Inverse potential problems, with applications to quasi-static electromagnetics

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Problèmes inverses de potentiel et applications à l'éléctromagnétique quasi-statique.

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Problèmes inverses de potentiel et applications à l'éléctromagnétique quasi-statique

Résumé : Les problèmes de potentiels inverses imprègnent de nombreuses branches des sciences et de l'ingénierie, en particulier en imagerie non destructive, leur étude est donc importante pour leur avancement. Les problèmes inverses que nous étudions dans la thèse sont régis par l'approximation quasi-statique des équations de Maxwell.

Dans la première partie de cette thèse, nous nous intéressons à la caractérisation de sources silencieuses pour des domaines homogènes connexes avec de faibles conditions de régularité. La caractérisation que nous donnons repose sur la décomposition de Helmholtz. Pour les domaines Lipschitz réguliers, on montre que les sources silencieuses ont une décomposition de Helmholtz. Pour ces domaines, des sources équivalentes minimisant la norme sont caractérisées, ce qui conduit à une décomposition des champs de vecteurs équivalente à la décomposition de Helmholtz lorsqué cette derniére existe. Pour les domaines lisses simplement connexes, un algorithme de minimisation alternée pour calculer la source équivalente de minimale norme à toute source donnée est présenté, dont l'implémentation numérique est faisable.

La problématique de l'imagerie cérébrale fonctionnelle et clinique via les problèmes inverses de l'électroencéphalographie (EEG), de la stéréo-EEG (sEEG) et de la magnétoencéphalographie (MEG) sur un modèle de tête non homogène avec des géométries réalistes est étudiée dans la deuxième partie de la thèse. Chacun de ces problèmes inverses est couplé au problème inverse de transmission. Le couplage du problème inverse de localisation de source inverse avec le problème inverse de transmission rend également relativement simple le couplage de modalités, c'est-à-dire la combinaison de données EEG, sEEG et/ou MEG lors de la résolution de problèmes inverses. Pour chaque problème inverse résultant, un problème de Tikhonov régularisé est résolu avec le régularisateur conçu pour exploiter les éléments structurels dans ces problèmes. Pour cela, un algorithme de minimisation alternée est utilisé pour résoudre le problème de Tikhonov en alternant entre le problème d'identification de source et le problème de transmission de potentiel électrique pour la source identifiée.

Dans la dernière partie de la thèse nous étudions les spectres des opérateurs de Toeplitz. Il est bien connu que pour les problèmes de point fixe, la vitesse de convergence des méthodes itératives est gouvernée par les rayons spectraux des opérateurs impliqués. De plus, la rapidité avec laquelle la méthode itérative atteint ce vitesse de convergence est régie par les résolvantes des opérateurs. Les méthodes itératives nécessitent généralement moins de mémoire que les méthodes directes. L'étude des spectres des opérateurs dans les problèmes de point fixe est donc primordiale pour la mise en œuvre de méthodes itératives. Nous avons montré que sous certaines conditions les spectres des opérateurs Hardy-Toeplitz (HT) et des opérateurs Bergman-Toeplitz (BT) sont les mêmes. Les opérateurs HT et BT sont utiles dans l'étude des problèmes potentiels inverses dans le plan.

Mots-clés : Analyse harmonique, équation de Poisson-Laplace, problème inverse de potentiel.

Inverse potential problems, with applications to quasi-static electromagnetics.

Abstract: Inverse potential problems permeate many branches of science and engineering especially in non-destructive imaging hence their study is important to the advancement of science and engineering. The inverse problems we study in the thesis are governed by the quasi-static approximation of Maxwell's equations.

In the first part of this thesis we look at characterising silent sources for connected homogeneous domains with mild smoothness conditions. The characterisation we give relies on the Helmholtz decomposition. For Lipschitz domains it is shown that silent sources have a Helmholtz decomposition and norm-minimising equivalent sources are characterised which leads to a decomposition of vector-fields that is equivalent to the Helmholtz decomposition when the the latter exists. For smooth simply connected domains an alternating minimisation algorithm for computing the norm-minimising equivalent source of any given source is presented whose numerical implementation is feasible.

The problem of functional and clinical brain imaging via the inverse problems of electroencephalography (EEG), stereo-EEG (sEEG) and magnetoencephalography (MEG) on non-homogeneous head model with realistic geometries is studied in the second part of the thesis. Each of these inverse problems is coupled with the inverse cortical mapping problem. The coupling of the inverse source localisation problem with the inverse cortical mapping problem makes it relatively straight forward to also couple modalities, that is, combining EEG, sEEG and/or MEG data when solving inverse problems. For each resulting inverse problems, a regularised Tikhonov problem is solved with the regulariser designed to exploit structural elements in the problems. To fully exploit this information an alternating minimisation algorithm is used to solve the Tikhonov problem by alternating between the source identification problem and the electrical potential transmission problem for the identified source.

In the last part of the thesis we study the spectra of Toeplitz operators. When solving fied point problems is well known that the rates of convergence of iterative methods are governed by the spectral radii of the operators involved. Further it is also known that how quickly the iterative method settles to these convergence rates is governed by the resolvents of the operators. Iterative methods typically have less memory requirement compared to direct methods. Hence a study the of spectra of the operators in the fixed point problems is paramount to the implementation of iterative methods. We showed that the under certain conditions the essential spectra of Hardy-Toeplitz (HT) operators and Bergman-Toeplitz (BT) operators coincide. Both HT and BT operators are useful in the study of inverse potential problems on the plane.

Keywords: harmonic analysis, Poisson-Laplace equation, inverse potential problem.

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Dedication

To my beautiful nephews Kundiso Zane and Nyashadzaishe Emmanuel

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Résumé long

L'étude des problèmes inverses est répandue parmi le monde scientifique, particulièrement dans les technologies ayant trait à l'imagerie non-destructive. Celle-ci peut être mise en œuvre de diverses manières, les plus communes relevant de l'éléctromagnétisme : CT scan (tomodensitométrie) avec le rayonnement X, l'imagerie par résonance magnétique (IRM), qui utilise le champ magnétique, ou la tomographie par impédance électrique (EIT), qui utilise le champ électrique. Ces exemples proviennent de l'imagerie médicale, ce qui semble raisonnable car c'est l'un des points de contact importants d'une partie de la société avec les questions d'imagerie. Ce n'est cependant pas une limitation, d'autres exemples proviennent des sciences de l'ingénieur (détection de défauts non-destructive dans les machines ou les structures), ou de la géologie, de la physique des planètes, du paléo-magnétisme (étude d'échantillons de roches pour comprendre l'apparition et l'évolution du champ magnétique de la Terre et de la Lune dans le passé).

Dans ce travail, nous nous concentrons sur certaines questions d'imagerie utilisant les mesures de champs éléctromagnétiques afin de retrouver les aimantations et/ou densités de courant qui les produisent, en lien avec le paléomagnétisme et l'imagerie cérébrale, fonctionnelle et clinique. En imagerie cérébrale, l'objectif est de retrouver une densité de courant dans le cerveau (qui indique les régions d'activité), étant données des enregistrements du potentiel éléctrique ou/et du champ magnétique générés par l'activité cérébrale. Ces zones d'activité peuvent être reliées à des tâches effectuées pour un examen fonctionnel ou à des évènements anormaux, comme des crises d'épilepsie, pour un examen clinique. Nous considérons ici les dispositifs tels que l'éléctoencéphalographie (EEG), la stéréo-éléctoencéphalographie (sEEG), et la magnétoencéphalographie (MEG). Les problèmes inverses sous-jacents sont modélisés par les équations de Maxwell et leur approximation quasi-statique, et sont ainsi fortement reliés entre eux. Un point commun particulièrement important entre ces problèmes est que, même dans la situation de mesures complètes et exactes hors du domaine contenant l'aimantation ou la densité de courant, l'inversion est fortement mal posée, et leur solution n'est pas unique ; nous résolvons ainsi des versions régularisées (par la méthode dite de Tikhonov) de ces problèmes.

Les deux premiers chapitres constituent le corps de la thèse, le troisième contient un résultat sur le spectre d'opérateurs de Toeplitz, qui a des applications à l'étude de la vitesse de convergence de méthodes itératives de résolution de systèmes linéaires. Dans le premier chapitre, nous caractérisons les sources silencieuses, à l'origine de la non-unicité des solutions des problèmes inverses, ainsi que les sources équivalentes (dont la différence est silencieuse) à une source donnée, de norme minimale. Le second chapitre

est dédié à l'étude des problèmes inverses EEG, sEEG and MEG, en particulier à la manière de les résoudre constructivement et d'exploiter les propriétés du potentiel électrique pour parvenir à une solution acceptable.

Aimantations L^p silencieuses

Nous supposons que $\Omega \subset \mathbb{R}^n$, $n \geq 3$, est un ensemble ouvert borné avec un nombre fini de composantes connexes dont le bord, $\partial\Omega$, n'est pas nécéssairement connexe. Nous supposons que les aimantations, M, ont un support contenu dans Ω areet sont des champs de vecteurs appartenant à $[L^p(\Omega)]^n$, $1 < p < \infty$ (chaque composante de M appartient à $L^p(\Omega)$). Étant donnée une aimantation $M \in [L^p(\Omega)]^n$, son potentiel magnétique, P_M, satisfait :

$$\operatorname{div} M = \Delta P_M.$$

Nous définissons une aimantation silencieuse comme $M \in [L^p(\Omega)]^n$ telle que P_M est constant presque partout dans chaque composante connexe de $\mathbb{R}^n \backslash \Omega$, les valeurs constantes pouvant différer selon les composantes connexes.

Pour $M \in [L^p(\Omega)]^n$, soit $\widetilde{M} \in [L^p(\mathbb{R}^n)]^n$ l'extension de M par 0 sur $\mathbb{R}^n$. En utilisant la décomposition de Helmholtz sur $[L^p(\mathbb{R}^n)]^n$ pour $1 < p < \infty$, et pour un ensemble ouvert général Ω dont le bord est de mesure nulle, nous avons prouvé que $M \in [L^p(\Omega)]^n$ est silencieuse si et seulement si $\widetilde{M} = \nabla\psi + D$ où D est à divergence nulle sur $\mathbb{R}^n$ et

$$\psi(x) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \widetilde{M}(y) \cdot \frac{(x-y)}{|x-y|^n} dy, \qquad x \in \mathbb{R}^n,$$

avec $\nabla\psi$ et D identiquement nulle sur $\mathbb{R}^n \backslash \Omega$.

Si le bord de Ω est de mesure nulle et si deux composantes connexes de $\mathbb{R}^n \backslash \Omega$ ont une intersection "épaisse" au sens de la capacité dans chaque composante, alors $M \in [L^p(\Omega)]^n$ est silencieuse si et seulement si $M = \nabla\psi + D$, où

$$\nabla\psi \in \overline{\{\nabla\varphi : \varphi \in C^\infty(\mathbb{R}^n), \nabla\varphi \in [C_c^\infty(\Omega)]^n\}}^{[L^p(\Omega)]^n} := G_0^p(\Omega)$$

et $\widetilde{D}$ est à divergence nulle sur $\mathbb{R}^n$.

Cette caractérisation provient du fait que, sous les hypothèses ci-dessus concernant le domaine, certains résultats de la théorie du potentiel permettent de montrer que si, pour $\Omega \subset \mathbb{R}^n$, nous définissons

$$G^p(\Omega) := \{\nabla\psi : \ \psi \in W^{1,p}_{loc}(\Omega), \ \nabla\psi \in [L^p(\Omega)]^n\},$$

alors

$$G_0^p(\Omega) = \{F \in G^p(\Omega) : \widetilde{F} \in G^p(\mathbb{R}^n)\}.$$

En supposant que Ω est un domaine Lipschitz régulier, nous avons prouvé que $M \in [L^p(\Omega)]^n$ est silencieuse si et seulement si $M = \nabla\psi + D$ avec $\nabla\psi \in G_0^p(\Omega)$ et

$$D \in \overline{\{u \in [C_c^\infty(\Omega)]^n : \operatorname{div} u = 0\}}^{[L^p(\Omega)]^n} := \operatorname{Div}_{p,0}(\Omega),$$

ce qui provient de l'équivalence bien connue sur les domaines Lipschitz réguliers :

$$\mathrm{Div}_{p,0}(\Omega) := \{u \in [L^p(\Omega)]^n : \mathrm{div}\, u = 0 \text{ and } u \cdot \nu = 0\},$$

où ν est le vecteur normal unitaire extérieur à $\partial\Omega$.

Les caractérisations ci-dessus sont reliées à la décomposition de Helmholtz. Lorsque $p = 2$ cette décomposition existe en général, pout tout $M \in [L^2(\Omega)]^n$ et tout Ω. Lorsque Ω est de classe C^1, tout $M \in [L^p(\Omega)]^n$ pour $1 < p < \infty$ admet une décomposition de Helmholtz. Finalement, lorsque Ω est Lipschitz régulier, tout $M \in [L^p(\Omega)]^n$ pour $\frac{3}{2} \leq p \leq 3$ admet aussi une décomposition de Helmholtz. La caractérisation des sources silencieuses sur des domaines Lipschitz réguliers établit que toute $M \in [L^p(\Omega)]^n$ silencieuse pour $1 < p < \infty$ admet une décomposition de Helmholtz, une extension des résultats précédemment connus concernant cette décomposition sur de tels domaines.

Deux champs de vecteurs $M_1, M_2 \in [L^p(\Omega)]^n$ sont dit Ω-équivalents si $P_{M_1} = P_{M_2} + \gamma$ p.p. sur $\mathbb{R}^n \backslash \Omega$ pour

$$\gamma \in span_{\mathbb{R}}\{\chi_O : O \text{ est une composante connexe de } \mathbb{R}^n \backslash \overline{\Omega}\}.$$

L'existence d'aimantations silencieuses soulève la question naturelle suivante : quelle est l'aimantation de norme minimale équivalente à une aimantation donnée $M \in [L^p(\Omega)]^n$? La question a d'autant plus d'intêret que les problèmes inverses sont résolus en pratique en utilisant leur régularisation de Tikhonov. Les données disponibles étant bruitées, les paramètres de régularisation doivent être choisis pour atténuer l'influence du bruit sur le problème. On peut montrer que lorsque le bruit et le paramètre de régularisation tendent vers zéro de manière appropriée et combinée, la solution du problème de Tikhonov converge vers la source équivalent de norme minimale. Dans ce but, nous avons pu prouver que si Ω est Lipschitz régulier, l'aimantation de norme minimale équivalente à $M \in [L^p(\Omega)]^n$ est la suivante :

$$\langle M, \nabla\phi \rangle |\nabla\phi|^{q-2} \nabla\phi, \tag{1}$$

où $\frac{1}{p} + \frac{1}{q} = 1$ et $\nabla\phi$ satisfont

$$\nabla\phi = \underset{\nabla\psi \in (G_0^p(\Omega))^\perp, \|\nabla\psi\|_{[L^q(\Omega)]^n} = 1}{\arg\max} |\langle M, \nabla\psi \rangle|.$$

Cette caractérisation des sources équivalentes de norme minimale provient de ce que pour trouver une aimantation de norme minimale équivalente à $M \in [L^p(\Omega)]^n$, nous devons trouver la projection de M sur S_p, l'ensemble des sources Ω-silencieuses dans $[L^p(\Omega)]^n$, c'est à dire :

$$M_S = \underset{M_0 \in S_p}{\arg\min} \|M - M_0\|_{[L^p(\Omega)]^n}.$$

Nous avons montré de plus que les $\nabla\psi \in (G_0^p(\Omega))^\perp$ tel que ψ est harmonique avec $\partial_\nu\psi$ de moyenne nulle sur chaque composante connexe de $\partial\Omega$ constituent exactement les éléments de $(S_p)^\perp$.

La caractérisation ci-dessus des aimantations équivalentes de norme minimale conduit à l'observation suivante. Soit Ω Lipschitz régulier, pour tout $M \in [L^p(\Omega)]^n$,

$\frac{1}{p}+\frac{1}{q}=1$, il existe un unique $\nabla\psi \in G_0^p(\Omega)$, $D \in \mathrm{Div}_{p,0}(\Omega)$ et un unique $\nabla h \in (G_0^p(\Omega))^\perp$ tels que

$$M = \nabla\psi + D + |\nabla h|^{q-2}\nabla h,$$

ce qui est une extension de la décomposition de Helmholtz, équivalente à celle-ci lorsqu'elle existe pour $M \in [L^p(\Omega)]^n$.

Lorsque Ω est Lipschitz régulier, nous pouvons exploiter les opérateurs de simple et double couche et les espaces de traces de Sobolev afin de caractériser la fonction harmonique ϕ dans (1), utilisant $P_M(x)$ pour $x \in \partial\Omega$. Ceci peut-être relié aux "duality mappings" sur les espaces de traces de Sobolev et utilisé pour calculer les aimantations équivalentes de norme minimale dans $[L^p(\Omega)]^n$.

Constructivement, si $n = 3$ et Ω est un domaine $C^{1,1}$ ou Lipschitz polyhédral, une procédure de minimisation alternée peut être utilisée pour calculer l'aimantation de norme minimale équivalente à une aimantation donnée $M \in [L^p(\Omega)]^3$ en résolvant alternativement un problème "q-curl-curl" et un problème de q-laplacien. les itérées convergent fortement vers l'aimantation de norme minimale équivalente dans $M \in [L^p(\Omega)]^3$.

Imagerie cérébrale et potentiels de couches

Nous avons examiné la question pratique de la localisation (récupération) de sources cérébrales (densités de courant) à l'aide de l'EEG, de la sEEG et/ou de la MEG. Nous avons supposé que la source est un champ de vecteur dont les composantes sont des fonctions ou des distributions intégrables et que la source est localisée dans le cerveau. Nous avons considéré la tête comme un conducteur formé de m couches non homogènes emboîtées, chaque couche ayant une conductivité électrique constante. Nous désignons par Σ_i, $i = 1, 2, \ldots, m+1$ les interfaces où la conductivité électrique change. Le potentiel électrique, ϕ, est localement Hölder continu et les courants normaux, $\sigma_i \partial_{\nu_i}\phi$, sont continus à travers chaque Σ_i, avec ν_i la normale unitaire pointant vers l'extérieur de Σ_i, puisque ϕ est gouverné par l'EDP elliptique

$$\nabla \cdot (\sigma \nabla \phi) = \nabla \cdot \boldsymbol{\xi},$$

$\boldsymbol{\xi}$ étant la densité de courant générant le potentiel électrique. Avec la régularité indiquée, le potentiel électrique associé à la densité de courant peut être exprimé comme une combinaison linéaire des expansions de potentiel à double couche. De plus, l'induction magnétique associée à la densité de courant peut être exprimée comme une combinaison linéaire des potentiels à simple couche. Ces expressions servent de "modèles directs" pour le potentiel électrique et la densité de flux magnétique. L'expression du potentiel électrique et de la densité de flux magnétique nécessite la connaissance de la densité de courant et des potentiels électriques de surface, ϕ_i, sur les Σ_i. Par conséquent, pour que le problème inverse de localisation de source soit fidèle aux données fournies, il est utile de résoudre également le problème inverse de transmission des données jusqu'au cortex, ce qui améliore la précision de la récupération. Le problème de la localisation des sources depuis le potentiel électrique ou l'induction magnétique

est connu pour être mal posé en raison de la non-unicité induite par l'existence de sources silencieuses. De plus, seule une poignéequantité restreinte de mesures bruitées sont disponibles pour être utilisées en vue localiser la source. Nous proposons donc de résoudre un problème régularisé par Tikhonov, en utilisant la norme de la source et les potentiels électriques sur les surfaces, Σ_i, $i = 1, 2, \ldots, m+1$, comme régularisateurs, c'est-à-dire :

Problem 1. *Étant donné des données mesurées f et $\alpha, \beta, \lambda_j > 0$, trouver $(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1})_\lambda$ tel que*

$$(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1})_\lambda = \underset{(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1})}{\arg\inf} \mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1}),$$

où

$$\mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1}) := \alpha \|\boldsymbol{\mathcal{F}}_1(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1}) - f\|^2$$
$$+ \beta \|\boldsymbol{\mathcal{F}}_2(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1})\|^2 + \lambda_0 R(\|\boldsymbol{\xi}\|) + \sum_{j=1}^{n+1} \lambda_j \|\phi_j\|^2_{L^2(\Sigma_j)},$$

où $R : [0, \infty) \to [0, \infty)$ est une fonction convexe, $\boldsymbol{\mathcal{F}}_1$ (modèle direct) et $\boldsymbol{\mathcal{F}}_2$ sont des opérateurs liés à la composante électromagnétique des données mesurées et à la régularité du potentiel électrique, respectivement. Nous montrons que pour tout ensemble donné de paramètres de régularisation positifs, $\alpha, \beta, \lambda_j > 0$, le problème de Tikhonov a une solution unique. Ceci est démontré pour Σ_i au moins Lipschitz réguliers et pour $\boldsymbol{\xi}$ des champs de vecteurs depuis des sous-espaces fermés ou faiblement fermés des espaces suivants : $[W^{1-\frac{1}{p},p}(\Sigma_0)]^3$, $[W^{1-\frac{1}{p},p}(\text{int}\Sigma_0)]^3$, $[(W^{1-\frac{1}{p},p}(\Sigma_0))^*]^3$ ou $[(W^{1,p}(\text{int}\Sigma_0))^*]^3$. La densité de courant étant membre des espaces précités, le potentiel électrique est membre de $W^{r,l}(\mathbb{R}^3)$ et l'induction magnétique est membre de $[W^{r,l}(\mathbb{R}^3)]^3$ pour différentes valeurs de r, l *dans*$\mathbb{R}_+$. La continuité des modèles directs pour le potentiel électrique et la densité de flux magnétique dans les topologies *faible* et/ou *faible** sur ces espaces permet alors de montrer l'existence et l'unicité des solutions aux problèmes de Tikhonov. Il est important de noter que la source peut être modélisée comme une *distribution* et que le potentiel électrique est toujours membre d'un *espaces de fonctions* ; par conséquent, les méthodes optimales de récupération des sources et du potentiel électrique ne sont pas toujours les mêmes pour un problème donné. Pour tenir compte de cela, nous avons proposé de résoudre le problème en utilisant une procédure de minimisation, qui résout alternativement pour la source et les potentiels électriques de surface ϕ_i's, c'est-à-dire, pour $l \in \mathbb{N}$ résoudre :

$$\boldsymbol{\xi}_\lambda^{\{l+1\}} = \underset{\boldsymbol{\xi}}{\arg\inf}\, \mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}(\boldsymbol{\xi}, \phi_1^{\{l\}}, \phi_2^{\{l\}}, \ldots, \phi_{m+1}^{\{l\}})$$
$$(\phi_1^{\{l+1\}}, \phi_2^{\{l+1\}}, \ldots, \phi_{m+1}^{\{l+1\}})_\lambda = \underset{(\phi_1, \ldots, \phi_{m+1})}{\arg\inf}\, \mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}(\boldsymbol{\xi}^{\{l\}}, \phi_1, \phi_2, \ldots, \phi_{m+1}).$$

Nous avons montré que la procédure de minimisation alternée converge linéairement vers le minimum de la fonction de Tikhonov $\mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}$. La convergence vers le minimum a lieu par construction. La convergence linéaire vers le minimum est due au fait que

les espaces de Banach que nous utilisons sont des espaces uniformément convexes et que les modèles directs induisent des opérateurs linéaires sur ces espaces, donc leurs dérivées de Fréchet sont Lipschitz régulières. Nous avons mis en œuvre la procédure de minimisation alternée avec des expressions des potentiels à double et simple couche qui sont exactes pour les domaines triangulés. Cela permet de contourner les imprécisions numériques associées aux approximations des intégrales singulières qui ont affecté les méthodes basées sur les potentiels à double couche. Cela permet également d'utiliser des géométries réalistes dans les calculs numériques.

Spectre des opérateurs de Toeplitz

Lors de la résolution de problèmes de points fixes à l'aide de méthodes itératives, le taux de convergence est déterminé par le rayon spectral de l'opérateur et la rapidité avec laquelle les méthodes itératives convergent est déterminée par la croissance du résolvant de l'opérateur. Il est donc intéressant d'étudier les spectres des opérateurs, ici nous étudions les spectres des opérateurs Toeplitz.

Soit $\mathbb{D}$ le disque unité dans le plan complexe $\mathbb{C}$ et soit $\mathbb{T} = \partial\mathbb{D}$. Nous montrons que le spectre essentiel d'un opérateur de Toeplitz de symbole $\phi \in W^{1,2}(\mathbb{T})$ est égal au spectre essentiel d'un opérateur de Bergman-Toeplitz avec le symbole qui est l'extension harmonique de ϕ à $\mathbb{D}$. Ceci a été réalisé en utilisant le théorème de Weyl sur les opérateurs de pertubation. En utilisant le fait que pour les opérateurs bornés leur spectre discret de multiplicité finie converge vers le spectre essentiel et un résultat de Favorov et Golinskii nous avons donné un résultat de vitesse de convergence pour le spectre discret d'un opérateur Toeplitz-Bergman dans la composante non bornée du domaine de Fredholm de l'opérateur de Hardy-Toeplitz associé.

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Summary

The study of inverse problems permeates throughout the scientific world especially in non-destructive imaging technologies. There are varied ways in which imaging is done but by far the most common techniques involve electromagnetism. One can think of computed tomography (CT) which uses X-ray radiation, magnetic resonance imaging (MRI) which uses magnetic fields or electrical impedance tomography (EIT) which uses electric fields. The example given are all from medical imaging which is reasonable as this is the point of contact much of society has with imaging. It turns out though that imaging is not limited to the medical field; in engineering sciences it is important for example in checking components of machinery for defects in a non destructive way. Other areas of science like geology, paleo-magnetism and planetary sciences have also benefited from imaging for instance in the study of magnetism in rock samples to track how the magnetic fields of the earth and the moon have evolved over time and in forming hypotheses on how planets developed their magnetic fields.

In the work presented here, we focus on imaging using electromagnetic fields to recover magnetisations and/or current densities that produce the observed fields. This makes contact with the imaging requirements of paleo-magnetism and brain imaging, both functional and clinical. In paleo-magnetism, the aim is to recover a magnetisation within a rock sample using measurements of the magnetic fields that are made at a distance away from the rock. In brain imaging, the aim is to find a current density in the brain (which indicates active regions of the brain) given measurements of the electric potential or magnetic flux density generated by brain activity. This activation of parts of the brain could be related to tasks being done for functional brain imaging or an abnormal neurological event such as an epileptic seizure for clinical brain imaging. For functional and clinical brain imaging we will focus on electroencephalography (EEG), stereo-electroencephalography (sEEG) and magnetoencephalography (MEG). All these inverse problems are governed by the quasi-static approximation of Maxwell's equation hence their studies have strong relations that they share. A key feature of the problems is that even with the knowledge of the entire measured field or component of the field outside the domain containing the magnetisation or current density, recovery of the magnetisation or current density whose effects are being observed is not unique. This makes these problems ill-posed hence Tikhonov regularised version of the problem is solved.

The first two chapters form the core of the thesis, the third chapter is a result on the spectrum of Toeplitz operators which have applications to studying convergence rates of iterative methods for solving linear problems. In the first chapter, we characterise

silent sources which result in the inverse source recovery problems have non-unique solutions. We also characterise norm-minimising equivalent sources to given sources. The second chapter, is devoted to studying EEG, sEEG and MEG inverse problems, particularly how to solve these problems and how to exploit certain properties of the electric potential to arrive at a reasonable solution.

Silent L^p magnetisations

We assume that $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a finitely connected bounded open set whose boundary, $\partial\Omega$, is not necessarily connected. We assume that the magnetisations, M, supported in Ω are $[L^p(\Omega)]^n$ vector-fields, $1 < p < \infty$, that is, each component is an element of $L^p(\Omega)$. For a given magnetisation $M \in [L^p(\Omega)]^n$, its magnetic potential, P_M, satisfies

$$\operatorname{div} M = \Delta P_M.$$

We define a silent magnetisation as $M \in [L^p(\Omega)]^n$ such that P_M is constant a.e. in each connected component of $\mathbb{R}^n \backslash \Omega$, we allow the constants to differ between the different connected components.

For $M \in [L^p(\Omega)]^n$ we let $\widetilde{M} \in [L^p(\mathbb{R}^n)]^n$ be the extension by zero of M on $\mathbb{R}^n$. Using the existence of the Helmholtz decomposition on $[L^p(\mathbb{R}^n)]^n$, for $1 < p < \infty$, we showed that for a general open set Ω with boundary of zero Lebesgue measure, $M \in [L^p(\Omega)]^n$ is silent if and only if $\widetilde{M} = \nabla\psi + D$ where D being divergence-free on $\mathbb{R}^n$ and

$$\psi(x) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \widetilde{M}(y) \cdot \frac{(x-y)}{|x-y|^n} dy, \qquad x \in \mathbb{R}^n,$$

with $\nabla\psi$ and D both identically zero on $\mathbb{R}^n \backslash \Omega$.

Under the mild conditions on Ω that its boundary be of zero Lebesgue measure and if two connected components of $\mathbb{R}^n \backslash \Omega$ have an intersection, the intersection is "thick in the capacitory sense" in each component then $M \in [L^p(\Omega)]^n$ is silent if and only if $M = \nabla\psi + D$ where

$$\nabla\psi \in \overline{\{\nabla\varphi : \varphi \in C^\infty(\mathbb{R}^n), \nabla\varphi \in [C_c^\infty(\Omega)]^n\}}^{[L^p(\Omega)]^n} := G_0^p(\Omega)$$

and $\widetilde{D}$ is divergence-free on $\mathbb{R}^n$. This characterisation is due to the fact that under these mild conditions of the domain it can be shown using results from non-linear potential theory that if for any $\Omega \subset \mathbb{R}^n$ we define

$$G^p(\Omega) := \{\nabla\psi : \ \psi \in W^{1,p}_{loc}(\Omega), \ \nabla\psi \in [L^p(\Omega)]^n\},$$

then

$$G_0^p(\Omega) = \{F \in G^p(\Omega) : \widetilde{F} \in G^p(\mathbb{R}^n)\}.$$

Assuming that Ω is a Lipschitz smooth domain, we showed that $M \in [L^p(\Omega)]^n$ is silent if and only if $M = \nabla\psi + D$ where $\nabla\psi \in G_0^p(\Omega)$ and

$$D \in \overline{\{u \in [C_c^\infty(\Omega)]^n : \operatorname{div} u = 0\}}^{[L^p(\Omega)]^n} := \operatorname{Div}_{p,0}(\Omega),$$

which follow from the well-known equivalence that hold on Lipschitz domains that

$$\mathrm{Div}_{p,0}(\Omega) := \{u \in [L^p(\Omega)]^n : \mathrm{div}\, u = 0 \text{ and } u \cdot \nu = 0\},$$

where ν is the unit outer normal to $\partial\Omega$.

The above characterisations of silent sources are reminiscent of the Helmholtz decomposition. In general when $p = 2$ the Helmholtz decomposition exists for any $M \in [L^2(\Omega)]^n$ for any Ω. When Ω is C^1 smooth, any $M \in [L^p(\Omega)]^n$ for $1 < p < \infty$ has a Helmholtz decomposition. Finally when Ω is Lipschitz smooth, any $M \in [L^p(\Omega)]^n$ for $\frac{3}{2} \leq p \leq 3$ also has a Helmholtz decomposition. The characterisation of silent sources on Lipschitz domains states that any silent $M \in [L^p(\Omega)]^n$ for $1 < p < \infty$ has a Helmholtz decomposition which extends what was known about the Helmholtz decomposition on Lipschitz domains.

We say that two vector-fields $M_1, M_2 \in [L^p(\Omega)]^n$ are Ω-equivalent if $P_{M_1} = P_{M_2} + \gamma$ a.e. on $\mathbb{R}^n \backslash \Omega$ for some

$$\gamma \in span_{\mathbb{R}}\{\chi_O : O \text{ is a connected component of } \mathbb{R}^n \backslash \overline{\Omega}\}.$$

Because of the existence of silent magnetisations the natural question is: what is the norm-minimising equivalent magnetisation to a given magnetisation $M \in [L^p(\Omega)]^n$. This is a question of interest since in practice inverse problems are solved using regularised Tikhonov problems. The data that is used is noisy hence regularisation parameters have to be chosen to mitigate the influence of noise on the problem. It can be shown that by taking the noise in the data and the regularisation parameters to zero in a specified combined fashion, the solution of the Tikhonov problem converge to the norm-minimising equivalent source. To that end, we managed to show that if Ω is Lipschitz smooth, the norm-minimising equivalent magnetisation to $M \in [L^p(\Omega)]^n$ is

$$\langle M, \nabla\phi\rangle |\nabla\phi|^{q-2}\nabla\phi, \tag{2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\nabla\phi$ satisfies

$$\nabla\phi = \underset{\nabla\psi \in (G_0^p(\Omega))^\perp, \|\nabla\psi\|_{[L^q(\Omega)]^n}=1}{\arg\max} |\langle M, \nabla\psi\rangle|.$$

The above characterisation of norm-minimising equivalent sources comes about as a result of noting that to find the norm-minimising equivalent magnetisation to $M \in [L^p(\Omega)]^n$ we need to find the projection of M in S_p, the set of Ω-silent sources in $[L^p(\Omega)]^n$, that is to find

$$M_S = \underset{M_0 \in S_p}{\arg\min} \|M - M_0\|_{[L^p(\Omega)]^n}.$$

It immediately follow that $M - M_S$ is the norm-minimising equivalent magnetisation to M. By looking at the problem dual to minimisation problem above and using the duality mappings of $[L^q(\Omega)]^n$, $\frac{1}{p} + \frac{1}{q} = 1$, we get that

$$M - M_S = \langle M, \nabla\phi_S\rangle |\nabla\phi_S|^{q-2}\nabla\phi_S.$$

Further, we showed that $\nabla\psi \in (G_0^p(\Omega))^\perp$ such that ψ is harmonic with $\partial_\nu\psi$ having mean value zero on each connected component of $\partial\Omega$ are exactly the elements of $(S_p)^\perp$.

The above characterisation of norm-minimising equivalent magnetisations leads to the following observation; let Ω be Lipschtz smooth, given any $M \in [L^p(\Omega)]^n$ with Ω, $\frac{1}{p} + \frac{1}{q} = 1$, there exists uniquely $\nabla\psi \in G_0^p(\Omega)$, $D \in \mathrm{Div}_{p,0}(\Omega)$ and $\nabla h \in (G_0^p(\Omega))^\perp$ such that

$$M = \nabla\psi + D + |\nabla h|^{q-2}\nabla h,$$

which is an extension of the Helmholtz decomposition and is equivalent to it when it exists for $M \in [L^p(\Omega)]^n$.

When Ω is Lipschitz smooth, we can exploit double and single layer potential operators and trace Sobolev spaces to give a characterisation of the harmonic function ϕ in (2) using $P_M(x)$ for $x \in \partial\Omega$. This can be related to duality mappings on trace Sobolev spaces and this can be used to compute the norm-minimising equivalent magnetisation in $[L^p(\Omega)]^n$.

Constructively, when $n = 3$ and Ω is $C^{1,1}$ smooth or Lipschitz polyhedral domain, an alternating minimisation procedure can be used to compute the norm-minimising equivalent magnetisation to a given $M \in [L^p(\Omega)]^3$ by solving a q-curl-curl problem and a q-Laplace problem alternatively. The iterates strongly converge to the norm-minimising equivalent magnetisation in $[L^p(\Omega)]^3$.

A layer potential approach to brain imaging

Here we looked at the practical issue of source (current density) localisation (recovery) using EEG, sEEG and/or MEG. We assumed that the source is a vector-field whose components are integrable function or distributions and that the source is supported with the brain. We took the head to be a nested non-homogeneous layered conductor of m layers with each layer having constant electric conductivity, we denote by Σ_i, $i = 1, 2, \ldots, m+1$ the interfaces where the electric conductivity changes. The electric potential, ϕ, is locally Hölder continuous and normal currents, $\sigma_i\partial_{\nu_i}\phi$, are continuous across each Σ_i, with ν_i, being the outward pointing unit normal to Σ_i, since ϕ is governed by the elliptic partial differential equation

$$\nabla \cdot (\sigma\nabla\phi) = \nabla \cdot \boldsymbol{\xi},$$

with $\boldsymbol{\xi}$ being the current density generating the electric potential. It is well known that an the electric potential with the stated regularity associated with the current density can be expressed as a linear combination of the well-known double layer potential expansions. Further the magnetic flux density associated with the current density can be expressed as a linear combination of the well-known single layer potentials. These expressions serve as *forward models* for the electric potential and magnetic flux density. Both the expression of the electric potential and the magnetic flux density require the knowledge of both the current density and the surface electric potentials, ϕ_i, on the Σ_i's. Hence for the inverse source localisation problem to be faithful to the data given it is useful for one to also solve the inverse cortical mapping problem there by improving accuracy in the recovery.

The problem of source localisation using the electric potential or magnetic flux density is known to be ill-posed due to the non-uniqueness induced by the existence of silent sources. Further only a handful of noisy measurement are available to use in the source localisation. Hence we propose to solve a Tikhonov regularised problem, we use the norm of the source and the electric potentials on the surfaces, Σ_i, $i = 1, 2, \ldots, m+1$, as regularisers, that is:

Problem 2. *Given measured data f and $\alpha, \beta, \lambda_j > 0$ find $(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1})_\lambda$ such that*

$$(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1})_\lambda = \underset{(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1})}{\arg\inf} \mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1}),$$

where

$$\begin{aligned}\mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1}) &:= \alpha\|\boldsymbol{\mathcal{F}}_1(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1}) - f\|^2 \\ &+ \beta\|\boldsymbol{\mathcal{F}}_2(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{n+1})\|^2 + \lambda_0 R(\|\boldsymbol{\xi}\|) + \sum_{j=1}^{n+1} \lambda_j \|\phi_j\|^2_{L^2(\Sigma_j)},\end{aligned}$$

where $R : [0, \infty) \to [0, \infty)$ is a convex function, $\boldsymbol{\mathcal{F}}_1$ (forward model) and $\boldsymbol{\mathcal{F}}_2$ are operators related to electromagnetic component of the measured data and the regularity of the electric potential, respectively. We show that for any given set of positive regularisation parameters, $\alpha, \beta, \lambda_j > 0$, the Tikhonov problem has a unique solution. This is shown for Σ_i that are at least Lipschitz smooth and for $\boldsymbol{\xi}$ that are vector-fields from closed or weakly closed subspaces of the following spaces $[W^{1-\frac{1}{p},p}(\Sigma_0)]^3$, $[W^{1-\frac{1}{p},p}(\text{int}\Sigma_0)]^3$, $[(W^{1-\frac{1}{p},p}(\Sigma_0))^*]^3$ or $[(W^{1,p}(\text{int}\Sigma_0))^*]^3$. With the current density being a member of the above stated spaces the electric potential is a member $W^{r,l}(\mathbb{R}^3)$ and magnetic flux density is a member of $[W^{r,l}(\mathbb{R}^3)]^3$ for various values of $r, l \in \mathbb{R}_+$. The continuity of the forward models for the electric potential and magnetic flux density in the *weak* and/or *weak** topologies on these spaces then allows to show the existence and uniqueness of the solutions to the Tikhonov problems. It is important to note that the source can be modelled as a *distribution* and the electrical potential are always member of *function spaces* hence the optimal methods for recovery of the sources and electrical potential may not always the same for a given problem. To account for this, we proposed to solve the problem using a minimisation procedure, that alternatively solves for the source and the surface electrical potentials ϕ_i's, that is, for $l \in \mathbb{N}$ solve

$$\begin{aligned}\boldsymbol{\xi}_\lambda^{\{l+1\}} &= \underset{\boldsymbol{\xi}}{\arg\inf}\, \mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}(\boldsymbol{\xi}, \phi_1^{\{l\}}, \phi_2^{\{l\}}, \ldots, \phi_{m+1}^{\{l\}}) \\ (\phi_1^{\{l+1\}}, \phi_2^{\{l+1\}}, \ldots, \phi_{m+1}^{\{l+1\}})_\lambda &= \underset{(\phi_1, \ldots, \phi_{m+1})}{\arg\inf}\, \mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}(\boldsymbol{\xi}^{\{l\}}, \phi_1, \phi_2, \ldots, \phi_{m+1}).\end{aligned}$$

We showed that the alternating minimisation procedure converges linearly to the minimum of the Tikhonov functional $\mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{n+1}}$. The convergence to the minimum is by construction. The linear convergence to the minimum is through the fact that the

Banach spaces we look using are uniformly convex spaces and the forward models are linear operators on these spaces hence they are Fréchet derivatives are Lipschitz.

We implemented the alternating minimisation procedure with expressions for the double and single layer potentials that are exact for triangulated domains. This circumvents the numerical inaccuracies that are associated with the approximations of the singular integrals that had plagued methods based on double layer potentials. This also allows for the use of realistic geometries in the numerical computations.

Spectra of Toeplitz operators

When solving fixed point problems using iterative methods, the rate of convergence is determined by the spectral radius of the operator and how quickly the iterative methods settle to the convergence rates is determined by the growth of the resolvent of the operator. It is therefore of interest to study the spectra of operators, here we study the spectra of Toeplitz operators.

Let $\mathbb{D}$ be the unit disk in the complex plane, $\mathbb{C}$ and let $\mathbb{T} = \partial\mathbb{D}$. We show that the essential spectrum of a Toeplitz operator with symbol $\phi \in W^{1,2}(\mathbb{T})$ is equal to the essential spectrum of a Bergman-Toeplitz operator with symbol that is the harmonic extension of ϕ in to $\mathbb{D}$. This was achieved using Wely's Theorem on pertubation operators. Using the fact that for bounded operators their discrete spectrum of finite multiplicity converge to the essential spectrum and a result of Favorov and Golinskii we gave a rate of convergence rate result for the discrete spectrum of a Toeplitz-Bergman operator in the unbounded component of the Fredholm domain of its Hardy-Toeplitz.

Chapter 1

Silent sources and equivalent L^p-magnetisations.

1.1 Introduction

Inverse source problems arising in static electromagnetism have various applications: for instance to medical and brain imaging, in particular, electroencephalography (EEG) and magnetoencephalography (MEG), where the primary cerebral current is to be estimated, or to geosciences where the remanent magnetisation of rocks and the Earth, or other celestial bodies, is studied to document their past history and study their structure. In magnetostatic aplications, one measures the magnetic field generated by a magnetisation away from its support and one seeks to reconstruct the magnetisation. Under the quasi-static assumption on Maxwell's equations [1], the scalar magnetic potential P_M generated by the magnetisation M is subject to the Poisson-Laplace partial differential equation in $\mathbb{R}^3$:

$$\Delta P_M = \text{div } M\,, \tag{1.1}$$

where P_M is zero at infinity (see Equation (1.39), Sec. 1.5), and the magnetic field is given by ∇P_M.

In EEG, one measures the electric potential P_M generated by the primary cerebral current M away from its support and one seeks to reconstruct that current. A common feature to both problems is that the source term is the divergence of a vector field: the magnetisation in the former case, and the primary current in the latter. Hereafter, we use the magnetostatic terminology to fix ideas, but everything applies without change to the electrostatic setting.

Such questions are severely ill-posed, due to the existence of nonzero *silent* magnetisations; *i.e.*, magnetisations that produce no field outside the body supporting them. Adding to a given magnetisation a silent one yields an *equivalent* magnetisation generating the same field as the original one, whence a fundamental uncertainty attaches to the solution of these inverse problems. As a consequence, further assumptions on the unknown magnetisation are needed in order to set up consistent regularising schemes. In order to derive such schemes, one needs a working characterisation of silent and equivalent magnetisations; this is the subject of the present chapter.

Silent magnetisations supported on a plane in $\mathbb{R}^3$ are described in [2] under weak (distributional) regularity assumptions, and a similar characterisation can be obtained on the 2D sphere by the same method, only replacing Riesz transforms by their spherical analogs defined in terms of layer potentials, see Sec. 1.4. In this connection, we note that silent magnetisations of L^2-class on the sphere have long been characterised in terms of their expansion in spherical harmonics, see for instance [3]. Because one deals here with closed surfaces (the plane being a sphere of infinite radius), one has to distinguish between silence from one side or from both sides. On the plane or the sphere, being silent from one side is equivalent to the property that, in the so-called Hardy-Hodge decomposition of the field representing the magnetisation, the component which is a harmonic gradient from the other side is identically zero. More generally, for a magnetisation carried by a closed Lipschitz surface, silence from one side is equivalent to the double layer potential of the normal component being equal to minus the single layer potential of the divergence of the tangential component; see [4, Thms 3.3 & 3.16] for an analysis in L^2-classes that does carry over, using results from [5], to magnetisations of L^p-class for restricted range of p around 2, and in fact for all $p \in (1, \infty)$ when the surface is C^1-smooth. Moreover, being silent from both sides is equivalent to being divergence-free, as a distribution in ambient Euclidean space. In another connection, silent magnetisations supported on a slender set (*i.e.* a set of measure zero whose complement has all its connected components of infinite measure) were characterised in [6] as being divergence-free, in the setting of vector-valued measures (that subsumes L^p-classes). Slender sets include general open surfaces, whose complement is connected and for which there is no notion of silence from one side. From this discussion, one can get a sense of what it means to be silent for magnetisations carried by a surface in $\mathbb{R}^3$.

In contrast, silent magnetisations supported in a volume seem not to have been studied systematically, and the goal of this chapter is to undertake such a study. Specifically, we show that silent magnetisations of L^p-class carried by a bounded open set Ω are, for $1 < p < \infty$, the (restriction to Ω of the) sum of a gradient and a divergence-free field on $\mathbb{R}^n$, each of which vanishes outside of Ω. If moreover Ω has a boundary which is locally a Lipschitz graph, then the previous characterisation amounts to say that silent magnetisations are the sum of a gradient vector field on Ω with zero tangential boundary component and of a divergence free field with zero normal boundary component.

As can be surmised from this description, the Helmholtz decomposition of vector fields plays a central role in our analysis. Using this description of silent sources, we can characterise the magnetisation of minimum L^p-norm equivalent to a given magnetisation. This gives rise to a non-classical decomposition of L^p-vector fields, as the sum of a gradient with zero tangential boundary component and a divergence-free field with zero normal boundary component, plus the duality mapping of a harmonic gradient. When $p = 2$ it coincides with the Helmholtz decomposition in degree 1, but unlike the Helmholtz decomposition the non-classical decomposition exists for all $p \in (1, \infty)$ on any bounded Lipschitz open set and it is nonlinear. Computing the magnetisation of minimum L^2-norm equivalent to a given one amounts to solve a Dirichlet problem for the Laplacian. When $p \neq 2$, computing an equivalent magnetisation of minimum L^p-norm is more difficult.

The chapter is organized as follows. In Sec. 1.2, we set up notation and recall a

number of properties of Sobolev functions on Lipschitz domains and Lipschitz surfaces, as well basic facts concerning divergence-free vector fields. Sec. 1.3 is a quick review of Helmholtz decompositions and Riesz transforms, while Sec. 1.4 compiles known facts on layer potentials. In Sec. 1.5 we characterise silent vector fields on bounded open sets with minimal assumptions on the regularity of the sets and use this in Sec. 1.6 to describe equivalent sources which are norm-minimal. The Appendix to this chapter contains some auxiliary information need to ground the ideas discussed in the present chapter.

1.2 Notation, definitions and preliminaries

Let $\mathbb{R}^n$ denote the Euclidean space of dimension n. Hereafter, we assume that $n \geq 3$. We write $x = (x_1, \cdots, x_n)^t$ to display the coordinates of $x \in \mathbb{R}^n$, with superscript "t" to mean "transpose", and $x \cdot y$ for the scalar product of $x, y \in \mathbb{R}^n$; $|x| = x \cdot x^{\frac{1}{2}}$ is the Euclidean norm of x. We let $B(x, r)$ be the open ball centered at x of radius r, and $S(x, r)$ the boundary sphere. We put χ_E for the characteristic function of E, and $\mathrm{d}(E_1, E_2)$ for the distance between $E_1, E_2 \subset \mathbb{R}^n$. We designate the set $\{x + e : e \in E\}$ by $x + E$, and for $\varepsilon > 0$ we set $E_\varepsilon := \{x \in \mathbb{R}^n : \mathrm{d}(x, E) < \varepsilon\}$ for the ε-neighborhoud of E.

1.2.1 Maxwell's equations

Central to the work that will be shown here are Maxwell's equations. There are a set of equations that state the relationship between primary currents ($J \in \mathbb{R}^3$), magnetisations ($M \in \mathbb{R}^3$), electric fields ($E \in \mathbb{R}^3$), magnetic fields ($H \in \mathbb{R}^3$), displacements ($D \in \mathbb{R}^3$), polarisation ($P \in \mathbb{R}^3$), charge density ($\rho \in \mathbb{R}$), magnetic permeability ($\mu \in \mathbb{R}_+$) and electric permittivity ($\varepsilon \in \mathbb{R}_+$). By taking the following constitutive relations,

$$D = \varepsilon E + P \qquad \text{and} \qquad B = \mu(H + M),$$

we have according to [7, Appendix on Units and Dimensions] that Maxwell's equations are given in differential form as follows

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times H = J + \frac{\partial D}{\partial t}, \quad \nabla \cdot D = \rho, \quad \nabla \cdot B = 0.$$

If the partial derivatives with respect to time in the above expressions are negligible then we can use a quasi-static approximation of the Maxwell's equations given as follows

$$\nabla \times E = 0 \qquad \text{and} \qquad \nabla \times H = J.$$

If we assume that $J = 0$ then we have that $H = -\nabla P_M$ where $P_M : \mathbb{R}^3 \to \mathbb{R}$. Hence the constitutive relations for B becomes

$$B = \mu(-\nabla P_M + M)$$

and taking divergence on both sides leads to (1.1).

1.2.2 Function spaces

In what follows $E \subset \mathbb{R}^n$ is Lebesgue-measurable and $1 \leq p \leq \infty$.

Definition 1. *$L^p(E)$ is the space of (equivalence class of a.e.. coinciding) $\mathbb{R}$-valued measurable functions on E whose absolute value to the p-th power is integrable, with norm*

$$\|g\|_{L^p(E)} = \Big(\int_E |g(y)|^p dy \Big)^{\frac{1}{p}} \qquad (ess.\ sup_E |g| \quad if \quad p = \infty).$$

For $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, the dual of $L^p(E)$ is $L^q(E)$, isometrically under the pairing

$$\langle f, g \rangle = \int_E fg.$$

We set $L^p_{loc}(E)$ to consist of functions f whose restriction $f_{|K}$ lies in $L^p(K)$ every compact $K \subset E$.

Given a functional space X, we write $[X]^m$ for the corresponding space of vector-fields with m components, each of which lies in X. Further, the dual of X will in general be denoted X^*, if the dual space is explicitly known it will be given as an explicit space. Of particular interest is the following space of vector-fields

Definition 2. *$[L^p(E)]^m$ is the space of $\mathbb{R}^m$-valued vector-fields M on E whose components belong to $L^p(E)$, with norm*

$$\|M\|_{[L^p(E)]^m} = \Big(\int_E |M|^p \, dy \Big)^{\frac{1}{p}} \qquad (ess.\ sup_E |M| \quad if \quad p = \infty), \tag{1.2}$$

and for $1 \leq p < \infty$ the dual of $[L^p(E)]^m$ is $[L^q(E)]^m$, $\frac{1}{p} + \frac{1}{q} = 1$, isometrically under the pairing

$$\langle F, G \rangle = \int_E F(y) \cdot G(y) \, dy. \tag{1.3}$$

If $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$, we designate by $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ the extension of f by zero outside E. The notation stands irrespective of m, n and E.

When solving partial differential equations, classical solutions are sort after. However, it is well-known that conditions that are set for classical solutions are typically too restrictive to be satisfied hence the concept of weak solutions has been introduced where the problem is converted into a variational problem. These variational problems have bigger solution sets that contains the classical solution set, hence when a classical solution exists it is also a weak solution. At the heart of weak solutions is a space of functions for which the notion of derivatives is weaker than the classical derivative. In what follows $\Omega \subset \mathbb{R}^n$ is an open set and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index with each $\alpha_i \geq 0$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

Definition 3. *Let $u : \Omega \to \mathbb{R}$ be a function, we use the notation*

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n}.$$

We denote by $C^k(\Omega)$ the linear space of all real functions, u, defined in Ω which together with all their derivatives $D^\alpha u$ of order $|\alpha| \leq k$ are continuous in Ω. We set

$$C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega).$$

Further, we set $C_c^k(\Omega)$ and $C_c^\infty(\Omega)$ to be linear subspaces of $C^k(\Omega)$ and $C^\infty(\Omega)$, respectively, of all functions having compact support in Ω.

Note that since $\Omega \subset \mathbb{R}^n$ is open, functions in $C^k(\Omega)$ need not be bounded. If $u \in C^k(\Omega)$ is bounded and uniformly continuous, then it possesses a unique, bounded continuous extension to the closure, $\overline{\Omega}$, of Ω. Hence we have the following definitions,

Definition 4. *$C^k(\overline{\Omega})$ is the linear subspace of $C^k(\Omega)$ of all functions u for which $D^\alpha u$ is bounded and uniformly continuous on Ω for all $|\alpha| \leq k$. Further for $0 < \lambda \leq 1$, $C^{k,\lambda}(\overline{\Omega})$ is the subspace of $C^k(\overline{\Omega})$ consisting of all functions u for which $D^\alpha u$ satisfies in Ω the Hölder condition with exponent λ, that is,*

$$|D^\alpha u(x) - D^\alpha u(y)| \leq K|x-y|^\lambda, \qquad x, y \in \Omega,$$

for all $|\alpha| \leq k$.

Remark 1. *Functions in $C^{0,1}(\overline{\Omega})$ are called Lipschitz functions.*

Definition 5. *Let $f \in L^1(\Omega)$, we say that f has an α-th distributional (weak) derivative, f_α, if and only if*

$$\int_\Omega f D^\alpha \psi = (-1)^{|\alpha|} \int_\Omega f_\alpha \psi \qquad \text{for all} \quad \psi \in C_c^\infty(\Omega).$$

Definition 6. *For $1 \leq p < \infty$, $W^{1,p}(\Omega)$ is the Sobolev space of functions lying in $L^p(\Omega)$ together with their first distributional derivatives, with the norm*

$$\|g\|_{W^{1,p}(\Omega)} = \left(\|g\|_{L^p(\Omega)}^p + \|\nabla g\|_{[L^p(\Omega)]^n)}^p \right)^{\frac{1}{p}},$$

where $\nabla g = (\partial_1 g, \cdots, \partial_n g)^t$ denotes the gradient of g and $\partial_j g$ the derivative with respect to the j-th variable. We let $W_0^{1,p}(\Omega)$ stand for the closure in $W^{1,p}(\Omega)$ of $C_c^\infty(\Omega)$ under the above norm.

Remark 2. *(i) $W^{1,p}(\Omega)$ is a Banach space under the given norm. A function belongs to $W_0^{1,p}(\Omega)$ if and only if its extension by zero outside Ω belongs to $W^{1,p}(\mathbb{R}^n)$, see [8, Thm 9.1.3].*

(ii) The definition above of Sobolev functions can be generalised to $g \in W^{m,p}(\Omega)$, $m \in \mathbb{N}$ where the p-integrability is required to hold for all distributional derivatives $D^\alpha g$ for all $|\alpha| \leq m$. $W^{m,p}$ is endowed with the norm

$$\|g\|_{W^{1,p}(\Omega)} = \left(\|g\|_{L^p(\Omega)}^p + \sum_{|\alpha| \leq m} \|D^\alpha g\|_{[L^p(\Omega)]^n)}^p \right)^{\frac{1}{p}}.$$

A special property of members of $W_0^{1,p}(\Omega)$ satisfy is an equivalence of the $W^{1,p}$-norm and the L^p-norm of the gradients. This is given in the following theorem, see [9, Thm. 15.4.1] for a more general result:

Theorem 1 (Poincaré inequality, [10, Ch. VI, Sec. 6.26])**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and let $1 < p < \infty$. Then*

$$\|\varphi\|_{L^p(\Omega)} \leq C\|\nabla\varphi\|_{[L^p(\Omega)]^n}, \tag{1.4}$$

where C depends only on the diameter of Ω.

We have the following result that gives the equivalence of the $W^{1,p}$-norm and the L^p-norm of the gradients:

Theorem 2 (Poincaré-Wirtinger inequality, [11, Comments on Ch. 9])**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded of class C^1 and let $1 < p < \infty$. Then*

$$\|\varphi - \overline{\varphi}\|_{L^p(\Omega)} \leq C\|\nabla\varphi\|_{[L^p(\Omega)]^n}, \tag{1.5}$$

where C depends only on the diameter of Ω and

$$\overline{\varphi} = \frac{1}{|\Omega|}\int_\Omega \varphi. \tag{1.6}$$

The space $W_{loc}^{1,p}(\Omega)$ is comprised of functions lying in $L_{loc}^p(\Omega)$ together with their first order derivatives; it is a Fréchet space, with seminorms the Sobolev norms on relatively compact open subsets of Ω exhausting the latter.

We put $\dot{W}^{1,p}(\Omega)$ for the quotient space, modulo constants, of distributions on Ω whose derivatives belong to $L^p(\Omega)$. Such a distribution, say ψ, necessarily lies in $W_{loc}^{1,p}(\Omega)$, see for example [9, Sec. 1.1.2], and we write $\dot{\psi} \in \dot{W}^{1,p}(\Omega)$ for the equivalence class of ψ. Endowed with the norm

$$\|g\|_{\dot{W}^{1,p}(\Omega)} := \|\nabla g\|_{[L^p(\Omega)]^n},$$

one can see that $\dot{W}^{1,p}(\Omega)$ is a Banach space called homogeneous Sobolev space (of index p), see [9, Sec. 1.1.13, Thm 1]. It can be shown that $\dot{W}^{1,p}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to $\|.\|_{\dot{W}^{1,p}(\mathbb{R}^n)}$ for $1 < p < \infty$, see the discussion in Sec. 1.3.

We introduce here a subspace of $[L^p(\Omega)]^n$ that depends on homogeneous Sobolev spaces.

Definition 7. *$G^p(\Omega)$ is the subspace of $[L^p(\Omega)]^n$ defined by*

$$G^p(\Omega) := \{\nabla\psi : \ \psi \in W_{loc}^{1,p}(\Omega), \ \nabla\psi \in [L^p(\Omega)]^n\}. \tag{1.7}$$

$G_0^p(\Omega)$ is the subspace of $[L^p(\Omega)]^n$ defined as closure of $[C_c^\infty(\Omega)]^n$ in $G^p(\Omega)$:

$$G_0^p(\Omega) := \overline{\{\nabla\psi : \ \psi \in C^\infty(\Omega), \ \nabla\psi \in [C_c^\infty(\Omega)]^n\}}^{[L^p(\Omega)]^n}. \tag{1.8}$$

In other words, $G^p(\Omega)$ comprises gradients of members of $\dot{W}^{1,p}(\Omega)$. When Ω is bounded, $G_0^p(\Omega)$ consists of vector fields in $[L^p(\Omega)]^n$ whose extension by zero outside Ω is the gradient of some member of $\dot{W}^{1,p}(\mathbb{R}^n)$, see Lemma 19 below.

For $1 < p < \infty$, the dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,q}(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$; it may be realised as the completion of $L^q(\Omega)$ with respect to the norm

$$\|\varphi\|_{W^{-1,q}(\Omega)} := \sup_{\|g\|_{W_0^{1,p}(\Omega)}=1} \left| \int_\Omega \varphi(y) g(y)\, dy \right|, \tag{1.9}$$

see [10, Sec. 3.12]. Note that members of $W^{-1,q}(\Omega)$ are generally not functions but rather distributions on Ω, and that $f \mapsto \nabla f$ is continuous from $L^p(\Omega)$ to $[W^{-1,q}(\Omega)]^n$, by (1.4). The support of a function or a distribution T will be denoted by $\operatorname{supp} T$.

Members of $L^p(\Omega)$ in general do not have point-wise values hence point-wise descriptions of these functions are meaningless in general. However under certain conditions point-wise values can be assigned to members of $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega) \subset L^p(\Omega)$, every $f \in W^{1,p}(\Omega)$ is such that a.e.. $x \in \Omega$ is a Lebesgue point where f can be defined in the strict sense:

$$f(x) = \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dy,$$

with m to indicate Lebesgue measure. Beyond the point-wise definition of Sobolev functions at Lebesgue points we can turn to the smoothness of Sobolev functions in order to assign point-wise values. To that end we recall the Sobolev embedding theorem, we will only quote the part of the theorem that we need for what follows immediately,

Theorem 3 (Sobolev embedding theorem, [10, Ch. V, Sec. 5.4])**.** *Let $\Omega \subset \mathbb{R}^n$ have a strong local Lipschitz property, see the Sec. 1.2.3, j and m be non-negative integers and let $1 \leq p < \infty$.*

(1) Suppose $mp > n > (m-1)p$. Then

$$W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\overline{\Omega}), \qquad 0 < \lambda \leq m - \frac{n}{p}.$$

(2) Suppose $n = (m-1)p$. Then

$$W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\overline{\Omega}), \qquad 0 < \lambda < 1.$$

By setting $j = 0$ and $m = 1$ we conclude that $f \in W^{1,p}(\Omega)$ for $p > n$ is continuous so that every point is Lebesgue hence the function has point-wise values. There however remains something to be said when $1 < p \leq n$ since non-Lebesgue points form a set of zero $(1,p)$-Bessel capacity. To define Bessel capacities we introduce the following space.

Definition 8. *$\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz class which consists of $C^\infty(\mathbb{R}^n)$ functions such that, for every pair of multi-indices (α, β), there exists a positive constant $C_{\alpha,\beta} < \infty$ for which*

$$\rho_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| \leq C_{\alpha,\beta}.$$

We need to state the following theorem as it will be used in what follows,

Theorem 4 (Young, [11, Thm 4.15]). *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Then for a.e. $x \in \mathbb{R}^n$ the function $y \to f(x-y)g(y)$ is integrable on $\mathbb{R}^n$ and we define*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

In addition $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.$$

For functions in the Schwartz class we define the following operator,

$$\mathcal{G}_\kappa = (I - \Delta)^{-\frac{\kappa}{2}}, \qquad \kappa \in \mathbb{R},$$

for which the Bessel potential representation of $f \in \mathscr{S}(\mathbb{R}^n)$,

$$f = \mathcal{G}_\kappa * g,$$

where $g \in \mathscr{S}(\mathbb{R}^n)$.

Definition 9. *$L^{\kappa,p}(\mathbb{R}^n)$ is the Bessel potential space defined by*

$$L^{\kappa,p}(\mathbb{R}^n) = \{f : f = \mathcal{G}_\kappa * g,\ g \in L^p(\mathbb{R}^n)\}, \qquad \kappa \in \mathbb{R}.$$

The Bessel potential space is endowed with the norm $\|f\|_{\kappa,p} = \|g\|_{L^p(\mathbb{R}^n)}$.

Definition 10. *Let $K \subset \mathbb{R}^n$ be compact and set*

$$w_K = \{\varphi \in \mathscr{S}(\mathbb{R}^n) : \varphi \geq 1 \text{ on } K\},$$

so that w_K is a convex subset of the Schwartz class. Let $\kappa > 0$ and $1 < p < \infty$, then the (κ, p)-Bessel capacity of K, $\mathcal{C}_{\kappa,p}(K)$, is defined as

$$\mathcal{C}_{\kappa,p}(K) = \inf\{\|\varphi\|_{\kappa,p}^p : \varphi \in w_K\}.$$

For $\Omega \subset \mathbb{R}^n$ an open set,

$$\mathcal{C}_{\kappa,p}(\Omega) = \sup\{\mathcal{C}_{\kappa,p}(K) : K \subset \Omega,\ K \text{ compact}\}.$$

Finally, if a property holds everywhere expect on a set of (κ, p)-Bessel capacity zero, the property is said to hold (κ, p)-quasi everywhere ((κ, p)-q.e.).

More details on Bessel capacities can be seen in [8, Ch. 2] and [8, Sec. 6.2] for capacitary properties of Lebesgue points of Sobolev functions. We turn to the following result of Calderon.

Theorem 5 ([12, Ch. V, Sec. 3, Thm 3]). *For $\kappa \in \mathbb{N}$, $W^{\kappa,p}(\mathbb{R}^n) = L^{\kappa,p}(\mathbb{R}^n)$, $1 < p < \infty$ with the equivalence of norms, that is, there is a constant A such that for all f*

$$A^{-1}\|f\|_{\kappa,p} \leq \|f\|_{W^{\kappa,p}(\mathbb{R}^n)} \leq A\|f\|_{\kappa,p}.$$

We rephrase here [8, Thm 6.2.1], we need the following definition first.

Definition 11. *A function f is said to be (κ,p)-quasi-continuous if for every $\varepsilon>0$ there exists an open set G such that $\mathcal{C}_{\kappa,p}(G)<\varepsilon$ and $f|_{G^c}\in C(G^c)$.*

Theorem 6 ([8, Thm 6.2.1])**.** *Let $f=\mathcal{G}_\kappa * g\in W^{\kappa,p}(\mathbb{R}^n)$, $1<p<\infty$, $0<\kappa p\le n$. Let q be such that $1\le q\le p^*$ where $p^*=\frac{np}{n-\kappa p}$ if $\kappa p<n$, and for all q, $1\le q<\infty$, if $\kappa p=n$. Then (κ,p)-q.e. x is a Lebesgue point for f in the L^q-sense, that is,*

$$\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)} f(y)dy=\tilde{f}(x)\qquad \text{exists},$$

and

$$\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)-\tilde{f}(x)|\,dy=0.$$

Moreover, the convergence is uniform outside an open set of arbitrarily small (κ,p)-Bessel capacity, $\tilde{f}$ is a (κ,p)-quasi-continuous representative for f, and

$$\tilde{f}(x)=\mathcal{G}_\kappa * g(x)\qquad (\kappa,p)\text{-q.e.}$$

The above results show that $(1,p)$-q.e., for $1<p\le n$, $f\in W^{1,p}(\mathbb{R}^n)$ has point-wise values. Sets of zero $(1,p)$-Bessel capacity are very thin. Thinness is defined as follows.

Definition 12. *Let $E\subset\mathbb{R}^n$ and let $1<p<\frac{n}{\kappa}$, $\frac{1}{p}+\frac{1}{q}=1$. Then E is (κ,p)- thin at a point $x\in\mathbb{R}^n$ if*

$$\int_0^1\Big(\frac{C_{\kappa,p}(E\cap B(x,r))}{r^{n-\kappa p}}\Big)^{q-1}\frac{dr}{r}<\infty.$$

The above definition is a generalisation of the following definition of thinness which is given for $\kappa=1$ and $p=2$,

Definition 13. *Let $E\subset\mathbb{R}^n$ be an arbitrary set. Then E is thin at $x\in\mathbb{R}^n$ if there exists a positive measure μ such that*

$$\mathcal{G}_2*\mu(x)<\liminf_{a\to x,\,a\in E\setminus\{x\}}\mathcal{G}_2*\mu(a).$$

Thinness plays the following role. Let $\Omega\subset\mathbb{R}^n$ be an open bounded domain and $\partial\Omega$ have finitely many connected components, say $\Gamma_1,\cdots,\Gamma_l$. Moreover, let the connected components of $\mathbb{R}^n\setminus\overline{\Omega}$ consist of l open sets $O_1,\cdots,O_l$, and with a suitable ordering O_1 is the exterior of Γ_1 while O_j is the interior of Γ_j for $j\neq 1$. If there exists a pair Γ_i,Γ_i^* $i\neq i^*$ such that $\Gamma_i\cap\Gamma_{i^*}\neq\emptyset$ with either O_i or O_{i^*} thin on this intersection or both then there exists a function $f\in W^{1,p}(\mathbb{R}^n)$ such that ∇f vanishes on each O_j but $\nabla f|_\Omega\notin G_0^p(\Omega)$. By setting $f=c_j$ on O_j we have that ∇f is compactly supported in Ω. Further $f=c_j$ on each Γ_j, however we can set $c_i\neq c_{i^*}$, because of the thinness of either O_i or O_{i^*} on $\Gamma_i\cap\Gamma_{i^*}$. It thus follows that there is no $C^\infty(\mathbb{R}^n)$ that approximates f. In Sec. 1.9.3 we give an example of a domain Ω that is open and bounded with a $\mathbb{R}^n\setminus\Omega$ that is connected and function $f\in W^{1,2}_{loc}(\mathbb{R}^n)$ such that ∇f vanishes on $\mathbb{R}^n\setminus\Omega$ but f attains two different constant values on $\mathbb{R}^n\setminus\Omega$.

1.2.3 Lipschitz open sets

The work we present here is valid for open sets in $\mathbb{R}^n$ with few restrictions on their geometry. However, for a particular class of open sets, their geometry makes it possible to have well behaved functions defined on their boundaries making computations possible. We discuss the geometry of such sets in this section. We will devote this section to sets that have the Lipschitz property, we will briefly discuss other classes of sets that may be referred to in later sections and chapters.

Definition 14. *Let Φ be a one-to-one transformation of a domain $\Omega \subset \mathbb{R}^n$ onto a domain $G \subset \mathbb{R}^n$, having inverse $\Psi = \Phi^{-1}$. We call Φ m-smooth if writing $y = \Phi(x)$ and $x = \Psi(y)$ as*

$$\begin{aligned} y_1 &= \phi_1(x_1, x_2, \ldots, x_n), & x_1 &= \psi_1(y_1, y_2, \ldots, y_n), \\ y_2 &= \phi_2(x_1, x_2, \ldots, x_n), & x_2 &= \psi_2(y_1, y_2, \ldots, y_n), \\ &\vdots & &\vdots \\ y_n &= \phi_n(x_1, x_2, \ldots, x_n), & x_n &= \psi_n(y_1, y_2, \ldots, y_n), \end{aligned}$$

the functions $\phi_1, \phi_2, \ldots, \phi_n \in C^m(\overline{\Omega})$ and the functions $\psi_1, \psi_2, \ldots, \psi_n \in C^m(\overline{G})$.

Definition 15. *Let $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ we write the Jacobian matrix (total derivative) of Φ at x as*

$$D\Phi(x) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(x) & \frac{\partial \phi_1}{\partial x_2}(x) & \ldots & \frac{\partial \phi_1}{\partial x_m}(x) \\ \frac{\partial \phi_2}{\partial x_1}(x) & \frac{\partial \phi_2}{\partial x_2}(x) & \ldots & \frac{\partial \phi_2}{\partial x_m}(x) \\ \vdots & \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1}(x) & \frac{\partial \phi_n}{\partial x_2}(x) & \ldots & \frac{\partial \phi_n}{\partial x_m}(x) \end{pmatrix}.$$

Given $X \subset \mathbb{R}^n$ and let $\Phi : U \subset \mathbb{R}^m \to X$ be a local parametrization around $x \in X$ where U is open. We may assume that $\Phi(0) = x$. The best approximation of $\Phi : U \subset \mathbb{R}^m \to X$ at 0 is the map

$$u \mapsto \Phi(0) + D\Phi(0)u = x + D\Phi(0)u.$$

The tangent space of X at x, T_xX, is range of the map $D\Phi(0) : \mathbb{R}^m \to \mathbb{R}^n$.

Definition 16. *For a smooth $\Phi : X \to Y$, a point $x \in X$ is called a regular point for Φ if $D\Phi(x) : T_xX \to T_yY$ is surjective. Otherwise, x is critical point. For a smooth $\Phi : X \to Y$, a point $y \in Y$ is called a regular value for Φ if $D\Phi(x) : T_xX \to T_yY$ is surjective at every point x such that $\Phi(x) = y$. Otherwise y is a critical value.*

Remark 3. *The moral of the above definition is y is a regular value if $\Phi^{-1}(y) = x$ is a regular point and y is a critical value if $\Phi^{-1}(y) = x$ is a critical point.*

Theorem 7 (Sard's theorem, [13, Ch. 1, Sec. 7].)**.** *If $\Phi : X \to Y$ is any smooth map of manifolds, then almost every point in Y is a regular value of Φ.*

Definition 17. *An open set $\Omega \subset \mathbb{R}^n$ is of class C^m, is there exists a locally finite open cover $\{U_j\}$ of the boundary of Ω, $\partial\Omega$, and a corresponding sequence $\{\Phi_j\}$ of m-smooth one-to-one transformations with Φ_j taking U_j onto $B(0,1) \subset \mathbb{R}^n$, such that:*

(i) *For some* $\delta > 0$, $\bigcup_{j=1}^{\infty} \Psi_j(B(0, \frac{1}{2})) \supset \Omega_\delta$, *where* $\Psi_j = \Phi_j^{-1}$.

(ii) *For some finite* R, *every collection of* $R+1$ *of the sets* U_j *has empty intersection.*

(iii) *For each* j, $\Phi_j(U_j \cap \Omega) = \{y \in B(0,1) : y_n > 0\}$.

(iv) *If* $(\phi_{1,j}, \phi_{2,j}, \ldots, \phi_{n,j})$ *and* $(\psi_{1,j}, \psi_{2,j}, \ldots, \psi_{n,j})$ *denote the components of* Φ_j *and* Ψ_j, *respectively, the there is a finite* M *such that for all* α, $|\alpha| \leq m$ *for every* $i, 1 \leq i \leq n$ *and for every* j, *we have*

$$|D^\alpha \phi_{j,i}(x)| \leq M, \qquad x \in U_j \quad \text{and} \quad |D^\alpha \psi_{j,i}(y)| \leq M, \qquad y \in B(0,1).$$

The rest of this section is devoted to the definition of Lipschitz sets. We take an approach slightly different from the one above in order to better get access to objects defined on $\partial\Omega$ that are useful for later computations. One of the objects that we will need access to is the Hausdorff measure defined on $\partial\Omega$

Definition 18. *Let* $E \subset \mathbb{R}^n$ *be an arbitrary set,* $0 \leq s < \infty$, $0 < \delta \leq \infty$. *We write*

$$\mathcal{H}^s_\delta(E) = \inf\Big\{ \sum_{j=1}^{\infty} \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)} \Big(\frac{diam\ C_j}{2}\Big)^s : E \subset \bigcup_{j=1}^{\infty} C_j,\ diam\ C_j \leq \delta \Big\}.$$

The s*-dimensional Hausdorff measure (Hausdorff* s*-measure) of* E *on* $\mathbb{R}^n$, $\mathcal{H}^s(E)$, *is defined as*

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

We say that an open set $\Omega \subset \mathbb{R}^n$ is Lipschitz if $\partial\Omega$ is compact and locally isometric to the graph of a Lipschitz function; see *e.g.* [10, 14]. More precisely, to each $x \in \partial\Omega$ there should exist an open set $U \subset \mathbb{R}^n$ containing x and a rigid tranformation T of $\mathbb{R}^n$ such that, for some open set $V \subset \mathbb{R}^{n-1}$ and some Lipschitz map $\Psi : V \to \mathbb{R}^+$, one has:

$$T(U) \cap \Omega = \{y \in \mathbb{R}^n, \quad (y_1, \cdots, y_{n-1})^t \in V, \quad 0 < y_n < \Psi(y_1, \cdots, y_{n-1})\}.$$

One can cover $\partial\Omega$ by finitely many (say $N \leq 1$) open sets U_j as above, with corresponding T_j, V_j and Ψ_j for $1 \leq j \leq N$. Now, if we define maps $\Phi_j : U_j \cap \partial\Omega \to V_j$ as $\Phi_j := P_{n-1} \circ T_j$ where P_{n-1} is the projection on the first $(n-1)$ components, we get an atlas on $\partial\Omega$ making it a compact Lipschitz manifold. Note that the parametrizations $\Phi_j^{-1} : V_j \to U_j \cap \partial\Omega \subset \mathbb{R}^n$ are themselves Lipschitz as they are given by $\Phi_j^{-1} = T_j^{-1} \circ (I_{n-1} \times \Psi_j)$, where I_{n-1} is the identity map on $\mathbb{R}^{n-1}$. Since $\partial\Omega$ is locally the image of an open subset of $\mathbb{R}^{n-1}$ by a Lipschitz function (*e.g.*, $\Phi_j^{-1} : V_j \to \mathbb{R}^n$), the volume measure σ coincides with the restriction to $\partial\Omega$ of Hausdorff $(n-1)$-measure, see [15, Sec. 3.2]. Integration of functions on $\partial\Omega$ is always understood with respect to σ.

For a Lipschitz open set $\Omega \subset \mathbb{R}^n$, we say that $x \in \partial\Omega$ is singular if there is $j \in \{1, \cdots, N\}$ such that $x \in U_j$ and $\Phi_j^{-1} : V_j \to \mathbb{R}^n$ is not differentiable at $\Phi_j(x)$. A point which is not singular is called regular. We denote the set of regular points by $\mathrm{Reg}\,\partial\Omega$

and put $\operatorname{Reg} V_j = \Phi_j(\operatorname{Reg}\partial\Omega \cap U_j)$. So defined, the set of regular points depends on the atlas, but we shall fix the latter. As Φ_j^{-1} is Lipschitz, the set $E_j \subset V_j$ where it is not differentiable has Lebesgue measure zero, by Rademacher's theorem [16, Thm 2.2.1]. Hence, $\sigma(\Phi_j(E_j)) = 0$ [16, Rem. 1.4.3], implying that singular points have σ-measure zero.

The tangent space of $\partial\Omega$ at x is the subspace $T_x\partial\Omega \subset \mathbb{R}^n$ equal to $\{0\}$ if x is singular and to $\operatorname{Ran} D\Phi_j^{-1}(\Phi_j(x))$ if $x \in U_j \cap \operatorname{Reg}\partial\Omega$, where $D\Phi_j^{-1}$ denotes the total derivative of Φ_j^{-1}. By the chain rule, the definition does not depend on j such that $x \in U_j$. At $x \in \operatorname{Reg}\partial\Omega$, the outward pointing unit normal to $\partial\Omega$ is well-defined, and we denote it as $\nu(x)$.

The connected components of $\partial\Omega$ are connected compact Lipschitz hypersurfaces in $\mathbb{R}^n$. For Γ any such hypersurface, $\mathbb{R}^n \setminus \Gamma$ has two connected components: its interior denoted by $\operatorname{int}\Gamma$ which is bounded, and its exterior denoted by $\operatorname{ext}\Gamma$ which is unbounded, see [17, Cor. 3.45].

The connected components of a Lipschitz open set $\Omega \subset \mathbb{R}^n$ are finite in number. Otherwise indeed, there would exist a sequence O_k of such components, $k \in \mathbb{Z}$, with $O_k \cap O_j = \emptyset$ for $k \neq j$. Then, we could construct a sequence $x_k \in O_k$ such that x_k remains at bounded distance from $\partial O_k \subset \partial\Omega$, hence x_k would be bounded and extracting a subsequence if necessary we might assume that x_k converges in $\mathbb{R}^n$ to some y. However, this is impossible for y cannot lie in Ω since the connected components are open, nor can it lie in $\mathbb{R}^n \setminus \overline{\Omega}$, and it cannot belong to $\partial\Omega$ either because, by definition of a Lipschitz open set, each member of $\partial\Omega$ has a neighborhood whose intersection with Ω is connected. For the same reason, distinct connected components of Ω cannot have a common boundary point, hence they lie at strictly positive distance from each other.

A Lipschitz domain is a *connected* Lipschitz open set. We record for later use an "obvious" topological lemma:

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then, $\partial\Omega$ has finitely many connected components, say $\Gamma_1, \cdots, \Gamma_l$. Moreover, the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$ consist of l open sets $O_1, \cdots, O_l$, and with a suitable ordering O_1 is the exterior of Γ_1 while O_j is the interior of Γ_j for $j \neq 1$.*

Proof. Since $\partial\Omega$ is a compact Lipschitz manifold which is locally a Lipschitz graph, each $x \in \partial\Omega$ has a neighborhood whose intersection with $\partial\Omega$ is connected. Consequently, by compactness, $\partial\Omega$ has finitely many connected components, say $\Gamma_1, \cdots, \Gamma_l$, and each Γ_j is a connected compact Lipschitz hypersurface in $\mathbb{R}^n$. As Ω is connected by assumption, for each $j \in \{1, \cdots, n\}$ either $\Omega \subset \operatorname{int}\Gamma_j$ and then $\operatorname{ext}\Gamma_j \subset \mathbb{R}^n \setminus \overline{\Omega}$, or else $\Omega \subset \operatorname{ext}\Gamma_j$ and then $\operatorname{int}\Gamma_j \subset \mathbb{R}^n \setminus \overline{\Omega}$. Since there is exactly one unbounded connected component of $\mathbb{R}^n \setminus \overline{\Omega}$, say O_1, it must contain $\operatorname{ext}\Gamma_j$ for all j such that $\Omega \subset \operatorname{int}\Gamma_j$; let us enumerate these j as $j_1, \cdots, j_m$. For $1 \leq i, k \leq m$, it holds that $\operatorname{int}\Gamma_{j_i} \cap \operatorname{int}\Gamma_{j_k} \neq \emptyset$ because Ω lies in this intersection, and since the Γ_j are disjoint one of these interiors is included in the other, say $\operatorname{int}\Gamma_{j_i} \subset \operatorname{int}\Gamma_{j_k}$. But if $j_i \neq j_k$, then $\Gamma_{j_k} \subset \operatorname{ext}\Gamma_{j_i}$ and the latter is contained in O_1, a contradiction. Consequently, $m = 1$ and Ω lies interior to exactly one of the Γ_j, say Γ_1. Necessarily then, $O_1 = \operatorname{ext}\Gamma_1$ because O_1 cannot strictly contain $\operatorname{ext}\Gamma_1$ without containing a point of Γ_1, which is impossible. Likewise, $\Omega \subset \operatorname{ext}\Gamma_j$ for $j \neq 1$ and then $\operatorname{int}\Gamma_j$ is a connected component of $\mathbb{R}^n \setminus \overline{\Omega}$. Finally, the closure of every

bounded connected component of $\mathbb{R}^n \setminus \overline{\Omega}$ must meet some Γ_j, and necessarily $j \neq 1$ for each point of Γ_1 has a neighborhood included in $O_1 \cup \Omega$, by the local Lipschitz graph property. Hence, this connected component meets $\operatorname{int}\Gamma_j$ for some $j \neq 1$, therefore it must coincide with $\operatorname{int}\Gamma_j$. □

Before stating the next result concerning Lipschitz domains we will state a result of Whitney that ensures the existence of a C^1 extension, $\tilde{f}$, of a given function f defined on a closed subset $E \subset \mathbb{R}^n$. Let $f : E \to \mathbb{R}$ and $d : E \to \mathbb{R}^n$ be given functions and

$$R(x,y) = \frac{f(x) - f(y) - d(y)\cdot(x-y)}{|x-y|} \qquad x, y \in E,\ x \neq y,$$

and for $K \subset E$ a compact set, and for $\delta > 0$

$$\rho_K(\delta) = \sup\{|R(x,y)| : 0 < |x-y| \leq \delta,\ x, y \in K\}.$$

Theorem 8 (Whitney extension theorem, [18, Thm 6.10])**.** *Assume that f, d are continuous, and for each compact set $K \subset E$,*

$$\rho_K(\delta) \to 0 \quad \text{as } \delta \to 0.$$

Then there exists a function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $\tilde{f} \in C^1$.

(ii) $\tilde{f} = f$ *and* $\nabla\tilde{f} = d$ *on* E.

At some point, we also need the following "trivial" fact:

Lemma 2. *For Ω a bounded Lipschitz domain and $\varepsilon > 0$, there exists a simply connected Lipschitz domain O such that $\overline{\Omega} \subset O \subset \overline{O} \subset \Omega_\varepsilon$.*

Proof. In fact, one can design O so that it is smooth. For this, we use the notation of Lemma 1 and observe that O_1 is the zero set of a non-negative C^∞-smooth function $f : \mathbb{R}^n \to \mathbb{R}$ from the Whitney extension theorem. Adding to f a smooth non-negative function which is 0 in a neighborhoud of $\overline{\Omega}$ and 1 in a neighborhoud of ∞, we may assume that $f(x) \geq 1$ for $|x|$ large enough. So, if $\eta > 0$ is a sufficiently small regular value of f, then $f^{-1}(\eta)$ is a smooth compact $(n-1)$-dimensional manifold and we can take $O := \operatorname{int}\partial V$ with V the unbounded component of $f^{-1}(\eta, +\infty)$; see, *e.g.* [13, Ch. 1, sec. 7] for a definition of regular values and Sard's theorem that non-regular values have 1-dimensional Lebesgue measure zero. □

Definition 19. *We say that a domain D is a special Lipschitz domain if there is a $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ which satisfies the Lipschitz condition*

$$|\varphi(x) - \varphi(x')| \leq M|x - x'|, \qquad \text{for all } x, x' \in \mathbb{R}^{n-1},$$

such that

$$D = \{(x,y) \in \mathbb{R}^n : y > \varphi(x)\}.$$

The smallest M such that the Lipschitz condition holds is called the bound of the special Lipschitz domain D.

Definition 20. *Let $\Omega \subset \mathbb{R}^n$ be open. We say that $\partial\Omega$ is minimally smooth, if there exists $\varepsilon > 0$, $N \in \mathbb{Z}$, $M > 0$ and a sequence $\{U_j\}$ of open sets such that:*

(i) For $x \in \partial\Omega$, there is a U_j such that $B(x, \varepsilon) \subset U_j$.

(ii) No point of $\mathbb{R}^n$ is contained in more than N of the U_j.

(iii) For each j there is a special Lipschitz domain D_j whose bound does not exceed M such that

$$U_j \cap \Omega = U_j \cap D_j.$$

1.2.4 Boundary traces of functions and fields

When $\Omega \subset \mathbb{R}^n$ is a Lipschitz open set, $W^{1,p}(\Omega)$ coincides with the restrictions to Ω of $W^{1,p}(\mathbb{R}^n)$-functions as a consequence of the following theorem.

Theorem 9 ([12, Ch. VI, Thm 5])**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with a minimally smooth boundary. Then there exists a linear operator $\mathfrak{E}$ mapping functions on Ω to functions on $\mathbb{R}^n$ with the properties*

(i) $\mathfrak{E}(f)|_\Omega = f$, that is, $\mathfrak{E}$ is an extension operator.

(ii) $\mathfrak{E}$ maps $W^{m,p}(\Omega)$ continuously into $W^{m,p}(\mathbb{R}^n)$, $1 \le p \le \infty$ and all non-negative $m \in \mathbb{Z}$.

Moreover, when $1 < p < \infty$, each $f \in W^{1,p}(\Omega)$ has a trace on $\partial\Omega$, say ψ. These traces are members of the so called fractional Sobolev spaces which we define as follows.

Definition 21. *A function $\psi \in W^{s,p}(\partial\Omega)$, $0 < s < 1$, $1 < p < \infty$ if*

$$\|\psi\|_{W^{s,p}(\partial\Omega)} := \|\psi\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{n-1+sp}} d\sigma(x) d\sigma(y) \right)^{\frac{1}{p}} \tag{1.10}$$

is finite.

For $f \in W^{1,p}(\Omega)$, $1 < p < \infty$ its trace ψ on $\partial\Omega$ lies in the fractional Sobolev space $W^{1-\frac{1}{p},p}(\partial\Omega)$. This means that the restriction to $\partial\Omega$, initially defined for functions in $C_c^\infty(\mathbb{R}^n)_{|\Omega}$, extends to a continuous map $W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$. Membership in $W^{1-\frac{1}{p},p}(\partial\Omega)$ characterises traces of Sobolev functions on the boundary of Lipschitz domains [19, Ch. VII, Thm 1], moreover the trace operator is surjective $W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$ [14, Thm. 1.5.1.3]. In particular, traces on $\partial\Omega$ of $C_c^\infty(\mathbb{R}^n)$-functions on $\mathbb{R}^n$ are dense in $W^{1-\frac{1}{p},p}(\partial\Omega)$. Also, $W_0^{1,p}(\Omega)$ coincides with the space of $W^{1,p}(\Omega)$-functions with zero trace [20, Prop. 3.3].

For $1 < l < \infty$, we let $W^{-\frac{1}{p},l}(\partial\Omega)$ indicate the fractional Sobolev space of negative order $-\frac{1}{p}$ and exponent l which is the dual space of $W^{\frac{1}{p},l'}(\partial\Omega)$, $\frac{1}{l} + \frac{1}{l'} = 1$; $W^{-\frac{1}{p},l}(\partial\Omega)$ may be realised as the completion of $L^l(\partial\Omega)$ with respect to the norm

$$\|\varphi\|_{W^{-\frac{1}{p},l}(\partial\Omega)} := \sup_{\|u\|_{W^{\frac{1}{p},l'}(\partial\Omega)}=1} \int_{\partial\Omega} \varphi u \, d\sigma. \tag{1.11}$$

Members of $W^{-\frac{1}{p},l}(\partial\Omega)$ are distributions on $\mathbb{R}^n$, supported on $\partial\Omega$.

Note that we defined the fractional Sobolev space $W^{s,p}(\partial\Omega)$ without resorting to the integral Sobolev space $W^{1,p}(\partial\Omega)$, that may be defined as those $f : \partial\Omega \to \mathbb{R}$ such that $f \circ \Phi_j^{-1} \in W^{1,p}(V_j)$ for all j, where V_j and Φ_j are as in Sec. 1.2.3. In fact $W^{s,p}(\partial\Omega) = [L^p(\partial\Omega), W^{1,p}(\partial\Omega)]_{s,p}$, where $[.,.]_{s,p}$ is the so-called real interpolation functor, see Appendix for a brief discussion. Functions $f \in W^{1,p}(\partial\Omega)$ have a well-defined gradient $\nabla_T f \in [L^p(\partial\Omega)]^n$, valued at a.e.. $x \in \Omega$ in the tangent space $T_x\partial\Omega$, see (1.78). Functions $\psi \in W^{1-\frac{1}{p},p}(\partial\Omega)$ also have a well-defined gradient $\nabla_T\psi$, but the latter is now a $(n-2)$-current on $\partial\Omega$ lying in the space $\mathcal{W}_1^{-\frac{1}{p},p}(\partial\Omega)$, see Appendix for details.

We are now in a position to introduce and study a second subspace of $[L^p(\Omega)]^n$.

Definition 22. *For Ω a bounded domain and $1 < p < \infty$, consider next the subspace $\mathrm{Div}_p(\Omega)$ of $[L^p(\Omega)]^n$ defined by*

$$\mathrm{Div}_p(\Omega) := \{u \in [L^p(\Omega)]^n : \ \mathrm{div}\, u \in L^p(\Omega)\};$$

here $\mathrm{div}\ u = \sum_{j=1}^n \partial_j u$ stands for the Euclidean (distributional) divergence of u, that we also denote sometimes as $\nabla \cdot u$.

Of particular interest is the following subspace of $\mathrm{Div}_p(\Omega)$:

$$\mathrm{Div}_{p,0}(\Omega) = \overline{\{u \in [C_c^\infty(\Omega)]^n : \mathrm{div}\, u = 0\}}^{[L^p(\Omega)]^n}, \tag{1.12}$$

that is, $\mathrm{Div}_{p,0}(\Omega)$ is the closure, with respect to the $\mathrm{Div}_p(\Omega)$-norm, (which coincides on $\mathrm{Div}_{p,0}(\Omega)$ with the $[L^p(\Omega)]^n$-norm) of those fields in $[C_c^\infty(\Omega)]^n$ that are divergence-free, see for example the exposition given in [22, Ch. III].

When Ω is taken to be Lipschitz more can be said about $\mathrm{Div}_p(\Omega)$ we state an important result here

Theorem 10 ([21, Lem. 1.2.2])**.** *Let Ω be Lipschitz, $\mathrm{Div}_p(\Omega)$ endowed with the norm*

$$\|u\|_{\mathrm{Div}_p(\Omega))} := (\|u\|^p_{[L^p(\Omega)]^n} + \|\mathrm{div}\, u\|^p_{L^p(\Omega)})^{\frac{1}{p}},$$

is a Banach space. There exists a bounded linear operator

$$\Gamma_\nu : u \mapsto \Gamma_\nu u, \qquad u \in \mathrm{Div}_p(\Omega),$$

from $\mathrm{Div}_p(\Omega)$ to $W^{-\frac{1}{p},p}(\partial\Omega)$ such that $\Gamma_\nu u$ coincides with the functional

$$\varphi \mapsto \langle \varphi, u \cdot \nu\rangle = \int_{\partial\Omega} (u \cdot \nu)\varphi\, d\sigma, \qquad \varphi \in W^{1,q}(\Omega).$$

In what follows whenever we use $u \cdot \nu$ we mean the normal trace operator, in each case ν is the exterior normal field on $\partial\Omega$ which coincides with the unit outer normal vector-field when it exists. The theorem above states that the normal component $u\cdot\nu$ of each $u \in \mathrm{Div}_p(\Omega)$, with Ω Lipschitz smooth, is well-defined as a member of $W^{-\frac{1}{p},p}(\partial\Omega)$ and the divergence formula holds:

$$\langle \varphi, u \cdot \nu\rangle = \int_\Omega \nabla\varphi(y) \cdot u(y)\, dy + \int_\Omega \varphi(y)\mathrm{div}\ u(y)\, dy, \quad \varphi \in W^{1,q}(\Omega). \tag{1.13}$$

Theorem 11 ([22, Thm III.2.3]). *If Ω Lipschitz smooth then the following equality is valid*

$$\mathrm{Div}_{p,0}(\Omega) = \{u \in [L^p(\Omega)]^n : \operatorname{div} u = 0 \text{ and } u \cdot \nu = 0\}. \tag{1.14}$$

Moreover, on domains Ω smooth enough such that (1.13) *holds for all $u \in [L^1_{loc}(\Omega)]^n \cap [L^p(\Omega)]^n$ with $\|\operatorname{div} u\|_{L^p(\Omega)} < \infty$ then equality* (1.14) *holds.*

Remark 4. *An example of domains that satisfy the conditions outlined in Theorem 11 is domains of finite perimeter. We can arrive at this conclusion by using [16, Thm 5.8.2], the density of $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ in $W^{1,p}(\Omega)$ and the density of $C^\infty_c(\Omega)$ in $L^p(\Omega)$.*

When $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we observe that $\mathrm{Div}_{p,0}(\Omega)$ and $G^q(\Omega)$ are the orthogonal spaces to each other *via* the pairing (1.3), for *every* open set $\Omega \subset \mathbb{R}^n$. Indeed, $\mathrm{Div}_{p,0}(\Omega)$ and $G^q(\Omega)$ are certainly orthogonal, since smooth compactly supported divergence-free fields are orthogonal to distributional gradients.

Moreover, if $F \in [L^1_{loc}(\Omega)]^n$ is orthogonal to all divergence free fields in $[C^\infty_c(\Omega)]^n$, then it is known that $F = \nabla\Psi$ where $\Psi \in W^{1,1}_{loc}(\Omega)$;

Theorem 12 ([22, Lem. III.1.1]). *Let $\Omega \subset \mathbb{R}^n$ be open and suppose $F \in [L^1_{loc}(\Omega)]^n$ verifies*

$$\int_\Omega F \cdot u = 0, \qquad \text{for all } u \in [C^\infty_c(\Omega)]^n \text{ with } \operatorname{div} u = 0.$$

Then, there exists a single-valued scalar function $\Psi \in W^{1,1}_{loc}(\Omega)$ such that $F = \nabla\Psi$.

We further have the following result.

Theorem 13 ([23, Thm 6.74]). *Let Ψ be a distribution on $\Omega \subset \mathbb{R}^n$ open, then*

$$\nabla\Psi \in L^p_{loc}(\Omega) \implies \Psi \in W^{1,p}_{loc}(\Omega).$$

Moreover, if Ω is bounded and of class C^1, then

$$\nabla\Psi \in L^p(\Omega) \implies \Psi \in W^{1,p}(\Omega).$$

Thus, if moreover, F in $[L^q(\Omega)]^n$, we get that Ψ in Theorem 12 is in $W^{1,q}_{loc}(\Omega)$, so that F belong to $G^q(\Omega)$. Thus, $G^q(\Omega) = (\mathrm{Div}_{p,0}(\Omega))^\perp$ and therefore, by the Hahn-Banach theorem, a member of $[L^p(\Omega)]^n$ which does not lie in the closed subspace $\mathrm{Div}_{p,0}(\Omega)$ cannot be orthogonal to $G^q(\Omega)$. Hence, it holds that $\mathrm{Div}_{p,0}(\Omega) = (G^q(\Omega))^\perp$, as announced.

1.3 The Helmholtz Decomposition

It was initially proved by Helmholtz that if $\mathbf{u} \in \mathbb{R}^3$ is a smooth vector-field that vanishes sufficiently fast at infinity, then it can be decomposed uniquely into the sum of a gradient and a curl, that is:

$$\mathbf{u} = \nabla\varphi + \nabla \times \mathbf{A}$$

where φ and $\mathbf{A}$ are the so-called scalar and vector potentials respectively, see [24]. This type of decomposition was subsequently extended to any dimension to function spaces that are useful for partial differential equations, see the account in [22, Ch. III].

Definition 23. *On a domain $\Omega \subset \mathbb{R}^n$, we say that $M \in [L^p(\Omega)]^n$ has a Helmholtz decomposition if there uniquely exist $\nabla\psi \in G^p(\Omega)$ and $D \in \mathrm{Div}_{p,0}(\Omega)$ such that*

$$M = D + \nabla\psi. \tag{1.15}$$

We say that a Helmholtz decomposition holds in $[L^p(\Omega)]^n$ if, for each $M \in [L^p(\Omega)]^n$, there uniquely exist $\nabla\psi \in G^p(\Omega)$ and $D \in \mathrm{Div}_{p,0}(\Omega)$ such that (1.15) is valid.

Note that when Ω is bounded, a decomposition like (1.15) is unique as soon as it exists. The existence of the Helmholtz decomposition is equivalent to the existence of a unique solution $\psi \in W^{1,p}(\Omega)$ to the following Neumann problem

$$\begin{aligned} \Delta\psi &= \operatorname{div} M, && \text{in } \Omega \\ \nabla\psi \cdot \nu &= M \cdot \nu, && \text{on } \partial\Omega. \end{aligned} \tag{1.16}$$

Since $[L^2(\Omega)]^n$ is a Hilbert space in which $G^2(\Omega)$ is the orthogonal space to $\mathrm{Div}_{2,0}(\Omega)$, a Helmholtz decomposition holds at exponent 2 for any domain Ω hence we have the topological direct sum

$$[L^2(\Omega)]^n = G^2(\Omega) \oplus \mathrm{Div}_{2,0}(\Omega).$$

Alternatively, it can shown that a solution to the Neumann problem exists always exists when $p = 2$. For $p \neq 2$ the situation becomes complicated. When Ω is Lipschitz, we turn to the following result,

Theorem 14 ([25, Thm 11.1])**.** *For each Lipschitz domain $\Omega \subset \mathbb{R}^n$, with arbitrary topology, there exists $\varepsilon(\Omega) > 0$ such that*

$$[L^p(\Omega)]^n = G^p(\Omega) \oplus \mathrm{Div}_{p,0}(\Omega),$$

is a topological direct sum for $p \in [\frac{3}{2} - \varepsilon(\Omega), 3 + \varepsilon(\Omega)]$. In the class of Lipschitz domains, this result is sharp. If however Ω of class C^1 we may take $1 < p < \infty$.

Moreover, if Ω is convex, then a Helmholtz decomposition exists for all $p \in (1, \infty)$, see [26, Thm 1.3]. More about domains on which a Helmholtz decomposition holds for $1 < p < \infty$ may be found in [27]. Note that if a Helmholtz decomposition exists in $[L^p(\Omega)]^n$, then

$$\|D\|_{[L^p(\Omega)]^n} + \|\nabla\psi\|_{[L^p(\Omega)]^n} \leq C(\Omega, p) \|M\|_{[L^p(\Omega)]^n}, \tag{1.17}$$

by the open mapping theorem. Hence, the Helmholtz decomposition is L^p-continuous whenever it exists. Observe also that if a Helmholtz decomposition holds in $[L^p(\Omega)]^n$ for some $p \in (1, \infty)$, then it holds in $[L^q(\Omega)]^n$ with $\frac{1}{p} + \frac{1}{q} = 1$, by duality.

For the remainder of this section we look at the Helmholtz decomposition on $\mathbb{R}^n$. On $\mathbb{R}^n$, a Helmholtz decomposition exists for $1 < p < \infty$. We review this classical result below, as it is important for our purposes. The standard proof is based on Riesz transforms:

Definition 24. *For $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, its j-th Riesz transform is defined as*

$$R_j(f)(x) = \lim_{\varepsilon \to 0} c_n \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy, \quad j \in \{1, 2, \ldots, n\}, \tag{1.18}$$

where

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

The pointwise limit in (1.18) exists for a.e.. $x \in \mathbb{R}^n$, moreover $R_j(f)$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Definition 25. *For $f \in \mathscr{S}(\mathbb{R}^n)$ the Fourier transform of f, denote $\hat{f}$ is defined as*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{2\pi i \xi \cdot y}\, dy, \qquad \xi \in \mathbb{R}^n.$$

We extend the notation $F \mapsto \hat{F}$ to designate the Fourier transform componentwise $[\mathscr{S}(\mathbb{R}^n)]^n \to [\mathscr{S}(\mathbb{R}^n)]^n$.

The Fourier transform, $f \mapsto \hat{f}$, maps $\mathscr{S}(\mathbb{R}^n)$ into itself. The Riesz transforms and the Fourier transform have a special relationship, the Riesz transform is what is called in [12] a multiplier of the Fourier transform, that is, for $j \in \{1, 2, \ldots, n\}$, it follows from see [12, Ch. II, III] that

$$\widehat{R_j(f)}(\xi) = i \frac{\xi_j}{|\xi|} \hat{f}(\xi), \qquad f \in \mathscr{S}(\mathbb{R}^n). \tag{1.19}$$

Much more closely related to the Riesz transforms are the Riesz potentials which are defined as follows :

Definition 26. *The Riesz potential of f, for f which is sufficiently smooth and for $0 < \kappa < n$ is defined by*

$$(I_\kappa f)(x) = \frac{1}{\gamma(\kappa)} \int_{\mathbb{R}^n} |x-y|^{-n+\kappa} f(y)\, dy, \tag{1.20}$$

where

$$\gamma(\kappa) = \frac{\pi^{\frac{n}{2}} 2^\kappa \Gamma(\frac{\kappa}{2})}{\Gamma(\frac{n}{2} - \frac{\kappa}{2})}.$$

From [12, Lem. 1] we have that if $f \in \mathscr{S}(\mathbb{R}^n)$, $n \geq 3$ and $2 \leq \kappa \leq n$ then $\Delta I_\kappa(f) = I_\kappa(\Delta f) = -I_{\kappa-2}(f)$. Thus if $\kappa = 2$ then $f = -I_2(\Delta f)$. Now, for $p \in (1, \infty)$, every $M \in [L^p(\mathbb{R})]^n$ has a Helmholtz decomposition:

$$M = D + \nabla \psi, \tag{1.21}$$

where $D \in [L^p(\mathbb{R}^n)]^n$ is divergence-free and $\psi \in \dot{W}^{1,p}(\mathbb{R}^n)$, and in fact

$$\nabla \psi = -\Big(R_j \Big(\sum_{k=1}^n R_k(M_k) \Big) \Big)^t \tag{1.22}$$

where M_k indicates the k-th component of M.

Indeed, the right-hand side of (1.22) lies in $[L^p(\mathbb{R}^n)]^n$ because of the L^p-boundedness of Riesz transforms. Moreover, when $M \in [L^2(\mathbb{R}^n)]^n$ and $f \in \mathscr{S}(\mathbb{R}^n)$, it follows from (1.19) and the isometric character of the Fourier transform in $L^2(\mathbb{R}^n)$ that

$$\begin{aligned} &\langle R_j\Big(\sum_{k=1}^n R_k(M_k)\Big)\,,\,\partial_l f\rangle - \langle R_l\Big(\sum_{k=1}^n R_k(M_k)\Big)\,,\,\partial_j f\rangle \\ &= \langle \frac{\xi_j}{|\xi|^2}\sum_{k=1}^n \xi_k \hat{M}_k\,,\, 2i\pi\xi_l \hat{f}\rangle - \langle \frac{\xi_l}{|\xi|^2}\sum_{k=1}^n \xi_k \hat{M}_k\,,\, 2i\pi\xi_j \hat{f}\rangle = 0, \end{aligned} \tag{1.23}$$

whence the first term in (1.23) is zero for $M \in [L^p(\mathbb{R}^n)]^n$, by density of $L^2(\mathbb{R}^n) \cap L^p(\Omega)$ in $L^p(\mathbb{R}^n)$. That is, the right hand side of (1.22) satisfies the (distributional) Schwarz rule and so it is a distributional gradient, which must be the gradient of some $W^{1,p}_{loc}$-function ψ. Consequently, this right hand side belongs to $G^p(\mathbb{R}^n)$. Finally, one verifies by an argument similar to the one in (1.23) that $M - \nabla\psi$ is orthogonal to gradients of Schwartz-functions, and therefore is divergence-free, see [28, Sec. 10.6]. Uniqueness of the decomposition comes from the fact that no nonconstant harmonic function on $\mathbb{R}^n$ can have a gradient in $[L^p(\mathbb{R}^n)]^n$.

The mechanism behind formula (1.22) is made transparent by the following, formal observation. Let ω_n denote the surface area of the unit ball in $\mathbb{R}^n$, and $\mathfrak{R}_2$ indicate the Riesz kernel of order 2:

$$\mathfrak{R}_2(x) := \frac{1}{(n-2)\omega_n |x|^{n-2}}. \tag{1.24}$$

If we put $U(x) = -\mathfrak{R}_2 * \mathrm{div} M$ for the harmonic potential of $\mathrm{div}\, M$, then $\Delta U = \mathrm{div} M$ where $\Delta := \sum_{j=1}^n \partial_j^2$ is the Euclidean Laplacian, whence $D := M - \nabla U$ is divergence-free and so $M = \nabla U + D$ is the Helmholtz decomposition, provided that $\nabla U \in [L^p(\mathbb{R}^n)]^n$. As the Fourier transform of $\mathrm{div} M$ is $-2i\pi \sum_{k=1}^n \xi_k \hat{M}_k(\xi)$ while $\hat{R}_2(\xi) = (2\pi|\xi|)^{-2}$ [12, Ch. V, Lem. 1], one has $\hat{U}(\xi) = i(2\pi)^{-1}|\xi|^{-2}\sum_{k=1}^n \xi_k \hat{M}_k(\xi)$ and therefore ∇U has Fourier transform $(\xi_1, \cdots, \xi_n)^t |\xi|^{-2}\sum_{k=1}^n \xi_k \hat{M}_k(\xi)$, which is equivalent to (1.22) in view of (1.19). What precedes suggests that $\psi = -\mathfrak{R}_2 * \mathrm{div} M$ is a natural candidate in (1.21), and the lemma below gives a rigorous argument to this effect when $M \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for some $q \in (1, n)$. However, we trade $\mathfrak{R}_2 * \mathrm{div} M$ for $\nabla \mathfrak{R}_2 * M$ (a formal integration by parts), as it will serve our purposes.

Lemma 3. *For $M \in [L^p(\mathbb{R}^n)]^n \cap [L^q(\mathbb{R}^n)]^n$ with $1 < p < \infty$ and $1 < q < n$, let us define*

$$\Psi_M(x) := \frac{1}{\omega_n}\int_{\mathbb{R}^n} M(y)\cdot \frac{(x-y)}{|x-y|^n} dy, \qquad x \in \mathbb{R}^n. \tag{1.25}$$

Then:

- *(i) the integral (1.25) converges absolutely for a.e.. x and $M \mapsto \Psi_M$ is continuous from $[L^q(\mathbb{R}^n)]^n$ into $L^{\frac{nq}{(n-q)}}(\mathbb{R}^n)$;*
- *(ii) $\nabla \Psi_M \in L^p(\mathbb{R}^n)$, and decomposition (1.21) holds with $\psi = \Psi_M$.*

Proof. Since $|\frac{x}{|x|^n}| \leq |x|^{1-n}$, assertion (i) follows from properties of Riesz potentials, see [12, Ch. V, Thm. 1]. *We claim* that it is enough to prove (ii) when $M \in [\mathscr{S}(\mathbb{R}^n)]^n$. Indeed, if $M \in [L^p(\mathbb{R}^n) \cap \mathrm{L}^q(\mathbb{R}^n)]^n$ then there is a sequence $F_n \in [\mathscr{S}(\mathbb{R}^n)]^n$ converging to M both in $[L^p(\mathbb{R}^n)]^n$ and in $[L^q(\mathbb{R}^n)]^n$, by mollification. Hence, if $M = D + \nabla\psi$ and $F_n = D_n + \nabla P_{F_n}$ are the Helmholtz decompositions of M and F_n respectively, we know from (1.17) that $\lim_n \nabla P_{F_n} = \nabla\psi$ in $[L^p(\mathbb{R}^n)]^n$ and from (i) that $\lim_n P_{F_n} = \Psi_M$ in $L^{nq/(n-q)}(\mathbb{R}^n)$. Because the latter limit implies that ∇P_{F_n} converges to $\nabla\Psi_M$ as a distribution, and since L^p-convergence implies convergence in the distributional sense, we conclude that $\nabla\Psi_M = \nabla\psi$, thereby *proving the claim.*

We now show that (ii) holds when $M \in [\mathscr{S}(\mathbb{R}^n)]^n$. In this case, the integral (1.25) converges absolutely for every x, and using Fubini's theorem and integration by parts one checks that $\Psi_M = -\mathfrak{R}_2 * \mathrm{div} M$, with $\mathfrak{R}_2$ as in (1.24). Hence, $\Delta\Psi_M = \mathrm{div} M$ and, by the discussion before the lemma, it remains to prove that $\nabla\Psi_M \in [L^p(\mathbb{R}^n)]^n$. Note that if $M \in [\mathscr{S}(\mathbb{R}^n)]^n$, then

$$\hat{\Psi}_M(\xi) = \frac{i}{2\pi|\xi|^2} \sum_{k=1}^{n} \xi_k \hat{M}_k(\xi) \tag{1.26}$$

as a tempered distribution, by [12, Ch. V, Lem. 1]. Let $\mathscr{S}_0(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$ consist of functions whose Fourier transform vanishes at 0; *i.e.*, functions in $\mathscr{S}(\mathbb{R}^n)$ with zero mean on $\mathbb{R}^n$. Let further $\Sigma(\mathbb{R}^n) \subset \mathscr{S}_0(\mathbb{R}^n)$ comprise those f such that $\hat{f}$ vanishes in a neighborhoud of the origin. For $M \in [\Sigma(\mathbb{R}^n)]^n$, it is clear from (1.26) that $\hat{\Psi}_M \in \mathscr{S}(\mathbb{R}^n)$, hence also $\Psi_M \in \mathscr{S}(\mathbb{R}^n)$ and *a fortiori* $\nabla\Psi_M \in [L^p(\mathbb{R}^n)]^n$, as desired. The conclusion in fact extends to every $M \in [\mathscr{S}_0(\mathbb{R}^n)]^n$, because $\Sigma(\mathbb{R}^n)$ is $L^p \cap L^q$-dense in $\mathscr{S}_0(\mathbb{R}^n)$ (see Lemma 8) and we may resort to a limiting argument resembling the one we used to reduce the proof to the case where $M \in [\mathscr{S}(\mathbb{R}^n)]^n$. Thus, in order to prove that $\nabla\Psi_M \in [L^p(\mathbb{R}^n)]^n$ for all $M \in \mathscr{S}^n$, it is enough to show this is true when $M = hv$ for any $v \in \mathbb{R}^n$ and some particular $h \in \mathscr{S} \setminus \mathscr{S}_0$, because the space of such functions complements $[\mathscr{S}_0]^n$ in $[\mathscr{S}]^n$. Since the function $\mathfrak{R}_2 * h$ is locally bounded and for $k \in \{1, \cdots, n\}$ the function $x \mapsto R_2(x - y)$ is absolutely continuous on each line $L_k := \{x : x_j = c_j,\, j \neq k\}$ except when $c_j = y_j$ for all $j \neq k$, we get on differentiating under the integral sign that $P_{hv} = -\sum_{k=1}^{n} v_k \partial_k (\mathfrak{R}_2 * h)$. So, we are left to check there exists $h \in \mathscr{S} \setminus \mathscr{S}_0$ such that $\mathfrak{R}_2 * h$ has all its second derivatives in $L^p(\mathbb{R}^n)$. Lemma 9 provides us with such a h, thereby concluding the proof.

□

As alluded to earlier, given a bounded open domain $\Omega \subset \mathbb{R}^n$, a Helmholtz decomposition does not always exist in $[L^p(\Omega)]^n$ for $1 < p < \infty$, even if Ω is Lipschitz. In Section 1.6, we will prove there is a related, three-term decomposition that exists in $[L^p(\Omega)]$ for all $1 < p < \infty$ as soon as Ω is Lipschitz. We state the result here.

Theorem 15. *Let Ω be a bounded Lipschitz open set and $M \in [L^p(\Omega)]^n$ a nonzero vector-field with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, there exists uniquely $\nabla\psi \in G_0^p(\Omega)$ and $D \in \mathrm{Div}_{p,0}(\Omega)$, together with a harmonic function h in Ω meeting $\int_{\Gamma_i} (\frac{\partial h}{\partial \nu})\, d\mathcal{H}^{n-1} = 0$ for each connected component Γ_i of $\partial\Omega$, such that*

$$M = \nabla\psi + D + |\nabla h|^{q-2}\nabla h. \tag{1.27}$$

The above theorem can be stated for a larger class of open sets as :

Corollary 1. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set whose boundary $\partial\Omega$ has zero Lebesgue measure, with $\{O_j\}_{j\in J}$ the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$. Assume that whenever $\overline{O}_j \bigcap \overline{\cup_{i\in I} O_i} \neq \emptyset$ for some index $j \in J$ and some subset of indices $I \subset J$, then $B_{1,p}\big(\overline{O}_j \bigcap \overline{\cup_{i\in I} O_i}\big) > 0$. Suppose in addition that O_j is non-thin at $B_{1,p}$-quasi every point of ∂O_i (which is no restriction if $n < p$) and that for all $u \in [L^1_{loc}(\Omega)]^n \cap [L^p(\Omega)]^n$ with $\|\operatorname{div} u\|_{L^p(\Omega)} < \infty$ the equality*

$$\langle \varphi, u \cdot \nu \rangle = \int_\Omega \nabla\varphi(y) \cdot u(y)\, dy + \int_\Omega \varphi(y) \operatorname{div} u(y)\, dy, \quad \varphi \in W^{1,q}(\Omega),$$

holds. Then for $M \in [L^p(\Omega)]^n$ a nonzero vector-field there exists uniquely $\nabla\psi \in G^p_0(\Omega)$, $D \in \operatorname{Div}_{p,0}(\Omega)$ and $h \in W^{1,q}(\Omega)$, a function harmonic in Ω satisfying $\nabla h \in (G^p_0(\Omega))^\perp$ such that

$$M = \nabla\psi + D + |\nabla h|^{q-2}\nabla h. \tag{1.28}$$

1.4 Double and Single layer potentials

In this section we look at layer potentials which will play some role in this work in relation to their use in the representation of solutions of problems such as (1.16). This section consists in mostly listing interesting properties of the layer potentials with the main references being two papers [29, 25]. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain whose boundary is either connected or disconnected. Hence, $\mathbb{R}^n \setminus \partial\Omega$ has at least two connected components. For any bounded connected set O we take the convention $O = \operatorname{int} \partial O$ and $O^- := \operatorname{ext} \partial O$.

Definition 27. *The double and single layer potentials of a function ψ defined on $\partial\Omega$, whose smoothness will be made precise later on, are defined by*

$$\mathcal{K}\psi(x) = -\frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y)\, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{1.29}$$

and

$$\mathcal{S}\psi(x) = -\frac{1}{(n-2)\omega_n} \int_{\partial\Omega} \psi(y) \frac{1}{|x-y|^{n-2}}\, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{1.30}$$

respectively. We call ψ the density of the double or single layer potential.

It is well-known that $\mathcal{K}\psi$ and $\mathcal{S}\psi$ are harmonic in $\mathbb{R}^n \setminus \partial\Omega$.

Definition 28. *For $x \in \partial\Omega$, the double layer potential is defined as the singular integral*

$$\begin{aligned} K\psi(x) &= p.v. - \frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y)\, d\sigma(y) \\ &= \lim_{\varepsilon \to 0} -\frac{1}{\omega_n} \int_{|x-y|>\varepsilon} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y)\, d\sigma(y). \end{aligned} \tag{1.31}$$

and

$$\begin{aligned} K^*\psi(x) &= p.v.\frac{1}{\omega_n}\int_{\partial\Omega}\psi(y)\frac{(x-y)}{|x-y|^n}\cdot\nu(x)\,d\sigma(y) \\ &= \lim_{\varepsilon\to 0}\frac{1}{\omega_n}\int_{|x-y|>\varepsilon}\psi(y)\frac{(x-y)}{|x-y|^n}\cdot\nu(x)\,d\sigma(y). \end{aligned} \tag{1.32}$$

Theorem 16 ([29, Thms 1.10, 1.11])**.** *If $\psi\in L^p(\partial\Omega)$ then $K\psi(x)$ and $K^*\psi(x)$ exist in $L^p(\partial\Omega)$ and point-wise for a.e.. $x\in\partial\Omega$. Also*

$$\lim_{y\to x}\mathcal{K}\psi(y)=\begin{cases}(\frac{1}{2}Id+K)\psi(x) & y\in\Omega\\ (-\frac{1}{2}Id+K)\psi(y) & y\in\mathbb{R}^n\backslash\Omega\end{cases},$$

$$\lim_{y\to x}\nu(x)\cdot\nabla\mathcal{S}\psi(y)=\begin{cases}(-\frac{1}{2}Id+K^*)\psi(x) & y\in\Omega\\ (\frac{1}{2}Id+K^*)\psi(y) & y\in\mathbb{R}^n\backslash\Omega\end{cases},$$

for almost every $x\in\partial\Omega$ where the convergence $y\to x$ is non-tangential.

It follows that the non-tangential limits on $\partial\Omega$ of the double layer potential from inside and outside, differ by the density ψ of the potential. For appropriate range of exponents, the double layer potential on Ω is a famous tool to solve the Dirichlet problem for the Laplace equation, which is to find $w:\Omega\to\mathbb{R}$ such that

$$\begin{aligned}\Delta w&=0 \text{ in } \Omega,\\ w&=g \text{ on } \partial\Omega.\end{aligned} \tag{1.33}$$

In fact, $(\frac{1}{2}Id+K):W^{\frac{1}{q},p}(\partial\Omega)\to W^{\frac{1}{q},p}(\partial\Omega)$ is invertible for $p\in[\frac{3}{2},3]$ and $\frac{1}{p}+\frac{1}{q}=1$, see [25, Thm 8.1]. Hence, the solution to (1.33) when $g\in W^{\frac{1}{q},p}(\partial\Omega)$ with $p\in[\frac{3}{2},3]$ is given by $w=\mathcal{K}(\frac{1}{2}Id+K)^{-1}g$ and belongs to $W^{1,p}(\Omega)$. Here, the boundary condition in (1.33) is satisfied both as a Sobolev trace and as a non-tangential limit a.e.. Likewise, the double layer potential on Ω^- is a tool to solve the exterior Dirichlet problem, which is to find $w:\Omega^-\to\mathbb{R}$ such that

$$\begin{aligned}\Delta w&=0 \text{ in } \mathbb{R}^n\backslash\overline{\Omega}\cup\{\infty\},\\ w&=g \text{ on } \partial\Omega,\end{aligned} \tag{1.34}$$

here, as $n\geq 3$, harmonicity at infinity means that $\lim_{|x|\to\infty}w(x)=0$ [30, Thm 4.8]. In fact, $(-\frac{1}{2}Id+K):W^{\frac{1}{q},p}(\partial\Omega)/\langle 1\rangle\to W^{\frac{1}{q},p}(\partial\Omega)/\langle 1\rangle$ is invertible for $p\in[\frac{3}{2},3]$, where the quotient by $\langle 1\rangle$ means "modulo constants", see [25, Thm 8.1]. Hence, $w=\mathcal{K}(-\frac{1}{2}Id+K)^{-1}g$ will solve the exterior Dirichlet problem up to a constant when $g\in W^{\frac{1}{q},p}(\partial\Omega)$ and $p\in[\frac{3}{2},3]$, with $w_{|\Omega^-\cap B(0,R)}\in W^{1,p}(\Omega^-\cap B(0,R))$ for all $R>0$ and $\nabla w\in[L^p(\Omega^-)]^n$. To account for constant boundary conditions, observe that the exterior Dirichlet problem with constant data on $\partial\Omega$ can be solved using the single layer potential of the Newtonian equilibrium measure of Ω [31, Ch. IV, Sec. 5, §20] (the latter has L^2 density with respect to σ after [32, Cor. to Thm 3], so that its single layer potential has gradient in

$[L^p(\Omega^-)]^n$). Altogether, the exterior Dirichlet problem with data $g \in W^{\frac{1}{q},p}(\partial\Omega)$ with $p \in [\frac{3}{2}, 3]$ can be solved using a combination of double and single layer potentials.

Unlike the double layer potential, the single layer potential is continuous across $\partial\Omega$, though its normal derivative is not. Note that the gradient of the single layer potential is given by

$$\nabla \mathcal{S}\psi(x) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{(x-y)}{|x-y|^n} \psi(y)\, d\sigma(y), \quad x \in \mathbb{R}^n \backslash \partial\Omega. \tag{1.35}$$

For $1 < p < \infty$ and $\psi \in L^p(\partial\Omega)$, it holds for a.e.. $y \in \partial\Omega$ that $\nu(y) \cdot \nabla \mathcal{S}\psi(x)$ converges to $-(\frac{1}{2}Id - K^*)\psi(y)$ (resp. $(\frac{1}{2}Id + K^*)\psi(y)$) as $x \to y$ non-tangentially in Ω (resp. Ω^-), where K^* operates on $L^q(\partial\Omega)$ for $1 < q < \infty$ and is the adjoint of K:

$$K^*\psi(x) = p.v. \frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(x)\, d\sigma(y)\,, \tag{1.36}$$

see [29, Thm 1.11] for a statement and further references.

For appropriate range of exponents, the single layer potential on Ω allows one to solve the Neumann problem for the Laplace equation:

$$\begin{aligned} \Delta w &= 0 \text{ in } \Omega, \\ \nabla w \cdot \nu &= g \text{ on } \partial\Omega, \end{aligned} \tag{1.37}$$

where the boundary condition in (1.37) is meant to satisfy the divergence formula; *i.e.* (1.13) holds when u gets replaced by ∇w and $(u \cdot \nu)$ by g. Likewise, the single layer potential on Ω^- can be used to solve the exterior Neumann problem:

$$\begin{aligned} \Delta w &= 0 \text{ in } \mathbb{R}^n \backslash \overline{\Omega} \cup \{\infty\}, \\ \nabla w \cdot \nu &= g \text{ on } \partial\Omega. \end{aligned} \tag{1.38}$$

More precisely, it follows from [25, Thm 8.1] that $(\pm\frac{1}{2}Id + K^*)$ extends to an invertible map $\tilde{W}^{-\frac{1}{p},p}(\partial\Omega) \to \tilde{W}^{-\frac{1}{p},p}(\partial\Omega)$ for $p \in [\frac{3}{2}, 3]$, where we have set

$$\tilde{W}^{-\frac{1}{p},p}(\partial\Omega) := \{ f \in W^{-\frac{1}{p},p}(\partial\Omega) \,:\, \int_{\partial\Omega} f(y)\, d\sigma(y) = 0 \}\,,$$

Thus, by [25, Thm 9.2], the solution to (1.37) (resp. (1.38)) can be written as $w = \mathcal{S}(\mp\frac{1}{2}Id + K^*)^{-1} g$, up to an additive constant, under the (necessary) condition that $\int_{\partial\Omega} g(y)\, d\sigma(y) = 0$. Moreover, w belongs to $W^{1,p}(\Omega)$ (resp. $w_{|\Omega^- \cap B(0,R)} \in W^{1,p}(\Omega^- \cap B(0,R))$ for all $R > 0$ and $\nabla w \in [L^p(\Omega^-)]^n$).

1.5 Silent magnetic sources

Let us represent a magnetisation carried by a bounded domain $\Omega \subset \mathbb{R}^3$ as a vector-field $M \in [L^p(\Omega)]^3$, with $1 < p < \infty$. Under the quasi-static assumption of Maxwell's equations, it is known (see [1]) that the scalar magnetic potential P_M generated by M is related to the latter by

$$\Delta P_M = \nabla \cdot M, \tag{1.39}$$

where P_M is zero at infinity. It follows from (1.39) and the vanishing of P_M at infinity (recall Ω is bounded by assumption) that

$$P_M(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \cdot M(y)}{|x-y|} dy, \tag{1.40}$$

that can be rewritten, since Ω is bounded, as

$$P_M(x) = \frac{1}{4\pi}\int_{\Omega} M(y) \cdot \frac{(x-y)}{|x-y|^3} dy. \tag{1.41}$$

A magnetisation is said to be Ω-silent, or a silent source in $\mathbb{R}^3 \setminus \Omega$, if it generates the zero field there. Although the main physical interest attaches to dimension 3, nothing gets simpler if we restrict to this setting, and perspective is gained if we extend the terminology to any dimension strictly greater than 2 by making the following, formal definition:

Definition 29. *For $\Omega \subset \mathbb{R}^n$ a bounded open set, we say that $M \in [L^p(\Omega)]^n$ is Ω-silent if*

$$\nabla P_M(x) = \nabla_x \left(\frac{1}{\omega_n} \int_{\Omega} M(y) \cdot \frac{(x-y)}{|x-y|^n} dy \right) = 0, \qquad \text{a.e. } x \in \mathbb{R}^n \setminus \Omega. \tag{1.42}$$

Note that P_M is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$. In particular, if $\partial\Omega$ has Lebesgue measure 0 then Definition 29 says that M is Ω-silent if and only if P_M is locally constant in $\mathbb{R}^n \setminus \overline{\Omega}$.

Note that P_M defined in (1.41) coincides with $\Psi_{\widetilde{M}}$ given by (1.25). As a consequence, Lemma 3 provides us at once with a criterion for M to be Ω-silent:

Theorem 17. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. A field $M \in [L^p(\Omega)]^n$ is Ω-silent if and only if the Helmholtz decomposition of $\widetilde{M}$ on $\mathbb{R}^n$, say $\widetilde{M} = \nabla\psi + D$, is such that both $\nabla\psi$ and D are zero a.e. on $\mathbb{R}^n \setminus \Omega$.*

Proof. Since Ω is bounded, $\widetilde{M}$ lies in $[L^q(\mathbb{R}^n)]^n$ for $1 < q \leq p$. Hence, in the Helmholtz decomposition $\widetilde{M} = \nabla\psi + D$, we may choose $\psi = \nabla\Psi_{\widetilde{M}}$, by Lemma 3. Thus, as $P_M = \Psi_{\widetilde{M}}$, the gradient term in the Helmholtz decomposition of $\widetilde{M}$ is ∇P_M, and therefore $\widetilde{M}$ is silent if and only if this gradient term is zero a.e.. on $\mathbb{R}^n \setminus \Omega$, by (1.42). Because $\widetilde{M}$ is zero on $\mathbb{R}^n \setminus \Omega$, this happens if and only if D is zero a.e. on $\mathbb{R}^n \setminus \Omega$, as desired. □

When Ω is a union of positively separated pieces, the question whether M is Ω-silent reduces to the corresponding question on each piece. This we record as a complement to Theorem 17, see *e.g.* [33] and [34, Cor. 3.5] for related, somewhat less general results.

Lemma 4. *Let $M \in [L^p(\Omega)]^n$ for some $p \in (1,\infty)$ with $\Omega = \cup_{j=1}^{l} \Omega_j$, where $\Omega_j \subset \mathbb{R}^n$ is a bounded open set whose boundary has zero measure and $\mathrm{d}(\Omega_j, \Omega_k) > 0$ for $j \neq k$. Then, M is Ω-silent if and only if the restriction $M_{|\Omega_j}$ is Ω_j-silent for each $j \in \{1, \cdots, l\}$.*

Proof. Let us put $O_1 := \cup_{j\neq 1}\Omega_j$. For each $i \in \{1, \cdots, n\}$, we have that $\partial_{x_i} P_{M_{|\Omega_1}}$ is harmonic in $\mathbb{R}^n \setminus \overline{\Omega_1}$ and $\partial_{x_i} P_{M_{|O_1}}$ is harmonic in $\mathbb{R}^n \setminus \overline{O_1}$, moreover $\partial_{x_i} P_{M_{|\Omega_1}} + \partial_{x_i} P_{M_{|O_1}} = 0$ on $\mathbb{R}^n \setminus \overline{\Omega}$. Because $\overline{\Omega}_1$ and $\overline{O}_1$ are positively separated, it entails that $\partial_{x_i} P_{M_{|\Omega_1}}$ extends harmonically to the whole of $\mathbb{R}^n$, and since it vanishes at infinity by inspection of (1.42) it must vanish identically, thanks to Liouville's theorem. Therefore $P_{M_{|\Omega_1}}$ is locally constant on $\mathbb{R}^n \setminus \overline{\Omega_1}$, as wanted. □

In the case where Ω is Lipschitz, Theorem 17 makes contact with the Helmholtz decomposition in $[L^p(\Omega)]^n$.

Definition 30. *Let Ω is a bounded Lipschitz domain of arbitrary topology with boundary $\partial\Omega$ and*

$$\mathbb{R}_{\partial\Omega} = span\{\chi_\omega : \omega \text{ is a connected component of } \partial\Omega\}$$

we shall denote by $W^{1,p}_{\mathbb{R}_{\partial\Omega}}(\Omega)$ the space of all members of $W^{1,p}(\Omega)$ that have trace in $\mathbb{R}_{\partial\Omega}$.

Theorem 18. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz open set, and $M \in [L^p(\Omega)]^n$ with $p \in (1, \infty)$. Then, M is Ω-silent if and only if $M = D_0 + \nabla\psi_0$, where $D_0 \in \mathrm{Div}_{p,0}(\Omega)$ and $\psi_0 \in W^{1,p}_{\mathbb{R}_{\partial\Omega}}(\Omega)$.*

Proof. Let $\Omega^{(1)} \cdots, \Omega^{(L)}$ be the connected components of Ω. We adapt the notation of Lemma 1 by writing $\Gamma^{(i)}_1, \cdots, \Gamma^{(i)}_{l_i}$ for the connected components of $\partial\Omega^{(i)}$ and $O^{(i)}_1, \cdots, O^{(i)}_{l_i}$ for the components of $\mathbb{R}^n \setminus \overline{\Omega^{(i)}}$, with $O^{(i)}_1 = \mathrm{ext}\,\Gamma^{(i)}_1$ and $O^{(i)}_j = \mathrm{int}\,\Gamma^{(i)}_j$ for $j \neq 1$.

Assume first that $M = D_0 + \nabla\psi_0$, with D_0 and ψ_0 as in the statement. Because D_0 has zero normal component on $\partial\Omega$ by assumption, one can see from (1.14) that $\widetilde{D}_0$ is divergence free on $\mathbb{R}^n$, as a distribution. Besides, as we suppose that ψ_0 is locally constant on $\partial\Omega$, to each $\Gamma^{(i)}_j$ there is a constant $c_{i,j}$ such that $(\psi_0)_{|\Gamma^{(i)}_j} = c_{i,j}$ for $i \in \{1, \cdots, N\}$ and $1 \leq j \leq l_i$. Replacing ψ_0 by $\psi_0 - c_{i,1}$ on $\Omega^{(i)}$, we may assume that $c_{i,1} = 0$ for all i and then one sees from Lemma 1 that the function ψ_1, equal to ψ_0 on Ω and to $c_{i,j}$ on $O^{(i)}_j$, lies in $W^{1,p}(\mathbb{R}^n)$ with $\nabla\psi_1 = \nabla\psi_0$ on Ω and $\nabla\psi_1 = 0$ outside Ω. Thus, $\widetilde{M} = \widetilde{D}_0 + \nabla\psi_1$ is the Helmholtz decomposition of $\widetilde{M}$ on $\mathbb{R}^n$, and both terms of the decomposition vanish outside Ω, as desired.

Conversely, assume that in the Helmholtz decomposition $\widetilde{M} = D + \nabla\psi$ on $[L^p(\mathbb{R}^n)]^n$, the summands D and $\nabla\psi$ vanish a.e.. outside Ω. Then, for any $\varphi \in C^\infty_c(\mathbb{R}^n)$, we have that $\int_\Omega D\cdot\nabla\varphi = \int_{\mathbb{R}^n} D\cdot\nabla\varphi = 0$ and therefore $\int_{\partial\Omega}(D\cdot\nu)\varphi d\sigma = 0$, by (1.13). Since traces of C^∞_c-functions are dense in $W^{1-\frac{1}{p},p}(\partial\Omega)$, we conclude that $D\cdot\nu = 0$ in $W^{-\frac{1}{p},p}(\partial\Omega)$ so that the restriction $D_{|\Omega}$ lies in $\mathrm{Div}_{p,0}(\Omega)$. Moreover, as $\nabla\psi = 0$ a.e.. on $\mathbb{R}^n \setminus \overline{\Omega}$, the function ψ is constant on each $O^{(i)}_j$ and so it has constant trace on each $\Gamma^{(i)}_j$. Hence, we can put $D_0 = D_{|\Omega}$ and $\psi_0|_{\partial\Omega} \in \mathbb{R}^n_{\partial\Omega}$. □

Before proceeding, we will discuss more the subspace $G^p_0(\Omega)$, in particular on the continuous extension of its members to $G^p(\mathbb{R}^n)$. The main interest in studying more this subspace is generalise Theorem 18. Recall the spaces $G^p(\Omega)$ and $G^p_0(\Omega)$ introduced in (1.7) and (1.8). We define a subspace $\widetilde{G}^p(\Omega) \subset G^p(\Omega)$ by

$$\widetilde{G}^p(\Omega) := \{F \in G^p(\Omega) : \ \widetilde{F} \in G^p(\mathbb{R}^n)\}. \tag{1.43}$$

Theorem 17 and Lemma 3 imply that $\widetilde{G}^p(\Omega)$ consists exactly of the magnetic fields, generated by a magnetisation in $L^p(\Omega)$, that vanish outside Ω; *i.e.*, the fields of silent magnetisations of L^p-class.

Unlike $G^p(\Omega)$ or $G^p_0(\Omega)$, the space $\widetilde{G}^p(\Omega)$ only depends on Ω modulo those $x \in \partial\Omega$ having a neighbourhood V_x such that $V_x \setminus \Omega$ has zero Lebesgue measure; for convenience, we say that such points are *Lebesgue isolated* in $\mathbb{R}^n \setminus \Omega$. The union of Ω and of the *Lebesgue-isolated* points in $\mathbb{R}^n \setminus \Omega$ is a bounded open set $\widetilde{\Omega} \supset \Omega$ such that $\widetilde{G}^p(\Omega) = \widetilde{G}^p(\widetilde{\Omega})$ and therefore, when studying $\widetilde{G}^p(\Omega)$, we may as well assume that $\mathbb{R}^n \setminus \Omega$ has no Lebesgue-isolated points. This hypothesis typically eliminates "cracks" from Ω.

We shall see that in fact $\widetilde{G}^p(\Omega)$ coincides with $G^p_0(\Omega)$ when $\partial\Omega$ has Lebesgue measure zero and $\mathbb{R}^n \setminus \Omega$ has no Lebesgue-isolated points, provided that the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$ are positively separated, meaning that the distance between any two of them is bigger than a strictly positive constant independent of the components; this is a consequence of Theorem 19 below. Of course, as Ω is bounded, the separation condition entails that the connected components are finite in number, and then separation reduces to the property that their closures do not meet. The separation condition cannot be dispensed with in general, as the example in Sec. 1.9.3 shows. More generally, in Theorem 19, we handle the situation where Ω is a bounded open set such that $\partial\Omega$ has Lebesgue measure zero and the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$ may only accumulate in a "thick" manner; the case of an arbitrary bounded Ω will not be considered here. We begin with a lemma:

Lemma 5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set whose boundary $\partial\Omega$ has zero Lebesgue measure. Let further $\{O_j\}_{j\in J}$ designate the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$. For $p \in (1,\infty)$, a vector field F lies in $\widetilde{G}^p(\Omega)$ if, and only if $\widetilde{F}$ is of the form ∇f for some (necessarily unique) $f \in W^{1,p}(\mathbb{R}^n)$ which is constant on each O_j.*

Proof. By definition, every $F \in G^p(\mathbb{R}^n)$ is of the form ∇f with $f \in W^{1,p}_{loc}(\mathbb{R}^n)$. If moreover $F = 0$ on $\mathbb{R}^n \setminus \Omega$, then f is constant in each O_j; conversely, since $\partial\Omega$ has Lebesgue measure 0, it is equivalent to say that F vanishes on $\mathbb{R}^n \setminus \Omega$ and that f is constant on each O_j. Let O_0 designate, without loss of generality, the unbounded connected component of $\mathbb{R}^n \setminus \overline{\Omega}$; the latter uniquely exists since Ω is bounded. Adding a constant to f if necessary, we may assume that it vanishes on O_0, and then $f \in W^{1,p}(\mathbb{R}^n)$. □

Note: if $f \in W^{1,p}(\mathbb{R}^n)$ is such that $\nabla f \in \widetilde{G}^p(\Omega)$, necessarily f vanishes in the unbounded component of $\mathbb{R}^n \setminus \overline{\Omega}$, for it is constant there and must lie in $L^p(\mathbb{R}^n)$.

Theorem 19. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set whose boundary $\partial\Omega$ has zero Lebesgue measure, with $\{O_j\}_{j\in J}$ the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$. Assume that whenever $\overline{O_j} \bigcap \overline{\cup_{i\in I} O_i} \neq \emptyset$ for some index $j \in J$ and some subset of indices $I \subset J$, then $B_{1,p}\big(\overline{O_j} \bigcap \overline{\cup_{i\in I} O_i}\big) > 0$. Suppose in addition that O_j is non-thin at $B_{1,p}$-quasi every point of ∂O_i (which is no restriction if $n < p$). Then, $G^p_0(\Omega) = \widetilde{G}^p(\Omega)$.*

Proof. Without loss of generality, we assume that O_0 is the unbounded component of $\mathbb{R}^n \setminus \overline{\Omega}$. Pick $\varphi \in C^\infty(\Omega)$ with $\nabla\varphi \in [C_c(\Omega)]^n$, and let the U_i enumerate, for

$i \in I \subset \mathbb{N}$, the connected components of $\mathbb{R}^n \setminus \operatorname{supp} \nabla\varphi$, with U_0 to denote the unbounded component. The $U_i \cap \Omega$ are nonempty open sets partitioning $\Omega \setminus \operatorname{supp} \nabla\varphi$. Assume first that Ω is connected, and define $B_i := \cup_{k\neq i} U_k \cup \Omega$ for $i \in I$. Clearly, B_i is a connected open set and $U_i \cup B_i = \mathbb{R}^n$. *We contend* that $U_i \cap B_i$ is connected. Indeed, as $\mathbb{R}^n$ is simply connected, the last portion of the Mayer-Vietoris sequence in homology [17, Sec. 2.2] yields (with H_l to indicate the l-th homology group):

$$0 = H_1(U_i \cup B_i) \to H_0(U_i \cap B_i) \to H_0(U_i) \oplus H_0(B_i) \to H_0(U_i \cup B_i) \to 0$$

from which it follows, since $H_0(U_i) = H_0(B_i) = H_0(U_i \cup B_i) = \mathbb{Z}$ (because U_i, B_i and $U_i \cup B_i$ are connected), that $H_0(U_i \cap B_i) = \mathbb{Z}$ as well. *This proves our contention*, and so $U_i \cap \Omega = U_i \cap B_i$ is in turn connected. Hence, φ is constant on $U_i \cap \Omega$ because $\nabla\varphi = 0$ there, say $\varphi_{|U_i\cap\Omega} \equiv c_i$. If we define Φ to be $\varphi - c_0$ on Ω and $c_i - c_0$ on U_i, it is readily checked that Φ is well-defined in $C_c^\infty(\mathbb{R}^n)$ with $\nabla\Phi = \widetilde{\nabla\varphi}$; in fact Φ is zero on U_0, in particular it vanishes outside every ball containing Ω. To recap, we showed that if Ω is connected and $\varphi \in C^\infty(\Omega)$ with $\nabla\varphi \in [C_c^\infty(\Omega)]^n$, then there is $\Phi \in C_c^\infty(\mathbb{R}^n)$ such that $\nabla\Phi = \widetilde{\nabla\varphi}$. *We claim* that the same holds even when Ω is not connected. Indeed, let $\{\Omega_l\}_{l\in L}$ denote the connected components of Ω, where $L = \{1, 2, \cdots\}$ is an initial segment of natural numbers (finite or infinite). If $\varphi \in C^\infty(\Omega)$ is such that $\nabla\varphi \in [C_c^\infty(\Omega)]^n$, then $\operatorname{supp} \nabla\varphi$ can meet only finitely many Ω_l, say $\Omega_1, \cdots, \Omega_N$. Thus, φ is constant on Ω_l for $l > N$, and we may as well assume it is zero there because this does not change $\nabla\varphi$. For $1 \le l \le N$, the function $\varphi_l := \varphi_{|\Omega_l}$ lies in $C^\infty(\Omega_l)$ and $\nabla\varphi_l = (\nabla\varphi)_{|\Omega_l}$ is compactly supported in Ω_l. Hence, by the first part of the proof, there is $\Phi_l \in C_c^\infty(\mathbb{R}^n)$ such that $\nabla\Phi_l = \widetilde{\nabla\varphi_l}$, and so $\Phi := \sum_{1\le l\le N} \Phi_l$ lies in $C_c^\infty(\mathbb{R}^n)$. By construction $\nabla\Phi = \sum_{1\le l\le N} \widetilde{\nabla\varphi_l}$ is zero outside $\cup_{1\le l\le N}\Omega_l$ and coincides with $\nabla\varphi_l$ on Ω_l, so that $\nabla\Phi = \widetilde{\nabla\varphi}$ *as claimed*.

Next, consider $F \in G_0^p(\Omega)$ and let φ_k be a sequence in $C^\infty(\Omega)$ such that $\nabla\varphi_k \in [C_c^\infty(\Omega)]^n$ with $\nabla\varphi_k \to F$ in $[L^p(\Omega)]^n$. Let further Φ_k be a sequence in $C_c^\infty(\mathbb{R}^n)$ with $\nabla\Phi_k = \widetilde{\nabla\varphi_k}$ and such that Φ_k is zero on O_0; such a sequence exists by what precedes. Applying (1.4) on a ball containing Ω, we deduce that Φ_k converges in $W^{1,p}(\mathbb{R}^n)$, as $k \to \infty$, to some f such that $\nabla f = \widetilde{F}$. Hence, $F \in \widetilde{G}^p(\Omega)$ so that $G_0^p(\Omega) \subset \widetilde{G}^p(\Omega)$.

Conversely, let $F \in \widetilde{G}^p(\Omega)$ so that, according to Lemma 5, $\widetilde{F} = \nabla f$ where $f \in W^{1,p}(\mathbb{R}^n)$ assumes a constant value $c_j(f)$ on O_j, for every $j \in J$. Note, since $c_0(f) = 0$ while O_0 is non-thin at quasi every point of ∂O_0 by assumption, that $f(x) = 0$ for $B_{1,p}$-quasi every $x \in \overline{O}_0$: it is so because f is continuous outside a set of arbitrary small $B_{1,p}$-capacity [8, Proposition 6.1.2]. Suppose for a while that $f \ge 0$ and fix $\delta > 0$, together with a sequence $\{\delta_i\}_{i\in\mathbb{N}}$ of strictly positive numbers such that $\sum_i \delta_i = \delta$. Pick $\varepsilon > 0$ and then $\varepsilon' \in (0, \varepsilon]$ small enough that $\int_{\{0\le f\le\varepsilon'\}} |\nabla f|^p dm \le \varepsilon^p$; since $\|\nabla f\|_{L^p(\mathbb{R}^n)} < \infty$, such a ε' exists, by the monotone convergence theorem. Let $I_0 \subset J$ be the set of indices i for which $c_i(f) \ge \varepsilon'$, and define $K_0 := \overline{\cup_{i\in I_0} O_i}$. Of necessity $K_0 \cap \overline{O}_0 = \emptyset$, otherwise $B_{1,p}(K_0 \cap \overline{O}_0) > 0$ by assumption and, since f is continuous outside a set of arbitrary small $B_{1,p}$-capacity, it would imply that $f \ge \varepsilon' > 0$ on a set $E \subset \partial O_0$ with $B_{1,p}(E) > 0$, contradicting that $f(x) = 0$ for $B_{1,p}$-quasi every $x \in \overline{O}_0$. Hence, by [8, Thm 9.1.3], there is a C^∞ function $\eta_0 : \mathbb{R}^n \to [0, 1]$ with $\eta_0 = 1$ on K_0 and $\eta = 0$ on $\overline{O}_0$ such that

$\|f-\eta f\|_{W^{1,p}(\mathbb{R}^n)} \leq \varepsilon'$. Now, if we let $f_0 := \max\{\eta f - \varepsilon', 0\}$, we find that $\nabla f_0 \in \widetilde{G}^p(\Omega)$ with $c_i(f_0) = c_i(f) - \varepsilon'$ for $i \in I$ and $c_i(f_0) = 0$ for $i \notin I$. Moreover, it follows from [8, Thm 3.3.1] that $\nabla f_0 = \nabla(\eta f)$ a.e.. on $\{x : \eta(x)f(x) > \varepsilon'\}$ and $\nabla f_0 = 0$ a.e.. on $\{x : \eta(x)f(x) \leq \varepsilon'\}$. Therefore,

$$\begin{aligned}
\|\eta f - f_0\|^p_{W^{1,p}(\mathbb{R}^n)} &= \textstyle\int_{\mathbb{R}^n} |\eta f - f_0|^p dm + \int_{\mathbb{R}^n} |\nabla(\eta f) - \nabla f_0|^p dm && (1.44)\\
&\leq (\varepsilon')^p m(\Omega) + \textstyle\int_{\{0\leq \eta f \leq \varepsilon'\}} |\nabla(\eta f)|^p dm && (1.45)\\
&\leq (\varepsilon')^p m(\Omega) + \left(\varepsilon' + (\textstyle\int_{\{0\leq f\leq \varepsilon'\}} |\nabla f|^p dm)^{\frac{1}{p}}\right)^p && (1.46)\\
&\leq (\varepsilon')^p m(\Omega) + (\varepsilon' + \varepsilon)^p . && (1.47)
\end{aligned}$$

Altogether, as $\varepsilon' \leq \varepsilon$, we get from the triangle inequality that

$$\|f - f_0\|_{W^{1,p(\mathbb{R}^n)}} \leq \left(1 + (m(\Omega) + 2^p)^{\frac{1}{p}}\right)\varepsilon \tag{1.48}$$

which is arbitrary small with ε. In the general case where f is not signed, we write $f = f^+ - f^-$ with $f^+ := \max\{f, 0\}$ and $f^- = (-f)^+$ and remark that $\nabla f^{\pm}$ satisfies the same assumptions as $\widetilde{F}$ [8, Thm. 3.3.1]. So, we can apply what precedes to f^+ and f^- to obtain functions $f_{0,+}$ and $f_{0,-}$; we then put $f_0 := f_{0,+} - f_{0,-}$. To recap, we constructed $f_0 \in W^{1,p}(\mathbb{R}^n)$ such that $\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)}$ is arbitrary small and f_0 is constant on a neighbourhood of $\overline{O}_0$, while it is constant on each O_j; in particular, we can make $\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)} < \delta_0$. Note also that if f was constant on a neighbourhood of $\overline{O}_j$ for some j, so is f_0 on the same neighbourhood (possibly with a different constant, though). We now proceed inductively: to complete the next step, pick $x_1 \in O_1$ and consider the inversion with center x_1 given by

$$\mathcal{I}_{x_1}(x) := \frac{x - x_1}{|x - x_1|^2}.$$

It is a smooth involution of the "sphere" $\mathbb{R}^n \cup \{\infty\} \sim \mathbb{S}^n$ that maps x_1 to ∞, with Jacobian determinant $-|x - x_1|^{-2n}$ at $x \neq x_1$; in fact, the Jacobian matrix at x is conjugate *via* a unitary matrix to $\operatorname{diag}\{-|x - x_1|^{-2}, |x - x_1|^{-2}, \cdots, |x - x_1|^{-2}\}$, see [30, Thm 4.2] (the unitary matrix depends on x, though). Clearly, $\Omega_1 := \mathcal{I}_{x_1}(\Omega)$ is a bounded open set, and the $U_i := \mathcal{I}_{x_1}(O_j)$ are the connected components of $\mathbb{R}^n \setminus \overline{\Omega}_1$, with U_1 being the unbounded component. From the change of variable formula and the definition of $B_{1,p}$-capacity, one checks that $\mathcal{I}_{x_1}$ preserves sets of $B_{1,p}$-capacity zero in $\mathbb{R}^n \setminus \{x_1, \infty\}$, consequently $B_{1,p}\big(\overline{U_j} \bigcap \overline{\cup_{i\in I} U_i}\big) > 0$ whenever $\overline{U_j} \bigcap \overline{\cup_{i\in I} U_i} \neq \emptyset$; also, by definition of thinness [8, Definition 6.3.7], $\mathcal{I}_{x_1}$ preserves thinness of a set at any $x \neq x_1$. Thus, U_j is non-thin at $B_{1,p}$-quasi every point of ∂U_j for each j. Furthermore, if U is an open set with compact closure in $\mathbb{R}^n \setminus \{x_1\}$ and we put $m_1 := \inf_{x\in U} |x - x_1|$, we get from the change of variable formula again that for $g \in W^{1,p}(U)$:

$$\|g\|_{W^{1,p}(U)} \leq \max\{m_1^{-2n/p}, m_1^{-2(n-p)/p}\}\|g \circ \mathcal{I}_{x_1}\|_{W^{1,p}(\mathcal{I}_{x_1}(U))}. \tag{1.49}$$

As the function $h_1 := f_0 \circ \mathcal{I}_{x_1} - c_1(f_0)$ lies in $W^{1,p}(\mathbb{R}^n)$ and is constant on U_j for each j as well as constant on a neighborhood of $\overline{U}_0$, we can argue as we did to construct f_0 from f, only with Ω_1 instead of Ω and h_1 instead of f.

This provides us with a function g_1 that can be made arbitrary close to h_1 in $W^{1,p}(\mathbb{R}^n)$ while being constant on each U_j and constant on a neighborhood of both $\overline{U}_0$ and $\overline{U}_1$. If λ_0 denotes the constant value assumed by g_1 on U_0, we get since $h_1 = -c_1(f_0)$ on U_0 that $|\lambda_0 + c_1(f_0)| m^{\frac{1}{p}}(U_0) \leq \|h_1 - g_1\|_{L^p(U_0)}$, whence $|\lambda_0 + c_1(f_0)|$ can be made arbitrary small with $\|h_1 - g_1\|_{W^{1,p}(\mathbb{R}^n)}$. Thus, letting $f_1 := (g_1 \circ \mathcal{I}_{x_1} - \lambda_0)$, we get in view of (1.49) since $f_0 = f_1 = 0$ on O_0:

$$\|f_0 - f_1\|_{W^{1,p}(\mathbb{R}^n)} = \|(h_1 - g_1) \circ \mathcal{I}_{x_1} + c_1(f_0) + \lambda_0\|_{W^{1,p}(\mathbb{R}^n \setminus U_1)} \tag{1.50}$$

$$\leq C\|h_1 - g_1\|_{W^{1,p}(\mathbb{R}^n)} + |\lambda_0 + c_1(f_0)| m^{\frac{1}{p}}(\mathbb{R}^n \setminus U_1) \tag{1.51}$$

$$\leq C'\|h_1 - g_1\|_{W^{1,p}(\mathbb{R}^n)}, \tag{1.52}$$

where C and C' are geometric constants depending on Ω and our choice of $x_1 \in O_1$. One deduces from (1.50) that $\|f_0 - f_1\|_{W^{1,p}(\mathbb{R}^n)}$ can be made arbitrary small, in particular smaller than δ_1. Iterating the argument, we construct a sequence of functions f_k, $0 \leq k$, such that $\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)} \leq \delta_0$ and $\|f_k - f_{k+1}\|_{W^{1,p}(\mathbb{R}^n)} \leq \delta_{k+1}$, with f_ℓ constant on a neighborhood V_k of $\overline{O}_k$ for $\ell \geq k$. Since $\sum_k \delta_k = \delta < \infty$, the sequence f_k is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$ that converges to some $\widetilde{f}$; clearly $\nabla \widetilde{f}$ is compactly supported in Ω, and $\|f - \widetilde{f}\|_{W^{1,p}(\mathbb{R}^n)} \leq \delta$. By mollification, we can now construct a $\varphi \in C^\infty(\mathbb{R}^n)$ which is constant on a neighborhood of O_j for each j (the constant being 0 when $j = 0$), and such that $\|f - \varphi\|_{W^{1,p}(\mathbb{R}^n)} \leq 2\delta$. Since δ was arbitrary, we find that $F \in G_0^p(\Omega)$, as desired. □

Hence more generally, we have the following extension of Theorem 18.

Theorem 20. *Let $M \in [L^p(\Omega)]^n$ for some $p \in (1, \infty)$, with $\Omega \subset \mathbb{R}^n$ a bounded finitely connected open set as in Theorem 19. Then, M is Ω-silent if and only if $M = D + \nabla\psi$ where $\widetilde{D}$ is divergence free and $\psi \in W^{1,p}(\Omega)$ is such that $\nabla\psi \in G_0^p(\Omega)$.*

Proof. The theorem is a direct consequence of Theorem 17 and Theorem 19. □

Theorem 20 raises the issue as to whether, under very mild conditions on Ω like those in that theorem, the divergence free character of $\widetilde{D}$ implies that $D \in \mathrm{Div}_{p,0}(\Omega)$; a positive answer would yield a generalization of Theorem 18 to very rough domains. We have the following partial answer:

Corollary 2. *Let $M \in [L^p(\Omega)]^n$ for some $p \in (1, \infty)$, with $\Omega \subset \mathbb{R}^n$ a bounded finitely connected open set as in Theorem 19 and the equality*

$$\langle \varphi, u \cdot \nu \rangle = \int_\Omega \nabla\varphi(y) \cdot u(y)\, dy + \int_\Omega \varphi(y) \mathrm{div}\, u(y)\, dy, \quad \varphi \in W^{1,q}(\Omega),$$

holds for all $u \in [L^1_{loc}(\Omega)]^n \cap [L^p(\Omega)]^n$ with $\|\mathrm{div}\, u\|_{L^p(\Omega)} < \infty$. Then, M is Ω-silent if and only if $M = D + \nabla\psi$ where $D \in \mathrm{Div}_{p,0}(\Omega)$ and $\psi \in W^{1,p}(\Omega)$ is such that $\nabla\psi \in G_0^p(\Omega)$.

Proof. The corollary is a direct consequence of Theorem 20 and Theorem 11. □

1.6 Equivalent Sources

We now wish to look at Ω-equivalent sources which are defined as

Definition 31. *Let $M_1, M_2 \in [L^p(\Omega)]^n$, we say that M_1 and M_2 are Ω-equivalent sources if*

$$P_{M_1}(x) = P_{M_2}(x) + \gamma \qquad \text{for a.e.} \qquad x \in \mathbb{R}^n \backslash \Omega,$$

with

$$\gamma \in \text{span}_{\mathbb{R}}\{\chi_O : O \text{ is a connected component of } \mathbb{R}^n \backslash \overline{\Omega}\}.$$

Among all sources $\{M_i\} \subset [L^p(\Omega)]^n$ Ω-equivalent to a given $M \in [L^p(\Omega)]^n$, the one with with minimum L^p-norm is called the norm-minimising equivalent source.

We have the following corollary of Theorem 18 for the characterisation of Ω-equivalent sources for Ω a Lipschitz domain.

Corollary 3. *Let Ω be a bounded Lipschitz set and $M_1, M_2 \in [L^p(\Omega)]^n$, $1 < p < \infty$, then M_1 and M_2 are Ω-equivalent sources if and only we can find a unique $\psi \in W^{1,p}_{\mathbb{R}_{\partial\Omega}}(\Omega)$ such that*

$$\int_\Omega (\nabla\psi - (M_1 - M_2)) \cdot \nabla\varphi = 0, \tag{1.53}$$

for all $\varphi \in W^{1,q}(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The above corollary is a direct consequence of Theorem 18, and the Helmholtz decomposition and the existence of a unique solution to the Neumann problem (1.16). □

Corollary 4. *Let Ω be a bounded Lipschitz set and $M_1, M_2 \in [L^p(\Omega)]^n$, $1 < p < \infty$, then M_1 and M_2 are Ω-equivalent sources if and only if for*

$$\psi_l(x) = -I_1\Big[\sum_{k=1}^n R_k\Big((M_l \vee 0)_k\Big)\Big](x), \tag{1.54}$$

$l = 1, 2$ we have $\psi_1(x) = \psi_2(x) + c_i$ for almost every $x \in \mathbb{R}^n \backslash \Omega \cap O_i$ where c_i is a constant for O_i a connected component of $\mathbb{R}^n \backslash \Omega$.

Proof. Suppose $M_1, M_2 \in [L^p(\Omega)]^n$ are Ω-equivalent we have from Theorem 18 that $M_1 - M_2$ has the decomposition as stated in the theorem, that is,

$$M_1 - M_2 = D + \nabla\psi \in [L^p(\Omega)]^n,$$

with $D \in \text{Div}_{p,0}(\Omega)$ and $\nabla\psi \in G^p_0(\Omega)$. Now we can extend $M_1 - M_2$ to $(M_1 - M_2) \vee \mathbf{0}$ on $\mathbb{R}^n$ so that we can apply the Riesz transform and get that the gradient, $\nabla\psi$, is obtained as in (1.22). Due to the continuity of Riesz transforms on $L^p(\mathbb{R}^n)$ we have that they vanish identically outside Ω. This implies that ψ is a constant on each connected component of $\mathbb{R}^n \backslash \overline{\Omega}$ up to the boundary. Now in view of the discussion in [12, Ch. V,

Sec. 2.3] and the fact that $\sum_{k=1}^{n} R_k^2 = -I$, where I is the identity operator, we have the formula,

$$\begin{aligned}\psi(x) &= I_1\Big[\sum_{j=1}^{n} R_j^2\Big(\sum_{k=1}^{n} R_k\Big((M_1 - M_2)_k\Big)\Big)\Big](x) \\ &= -I_1\Big[\sum_{k=1}^{n} R_k\Big((M_1 \vee 0)_k\Big)\Big](x) + I_1\Big[\sum_{k=1}^{n} R_k\Big((M_2 \vee 0)_k\Big)\Big](x),\end{aligned} \tag{1.55}$$

where we have abused notation by denoting the components of $M_1 - M_2$ extended to $(M_1 - M_2) \vee \mathbf{0}$ on $\mathbb{R}^n$ by $(M_1 - M_2)_k$ $k = 1, 2, 3$. Now using the linearity of the Riesz transform and Riesz potential, and the fact that ψ is constant on each connected component of $\mathbb{R}^n \backslash \Omega$ up to the boundary we have the desired result.

Conversely, we begin by noting that the Helmholtz decomposition is valid on $\mathbb{R}^n$ therefore representing $M_1 \vee \mathbf{0}$ and $M_2 \vee \mathbf{0}$ by M_1 and M_2, respectively, we have that $\nabla(\psi_1 - \psi_2) = (M_1 - M_2) - (D_1 - D_2)$, ψ_l as defined in (1.54) and $D_l \in \mathrm{Div}_p(\mathbb{R}^n)$, $l = 1, 2$. Now letting $\nabla\psi = \nabla(\psi_1 - \psi_2)$ and $D = D_1 - D_2$ we that ψ constant on each connected component of $\mathbb{R}^n \backslash \Omega$ so that we can extend D by $\mathbf{0}$ outside Ω so that by taking restrictions, $D \in \mathrm{Div}_{p,0}(\Omega)$ by [22, Exercise III.2.3]. Now from Theorem 18 we have $M_1 - M_2$ is Ω-silent as desired. □

The proof of the following result serves to show the use of double layer potentials in the characterisation of silent sources. The proof relies simply on the isomorphism of the double layer potential on Lipschitz domains as outlined in [25].

Corollary 5. *Let $\Omega \subset \mathbb{R}^n$ be Lipschitz, for $p \in [\frac{3}{2}, 3]$, let $M_1 = D_1 + \nabla\psi_1 \in [L^p(\Omega)]^n$ and $M_2 = D_2 + \nabla\psi_2 \in [L^p(\Omega)]^n$ with $D_1, D_2 \in \mathrm{Div}_{p,0}(\Omega)$ and $\psi_1, \psi_2 \in W^{1,p}(\Omega)$. We have that M_1 and M_2 are Ω-equivalent if and only if the trace of ψ_1 and the trace of ψ_2 differ by a member of $\mathbb{R}_{\partial\Omega}$, that is, they differ by constants on each connected component of the boundary.*

Proof. We begin by noting that the potential produced by the divergence of M_i outside Ω is determined only by the trace of the ψ_i, hence the potential is a double layer potential. This is easily seen from taking $x \in \mathbb{R}^n \backslash \Omega$ and letting $P_{M_i}(x)$ be the potential generated by the divergence of M_i at x we have

$$\begin{aligned}P_{M_i}(x) &= \frac{1}{(n-2)\omega_n} \int_\Omega (D_i + \nabla\psi_i)(y) \cdot \nabla_y\Big(\frac{1}{|x-y|^{(n-2)}}\Big)\, dy \\ &= \frac{1}{(n-2)\omega_n} \int_{\partial\Omega} \psi_i(y) \nabla_y\Big(\frac{1}{|x-y|^{(n-2)}}\Big) \cdot \nu(y)\, d\sigma(y),\end{aligned}$$

where we have used that $D_i \in \mathrm{Div}_{p,0}(\Omega)$ and Green's identity. It follows from the remark in [25] after the proof of Theorem 8.1 that if the trace of ψ_1 and the trace of ψ_2 differ by a member of $\mathbb{R}_{\partial\Omega}$ then for almost every $x \in \partial\Omega$

$$P_{M_1}(x) - P_{M_2}(x) = \frac{1}{(n-2)\omega_n} \int_{\partial\Omega} (\psi_1 - \psi_2)(y) \nabla_y\Big(\frac{1}{|x-y|^{(n-2)}}\Big) \cdot \nu(y)\, d\sigma(y)$$

is a member of $\mathbb{R}_{\partial\Omega}$ hence we have that P_{M_1} and P_{M_2} differ by a constant in each connected component of $\mathbb{R}^n\backslash\Omega$ hence M_1 and M_2 are Ω-equivalent.

The converse can be shown using a direct computation or arguing directly from Theorem 18. □

Remark 5. *It suffices that the traces the ψ's in the above corollaries be constants.*

We have just shown that the recovery of the vector field $M \in [L^p(\Omega)]^n$ from the knowledge of the potential outside Ω is not unique because of the existence of Ω-equivalent vector-fields. Popular methods for solving inverse potential problems that use regularied Tikhonov methods have a feature that if the regularisation parameter and noise level approach zero in a combined manner the solution that is recovered with the method is of minimum-norm among all the equivalent solutions, see for example [35]. We now study norm-miniming equivalent sources given a vector-field. To achieve this given a vector-field $M \in [L^p(\Omega)]^n$, we want to find a silent source M_S that satisfies the following

$$M_S = \underset{M_0 \in S_p}{\arg\min} \left\| M - M_0 \right\|_{[L^p(\Omega)]^n}, \tag{1.56}$$

where S_p is the set of Ω-silent sources in $[L^p(\Omega)]^n$.

Lemma 6. *S_p is closed in $[L^p(\Omega)]^n$ for $p \in (1, \infty)$.*

Proof. Let $\{M_n\}$ be a sequence in S_p that converges to M. We show that $M \in S_p$. Using notations and definitions from Lemma 3, we have that $\{M_n\}$ converges in $[L^q(\Omega)]^n$ and due to the continuity asserted in Lemma 3 it then follows that

$$\left\| \Psi_{M_n} - \Psi_M \right\|_{L^{nq/(n-q)}(\mathbb{R}^n)} \leq C \left\| M_n - M \right\|_{[L^q(\mathbb{R}^n)]^n}. \tag{1.57}$$

Due to the silence of the sequence $\{M_n\}$ we have that Ψ_M is zero a.e. on $\mathbb{R}^n\backslash\Omega$ hence $M \in S_p$. □

Now that we have established that S_p is closed we will state the characterisation of norm-minimising Ω-equivalent sources given a vector-field M. We begin by noting that from [37, Cor. of Thm 2], $L^p(\Omega)$ is uniformly convex hence strictly convex. Further from [37, Thm 1] we have that $[L^p(\Omega)]^n$ endowed with the norm given in (1.2) is also uniformly convex hence it is strictly convex. Also note that the set of Ω-silent sources, S_p, is convex since any linear combination of elements in S_p is also in S_p. Thus from [38, Part 3, Ch. II, Prop. 5] there is a best approximation projection of M on S_p, that is if we let $Proj$ be the projection from $[L^p(\Omega)]^n$ to S_p then the norm-minimising Ω-equivalent sources given a vector-field M is $M - Proj(M)$. Since the objective function in (1.56) is strictly convex we have that $M_S = Proj(M)$ is unique. From this, it follows that $M - M_S$ is the norm-minimising Ω-equivalent source.

We note that for $p = 2$ the Helmholtz decomposition we introduced is an orthogonal decomposition, hence the sum is an orthogonal sum as can be seen from the following result

Theorem 21 ([21, Lem. 2.5.1])**.** *Let $\Omega \subset \mathbb{R}^n$ be any domain. Then for each $M \in [L^2(\Omega)]^n$ there uniquely exist $\nabla\psi \in G^2(\Omega)$ and $D \in \mathrm{Div}_{2,0}(\Omega)$ such that*

$$M = D + \nabla\psi.$$

Further,

$$\|M\|^2_{[L^2(\Omega)]^n} = \|D\|^2_{[L^2(\Omega)]^n} + \|\nabla\psi\|^2_{[L^2(\Omega)]^n}.$$

When $p = 2$ from the above theorem, given $M \in [L^2(\Omega)]^n$ with Helmholtz decomposition $M = D+\nabla\psi$, the magnetisation of minimum-norm Ω-equivalent to M is the same as the one Ω-equivalent to $\nabla\psi$; it is so because a $\mathrm{Div}_{p,0}(\Omega)$-field is a Ω-silent source. Moreover, by the same reason, the magnetisation of minimum-norm Ω-equivalent to $\nabla\psi$ is the gradient of a Sobolev function and therefore, the norm-minimising vector field Ω-equivalent to M is $\nabla\psi - \nabla u$, where ∇u is the projection of $\nabla\psi$ on $\widetilde{G}^2(\Omega)$, defined in (1.43). The latter is not so easy to characterise for general open sets: in this connection, we shall find it convenient to make the following definition.

Definition 32. *For $p \in (1,\infty)$, we say that the complement of a bounded open set $\Omega \subset \mathbb{R}^n$ is* p-unstretched *if $\partial\Omega$ has zero Lebesgue measure and if, letting $\{O_j\}_{j\in J}$ denote the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$, the following two properties hold:*

(i) O_j is non-thin at $B_{1,p}$-quasi every point of ∂O_j for all j;

(ii) whenever $\overline{O}_j \bigcap \overline{\cup_{i\in I} O_i} \neq \emptyset$ for some index $j \in J$ and some subset of indices $I \subset J$ then $B_{1,p}\big(\overline{O}_j \bigcap \overline{\cup_{i\in I} O_i}\big) > 0$.

Note that when $p > n$, conditions (i) and (ii) above are always met and so $\mathbb{R}^n \setminus \Omega$ is unstretched if and only if $\partial\Omega$ has Lebesgue measure zero. Observe also that $\mathbb{R}^n \setminus \Omega$ is unstretched as soon as $\mathbb{R}^n \setminus \overline{\Omega}$ has finitely many connected components satisfying the segment condition. In particular, this is the case when Ω is Lipschitz.

When $\mathbb{R}^n \setminus \Omega$ is unstretched, we can characterise $[L^2(\Omega)]^n$-norm minimising Ω-equivalent sources as solutions to a boundary-value problem:

Theorem 22. *Let Ω be a bounded open set and $M \in [L^2(\Omega)]^n$. If we write $M = D + \nabla\psi \in [L^2(\Omega)]^n$ for the Helmholtz decomposition, then the unique norm-minimising vector field Ω-equivalent to M is $\nabla\psi - \nabla u$, where ∇u is the projection of $\nabla\psi$ on $\widetilde{G}^2(\Omega)$. Moreover, $\psi_{nm} := \psi - u$ is a $W^{1,2}(\Omega)$-solution to the following boundary value problem:*

$$\begin{aligned} \Delta\psi_{nm} &= 0 \text{ in } \Omega, \\ \nabla\psi_{nm} - \nabla\psi &\in \widetilde{G}^2(\Omega), \\ \int_\Gamma \frac{\partial\psi_{nm}}{\partial\nu} d\mathcal{H}^{n-1} &= 0 \text{ for all compact } C^\infty\text{-hypersurfaces } \Gamma \subset \Omega\ , \end{aligned} \tag{1.58}$$

where $\mathcal{H}^{n-1}$ indicates $(n-1)$-Hausdorff measure. Conversely, if $\mathbb{R}^n \setminus \Omega$ is unstretched, then ψ_{nm} is the unique solution to (1.58)*, up to an additive constant.*

Proof. As mentioned before the theorem, the norm-minimising vector field Ω-equivalent to M is $\nabla\psi - \nabla u$, where ∇u is the projection of $\nabla\psi$ on $\widetilde{G}^2(\Omega)$. Putting $\psi_{nm} := \psi - u$, we get that $\nabla\psi_{nm}$ is orthogonal to $\widetilde{G}^2(\Omega)$ and in particular to $\nabla\varphi$ for all $\varphi \in C_c^\infty(\Omega)$. It implies that ψ_{nm} is harmonic. Moreover, if $\Gamma \subset \Omega$ is a compact C^∞-hypersurface, we can find another smooth compact hypersurface $\Gamma_1 \subset \Omega$ with $\Gamma_1 \cap \Gamma = \emptyset$, close enough to Γ that the open "shell" S between Γ and Γ_1 is contained in Ω (using for instance a tubular neighborhood of Γ, see [13, Ch. 2, Sec. 3]). Assume for definiteness that S lies inside Γ (if it lies outside the argument is similar). We can find a function $g \in C^\infty(\mathbb{R}^n)$ which is equal to 1 on $\operatorname{ext}\Gamma$ and equal to zero on a neighborhood of $\Gamma_1 \cup \operatorname{int}\Gamma_1$. Then, $\nabla g \in G_0^2(\Omega) \subset \widetilde{G}_2$ so that $\int \nabla\psi_{nm} \cdot \nabla g dm = 0$, and by the divergence formula the later is equal to $\int_\Gamma \frac{\partial\psi_{nm}}{\partial\nu} d\mathcal{H}^{n-1}$. Hence, ψ_{nm} satisfies (1.58).

Conversely, assume that (1.58) holds and pick $\nabla g \in G_0^2(\Omega)$. Let further $K \subset \Omega$ be the support of $|\nabla g|$. Since K is compact, there is $h \in C^\infty(\mathbb{R}^n)$ such that $h \geq 0$ and K is the zero set of h; this follows easily from a combination of [12, Chapter VI, Theorem 2] and [36, Theorem I]. Replacing h with h^2 we may assume that $h \geq 0$, and redefining $h(x)$ for $|x|$ large we can arrange things so that $h \to 1$ at infinity. Putting $L := h^{-1}(\varepsilon)$ for ε a sufficiently small regular value of h (almost every positive number is a regular value by Sard's theorem), we find that L is a finite union of smooth compact hypersurfaces $\Gamma_1, \cdots, \Gamma_N$ included in Ω, and we can find $\Gamma_{i_1}, \cdots \Gamma_{i_k}$, each of which lies exterior to the others, with $K \subset \cup_\ell \operatorname{int}\Gamma_{i_\ell}$; moreover, if for some i_ℓ it holds that $\operatorname{int}\Gamma_{i_\ell} \not\subset \Omega$, there are $\Gamma_{j_1}, \cdots, \Gamma_{j_m}$, each of which lies interior to $\operatorname{int}\Gamma_{i_\ell}$, such that $K \subset \cap_k \operatorname{ext}\Gamma_{j_k}$ (recall that $h > \eta > 0$ on $\mathbb{R}^n \setminus \Omega$). By construction g is equal to a constant c_j on Γ_j, and therefore

$$\int_\Omega \nabla\psi_{nm} \cdot \nabla g \, dm = \sum_j c_j \int_{\Gamma_j} \frac{\partial\psi_{nm}}{\partial\nu} d\mathcal{H}^{n-1} = 0,$$

by (1.58). Thus, $\nabla\psi_{nm}$ is orthogonal to $G_0^2(\Omega)$, and by Theorem 19 the latter coincides with $\widetilde{G}^2(\Omega)$ when $\mathbb{R}^n \setminus \Omega$ is unstretched. We now see from the second equation in (1.58) that $\nabla\psi_{nm} = \nabla\psi - \nabla u$ where ∇u is the projection of $\nabla\psi$ on $\widetilde{G}^2(\Omega)$, as desired. □

When Ω is Lipschitz, Theorem 22 yields a fairly explicit characterization of norm-minimising Ω-equivalent sources in $[L^2(\Omega)]^n$. By Lemma 4, it is enough to consider the case where Ω is connected:

Theorem 23. *Let Ω be a bounded Lipschitz domain and $\Gamma_1, \cdots, \Gamma_l$ the connected components of $\partial\Omega$. Let further $M \in [L^2(\Omega)]^n$ and write the Helmholtz decomposition as $M = D + \nabla\psi \in [L^2(\Omega)]^n$. Then, the norm-minimising vector field Ω-equivalent to M is $\nabla\psi_{nm}$, where $\psi_{nm} = u + \sum_{j=1}^l c_j \omega(\Gamma_j)$ with u the $W^{1,2}(\Omega)$ solution to the Dirichlet problem:*

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= \psi \text{ on } \partial\Omega, \end{aligned} \tag{1.59}$$

and $\omega(\Gamma_j)$ the harmonic measure of Γ_j, while the vector $(c_1, \cdots, c_l)^t \in \mathbb{R}^l$ is determined, up to a multiple of $(1, \cdots, 1)^t$ (which will only alter ψ_{nm} by a constant and therefore respect $\nabla\psi_{nm}$) by the property that $\int_{\Gamma_j} \frac{\partial\psi_{nm}}{\partial\nu} d\mathcal{H}^{n-1} = 0$ for $1 \leq i \leq l$.

Proof. As Lipschitz open sets have unstretched complement, Theorem 22 and Corollary 5 imply that the norm-minimising vector field Ω-equivalent to M is $\nabla\psi_{nm}$ with $\psi_{nm} = u + \sum_{j=1}^{l} c_j\omega(\Gamma_j)$, where $u \in W^{1,2}(\Omega)$ solves (1.59) and $\omega(\Gamma_j)$ is the harmonic measure of Γ_j, while the c_j are real numbers. Using the notation of Lemma 1 and appealing to the Whitney and Sard theorems as in the proof of Theorem 22 to construct non-negative functions $h_j \in C^\infty(\mathbb{R}^n)$ whose zero set is $\overline{\operatorname{int} O_1}$ for $j = 1$ and $\overline{\operatorname{ext} O_j}$ for $j \neq 1$, we construct smooth compact hypersurfaces $\Sigma_j \subset \Omega$ such that the shell between Γ_j and Σ_j is included in Ω. Since $\int_{\Sigma_j} \frac{\partial\psi_{nm}}{\partial\nu} d\mathcal{H}^{n-1} = 0$ by (1.59), we get from the Green formula that $\int_{\Gamma_j} \frac{\partial\psi_{nm}}{\partial\nu} d\mathcal{H}^{n-1} = 0$ for $1 \leq j \leq l$. Hence, it holds for $j \in \{1, \cdots, l\}$ that

$$-\int_{\Gamma_j} \frac{\partial u}{\partial\nu} d\mathcal{H}^{n-1} = \sum_{\ell=1}^{L} c_\ell \int_{\Gamma_j} \frac{\partial\omega(\Gamma_\ell)}{\partial\nu} d\mathcal{H}^{n-1}. \tag{1.60}$$

To see that (1.60) determines $(c_1, \cdots, c_l)$ up to a multiple of $(1, \cdots, 1)$, observe that if we put $v = \sum_\ell a_\ell\omega(\Gamma_\ell)$ then

$$\int_\Omega \nabla v \cdot \nabla v \, dm = \sum_{j=1}^{l} \int_{\Gamma_j} v \frac{\partial v}{\partial\nu} d\mathcal{H}^{n-1} = \sum_{j,\ell=1}^{l} a_j a_\ell \int_{\Gamma_j} \frac{\partial\omega(\Gamma_\ell)}{\partial\nu} d\mathcal{H}^{n-1},$$

so that the quadratic form on $\mathbb{R}^l$ whose matrix has (j,ℓ)-entry $\int_{\Gamma_j} \frac{\partial\omega(\Gamma_\ell)}{\partial\nu} d\mathcal{H}^{n-1}$ is non-negative with kernel the multiples of $(1, \cdots, 1)^t$ (corresponding to a constant v). □

When $p \neq 2$, we shall not be able to characterize Ω-equivalent sources of minimum L^p-norm in such an explicit manner. However, when $\mathbb{R}^n \setminus \Omega$ is Lipschitz at least, the problem can be approached *via* duality.

For this, recall that S_p is a closed subspace of $[L^p(\Omega)]^n$ by Lemma 6, and let $S_p^\perp \subset [L^q(\Omega)]^n$, $\frac{1}{p} + \frac{1}{q} = 1$, be the space of annihilators of S_p; that is: the space of all $\Phi \in [L^q(\Omega)]^n$ such that $\langle M_0, \Phi\rangle = 0$ for all $M_0 \in S_p$.

Lemma 7. *Let Ω be a bounded Lipschitz open set, and $p \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, $S_p^\perp$ consists of those vector-fields $\nabla\phi \in [L^q(\Omega)]^n$ such that $\phi \in W^{1,q}(\Omega)$ is harmonic in Ω and $\langle \nabla\phi \cdot \nu, \chi\rangle = 0$ for all $\chi \in \mathbb{R}_{\partial\Omega}$; in other words, $\int_{\Gamma_i} \frac{\partial\phi}{\partial\nu} d\mathcal{H}^{n-1} = 0$ for each connected component Γ_i of $\partial\Omega$.*

Proof. By Lemma 4, we may assume that Ω is connected. Let $\Phi \in [L^q(\Omega)]^n$ be such that $\langle M_0, \Phi\rangle = 0$ for all $M_0 \in S_p$. By Theorem 18, it means that $\langle \Phi, \nabla\psi_0 + D_0\rangle = 0$ for $\psi_0 \in W^{1,p}_{\mathbb{R}_{\partial\Omega}}(\Omega)$ and $D_0 \in \mathrm{Div}_{p,0}$. In particular Φ is orthogonal to $\mathrm{Div}_{p,0}$ whence it is a gradient, say $\Phi = \nabla\phi$ with $\phi \in W^{1,q}(\Omega)$. Moreover, as $\nabla\phi = \Phi$ is orthogonal to all gradients of functions in $C_c^\infty(\Omega) \subset W^{1,p}_{\mathbb{R}_{\partial\Omega}}(\Omega)$, it follows that $\nabla\phi$ is divergence-free as a distribution; *i.e.*, ϕ is harmonic in Ω. Using the notation of Lemma 1 so that the O_j are the connected of components $\mathbb{R}^n \setminus \overline{\Omega}$ and the Γ_j are the connected components of $\partial\Omega$ for $1 \leq j \leq l$, we can argue as in the proof of Theorem 23 and construct for each j a closed smooth hypersurface $\Sigma_j \subset \Omega$ such that the shell between Γ_j and Σ_j is included in Ω. Then, we can find $h_j \in C^\infty(\mathbb{R}^n)$ such that $h_j = 1$ on $\overline{O}_j$ and zero outside a neighborhood of Γ_j containing Σ_j. Then $h_j \in W^{1,p}_{\mathbb{R}_{\partial\Omega}}(\Omega)$, and

we get from the divergence formula applied to $h_j \nabla\phi$ on the shell between Σ_j and Γ_j that $\int_{\Gamma_j} \frac{\partial\phi}{\partial\nu} d\mathcal{H}^{n-1} = 0$ for $1 \leq j \leq l$. This proves that the elements of $S_p^\perp$ are of the announced type, and the previous arguments are easily reverted to yield the converse. □

Remark 6. *Whenever Ω is such that $G_0^p(\Omega)$, coincides with $\widetilde{G}^p(\Omega)$ and $\mathrm{Div}_{p,0}(\Omega)$ coincides with divergence-free fields in $[L^p(\mathbb{R}^n)]^n$ that vanish off Ω, the reasoning in Lemma 7 is easily adapted to yield that $S_p^\perp$ consists of those $\nabla\phi \in [L^q(\Omega)]^n$ such that ϕ is a harmonic function in Ω and $\int_\Gamma \frac{\partial\phi}{\partial\nu} d\mathcal{H}^{n-1} = 0$ for all closed C^∞-hypersurfaces $\Gamma \subset \Omega$. In particular, if we knew that $\mathrm{Div}_{p,0}(\Omega)$ coincides with divergence-free fields in $[L^p(\mathbb{R}^n)]^n$ vanishing off Ω when $\mathbb{R}^n \setminus \Omega$ is unstretched, then the above characterisation of $S_p^\perp$ would hold under very general assumptions. Such is the case for domains Ω as in Corollary 2 with $\mathbb{R}^n \setminus \Omega$ unstretched .*

We note that S_p is a closed subspace of $[L^p(\Omega)]^n$ and let $S_p^\perp$ be the set of annihilators of S_p, that is, the set of all $\Phi \in [L^q(\Omega)]^n$ such that $\langle M_0, \Phi\rangle = 0$ for all $M_0 \in S_p$.

Theorem 24 ([39, Thm 7.2])**.** *Let X be a Banach space and S be a closed subspace of X. Then the space $(X/S)^*$ is isometrically isomorphic to $S^\perp$. Furthermore, for each fixed $x \in X$,*

$$\max_{\psi\in S^\perp,\, \|\psi\|\leq 1} |\psi(x)| = \inf_{y\in S} \|x + y\| .$$

From Theorem 24 we have that

$$\begin{aligned} \inf_{M_0\in S_p} \|M - M_0\|_{[L^p(\Omega)]^n} &= \max_{\|\Phi\|_{[L^q(\Omega)]^n}\leq 1, \Phi\in S_p^\perp} |\langle M, \Phi\rangle| \\ &= \max_{\|\nabla\phi\|_{[L^q(\Omega)]^n}=1, \nabla\phi\in(G_0^p(\Omega))^\perp} |\langle M, \nabla\phi\rangle|. \end{aligned} \tag{1.61}$$

Remark 7. *Note that in* (1.61) *in the last equality the constraint on the norm of the gradients of functions harmonic in Ω is saturated. This is easily shown as follows, suppose the ϕ is such that $\|\nabla\phi\|_{[L^q(\Omega)]^n} < 1$ then for $\varepsilon > 0$ small enough $\|\nabla(1+\varepsilon)\phi\|_{[L^q(\Omega)]^n} \leq 1$ and $|\langle M, \nabla\phi\rangle| < (1+\varepsilon)|\langle M, \nabla\phi\rangle|$, contradicting that ϕ is a maximiser.*

The existence and uniqueness of a harmonic function $\phi_S \in W^{1,q}(\Omega)$ with $\nabla\phi_S \in (G_0^p(\Omega))^\perp$ that maximised (1.61) is inherited from the existence and uniqueness of M_S. Further, ϕ_S satisfies the following

$$\langle M - M_S, \nabla\phi_S\rangle = \|M - M_S\|_{[L^p(\Omega)]^n} , \tag{1.62}$$

and a simple argument based on the equality in Hölder's inequality shows that

$$\nabla\phi_S = \frac{|M - M_S|^{p-2}(M - M_S)}{\|M - M_S\|^{p-1}_{[L^p(\Omega)]^n}}. \tag{1.63}$$

It thus follows that given $M \in [L^p(\Omega)]^n$ then the norm-minimising Ω-equivalent source is given by

$$M - M_S = \langle M, \nabla\phi_S\rangle |\nabla\phi_S|^{q-2}\nabla\phi_S. \tag{1.64}$$

Above we have proven the following theorem which is a generalisation of Theorem 23,

Theorem 25. *Let Ω be a bounded Lipschitz open set, given a vector-field $M \in [L^p(\Omega)]^n$, $1 < p < \infty$, the norm-minimising Ω-equivalent vector-field is given by*

$$\langle M, \nabla\phi_S\rangle|\nabla\phi_S|^{q-2}\nabla\phi_S,$$

where ϕ_S is, up to an additive constant, the unique harmonic function on Ω with $\|\nabla\phi_S\|_{[L^q(\Omega)]^n} = 1$ and $\nabla\phi_S \in (G_0^p(\Omega))^\perp$ that satisfies

$$\langle M, \nabla\phi_S\rangle = \max_{\|\nabla\phi\|_{[L^q(\Omega)]^n}=1, \nabla\phi_S\in(G_0^p(\Omega))^\perp} \langle M, \nabla\phi\rangle. \tag{1.65}$$

From Remark 6 we have the following corollary to Theorem 25:

Corollary 6. *Let Ω be as in Theorem 19 such that for all $u \in [L^1_{loc}(\Omega)]^n \cap [L^p(\Omega)]^n$ with $\|\operatorname{div} u\|_{L^p(\Omega)} < \infty$ the equality*

$$\langle \varphi, u\cdot\nu\rangle = \int_\Omega \nabla\varphi(y)\cdot u(y)\,dy + \int_\Omega \varphi(y)\operatorname{div} u(y)\,dy, \quad \varphi \in W^{1,q}(\Omega),$$

holds then given a vector-field $M \in [L^p(\Omega)]^n$, $1 < p < \infty$, the norm-minimising Ω-equivalent vector-field is given by

$$\langle M, \nabla\phi_S\rangle|\nabla\phi_S|^{q-2}\nabla\phi_S,$$

where ϕ_S is, up to an additive constant, the unique harmonic function on Ω with $\|\nabla\phi_S\|_{[L^q(\Omega)]^n} = 1$ and $\nabla\phi_S \in (G_0^p(\Omega))^\perp$ that satisfies

$$\langle M, \nabla\phi_S\rangle = \max_{\|\nabla\phi\|_{[L^q(\Omega)]^n}=1, \nabla\phi_S\in(G_0^p(\Omega))^\perp} \langle M, \nabla\phi\rangle. \tag{1.66}$$

We are now in a position to prove Theorem 15 and its corollary.

Proof of Theorem 15. It follows from (1.56) and the theorem above that given any $M \in [L^p(\Omega)]^n$, $1 < p < \infty$, the norm-minimising Ω-equivalent source to M can be written uniquely in the form

$$\langle M, \nabla\phi_S\rangle|\nabla\phi_S|^{q-2}\nabla\phi_S = M - \nabla\phi_0 - D_0,$$

where ϕ_S is harmonic with $\nabla\phi_s \in [L^q(\Omega)]^n$ while $\phi_0 \in W^{1,p}(\Omega)$ with $\nabla\phi_0 \in G_0^p(\Omega)$ and $D_0 \in \operatorname{Div}_{p,0}(\Omega)$. Letting $h = \langle M, \nabla\phi_S\rangle^{\frac{1}{q-1}}\phi_S$ together with $\psi = \phi_0$ and $D = D_0$, we get the decomposition (1.27). □

For the remainder of the section we take Ω to be a Lipschitz domain. Note that we can express the ϕ in (1.65) as a single layer potential, that is,

$$\phi(x) = \mathcal{S}f(x),$$

for some $f \in \hat{W}^{-\frac{1}{q},q}(\partial\Omega)$ where

$$\hat{W}^{-\frac{1}{q},q}(\partial\Omega) = \{f \in W^{-\frac{1}{q},q}(\partial\Omega) : \langle f, \chi\rangle = 0 \text{ for all } \chi \in \mathbb{R}_{\partial\Omega}\}$$

hence

$$\nabla\phi\cdot\nu=-\Big(\frac{1}{2}Id-K^*\Big)f.$$

Thus using Fubini's theorem we can rewrite the RHS of the last equality of (1.65) as

$$\begin{aligned}\langle M,\nabla\mathcal{S}f\rangle&=\int_\Omega M(x)\cdot\int_{\partial\Omega}\frac{(x-y)}{|x-y|^n}f(y)\,d\sigma(y)\,dx\\&=\int_{\partial\Omega}\int_\Omega M(x)\cdot\frac{(x-y)}{|x-y|^n}\,dx\,f(y)\,d\sigma(y)\\&=\int_{\partial\Omega}P_M(y)f(y)\,d\sigma(y).\end{aligned}\tag{1.67}$$

In (1.67) note that $P_M(y)$ is the potential associated with M for $y\in\partial\Omega$ when approaching the boundary non-tangentially from outside Ω, hence we can express (1.67) as

$$\max_{f\in\hat{W}^{-\frac{1}{q},q}(\partial\Omega),\|\nabla\mathcal{S}f\|_{[L^q(\Omega)]^n}=1}\int_{\partial\Omega}P_M(y)f(y)\,d\sigma(y).\tag{1.68}$$

Note that in (1.68) the potential $P_M\in W^{\frac{1}{q},p}(\partial\Omega)$ is acting up to additive constants on each connected component of the boundary; *i.e.*, $P_M\in W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}$. Note that the space $\hat{W}^{-\frac{1}{q},q}(\partial\Omega)$ is the dual of $W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}$ using the arguments of Sec. 1.2. Due to [25, Thm 3.1], we can renorm $\hat{W}^{-\frac{1}{q},q}(\partial\Omega)$ with the norm

$$\|f\|_{\hat{W}^{-\frac{1}{q},q}(\partial\Omega)}=\|\nabla\mathcal{S}f\|_{[L^q(\Omega)]^n}.$$

We can then renorm $W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}$ with the norm

$$\begin{aligned}\|u\|_{W^{\frac{1}{q},p}(\partial\Omega)}&:=\sup_{f\in\hat{W}^{-\frac{1}{q},q}(\partial\Omega),\|\nabla\mathcal{S}f\|_{[L^q(\Omega)]^n}=1}\int_{\partial\Omega}uf\,d\sigma\\&=\sup_{f\in\hat{W}^{-\frac{1}{q},q}(\partial\Omega),\|\nabla\mathcal{S}f\|_{[L^q(\Omega)]^n}=1}\langle u,f\rangle.\end{aligned}$$

Under the new norm we can relate (1.68) with the norm of P_M on $\partial\Omega$. This leads to the notion of duality mappings.

Definition 33. *A duality mapping with gauge function ϕ, $J^X_\phi:X\to 2^{X^*}$, from X to subsets of X^*, with $\phi:\mathbb{R}_+\to\mathbb{R}_+$ a continuous and strictly increasing function such that $\phi(0)=0$ and $\lim_{t\to\infty}\phi(t)=\infty$ is the set valued map*

$$J^X_{\phi(t)}x:=\{x^*\in X^*:\langle x,x^*\rangle=\|x\|_X\|x^*\|_{X^*},\|x^*\|_{X^*}=\phi(\|x\|_X)\}.$$

Thus given a gauge function $\phi(t)=t$ we define the

$$J_t^{W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}}:W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}\to 2^{\hat{W}^{-\frac{1}{q},q}(\partial\Omega)}$$

by

$$J_t^{W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}} u := \{ f \in \hat{W}^{-\frac{1}{q},q}(\partial\Omega) : \langle u, f \rangle = \|u\|_{W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}} \|f\|_{\hat{W}^{-\frac{1}{q},q}(\partial\Omega)}, \\ \|f\|_{\hat{W}^{-\frac{1}{q},q}(\partial\Omega)} = \|u\|_{W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}} \}.$$

When the gauge function $\phi(t) = t$ the duality mapping J_ϕ^X is called the normalised duality mapping. We refer the interested reader to [40, Ch. 3] and [41, Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications] for an exploratory study of duality mappings. Note that inasmuch as the duality mapping introduced above is set-valued, due to the reflexivity of the spaces we consider, the duality mapping in this case is a singleton. This leads to the conclusion that the maximiser of (1.68) is a constant multiple of the duality mapping of $P_M \in W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}$ under the duality mapping above. It should be noted though that describing the duality mapping above of $u \in W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}_{\partial\Omega}$ is not a trivial exercise which limits the practical application of this method to obtaining the maximiser of (1.68). Cases where this is practical is for example in (1.63) in which the RHS is the $[L^p(\Omega)]^n$ normalised duality mapping of $M - M_S$.

However, an interesting property of duality mappings is that the duality mapping $J_\phi^{X^*} : X^* \to X$ is the inverse of the duality mapping J_ϕ^X. Hence we can look at the problem in (1.68) as looking for $f \in \hat{W}^{-\frac{1}{q},q}(\partial\Omega)$ whose duality mapping under $J_t^{\hat{W}^{-\frac{1}{q},q}(\partial\Omega)}$ is P_M. To begin, it is fairly direct to observe that for a given $M \in [L^p(\Omega)]^n$ a necessary and sufficient condition for

$$f \in \hat{W}^{-\frac{1}{q},q}(\partial\Omega) \text{ with } \|\nabla \mathcal{S} f\|_{[L^q(\Omega)]^n} = 1$$

to be the maximiser of (1.68) is that

$$\frac{\langle M, \nabla \mathcal{S} f \rangle}{\omega_n} \int_\Omega |\nabla \mathcal{S} f|^{q-2} \nabla \mathcal{S} f(y) \cdot \frac{(x-y)}{|x-y|^n} dy = P_M(x), \qquad x \in \partial\Omega, \tag{1.69}$$

and in general it is true for $x \in \mathbb{R}^n \backslash \Omega$. Therefore, the LHS of (1.69) is up a multiplicative constant $J_t^{\hat{W}^{-\frac{1}{q},q}(\partial\Omega)}$. In this case Galerkin methods can be applied to solve the problem by noting that using (1.69) and a frame for $W^{-\frac{1}{q},q}(\partial\Omega)$, the maximiser f^* for (1.68) can be computed up to additive constants. A frame for the space of distributions $W^{-\frac{1}{q},q}(\partial\Omega)$ can be generated using methods such as the one introduced in [42] in which one perturbs a frame of a separable Hilbert space which has a as a subspace the space of Schwartz functions.

1.7 Approximation problem when $n = 3$

In this section we restrict to domains, Ω, that are simply connected and at least $C^{1,1}$ or Lipschitz polyhedron domains. Given a vector-field $M \in [W^{1,p}(\Omega)]^3$ we study how one

can approximate M by a silent gradient and a silent divergence-free vector-field, that is, given M we wish to solve the following problems

$$\begin{aligned} &\inf_{\nabla\phi_0,\, \phi_0 \in W_0^{1,p}(\Omega)} \|M - \nabla\phi_0\|_{[L^p(\Omega)]^3}\,, \\ &\inf_{D \in \mathrm{Div}_{p,0}(\Omega)} \|M - D\|_{[L^p(\Omega)]^3}\,, \end{aligned} \tag{1.70}$$

respectively. We will set up these problems in such a way that Galerkin type methods can be used to solve each problem hence making numerical solution feasible using well-known techniques.

1.7.1 Approximating by silent gradient

It can be shown that given an M such as above the best approximate silent gradient, say $\nabla\phi_0$ with $\phi_0 \in W_0^{1,p}(\Omega)$, has the following property

$$M - \nabla\phi_0 = |\mathfrak{D}|^{q-2}\mathfrak{D}, \tag{1.71}$$

for some $\mathfrak{D} \in \mathrm{Div}_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. This follows from a duality argument Theorem 24. If we then rewrite $\mathfrak{D} = \nabla \times h$ for some $h \in [W^{1,q}(\Omega)]^3 \cap \mathrm{Div}_{q,0}$ and take curl in (1.71) and impose a tangential boundary condition we have the following q-curl-curl problem

$$\begin{aligned} \nabla \times M &= \nabla \times \Big(|\nabla \times h|^{q-2}\nabla \times h\Big) \text{ in } \Omega, \\ M \times \nu &= \Big(|\nabla \times h|^{q-2}\nabla \times h\Big) \times \nu \text{ on } \partial\Omega. \end{aligned} \tag{1.72}$$

It follows from [43, Proposition 2.1] that given $M \in [W^{1,p}(\Omega)]^3$ with $1 < p \leq 6$ (1.72) can be solved weakly , that is, there exists uniquely $h \in [W^{1,q}(\Omega)]^3 \cap \mathrm{Div}_{q,0}$ such that

$$\int_\Omega |\nabla \times h|^{q-2}\nabla \times h \cdot \nabla \times \varphi \, d\Omega = \int_\Omega \nabla \times M \cdot \varphi \, d\Omega + \int_{\partial\Omega} M \times \nu \cdot \varphi \, d\sigma, \tag{1.73}$$

for all $\varphi \in [W^{1,q}(\Omega)]^3$. The proof of the existence of the solution of (1.73) in [43] used the the well-known property of monotone operators [44, Thm 2.1]. It was achieved by considering the function $a : \mathbb{R}^3 \to \mathbb{R}^3$,

$$a(u) = |u|^{q-2}u,$$

that has the following properties

(i) $a(u) \cdot u \geq a_*|u|^q$

(ii) $|a(u)| \leq a^*|u|^{q-1}$

(iii) $(a(u) - a(v)) \cdot (u - v) > 0$ for all $u \neq v$.

Note we can rewrite (1.73) as

$$\begin{aligned} \int_\Omega a(\nabla \times h) \cdot \nabla \times \varphi \, d\Omega = \int_\Omega \nabla \times M \cdot \varphi \, d\Omega + \int_{\partial\Omega} M \times \nu \cdot \varphi \, d\sigma \\ \langle \mathcal{A}h, \varphi \rangle = \langle L, \varphi \rangle \qquad \text{for all } \varphi \in W^{1,q}(\Omega). \end{aligned} \tag{1.74}$$

For what follows, we need some definitions.

Definition 34. *An operator $\mathcal{A} : X \to X^*$ is hemi-continuous if for all $h, g, f \in X$ and $\lambda \in \mathbb{R}$ we that*

$$\langle \mathcal{A}(h + \lambda g), f \rangle$$

is continuous in λ.

The operator $\mathcal{A}$ is said to be monotone if for all $h, g \in X$

$$\langle \mathcal{A}h - \mathcal{A}g, h - g \rangle \geq 0.$$

Further $\mathcal{A}$ is said to be coercive if

$$\frac{\langle \mathcal{A}h, h \rangle}{\|h\|} \to \infty \quad as \quad \|h\| \to \infty.$$

It follows that $\mathcal{A} : [W^{1,q}(\Omega)]^3 \to [L^p(\Omega)]^3 \subset [(W^{1,q}(\Omega))^*]^3$ as defined in (1.74) is hemi-continuous, monotone and coercive as soon as the semi-norm $\|\nabla \times h\|_{[L^q(\Omega)]^3}$ is equivalent to the $\|h\|_{[W^{1,q}(\Omega)]^3}$ and from [43, Thm 2.1] this is true for $q > \frac{6}{5}$. Hence in this case we can solve 1.71 for $1 < p \leq 6$.

We note that the result in [43] was given for simply connected C^2 domains however it remains true for $C^{1,1}$ and bounded convex Lipschitz polyhedron domains. We arrived at this conclusion as follows, if we consider the following Banach spaces

$$\begin{aligned} W^p(\nabla\times, \Omega) &= \{h \in [L^p(\Omega)]^3 : \nabla \times h \in [L^p(\Omega)]^3\}, \\ W^p(\mathrm{div}, \Omega) &= \{h \in [L^p(\Omega)]^3 : \mathrm{div}\, h \in L^p(\Omega)\}, \\ X^p(\Omega) &= W^p(\nabla\times, \Omega) \cap W^p(\mathrm{div}, \Omega), \\ X^p_T(\Omega) &= \{h \in X^p(\Omega) : h \cdot \nu = 0\}, \\ X^p_T(\Omega, \mathrm{div}, 0) &= \{h \in X^p_T(\Omega) : \mathrm{div}\, h = 0\}, \end{aligned}$$

we have that if Ω is $C^{1,1}$ and simply connected then on $X^p_T(\Omega)$ the semi-norm

$$\|\nabla \times h\|_{[L^p(\Omega)]^3},$$

is equivalent to the graph norm

$$\|h\|_{[L^p(\Omega)]^3} + \|\nabla \times h\|_{[L^p(\Omega)]^3} + \|\mathrm{div}\, h\|_{L^p(\Omega)},$$

and further for all $h \in [W^{1,q}(\Omega)]^3$ with $h \cdot \nu = 0$ then there is $C > 0$ such that

$$\|h\|_{[W^{1,q}(\Omega)]^3} \leq C\Big(\|\nabla \times h\|_{[L^p(\Omega)]^3} + \|\mathrm{div}\, h\|_{L^p(\Omega)} \Big)$$

hence as well as on $X^p_T(\Omega)$, see [45, Cor. 3.4]. Thus, if we take $h \in X^p_T(\Omega, \mathrm{div}, 0)$ which has $\mathrm{div}\, h = 0$ then Proposition 2.1 of [43] can be extended to show that (1.72) can be solved weakly given any $M \in [W^{1,p}(\Omega)]^3$ for $1 < p < \infty$ as in this case $\mathcal{A}$ can be shown to be hemi-continuous, monotone and coercive for $1 < q < \infty$.

1.7.2 Approximating by silent divergence-free vector-field

In this case the best approximate silent divergence-free vector-field, $D \in \mathrm{Div}_{p,0}(\Omega)$ is such that

$$M - D = |\nabla\varphi|^{q-2}\nabla\varphi, \tag{1.75}$$

for some $\varphi \in W^{1,q}(\Omega)/\langle 1\rangle$ again by Theorem 24 and $\frac{1}{p} + \frac{1}{q} = 1$. We then have the following Neumann problem for the q-Laplacian by taking divergence and imposing a Neumann boundary condition

$$\begin{aligned} \operatorname{div} M &= \operatorname{div}\left(|\nabla\varphi|^{q-2}\nabla\varphi\right) \text{ in } \Omega, \\ M\cdot\nu &= |\nabla\varphi|^{q-2}\nabla\varphi\cdot\nu \quad \text{ on } \partial\Omega, \end{aligned} \tag{1.76}$$

for which a weak solution φ exists uniquely, that is, there exists uniquely $\varphi \in W^{1,q}(\Omega)/\langle 1\rangle$ such that

$$\begin{aligned} \int_\Omega |\nabla\varphi|^{q-2}\nabla\varphi\cdot\nabla\psi\, d\Omega = \int_\Omega \operatorname{div} M\psi\, d\Omega + \int_{\partial\Omega} M\cdot\nu\psi\, d\sigma, \\ \langle \mathcal{A}\varphi, \psi\rangle = \langle L, \psi\rangle. \end{aligned} \tag{1.77}$$

for all $\psi \in W^{1,q}(\Omega)/\langle 1\rangle$. To see this we have from (1.77) that

$$\mathcal{A} : [W^{1,q}(\Omega)/\langle 1\rangle]^3 \to [L^p(\Omega)]^3 \subset [(W^{1,q}(\Omega)/\langle 1\rangle)^*]^3$$

is hemi-continuous, monotone and coercive. To show the coercivity we appeal to the Poincaré-Wirtinger inequality and by taking each $\varphi \in W^{1,q}(\Omega)/\langle 1\rangle$ as $\varphi - \overline{\varphi}$ with $\overline{\varphi}$ as defined in (1.6).

1.7.3 Iterative procedure to approximate equivalent norm-minimising source

Given M we iteratively approximate $M - M_S$ by switching between approximating the iterates with a silent gradient and a silent divergence-free vector-field. As a starting point one chooses to either begin with approximating with a silent gradient or a silent divergence-free vector-field. We will start with approximating with a silent gradient.

Algorithm 1: Finding norm-minimising equivalent source

Result: $M - M_S$.

$i = 0$;
$M^{(0)} = M$;
while $\nabla\times(|M^{(i)}|^{p-2}M^{(i)}) \neq 0$ *or* $\nabla\cdot(|M^{(i)}|^{p-2}M^{(i)}) \neq 0$ *in* Ω **do**
  $M^{(i+1)} = |\mathfrak{D}^{(i)}|^{q-2}\mathfrak{D}^{(i)}$ by solving (1.72) for h with $M = M^{(i)}$ and setting $\mathfrak{D} = \nabla\times h$;
  $M^{(i+2)} = |\nabla\varphi^{(i+1)}|^{q-2}\nabla\varphi^{(i+1)}$by solving (1.76) for φ with $M = M^{(i+1)}$;
  $i = i + 2$;
end
$M^{(i)} = M - M_S$.

The previous algorithm is convergent with $M \in [W^{1,p}(\Omega)]^3$.

Theorem 26. *Given $M \in [W^{1,p}(\Omega)]^3$, for $1 < p < \infty$, the iterative scheme above converges to the norm-minimising Ω-equivalent source in $[L^p(\Omega)]^3$.*

Proof. Note that at each iterative step we solve one of the problems in (1.70) therefore at each step the norm of the iterates decreases. The sequence of iterates is bounded by M and since $[L^p(\Omega)]^3$, for $1 < p < \infty$, are reflexive Banach spaces we have that the sequence of iterates has a weakly convergent subsequence, see for example [11, Thm 3.18]. Further, since $[L^p(\Omega)]^3$, for $1 < p < \infty$, are uniformly convex Banach spaces we have from the boundedness and monotonicity of the sequence of norms of the weakly convergent subsequence that the subsequence converges strongly, see for example [11, Proposition 3.32]. This subsequence of iterates converges strongly to the norm-minimising Ω-equivalent source otherwise we have that the sequence of the norms of the iterates does not converge to the norm of the norm-minimising Ω-equivalent source. □

The iterative procedure above can be implemented numerically by using frames of the spaces $W^{1,p}(\mathbb{R}^3)$ for $1 < p < \infty$, see for example [46] for their construction. Since we consider Ω that is class $C^{1,1}$ in this section we have from [23, Thm 2.75] that we can extend functions in $W^{1,p}(\Omega)$ to functions in $W^{1,p}(\mathbb{R}^3)$. Due to the properties of the functions and vector-fields we seek in the iterative procedure these are extensions by zero. By expressing functions in $W^{1,p}(\Omega)$ as finite linear combinations of frames we are able to build systems of non-linear equations, via Galerkin methods, that can be solved to approximate solutions to (1.73) and (1.77) hence approximating the iterative procedure. This Galerkin approximation indeed provides approximate solutions to approximate problems of (1.73) and (1.77) that have to be solved at each step of the iterative procedure, see [47, Ch. II, Thm 2.1], [44, Ch. 2, Thm 2.1, Thm 2.2].

1.8 Conclusion

We have managed to provide a characterisation of silent $[L^p(\Omega)]^n$ magnetisations using the Helmholtz decomposition on $\mathbb{R}^n$. Further, we have done this with minimal assumptions on the set Ω. Using this characterisation of silent sources we arrived at a characterisation of norm-minimising equivalent magnetisation to any given magnetisation. This allowed to have a decomposition which can be viewed as an extension of the Helmholtz decomposition on more general domain. On Lipschitz domains we have managed to propose a method for computing the norm-minimising equivalent magnetisation to any given one that makes contact with the duality mapping on the trace Sobolev dual space, $\hat{W}^{-\frac{1}{q},q}(\partial\Omega)$, in a sense we have managed to characterise the duality mapping via Newton potentials, see (1.69). The numerical application of this method is possible but may not be easily achieved. For $n = 3$ and smooth domains we provided an iterative procedure that reformulates the problem to problems that are well studied and for which numerical schemes already exist.

What we presented here can be applied to magnetisations that are assumed to be distribution vector-fields with the distributions in the Sobolev dual spaces. These spaces of distributions include spaces of measures. Preliminary thoughts suggest that

using a decomposition in these spaces that are based on the Stokes problem, see [48], leads to a characterisation of silent distributions that is similar to the one presented in this work. In the Stokes decomposition one take advantage of the fact that the Stokes volume potential is equivalent to the Newton potential, see [22, Ch. IV]. Future work can be devoted to answering this question fully.

Another extension of this work would be to give a characterisation of silent $[L^p(\Omega)]^n$ magnetisations while allowing the connected components of $\mathbb{R}^n\backslash\Omega$ that have intersections to be thin at those intersections. This would require to make changes to the definition of $G_0^p(\Omega)$ such that the degenerate cases can be accounted for in the set.

1.9 Appendix

1.9.1 More on Sobolev functions

Let $\Omega \subset \mathbb{R}^n$ be open. For $f \in W^{1,p}(\Omega)$ and $g \in W_0^{1,q}(\Omega)$, it holds for $1 \leq i \leq n$ that $\int_\Omega \partial_{x_i} f g = -\int_\Omega \partial_{x_i} g f$, by absolute continuity on a.e. line of Sobolev functions [16, Thm. 2.1.4]. Thus, $\|\nabla f\|_{[W^{-1,p}(\Omega)]^n} \leq n^{\frac{1}{p}} \|f\|_{L^p(\Omega)}$, and since $W^{1,p}(\Omega)$ is dense in $L^p(\Omega)$ it follows that $f \mapsto \nabla f$ extends to a continuous map from $L^p(\Omega)$ into $[W^{-1,p}(\Omega)]^n$.

When Ω is a Lipschitz open set and $(U_j \cap \partial\Omega, \Phi_j)$ an atlas for $\partial\Omega$ as in Sec. 1.2.3, the Sobolev space $W^{1,p}(\partial\Omega)$ comprises those $f : \partial\Omega \to \mathbb{R}$ such that $f \circ \Phi_j^{-1} \in W^{1,p}(V_j)$ for all j, with $V_j = \Phi_j(U_j \cap \partial\Omega)$. The definition does not depend on the atlas, since Lipschitz changes of variables preserve Sobolev functions [16, Thm 2.2.2]. The tangential gradient $\nabla_T f \in [L^p(\partial\Omega)]^n$ is given on $U_j \cap \partial\Omega$ by

$$\nabla_T f \circ \Phi_j^{-1} = D\Phi_j^{-1}\big((D\Phi_j^{-1})^t D\Phi_j^{-1}\big)^{-1} \nabla(f \circ \Phi_j^{-1}); \tag{1.78}$$

note that the definitions agree a.e. on $U_{j_1} \cap U_{j_2}$ and that $\nabla_T f(x) \in T_x\partial\Omega$ for σ-a.e. $x \in \partial\Omega$. For a.e. $x \in U_j \cap \partial\Omega$ and each $X \in T_x\partial\Omega$, we have $\nabla_T f(x) \cdot X = \mathrm{d}f(x)(X)$ where the differential $\mathrm{d}f(x)$ of f at x is the 1-form given by $\mathrm{d}f(x)(X) = \nabla(f \circ \Phi_j^{-1})(\Phi_j(x)) \cdot D\Phi_j(x)(X)$. One sees that $W^{1,p}(\partial\Omega)$ is a Banach space for the norm

$$\begin{aligned} &\|f\|_{W^{1,p}(\partial\Omega)} = \Big(\|f\|^p_{L^p(\partial\Omega)} + \|\nabla_T f\|^p_{[L^p(\partial\Omega)]^n)}\Big)^{\frac{1}{p}} \\ &(\max\{\|f\|_{L^\infty(\partial\Omega)}, \|\nabla_T f\|_{[L^\infty(\partial\Omega)]^n)}\} \quad \text{if} \quad p = \infty) \end{aligned}$$

which is equivalent to $\sum_{j=1}^N \|f \circ \Phi_j^{-1}\|_{W^{1,p}(V_j)}$. Observe that $W^{1,\infty}(\partial\Omega)$ identifies with Lipschitz functions on $\partial\Omega$.

When $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, the dual space of $W^{1,p}(\partial\Omega)$, denoted by $W^{-1,q}(\partial\Omega)$, can be realised as the completion of $L^q(\partial\Omega)$ for the norm

$$\|\varphi\|_{W^{-1,q}(\partial\Omega)} := \sup_{\|g\|_{W^{1,p}(\partial\Omega)}=1} \int_{\partial\Omega} \varphi g \, d\sigma \tag{1.79}$$

by the same argument that leads to $(W_0^{1,p}(\Omega))^* = W^{-1,q}(\Omega)$.

Forms on $\partial\Omega$ proceed as in the smooth case: for $1 \leq k \leq n-1$, a k-form is a map $x \mapsto \omega(x)$ where $\omega(x)$ is an alternating k-linear map on $T_x\partial\Omega$ (a 0-form is simply

a function). Its local representative in the chart $\Phi_j : U_j \cap \partial\Omega \to V_j$ is the k-form $(\Phi_j^{-1})^*(\omega)$ on V_j which the pullback of ω under Φ_j^{-1}:

$$(\Phi_j^{-1})^*(\omega)(y) := \omega(\Phi_j^{-1}(y)) \circ (D\Phi_j^{-1}(y) \times \cdots \times D\Phi_j^{-1}(y)), \quad y \in \operatorname{Reg} V_j. \tag{1.80}$$

Rearranging (1.80), we get an expression of the form

$$(\Phi_j^{-1})^*(\omega)(y) = \sum_{i_1<i_2,\cdots,<i_k} a^{\{\Phi_j\}}_{i_1,\cdots,i_k}(y)\ dy_{i_1} \wedge \cdots \wedge dy_{i_k}, \qquad y \in \operatorname{Reg} V_j,$$

where the coefficients $a^{\{\Phi_j\}}_{i_1,\cdots,i_k}$ are functions on V_j that transform naturally under changes of coordinates. We identify forms that agree a.e.. and say that ω is of L^p-class if the $a^{\{\Phi_j\}}_{i_1,\cdots,i_k}$ lie in $L^p(V_j)$ for each j. In this case we write $\omega \in F^p_k(\partial\Omega)$ and

$$\|\omega\|_{F^p_k(\partial\Omega)} := \sum_{j=1}^N \sum_{i_1<i_2<\cdots<i_k} \|a^{\{\Phi_j\}}_{i_1,\cdots,i_k}\|_{L^p(V_j)}\,.$$

A change of atlas yields an equivalent norm. The image under ϕ_j of the restriction $\sigma_{|U_j}$ is the measure on V_j, absolutely continuous with respect to Lebesgue measure, with differential $(1+|\nabla\Psi_j(y)|^2)^{1/2}dy$. Since $\Psi_j \in L^\infty(V_j)$, it follows that $f \in L^p(\partial\Omega)$ if and only if it is of L^p-class as a 0-form, and $\|f\|_{L^p(\partial\Omega)}$ is equivalent to $\|f\|_{F^p_0(\partial\Omega)}$.

Integrating $(n-1)$-forms on $\partial\Omega$ goes as in the smooth case on an oriented Riemannian manifold [49, Sec. 4.10] (note that $\partial\Omega$ is oriented by construction). That is: for ω a $(n-1)$-form of L^1-class on $\partial\Omega$ and (φ_j) a Lipschitz partition of unity subordinated to the $U_j \cap \partial\Omega$, if one writes

$$(\Phi_j^{-1})^*(\varphi_j\omega)(y) = a(y)dy_1 \wedge \cdots \wedge y_{n-1}$$

then

$$\int_{\partial\Omega} \omega = \sum_{j=1}^N \int_{V_j} a_j(y)dy\,.$$

The latter is independent from the atlas and the partition of unity, thanks to the change of variable formula which is valid for Lipschitz reparametrizations. In particular, if we define on $U_j \cap \partial\Omega$ the $(n-1)$-form

$$\omega^j(x)(\sum_{i=1}^{n-1} \lambda_i \partial_{y_i}\Phi_j^{-1}) := \lambda_1\lambda_2\cdots\lambda_{n-1}J_k(\Phi_j(x))\,,$$

where J_k is the square root of the sum of the squares of the $(n-1)\times(n-1)$ minors of the Jacobian matrix $D\Phi_j^{-1}$, then $\omega^{j_1} = \omega^{j_2}$ a.e. on $U_{j_1} \cap U_{j_2} \cap \partial\Omega$ and the $(n-1)$-form ω_{vol} on $\partial\Omega$ whose restriction to $U_j \cap \partial\Omega$ is ω^j (the so-called volume form) satisfies $\int_{\partial\Omega} f d\sigma = \int_{\partial\Omega} f\omega_{vol}$ for every $f \in L^1(\partial\Omega)$.

Let us define $\mathcal{W}^{1,p}_k(\partial\Omega)$ to consist of k-forms $\omega \in F^p_k(\partial\Omega)$ for which there exists a $(k+1)$-form $\mathrm{d}\omega \in F^p_{k+1}(\partial\Omega)$ with the property that, for each $j \in \{1,\cdots,N\}$,

$$\int_{V_j} (\Phi_j^{-1})^*(\omega) \wedge \mathrm{d}\mu_j = (-1)^{k+1} \int_{V_j} (\Phi_j^{-1})^*(\mathrm{d}\omega) \wedge \mu_j$$

whenever μ_j is a C^∞-smooth $(n-k-1)$-form compactly supported on V_j. Here, the exterior derivative $\mathrm{d}\mu_j$ is the usual one for smooth forms. Though not obvious at first glance, this definition is consistent, for if $\mathrm{d}\omega$ exists then $(\Phi_{j_1}\circ\Phi_{j_2}^{-1})^*\Big((\Phi_{j_1}^{-1})^*(\omega)\Big) \in \mathcal{W}_k^{1,p}(\Phi_{j_2}(U_{j_1}\cap U_{j_2}\cap\partial\Omega))$ and on $\Phi_{j_2}(U_{j_1}\cap U_{j_2}\cap\partial\Omega)$ it holds a.e. that

$$\mathrm{d}\Big((\Phi_{j_1}\circ\Phi_{j_2}^{-1})^*\Big((\Phi_{j_1}^{-1})^*(\omega)\Big)\Big) = (\Phi_{j_1}\circ\Phi_{j_2}^{-1})^*\Big((\Phi_{j_1}^{-1})^*(\mathrm{d}\omega)\Big),$$

see [50, Thm 2.2]. We endow $\mathcal{W}_k^{1,p}(\partial\Omega)$ with the norm

$$\|\omega\|_{\mathcal{W}_k^{1,p}(\partial\Omega)} = (\|\omega\|^p_{F_k^p(\partial\Omega)} + \|\mathrm{d}\omega\|^p_{F_{k+1}^p(\partial\Omega)})^{\frac{1}{p}}.$$

It is easy to see that $f\in W^{1,p}(\partial\Omega)$ if and only if $f\in\mathcal{W}_0^{1,p}(\partial\Omega)$ and that

$$(\Phi_j^{-1})^*(\mathrm{d}f) = \sum_i \partial_{y_i}(f\circ\Phi_j^{-1})\,dy_i\,.$$

We let $\mathcal{W}_k^{-1,q}(\partial\Omega)$ be the completion of $F_k^q(\partial\Omega)$ for the norm

$$\|\omega\|_{\mathcal{W}_k^{-1,q}(\partial\Omega)} := \sup_{\|\mu\|_{\mathcal{W}^{1,p}_{n-1-k}(\partial\Omega)}=1} \int_{\partial\Omega}\omega\wedge\mu,$$

so that $\mathcal{W}_k^{-1,q}(\partial\Omega)$ consists of linear forms on $\mathcal{W}^{1,p}_{n-1-k}$; *i.e.*, of $(n-1-k)$-currents on $\partial\Omega$. When $f\in W^{1,p}(\partial\Omega)$ and $\mu\in\mathcal{W}^{1,q}_{n-2}(\partial\Omega)$, it holds that $\int_{\partial\Omega}\mathrm{d}f\wedge\mu = -\int_{\partial\Omega} f\mathrm{d}\mu$, as can be checked in local coordinates from the absolute continuity on a.e. line of Sobolev functions [16, Thm 2.1.4]. Thus, $\|\mathrm{d}f\|_{\mathcal{W}_1^{-1,p}(\partial\Omega)}\le C\|f\|_{L^p(\Omega)}$, and since $W^{1,p}(\partial\Omega)$ is dense in $L^p(\partial\Omega)$ it follows that $f\mapsto\mathrm{d}f$ extends to a continuous map from $L^p(\partial\Omega)$ into $\mathcal{W}_1^{-1,p}(\partial\Omega)$.

Since the gradient defines a continuous map $\nabla: L^p(V_j)\to[W^{-1,p}(V_j)]^{n-1}$ as pointed out at the begining of this section, (1.78) implies on using a Lipschitz partition of unity subordinated to the U_j that $\nabla_T: L^p(\partial\Omega)\to\mathcal{W}_1^{-1,p}(\partial\Omega)$ is continuous.

It is known that $W^{s,p}(\partial\Omega) = [L^p(\partial\Omega), W^{1,p}(\partial\Omega)]_{s,p}$, where $[.,.]_{s,p}$ is the so-called real interpolation functor, see [51, Ch.4] for a definition of the latter. This fact follows from the analogous result on $\mathbb{R}^{n-1}$ [51, Ch. 4, Cor. 4.13] and [25, Lem. 1.1] which allows one to localize the statement in the charts (U_j,Φ_j), using a Lipschitz partition of unity on $\partial\Omega$. Hence, by duality [52, Thm 3.7.1], we get that $W^{-\frac{1}{p},p}(\partial\Omega)\sim[L^p(\partial(\Omega), W^{-1,p}(\partial\Omega)]_{\frac{1}{p},p} = [W^{-1,p}(\partial\Omega), L^p(\partial(\Omega)]_{1-\frac{1}{p},p}$ with equivalence of norms, and since taking the gradient maps $L^p(\partial\Omega)$ into $[W^{-1,p}(\partial\Omega)]^n$ and $W^{1,p}(\partial\Omega)$ into $[L^p(\partial\Omega)]^n$ continuously, we see by interpolation upon setting $s=1-\frac{1}{p}$ that the tangential gradient $\nabla_T\psi$ of $\psi\in W^{1-\frac{1}{p},p}(\partial\Omega)$ exists as a member of $\mathcal{W}_1^{-\frac{1}{p},p}(\partial\Omega)$.

1.9.2 Auxiliary lemmas

Lemma 8. *To every $f\in\mathscr{S}$ on $\mathbb{R}^n$, there is a sequence $\phi_n\in\mathscr{S}$ such that*

- $\lim_{n\to\infty} \phi_n = f$ *in* $L^p(\mathbb{R}^n)$ *for all* $p \in [1,\infty)$,
- *to each* n *there is a nonempty neighborhoud* V_n *of* 0 *with the property that* $\hat{\phi}_n(\xi) = \hat{f}(0)$ *for* $\xi \in V_n$.

Proof. We adapt the proof of [53, Lem. 9.2.]. Pick $g \in \mathscr{S}$ such that $\hat{g} = 1$ in some ball $B(0,r)$, $r > 0$. For $\lambda > 0$, put $g_\lambda(x) := \lambda^{-n} g(x/\lambda)$ and define

$$h_\lambda(x) := \hat{f}(0) g_\lambda(x) - f * g_\lambda(x), \qquad x \in \mathbb{R}^n. \tag{1.81}$$

Clearly, $h_\lambda \in \mathscr{S}$, and since $\hat{g}_\lambda = 1$ in some neighborhood V_λ of 0 we see from (1.81) that $\hat{h}_\lambda(\xi) = \hat{f}(0) - \hat{f}(\xi)$ for $\xi \in V_\lambda$. Thus, letting $\phi_\lambda := f + h_\lambda$, we get that $\hat{\phi}_\lambda(\xi) = \hat{f}(0)$ for $\xi \in V_\lambda$. It remains to show that $\lim_{\lambda\to+\infty} h_\lambda = 0$ in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, for then ϕ_n will satisfy our requirements. Now, since $|f|$ is summable because $f \in \mathscr{S}$, we get by convexity that

$$|h_\lambda(x)|^p = \left| \int_{\mathbb{R}^n} f(y) \big(g_\lambda(x) - g_\lambda(x-y) \big) dy \right|^p \le \int_{\mathbb{R}^n} |f(y)| \big| g_\lambda(x) - g_\lambda(x-y) \big|^p dy.$$

Therefore, by Fubini's theorem and the change of variable $x = \lambda z$,

$$\|h_\lambda\|^p_{L^p(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} \big| g(z) - g(z - y/\lambda) \big|^p dz \right) dy. \tag{1.82}$$

The inner integral in (1.82) is at most $2\|g\|^p_{L^p(\mathbb{R}^n)}$ and it tends to zero for fixed y as $\lambda \to +\infty$, by the continuity of argument translation in $L^p(\mathbb{R}^n)$. Hence, the right hand side of (1.82) goes to zero when $\lambda \to +\infty$, by dominated convergence. □

Lemma 9. *Let* $\varphi : [0,\infty) \to [0,\infty)$ *be* C^∞*-smooth and supported in* $(0,1)$. *Then,* $h(x) := \varphi(|x|^n)$ *lies in* $\mathscr{S}(\mathbb{R}^n)$ *and its Newton potential*

$$N(x) := -\frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{\varphi(|y|^n)}{|x-y|^{n-2}} dy, \qquad x \in \mathbb{R}^n,$$

is a C^∞*-smooth function with gradient given by*

$$\nabla N(x) = -\frac{x}{|x|^n} \frac{\Phi(|x|^n)}{n}, \tag{1.83}$$

where $\Phi : [0,\infty) \to \mathbb{R}$ *is the indefinite integral of* φ *satisfying* $\Phi(0) = 0$. *Moreover, the second derivatives* $\partial^2_{i,j} N$ *lie in* $L^p(\mathbb{R}^n)$ *for* $1 \le i, j \le n$.

Proof. Since φ is C^∞-smooth and vanishes in a neighborhood of 0 and ∞, it is clear that $h \in \mathscr{S}$. Integrating in polar coordinates, we get

$$N(x) = \frac{1}{(n-2)\omega_n} \int_0^1 r^{n-1} \varphi(r^n) dr \int_{S(0,1)} \frac{d\sigma(\zeta)}{|x - r\zeta|^{n-2}}.$$

When $|x| > r$ the mean value property for harmonic functions yields that

$$\frac{1}{\omega_n} \int_{S(0,1)} \frac{d\sigma(\zeta)}{|x - r\zeta|^{n-2}} = \frac{1}{|x|^{n-2}},$$

and when $|x| \leq r$ it holds that

$$\frac{1}{\omega_n}\int_{S(0,1)}\frac{d\sigma(\zeta)}{|x-r\zeta|^{n-2}} = \frac{1}{r^{n-2}\omega_n}\int_{S(0,1)}\frac{d\sigma(\zeta)}{|x/r-\zeta|^{n-2}} = \frac{1}{r^{n-2}} \tag{1.84}$$

because σ/ω_n is the Newtonian equilibrium measure of $\overline{B(0,1)}$ which is a regular set, whence (1.84) is independent of $x \in \overline{B(0,1)}$, see [31, Ch. II, Sec. 13]. Altogether, we obtain:

$$N(x) = \frac{1}{n-2}\int_0^{\min\{|x|,1\}} r^{n-1}\varphi(r^n)\frac{dr}{|x|^{n-2}} + \frac{1}{n-2}\int_{\min\{|x|,1\}}^1 r\varphi(r^n)dr,$$

implying that

$$N(x) = \frac{1}{n(n-2)|x|^{n-2}}\Phi(|x|^n) + \frac{1}{(n-2)}\int_{|x|}^1 r\varphi(r^n)dr, \tag{1.85}$$

where the integral is interpreted as zero for $|x| \geq 1$. Since φ and Φ vanish in a neighborhoud of 0 while φ also vanishes in a neighborhood of 1, one can see that $N \in C^\infty(\mathbb{R}^n)$. Differentiating (1.85) we get that

$$\nabla N(x) = -\frac{x\Phi(|x|^n)}{n|x|^n} + \frac{1}{n-2}\frac{x}{|x|}\left(|x|\varphi(|x|^n) - |x|\varphi(|x|^n)\right),$$

which is (1.83). Differentiating once more gives us:

$$\partial^2_{i,j}N(x) = \frac{\Phi(|x|^n)}{n}\left(-\frac{\delta_{i,j}}{|x|^n} + n\frac{x_ix_j}{|x|^{n+2}}\right) - \frac{x_ix_j}{|x|^2}\varphi(|x|^n),$$

and using that φ is compactly supported while Φ is bounded and vanishes in a neighborhoud of 0, one verifies that $\partial^2_{i,j}N \in L^p(\mathbb{R}^n)$. □

1.9.3 Example of Sobolev function on a thin set

We begin by considering the set $E \subset \mathbb{R}^3$ that we define as

$$E = \{(x,y,z) : z \geq 2 - e^{-(x^2+y^2)^\alpha}\} \cap B(0,\sqrt{2})$$

with $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^3$ be the set

$$\Omega = (B(0,\sqrt{2})\backslash E)\backslash\overline{B(0,1)},$$

we let $O_0 = (\mathbb{R}^3\backslash\overline{\Omega})\backslash B(0,1)$ and $O_1 = B(0,1)$. We will call Ω_0 the intersection of Ω and the closed cone with vertex at $(0,0,0)$ that passes through the points (x,y,z) on $S(0,\sqrt{2})$ that satisfy $x^2+y^2+z^2 = x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as follows

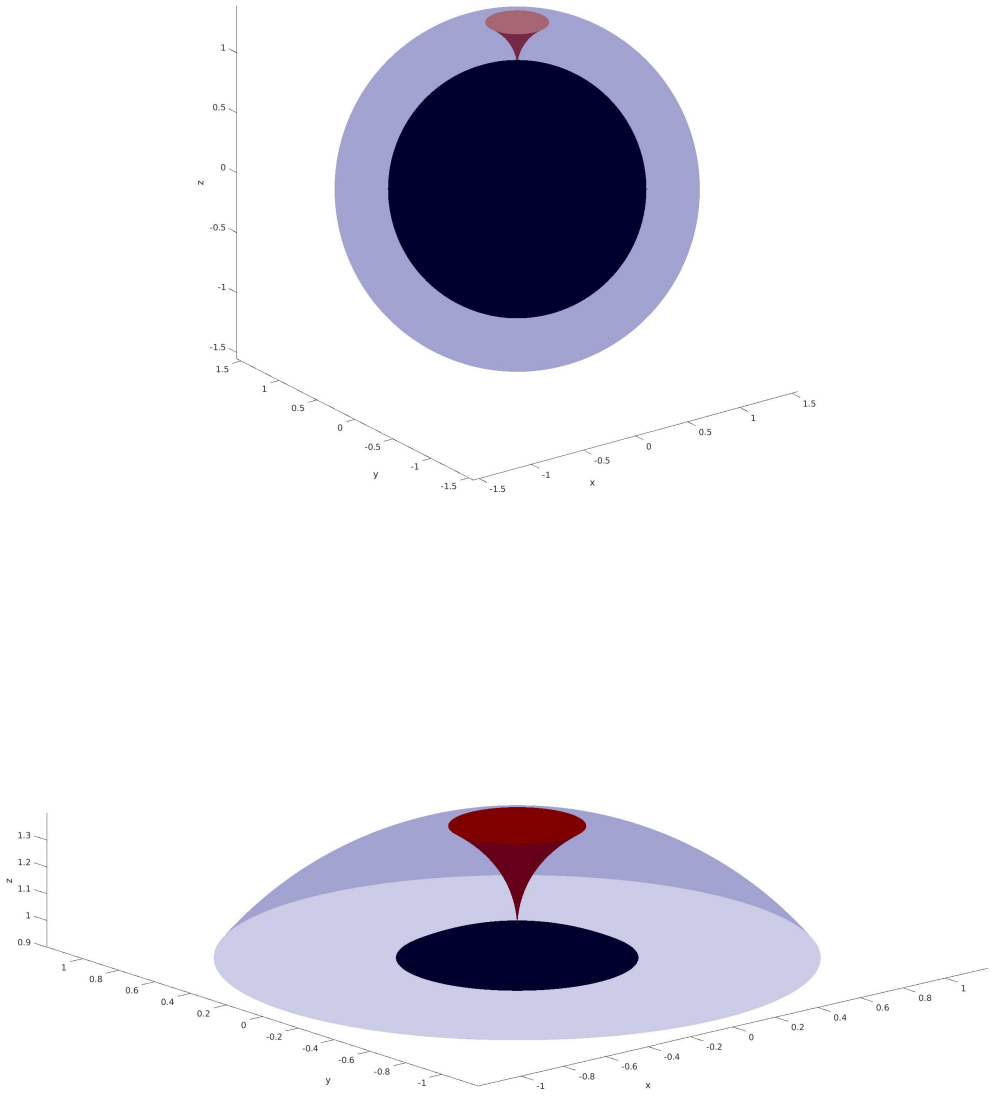

Figure 1.1: The domain Ω is contained in the grey part, $\alpha = \frac{1}{4}$. The subset of O_0 in red thin at $(0, 0, 1)$. Below is close up of the region around $(0, 0, 1)$

$$f(x,y,z)=\begin{cases}1 & \text{on } \overline{O_0}\backslash\{(0,0,1)\}\\ x^2+y^2+z^2-1 & \text{on } \overline{\Omega\backslash\Omega_0}\\ \dfrac{x^2+y^2+z^2-1}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1} & \text{on } \Omega_0\\ 0 & \text{on } \overline{O_1}\end{cases}.$$

We claim that $f\in W^{1,2}(\mathbb{R}^3)$ for some α. We note that the gradient of f on Ω_0 is given by

$$\nabla f=\begin{pmatrix}\frac{2x}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1}-\frac{4\alpha x(x^2+y^2)^{\alpha-1}e^{-(x^2+y^2)^\alpha}(2-e^{-(x^2+y^2)^\alpha})+2x}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2(x^2+y^2+z^2-1)^{-1}}\\ \frac{2y}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1}-\frac{4\alpha y(x^2+y^2)^{\alpha-1}e^{-(x^2+y^2)^\alpha}(2-e^{-(x^2+y^2)^\alpha})+2y}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2(x^2+y^2+z^2-1)^{-1}}\\ \frac{2z}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1}\end{pmatrix}.$$

To show that $f\in W^{1,2}_{loc}(\mathbb{R}^3)$ we need only show that the following integrals associated with the L^2-norm of the partial derivatives are finite for some α. In what follows we need to recall the following series representations

$$\begin{aligned}2-e^{-r^{2\alpha}}&=1+r^{2\alpha}-\frac{r^{4\alpha}}{2}+\frac{r^{6\alpha}}{6}-\frac{r^{8\alpha}}{24}+\frac{r^{10\alpha}}{120}+O(r^{12\alpha})\\ \sqrt{1-r^2}&=1-\frac{r^2}{2}-\frac{r^4}{8}-\frac{r^6}{16}+O(r^8)\end{aligned}. \tag{1.86}$$

We begin with the following integrals associated with the partial derivatives with respect to x and y:

$$\begin{aligned}&\int\limits_{(x^2+y^2)<R^2}\int_{z=\sqrt{1-(x^2+y^2)}}^{z=2-e^{-(x^2+y^2)^\alpha}}\frac{4(x^2+y^2)}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2}\,dz\,dx\,dy\\ &=4\int_0^{2\pi}\int_0^R\frac{r^2(2-e^{-r^{2\alpha}}-\sqrt{1-r^2})}{(r^2+(2-e^{-r^{2\alpha}})^2-1)^2}\,r\,dr\,d\theta\\ &=4\int_0^{2\pi}\int_0^R\frac{r^2}{(2-e^{-r^{2\alpha}}-\sqrt{1-r^2})(2-e^{-r^{2\alpha}}+\sqrt{1-r^2})^2}\,r\,dr\,d\theta,\end{aligned}$$

using the series expansions in (1.86) that near $r=0$ the integrand is dominated by $r^{3-2\alpha}$, hence if $3-2\alpha>-1$ the integral is finite, which is necessarily true for $0<\alpha<1$. To finish the discussion for integrals associated with the partial derivatives with respect to

x and y we look at

$$\int\limits_{(x^2+y^2)<R^2} \int_{z=\sqrt{1-(x^2+y^2)}}^{z=2-e^{-(x^2+y^2)^\alpha}} \frac{(4\alpha(x^2+y^2)^{\alpha-1}e^{-(x^2+y^2)^\alpha}(2-e^{-(x^2+y^2)^\alpha})+2)^2}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^4}$$
$$(x^2+y^2)(x^2+y^2+z^2-1)^2\,dz\,dx\,dy$$
$$= \int_0^{2\pi}\int_0^R \frac{r^2(4\alpha r^{2\alpha-2}e^{-r^{2\alpha}}(2-e^{-r^{2\alpha}})+2)^2}{(r^2+(2-e^{-r^{2\alpha}})^2-1)^4}$$
$$\Big[(2-e^{-r^{2\alpha}}-\sqrt{1-r^2})(r^4-2r^2+1)+$$
$$\frac{(2-e^{-r^{2\alpha}})^3-(\sqrt{1-r^2})^3}{3}(2r^2-2)+\frac{(2-e^{r^{2\alpha}})^5-(\sqrt{1-r^2})^5}{5}\Big]\,r\,dr\,d\theta,$$

using the series expansions in (1.86) we have that

$$\Big[(2-e^{-r^{2\alpha}}-\sqrt{1-r^2})(r^4-2r^2+1)+$$
$$\frac{(2-e^{-r^{2\alpha}})^3-(\sqrt{1-r^2})^3}{3}(2r^2-2)+\frac{(2-e^{r^{2\alpha}})^5-(\sqrt{1-r^2})^5}{5}\Big]$$

is dominated by $r^{6\alpha}$ near $r=0$. We can show that the integrand near $r=0$ is dominated by $r^{6\alpha-1}$ hence integrable for $0<\alpha<1$. Using the fact that for $a,b\in\mathbb{R}$, $(a+b)^2\le 2(a^2+b^2)$ and the above integrals we have that the partial derivatives with respect to x and y are in $L^2(\mathbb{R}^3)$. Finally the integral associated with the partial derivative with respect to z:

$$\int\limits_{(x^2+y^2)<R^2} \int_{z=\sqrt{1-(x^2+y^2)}}^{z=2-e^{-(x^2+y^2)^\alpha}} \frac{4z^2}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2}\,dz\,dx\,dy$$
$$= \frac{4}{3}\int_0^{2\pi}\int_0^R \frac{(2-e^{-r^{2\alpha}})^3-(\sqrt{1-r^2})^3}{(r^2+(2-e^{-r^{2\alpha}})^2-1)^2}\,r\,dr\,d\theta,$$

using the series expansions in (1.86) the integrand is dominated by $r^{1-2\alpha}$ near $r=0$ hence for $0<\alpha<1$ the integral is finite. Hence we have shown that for any α with $0<\alpha<1$, $f\in W^{1,2}_{loc}(\mathbb{R}^3)$ and is not constant on $\mathbb{R}^3\backslash\Omega$.

It remains only to show that the set O_0 is thin at (0,0,1). We use [54, Thm 7.2.5] to show that the set E_0 such that $(x,y,z)\in E_0$ with $0<z<1-\varepsilon(\alpha)$ for some $\varepsilon(\alpha)>0$ satisfy $\sqrt{(x^2+y^2)}<\ln(\frac{1}{1-z})^{\frac{1}{2\alpha}}$ is thin at (0,0,0). By letting

$$f(t)=\ln\Big(\frac{1}{1-t}\Big)^{\frac{1}{2\alpha}} \quad \text{for} \quad 0<t<1-\varepsilon(\alpha)$$

then continuously and boundedly extending f in $1-\varepsilon(\alpha)\le t\le 1$ we study the following integral

$$\int_0^1 \frac{1}{t\Big(1+\ln^+\Big(\frac{t}{f(t)}\Big)\Big)}\,dt.$$

We can choose $\varepsilon(\alpha)$ to be such that $\frac{t}{f(t)} > 1$ in $0 < t < 1 - \varepsilon(\alpha)$, such an $\varepsilon(\alpha)$ exits since $\frac{t}{f(t)} \to \infty$ as $t \to 0$. Further we can show that for $t \ln\left(\frac{t}{f(t)}\right) \to 0$ as $t \to 0$. It thus follows that

$$\ln\left(t\Big(1 + \ln^+\left(\frac{t}{f(t)}\right)\Big)\right) > \ln(t), \quad \text{for all } 0 < t < 1 - \varepsilon(\alpha).$$

Since $\ln(t) < 0$ in $0 < t < 1 - \varepsilon(\alpha)$ it follows that we can find ρ with $0 < \rho < 1$ such that

$$\ln\left(t\Big(1 + \ln^+\left(\frac{t}{f(t)}\right)\Big)\right) > \rho \ln(t) > \ln(t), \quad \text{for all } 0 < t < 1 - \varepsilon(\alpha).$$

Hence, for $0 < t < 1 - \varepsilon(\alpha)$ the integrand is dominated by $\frac{1}{t^\rho}$ for $0 < \rho < 1$ and the integral is finite implying that E_0 is thin at $(0, 0, 0)$. By noting that $(x, y, z) \in O_0$ with $\sqrt{(x^2 + y^2)} < 1$ and $1 < z < 2 - \varepsilon$ is just $(0, 0, 1) + E_0$ we conclude the that O_0 is thin at $(0, 0, 1)$.

Chapter 2

A layer potential approach to inverse problems in brain imaging

2.1 Introduction

In this chapter, we look at the problem of identifying active regions of the brain using the electric potential and/or the magnetic flux density associated with brain activity. This is done using the modalities of EEG, MEG and sEEG. To further study how EEG, MEG and sEEG work we invite the reader to look into [55], [56] and [57], respectively. We will give a brief explanation of these modalities here. EEG and sEEG are brain imaging modalities that use the electric potential associated with brain activity. In EEG the electrical potential is measured on the scalp whereas in sEEG the electrical potential is measured inside the head. MEG is a modality that used the magnetic flux density measured typically using Superconducting QUantum Interference Devices (SQUIDs) placed at a distance away from the head and around the head. Recently, Optically Pumped Magnetometers (OPMs) have been used to measure the magnetic flux density, see for example, [58].

The inverse source localisation problem using these modalities entails a lot into how the electric potential and magnetic flux density are transmitted in the head. However, the problem of source localisation and the inverse transmission problem have largely been solved separately. For source localisation see for example [59], and [60]. The problem of the transmission of the electric potential in the head has been studied before notably in [61] and [62] where the so-called boundary elements symmetric method (sBEM) was employed. The sBEM uses the single and double layer potentials, the normal derivative of the double layer potential and the adjoint of the trace of the double layer potential, and an application of Galerkin methods to solve the problem of the transmission of electromagnetic fields associated with brain activity. Instead, we aim to use only the single and double layer potentials with an application of boundary elements methods (BEM). In some cases the application of the BEM can be replaced with a method of fundamental solutions, see for example [63], [64] for details on the theory and applications of the method of fundamental solutions. We model brain activity, which may also be referred to as a source or primary cerebral current, as a vector-field whose components are elements of a Banach space supported on the grey/white mat-

ter interface, which is in accordance with the neurophysiological structure of pyramidal neurons in the cerebral cortex, see for example [57]. We give a special focus to $[W^{1/2,2}]^3$-vector-fields as sources as a way to clearly demonstrate the use of layer potentials in this endeavour. We use the formulations introduced by Geselowitz for the electric potential and magnetic flux density, see for example [65], which results in expressions of the forward models of the electric potential and the magnetic flux density that have strong relations with single and double layer potentials. When considering simple approximations of the head geometry such as a spherical head model, the expressions for the electrical potential and magnetic flux density have explicit analytic expressions. Such analytic expressions are difficult to obtain when considering realistic head geometries such as those that can be obtained via segmentation of MRI images, see for example [66]. In this case appropriate discretisations of the brain structures and the vector-fields are required. On any meshed surface we can obtain exact expressions for the single and double layer potentials associated defined on the surface hence exact expressions for the electric potential and magnetic flux density, see [67]. This offers improved numerical accuracy and the versatility of applying the forward model of electric potential to both EEG and intra-cranial recordings as in sEEG. When considering the inverse source localisation problem, we solve a Tikhonov regularised problem where we find the source that minimises a functional which involves the forward model. Hence, improved accuracy of the forward models in turn improves the source identification. In addition to source identification, the forward models that we employ can be co-opted to solve the so-called inverse cortical mapping problem, see [68], which is an inverse problem of the transmission of the electric potential within the head. In [68] a method based on the sBEM for solving the inverse cortical mapping problem was presented. The key difference between the method in [68] and the one we present here is that the unknown source is required in the computations. Further, the method we present is such that the inverse cortical mapping problem can be solved given either electric or magnetic data associated with brain activity. Coupling the problems of source localisation and cortical mapping should in principle improve the accuracy of the source localisation. The forward models we use provide a natural coupling of electric potential and magnetic flux density and we use this coupling to solve the inverse source localisation problem with simultaneous sEEG and MEG data which has interesting practical applications.

The chapter is organised as follows. In Sec. 2.2, we give an overview on layer potentials and Maxwell's equation to motivate their later applications. In Sec. 2.3, we look at the forward models for the electric and magnetic potential in inhomogeneous domains, which forms the bedrock of this paper. In Sec. 2.4, we apply the forward models to $[W^{1/2,2}]^3$-vector-fields normally oriented to the grey/white matter interface to illustrate the use of the layer potentials. We will also look at how to build the discrete version of the problem which is largely applicable with minor changes when the source is taken from other Banach spaces. In Sec. 2.5, we discuss the inverse problems for sEEG, EEG and MEG together with the inverse cortical mapping problem. We show that a solution always exists and propose an algorithm to solve the problem. In Sec. 2.6, we present numerical examples of the algorithm with the sources taken as $[W^{1/2,2}]^3$-vector-fields and as dipoles to show the versatility of the algorithm and what we are aiming for. We provide a conclusion and outlook in Sec. 2.7.

2.2 Preliminaries

Notation and definitions introduced in Chapter 1 will be carried over into the present chapter with new notation and definitions being introduced as needed. In this chapter $n = 3$.

2.2.1 Maxwell's equations

Recall the discussion in Section 1.2.1. We make the notational change of bolding all $\mathbb{R}^3$-valued quantities. In the constitutive relationships we assume that $\mathbf{P} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$. For EEG and MEG the quasi-static approximation can be made for the electromagnetic dynamics. In the quasi-static regime $\nabla \times \mathbf{E} = 0$, thus it follows that $\mathbf{E} = -\nabla\phi$ for some scalar potential ϕ. The total current density is of the form $\mathbf{J} = \mathbf{J}^i - \sigma\nabla\phi$ where $\mathbf{J}^i$ is the impressed primary current, $-\sigma\nabla\phi$ is the ohmic current and σ is the electric conductivity which needs not be uniform in the domain. The elliptic equation

$$\nabla \cdot (\sigma\nabla\phi) = \nabla \cdot \mathbf{J}^i, \tag{2.1}$$

can be derived from the quasi-static approximation and we shall use it in this work. The similarities of (1.1) and (2.1) ensures that the analysis done in Chapter 1 remains valid for problems governed by (2.1) when σ is uniform in the domain.

2.3 Forward Models

2.3.1 Unbounded homogeneous domain

Suppose that the source is an element of $[\Xi]^3$ for some Banach space Ξ, that will be precised later. Thus, it follows from (2.1) that the electrical potential associated with the source $\boldsymbol{\xi} \in [\Xi]^3$ satisfies

$$\nabla \cdot (\sigma\nabla\phi) = \nabla \cdot \boldsymbol{\xi}. \tag{2.2}$$

From (2.2) in an infinite homogeneous medium the electric potential, ϕ, is given by

$$\sigma\phi(\boldsymbol{\xi})(x) = \frac{1}{4\pi}\int \frac{\nabla \cdot \boldsymbol{\xi}(y)}{|x-y|}\,dy, \quad x \notin \operatorname{supp}\boldsymbol{\xi}, \tag{2.3}$$

and the gradient of the potential, $\nabla\phi$, is given by

$$\sigma\nabla\phi(\boldsymbol{\xi})(x) = \frac{1}{4\pi}\int \frac{\boldsymbol{\xi}(y)}{|x-y|^3}\,dy - 3\int (x-y)\frac{(x-y)}{|x-y|^5} \cdot \boldsymbol{\xi}(y)\,dy, \quad x \notin \operatorname{supp}\boldsymbol{\xi}. \tag{2.4}$$

Since the magnetic flux density, $\mathbf{B}$, is divergence free it is the curl of the vector magnetic potential which we denote by $\mathbf{A}$, that is,

$$\mathbf{B} = -\nabla \times \mathbf{A}.$$

Given a primary current $\boldsymbol{\xi}$ then

$$\mathbf{A}(\boldsymbol{\xi})(x) = \frac{\mu}{4\pi}\int \frac{\boldsymbol{\xi}(y)}{|x-y|}\,dy, \quad x \notin \operatorname{supp}\boldsymbol{\xi}. \tag{2.5}$$

Using the relationship above between the vector magnetic potential and the magnetic flux density we have that

$$\mathbf{B}(\boldsymbol{\xi})(x) = \frac{\mu}{4\pi} \int \boldsymbol{\xi}(y) \times \nabla_y \frac{1}{|x-y|}\, dy, \quad x \notin \operatorname{supp} \boldsymbol{\xi}. \tag{2.6}$$

2.3.2 Non-homogeneous bounded domains

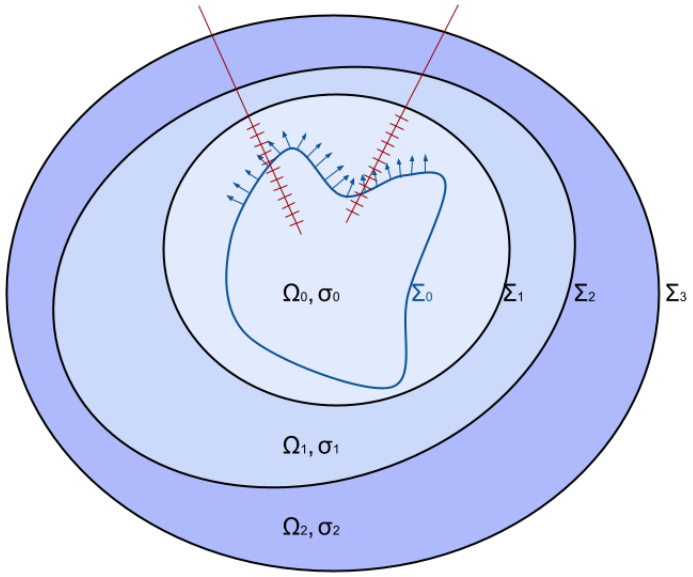

Figure 2.1: An example cross-section of a non-homogeneous domain and the placement of intra-cranial electrodes for sEEG, in red; the blue arrows represent the normal orientation of the source term on its support.

We now study the transmission of the electric potential and magnetic flux density in a bounded non-homogeneous conductor. Figure 2.1 is an example of such a conductor. To that end we consider the following bounded domain; let $\Omega \subset \mathbb{R}^3$ be a nested non-homogeneous bounded Lipschitz domain such that $\Omega_0, \Omega_1, \ldots, \Omega_m$ are nested Lipschitz domains with $\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_m$ and $\Omega = \cup_i \Omega_i$. We let the boundaries $\partial\Omega_i \cap \partial\Omega_{i+1} = \Sigma_{i+1}$ with ν_{i+1}, $i = 0, 1, 2, \ldots, m-1$ being the outward pointing unit normal to Σ_{i+1}, $i = 0, 1, 2, \ldots, m-1$, respectively. For consistency of notation the outer boundary of Ω_m shall be called Σ_{m+1} with ν_{m+1} as the outward pointing unit normal to Σ_{m+1}. We will assume that the support of $\boldsymbol{\xi} \in [\Xi]^3$ is a proper subset of $\overline{\Omega}_0$. The electric conductivities of the different domains are constant in each domain but different between domain, we call them $\sigma_0, \sigma_1, \ldots, \sigma_m$ for $\Omega_0, \Omega_1, \cdots, \Omega_m$, respectively. The conductivity outside Σ_{m+1} will be set to zero, hence $\sigma_{m+1} = 0$. On any of the interfaces Σ_i, $i = 1, 2, \ldots, m+1$, let σ_i^- and σ_i^+ be the conductivities inside and outside, respectively.

The electric potential at the interfaces is denoted by ϕ_i^- and ϕ_i^+ depending on whether the electric potential is taken as a non-tangential limit approaching the interface Σ_i, $i = 1, 2, \ldots, m+1$ from inside or outside, respectively. On each interface Σ_i,

the electric potential satisfies

$$\begin{aligned} \phi_i^- &= \phi_i^+, \\ \sigma_i^- \, \partial_{\nu_i} \phi^- &= \sigma_i^+ \, \partial_{\nu_i} \phi^+, \end{aligned} \tag{2.7}$$

since ϕ is a solution to the elliptic problem (2.2), see for example [69, Chap. II, Sec. 8.3, Prop. 9]. Henceforth we will denote by ϕ_i the electric potential on the surface Σ_i.

This regularity of the electric potential and normal currents leads to the well-known fact that the electric potential at any point $x \in \mathbb{R}^3$ is given by :

$$\sigma(x)\phi(x) = \sigma_0 \phi(\boldsymbol{\xi})(x) - \sum_{i=1}^{m+1} \frac{\sigma_i^- - \sigma_i^+}{4\pi} \int_{\Sigma_i} \phi_i(y) \nu_i(y) \cdot \nabla_y \Big(\frac{1}{|x-y|} \Big) \, d\mathcal{H}_i(y) \tag{2.8}$$

where $\mathcal{H}_i$ is the 2-dimensional Hausdorff measure on the surface Σ_i, see for example [65]. Note the above formula is valid for $x \in \Sigma_i$ and whenever $x \in \Sigma_i$ we take the non-tangential limit approaching from $\operatorname{int} \Sigma_i$. Using (2.8) we have that on each interface $\Sigma_k, k = 1, 2, \ldots, m+1$ the first regularity condition of (2.7) can be rewritten as

$$\frac{\sigma_k^+ + \sigma_k^-}{2} \phi_k^{\pm}(x) = \sigma_0 \phi(\boldsymbol{\xi})(x) - \sum_{i=1, i \neq k}^{m+1} (\sigma_i^- - \sigma_i^+) \mathcal{K}_i \phi_i(x) - (\sigma_k^- - \sigma_k^+) K_k \phi_k(x), \tag{2.9}$$

where $\mathcal{K}_i$ is the double layer potential defined on the interface Σ_i. Once (2.9) is satisfied the continuity of the normal derivatives of the double layer potentials across the interfaces ensures that the second condition of (2.7) is also satisfied. Due to the condition that the electric conductivity outside Ω is zero, the electric potential has to satisfy,

$$\sigma_m \partial_{\nu_{m+1}} \phi^-(x) = 0, \tag{2.10}$$

for $x \in \Sigma_{m+1}$.

Remark 8. *It is important to note that for one to be able to obtain the electric potential associated with $\boldsymbol{\xi}$, it is sufficient to solve a system of equations with (2.9) on the surfaces Σ_k, $k = 1, 2, \ldots, m+1$. Alternatively, the equality (2.10) can be used on the surface Σ_{m+1} instead.*

Note that the electric potentials on the surfaces Σ_i, also produce magnetic fields that have to be considered when looking at the magnetic flux density associated with the primary current $\boldsymbol{\xi}$. At any point $x \in \mathbb{R}^3$, have the following expression for the magnetic flux density associated with the primary current $\boldsymbol{\xi}$,

$$\mathbf{B}(x) = \mathbf{B}(\boldsymbol{\xi})(x) - \mu \sum_{i=1}^{m+1} \frac{\sigma_i^- - \sigma_i^+}{4\pi} \int_{\Sigma_i} \nu_i(y) \times \nabla_y \frac{1}{|x-y|} \phi_i(y) \, d\mathcal{H}_i(y), \tag{2.11}$$

see for example [65, Eq. (17)], where the ϕ_i's on the surfaces, Σ_i, $i = 1, 2, \ldots, m+1$, are the same as the surface potentials in (2.8). Note that in (2.11) care need to be taken when $x \in \Sigma_i$, see for example in the proof of Proposition 1 below.

We briefly discuss about the sBEM to highlight the main differences with BEM approach we are going to take here. We first need to recall that for each point $x \in \mathbb{R}^3$, we can represent it using the spherical coordinate system (r, θ, ϕ) with $r = |x|$. Then sBEM is based on the following observation:

Theorem 27 ([70, Thm 3.1.1], Representation Theorem)**.** *Let u be a harmonic function in $\mathbb{R}^3\backslash\partial\Omega$ satisfying the decay condition*

$$\begin{cases} \lim_{r\to\infty} r|u(x)| < \infty, \\ \lim_{r\to\infty} r\partial_r u(x) = 0, \end{cases}$$

where $\partial_r u(x)$ is the partial derivative of u in the radial direction. Then on $\partial\Omega$

$$\begin{aligned} -\partial_\nu u^\pm &= \pm\frac{[\partial_\nu u]_{\partial\Omega}}{2} + \mathcal{N}[u]_{\partial\Omega} - K^*[\partial_\nu u]_{\partial\Omega} \\ u^\pm &= \mp\frac{[u]_{\partial\Omega}}{2} - K[u]_{\partial\Omega} + \mathcal{S}[\partial_\nu u]_{\partial\Omega} \end{aligned},$$

where $[u]_{\partial\Omega} = u^- - u^+$ of $u : \mathbb{R}^3 \to \mathbb{R}$ on $\partial\Omega$ and $\mathcal{N}$ is the normal derivative of $\mathcal{K}$ on $\partial\Omega$.

In the sBEM the source is decomposed as follows

$$\nabla\cdot\boldsymbol{\xi} = \sum_{i=0}^{m} \nabla\cdot\boldsymbol{\xi}_{\Omega_i} \quad \text{such that} \quad \nabla\cdot\boldsymbol{\xi}_{\Omega_i} = (\nabla\cdot\boldsymbol{\xi})\chi_{\Omega_i}$$

and on each Ω_i we define for each $i = 0, 1, \ldots, m$

$$v_{\Omega_i} = -\mathfrak{R}_2 * (\nabla\cdot\boldsymbol{\xi}_{\Omega_i}).$$

It follows that each v_{Ω_i} is harmonic in $\mathbb{R}^3\backslash\overline{\Omega}_i$, $i = 0, 1, \ldots, m$. Further, define on each Ω_i the harmonic function

$$u_{\Omega_i} = \begin{cases} \phi - \dfrac{v_{\Omega_i}}{\sigma_i} & \text{in } \Omega_i \\ -\dfrac{v_{\Omega_i}}{\sigma_i} & \text{in } \mathbb{R}^3\backslash\overline{\Omega}_i \end{cases}.$$

For each u_{Ω_i}, $i = 0, 1, \ldots, m$ the following is true,

$$[u_{\Omega_i}]_{\Sigma_i} = -\phi_i, \qquad [u_{\Omega_i}]_{\Sigma_{i+1}} = \phi_{i+1}$$

and

$$[\partial_{\nu_i} u_{\Omega_i}]_{\Sigma_i} = -\partial_{\nu_i}\phi_i, \qquad [\partial_{\nu_{i+1}} u_{\Omega_i}]_{\Sigma_{i+1}} = \partial_{\nu_{\imath+1}}\phi_{i+1}.$$

Using the Representation Theorem, one obtains on each Σ_i, $i = 1, 2, \ldots, m+1$ the expressions for

$$u^-_{\Omega_{i-1}}, \quad u^+_{\Omega_i}, \quad (\partial_{\nu_i} u_{\Omega_{i-1}})^-, \quad (\partial_{\nu_i} u_{\Omega_i})^+$$

then by taking

$$u^-_{\Omega_{i-1}} - u^+_{\Omega_i} \quad \text{and} \quad \sigma_{i-1}(\partial_{\nu_i} u_{\Omega_{i-1}})^- - \sigma_i(\partial_{\nu_i} u_{\Omega_i})^+,$$

on each Σ_i expressions for ϕ_i and $\partial_{\nu_i}\phi$ are obtained. These expression are then used to build the linear system to be solved to obtain the surface electric potentials, ϕ_i, and normal currents, $\partial_{\nu_i}\phi$. See [61] for more details on the method. The method we implement in this paper is equivalent to the "Double-Layer Approach" of the paper [61]. The approach we take differs from the sBEM in that our approach only computes the surface potentials while the sBEMs computes both the surface potentials and the normal currents.

2.4 $[W^{1/2,2}]^3$ sources

In what follows Σ_0 denotes a closed Lipschitz surface inside Ω_0, which is the support of the sources. We now take $[\Xi]^3$ to be the subspace of $[W^{1/2,2}(\Sigma_0)]^3$ composed of vector-fields normally oriented to Σ_0, that is we take vector-fields $\mathbf{M}_{\Sigma_0} \in [W^{1/2,2}(\Sigma_0)]^3$ of the form $\mathbf{M}_{\Sigma_0} = M_{\Sigma_0}\nu_0$, with $M_{\Sigma_0} \in W^{1/2,2}(\Sigma_0)$. Both the orientation and magnitude of $\mathbf{M}_{\Sigma_0}$ are encoded by $M_{\Sigma_0} \in W^{1/2,2}(\Sigma_0)$, we take $\mathbf{M}_{\Sigma_0}$ to be oriented in the same direction as ν_0 when M_{Σ_0} is positive and in the opposite direction otherwise. Note that (2.3) can be rewritten as

$$\begin{aligned}\sigma_0\phi(\mathbf{M}_{\Sigma_0})(x) &= \frac{1}{4\pi}\int_{\Sigma_0} \mathbf{M}_{\Sigma_0}(y)\cdot\frac{(x-y)}{|x-y|^3}\,d\mathcal{H}_0(y),\\ &= \frac{1}{4\pi}\int_{\Sigma_0} M_{\Sigma_0}(y)\frac{(x-y)}{|x-y|^3}\cdot\nu_0(y)\,d\mathcal{H}_0(y),\\ &= \mathcal{K}_0 M_{\Sigma_0}(x),\end{aligned} \tag{2.12}$$

for $x \in \mathbb{R}^3\backslash\Sigma_0$. From [25, Thm 4.1] we have that $\phi \in W^{1,2}(\text{int}\Sigma_0)$ and $\phi \in W^{1,2}(\text{ext}\Sigma_0)$. For $x \in \Sigma_0$, we have that by approaching x non-tangentially

$$\sigma_0\phi(\mathbf{M}_{\Sigma_0})(x) = \pm\frac{M_{\Sigma_0}(x)}{2} + K_0 M_{\Sigma_0}(x)\,, \tag{2.13}$$

where the $-$ and $+$ are from approaching the boundary from interior and exterior of Σ_0, respectively.

2.4.1 Forward model for electric potential

We can now rewrite (2.8) as

$$\begin{aligned}\sigma(x)\phi(x) &= \sigma_0\phi(\mathbf{M}_{\Sigma_0})(x) - \sum_{i=1}^{m+1}\frac{\sigma_i^- - \sigma_i^+}{4\pi}\int_{\Sigma_i}\phi_i(y)\nu_i(y)\cdot\nabla_y\Big(\frac{1}{|x-y|}\Big)\,d\mathcal{H}_i(y)\\ &= \mathcal{K}_0 M_{\Sigma_0}(x) - \sum_{i=1}^{m+1}(\sigma_i^- - \sigma_i^+)\mathcal{K}_i\phi_i(x),\end{aligned} \tag{2.14}$$

We can therefore rewrite (2.9) as

$$\frac{\sigma_k^+ + \sigma_k^-}{2}\phi_k^{\pm}(x) = \mathcal{K}_0 M_{\Sigma_0}(x) - \sum_{i=1, i\neq k}^{m+1}(\sigma_i^- - \sigma_i^+)\mathcal{K}_i\phi_i(x) - (\sigma_k^- - \sigma_k^+)K_k\phi_k(x). \tag{2.15}$$

We now discuss the numerical implementation of (2.14) and (2.15) on triangular meshes of the surfaces Σ_i, $i = 0, 1, \ldots, m+1$. Since we have made the assumption that $\Omega_i, i = 0, 1, \ldots, m$ is a Lipschitz domain then $K : L^2(\Sigma_{i+1}) \to L^2(\Sigma_{i+1})$ fails generally to be compact, see for example [29]. When $K : L^2(\Sigma_{i+1}) \to L^2(\Sigma_{i+1})$ is compact we have that $(\pm\frac{1}{2}I + K) : L^2(\Sigma_{i+1}) \to L^2(\Sigma_{i+1})$ has a canonical representation, that is,

$$\Big(\pm\frac{1}{2}I + K\Big)f = \sum_{j\geq 1}\alpha_j\langle f, u_j\rangle u_j, \tag{2.16}$$

where $\{\alpha_j\}$ and $\{u_j\}$ are the eigenvalues and an orthonormal basis of eigenfunctions of $(\pm\frac{1}{2}I + K)$, respectively. This is true for example when Σ_{i+1} is a sphere in which case the u_j are spherical harmonics, see for example [71], hence we can derive explicit expressions for (2.14). In general, if Σ_{i+1} is C^1 smooth then K is a compact operator, see for example [29], hence a canonical representation as was given above is achievable. It then is possible to numerically approximate the eigenvalues and eigenfunctions in these smooth cases, see for example [72]. In these cases we can use the *method of fundamental solutions* to build (2.14) and (2.15), see [64] for an example of this application of the method.

For Σ a Lipschitz surface we propose the following use of BEM for (2.14) and (2.15). Given a surface Σ, we begin by triangulating the surface to obtained Σ_T. On Σ_T we will represent each function $f \in W^{1/2,2}(\Sigma)$ by considering its values on the vertices of Σ_T. We assume that on each triangle the function can be represented by linear shape functions, that is, on each triangle there are three linear basis functions each of which has value one on one vertex and zero on the other two. Thus, given a function $f \in W^{1/2,2}(\Sigma)$, on each triangle T_k of Σ_T we write,

$$f(y) = \sum_{j=1}^{3} f_{kj}\psi_{kj}(y), \tag{2.17}$$

where $y \in T_k$, f_{kj} is the value of f on the j-th vertex of the triangle T_k and ψ_{kj} is the linear shape function on T_k that has value one on the j-th vertex of the triangle. Now given a point $x \in \mathbb{R}^3$ we wish to compute the double layer potential $\mathcal{K}f(x)$ for $f \in W^{1/2,2}(\Sigma)$. We use the analytic formulation proposed in [67], which enables us to numerically compute exactly the double layer potential defined on Σ_T even when $x \in \Sigma_T$. With this formulation we write

$$\mathcal{K}f(x) = \mathbf{H}(x)\mathbf{f}, \tag{2.18}$$

where

1) $\mathbf{H}(x)$ is a row vector in which the l-th element of $\mathbf{H}(x)$ is the sum of the contribution to the double layer potential of each triangle that has the l-th vertex of the triangulation as a vertex

2) $\mathbf{f}$ is a column vector of the values of f on the vertices of Σ_T with the l-th element of $\mathbf{f}$ is the value of f on the l-th vertex of the triangulation.

Note that depending on where x is located, $\mathbf{H}(x)\mathbf{f}$ is either $\mathcal{K}f(x)$ or $(-\frac{1}{2}I + K)f(x)$, with a "$-$" for $x \in \Sigma_T$ as we assume that the approach is from the interior. Using this notation, (2.14) can then be written as

$$\sigma(x)\phi(x) = \sigma_0\mathbf{H}_0(x)\mathbf{\Phi}_0 - \sum_{i=1}^{m+1}(\sigma_i^- - \sigma_i^+)\mathbf{H}_i(x)\mathbf{\Phi}_i, \tag{2.19}$$

where $\mathbf{H}_i(x)$ is as described above and also depends on the surface $\Sigma_{i,T}$, the triangulation of Σ_i and $\mathbf{\Phi}_i$ are the values of ϕ_i at the vertices of $\Sigma_{i,T}$, $i = 0, 1, 2, \ldots, m+1$, respectively.

For $x \in \Sigma_{k,T}$, $k = 1, 2, \ldots, m+1$, (2.15) becomes

$$\sigma_k^-(x)\phi(x) + (\sigma_k^- - \sigma_k^+)\mathbf{H}_k(x)\mathbf{\Phi}_k = \sigma_0\mathbf{H}_0(x)\mathbf{\Phi}_0 - \sum_{i=1, i\neq k}^{m+1}(\sigma_i^- - \sigma_i^+)\mathbf{H}_i(x)\mathbf{\Phi}_i. \quad (2.20)$$

We can use the same idea as above to compute the gradient of the double layer potential at any point x for any function $f \in W^{1/2,2}(\Sigma)$. To be able to solve the forward cortical mapping problem we only need the normal derivatives of the double layer potential at the outer most surface. We will have the following formulation

$$\partial_\nu \mathcal{K} f(x) = \mathbf{N}(x)\mathbf{f}, \quad (2.21)$$

where $\mathbf{N}(x)$ is constructed in the same manner as $\mathbf{H}(x)$. It then follows that the normal derivative of (2.14) for $x \in \Sigma_{m+1,T}$ becomes

$$\sigma_0\mathbf{N}_0(x)\mathbf{\Phi}_0 = \sum_{i=1}^{m+1}(\sigma_i^- - \sigma_i^+)\mathbf{N}_i(x)\mathbf{\Phi}_i, \quad (2.22)$$

In view of Remark 8, given the surfaces $\Sigma_{i,T}$, $i = 1, 2, \ldots, m+1$, $\mathbf{\Phi_0}$ and letting X_i the set of all vertices on all surfaces $\Sigma_{i,T}$, we can build the linear system

$$\left\{ \Big(\sigma_k^-\mathbf{I} + (\sigma_k^- - \sigma_k^+)\mathbf{H}_k(X_k)\Big)\mathbf{\Phi}_k + \sum_{i=1, i\neq k}^{m+1}(\sigma_i^- - \sigma_i^+)\mathbf{H}_i(X_k)\mathbf{\Phi}_i = \sigma_0\mathbf{H}_0(X_k)\mathbf{\Phi}_0 \right\}_{k=1,2,\ldots,m+1} \quad (2.23)$$

or

$$\left\{ \Big(\sigma_k^-\mathbf{I} + (\sigma_k^- - \sigma_k^+)\mathbf{H}_k(X_k)\Big)\mathbf{\Phi}_k + \sum_{i=1, i\neq k}^{m+1}(\sigma_i^- - \sigma_i^+)\mathbf{H}_i(X_k)\mathbf{\Phi}_i = \sigma_0\mathbf{H}_0(X_k)\mathbf{\Phi}_0 \right\}_{k=1,2,\ldots,m}$$

$$\sum_{i=1}^{m+1}(\sigma_i^- - \sigma_i^+)\mathbf{N}_i(X_{n+1})\mathbf{\Phi}_i = \sigma_0\mathbf{N}_0(X_{m+1})\mathbf{\Phi}_0.$$

and solving either of the above linear systems for $\mathbf{\Phi}_k$, $k = 1, 2, \ldots, m+1$, solves the forward cortical mapping problem.

As alluded to earlier to compute the matrices $\mathbf{H}$ and $\mathbf{N}$ we will implement a method suggested in [67] which results in exact expressions for the quantities that are required to build the matrices *for the meshed surfaces*, $\Sigma_{i,T}$. This allows to compute these matrices for any arbitrary point $x \in \mathbb{R}^3$ hence allowing the numerical approximations required for either EEG or sEEG with a high accuracy.

2.4.2 Forward model for magnetic flux density

We now turn our attention to the magnetic flux density associated with vector-fields in $[W^{1/2,2}(\Sigma_0)]^3$ that were introduced in the previous subsection. It follows from (2.5) that the magnetic vector potential of $\mathbf{M}_{\Sigma_0}$ is given by

$$\mathbf{A}(\mathbf{M}_{\Sigma_0})(x) = \frac{\mu}{4\pi}\int_{\Sigma_0}\frac{\mathbf{M}_{\Sigma_0}(y)}{|x-y|}\,d\mathcal{H}(y) = \mu\mathcal{S}\mathbf{M}_{\Sigma_0}(x), \quad x\in\mathbb{R}^3\backslash\Sigma_0, \tag{2.24}$$

with the integral being taken in the principal value sense for $x\in\Sigma_0$, see for example [7, Eq. (5.32)]. Note that the last equality of (2.24) is a direct application of the single layer potential hence $\mathbf{A}(\mathbf{M}_{\Sigma_0})$ has components that are equal to the single layer potentials of the corresponding components of $\mathbf{M}_{\Sigma_0}$. Since $W^{1/2,2}(\Sigma_0)\subset L^2(\Sigma_0)\subset W^{-1/2,2}(\Sigma_0)$ we have from [25, Thm 3.1] $\mathbf{A}(\mathbf{M}_{\Sigma_0})\in[W^{1,2}(\Omega)]^3$ and $\mathbf{A}(\mathbf{M}_{\Sigma_0})\in[W^{1,2}(\mathbb{R}^3\backslash\overline{\Omega})]^3$.

Since $\mathbf{B}=-\nabla\times\mathbf{A}$ it follows that $\mathbf{B}(\mathbf{M}_{\Sigma_0})\in[L^2(\Omega)]^3$ and $\mathbf{B}(\mathbf{M}_{\Sigma_0})\in[L^2(\mathbb{R}^3\backslash\overline{\Omega})]^3$ and the magnetic flux density associated with a vector-field $\mathbf{M}_{\Sigma_0}\in[W^{1/2,2}(\Sigma_0)]^3$ is given by

$$\begin{aligned}\mathbf{B}(\mathbf{M}_{\Sigma_0})(x) &= \frac{\mu}{4\pi}\int_{\Sigma_0}\mathbf{M}_{\Sigma_0}(y)\times\nabla_y\frac{1}{|x-y|}\,d\mathcal{H}(y)\\ &= \mu\nabla\times\mathcal{S}\mathbf{M}_{\Sigma_0}(x),\end{aligned} \tag{2.25}$$

with $x\in\mathbb{R}^3\backslash\Sigma_0$ .Since $\mathbf{M}_{\Sigma_0}=M_{\Sigma_0}\nu_0$ we can rewrite the above equation as

$$\mathbf{B}(\mathbf{M}_{\Sigma_0})(x) = \frac{\mu}{4\pi}\int_{\Sigma_0}\nu_0(y)\times\nabla_y\frac{1}{|x-y|}M_{\Sigma_0}(y)\,d\mathcal{H}(y) = \mu\mathbf{S}M_{\Sigma_0}(x), \tag{2.26}$$

with $x\in\mathbb{R}^3\backslash\Sigma_0$.

Note that the last equality of (2.26) is how we define the operator $\mathbf{S}:W^{1/2,2}(\Sigma_0)\to\mathbb{R}^3$, which will be carried over to the discrete case. It follows from (2.11) that

$$\begin{aligned}\mathbf{B}(x) &= \mathbf{B}(\mathbf{M}_{\Sigma_0})(x) - \mu\sum_{i=1}^{m+1}\frac{\sigma_i^- - \sigma_i^+}{4\pi}\int_{\Sigma_i}\nu_i(y)\times\nabla_y\frac{1}{|x-y|}\phi_i(y)\,d\mathcal{H}_i(y)\\ &= \mathbf{S}M_{\Sigma_0}(x) - \mu\sum_{i=1}^{m+1}\frac{\sigma_i^- - \sigma_i^+}{4\pi}\int_{\Sigma_i}\nu_i(y)\times\nabla_y\frac{1}{|x-y|}\phi_i(y)\,d\mathcal{H}_i(y),\end{aligned} \tag{2.27}$$

where the ϕ_i's on the surfaces are surface potentials obtained from the forward model of the electric potential associated with M_{Σ_0}.

Remark 9. *Note that MEG measures $\mathbf{B}\cdot\mathbf{v}$, for some known vector-field $\mathbf{v}$. Typically, $\mathbf{v}$ is taken to be a radial vector-field, hence if all of the Σ_i above are spherical, all the vector-field of the form $M_{\Sigma_0}\nu_0$ will result in null MEG measurements, see [65, Eq. (20)], which make a spherical head model uninteresting for MEG in this context.*

We now look at how to numerically compute (2.27) for Lipschitz surfaces Σ_i. To that end we need to be able to express $\mathbf{S}M_{\Sigma_0}$ as a linear combination of the $\mathbf{S}M_{\Sigma_{0_i}}$'s where the $M_{\Sigma_{0_i}}\in W^{1/2,2}(\Sigma_0)$ are elements of the basis of $W^{1/2,2}(\Sigma_0)$. Note that it only suffices

that we look at how to numerically compute the expression given in (2.26) because to compute (2.27) we repeatedly apply the same idea. We begin with a triangulation of the surfaces and discretisation of $M_\Sigma \in W^{1/2,2}(\Sigma)$ as was done in the previous subsection. Note that (2.26) on Σ_T is given by

$$\mathbf{B}(x) = \sum_{T_k} \frac{\mu}{4\pi} \int_{T_k} \nu_{T_k}(y) \times \nabla_y \frac{1}{|x-y|} M_\Sigma(y)\, d\mathcal{H}(y), \tag{2.28}$$

since ν_{T_k} is taken to be constant on each triangle hence we have that

$$\begin{aligned} \mathbf{B}(x) &= \mu \sum_{T_k} \nu_{T_k} \times \frac{1}{4\pi} \int_{T_k} \nabla_y \frac{1}{|x-y|} M_\Sigma(y)\, d\mathcal{H}(y) \\ &= \mathbf{S}(x) M_\Sigma, \end{aligned} \tag{2.29}$$

where $\mathbf{S}(x)$ is a row vector which is generated in a similar manner as $\mathbf{H}(x)$ and M_Σ is a column vector of the values of M_Σ on the vertices of Σ_T, where we use the formula given in [67] to compute the integral above on each triangle. Hence (2.27) is written in discrete form as

$$\mathbf{B}(x) = \mathbf{S}_0(x)\mathbf{\Phi}_0 - \sum_{i=1}^{m+1} (\sigma_i^- - \sigma_i^+)\mathbf{S}_i(x)\mathbf{\Phi}_i, \tag{2.30}$$

where the Φ_i, $i = 1, 2, \ldots, m+1$, are obtained from the forward model for electrical potential.

2.4.3 Silent Sources

We will conclude this section by discussing *silent sources*. We begin with a definition:

Definition 35. *Let $E \subset \mathbb{R}^3 \backslash \text{supp}(\boldsymbol{\xi})$ then a non-zero vector-field $\boldsymbol{\xi} \in [\Xi]^3$ is electrically (magnetically) silent in E if it produces an identically zero electric potential (magnetic flux density) in E.*

The existence of silent sources results in the non-uniqueness of solutions to the source identification problems we aim to solve. It is therefore important to understand the nature of the silent sources as this may help to mitigate their impact on the uniqueness of solutions.

As an example, take $[\Xi]^3$ to be the subspace of $[W^{1/2,2}(\Sigma_0)]^3$ composed of vector-fields of the form $M_{\Sigma_0}\nu_0$, with $M_{\Sigma_0} \in W^{1/2,2}(\Sigma_0)$. We will now explore the silent sources among these vector-fields.

Proposition 1. *Vector-fields of the form $M_{\Sigma_0}\nu_0$, with $M_{\Sigma_0} \in W^{1/2,2}(\Sigma_0)$ such that M_{Σ_0} is constant are both electrically and magnetically silent in $\mathbb{R}^3 \backslash \Omega_0$.*

Proof. From (2.12) we have that

$$\sigma_0 \phi(\mathbf{M}_{\Sigma_0})(x) = K_0 M_{\Sigma_0}(x), \quad x \in \mathbb{R}^3 \backslash \Omega_0,$$

and from [73, Ex. 6.14] we have that the above equation is identically zero in $\mathbb{R}^3\backslash\Omega_0$ if M_{Σ_0} is a constant. Hence the proposition is proved for the electric silence.

It also follows from [74, Lem. 4.3] that

$$\mathbf{B}(\mathbf{M}_{\Sigma_0})(x) = \mathbf{S}M_{\Sigma_0}(x) = \nabla \times \mathcal{S}(M_{\Sigma_0}\nu)(x) = \mathcal{S}(\nu \times \nabla M_{\Sigma_0})(x), \tag{2.31}$$

where ∇M_{Σ_0} is the gradient of M_{Σ_0} on Σ_0. It follows from [29, Thm 3.3] that all $\mathcal{S} : L^2(\Sigma_0) \to L^2(\Sigma_0)$ is injective. Hence, if $\mathcal{S}(M)$ is identically zero in $\mathbb{R}^3\backslash\Omega_0$ then its non-tangential limit to Σ_0 is identically zero on Σ_0 thus $M = 0$ on Σ_0. We can therefore conclude that, if $M_{\Sigma_0} \in L^2(\Sigma_0)$ is such that ∇M_{Σ_0} vanishes on Σ_0 then the resulting magnetic flux density is identically zero. Hence, all $M_{\Sigma_0}\nu_0$ such that M_{Σ_0} is constant are magnetically silent in $\mathbb{R}^3\backslash\Omega_0$. □

Remark 10. *From [73, Ex. 6.14] we observe that if $M_{\Sigma_0} \in W^{1/2,2}(\Sigma_0)$ is constant then*

$$\sigma_0\phi(\mathbf{M}_{\Sigma_0})(x) = K_0 M_{\Sigma_0}(x) = M_{\Sigma_0}, \quad x \in \text{int } \Sigma_0,$$

hence these vector-fields, $M_{\Sigma_0}\nu_0$, are not silent in Ω_0. We can exploit this fact by combining EEG or MEG data with SEEG data. The SEEG data will in principle allow to eliminate the silent sources outlined in Proposition 1 in the source recovery.

As a second example take a closed set $S \subset \text{int}\,\Sigma_0$ and let S be a slender set, that is, $m_3(S) = 0$, where m_3 is the Lebesgue measure on $\mathbb{R}^3$ and $m_3(\mathbb{R}^3\backslash S) = \infty$. We will take $[\Xi]^3 = [\mathcal{M}(S)]^3 \subset [\mathcal{M}(\text{int}\,\Sigma_0)]^3$, that is, vector-valued measures supported on the slender set S. For $\boldsymbol{\xi} \in [\mathcal{M}(\text{int}\,\Sigma_0)]^3$ we define $\nabla \cdot \boldsymbol{\xi}$ in the sense of distributions and when $\nabla \cdot \boldsymbol{\xi} = 0$ (divergence-free) we mean

$$\int_{\text{int}\,\Sigma_0} \nabla u \cdot d\boldsymbol{\xi} = \int_{\text{int}\,\Sigma_0} \sum_{j=1}^{3} \frac{\partial u}{\partial x_j}\, d\xi_j = 0,$$

for all $u \in C_c^\infty(\text{int}\,\Sigma_0)$.

Proposition 2. *Every $\boldsymbol{\xi} \in [\mathcal{M}(\text{int}\,\Sigma_0)]^3$ that has a slender support and is divergence-free is electrically silent in $\mathbb{R}^3\backslash\Omega_0$.*

Proof. This is a direct application of [6, Thm 2.2]. □

Examples of divergence-free vector-fields on slender sets in $\mathbb{R}^3$ can be constructed as follows. Let $\boldsymbol{\gamma} : [0, l] \to \mathbb{R}^3$ be a Lipschitz mapping and let $S := \boldsymbol{\gamma}([0, l])$. If $\boldsymbol{\gamma}$ is such that

$$\mathcal{H}^1(\boldsymbol{\gamma}([a, b])) = b - a, \quad \forall [a, b] \subset [0, l],$$

then $\boldsymbol{\gamma}$ is an orientable rectifiable curve. Note that S is slender and on S define the vector measure $\mathbf{R}_\gamma$ through the relation

$$\langle \mathbf{R}_\gamma, \mathbf{f} \rangle = \int_0^l \mathbf{f}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt, \quad \text{for } \mathbf{f} \in [C_c(\mathbb{R}^3)]^3,$$

where $\boldsymbol{\gamma}'$ is the unit tangent vector of $\boldsymbol{\gamma}$. Let the endpoints of S be $s, t \in \mathbb{R}^3$, it follows that $\mathbf{R}_{\boldsymbol{\gamma}} = \boldsymbol{\gamma}'\mathcal{H}^1$ and the divergence of $\mathbf{R}_{\boldsymbol{\gamma}}$ is given as

$$\int_0^l \nabla u(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = \int_{\boldsymbol{\gamma}} \dot{u}\, d\mathcal{H}^1 = u(s) - u(t), \quad \forall u \in C_c^\infty(\operatorname{int} \Sigma_0),$$

where $\dot{u}$ is the derivative of u along $\boldsymbol{\gamma}$. Hence if $s = t$ then $\mathbf{R}_{\boldsymbol{\gamma}}$ is divergence free and by the proposition above is electrically silent. Concretely, take S to be a circle embedded on $\mathbb{R}^3$ by taking

$$\boldsymbol{\gamma} := \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix},$$

with $t \in [0, 2\pi]$. More details and examples can be found in [75].

These two examples show the particularity of silent sources to the assumptions on the sources.

2.5 Inverse problems

In this section we consider the general situation when elements of $[\Xi]^3$ are vector-fields with components that are elements of a Banach space, Ξ, supported in $\overline{\Omega}_0$. When solving the inverse problems for MEG, EEG and sEEG, we need that the electric potential, ϕ, associated with the recovered source satisfies the conditions (2.7) and (2.10). We recall (2.9) and (2.10) here

$$\begin{aligned} \Big\{ -\sigma_0\phi(\boldsymbol{\xi})(x) + \frac{\sigma_k^+ + \sigma_k^-}{2}\phi_k + \sum_{i=1, i\neq k}^{m+1} (\sigma_i^- - \sigma_i^+)\mathcal{K}_i\phi_i(x) \\ - (\sigma_k^- - \sigma_k^+)K_k\phi_k(x) = 0 \Big\}_{k=1,2,\ldots,m+1} \end{aligned}, \tag{2.32}$$

or

$$\begin{aligned} \Big\{ -\sigma_0\phi(\boldsymbol{\xi})(x) + \frac{\sigma_k^+ + \sigma_k^-}{2}\phi_k + \sum_{i=1, i\neq k}^{m+1} (\sigma_i^- - \sigma_i^+)\mathcal{K}_i\phi_i(x) \\ - (\sigma_k^- - \sigma_k^+)K_k\phi_k(x) = 0 \Big\}_{k=1,2,\ldots,m} \\ \sigma_m\partial_{\nu_{m+1}}\phi(x) = 0 \end{aligned}, \tag{2.33}$$

and let

$$\mathcal{C} : [\Xi]^3 \times L^2(\Sigma_1) \times \cdots \times L^2(\Sigma_{m+1}) \to L^2(\Sigma_1) \times L^2(\Sigma_2) \times \cdots \times L^2(\Sigma_{m+1})$$

be the LHS of (2.32). Note that the null space of $\mathcal{C}$, which we denote $\mathcal{C}_0$, consists of those surface electric potentials and their associated sources that satisfy the conditions

(2.7) and (2.10). In other words, we look for solutions to inverse problem of MEG, EEG and/or sEEG in $\mathcal{C}_0$. For the discrete setting we rewrite (2.23) as

$$\left\{ -\sigma_0 \mathbf{\Phi}(\boldsymbol{\xi})(X_k) + \Big(\sigma_k^- \mathbf{I} + (\sigma_k^- - \sigma_k^+)\mathbf{H}_k(X_k)\Big)\mathbf{\Phi}_k + \sum_{i=1, i\neq k}^{m+1} (\sigma_i^- - \sigma_i^+)\mathbf{H}_i(X_k)\mathbf{\Phi}_i = \mathbf{0} \right\}_{k=1,2,\ldots,m+1} \tag{2.34}$$

or

$$\begin{aligned} &\left\{ -\sigma_0 \mathbf{\Phi}(\boldsymbol{\xi})(X_k) + \Big(\sigma_k^- \mathbf{I} + (\sigma_k^- - \sigma_k^+)\mathbf{H}_k(X_k)\Big)\mathbf{\Phi}_k + \sum_{i=1, i\neq k}^{m+1} (\sigma_i^- - \sigma_i^+)\mathbf{H}_i(X_k)\mathbf{\Phi}_i = \mathbf{0} \right\}_{k=1,2,\ldots,m} \\ &-\sigma_0 \partial_{\nu_{m+1}} \mathbf{\Phi}(\boldsymbol{\xi})(X_{m+1}) + \sum_{i=1}^{m+1} (\sigma_i^- - \sigma_i^+)\mathbf{N}_i(X_{m+1})\mathbf{\Phi}_i = \mathbf{0}, \end{aligned}$$

where $\sigma_0 \mathbf{\Phi}(\boldsymbol{\xi})(X_k)$ and $\sigma_0 \partial_{\nu_{m+1}} \mathbf{\Phi}(\boldsymbol{\xi})(X_{m+1})$ are appropriate discretisation of (2.3) and $\sigma_0 \partial_{\nu_{m+1}} \phi(x)$, respectively. Hence, (2.34) gives the discrete version of $\mathcal{C}$ on the discretised surfaces $\Sigma_{i,T}$, $i = 1, 2, \ldots, m+1$. The LHS of (2.34) will be denoted $\mathbf{C}$ and the null space of $\mathbf{C}$ by $\mathbf{C}_0$.

We briefly discuss the sBEM for the inverse cortical mapping problem so as to compare it with what we have just discussed above. The fact that the electric potential is harmonic outside the support of the source $\boldsymbol{\xi}$, allows Theorem 27 to be used to express the electric potential in a manner that excludes the unknown source. A linear system can be built from the Theorem 27, see [68], that imposed the required regularity on the electric potential and the normal currents. In [68] EEG data is used to perform the inverse cortical mapping problem and in order to use this data they introduce an operator that can reproduce the EEG data using the surface potentials, ϕ_i, and normal currents, $\partial_{\nu_i}\phi$, $i = 1, 2, \ldots, m+1$. Note that this operator excludes the unknown source as well.

The approach we have taken to impose the desired regularity of the electric potential and the normal currents involves the unknown source. Hence, we can use forward models (2.14) and (2.27) in solving the inverse cortical mapping problem to express the electric potential and magnetic flux density, respectively, at the points where data is captured.

2.5.1 EEG and sEEG problems

In the inverse source localisation problems that use the electric potential, there are two regimes, one that uses the electrical data measured on the scalp as is done for EEG and another that uses intra-cranial electric potential recording as is done for sEEG. In both these instances, we wish to solve the problem that given point-wise measurements

of the electric potential, ϕ, in some subset of $\mathbb{R}^3$, find $\boldsymbol{\xi} \in [\Xi]^3$ and $\phi_i \in L^2(\Sigma_i)$, $i = 1, 2, \ldots, m+1$, such that (2.8) is satisfied.

Due to the existence of silent sources for EEG and sEEG as highlighted in Sec. 2.4.3, we can only recover $\boldsymbol{\xi} \in [\Xi]^3$ up to silent sources. Further, since we only have point-wise data for EEG and sEEG, the unique recovery of $\boldsymbol{\xi} \in [\Xi]^3$ is also negatively impacted by the existence of $\boldsymbol{\xi} \in [\Xi]^3$ that generate an electric potential that vanishes at the measurement points. Due to the non-uniqueness of solutions highlighted here these inverse problems are ill-posed, hence we solve Tikhonov regularised problems.

Keeping in mind that the electric potential that is reconstructed from the point-wise measurements has to have the regularity stipulated in (2.7) and (2.10). As indicated earlier we look for solutions in the null space of $\mathcal{C}_0$, which is exactly solving the cortical mapping problem. We build the problems in such a way that they incorporate the regularity requirements of the potential and normal current. There are a multitude of ways this can be achieved, for example, in [68] the authors used a projector onto $\mathbf{C}_0$ and looked for solutions directly in $\mathbf{C}_0$. In our case we will use the projection onto $\mathbf{C}_0$ as a regulariser in the associated Tikhonov problems.

2.5.2 MEG problem

In the simplest terms the inverse source localisation problem of MEG can be stated as follows, given $\mathbf{B}$ on $\mathbb{R}^3 \backslash \overline{\Omega}$ find $\boldsymbol{\xi} \in [\Xi]^3$ and ϕ in $\overline{\Omega}$ such that (2.27) holds. Again in this case, due to the existence of silent sources for MEG as highlighted in Sec. 2.4.3, we can only recover the source, $\boldsymbol{\xi} \in [\Xi]^3$, up to silent sources. In practice, $\mathbf{B}$ is known only point-wise and only a component of it is considered with the additional issue that the electric potential ϕ in $\overline{\Omega}$ is also unknown. Similar observation and consideration about cortical mapping can be made here and appropriate changes akin to those made to the source localisation problem for EEG and/or sEEG can also be made to the MEG source localisation problem. All these considerations greatly increases the non-uniqueness of the problem and hence we have to solve a Tikhonov regularised problem as well.

2.5.3 Existence of solutions to the sEEG, EEG and MEG problems

As highlighted above when solving the inverse problems for sEEG, EEG and MEG we aim to recover the source $\boldsymbol{\xi} \in [\Xi]^3$ and by extension the surface electric potentials $\phi \in L^2(\Sigma_i)$, $i = 1, 2, \ldots, m+1$. We now mathematically set up these problems and show that the solutions exist. To that end let $\mathfrak{S}$ be the product Banach space

$$[\Xi]^3 \times L^2(\Sigma_1) \times L^2(\Sigma_2) \times \cdots \times L^2(\Sigma_{m+1})$$

endowed with the norm

$$\|\cdot\|_{[\Xi]^3} + \|\cdot\|_{L^2(\Sigma_1)} + \|\cdot\|_{L^2(\Sigma_3)} + \cdots + \|\cdot\|_{L^2(\Sigma_{m+1})},$$

and let

$$\mathfrak{D}_1 = L^2(D),$$

be the Hilbert space which corresponds to the data measured with EEG, MEG and sEEG measurements with $D \subset \mathbb{R}^3$ depending on the measurement modalities used. Finally we would let the product Hilbert space

$$\mathfrak{D}_2 = L^2(\Sigma_1) \times L^2(\Sigma_2) \times \cdots \times L^2(\Sigma_m) \times L^2(\Sigma_{m+1}),$$

endowed with the norm

$$(\|\cdot\|^2_{L^2(\Sigma_1)} + \|\cdot\|^2_{L^2(\Sigma_3)} + \cdots + \|\cdot\|^2_{L^2(\Sigma_{m+1})})^{1/2},$$

which is useful in the study of (2.32). Let $\boldsymbol{\mathcal{F}}_i : \mathfrak{S} \to \mathfrak{D}_i$ for $i = 1, 2$ be linear operators with $\boldsymbol{\mathcal{F}}_1$ as the forward model of EEG, MEG and/or sEEG, that is, the formulas given in by (2.8), (2.11) and/or (2.8), respectively and $\boldsymbol{\mathcal{F}}_2 = \mathcal{C}$. Thus, given f as data and $R : [0, \infty) \to [0, \infty)$ a convex function, when solving the inverse problems for MEG, EEG and/or sEEG we consider the appropriate functional

$$\begin{aligned}\mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{m+1}}(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1}) &:= \alpha\|\boldsymbol{\mathcal{F}}_1(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1}) - f\|^2_{\mathfrak{D}_1} \\ &+ \beta\|\boldsymbol{\mathcal{F}}_2(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1})\|^2_{\mathfrak{D}_2} + \lambda_0 R(\|\boldsymbol{\xi}\|_{[\Xi]^3}) + \sum_{j=1}^{m+1} \lambda_j \|\phi_j\|^2_{L^2(\Sigma_j)}.\end{aligned} \tag{2.35}$$

Remark 11. *Since it is cumbersome to write $\mathcal{T}_{f,\alpha,\beta,\lambda_0,\ldots,\lambda_{m+1}}$ here after we will simply write $\mathcal{T}$ keeping in mind the dependence of $\mathcal{T}$ on $f, \alpha, \beta, \lambda_0, \ldots, \lambda_{m+1}$.*

Thus we solve the following problem :

Problem 3. *Given data $f \in \mathfrak{D}_1$ and $\alpha, \beta, \lambda_j > 0$ find $(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1})_\lambda \in \mathfrak{S}$ such that*

$$(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1})_\lambda = \underset{(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1}) \in \mathfrak{S}}{\arg\inf} \mathcal{T}(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1}). \tag{2.36}$$

Remark 12. *We will prove the existence of a solution to Problem 3 in Theorem 28 below keeping in mind that we are mostly interested in $[\Xi]^3$ that is either $[W^{1/2,2}(\Sigma_0)]^3$ or $[\mathcal{M}(\Sigma_0)]^3$ or $[\mathcal{M}(\text{int } \Sigma_0)]^3$, with $\mathcal{M}$ being the Banach space space of measures endowed with the total variation norm. By using embeddings of various Sobolev space into the space of continuous function we view spaces of measures as being contained in the duals of certain Sobolev spaces. We take the view that $[\mathcal{M}(\text{int } \Sigma_0)]^3 \subset (W^{1,q}[(\text{int } \Sigma_0))^*]^3$ for $q > 3$ from the Sobolev embedding theorem, see for example [10, Thm 5.4, Part II] hence we will discuss about the Newton potential in these spaces. We also have $[\mathcal{M}(\Sigma_0)]^3 \subset [(W^{1-\frac{1}{q},q}(\Sigma_0))^*]^3$ for appropriate choices of q, see for example [23, Thm 4.57, Thm 4.58] states that $W^{1,q}(\Omega)$ embeds into a space of functions continuous on $\overline{\Omega}$ hence the traces $W^{1-\frac{1}{q},q}(\Sigma_0)$ are continuous. In the case of $[W^{1/2,2}(\Sigma_0)]^3$ and $[\mathcal{M}(\Sigma_0)]^3$ we need only look at how the layer potentials behave.*

Theorem 28. *A unique solution to Problem 3 exists.*

Proof. We use the result [35, Thm 3.1] to make this conclusion hence we need only show that Problem 3 satisfies the assumptions of [35, Thm 3.1].

(1) We note that the duals of $\mathfrak{S}$ and $\mathfrak{D}_2$ are isometric to $([\Xi]^3)^* \times L^2(\Sigma_1) \times L^2(\Sigma_2) \times \cdots \times L^2(\Sigma_{m+1})$ and $L^2(\Sigma_1) \times L^2(\Sigma_2) \times \cdots \times L^2(\Sigma_{m+1})$, respectively, and we will associate $\mathfrak{S}$ with its *weak** topology, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ with their *weak* topologies.

(2) From [11, Prop. 3.5] we have that norms are weakly lower semi-continuous hence the norm of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are weakly lower semi-continuous.

(3) We will discuss the continuity of $\boldsymbol{\mathcal{F}_1}$ and $\boldsymbol{\mathcal{F}_2}$.

(i) We discuss the continuity of $\boldsymbol{\mathcal{F}}_1$ corresponding to (2.8) and $\boldsymbol{\mathcal{F}}_2$ which corresponds to (2.32) because of the similarities.

(a) When $\Xi = W^{s,p}(\partial\Omega)$ for $0 < s < 1$, $1 < p < \infty$, the double layer potential from $W^{s,p}(\partial\Omega)$ into $W^{1+\frac{1}{p}-s,p}(\Omega)$ for $0 < s < 1$, $1 < p < \infty$, is continuous, see [25, Thm 4.1].

(b) For the case $\Xi = (W^{1-s,p}(\partial\Omega))^*$, we begin by noting that (2.3) can be rewritten

$$\sigma\phi(\boldsymbol{\xi})(x) = \frac{1}{4\pi}\int_{\partial\Omega} \frac{\boldsymbol{\xi}(y)\cdot(x-y)}{|x-y|^3}\, d\mathcal{H}(y),$$

which can be seen as the sum of the quantities

$$\frac{1}{4\pi}\int_{\partial\Omega} \frac{\xi_i(y)(x-y)_i}{|x-y|^3}\, d\mathcal{H}(y),$$

for $i = 1, 2, 3$, each of the which is the i-th component of the gradient of the single layer potential $\mathcal{S}\xi_i(x)$. The single layer potential maps $W^{-s,p}(\partial\Omega)$ into $W^{1+\frac{1}{p}-s,p}(\Omega)$ for $0 < s < 1$, $1 < p < \infty$ continuously and ∇ maps $W^{s,p}(\Omega)$ to $W^{s-1,p}(\Omega)$ continuously for $s > 0$, $1 < p < \infty$. Hence $\phi(\boldsymbol{\xi})$ maps $(W^{1-s,p}(\partial\Omega))^*$ to $W^{\frac{1}{q}-s,q}(\Omega)$ for $0 < s < 1$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, continuously.

(c) When $\Xi = (W^{1+s,q}(\Omega))^*$ for $-1 \le s \le 2$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, from the continuity of the Newton potential from the distributions $(W^{1+s,q}(\Omega))^*$ to $W^{1-s,p}(\Omega)$, see [25, Prop. 2.1], and the continuity of the double layer potential from $W^{s,p}(\partial\Omega)$ into $W^{1+\frac{1}{p}-s,p}(\Omega)$ with Ω a bounded Lipschitz domain.

(ii) We now discuss the continuity of $\boldsymbol{\mathcal{F}_1}$ corresponding to (2.11).

(a) When $\Xi = W^{s,p}(\partial\Omega)$ for $0 \le s \le 1$ and $1 < p < \infty$, we begin by noting that

$$\{(\nu \times \nabla)f : f \in W^{1-s,p}(\partial\Omega)\} \subseteq W^{-s,p}(\partial\Omega),$$

see [74, (4.7)]. From [74, Lem. 4.3] and its proof we conclude that each component of the terms that appear in the sum of (2.11) is continuous since [25, Thm 3.1] shows the single layer potential is continuous from $W^{-s,p}(\partial\Omega)$ into $W^{1+\frac{1}{p}-s,p}(\Omega)$ for $0 < s < 1$, $1 < p < \infty$.

(b) When $\Xi = (W^{1-s,p}(\partial\Omega))^*$, we note that (2.5) can be rewritten as

$$\mathbf{A}(\boldsymbol{\xi})(x) = \frac{\mu}{4\pi}\int_{\partial\Omega}\frac{\boldsymbol{\xi}(y)}{|x-y|}\,d\mathcal{H}(y),$$

thus each component is a single layer potential. Since the single layer potential maps $W^{-s,p}(\partial\Omega)$ into $W^{1+\frac{1}{p}-s,p}(\Omega)$ for $0 < s < 1$, $1 < p < \infty$ continuously, we have that $\mathbf{B}(\boldsymbol{\xi})$ maps $(W^{1-s,p}(\partial\Omega))^*$ to $W^{\frac{1}{q}-s,q}(\Omega)$ for $0 < s < 1$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, continuously in each component.

(c) For the case $\Xi = (W^{1+s,q}(\Omega))^*$ for $-1 \le s \le 2$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, note that the first term on the RHS of (2.11) is the curl of the Newton potential. From (2.5) and (2.6) it follows that $\mathbf{B}(\boldsymbol{\xi})$ is continuous from $(W^{1+s,q}(\Omega))^*$ to $W^{-s,p}(\Omega)$ in each component for $-1 \le s \le 2$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

By choosing s and p appropriately we can conclude that $\boldsymbol{\mathcal{F}_1}$ and $\boldsymbol{\mathcal{F}_2}$, are continuous from $\mathfrak{S}$ to $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively in their *weak* topologies precised in (1).

(4) In what follows we denote by $\mathrm{dom}(\boldsymbol{\mathcal{F}_1}, \boldsymbol{\mathcal{F}_2})$, $\mathrm{dom}(\boldsymbol{\mathcal{F}_1}) \cap \mathrm{dom}(\boldsymbol{\mathcal{F}_2}) \subset \mathfrak{S}$, where $\mathrm{dom}(\boldsymbol{\mathcal{F}})$ denotes the domain of $\boldsymbol{\mathcal{F}}$ in $\mathfrak{S}$. Note that

$$\mathfrak{R} = \lambda_0 R(\|\boldsymbol{\xi}\|_{[\Xi]^3}) + \sum_{j=1}^{m+1}\lambda_j\|\phi_j\|^2_{L^2(\Sigma_j)}$$

is *weak** lower semi-continuous in $\mathfrak{S}$ from [11, Prop. 3.13]. Note that the zero element, $\mathbf{0}$, of $\mathfrak{S}$ is in $\mathrm{dom}(\boldsymbol{\mathcal{F}_1}, \boldsymbol{\mathcal{F}_2})$, hence we have that $\mathfrak{R}(\mathbf{0}) = 0$ hence $\mathfrak{R}$ is finite for at least one element of $\mathrm{dom}(\boldsymbol{\mathcal{F}_1}, \boldsymbol{\mathcal{F}_2})$, that is, $\mathfrak{R}$ is proper. Finally, $\mathfrak{R}$ is convex since it is sum of convex function of norms.

(5) We now show the that $\mathrm{dom}(\boldsymbol{\mathcal{F}_1}, \boldsymbol{\mathcal{F}_2})$ is *weak** closed. We will again do this in two steps, first for the electrical potential and then for the magnetic flux density.

(i) Note that for the electric potential, the domain $[\Xi]^3$ is $[(W^{1+s,q}(\Omega))^*]^3$ or $[(W^{-s-\frac{1}{p},p}(\partial\Omega))]^3$ for $-1 \le s \le 2$, $1 < p, q < \infty$ and these are *weak** closed, see for example [11, Thm 3.33].

(ii) For the magnetic flux density we require that $[\Xi]^3$ is the set of elements $\boldsymbol{\xi} \in [(W^{1+s,q}(\Omega))^*]^3$ for $-1 \le s \le 2$, $1 < p, q < \infty$ with $\nabla \times \boldsymbol{\xi} \in [(W^{1+s,q}(\Omega))^*]^3$ which is a closed subspace of $[(W^{1+s,q}(\Omega))^*]^3$ or $[\Xi]^3$ is $\boldsymbol{\xi} \in [(W^{-s-\frac{1}{p},p}(\partial\Omega))]^3$. In either case we have *weak** closed sets, see for example [11, Thm 3.33].

Hence $\mathrm{dom}(\boldsymbol{\mathcal{F}_1}, \boldsymbol{\mathcal{F}_2}) \subset \mathfrak{S}$ is *weak** closed.

(6) Let $\gamma > 0$ and $\kappa > 0$ and set one of $\alpha, \beta, \lambda_0, \ldots, \lambda_{m+1}$ equal to γ define the set

$$\mathfrak{D} = \{\mathbf{u} \in \mathrm{dom}(\boldsymbol{\mathcal{F}_1}, \boldsymbol{\mathcal{F}_2}) \subset \mathfrak{S} : \mathcal{T}(\mathbf{u}) \le \kappa\}.$$

Since $\mathcal{T}(\mathbf{u})$ is continuous on $\text{dom}(\boldsymbol{\mathcal{F}_1}, \boldsymbol{\mathcal{F}_2}) \subset \mathfrak{S}$ it follows that $\mathfrak{D}$ is a bounded set hence *weak** compact from Banach-Alaoglu-Bourbaki Thm, see for example [11, Thm 3.16].

Conditions (1)-(6) above show that Problem 3 satisfies the assumption of [35, Thm 3.1] hence a solution exists and the solution is unique due to the convexity of $\mathcal{T}$. □

Remark 13. *Note that Problem 3 does not discuss the contamination of the data with noise. In practice $f \in \mathfrak{D}_1$ is contaminated with noise and we assume that the noisy data $f^\delta \in \mathfrak{D}_1$ and $\|f - f^\delta\|_{\mathfrak{D}_1} \leq \delta$, $\delta > 0$. The stability and convergence of Problem 3 with respect to noise and regularisers $\alpha, \beta, \lambda_0, \ldots, \lambda_{m+1}$ is also discussed in [35], see especially [35, Thm 3.4 and 3.5].*

2.5.4 Alternating minimisation algorithm for solving inverse problems

We wish to solve Problem 3 by splitting it into a problem that solves for the source $\boldsymbol{\xi} \in [\Xi]^3$ and a problem that solves for the surface electric potentials $\phi \in L^2(\Sigma_i)$. We iteratively solves these problems to obtain the solution we desire as discussed in the introduction of [76]. The alternating minimisation procedure is as follows, beginning with some initial guess

$$(\boldsymbol{\xi}^{\{0\}}, \phi_1^{\{0\}}, \phi_2^{\{0\}}, \ldots, \phi_{m+1}^{\{0\}}),$$

then generate a sequence of solutions

$$\left\{(\boldsymbol{\xi}^{\{l\}}, \phi_1^{\{l\}}, \phi_2^{\{l\}}, \ldots, \phi_{m+1}^{\{l\}})_\lambda\right\}_{l \in \mathbb{N}},$$

by solving the following problems

$$\begin{aligned} \boldsymbol{\xi}_\lambda^{\{l+1\}} &= \underset{\boldsymbol{\xi} \in [\Xi]^3}{\arg\inf}\, \mathcal{T}(\boldsymbol{\xi}, \phi_1^{\{l\}}, \phi_2^{\{l\}}, \ldots, \phi_{m+1}^{\{l\}}) \\ (\phi_1^{\{l+1\}}, \phi_2^{\{l+1\}}, \ldots, \phi_{m+1}^{\{l+1\}})_\lambda & \\ &= \underset{(\phi_1, \ldots, \phi_{m+1}) \in \boldsymbol{\mathfrak{D}}_2}{\arg\inf}\, \mathcal{T}(\boldsymbol{\xi}^{\{l\}}, \phi_1, \phi_2, \ldots, \phi_{m+1}). \end{aligned} \tag{2.37}$$

A closer inspection of $\boldsymbol{\mathcal{F}}_1$ and $\boldsymbol{\mathcal{F}}_2$ as we proposed be taken reveals that

$$\begin{aligned} \boldsymbol{\mathcal{F}}_1(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1}) &= \mathcal{F}_{1,1}(\boldsymbol{\xi}) + \mathcal{F}_{1,2}(\phi_1, \phi_2, \ldots, \phi_{m+1}) \\ \boldsymbol{\mathcal{F}}_2(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1}) &= \mathcal{F}_{2,1}(\boldsymbol{\xi}) + \mathcal{F}_{2,2}(\phi_1, \phi_2, \ldots, \phi_{m+1}), \end{aligned} \tag{2.38}$$

hence the problems that we solves in (2.37) have the variables $\boldsymbol{\xi} \in [\Xi]^3$ and $(\phi_1, \phi_2, \ldots, \phi_{m+1}) \in \boldsymbol{\mathfrak{D}}_2$ well separated in $\boldsymbol{\mathcal{F}}_1$ and $\boldsymbol{\mathcal{F}}_2$ hence we can implement methods that best recover each variable.

We now show that the sequence

$$\left\{(\boldsymbol{\xi}^{\{l\}}, \phi_1^{\{l\}}, \phi_2^{\{l\}}, \ldots, \phi_{m+1}^{\{l\}})_\lambda\right\}_{l \in \mathbb{N}},$$

generated by the above alternating minimisation algorithm results in the objective function converging linearly to the minimum of Problem 3. The result below is valid for all $[\Xi]^3$ outline in Remark 12 and Theorem 28.

Theorem 29. *The sequence generated by the alternating minimisation algorithm converges linearly to the minimum of the Tikhonov functional $\mathcal{T}$.*

Proof. Take

$$\mathfrak{B}_1 = [\Xi]^3,$$

with

$$\|\cdot\|_1 = \|\cdot\|_{[\Xi]^3},$$

and

$$\mathfrak{B}_2 = \mathfrak{D}_2,$$

with

$$\|\cdot\|_2 = (\|\cdot\|^2_{L^2(\Sigma_1)} + \|\cdot\|^2_{L^2(\Sigma_3)} + \cdots + \|\cdot\|^2_{L^2(\Sigma_{m+1})})^{1/2}.$$

Further take functions h, g_1, g_2 defined on $\mathfrak{S}$ as

$$h = \alpha\|\boldsymbol{\mathcal{F}}_1(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1}) - f\|^2_{\mathfrak{D}_1} + \beta\|\boldsymbol{\mathcal{F}}_2(\boldsymbol{\xi}, \phi_1, \phi_2, \ldots, \phi_{m+1})\|^2_{\mathfrak{D}_2},$$

$$g_1 = \lambda_0 R(\|\boldsymbol{\xi}\|_{[\Xi]^3}),$$

$$g_2 = \sum_{i=1}^{m+1} \lambda\|\phi_i\|^2_{L^2(\Sigma_i)}.$$

Note that the above problem (2.35) can equivalently be written as the following problem

$$\min\Big\{H(z_1, z_2) \equiv h(z_1, z_2) + g_1(z_1) + g_2(z_2) | (z_1, z_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2\Big\}, \tag{2.39}$$

where $\mathfrak{B}_1, \mathfrak{B}_2, h, g_1, g_2$ satisfy the following conditions from [77]:

(P1) The feasible sets $(\mathfrak{B}_i, \|\cdot\|_i)$ are Banach spaces with duals $(\mathfrak{B}_i^*, \|\cdot\|_{i,*})$ and the duality pairing $\langle\cdot,\cdot\rangle_i$, $i = 1, 2$.

(P2) Since norms of uniformly convex spaces are (Fréchet) differentiable, see (A2) below, the function $g_i : \mathfrak{B}_i \to \mathbb{R} \cup \{\infty\}$ is proper convex, (Fréchet) subdifferentiable with subdifferential ∂g_i on dom g_i, $i = 1, 2$. Let $\mathcal{D} :$ dom $g_1 \times$ dom g_2.

(P3) The function $h : \mathfrak{B}_1 \times \mathfrak{B}_2 \to \mathbb{R}$ is convex and (Fréchet) differentiable over $\mathcal{D}$. For our problems this follows from the fact that norms are convex and norms of Hilbert spaces are (Fréchet) differentiable. Let ∇h denote the (Fréchet) derivative of h.

(P4) Since a solution to Problem 3 exists, the optimal set of the problem (2.39), denoted by $\mathcal{O}^* \subset \mathfrak{B}_1 \times \mathfrak{B}_2$, is non-empty, and the corresponding optimal value is denoted by H^*.

(A1) $\mathfrak{B}_1 \times \mathfrak{B}_2$ is equipped with a separate norm $\|\cdot\|$ which we can take to be the graph norm and $\beta_1, \beta_2 \geq 0$, satisfying

$$\|(z_1, z_2)\|^2 \geq \beta_i\|z_i\|_i^2 \quad \text{for all } (z_1, z_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2,$$

by definitions given above. Furthermore, we equip $\boldsymbol{\mathfrak{B}}_1 \times \boldsymbol{\mathfrak{B}}_2$ with the duality pairing $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2$.

(P5) For any $(\tilde{z}_1, \tilde{z}_2) \in \mathcal{D}$, the following problems have minimizers

$$\min_{z_1 \in \mathfrak{B}_1} H(z_1, \tilde{z}_2), \quad \text{and} \quad \min_{z_2 \in \mathfrak{B}_2} H(\tilde{z}_1, z_2),$$

as a consequence of Theorem 28.

(A2) The partial (Fréchet) derivative of h with respect to the i-th component, denoted by $\nabla_i h \in \mathfrak{B}_i^*$, is Lipschitz continuous with Lipschitz constant $L_i \in (0, \infty]$, $i = 1, 2$. with $\min\{L_1, L_2\} < \infty$; exemplarily, for $i = 1$ (analogously for $i = 2$) it holds that $\|\nabla_1 h(z_1 + y_1, z_2) - \nabla_1 h(z_1, z_2)\|_{1,*} \leq L_1 \|y_1\|_1$ for all $(z_1, z_2) \in \mathcal{D}, y_1 \in \mathfrak{B}_1$, such that $z_1 + y_1 \in \text{ dom } g_1$,
as a consequence of the chain rule of Fréchet differentiation and the fact that L^p, $1 < p < \infty$ are uniformly convex hence uniformly smooth, see for example [38, Part 3, Chap. II, Sec. 1, Prop. 8 and Part 3, Chap. II, Sec. 2, Prop. 2], hence their norms are *uniformly* Fréchet differentiable [38, Part 3, Chap. II, Sec. 2, Prop. 1] and that $\boldsymbol{\mathcal{F}}_1$, $\boldsymbol{\mathcal{F}}_2$ are bounded linear operators hence they are their Fréchet derivatives with respect to $\boldsymbol{\xi}$ and $(\phi_1, \phi_2, \ldots, \phi_{n+1})$.

(A3a) Recall that a function h is *strongly convex* if there exists $\sigma > 0$ such that

$$\langle \nabla h(z) - \nabla h(\overline{z}), \overline{z} - z \rangle \geq \sigma \|z - \overline{z}\|^2,$$

for all $z \in \text{ dom } h$. The function $h : \mathfrak{B}_1 \times \mathfrak{B}_2 \to \mathbb{R}$ is *quasi-strongly convex* with respect to $\mathcal{O}^*$, with modulus $\sigma > 0$, that is, for all $z \in \mathcal{D}$ and $\overline{z} := \arg\min\{\|z - y\| | y \in \mathcal{O}^*\}$, the projection of z onto $\mathcal{O}^*$, it holds

$$h(\overline{z}) \geq h(z) + \langle \nabla h(z), \overline{z} - z \rangle + \frac{\sigma}{2} \|z - \overline{z}\|^2,$$

as a consequence of h being the sum of squares of Hilbert norms each of which is *strongly convex* function with modulus $\sigma > 0$ and the boundedness of $\boldsymbol{\mathcal{F}}_1$, $\boldsymbol{\mathcal{F}}_2$ in $\boldsymbol{\xi}$ and $(\phi_1, \phi_2, \ldots, \phi_{m+1})$.

We have shown that Problem 3 satisfies the assumption of [77, Thm 1], hence the sequence generated by the alternating minimisation algorithm converges linearly to the minimum of the Tikhonov functional $\mathcal{T}$. □

Remark 14. *For the discrete version of Problem 3 we have that the alternating minimisation sequence converges to a solution of the problem as a consequence of the results provided in [78].*

2.5.5 EEG, sEEG and cortical mapping

Numerically, we solve discretised versions of the continuous problems of the previous subsection. Note that in what follows $\boldsymbol{\xi}$ is discretised. In the discrete problem we introduce regularisation matrices $\boldsymbol{\Gamma}_j$ that discretise the regularisers in the continuous problem. For EEG or sEEG we solve the following problem:

Problem 4. *Given point-wise recordings of electric potential,* $\{\phi(Y)\}$*,* $\alpha, \beta, \lambda_j > 0$*, find* $(\boldsymbol{\xi}^*, \boldsymbol{\Phi}^*)$ *such that*

$$(\boldsymbol{\xi}^*, \boldsymbol{\Phi}^*) = \underset{\boldsymbol{\xi}\in[\Xi]^3, \boldsymbol{\Phi}\in\mathbb{R}^{m^*}}{\arg\min} \Big(\alpha\|\mathbf{F}_E(Y)(\boldsymbol{\xi}, \boldsymbol{\Phi}) - \sigma(Y)\phi(Y)\|_2^2 + \beta\|\mathbf{C}(\boldsymbol{\xi}, \boldsymbol{\Phi})\|_2^2 + \lambda_0 R(\|\boldsymbol{\Gamma}_0\boldsymbol{\xi}\|_{[\Xi]^3}) + \sum_{j=1}^{m+1} \lambda_j \|\boldsymbol{\Gamma}_j\boldsymbol{\Phi}\|_2^2\Big), \tag{2.40}$$

where $\mathbf{F}_E$ is an appropriate discretisation of the forward model of the electric potential (2.8) and $\phi(Y)$ is a vector of the potentials at the points $Y = \{y_j\}$ and m^* is the total number of vertices in the triangulation of the cortex, skull and scalp, $\Sigma_{1,T}, \Sigma_{2,T}, \Sigma_{3,T}$, respectively.

2.5.6 MEG and cortical mapping

Numerically, for MEG we will solve the following problem:

Problem 5. *Given point-wise measurements of components of the magnetic flux density,* $\mathbf{B}(Y)\cdot\mathbf{v}(Y)$*,* $\alpha, \beta, \lambda_j > 0$ *find* $(\boldsymbol{\xi}^*, \boldsymbol{\Phi}^*)$ *such that*

$$(\boldsymbol{\xi}^*, \boldsymbol{\Phi}^*) = \underset{\boldsymbol{\xi}\in[\Xi]^3, \boldsymbol{\Phi}\in\mathbb{R}^{m^*}}{\arg\min} \Big(\alpha\|\mathbf{F}_B(Y)(\boldsymbol{\xi}, \boldsymbol{\Phi})\cdot\mathbf{v}(Y) - \mathbf{B}(Y)\cdot\mathbf{v}(Y)\|_2^2 + \beta\|\mathbf{C}(\boldsymbol{\xi}, \boldsymbol{\Phi})\|_2^2 + \lambda_0 R(\|\boldsymbol{\Gamma}_0\boldsymbol{\xi}\|_{[\Xi]^3}) + \sum_{j=1}^{m+1} \lambda_j \|\boldsymbol{\Gamma}_j\boldsymbol{\Phi}\|_2^2\Big), \tag{2.41}$$

where $\mathbf{F}_B$ is an appropriate discretisation of the forward model of the magnetic flux density (2.11) and $\mathbf{B}(Y)\cdot\mathbf{v}(Y)$ is a vector of the $\mathbf{v}$ components of the magnetic flux density at the points $Y = \{y_j\}$ and m^* is the total number of vertices in the triangulation of the cortex, skull and scalp, $\Sigma_{1,T}, \Sigma_{2,T}, \Sigma_{3,T}$, respectively.

2.5.7 EEG, sEEG, MEG and cortical mapping

There is an obvious connections among the EEG, sEEG and MEG source localisation problem and cortical mapping as can be seen in (2.14) and (2.27) hence these problems can be solved in a unified way by making appropriate changes to either (2.40) and (2.41) if simultaneous recordings of EEG, sEEG and MEG are available.

2.5.8 Resolving the discrete problems

In [78] weaker assumptions than those stated in Theorem 29 are given that ensure the discrete problems have unique solutions, in particular the omission of the Fréchet subdifferentiability of g_1 and g_2 in (P2). This weaker set of assumptions ensures sublinear convergence of the objective function to the minimum when the Banach spaces $\mathfrak{B}_i$ are $\mathbb{R}^{n_i}$ for some $n_i \in \mathbb{N}$, $i = 1, 2$. This is particularly useful once we have discretised

the problem such as in the case measure being discretised, as a collection of dipoles. This discretisation of measures results in the *total variation norm* of the measures being the $\|\cdot\|_{[\ell_1(\mathbb{N})]^3}$ of the sequence of Euclidean norms of the dipole moments and this norm is not Fréchet subdifferentiable. This leads to the conclusion that applying an alternating minimisation procedure to the discrete problems for sEEG, EEG and MEG results in obtaining the minimisers that we seek. Note that the solutions we obtain for these problems depend on the regularisation parameters α, β, λ_j. The parameters, α, β, λ_j are chosen such that the influence of the noise in the measured data on the solution is minimimal. Hence, the choice of the parameters α, β, λ_j is of great importance. Since in practise there may be no information on the properties of the noise corrupting data we propose the use of the L-hypersurface technique for choosing appropriate regularisation parameters, see for example [79], for an exploratory study of this technique. The L-hypersurface approach is a generalisation of the L-curve technique, see for example [80]. In short, in the L-hypersurface technique one aims to find the point of maximum Gaussian curvature on a hypersurface that is generated by plotting the data discrepancy of a solution against the regularisation parameters generating the solution; the plotted values are scaled appropriately. It has to be noted that this a computationally expensive technique and other less computationally expensive techniques can be implemented such as one called the minimal distance function technique which is studied in [79].

2.6 Numerical Results

We now present some numerical results of the inverse source localisation problem using EEG, sEEG and MEG data, we also used combined sEEG and MEG data. These numerical results were obtained using code written in MATLAB and the graphics were produced using a MATLAB add-on Toolbox Graph [81]. The meshes used were processed using the MATLAB add-on Iso2Mesh [82]. Iso2Mesh was used to fix defects within the mesh structures such as intersection between meshes and within meshes. The forward problem can be solved satisfactorily with disregard to such defects but the inverse problem is very sensitive to such defects. The processing done in Iso2Mesh resulted in meshes that differed from the ones used in the forward problem thus helping in avoiding the inverse crime. The data we use was generated using OpenMEEG, see [61] and [83], which is based on the boundary elements symmetric method. OpenMEEG uses current dipoles as the elementary electromagnetic object. The dipoles used in OpenMEEG to generate the data were outwardly normally oriented to the grey/white matter interface. In the source recovery problem we attempt to recover the locations of the current dipoles associated with these data by using either $W^{1/2,2}$ functions or a finite collection of dipoles. We attempt the recovery with $[W^{1/2,2}]^3$ vector-field that are normally oriented to the grey/white matter interface or with a collection of dipoles whose locations can only possibly be on the barycentres of the triangles of the meshes hence we need only recover the moments of the dipoles placed on the barycentres all triangles. With these source recovery problems we also solve the cortical mapping problem. Note that for the $W^{1/2,2}$ recovery we can use direct inversion methods to solve the inverse problems however for completeness we used the alternating minimisation

procedure to solve these problems as well.

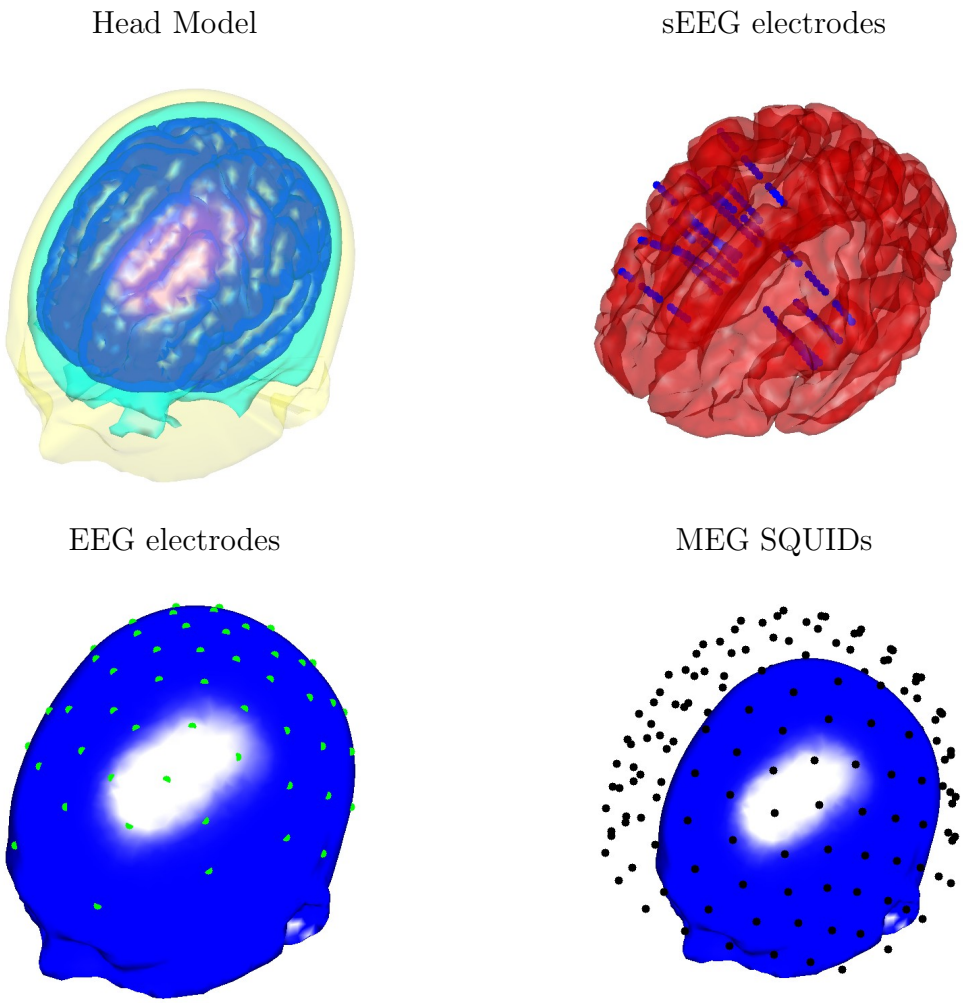

Figure 2.2: Head model and sensor positioning.

In Figure 2.2, we show a head model with4 surfaces, the inner most red surface represents the grey/white matter interface, the dark blue surface represents the cortical surface, the light blue surface represents the outer surface of the skull and the outer most yellow surface represents the scalp. In the same figure we show the location of the 198 sEEG electrodes as blue dots, the majority of them being in the region enclosed by the grey/white matter interface. The 64 EEG electrodes are represented as green dots on the scalp and the locations of the 151 Superconducting QUantum Interference Devices (SQUIDs) for recoding MEG measurements are shown as black dots outside the head.

Figures 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9 and 2.10 show the ground truth (OpenMEEG data) and recovery results obtained from using sEEG, MEG , combined sEEG and MEG and EEG data, respectively. In each figure the first column shows the ground truth data generated by OpenMEEG using the dipoles represented as red dots in the bottom most figure of the first column. The second column of each Figure shows the recovery

results when the source is assumed to be $[W^{1/2,2}]^3$ vector-field normally oriented to the grey/white matter interface and the third column shows the recovery results when the source is assumed to a finite collection of dipoles, the dots representing the dipoles are colour coded by the magnitude of their moments. In the case of $[W^{1/2,2}]^3$ sources, the source is interpreted as being located within a neighbourhood of the maximal valued of the recovered $W^{1/2,2}$ function supported on the grey/white matter interface. For dipoles, the area with the highest concentration of dipoles with moments of largest magnitude is taken to be the area supporting the true source.

From Figures 2.3 and 2.7 we see that the placement of sEEG electrodes has a higher influence on the source recovery compared to EEG and MEG. The figures suggest that the closer the sEEG electrodes are to the true source the better the recovery of the source. We also observe that the recovered surface electrical potentials are overestimated on the skull and the scalp while being underestimated on the cortex however their distribution is in accordance with the recovered source. It seems that the sharper the recovered source the more accurate the recovered surface electrical potentials in terms of their distribution of the surfaces.

In Figures 2.4 and 2.8 we immediately note that the surface electrical potential are underestimated however their distributions are more representative of the ground truth compared to the recovery managed using sEEG data. With MEG data the localisation of the sources is closer to the true sources and also more spatially localised than the ones recovered with sEEG data when the sEEG electrodes are far from the true source. This suggests that for source localisation MEG is more robust than sEEG. That we managed to recover surfaces electric potentials that have the correct distribution with MEG data is short of impressive, especially when look at the recovery of the surface electric potentials with a $[W^{1/2,2}]^3$ source in both figures.

Observing that a better source localisation leads to be a better recovery of the surface electric potentials, it is of interest to see if the robustness of the source localisation of MEG can be complimented with the electric data of sEEG for a better recovery of the source and the surface electric potentials. Figures 2.5 and 2.9 show results obtained by using simultaneous MEG and sEEG data. We can observe that the recovered source and the recovered surface electrical potentials have properties that are shared between the solutions from each modality hence represent a better and more robust recovery. The source recovered with combined MEG and sEEG data is more spatially localised than with standalone modalities and it is located closer to the true source. Further, the recovered surface potential are much more representative of the ground truth than had been previously seen with standalone modalities.

It is evident from Figures 2.6 and 2.10 that the recovery done with EEG data outperforms the other recoveries. The method we implemented here makes the recovery with EEG data robust as it managed a near perfect recovery of both sources and surface electrical potential. In Figure 2.10 we see that the source recovered for $[W^{1/2,2}]^3$ is not as spatially localised as the dipolar source, however, with the interpretation of source we are using, the source location of the $[W^{1/2,2}]^3$ overlaps with the true source location. We think that the "depth" of the source in this case resulted in the poor spatial location we observed here.

What is evident from the numerical results presented here is that the surface electri-

cal potential on the cortex was poorly recovered across all the modalities. We put this to the complicated geometry of the surface in which it folds on itself and when meshed these folds have the potential to creates self intersections in the mesh. Even though we could not show it in the graphics, the maxima of the surface electric potential on the cortex are located on the "sulci" of the cortex in these examples. We managed to recreate these maxima in approximately the correct locations on the sulci of the cortex. The above mentioned self intersections of the meshed cortex introduce numerical errors that the methods we implemented here struggled with handling. Much of the work we put into the numerical implementation of inverse problems was to correct these defects in the meshes which helped in solving the inverse problems; the regularisation helped in minimising the impact of these self intersections that could not be corrected. Some of the intersections of the meshes occur between meshes, in our case there were some between the surface carrying the sources and the cortex and these where much more difficult to correct and had to be taken care of with the regularisation. The extremely negative values of the surface electric potential observed on the cortex in the recovery is due to the intersection of meshes of the grey/white matter interface and the cortex. We did not discuss much about the numerical implementation of the forward model as it was not the major focus of this work, we however observed though that the forward problem is much more resilient to such defects.

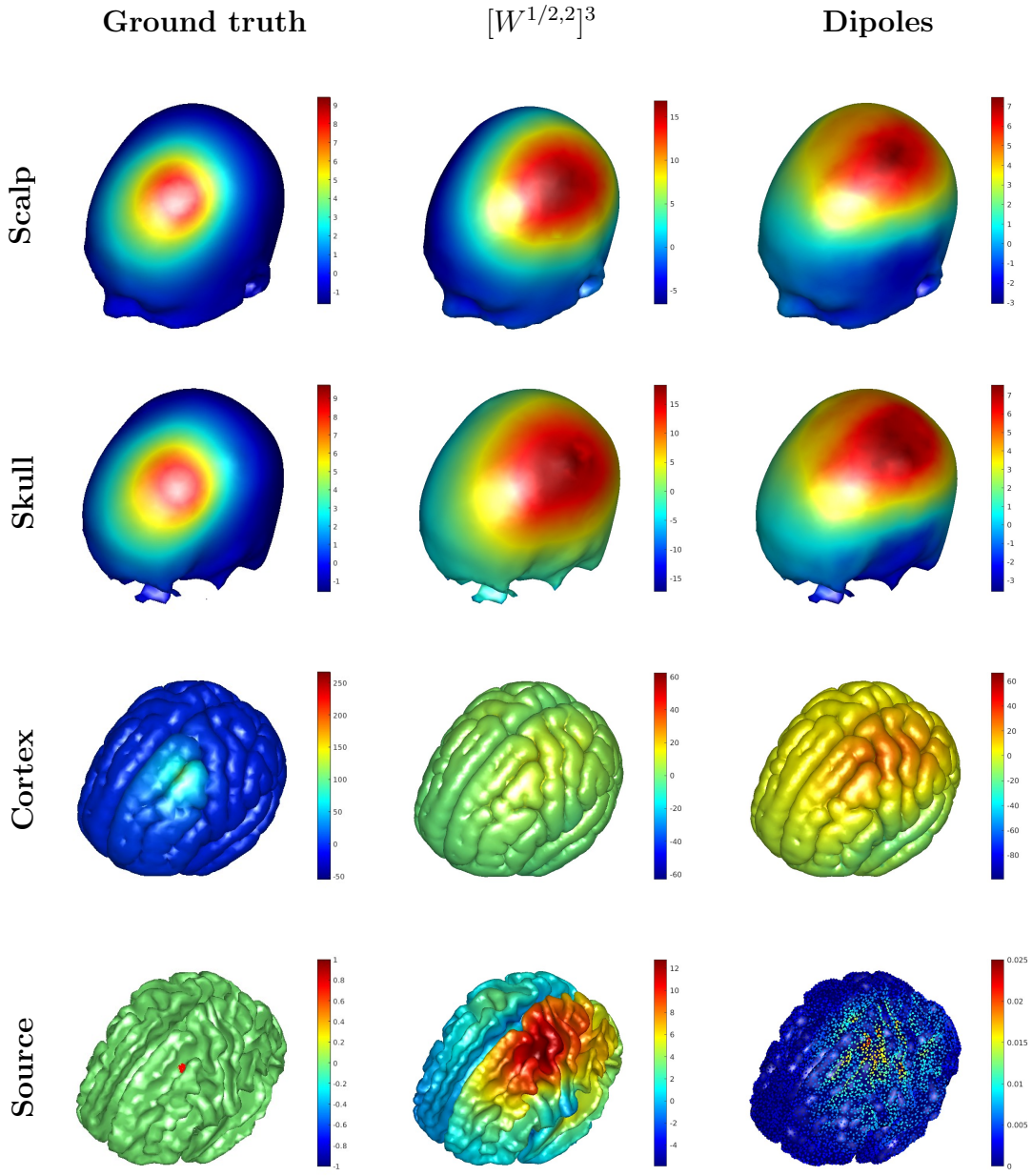

Figure 2.3: Recovery with sEEG data.

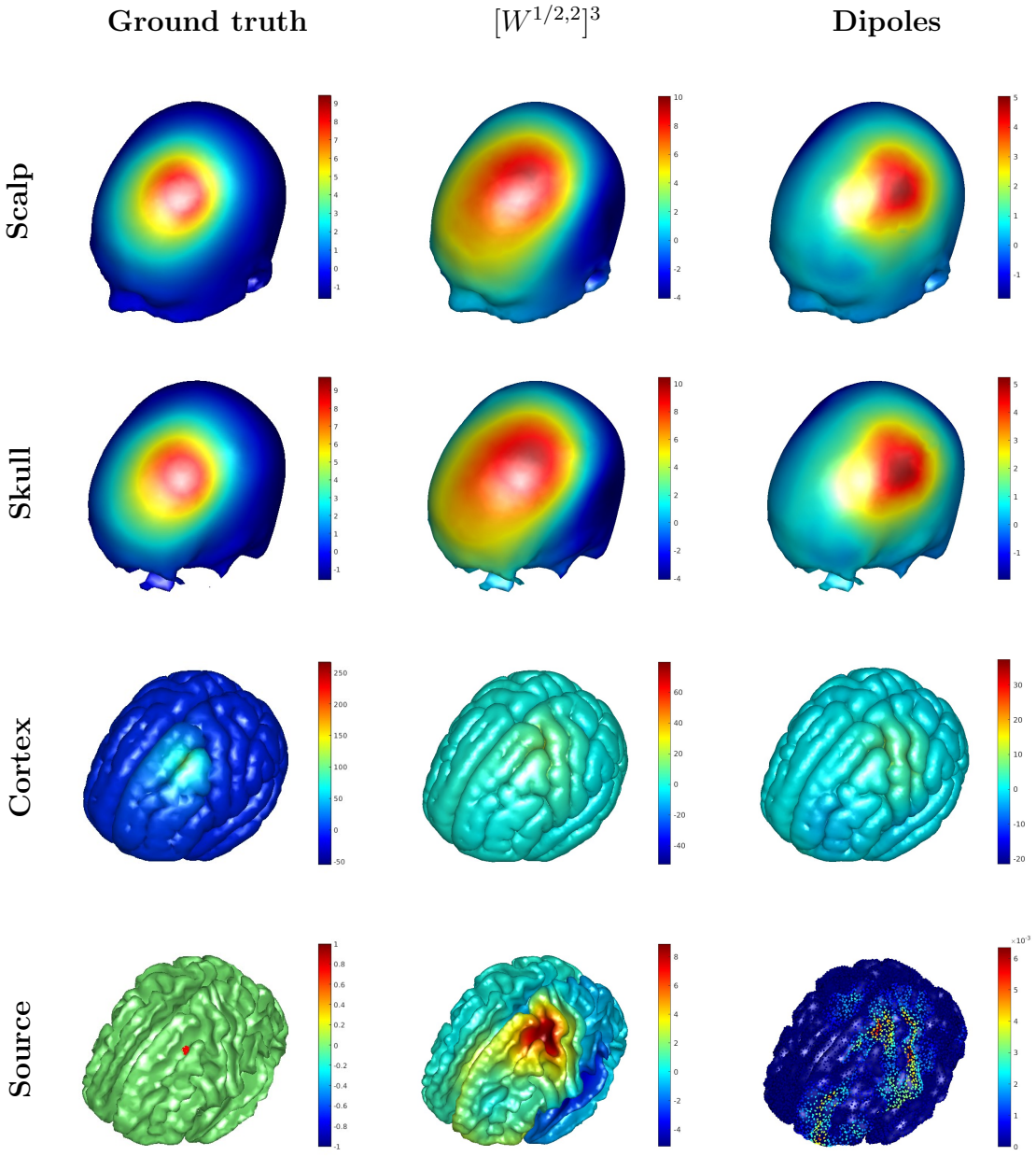

Figure 2.4: Recovery with MEG data.

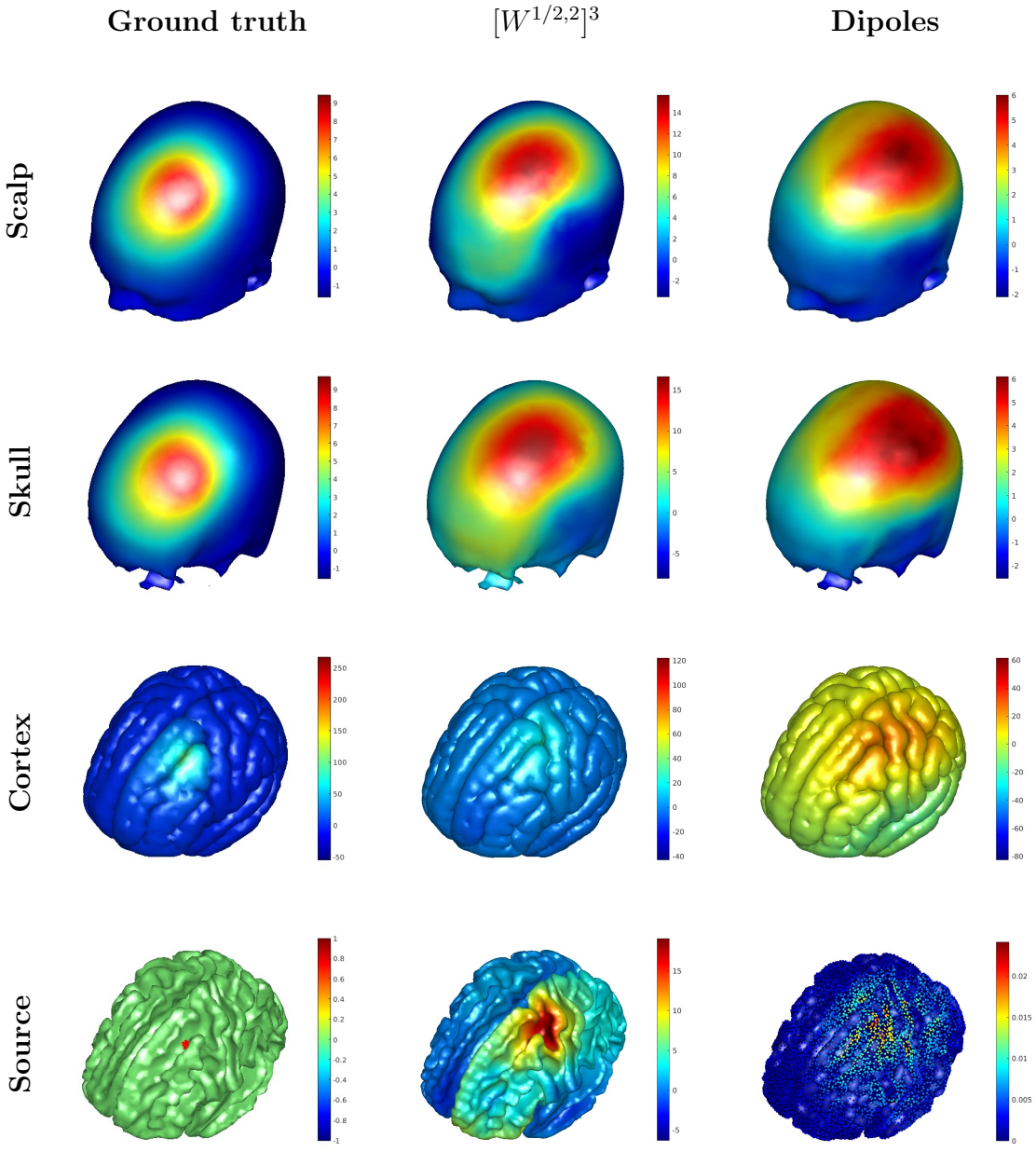

Figure 2.5: Recovery with combined MEG and sEEG data.

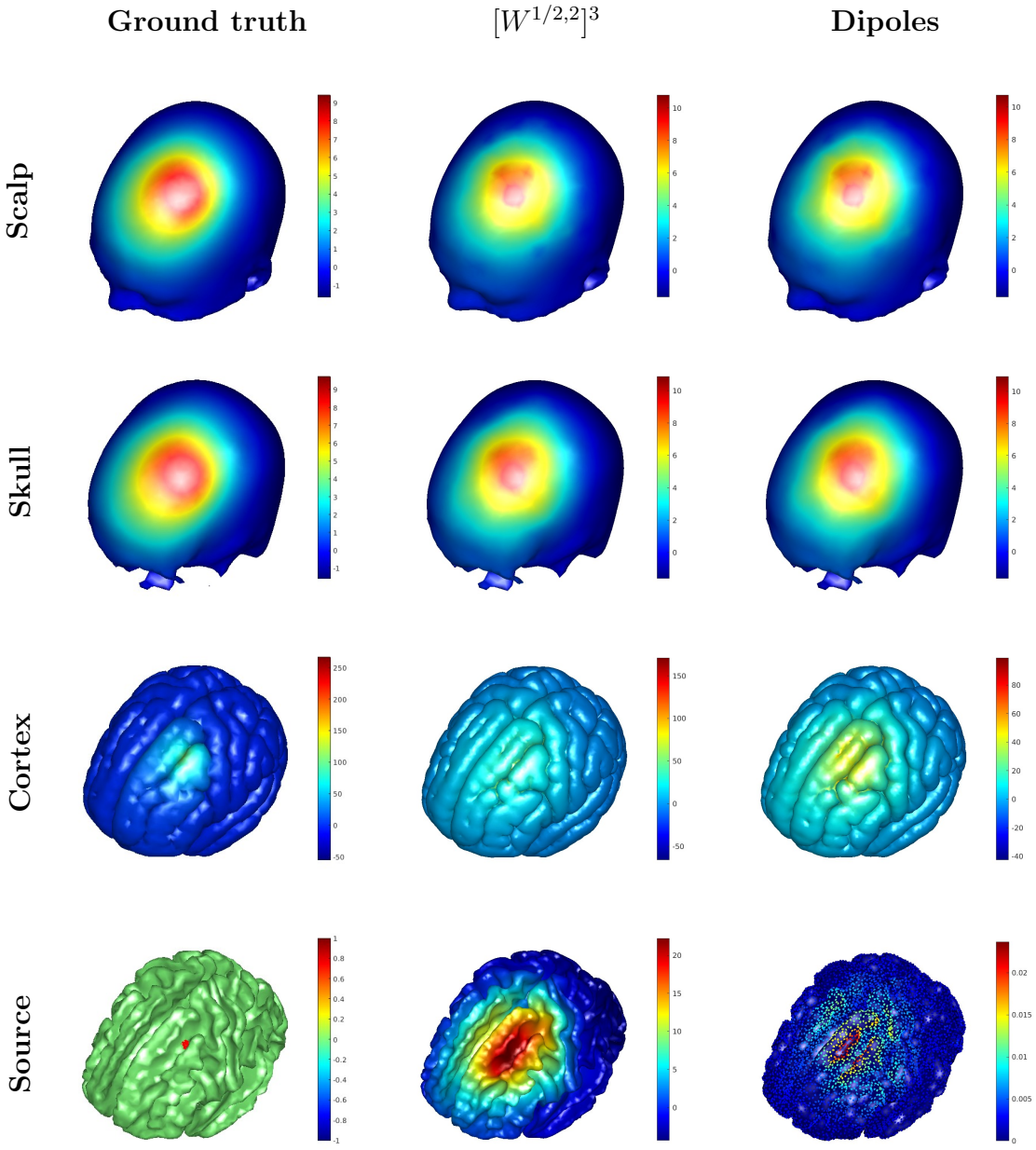

Figure 2.6: Recovery with EEG data.

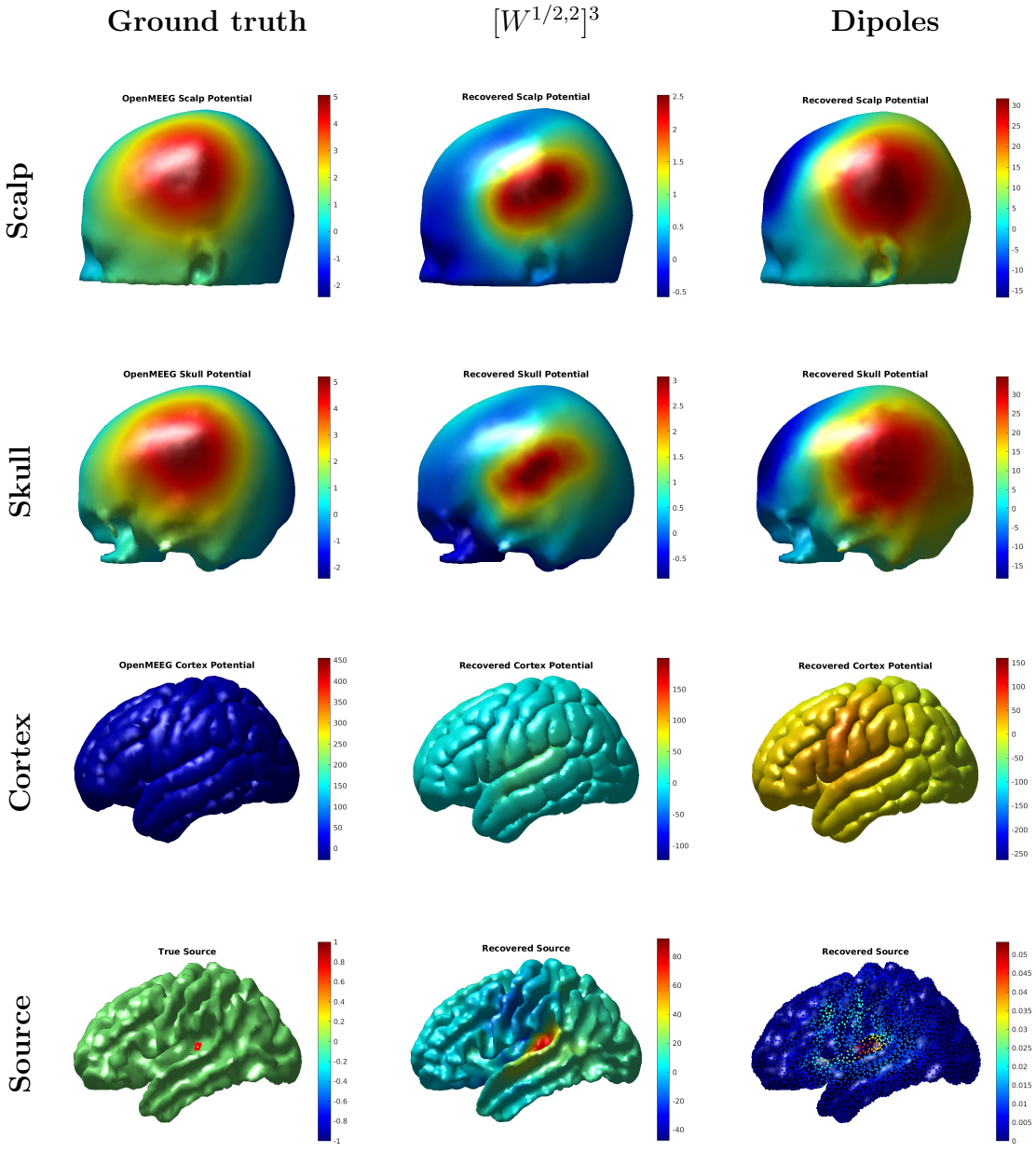

Figure 2.7: Recovery with SEEG data.

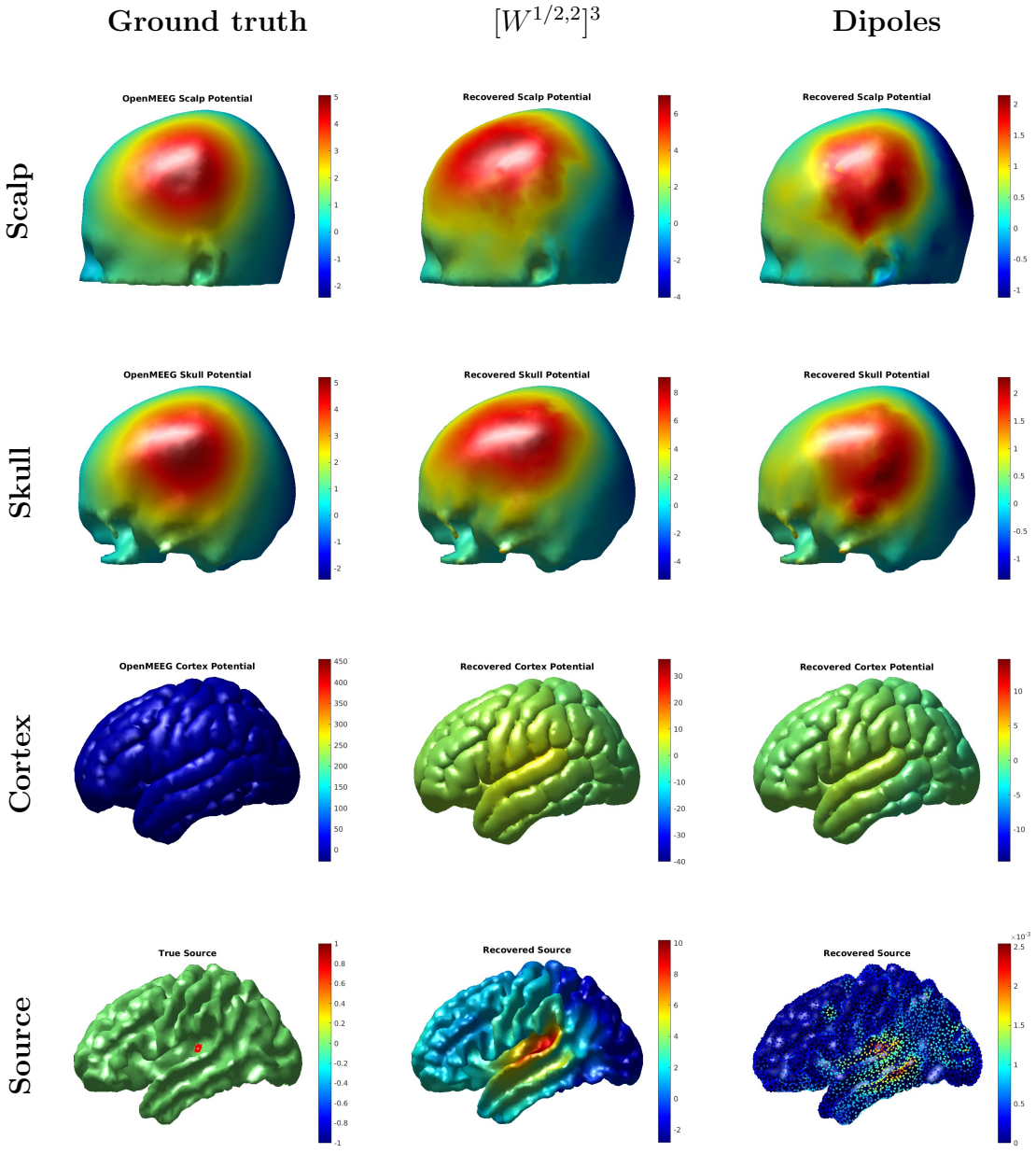

Figure 2.8: Recovery with MEG data.

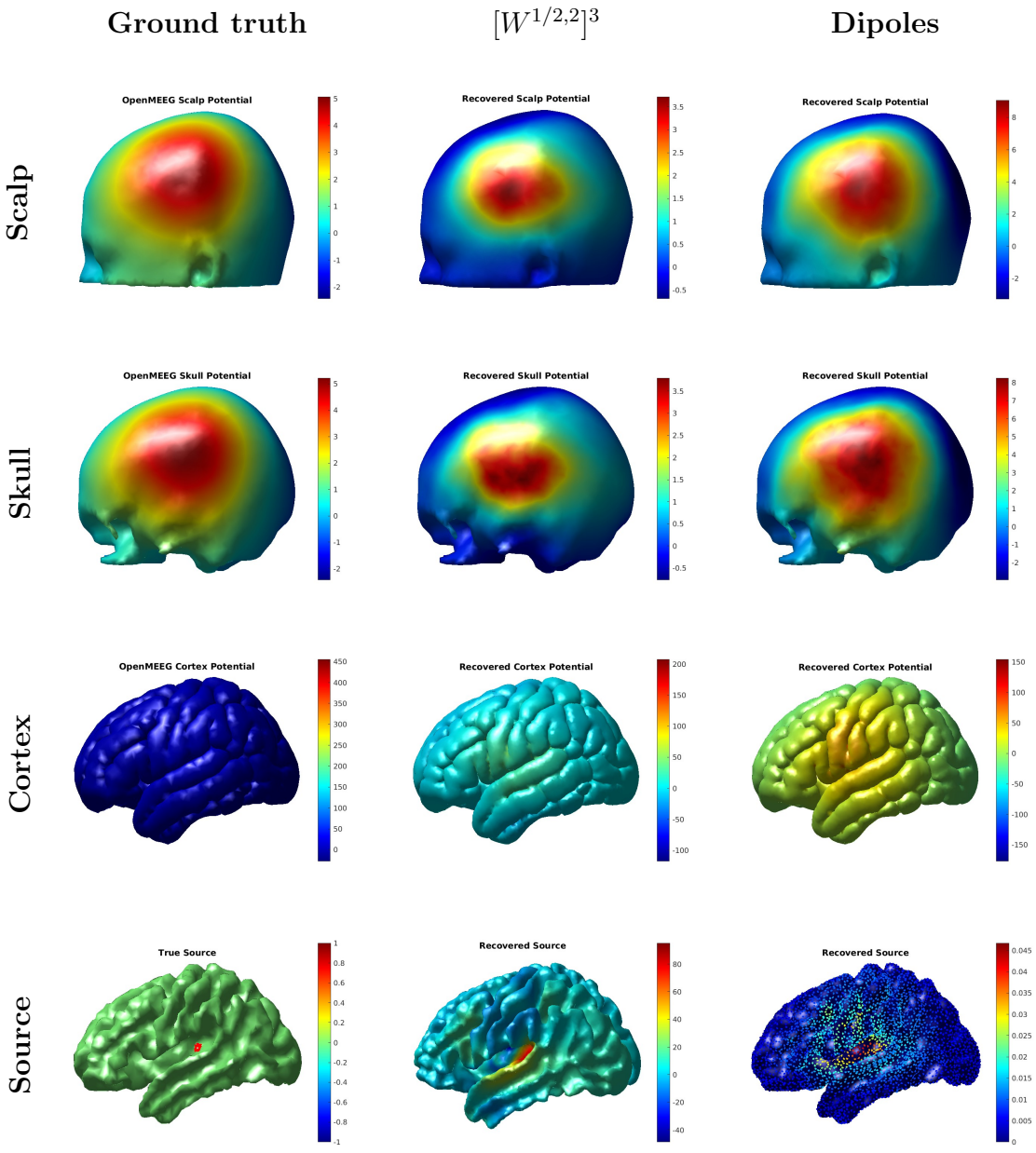

Figure 2.9: Recovery with combined MEG and sEEG data.

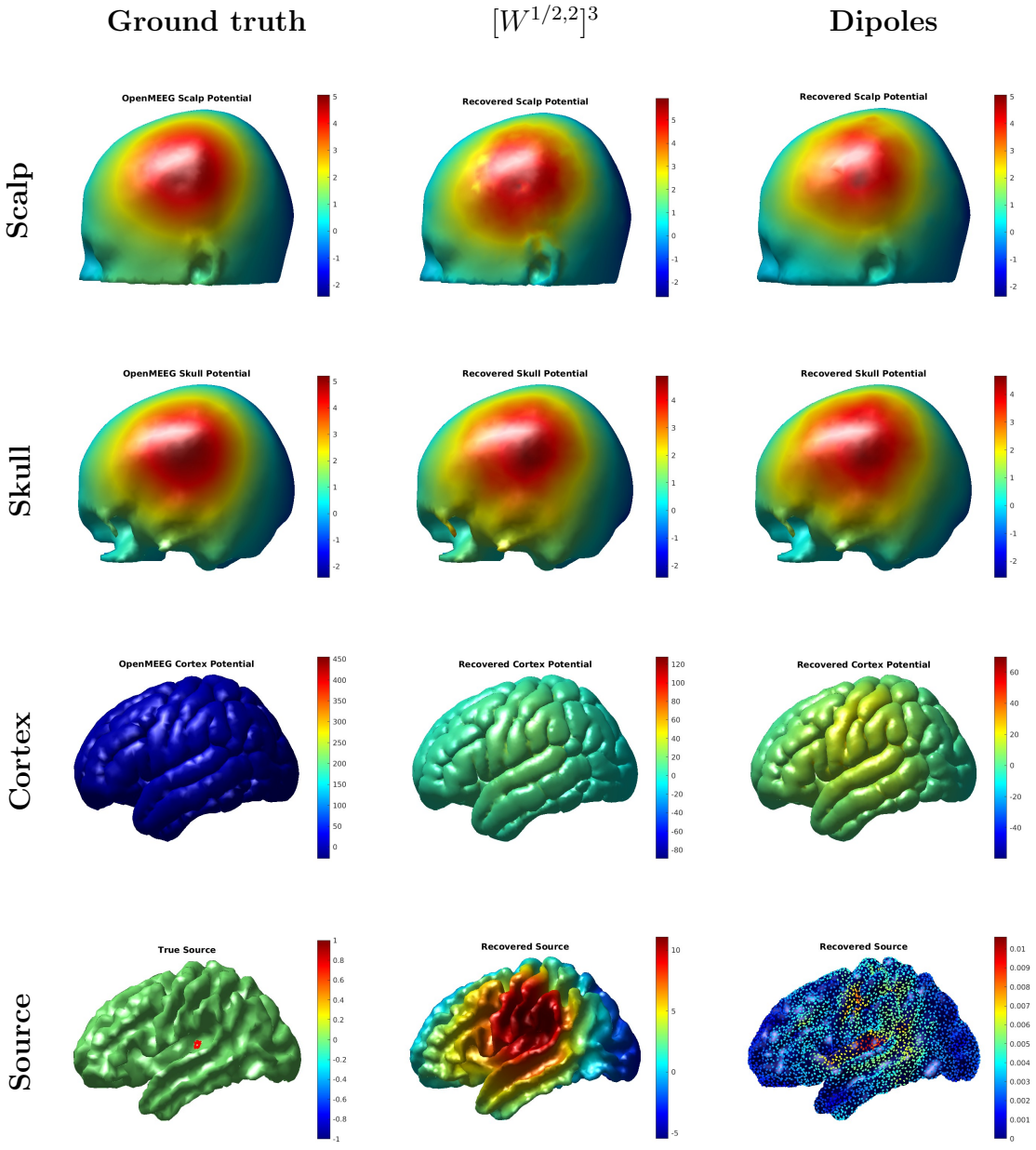

Figure 2.10: Recovery with EEG data.

2.7 Conclusions

We have presented a method that uses the double and single layer potentials only in the transmission of the electric potential and the magnetic flux density associated with a source which allows to solve the inverse source recovery problem and the inverse cortical mapping problem using either electric potential data or magnetic flux density data. The simultaneous resolution of these problems allow to capture the full behaviour of the source which aids in having better solutions for the inverse problems. The method is also less computationally complex as it involves fewer boundary integral operators than the symmetric method that has been used previously in inverse cortical mapping. The method also allows for using realistic geometries of the head and making exact computations at arbitrary points in space which allows for exact placement of sensors in the models. The method also takes advantage of the formulas of [67] for improved numerical accuracy for building and solving the discretised problems. The manner in which the discretised problems are built allows for the use of numerical frames/bases of Banach spaces defined on surfaces hence allowing the numerical implementation of the method to the Banach spaces in which the problem is solvable as was demonstrated in Theorem 28. The alternating minimisation procedure employed allows to use discretisations that offer the best accuracy for the source coupled with the freedom to use the best and efficient methods for recovering the source. This in principle should offer a better recovery of the source and the surface potentials. Further the method offers a natural coupling of the electric data and magnetic flux density making it easier to combine these data for the recovery as demonstrated in the combined use of sEEG and MEG data.

Future work on this subject includes making performance comparisons with existing methods for source recovery and inverse cortical mapping, testing the method on real data and implementing this method on a wider range of source classes such as models for sources that are assumed to be supported on the white matter fibres.

Chapter 3

On discrete spectra of Bergman–Toeplitz operators with harmonic symbols

3.1 Introduction

Various problems of modern analysis require the study of certain classes of "model" operators. One of the important families is the class of Toeplitz operators and operators related to them. Probably, the most classical objects of this kind are Toeplitz operators on Hardy spaces of analytic functions, see Nikolski [92] for a complete account on the subject. The applications of operators of this class can be found in Nikolski [93]. Another "similar" class is the family of Toeplitz operators on Bergman spaces. Their study started in late 80's of the last century, see Zhu [99] for a nice overview of the topic. In connection to inverse problems, this chapter deals with iterative methods for solving forward and inverse problems. When solving fixed point problems using iterative methods, the rate of convergence is determined by the spectral radius of the operator and how quickly the iterative methods settle to the convergence rates is determined by the growth of the resolvent of the operator, see for example. It is therefore of interest to study the spectra of operators, here we study the spectra of Toeplitz operators.

We proceed with some definitions. Let the complex plane be denote by $\mathbb{C}$, for each $z \in \mathbb{C}$ we can write $z = x + iy$ and the conjugate of z as $\overline{z} = x - iy$ then let $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ be the absolute value of $z \in \mathbb{C}$. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ with its boundary $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Definition 36. *$L^p(\mathbb{T})$, $1 \leq p \leq \infty$ is the space of $\mathbb{C}$-valued measurable functions, φ, whose absolute value is p-integrable with respect of the normalised Lebesgue measure on $\mathbb{T}$, with norm,*

$$\|\varphi\|_p^p = \int_{\mathbb{T}} |\varphi|^p \, dm, \qquad (\|\varphi\|_\infty = ess.\ sup_{z\in\mathbb{T}}\{|\varphi(z)|\} < \infty),$$

where m is the normalised Lebesgue measure on $\mathbb{T}$.

Definition 37. *Let $\varphi, \psi \in L^2(\mathbb{T})$ the inner product of φ, ψ is*

$$(\varphi, \psi)_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \varphi \overline{\psi}\, dm.$$

Using the inner product we can show that $\{z^n\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{T})$ and each function $\varphi \in L^2(\mathbb{T})$ may be represented by its Fourier series with respect to $\{z^n\}_{n\in\mathbb{Z}}$ as

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n) z^n,$$

where

$$\hat{f}(n) = (f, z^n)_{L^2(\mathbb{T})}.$$

This allows to identify $L^2(\mathbb{T})$ with the space of sequences

$$l^2(\mathbb{Z}) = \Big\{(a_n)_{n\in\mathbb{Z}} : a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty\Big\}.$$

Definition 38. *$H^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ defined as*

$$H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \hat{f}(n) = 0, n < 0\}.$$

It is called the Hardy space of the unit circle.

Definition 39. *For a function $\varphi \in L^\infty(\mathbb{T})$, the* Hardy–Toeplitz operator *$T_\varphi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is defined as*

$$T_\varphi h = P_+(\varphi h), \quad h \in H^2(\mathbb{T}), \tag{3.1}$$

where P_+ is the well-known Riesz othogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$, see Garnett [88]. The function φ is called a symbol *of the operator. For the sake of brevity, we call operator T_φ (3.1) an HT-operator.*

The definition of a *Bergman–Toeplitz operator* T_ψ (a BT-operator, for short), is rather similar to the above one.

Definition 40. *$L^p(\mathbb{D})$, $1 \le p \le \infty$ is the space of $\mathbb{C}$-valued measurable functions, h, whose absolute value is p-integrable with respect of the normalised Lebesgue measure on $\mathbb{D}$, with norm,*

$$\|h\|_p^p = \int_{\mathbb{D}} |h|^p \,\frac{dxdy}{2\pi}, \qquad (\|h\|_\infty = ess.\ sup_{z\in\mathbb{D}}\{|\varphi(z)|\} < \infty).$$

Definition 41. *Let $h, h \in L^2(\mathbb{D})$ the inner product of h, g is*

$$(h, g)_{L^2(\mathbb{D})} = \int_{\mathbb{D}} h\overline{g}\,\frac{dxdy}{2\pi}.$$

Definition 42. *Let $L^2_a(\mathbb{D})$ be the closed subspace in $L^2(\mathbb{D})$ of analytic on $\mathbb{D}$ functions. Given $\psi \in L^\infty(\mathbb{D})$, set*

$$T_\psi : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D}), \quad T_\psi h = \hat{P}_+(\psi h), \tag{3.2}$$

where $\hat{P}_+$ is the orthogonal projection acting from $L^2(\mathbb{D})$ to $L^2_a(\mathbb{D})$, see Zhu [99, Ch. 7].

For a function $\varphi \in L^\infty(\mathbb{T})$, it is sometimes convenient to consider a harmonic function $\hat{\varphi}$ on $\mathbb{D}$, the harmonic extension of φ to $\mathbb{D}$, given by

$$\hat{\varphi}(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|t-z|^2} \varphi(t)\, dm(t), \quad z \in \mathbb{D}. \tag{3.3}$$

Certainly, $\hat{\varphi} \in L^\infty(\mathbb{D})$ and $\|\hat{\varphi}\|_\infty \le \|\varphi\|_\infty$.

Given an HT-operator T_φ (with its symbol defined on $\mathbb{T}$), we consider its *associated BT-operator* $T_{\hat{\varphi}}$ with symbol $\hat{\varphi}$ given by (3.3). Notice that even though we use "similar-looking" notation for an HT-operator T_φ and a BT-operator $T_{\hat{\varphi}}$, the confusion is not possible since the functions φ and $\hat{\varphi}$ are defined on $\mathbb{T}$ and $\mathbb{D}$, respectively. The domains of definitions of corresponding symbols will be always clear from the context of the discussion.

We shall be concerned with Toeplitz operators $T_\varphi, T_{\hat{\varphi}}$ having symbols defined as follows. Set

$$\varphi(t) := \overline{g(t)} + f(t), \quad t \in \mathbb{T}, \quad f, g \in H^\infty(\mathbb{T}). \tag{3.4}$$

Clearly, we have

$$\psi(z) := \hat{\varphi}(z) = \overline{g(z)} + f(z), \quad z \in \mathbb{D}. \tag{3.5}$$

It is plain that $\hat{\varphi}$ has the non-tangential boundary values on the unit circle

$$\hat{\varphi}(t) = \lim_{r \to 1^-} \hat{\varphi}(rt), \quad \text{for a. e. } t \in \mathbb{T},$$

and these boundary values coincide with φ a.e. on $\mathbb{T}$.

Despite the similarity of definitions (3.1), (3.2), the BT- operators exhibit considerably reacher spectral behavior as compared to HT-operators. For instance, the essential spectrum of HT-operator T_φ is connected, see Widom [97], while, in general, the essential spectrum of BT-operator T_ψ is not. There are non-trivial compact BT-operators with quite simple (even radial) symbols [99, Sec.s 7.2, 7.3], while a compact HT-operator is necessarily zero [92, Part B, Ch. 4].

Sundberg–Zheng [96] showed that there are BT-operators with harmonic symbols having isolated eigenvalues in their spectrum. In subsequent papers, Zhao–Zheng [98], Guan–Zhao [89] and Guo–Zhao–Zheng [90] presented a class of BT-operators with harmonic symbols posessing "rather big" *discrete spectrum*, that is, the set of isolated eigenvalues of finite algebraic multiplicity.

So, in contrast to HT-operators, the notion of the discrete spectrum of a BT-operator with harmonic symbol makes sense. The study of the properties of the discrete spectrum for BT-operators with symbols (3.5) is the core of the present paper. Unlike the articles [89, 90, 98], our results are essentially based on the perturbation techniques from operator theory and function-theoretic results of Borichev–Golinskii–Kupin [85, 86] and Favorov–Golinskii [87].

Definition 43. *We say that a function $h : \mathbb{T} \to \mathbb{C}$ is absolutely continuous if h is an indefinite integral of a locally Lebesgue-integrable function, we write $h \in AC$.*

The Sobolev space $W^{1,2}(\mathbb{T})$ of absolutely continuous functions on the unit circle $\mathbb{T}$ with derivative in L^2:

$$W^{1,2}(\mathbb{T}) := \{h : \mathbb{T} \to \mathbb{C},\ h \in AC,\ h' \in L^2(\mathbb{T})\}.$$

For further purposes, we would like to introduce two closely related characteristics of compact sets on the complex plane $\mathbb{C}$. The following definitions are borrowed from Perkal [95, Sec. 2] and Peller [94, Sec. 4], respectively.

Definition 44. *Let $r > 0$. A closed set $E \subset \mathbb{C}$ is called r-convex if*

$$\mathbb{C}\backslash E = \bigcup \Big\{ B(x,r) : \ B(x,r) \subset \mathbb{C}\backslash E \Big\},$$

that is, the complement to E can be covered with open disks of a fixed radius $r > 0$, which lie in that complement.

Definition 45. *A compact set $K \subset \mathbb{C}$ is called circularly convex, if there is $r > 0$ such that for each $\lambda \in \mathbb{C}\backslash K$ with $\operatorname{dist}(\lambda, K) < r$ there are points $\mu \in \partial K$ and $\nu \in \mathbb{C}\backslash K$ so that*

$$|\mu - \nu| = r, \quad \lambda \in (\mu, \nu], \quad \{\zeta : \ |\nu - \zeta| < r\} \subset \mathbb{C}\backslash K.$$

For example, if K is a convex set, or the boundary ∂K is of C^2-class (without intersections and cusps), then K is a circularly convex set.

Note that the later definition is a bit more stringent than the former one. When K is a (closed) Jordan curve (a rectifiable continuous curve with no self-intersections), one can also see that the above definitions are equivalent.

A short reminder on standard notions and notations from operator theory is given in Subsection 3.2.1 below. For instance, see (3.9) for the notion of the unbounded (outer) open component of the Fredholm domain $\mathcal{F}_0(T)$.

The main result of this note is the following theorem.

Theorem 30. *Let T_φ be an HT-operator with the symbol φ (3.4) from $W^{1,2}(\mathbb{T})$, $\hat{\varphi}$ its harmonic extension (3.3), and $T_{\hat{\varphi}}$ be the BT-operator associated to T_φ. Assume that the spectrum $\sigma(T_\varphi)$ is a circularly convex set. Then, for each $\varepsilon > 0$*

$$\sum_{\lambda \in \sigma_d(T_{\hat{\varphi}}) \cap \mathcal{F}_0(T_\varphi)} \operatorname{dist}^{3+\varepsilon}(\lambda, \sigma(T_\varphi)) \le C(\varphi, \varepsilon) \, \|\varphi'\|_2^2. \tag{3.6}$$

Corollary 7. *Let q and p be algebraic polynomials, $\varphi = \overline{q} + p$ be a harmonic polynomial, and assume that the image $\varphi(\mathbb{T})$ is a Jordan curve without cusps. Then (3.6) holds for the discrete spectrum of BT-operator $T_{\hat{\varphi}}$.*

3.2 Some preliminaries

3.2.1 Generalities from operator theory

In this section, we recall some well-known notions of the classical operator theory, see Kato [91, Sec. IV. 5].

Let T be a bounded linear operator on a (separable) Hilbert space H. As usual, the resolvent set of T is

$$\rho(T) := \{\lambda \in \mathbb{C} : \ (T - \lambda) : H \to H \text{ is bijective } \}. \tag{3.7}$$

It follows that $(T-\lambda)^{-1}$ is bounded for $\lambda \in \rho(T)$. The spectrum of T is defined as

$$\sigma(T) = \mathbb{C}\backslash\rho(T). \tag{3.8}$$

Furthermore, we say that a bounded operator T is *Fredholm*, if its kernel and co-kernel are of finite dimension where the co-kernel of T is defined as

$$\text{co-kernel}\, T = H\backslash T(H).$$

The essential spectrum of T is defined as

$$\sigma_{ess}(T) = \{\lambda \in \mathbb{C} : (T-\lambda) \text{ is not Fredholm}\}.$$

One can see that $\sigma_{ess}(T)$ is a closed subset of $\sigma(T)$. One considers also the Fredholm domain of T, $\mathcal{F}(T) = \mathbb{C}\backslash\sigma_{ess}(T)$. Clearly, $\rho(T) \subset \mathcal{F}(T)$. We represent $\mathcal{F}(T)$ as

$$\mathcal{F}(T) = \bigcup_{j=0}^{\infty} \mathcal{F}_j(T), \tag{3.9}$$

where $\mathcal{F}_j(T)$ are disjoint (open) connected components of the set. We agree that $\mathcal{F}_0(T)$ stays for the unbounded connected component of $\mathcal{F}(T)$.

The discrete spectrum $\sigma_d(T)$ of T is the set of all isolated eigenvalues of T of finite algebraic multiplicity. For convenience, we put

$$\sigma_0(T) := \sigma_d(T) \cap \mathcal{F}_0(T) \subset \sigma_d(T). \tag{3.10}$$

Let A_0, A be bounded operators on a Hilbert space such that $A - A_0$ is compact. The operators A and A_0 are called compact perturbations of each other. The celebrated Weyl's theorem states that

$$\sigma_{ess}(A) = \sigma_{ess}(A_0), \tag{3.11}$$

see Kato [91, Sec. IV.5.6].

We shall be interested in the situation when $\sigma_0(A)$ is at most countable set, $\sigma_0(A) = \{\lambda_j\}_{j\geq 1}$ and it accumulates to the essential spectrum $\sigma_{ess}(A_0)$ only.

3.2.2 Reminder on Hilbert-Schmidt operators

In this subsection, we recall briefly the notion of a Hilbert-Schmidt operator and its simplest properties, see Birman-Solomyak [84, Sec. 11.3].

Let A be a compact operator. The sequence of *singular values* $\{s_j(A)\}_{j\geq 1}$ is defined as

$$s_j(A) = \lambda_j(A^*A)^{1/2}, \qquad s_j(A) \geq 0,$$

where $\lambda_j(A^*A)$ are eigenvalues of the compact operator A^*A. Without loss of generality one can suppose that $\{s_j(A)\}_{j\geq 1}$ forms a decreasing sequence, and, moreover

$$\lim_{j\to+\infty} s_j(A) = 0.$$

One says that $A \in \mathcal{S}_2$, the Hilbert-Schmidt class of compact operators, iff

$$\|A\|_{\mathcal{S}_2}^2 := \sum_{j\geq 1} s_j(A)^2 < \infty.$$

Equivalently, $A \in \mathcal{S}_2$ if and only if $\{s_j(A)\}_{j\geq 1} \in \ell^2$. Alternatively, the $\mathcal{S}_2$-norm of the operator can be computed as

$$\|A\|_{\mathcal{S}_2}^2 = \sum_{j,k\geq 1} |(Ae_j, e_k)|^2,$$

where $\{e_j\}_{j\geq 1}$ is an arbitrary orthonormal basis in the given Hilbert space.

3.2.3 On the discrete spectrum of a perturbed operator: a result of Favorov–Golinskii

Some useful quantitative bounds for the rate of convergence of the discrete spectrum of a perturbed operator are given in Favorov–Golinskii [87, Sec. 5]. A special case of [87, Thm 5.1] (cf. a remark right after its proof and formula (5.8)) looks as follows.

Theorem 31 ([87])**.** *Let A_0 be a bounded linear operator on a Hilbert space, which satisfies the conditions:*

1. *The spectrum $\sigma(A_0)$ is an r-convex set.*
2. *The resolvent $R(z, A_0) = (A_0 - z)^{-1}$ is subject to the bound*

$$\|R(z, A_0)\| \leq \frac{C(A_0)}{\operatorname{dist}^p(z, \sigma(A_0))}, \qquad p > 0, \quad z \in \mathcal{F}_0(A_0). \tag{3.12}$$

Let B be a Hilbert–Schmidt operator, and $A = A_0 + B$. Then for each $\varepsilon > 0$

$$\sum_{\lambda\in\sigma_d(A)\cap\mathcal{F}_0(A_0)} \operatorname{dist}^{2p+1+\varepsilon}(\lambda, \sigma(A_0)) \leq C(\sigma(A_0), p, \varepsilon)\, \|B\|_2^2. \tag{3.13}$$

If $\sigma_{ess}(A_0)$ does not split the plane, and (3.12) holds for all $\lambda \in \mathbb{C}\backslash\sigma(A_0)$, then (3.13) is true for the whole discrete spectrum $\sigma_d(A)$. For the class of (non-selfadjoint) HT-operators $A_0 = T_\varphi$, φ in (3.4), the essential spectrum, in general, splits the plane.

3.3 Proof of the main result

Let φ be as in (3.4). Consider the HT-operator T_φ (3.1) and the associated BT-operator $T_{\hat{\varphi}}$ (3.2). The technical way to compare these operators is to look at their matrices in appropriately chosen bases. Namely, define

$$e_{H,n}(t) = t^n, \quad e_{B,n}(z) = \sqrt{n+1}\, z^n, \quad n \geq 0. \tag{3.14}$$

It is plain that the systems $\{e_{H,n}\}_{n\geq 0}$ and $\{e_{B,n}\}_{n\geq 0}$ are the orthonormal bases in $H^2(\mathbb{T})$ and $L^2_a(\mathbb{D})$, respectively. Set

$$\mathcal{T}_\varphi = [(T_\varphi e_{H,i}, e_{H,j})_{H^2(\mathbb{T})}]_{i,j\geq 0}, \quad \mathcal{T}_{\hat{\varphi}} = [(T_{\hat{\varphi}} e_{B,i}, e_{B,j})_{L^2_a(\mathbb{T})}]_{i,j\geq 0}. \tag{3.15}$$

The operators $\mathcal{T}_\varphi$ and $\mathcal{T}_{\hat{\varphi}}$ are unitarily equivalent to original operators T_φ and $T_{\hat{\varphi}}$, and they both act on $\ell^2(\mathbb{Z}_+)$. So, one can argue on the operators $\mathcal{T}_\varphi$ and $\mathcal{T}_{\hat{\varphi}}$ being "close" in a certain sense.

Rewriting relation (3.4) in more detailed form, we have

$$\begin{aligned} &\hat{\varphi}(z) = \overline{g(z)} + f(z), \\ &f(z) = \sum_{k=0}^{\infty} f_k z^k \in H^\infty(\mathbb{T}), \qquad g(z) = \sum_{k=1}^{\infty} g_k z^k \in H^\infty(\mathbb{T}), \\ &\varphi(e^{i\theta}) = \hat{\varphi}(e^{i\theta}) = \sum_{j\in\mathbb{Z}} b_j e^{ij\theta}, \quad b_j = \begin{cases} f_j, & j \geq 0, \\ \overline{g_{-j}}, & j < 0. \end{cases} \end{aligned} \tag{3.16}$$

Proposition 3. *Assume that the symbol φ (3.4) belongs to $W^{1,2}(\mathbb{T})$. Then $\mathcal{T}_{\hat{\varphi}} - \mathcal{T}_\varphi$ is a Hilbert-Schmidt operator and*

$$\|\mathcal{T}_{\hat{\varphi}} - \mathcal{T}_\varphi\|^2_{\mathcal{S}_2} \leq \frac{\pi^2}{24} \|\varphi'\|^2_2. \tag{3.17}$$

Proof. The matrix representation of $\mathcal{T}_\varphi$ is obvious: $\mathcal{T}_\varphi = [b_{i-j}]_{i,j\geq 0}$. So let us compute the matrix $\mathcal{T}_{\hat{\varphi}} = [\tau_{i,j}]_{i,j\geq 0}$ in the orthonormal basis $\{e_{B,n}\}_{n\geq 0}$ (3.14). For $l, k \geq 0$

$$\begin{aligned} \tau_{k,k+l} &= (T_{\hat{\varphi}} e_{B,k+l}, e_{B,k})_{L^2_a(\mathbb{D})} = \frac{1}{\pi} \int_{\mathbb{D}} \hat{\varphi}(z) e_{B,k+l}(z)\, \overline{e_{B,k}(z)}\, dxdy \\ &= \frac{\sqrt{(k+l+1)(k+1)}}{\pi} \int_{\mathbb{D}} \hat{\varphi}(z) z^{k+l}\, \overline{z^k}\, dxdy, \end{aligned}$$

or, in polar coordinates,

$$\tau_{k,k+l} = \frac{\sqrt{(k+l+1)(k+1)}}{\pi} \int_0^1 \int_0^{2\pi} \sum_{n\in\mathbb{Z}} b_n r^{|n|+2k+l+1}\, e^{i(n+l)\theta}\, drd\theta.$$

Finally,

$$\tau_{k,k+l} = \sqrt{\frac{k+1}{k+l+1}}\, b_{-l}, \qquad k, l \geq 0.$$

The same formula holds for $\tau_{k+l,k}$, $k, l \geq 0$, and so

$$\tau_{i,j} = \sqrt{\frac{\min(i,j)+1}{\max(i,j)+1}}\, b_{i-j}, \qquad i, j \geq 0. \tag{3.18}$$

Let us estimate the Hilbert-Schmidt norm

$$\|\mathcal{T}_{\hat{\varphi}} - \mathcal{T}_\varphi\|^2_{\mathcal{S}_2} = \sum_{i,j\geq 0} |\tau_{i,j} - b_{i-j}|^2 = \sum_{i,j\geq 0} |b_{i-j}|^2 \left| 1 - \sqrt{\frac{\min(i,j)+1}{\max(i,j)+1}} \right|^2.$$

As

$$\sum_{i,j\geq 0} a_{ij} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} a_{k,k+l} + \sum_{l=1}^{\infty}\sum_{k=0}^{\infty} a_{k+l,k},$$

we have

$$\begin{aligned}
\|\mathcal{T}_{\hat{\varphi}} - \mathcal{T}_{\varphi}\|_{\mathcal{S}_2}^2 &= \sum_{l=0}^{\infty} l^2 \left[|b_l|^2 + |b_{-l}|^2\right] \sum_{k=0}^{\infty} \frac{1}{(k+l+1)\left(\sqrt{k+l+1}+\sqrt{k+1}\right)^2} \\
&= \sum_{l\in\mathbb{Z}} |l\, b_l|^2 \sum_{k=0}^{\infty} \frac{1}{(k+|l|+1)\left(\sqrt{k+|l|+1}+\sqrt{k+1}\right)^2} \\
&\leq \left(\sum_{l\in\mathbb{Z}} |l b_l|^2\right) \left(\sum_{k=0}^{\infty} \frac{1}{(k+1)\,(2\sqrt{k+1})^2}\right) \\
&= \sum_{l\in\mathbb{Z}} |l b_l|^2 \sum_{k=0}^{\infty} \frac{1}{4\,(k+1)^2} = \frac{\pi^2}{4\cdot 6}\,\|\varphi'\|_2^2 = \frac{\pi^2}{24}\,\|\varphi'\|_2^2.
\end{aligned}$$

as claimed. □

Weyl's theorem concerning spectra of compact perturbations, mentioned above, leads to the following

Corollary 8. *Let the symbol φ satisfy hypothesis of the above Proposition. Then the essential spectrum of BT-operator $T_{\hat{\varphi}}$ is*

$$\sigma_{ess}(T_{\hat{\varphi}}) = \sigma_{ess}(T_{\varphi}) = \varphi(\mathbb{T}) =: \Gamma,$$

and the discrete spectrum $\sigma_d(T_{\hat{\varphi}})$ is at most countable set of eigenvalues of finite algebraic multiplicity with all its accumulation points on Γ.

We go on with the proof the quantitative version of the above corollary.

Theorem 30. Let $A_0 = \mathcal{T}_{\varphi}$, $A = \mathcal{T}_{\hat{\varphi}}$, see (3.15). We only have to ensure that the conditions of Theorem 31 are met.

It is clear that $\varphi \in W^{1,2}(\mathbb{T})$ implies that $\varphi \in W$, the Wiener algebra of absolutely convergent Fourier series. By [94, Thm 4], the resolvent $(A_0 - z)^{-1}$ admits the linear growth, that is, (3.12) holds with $p = 1$. Next, by Proposition 3, the difference $A - A_0$ is the Hilbert-Schmidt operator with the norm bound (3.17). The proof is complete. □

3.4 Conclusion

We have just shown that the essential spectra of a BT-operator and an HT-operator with symbols that coincide on $\mathbb{T}$ and in $W^{1,2}(\mathbb{T})$ are identical. This allow to use well known properties of the essential spectrum of HT-operators to make conclusions on the essential spectrum of BT-operators and hence on the entire spectrum of BT-operators.

We also gave a convergence rate for the discrete spectrum of the BT-operator that is contained in the unbounded component of the Fredholm domain of the the HT-operator.

Contributions

In this work we made the following main contributions:

- In the first chapter, we provided a characterisation of silent $[L^p(\Omega)]^n$ vector-fields, $n \geq 3$, $1 < p < \infty$, for $\Omega \subset \mathbb{R}^n$ of mild smoothness, see Theorem 20. This characterisation and the methods used can be a launching point to characterise silent $[W^{-1,p}(\Omega)]^n$ vector-fields for $\Omega \subset \mathbb{R}^n$ Lipschitz smooth using the decompositions of such vector-fields given in [100]. The characterisation of silent sources led to a characterisation of norm-minimising equivalent vector-fields in $[L^p(\Omega)]^n$, for $\Omega \subset \mathbb{R}^n$ which are domains of finite perimeter with mild smoothness, see Theorem 25 and Corollary 6. In turn this characterisation led to a decomposition of vector-fields on Ω which is valid for $[L^p(\Omega)]^n$, $1 < p < \infty$, see Theorem 15 and Corollary 1.
  This work is in preparation for publicaton.

- In the second chapter we introduced an alternating minimisation procedure to solve the inverse source localisation for sEEG, EEG and/or MEG simultaneously with the inverse cortical mapping problem using realistic head geometries, see Section 2.5.4. We managed to show that this method can be implemented for sources that are assumed to be distributions as well rather than functions only, see Theorems 28 and 29. This opens up the possibility of using sources in the inverse source localisation problem that can closely respects known biological constraints such as brain connectivity by using the white matter tractography as a support for the sources.
  This work is in preparation for publication.

- We gave a convergence rate result for the discrete spectrum of Bergman-Toeplitz operators in the unbounded component of the Fredholm domain of their corresponding Hardy-Toeplitz operators. This result was given for those Bergman-Toeplitz operators whose essential spectrum coincides with the essential spectrum of Hardy-Toeplitz operators.
  This work is to appear as On discrete spectra of Bergman-Toeplitz operators with harmonic symbols in Birkhauser Memorial Volume in honor of Sergey Naboko.

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