(\tau^{-1}(\omega))) & \longmapsto & g_{\omega} \cdot \hat{v}(\tau^{-1}(\omega)), \end{array}$$

which thanks to [Lemma 4.3.3](#), are measurable. Moreover, since  $\mathbf{G}$  acts by isometries on  $\mathbb{V}$ , a direct verification shows that  $\|(\alpha, g, \tau, \beta) \cdot (\beta, \hat{v})\|_2 = \|(\beta, \hat{v})\|_2$ . We obtain in this manner a linear action by isometries  $(\mathbf{G} \wr \Omega)_{\pi} \curvearrowright L_2(\Omega, \mathbb{V})_{\pi}$ .

Now suppose that  $(X, \rho)$  is a compact, metrizable  $\mathbf{G}$ -space over  $2^{\mathbb{N}}$ . Assume that the action admits a representation on a field of Banach spaces  $(\mathbb{V}, \Gamma)$ . Let  $\xi : X \rightarrow B(\mathbb{V}^*)$  be the weak\* continuous map of the representation. Take  $(\alpha, \lambda) \in S(\Omega, X)$  and define  $\xi^R(\alpha, \lambda) : L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha}) \rightarrow \mathbb{R}$  by

$$\langle (\alpha, \hat{v}), \xi^R(\alpha, \lambda) \rangle = \int_{\Omega \times_{\alpha} X} \langle \hat{v}(\omega), \xi(x) \rangle d\lambda(\omega, x),$$

for  $(\alpha, \hat{v}) \in L_2(\{V_{\pi(\alpha)(\omega)} ; \omega \in \Omega\}, \Lambda_{\alpha})$ .  $\xi^R(\alpha, \lambda)$  is clearly linear and from Hölder's inequality we deduce that  $\|\xi^R(\alpha, \lambda)\| \leq 1$ . We thus get a weakly\* continuous map  $\xi^R : S(\Omega, X) \rightarrow B(L_2(\Omega, \mathbb{V})_{\pi}^*)$ . Consequently, we get a representation of  $(\mathbf{G} \wr \Omega)_{\pi} \curvearrowright S(\Omega, X)$  on  $(L_2(\Omega, \mathbb{V})_{\pi}, \Lambda)$ .

Now suppose that  $\mathbf{f} \in C(X)$  comes from the original representation. This is, there is  $\mathbf{v} \in \Gamma$  such that  $\mathbf{f}(x) = \langle \mathbf{v}, \xi(x) \rangle$  for  $x \in X$ , as in [Proposition 3.6.6](#). Then, the map in  $\mathbf{f}^R \in C(S(\Omega, X))$  coming from  $\hat{\mathbf{v}} \in \Lambda$  is given by

$$\mathbf{f}^R(\alpha, \lambda) = \langle \hat{\mathbf{v}}, \xi^R(\alpha, \lambda) \rangle = \int_{\Omega \times_{\alpha} X} \langle \mathbf{v}(\pi(\alpha)(\omega)), \xi(x) \rangle d\lambda(\omega, x) = \int_{\Omega \times_{\alpha} X} \mathbf{f}(x) d\lambda(\omega, x)$$

for  $(\alpha, \lambda) \in S(\Omega, X)$ . In particular when  $X$  is the space of types  $S_{D_T \times F}(T)$ ,  $\mathbf{f}$  is the function on the space of types given by a formula  $\varphi(y, u)$ , then the map  $\mathbf{f}^R$  is defined by the formula  $\mathbb{E}[\varphi(y, u)]$ . Consequently, if the field  $(\mathbb{V}, \Gamma)$  consists of reflexive spaces and  $(L_2(\Omega, \mathbb{V})_{\pi}, \Lambda)$  is also reflexive, then [Theorem 4.2.13](#) would imply that  $\mathbb{E}[\varphi(y, u)]$  is stable.

## 4.4 Further questions

We expect to show that if  $(\mathbb{V}, \Gamma)$  is a field of reflexive Banach spaces then  $(L_2(\Omega, \mathbb{V})_{\pi}, \Lambda)$  is also a field of reflexive Banach spaces. The analogous property for Banach spaces holds, see [\[DU77, Corollary IV.1.2\]](#). However, the proof of this result uses the fact that reflexive spaces have the Radon-Nikodym property, for which we do not have an analog in the context of Banach fields.

In [Section 3.7](#) we mentioned that we would like to study representations of tame functions on Rosenthal spaces in the context of Banach fields. In the  $\omega$ -categorical case, T. Ibarlucía in [\[Iba16\]](#) established the correspondence between tame functions and NIP formulas. Subsequently in [\[Iba17\]](#), he showed how this property is preserved by the randomization. We would like to explore this route in the context of theories admitting a universal Skolem sort. Namely, find a

dynamical analog of NIP and use representations on Banach fields to show that NIP is preserved by randomization.



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# Dynamique topologique de groupoïdes pour l'étude des aléatorisations

**Résumé:** L'objectif principal de cette thèse est l'étude de la préservation des propriétés modèle-théoriques par l'aléatorisation en utilisant des méthodes dynamiques, en se basant notamment sur les travaux de T. Ibarlucía. Les aléatorisations sont des structures continues dont les éléments sont des fonctions aléatoires d'un espace de probabilité sans atomes dans une famille de structures. Étant donnée une théorie  $T$ , son aléatorisation  $T^R$ , est la théorie commune aux aléatorisations de modèles de  $T$ . La nouvelle théorie préserve plusieurs propriétés de  $T$ , parmi elles stabilité et NIP. Récemment, Ben Yaacov a montré qu'à toute théorie dénombrable, complète admettant une sorte de Skolem universelle, on peut associer un groupoïde topologique qui détermine la théorie à bi-interprétation près. On prouve que si  $T$  admet une telle sorte alors son aléatorisation aussi. Ensuite, on étudie les champs d'espaces de Banach et les actions de groupoïdes. On montre que les fonctions faiblement presque-périodiques (ou wap par leur nom en anglais) sont précisément celles provenant de représentations sur des champs d'espaces réflexifs. Finalement, on étudie l'action du groupoïde associé à une théorie sur l'espace de types partiels. On montre que les formules stables correspondent aux fonctions wap. De plus, on montre comment construire une représentation de l'aléatorisation de l'action à partir d'une représentation de l'action originale.

**Mots clés :** Aléatorisation; sorte de Skolem universelle; représentation des actions de groupoïdes; stabilité.

## Topological dynamics of groupoids for the study of randomizations

**Abstract:** The primary objective of this thesis is the study of the preservation of model-theoretic properties by the randomization using dynamical methods, building on the work done by T. Ibarlucía. Randomizations are continuous structures consisting of random functions from an atomless probability space into a family of structures. Given a theory  $T$ , its randomization  $T^R$ , is the common theory of the randomizations of models of  $T$ . The new theory preserve many of the properties of  $T$ , among them stability and NIP. Recently, Ben Yaacov showed that to any complete, countable theory admitting a universal Skolem sort, we can associate a topological groupoid that determines the theory up to bi-interpretation. We show that if  $T$  admits such a sort then so it does its randomization. Subsequently, we study fields of Banach spaces and groupoid actions. We show that weakly almost periodic functions (wap for short) are those coming from representations on fields of reflexive spaces. Finally, we study the action of the groupoid associated to a theory on the space of partial types. We prove that stable formulas correspond to wap functions. Furthermore, we show how to build a representation of the randomization of the action starting from a representation of the original action.

**Keywords:** Randomizations; universal Skolem sort; representation of groupoid actions; stability.

**Image en couverture :** Alejandro Obregón, *Condor* (1979).

