L-functions of Kloosterman sheaves

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献给我的父母

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Contents

Introduction (en français)

La théorie des fonctions L joue un rôle crucial dans la théorie des nombres. De manière générale, une fonction L est une série de Dirichlet $L(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ qui converge lorsque $\mathrm{Re}(s)$ est assez grand et se prolonge en une fonction méromorphe sur le plan complexe, satisfaisant à une équation fonctionnelle et à la formule du produit d'Euler. L'exemple le plus célèbre est la fonction zêta de Riemann $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$, une fonction méromorphe sur $\mathbb{C}$ qui satisfait à l'équation fonctionnelle

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s),$$

prédite par Euler en 1748. De plus, elle satisfait la formule du produit d'Euler $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$. Comme les fonctions L sont des objets analytiques, elles nous permettent d'étudier des objets arithmétiques à l'aide d'outils de l'analyse. Par exemple, le fait que $\zeta(s)$ ait un pôle simple implique qu'il existe une infinité de nombres premiers, et le fait que les valeurs des fonctions L de Dirichlet soient non nulles en 1 implique le théorème de Dirichlet sur les progressions arithmétiques. En outre, les valeurs spéciales des fonctions L en des nombres entiers pourraient être liées aux périodes [KZ01, §1.1].

On peut associer une fonction L à certaines sommes exponentielles. Les *sommes de Kloosterman* sont des sommes exponentielles sur des corps finis, définies pour chaque puissance de nombres premiers $q = p^r$ et $a \in \mathbb{F}_q^\times$, par

$$\mathrm{Kl}_2(a;q) := \sum_{x\in\mathbb{F}_q^\times} \exp\left(2\pi i/p \cdot \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\left(x + \frac{a}{x}\right)\right),$$

où $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)$ est la trace de $\mathbb{F}_q$ à $\mathbb{F}_p$. Si a n'est pas nul, Weil a montré dans [Wei48] que $\mathrm{Kl}_2(a;q) = -(\alpha_a + \beta_a)$ pour deux nombres algébriques α_a et β_a de norme $p^{1/2}$. En particulier, on obtient une borne supérieure $|\mathrm{Kl}_2(a,q)| \leq 2p^{1/2}$. Pour chaque entier $k \geq 1$, les *k-ièmes moments de puissance symétrique des sommes de Kloosterman* sont des entiers de la forme $m_2^k(q) = \sum_{a\in\mathbb{F}_q} \sum_{i=0}^k \alpha_a^i \beta_a^{k-i}$. Dans [Bro16, Bro17], les *fonctions L attachées aux k-ièmes moments de puissance symétrique des sommes de Kloosterman*, notées $L_k(s)$, sont définies comme des produits d'Euler, où les facteurs locaux aux nombres premiers p sont produits à partir de la série génératrice

$$\exp\left(\sum_{r\geq 1} \frac{m_2^k(p^r)}{r} \cdot T^r\right).$$

Ces fonctions L sont définies à priori sur un domaine $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1 + \frac{k+1}{2}\}$ par construction et d'après les travaux de Fu-Wan [FW05]. Broadhurst et Roberts ont conjecturé que ces fonctions L se prolongent méromorphiquement à $\mathbb{C}$, et ils ont prédit les formules précises des équations fonctionnelles.

Il n'est pas toujours facile de montrer qu'une fonction L a un prolongement méromorphe dans le plan complexe. Par exemple, on peut montrer que les fonctions L des formes modulaires sont des fonctions holomorphes sur $\mathbb{C}$ directement par des outils de l'analyse, comme dans le cas de la fonction zêta de Riemann. Cependant, pour montrer les mêmes choses pour les fonctions L des courbes elliptiques, le seul moyen connu est de les identifier à des fonctions L de formes modulaires. Le lien entre ces deux types de fonctions L est établi par le théorème de modularité (la conjecture de Taniyama-Shimura)

[BCDT01], dont la version plus faible a conduit à la preuve du dernier théorème de Fermat par Wiles [Wil95].

Dans [FSY22], Fresán, Sabbah et Yu ont construit une famille de représentations galoisiennes provenant de la géométrie, dont les fonctions L sont exactement $L_k(s)$. Ensuite, ils ont utilisé la philosophie de Langlands pour prouver la conjecture de Broadhurst et Roberts, en montrant que les fonction L des représentations galoisiennes sont égales à des fonctions L automorphes via un théorème de Patrikis-Taylor [PT15]. Pour résumer, ils ont prouvé le théorème suivant.

Theorem (Fresán-Sabbah-Yu)**.** *La fonction $L_k(s)$ a un prolongement méromorphe dans le plan complexe. De plus, on peut compléter $L_k(s)$ en une fonction L complète $\Lambda_k(s)$ telle que*

$$\Lambda_k(s) = \epsilon_k \Lambda_k(k+2-s),$$

où $\epsilon_k \in \{\pm 1\}$ et ϵ_k est égal à 1 si k est impair.

Dans un article suivant de Fresán, Sabbah, et Yu [FSY20a], ils expliquent également une certaine relation entre les valeurs spéciales des fonctions L aux entiers critiques et certains déterminants des moments de Bessel

$$\int_0^\infty I_0(z)^a K_0(z)^{k-a} z^b \frac{\mathrm{d}z}{z},$$

où $I_0(z)$ et $K_0(z)$ sont les fonctions de Bessel modifiées de première et deuxième espèce (solutions de $(z\partial_z)^2 - z^2 = 0$).

Cette thèse vise à généraliser le théorème ci-dessus aux fonctions L attachées aux moments (pas nécessairement ceux de puissance symétrique) des sommes de Kloosterman en plusieurs variables.

Résultats principaux

Les fonctions L des faisceaux de Kloosterman

Les sommes de Kloosterman en n variables sont les sommes exponentielles sur les corps finis définies, pour chaque puissance de nombres premiers $q = p^r$ et chaque $a \in \mathbb{F}_q^\times$, par

$$\mathrm{Kl}_{n+1}(a;q) := \sum_{x_1,\ldots,x_n \in \mathbb{F}_q^\times} \exp\left(2\pi i/p \cdot \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\left(x_1 + \ldots + x_n + \frac{a}{x_1 \cdots x_n}\right)\right).$$

Elles sont les analogues sur un corps fini des fonctions de Bessel

$$\oint_{(S^1)^n} \exp\left(x_1 + \ldots + x_n + \frac{z}{x_1 \cdots x_n}\right) \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n},$$

satisfaisant aux équations différentielles de Bessel $(z\partial_z)^{n+1} - z = 0$.

En fixant un nombre premier $\ell \neq p$ et un plongement $\iota\colon \bar{\mathbb{Q}}_\ell \to \mathbb{C}$, Deligne a construit dans [Del77, Sommes. Trig. Th. 7.8] des faisceaux ℓ-adiques lisses Kl_{n+1} de rang $n+1$, appelés les *faisceaux de Kloosterman*, tels que pour chaque $q = p^r$, $a \in \mathbb{F}_q^\times = \mathbb{G}_{m,\mathbb{F}_q}(\mathbb{F}_q)$, et chaque point géométrique $\bar{a}$ au-dessus de a, les normes complexes des valeurs propres de Frob_q soient $q^{n/2}$ via l'immersion ι, et qu'on ait

$$\iota \circ \mathrm{Tr}(\mathrm{Frob}_q, (\mathrm{Kl}_{n+1})_{\bar{a}}) = (-1)^n \mathrm{Kl}_{n+1}(a;q).$$

Comme conséquence directe, on a la borne supérieure $|\mathrm{Kl}_{n+1}(a;q)| \leq (n+1)q^{n/2}$.

Si l'on considère les faisceaux ℓ-adiques lisses Kl_{n+1} sur $\mathbb{G}_m$ comme des représentations ℓ-adiques continues $\rho_{\mathrm{Kl}_{n+1}}$ du groupe fondamental étale $\pi_1^{ét}(\mathbb{G}_{m,\mathbb{F}_q})$, le *groupe de monodromie arithmétique (resp. géométrique)* est défini comme la fermeture de Zariski de l'image de $\pi_1^{ét}(\mathbb{G}_{m,\mathbb{F}_q})$ (resp. $\pi_1^{ét}(\mathbb{G}_{m,\bar{\mathbb{F}}_q})$) dans

$\mathrm{GL}_{n+1}(\bar{\mathbb{Q}}_\ell)$ sous $\rho_{\mathrm{Kl}_{n+1}}$. Dans [Kat88], Katz a montré que les groupes de monodromie arithmétique et les groupes de monodromie géométrique de Kl_{n+1} coïncident, et égaux aux groupes

$$\begin{cases} \mathrm{SP}_{n+1} & n+1 \text{ pair}, \\ \mathrm{SL}_{n+1} & p(n+1) \text{ impair}, \\ \mathrm{SO}_{n+1} & p=2,\ n+1 \text{ impair et } n\neq 6, \\ G_2 & p=2, n=6. \end{cases}$$

L'apparition du groupe exceptionnel G_2 semble assez surprenante. Katz a demandé dans [Kat90] si tous les groupes semisimples apparaissent comme des groupes de monodromie de faisceaux ℓ-adiques lisses.

Dans [HNY13], Heinloth, Ngô et Yun ont construit des *faisceaux de Kloosterman pour les groupes réductifs* de manière uniforme en utilisant des méthodes du programme de Langlands géométrique. Plus précisément, on peut attacher à toute paire, constituée d'un groupe réductif scindé G et d'une représentation V de dimension finie de G, un faisceau ℓ-adic lisse sur $\mathbb{G}_{m,\mathbb{F}_q}$, d'une manière fonctorielle. Ils ont trouvé que les groupes exceptionnels E_7, E_8, G_2, F_4 apparaissent parmi les groupes de monodromie des faisceaux de Kloosterman pour les groupes réductifs. Comme cas particulier, on retrouve le faisceau de Kloosterman classiques $\mathrm{Kl}_{n+1}(-\frac{n}{2})$ en prenant $G=\mathrm{SL}_{n+1}$ et V la représentation standard de SL_{n+1}.

On s'intéresse uniquement aux cas où $G=\mathrm{SL}_{n+1}$ et V est la représentation de plus haut poids λ de SL_{n+1}. Pour un poids λ, on désigne par $|\lambda|=\sum_{i=1}^n \lambda_i$, et par $\mathrm{Kl}_{n+1}^\lambda$ le faisceau $\mathrm{Kl}_{\mathrm{SL}_{n+1}}^{V_\lambda}(-\frac{nk}{2})$ pour simplifier. Si V est la k-ième puissance symétrique de la représentation standard, c'est-à-dire $\lambda=(k,0,\dots,0)$ (on identifie λ et k dans ce cas), le faisceaix $\mathrm{Kl}_{n+1}^\lambda$ coïncide avec la k-ième puissance symétrique de Kl_{n+1}, notée $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$.

Pour chaque faisceau de Kloosterman $\mathscr{F}=\mathrm{Kl}_{n+1}^\lambda$, en suivant la construction dans [FSY22], on introduit une fonction L comme produit d'Euler, où les facteurs locaux sont construits en utilisant les *moments de* $\mathrm{Kl}_{n+1}^\lambda$, c'est-à-dire les entiers de la forme.

$$m_{n+1}^\lambda(q) := -\sum_{a\in\mathbb{F}_q^\times} \mathrm{Tr}(\mathrm{Frob}_q, (\mathrm{Kl}_{n+1}^\lambda)_{\bar{a}}).$$

La série génératrice $Z(\lambda, n+1, p; T)$ de $m_{n+1}^\lambda(q)$ est définie comme $\exp(\sum_{r\geq 1} m_{n+1}^\lambda(p^r)/r\cdot T^r)$. Par la formule des traces de Grothendieck [Del77, Rapport. Th. 3.1], on a

$$Z(\lambda, n+1, p; T) = \prod_{i=0}^{2} \det(1-\mathrm{Frob}_p T \mid \mathrm{H}^i_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda))^{(-1)^{i+1}},$$

qui est une fonction rationnelle. Cependant, les normes complexes des racines et des pôles de $Z(\lambda, n+1, p; T)$ appartiennent à l'ensemble $\{p^{-i/2} \mid 0\leq i\leq nk\}$. C'est pourquoi $Z(\lambda, n+1, p; T)$ n'est donc pas un candidat idéal pour les facteurs locaux du produit d'Euler. De manière similaire au travail de Fu-Wan [FW05, FW08] pour les faisceaux $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$, on peut définir

$$M(\lambda, n+1, p; T) = \det(1-\mathrm{Frob}_p T \mid \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda)),$$

où

$$\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda) = \mathrm{im}(\mathrm{H}^1_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda) \to \mathrm{H}^1_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda))$$

est la cohomologie ℓ-adique intermédiaire. Le polynôme $M(\lambda, n+1, p; T)$ est, en fait, un facteur de $Z(\lambda, n+1, p; T)$, et on va prouver dans le théorème 5.4 qu'il existe un ensemble fini $S(\lambda, n+1)$ de nombres premiers tel que le degré de $M(\lambda, n+1, p; T)$ est constant pour $p\notin S(\lambda, n+1)$.

On construit maintenant une fonction L partielle $L^S(\lambda, n+1; s)$ de $\mathscr{F}$ en considérant le produit d'Euler,

$$L^S(\lambda, n+1; s) := \prod_{s\notin S} M(\lambda, n+1, p; p^{-s})^{-1},$$

qui est a priori une fonction holomorphe sur le domaine $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1 + \frac{n|\lambda|+1}{2}\}$. Mais la définition ne nous donne pas plus d'information. Un cas particulier de cette construction, celui où $\mathscr{F} = \mathrm{Sym}^k \mathrm{Kl}_2$, n'est rien d'autre que la fonction attachée à la k-ième puissance symétrique des sommes de Kloosterman.

On peut se demander si la fonction L partielle $L^S(\lambda, n+1; s)$ peut se prolonger en une fonction méromorphe sur le plan complexe et satisfait à une équation fonctionnelle. Le théorème de Fresán-Sabbah-Yu répond aux cas des faisceaux $\mathrm{Sym}^k \mathrm{Kl}_2$. Voici le résultat principal de cette thèse.

Théorème A. *Soit $\mathscr{F}$ dans l'un des deux ensembles suivants :*

(1). $\{\mathrm{Sym}^k \mathrm{Kl}_3 \mid k \leq 9\}$,

(2). $\{\mathrm{Sym}^4 \mathrm{Kl}_3, \mathrm{Sym}^3 \mathrm{Kl}_4, \mathrm{Sym}^4 \mathrm{Kl}_4, \mathrm{Sym}^3 \mathrm{Kl}_5, \mathrm{Kl}_3^{(2,1)}, \mathrm{Kl}_3^{(2,2)}\}$.

Alors la fonction L partielle $L^S(\lambda, n+1; s)$ de $\mathscr{F}$ se prolonge en une fonction méromorphe sur $\mathbb{C}$, et peut être complétée en une fonction $\Lambda(\lambda, n+1; s)$ qui satisfait à l'équation fonctionnelle

$$\Lambda(\lambda, n+1; s) = \pm \Lambda(\lambda, n+1; n|\lambda| + 2 - s).$$

Conjectures de type Evans

Le deuxième cas du théorème A est lié aux conjectures de type Evans de [YV15], qui prédit certaines relations entre les moments de certains faisceaux de Kloosterman $\mathrm{Kl}_{n+1}^{\lambda}$ et les coefficients de Fourier de certaines formes propres cuspidales normalisées. Lorsque $\mathscr{F} = \mathrm{Sym}^k \mathrm{Kl}_2$ pour $5 \leq k \leq 8$, on a les exemples connus suivants.

- Lorsque $k = 5$, il existe une forme propre cuspidale normalisée $f \in S_3(\Gamma_0(15), \left(\frac{\cdot}{15}\right))$ telle que

$$a_f(p) = -\frac{1}{p^2}(m_2^5(p) + 1)$$

si $p \nmid 15$, prouvée par Peters *et al.* [PTvdV92] et Livné [Liv95].

- Lorsque $k = 6$, il existe une forme propre cuspidale normalisée $f \in S_4(\Gamma_0(6))$ telle que

$$a_f(p) = -\frac{1}{p^2}(m_2^6(p) + 1)$$

si $p \nmid 6$, prouvée par Helek *et al.* [HSvGvS01].

- Lorsque $k = 7$, il existe une forme propre cuspidale normalisée $f \in S_3(\Gamma_0(525), \epsilon_f)$, où $\epsilon_f = \left(\frac{\cdot}{21}\right) \cdot \epsilon_5$ et ϵ_5 est un caractère quartique avec conducteur 5, telle que

$$a_f(p)^2 \epsilon_f(p)^{-1} - p^2 = -\frac{1}{p^2}\left(\frac{p}{105}\right)(m_2^7(p) + 1)$$

si $p > 7$, conjecturée par Evans [Eva10a] et prouvée par Yun [YV15].

- Lorsque $k = 8$, il existe une forme propre cuspidale normalisée $f \in S_6(\Gamma_0(6))$, telle que

$$a_f(p) = -\frac{1}{p^2}(m_2^8(p) + 1)$$

si $p \nmid 6$, conjecturée par Evans [Eva10b] et prouvée par Yun et Vincent [YV15].

En particulier, on peut déterminer toutes ces formes modulaires en calculant suffisamment de valeurs de moments. En raison des identités entre moments et coefficients de Fourier des formes modulaires, on peut identifier les fonctions L de $\mathrm{Sym}^k\mathrm{Kl}_2$ et les fonctions L des formes modulaires correspondantes. Ainsi les fonctions L se prolongent en des fonctions méromorphes sur $\mathbb{C}$ et satisfont des équations fonctionnelles.

Yun a prédit de nouveaux résultats de cette forme dans [YV15] sur les faisceaux de Kloosterman pour les groupes réductifs. On prouvera ces conjectures de type Evans pour les faisceaux dans le deuxième cas du théorème A.

Théorème B. *Les fonctions L des faisceaux de Kloosterman $\mathrm{Sym}^4\mathrm{Kl}_3$, $\mathrm{Sym}^3\mathrm{Kl}_4$, $\mathrm{Sym}^4\mathrm{Kl}_4$, $\mathrm{Sym}^3\mathrm{Kl}_5$, $\mathrm{Kl}_3^{(2,1)}$ et $\mathrm{Kl}_3^{(2,2)}$ proviennent de formes modulaires. On présente un résumé des informations sur ces formes modulaires $f \in S_k(\Gamma_0(N), \epsilon)$ dans le tableau suivant. Ici LMFDB [LMF22] est la base de données des fonctions L et des formes modulaires.*

faisceaux	N	k	ϵ	*étiquettes dans LMFDB*
$\mathrm{Sym}^4\mathrm{Kl}_3$	14	4	1	*14.4.a.b*
$\mathrm{Sym}^3\mathrm{Kl}_4$	15	3	$\left(\frac{\bullet}{15}\right)$	*15.3.a.b.*
$\mathrm{Sym}^4\mathrm{Kl}_4$	10	6	1	*10.6.a.a.*
$\mathrm{Sym}^3\mathrm{Kl}_5$	33	4	1	33.4.*a.b*
$\mathrm{Kl}_3^{(2,1)}$	14	2	1	*14.2.a.a*
$\mathrm{Kl}_3^{(2,2)}$	6	4	1	*6.4.a.a*

En particulier, on a des égalités similaires entre moments des faisceaux de Kloosterman et les coefficients de Fourier des formes modulaires dans les théorèmes 5.22, 5.23, 5.26, 5.28, 5.30 et 5.31, qui implique que les fonctions L de ces faisceaux sont les fonctions L des formes modulaires correspondantes. Ainsi le théorème B fournit une preuve pour le second cas du théorème A.

Remarque. Un corollaire intéressant du théorème ci-dessus est que pour $p \nmid 6$, les moments des faisceaux $\mathrm{Sym}^6\mathrm{Kl}_2$ et $\mathrm{Kl}_3^{(2,2)}$ sont les mêmes, car ils sont égaux aux coefficients de Fourier de la forme modulaire de l'étiquette 6.4.a.a.

Idée de la preuve

Représentations galoisiennes provenant de la géométrie

La stratégie pour prouver le théorème A est de construire des familles de représentations galoisiennes (d'origine géométrique) et de montrer que leurs fonctions L sont automorphes.

Dans [FSY22], inspirés par l'analogie entre les sommes de Kloosterman et les fonctions de Bessel, les auteurs considèrent la *connexion de Kloosterman*, aussi notée Kl_{n+1}, qui est une connexion sur $\mathbb{G}_{m,\mathbb{C}}$ de rang $n+1$ correspondant à l'équation différentielle de Bessel $(z\partial_z)^{n+1} - z = 0$. Ils interprètent la cohomologie de de Rham intermédiaire de la connexion Kl_{n+1}, c'est-à-dire l'image du morphisme d'oubli des supports

$$\mathrm{im}(\mathrm{H}^1_{\mathrm{dR},c}(\mathbb{G}_{m,\mathbb{C}}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}) \to \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{m,\mathbb{C}}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})),$$

comme la réalisation de de Rham d'un motif exponentiel sur $\mathbb{Q}$ au sens de [FJ]. Ce motif exponentiel est classique, c'est-à-dire isomorphe à un motif de Nori M^k_{n+1} sur $\mathbb{Q}$. Ce motif est isomorphe à un sous-quotient de $\mathrm{H}_c^{nk-1}(\mathcal{K})(-1)$, où $\mathcal{K}$ est l'hypersurface définie par le polynôme de Laurent[1] $g_{n+1}^{\boxplus k} = \sum_{i=1}^k(\sum_{j=1}^n y_{i,j} + 1/\prod_{j=1}^n y_{i,j})$ dans le tore $\mathbb{G}_m^{nk}$.

[1] C'est la somme de Thom-Sebastiani itérée k fois du polynôme de Laurent $g_{n+1} = \sum_{j=1}^n y_j + 1/\prod_{j=1}^n y_j$.

En suivant la construction de [FSY22], pour chaque λ, on construit dans la définition 2.1 un motif $\mathrm{M}^{\lambda}_{n+1}$ comme un sous-quotient de $\mathrm{H}_c^{n|\lambda|-1}(\mathcal{K})(-1)$, où $\mathcal{K}$ est l'hypersurface définie par le polynôme de Laurent $g_{n+1}^{\boxplus|\lambda|} = \sum_{i=1}^{|\lambda|}(\sum_{j=1}^{n} y_{i,j} + 1/\prod_{j=1}^{n} y_{i,j})$ dans le tore $\mathbb{G}_m^{n|\lambda|}$.

De manière similaire au cas des faisceaux de Kloosterman pour les groupes réductifs, on peut attacher aux paires (G, V) des connexions sur $\mathbb{G}_{m,\mathbb{C}}$ de manière fonctorielle, voir [FG09] et la remarque 2.7. Si $G = \mathrm{SL}_{n+1}$ et V est la représentation de plus haut poids λ, la connection est également notée $\mathrm{Kl}^{\lambda}_{n+1}$. Si on prend $\lambda = (k, 0, \ldots, 0)$, on retrouve $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$.

Ensuite, pour chaque motif $\mathrm{M}^{\lambda}_{n+1}$,

- sa réalisation de de Rham $(\mathrm{M}^{\lambda}_{n+1})_{\mathrm{dR}}$ est isomorphe à la cohomologie de de Rham intermédiaire de $\mathrm{Kl}^{\lambda}_{n+1}$, c'est-à-dire,
$$(\mathrm{M}^{\lambda}_{n+1})_{\mathrm{dR}} \simeq \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{m,\mathbb{C}}, \mathrm{Kl}^{\lambda}_{n+1}),$$
- ses réalisations ℓ-adiques $(\mathrm{M}^{\lambda}_{n+1})_{\ell}$ sont des représentations ℓ-adiques continues de $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ avec coefficients dans $\mathbb{Q}_\ell$, isomorphes à des sous-quotients de $\mathrm{H}^{n|\lambda|-1}(\mathcal{K}_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)(-1)$.

Le théorème suivant nous indique que ces motifs $\mathrm{M}^{\lambda}_{n+1}$ conviennent à notre but.

Théorème C. *Soient $p \neq \ell$ des nombres premiers, et ζ_p une racine de l'unité primitive dans $\bar{\mathbb{Q}}_\ell$. Il existe un ensemble fini $S(\lambda, n+1)$ de nombres premiers tel que, si $p \notin S(\lambda, n+1)$, la réalisation ℓ-adique est non ramifiée et il existe un isomorphisme des représentations de* $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

$$(\mathrm{M}^{\lambda}_{n+1})_{\ell}[\zeta_p] \simeq \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^{\lambda}_{n+1}).$$

Grâce au théorème ci-dessus, on appelle $\mathrm{M}^{\lambda}_{n+1}$ le *motif attaché au faisceau* $\mathrm{Kl}^{\lambda}_{n+1}$. Cet ensemble $S(\lambda, n+1)$ est le même que celui que l'on a utilisé pour définir les fonctions L. Les réalisations ℓ-adiques des motifs $\mathrm{M}^{\lambda}_{n+1}$ sont des familles de représentations galoisiennes, dont les fonctions L partielles $L^S\left((\mathrm{M}^{\lambda}_{n+1})_\ell, s\right)$ sont les fonctions L des $\mathrm{Kl}^{\lambda}_{n+1}$, car les facteurs locaux en $p \notin S$ des premières coïncident avec les facteurs locaux des secondes. Yun a également construit des motifs dans [YV15, §3.2], dont les réalisations ℓ-adiques ont les mêmes fonctions L, d'une manière différente en utilisant des Grassmanniennes affines.

La preuve du théorème C se déroule comme suit. Par le théorème 2.17, on peut exprime la cohomologie ℓ-adique $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^{\lambda}_{n+1})$ comme un sous-quotient de $\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell(\zeta_p))(-1)$, de la même manière qu'on exprime $(\mathrm{M}^{\lambda}_{n+1})_\ell$ comme un sous-quotient de $\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell(\zeta_p))(-1)$. Via le morphisme de spécialisation,

$$\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\mathcal{K}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)(-1) \to \mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\mathcal{K}_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell)(-1)$$

on peut en déduire un morphisme

$$\iota\colon \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^{\lambda}_{n+1}) \to (\mathrm{M}^{\lambda}_{n+1})_\ell[\zeta_p].$$

On peut montrer que ι est une inclusion. De plus, on déduit on peut déduire des dimensions de la source et du but de ι que ι est un isomorphisme tant que $\mathcal{K}_{\bar{\mathbb{F}}_p}$ est lisse en p. On prend alors S comme un ensemble fini de nombres premiers contenant tous les nombre premiers p tels que ι ne soit pas un isomorphisme.

On peut montrer que les représentations galoisiennes $(\mathrm{M}^{\lambda}_{n+1})_\ell$ sont au plus faiblement ramifiées aux nombres premiers $p \in S$ lorsque $p \nmid n+1$ dans le théorème 5.12. Le point clé est que la réduction mod p de l'hypersurface $\mathcal{K}$ ne possède que des singularités isolées, qui sont quadratiques ordinaires si $p \nmid n+1$. On peut donc utiliser la formule de Picard-Lefschetz [DK73, Exp. XV., Thm. 3.4] pour $\mathcal{K}_{\mathbb{Z}_p} \to \mathrm{Spec}(\mathbb{Z}_p)$.

Lorsque l'on traite des conjectures de type Evans, on a besoin de mieux comprendre l'ensemble S. Dans le cas de $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$, on peut décider si $p \notin S$ lorsque $p \nmid n+1$ en calculant les dimensions de la

source et du but de ι. Grâce aux travaux de Fu-Wan [FW05, FW08], on peut calculer la dimension de $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})$ lorsque $p \nmid n+1$, voir Proposition 4.27. Quant à la dimension de $(\mathrm{M}^k_{n+1})_\ell$, puisqu'elle est indépendante de p, on peut choisir un nombre premier p suffisamment grand pour que la dimension de $(\mathrm{M}^k_{n+1})_\ell$ soit égale à celle de $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})$.

Remark. Dans ce qui précède, on impose la condition $p \nmid n+1$ car on n'a pas pu calculer la dimension de $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})$. Lorsque $n=1$, Yun a calculé la dimension de $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_2}, \mathrm{Sym}^k\mathrm{Kl}_2)$ [YV15, Cor. 4.3.5]. Suivant sa méthode, on donne une formule de dimension de $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^k\mathrm{Kl}_3)$ dans le corollaire 4.40.

L'idée du calcul est d'utiliser la classification complète des sous-groupes finis de SL_3 pour trouver le groupe de monodromie local à ∞ de Kl_3, ce qui nous permet de calculer le conducteur de Swan de $\mathrm{Sym}^k\mathrm{Kl}_3$ à ∞, voir Théorème 4.30. Par la formule de Grothendieck-Ogg-Shafarevich [Gro77, X. Théorème 7.1], la dimension de $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^k\mathrm{Kl}_3)$ est égale au conducteur de Swan de $\mathrm{Sym}^k\mathrm{Kl}_3$. En conséquence, le nombre premier $p=3$ n'est pas dans $S(k,3)$ si $3 \nmid k$. On utilisera cette conséquence dans la preuve de théorème 5.12 et celle du théorème 5.22.

La filtration de Hodge irrégulière

Pour appliquer le théorème de Patrikis-Taylor à la famille $\{(\mathrm{M}^\lambda_{n+1})_\ell\}_\ell$, la seule condition non triviale à vérifier est appelée *régularité*, c'est-à-dire que tous les nombres de Hodge des réalisations de de Rham $(\mathrm{M}^\lambda_{n+1})_{\mathrm{dR}}$ sont 0 ou 1. Contrairement au cas des motifs M^k_2 dans [FSY22], où la condition de régularité est toujours satisfaite, on a les résultats suivants.

Théorème D. *(1). La réalisation de de Rham $(\mathrm{M}^k_3)_{\mathrm{dR}}$ est l'espace sous-jacent à une structure de Hodge pure de poids $2k+1$, dont les nombres de Hodge sont*

$$h^{p,2k+1-p} = \begin{cases} \left\lfloor \frac{\min(p,2k+1-p)}{6} \right\rfloor - d_{p,k} & p \equiv 0,1,2,4 \bmod 6; \\ \left\lfloor \frac{\min(p,2k+1-p)}{6} \right\rfloor + 1 - d_{p,k} & p \equiv 3,5 \bmod 6, \end{cases}$$

où $d_{p,k}$ est 1 si $3 \mid k$ et $p=k$, et 0 sinon. En particulier, tous les nombres de Hodge de $(\mathrm{M}^k_3)_{\mathrm{dR}}$ sont 0 ou 1 si $k \leq 9$.

(2). Soit M l'un des motifs M^3_4, M^3_5 et $\mathrm{M}^{(2,1)}_3$. La réalisation de de Rham $(\mathrm{M})_{\mathrm{dR}}$ est l'espace sous-jacent à une structure de Hodge pure, dont les nombres de Hodge sont soit 0 soit 1.

Remarque.

1. Le premier cas ayant des nombres de Hodge supérieurs à 1 que l'on a trouvé est le cas de M^{10}_3, où les nombres de Hodge sont donnés par

$$(h^{p,21-p})_{0\leq p\leq 21} = (0,0,0,1,0,1,1,1,1,2,1,1,2,1,1,1,1,0,1,0,0,0).$$

   Ceci explique pourquoi on ajoute la restriction $k \leq 9$ dans l'énoncé du théorème A. Bien que l'on ne puisse pas appliquer le théorème de Patrikis-Taylor lorsque $k \geq 10$, il serait très intéressant de voir si certains des développements récents sur l'automorphie (potentielle), par exemple, un travail de Boxer-Calegari-Gee-Pilloni [BCGP21], nous permettent d'établir de nouveaux cas au-delà de cette limite sur les nombres de Hodge.

2. Les nombres de Hodge des réalisations de de Rham de M^4_4 et $\mathrm{M}^{(2,2)}_3$ sont également soit 0 soit 1, ce qui peut être déduit du théorème 5.26 et du théorème 5.31, voir remarque 5.27 et remarque 5.32. En particulier, notre méthode de calcul des nombres de Hodge n'est pas valable pour le cas de $\mathrm{M}^{(2,2)}_3$.

3. En utilisant les mêmes méthodes de calcul des nombres de Hodge irréguliers, on peut trouver une alternative au travail de Sabbah-Yu [SY19] pour calculer les nombres de Hodge irréguliers des connexions hypergéométriques [QX].

Pour prouver le théorème D, on suit les méthodes de [FSY22].

- En utilisant l'isomorphisme

$$(\mathrm{M}_{n+1}^{\lambda})_{\mathrm{dR}} \simeq \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda}),$$

  la réalisation de de Rham est l'espace sous-jacent à une *structure de Hodge mixte exponentielle*, défini par Kontsevich et Soibelman [KS11], désignée par $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$. En étendant une idée de Deligne dans [Del07], Sabbah a associé une *filtration de Hodge irrégulière* à chaque structure de Hodge mixte exponentielle dans [Sab10].

- Puisque le motif $\mathrm{M}_{n+1}^{\lambda}$ est un motif de Nori, la réalisation de de Rham est également l'espace sous-jacent à une structure de Hodge classique. Par [FSY22, A.13], les deux filtrations de Hodge sur $(\mathrm{M}_{n+1}^{\lambda})_{\mathrm{dR}}$ coïncident. Par conséquent, pour prouver le théorème D, il suffit de calculer la filtration de Hodge irrégulière sur $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$.

- On plonge la cohomologie de Rham $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$ dans la cohomologie de Rham tordue $\mathrm{H}^{n|\lambda|+1}_{\mathrm{dR}}(\mathbb{G}_m^{n|\lambda|+1}, f_{|\lambda|})$, de manière compatible à la filtration de Hodge irrégulière. Ici

$$f_{|\lambda|} = \sum_{i=1}^{|\lambda|} \left(\sum_{j=1}^{n} x_{i,j} + \frac{z}{\prod_j x_{i,j}} \right).$$

- On peut obtenir une base $\{\omega\}$ de la cohomologie de Rham $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$ pour les cas du théorème D, voir le théorème 3.13 et la section 3.4.1.

- Si $\mathrm{pgcd}(|\lambda|, n+1) = 1$ (dans le théorème D, seul les M_3^{3k} ne satisfont pas cette condition), $f_{|\lambda|}$ est non dégénéré par rapport au polyèdre de Newton $\Delta(f_{|\lambda|})$, voir la section 1.3. Par les travaux d' Adolphson-Sperber [AS97], Esnault-Sabbah-Yu [ESY17], et Yu [Yu14], la filtration sur la cohomologie de de Rham tordue a une interprétation géométrique en termes de ce que l'on appelle la *filtration du polyèdre de Newton*, qui peut être calculée en utilisant la combinatoire des polyèdres. Il s'ensuit que les éléments ω dans la base se trouvent dans

$$F^{p(\omega)}\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$$

  pour certains entiers $p(\omega)$ dépendant de ω. Grâce à nos choix particuliers de bases, on conclut par la symétrie de Hodge que chaque ω a une classe non nulle dans $\mathrm{gr}_F^{p(\omega)}\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$. Par conséquent, les éléments des bases sont adaptés à la filtration de Hodge.

- Cependant, si $\mathrm{pgcd}(|\lambda|, n+1) > 1$, comme dans le cas où k est pair dans [FSY22, SY21], on a besoin d'informations supplémentaires pour terminer le calcul, voir les sections 3.3.1, 3.3.2 et 3.3.5.

Remarque. Par rapport à la preuve de [FSY22, Th. 1.8], la difficulté principale dans celle du théorème D est l'étape de calcul des bases. Ceci est dû à la complexité des actions de monodromie locale des connexions de Kloosterman $\mathrm{Kl}_{n+1}^{\lambda}$ en 0. Par exemple, la partie nilpotente de l'action de monodromie locale en 0 de $\mathrm{Sym}^k\mathrm{Kl}_2$ n'a qu'un seul bloc de Jordan pour chaque k, mais celles de $\mathrm{Sym}^k\mathrm{Kl}_3$, $\mathrm{Sym}^3\mathrm{Kl}_4$, $\mathrm{Sym}^4\mathrm{Kl}_4$, $\mathrm{Sym}^3\mathrm{Kl}_5$, $\mathrm{Kl}_3^{(2,1)}$ ont plusieurs blocs de Jordan (de tailles différentes). Ce phénomène cause les difficultés principales pour trouver des bases.

Preuve des conjectures de type Evans

Soit M un motif attaché à l'un des faisceaux du théorème B. Pour prouver les conjectures de type Evans, il suffit de montrer que $\mathrm{M}_\ell \simeq \rho_{f,\ell}(h)$ pour une forme propre de Hecke cuspidale normalisée f et un nombre entier h. Ici, $\rho_{f,\ell}$ est une représentation galoisienne de dimension deux de $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ attachée à f, construite par les travaux de Eichler-Shimura pour $k=2$, Deligne pour $k\geq 2$ [Del71], et Deligne-Serre [DS74] pour $k=1$.

L'argument de la preuve est dû originellement à Serre [Ser87, §4.8], et on peut trouver des arguments similaires dans [Kis07, Th. 1.4.3] et [YV15, Th. 4.6.1]. On esquisse une preuve dans le théorème 1.17 du chapitre 1. L'ingrédient clé de la preuve est la conjecture de modularité de Serre [Ser87] (prouvée dans [KW09a] et [KW09b]).

Pour appliquer les théorèmes ci-dessus, on doit vérifier que M_ℓ est impair, c'est-à-dire que la valeur du déterminant de la conjugaison complexe est -1, ce qui est garanti par le théorème ci-dessous et le théorème de densité de Chebotarev.

Théorème E. *Soit M^λ_{n+1} l'un des motifs apparaissent dans le Théorème A.*

(1). Lorsque $n|\lambda|$ est pair et $p\notin S\cup\{\ell\}$, on a

$$\det((\mathrm{M}^\lambda_{n+1})_\ell)(\mathrm{Frob}_p)=p^{\frac{n|\lambda|+1}{2}\dim \mathrm{M}^\lambda_{n+1}}.$$

(2). Lorsque $p\notin\{2,3,5\}\cup\ell$, on a

$$\det((\mathrm{M}^3_4)_\ell)(\mathrm{Frob}_p)=\Big(\frac{p}{15}\Big)p^{10}.$$

La première partie du théorème ci-dessus est facile à prouver, car la représentation $(\mathrm{M}^\lambda_{n+1})_\ell$ est symplectique par la proposition 2.3. En revanche, la deuxième partie nécessite plus d'efforts. On esquisse la preuve comme suit.

- Grâce au théorème C, les valeurs des déterminants des représentations galoisiennes de Frob_p lorsque $p\notin S\cup\{\ell\}$ sont les facteurs ε du faisceau de Kloosterman $\mathrm{Sym}^3\mathrm{Kl}_4$, c'est-à-dire,

$$\varepsilon\big(\mathbb{P}^1_{\bar{\mathbb{F}}_p},j_*\mathrm{Sym}^3\mathrm{Kl}_4\big):=\prod_{i=0}^{2}\det\big(-\mathrm{Frob}_k\mid \mathrm{H}^i_{\acute{e}t}\big(\mathbb{P}^1_{\bar{\mathbb{F}}_p},j_*\mathrm{Sym}^3\mathrm{Kl}_4\big)\big)^{(-1)^{i+1}},$$

  où $j\colon\mathbb{G}_m\hookrightarrow\mathbb{P}^1$ est l'inclusion.

- On calcule le facteur ε en utilisant un argument similaire à celui de [FW10] pour les faisceaux $\mathrm{Sym}^k\mathrm{Kl}_2$. L'idée est d'utiliser la formule du produit de Laumon [Lau87] pour obtenir

$$\varepsilon\big(\mathbb{P}^1_{\bar{\mathbb{F}}_p},j_*\mathrm{Sym}^3\mathrm{Kl}_4\big)=p^{15}\prod_{x\in|\mathbb{P}^1_{\bar{\mathbb{F}}_p}|}\varepsilon\big(\mathbb{P}^1_{(x)},j_*\mathrm{Sym}^3\mathrm{Kl}_4|_{\mathbb{P}^1_{(x)}},\mathrm{d}t|_{\mathbb{P}^1_{(\infty)}}\big),$$

  où $\mathbb{P}^1_{(x)}$ est la localisation stricte de $\mathbb{P}^1$ en x, $\mathrm{d}t$ est une forme différentielle méromorphe sur $\mathbb{P}^1$, et $\varepsilon\big(\mathbb{P}^1_{(x)},\mathrm{Sym}^3\mathrm{Kl}_4|_{\mathbb{P}^1_{(x)}},\mathrm{d}t|_{\mathbb{P}^1_{(x)}}\big)$ sont des facteurs epsilon locaux, définis dans [Del73, 4.1].

- On peut calculer les facteurs epsilon locaux par les axiomes et les propriétés des facteurs epsilon locaux [Lau87, (3.1.3.2), (3.1.5.4), (3.1.5.5)&(3.1.5.6)], voir la section 1.4.4. On a donc besoin des structures locales du faisceau $\mathrm{Sym}^3\mathrm{Kl}_4$ en 0 et en ∞, qui peuvent être déduites de celles de Kl_4.

- Katz a déterminé la structure locale de Kl_4 en 0 dans [Kat88, 7.0,7]. Cependant, les structures locales de Kl_4 en ∞ ne sont déterminées qu'après un changement de base vers une certaine extension finie de $\mathbb{F}_p$ par Fu et Wan [FW05], car la racine de l'unité ζ_4 n'est pas toujours dans $\mathbb{F}_p$. Notre solution s'inspire de la méthode de [FW10] pour la structure local du faisceau Kl_2 en ∞, qui utilise le principe de la phase stationnaire [Lau87, Proposition 2.3.2.1], c'est-à-dire que la structure locale de la transformée de Deligne-Fourier d'un faisceau $\mathscr{F}$ sur $\mathbb{A}^1$ à ∞ est déterminée par les structures locales de $\mathscr{F}$ en ses points de non-lissités et en ∞.
- Pour notre propos, il suffit de caractériser les structures locales du faisceau $\mathrm{R}^{n-1}(g_{n+1})_!\bar{\mathbb{Q}}_\ell$ aux valeurs critiques de g_{n+1}, où $g_{n+1} = \sum_{i=1}^n y_i + 1/\prod y_i$. Lorsque $p \nmid n+1$, tous les points critiques de g sont quadratiques ordinaires. On utiliser la formule de Picard-Lefschetz [DK73, Exp. XV., Th. 3.4] pour montrer que $\mathrm{R}^{n-1}g_!\bar{\mathbb{Q}}_\ell|_{\mathbb{P}^1_{(x)}}$ est une extension d'un faisceau localement constant et de la somme directe de plusieurs faisceaux de rang 1 supportés au point générique de $\mathbb{P}^1_{(x)}$.

Après avoir montré la modularité, on connaît l'existence des formes modulaires. Il ne reste plus qu'à déterminer les formes modulaires de manière aussi explicite que possible. On calcule les traces de Frobenius de M_ℓ pour un nombre suffisant de nombres premiers (les coefficients de Fourier de la forme modulaire correspondante) en utilisant Sagemath [The22]. Ensuite, on utilise les informations de la géométrie (poids, conducteur, déterminant) et les calculs des traces de Frobenius pour rechercher la forme modulaire dans la base de données LMFDB.

Organisation de la thèse

- Dans le chapitre 1, nous rappelons quelques résultats qui seront utilisés dans les preuves des théorèmes principales.
- Dans le chapitre 2, on construit les motifs attachés aux faisceaux de Kloosterman et étudie leurs réalisations. On définit les motifs dans la section 2.1. Puis, dans la section 2.2, on montre que les réalisations de de Rham de tels motifs sont les espaces sous-jacents à des structures de Hodge mixtes exponentielles, isomorphes à des structures de Hodge classiques. Dans la section 2.3, on résume les propriétés d'autres réalisations en caractéristique $p > 0$.
- Dans le chapitre 3, on prouve théorème D. Dans la section 3.1, on étudie la structure formelle locale des connexions $\mathrm{Sym}^k\mathrm{Kl}_3$ en 0 et en ∞ et donne la formule pour la dimension de la cohomologie de de Rham intermédiaire. Ensuite, dans la section 3.2, on détermine une base de classes de cohomologie de la réalisation de de Rham de $(\mathrm{M}_3^k)_{\mathrm{dR}}$ dans le théorème 3.13. Ensuite, dans la section 3.3, on prouve le théorème 3.33, qui implique la première partie du théorème D. Enfin, dans la section 3.4, les cas de la deuxième partie du théorème D sont prouvés au cas par cas.
- Dans le chapitre 4, on étudie les propriétés des faisceaux de Kloosterman (principalement celles qui apparaissent dans le théorème A), par exemple, les structures locales à 0 et à ∞ dans la section 4.2 et la section 4.3. Dans la section 4.4, on donne la formule de dimension de la cohomologie ℓ-adique intermédiaire de tels faisceaux lorsque $\gcd(p, n+1) = 1$, et la formule pour la dimension de cohomologie ℓ-adic intermédiaire de $\mathrm{Sym}^k\mathrm{Kl}_3$ lorsque $p = 3$. Dans la section 4.5, on prouve le théorème E.
- Dans le chapitre 5, on étudie les rafimications des representations galoisiennes $(\mathrm{M}_{n+1}^\lambda)_\ell$ dans la section 5.1 et on prouve le théorème C. Dans la section 5.2, on prouve la première partie du théorème A. Dans la section 5.3, on prouve le théorème B. En conséquence, le théorème A est prouvée.
- Dans l'appendice A, on explique le calcul des moments des faisceaux de Kloosterman et en inclut le code de Sagemath [The22].

Introduction (in English)

The theory of L-functions plays a crucial role in number theory. Generally speaking, an L-function is a Dirichlet series $L(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ converging for $\mathrm{Re}(s)$ big enough, extending to a meromorphic function on $\mathbb{C}$, satisfying a functional equation, and having an Euler product. The most famous example is the Riemann zeta function $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$, a meromorphic function on $\mathbb{C}$ (with a simple pole at $s = 1$) [Rie59] that satisfies a functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s),$$

predicted by Euler in 1748. Moreover, it has the Euler product $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$. Since L-functions are analytic objects, they provide us with a way to study arithmetic objects using tools from analysis. For example, the fact that $\zeta(s)$ has a simple pole implies that there are infinitely many prime numbers, and the nonvanishing of the values of Dirichlet L-functions at 1 implies the Dirichlet's theorem on arithmetic progressions. Besides, the special values at integers of L-functions could be related to periods [KZ01, §1.1].

One can associate L-functions with certain exponential sums. The *Kloosterman sums* are exponential sums over finite fields, defined for each power of prime numbers $q = p^r$ and each $a \in \mathbb{F}_q$, by

$$\mathrm{Kl}_2(a; q) := \sum_{x\in\mathbb{F}_q^\times} \exp\left(2\pi i/p \cdot \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\left(x + \frac{a}{x}\right)\right),$$

where $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ is the trace from $\mathbb{F}_q$ to $\mathbb{F}_p$. If $a \neq 0$, Weil showed in [Wei48] that $\mathrm{Kl}_2(a; q) = -(\alpha_a+\beta_a)$ for some algebraic numbers α_a, β_a of norm $p^{1/2}$. In particular, one gets an upper bound $|\mathrm{Kl}_2(a, q)| \leq 2p^{1/2}$. For each $k \geq 1$, the *k-th symmetric power moments of Kloosterman sums* are integers of the form $m_2^k(q) = \sum_{a\in\mathbb{F}_q} \sum_{i=0}^k \alpha_a^i\beta_a^{k-i}$. In [Bro16, Bro17], the *L-functions $L_k(s)$ attached to the k-th symmetric power moments of Kloosterman sums* are defined as Euler products, where the local factors at p are made from the generating series

$$\exp\left(\sum_{r\geq 1} \frac{m_2^k(p^r)}{r} \cdot T^r\right).$$

These L-functions are a priori defined on the domain $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1 + \frac{k+1}{2}\}$ by construction and the work of Fu-Wan [FW05]. Broadhurst and Roberts conjectured that these L-functions extend meromorphically to $\mathbb{C}$, and they predicted the precise formulas of the functional equations of $L_k(s)$.

It is not always easy to show that an L-function has meromorphic continuation to the complex plane. For example, one can show that the L-functions of modular forms are holomorphic functions on $\mathbb{C}$ directly by tools from analysis, similar to the case of the Riemann zeta function. However, to show the same things for L-functions of elliptic curves, the only know way is to identify them with L-functions of modular forms. The link between two kinds of L-functions is established by the modularity theorem (Taniyama-Shimura Conjecture) [BCDT01], whose weaker form led to the proof of Fermat's last theorem by Wiles [Wil95].

In [FSY22], Fresán, Sabbah, and Yu constructed a family of Galois representations of geometric origin, whose L-functions are exactly $L_k(s)$. Then, they used the Langlands philosophy to prove the

conjecture of Broadhurst and Roberts, by showing that the L-functions of Galois representations are equal to automorphic L-functions via a theorem of Patrikis-Taylor [PT15]. To summarize, they proved the following theorem.

Theorem (Fresán-Sabbah-Yu)**.** *The L-function $L_k(s)$ can be extended meromorphically to the complex plane. And we can complete $L_k(s)$ to a complete L-function $\Lambda_k(s)$ such that*

$$\Lambda_k(s) = \epsilon_k \Lambda_k(k+2-s),$$

where $\epsilon_k \in \{\pm 1\}$ and ϵ_k is 1 if k is odd.

In a sequel paper of Fresán, Sabbah, and Yu [FSY20a], they also explain some relations between special values of the L-functions at critical integers and certain determinants of Bessel moments

$$\int_0^\infty I_0(z)^a K_0(z)^{k-a} z^b \frac{\mathrm{d}z}{z},$$

where $I_0(z)$ and $K_0(z)$ are the modified Bessel functions of the first and the second kind (solutions of $(z\partial_z)^2 - z^2 = 0$).

This thesis aims to generalize the above theorem to L-functions attached to moments (not necessarily symmetric power moments) of Kloosterman sums of many variables.

Main results

The L-functions of Kloosterman sheaves

The Kloosterman sums in n variables are the exponential sums over finite fields defined, for each power of prime numbers $q = p^r$ and each $a \in \mathbb{F}_q^\times$, by

$$\mathrm{Kl}_{n+1}(a;q) := \sum_{x_1,\ldots,x_n \in \mathbb{F}_q^\times} \exp\left(2\pi i/p \cdot \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\left(x_1 + \ldots + x_n + \frac{a}{x_1 \cdots x_n}\right)\right).$$

They are the finite field analogs of the Bessel functions

$$\oint_{(S^1)^n} \exp\left(x_1 + \ldots + x_n + \frac{z}{x_1 \cdots x_n}\right) \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n},$$

satisfying the Bessel differential equations $(z\partial_z)^{n+1} - z = 0$.

Fixing a prime number $\ell \neq p$ and an embedding $\iota\colon \bar{\mathbb{Q}}_\ell \to \mathbb{C}$, Deligne constructed in [Del77, Sommes. Trig. Thm. 7.8] some lisse ℓ-adic sheaves Kl_{n+1} of rank $n+1$ over $\mathbb{G}_{m,\mathbb{F}_q} = \mathbb{A}^1_{\mathbb{F}_q} \backslash \{0\}$, called the *Kloosterman sheaves*, such that for every $a \in \mathbb{F}_q^\times = \mathbb{G}_{m,\mathbb{F}_q}(\mathbb{F}_q)$ and every geometric point $\bar{a}$ localized at a, the complex norms of eigenvalues of Frob_q are $q^{n/2}$ via the embedding ι, and we have

$$\iota \circ \mathrm{Tr}(\mathrm{Frob}_q, (\mathrm{Kl}_{n+1})_{\bar{a}}) = (-1)^n \mathrm{Kl}_{n+1}(a;q).$$

As a direct consequence, we have the upper bound $|\mathrm{Kl}_{n+1}(a;q)| \leq (n+1)q^{n/2}$.

Viewing lisse ℓ-adic sheaves Kl_{n+1} on $\mathbb{G}_m$ as continuous ℓ-adic representations $\rho_{\mathrm{Kl}_{n+1}}$ of the étale fundamental group $\pi_1^{\acute{e}t}(\mathbb{G}_{m,\mathbb{F}_q})$, the *arithmetic (resp. geometric) monodromy groups* are defined as the Zariski closures of the images of $\pi_1^{\acute{e}t}(\mathbb{G}_{m,\mathbb{F}_q})$ (resp. $\pi_1^{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_q})$) in $\mathrm{GL}_{n+1}(\bar{\mathbb{Q}}_\ell)$ under $\rho_{\mathrm{Kl}_{n+1}}$. In [Kat88], Katz showed that the arithmetic monodromy groups and the geometric monodromy groups of Kl_{n+1} coincide, and are equal to groups

$$\begin{cases} \mathrm{SP}_{n+1} & n+1 \text{ even}, \\ \mathrm{SL}_{n+1} & p(n+1) \text{ odd}, \\ \mathrm{SO}_{n+1} & p = 2,\ n+1 \text{ odd and } n \neq 6, \\ G_2 & p = 2, n = 6. \end{cases}$$

The appearance of the exceptional group G_2 seems quite surprising. Katz raised the question [Kat90] that if all semisimple groups appear as monodromy groups of lisse ℓ-adic sheaves.

In [HNY13], Heinloth, Ngô, and Yun constructed *Kloosterman sheaves for reductive groups* in a uniform way using methods from the geometric Langlands program. More precisely, we can attach to each pair (G, V), consisting of a split reductive group G and a finite-dimensional representation V of G, a lisse ℓ-adic sheaf Kl_G^V on $\mathbb{G}_m$ functorially. They found that the exceptional groups E_7, E_8, G_2, F_4 appear among the monodromy groups of Kloosterman sheaves for reductive groups. As a special case, we recover the classical Kloosterman sheaf $\mathrm{Kl}_{n+1}(-\frac{n}{2})$ by taking $G = \mathrm{SL}_{n+1}$ and V to be the standard representation of SL_{n+1}.

We will mainly be interested in the case that $G = \mathrm{SL}_{n+1}$ and $V = V_\lambda$ is the finite-dimensional representation of the highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$ of SL_{n+1}. For a weight λ, we denote by $|\lambda| = \sum_{i=1}^n \lambda_i$, and by $\mathrm{Kl}_{n+1}^\lambda$ the sheaf $\mathrm{Kl}_{\mathrm{SL}_{n+1}}^{V_\lambda}(-\frac{nk}{2})$ for simplicity. If V is the k-th symmetric power of the standard representation, i.e., $\lambda = (k, 0, \dots, 0)$ (we identify λ and k in this case), the sheaf $\mathrm{Kl}_{n+1}^\lambda$ is the k-th symmetric power of Kl_{n+1}, denoted by $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$.

For each Kloosterman sheaf $\mathscr{F} = \mathrm{Kl}_{n+1}^\lambda$, following the construction in [FSY22], we introduce an L-function $L(s)$ as an Euler product, where the local factors are constructed using the *moments* of $\mathrm{Kl}_{n+1}^\lambda$, i.e., the integers of the form

$$m_{n+1}^\lambda(q) := -\sum_{a \in \mathbb{F}_q^\times} \mathrm{Tr}(\mathrm{Frob}_q, (\mathrm{Kl}_{n+1}^\lambda)_{\bar{a}}).$$

The generating series $Z(\lambda, n+1, p; T)$ of $m_{n+1}^\lambda(q)$ is defined as $\exp(\sum_{r\geq 1} m_{n+1}^\lambda(p^r)/r \cdot T^r)$. By the Grothendieck trace formula [Del77, Rapport. Thm. 3.1], we have

$$Z(\lambda, n+1, p; T) = \prod_{i=0}^{2} \det(1 - \mathrm{Frob}_p T \mid \mathrm{H}^i_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda))^{(-1)^{i+1}},$$

which is a rational function. However, the complex norms of roots and poles of $Z(\lambda, n+1, p; T)$ ranges within the set $\{p^{-i/2} \mid 0 \leq i \leq n|\lambda|\}$. So $Z(\lambda, n+1, p; T)$ is not an ideal candidate for the local factors of the Euler product. Similar to the work of Fu-Wan [FW05, FW08] for sheaves $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$, we can define

$$M(\lambda, n+1, p; T) = \det(1 - \mathrm{Frob}_p T \mid \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda)),$$

where

$$\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda) = \mathrm{im}(\mathrm{H}^1_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda) \to \mathrm{H}^1_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_{n+1}^\lambda))$$

is the middle ℓ-adic cohomology. The polynomial $M(\lambda, n+1, p; T)$ is, in fact, a factor of $Z(\lambda, n+1, p; T)$, and we will prove in Theorem 5.4 that there exists a finite set $S(\lambda, n+1)$ of primes such that the degree of $M(\lambda, n+1, p; T)$ is constant for $p \notin S$. For simplicity, we denote by S the set $S(\lambda, n+1)$ if there is no confusion.

We now build a partial L-function $L^S(\lambda, n+1; s)$ of $\mathscr{F}$, by considering the Euler product

$$L^S(\lambda, n+1; s) := \prod_{s \notin S} M(\lambda, n+1, p; p^{-s})^{-1},$$

which is a priori a holomorphic function on the domain $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1 + \frac{n|\lambda|+1}{2}\}$. But we do not know more information from the definition. In particular, when $\mathscr{F} = \mathrm{Sym}^k\mathrm{Kl}_2$, the function $L^S(\lambda, n+1; s)$ is nothing but the function $L_k^S(\lambda, n+1; s)$ attached to k-th symmetric power of Kloosterman sum.

One can ask if the partial L-function $L^S(\lambda, n+1; s)$ can be extended meromorphically to the complex plane and satisfies a functional equation. The Theorem of Fresán-Sabbah-Yu answers the cases of $\mathrm{Sym}^k\mathrm{Kl}_2$. Here is the main result of this thesis.

Theorem A. *Let $\mathscr{F}$ be in one of the following two set:*

(1). $\{\mathrm{Sym}^k\mathrm{Kl}_3 \mid k \leq 9\}$,

(2). $\{\mathrm{Sym}^4\mathrm{Kl}_3, \mathrm{Sym}^3\mathrm{Kl}_4, \mathrm{Sym}^4\mathrm{Kl}_4, \mathrm{Sym}^3\mathrm{Kl}_5, \mathrm{Kl}_3^{(2,1)}, \mathrm{Kl}_3^{(2,2)}\}$.

Then the partial L-function $L^S(\lambda, n+1; s)$ of $\mathscr{F}$ can be extended meromorphically to the complex plane, and can be completed to a function $\Lambda(\lambda, n+1; s)$ which satisfies a functional equation

$$\Lambda(\lambda, n+1; s) = \pm\Lambda(\lambda, n+1; n|\lambda| + 2 - s).$$

Conjectures of Evans type

Theorem A.2 is related to the conjectures of Evans type from [YV15], which predicts some relations between moments of some Kloosterman sheaves $\mathrm{Kl}_{n+1}^{\lambda}$ and Fourier coefficients of some normalized Hecke eigenforms. When $\mathscr{F} = \mathrm{Sym}^k\mathrm{Kl}_2$ for $5 \leq k \leq 8$, we have the following known examples.

- When $k = 5$, there exists a normalized cuspidal Hecke eigenform $f \in S_3\big(\Gamma_0(15), \big(\frac{\cdot}{15}\big)\big)$ such that

  $$a_f(p) = -\frac{1}{p^2}(m_2^5(p) + 1)$$

  if $p \nmid 15$ proved by Peters *et al.* [PTvdV92] and Livné [Liv95].

- When $k = 6$, there exists a normalized cuspidal Heck eigenform $f \in S_4(\Gamma_0(6))$ such that

  $$a_f(p) = -\frac{1}{p^2}(m_2^6(p) + 1)$$

  if $p \nmid 6$, proved by Helek *et al.* [HSvGvS01].

- When $k = 7$, there exists a normalized cuspidal Hecke eigenform $f \in S_3(\Gamma_0(525), \epsilon_f)$, where $\epsilon_f = \big(\frac{\cdot}{21}\big) \cdot \epsilon_5$ and ϵ_5 is a quartic character with conductor 5, such that

  $$a_f(p)^2\epsilon_f(p)^{-1} - p^2 = -\frac{1}{p^2}\left(\frac{p}{105}\right)(m_2^7(p) + 1)$$

  for $p > 7$, conjectured by Evans [Eva10a] and proved by Yun [YV15].

- When $k = 8$, there exists a normalized cuspidal Hecke eigenform $f \in S_6(\Gamma_0(6))$, such that

  $$a_f(p) = -\frac{1}{p^2}(m_2^8(p) + 1)$$

  for $p \nmid 6$, conjectured by Evans [Eva10b] and proved by Yun and Vincent [YV15].

In particular, we can determine all these modular forms by computing enough values of moments. Because of the identities between moments and Fourier coefficients of modular forms, we can identify L-functions of $\mathrm{Sym}^k\mathrm{Kl}_2$ and L-functions of corresponding modular forms. So the L-functions of $\mathrm{Sym}^k\mathrm{Kl}_2$ for $5 \leq k \leq 8$ have meromorphic continuations to $\mathbb{C}$ and satisfy functional equations.

Yun predicted some new results of this form in [YV15] for Kloosterman sheaves for reductive groups. We will prove these conjectures of Evans type for the sheaves listed in the second case of Theorem A.

Theorem B. *The L-functions of the Kloosterman sheaves* $\mathrm{Sym}^4\mathrm{Kl}_3$, $\mathrm{Sym}^3\mathrm{Kl}_4$, $\mathrm{Sym}^4\mathrm{Kl}_4$,$\mathrm{Sym}^3\mathrm{Kl}_5$, $\mathrm{Kl}_3^{(2,1)}$ *and* $\mathrm{Kl}_3^{(2,2)}$ *arise from modular forms. We summarize the information of these modular forms $f \in S_k(\Gamma_0(N), \epsilon)$ in the following table. Here LMFDB [LMF22] is the database of L-functions and modular forms.*

Sheaves	N	k	ϵ	*labels in LMFDB*
$\mathrm{Sym}^4\mathrm{Kl}_3$	14	4	1	*14.4.a.b*
$\mathrm{Sym}^3\mathrm{Kl}_4$	15	3	$\left(\frac{\bullet}{15}\right)$	*15.3.a.b.*
$\mathrm{Sym}^4\mathrm{Kl}_4$	10	6	1	*10.6.a.a.*
$\mathrm{Sym}^3\mathrm{Kl}_5$	33	4	1	33.4.*a.b*
$\mathrm{Kl}_3^{(2,1)}$	14	2	1	*14.2.a.a*
$\mathrm{Kl}_3^{(2,2)}$	6	4	1	*6.4.a.a*

In particular, we have similar equalities between moments of Kloosterman sheaves and Fourier coefficients of modular forms in Theorems 5.22, 5.23, 5.26, 5.28, 5.30 and 5.31, which implies that the L-functions of these sheaves are L-functions of the corresponding modular forms. So Theorem B provides a proof for Theorem A.2.

Remark. One interesting corollary of the above theorem is that for $p \nmid 6$, the moments of the sheaves $\mathrm{Sym}^6\mathrm{Kl}_2$ and $\mathrm{Kl}_3^{(2,2)}$ are the same, because they are both equal to the Fourier coefficient of the modular form of label 6.4.a.a.

Idea of the proof

As explained before, the strategy to prove Theorem A is to construct families of Galois representations (of geometric origins), and show that their L-functions are automorphic.

Galois representations arising from geometry

In [FSY22], inspired by the analogy between Kloosterman sums and Bessel functions, the authors consider the *Kloosterman connection*, also denoted by Kl_{n+1}, which is the rank $n+1$ connection on $\mathbb{G}_{m,\mathbb{C}}$ corresponding to the Bessel differential equation $(z\partial_z)^{n+1} - z = 0$. They interpret the middle de Rham cohomology of the connection $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$, i.e., image of the forget supports morphism from the cohomology with compact support to the usual cohomology

$$\mathrm{im}(\mathrm{H}^1_{\mathrm{dR},c}(\mathbb{G}_{m,\mathbb{C}}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}) \to \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{m,\mathbb{C}}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})),$$

as the de Rham realization of an exponential motive over $\mathbb{Q}$ in the sense of [FJ]. This exponential motive is classical, i.e., isomorphic to a Nori motive M^k_{n+1} over $\mathbb{Q}$. This motive is isomorphic to a subquotient of $\mathrm{H}_c^{nk-1}(\mathcal{K})(-1)$, where $\mathcal{K}$ is the hypersurface defined by the Laurent polynomial[2] $g_{n+1}^{\boxplus k} = \sum_{i=1}^k(\sum_{j=1}^n y_{i,j} + 1/\prod_{j=1}^n y_{i,j})$ in the torus $\mathbb{G}_m^{nk}$.

Following the construction in [FSY22], for each λ, we define in definition 2.1 a motive M^λ_{n+1} as a subquotient of $\mathrm{H}_c^{n|\lambda|-1}(\mathcal{K})(-1)$, where $\mathcal{K}$ is the hypersurface defined by the Laurent polynomial $g_{n+1}^{\boxplus|\lambda|} = \sum_{i=1}^{|\lambda|}(\sum_{j=1}^n y_{i,j} + 1/\prod_{j=1}^n y_{i,j})$ inside the torus $\mathbb{G}_m^{n|\lambda|}$.

Similar to the case of Kloosterman sheaves for reductive groups, one can attach connections on $\mathbb{G}_{m,\mathbb{C}}$ to pairs (G, V) consisting of a simple group G and a finite dimensional representation V of G in a functorial way, see [FG09] and Remark 2.7. If $G = \mathrm{SL}_{n+1}$ and V is the representation of the highest weight λ, the connection is also denoted by $\mathrm{Kl}^\lambda_{n+1}$. If we take $\lambda = (k, 0, \ldots, 0)$, we recover $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$.

Then, for each motive M^λ_{n+1},

- its de Rham realization $(\mathrm{M}^\lambda_{n+1})_{\mathrm{dR}}$ is isomorphic to the middle de Rham cohomology of $\mathrm{Kl}^\lambda_{n+1}$.
- its ℓ-adic realizations $(\mathrm{M}^\lambda_{n+1})_\ell$ are continuous ℓ-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ with coefficients in $\mathbb{Q}_\ell$, isomorphic to subquotients of $\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)(-1)$.

[2] This is the k-th iterated Thom-Sebastiani sum of the Laurent polynomial $g_{n+1} = \sum_{j=1}^n y_j + 1/\prod_{j=1}^n y_j$.

The following theorem tells us that these motives $\mathrm{M}^{\lambda}_{n+1}$ satisfy our need.

Theorem C. *Let $p \neq \ell$ be two prime numbers, and ζ_p be a primitive p-th root of unity in $\bar{\mathbb{Q}}_\ell$. There exists a finite set $S(\lambda, n+1)$ of primes such that, if $p \notin S(\lambda, n+1)$, the ℓ-adic realization $(\mathrm{M}^{\lambda}_{n+1})_\ell$ is unramified, and there is an isomorphism of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations*

$$(\mathrm{M}^{\lambda}_{n+1})_\ell[\zeta_p] \simeq \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^{\lambda}_{n+1}).$$

Thanks to Theorem C, we also call $\mathrm{M}^{\lambda}_{n+1}$ the *motive attached to the sheaf* $\mathrm{Kl}^{\lambda}_{n+1}$. Here the set $S(\lambda, n+1)$ is the same as the set that we used to define the L-functions. The ℓ-adic realizations of the motives $\mathrm{M}^{\lambda}_{n+1}$ form families of Galois representations, whose partial L-functions $L^S\left((\mathrm{M}^{\lambda}_{n+1})_\ell, s\right)$ are the L-functions $L^S(\lambda, n+1; s)$ of $\mathrm{Kl}^{\lambda}_{n+1}$, because the local factors at $p \notin S$ of $L^S\left((\mathrm{M}^{\lambda}_{n+1})_\ell, s\right)$ agree with local factors of $L^S(\lambda, n+1; s)$. Yun also constructed motives in [YV15, §3.2], whose ℓ-adic realizations have the same L-functions, differently using affine Grassmannians.

The proof of Theorem C goes as follows. By Theorem 2.17, we can express the ℓ-adic cohomology of Kloosterman sheaves $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^{\lambda}_{n+1})$ as a subquotient of $\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell(\zeta_p))(-1)$, in the same way as expressing $(\mathrm{M}^{\lambda}_{n+1})_\ell$ as a subquotient of $\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell(\zeta_p))(-1)$. Via the specialization morphism,

$$\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\mathcal{K}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)(-1) \to \mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\mathcal{K}_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell)(-1)$$

we can derive a morphism

$$\iota\colon \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^{\lambda}_{n+1}) \to (\mathrm{M}^{\lambda}_{n+1})_\ell[\zeta_p].$$

We can show that ι is an inclusion. Moreover, we can infer from the dimensions of the source and the target of ι that ι is an isomorphism as long as $\mathcal{K}_{\bar{\mathbb{F}}_p}$ is smooth at p. We then take S to be a finite set of primes containing all primes p such that ι fails to be an isomorphism.

We can show that the Galois representation $(\mathrm{M}^{\lambda}_{n+1})_\ell$ are at most tamely ramified at primes $p \in S$ when $p \nmid n+1$ in Theorem 5.12. The key point is that the mod p reduction of the hypersurface $\mathcal{K}$ only has isolated singularities, which are ordinary quadratic if $p \nmid n+1$. So we can use the Picard-Lefschetz formula [DK73, Exp. XV., Thm. 3.4] to $\mathcal{K}_{\mathbb{Z}_p} \to \mathrm{Spec}(\mathbb{Z}_p)$.

When we deal with conjectures of Evans type, we need to know the set S better. In the cases of $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$, we can decide if $p \notin S$ when $p \nmid n+1$ by computing the dimensions of both the source and the target of ι. By the work of Fu-Wan [FW05, FW08], we can compute the dimension of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})$ when $p \nmid n+1$, see Proposition 4.27. As for the dimension of $(\mathrm{M}^{k}_{n+1})_\ell$, since it is independent of p, we can choose a prime p large enough such that the dimension of $(\mathrm{M}^{k}_{n+1})_\ell$ equals that of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})$.

Remark. In the above, we impose the condition $p \nmid n+1$ because we could not compute the dimension of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})$. When $n=1$, Yun computed the dimension of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_2}, \mathrm{Sym}^k\mathrm{Kl}_2)$ [YV15, Cor. 4.3.5]. Following his method, we give a dimension formula of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^k\mathrm{Kl}_3)$ in Corollary 4.40.

The idea of the computation is to use the complete classification of finite subgroups of SL_3 to find the local monodromy group at ∞ of Kl_3, enabling us to compute the Swan conductor of $\mathrm{Sym}^k\mathrm{Kl}_3$ at ∞, see Theorem 4.30. By the Grothendieck-Ogg-Shafarevich formula [Gro77, X. Thm. 7.1], the dimension of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^k\mathrm{Kl}_3)$ is equal to the Swan conductor Kl_3. As a consequence, the prime $p=3$ is not in $S(k,3)$ if $3 \nmid k$. We will use this consequence in the proof of Theorem 5.12 and that of Theorem 5.22.

Irregular Hodge filtration

To apply the theorem of Patrikis-Taylor to the family $\{(\mathrm{M}^{\lambda}_{n+1})_\ell\}_\ell$, the only nontrivial condition to check is called *regularity*, i.e., all Hodge numbers of the de Rham realizations $(\mathrm{M}^{\lambda}_{n+1})_{\mathrm{dR}}$ are 0 or 1. Different from the case of motives M^{k}_{2} in [FSY22], where the Hodge numbers of the de Rham realizations are always 0 or 1, we have the following results.

Theorem D. *(1). The de Rham realization* $(\mathrm{M}_3^k)_{\mathrm{dR}}$ *underlies a pure Hodge structure of weight* $2k+1$*, whose Hodge numbers are*

$$h^{p,2k+1-p} = \begin{cases} \left\lfloor \frac{\min(p,2k+1-p)}{6} \right\rfloor - d_{p,k} & p \equiv 0,1,2,4 \bmod 6; \\ \left\lfloor \frac{\min(p,2k+1-p)}{6} \right\rfloor + 1 - d_{p,k} & p \equiv 3,5 \bmod 6, \end{cases}$$

where $d_{p,k}$ *is* 1 *if* $3 \mid k$ *and* $p = k$*, and* 0 *otherwise. In particular, all Hodge numbers of* $(\mathrm{M}_3^k)_{\mathrm{dR}}$ *are* 0 *or* 1 *if* $k \leq 9$*.*

(2). Let M *be one of the motives* M_4^3*,* M_5^3 *and* $\mathrm{M}_3^{(2,1)}$*. The de Rham realization* $(\mathrm{M})_{\mathrm{dR}}$ *underlies a pure Hodge structure, whose Hodge numbers are either* 0 *or* 1*.*

Remark. 1. The first case having Hodge numbers bigger than 1 that we found is the case of M_3^{10}, where the Hodge numbers are given by

$$(h^{p,21-p})_{0\leq p\leq 21} = (0,0,0,1,0,1,1,1,1,2,1,1,2,1,1,1,1,0,1,0,0,0).$$

This explains why we add the restriction $k \leq 9$ in the statement of Theorem A. Although we cannot apply the theorem of Patrikis-Taylor when $k \geq 10$, it would be very interesting to see if some of the recent developments on (potential) automorphy, for example, a work of Boxer-Calegari-Gee-Pilloni [BCGP21], allow us to establish some new cases beyond this bound on Hodge numbers.

2. The Hodge numbers of the de Rham realizations of M_4^4 and $\mathrm{M}_3^{(2,2)}$ are also either 0 or 1, which can be deduced from Theorem 5.26 and 5.31, see Remark 5.27 and Remark 5.32. In particular, our methods for computing Hodge numbers is not valid for the case of $\mathrm{M}_3^{(2,2)}$.

3. Using the same methods of computing irregular Hodge numbers, we can find an alternative way to the work of Sabbah-Yu [SY19] to calculate the irregular Hodge numbers of hypergeometric connections in [QX].

To prove Theorem D, we use the irregular Hodge filtration, following the methods in [FSY22].

- Using the isomorphism
$$(\mathrm{M}_{n+1}^{\lambda})_{\mathrm{dR}} \simeq \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda}),$$
the de Rham realization underlies an *exponential mixed Hodge structure*, as defined by Kontsevich and Soibelman [KS11], denoted by $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$. By extending an idea of Deligne in [Del07], Sabbah associated an *irregular Hodge filtration* to each exponential mixed Hodge structure in [Sab10].

- Since the motive $\mathrm{M}_{n+1}^{\lambda}$ is a Nori motive, the de Rham realization also underlies a classical Hodge structure. By [FSY22, A.13], the two Hodge filtration on $(\mathrm{M}_{n+1}^{\lambda})_{\mathrm{dR}}$ coincide. Therefore, to prove Theorem D, it suffices to compute the irregular Hodge filtration on $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$.

- Then we embed the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$ into the twisted de Rham cohomology $\mathrm{H}^{n|\lambda|+1}_{\mathrm{dR}}(\mathbb{G}_m^{n|\lambda|+1}, f_{|\lambda|})$, compatible with the irregular Hodge filtration. Here
$$f_{|\lambda|} = \sum_{i=1}^{|\lambda|} \left(\sum_{j=1}^{n} x_{i,j} + \frac{z}{\prod_j x_{i,j}} \right).$$

- We can get a basis $\{\omega\}$ of the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\lambda})$ for the cases in Theorem D, see Theorem 3.13 and Section 3.4.1.

- If $\gcd(|\lambda|, n+1) = 1$ (in Theorem D only these motives M_3^{3k} do not satisfy this condition), $f_{|\lambda|}$ is non-degenerate with respect to the Newton polytope $\Delta(f_{|\lambda|})$, see Section 1.3. By the work of Adolphson-Sperber [AS97], Esnault-Sabbah-Yu [ESY17], and Yu [Yu14], the filtration on twisted de Rham cohomology has a geometric interpretation in terms of the so-called *Newton polyhedron filtration*, which can be computed using polyhedral combinatorics. It follows that the elements ω in the basis lie in
$$F^{p(\omega)}\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}^{\lambda}_{n+1})$$
for some integers $p(\omega)$ depending on ω. Thanks to our particular choices of bases, we conclude by the Hodge symmetry that each ω has a nonzero class in $\mathrm{gr}_F^{p(\omega)}\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}^{\lambda}_{n+1})$. Hence, the elements of the bases are adapted to the Hodge filtration.

- If $\gcd(|\lambda|, n+1) > 1$, $f_{|\lambda|}$ is not non-degenerate with respect to the Newton polytope $\Delta(f_{|\lambda|})$. Similar to the case that k is even in [FSY22, SY21], we need extra information to finish the calculation, see Sections 3.3.1, 3.3.2 et 3.3.5.

Remark. Compared to the proof of [FSY22, Thm. 1.8], the main difficulties in that of Theorem D are to compute bases, given in Theorem 3.13 and Section 3.4.1. The reason is that the local monodromy actions of Kloosterman connections $\mathrm{Kl}^{\lambda}_{n+1}$ at 0 become more complicated as soon as the size of n and $|\lambda|$ grows. For example, the nilpotent part of the local monodromy action at 0 of $\mathrm{Sym}^k\mathrm{Kl}_2$ only has a single Jordan block for each k. However, those of $\mathrm{Sym}^k\mathrm{Kl}_3$, $\mathrm{Sym}^3\mathrm{Kl}_4$, $\mathrm{Sym}^4\mathrm{Kl}_4$, $\mathrm{Sym}^3\mathrm{Kl}_5$, $\mathrm{Kl}_3^{(2,1)}$ have several Jordan blocks (of different sizes).

Proof of conjectures of Evans type

Let M be a motive attached to one of the sheaves in Theorem B. To prove the claimed conjectures of Evans type, it suffices to show that $\mathrm{M}_\ell \simeq \rho_{f,\ell}(h)$ for some normalized cuspidal Hecke eigenform f and some integer h. Here $\rho_{f,\ell}$ is a two-dimensional Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ attached to f, constructed by the work of Eichler-Shimura for $k = 2$, Deligne for $k \geq 2$ [Del71], and Deligne-Serre [DS74] for $k = 1$.

The argument is originally due to Serre [Ser87, §4.8], and one can find similar arguments in [Kis07, Thm. 1.4.3] and [YV15, Thm. 4.6.1]. We sketch a proof in Theorem 1.17 in Chapter 1. The key ingredient in the proof is Serre's modularity conjecture [Ser87] (proved in [KW09a] and [KW09b]).

To apply the above theorems, we need to check that M_ℓ is odd, i.e. the value of the determinant at the complex conjugation is -1, guaranteed by the Theorem below and Chebotarev's density theorem.

Theorem E. *Let $\mathrm{M}^{\lambda}_{n+1}$ be one of the motives in Theorem A.*

(1). When $n|\lambda|$ is even and $p \notin S \cup \{\ell\}$, we have
$$\det((\mathrm{M}^{\lambda}_{n+1})_\ell)(\mathrm{Frob}_p) = p^{\frac{n|\lambda|+1}{2}\cdot \dim \mathrm{M}^{\lambda}_{n+1}}.$$

(2). When $p \notin \{2, 3, 5\} \cup \ell$, we have
$$\det((\mathrm{M}^{3}_{4})_\ell)(\mathrm{Frob}_p) = \left(\frac{p}{15}\right) p^{10}.$$

The first part of the above Theorem is easy to prove, because the representation $(\mathrm{M}^{\lambda}_{n+1})_\ell$ is symplectic by Proposition 2.3. On the contrary, the second part needs more efforts. We sketch the proof as follows.

- By our construction, the values of the determinants of the Galois representations at Frob_p when $p \notin S \cup \{\ell\}$ are the ε-factors of the Kloosterman sheaf $\mathrm{Sym}^3\mathrm{Kl}_4$, i.e.,

$$\varepsilon\big(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, j_*\mathrm{Sym}^3\mathrm{Kl}_4\big) := \prod_{i=0}^{2} \det\big(-\mathrm{Frob}_k \mid \mathrm{H}^i_{\acute{e}t}\big(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, j_*\mathrm{Sym}^3\mathrm{Kl}_4\big)\big)^{(-1)^{i+1}},$$

  where $j\colon \mathbb{G}_m \hookrightarrow \mathbb{P}^1$ is the inclusion.

- We compute the ε-factors using a similar argument as that in [FW10] for sheaves $\mathrm{Sym}^k\mathrm{Kl}_2$. The idea is to use Laumon's product formula [Lau87] to get

$$\varepsilon\big(\mathbb{P}^1_{\bar{\mathbb{F}}_p}, j_*\mathrm{Sym}^3\mathrm{Kl}_4\big) = p^{15} \prod_{x \in |\mathbb{P}^1_{\mathbb{F}_p}|} \varepsilon\big(\mathbb{P}^1_{(x)}, j_*\mathrm{Sym}^3\mathrm{Kl}_4|_{\mathbb{P}^1_{(x)}}, \mathrm{d}t|_{\mathbb{P}^1_{(\infty)}}\big),$$

  where $\mathbb{P}^1_{(x)}$ is the strict localization of $\mathbb{P}^1_{\mathbb{F}_p}$ at x, $\mathrm{d}t$ is a meromorphic differential form on $\mathbb{P}^1_{\mathbb{F}_p}$, and $\varepsilon\big(\mathbb{P}^1_{(x)}, \mathrm{Sym}^3\mathrm{Kl}_4|_{\mathbb{P}^1_{(x)}}, \mathrm{d}t|_{\mathbb{P}^1_{(x)}}\big)$ are local epsilon factors, define in [Del73, 4.1].

- We can compute the local epsilon factors by axioms and properties of local epsilon factors [Lau87, (3.1.3.2), (3.1.5.4), (3.1.5.5)&(3.1.5.6)], see Section 1.4.4. So we need the local structures of the sheaf $\mathrm{Sym}^3\mathrm{Kl}_4$ at 0 and at ∞ which can be deduced from those of Kl_4.

- Katz determined the local structure of Kl_4 at 0 in [Kat88, 7.0,7]. However, the local structures of Kl_4 at ∞ were only determined after a base change to some finite extension of $\mathbb{F}_p$ by Fu and Wan [FW05], because the root of unity ζ_4 is not always in $\mathbb{F}_p$. Our solution is inspired by the method in [FW10] for the local structure of the sheaf Kl_2 at ∞, which uses the stationary phase principle [Lau87, Prop. 2.3.2.1], i.e., the local structure of the Deligne-Fourier transform of a sheaf $\mathscr{F}$ on $\mathbb{A}^1$ at ∞ is determined by the local structures of $\mathscr{F}$ at its non-lisse points in $\mathbb{A}^1$ and infinity.

- For our purpose, it suffices to characterize the local structures of the sheaf $\mathrm{R}^{n-1}(g_{n+1})_!\bar{\mathbb{Q}}_\ell$ at critical values of g_{n+1}, where $g_{n+1} = \sum_{i=1}^n y_i + 1/\prod y_i$. When $p \nmid n+1$, all critical points of g are ordinary quadratic. So we can use the Picard-Lefschetz formula [DK73, Exp. XV., Thm. 3.4] to show that $\mathrm{R}^{n-1}g_!\bar{\mathbb{Q}}_\ell|_{\mathbb{P}^1_{(x)}}$ is an extension of a locally constant sheaf and the direct sum of several rank one sheaves supported on the generic point of $\mathbb{P}^1_{(x)}$.

After showing the modularity, we know the existence of modular forms, whose Fourier coefficients are related to moments of Kloosterman sheaves. All that remains is to determine the information of modular forms as explicitly as possible. We compute the traces of Frobenius of M_ℓ at sufficiently many primes (the Fourier coefficients of the corresponding modular form) using Sagemath [The22]. Then we use the information from geometry (weight, conductor, determinant) and the computations of traces of Frobenius to search the modular form in the database LMFDB.

Organization of the thesis

- In Chapter 1, we recall some results that will be used in the proofs of main theorems.

- In Chapter 2, we construct the motives attached to Kloosterman sheaves and study their realizations. We define the motives in Section 2.1. Then, in Section 2.2 we show that the de Rham realizations of such motives underlie exponential mixed Hodge structures, and is isomorphic to a classical Hodge structure. In Section 2.3, we summarize properties of other realizations in characteristic $p > 0$.

- In Chapter 3, we give the proof of Theorem D. In Section 3.1, we study the formal local structure of connections $\mathrm{Sym}^k\mathrm{Kl}_3$ at 0 and at ∞ and provide a dimension formula of the de Rham cohomology. Next, in Section 3.2, we determine a basis of cohomology classes of the de Rham realization $(\mathrm{M}_3^k)_{\mathrm{dR}}$ in Theorem 3.13. Then, in Section 3.3, we prove Theorem 3.33, which implies the first part of Theorem D. Finally, in Section 3.4, the cases in the second part of Theorem D are proved case by case.

- In Chapter 4, we will study the properties of Kloosterman sheaves (mainly for those appearing in Theorem A), e.g., local structures at 0 and ∞ in Section 4.2 and Section 4.3. In Section 4.4, we give the dimension formulas of the ℓ-adic cohomology of such sheaves when $\gcd(p, n+1) = 1$, and dimension formulas for $\mathrm{Sym}^k\mathrm{Kl}_3$ when $p = 3$. In Section 4.5, we prove Theorem E.

- In Chapter 5, we study the ramifications of the Galois representations $(\mathrm{M}_{n+1}^{\lambda})_{\ell}$ in Section 5.1 and prove Theorem C. In Section 5.2, we prove Theorem A.1. In Section 5.3, we prove Theorem B. Then the Theorem A is completely proven.

- In the Appendix A, we explain the calculation of moments of Kloosterman sheaves and include the code of Sagemath [The22].

Chapter 1

Preliminaries

In this chapter, we recall some results that will be used in the proof of the main theorems.

1.1 Weyl's construction

We recall some preliminaries from [FH04, §6, §15 & §17].

1.1.1 Young diagrams

A partition of an integer N is a sequence of non-negative integers of the form $\mu := (\mu_1, \mu_2, \ldots, \mu_m)$ such that $\mu_1 \geq \mu_2 \cdots \geq \mu_m$ and $\sum_i \mu_i = N$. For such a partition, we can associate a Young diagram, such that μ_i are the lengths of the i-th rows. For example the Young diagram of the partition $(3, 2, 1)$ is as follows:

1	2	3
4	5	
6		

1.1.2 Schur functors

For a partition μ as above, we will define two elements a_μ and b_μ in the permutation group S_N in N symbols. First we label each block in the Young diagram by indexes in $\{1, \ldots, N\}$. Then we take $P_\mu := \{\sigma \in S_N \mid \sigma \text{ preserves each row}\}$ and $Q_\mu := \{\tau \in S_N \mid \tau \text{ preserves each column}\}$. Let $\operatorname{sign}\colon S_N \to \{\pm 1\}$ be the sign character of S_N. Then we define

$$a_\mu := \sum_{\sigma \in P_\mu} \sigma \text{ and } b_\mu := \sum_{\tau \in Q_\mu} \operatorname{sign}(\tau)\tau$$

in the group ring $\mathbb{Z}[S_N]$.

Let K be a field of characteristic 0 and V be a K-vector space. The group S_N acts on the tensor product $V^{\otimes N}$ by

$$\sigma v_1 \otimes \cdots \otimes v_N := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)}.$$

Then we have the endofunctor $\mathbb{S}_\mu$ of the category of finite dimensional representations of SL_{n+1}

$$\mathbb{S}_\mu V := a_\mu b_\mu V^{\otimes N} = (V^{\otimes N})^{P_\mu \times Q_\mu, \mathrm{id}, \mathrm{sign}},$$

where the exponent $(P_\mu \times Q_\mu, \mathrm{id}, \mathrm{sign})$ means taking the isotypic component.

1.1.3 Construction of irreducible representations of SL_{n+1}

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a sequence of nonnegative integers. Let V be the standard representation K^{n+1} of SL_{n+1}, i.e., equipped with the action of SL_{n+1} by left multiplication, and V_λ be the unique irreducible subrepresentation of

$$\mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} \wedge^2 V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n} \wedge^n V,$$

whose highest weight is $\sum_i \lambda_i(L_1 + \ldots + L_i)$. In the case of SL_3, the representation with the highest weight $\lambda_1 L_1 + \lambda_2(L_1 + L_2)$ can be described as

$$\ker(\mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} \wedge^2 V \xrightarrow{\pi_{\lambda_1,\lambda_2}} \mathrm{Sym}^{\lambda_1 - 1} V \otimes \mathrm{Sym}^{\lambda_2 - 1} \wedge^2 V), \tag{1.1}$$

where $\pi_{\lambda_1,\lambda_2}$ sends $v_1 \cdots v_{\lambda_1} \otimes w_1 \otimes w_{\lambda_2}$ to

$$\frac{1}{(\lambda_1)!(\lambda_2)!} \sum_{\sigma \in S_{\lambda_1}, \tau \in S_{\lambda_2}} < v_{\sigma(1)}, w_{\tau(1)} > \cdot v_{\sigma(2)} \cdots v_{\sigma(a_1)} \otimes w_{\tau(2)} \cdots w_{\tau(\lambda_2)}$$

where $< \cdot, \cdot >: V \times \wedge^2 V \to K$ is the natural pairing.

In general, we can construct the representation V_λ by Schur functors. Let

$$\mu(\lambda) = (\lambda_1 + \ldots + \lambda_n, \lambda_2 + \ldots + \lambda_n, \ldots, \lambda_n)$$

and apply $\mathbb{S}_{|\mu(\lambda)|}$ to $V^{\otimes |\lambda|}$. The resulting representation $\mathbb{S}_{|\mu(\lambda)|} V$ is nothing but V_λ, i.e., the (unique) finite dimensional irreducible representation of SL_{n+1} of the highest weight $\sum_i \lambda_i(L_1 + \ldots + L_i)$, see [FH04, Prop 15.15].

1.2 A review of Exponential mixed Hodge structures

1.2.1 Notation for $\mathcal{D}$-modules

We recall some facts in the theory of $\mathscr{D}$-modules. For detail, see for example [HTT07].

Let X be a complex variety. We denote by $\mathscr{D}_X$ the *sheaf of differential operators* on X, which is a subsheaf of $\mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X)$, generated by $\mathcal{O}_X$ (acting on $\mathcal{O}_X$ by left multiplication) and the sheaf of vector fields $\Theta_X = (\Omega^1_X)^\vee$. For example, if $X = \mathbb{A}^1$, the sheaf $\mathscr{D}_X$ is associated with the $\mathbb{C}$-algebra $\mathbb{C}[t]\langle\partial_t\rangle$, satisfying $\partial_t \cdot t - t \cdot \partial_t = 1$. A left (resp. right) $\mathscr{D}_X$-module is an $\mathcal{O}_X$-module with a left (resp. right) action of $\mathscr{D}_X$. When X is affine, we identify $\mathscr{D}$-modules with their global sections. For example, $\mathscr{D}_{\mathbb{A}^1}$-modules are identified with $\mathbb{C}[t]\langle\partial_t\rangle$-modules. We denote by $\mathrm{Mod}(\mathscr{D}_X)$ the category of (left)-$\mathscr{D}_X$-modules on X, and by $\mathrm{D}^b(\mathscr{D}_X)$ the bounded derived category of $\mathscr{D}_X$-modules.

Let $f: X \to Y$ be a morphism between smooth complex varieties. Let $\mathscr{D}_{X \to Y}$ and $\mathscr{D}_{Y \leftarrow X}$ be transfer modules. For a complex of $\mathcal{D}_X$-modules M and a complex of $\mathcal{D}_Y$-modules N, the direct image and inverse image are defined by

$$f_+ M = \mathrm{R}f_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X} M) \in \mathrm{D}^b(\mathcal{D}_Y)$$

and

$$f^+ N = \mathcal{D}_{X \to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1} N \in \mathrm{D}^b(\mathcal{D}_X)$$

respectively. Let $\mathbb{D}_X$ and $\mathbb{D}_Y$ be the duality functor. We put $f_\dagger = \mathbb{D}_Y \circ f_+ \circ \mathbb{D}_X$. Then there is a canonical forget supports morphism $f_\dagger M \to f_+ M$, which is an isomorphism if f is proper.

A $\mathscr{D}_X$-modules is called *holonomic* if the dimension of its characteristic variety is smaller or equal to $\dim X$. We denote by $\mathrm{D}^b_h(\mathscr{D}_X)$ the full subcategory of $\mathrm{D}^b(\mathscr{D}_X)$, whose objects are complexes with holonomic cohomologies. The categories $\mathrm{D}^b_h(\mathscr{D}_X)$ are stable under the six functor formalism of $\mathscr{D}$-modules.

De Rham cohomology

For a $\mathscr{D}_X$-module, we denote by $\mathrm{DR}(M)$ the (analytic) de Rham complex $\Omega_X^{\mathrm{an}} \otimes^{\mathrm{L}}_{\mathcal{O}_X} M^{\mathrm{an}}$. The de Rham cohomology $\mathrm{H}^r_{\mathrm{dR}}(X, M)$ of a $\mathscr{D}$-module M is the hypercohomology of the de Rham complex of M, which is finite dimensional if M is holonomic.

Now let $j\colon U \hookrightarrow X$ be an inclusion of an open subvariety U in a projective smooth variety X, whose complement is a divisor D. For a connection on U, i.e., a locally free $\mathscr{D}_U$-module M, the de Rham cohomology with compact support $\mathrm{H}^r_{\mathrm{dR,c}}(U, M)$ is the hypercohomology of $\mathrm{DR}(j_\dagger M)$. We denote by $j_{\dagger+}M$ the intermediate extension of M, i.e. a $\mathscr{D}_X$-module N such that $j^+N = M$ and N admits no subobjects or quotient objects supported on $X\backslash U$.

If X is a curve, we denote by $\mathrm{H}^r_{\mathrm{dR,mid}}(U, M)$ the *middle de Rham cohomology* of M, i.e., the image of the de Rham cohomology with compact support in the usual de Rham cohomology. It is identified with the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(X, j_{\dagger+}M)$ of the intermediate extension $j_{\dagger+}M$ [FSY20b, (3.1)].

The Fourier transform

Let pr_t and pr_τ be the projections of $\mathbb{A}^1_t \times \mathbb{A}^1_\tau$ to the first and the second factors respectively. We denote by $\mathcal{E}^{t\tau} = (\mathcal{O}_{\mathbb{A}^1_t \times \mathbb{A}^1_\tau}, \mathrm{d} + \mathrm{d}(t\tau))$ the rank 1 connection on $\mathbb{A}^1_t \times \mathbb{A}^1_\tau$. Then the *Fourier transform* of a $\mathscr{D}$-module M on $\mathbb{A}^1_t$ is given by

$$\mathrm{FT}M = \mathrm{pr}_{\tau+}\big(\mathrm{pr}_t^+ M \otimes_{\mathcal{O}_{\mathbb{A}^1 \times \mathbb{A}^1}} \mathcal{E}^{t\tau}\big).$$

If M is a holonomic $\mathscr{D}$-module on $\mathbb{A}^1_t$, its Fourier transform is a complex with holonomic cohomologies, concentrated on degree 0.

The nearby cycle and the vanishing cycle functors

Let $f\colon X \to \mathbb{A}^1$ be a regular function. If M is a holonomic $\mathscr{D}$-module, the nearby cycle functor ψ_f and the vanishing cycle functor ϕ_f send M to holonomic $\mathscr{D}_X$-modules supported on $f^{-1}(0)$, which are defined in terms of the *Kashiwara-Malgrange filtration* of M. Up to a shift, the de Rham functor sends $\psi_f M$ and $\phi_f M$ to the nearby cycle and vanishing cycle of the perverse sheaf $\mathrm{DR}M[\dim X]$.

By construction, there is an automorphism T on both $\psi_f M$ and $\phi_f M$. So we can decompose them, and denote by $\psi_{f,\lambda}M$ and $\phi_{f,\lambda}$ the generalized eigencomponents corresponding to an eigenvalue $\lambda \in \mathbb{C}^\times$. If $\lambda \neq 1$, there is an isomorphism $\psi_{f,\lambda}M \simeq \phi_{f,\lambda}M$ compatible with the automorphism T, and if $\lambda = 1$, there is a quiver

$$\psi_{f,1}M \underset{\mathrm{var}}{\overset{\mathrm{can}}{\rightleftarrows}} \phi_{f,1}M$$

such that the maps $\exp(2\pi i \mathrm{var} \circ \mathrm{can})$ and $\exp(2\pi i \mathrm{can} \circ \mathrm{var})$ equal to the unipotent automorphism T on $\psi_{f,1}M$ and $\phi_{f,1}M$ respectively. We denote by N the nilpotent part of T.

1.2.2 Exponential mixed Hodge structure

We recall the basic definitions of exponential mixed Hodge structures from [FSY22, Appx.].

The category $\mathrm{MHM}(X)$ of *mixed Hodge modules* on a smooth complex variety X is an abelian category. In particular, if X is a point, the category is nothing but the category of mixed Hodge structures MHS. The bounded derived categories $\mathrm{D}^b(\mathrm{MHM}(X))$ admit the six functor formalism. We put a subscript for functors of Mixed Hodge modules to distinguish them with functors of $\mathscr{D}$-modules, for example ${}_{\mathrm{H}}f^*$. For more details about mixed Hodge modules, see [Sai90].

The category EMHS of *exponential mixed Hodge structures* is defined in [KS11] as the full subcategory of $\mathrm{MHM}(\mathbb{A}^1)$, whose objects N^{H} have vanishing cohomology on $\mathbb{A}^1$, i.e., satisfying ${}_{\mathrm{H}}\pi_* N^{\mathrm{H}} = 0$ for

$\pi\colon \mathbb{A}^1 \to \mathrm{Spec}(\mathbb{C})$ the structure morphism. There is a projector $\Pi\colon \mathrm{MHM}(\mathbb{A}^1) \to \mathrm{EMHS}$ defined by

$$N^{\mathrm{H}} \mapsto {}_{\mathrm{H}}s_*(N^{\mathrm{H}} \boxtimes {}_{\mathrm{H}}j_!\mathcal{O}^{\mathrm{H}}_{\mathbb{G}_m}) \tag{1.2}$$

where $j\colon \mathbb{G}_m \to \mathbb{A}^1$ is the inclusion and $s\colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the summation map. With the help of this functor, if we view the category of mixed Hodge structures MHS as $\mathrm{MHM}(\{0\})$, the category of mixed Hodge modules at the point 0, it can be seen as a full subcategory of EMHS, by the functor

$$V \mapsto \Pi\,({}_{\mathrm{H}}i_{0,!}V)\,. \tag{1.3}$$

We can define for each object N^{H} in the category EHMS a weight filtration W^{EMHS} on N^{H} in terms of that on N^{H}, viewed as a mixed Hodge module on $\mathbb{A}^1$.

The *de Rham fiber* functor from EMHS to $\mathrm{Vect}_{\mathbb{C}}$ is defined by

$$\Pi(N^{\mathrm{H}}) \mapsto \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, \Pi(N) \otimes \mathcal{E}^{\theta}), \tag{1.4}$$

where θ is the coordinate of $\mathbb{A}^1$, and $\mathcal{E}$ denotes the rank 1 connection $(\mathcal{O}_{\mathbb{A}^1}, \mathrm{d} + \mathrm{d}\theta)$. Each de Rham fiber is equipped with a weight filtration $W_\bullet$. Moreover, one can associate an *irregular Hodge filtration* $F^\bullet_{irr}$ on every de Rham fiber by methods in [Sab10, SY15, ESY17].

For a regular function $f\colon U \to \mathbb{A}^1$ on an affine complex smooth variety U of dimension d, we define exponential mixed Hodge structures

$$\mathrm{H}^r(U,f) := \Pi\big(\mathcal{H}^{r-d}\,{}_{\mathrm{H}}f_*\mathbb{Q}^{\mathrm{H}}_U\big),\ \mathrm{H}^r_c(U,f) := \Pi\big(\mathcal{H}^{r-d}\,{}_{\mathrm{H}}f_!\mathbb{Q}^{\mathrm{H}}_U\big)$$

and

$$\mathrm{H}^r_{\mathrm{mid}}(U,f) := \Pi\big(\mathrm{im}\big(\mathcal{H}^{r-d}\,{}_{\mathrm{H}}f_!\mathbb{Q}^{\mathrm{H}}_U \to \mathcal{H}^{r-d}\,{}_{\mathrm{H}}f_*\mathbb{Q}^{\mathrm{H}}_U\big)\big).$$

In particular, the de Rham fiber of $\mathrm{H}^r_?(U,f)$ is $\mathrm{H}^r_{\mathrm{dR},?}(U,f)$.

1.3 The irregular Hodge filtration on twisted de Rham cohomology

We recall the definition of the irregular Hodge filtration on twisted de Rham cohomologies from [Yu14], compatible with the irregular filtration mentioned in §1.2.2. Let U be a complex smooth quasi-projective variety and f be a regular function on U. Let $j\colon U \to X$ be a compactification and $S := X\backslash U$ be the boundary divisor. The pair (X,S) is called a *good compactification* of the pair (U,f), if S is normal crossing and f extends to a morphism $\bar{f}\colon X \to \mathbb{P}^1_{\mathbb{C}}$.

Let P be the pole divisor of f. For the twisted de Rham complex $(\Omega(X)^\bullet(*S), \nabla = \mathrm{d} + \mathrm{d}f)$, there is a decreasing filtration $F^\lambda(\nabla) := F^0(\lambda)^{\geq\lceil\lambda\rceil}$, indexed by non-negative real numbers, where $F^0(\lambda)$ is the complex

$$\mathcal{O}(\lfloor -\lambda P\rfloor) \xrightarrow{\nabla} \Omega^1(\log S)(\lfloor(1-\lambda)P\rfloor) \to \cdots \to \Omega^p(\log S)(\lfloor(p-\lambda)P) \to \cdots.$$

Definition 1.5. The *irregular Hodge filtration of the de Rham cohomology* $\mathrm{H}^i_{\mathrm{dR}}(U,\nabla)$ is defined by

$$F^\lambda \mathrm{H}^i_{\mathrm{dR}}(U,\nabla) := \mathrm{im}(\mathbb{H}^i(X, F^\lambda(\nabla)) \to \mathrm{H}^i_{\mathrm{dR}}(U,\nabla)).$$

The filtration is independent of the choice of the good compactification (X,S) (see [Yu14, Thm. 1.7]).

Newton polyhedral filtration

When U is $(\mathbb{C}^\times)^n$, a regular function $f = \sum_\alpha c(\alpha)x^\alpha$ on U is a Laurent polynomial. We say that f is *non-degenerate with respect to* $\Delta(f)(:=$ convex hull of $\{0\} \cup \mathrm{Supp}(f))$, if for each face δ of $\Delta(f)$ not containing 0, the function $f_\delta = \sum_{\alpha\in\delta} c(\alpha)x^\alpha$ has no critical point in U. In this case the irregular Hodge filtration can be computed differently (see [AS97, Thm 1.4], [Yu14, Thm 4.6]).

We view $\Delta(f)$ as a subset of the character lattice $M := \mathrm{Hom}(U, \mathbb{C}^\times)$, and denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_{\mathbb{R}}$. Let $F := F(\Delta)$ be the normal fan of the Newton polytope Δ on the dual space $N_{\mathbb{R}} = \mathrm{Hom}(M_{\mathbb{R}}, \mathbb{R})$, i.e. each ray of the normal fan corresponds to a codimension 1 face of Δ, pointing inward with respect to the paring $N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$. We refine the fan F to $\widetilde{F}$ such that the corresponding toric variety X_{tor} is smooth proper and the toric boundary $S := X_{\mathrm{tor}} \backslash U$ is simple normal crossing. We again denote by P the pole divisor of f. Then each ray in $\widetilde{F}$ (resp. F) corresponds to an irreducible component of S (resp. P).

In [Yu14], the filtration $F^{\lambda}_{\mathrm{NP}}(\nabla)$ and the filtration $F^{\lambda}_{\mathrm{NP}}(\mathrm{H}^1_{\mathrm{dR}}(U, \nabla))$ are defined similarly as that in definition 1.5 by replacing the good compactification (X, S) with (X_{tor}, S), and agree with those defined by a good compactification. In particular, we have

$$\mathbb{H}^i\big(X_{\mathrm{tor}}, F^{\lambda}_{\mathrm{NP}}(\nabla)\big) = \mathrm{H}^i\big(\Gamma\big(X_{\mathrm{tor}}, F^{\lambda}_{\mathrm{NP}}(\nabla)\big)\big),$$

which makes the computation easier.

1.4 Some results on ℓ-adic sheaves

Let $p \neq \ell$ be two prime numbers, q a power of p, and X a variety over $\mathbb{F}_q$. We denote by E be either $\bar{\mathbb{Q}}_\ell$ or a finite extension of $\mathbb{Q}_\ell$ inside $\bar{\mathbb{Q}}_\ell$. We can define ℓ-adic sheaves with coefficients in E (or E-sheaves) on X, as well as the bounded category of constructible E-sheaves $D^b_c(X, E)$, see [Del80, §1.1].

The ℓ-adic sheaves behave like constructible étale sheaves with coefficients in E, and lisse ℓ-adic sheaves $\mathscr{F}$ are like locally constant étale sheaves with coefficients in E on X. Assume that X is irreducible, and we take a geometric point $\bar{\eta}$ above the generic point of X. There is an equivalence between the category of lisse ℓ-adic sheaves with coefficients in E on X and the category of continuous ℓ-adic representations of the étale fundamental group $\pi_1^{\acute{e}t}(X, \bar{\eta})$ with coefficients in E.

1.4.1 Grothendieck-Ogg-Shafarevich formula

We recall the statement of the formula from [Lau87, (2.2)], see also [Gro77, X. Théorème 7.1]. Let C be a geometrically connected smooth projective curve over $\mathbb{F}_q$. For each closed point $x \in |C|$, we denote by $C_{(x)}$ (resp. $C_{(\bar{x})}$) the localization (resp. striction localization) of C at x (resp. $\bar{x}$). The special points and generic points of $C_{(x)}$ and $C_{(\bar{x})}$ are denoted by s_x, η_x and $s_{\bar{x}}, \eta_{\bar{x}}$ respectively.

Let $\mathscr{F}$ be an ℓ-adic sheaf on C, then $\mathscr{F}|_{\eta_{(x)}}$ is a continuous ℓ-adic representation of $\mathrm{Gal}(\bar{\eta}_x/\eta_x)$, to which one can associate the Swan conductors. We denote by

- $\mathrm{rk}(\mathscr{F}) = \mathrm{rk}(\mathscr{F}_{\bar{\eta}})$ and $\mathrm{rk}_x(\mathscr{F}) = \mathrm{rk}(\mathscr{F}_{\bar{s}_x})$,
- $\mathrm{Sw}_x(\mathscr{F}) = \mathrm{Sw}(\mathscr{F}_{\bar{\eta}_x})$,
- $a_x(\mathscr{F}) = \mathrm{rk}(\mathscr{F}) - \mathrm{rk}_x(\mathscr{F}) + \mathrm{Sw}_x(\mathscr{F})$.

Theorem 1.6. *If $\mathscr{F}$ is an ℓ-adic sheaf on C, then*

$$\chi(C_{\bar{\mathbb{F}}_q}, \mathscr{F}) = (2 - 2g) \cdot \mathrm{rk}(\mathscr{F}) - \sum_{x \in |C|} \deg(x) \cdot a_x(\mathscr{F}).$$

1.4.2 An exact sequence

Let C be a smooth projective curve over $\mathbb{F}_q$, U be an open subvariety, $j\colon U \hookrightarrow C$ be the open immersion, and η be the generic point of C. For a lisse ℓ-adic sheaf $\mathscr{F}$ on U, we denote by $\rho_{\mathscr{F}}$ the corresponding continuous ℓ-adic representation of $\pi_1^{\acute{e}t}(U, \bar{\eta})$, and by G_{geom} be the geometric monodromy

group of $\mathscr{F}$, i.e., the Zariski closure of the image of $\pi_1^{\acute{e}t}(U_{\bar{\mathbb{F}}_q}, \bar{\eta})$ in $\mathrm{GL}(\mathscr{F}_{\bar{\eta}})$ under $\rho_{\mathscr{F}}$. By [Kat88, 2.0,7], we have a long exact sequence

$$\begin{aligned} 0 \to \mathscr{F}^{G_{\mathrm{geom}}} \to \oplus_{x\in|C\setminus U|,\ \bar{x} \text{ over } x} (\mathscr{F}|_{\eta_x})^{I_{\bar{x}}} \to \mathrm{H}^1_{\acute{e}t,c}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathscr{F}\big) \\ \to \mathrm{H}^1_{\acute{e}t}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathscr{F}\big) \to \oplus_{x\in|C\setminus U|,\ \bar{x} \text{ over } x} (\mathscr{F}|_{\eta_x})_{I_{\bar{x}}}(-1) \to \mathscr{F}_{G_{geom}}(-1) \to 0 \end{aligned} \tag{1.7}$$

where $\eta_{\bar{x}}$ are the generic point of the strict henselization of $\mathbb{P}^1$ at $\bar{x}$, the groups $I_{\bar{x}}$ are the inertia groups at $\bar{x}$, $(\mathscr{F}|_{\eta_x})^{I_{\bar{x}}}$ is the invariants of $I_{\bar{x}}$ and $(\mathscr{F}|_{\eta_x})_{I_{\bar{x}}}$ is the coinvariants of $I_{\bar{x}}$. Assume that $\mathscr{F}$ is pure of weight w. By considering the Frobenius weight, we conclude that

$$\mathrm{gr}^W_{w+1}\mathrm{H}^1_{\acute{e}t,c}\big(U_{\bar{\mathbb{F}}_q}, \mathscr{F}\big) \simeq \mathrm{gr}^W_{w+1}\mathrm{H}^1_{\acute{e}t}\big(U_{\bar{\mathbb{F}}_q}, \mathscr{F}\big)$$

is isomorphic to the middle ℓ-adic cohomology $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(U_{\bar{\mathbb{F}}_q}, \mathscr{F})$, i.e., the image of the cohomology with compact support in the usual cohomology. The dimension of the middle ℓ-adic cohomology is given by

$$\dim \mathrm{H}^1_{\acute{e}t,c}\big(U_{\bar{\mathbb{F}}_q}, \mathscr{F}\big) - \sum_{x\in|C\setminus U|,\ \bar{x} \text{ over } x} \dim(\mathscr{F}|_{\eta_x})^{I_{\bar{x}}} + \dim \mathscr{F}^{G_{\mathrm{geom}}}. \tag{1.8}$$

1.4.3 Laumon's Product formula of ε-factors

Let $\mathscr{F}$ be an ℓ-adic sheaf over a variety X over $\mathbb{F}_q$, the epsilon factor $\varepsilon(X, \mathscr{F})$ is the ℓ-adic number

$$\varepsilon(X, \mathscr{F}) := \prod_{i=0}^{2} \det\big(-\mathrm{Frob}_q \mid \mathrm{H}^i_{\acute{e}t}\big(X_{\bar{\mathbb{F}}_q}, \mathscr{F}\big)\big)^{(-1)^{i+1}}.$$

Inspired by Langland's program for function fields, i.e. a correspondence between certain ℓ-adic Galois representations and certain automorphic forms over function fields, Deligne constructed in [Del73] local ε-factors satisfying some reasonable axioms. These local factors are analogs of local ε-factors of automorphic representations. Moreover, he conjectured that the global ε-factor was product of local ε-factors and some power of p, and gave a geometrical proof for tamely ramified ℓ-adic sheaves. The complete proof was given by Laumon [Lau87].

Theorem 1.9 (Laumon's product formula)**.** *Let X be a smooth geometrically connected proper curve over $k = \mathbb{F}_q$, ω be a meromorphic 1-form, not identically zero on X, and $\mathscr{F}$ be an ℓ-adic sheaf on X. Then the global ε-factor $\varepsilon(X, K)$ verifies the product formula*

$$\varepsilon(X, \mathscr{F}) = q^{(1-g)\mathrm{rk}(\mathscr{F})} \prod_{x\in|X|} \varepsilon\big(X_{(x)}, \mathscr{F}|_{X_{(x)}}, \omega|_{X_{(x)}}\big),$$

where g is the genus of X, and $\varepsilon\big(X_{(x)}, \mathscr{F}|_{X_{(x)}}, \omega|_{X_{(x)}}\big)$ are local epsilon factors.

1.4.4 Properties of local epsilon-factors

Now we gather some results about local epsilon factors from [Lau87], especially (3.1.3.2), (3.1.5.4), (3.1.5.5), and (3.1.5.6). Let (T, K, ω) be a triple consisting of a henselian trait T of equal characteristic $p > 0$ with finite residue field $k \supset \mathbb{F}_q$, an object K in $D^b_c(T, \bar{\mathbb{Q}}_\ell)$, and a nonzero meromorphic differential 1-form $\omega \in \Omega^1_{k(\eta)/k} - \{0\}$. To each triple, there is an associated an ℓ-adic number $\varepsilon_\psi(T, K, \omega) \in \bar{\mathbb{Q}}_\ell^\times$ satisfying axioms (1)-(5) below, called the *local ε-factor* [Del73, Thm. 4.1]. We summarize the axioms and some properties of local ε-factors here.

(1). $\varepsilon_\psi(T, K, \omega)$ only depends on the isomorphism class of triplets.

(2). For a distinguished triangle

$$K' \to K \to K'' \to K'[1]$$

in $D_c^b(T, \bar{\mathbb{Q}}_\ell)$, we have

$$\varepsilon_\psi(T, K, \omega) = \varepsilon_\psi(T, K', \omega) \cdot \varepsilon_\psi(T, K'', \omega). \tag{1.10}$$

(3). If K is supported on the closed point s of T, we have

$$\varepsilon_\psi(T, K, \omega) = \det(-\mathrm{Frob}_s, K)^{-1}. \tag{1.11}$$

(4). Let η be the generic point of T. If η_1/η is a finite separable extension, $f : T_1 \to T$ is the normalization of T in η_1, and K_1 is an object in $D_c^b(T_1, \bar{\mathbb{Q}}_\ell)$ of generic rank 0, we have

$$\varepsilon_\psi(T_1, K_1, f^*\omega) = \varepsilon_\psi(T, f_*K_1, \omega). \tag{1.12}$$

(5). We denote by K_t the completion of function field $k(\eta)$ of T, by $v_t : K_t^\times \to \mathbb{Z}$ its discrete valuation, and by $\mathcal{O}$ its integral ring. Then the valuation induces a function $v_t : \Omega^1_{K_t} - \{0\} \to \mathbb{Z}$ by sending $a \cdot \mathrm{d}b$ to $v_t(a)$ if $a, b \in K_t^\times$ and $v_t(b) = 1$. If V is a lisse ℓ-adic sheaf of rank 1 on η, it induces a character $\chi : K_t^\times \to \bar{\mathbb{Q}}_\ell^\times$ via the reciprocity map $i_{K_t} : K_t^\times \to G^{\mathrm{ab}}$ from local field theory, whose conductor is $a(V) = 1 + \mathrm{Sw}(V)$. Let $j\colon \eta \hookrightarrow T$ be the inclusion. Then we have

$$\varepsilon_\psi(T, j_*V, \omega) = \varepsilon(\chi, \psi_\omega) \tag{1.13}$$

where the character $\Psi_\omega : K_t^\times \to G^{\mathrm{ab}}$ is defined by the formula

$$\Psi_\omega(a) = \psi \circ \mathrm{Tr}_{k/\mathbb{F}_q}(\mathrm{Res}(a \cdot \omega))$$

whose conductor $c(\Psi_\omega)$ is $v_t(\omega)$ and where $\varepsilon(\chi, \Psi_\omega)$ is the local constant of Tate

$$\varepsilon(\chi, \Psi_\omega) = \int_{(\pi_x)^{-a(V)-c(\Psi_\omega)}\mathcal{O}^\times} \chi^{-1}(z)\Psi_\omega(z)\mathrm{d}z$$

where $\mathrm{d}z$ is the normalized Haar measure on K_t, such that the volume of $\mathcal{O}$ is 1.

(6). For all $a \in k(\eta)^\times$, we have

$$\varepsilon(T, K, a \cdot \omega) = \chi_K(a) \cdot \#k^{\mathrm{rk}(K_{\bar{\eta}}) \cdot v_t(a)} \cdot \varepsilon(T, K, \omega), \tag{1.14}$$

where $\chi_K : K_t^\times \to \bar{\mathbb{Q}}_\ell^\times$ is the character induced from the lisse ℓ-adic sheaf $\det(K|_\eta)$ of rank 1 on η, via the reciprocity map.

(7). If F is a lisse ℓ-adic sheaf of rank $\mathrm{rk}(F)$ on T, we have

$$\varepsilon(T, K \otimes F, \omega) = \det(\mathrm{Frob}_t, F)^{a(T,K,\omega)} \cdot \varepsilon(T, K, \omega)^{\mathrm{rk}(F)}, \tag{1.15}$$

where $a(T, K, \omega) = \mathrm{rk}(K_{\bar{\eta}}) + \mathrm{Sw}(K_{\bar{\eta}}) - \mathrm{rk}(K_{\bar{s}}) + \mathrm{rk}(K_{\bar{\eta}}) \cdot v_t(\omega)$.

(8). The local ε-factor is 1 if K is lisse at x and ω is holomorphic at x.

1.5 Some results about Galois representations

1.5.1 Galois representations attached to modular forms

One can attach 2-dimensional Galois representations to normalized cuspidal Hecke eigenform $f \in S_k(\Gamma_1(N))$ of weight k.

Theorem. *Let $N \geq 1$ and $k \geq 1$, $f \in S_k(\Gamma(N))$ be a normalized cuspidal Hecke eigenform, and $K_f = \mathbb{Q}(a_f(p))$ be the number field generated by the Fourier coefficients of f. Then for any place λ of K_f over a prime $\ell \nmid N$, there exists a continuous, odd, irreducible Galois representation*

$$\rho_{f,\lambda} \colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\lambda}), \tag{1.16}$$

unramified if $p \nmid N$, such that for $p \nmid N\ell$, the trace of the arithmetic Frobenius Frob_p^{-1} at p is $a_p(f)$.

Notice that $\rho_{f,\lambda}$ has conductor N and Hodge-Tate weight $(0, k-1)$. Also, it is odd, i.e., $\det(\rho_{f,\lambda})(c) = -1$, where c is the complex conjugation.

Given such a $\rho_{f,\lambda}$, we denote by $\bar{\rho}_{f,\lambda} \colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_\ell)$ its mod ℓ reduction. It is obtained by choosing a Galois stable $\mathcal{O}_\lambda$-lattice in $K_{f,\lambda}^2$ and reducing modulo the maximal ideal of $\mathcal{O}_\lambda$, where $\mathcal{O}_\lambda$ is the ring of integers of K_λ. Although $\bar{\rho}_{f,\lambda}$ depends on the choice of the lattice, its semisimplification does not.

1.5.2 Serre's modularity conjecture

Serre's modularity conjecture says that a continuous absolutely irreducible odd mod ℓ representation $\bar{\rho} \colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_\ell)$ is isomorphic to $\bar{\rho}_{f,\ell}$ for some modular form f, where $\bar{\rho}_{f,\ell}$ is the mod ℓ reduction of Galois representations from (1.16).

There is a stronger version, which also predicts the weight and the level of f. Here we will recall the definition of $k(\bar{\rho})$ and $N(\bar{\rho})$ from [Kis07], which is slightly different from that of [Ser87].

The weight

We denote by $\omega_i \colon I_\ell \to \bar{\mathbb{F}}_\ell^\times$ the character, which sends σ to $\sigma(p^{1/(p^i-1)})/p^{1/(p^i-1)} (\mathrm{mod}\, \ell)$, called the fundamental character of level i. If $i = 1$, write ω_1 as the mod ℓ reduction of the ℓ-adic cyclotomic character. By abuse of notation, we denote by ω for both ω_1 and the ℓ-adic cyclotomic character.

Suppose we are given a representation $\bar{\rho} \colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_\ell)$, the restriction $\bar{\rho}|I_\ell$ is either of the form $\begin{pmatrix} \omega^i & * \\ 0 & 1 \end{pmatrix} \otimes \omega^j$ for $i, j \in \mathbb{Z}$, or $\begin{pmatrix} \omega_2^i & 0 \\ 0 & \omega_2^{\ell i} \end{pmatrix} \otimes \omega^j$ for some $i, j \in \mathbb{Z}$ and $p+1 \nmid i$. When $\bar{\rho}|_{I_\ell}$ is semisimple (i.e., tamely ramified), we can choose $i, j \geq 0$, and such that $j \in [0, \ell-2]$ and $i+j \in [1, \ell-1]$. When $\bar{\rho}|_{I_\ell}$ is wildly ramified, $i, j \in [0, \ell-2]$ are uniquely determined. We set $k(\bar{\rho}) = 1 + i + (\ell+1)j$, unless $\bar{\rho}|_{I_\ell} = \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \omega^j$ and this $*$ is très ramifiée [Ser87, p. 186]. We put $k(\bar{\rho}) = (\ell+1)(j+1)$ in the exceptional case.

The level

Let V be the underlying space of $\bar{\rho}$. For each $p \neq \ell$, the Artin conductor of $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is

$$a(\bar{\rho}|_{G_{\mathbb{Q}_\ell}}) = \sum_{i=0} \frac{1}{[G_0 : G_i]} \dim V/V^{G_i},$$

where $\{G_i\}$ are the ramification subgroups of $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Then we set

$$N(\bar{\rho}) = \prod_{p \neq \ell} p^{a(\bar{\rho}|_{G_{\mathbb{Q}_\ell}})}.$$

The Strong Serre conjecture

Theorem. *Let $\bar{\rho}\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_\ell)$ be odd and absolutely irreducible. Then $\bar{\rho}^{ss} \simeq \bar{\rho}^{ss}_{f,\ell}$ for an eigenform f of weight $k(\bar{\rho})$ and level $N(\bar{\rho})$.*

1.5.3 Modularity

In this section, we sketch a proof of the modularity of some Galois representations associated with certain two-dimensional motives, i.e., isomorphic to one $\rho_{f,\ell}$ in (1.16). The argument is originally due to Serre [Ser87, §4.8], and one can find similar arguments in [Kis07, Thm. 1.4.3] and [YV15, Thm. 4.6.1]. The key ingredient in the proof is Serre's modularity conjecture [Ser87] (proved in [KW09a] and [KW09b]). In Section 5.3 we will use this result to prove Theorem B.

Theorem 1.17. *Let M be an irreducible pure motive of dimension 2 over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$. Assume that the nonzero Hodge numbers of the de Rham realization of M are $h^{r,s} = h^{s,r} = 1$ for some $0 \le r < s$, and the ℓ-adic Galois representations M_ℓ are odd. Then for some $N \ge 1$ and some Dirichlet character $\varepsilon\colon \mathbb{Z}/N\mathbb{Z}^\times \to \mathbb{C}^\times$, there exists an eigenform $f \in S_{s-r+1}(\Gamma_0(N), \varepsilon)$ such that $\rho_{f,\ell} \simeq \mathrm{M}_\ell(s)$.*

Proof. We first give bounds for the possible weights and conductors of $\rho_{\mathrm{M},\ell}$.

Let $j = s - r$. We write $\rho_{\mathrm{M},\ell}$ for $\mathrm{M}_\ell(s)$. The Hodge-Tate number of $\rho_{\mathrm{M},\ell}$ is thus $(0, j)$. By Fontaine-Messing [FM87, 2.3] and Fontaine-Laffaille theory [FL82, §8, Thm. 5.3], if $j + 1 < \ell$, the restriction of $\bar{\rho}_{\mathrm{M},\ell}$ to the inertia group I_ℓ is either of the form $\begin{pmatrix} \omega_2^j & 0 \\ 0 & \omega_2^{\ell j} \end{pmatrix}$ or $\begin{pmatrix} \omega^j & * \\ 0 & 1 \end{pmatrix}$. Moreover, if $j = 1$, then $\bar{\rho}_{M,\ell}|_{G_{\bar{\mathbb{Q}}/\mathbb{Q}_p}}$ arises from a finite flat group scheme [FL82, §9], so in the second case the item $*$ is peu-ramifiée [Ser87, p. 186]. It follows that $k(\bar{\rho}_{\mathrm{M},\ell}) = s - r + 1$.

As for the conductor, let S be a finite set such that for each $p \notin S$, the representation $\rho_{M,\ell}$ is unramified if $p \neq \ell$. We denote by $n(r, \ell)$ the Artin conductor of $\bar{\rho}_{M,\ell}$ at the prime $r \in S$. By [Ser87, 4.8.8], assuming that

$$\begin{cases} \ell \not\equiv \pm 1 (\mathrm{mod}\, 8) & \text{if } 2 \in S, \\ \ell \not\equiv \pm 1 (\mathrm{mod}\, 9) & \text{if } 3 \in S, \\ \ell \not\equiv \pm 1 (\mathrm{mod}\, r) & \text{if } p \in S \text{ and } p \ge 5, \end{cases} \tag{1.18}$$

we have

$$\begin{cases} n(2, \ell) \le 9 & \text{if } 2 \in S, \\ n(3, \ell) \le 5 & \text{if } 3 \in S, \\ n(p, \ell) \le 2 & \text{if } p \in S \text{ and } p \ge 5. \end{cases}$$

It follows that $N(\bar{\rho}_{\mathrm{M},\ell})$ is a divisor of $N := \prod_{r \in S} r^{n(r,\ell)}$.

Now assume that there are infinitely many primes ℓ such that $\bar{\rho}_{\mathrm{M},\ell}$ is absolutely irreducible. We let L be the set of primes ℓ such that

- $\ell > s - r + 1$ and $\ell \notin S$,
- ℓ satisfies (1.18),
- $\bar{\rho}_{\mathrm{M},\ell}$ is absolutely irreducible.

Then L is an infinite set. For each $\ell \in L$, there exists a modular form $f_\ell \in \Gamma_{s-r+1}(\Gamma_0(N), \varepsilon)$ for some Dirichlet character ε such that $\bar{\rho}^{ss}_{\mathrm{M},\ell} \simeq \bar{\rho}^{ss}_{f_\ell,\ell}$. Since the choices of Dirichlet characters of $\mathbb{Z}/N\mathbb{Z}^\times$ are finite, and the vector space $\Gamma_{s-r+1}(\Gamma_0(N), \varepsilon)$ is finite dimensional, the choices of f_ℓ lie in a finite set. It follows that there exists an infinite subset $\tilde{L}$ of L, such that the modular forms $f := f_\ell$ for $\ell \in \tilde{L}$ are the same. Then for each $p \nmid N\ell$, we have

$$\mathrm{Tr}(\mathrm{Frob}_p^{-1} \mid \rho_{\mathrm{M},\ell}) \equiv \mathrm{Tr}(\mathrm{Frob}_p^{-1} \mid \bar{\rho}_{\mathrm{M},\ell}) = \mathrm{Tr}(\mathrm{Frob}_p^{-1} \mid \bar{\rho}_{f,\ell}) \equiv a_f(p) (\mathrm{mod}\, \ell).$$

Notice that the left-hand side only depends on p. Since the above identity holds for any $\ell \in \tilde{L}$ and $p \nmid N\ell$, and any nonzero algebraic integer in $\mathbb{Q}$ is only divisible by finitely many primes, conclude that

$$\mathrm{Tr}(\mathrm{Frob}_p^{-1} \mid \rho_{\mathrm{M},\ell}) = a_f(p).$$

By Chebotarev density theorem, we have $\rho_{\mathrm{M},\ell}^{ss} \simeq \rho_{f,\ell}^{ss}$. Since both $\rho_{\mathrm{M},\ell}$ and $\rho_{f,\ell}$ are irreducible, they are isomorphic.

At last, we show that there are finitely many ℓ such that $\bar{\rho}_{\mathrm{M},\ell}$ are absolutely irreducible. Assuming this is false, then the semisimplification of $\bar{\rho}_{\mathrm{M},\ell}$ would be the mod ℓ reduction of $\varepsilon_1 \oplus \varepsilon_2 \chi_{cyc}^j$, where χ is the ℓ-adic cyclotomic character, and ε_i are Dirichlet characters of conductor dividing N. A finiteness argument as above implies that the semisimplification of $\rho_{\mathrm{M},\ell}$ is again of the form $\varepsilon_1 \oplus \varepsilon_2 \chi_{cyc}^j$. However, this contradicts the fact that $\rho_{\mathrm{M},\ell}(\mathrm{Frob}_p^{-1})$ only has roots of complex norm $p^{j/2}$. □

1.5.4 Generalities on Deligne-Weil representations

We recall the definition of Weil-Deligne (or simply, WD-) representations from [Tay04].

For each prime p, there is an exact sequence

$$1 \to I_p \to \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \simeq \hat{\mathbb{Z}} \to \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \to 1,$$

where I_p is the *inertial group* at p. Moreover, there is a surjection $t_\ell \colon I_p \to \mathbb{Z}_\ell$. Let $W_{\mathbb{Q}_p}$ be the *Weil group* of $\mathbb{Q}_p$, i.e., the inverse image of the subgroup generated by Frobenius of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \simeq \hat{\mathbb{Z}}$ in $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ equipped with the induced topology.

A *WD*-representation on an E-vector space V (with discrete topology) is a pair (r, N), consisting of a representation

$$r \colon W_{\mathbb{Q}_p} \to \mathrm{GL}(V)$$

with open kernel, and an endormophism $N \in \mathrm{End}(V)$, such that

$$r(\phi) N r(\phi^{-1}) = p^{-1} N$$

for every lift $\phi \in W_{\mathbb{Q}_p}$ of Frob_p. It is called *unramified* if $N = 0$ and $r(I_p) = 1$. It is called *Frobenius semisimple* if r is semisimple. For a lift ϕ of Frobenius, we can decompose $r(\phi) = r(\phi)^{ss} r(\phi)^u = r(\phi)^u r(\phi)^{ss}$, where $r(\phi)^{ss}$ is semisimple and $r(\phi)^u$ is unipotent. Any WD-representation (r, N) has a canonical Frobenius semisimplification $(r, N)^{ss}$, by keeping N and $r|_{I_p}$ unchanged, and replacing $r(\phi)$ by $r(\phi)^{ss}$. In the case that $E = \bar{\mathbb{Q}}_\ell$, we call (r, N) *ℓ-integral* if all the eigenvalues of $r(\phi)$ have absolute value 1. This property is independent of the choice of ϕ.

If $\ell \neq p$, there is a canonical way to attach a WD-representation $\mathrm{WD}_p(\rho)$ to an ℓ-adic representation ρ of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ as follows: By Grothendieck's quasi-unipotency theorem, there exists an open subgroup H of I_p of finite index, and a unique nilpotent endomorphism N satisfying $r(\sigma) = \exp(t_\ell(\sigma) N)$ for all $\sigma \in H$. Let ϕ be a lift of Frob_p and $\sigma \in I_p$, one sets

$$r(\phi^n \sigma) := \rho(\phi^n \sigma) \exp(t_\ell(\sigma) N). \tag{1.19}$$

Notice that $\mathrm{WD}_p(\rho)$ is unramified if and only if $\rho(I_p) = 1$, i.e., ρ is unramified.

A WD-representation (r, N) on $\bar{\mathbb{Q}}_\ell$ is called *pure of weight w* [BLGGT14, p. 528] if there is an exhaustive and separated ascending monodromy filtration M_i of V such that

- each $F_i V$ is invariant under r,
- for each lift ϕ of Frob_p, all eigenvalues $r(\phi^m)$ on $\mathrm{gr}_i^M V$ are Weil-numbers of weight $m \cdot i$,
- the endormorphism N sends $M_i V$ into $M_{i-2} V$, and induces isomorphisms $N^j \colon \mathrm{gr}_{w+j}^M \simeq \mathrm{gr}_{w-j}^M V$ for each $j \geq 1$.

1.5.5 Potential automorphy

We first recall the Theorem of Patrikis-Taylor [PT15].

Definition 1.20. A *weakly compatible system* $\mathscr{R} = \{\rho_\ell\}$ of n-dimensional ℓ-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{Q}$ and unramified outside S is a family of continuous semisimple representations

$$\rho_\ell \colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(V_\ell)$$

for each prime number ℓ, with the following properties.

(1). If $p \notin S$, for all $\ell \neq p$, the representation ρ_ℓ is unramified at p and the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_p)$ is a polynomial with coefficients in $\mathbb{Q}$, independent of the choice of ℓ,

(2). Each representation ρ_ℓ is de Rham at ℓ, and is crystalline if $\ell \notin S$,

(3). The Hodge-Tate number of ρ_ℓ is independent of ℓ.

To a weakly compatible system of ℓ-adic representations, we can attach a partial L-function

$$L^S(\mathscr{R}, s) = \prod_{p \notin S} \det(1 - \rho_\ell(\mathrm{Frob}_p) p^{-s})^{-1}.$$

Moreover, we call $\mathscr{R}$ *strictly compatible* if for each p, there exists a WD-representation $\mathrm{WD}_p(\mathscr{R})$ of $W_{\mathbb{Q}_p}$ over $\bar{\mathbb{Q}}$ such that for each $\ell \neq p$ and each $\iota \colon \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$, the push forward $\iota\mathrm{WD}_p(\mathscr{R})$ is isomorphic to $\mathrm{WD}_p(\rho_\ell)^{ss}$. To a strictly compatible family $\mathscr{R}$, we can attach an L-function

$$L(\mathscr{R}, s) = \prod_p \det\left(1 - \mathrm{Frob}_p \cdot p^{-s} \mid \mathrm{WD}_p(\mathscr{R})^{I_p, N=0}\right)^{-1},$$

which differs from $L^S(\mathscr{R}, s)$ only by finitely many Euler factors at $p \in S$.

Theorem 1.21 ([PT15, Thm. A])**.** *Let $\{\rho_\ell\}$ be a weakly compatible system of n-dimensional ℓ-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ defined over $\mathbb{Q}$ and unramified outside S. Suppose that $\{\rho_\ell\}$ satisfies the following properties.*

(1). (Purity) There exists an integer w such that, for each prime $p \notin S$, the roots of the common characteristic polynomial of $\rho_\ell(\mathrm{Frob}_p)$ are Weil numbers of weight w.

(2). (Regularity) The representation ρ_ℓ has m distinct Hodge-Tate weights.

(3). (Odd essential self-duality) Either each ρ_ℓ factors through a map to $\mathrm{GO}_m(\overline{\mathbb{Q}}_\ell)$ with even similitude character, or each ρ_ℓ factors through a map to $\mathrm{GSp}_m(\overline{\mathbb{Q}}_\ell)$ with odd similitude character. Moreover, in either case similitude characters form a weakly compatible system.

Then there exists a finite Galois totally real extension $F/\mathbb{Q}$, over which all the ρ_ℓ become automophic. In particular, the partial L-function $L^S(\mathscr{R}, s)$ has meromorphic continuation to the whole complex plane and satisfies the expected functional equation.

To describe the complete L-function, we still need the gamma factor at ∞. Serre described the conjectural shape of the gamma factors at infinity of the complete L-function of a pure motive over $\mathbb{Q}$ in [Ser70, § 3]. We denote by $L_\infty(\mathscr{R}, s)$ the gamma factor attached to $\mathscr{R}$.

Corollary 1.22 ([PT15, Cor. 2.2])**.** *Let $\mathscr{R} = \{\rho_\ell\}$ be a weakly compatible system that is pure of weight w, regular, and odd essential self-dual. Then, for any distinct primes p and ℓ, the Weil-Deligne representation $\mathrm{WD}_p(\mathscr{R})$ of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ associated with ρ_ℓ is pure of weight w. Moreover, the completed L-function*

$$\Lambda(\mathscr{R}, s) = L_\infty(\mathscr{R}, s) \cdot L(\mathscr{R}, s)$$

satisfies the functional equation $\Lambda(\mathscr{R}, s) = \varepsilon(\mathscr{R}, s)\Lambda(\mathscr{R}^\vee, 1 - s)$.

Chapter 2

Motives attached to Kloosterman sheaves

In this chapter, we give the construction of motives attached to Kloosterman sheaves following [FSY22]. Then we study their de Rham realizations, ℓ-adic realizations, and other realizations in characteristic $p > 0$.

2.1 The construction of motives

In [FSY22, §3], the authors constructed the motives M_{n+1}^{k}. In this section, we use the Weyl construction (see Section 1.1) to construct the motives $\mathrm{M}_{n+1}^{\lambda}$ (Nori motives over $\mathbb{Q}$ with rational coefficients).

Let n be an integer, and V_λ be the irreducible representation of the highest weight $\sum_i \lambda_i(L_1+\ldots+L_i)$. Let $\mathcal{K} \subset \mathbb{G}_m^{n|\lambda|}$ be the hypersurface defined by the equation

$$\sum_{i=1}^{|\lambda|}\Big(\sum_{j=1}^{n} x_{i,j} + \frac{1}{\prod_{j=1}^{n} x_{i,j}}\Big) = 0.$$

The group $S_{|\lambda|} \times \mu_{n+1}$ acts on $\mathcal{K}$ by $(\sigma \times \mu) \cdot x_{i,j} := \mu \cdot x_{\sigma(i),j}$. By abuse of notation, we denote P_λ and Q_λ the groups $P_{\mu(\lambda)}$ and $Q_{\mu(\lambda)}$. We define the group $G_\lambda = P_\lambda \times Q_\lambda$ and the character

$$\chi_\lambda \colon P_\lambda \times Q_\lambda \times \mu_{n+1} \xrightarrow{\mathrm{pr}_{1,2}} P_\lambda \times Q_\lambda \xrightarrow{\mathrm{Sign}^n \times \mathrm{Sign}^{n+1}} \{\pm 1\}.$$

Definition 2.1. The *motives attached to* $\mathrm{Kl}_{n+1}^{\lambda}$ are the Nori motives over $\mathbb{Q}$ with rational coefficients, of the form

$$\mathrm{M}_{n+1}^{\lambda} := \mathrm{gr}^{W}_{n|\lambda|+1}(\mathrm{H}_c^{n|\lambda|-1}(\mathcal{K})^{G_\lambda \times \mu_{n+1}, \chi_\lambda}(-1)),$$

where the exponent $(G_\lambda \times \mu_{n+1}, \chi_\lambda)$ means taking the χ_λ-isotypic component, and $W_\bullet$ is the (motivic) weight filtration [HMS17, Thm. 10.2.5].

Remark 2.2. (1). In the first version of [FSY22], the motives M_{n+1}^{k} were constructed as exponential motives

$$\mathrm{H}_{\mathrm{mid}}^{nk+1}(\mathbb{G}_m^{nk+1}, f_k)^{S_k \times \mu_{n+1}, \chi_n},$$

which were shown to be isomorphic to classical motives in Definition 2.1.

(2). As explained in the introduction, the name is justified by Theorem C, because the L-functions of $\mathrm{M}_{n+1}^{\lambda}$ coincide with the L-functions attached to Kloosterman sheaves $\mathrm{Kl}_{n+1}^{\lambda}$.

(3). The action of $\zeta_{n+1} \in \mu_{n+1}$ on $\mathcal{K}$ is not an automorphism defined over $\mathbb{Q}$ (only defined over $K = \mathbb{Q}(\zeta_{n+1})$). But taking the invariants of μ_{n+1} on $\mathrm{N} := \mathrm{gr}^{W}_{n|\lambda|+1}\big(\mathrm{H}_c^{n|\lambda|-1}(\mathcal{K})^{G_\lambda \times 1, \chi_\lambda}(-1)\big)$ still gives rise to a Nori motive over $\mathbb{Q}$. In fact, one can see the Nori motive N as a $\mathbb{Q}$-vector space

together with an action of the motivic Galois group $G_{\mathrm{mot}}(\mathbb{Q})$. We restrict N to a Nori motive N_K over K, i.e., a $\mathbb{Q}$-vector space with an action of the motivic Galois group $G_{\mathrm{mot}}(K)$. Then one can consider a Nori motive over K

$$\mathrm{N}^{\mu_{n+1}} := \mathrm{im}(\mathrm{N}_K \xrightarrow{\varphi} \mathrm{N}_K),$$

where $\varphi = \frac{1}{\#\mu_{n+1}} \sum_{\zeta \in \mu_{n+1}} \zeta$. One can check that $\mathrm{N}^{\mu_{n+1}}$ is stable under the action of $\mathrm{Gal}(K/\mathbb{Q})$. By [HMS17, Thm. 9.1.16], the motive $\mathrm{N}^{\mu_{n+1}}$ comes from a Nori motive over $\mathbb{Q}$.

When the representation V_λ is the k-th symmetric power of the standard representation of SL_{n+1}, i.e., $V_{(k,0,\dots,0)}$. The group G_λ is S_k and the character χ_λ is sign^n. For simplicity, we denote the motive M^λ_{n+1} by M^k_{n+1}, and the character sign^n by χ_n.

Proposition 2.3. *The motives M^λ_{n+1} are pure of weight $n|\lambda|+1$. Moreover, they are equipped with $(-1)^{n|\lambda|+1}$-symmetric perfect pairings*

$$\mathrm{M}^\lambda_{n+1} \times \mathrm{M}^\lambda_{n+1} \to \mathbb{Q}(-n|\lambda|-1).$$

Proof. The motives $\mathrm{gr}^W_{n|\lambda|+1}(\mathrm{H}_c^{n|\lambda|-1}(\mathcal{K})(-1))$ are pure of weight $n|\lambda|+1$. Moreover, they are equipped with $(-1)^{n|\lambda|+1}$-symmetric perfect pairings, using a similar proof in [FSY22, Thm. 3.2] for exponential mixed Hodge structures. Since the motives M^λ_{n+1} are subquotients of $\mathrm{gr}^W_{n|\lambda|+1}(\mathrm{H}_c^{n|\lambda|-1}(\mathcal{K})(-1))$, they are also pure of weight $n|\lambda|+1$, and have induced $(-1)^{n|\lambda|+1}$-symmetric pairings. □

2.2 Realizations in characteristic 0

2.2.1 The de Rham realizations

Kloosterman connections

Consider the diagram

$$\begin{array}{ccccc} & & \mathbb{G}_m^{n+1} & & \\ & \swarrow^{\pi} & & \searrow^{\sigma} & \\ \mathbb{G}_{m,z} & & & & \mathbb{A}^1_y \end{array} \tag{2.4}$$

where σ and π are the sum and the product map respectively. The *Kloosterman connection* is defined by

$$\mathrm{Kl}_{n+1} := \mathcal{H}^0 \pi_+ \sigma^+ \mathcal{E}^y,$$

where π_+ and σ^+ are direct image and pullback of $\mathcal{D}$-modules, and $\mathcal{E}^y$ denotes the rank 1 vector bundle with connection $(\mathcal{O}_{\mathbb{A}^1}, \mathrm{d}+\mathrm{d}y)$. It is shown in [FSY22, Prop. 2.4] that Kl_{n+1} is a free $\mathcal{O}_{\mathbb{G}_m}$-module of rank $n+1$, whose connection, in terms of some basis $\{v_0, v_1, \dots, v_n\}$, is given by

$$\mathrm{d} + N\frac{\mathrm{d}z}{z} + E\mathrm{d}z = \mathrm{d} + \begin{pmatrix} 0 & 0 & \cdots & 0 & z \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \frac{\mathrm{d}z}{z} \tag{2.5}$$

where N is the lower triangular Jordan block of size $n+1$ with eigenvalue 0, and E is the matrix with 1 at row 1 and column $n+1$ and 0 elsewhere. Moreover, we have an isomorphism

$$\mathrm{Kl}^\vee_{n+1} \simeq \iota^+_{n+1}\mathrm{Kl}_{n+1}$$

where $\mathrm{Kl}^\vee_{n+1}$ is the dual of Kl_{n+1}, and ι_{n+1} is the involution $z \mapsto (-1)^{n+1}z$.

The connection Kl_{n+1} has a regular singularity at 0, and an irregular singularity at ∞, with slope $\frac{1}{n+1}$. To turn it into a connection with slope 0 or 1 at ∞, we consider the pull-back of Kl_{n+1} along the map $[n+1]\colon \mathbb{G}_{m,t} \to \mathbb{G}_{m,z}$ given by $a \mapsto a^{n+1}$, denoted by

$$\widetilde{\mathrm{Kl}}_{n+1} := [n+1]^{+}\mathrm{Kl}_{n+1}. \tag{2.6}$$

Moreover, there is an action of μ_{n+1} on $[n+1]_{+}\widetilde{\mathrm{Kl}}_{n+1}$, and the invariant part is exactly Kl_{n+1}.

For the representation of the highest weight $(\lambda_1, \ldots, \lambda_n)$ of $\mathrm{SL_{n+1}}$, we denote by $\mathrm{Kl}^{\lambda}_{n+1}$ the connection $(\mathrm{Kl}^{\otimes k}_{n+1})^{G_\lambda, (\mathrm{id}\times\mathrm{sign})}$. If $V_\lambda = \mathrm{Sym}^k\mathrm{Std}$, i.e. $\lambda = (k, 0, \ldots, 0)$, we get $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$, the k-th symmetric power of Kl_{n+1} of rank $\binom{n+k}{k}$. There are similar constructions for $\widetilde{\mathrm{Kl}}_{n+1}$.

Remark 2.7. In [FG09, 6.1], there is a functorial way to associate connections ∇ on $\mathbb{G}_m$ to pairs (G, V), where G are simple reductive groups and V are finite dimensional representations of G with coefficients in $\mathbb{C}$.

Let Φ_+ be a choice of positive roots of G and Δ be the set of positive simple roots. Then the Lie algebra of G can be decomposed as $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha \oplus \mathfrak{h}$. We choose a generator $X_{-\alpha}$ of $\mathfrak{g}_{-\alpha}$ for each simple root α, and a generator X_θ of $\mathfrak{g}_\theta$ for the maximal root θ. Let $N = \sum_\alpha X_{-\alpha}$ and $E = X_\theta$. Then the Frenkel-Gross connection associated with the pair (G, V) is

$$\nabla(V) = \mathrm{d} + N(V)\frac{\mathrm{d}z}{z} + E(V)\mathrm{d}z,$$

where $N(V)$ and $E(V)$ are the corresponding nilpotent endomorphisms of V induced by N and E respectively.

The construction of Frenkel-Gross connection is functorial, i.e., we have a tensor functor

$$\mathrm{Rep}(G) \to \mathrm{D.E.}(\mathbb{G}_m)$$

from the category of finite dimensional representations with coefficients in $\mathbb{C}$ of G, to the category of flat connections on $\mathbb{G}_m$. One can verify that the Frenkel-Gross connections attached to the pairs $(\mathrm{SL}_{n+1}, V_\lambda)$ are the connections $\mathrm{Kl}^{\lambda}_{n+1}$ in the above.

The de Rham cohomologies

In the next proposition, we gather some results about the de Rham cohomologies of $\mathrm{Kl}^{\lambda}_{n+1}$ from [FSY22, Cor. 2.11, Cor. 2.15, Thm. 3.8], which can be proved simply by replacing $(S_k \times \mu_{n+1}, \chi_n)$ with $(G_\lambda \times \mu_{n+1}, \chi_\lambda)$.

Proposition 2.8. *Let $\{x_{i,j} | 1 \le i \le |\lambda|, 1 \le j \le n\}$ and z be the coordinates of $\mathbb{G}_m^{n|\lambda|+1}$. The group G_λ acts on $\mathbb{G}_m^{n|\lambda|+1}$ by $(\sigma \times \mu)\cdot x_{i,j} = x_{\sigma(i),j}$ and $(\sigma \times \mu)\cdot z = \mu \cdot z$. Let $\widetilde{f}_{|\lambda|}\colon \mathbb{G}^{n|\lambda|+1}_{m,(x_{i,j},t)} \to \mathbb{A}^1_t$ be the Laurent polynomial $\sum_i \left(\sum_j x_{i,j} + \frac{t^{n+1}}{\prod_j x_{i,j}}\right)$. Then for $? \in \{\varnothing, \mathrm{c}\}$, we have*

$$\mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_m, \mathrm{Kl}^{\lambda}_{n+1}) \simeq \mathrm{H}^{n|\lambda|+1}_{\mathrm{dR},?}(\mathbb{G}_m^{n|\lambda|+1}, \widetilde{f}_{|\lambda|})^{G_\lambda \times \mu_{n+1}, \chi_n}$$

and

$$\mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_m, \widetilde{\mathrm{Kl}}^{\lambda}_{n+1}) \simeq \mathrm{H}^{n|\lambda|+1}_{\mathrm{dR},?}(\mathbb{G}_m^{n|\lambda|+1}, \widetilde{f}_{|\lambda|})^{G_\lambda, \chi_n}.$$

Remark 2.9. The meaning of the above isomorphisms is that the de Rham realization $(\mathrm{M}^{\lambda}_{n+1})_{\mathrm{dR}}$ of the motive $\mathrm{M}^{\lambda}_{n+1}$ is the middle de Rham cohomology of $\mathrm{Kl}^{\lambda}_{n+1}$.

2.2.2 Exponential mixed Hodge structures

By [FSY22, Thm. 3.2], for $? \in \{\varnothing, \mathrm{c}, \mathrm{mid}\}$ the (twisted) de Rham cohomologies

$$\mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_m, \mathrm{Kl}^\lambda_{n+1}) \text{ and } \mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_m, \widetilde{\mathrm{Kl}}^\lambda_{n+1})$$

underlie the exponential mixed Hodge structures

$$\mathrm{H}^1_?(\mathbb{G}_m, \mathrm{Kl}^\lambda_{n+1}) := \mathrm{H}^{n|\lambda|+1}_?(\mathbb{G}_m^{n|\lambda|+1}, \widetilde{f}_{|\lambda|})^{G_\lambda \times \mu_{n+1}, \chi_n}$$

and

$$\mathrm{H}^1_?(\mathbb{G}_m, \widetilde{\mathrm{Kl}}^\lambda_{n+1}) := \mathrm{H}^{n|\lambda|+1}_?(\mathbb{G}_m^{n|\lambda|+1}, \widetilde{f}_{|\lambda|})^{G_\lambda, \chi_n}$$

respectively. By [FSY22, Thm. 3.8], the exponential mixed Hodge structures $\mathrm{H}^1_?(\mathbb{G}_m, \mathrm{Kl}^\lambda_{n+1})$ and $\mathrm{H}^1_?(\mathbb{G}_m, \widetilde{\mathrm{Kl}}^\lambda_{n+1})$ are isomorphic to (classical) mixed Hodge structures, i.e., lie in the essential image of (1.3). More precisely, we have isomorphisms of pure Hodge structures

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Kl}^\lambda_{n+1}) \simeq \mathrm{gr}^W_{n|\lambda|+1}\left(\mathrm{H}^{n|\lambda|-1}_{\mathrm{c}}(\mathcal{K})^{G_\lambda \times \mu_{n+1}, \chi_n}(-1)\right),$$

and

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \widetilde{\mathrm{Kl}}^\lambda_{n+1}) \simeq \mathrm{gr}^W_{n|\lambda|+1}\left(\mathrm{H}^{n|\lambda|-1}_{\mathrm{c}}(\mathcal{K})^{G_\lambda, \chi_n}(-1)\right).$$

Moreover, they are (mixed) of weights $\geq n|\lambda|+1$, $\leq n|\lambda|+1$ and $n|\lambda|+1$ respectively. And the natural forget support morphisms induce isomorphisms

$$\mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^1_{\mathrm{c}}(\mathbb{G}_m, \mathrm{Kl}^\lambda_{n+1}) \to \mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^1(\mathbb{G}_m, \mathrm{Kl}^\lambda_{n+1})$$

and

$$\mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^1_{\mathrm{c}}(\mathbb{G}_m, \widetilde{\mathrm{Kl}}^\lambda_{n+1}) \to \mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^1(\mathbb{G}_m, \widetilde{\mathrm{Kl}}^\lambda_{n+1}).$$

2.2.3 The ℓ-adic realizations

For a prime ℓ, the ℓ-adic realization

$$(\mathrm{M}^\lambda_{n+1})_\ell := \mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\mathcal{K}_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell)^{G_\lambda \times \mu_{n+1}, \chi_n}(-1) \tag{2.10}$$

of M^λ_{n+1} is a continuous ℓ-adic representation of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, which is pure of weight $n|\lambda|+1$ and is equipped with a $(-1)^{n|\lambda|+1}$-symmetric pairing by Proposition 2.3.

Similar to the situation for motives, we get indeed a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. As explained in Remark 2.2.(3), although the action of μ_{n+1} does not commute with $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the invariants of μ_{n+1} is stable under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

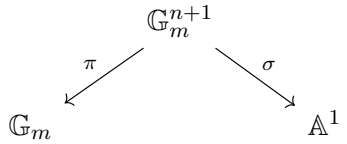
2.3 Other realizations in characteristic $p > 0$

2.3.1 The ℓ-adic case

Kloosterman sheaves

Let p be a prime number and ℓ a prime number different from p. We fix a primitive p-th root of unity ζ_p in $\bar{\mathbb{Q}}_\ell$, an $(n+1)$-th root of unity ζ_{n+1} in $\bar{\mathbb{F}}_p$, and a nontrivial additive character $\psi_p \colon \mathbb{F}_p \to \mathbb{Q}_\ell(\zeta_p)^\times$. The *Artin-Schreier sheaf* $\mathscr{L}_{\psi_p}$ is a lisse ℓ-adic sheaf with coefficients in $\mathbb{Q}_\ell(\zeta_p)^\times$ defined on $\mathbb{A}^1_{\mathbb{F}_p}$, whose trace function is given by ψ_p. We denote by $\mathscr{L}_{\psi_p(f)}$ the inverse image $f^*\mathscr{L}_{\psi_p}$ of the Artin-Schreier sheaf along a regular function $f \colon X \to \mathbb{A}^1_{\mathbb{F}_p}$.

Consider the diagram



where σ is the sum of coordinates and π is the product of coordinates. We define the *Kloosterman sheaf* on $\mathbb{G}_{m,\mathbb{F}_p}$ by

$$\mathrm{Kl}_{n+1} := \mathrm{R}^n\pi_!\sigma^*\mathscr{L}_{\psi_p}, \tag{2.11}$$

which is a lisse ℓ-adic sheaf on $\mathbb{G}_{m,\mathbb{F}_p}$ of rank $n+1$ and pure of weight n [Del77, Sommes. Trig. Thm. 7.8].

In [HNY13], the authors constructed *Kloosterman sheaves for reductive groups*. For a connected split reductive group G, we get a tensor functor

$$\mathrm{Kl}_G\colon \mathrm{Rep}(G) \to \mathrm{Loc}(\mathbb{G}_{m,\mathbb{F}_q}, \bar{\mathbb{Q}}_\ell)$$

from the category of finite dimensional representations of G to the category of ℓ-adic local systems on $\mathbb{G}_m$. In particular, if we take $G = \mathrm{SL}_{n+1}$, and V the standard representation, we get the Kloosterman sheaves $\mathrm{Kl}_{n+1}(\frac{n}{2})$. If we take V as the irreducible representation of the highest weight λ, we get the local system $(\mathrm{Kl}_{n+1}^{\otimes k})^{G_\lambda,\mathrm{id}\times\mathrm{sign}}(\frac{nk}{2})$. For simplicity, we denote the local system $(\mathrm{Kl}_{n+1}^{\otimes k})^{G_\lambda,\mathrm{id}\times\mathrm{sign}}$ by $\mathrm{Kl}_{n+1}^{\lambda}$.

Similar to the computation in [FSY22, Prop. 2.10], we give an alternative description of these sheaves. Let $g\colon \mathbb{G}_{m,\mathbb{F}_p}^n \to \mathbb{A}^1_{\mathbb{F}_p}$ be the Laurent polynomial $\sum_{i=1}^n y_i + \frac{1}{\prod_i y_i}$. Then we have the following lemma.

Lemma 2.12. *We have the isomorphism of ℓ-adic sheaves*

$$[n+1]^*\mathrm{Kl}_{n+1} \simeq \mathrm{FT}_{\psi_p}(\mathrm{R}^{n-1}g_!\bar{\mathbb{Q}}_\ell)|_{\mathbb{G}_m},$$

where FT_{ψ_p} is the Deligne-Fourier transform [Lau87], and $[n+1]\colon \mathbb{G}_{m,\mathbb{F}_p} \to \mathbb{G}_{m,\mathbb{F}_p}$ be the $(n+1)$-th power map.

Proof. Let $x_1, \dots, x_{n+1}$ be the coordinates of $\mathbb{G}_{m,\mathbb{F}_p}^{n+1}$ in the diagram for (2.11). We do a change of variable $z = \prod_{i=1}^n x_i$. Let $j\colon \mathbb{G}_{m,\mathbb{F}_p} \to \mathbb{A}^1_{\mathbb{F}_p}$. Then we can rewrite (2.11) as

$$\mathrm{Kl}_{n+1} = j^*\mathrm{R}(\mathrm{pr}_z)_!\mathscr{L}_{\psi_p\left(\sum_{i=1}^n x_i + \frac{z}{\prod_{i=1}^n x_i}\right)}[n].$$

Let t be the coordinate of the source of the map $[n+1]$. Then

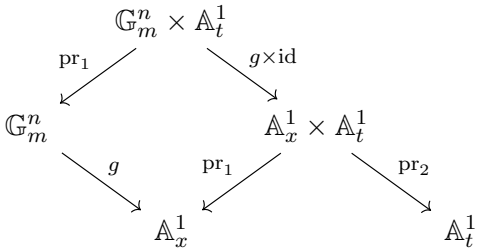
$$[n+1]^*\mathrm{Kl}_{n+1} \simeq j^*\mathrm{R}(\mathrm{pr}_z)_!\mathscr{L}_{\psi_p\left(\sum_{i=1}^n x_i + \frac{t^{n+1}}{\prod_{i=1}^n x_i}\right)}[n] \simeq \mathrm{R}(\mathrm{pr}_t)_!\mathscr{L}_{\psi_p(tg)}[n], \tag{2.13}$$

where we did a change of variable $y_i = x_i/t$.

By a calculation of Deligne Fourier transform, we know that

$$\begin{aligned}
\mathrm{FT}_{\psi_p}(\mathrm{R}g_!\bar{\mathbb{Q}}_\ell) \simeq& \mathrm{R}(\mathrm{pr}_2)_!(\mathrm{pr}_1^*\mathrm{R}g_!\bar{\mathbb{Q}}_\ell \otimes \mathscr{L}_{\psi_p(xt)}[1]) \\
\simeq& \mathrm{R}(\mathrm{pr}_2)_!(\mathrm{R}(g\times\mathrm{id})_!\mathrm{pr}_1^*\bar{\mathbb{Q}}_\ell \otimes \mathscr{L}_{\psi_p(xt)}[1]) \\
\simeq& \mathrm{R}(\mathrm{pr}_2)_!\mathrm{R}(g\times\mathrm{id})_!(\mathrm{pr}_1^*\bar{\mathbb{Q}}_\ell \otimes \mathscr{L}_{\psi_p(tg)}[1]) \\
\simeq& \mathrm{R}(\mathrm{pr}_t)_*\mathscr{L}_{\psi_p(tg)}[1],
\end{aligned}$$

where we used the base change theorem in the second isomorphism, and the projection formula in the third isomorphism. The morphisms in the above calculation are illustrated as follows.



We conclude from the above isomorphisms that $[n+1]^*\mathrm{Kl}_{n+1} \simeq j^*\,\mathrm{FT}_{\psi_p}(\mathrm{R}g_!\bar{\mathbb{Q}}_\ell)[n-1]$. □

Consider the torus $\mathbb{G}_{m,\mathbb{F}_p}^{n|\lambda|+1}$ with coordinates $\{x_{i,j} \mid 1 \le i \le |\lambda|, 1 \le j \le n\}$ and z. Let $f_{|\lambda|}\colon \mathbb{G}_{m,\mathbb{F}_p}^{n|\lambda|+1} \to \mathbb{A}_{\mathbb{F}_p}^1$ be the Laurent polynomial $\sum_{i=1}^{|\lambda|}(\sum_{j=1}^n x_{i,j} + \frac{z}{\prod_j x_{i,j}})$ and pr_z be the projection from $\mathbb{G}_{m,\mathbb{F}_p}^{n|\lambda|+1}$ to its z-coordinate. Similarly, consider the torus $\mathbb{G}_{m,\mathbb{F}_p}^{n|\lambda|+1}$ with coordinates $\{x_{i,j} \mid 1 \le i \le |\lambda|, 1 \le j \le n\}$ and z. We let $\tilde{f}_{|\lambda|}\colon \mathbb{G}_{m,\mathbb{F}_p}^{n|\lambda|+1} \to \mathbb{A}_{\mathbb{F}_p}^1$ be the Laurent polynomial $\sum_{i=1}^{|\lambda|}(\sum_{j=1}^n x_{i,j} + \frac{t^{n+1}}{\prod_j x_{i,j}})$ and pr_t be the projection from $\mathbb{G}_{m,\mathbb{F}_p}^{n|\lambda|+1}$ to its t-coordinate.

Proposition 2.14. *We have the isomorphism of ℓ-adic sheaves*

$$\mathrm{Kl}_{n+1}^\lambda \simeq j^*\Big(\mathrm{R}^{n|\lambda|}\mathrm{pr}_{z*}\mathscr{L}_{\psi_p(f_{|\lambda|})}\Big)^{G_\lambda\times 1,\chi_\lambda}$$

and

$$[n+1]^*\mathrm{Kl}_{n+1}^\lambda \simeq j^*\Big(\mathrm{R}^{n|\lambda|}\mathrm{pr}_{t*}\mathscr{L}_{\psi_p(\tilde{f}_{|\lambda|})}\Big)^{G_\lambda\times 1,\chi_\lambda}.$$

Proof. By [Lau87, (1.2.2.7)], the Deligne-Fourier transform interchanges tensor product and the exterior product. So

$$\begin{aligned}
([n+1]^*\mathrm{Kl}_{n+1})^{\otimes|\lambda|} \simeq& j^*\mathrm{FT}_{\psi_p}\Big(((\mathrm{R}g_!\bar{\mathbb{Q}}_\ell)[n-1])^{\boxtimes|\lambda|}\Big)[|\lambda|]\\
\simeq& j^*\mathrm{FT}_{\psi_p}\big((\mathrm{R}g_!^{\boxplus|\lambda|}\bar{\mathbb{Q}}_\ell)\big)[n|\lambda|-1]\\
\simeq& j^*\mathrm{R}(\mathrm{pr}_t)_!\mathscr{L}_{\psi_p(t\cdot g^{\boxplus|\lambda|})}[n|\lambda|-1]\\
\simeq& j^*\mathrm{R}(\mathrm{pr}_t)_!\mathscr{L}_{\psi_p(\tilde{f}_{|\lambda|})}[n|\lambda|-1],
\end{aligned}$$

where we used the Künneth formula in the third isomorphism, and we did a change of variable $x_{i,j} = t\cdot y_{i,j}$ in the last isomorphism.

Notice that the Deligne-Fourier transform preserves the action of symmetric group $S_{|\lambda|}$, and there are some extra signs coming from the Künneth formula on the right-hand side. By taking the corresponding isotypic component on both sides, we get the second isomorphism.

As for the first isomorphism, similar to Remark 2.2.(3), one has

$$\begin{aligned}
\mathrm{Kl}_{n+1}^{\otimes|\lambda|} \simeq& \big([n+1]_*([n+1]^*\mathrm{Kl}_{n+1})^{\otimes k}\big)^{\mu_{n+1}}\\
\simeq& j^*\Big(\mathrm{R}(\mathrm{pr}_t)_!\mathscr{L}_{\psi_p(\tilde{f}_{|\lambda|})}\Big)^{\mu_{n+1}}[n|\lambda|-1]\\
\simeq& j^*\mathrm{R}(\mathrm{pr}_z)_!\mathscr{L}_{\psi_p(f_{|\lambda|})}[n|\lambda|-1].
\end{aligned}$$

At last, we add the corresponding isotypic components to both sides, we get the first isomorphism. □

ℓ-adic Cohomologies

Proposition 2.15. *We have*

$$\mathrm{H}^i_{\acute{e}t,?}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1}) \simeq \mathrm{H}^{n|\lambda|+i}_{\acute{e}t,?}\left(\mathbb{G}^{n|\lambda|+1}_{m,\bar{\mathbb{F}}_p}, \mathscr{L}_{\psi(\tilde{f}_{|\lambda|})}\right)^{G_\lambda \times \mu_{n+1}, \chi_\lambda},$$

for $i \in \{0,1,2\}$. Here the action of μ_{n+1} is induced by that on $\mathbb{G}_{m,\mathbb{F}_p(\zeta_{n+1})}$, and we can understand the μ_{n+1}-invariants similarly as in Remark 2.2.(3).

Proof. We only give the proof for the usual cohomology. The properties for the cohomology with compact support and the middle cohomology can be proved similarly.

Let pr_z be the projection from $\mathbb{G}_m^{n|\lambda|} \times \mathbb{G}_{m,z}$ to the last factor $\mathbb{G}_{m,z}$. The projection pr_t is defined in a parallel way to pr_z. By the isomorphism $\left([n+1]_*\mathscr{L}_{\psi_p(\tilde{f}_{|\lambda|})}\right)^{\mu_{n+1}} \simeq \mathscr{L}_{\psi_p(f_{|\lambda|})}$, we have $\mathrm{Kl}^\lambda_{n+1} \simeq ([n+1]_*[n+1]^*\mathrm{Kl}^\lambda_{n+1})^{\mu_{n+1}}$. Then

$$\begin{aligned}
\mathrm{H}^i_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1}) \simeq & \mathrm{H}^i_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, ([n+1]_*[n+1]^*\mathrm{Kl}^\lambda_{n+1})^{\mu_{n+1}}) \\
\simeq & \mathrm{H}^i_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, [n+1]^*\mathrm{Kl}^\lambda_{n+1})^{\mu_{n+1}} \\
\simeq & \mathrm{H}^{n|\lambda|+i}_{\acute{e}t}\left(\mathbb{G}^{n|\lambda|+1}_{m,\bar{\mathbb{F}}_p}, \mathscr{L}_{\psi_p(\tilde{f}_{|\lambda|})}\right)^{G_\lambda \times \mu_{n+1}, \chi_\lambda}.
\end{aligned}$$

□

Corollary 2.16. *There is a $(-1)^{n|\lambda|+1}$-symmetric perfect self-pairing on $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1})$.*

Proof. Similar to the construction of the pairings for de Rham cohomologies [FSY22, (2.12)], we can construct a pairing on $\mathrm{H}^{n|\lambda|+1}_{\acute{e}t,\mathrm{mid}}\left(\mathbb{G}^{n|\lambda|+1}_{m,\bar{\mathbb{F}}_p}, \mathscr{L}_{\psi(\tilde{f}_{|\lambda|})}\right)$, which induces one on its $(G_\lambda \times \mu_{n+1}, \chi_\lambda)$-isotypic component. By Proposition 2.15, we get a $(-1)^{n|\lambda|+1}$-symmetric perfect self-pairing on $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1})$. □

Theorem 2.17. *Assume that $n|\lambda| \geq 3$. We have isomorphisms of ℓ-adic cohomologies*

$$\mathrm{gr}^W_{n|\lambda|+i}\mathrm{H}^i_{\acute{e}t,\mathrm{c}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1}) \simeq \mathrm{gr}^W_{n|\lambda|+i}\mathrm{H}^{n|\lambda|-2+i}_{\acute{e}t,\mathrm{c}}(\mathcal{K}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell(\zeta_p))(-1)^{G_\lambda \times \mu_{n+1}, \chi_\lambda}$$

for $i \in \{0,1,2\}$, and

$$\begin{aligned}
\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1}) \simeq & \mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,\mathrm{c}}\left(\mathcal{K}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell(\zeta_p)\right)(-1)^{G_\lambda \times \mu_{n+1}, \chi_\lambda} \\
\simeq & \mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^{n|\lambda|+1}_{\acute{e}t,\mathcal{K}_{\bar{\mathbb{F}}_\mathrm{p}}}\left(\mathbb{G}^{n|\lambda|}_{m,\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell(\zeta_p)\right)^{G_\lambda \times \mu_{n+1}, \chi_\lambda},
\end{aligned}$$

which is also isomorphic to $\mathrm{gr}^W_{n|\lambda|+i}\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}\left(\mathcal{K}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell(\zeta_p)\right)(-1)^{G_\lambda \times \mu_{n+1}, \chi_\lambda}$ when $\mathcal{K}$ is smooth.

Proof. By doing a change of variable $(t, x_{i,j}) \mapsto (t, x_{i,j}/t)$, for $i \in \{0,1\}$, we have

$$\mathrm{H}^{n|\lambda|+i}_{\acute{e}t,c}\left(\mathbb{G}^{n|\lambda|+1}_{m,\bar{\mathbb{F}}_p}, \mathscr{L}_{\psi(\tilde{f}_{|\lambda|})}\right) \simeq \mathrm{H}^{n|\lambda|+i}_{\acute{e}t,c}\left(\mathbb{G}^{n|\lambda|+1}_{m,\bar{\mathbb{F}}_p}, \mathscr{L}_{\psi(t \cdot g^{\boxplus|\lambda|})}\right).$$

Then considering the localization sequence for the triple

$$\left((\mathbb{A}^1 \times \mathbb{G}^{n|\lambda|}_m, t \cdot g^{\boxplus|\lambda|}), (\mathbb{G}^{n|\lambda|+1}_m, t \cdot g^{\boxplus|\lambda|}), (0 \times \mathbb{G}^{n|\lambda|}_m, 0)\right),$$

we have exact sequences

$$\begin{aligned}\mathrm{H}^{n|\lambda|-1+i}_{\acute{e}t,c}\Big(\mathbb{G}^{n|\lambda|}_{m,\bar{\mathbb{F}}_p},\mathbb{Q}(\zeta_p)\Big)\to&\mathrm{H}^{n|\lambda|+i}_{\acute{e}t,c}\Big(\mathbb{G}^{n|\lambda|+1}_{m,\bar{\mathbb{F}}_p},\mathscr{L}_{\psi_p(t\cdot g^{\boxplus|\lambda|})}\Big)\\ \to&\mathrm{H}^{n|\lambda|+i}_{\acute{e}t,c}\Big(\mathbb{A}^1\times\mathbb{G}^{n|\lambda|}_{m,\bar{\mathbb{F}}_p},\mathscr{L}_{\psi_p(t\cdot g^{\boxplus|\lambda|})}\Big)\to\mathrm{H}^{n|\lambda|+i}_{\acute{e}t,c}\Big(\mathbb{G}^{n|\lambda|}_{m,\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta_p)\Big).\end{aligned}\tag{2.18}$$

for $i\in\{0,1,2\}$. Next, we consider another triple

$$\Big((\mathbb{A}^1\times\mathbb{G}^{n|\lambda|}_m,t\cdot g^{\boxplus k}),(\mathbb{A}^1\times(\mathbb{G}^{n|\lambda|}_m\backslash\mathcal{K}),t\cdot g^{\boxplus|\lambda|}),(\mathbb{A}^1\times\mathcal{K},0)\Big).\tag{2.19}$$

Notice that for any $r\geq 0$, we have

$$\begin{aligned}\mathrm{H}^r_{\acute{e}t,c}\Big(\mathbb{A}^1_{\bar{\mathbb{F}}_p}\times(\mathbb{G}^{n|\lambda|}_m\backslash\mathcal{K})_{\bar{\mathbb{F}}_p},\mathscr{L}_{\psi_p(t\cdot g^{\boxplus|\lambda|})}\Big)=&\mathrm{H}^r_{\acute{e}t,c}\Big(\mathbb{A}^1_{\bar{\mathbb{F}}_p}\times(\mathbb{G}^{n|\lambda|}_m\backslash\mathcal{K})_{\bar{\mathbb{F}}_p},\mathscr{L}_{\psi_p(t)}\Big)\\ =\oplus_{a+b=r}\,&\mathrm{H}^a_{\acute{e}t,c}\Big(\mathbb{A}^1_{\bar{\mathbb{F}}_p},\mathscr{L}_{\psi_p}\Big)\otimes\mathrm{H}^b_{\acute{e}t,c}\Big((\mathbb{G}^{n|\lambda|}_m\backslash\mathcal{K})_{\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta_p)\Big)=0,\end{aligned}$$

where we did a change of variable in the first identity by $(t,x_{i,j})\mapsto(t\cdot(g^{\boxplus|\lambda|})^{-1},x_{i,j})$. So by the long exact sequences associated with the triple (2.19), we have

$$\mathrm{H}^{n|\lambda|+i}_{\acute{e}t,c}\Big(\mathbb{A}^1_{\bar{\mathbb{F}}_p}\times\mathbb{G}^{n|\lambda|}_{m,\bar{\mathbb{F}}_p},\mathscr{L}_{\psi_p(t\cdot g^{\boxplus|\lambda|})}\Big)\simeq\mathrm{H}^{n|\lambda|-2+i}_{\acute{e}t,c}\Big(\mathcal{K}_{\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta_p)\Big)(-1).\tag{2.20}$$

Now we combine (2.18) and (2.20), we get exact sequences for $i\in\{0,1,2\}$. Then we take the $(G_\lambda\times\mu_{n+1},\chi_\lambda)$-isotypic component of the sequences, and use Proposition 2.15 to conclude

$$\begin{aligned}\mathrm{H}^{n|\lambda|-1+i}_{\acute{e}t,c}\Big(\mathbb{G}^{n|\lambda|}_{m,\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta_p)\Big)\to&\,\mathrm{H}^i_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Kl}^\lambda_{n+1})\\ \to&\mathrm{H}^{n|\lambda|-2+i}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta_p))(-1)^{G_\lambda\times\mu_{n+1},\chi_\lambda}\to\mathrm{H}^{n|\lambda|+i}_{\acute{e}t,c}\Big(\mathbb{G}^{n|\lambda|}_{m,\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta_p)\Big).\end{aligned}\tag{2.21}$$

Taking the graded quotient $\mathrm{gr}^W_{n|\lambda|+i}$ of the sequence (2.21), and by the Frobenius weights of terms in the sequence, we have

$$\mathrm{gr}^W_{n|\lambda|+i}\mathrm{H}^i_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Kl}^\lambda_{n+1})\simeq\mathrm{gr}^W_{n|\lambda|+i}\mathrm{H}^{n|\lambda|-2+i}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta_p))(-1)^{G_\lambda\times\mu_{n+1},\chi_\lambda}.$$

For the usual cohomology, we use similar localization sequences to get

$$\mathrm{gr}^W_{n|\lambda|+i}\mathrm{H}^i_{\acute{\mathrm{e}}\mathrm{t}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Kl}^\lambda_{n+1})\simeq\mathrm{gr}^W_{n|\lambda|+i}\mathrm{H}^{n|\lambda|+i}_{\acute{\mathrm{e}}\mathrm{t},\mathcal{K}_{\bar{\mathbb{F}}_\mathrm{p}}}\Big(\mathbb{G}^{n|\lambda|}_{m,\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta_p)\Big)^{G_\lambda\times\mu_{n+1},\chi_\lambda},$$

which is also isomorphic to $\mathrm{gr}^W_{n|\lambda|+i}\mathrm{H}^{n|\lambda|-i}_{\acute{\mathrm{e}}\mathrm{t}}(\mathcal{K}_{\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta_p))(-1)^{G_\lambda,\chi_\lambda}$ when $\mathcal{K}$ is smooth. □

2.3.2 The p-adic case.

Bessel F-isocrystal

Let $\bar{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$, and we choose an element ϖ such that $\varpi^{p-1}=-p$. This gives rise to a unique nontrivial additive character $\psi\colon\mathbb{F}_p\to\bar{\mathbb{Q}}_p^\times$, satisfying $\psi(1)\equiv 1+\varpi \mod \varpi^2$. The Dwork's F-isocrystal $\mathscr{L}_\varpi$ is a rank 1 connection $\mathrm{d}+\varpi\mathrm{d}z$ with Frobenius structure $\exp(\varpi(z^p-z))$ on the overconvergent structure sheaf of $\mathbb{A}^1$ over $K=\mathbb{Q}_p(\varpi)$. We denote by $\mathscr{L}_{\varpi h}$ to be the inverse image of $\mathscr{L}_\varpi$ along a regular function $h\colon X\to\mathbb{A}^1$.

The *Kloosterman crystal* is an overconvergent F-isocrystal defined using again the same diagram as before:

$$\mathrm{Kl}_{n+1}:=\mathrm{R}\pi_{\mathrm{rig}*}\mathscr{L}_{\varpi\sigma}[n].$$

Some basic properties of Kl_{n+1} are summarized in [Cre94, 1.5]. In particular, the connection of Kl_{n+1} [Cre94, 1.1.5] is

$$\nabla = -N\frac{\mathrm{d}z}{z} - \varpi^n E\mathrm{d}z,$$

where N is lower triangular Jordan block of size $n+1$ with eigenvalue 0, and E is the matrix $E_{1,n+1}$, i.e. 1 at row 1 column $n+1$ and 0 at other places. Similar to the connections from [FG09], there are Bessel F-crystals for reductive groups from [XZ22]. The connection associated with $G = \mathrm{SL}_{n+1}$ and $V = V_\lambda$ is $(\mathrm{Kl}_{n+1}^{\otimes k})^{G_\lambda,(\mathrm{id}\times\mathrm{sign})}(\frac{nk}{2})$. By abuse of notation, we denote by $\mathrm{Kl}_{n+1}^\lambda$ the F-crystal $(\mathrm{Kl}_{n+1}^{\otimes k})^{G_\lambda,(\mathrm{id}\times\mathrm{sign})}$.

Rigid cohomologies

Similar to the ℓ-adic case, for $? = \varnothing, \mathrm{c}, \mathrm{mid}$, we have

$$\mathrm{H}^1_{\mathrm{rig},?}(\mathbb{G}_m/K, \mathrm{Kl}^\lambda_{n+1}) = \mathrm{H}^{n|\lambda|+1}_{\mathrm{rig},?}\Big(\mathbb{G}_m^{n|\lambda|+1}, \mathscr{L}_{\varpi\tilde{f}_{|\lambda|}}\Big)^{G_\lambda\times\mu_{n+1},\chi_\lambda}[\varpi].$$

There is also a version of the long exact sequence (1.7) for F-isocrystals, see [Ked06, (2.5.1)], where the canonical morphism from the cohomology with compact support to the usual cohomology is in the middle. Use [Ked06, Prop. 5.1.4 and Thm. 5.2.3], we have

$$\begin{aligned}\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_m/K, \mathrm{Kl}^\lambda_{n+1}) \simeq &\mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^{n|\lambda|+1}_{\mathrm{rig,c}}\Big(\mathbb{G}_m^{n|\lambda|+1}/K, \mathscr{L}_{\varpi\tilde{f}_{|\lambda|}}\Big)^{G_\lambda\times\mu_{n+1},\chi_\lambda}[\varpi]\\ &\mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^{n|\lambda|+1}_{\mathrm{rig}}\Big(\mathbb{G}_m^{n|\lambda|+1}/K, \mathscr{L}_{\varpi\tilde{f}_{|\lambda|}}\Big)^{G_\lambda\times\mu_{n+1},\chi_\lambda}[\varpi].\end{aligned}$$

In the end, we make a change of variable to pass from $\tilde{f}_{|\lambda|}$ to $t\cdot g^{\boxplus|\lambda|}$, and use the localization exact sequences for rigid cohomologies [Stu07, Prop. 8.2.18(ii)] and the Gysin isomorphism [Tsu99, Thm. 4.1.1] to conclude that

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_m/K, \mathrm{Kl}^\lambda_{n+1}) \simeq \mathrm{gr}^W_{n|\lambda|-1}\mathrm{H}^{n|\lambda|-1}_{\mathrm{rig,c}}(\mathcal{K}/K)(-1)^{G_\lambda\times\mu_{n+1},\chi_\lambda}[\varpi], \tag{2.22}$$

which is also isomorphic to $\mathrm{gr}^W_{n|\lambda|-1}\mathrm{H}^{n|\lambda|-1}_{\mathrm{rig}}(\mathcal{K}/K)(-1)^{G_\lambda\times\mu_{n+1},\chi_\lambda}$ when $\mathcal{K}$ is smooth.

Chapter 3

Computation of Hodge numbers

In this chapter, we will prove Theorem D. In particular, the first part is proved in Theorem 3.33, and the second part is proved in Propositions 3.57, 3.58 and 3.59. Except in Section 3.4, our primary interests will be in $\mathrm{Sym}^k \mathrm{Kl}_3$.

3.1 Properties of Kloosterman connections

The connection $\mathrm{Sym}^k \mathrm{Kl}_3$ (resp. $\mathrm{Sym}^k \widetilde{\mathrm{Kl}_3}$) on $\mathbb{G}_{m,z}$ (resp. $\mathbb{G}_{m,t}$) is of rank $\binom{k+2}{2}$. In this section, we study their formal local structures at 0 and ∞, and compute the dimensions of their de Rham cohomologies and their middle de Rham cohomologies.

3.1.1 The local structure at 0

Let $V = \mathbb{C}^3$ be the standard representation of SL_3, and $V_k := \mathrm{Sym}^k V$ the k-th symmetric power of the representation V. We denote by $\{v_0, v_1, v_2\}$ the standard basis of V, and by N the matrix

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

By construction, we have $Nv_i = v_{i+1}$ for $i = 0, 1$ and $Nv_2 = 0$. The action of N on V can be enhanced to a Lie algebra representation $\rho\colon \mathrm{sl}_2 \to \mathrm{End}(V)$, such that $\rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = N$ and $\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Using ρ we can also view V_k as a representation of sl_2, such that $\rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathrm{Sym}^k N$. For simplicity, we denote by $N_k := \mathrm{Sym}^k N$.

Recall that all irreducible representations of sl_2 are of the form $\mathrm{Sym}^m(\mathbb{C}^2)$. We can decompose V_k as the direct sum of some irreducible representations of sl_2 [FH04, Ex. 11.14]. In the following lemma, we make the decomposition more explicit.

Lemma 3.1. *As representations of* sl_2*, we have*

$$V_k = \bigoplus_{m=0}^{\lfloor \frac{k}{2} \rfloor} \mathrm{Sym}^{2k-4m}(\mathbb{C}^2),$$

where each $\mathrm{Sym}^{2k-4m}(\mathbb{C}^2)$ *is of the form* $\oplus_{i=0}^{2k-4m} N_k^i v_0^{k-2m}(v_1^2 - 2v_0v_2)^m$*. In particular, the cokernel of* N_k *on* V_k *has dimension* $\lfloor \frac{k+2}{2} \rfloor$*, spanned by the set* $\{v_0^{k-2m}(v_1^2 - 2v_0v_2)^m\}$ *for* $m \in \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$*.*

Proof. Observe that we have $N_k(v_1^2 - 2v_0v_2) = 0$. For each $m = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$, the subspace $W_m := \oplus_{i=0}^{2k-4m} \mathbb{C} N_k^i v_0^{k-2m}(v_1^2 - 2v_0v_2)^m$ of V_k is stable under the action of N_k. One can check that W_k is also stable under the action of sl_2. So W_m can be seen as a representation of sl_2, and is isomorphic to the irreducible representation $\mathrm{Sym}^{2k-4m}(\mathbb{C}^2)$ of dimension $2k - 4m + 1$.

If $m \neq m'$, the representations W_m and $W_{m'}$ have different dimensions. So the intersection of W_m and $W_{m'}$ is 0. Hence, the direct sum $W := \bigoplus_{m=0}^{\lfloor \frac{k}{2} \rfloor} W_m$ is a subrepresentation of V_k, of dimension $\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor}(2k - 4m + 1)$. Notice that the dimension of V_k is $\binom{k+2}{k}$, equal to the rank of $\mathrm{Sym}^k \mathrm{Kl}_3$. By the equality

$$\binom{k+2}{2} = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor}(2k - 4m + 1),$$

we prove that $\dim W = \dim V_k$. Therefore, $W = V_k$.

At last, for each $m \in \{0, 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$, the cokernel of N_k on $W_m = \mathrm{Sym}^{2k-4m}(\mathbb{C}^2)$ is one-dimensional, spanned by $v_0^{k-2m}(v_1^2 - 2v_0v_2)^m$. So the dimension of the cokernel of N_k on V_k is equal to $\lfloor \frac{k+2}{2} \rfloor$, and is generated by elements listed in the lemma. □

The formal structures of the connections $\mathrm{Sym}^k \mathrm{Kl}_3$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3$ at 0 are isomorphic to

$$\mathbb{C}((z)) \otimes_{\mathbb{C}[z,z^{-1}]} \mathrm{Sym}^k \mathrm{Kl}_3 \simeq \left(\mathcal{O}_{\mathbb{G}_m}^{\binom{k+2}{2}}, \mathrm{d} - N_k \frac{\mathrm{d}z}{z}\right)$$

and

$$\mathbb{C}((t)) \otimes_{\mathbb{C}[t,t^{-1}]} \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3 \simeq \left(\mathcal{O}_{\mathbb{G}_m}^{\binom{k+2}{2}}, \mathrm{d} - 3N_k \frac{\mathrm{d}t}{t}\right)$$

respectively. Using Lemma 3.1, we have the following corollary.

Corollary 3.2. *The formal structures of the connections* $\mathrm{Sym}^k \mathrm{Kl}_3$ *and* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3$ *at* 0 *are isomorphic to*

$$\bigoplus_{m=0}^{\lfloor \frac{k}{2} \rfloor} \left(\mathcal{O}_{\mathbb{G}_m}^{2k-4m+1}, \mathrm{d} - J_{2k-4m+1}(0)\frac{\mathrm{d}z}{z}\right) \text{ and } \bigoplus_{m=0}^{\lfloor \frac{k}{2} \rfloor} \left(\mathcal{O}_{\mathbb{G}_m}^{2k-4m+1}, \mathrm{d} - 3J_{2k-4m+1}(0)\frac{\mathrm{d}t}{t}\right)$$

respectively, where each matrix $J_{2k-4m+1}(0)$ *represents a Jordan block of size* $2k-4m+1$ *with eigenvalue* 0.

3.1.2 The local structure at ∞

Let $[3]\colon \mathbb{G}_{m,t} \to \mathbb{G}_{m,z}$ be the cubic map, and ζ be a primitive third root of unity. The formal local structures of Kl_3 and $\widetilde{\mathrm{Kl}}_3$ are determined in the following lemma.

Lemma 3.3. *We have isomorphisms of formal connections*

$$\mathbb{C}((z^{-1})) \otimes_{\mathbb{C}[z,z^{-1}]} \mathrm{Kl}_3 \simeq [3]_+\mathcal{E}^{3t}, \tag{3.4}$$

and

$$\mathbb{C}((t^{-1})) \otimes_{\mathbb{C}[t,t^{-1}]} \widetilde{\mathrm{Kl}}_3 \simeq \bigoplus_{i=0}^{2} \mathcal{E}^{3\zeta^i t}. \tag{3.5}$$

Proof. Let $\mathrm{inv}\colon \mathbb{G}_{m,z} \to \mathbb{G}_{m,z}$ be the inversion map $z \mapsto z^{-1}$, $j\colon \mathbb{G}_{m,z} \hookrightarrow \mathbb{A}^1_z$ the inclusion map, and FT the Fourier transform of $\mathcal{D}$-modules on $\mathbb{A}^1_z$. By [FSY22, Lem. 2.5], we have

$$\mathrm{Kl}_3 = j^+\mathrm{FT}(j_+\mathrm{inv}^+\mathrm{Kl}_2).$$

Let $[2]\colon \mathbb{G}_{m,t} \to \mathbb{G}_{m,z}$ be the square map, and L_{-1} the connection $(\mathbb{C}((z^{-1})), \mathrm{d} + \frac{1}{2}\frac{\mathrm{d}z^{-1}}{z^{-1}})$. By the equation (4.3) in *loc. cit.*, we have an isomorphism of formal connections

$$\mathbb{C}((z^{-1})) \otimes_{\mathbb{C}[z,z^{-1}]} \mathrm{Kl}_2 = [2]_+\mathcal{E}^{2t} \otimes \mathrm{L}_{-1}.$$

Let $j_\infty\colon \mathbb{G}_m \hookrightarrow \mathbb{P}^1\backslash\{0\}$ be the inclusion. Applying the formal stationary phase formula [Sab08, Thm 5.1] to $j_{\infty+}\mathrm{Kl}_3 = j_{\infty+}j^+\mathrm{FT}(j_+\mathrm{inv}^+\mathrm{Kl}_2)$, we have the formula (3.4). Then use the identity $[3]^+[3]_+\mathcal{E}^{3t} = \bigoplus_{i=0}^2 \mathcal{E}^{3\zeta^i t}$ in [Sab08, Lem. 2.4], we get (3.5). □

Now we determine the formal structures of $\mathrm{Sym}^k\mathrm{Kl}_3$. For each $\underline{I} = (I_0, I_1, I_2) \in \mathbb{N}^3$, we denote by $C_{\underline{I}}$ the complex number $\sum_{i=0}^2 I_i\zeta^i$. The element $\sigma = (0,1,2) \in S_3$ acts on the set of multi-indices by $\sigma\underline{I} = (I_2, I_0, I_1)$. Let B_k be the set of orbits of multi-indices $\underline{I}$ such that $|\underline{I}| = \sum_{i=0}^2 I_i = k$ under the action of σ.

For each $\underline{I}$, if $C_{\underline{I}} \neq 0$, we have

$$\left(\mathcal{E}^{3C_{\underline{I}}t} \oplus \mathcal{E}^{3C_{\underline{I}}\zeta t} \oplus \mathcal{E}^{3C_{\underline{I}}\zeta^2 t}\right)^{\mu_3} = [3]_+\mathcal{E}^{3C_{\underline{I}}t}, \tag{3.6}$$

where the action of $\mu \in \mu_3$ is given by $\mu\cdot t := \mu t$. Since $C_{\sigma\underline{I}} = \zeta C_{\underline{I}}$, we deduce that $[3]_+\mathcal{E}^{3C_{\underline{I}}t} = [3]_+\mathcal{E}^{3C_{\sigma\underline{I}}t}$, i.e. $[3]_+\mathcal{E}^{3C_{\underline{I}}t}$ only depends on the orbit of $\underline{I}$ in B_k.

Proposition 3.7. *We have isomorphisms of formal connections*

$$\mathbb{C}((z^{-1})) \otimes_{\mathbb{C}[z,z^{-1}]} \mathrm{Sym}^k\mathrm{Kl}_3 = \begin{cases} \bigoplus_{[\underline{I}]\in B_k} [3]_+\mathcal{E}^{3C_{\underline{I}}t} & 3 \nmid k \\ \bigoplus_{[\underline{I}]\in B_k,\, C_{\underline{I}}\neq 0} [3]_+\mathcal{E}^{3C_{\underline{I}}t} \oplus (\mathbb{C}((z^{-1})), \mathrm{d}(z^{-1})) & 3 \mid k \end{cases}$$

and

$$\mathbb{C}((t^{-1})) \otimes_{\mathbb{C}[t,t^{-1}]} \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3 = \begin{cases} \bigoplus_{\underline{I}} \mathcal{E}^{3C_{\underline{I}}t} & 3 \nmid k \\ \bigoplus_{C_{\underline{I}}\neq 0} \mathcal{E}^{3C_{\underline{I}}t} \oplus (\mathbb{C}((t^{-1})), \mathrm{d}(t^{-1})) & 3 \mid k \end{cases}$$

where $[3]_+\mathcal{E}^{3C_{\underline{I}}t}$ *only depends on the orbit of* $\underline{I}$ *in* B_k.

Proof. By taking the symmetric power of (3.5), we have

$$\mathbb{C}((t^{-1})) \otimes_{\mathbb{C}[t,t^{-1}]} \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3 = \mathrm{Sym}^k\left(\mathcal{E}^{3t} \oplus \mathcal{E}^{3\zeta t} \oplus \mathcal{E}^{3\zeta^2 t}\right) = \oplus_{\underline{I}}\mathcal{E}^{3C_{\underline{I}}t}.$$

As long as $C_{\underline{I}} = 0$, we get a trivial component $(\mathbb{C}((t^{-1})), \mathrm{d}(t^{-1}))$. This only happens when $3 \mid k$ and $\underline{I} = (k/3, k/3, k/3)$. Hence, we prove the second isomorphism.

Let μ_3 acts on t by multiplication. The local structure of $\mathrm{Sym}^k\mathrm{Kl}_3$ is the μ_3-invariant part of that of $\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$. If $C_{\underline{I}} \neq 0$, by (3.6), the μ_3-invariant part of

$$\mathcal{E}^{3C_{\underline{I}}t} \oplus \mathcal{E}^{3C_{\underline{I}}\zeta t} \oplus \mathcal{E}^{3C_{\underline{I}}\zeta^2 t}$$

is $[3]_+\mathcal{E}^{3C_{\underline{I}}t}$. If $C_{\underline{I}} = 0$, then $\mathcal{E}^{3C_{\underline{I}}t} = (\mathbb{C}((t^{-1})), \mathrm{d}(t^{-1}))$. And the μ_3-invariant part of $(\mathbb{C}((t^{-1})), \mathrm{d}(t^{-1}))$ is $(\mathbb{C}((z^{-1})), \mathrm{d}(z^{-1}))$. Therefore, we deduce the first isomorphism. □

Corollary 3.8. *The irregularities of* $\mathrm{Sym}^k\mathrm{Kl}_3$ *and* $\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$ *at* ∞ *are* $\left(\binom{k+2}{2} - d(k,3)\right)/3$ *and* $\binom{k+2}{2} - d(k,3)$ *respectively, where* $d(k,3)$ *is* 0 *if* $3 \nmid k$ *and* 1 *if* $3 \mid k$.

Proof. The irregularity of $\mathcal{E}^{\lambda t}$ at ∞ is 1 if $\lambda \neq 0$ and 0 if $\lambda = 0$. Since $C_{\underline{I}} = 0$ only happens when $3 \mid k$ and $\underline{I} = (k/3, k/3, k/3)$, the irregularity of $\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$ at ∞ is $\mathrm{rk}\,\mathrm{Sym}^k\mathrm{Kl}_3 - d(k,3)$, by Proposition 3.7.

For a $\mathcal{D}$-module V, by [Del70, II. Thm. 2.12], the slopes of $[3]_+V$ are $1/3$ times the slopes of V. Therefore, the irregularity of $[3]_+\mathcal{E}^{3C_{\underline{I}}t}$ is 1 if $C_{\underline{I}} \neq 0$. By Proposition 3.7 again, the irregularity of $\mathrm{Sym}^k\mathrm{Kl}_3$ at ∞ is $1/3(\mathrm{rk}\,\mathrm{Sym}^k\mathrm{Kl}_3 - d(k,3))$. □

3.1.3 The dimension of the de Rham cohomologies

Proposition 3.9. *The dimensions of* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$ *and* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)$ *are*

$$\frac{1}{3}\left(\binom{k+2}{k} - d(k,3)\right) \text{ and } \left(\binom{k+2}{k} - d(k,3)\right)$$

respectively, where $d(k,3)$ *is* 1 *if* $3 \mid k$ *and* 0 *otherwise.*

Proof. Let $\mathcal{M}$ be either $\mathrm{Sym}^k \mathrm{Kl}_3$ or $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3$. By the analogue of the Grothendieck-Ogg-Shafarevich formula [Kat90, 2.9.8.2], we have

$$\chi(\mathbb{G}_m, \mathcal{M}) = -\mathrm{Irr}_\infty(\mathcal{M}).$$

Since $\mathbb{G}_m$ is affine, we have $\mathrm{H}^2_{\mathrm{dR}}(\mathbb{G}_m, \mathcal{M}) = 0$. By [FSY22, Prop. 2.7], $\mathcal{M}$ is irreducible, which implies that $\mathrm{H}^0_{\mathrm{dR}}(\mathbb{G}_m, \mathcal{M}) = 0$. So the only non-vanishing cohomology of $\mathcal{M}$ is $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathcal{M})$. Therefore, the dimension of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathcal{M})$ is equal to the irregularity of $\mathcal{M}$ at ∞, which is computed in Corollary 3.8. □

Corollary 3.10. *The dimensions of* $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$ *and* $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)$ *are*

$$\frac{1}{3}\left(\binom{k+2}{k} - d(k,3)\right) - d(k,3) - \left\lfloor \frac{k+2}{2} \right\rfloor$$

and

$$\left(\binom{k+2}{k} - d(k,3)\right) - d(k,3) - \left\lfloor \frac{k+2}{2} \right\rfloor$$

respectively.

Proof. Let $\mathcal{M}$ be either $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ or $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$, and t the coordinate of $\mathbb{G}_m$ by abuse of notation. The number Soln_0 and Soln_∞ are the dimensions of the spaces of formal meromorphic solutions at 0 and ∞ respectively. More precisely, they are the dimensions

$$\dim \mathrm{Hom}_{\mathcal{D}}\left(\mathbb{C}((t)) \otimes_{\mathbb{C}[t,t^{-1}]} \mathcal{M}, (\mathbb{C}((t)), \mathrm{d}(t))\right)$$

and

$$\dim \mathrm{Hom}_{\mathcal{D}}\left(\mathbb{C}((t^{-1})) \otimes_{\mathbb{C}[t,t^{-1}]} \mathcal{M}, (\mathbb{C}((t^{-1})), \mathrm{d}(t^{-1}))\right)$$

respectively. Applying [Kat90, Cor. 2.9.8.1] to $j_{\dagger+}\mathcal{M}$, where $j\colon \mathbb{G}_m \hookrightarrow \mathbb{P}^1$ is the inclusion, we have

$$\chi(\mathbb{P}^1, j_{\dagger+}\mathcal{M}) = \chi(\mathbb{G}_m, \mathcal{M}) + \mathrm{Soln}_0 + \mathrm{Soln}_\infty.$$

Let $j_0\colon \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the inclusion. By the Lemma 3.1 and the Corollary 3.2, we know that the cokernel of

$$j_{0\dagger+}\mathcal{M} \to j_{0+}\mathcal{M}$$

is supported at 0, of rank $\lfloor \frac{k+2}{2} \rfloor$. Hence, $\mathrm{Soln}_0 = \lfloor \frac{k+2}{2} \rfloor$.

Let $j_\infty\colon \mathbb{G}_m \hookrightarrow \mathbb{P}^1 \backslash \{0\}$ be the inclusion. The rank of the cokernel of the injective morphism

$$j_{\infty\dagger+}\mathcal{M} \to j_{\infty+}\mathcal{M}$$

is $d(k,3)$ by Proposition 3.7. Hence, $\mathrm{Soln}_\infty = d(k,3)$. □

3.2 Bases of the de Rham cohomologies

The aim of this section is to get bases of the de Rham cohomologies of $\mathrm{Sym}^k\mathrm{Kl}_3$ and $\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$. Recall that Kl_3 is the connection $(\mathcal{O}^3, \mathrm{d} + N\frac{\mathrm{d}z}{z} + E\mathrm{d}z)$ defined in (2.5). We can choose a basis v_0, v_1 and v_2 of Kl_3 (as a $\mathbb{C}[z, z^{-1}]\langle\partial_z\rangle$-module) such that

$$z\partial_z(v_i) = v_{i+1} \text{ for } i = 0, 1 \text{ and } z\partial_z(v_2) = zv_0,$$

i.e., v_0, v_1 and v_2 generates $\mathbb{C}[z, z^{-1}]\langle\partial_z\rangle/((z\partial_z)^3 - z)$. The set $\{v_i\}$ also serve as a basis of the connection $\widetilde{\mathrm{Kl}}_3 = (\mathcal{O}^3, \mathrm{d} + 3N\frac{\mathrm{d}t}{t} + 3t^2E\mathrm{d}t)$, and satisfies similar conditions

$$t\partial_t(v_i) = 3v_{i+1} \quad \text{for } i = 0, 1 \text{ and } t\partial_t(v_2) = 3t^3v_0.$$

In this way, elements $\{v^{\underline{I}} := v_0^{I_0}v_1^{I_1}v_2^{I_2} || \underline{I}| = I_0 + I_1 + I_2 = k\}$ for a basis of $\mathrm{Sym}^k\mathrm{Kl}_3$ (resp. $\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$) as a $\mathbb{C}[z]$-module (resp. $\mathbb{C}[t]$-module).

Let deg be a grading on Kl_3 (resp. $\widetilde{\mathrm{Kl}}_3$), defined by $\deg(z^a v_i) = 3a + i$ (resp. $\deg(t^a v_i) = a + i$). It induces a grading on the symmetric power $\mathrm{Sym}^k\mathrm{Kl}_3$ (resp. $\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$). More precisely, the degree is given by

$$\deg(z^a v_0^{I_0}v_1^{I_1}v_2^{I_2}) = 3a + I_1 + I_2 \text{ (resp. } \deg(t^a v_0^{I_0}v_1^{I_1}v_2^{I_2}) = a + I_1 + I_2). \tag{3.11}$$

The main results of this section are the following theorems.

Theorem 3.12. *Assume that $k \geq 4$. There exist finite subsets $W_d \subset \mathrm{Sym}^k\mathrm{Kl}_3$ (resp. $\widetilde{W}_d \subset \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$) for $0 \leq d \leq 2k + 1$ such that*

(1). the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3)$ (resp. $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$) is spanned by the set $\cup_d W_d$ (resp. $\cup\widetilde{W}_d$),

(2). $W_d \subset \widetilde{W}_d$,

(3). each element in W_d (resp. $\widetilde{W}_d$) has degree d,

(4). if $d \leq k$, the cardinality of W_d (resp. $\widetilde{W}_d$) is

$$n_d = \begin{cases} \lfloor \frac{d}{6} \rfloor + 1 & 6 \nmid d - 1 \\ \lfloor \frac{d}{6} \rfloor & 6 \mid d - 1 \end{cases}$$

(resp, $\tilde{n}_d = \lfloor \frac{d}{2} \rfloor + 1$),

(5). if $k + 1 \leq d \leq 2k + 1$ and $3 \nmid k$, the cardinality of W_d (resp. $\widetilde{W}_d$) is $n_{2k+1-d} - (d + 1 - 2\lfloor \frac{d+1}{2} \rfloor)$ (resp. $\tilde{n}_{2k+1-d} - (d + 1 - 2\lfloor \frac{d+1}{2} \rfloor)$).

Moreover, we can choose subsets of the bases above to generate the middle de Rham cohomologies.

Theorem 3.13. *Assume that $k \geq 4$. There exist finite subsets $W'_d \subset \mathrm{Sym}^k\mathrm{Kl}_3$ (resp. $\widetilde{W}'_d \subset \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$) for $0 \leq d \leq 2k + 1$ such that*

(1). the middle de Rham cohomology $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3)$ (resp. $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$) is spanned by the set $\cup_d W'_d$ (resp. $\cup\widetilde{W}'_d$),

(2). each element in W'_d (resp. $\widetilde{W}'_d$) has degree d if $3 \nmid k$,

(3). $W'_d \subset \widetilde{W}'_d$,

(4). if $d \le k$, the cardinality of W'_d (resp. $\widetilde{W}'_d$) is

$$n'_d = -\delta_{d,k}\cdot d(k,3) + \begin{cases} \lfloor \frac{d}{6} \rfloor & d \equiv 0,1,2,4 \bmod 6, \\ \lfloor \frac{d}{6} \rfloor + 1 & p \equiv 3,5 \bmod 6, \end{cases}$$

(resp, $\tilde{n}_d = \lfloor \frac{d+1}{2} \rfloor - \delta_{d,k}\cdot d(k,3)$), where $\delta_{a,b}$ be the Kronecker delta symbol,

(5). if $k+1 \le d \le 2k+1$ and $3 \nmid k$, the cardinality of W'_d (resp. $\widetilde{W}'_d$) is n'_{2k+1-d} (resp. $\tilde{n}'_{2k+1-d}$) if $k+1 \le d \le 2k+1$.

Remark 3.14. The middle cohomology of $\mathrm{Sym}^k \mathrm{Kl}_3$ is trivial when $k \le 3$ by Corollary 3.10. So we only need to deal with the case that $k \ge 4$. We also omitted the cardinalities of the sets W'_d and $\widetilde{W}'_d$ when $d \ge k+1$ and $3 \mid k$, because we will not use them in the computation of Hodge numbers.

3.2.1 Cohomology classes of the de Rham cohomologies

Some preparations

Let $V_k = \mathrm{Sym}^k V$ be the k-th symmetric power of the standard representation V of SL_3. Recall that in Lemma 3.1, we can decompose V_k as the direct sum of sl_2-representations

$$\bigoplus_{m=0}^{\lfloor \frac{k}{2} \rfloor} \bigoplus_{i=0}^{2k-4m} N_k^i \left(v_0^{k-2m} \omega_2^m \right),$$

where $\omega_2 = v_1^2 - 2v_0v_2$. We denote by $N_k = \mathrm{Sym}^k N$ and $E_k = \mathrm{Sym}^k E$ the endormorphisms of V_k. Let $G = G_k$ be $V_k[z,z^{-1}] := V_k \otimes_{\mathbb{C}} \mathbb{C}[z,z^{-1}]$, on which $\theta_z := z\partial_z$ acts by

$$\theta_z(z^\ell v_0^{I_0} v_1^{I_1} v_2^{I_2}) = \ell z^\ell v_0^{I_0} v_1^{I_1} v_2^{I_2} + z^\ell (N_k + zE_k)(v_0^{I_0} v_1^{I_1} v_2^{I_2}).$$

Similarly, the operator $\theta_t := t\partial_t$ acts on $G[t] = G \otimes_{\mathbb{C}[z]} \mathbb{C}[t]$ $(z = t^3)$ by

$$\theta_t(t^\ell v_0^{I_0} v_1^{I_1} v_2^{I_2}) = \ell t^\ell v_0^{I_0} v_1^{I_1} v_2^{I_2} + t^\ell 3(N_k + zE_k)(v_0^{I_0} v_1^{I_1} v_2^{I_2}).$$

The first de Rham cohomologies $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$ and $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)$ are identified with the cokernels of the two-term complexes

$$G \xrightarrow{\theta_z} G \text{ and } G[t] \xrightarrow{\theta_t} G[t] \tag{3.15}$$

respectively. We do a reduction following [FSY22, Lem. 4.15].

Lemma 3.16. *Let $G^+ := V_k \otimes_{\mathbb{C}} \mathbb{C}[z]$ and $G^+[t] = G^+ \otimes_{\mathbb{C}[z]} \mathbb{C}[t]$. Then the two-term complexes*

$$G^+ \xrightarrow{\theta_z} G^+ \text{ and } G^+[t] \xrightarrow{\theta_t} G^+[t]$$

are quasi-isomorphic to complexes in (3.15) respectively.

Proof. We only give the proof for G^+, and the proof for $G^+[t]$ is the same. Observe that $G = \bigcup_{r\ge 0} z^{-r} G^+$. It suffices to show that θ_z is an invertible operator on the quotient $z^{-r-1}G^+/z^{-r}G^+$ for $r > 0$.

In fact, the induced operator θ_z on the quotient $z^{-r-1}G^+/z^{-r}G^+$ is now $z\partial_z + N$. Using the decomposition with respect to N_k in Lemma 3.1, we have

$$z^{-r-1}G^+/z^{-r}G^+ = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} z^{-r-1}(\mathrm{Sym}^{2m}(\mathbb{C}^2)).$$

Here the operator θ_z acts on $z^{-r-1}\mathrm{Sym}^{2m}(\mathbb{C}^2)$ by an invertible matrix $J_{2m+1}(-r-1)$, where $J_{2m+1}(-r-1)$ is the lower Jordan block with diagonal $-r-1$ of size $2m+1$. □

Let deg be the grading (3.11). Then θ_z and θ_t are (inhomogeneous) of degree 1. We denote by $\bar{\theta}_z = (N + zE)$ and $\bar{\theta}_t = 3(N + t^3E)$ the induced map on the graded quotient $\operatorname{gr} G^+$ (resp. $\operatorname{gr} G^+[t]$). We can identify $\operatorname{gr} G^+$ (resp. $\operatorname{gr} G^+[t]$) with G^+ (resp. $G^+[t]$).

Recall that $t^3 = z$. For $i = 0, 1, 2$, the elements $f_i = \sum_{j=0}^{2} \zeta^{i(2-j)} t^{2-j} v_j$ are eigenvectors of $\bar{\theta}_t = 3(N_k + t^3 E_k)$, with eigenvalues $3t \cdot \zeta^i$. Moreover, the set $\{f^{\underline{I}} := \prod_{j=0}^{2} f_j^{I_j}\}_{\underline{I} \in \mathbb{N}^3, |\underline{I}|=k}$ is a basis of $G^+[t]$ as $\mathbb{C}[t]$-module, and $\bar{\theta}_t(f^{\underline{I}}) = 3t \cdot C_{\underline{I}} f^{\underline{I}}$, where $C_{\underline{I}}$ is the sum $\sum \zeta^j \cdot I_j$.

Lemma 3.17. *If $3 \nmid k$, the map $\bar{\theta}_z$ (resp. $\bar{\theta}_t$) is injective. If $3 \mid k$, the kernel $\ker \bar{\theta}_z$ (resp. $\ker \bar{\theta}_t$) is a free $\mathbb{C}[z]$-module (resp. $\mathbb{C}[t]$-module) of rank 1 generated by $\eta^{k/3}$, where $\eta = f^{(1,1,1)} = t^6 v_0^3 + t^3 v_1^3 + v_2^3 - 3t^3 v_0 v_1 v_2$.*

Proof. By the commutative diagram

$$\begin{array}{ccc} G^+ & \xrightarrow{\bar{\theta}_z} & G^+ \\ \downarrow & & \downarrow \\ G^+[t] & \xrightarrow{\bar{\theta}_t} & G^+[t] \end{array}$$

we observe that $\ker \bar{\theta}_t \cap G^+ = \ker \bar{\theta}_z$. So it suffices to study $\ker \theta_t$.

Let $\sum_{\underline{i}} g_{\underline{I}}(t) f^{\underline{I}}$ be an element in $\ker \bar{\theta}_t$, then

$$0 = \bar{\theta}_t \left(\sum_{\underline{I}} g_{\underline{I}}(t) f^{\underline{I}} \right) = \sum_{\underline{I}} 3t \cdot C_{\underline{I}} \cdot g_{\underline{I}}(t) f^{\underline{I}}.$$

Since $f^{\underline{I}}$ are linearly independent, the coefficients $C_{\underline{I}}$ are 0 as long as $g_{\underline{I}} \neq 0$. So $\ker \bar{\theta}_t$ is $\bigoplus_{C_{\underline{I}}=0} \mathbb{C}[t] f^{\underline{I}}$. If $3 \nmid k$, the numbers $C_{\underline{I}}$ are always nonzero. If $3 \mid k$, the only $\underline{I}$ such that $C_{\underline{I}} = 0$ is $(k/3, k/3, k/3)$. So the element $\eta^{k/3} = f^{(k/3,k/3,k/3)}$ generates the kernel of $\bar{\theta}_t$. □

Now we determine the cokernel of $\bar{\theta}_t$.

Proposition 3.18. *Assume that $k \geq 4$. There exist finite subsets $W_i \subset G^+$ (resp. $\widetilde{W}_i \subset G^+[t]$) for $0 \leq i \leq 2k+1$ such that*

(1). if $3 \nmid k$, then $\operatorname{coker} \bar{\theta}_z = \operatorname{span}(W_i \mid 0 \leq i \leq 2k+1)$ (resp. $\operatorname{coker} \bar{\theta}_t = \operatorname{span}(\widetilde{W}_i \mid 0 \leq i \leq 2k+1)$),

(1)′. if $3 \mid k$, then $\operatorname{coker} \bar{\theta}_z = \operatorname{span}(W_i \mid 0 \leq i \leq 2k+1) \oplus \mathbb{C}[z]\eta^{k/3}$ (resp. $\operatorname{coker} \bar{\theta}_t = \operatorname{span}(\widetilde{W}_i \mid 0 \leq i \leq 2k+1) \oplus \mathbb{C}[t]\eta^{k/3}$),

(2). $W_d \subset \widetilde{W}_d$,

(3). each element in W_d (resp. $\widetilde{W}_d$) has degree d,

(4). if $d \leq k$, the cardinality of W_d (resp. $\widetilde{W}_d$) is

$$n_d = \begin{cases} \lfloor \frac{d}{6} \rfloor + 1 & 6 \nmid d-1 \\ \lfloor \frac{d}{6} \rfloor & 6 \mid d-1 \end{cases} \quad \text{(resp. } \tilde{n}_d = \lfloor \tfrac{d}{2} \rfloor + 1\text{)},$$

(5). if $k+1 \leq d \leq 2k+1$ and $3 \nmid k$, the cardinality of W_d (resp. $\widetilde{W}_d$) is $n_{2k+1-d} - (d + 1 - 2\lfloor \frac{d+1}{2} \rfloor)$ (resp. $\tilde{n}_{2k-d} - (d + 1 - 2\lfloor \frac{d+1}{2} \rfloor)$).

Proof. We can decompose $G^+ = \mathrm{gr}G^+$ as $\oplus_{d\geq 0} G_d$ where G_d is generated by elements of degree d, i.e., linear combination of $z^\ell v_0^{I_0} v_1^{I_1} v_2^{I_2}$ such that $3\ell + I_1 + 2I_2 = d$. Since $\bar\theta_z$ is homogeneous of degree 1 with respect to the degree function (3.11), the kernel and cokernel of $\bar\theta_z$ can be decomposed as

$$\ker \bar\theta_z = \oplus_d \ker \bar\theta_z \cap G_d$$

and

$$\mathrm{coker}\, \bar\theta_z = \oplus_d G_d/\bar\theta_z(G_{d-1})$$

respectively.

Step 1: We prove the statements (1), (3), (4) and (5) when $3 \nmid k$ for z coordinate.

In this case $\ker \bar\theta_z = 0$ by Lemma 3.17. We take $W_d \subset G_d$ such that the image of W_d in $G_d/\bar\theta_z(G_{d-1})$ is a basis, which verifies (3). Hence, $n_d = \#W_d = \dim G_d - \dim G_{d-1}$ because $\bar\theta_z$ is injective. By the construction of G_d, we have

$$\dim G_d = \#\{(a, I_0, I_1, I_2) \mid I_0 + I_1 + I_2 = k,\ 3a + I_1 + 2I_2 = d\}.$$

Notice that if $d \geq 3$, the number $\dim G_d - \dim G_{d-3} = \dim G_d/zG_{d-3}$ is equal to

$$\#\{i_0 \mid \max\{0, k-d\} \leq i_0 \leq k - \lceil \tfrac{d}{2} \rceil\}$$

because G_d/zG_{d-3} can be identified with the vector space spanned by elements in G_d whose degree of z is 0. It follows that

$$\dim G_d - \dim G_{d-3} = \begin{cases} \lfloor \frac{d}{2} \rfloor + 1 & d \leq k, \\ k - \lceil \frac{d}{2} \rceil + 1 & k+1 \leq d \leq 2k, \\ 0 & 2k+1 \leq d. \end{cases} \tag{3.19}$$

If $d \geq 6$, to compute $n_d = \dim G_d - \dim G_{d-1}$, we can write $n_d - n_{d-6}$ as

$$(\dim G_d - \dim G_{d-3} + \dim G_{d-3} - \dim G_{d-6}) - (\dim G_{d-1} - \dim G_{d-4} + \dim G_{d-4} - \dim G_{d-7}).$$

Using (3.19), we deduce that

$$n_d - n_{d-6} = \begin{cases} 1 & d \leq k, \\ 0 & k+1 \leq d \leq k+3, \\ -1 & k+4 \leq d \leq 2k+2 \text{ or } d = 2k+4, \\ 0 & d = 2k+3 \text{ or } 2k+5 \leq d. \end{cases} \tag{3.20}$$

If $d \leq 5$ and $d \leq k$, we can check directly that $n_d = \#W_d = \begin{cases} 1 & d \neq 1 \\ 0 & d = 1 \end{cases}$. It follows that

$$n_d = \#W_d = \begin{cases} \lfloor \frac{d}{6} \rfloor + 1 & 6 \nmid d-1 \\ \lfloor \frac{d}{6} \rfloor & 6 \mid d-1 \end{cases}$$

if $d \leq k$. As for the case that $k+1 \leq d \leq 2k+1$, we have the following lemma.

Lemma 3.21. *If $k+1 \leq d \leq 2k+1$, then $n_d = n_{2k+1-e} - (e + 1 - 2\lfloor \frac{e+1}{2} \rfloor)$.*

Proof. We write $e = 2k+1-d = 6\ell + s \leq k$ and $k - e = 6m + r$ for $0 \leq s, r \leq 5$. Then we deduce from (3.20) that

$$n_e = \begin{cases} \ell + 1 & s \not\equiv 1 \pmod 6, \\ \ell & s \equiv 1 \pmod 6. \end{cases}$$

On the other hand, we have $k = 6(\ell+m)+s+r$, and

$$n_d = n_{2k+1-e} = n_{k+1+6m+r} = n_{k+1+r} - m = \begin{cases} n_{k+r-5} - m & 0 \le r \le 2 \\ n_{k+r-5} - m - 1 & 3 \le r \le 5 \end{cases} \tag{3.22}$$

by (3.20). Since $k \ge k+r-5 = 6(\ell+m)+2r+s$, we have

$$n_{k+r-5} = \begin{cases} \ell+m+\lfloor \frac{2r+s+1}{6} \rfloor & 2r+s \not\equiv 1 \,(\mathrm{mod}\, 6), \\ \ell+m+\lfloor \frac{2r+s+1}{6} \rfloor - 1 & 2r+s \equiv 1 \,(\mathrm{mod}\, 6). \end{cases} \tag{3.23}$$

By (3.22) and (3.23), we can check that $n_d = n_e - (e - 2\lfloor \frac{e}{2} \rfloor)$ by listing each possibility of r and s. □

So we deduce the statements (4) and (5) from the above lemma. To prove statement (1), it suffices to show that $\dim G_d = \dim G_e = \cdots$ for any $d, e \ge 2k+2$, or equivalently $n_d = 0$ for $d \ge 2k+2$. By the recursive formula of $\dim G_d$, we have $\dim G_d - \dim G_{d-3} = 0$ if $2k+1 \le d$. So we only need to check that $n_{2k+2} = n_{2k+3} = n_{2k+4} = 0$, which can be checked by (3.20) and Lemma 3.21.

Step 2: We prove the statements (1)' (3) and (4) when $3 \mid k$ for z coordinate.

In this case, the kernel of $\bar{\theta}_z$ is of rank 1 generated by $\eta^{k/3}$, where $\eta = z^2 v_0^3 + z v_1^3 + v_2^3 - 3 z v_0 v_1 v_2$ has degree $2k$. We take $W_d \subset G_d$ such that their image in $G_d/(\bar{\theta}_z(G_{d-1}) + G_d \cap \mathbb{C}[z]\eta^{k/3})$ is a basis when $d \le 2k+1$. These sets W_d satisfy (3).

The formula of $\dim G_d - \dim G_{d-3}$ is the same as that in (3.19). Since $\deg(\eta^{k/3}) = 2k$, the number n_d and the formula of $n_d - n_{d-6}$ are the same as those in the case $3 \nmid k$ if $d \le k$. So we have again

$$n_d = \#W_d = \begin{cases} \lfloor \frac{d}{6} \rfloor + 1 & 6 \nmid d-1 \\ \lfloor \frac{d}{6} \rfloor & 6 \mid d-1 \end{cases}$$

if $d \le k$, which verifies the statement (4). To conclude the statement (1), it suffices to show that

$$\dim G_{2k+3e+1} = \dim G_{2k+3f+2} = \dim G_{2k+3g+3} - 1$$

for any $e, f, g \ge 0$. By (3.19), it suffices to show that $\dim G_{2k+1} = \dim G_{2k+2} = \dim G_{2k+3} - 1$. We can check this by writing $k = 6\ell$ or $6\ell+3$ and use (3.19).

Step 3: We prove the statements (1) (1)' (3) (4) and (5) for coordinate t, and show statement (2).

As before, we decompose $G^+[t]$ as $\oplus \tilde{G}_d$, and take $\widetilde{W}_d \subset \tilde{G}_d$ such that its image in $\tilde{G}_d/\bar{\theta}_t(\tilde{G}_{d-1})$ or $\tilde{G}_d/(\bar{\theta}_t(\tilde{G}_{d-1}) + \tilde{G}_d \cap \mathbb{C}[t]\eta^{k/3})$ is a basis. Since $G_d/\bar{\theta}_z(G_{d-1}) \subset \tilde{G}_d/\bar{\theta}_t(\tilde{G}_{d-1})$, we can choose $\widetilde{W}_d$ containing W_d. So (2) and (3) are satisfied. The proof of (1) (1)' (4) and (5) are similar to those in the above, by establishing a recursive formula for $n_d = \dim G_d - \dim G_{d-1}$. □

Proof of Theorem 3.12

It suffices to compute the basis of the de Rham cohomologies of $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3$, because by our choice of W_d and $\tilde{W}_d$ we have $\widetilde{W}_d^{\mu_3} = \{w \in \widetilde{W}_d \mid w \in W_d\} = W_d$, and

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)^{\mu_3},$$

where the action of μ_3 is given by $\mu \cdot t^s v^{\underline{I}} := (\mu t)^s v^{\underline{I}}$.

We have the following spectral sequence

$$E_1^{p,q} = \mathrm{H}^{-p+q}(\mathrm{gr}_p G^+[t] \xrightarrow{\bar{\theta}_t} \mathrm{gr}_{p+1} G^+[t]) \Longrightarrow \mathrm{H}^{-p+q}(G^+[t], \theta_t) \quad (p \ge 0, -p+q \in \{0,1\})$$

associated with the (increasing) filtration by deg on $G^+[t]$. It degenerates at the E_2-page.

If $3 \nmid k$, by Proposition 3.18.(1), we have the identification

$$\mathrm{coker}\, \bar{\theta}_t = \oplus_d \oplus_{w \in \widetilde{W}_d} \mathbb{C} w.$$

Since $\bar{\theta}_t$ is injective by Lemma 3.17, the spectral sequence is already degenerated at the E_1-page. So there exists a decreasing filtration $F^\bullet$ on $\operatorname{coker}\theta_t$ such that the graded pieces of $\operatorname{coker}\theta_t$ are $\oplus_{w\in\widetilde{W}_d}\mathbb{C}w$. So we can take the union of these finite sets $\widetilde{W}_d$ as a basis $\operatorname{coker}(\theta_t)$.

If $3 \mid k$, the kernel of $\bar{\theta}_t$ on $G^+[t]$ is $\mathbb{C}[t]\eta^{k/3}$ by Lemma 3.17, and we have the identification

$$\operatorname{coker}\bar{\theta}_t = \oplus_d \oplus_{w\in\widetilde{W}_d}\mathbb{C}w \oplus \mathbb{C}[t]\eta^{k/3}$$

by Proposition 3.18.$(1)'$.

Lemma 3.24. *The morphism θ_t induces an injective morphism*

$$\theta_t\colon \ker\bar{\theta}_t \to \operatorname{coker}\bar{\theta}_t.$$

Moreover, we have $\theta_t(t^r\eta^{k/3}) \equiv \mu t^r\eta^{k/3}$ in $\operatorname{coker}\bar{\theta}_t$ for some $\mu \neq 0$.

Proof. Write $k = 3\ell$. For $r \geq 0$, we have

$$\theta_t(t^r\eta^\ell) = rt^r\eta^\ell + 3\ell t^r(2t^6v_0^3 + t^3v_1^3 - 3t^3v_0v_1v_2)\eta^{\ell-1}.$$

Since

$$2t^6v_0^3 + t^3v_1^3 - 3t^3v_0v_1v_2 - \eta = t^6v_0^3 - v_2^3 = \bar{\theta}_t(t^3v_0^2v_2 - v_1v_2^2),$$

it follows that

$$(2t^6v_0^3 + t^3v_1^3 - 3t^3v_0v_1v_2)\eta^{\ell-1} = \eta^\ell + \bar{\theta}_t(t^3v_0^2v_2 - v_1v_2^2)\eta^{\ell-1} \overset{(*)}{=} \eta^\ell + \bar{\theta}_t((t^3v_0^2v_2 - v_1v_2^2)\eta^{\ell-1}).$$

Here $(*)$ is because $\bar{\theta}_t(g\cdot\eta^{\ell-1}) = \bar{\theta}_t(g)\eta^{\ell-1} + (\ell-1)g\cdot\bar{\theta}_t(\eta)\eta^{\ell-2} = \bar{\theta}_t(g)\eta^{\ell-1}$. Therefore, we deduce that

$$\theta_t(t^r\eta^\ell) = (r+3\ell)\,t^r\eta^\ell + \bar{\theta}_t(3\ell t^r(t^3v_0^2v_2 - v_1v_2^2)\eta^{\ell-1}),$$

and θ_t is injective. □

The spectral sequence is degenerated at the E_2-page, because all morphisms $d_2^{p,q}$ are already 0 by Lemma 3.24. So the two vector spaces $\ker\theta_t$ and $\operatorname{coker}\theta_t$ are respectively the kernel and the cokernel of the induced map $\theta_t\colon \ker\bar{\theta}_t \to \operatorname{coker}\bar{\theta}_t$. Therefore, $\ker\theta_t = 0$ and the composition of morphisms

$$W := \oplus_d \oplus_{w\in\widetilde{W}_d}\mathbb{C}w \hookrightarrow \oplus_d \oplus_{w\in\widetilde{W}_d}\mathbb{C}w \oplus \mathbb{C}[t]\eta^{k/3} = \operatorname{coker}\bar{\theta}_t \twoheadrightarrow \operatorname{coker}\theta_t$$

is an isomorphism. Hence, $\cup_d\widetilde{W}_d$ form a basis of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$.

3.2.2 Cohomology classes of the middle cohomologies

Now we want to determine bases for the middle de Rham cohomologies. Since

$$\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3) = \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)^{\mu_3},$$

we only need to determine bases for the latter cohomologies. A key tool for determining the cohomology classes is the following:

Proposition 3.25 ([FSY20b, Cor. 3.5])**.** *The cohomology with compact support $\mathrm{H}^1_{\mathrm{dR},c}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$ can be identified with the quotient of the $\mathbb{C}$-vector space*

$$\{(m_0, m_\infty, \omega) \in G^+((t)) \times G^+((t^{-1})) \times G^+[t] \mid \theta_t(m_0, m_\infty) = (\omega|_0, \omega|_\infty)\}$$

by the $\mathbb{C}$-vector space $\{(h|_0, h|_\infty, \theta(h)) \mid h \in G^+[t]\}$.

Lemma 3.26. *The cokernel of* $\theta_t\colon G^+((t)) \to G^+((t))$ *is* $\operatorname{coker} N_k$. *In particular, each element in* $t\cdot G^+[t]$ *is in the image of* θ_t.

Proof. By an abuse of notation, we denote by $\bar{\theta}_t$ the operator $3(N_k + E_k t^3)$ on $G^+((t))$. Similar to Lemma 3.16, the complex $G^+((t)) \xrightarrow{\theta_t} G^+((t))$ is quasi-isomorphic to $G^+[[t]] \xrightarrow{\theta_t} G^+[[t]]$.

We first show that elements in $t\cdot G^+[t]$ are in the image of θ_t. Consider elements of the form $\omega = \sum_{i=1}^{d_0} t^i\omega_{i,0} \in t\cdot G^+[t]$, where $\omega_{i,0}\in V_k$. Then

$$\sum_{i=1}^{d_0} t^i\omega_{i,0} = \theta_t\Big(\sum_{i=1}^{d_0}\frac{1}{i}t^i\omega_{i,0}\Big) - \bar{\theta}_t\Big(\sum_{i=1}^{d_0}\frac{1}{i}t^i\omega_{i,0}\Big).$$

Now write $\bar{\theta}_t(\sum_{i=1}^{d_0}\frac{1}{i}t^i\omega_{i,0})$ as $\sum_{i=1}^{d_1} t^i\omega_{i,1}$, we have again

$$\sum_{i=1}^{d_1} t^i\omega_{i,1} = \theta_t\Big(\sum_{i=1}^{d_1}\frac{1}{i}t^i\omega_{i,1}\Big) - \bar{\theta}_t\Big(\sum_{i=1}^{d_1}\frac{1}{i}t^i\omega_{i,1}\Big).$$

Continuing this operation, we have

$$\sum_{i=1}^{d_s} t^i\omega_{i,s} = \theta_t\Big(\sum_{i=1}^{d_s}\frac{1}{i}t^i\omega_{i,s}\Big) - \bar{\theta}_t\Big(\sum_{i=1}^{d_s}\frac{1}{i}t^i\omega_{i,s}\Big),$$

and we write

$$\bar{\theta}_t\Big(\sum_{i=1}^{d_s}\frac{1}{i}t^i\omega_{i,s}\Big) = \sum_{i=1}^{d_{s+1}} t^i\omega_{s+1}.$$

Since $\bar{\theta}_t$ is homogeneous of degree 1 with respect to the degree on $G^+[t]$, the element $\sum_{i=1}^{d_s} t^i\omega_{i,s}$ is in $t^sG^+[t]$ as long as $i > 2k+1+s$. In fact, if $i > 2k+1+s$ and $\omega_{i,s}\neq 0$, then

$$2k+1+s \le \deg(t^i\omega_{i,s}) = i + \deg(\omega_{i,s}) \le i + 2k + 1,$$

which forces $s \le i$. So the sum

$$m_0 = \sum_{s\ge 1}(-1)^s\sum_{i=1}^{d_s} t^i\omega_{i,s}$$

is well-defined as an element in $G^+[[t]]$, and we have $\theta_t(m_0) = \omega$. So each element in $tG^+[[t]]$ is 0 in the cokernel.

At last, assume that $v\in V_k = G^+[[t]]/tG^+[[t]]$. On the one hand, if $v = 3N_k(w)$ for some $w\in V_k$, then $v = \theta_t(w) - 3t^3E_k(w)$. Since $t^3E_k(w)$ is also in the image of θ_t by the above argument, v is in the image of θ_t. On the other hand, if v is in the image of θ_t, i.e., $v = \theta(\sum_{i\ge 0} t^iw_i)$. Then by comparing the constant term with respect to t, we have $v = 3N_kw_0$. Therefore, the cokernel of θ_t is $\operatorname{coker} N$. □

Lemma 3.27. *The cokernel of* $\theta_t\colon G^+((t^{-1})) \to G^+((t^{-1}))$ *is* 0 *if* $3\nmid k$ *and is* $\mathbb{C}t^kv_0^k$ *if* $3\mid k$.

Proof. Let $u = t^{-1}$. Recall that in $G^+((u))$, the subspace of elements of degree 0 with respect to t is V_k. If $3\nmid k$, it suffices to show that for each $\underline{I}$, the element $u^af^{\underline{I}}$ is in $\theta_t(G^+((u)))$. Since

$$\theta_t(u^{a+1}f^{\underline{I}}) = (-a-1)u^{a+1}f^{\underline{I}} - 3C_{\underline{I}}u^af^{\underline{I}},$$

and $C_{\underline{I}}\neq 0$ for any $\underline{I}$, we have

$$u^af^{\underline{I}} = \theta_t(g_1) + h_1,$$

for some g_1 and h_1 in $u^{a+1}V_k$. We repeat this process inductively to get

$$h_i = \theta_t(g_{i+1}) + h_{i+1}$$

for $i \geq 1$ and some g_{i+1} and h_{i+1} in $u^{a+i+1}V_k$. Therefore, we conclude that the sum $\sum_{i\geq 1} g_i$ is well-defined in $G^+((u))$ and $u^a f^{\underline{I}} = \theta_t(\sum_{i\geq 1} g_i)$.

If $3 \mid k$, write $k = 3\ell$. Let $\pi\colon V_k \to \oplus_{C_{\underline{I}}\neq 0}\mathbb{C}f^{\underline{I}}$ be the projection. For each $\underline{I}$ such that $C_{\underline{I}} \neq 0$, we have

$$u^a f^{\underline{I}} = \theta_t(g_1) + h_1 \tag{3.28}$$

for some g_1 and h_1 in $u^{a+1}V_k$. Moreover, we have $h_1 - \pi(h_1) \in u^{a+1}\eta^\ell \cdot V_k$. Then we do similar operations as those in (3.28) inductively to get

$$\pi(h_i) = \theta_t(g_{i+1}) + h_{i+1}$$

for $i \geq 1$ and some g_{i+1} and h_{i+1} in $u^{a+i+1}V_k$. Again, $h_{i+1} - \pi(h_{i+1}) \in u^{a+i+1}\eta^\ell V_k$. Therefore, we conclude that the sum $\sum_{i\geq 1} g_i$ is well-defined in $G^+((u))$ and

$$u^a f^{\underline{I}} = \theta_t\left(\sum_{i\geq 1} g_i\right) + R(u)\cdot\eta^\ell \tag{3.29}$$

for $R(u) = \sum_{i=a+1} u^i(h_{i-a} - \pi(h_{i-a})) \in u^{a+1}\mathbb{C}[[u]]$, i.e. in the cokernel $u^a f^{\underline{I}}$ has the same class as $R(u)\eta^\ell$.

For $u^a\eta^\ell$, we have

$$\begin{aligned}\theta_t\left(u^a\eta^\ell\right) &= -au^a\eta^\ell + 3\ell u^a\eta^{\ell-1}(2t^6v_0^3 + t^3v_1^3 - 3t^3v_0v_1v_2)\\ &= -au^a\eta^\ell + 3lu^a\eta^{\ell-1}\left(\eta + \sum_{|\underline{J}|=3, C_{\underline{J}}\neq 0}\lambda_{\underline{J}} f^{\underline{J}}\right)\\ &= (-a+3\ell)u^a\eta^\ell + u^a\sum_{|\underline{I}|=k, C_{\underline{I}}\neq 0}\mu_{\underline{I}} f^{\underline{I}}\end{aligned} \tag{3.30}$$

for some complex numbers $\lambda_{\underline{J}}$ and $\mu_{\underline{I}}$. If $-a+3\ell \neq 0$, let

$$G_1 = \frac{1}{-a+3\ell}u^a\eta^\ell \text{ and } H_1 = \frac{-1}{-a+3\ell}u^a\sum_{|\underline{I}|=k, C_{\underline{I}}\neq 0}\mu_{\underline{I}} f^{\underline{I}}.$$

Then we have

$$u^a\eta^\ell = \theta_t(G_1) + H_1, \tag{3.31}$$

for $G_1, H_1 \in u^aV_k$. We do the same operation as that for (3.29) to the component H_1 and get

$$H_1 = \theta_t(\tilde{G}_1) + C_1u^{a+1}\eta^\ell$$

for $\tilde{G}_1 \in u^{a+1}G^+[[u]]$ and $C_1 \in \mathbb{C}[[u]]$. And then repeat the same process in (3.31) to $C_1u^{a+1}\eta^\ell$ again, as long as $-a+3\ell \neq 0$.

Doing similar operations to those in (3.29) and (3.31) inductively, we can find $G_i, H_i, \tilde{G}_i$ and C_i inductively, and finally get an identity

$$u^a\eta^\ell = \theta_t(G) + Cu^{3\ell}\eta^\ell$$

for some $G \in G^+[[u]]$ and $C \in \mathbb{C}$.

At last, notice that

$$(3v_0)^k = (f_0+f_1+f_2)^k = \delta\eta^\ell + \sum_{\underline{I}\neq(\ell,\ell,\ell)}\delta_{\underline{I}} f^{\underline{I}}$$

for a nonzero δ, and numbers $\delta_{\underline{I}}$. Then we have $u^{3\ell}\eta^\ell - \delta t^k v_0^k \in \mathrm{im}(\theta_t)$. Hence, each element is cohomologous to an element in $\mathbb{C}[t]t^kv_0^k$. Conversely, by (3.29), (3.30) and (3.31), $t^kv_0^k$ is not in the image of $\bar{\theta}_t$. Otherwise, $u^{3\ell}\eta^\ell$ will be in the image of $\bar{\theta}_t$. This verifies our claim when $3 \mid k$. □

Proof of Theorem 3.13. There are commutative diagrams

$$\begin{array}{ccccc} G^+[t] & \xrightarrow{\theta_t} & G^+[t] & \longrightarrow & \operatorname{coker}\theta_t \\ \downarrow & & \downarrow & & \downarrow \\ G^+[[t]] & \xrightarrow{\theta_t} & G^+[[t]] & \longrightarrow & \operatorname{coker} N_k \end{array}$$

and

$$\begin{array}{ccccc} G^+[t] & \xrightarrow{\theta_t} & G^+[t] & \longrightarrow & \operatorname{coker}\theta_t \\ \downarrow & & \downarrow & & \downarrow \\ G^+[[t^{-1}]] & \xrightarrow{\theta_t} & G^+[[t^{-1}]] & \longrightarrow & \mathbb{C}t^k v_0^k \text{ or } 0 \end{array}$$

by Lemma 3.26 and Lemma 3.27. We conclude that there exist finite sets $\Sigma_d \subset G_d$ such that the image of $\cup_d \Sigma_d$ in $\operatorname{coker}\theta_t$ form a basis of $\operatorname{coker} N_k$ (resp. $\operatorname{coker} N_k \oplus \mathbb{C}t^k v_0^k$) if $3 \nmid k$ (resp. $3 \mid k$).

When $3 \nmid k$ and $w = \sum_{i=0}^d t^i w_i \in \tilde{G}_d$, we can decompose w as $w' + w''$, where w' is the constant term w_0 with respect to t, and $w'' = \sum_{i=1}^d t^i w_i$. By Lemma 3.26 and Lemma 3.27, we know that w'' is in the image of both $\theta_t \colon G^+((t)) \to G^+((t))$ and $\theta_t \colon G^+((t^{-1})) \to G^+((t^{-1}))$.

When $3 \mid k$ and $w = \sum_{i=0}^d t^i w_i \in \tilde{G}_d$, we can decompose w as $w' + w''$, where $w' = w_0 + \gamma t^k v_0^k$ for some $\gamma \in \mathbb{C}$, and $w'' = \sum_{i=1}^d t^i w_i - \gamma t^k v_0^k$. By Lemma 3.26 and Lemma 3.27, we can choose γ such that w'' is in the image of both $\theta_t \colon G^+((t)) \to G^+((t))$ and $\theta_t \colon G^+((t^{-1})) \to G^+((t^{-1}))$.

As a consequence, one can choose finite sets $\widetilde{W}'_d \subset \tilde{G}_d$ such that

- each element in $\widetilde{W}'_d$ has degree d when $3 \nmid k$,
- $\mathrm{span}(\widetilde{W}_d) = \mathrm{span}(\widetilde{W}'_d) \oplus \mathrm{span}(\Sigma_d)$,
- elements in $\widetilde{W}'_d$ are in the image of both $\theta_t \colon G^+((t)) \to G^+((t))$ and $\theta_t \colon G^+((t^{-1})) \to G^+((t^{-1}))$.

We can choose the sets W'_d similarly. Since $\operatorname{coker} N$ and $t^k v_0^k = z^{k/3} v_0^k$ (if $3 \mid k$) are contained in G^+, we can choose $\widetilde{W}'_d$ containing W'_d. This verifies statements (2) and (3).

By Proposition 3.25, $\mathrm{span}(\cup \widetilde{W}'_d) \subset \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)$. We use the computation of dimensions from Lemma 3.10 to conclude that the above inclusions are in fact identities. If we take the μ_3 invariance, we also deduce that $\mathrm{span}(\cup W'_d) \subset \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$. This verifies statement (1).

At last, we give the formula for the numbers n'_d and $\tilde{n}'_d$ in statements (4) and (5). Since the degree d part of $\operatorname{coker} N$ has dimension 1 for $d = 0, 2, \ldots, 2\lfloor \frac{k}{2} \rfloor$, and 0 otherwise. The degree d part of $\mathbb{C}t^k v_0^k$ has dimension 1 if $d = k$ and 0 otherwise. By Proposition 3.18, the cardinalities $\#\widetilde{W}'_d$ are thus numbers stated in the Theorem. □

Remark 3.32. From the above proof, we see that $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)/\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)$ is isomorphic to $\operatorname{coker} N_k$ if $3 \nmid k$ and $\operatorname{coker} N_k \oplus t^k v_0^k$ if $3 \mid k$. The same conclusion holds for $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)/\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$.

3.3 Calculation of the Hodge numbers of $\mathrm{Sym}^k\mathrm{Kl}_3$

In this section, we will compute the Hodge filtration on the de Rham realizations of some motives M^λ_{n+1}, using the cohomology classes in Theorems 3.12 and 3.13. The main result in this section is the following:

Theorem 3.33. *The mixed Hodge structure* $\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$ *is mixed of weight* $\geq 2k+1$*, whose Hodge numbers are*

$$h^{p,q} = \begin{cases} \lfloor \frac{\min(p,q)}{6} \rfloor - (\delta_{p,k} + \delta_{p,k+1}) d(k,3) & p \equiv 0,1,2,4 \bmod 6,\ p+q = 2k+1, \\ \lfloor \frac{\min(p,q)}{6} \rfloor + 1 - (\delta_{p,k} + \delta_{p,k+1}) d(k,3) & p \equiv 3,5 \bmod 6,\ p+q = 2k+1, \\ (1 + 2\lfloor \frac{k}{2} \rfloor - k) + d(k,3) & p = q = k+1, \\ 1 & k+2 \leq p = q \leq 2k+1,\ 2 \nmid p \\ 0 & \textit{else}, \end{cases}$$

where $\delta_{a,b}$ *is the Kronecker symbol. In particular, the first and second line give the Hodge numbers of the pure Hodge structure* $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$.

Remark 3.34. By Remark 3.32, we know that $\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)/\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$ is isomorphic to $\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)/\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)$. So the third and forth line of the Hodge numbers of $\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$ is deduced from the second and third line of the Hodge numbers of $\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)$ in Proposition 3.41. As for the first and second line of the Hodge numbers, they are calculated in Propositions 3.47 and 3.53.

3.3.1 The inverse Fourier transform

Let $j_0 \colon \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the inclusion. There is an endofunctor Π on the category of regular holonomic $\mathcal{D}_{\mathbb{A}^1}$-modules, defined by sending N to $N \star j_{0+}\mathcal{O}_{\mathbb{G}_m}$, where $\star$ is the additive convolution on the affine line. The essential image of this functor is the category of holonomic $\mathcal{D}$-modules with vanishing global de Rham cohomologies. In particular, the projector Π in (1.2) is the lifting of the endofunctor Π. Recall that the Fourier transform FT sends $\mathcal{D}_{\mathbb{A}^1_t}$-modules on the affine line $\mathbb{A}^1_t$ to $\mathcal{D}_{\mathbb{A}^1_\tau}$-modules on the dual affine line $\mathbb{A}^1_\tau$. In the proof of [FSY22, Prop. 2.10], we have an isomorphism of $\mathcal{D}$-modules

$$\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3 \simeq j_0^+ \mathrm{FT}(g_{3+}^{\boxplus k}\mathcal{O})^{S_k},$$

where $g_3^{\boxplus k} \colon \mathbb{G}_m^{2k} \to \mathbb{A}^1$ is defined by sending $(y_{i,j})$ to $\sum_i (\sum_j y_{i,j} + \frac{1}{\prod_j y_{i,j}})$, and the action of the symmetric group S_k on the coordinates is given by $\sigma \cdot y_{i,j} = y_{\sigma(i),j}$. By the definition of intermediate extension, we have a short exact sequence

$$0 \to j_{0\dagger+}\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3 \to j_{0+}\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3 \to \widetilde{C}_k \to 0,$$

where $\widetilde{C}_k$ is supported at the origin. Let $\widetilde{M} = \mathrm{FT}^{-1} j_{0\dagger+}\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3$. By the isomorphism of functors

$$\Pi \circ \mathrm{FT}^{-1} \simeq \mathrm{FT}^{-1} \circ j_{0+} j_0^+$$

[Kat90, Prop. 12.3.5], we deduce that $\Pi(\widetilde{M}) \simeq \mathrm{FT}^{-1}(j_{0+}\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_3)$. We get an exact sequence of regular holonomic $\mathcal{D}$-modules on the dual affine line $\mathbb{A}^1_\tau$

$$0 \to \widetilde{M} \to \Pi(\widetilde{M}) \to \widetilde{M}' \to 0. \tag{3.35}$$

The sequence (3.35) can also be lifted to a short exact sequence in $\mathrm{MHM}(\mathbb{A}^1_\tau)$

$$0 \to \widetilde{M}^{\mathrm{H}} \to \Pi(\widetilde{M})^{\mathrm{H}} \to \widetilde{M}'^{\mathrm{H}} \to 0,$$

where $\widetilde{M}^{\mathrm{H}}$ is a pure Hodge module of weight $2k$, and $\widetilde{M}'^{\mathrm{H}}$ is a constant mixed Hodge module of weight $\geq 2k+1$, see [FSY22, Prop. 2.21]. Moreover, the structure of $\widetilde{M}'$ can be made precise, given in Corollary 3.37.

Proposition 3.36. *Let $S \subset \mathbb{A}^1$ be the set of points x such that there exists a triple $\underline{I} \in \mathbb{N}^3$ such that $x = 3C_{\underline{I}} = 3(I_0 + \zeta I_1 + \zeta^2 I_2)$ and $|\underline{I}| = k$. Then:*

(1). The inverse Fourier transform $\widetilde{M}$ is a regular holonomic $\mathcal{D}_{\mathbb{A}^1_\tau}$-module. Its generic rank is $\frac{(k+1)(k+2)}{2} - \lfloor \frac{k+2}{2} \rfloor$ and its singularity set is S. The vanishing cycle space at a singularity $x \in S$ has dimension $\#\{\underline{I} \mid x = 3C_{\underline{I}} \text{ and } |\underline{I}| = k\}$ with trivial monodromy. Moreover, the monodromy at ∞ is unipotent, with one Jordan block of size $2k - 4m$ for each $m = 0, \dots, \lfloor \frac{k}{2} \rfloor$;

(2). $\Pi(\widetilde{M})$ is a regular holonomic $\mathcal{D}_{\mathbb{A}^1_\tau}$-module. Its generic rank is $\frac{(k+1)(k+2)}{2}$ and its singularity set is S. The vanishing cycle space at a singularity $x \in S$ has dimension $\#\{\underline{I} \mid x = 3C_{\underline{I}} \text{ and } |\underline{I}| = k\}$ with trivial monodromy. Moreover, the monodromy at ∞ is unipotent, with one Jordan block of size $2k - 4m + 1$ for each $m = 0, \dots, \lfloor \frac{k}{2} \rfloor$;

Proof. Let $\psi_t \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$ be the nearby cycle module at 0 of $j_{0+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$. Recall that in Section 3.1.1, we enhanced the action of the nilpotent part N_k of the local monodromy on $\psi_t \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$ to an sl_2-action. From Corollary 3.2, we conclude that the monodromy of $\psi_t \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$ is unipotent, with one Jordan block of size $2k - 4m + 1$ for each $m \in \{0, \dots, \lfloor \frac{k}{2} \rfloor\}$. By [Sai88, Lem. 5.1.4], the vanishing cycle module $\phi_t j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$ of the intermediate extension $j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$ is identified with $\mathrm{im}\, N_k$. So the monodromy action of $\phi_t j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$ is unipotent, with one Jordan block of size $2k - 4m$ for each $m \in \{0, \dots, \lfloor \frac{k}{2} \rfloor\}$.

By construction, we have $\widetilde{M} = \mathrm{FT}^{-1} j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$. Applying the (inverse) stationary phase principle [Sab08] to $\widetilde{M}$, it follows that the nearby cycle $\psi_{1/\tau}\widetilde{M}$ is isomorphic to $\phi_t j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3 \simeq \mathrm{im}\, N_k$. Therefore, the monodromy of $\psi_{1/\tau}\widetilde{M}$ is unipotent, with one Jordan block of size $2k - 4m$ for each $0 \leq m \leq \lfloor \frac{k}{2} \rfloor$. The primitive parts of the Lefschetz decomposition of $\psi_{1/\tau}\widetilde{M}$ are

$$\mathrm{P}_{2k-1}, \mathrm{P}_{2k-5}, \dots,$$

and each $\mathrm{P}_{2k-1-4m}$ is 1-dimensional. By the property of the Lefschetz decomposition, the rank of $\widetilde{M}$ is

$$\sum_{2k-1-4m \geq 0} (2k - 4m) = \frac{(k+1)(k+2)}{2} - \left\lfloor \frac{k+2}{2} \right\rfloor.$$

In the end, by applying the stationary phase principle to $j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3 = \mathrm{FT}(\widetilde{M})$, we conclude from the local structure of $j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$ at ∞ in Proposition 3.7 that the singular set of $\widetilde{M}$ is equal to S. In fact, each singular point $a \in S$ corresponds to a component $\mathcal{E}^{at}$. So we proved all claims in the first statement. The proof for the second statement is similar. □

Corollary 3.37. *The graded pieces of $\widetilde{M}'^{\mathrm{H}}$ with respect to the weight filtration have rank 1 and are of weight $4k - 4m$ for each $m = 0, 1, \dots, \lfloor \frac{k}{2} \rfloor$.*

Proof. By applying the nearby cycle functor to the exact sequence (3.35), and using the rank formula of $\widetilde{M}^{\mathrm{H}}$ and $\Pi\widetilde{M}^{\mathrm{H}}$ in Proposition 3.36, we know that $\widetilde{M}'^{\mathrm{H}}$ has rank $\lfloor \frac{k+2}{2} \rfloor$.

To prove the weight property of $\widetilde{M}'^{\mathrm{H}}$, it suffices to compute that of $\psi_{1/\tau}\widetilde{M}'^{\mathrm{H}}$, because the constant mixed Hodge module $\widetilde{M}'^{\mathrm{H}}$ extends smoothly at ∞. By a similar argument in the proof of Proposition 3.36, we know that the primitive parts in the Lefschetz decomposition of $\psi_{1/\tau}\Pi\widetilde{M}^{\mathrm{H}}$ are

$$\mathrm{P}_{2k+1}, \mathrm{P}_{2k+3}, \dots,$$

all being 1-dimensional. Since the weight filtration on $\psi_{1/\tau}\Pi\widetilde{M}^{\mathrm{H}}$ is identified with the monodromy filtration centered at $2k-1$ by [MT13, Prop. A.1], it follows that $\mathrm{P}_{2k+1-4m}$ is pure of weight $4k-4m$, of rank 1. At last, by the Equation (A.2) in the proof of [MT13, Prop. A.3], the graded pieces of $\psi_{1/\tau}\widetilde{M}'^{\mathrm{H}}$ are identified with the primitive parts in the Lefschetz decomposition of $\psi_{1/\tau}\Pi\widetilde{M}^{\mathrm{H}}$. Therefore, the graded pieces of $\mathrm{gr}^W_{4k-4m}\widetilde{M}'^{\mathrm{H}}$ with respect to the weight filtration are $P_{2k-1-4m}(-1)$, of rank 1 and weight $4k - 4m$. □

3.3.2 The Hodge filtration for $\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3$

Before computing the Hodge filtration on $\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3)$, we calculate the Hodge filtration on $\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$, which will be useful in the case that $3 \mid k$ in §3.3.5.

Applying [FSY22, Exe. A.2 & Thm. A.30] to the short exact sequence (3.35), we have a short exact sequence of mixed Hodge structures

$$0 \to \mathrm{H}^1(\mathbb{A}^1, j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3) \xrightarrow{\iota} \mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3) \to V^{\mathrm{H}} \to 0, \tag{3.38}$$

where the mixed Hodge structure $\mathrm{H}^1(\mathbb{A}^1, j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$ is

$$\mathrm{coker}\,(N_k \colon \psi_{\tau,1}\widetilde{M}^{\mathrm{H}} \to \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}(-1)), \tag{3.39}$$

and V^{H} is the cokernel of ι. By applying [FSY22, Cor. A31] to the pure Hodge module $\widetilde{M}^{\mathrm{H}}$, we know that V^{H} is mixed of weight $\geq 2k+2$, and satisfies

$$\dim \mathrm{gr}_F^p \mathrm{gr}_\ell^W V^{\mathrm{H}} = \mathrm{rk}\,\mathrm{gr}_F^{p-1}\mathrm{gr}_{\ell-1}^W \widetilde{M}'^{\mathrm{H}} \tag{3.40}$$

for all $p, \ell \in \mathbb{Z}$. In particular, by the above and [FSY22, Prop. 2.21], the weights of the mixed Hodge structure $\mathrm{H}^1(\mathbb{A}^1, j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$ are at least $2k+1$, and we have

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3) = \mathrm{gr}_{2k+1}^W \mathrm{H}^1(\mathbb{A}^1, j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3).$$

Proposition 3.41. *The Hodge numbers of* $\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$ *are given by*

$$h^{p,q} = \begin{cases} \left\lfloor \frac{\min\{p,q\}+1}{2} \right\rfloor - (\delta_{p,k} + \delta_{p,k+1}) \cdot d(k,3) & p+q = 2k+1 \\ (1 + 2\lfloor \frac{k}{2} \rfloor - k) + d(k,3) & p = q = k+1 \\ 1 & k+2 \leq p = q \leq 2k+1,\ 2 \nmid p \\ 0 & \textit{else} \end{cases}$$

where $d(k,3)$ *is* 1 *if* $3 \mid k$ *and* 0 *otherwise, and* $\delta_{a,b}$ *is the Kronecker symbol, which is* 1 *if* $a = b$ *and* 0 *otherwise. In particular, the first line of Hodge numbers gives the Hodge numbers of* $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$.

Proof. We proceed in three steps.

Step 1: We compute the rank of the Hodge filtration on $\widetilde{M}^H$. Using a similar argument as in the proof of Proposition 3.36, the monodromy of the nearby cycle $\psi_{1/\tau}\widetilde{M}^{\mathrm{H}}$ at ∞ is unipotent, with one Jordan block of size $2k-4m$ for each $m = 0, \ldots, \lfloor \frac{k}{2} \rfloor$. Therefore, the primitive parts $P_{2k-4m-1}$ in the Lefschetz decomposition have dimension one and are of Hodge-Tate type. Since $\widetilde{M}$ is pure of weight $2k$, the primitive part $P_{2k-4m-1}$ is pure of weight $4k-4m-2$. Hence, $P_{2k-4m-1} = \mathrm{gr}_F^{4k-4m-2} P_{2k-4m-1}$. Then by the Lefschetz decomposition, for each ℓ in $\mathbb{Z}$, the graded quotient $\mathrm{gr}_{2k+\ell-1}^W \psi_{1/\tau}\widetilde{M}^{\mathrm{H}}$ is Hodge-Tate of dimension

$$\#\{m \mid 0 \leq 2m \leq k,\ 4m \leq 2k-\ell-1\}.$$

By the compatibility of [Sai88, 3.2.1] between the Hodge filtration and the Kashiwara-Malgrange filtration of the filtered $\mathcal{D}$-modules underlying Hodge modules, we have

$$\mathrm{rk}\,\mathrm{gr}_F^p \widetilde{M}^{\mathrm{H}} = \dim \mathrm{gr}_F^p \mathrm{gr}^W \psi_{1/\tau}\widetilde{M}^{\mathrm{H}}$$

in the case of smooth curves. From this formula, we conclude that

$$\mathrm{rk}\,\mathrm{gr}_F^p \widetilde{M}^{\mathrm{H}} = \left\lfloor \frac{\min\{p, 2k-1-p\}+2}{2} \right\rfloor. \tag{3.42}$$

Step 2: We compute the Hodge filtration on $\mathrm{H}^1(\mathbb{A}^1, j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}_3})$.

If $3\nmid k$, by Proposition 3.36, 0 is not a singular point of $\widetilde{M}$. Therefore, the Hodge numbers can be computed using [FSY22, Cor. A31(ii)]. More precisely, the nilpotent part of the monodromy operator acts on $\psi_{\tau,1}\widetilde{M}^{\mathrm{H}} = \psi_\tau\widetilde{M}^{\mathrm{H}}$ by 0. So $\mathrm{H}^1(\mathbb{A}^1, j_{0\dagger+}\mathrm{Sym}^k\widetilde{\mathrm{Kl}_3}) = \psi_\tau\widetilde{M}^{\mathrm{H}}(-1)$ is pure of weight $2k+1$, which coincides with $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}_3})$. Its Hodge numbers $h^{p,2k+1-p}$ are the ranks of $\mathrm{gr}_F^{p-1}\widetilde{M}^{\mathrm{H}}$ given by (3.42). This gives the first line of the Hodge numbers when $3\nmid k$.

If $3\mid k$, the dimension of the formal regular component of $\mathrm{Sym}^k\widetilde{\mathrm{Kl}_3}$ at ∞ is 1. Therefore, $\tau=0$ is a singular point of $\widetilde{M}$, and the dimension of the vanishing cycle $\phi_{t,1}\widetilde{M}^{\mathrm{H}}$ is 1. By (3.39), it suffices to study the monodromy on the nearby cycle $\psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$. Since $\widetilde{M}^{\mathrm{H}}$ is an intermediate extension at $\tau=0$ and $\phi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ has dimension 1, the nilpotent part of the monodromy operator, denoted by N_k, acting on $\psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ satisfies $N_k^2=0$. This forces the primitive parts of the Lefschetz decomposition of $\mathrm{gr}^W\psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ to be $P_1 = \mathrm{gr}^W_{2k}\psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ and $P_0 = \mathrm{gr}^W_{2k-1}\psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$, where $\dim P_1 = \dim N_kP_1 = 1$, and $\dim P_0 = \mathrm{rk}\,\widetilde{M} - 2$. In conclusion,

$$\mathrm{gr}^W\mathrm{coker}N = P_0(-1)\oplus P_1(-1).$$

Hence, the direct summand $P_0(-1)$ is $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}_3})$. Also, the direct summand $P_1(-1)$ is $\mathrm{gr}^W_{2k+2}\mathrm{H}^1(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}_3})$, whose rank is 1. It is of Hodge-Tate type $(k+1,k+1)$. Then using the equality derived from the Lefschetz decomposition

$$\mathrm{rk}\,\mathrm{gr}_F^{p-1}\widetilde{M}^{\mathrm{H}} = \mathrm{rk}\,\mathrm{gr}_F^{p-1}\psi_{\tau,1}\,\widetilde{M}^{\mathrm{H}} = \dim\mathrm{gr}_F^p(P_0(-1)) + \mathrm{gr}_F^p(P_1(-1)) + \mathrm{gr}_F^p(NP_1(-1)),$$

we proved the first line of the Hodge numbers when $3\mid k$, as well as the $d(k,3)$ part in the second line of the Hodge numbers.

Step 3: We compute the Hodge filtration on V^{H}. By Corollary 3.37, since the graded pieces of $\widetilde{M}'^{\mathrm{H}}$ with respect to the weight filtration have rank 1, of weight $4k-4m$ for $m=0,1,\dots,\lfloor\frac{k}{2}\rfloor$, they are all of Hodge-Tate type $(2k-2m, 2k-2m)$. Thus, by (3.40), we know that for $m=0,\dots,\lfloor\frac{k}{2}\rfloor$, we have $\dim\mathrm{gr}_F^{2k-2m+1}\mathrm{gr}^W_{4k-4m+2}V^{\mathrm{H}} = 1$, and the remaining graded pieces are 0. Therefore, we get the third line for the Hodge numbers by using the exact sequence (3.38). □

3.3.3 The irregular Hodge filtration for twisted de Rham cohomologies

To do the concrete calculation, we use the irregular Hodge filtration on twisted de Rham cohomologies from [Yu14], see Section 1.3. The Kloosterman connection Kl_3, as a $\mathbb{C}[z,z^{-1}]$-module, equals to the cokernel of the complex

$$\mathbb{C}[x_1^\pm,x_2^\pm,z^\pm]\mathrm{d}z\wedge\mathrm{d}x_1\oplus\mathbb{C}[x_1^\pm,x_2^\pm,z^\pm]\mathrm{d}z\wedge\mathrm{d}x_2\xrightarrow{\mathrm{d}+\partial_{x_1}f\mathrm{d}x_1+\partial_{x_2}f\mathrm{d}x_2}\mathbb{C}[x_1^\pm,x_2^\pm,z^\pm]\mathrm{d}z\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2.$$

In this way, the classes v_0, v_1 and v_2 are sent to

$$\frac{\mathrm{d}z}{z}\wedge\frac{\mathrm{d}x_1}{x_1}\wedge\frac{\mathrm{d}x_2}{x_2}, x\frac{\mathrm{d}z}{z}\wedge\frac{\mathrm{d}x_1}{x_1}\wedge\frac{\mathrm{d}x_2}{x_2}\text{ and }xy\frac{\mathrm{d}z}{z}\wedge\frac{\mathrm{d}x_1}{x_1}\wedge\frac{\mathrm{d}x_2}{x_2},$$

and we have

$$(z\partial_z)^3\Big(\frac{\mathrm{d}z}{z}\wedge\frac{\mathrm{d}x_1}{x_1}\wedge\frac{\mathrm{d}x_2}{x_2}\Big) = z\cdot\frac{\mathrm{d}z}{z}\wedge\frac{\mathrm{d}x_1}{x_1}\wedge\frac{\mathrm{d}x_2}{x_2}.$$

Let $f_k\colon\mathbb{G}_m^{2k+1}\to\mathbb{A}^1$ be the function $(x_{i,1},x_{i,2},z)\mapsto\sum_i\Big(x_{i,1}+x_{i,2}+\frac{z}{x_{i,1}x_{i,2}}\Big)$. Then we have inclusions

$$\begin{aligned}\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m,\mathrm{Sym}^k\mathrm{Kl}_3)\hookrightarrow\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m,\mathrm{Sym}^k\mathrm{Kl}_3) &= \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m,\mathrm{Kl}_3^{\otimes k})^{S_k}\\ &\hookrightarrow\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m,\mathrm{Kl}_3^{\otimes k})\simeq\mathrm{H}^{2k+1}_{\mathrm{dR}}(\mathbb{G}_m^{2k+1},f_k).\end{aligned}\tag{3.43}$$

The first morphism is compatible with the irregular Hodge filtration, because it is compatible with the classical Hodge filtration, and by [FSY22, Thm. A.24], the irregular Hodge filtration coincide with the classical Hodge filtration. The second morphism corresponds to the inclusion of $S_k \times \mu_3$ invariant parts in $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Kl}_{n+1}^{\otimes k})$, which is compatible with the irregular Hodge filtration by [Yu14, Prop. 1.9(i)].

3.3.4 The Hodge filtration for $\mathrm{Sym}^k\mathrm{Kl}_3$ when $3 \nmid k$

Under the last isomorphism in (3.43), an element $z^j v_0^{I_0} v_1^{I_1} v_2^{I_2}$ is sent to

$$z^j \frac{1}{k!} \sum_{\sigma \in S_k} \prod_{i=I_1+1}^{k} x_{\sigma(i),1} \prod_{i=I_2+1}^{k} x_{\sigma(i),2} \frac{\mathrm{d}z}{z} \frac{\mathrm{d}x_{1,1}}{x_{1,1}} \frac{\mathrm{d}x_{1,2}}{x_{1,2}} \cdots \frac{\mathrm{d}x_{k,1}}{x_{k,1}} \frac{\mathrm{d}x_{k,2}}{x_{k,2}} \tag{3.44}$$

in $\mathrm{H}^{2k+1}_{\mathrm{dR}}(\mathbb{G}_m^{2k+1}, f_k)^{S_k}$. So each element $w \in W'_d$ from Theorem 3.13 is sent to

$$g(w) \frac{\mathrm{d}z}{z} \frac{\mathrm{d}x_{1,1}}{x_{1,1}} \frac{\mathrm{d}x_{1,2}}{x_{1,2}} \cdots \frac{\mathrm{d}x_{k,1}}{x_{k,1}} \frac{\mathrm{d}x_{k,2}}{x_{k,2}}$$

for a polynomial in z, x_i, y_i such that $\deg(g(w)) = d$. By abuse of notation, we still denote by w the image of w under (3.43).

Lemma 3.45. *Assume $3 \nmid k$. Then $W'_d \subset F^p\mathrm{H}^{2k+1}_{\mathrm{dR}}(\mathbb{G}_m^{2k+1}, f_k)$ if $p \leq 2k+1-d$.*

Proof. The Newton polytope $\Delta(f_k)$ defined by f_k has only one facet that does not contain the origin, which is lying on the hyperplane $3z + \sum_i (x_{i,1} + x_{i,2}) = 1$. So the condition that f_k is non-degenerate with respect to $\Delta(f_k)$ is equivalent to the condition that the Laurent polynomial f_k has no critical points in $\mathbb{G}_m^{2k+1}$. We can check that the latter condition holds $3 \nmid k$. So the irregular Hodge filtration on $\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_m^{2k+1}, f_k)$ can be computed via the Newton filtration on monomials in $\mathbb{R}_{\geq 0}\Delta(f_k)$ by [Yu14, Thm. 4,6].

The cone $\mathbb{R}_{\geq 0}\Delta(f_k)$ is given by inequalities

$$\beta + \sum_{i=1}^{k} \epsilon_i \alpha_{i,1} + \eta_i \alpha_{i,2} \geq 0,$$

where $(\epsilon_i, \eta_i) \in \{(0,0), (1,0), (0,1)\}$. We take the fan F generated by rays

$$\mathbb{R}_{\geq 0} \cdot \pm(\beta + \sum_{i=1}^{k} \epsilon_i \alpha_{i,1} + \eta_i \alpha_{i,2}) \text{ and } \mathbb{R}_{\geq 0} \pm (3\beta + \sum_i \alpha_{i,1} + \alpha_{i,2})$$

where $(\epsilon_i, \eta_i) \in \{(0,0), (1,0), (0,1)\}$. The fan F is a simplicial polytopal fan. So the corresponding toric variety X_{tor} is smooth projective. In particular, each irreducible component of the pole divisor P of f_k has multiplicity 1.

Observe that an element $z^a \prod_{i=1}^k x_{i,1}^{b_i} x_{i,2}^{c_i} \cdot \frac{\mathrm{d}z}{z} \frac{\mathrm{d}x_{1,1}}{x_{1,1}} \frac{\mathrm{d}x_{1,2}}{x_{1,2}} \cdots \frac{\mathrm{d}x_{k,1}}{x_{k,1}} \frac{\mathrm{d}x_{k,2}}{x_{k,2}} \in F^p\mathrm{H}^{2k+1}_{\mathrm{dR}}(\mathbb{G}_m^{2k+1}, f_k)$ if it belongs to $\Omega^{2k+1}(\log S)(\lfloor (2k+1-\lambda)P \rfloor)$, which is equivalent to

$$\mathrm{ord}_D(z^a \prod_{i=1}^{k} x_{i,1}^{b_i} x_{i,2}^{c_i}) \geq -(2k+1-p) \tag{3.46}$$

for all irreducible components D of P. By [Ful93, p.61], the condition (3.46) is equivalent to

$$-\xi(z^a \prod_{i=1}^{k} x_{i,1}^{b_i} x_{i,2}^{c_i}) \geq -(2k+1-p),$$

where $\xi(z^a\prod_{i=1}^k x_{i,1}^{b_i}x_{i,2}^{c_i}) = 3a+\sum_i(b_i+c_i)$.

For $w\in W_d'$, we have $\xi(g(w))=d$, where $g(w)$ is a polynomial such that the differential form $g(w)\frac{\mathrm{d}z}{z}\frac{\mathrm{d}x_{1,1}}{x_{1,1}}\frac{\mathrm{d}x_{1,2}}{x_{1,2}}\cdots\frac{\mathrm{d}x_{k,1}}{x_{k,1}}\frac{\mathrm{d}x_{k,2}}{x_{k,2}}$ represents the image of $w\in W_d$ in $\mathrm{H}^{2k+1}_{\mathrm{dR}}(\mathbb{G}_m^{2k+1},f_k)^{S_k}$. Therefore, we conclude the assertion. □

Proposition 3.47. *Assume that $3\nmid k$. The Hodge numbers of $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m,\mathrm{Sym}^k\mathrm{Kl}_3)$ are given by*

$$h^{p,2k+1-p}=\begin{cases}\lfloor\frac{p}{6}\rfloor & p\not\equiv 3,5 \mod 6;\\ \lfloor\frac{p}{6}\rfloor+1 & \textit{else},\end{cases}$$

if $p\le k$, and $h^{p,2k+1-p}=h^{2k+1-p,p}$ if $k<p$.

Proof. Let W_d' be the finite sets in Theorem 3.13. We construct an auxiliary filtration $G^\bullet$ on $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m,\mathrm{Sym}^k\mathrm{Kl}_3)$ by letting the subspace $G^p\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m,\mathrm{Sym}^k\mathrm{Kl}_3)$ be generated by elements $w\in W_d'$ such that $p\le 2k+1-d$. Lemma 3.45 shows that

$$G^p\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m,\mathrm{Sym}^k\mathrm{Kl}_3)\subset F^p\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m,\mathrm{Sym}^k\mathrm{Kl}_3).$$

Let $d_p=\dim\mathrm{gr}_F^p\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m,\mathrm{Sym}^k\mathrm{Kl}_3)$ and $\delta_p=\dim\mathrm{gr}_G^p\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m,\mathrm{Sym}^k\mathrm{Kl}_3)$ be the dimensions of the graded quotients with respect to the two filtration. Then

$$\sum_{q\ge p}\delta_p\le\sum_{q\ge p}d_p, \tag{3.48}$$

where the equality holds if $q=0$ and $q=2k+1$. By Hodge symmetry, we have $d_p=d_{2k+1-p}$. Since δ_p are the numbers n'_{2k+1-p} in Theorem 3.13, we have $\delta_p=\delta_{2k+1-p}$.

Combining the symmetry properties of d_p and δ_p, and (3.48), we find

$$\sum_{q\ge p}\delta_p\le\sum_{q\ge p}d_p=\sum_{2k+1-q\le p}d_p\le\sum_{2k+1-q\le p}\delta_p=\sum_{q\ge p}\delta_p.$$

This implies that $\delta_p=d_p$ for each p. □

3.3.5 The Hodge filtration for $\mathrm{Sym}^k\mathrm{Kl}_3$ when $3\mid k$

We want to find another way to compute the irregular Hodge filtration when $3\mid k$, because the function f_k is not non-degenerate with respect to its Newton polytope $\Delta(f_k)$.

A compactification

Definition 3.49 ([Moc15, Def. 2.6])**.** Let K be a field of characteristic 0. Let U be a smooth quasi-projective variety over K and $f\in\mathcal{O}_U(U)$. A *non-degenerate compactification* of a pair (U,f) is a compactification X of U such that

- $D=X\backslash U$ is a strict normal crossing divisor;
- f extends to a rational morphism $f\colon X\dashrightarrow\mathbb{P}^1$;
- étale locally or analytical locally near each point in the pole divisor P, there is a coordinate system $\{x_1,\dots,x_r,y_1,\dots,y_m,z_1\dots,z_l\}$ of X such that

$$D=V(\prod x_i\cdot\prod y_j),\quad f=\frac{1}{\prod x_i^{e_i}}\text{ or }\frac{z_1}{\prod x_i^{e_i}}$$

for some $e_i\in\mathbb{Z}_{>0}$.

We begin with the pair $\left(\mathbb{G}_m^{2k}, g_3^{\boxplus k} := \sum_{i=1}^k y_{i,1} + y_{i,2} + \frac{1}{y_{i,1}y_{i,2}}\right)$. Let $M = \oplus_i (\mathbb{Z}y_{i,1} \oplus \mathbb{Z}y_{i,2})$ be the lattice of monomials on $\mathbb{G}_m^{2k}$ and $N = \oplus_i (\mathbb{Z}e_{i,1} \oplus \mathbb{Z}e_{i,2})$ be the dual lattice where $e_{i,j}$ is dual to $y_{i,j}$. We consider the toric compactification X of $\mathbb{G}_m^{2k}$ attached to the simplicial fan F in $N_{\mathbb{R}}$ generated by the rays

$$\mathbb{R}_{\geq 0} \cdot \textstyle\sum_i \epsilon_i e_{i,1} + \eta_i e_{i,2}$$

where $\epsilon_i, \eta_i \in \{0, \pm 1\}$ and $(\epsilon_i, \eta_i)_i \neq 0$. Each simplicial cone of maximal dimension $2k$ in F provides an affine chart of X, which is isomorphic to $\mathbb{A}^{2k}$. On each chart, the function $g_3^{\boxplus k}$ has the same structure. For example, we can consider the maximal cone generated by

$$\gamma_{i_0,1} := \sum_{i \leq i_0 - 1, j} e_{i,j} + e_{i_0,1} \text{ and } \gamma_{i_0,2} = \sum_{i \leq i_0, j} e_{i,j}$$

for $1 \leq i_0 \leq k$, where the affine ring associated with the dual cone is the polynomial ring $\mathbb{Q}[u_i, v_i]$ such that $u_{i,j} = \begin{cases} y_{i,1}/y_{i,2} & j = 1, \\ y_{i,2}/y_{i+1,1} & i < k, j = 2, \\ y_{k,2} & i = k, j = 2. \end{cases}$ In this chart, we can rewrite $g_3^{\boxplus k}$ as $g_1/\left(u_{1,1}u_{1,2}^2 \cdot \prod_{2 \leq i \leq k, j} u_{2,j}^2\right)$, where

$$g_1 = 1 + \sum_{e=1}^{k-1} u_{1,1}u_{1,2}^2 \cdot \prod_{2 \leq i \leq e, j} u_{i,j}^2 \cdot u_{e+1,1} + u_{1,1}u_{1,2}^2 \cdot \prod_{2 \leq i \leq k, j} u_{i,j}^2 \cdot h$$

for a polynomial h. So the toric variety X provides a non-degenerate compactification of $(\mathbb{G}^{2k}, g_3^{\boxplus k})$, where the closure of the zero locus of $g_3^{\boxplus k}$, and $X \backslash \mathbb{G}_m^{2k}$ form a strict normal crossing divisor.

The product $\mathbb{P}_t^1 \times X$ is a compactification of $(\mathbb{G}_m^{2k+1}, t \cdot g_3^{\boxplus k}) \simeq (\mathbb{G}_m^{2k+1}, \widetilde{f}_k)$. Let $S_1, \ldots, S_N$ be the irreducible components of $X \backslash \mathbb{G}_m^{2k}$. We first do the blow-up along the intersection of $0 \times X$ and S_1. And then we do blow-ups along the intersection of the proper transform of $0 \times X$ and the proper transform of S_i for $2 \leq i \leq N$ successively. The resulting variety $\bar{X}$ is a compactification of $\mathbb{G}_m^{2k+1}$.

We can verify that $(\bar{X}, \tilde{f}_k)$ is a non-degenerate compactification of $(\mathbb{G}_m^{2k+1}, t \cdot g_3^{\boxplus k})$ if $3 \nmid k$. Otherwise, we need to do two more blow-ups. First at each closed points of $\infty \times V(y_{i,1}^3 - 1, y_{i,2}^3 - 1, \sum_i y_{i,1} + y_{i,2} + \frac{1}{y_{i,1}y_{i,2}})$. Then on the intersection of the exceptional divisors from the previous step and the proper transform of $\infty \times X$. We denote by E_1, E_2 the exceptional divisors of the two steps respectively. We denote by $\widetilde{X}$ the resulting variety. By a direct computation, we can verify that $\operatorname{ord}_{E_1} \tilde{f}_k = 1$ and $\operatorname{ord}_{E_2} \tilde{f}_k = 0$, and $(\widetilde{X}, \widetilde{f}_k)$ is a non-degenerate compactification of $(\mathbb{G}_m^{2k+1}, \widetilde{f}_k)$.

A lemma

By abuse of notation, for $w \in \widetilde{W}_d$, we still denote by w its image in $\mathrm{H}_{\mathrm{dR}}^{2k+1}(\mathbb{G}_m^{2k+1}, \tilde{f}_k)$ under the inclusion (3.43).

Lemma 3.50. *Assume $3 \mid k$. Then $\widetilde{W}_d \subset F^p \mathrm{H}_{\mathrm{dR}}^{2k+1}(\mathbb{G}_m^{2k+1}, \tilde{f}_k)$ if $p \leq 2k + 1 - d$ and $k - d \geq 0$.*

Proof. Let $\widetilde{X}$ be the compactification as above and $D = \widetilde{X} \backslash \mathbb{G}_m^{2k+1}$ is the boundary divisor. Because the indeterminacy locus of the rational map $\tilde{f}_k \colon \widetilde{X} \dashrightarrow \mathbb{P}^1$ has codimension at least 2 in $\widetilde{X}$, we can take the pole divisor P as the closure of a representative of $\widetilde{f}_k$. The exceptional divisors E_1 and E_2 are not contained in the support of the pole divisor, because $\operatorname{ord}_{E_1} \tilde{f}_k = 1$ and $\operatorname{ord}_{E_2} \tilde{f}_k = 0$.

Similar to (3.44), the image of an element $t^a v_0^{I_0} v_1^{I_1} v_2^{I_2}$ of degree d in $\mathrm{H}_{\mathrm{dR}}^{2k+1}(\mathbb{G}_m^{2k+1}, \tilde{f}_k)^{S_k} = \mathrm{H}_{\mathrm{dR}}^{2k+1}(\mathbb{G}_m^{2k+1}, t \cdot g_3^{\boxplus k})^{S_k}$ is of the form

$$t^d \frac{1}{k!} \sum_{\sigma \in S_k} \prod_{i=I_1+1}^{k} y_{\sigma(i),1} \prod_{i=I_2+1}^{k} y_{\sigma(i),2} \frac{\mathrm{d}t}{t} \frac{\mathrm{d}y_{1,1}}{y_{1,1}} \frac{\mathrm{d}y_{1,2}}{y_{1,2}} \cdots \frac{\mathrm{d}y_{k,1}}{y_{k,1}} \frac{\mathrm{d}y_{k,2}}{y_{k,2}}$$

up to a nonzero constant (compared to (3.44) we did a change of variable $x_{i,j} = ty_{i,j}$). By a direct computation, one finds that $\mathrm{ord}_{E_1}(t^a v_0^{I_0} v_1^{I_1} v_2^{I_2}) = 2k - d$ and $\mathrm{ord}_{E_2}(t^a v_0^{I_0} v_1^{I_1} v_2^{I_2}) = 2k - 2d$. Since each $w \in \widetilde{W}_d$ is a linear combination of terms $t^a v_0^{I_0} v_1^{I_1} v_2^{I_2}$ of degree d, we find

$$\widetilde{W}_d \subset \Gamma(\widetilde{X}, \Omega^{2k+1}(\log D)(dP - (2k-d)E_1 - (2k-2d)E_2)).$$

We further have

$$\widetilde{W}_d \subset \Gamma(\widetilde{X}, \Omega^{2k+1}(\log D)(\lfloor 2k+1-\eta \rfloor P)),$$

if $k - d \geq 0$ and $d \leq 2k + 1 - \eta$. In this case, there is a natural map

$$\Gamma(\widetilde{X}, \Omega^{2k+1}(\log D)(\lfloor 2k+1-\eta \rfloor P)) \to F^\eta \mathrm{H}^{2k+1}(\mathbb{G}_m^{2k+1}, \tilde{f}_k),$$

see [Yu14, §4 b)] or [FSY22, § 4.3.3]. This finishes the proof of the lemma. □

Proposition 3.51. *When $p \geq k+1$, the vector space $F^p\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$ is*

$$\mathrm{span}\left(\widetilde{W}_d \mid p \leq 2k+1-d\right).$$

Proof. We construct a filtration $G^\bullet$ on the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$ by letting the subspace $G^p\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$ be generated by $\widetilde{W}_d$ satisfying $p \leq 2k+1-d$. Using the map in (3.43) and Lemma 3.50, we deduce that

$$G^p\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3) \subset F^p\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_3)$$

if $p \geq k+1$.

Let $d_p = \dim \mathrm{gr}_F^p$ and $\delta_p = \dim \mathrm{gr}_G^p$. Then

$$\sum_{q \geq p} \delta_p \leq \sum_{q \geq p} d_p \tag{3.52}$$

and $d_p = \sum_q h^{p,q}$ if $p \geq k+1$, where $h^{p,q}$ is defined by Proposition 3.41. By Theorem 3.12, we know that the number δ_p is $\tilde{n}_{2k+1-p} = \lfloor \frac{2k+1-p}{2} \rfloor + 1 = d_p$ for any $p \geq k+1$. This shows that the filtration $G^\bullet$ coincides with the Hodge filtration $F^\bullet$ when $p \geq k+1$. □

The Hodge numbers

Proposition 3.53. *Assume that $3 \mid k$. For $p \leq k$, the Hodge numbers of the mixed Hodge structure $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3)$ are given by*

$$h^{p,2k+1-p} = h^{2k+1-p,p} = -\delta_{k,p} + \begin{cases} \lfloor \frac{p}{6} \rfloor & p \not\equiv 3,5 \pmod 6; \\ \lfloor \frac{p}{6} \rfloor + 1 & p \equiv 3,5 \pmod 6. \end{cases}$$

Proof. The set $\cup W_d$ in Theorem 3.12 serve as a basis of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3)$. By Lemma 3.50, we know that $w \in W_d$ is in $F^p\mathrm{gr}^W_{2k+1}\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3)$ if

$$p \leq 2k+1-d \text{ and } k-d \geq 0.$$

By Proposition 3.51 we know that w is a nonzero element in the graded quotient

$$\mathrm{gr}_F^{2k+1-d}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3)$$

if $k - d \geq 0$. Moreover, by Proposition 3.41 and Remark 3.34, we have

$$\dim \mathrm{gr}_F^{2k+1-d}\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3) = \dim \mathrm{gr}_F^{2k+1-d}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^k\mathrm{Kl}_3) - \delta_{d,k} - (1 + 2\lfloor \tfrac{d}{2} \rfloor - d),$$

which is exactly n'_{2k+1-d} by Theorem 3.13. Therefore, if $k-d \geq 0$, the Hodge numbers

$$h^{d,2k+1-d} = h^{2k+1-d,d} = \dim \mathrm{gr}_F^{2k+1-d} \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$$

are bigger than or equal to n'_d. Then

$$\dim \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3) = \sum_{d=0}^{2k+1} h^{d,2k+1-d} \geq 2 \sum_{d=0}^{k} n'_d \overset{(*)}{=} \dim \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3),$$

where (*) can be checked by writing $k = 6\ell + r$ for $r \in \{0, 3\}$. Therefore, we have $h^{d,2k+1-d} = h^{2k+1-d,d} = n'_d$ for $d \leq k$, which are exactly numbers stated in the proposition. □

3.4 More examples of computations of Hodge numbers

In this section, we compute more examples of Hodge numbers of motives attached to Kloosterman sheaves.

3.4.1 More examples of cohomology classes

Similarly to the proof of Theorem 3.12, we can also get expressions of de Rham cohomologies for three more cases.

The case of the connection $\mathrm{Sym}^3\mathrm{Kl}_4$

When $n = 3$ and $k = 3$, the kernel and cokernel of the nilpotent part of the local monodromy of $\mathrm{Sym}^3\mathrm{Kl}_4$ at 0 have dimension 3. They are generated by

$$\{v_0^3, 3v_0^2v_2 - 2v_0v_1^2, 9v_0v_1v_2 - 9v_0^2v_3 - 4v_1^3\}$$

and

$$\{v_3^3, v_3(v_2^2 - 2v_1v_3), v_2^3 - 3v_1v_2v_3 + 3v_0v_3^2\}$$

respectively. Therefore, we have

$$\mathbb{C}((z)) \otimes_{\mathbb{C}[z,z^{-1}]} \mathrm{Sym}^3\mathrm{Kl}_4 = (\mathcal{O}, \mathrm{d}) \oplus (\mathcal{O}^2, \mathrm{d} + J_2(0)\tfrac{\mathrm{d}z}{z}) \oplus (\mathcal{O}^3, \mathrm{d} + J_3(0)\tfrac{\mathrm{d}z}{z}).$$

Similar to the case of $\mathrm{Sym}^k\mathrm{Kl}_3$ in Section 3.2, we can use the degree $\deg(z^i v_0^{I_0} v_1^{I_1} v_2^{I_2} v_3^{I_3}) = 4i + I_1 + 2I_2 + 3I_3$ to obtain the following proposition.

Proposition 3.54. *There exists finite subset $W_d, W'_d \subset G^+$ of degree d, such that*

- *the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^3\mathrm{Kl}_4)$ is spanned by $W_0 \cup W_2 \cup W_3 \cup W_4 \cup W_6$ with $\#W_0 = \#W_2 = \#W_3 = \#W_4 = \#W_6 = 1$,*
- *the middle de Rham cohomology $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^3\mathrm{Kl}_4)$ is spanned by $W'_4 \cup W'_6$ with $\#W'_4 = \#W'_6 = 1$.*

The case of the connection $\mathrm{Sym}^3\mathrm{Kl}_5$

When $n = 4$ and $k = 3$, the cokernel of the nilpotent part of the local monodromy of $\mathrm{Sym}^3\mathrm{Kl}_5$ at 0 has dimension 5. It is generated by

$$\Sigma = \{v_0^3, c_2, 4v_0^2v_3 - 6v_0v_1v_2 + 3v_1^3, c_4, c_5\},$$

where $c_2 = 4v_0^2v_2 - 3v_0v_1^2$, $c_4 = 2v_0^2v_4 - 2v_0v_1v_3 + v_0v_2^2$, and $c_5 = 12v_0v_2v_4 - 9v_1^2v_4 - 6v_0v_3^2 + 6v_1v_2v_3 - 2v_2^3$. Therefore, the formal connection $\mathbb{C}((z)) \otimes_{\mathbb{C}[z,z^{-1}]} \mathrm{Sym}^3\mathrm{Kl}_5$ is isomorphic to

$$(\mathcal{O}, \mathrm{d}) \oplus (\mathcal{O}^2, \mathrm{d} + J_2(0)\tfrac{\mathrm{d}z}{z}) \oplus (\mathcal{O}^3, \mathrm{d} + J_3(0)\tfrac{\mathrm{d}z}{z}) \oplus (\mathcal{O}^4, \mathrm{d} + J_4(0)\tfrac{\mathrm{d}z}{z}) \oplus (\mathcal{O}^6, \mathrm{d} + J_6(0)\tfrac{\mathrm{d}z}{z}).$$

Similar to the case of $\mathrm{Sym}^k\mathrm{Kl}_3$ in Section 3.2, we can use the degree $\deg(z^i v_0^{I_0} v_1^{I_1} v_2^{I_2} v_3^{I_3} v_4^{I_4}) = 5i + I_1 + 2I_2 + 3I_3 + 4I_4$ to obtain the following proposition.

Proposition 3.55. *There exists finite subset $W_d, W_d' \subset G^+$ of degree d, such that*

- *the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Sym}^3\mathrm{Kl}_5)$ is spanned by $W_0 \cup W_2 \cup W_3 \cup W_4 \cup W_5 \cup W_8$ with $\#W_0 = \$W_2 = \#W_3 = \#W_4 = \#W_8 = 1$ and $\#W_5 = 2$,*
- *the middle de Rham cohomology $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^3\mathrm{Kl}_5)$ is spanned by $W_5' \cup W_8'$ with $\#W_5' = \#W_8' = 1$.*

The case of the connection $\mathrm{Kl}_3^{(3,1)}$

Let V be the standard representation of SL_3, and $V_{2,1}$ be the representation of the highest weight $2L_1 + (L_1 + L_2)$. By the Weyl construction in Section 1.1, $V_{2,1}$ is $(V^{\otimes 4})^{(P\times Q,\chi)}$, where P and Q are two subgroups of S_4 generated by $\{(12), (123)\}$ and $\{(14)\}$ respectively, and χ is the character $\chi\colon P \times Q \xrightarrow{\mathrm{pr}_2} Q \xrightarrow{\mathrm{sign}} \{\pm 1\}$. Let $N(V_{2,1})$ and $E(V_{2,1})$ be the corresponding nilpotent endomorphisms of N and E in (2.5).

The associated Kloosterman connection $\mathrm{Kl}_3^{(2,1)}$ is the vector bundle with connection

$$\Big(\mathbb{G}_m^{\dim V_{2,1}}, d - N(V_{2,1})\frac{\mathrm{d}z}{z} - E(V_{2,1})\mathrm{d}z\Big).$$

This connection has an extra sign compared to that in [FG09].

Let v_0, v_1, v_2 be the basis of Kl_3 such that $Nv_0 = v_1$, $Nv_1 = v_2$ and $Nv_2 = 0$. We also let $w_0 = v_0 \wedge v_1, w_1 = v_0 \wedge v_2$ and $w_2 = v_1 \wedge v_2$ be the basis of $\wedge^2\mathrm{Kl}_3$. Viewed as the subconnection of $\mathrm{Sym}^2\mathrm{Kl}_3 \otimes \wedge^2\mathrm{Kl}_3$, the cokernel of the nilpotent part of the monodromy operator of $\mathrm{Kl}_3^{(3,1)}$ at 0 has dimension 3, whose basis can be taken as

$$\{v_0^2 \otimes w_0, v_0v_1 \otimes w_0 - v_0^2 \otimes w_1, c_3\},$$

where $c_3 = v_1^2 \otimes w_0 - v_0v_2 \otimes w_0 - v_0v_1 \otimes w_1 + v_0^2 \otimes w_2$. Similar to the case of $\mathrm{Sym}^k\mathrm{Kl}_3$ in Section 3.2, we can use the degree $\deg(z^i v_0^{I_0} v_1^{I_1} v_2^{I_2} \otimes w_0^{J_0} w_1^{J_1} w_2^{J_2}) = 3i + I_1 + 2I_2 + J_0 + 2J_1 + 3J_2$ to obtain the following proposition.

Proposition 3.56. *There exists finite subset $W_d, W_d' \subset G^+$ of degree d, such that*

- *the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, \mathrm{Kl}_3^{(2,1)})$ is spanned by $W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5$ with $\#W_1 = \#W_2 = \#W_3 = \#W_4 = \#W_5 = 1$,*
- *the middle de Rham cohomology $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Kl}_3^{(3,1)})$ is spanned by $W_4' \cup W_5'$ with $\#W_4' = \#W_5' = 1$.*

3.4.2 More examples of Hodge filtration

In the end, we give more examples of Hodge filtration on de Rham realization of motives attached to Kloosterman sheaves. The computations are almost identical to that in Lemma 3.45 and Proposition 3.47.

The case of the motive M_4^3

Let $f\colon \mathbb{G}_m^{10} \to \mathbb{A}^1$ be the function $(x_{1,i}, x_{2,i}, x_{3,i}, z) \mapsto \sum_{i=1}^3 x_{1,i} + x_{2,i} + x_{3,i} + \frac{z}{x_{1,i}x_{2,i}x_{3,i}}$. The group S_3 acts on the coordinate by $\sigma x_{i,j} = x_{\sigma(i),j}$. Under a map analogous to (3.43), elements in W_4' and W_6' are mapped to

$$g\frac{\mathrm{d}z}{z}\prod_{i,j}\frac{\mathrm{d}x_{i,j}}{x_{i,j}} \text{ and } h\frac{\mathrm{d}z}{z}\prod_{i,j}\frac{\mathrm{d}x_{i,j}}{x_{i,j}}$$

in $\mathrm{H}^{10}_{\mathrm{dR}}(\mathbb{G}_m^{10}, f)$, for polynomials g, h in $z, x_{i,j}$ of degree 4 and 6 respectively. Because f is non-degenerate with respect to $\Delta(f)$, we can use a similar argument as in the proof of Lemma 3.45 and Proposition 3.47 to do the computation

Proposition 3.57. *We have*

$$W_6' \subset \mathrm{gr}_F^4\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^3\mathrm{Kl}_4)$$

and

$$W_4' \subset \mathrm{gr}_F^6\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^3\mathrm{Kl}_4).$$

In particular, the nonzero Hodge numbers of the de Rham realization of $(\mathrm{M}_4^3)_{\mathrm{dR}}$ *are* $h^{4,6} = h^{6,4} = 1$.

The Hodge filtration for the motive M_5^3

Let $f\colon \mathbb{G}_m^{13} \to \mathbb{A}^1$ be the function $(x_{1,i}, x_{2,i}, x_{3,i}, x_{4,i}, z) \mapsto \sum_i x_{1,i} + x_{2,i} + x_{3,i} + x_{4,i} + \frac{z}{x_{1,i}x_{2,i}x_{3,i}x_{4,i}}$. The group S_3 acts on the coordinate by $\sigma x_{i,j} = x_{\sigma(i),j}$. Under a map analogous to (3.43), the elements in W_5' and W_8' are mapped to

$$g\frac{\mathrm{d}z}{z}\prod_{i,j}\frac{\mathrm{d}x_{i,j}}{x_{i,j}} \text{ and } h\frac{\mathrm{d}z}{z}\prod_{i,j}\frac{\mathrm{d}x_{i,j}}{x_{i,j}}$$

in $\mathrm{H}^{13}_{\mathrm{dR}}(\mathbb{G}_m^{13}, f)$, for some polynomial g, h in $z, x_{i,j}$ of degree 5 and 8 respectively. Because f is non-degenerate with respect to $\Delta(f)$, we can use a similar argument as that in the proof of Lemma 3.45 and Proposition 3.47 to do the computation.

Proposition 3.58. *We have*

$$W_8' \subset \mathrm{gr}_F^5\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^3\mathrm{Kl}_5)$$

and

$$W_5' \subset \mathrm{gr}_F^8\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_m, \mathrm{Sym}^3\mathrm{Kl}_5).$$

In particular, the nonzero Hodge numbers of the de Rham realization of $(\mathrm{M}_5^3)_{\mathrm{dR}}$ *are* $h^{5,8} = h^{8,5} = 1$.

The Hodge filtration of the motive $\mathrm{M}_3^{(2,1)}$

Let $f\colon \mathbb{G}_m^9 \to \mathbb{A}^1$ be the function $(x_i, y_i, z) \mapsto \sum_{i=1}^4 x_i + y_i + \frac{z}{x_i y_i}$. The permutation group S_4 acts on the subindices of the coordinates. Similar to Proposition 2.8, we have

$$\mathrm{H}^1_{\mathrm{dR}}\left(\mathbb{G}_m, \mathrm{Kl}_3^{(2,1)}\right) \simeq \mathrm{H}^9_{\mathrm{dR}}(\mathbb{G}_m^9, f)^{P\times Q,\chi}.$$

By considering an analogue of the inclusion (3.43), the elements in W_4' and W_5' are mapped to

$$g\frac{\mathrm{d}z}{z}\prod_{i,j}\frac{\mathrm{d}x_{i,j}}{x_{i,j}} \text{ and } h\frac{\mathrm{d}z}{z}\prod_{i,j}\frac{\mathrm{d}x_{i,j}}{x_{i,j}}$$

in $\mathrm{H}^9_{\mathrm{dR}}(\mathbb{G}_m^9, f)$, for some polynomials in z, x_i, y_i of degree 4 and 5 respectively.

Because f is non-degenerate with respect to $\Delta(f)$, we can use a similar argument as in the proof of Lemma 3.45 and Proposition 3.47 to get the Hodge numbers.

Proposition 3.59. *We have*

$$W_5' \subset \mathrm{gr}_F^4 \mathrm{H}^1_{\mathrm{dR,mid}}\Big(\mathbb{G}_m, \mathrm{Kl}_3^{(2,1)}\Big)$$

and

$$W_4' \subset \mathrm{gr}_F^5 \mathrm{H}^1_{\mathrm{dR,mid}}\Big(\mathbb{G}_m, \mathrm{Kl}_3^{(2,1)}\Big).$$

In particular, the nonzero Hodge numbers of the de Rham realization of the motive $\mathrm{M}_3^{(2,1)}$ *are* $h^{4,5} = h^{5,4} = 1$.

Chapter 4

Properties of Kloosterman sheaves

In this chapter, we will gather some results about some Kloosterman sheaves, mainly about sheaves appearing in Theorem A and B. For simplicity, we regard all Kloosterman sheaves as $\bar{\mathbb{Q}}_\ell$-sheaves.

We will first study the local structure of Kloosterman sheaves at 0 and ∞. Then we compute the dimensions of the ℓ-adic cohomologies of $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ in Proposition 4.27 when $p \nmid n+1$, and the dimensions of the ℓ-adic cohomologies of $\mathrm{Sym}^k\mathrm{Kl}_3$ in Corollary 4.40 when $p = 3$. In Section 4.5, we will prove Theorem E.

4.1 Notation

Let $p \neq \ell$ be two prime numbers, and $\zeta = \zeta_{n+1}$ be a primitive $(n+1)$-th root of unity in $\bar{\mathbb{F}}_p$.

(1). (a) We fix a nontrivial additive character $\psi_p \colon \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^\times$.

(b) If $k = \mathbb{F}_q$, we denote by ψ_q or ψ_k the additive character $\psi_p \circ \mathrm{Tr}_{k/\mathbb{F}_p}$.

(c) Let $\chi \colon \{\pm 1\} \to \bar{\mathbb{Q}}_\ell^\times$ be the unique nontrivial quadratic character, which can also be seen as characters $\chi_{\mathbb{F}_q} = \chi_q \colon \mathbb{F}_p^\times \to \{\pm 1\} \xrightarrow{\chi} \bar{\mathbb{Q}}_\ell^\times$.

(d) We also define $\theta_g \colon \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \to \bar{\mathbb{Q}}_\ell$ to be the character

$$\mathrm{Frob}_p \mapsto g(\chi, \psi_p) = -\sum_{a \in \mathbb{F}_p} \chi(a)^{-1}\psi_p(a).$$

(2). (a) The Artin-Schreier sheaf $\mathscr{L}_{\psi_q}$ is the lisse ℓ-adic sheaf of rank 1 on $\mathbb{A}^1_{\mathbb{F}_q}$ having trace function ψ_q.

(b) Let $[a] \colon \mathbb{A}^1_{\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q}$ be the multiplication by $a \in \mathbb{F}_q^\times$, and we denote by $\mathscr{L}_{\psi_q(at)}$ the sheaf $[a]^*\mathscr{L}_{\psi_q}$, where t is the coordinate of $\mathbb{A}^1_{\mathbb{F}_q}$.

(c) Let χ_q be the character as above. The Kummer sheaf $\mathscr{L}_{\chi_q}$ is a lisse ℓ-adic sheaf of rank 1 on $\mathbb{G}_{m,\mathbb{F}_q}$ whose trace function is χ_q.

(d) The pullback of θ_g along the structure morphism $\mathbb{P}^1_{\mathbb{F}_p} \to \mathrm{Spec}(\mathbb{F}_p)$ is a lisse sheaf of rank 1 on the projective line, denoted by $\mathscr{L}_{\theta_g}$.

(3). For multi-indices $\underline{I} \in \mathbb{N}^{n+1}$, we denote by

(a) $C_{\underline{I}} = \sum_i I_i \cdot \zeta^i$.

(b) $m_{\underline{I}} = \sum_{i=0}^n i \cdot I_i$.

(4). Let $v_0, \ldots, v_n$ be a basis of $\bar{\mathbb{Q}}_\ell^{n+1}$. We denote by $\sigma = (01\cdots n) \in S_{n+1}$, acting on v_i by $\sigma v_i := v_{\sigma(i)}$. For a multi-index $\underline{I} \in \mathbb{N}^{n+1}$, we denote by $v^{\underline{I}} = v_0^{I_0} \cdots v_n^{I_n}$.

(a) Let $d(k,n+1,p)$ be the cardinality of the set $A_k^0 := \{\underline{I} \mid |\underline{I}| = k,\ C_{\underline{I}} = 0\}$.

(b) We denote by $a(k,n+1,p)$ the cardinality of the dimension of the $\bar{\mathbb{Q}}_\ell$-vector space spanned by the set $\{\sum_i v^{\sigma_{\underline{I}}} \mid \underline{I} \in A_k^0\}$.

(c) We denote by $b(k,n+1,p)$ the cardinality of the $\bar{\mathbb{Q}}_\ell$-vector space spanned by the set $\{\sum_i (-1)^{I_i} v^{\sigma_{\underline{I}}} \mid \underline{I} \in A_k^0\}$.

(d) We denote by $d(k,n+1)$, $a(k,n+1)$ and $b(k,n+1)$ the minimal values of $d(k,n+1,p)$, $a(k,n+1,p)$ and $b(k,n+1,p)$ as p varies respectively.

(5). (a) Let C be the set of orbits of $\{1,\zeta,\dots,\zeta^n\}$ under the action of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$.

(b) For each $a \in \mathbb{Z}$, we denote by p_a the point $(\zeta^a,\dots,\zeta^a) \in \mathbb{G}_m^n(\bar{\mathbb{F}}_p)$.

(c) the field

$$F_{a,n} := \begin{cases} \mathbb{F}_p(\zeta^a)\left(\sqrt{(-1)^{\frac{n-1}{2}} \cdot (\frac{n+1}{2}) \cdot \zeta^a}\right) & 2 \nmid n, \\ \mathbb{F}_p(\zeta^a)\left(\sqrt{(-1)^{\frac{n}{2}} \cdot (n+1) \cdot \zeta^a}\right) & 2 \mid n. \end{cases}$$

(d) the character $\theta_{a,n} \colon \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p(\zeta^a)) \to \bar{\mathbb{Q}}_\ell^\times$ is trivial if $F_{a,n} = \mathbb{F}_p(\zeta^a)$, otherwise it is the unique nontrivial character of the Galois group of the quadratic extension $F_{a,n}$ over $\mathbb{F}_p(\zeta^a)$.

(e) the lisse ℓ-adic sheaves $\mathscr{L}_{\theta_{a,n}}$ on $\mathbb{P}^1_{\mathbb{F}_p(\zeta^a)}$, defined by the pull-back of the $\theta_{a,n}$ via the structure morphism $\mathbb{P}^1_{\mathbb{F}_p(\zeta^a)} \to \mathrm{Spec}(\mathbb{F}_p(\zeta^a))$.

4.2 The local structures of Kloosterman sheaves at 0

In this section, we assume that the base field k is the finite field $\mathbb{F}_q$ of characteristic p.

4.2.1 The Frobenius structure

Let $\mathbb{A}^1_{(0)} = \{s_0, \eta_0\}$ be the henselization of $\mathbb{A}^1_k$ at 0. The inertial group $I_{\bar{0}}$ acts on the generic fiber $(\mathrm{Kl}^\lambda_{n+1})_{\bar{\eta}_0}$. By a special case of [Kat88, 7.0.7], the generic fiber $V = (\mathrm{Kl}_{n+1})_{\bar{\eta}_0}$ is an ℓ-adic representation of $\mathrm{Gal}(\bar{\eta}_0/\eta_0)$, which is tamely ramified, and the inertia group $I_{\bar{0}}$ acts on V unipotently by a single Jordan block. Moreover, there exists a basis $\{v_0, v_1, \dots, v_n\}$ on which the nilpotent part of the monodromy operator $N \colon V \to V(-1)$ and Frob_0 act by

$$\mathrm{Frob}_0(v_i) = q^{n-i} v_i \quad \text{and} \quad N(v_i) = v_{i+1}$$

for $i = 0,\dots,n$ (for convenience, we let $v_{n+1} = 0$).

The sheaves $(\mathrm{Kl}^\lambda_{n+1})_{\bar{\eta}_0}$ can be identified with $(V^{\otimes|\lambda|})^{G_\lambda, \mathrm{id}\times\mathrm{sign}}$.

Symmetric power

Consider the symmetric power $V_k := \mathrm{Sym}^k V$. We can take the set

$$\{v^{\underline{I}} \mid \underline{I} \in \mathbb{N}^{n+1} \text{ and } |\underline{I}| = \sum_i I_i = k\}$$

as a basis of V_k. This basis is ordered by the lexicographical order. We define a filtration on V_k, indexed by $\underline{I}$, by setting $V_k^{\underline{I}} := \mathrm{Span}\{v^{\underline{J}} \mid \underline{I} \le \underline{J}\}$. We also denote by

$$V_k^{\underline{I},+} := \mathrm{Span}\{v^{\underline{J}} \mid \underline{I} < \underline{J}\} = \cup_{\underline{I}<\underline{J}} V_k^{\underline{J}}.$$

Let $m_{\underline{I}}$ be the integer $\sum_i i \cdot I_i$. The Frobenius and the monodromy operator N act by

$$\mathrm{Frob}_0(v^{\underline{I}}) = q^{nk-m_{\underline{I}}} v^{\underline{I}} \quad \text{and} \quad N(v^{\underline{I}}) = \Big(\sum_{i=0}^{n} I_i v_i^{-1} v_{i+1}\Big) v^{\underline{I}} \in V_k^{\underline{I},+}.$$

Combining the above discussion, we conclude that

$$V_k^{\underline{I}}/V_k^{\underline{I},+} \simeq \bar{\mathbb{Q}}_\ell(-nk + m_{\underline{I}}),$$

where $m_{\underline{I}} = \sum_{i=0}^{n} i \cdot I_i$. To summarize, we showed the following proposition.

Proposition 4.1. *There exists a decreasing filtration on* $(\mathrm{Sym}^k \mathrm{Kl}_{n+1})_{\bar{\eta}_0}$*, indexed by multi-indices* $\{\underline{I} \in \mathbb{N}^{n+1} \mid |\underline{I}| = k\}$*, such that the graded quotients at* $\underline{I}$ *are isomorphic to* $\bar{\mathbb{Q}}_\ell(-nk + m_{\underline{I}})$*.*

Two more examples

The representation $V_{2,1}$ and $V_{2,2}$ of SL_3 can be seen as subrepresentations of tensor products of symmetric powers of standard representations, see Section 1.1.3. It follows that $\mathrm{Kl}_3^{(2,1)}$ and $\mathrm{Kl}_3^{(2,2)}$ are subsheaves of $\mathrm{Kl}_3^{\otimes 4}$ and $\mathrm{Kl}_3^{\otimes 6}$ respectively. By a direct computation, we have the following proposition.

Proposition 4.2. *(1). The semisimplification of* $\mathrm{Kl}_3^{(2,1)}|_{\eta_0}$ *is isomorphic to*

$$\oplus_{i=1}^{7} \bar{\mathbb{Q}}_\ell(-i) \oplus \oplus_{i=2}^{6} \bar{\mathbb{Q}}_\ell(-i) \oplus \oplus_{i=3}^{5} \bar{\mathbb{Q}}_\ell(-i),$$

and as a Frob_0*-module, the* $I_{\bar{0}}$*-invariants of* $\mathrm{Kl}_3^{(2,1)}|_{\eta_0}$ *is isomorphic to*

$$\bar{\mathbb{Q}}_\ell(-1) \oplus \bar{\mathbb{Q}}_\ell(-2) \oplus \bar{\mathbb{Q}}_\ell(-3).$$

(2). As a Frob_0*-module, the semisimplification of* $\mathrm{Kl}_3^{(2,2)}|_{\eta_0}$ *is isomorphic to*

$$\oplus_{i=2}^{10} \bar{\mathbb{Q}}_\ell(-i) \oplus \oplus_{i=3}^{9} \bar{\mathbb{Q}}_\ell(-i) \oplus \oplus_{i=4}^{8} \bar{\mathbb{Q}}_\ell(-i)^{\oplus 2} \oplus \bar{\mathbb{Q}}_\ell(-6),$$

and the $I_{\bar{0}}$*-invariants of* $\mathrm{Kl}_3^{(2,2)}|_{\eta_0}$ *is isomorphic to*

$$\bar{\mathbb{Q}}_\ell(-2) \oplus \bar{\mathbb{Q}}_\ell(-3) \oplus \bar{\mathbb{Q}}_\ell(-4)^{\oplus 2} \oplus \bar{\mathbb{Q}}_\ell(-6).$$

4.2.2 The dimension of invariants of the inertia group

Observe that the local monodromy of $\mathrm{Kl}_{n+1}^{\lambda}|_{\eta_0}$ does not depend on the characteristic p of the base field $\mathbb{F}_p$. So the dimension of the $I_{\bar{0}}$-invariants of $\mathrm{Kl}_{n+1}^{\lambda}|_{\eta_0}$ does not depend on p.

In the case of symmetric power, the dimension of the $I_{\bar{0}}$-invariants of $(\mathrm{Sym}^k \mathrm{Kl}_{n+1})_{\bar{\eta}_0}$ are computed in [FW08, Thm 0.1].

Proposition 4.3 (Fu-Wan)**.** *As a* Frob_k*-module, the* $I_{\bar{0}}$*-invariants* $(\mathrm{Sym}^k \mathrm{Kl}_{n+1})_{\bar{\eta}_0}^{I_{\bar{0}}}$ *is isomorphic to* $\bigoplus_{u=0}^{\lfloor \frac{nk}{2} \rfloor} \bar{\mathbb{Q}}_\ell(-u)^{\oplus m_k(u)}$*, where* $m_k(u)$ *are numbers characterized by*

$$\sum_{u=0} m_k(u) x^u = \prod_{n+1}^{n+k} (1 - x^i) \cdot \prod_{2}^{k} (1 - x^i)^{-1}. \tag{4.4}$$

In particular, the dimension of $(\mathrm{Sym}^k \mathrm{Kl}_{n+1})_{\bar{\eta}_0}^{I_{\bar{0}}}$ *is* $\sum_{u=0}^{\lfloor \frac{nk}{2} \rfloor} m_k(u)$*.*

4.3 The local structures of Kloosterman sheaves at ∞

In [FW05], Fu and Wan computed the local structures of Kl_{n+1} at infinity over $\mathbb{F}_q$ for $n+1 \mid q-1$ up to a character. Later they completely determined the case when $n=2$, as well as the local structures of $\mathrm{Sym}^k\mathrm{Kl}_2$ over $\mathbb{F}_p$ in [FW10]. In this section, our main goal is to prove Propositions 4.23 and 4.25.

4.3.1 Local structures of the pullback $[n+1]^*\mathrm{Kl}_{n+1}$

We denote by π_a the morphism $\eta_\infty \otimes \mathbb{F}_p(\zeta^a) \to \eta_\infty$ for $a \in \mathbb{N}$.

Proposition 4.5. *Assume that $p \nmid n+1$. The pullbacks $[n+1]^*\mathrm{Kl}_{n+1}$ are isomorphic to*

$$\begin{cases} \bigoplus_{[a]\in C}(\pi_a)_*\left(\mathscr{L}_{\psi_{\mathbb{F}_p(\zeta^a)}((n+1)\zeta^a t)} \otimes \mathscr{L}_{\chi_{\mathbb{F}_p(\zeta^a)}} \otimes \mathscr{L}_{\theta_g} \otimes \mathscr{L}_{\theta_{a,n}}\right)\left(\frac{1-n}{2}\right)|_{\eta_\infty} & 2 \nmid nk \\ \bigoplus_{[a]\in C}(\pi_a)_*\left(\mathscr{L}_{\psi_{\mathbb{F}_p(\zeta^a)}((n+1)\zeta^a t)} \otimes \mathscr{L}_{\theta_{a,n}}\right)\left(\frac{-n}{2}\right)|_{\eta_\infty} & 2 \mid nk \end{cases}$$

as $\mathrm{Gal}(\bar\eta_\infty/\eta_\infty)$-modules, where $\theta_{a,n}$ are characters defined in Section 4.1.

Remark 4.6. We can get similar results for $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ by replacing g_{n+1} by $g_{n+1}^{\boxplus kb}$ in the proof below, which is again a nondegenerate Laurent polynomial. However, some of the characters $\theta_{a,n}$ might not be determined in this way. To determine these characters, we can use similar arguments as those in Lemma 4.19 and Lemma 4.20 for the sheaf $\mathrm{Sym}^3\mathrm{Kl}_4$.

Recall that by Lemma 2.12, we have

$$[n+1]^*\mathrm{Kl}_{n+1} \simeq \mathrm{FT}_{\psi_p}(\mathrm{R}^{n-1}g_!\bar{\mathbb{Q}}_\ell)|_{\mathbb{G}_m}, \tag{4.7}$$

where $g = g_{n+1}\colon \mathbb{G}_m^n \to \mathbb{A}^1_{\mathbb{F}_p}$ is the function $(x_i) \mapsto \sum_i y_i + \frac{1}{\prod_i y_i}$. For each a, we denote by p_a the point $(\zeta^a)_i \in \mathbb{G}_m^n(\bar{\mathbb{F}}_p)$. We can check that all critical points of g are exactly p_a.

Lemma 4.8. *We can find a compactification $(X, \tilde g\colon X \to \mathbb{A}^1)$ of the pair $(\mathbb{G}_m^n, g)$, such that*

(1). $\tilde g$ is proper, and is smooth along $X\backslash\mathbb{G}_m^n$,

(2). the complex $\mathrm{R}\left(\tilde g\,|_{X\backslash\mathbb{G}_m^n}\right)_*\bar{\mathbb{Q}}_\ell$ is constant,

(3). the complex $\mathrm{R}\tilde g_*\bar{\mathbb{Q}}_\ell$ is tame at ∞.

Proof. When $n=1$, the map g already satisfies the above requirements. We assume that $n \geq 2$ in the following. The Laurent polynomial g is nondegenerate with respect to its Newton Polytope $\Delta(g) := \mathrm{Covex}(\mathrm{Supp}(g))$. It is also convenient, i.e. 0 is an interior point of the Newton Polytope.

Following [DL91, (3.3) & (3.6)], we get a toroidal compactification $\tilde g\colon X \to \mathbb{A}^1$ of g. In fact, let $A = k[t]$. Since the Newton polytope of $g-t$ as a Laurent polynomial in coefficient A agrees with the Newton polytope of g, we have a toric compactification $X_A(\Delta(g-t)) \simeq X_k(\Delta(g)) \times \mathbb{A}^1_k$ of $\mathbb{G}^n_{m,A}$ with respect to the Newton polytope. For example, we can take Σ as a refinement of the dual fan of $\Delta(g)$, generated by the rays

$$\mathbb{R}_{\geq 0}\cdot\textstyle\sum_i \epsilon_i e_i$$

where $\epsilon_i \in \{0, \pm 1\}$ and $(\epsilon_i)_i \neq 0$. Then take X as the scheme-theoretic closure of $\{t=0\}$ inside $X_A(\Delta(g-t))$, and $\tilde g$ as the map $X \hookrightarrow X_k(\Delta(g)) \times \mathbb{A}^1_k \to \mathbb{A}^1_k$.

Each simplicial cone of maximal dimension n in Σ provides an affine chart of X, which is isomorphic to $\mathbb{A}^n_A = \mathbb{A}^n_k \times \mathbb{A}^1_{k,t}$. On each chart, the function g has the same structure. For example, we can consider the maximal cone generated by $\gamma_i := e_1 + \ldots + e_i$, where the affine ring associated with the dual cone is the polynomial ring $\mathbb{Q}[u_i]$ such that $u_i = \begin{cases} y_i/y_{i+1} & 1 \leq i < n, \\ y_n & i = n. \end{cases}$

In this chart X_1, we can rewrite $g-t$ as $g_1/\big(\prod_{1\leq i\leq n} u_i^i\big)$, where

$$g_1 = 1 + \prod_{1\leq j\leq n} u_i^i \cdot \Big(\sum_{e=1}^{n}\prod_{i\geq e} u_i - t\Big).$$

In particular, the intersection $X_1 \cap X$ is the zero locus of g_1 and the morphism $\tilde{g}$ is the composition $Z(g_1) \to X_1 \subset X_A(\Delta(g-t)) \to \mathrm{Spec}\,(A) = \mathbb{A}^1_t$. One can check that

$$Z(\partial g_1/\partial u_1) \cap Z(u_2) = Z(1) = \varnothing,$$

which implies that $\tilde{g}$ has no critical point outside the dense torus $\mathbb{T}^n \subset X_1 \cap X$. Hence, we deduce that $\tilde{g}$ is smooth along $X\backslash\mathbb{G}_m^n$.

Moreover, using (3.4), (3.9)(c) and (4.4) in loc. cit., the sheaf $\mathrm{R}(\tilde{g}\,|_{X\backslash\mathbb{G}_m^n})_*\bar{\mathbb{Q}}_\ell$ is constant, and the sheaf $\mathrm{R}\tilde{g}_*\bar{\mathbb{Q}}_\ell|_{\eta_\infty}$ is tame. □

Lemma 4.9. *We have an isomorphism of ℓ-adic sheaves on $\mathbb{G}_m$*

$$\mathrm{FT}_{\psi_p}(\mathrm{R}^{n-1}g_!\bar{\mathbb{Q}}_\ell)|_{\mathbb{G}_m} \simeq \mathrm{FT}_{\psi_p}(\mathrm{R}^{n-1}\tilde{g}_*\bar{\mathbb{Q}}_\ell)|_{\mathbb{G}_m}.$$

In particular, we can replace the map g by $\tilde{g}$ in (4.7).

Proof. Consider the distinguished triangle over $\mathbb{A}^1$

$$\mathrm{R}g_!\bar{\mathbb{Q}}_\ell \to \mathrm{R}\tilde{g}_*\bar{\mathbb{Q}}_\ell \to \mathrm{R}(\tilde{g}|_{X\backslash\mathbb{G}_m^n})_*\bar{\mathbb{Q}}_\ell \xrightarrow{+1}.$$

Applying the Deligne-Fourier transform, we have

$$\mathrm{FT}_{\psi_p}\left(\mathrm{R}g_!\bar{\mathbb{Q}}_\ell\right) \to \mathrm{FT}_{\psi_p}\left(\mathrm{R}\tilde{g}_*\bar{\mathbb{Q}}_\ell\right) \to \mathrm{FT}_{\psi_p}\left(\mathrm{R}(\tilde{g}|_{X\backslash\mathbb{G}_m^n})_*\bar{\mathbb{Q}}_\ell\right) \xrightarrow{+1}.$$

Since $\mathrm{R}(\tilde{g}|_{X\backslash\mathbb{G}_m^n})_*\bar{\mathbb{Q}}_\ell$ is constant, and the Deligne-Fourier transforms of constant sheaves are supported at the origin [Lau87, 1.2.2.2], we get the isomorphism by restricting the distinguished triangle to $\mathbb{G}_m$. □

Let p_a be the point $(\zeta^a)_i$. Since $g(p_a) = (n+1)\zeta^a$, the critical values of g are elements of the set

$$\{(n+1)\zeta^a \mid 0 \leq a \leq n\}.$$

We can check that each critical point p_a is ordinary quadratic, under the assumption $p \nmid n+1$. For each critical value $c = (n+1)\zeta^a$, we apply the Picard-Lefschetz formula [DK73, Exp. XV., Thm. 3.4] to $\tilde{g}\colon X \times_{\mathbb{A}^1} \mathbb{A}^1_{(c)} \to \mathbb{A}^1_{(c)}$, where $\mathbb{A}^1_{(c)} = \{s_c, \eta_c\}$ is the henselization of $\mathbb{A}^1$ at c. The generic fiber of $\tilde{g}$ is smooth, and the special fiber has one (ordinary quadratic) singular point p_a. So we get an exact sequence

$$0 \to \mathrm{H}^{n-1}_{\acute{e}t}(X_{\bar{s}_c}, \bar{\mathbb{Q}}_\ell) \xrightarrow{\varphi} \mathrm{H}^{n-1}_{\acute{e}t}(X_{\bar{\eta}_c}, \bar{\mathbb{Q}}_\ell) \xrightarrow{(\ ,\delta_a)} \bar{\mathbb{Q}}_\ell(-\lfloor\tfrac{nk}{2}\rfloor), \tag{4.10}$$

where δ_a is the vanishing cycle class attached to the critical points p_a.

Lemma 4.11. *The sequence (4.10) is right exact.*

Proof. The set of critical values of $\tilde{g}$ is a finite subset of $\bar{\mathbb{F}}_p$, on which the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ acts. We denote by $[c]$ the class of c under this action, and by G_c the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p(c))$. At each critical value, we denote by M_c the cokernel of the morphism φ. By (4.10), the dimension of M_c is at most 1.

By a modified version of the stationary phase principle [Kat90, Cor. 7.4.2], the $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$-module $[n+1]^*\mathrm{Kl}_{n+1}\mid_{\eta_\infty}$ is isomorphic to

$$\oplus_{[c]}\mathrm{Ind}_{G_c}^{G_{\mathbb{F}_p}}\left(\mathrm{FTloc}(0,\infty')\left(M_c\right)\otimes\mathscr{L}_{\psi(at)}\right)\oplus\mathrm{FTloc}(\infty,\infty')(\mathrm{R}^{n-1}\tilde{g}_*\bar{\mathbb{Q}}_\ell) \tag{4.12}$$

where $\mathrm{FTloc}(0,\infty')$ and $\mathrm{FTloc}(\infty,\infty')$ are the local Fourier transformations. Moreover, by [Lau87, Thm. 2.4.3(i)b) & (iii)b)], we know that

$$\mathrm{rk}\left(\mathrm{FTloc}(0,\infty')\left(M_c\right)\right)=\mathrm{rk}\left(M_c\right)$$

and

$$\mathrm{FTloc}(\infty,\infty')(\mathrm{R}\tilde{g}_!\bar{\mathbb{Q}}_\ell)=0.$$

Since $\mathrm{rk}(\mathrm{Kl}_{n+1})=(n+1)$, we know that

$$(n+1)=\mathrm{rk}\ \mathrm{Kl}_{n+1}=\sum_{c=(n+1)\zeta^a}\dim(M_c)_{\bar{\eta}_c}\leq(n+1).$$

It follows that the dimension of the generic fiber of M_c equals to 1. Therefore, the sequence (4.10) is right-exact. □

Now we want to determine the Galois action on M_c.

Lemma 4.13. *For a critical value $c=(n+1)\zeta^a$ the $I_{\bar{c}}$-module M_c is isomorphic to*

$$\begin{cases}\mathscr{L}_{\theta_{a,n}}\otimes\mathscr{L}_{\chi_{\mathbb{F}_p(\zeta^a)}}\Big|_{\eta_c}\left(-\frac{n-1}{2}\right) & 2\mid n\\ \mathscr{L}_{\theta_{a,n}}\Big|_{\eta_c}\left(-\frac{n}{2}\right) & 2\nmid n\end{cases},$$

Proof. Locally around p_a, the morphism $\tilde{g}$ is of the form

$$(n+1)\zeta^a+\zeta^{-a}\sum_{i,j}(x_i-\zeta^a)(x_j-\zeta^a)+\text{higher degree terms on }(x_i-\zeta^a).$$

Step 1: When n is odd. From the Picard-Lefschetz formula, the vanishing cycle class δ_a satisfy

$$(\delta_a,\delta_a)=(-1)^{\frac{n-1}{2}}2.$$

So the map $(\ ,\delta_a)$ admits a section. It follows that $\mathrm{H}^{n-1}_{\acute{e}t}(X_{\bar{\eta}_c},\bar{\mathbb{Q}}_\ell)$ is the direct sum of $\mathrm{H}^{nk-1}_{\acute{e}t}(X_{\bar{s}_c},\bar{\mathbb{Q}}_\ell)$ and $\bar{\mathbb{Q}}_\ell\delta_a$. Therefore, M_c is identified with $\bar{\mathbb{Q}}_\ell(-\frac{n-1}{2})\delta_a$.

Let π_c be the uniformizer of $\mathcal{O}(\mathbb{A}^1_{(c)})$. By [DK73, Exp. XV., Prop. 2.2.3], the Galois module $\bar{\mathbb{Q}}_\ell(-\frac{n-1}{2})\bar{\mathbb{Q}}_\ell\delta_x$ is identified with the cohomology $\mathrm{H}^{n-1}_{\acute{e}t,c}((D\backslash C)_{\bar{\mathbb{F}}_p((\pi_c))})$, where

$$D=Z\Big(-\pi_a\cdot w^2+\zeta^{-a}\sum_{i,j}y_iy_j\Big)\subset\mathbb{P}^n_{\mathbb{F}_p[[\pi_c]]}$$

and $C=D\cap Z(w)$. By the computation of cohomology of quadrics in projective spaces [DK73, Exp. XII., Thm. 3.3], then $\bar{\mathbb{Q}}_\ell(-\frac{n-1}{2})\delta_a$ is the primitive part of $\mathrm{H}^{n-1}_{\acute{e}t,c}(D_{\bar{\mathbb{F}}_p((\pi_c))})$, identified with the determinant $\det\mathrm{H}^{n-1}_{\acute{e}t,c}(D_{\bar{\mathbb{F}}_p((\pi_c))})$. According to [Sai12], the Galois action on $\det\left(\mathrm{H}^{n-1}_{\acute{e}t,c}(D_{\bar{\mathbb{F}}_p((\pi_c))})\left(\frac{n-1}{2}\right)\right)$ is given by the character corresponding to the quadratic extension

$$\mathbb{F}_p(\zeta^a)((\pi_c))\left(\sqrt{(-1)^{\frac{n-1}{2}}\cdot\pi_c\cdot\tfrac{(n+1)}{2^n}\cdot\zeta^{-nm_{\underline{I}}}}\right)\Big/\mathbb{F}_p(\zeta^a)((\pi_c)). \tag{4.14}$$

It follows that $\bar{\mathbb{Q}}_\ell(-\frac{n-1}{2})\delta_a \simeq \mathscr{L}_{\theta_{a,n}}\left(-\frac{n-1}{2}\right)\otimes \mathscr{L}_{\chi_{\mathbb{F}_p(\zeta^a)}}\big|_{\eta_c}$.

Step 2: When n is even. By the Picard-Lefschetz formula [DK73, Exp. XV. Thm. 3.4], the intersection pairing for the vanishing cycle classes δ_a is antisymmetric. By Lemma 4.11, the map $(\ ,\delta_a)$ is surjective. So we can find an element δ'_a being orthogonal to $\mathrm{H}^{n-1}_{\acute{e}t}(X_{\bar{s}_a},\bar{\mathbb{Q}}_\ell)$ such that $(\delta_a,\delta'_a)=1$. It follows that $\mathrm{H}^{n-1}_{\acute{e}t}(X_{\bar{\eta}_c},\bar{\mathbb{Q}}_\ell)$ is the extension of $\mathrm{H}^{n-1}_{\acute{e}t}(X_{\bar{s}_c},\bar{\mathbb{Q}}_\ell)$ and $\bar{\mathbb{Q}}_\ell(-\frac{n}{2})\delta'_a$. Therefore, M_c is identified with $\bar{\mathbb{Q}}_\ell(-\frac{n}{2})\delta'_a$.

Let π_c be the uniformizer of $\mathcal{O}(\mathbb{A}^1_{(c)})$. By [DK73, Exp. XV., Prop. 2.2.3], the Galois module $\bar{\mathbb{Q}}_\ell(-\frac{n-2}{2})\delta_a$ is identified with the cohomology $\mathrm{H}^{n-1}_{\acute{e}t,c}((D\backslash C)_{\bar{\mathbb{F}}_p((\pi_c))})$, where

$$D=Z\Big(-\pi_c\cdot w^2+\zeta^{-a}\sum_{i,j}y_iy_j\Big)\subset\mathbb{P}^n_{\mathbb{F}_p[[\pi_c]]}$$

and $C=D\cap Z(w)$. By the computation of cohomology of quadrics in projective spaces [DK73, Exp. XII., Thm. 3.3], $\bar{\mathbb{Q}}_\ell(-\frac{n-2}{2})\delta_a$ is the primitive part of $\mathrm{H}^{n-2}_{\acute{e}t,c}(C_{\bar{\mathbb{F}}_p((\pi_c))})$, identified with $\det \mathrm{H}^{n-2}_{\acute{e}t,c}(C_{\bar{\mathbb{F}}_p((\pi_c))})$. According to [Sai12], the Galois action on $\det\left(\mathrm{H}^{n-2}_{\acute{e}t,c}(C_{\bar{\mathbb{F}}_p((\pi_c))})(\frac{n-2}{2})\right)$ is given by the character corresponding to the quadratic extension

$$\mathbb{F}_p(\zeta^a)((\pi_c))\left(\sqrt{(-1)^{\frac{n}{2}}\cdot\tfrac{(n+1)}{2^n}\cdot\zeta^{-nm_{\underline{I}}}}\right)\Big/\mathbb{F}_p(\zeta^a)((\pi_c)). \tag{4.15}$$

Hence, $\bar{\mathbb{Q}}_\ell\left(-\frac{n-2}{2}\right)\delta_a\simeq\left(\mathscr{L}_{\theta_{a,n}}\right)\left(-\frac{n-2}{2}\right)\big|_{\eta_c}$. Since the intersection pairing is equivariant with respect to the Galois group G_c, it follows that $\bar{\mathbb{Q}}_\ell\left(-\frac{n}{2}\right)\delta'_a\simeq\left(\mathscr{L}_{\theta_{a,n}}\right)\left(-\frac{n}{2}\right)\big|_{\eta_c}$. □

Proof of Proposition 4.5. By [Lau87, 2.5.3.1], the local Fourier transform of $j_!\bar{\mathbb{Q}}_\ell$ and $j_!\mathscr{L}_{\chi_{\mathbb{F}_p(\zeta^a)}}|_{\eta_c}$ are given

$$\mathrm{FTloc}(0,\infty')(\bar{\mathbb{Q}}_\ell|_{\eta_c})=\bar{\mathbb{Q}}_\ell|_{\eta_\infty\otimes\mathbb{F}_p(\zeta^a)}$$

and

$$\mathrm{FTloc}(0,\infty')(\mathscr{L}_{\chi_{\mathbb{F}_p(\zeta^a)}})=\mathscr{L}_{\chi_{\mathbb{F}_p(\zeta^a)}}\otimes\mathscr{L}_{\theta_g}|_{\eta_\infty\otimes\mathbb{F}_p(\zeta^a)}.$$

Step 1: When n is odd. Combining (4.12) and the computation of local Fourier transformations, we have

$$\begin{aligned}
[n+1]^*\mathrm{Kl}_{n+1}|_{\eta_\infty}=&\bigoplus_{[c]}\mathrm{Ind}_{G_c}^{G_{\mathbb{F}_p}}\mathrm{FTloc}\left(0,\infty'\right)\left(M_c|_{\eta_c}\right)\otimes\mathscr{L}_{\psi_{\mathbb{F}_p(c)}(ct)}|_{\eta_\infty}\\
&\bigoplus_{[c]}\mathrm{Ind}_{G_c}^{G_{\mathbb{F}_p}}\mathrm{FTloc}\left(0,\infty'\right)\left(\mathscr{L}_{\chi_{\mathbb{F}_p(\zeta^a)}}\right)\otimes\mathscr{L}_{\psi_{\mathbb{F}_p(c)}(ct)}\otimes\mathscr{L}_{\theta_{a,n}}\left(\tfrac{1-n}{2}\right)|_{\eta_\infty}\\
=&\bigoplus_{[a]\in C}(\pi_a)_*\left(\mathscr{L}_{\psi_{\mathbb{F}_p(\zeta^a)}((n+1)\zeta^at)}\otimes\mathscr{L}_{\chi_{\mathbb{F}_p(\zeta^a)}}\otimes\mathscr{L}_{\theta_g}\otimes\mathscr{L}_{\theta_{a,n}}\right)\left(\tfrac{1-n}{2}\right)|_{\eta_\infty}.
\end{aligned}$$

Step 2: When nk is even. Similarly, we have

$$\begin{aligned}
[n+1]^*\mathrm{Kl}_{n+1}|_{\eta_\infty}=&\bigoplus_{[c]}\mathrm{Ind}_{G_c}^{G_{\mathbb{F}_p}}\mathrm{FTloc}\left(0,\infty'\right)(\bar{\mathbb{Q}}_\ell)\otimes\mathscr{L}_{\psi_{\mathbb{F}_p(c)}(ct)}\otimes\mathscr{L}_{\theta_{a,n}}\left(\tfrac{-n}{2}\right)|_{\eta_\infty}\\
=&\bigoplus_{[a]\in C}(\pi_a)_*\left(\mathscr{L}_{\psi_{\mathbb{F}_p(\zeta^a)}((n+1)\zeta^at)}\otimes\mathscr{L}_{\theta_{a,n}}\right)\left(\tfrac{-n}{2}\right)|_{\eta_\infty}.
\end{aligned}$$

□

4.3.2 Local structures of some $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$

The case of Kl_{n+1}

Lemma 4.16. *If $p \nmid n+1$, the $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$-module $\mathrm{Kl}_{n+1}|_{\eta_\infty}\left(\lfloor\frac{n}{2}\rfloor\right)$ is isomorphic to*

$$\begin{cases} [n+1]_*\left(\mathscr{L}_{\psi((n+1)t)}\otimes\mathscr{L}_\chi\otimes\mathscr{L}_{\theta_g\cdot\theta_{0,n}}\right)\Big|_{\eta_\infty} & 2\nmid n \\ [n+1]_*\left(\mathscr{L}_{\psi((n+1)t)}\otimes\mathscr{L}_{\theta_{0,n}}\right)\Big|_{\eta_\infty} & 2\mid n \end{cases}$$

where $\theta_{0,n}$ is the character defined in Section 4.1.

Proof. We can view these representations of $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$ as representations of Galois groups of some finite extensions, and view $[n+1]^*$, $[n+1]_*$ as restrictions and inductions of representations of finite groups.

Let $\mathscr{F}$ be $\mathscr{L}_{\psi_p((n+1)t)}\otimes\mathscr{L}_\chi\otimes\mathscr{L}_{\theta_g\cdot\theta_{0,n}}$ if $2\nmid n$ and be $\mathscr{L}_{\psi_p((n+1)t)}\otimes\mathscr{L}_{\theta_{0,n}}$ if $2\mid n$. By Frobenius reciprocity [Ser77, §7.2 Thm. 13], we have

$$1\le\left\langle[n+1]^*\mathrm{Kl}_{n+1}|_{\eta_\infty},\mathscr{F}\left(-\lfloor\tfrac{n}{2}\rfloor\right)\Big|_{\eta_\infty}\right\rangle=\left\langle\mathrm{Kl}_{n+1}|_{\eta_\infty},[n+1]_*\mathscr{F}\left(-\lfloor\tfrac{n}{2}\rfloor\right)\Big|_{\eta_\infty}\right\rangle. \tag{4.17}$$

By [Kat88, 1.13.2], the representation $[n+1]_*\mathscr{F}|_{\eta_\infty}(-\lfloor\frac{n}{2}\rfloor)$ is irreducible, because it is totally wild and its Swan conductor is 1. Since $[n+1]_*\mathscr{F}|_{\eta_\infty}(-\lfloor\frac{n}{2}\rfloor)$ is of rank $n+1$, it coincides with $\mathrm{Kl}_{n+1}|_{\eta_\infty}$. □

Remark 4.18. We can also deduce this Lemma by [Fu10, Thm. 0.1].

The case of $\mathrm{Sym}^3\mathrm{Kl}_4$

Let ζ_4 be a primitive forth root of unity, and we choose ζ_8 such that $\zeta_8^2=\zeta_4$.

Lemma 4.19. *Let q be the cardinality of $\mathbb{F}_p(\zeta_8)$. Assume that $p\neq 2,3,5$. Then $\mathrm{Sym}^3\mathrm{Kl}_4|_{\eta_\infty\otimes\mathbb{F}_q}(4)$ is isomorphic to*

$$\begin{aligned}&[4]_*\left(\mathscr{L}_{\psi(12t)}\otimes\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\theta_g}\right)\oplus[4]_*\left(\mathscr{L}_{\psi(-4t)}\otimes\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\theta_g}\right)^{\oplus 2}\\ \oplus&[4]_*\left(\mathscr{L}_{\psi((8+4\zeta_4)t)}\otimes\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\theta_g}\right)\oplus[4]_*\left(\mathscr{L}_{\psi((8-4\zeta_4)t)}\otimes\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\theta_g}\right).\end{aligned}$$

as $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty\otimes\mathbb{F}_q)$-modules.

Proof. **Step 1**: Let $\mathbb{F}_q(z)$ be the extension $\mathbb{F}_q(t)[z]/(z^4-t)$ of $k(t)$, and $\mathbb{F}_q(y,w)$ be the extension $\mathbb{F}_q(z)[y,w]/(y^q-y-z,w^2-z)$ of $\mathbb{F}_q(z)$. The Galois group $H=\mathrm{Gal}(k(y,w)/k(z))$ is isomorphic to $\mathbb{F}_q\times\mathbb{Z}/2\mathbb{Z}$. For each $(a,\mu)\in\mathbb{F}_q\times\mathbb{Z}/2\mathbb{Z}$, we denote by $g_{a,\mu}$ the corresponding element in H. The action of $g_{a,\mu}$ on $k[y,w]$ is given by

$$g_{a,\mu}\cdot y=y-a \text{ and } g_{a,\mu}\cdot w=\mu w.$$

Notice that H is a normal subgroup of $G=\mathrm{Gal}(k(y,w)/k(t))$, and the quotient

$$G/H=\mathrm{Gal}(\mathbb{F}_q[z]/\mathbb{F}_q[t])=\mathbb{Z}/4\mathbb{Z}.$$

We choose an element $g\in G$ such that $g\cdot z=\zeta_4 z$. It follows that the image of g in G/H is 1.

Let W be a one-dimensional $\bar{\mathbb{Q}}_\ell$-vector space and choose v_0 as a basis. We define the action of H on W by

$$g_{a,\mu}\cdot v_0=\psi_q(-4a)\chi_q(\mu^{-1})v_0.$$

By the construction, W is isomorphic to $\left(\mathscr{L}_{\psi_q(4t)}\otimes\mathscr{L}_{\chi_q}\right)\big|_{\eta_\infty\otimes\mathbb{F}_q}$.

Consider the vector space

$$V:=\mathrm{Ind}_H^G W=\oplus_{g=0}^3 g^iW,$$

which is identified with $[4]_*\left(\mathscr{L}_{\psi_q(4t)}\otimes\mathscr{L}_{\chi_q}\right)\big|_{\eta_\infty\otimes\mathbb{F}_q}$. Let $v_i := g^i v_0$ for $i=0,\dots,3$. The set $\{v_i\}$ form a basis of V, and the action of G on V is given by

$$\begin{aligned} g_{a,\mu}\cdot v_i &= \psi(-4\zeta_4^i a)\chi_q(\mu^{-1})v_i \\ g\cdot v_i &= v_{i+1}\quad (v_{n+1}:=-v_0). \end{aligned}$$

Step 2: The symmetric power $V_3 := \mathrm{Sym}^3 V$ of the vector space V with the induced Galois action is identified with $\mathrm{Sym}^k[4]_*\left(\mathscr{L}_{\psi_q(4t)}\otimes\mathscr{L}_{\chi}\right)|_{\eta_\infty\otimes\mathbb{F}_q}$. We choose the set $\{v^{\underline{I}} \mid \underline{I}\in\mathbb{N}^4 \text{ and } |\underline{I}|=3\}$ as a basis of V_3. With respect to this basis, the action of G on V_3 is given by

$$\begin{aligned} g_{a,\mu}\cdot v^{\underline{I}} &= \psi(-4C_{\underline{I}})\chi_q(\mu^{-1})v^{\underline{I}}, \\ g\cdot v^{\underline{I}} &= v^{\sigma(\underline{I})} \end{aligned}$$

where $\sigma=(0123)\in S_4$ acts on multi-indices by permuting the sub-indexes.

Now for each $\underline{I}$, the set $\{g^i v^{\underline{I}} \mid 0\le i\le 3\}$ is the basis of

$$[4]_*\big(\mathscr{L}_{\psi_q(4C_{\underline{I}}t)\otimes\mathscr{L}_{\chi_q}}\big)|_{\eta_\infty\otimes\mathbb{F}_q},$$

which only depends on the classes of $\underline{I}$ modulo the action of σ. We divide $\{v^{\underline{I}} \mid \underline{I}\in\mathbb{N}^4 \text{ and } |\underline{I}|=3\}$ properly into 5 sets corresponding to $\{(1,0,0,0),(0,1,1,1),(1,0,2,0),(2,1,0,0),(2,0,0,1)\}$. Adding back the Tate twist, we get the expression in the lemma. □

Lemma 4.20. *Let π be the map $\eta_\infty\otimes\mathbb{F}_p(\zeta_4)\to\eta_\infty$ and assume that $p\neq 2,3,5$.*

(1). If $p\equiv 1\,(\mathrm{mod}\,4)$, the $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$-module $\mathrm{Sym}^3\mathrm{Kl}_4(4)|_{\eta_\infty}$ is

$$\begin{aligned} &[4]_*\big(\mathscr{L}_{\psi_p(12t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\theta_{\eta_1}}\big)\oplus[4]_*\big(\mathscr{L}_{\psi_p(-4t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\theta_{\eta_2}}\big)\\ \oplus&[4]_*\big(\mathscr{L}_{\psi_p(-4t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\theta_{\eta_3}}\big)\oplus[4]_*\big(\mathscr{L}_{\psi_p((8+4\zeta_4)t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\theta_{\eta_4}}\big).\\ \oplus&[4]_*\big(\mathscr{L}_{\psi_p((8-4\zeta_4)t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\theta_{\eta_5}}\big) \end{aligned}$$

where η_i are some characters of $\mathrm{Gal}(\mathbb{F}_p(\zeta_8)/\mathbb{F}_p)$.

(2). If $p\equiv -1\,(\mathrm{mod}\,4)$, the $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$-module $\mathrm{Sym}^3\mathrm{Kl}_4(4)|_{\eta_\infty}$ is

$$\begin{aligned} &[4]_*\big(\mathscr{L}_{\psi_p(12t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\theta_{\eta_1}}\big)\oplus[4]_*\big(\mathscr{L}_{\psi_p(-4t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\theta_{\eta_2}}\big)\\ \oplus&[4]_*\big(\mathscr{L}_{\psi_p(-4t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\eta_3}\big)\oplus\pi_*[4]_*\big(\mathscr{L}_{\psi_{p^2}((8+4\zeta_4)t)}\otimes\mathscr{L}_{\chi_{p^2}}\otimes\mathscr{L}_{\theta_g\cdot\theta_{\eta_4}}\big), \end{aligned}$$

where η_i are some characters of $\mathrm{Gal}(\mathbb{F}_p(\zeta_8)/\mathbb{F}_p)$.

Proof. If $\underline{I}$ is a multi-index such that $C_{\underline{I}}\in\mathbb{F}_p$, then $[4]_*(\mathscr{L}_{\psi_p(4C_{\underline{I}}t)})|_{\eta_\infty}$ is an irreducible representation of $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$, because $[4]_*(\mathscr{L}_{\psi_p(4C_{\underline{I}}t)})|_{\eta_\infty}$ is totally wild with Swan conductor 1. If $\underline{I}$ is a multi-index such that $C_{\underline{I}}\zeta_4^i\notin\mathbb{F}_p$ for any $0\le i\le 3$, for similar reason, $[4]_*(\mathscr{L}_{\psi_p(4C_{\underline{I}}t)})|_{\eta_\infty}$ is an irreducible representation of $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty\otimes\mathbb{F}_p(\zeta_4))$. By the property of induced representations [Ser87, Prop. 24 in p.61], the representation

$$\pi_*[4]_*(\mathscr{L}_{\psi_p(4C_{\underline{I}}t)})|_{\eta_\infty} \tag{4.21}$$

is also irreducible.

Let $\tilde{\pi}$ be the map $\eta_\infty\otimes\mathbb{F}_p(\zeta_8)\to\eta_\infty$, and $\mathscr{F}$ be one of the direct summands in the lemma but without the characters η_i (for example $\mathscr{F}=[4]_*\big(\mathscr{L}_{\psi_p(12t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g}\big)$). Notice that $\mathscr{F}$ is irreducible. Using the Frobenius reciprocity,

$$1\le\left\langle\tilde{\pi}^*\mathscr{F},\tilde{\pi}^*(\mathrm{Sym}^3\mathrm{Kl}_4(4)|_{\eta_\infty})\right\rangle=\langle\tilde{\pi}_*\tilde{\pi}^*\mathscr{F},\mathrm{Sym}^3\mathrm{Kl}_4(4)|_{\eta_\infty}\rangle.$$

Notice that $\tilde{\pi}_*\tilde{\pi}^*\mathscr{F}=\oplus_\eta\mathscr{F}\otimes\mathscr{L}_\eta$, where η runs through all characters of $\mathrm{Gal}(\mathbb{F}_p(\zeta_8)/\mathbb{F}_p)$. We conclude that for each $\mathscr{F}$, there exists an η such that $\mathscr{F}\otimes\mathscr{L}_\eta$ is a direct summand of $\mathrm{Sym}^3\mathrm{Kl}_4(4)|_{\eta_\infty}$.

Hence, for some appropriate characters η_i, the direct sum in the lemma is a subrepresentation of $\mathrm{Sym}^3\mathrm{Kl}_4|_{\eta_\infty}(4)$. So the direct sums in the lemma is a submodule of $\mathrm{Sym}^k\mathrm{Kl}_4$ and have the same ranks as $\mathrm{Sym}^k\mathrm{Kl}_4$. Therefore, we have an equality. □

Now it suffices to specify these characters η_i to finish conclude the local structure of $\mathrm{Sym}^3\mathrm{Kl}_4$ at ∞.

Lemma 4.22. *Let $k \geq 1$ be an integer and $q = p^2$. Assume that $p \equiv -1 (\mathrm{mod}\, 4)$ and $p > k$. We denote by π the map $\eta \otimes \mathbb{F}_p(\zeta_4) \to \eta_\infty$. Then $\mathrm{Sym}^k \pi_* \big(\mathcal{L}_{\psi_{\mathbb{F}_p(\zeta_4)}(4\zeta_4 t)} \otimes \mathscr{L}_\chi\big)$ are isomorphic to*

$$\begin{cases} \bigoplus_{2b \leq k} \pi_* \big(\mathscr{L}_{\psi_{\mathbb{F}_p(\zeta_4)}(4(2b-k)\zeta_4 t)} \otimes \mathscr{L}_{\chi_q^k|_{\eta_\infty \otimes \mathbb{F}_q}}\big) & 2 \nmid k \\ \bigoplus_{2b < k} \pi_* \big(\mathscr{L}_{\psi_{\mathbb{F}_p(\zeta_4)}(4(2b-k)\zeta_4 t)} \otimes \mathscr{L}_{\chi_q^k|_{\eta_\infty \otimes \mathbb{F}_q}}\big) \oplus \bar{\mathbb{Q}}_\ell & 2 \mid k \end{cases}$$

as $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$-modules.

Proof. **Step 1**: Consider the extension

$$\mathbb{F}_p(\zeta_4)(y,w) = \mathbb{F}_p(\zeta_4)(z)[y,w]/(y^q - y - z, w^2 - z)$$

of $\mathbb{F}_p(\zeta_4)(z)$. The Galois group $H = \mathrm{Gal}(\mathbb{F}_p(\zeta_4)(y,w)/\mathbb{F}_p(\zeta_4)(z))$ is isomorphic to $\mathbb{F}_q \times \{\pm 1\}$. For each $(a,\mu) \in \mathbb{F}_q \times \{\pm 1\}$, we denote by $g_{a,\mu}$ the element in H, such that

$$g_{a,\mu} \cdot y = y - a, \text{ and } g_{a,\mu} \cdot w = \mu w.$$

Notice that H is a normal subgroup of $G = \mathrm{Gal}(\mathbb{F}_p(\zeta_4)(y,w)/\mathbb{F}_p(z))$, and the quotient

$$G/H = \mathrm{Gal}(\mathbb{F}_p(\zeta_4)[z]/\mathbb{F}_p[z]) = \mathbb{Z}/2\mathbb{Z}.$$

We choose an element $g \in G$ such that $g \cdot a = a^q$ for $a \in k$. It follows that $g^2 = g_{0,0}$.

Let W be a one-dimensional $\bar{\mathbb{Q}}_\ell$-vector space and choose e_1 as a basis. We define the action of H on W by

$$g_{a,\mu,\mu'} \cdot v_1 = \psi_q(-4\zeta_4 a)\chi_q(\mu^{-1})v_0.$$

By the construction, W is isomorphic to $\mathscr{L}_{\psi_q(4\zeta_4 t)} \otimes \mathscr{L}_{\chi_q}\big|_{\eta_\infty \otimes \mathbb{F}_q}$.

Consider the vector space

$$V := \mathrm{Ind}_H^G W = \oplus_{i=0}^1 g^i W,$$

which is identified with $[\pi]_*\big(\mathscr{L}_{\psi_q(4\zeta_4 t)} \otimes \mathscr{L}_{\chi_q}\big|_{\eta_\infty \otimes \mathbb{F}_q}\big)$. Let $v_3 := gv_1$. The set $\{v_1, v_3\}$ form a basis of V. The action of H on v_3 is given by

$$g_{a,\mu} \cdot v_3 = g \cdot g^{-1} g_{a,\mu} g \cdot v_1 = g \cdot g_{a^p,\mu} \cdot v_1 = \psi_q(-4\zeta_4 a^p)\chi_q(\mu^{-1})v_3 = \psi_q(-4\zeta_4 a)\chi_q(\mu^{-1})v_3,$$

and the action of g on V is given by

$$gv_1 = v_3 \text{ and } gv_3 = v_1.$$

Step 2: The symmetric power $V_k := \mathrm{Sym}^k V$ of the vector space V with the induced Galois action is identified with $\mathrm{Sym}^k[\pi]_*\left(\mathscr{L}_{\psi_q(4\zeta_4 t)} \otimes \mathscr{L}_{\chi_q|_{\eta_\infty \otimes \mathbb{F}_q}}\right)$. We choose the set $\{v_1^b v_3^{k-b} \mid 0 \leq b \leq k\}$ as a basis of V_k. In particular, the action of G is

$$\begin{aligned} g_{a,\mu} \cdot v_1^b v_3^{k-b} &= \psi(-4\zeta_4 a(2b-k))\chi_q^k(\mu^{-1})v_1^b v_3^{k-b}, \\ g \cdot v_1^b v_3^{k-b} &= v_1^{k-b} v_3^b. \end{aligned}$$

Now we have the following cases:

- If $2b - k \neq 0$, the set $\{v_1^b v_3^{k-b}, g(v_1^b v_3^{k-b})\}$ is the basis of

$$[\pi]_*\big(\mathscr{L}_{\psi_q(4\zeta_4(2b-k)t)} \otimes \mathscr{L}_{\chi_q^k|_{\eta_\infty \otimes \mathbb{F}_q}}\big) = [\pi]_*\big(\mathscr{L}_{\psi_q(4C_{(0,b,0,k-b)}t)} \otimes \mathscr{L}_{\chi_q^k|_{\eta_\infty \otimes \mathbb{F}_q}}\big),$$

  where $\zeta_4 a(2b-k)$ is exactly $C_{(0,b,0,k-b)}$.

- If $2b = k$, the action G on the single element $v_1^b v_3^b$ is trivial. So this element generates $\bar{\mathbb{Q}}_\ell|_{\eta_\infty \otimes \mathbb{F}_p}$;

So we get a direct summand $\pi_*\big(\mathscr{L}_{\psi_{\mathbb{F}_p(\zeta_4)}(4(2b-k)\zeta_4 t)} \otimes \mathscr{L}_{\chi_q^k}|_{\eta_\infty \otimes \mathbb{F}_q}\big)$ for each $b \leq k/2$, and a direct summand $\bar{\mathbb{Q}}_\ell$ for $b = k/2$. Since the direct sums in the Lemma has rank $k+1$, and is a submodule of $\mathrm{Sym}^k \pi_*\big(\mathcal{L}_{\psi_{\mathbb{F}_p(\zeta_4)}(4\zeta_4 t)} \otimes \mathscr{L}_\chi\big)$, we have an inclusion up to some Tate twists. □

Proposition 4.23. *Assume that $p \neq 2, 3, 5$. We denote by η_1, η_2, η_3 characters of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ such that $\eta_1(\mathrm{Frob}_p) = \eta_2(\mathrm{Frob}_p) = \left(\frac{2}{p}\right)$ and $\eta_3(\mathrm{Frob}_p) = \left(\frac{-2}{p}\right)$ respectively.*

(1). If $p \equiv 1 \pmod 4$, the $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$-module $\mathrm{Sym}^3\mathrm{Kl}_4(4)|_{\eta_\infty}$ is

$$\begin{aligned}&[4]_*\big(\mathscr{L}_{\psi_p(12t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g \cdot \eta_1}\big) \oplus [4]_*\big(\mathscr{L}_{\psi_p(-4t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g \cdot \eta_2}\big)\\ \oplus&[4]_*\big(\mathscr{L}_{\psi_p(-4t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g \cdot \eta_3}\big) \oplus [4]_*\big(\mathscr{L}_{\psi_p((8+4\zeta_4)t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g}\big).\\ \oplus&[4]_*\big(\mathscr{L}_{\psi_p((8-4\zeta_4)t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g}\big).\end{aligned}$$

(2). If $p \equiv -1 \pmod 4$, the $\mathrm{Gal}(\bar{\eta}_\infty/\eta_\infty)$-module $\mathrm{Sym}^3\mathrm{Kl}_4(4)|_{\eta_\infty}$ is

$$\begin{aligned}&[4]_*\big(\mathscr{L}_{\psi_p(12t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g \cdot \eta_1}\big) \oplus [4]_*\big(\mathscr{L}_{\psi_p(-4t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g \cdot \eta_2}\big)\\ \oplus&[4]_*\big(\mathscr{L}_{\psi_p(-4t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g \cdot \eta_3}\big) \oplus \pi_*[4]_*\big(\mathscr{L}_{\psi_{p^2}((8+4\zeta_4)t)} \otimes \mathscr{L}_{\chi_{p^2}} \otimes \mathscr{L}_{\theta_g}\big).\end{aligned}$$

Proof. We give the proof when $p \equiv -1 (\mathrm{mod}\, 4)$. The proof of the other case is similar. Let π be the map $\mathrm{Spec}\,\mathbb{F}_p(\zeta_4)((t^{-1})) \to \mathrm{Spec}\,\mathbb{F}_p((t^{-1}))$. Let $\mathscr{F}$ one of the components in the decomposition in Lemma 4.20. It suffices to specify these characters η_i.

Notice that we have inclusions

$$\mathscr{L}_{\psi_p(at)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_\eta \subset [4]^*[4]_*\left(\mathscr{L}_{\psi_p(at)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_\eta\right)$$

and

$$\pi_*\big(\mathscr{L}_{\psi_{p^2}(at)} \otimes \mathscr{L}_{\chi_{p^2}} \otimes \mathscr{L}_\eta\big) \subset [4]^*\pi_*[4]_*\big(\mathscr{L}_{\psi_{p^2}(at)} \otimes \mathscr{L}_{\chi_{p^2}} \otimes \mathscr{L}_\eta\big).$$

So to determine these characters, we can search terms of the form

$$\mathscr{L}_{\psi_p(at)} \otimes \mathscr{L}_\chi \otimes \mathscr{L}_\eta \text{ or } \pi_*(\mathscr{L}_{\psi_{p^2}(at)} \otimes \mathscr{L}_\chi \otimes \mathscr{L}_\eta) \tag{4.24}$$

for some characters η within the expression of $[4]^*\mathrm{Sym}^3\mathrm{Kl}_4(4)$ for $a = 12, -4$ and $8 \pm 4\zeta_4$.

Since $[4]^*\mathrm{Kl}_4(1)$ is the direct sum of

$$\mathscr{F}_1 := \mathscr{L}_{\psi_p(4t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g \cdot \theta_{0,3}} \oplus \mathscr{L}_{\psi_p(-4t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g \cdot \theta_{2,3}}$$

and

$$\mathscr{F}_2 := \pi_*\big(\mathscr{L}_{\psi_{p^2}(4\zeta_4 t)} \otimes \mathscr{L}_{\chi_{p^2}} \otimes \mathscr{L}_{\theta_g}\big) \simeq \pi_*\big(\mathscr{L}_{\psi_{p^2}(4\zeta_4 t)} \otimes \mathscr{L}_{\chi_{p^2}}\big) \otimes \mathscr{L}_{\theta_g},$$

where we used the projection formula and the fact that $\theta_{1,3}$ is trivial. So the symmetric power $[4]^*(\mathrm{Sym}^3\mathrm{Kl}_4(1)) = \mathrm{Sym}^3([4]^*\mathrm{Kl}_4(1))$ is direct sum

$$\mathrm{Sym}^3\mathscr{F}_1 \oplus \mathrm{Sym}^2\mathscr{F}_1 \otimes \mathscr{F}_2 \oplus \mathscr{F}_1 \otimes \mathrm{Sym}^2\mathscr{F}_2 \oplus \mathrm{Sym}^3\mathscr{F}_2.$$

Combining Lemma 4.22, and use the projection formula, we find direct sums of the form 4.24 as follows:

- only one direct summand $\mathscr{L}_{\psi(12t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g^3 \cdot \theta_{0,3}^3}$ corresponding to $a = 12$,
- two direct summands $\mathscr{L}_{\psi_p(-4t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g^3 \cdot \theta_{0,3} \cdot \theta_{2,3}^2}$ and $\mathscr{L}_{\psi_p(-4t)} \otimes \mathscr{L}_{\chi_p} \otimes \mathscr{L}_{\theta_g^3 \cdot \theta_{2,3}}$ for $a = -4$,
- only one direct summand $\pi_*\big(\mathscr{L}_{\psi_{p^2}((8+4\zeta_4)t)} \otimes \mathscr{L}_{\chi_{p^2}} \otimes \mathscr{L}_{\theta_g^3 \cdot \theta_{0,3}^2 \cdot \theta_{1,3}}\big)$ for $a = 8 + 4\zeta_4$.

Notice that the characters $\theta_{0,3}$ and $\theta_{2,3}$ send Frob_p to $\left(\frac{-2}{p}\right)$ and $\left(\frac{2}{p}\right)$ respectively. We can check that $\theta_g^2 \cdot \theta_{0,3}^3 = \eta_1$, $\theta_g^2 \cdot \theta_{0,3} \cdot \theta_{2,3}^2 = \eta_2$, and $\theta_g^2 \cdot \theta_{2,3} = \eta_3$. So the characters η_i are determined. □

4.3.3 More examples

By a similar argument in Lemma 4.19, and use the description of irreducible representations of SL_3 in (1.1), we have the local structure of the sheaves $\mathrm{Kl}_3^{(2,1)}$ and $\mathrm{Kl}_3^{(2,2)}$ over $\mathbb{F}_p(\zeta_3)$.

Proposition 4.25. *Let p be a prime, and ψ be the character $\psi_p \circ \mathrm{Tr}_{\mathbb{F}_p(\zeta_3)/\mathbb{F}_p}$.*

(1). If $p \neq 3$, the $\mathrm{Gal}(\bar{\eta}_\infty/\eta \otimes \mathbb{F}_p(\zeta_3))$-module $\mathrm{Kl}_3^{(2,1)}|_{\eta_\infty \otimes \mathbb{F}_p(\zeta_3)}(4)$ is

$$[3]_*\mathscr{L}_{\psi(3t)}^{\oplus 2} \oplus [3]_*\mathscr{L}_{\psi(-6t)} \oplus [3]_*\mathscr{L}_{\psi(3(3+2\zeta_3)t)} \oplus [3]_*\mathscr{L}_{\psi(3(3-2\zeta_3)t)}.$$

In particular, the Swan conductor of $\mathrm{Kl}_3^{(2,1)}$ at ∞ is 5 if $p \neq 2, 3, 7$, is 4 if $p = 2$ and 3 if $p = 7$.

(2). If $p \neq 2, 3$, the $\mathrm{Gal}(\bar{\eta}_\infty/\eta \otimes \mathbb{F}_p(\zeta_3))$-module $\mathrm{Kl}_3^{(2,2)}|_{\eta_\infty \otimes \mathbb{F}_p(\zeta_3)}(6)$ is

$$\begin{aligned}[3]_*\bar{\mathbb{Q}}_\ell \oplus [3]_*\mathscr{L}_{\psi(3(4+2\zeta_3)t)} \oplus [3]_*\mathscr{L}_{\psi(9t)} \oplus [3]_*\mathscr{L}_{\psi(3(2+\zeta_3)t)}^{\oplus 2} \\ \oplus [3]_*\mathscr{L}_{\psi(3(4-2\zeta_3)t)} \oplus [3]_*\mathscr{L}_{\psi(-9t)} \oplus [3]_*\mathscr{L}_{\psi(3(2-\zeta_3)t)}^{\oplus 2}.\end{aligned}$$

In particular, the Swan conductor of $\mathrm{Kl}_3^{(2,2)}$ at ∞ is 8.

4.3.4 The invariants of the inertia group at ∞

The dimension of $(\mathrm{Sym}^k\mathrm{Kl}_{n+1}|_{\eta_\infty})^{I_\infty}$ is computed in [FW05, Thm. 2.5].

Proposition 4.26 (Fu-Wan)**.** *If $p \nmid n+1$ and $2n \mid q-1$, we have an isomorphism*

$$(\mathrm{Sym}^k\mathrm{Kl}_{n+1}|_{\eta_\infty \otimes \mathbb{F}_q})^{I_{\overline{\infty}}}\left(\tfrac{nk}{2}\right) \simeq \begin{cases} \bar{\mathbb{Q}}_\ell^{\oplus a(k,n+1,p)} & 2 \mid n; \\ 0 & 2 \nmid nk; \\ \bar{\mathbb{Q}}_\ell^{\oplus b(k,n+1,p)} & 2 \nmid n \text{ and } 2 \mid k; \end{cases}$$

as $\mathrm{Frob}_{\infty,q}$-modules.

4.4 The dimensions of the middle ℓ-adic cohomology

Let p and ℓ be two different primes. In this subsection, we want to compute the dimension of the middle ℓ-adic cohomology $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_3)$. The Corollary 4.27 gives the dimensions when p is coprime to $n+1$.

However, the case that $p \mid n+1$ remains mysterious. When $n = 1$, the dimension at $p = 2$ was computed in [YV15, Cor. 4.3.5]. Following his method, we give a dimension formula when $n = 2$ and $p = 3$. The key idea is to use the complete classification of finite subgroups of SL_3 to find the local monodromy group at ∞ of Kl_3.

4.4.1 When $\gcd(p, n+1) = 1$

Proposition 4.27. *When p is coprime to $n+1$, the dimension of the middle ℓ-adic cohomology $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})$ is*

$$\frac{1}{n+1}\left(\binom{k+n}{n} - d(k,n+1,p)\right) - \sum_{u=0}^{\lfloor \frac{nk}{2} \rfloor} m_k(u) - b(k,n+1,p) + \delta(k,n+1),$$

where the number $\delta(k,n+1)$ is $\begin{cases} 1 & p = 2 \text{ and } 2 \mid k, \\ 0 & \text{else}, \end{cases}$ the numbers $m_k(u)$ are defined in (4.4), the numbers $d(k,n+1,p)$ and $b(k,n+1,p)$ are defined in Section 4.1.

Proof. By the long exact sequence (1.7), the dimension of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})$ is

$$\mathrm{H}^1_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}) + \dim \mathrm{Sym}^k\mathrm{Kl}_{n+1}^{G_{\mathrm{geom}}} - \dim \mathrm{Sym}^k\mathrm{Kl}_{n+1}^{I_0} - \dim \mathrm{Sym}^k\mathrm{Kl}_{n+1}^{I_\infty}.$$

To compute the dimension of the middle cohomology, it suffices to compute dimension of the invariants of global monodromy and of inertia groups.

First, combining [Kat88, Thm. 11.1] and [FW08, Lem. 0.2], the invariants of the monodromy group is $\bar{\mathbb{Q}}_\ell(nk/2)$ if $p = 2$ and n even, and is 0 otherwise.

At 0, the dimension of the invariants of the inertia group at 0 is summarized in Proposition 4.3.

At ∞, the dimension of the invariants of the inertia group at ∞ is summarized in Proposition 4.26 if $p \nmid n+1$. □

4.4.2 When $n = 2$ and $p = 3$

The classification of finite subgroups of SL_3

Let ζ_9 be a primitive ninth root of unity, and we put $\omega = \zeta_9^6$ and $\varepsilon = \zeta_9^4$. We define the following matrices in $\mathrm{SL}_3(\mathbb{C})$

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \; T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega \end{pmatrix}, \; V = \frac{1}{\omega - \omega^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

Let

$$G_{108} =< S, T, V > \subset \mathrm{SL}_3,$$

$$G_{216} =< S, T, V, UVU^{-1} > \subset \mathrm{SL}_3,$$

and

$$G_{648} =< S, T, V, U > \subset \mathrm{PGL}_3.$$

We summarize the complete classification of solvable finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ from [MBD61, Ch. XII] in the following Theorem.

Theorem 4.28. *If G is a finite solvable subgroup of $\mathrm{SL}_3(\mathbb{C})$, it is isomorphic to one of the following groups:*

(A). Diagonal abelian groups,

(B). Groups arsing from finite subgroups of GL_2,

(C). Groups generated by groups of type (A) and the element T,

(D). Groups generated by groups of type (C) and a matrix of the form

$$Q_{a,b,c} := \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}$$

for some roots of unity satisfying $abc = -1$,

(E). The group G_{108},

(F). The group G_{216},

(G). The group G_{648}.

Remark 4.29. The rest possibilities in the classification of finite subgroups of SL_3 are groups of types (H) (I) (J) (K) and (L) in [MBD61], which are not solvable. In particular, the groups of type (H) and type (I) are the well-known finite simple groups of order 60 and 168 respectively.

The local monodromy at ∞ when $p = 3$

Restricting the local system Kl_3 to η_∞, we have a representation $\rho\colon I_\infty \to \mathrm{SL}_3(\bar{\mathbb{Q}}_\ell)$ of the inertia group at ∞. We want to determine the group $D_0 = \rho(I_\infty)$, which is a finite solvable subgroup of SL_3. The group admits a lower numbering filtration $\{D_i\}$ terminating at D_N, such that $\#D_0/D_1$ is coprime to 3, D_1 is the 3-Sylow subgroup of D_0, and D_i/D_{i+1} are cyclic abelian of order 3 for $i \geq 1$.

Theorem 4.30. *The image of I_∞ under ρ is isomorphic G_{108}, whose lower numbering filtration is given by*

$$D_0 \rhd D_1 =< S, T > \rhd D_2 = \cdots = D_4 =< \omega \mathrm{I}_3 > \rhd \{1\}.$$

Proof. By [Kat88, 11.5.1], the local monodromy group $D_0 = \rho(\infty)$ satisfies the following conditions:

(a). D_0 acts on $V = \mathrm{Kl}_3 \mid_{\bar{\eta}_\infty}$ irreducibly,

(b). D_0 admits no faithful $\bar{\mathbb{Q}}_\ell$-linear representation of dimension smaller than 3,

The groups of type (A) are abelian groups. Since all irreducible representations are all 1-dimensional. Thanks to the condition (a), the image D_0 is not isomorphic to the groups of type (A).

The groups of type (B) are groups induced from subgroups of GL_2, which admit faithful $\bar{\mathbb{Q}}_\ell$-linear representations of dimension 2. The image D_0 is not isomorphic to the groups of type (B) because of the condition (b). Now assume that D_0 is of type (C) or type (D). To rule out these possibilities, we first prove the following lemma.

Lemma 4.31. *Let $V = \mathrm{Kl}_3 \mid_{\bar{\eta}_\infty}$ be the geometric fiber of Kl_3 at ∞. The Swan conductor of $\mathrm{Sym}^3 V$ is $2 + \dim(\mathrm{Sym}^3 V)^{I_\infty}$.*

Proof. Since the symmetric power of the standard representation of SL_3 is irreducible, the invariants $(\mathrm{Sym}^3\mathrm{Kl}_3)^{\mathrm{SL}_3}$ (which is isomorphic to $\mathrm{H}^0_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^k\mathrm{Kl}_3)$) is 0. By the Grothendieck-Ogg-Shafarevich formula, the dimension of $\mathrm{H}^1_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^3\mathrm{Kl}_3)$ is equal to the Swan conductor of the fiber $\mathrm{Sym}^3 V = (\mathrm{Sym}^3\mathrm{Kl}_3)_{\bar{\eta}_\infty}$, The Swan conductor is smaller or equal to $\lfloor \frac{1}{3} \cdot \mathrm{rk}\,\mathrm{Sym}^3\mathrm{Kl}_3 \rfloor = 3$ because the breaks of $\mathrm{Sym}^3\mathrm{Kl}_3$ are at most $\frac{1}{3}$.

Consider the long exact sequence (1.7), we have

$$\begin{aligned} 0 \to (\mathrm{Sym}^3\mathrm{Kl}_3)^{\mathrm{SL}_3} &\to (\mathrm{Sym}^3\mathrm{Kl}_3 \mid_{\eta_0})^{I_{\bar{0}}} \oplus (\mathrm{Sym}^3\mathrm{Kl}_3 \mid_{\eta_\infty})^{I_\infty} \\ &\to \mathrm{H}^1_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^3\mathrm{Kl}_3) \to \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^3\mathrm{Kl}_3) \to 0. \end{aligned}$$

Recall that the dimension of $(\mathrm{Sym}^3\mathrm{Kl}_3 \mid_{\eta_0})^{I_{\bar{0}}}$ is 2 by Proposition 4.3. We deduce from the exact sequence that

$$3 \geq \dim \mathrm{H}^1_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^3\mathrm{Kl}_3) = 2 + \dim \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^3\mathrm{Kl}_3) + \dim(\mathrm{Sym}^3\mathrm{Kl}_3 \mid_{\eta_\infty})^{I_\infty}.$$

If the middle cohomology is nonzero, it is 1-dimensional. By the computations in Appendix A.1.1, we know that

$$\mathrm{Tr}(\mathrm{Frob} \mid \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^3\mathrm{Kl}_3)) = -(m_3^3(3) + 1 + p^2) = 0.$$

We get a contradiction, because $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^3\mathrm{Kl}_3)$ is pure of weight 10. Therefore, the Swan conductor is $2 + \dim(\mathrm{Sym}^3 V)^{I_\infty}$. □

Now assume that D_0 is of type (C) or type (D). The representation $\mathrm{Sym}^3 V$ is the direct sum of three subrepresentations $V_1 = \mathrm{span}\{v_0^3, v_1^3, v_2^3\}$, $V_2 = \mathrm{span}\{v_i^2 v_j\}_{i\neq j}$ and $V_3 = \mathrm{span}\{v_0 v_1 v_2\}$.

- If D_0 is of type (C), the action of D_0 has fixed vectors in each V_i. So $\dim(\mathrm{Sym}^3 V)^{I_\infty} \geq 3$. Hence, we deduce from Lemma 4.31 that
$$5 = \mathrm{Sw}(\mathrm{Sym}^3 V) \leq 3,$$
a contradiction.
- If D_0 is of type (D), the operators T and $Q_{a,b,c}$ admit no fixed vectors in each V_i. So the invariant subspace $(\mathrm{Sym}^3 V)^{I_\infty}$ has dimension 0 and V_i are all totally wild. Hence, we deduce from Lemma 4.31 that
$$2 = \mathrm{Sw}(\mathrm{Sym}^3 V) = \sum_{i=1}^{3} \mathrm{Sw}(V_i) \geq 3,$$
a contradiction.

The group D_0 is also not isomorphic to groups of type (G), because G_{648} has no normal subgroup of order 81, i.e. a normal 3-Sylow subgroup. The possible orders of normal subgroups of G_{648} are $1, 3, 27, 54, 216$ and 648 by a group theoretic computation.

Now the remaining cases are the groups of type (E) and (F). We need the following lemma to analyze their ramification filtration.

Lemma 4.32. *If D_0 is of type (E) or (F), the Swan conductor of $\mathrm{Sym}^6 V$ is 6.*

Proof. From the above discussion, the group D_0 is either the group G_{108} or G_{216}. In both cases, the 3-Sylow subgroup D_1 of D_0 is generated by matrices S and T, of order 27. The group D_1 has only 3 subgroups of order 9. They are
$$H_1 =< S, \omega I_3 >, \quad H_2 =< T, \omega I_3 > \quad \text{and} \quad H_3 =< ST, \omega I_3 > .$$
Because V is totally wilde, the last nontrivial group D_N in the ramification filtration has no invariant vectors, i.e. $V^{D_N} = 0$. Since S, ST and T have nonzero fixed vectors $v_1, v_1 + w^2 v_2 + v_3$ and $v_1 + v_2 + v_3$ respectively, the group D_N is H_i or $< \omega I_3 >$.

There exists nonnegative integers a, b, c such that the lower numbering filtration is of the form
$$D_0 \rhd D_1 = \cdots = D_a \rhd \cdots \rhd D_{a+b} \rhd \cdots = D_{a+b+c} \rhd \{1\},$$
where $D_{a+1} = \cdots D_{a+b}$ is H_1, H_2 or H_3 if $b \neq 0$, and $D_{a+b+1} = \cdots = D_{a+b+c} =< \omega I_3 >$ if $c \neq 0$.

We know that $\mathrm{Sw}(V) = 1$ and $\mathrm{Sw}(\mathrm{Sym}^3 V) = 2$ or 3 according to (4.31). Then
$$1 = \mathrm{Sw}(V) = \sum_{i=1}^{\infty} \frac{\dim V - \dim V^{D_i}}{[D_0 : D_i]} = \frac{1}{[D_0 : D_1]} \left(3 * a + 3 * \frac{b}{3} + 3 * \frac{c}{9}\right)$$
and
$$2 \text{ or } 3 = \mathrm{Sw}(\mathrm{Sym}^3 V) = \sum_{i=1}^{\infty} \frac{\dim \mathrm{Sym}^3 V - \dim \mathrm{Sym}^3 V^{D_i}}{[D_0 : D_i]} = \frac{1}{[D_0 : D_1]} \left(8 * a + 6 * \frac{b}{3} + 0 * \frac{c}{9}\right).$$
If $D_0 = G_{108}$, the only possibility is $(a, b, c) = (1, 0, 3)$. If $D_0 = G_{216}$, we have two possibilities $(a, b, c) = (2, 0, 6)$ or $(1, 4, 3)$. Since the number c is nonzero, the last nontrivial group $D_N = D_{a+b+c}$ is $< \omega I_3 >$ of order 3 for all cases. Also, in all cases $\mathrm{Sw}(\mathrm{Sym}^3 V) = 2$.

Now consider the Swan conductor of $\mathrm{Sym}^6 V = \mathrm{Sym}^6(\mathrm{Kl}_3)_{\bar{\eta}_\infty}$. In this case D_N acts trivially on $\mathrm{Sym}^6 V$, so it suffices to compute $\dim \mathrm{Sym}^6 V^{D_1}$ and $\dim \mathrm{Sym}^6 V^{H_i}$ (if $b \neq 0$, $D_{a+1} = H_i$ for some

$i \in \{1,2,3\}$). Let $\{v_i\}_{i=0,1,2}$ be the canonical basis of V and $f_i = v_0 + \omega^i v_1 + \omega^{2i} v_2$ for $i = 0,1,2$. Then the actions of S and T on the basis $\{f_i\}$ are

$$Sf_i = f_{i+1} \quad \text{and} \quad Tf_i = \omega^{-i} f_i$$

where $f_3 := f_0$. Consider the set of multi-indexes $A := \{\underline{I} = (I_0, I_1, I_2) \in \mathbb{Z}^3_{\geq 0} \mid |\underline{I}| := I_0 + I_1 + I_2 = 6\}$, on which $\sigma = (123) \in S_3$ acts. For any vector $f = \sum_{\underline{I} \in A} a_{\underline{I}} f^{\underline{I}}$ in $\mathrm{Sym}^k V$, we have

$$Sf = \sum_{\underline{I} \in A} a_{\sigma^{-1}\underline{I}} f^{\underline{I}} \quad \text{and} \quad Tf = \sum_{\underline{I} \in A} a_{\underline{I}} \omega^{I_2 - I_1} f^{\underline{I}}.$$

So if $f \in \mathrm{Sym}^6 V^{D_1}$, i.e., $Sf = Tf = f$, the vector f will be contained in the span of $\{\sum_{i=0}^2 f^{\sigma \underline{I}} \mid I_0 \equiv I_1 \equiv I_2 \mod 3\}$. The dimension of the invariants $(\mathrm{Sym}^3 V)^{D_1}$ is 4. Similarly, we can compute that $\dim \mathrm{Sym}^6 V^{H_i} = 10$ for $i \in \{1,2,3\}$.

In conclusion, the Swan conductor is

$$\frac{1}{4}\left(24 * 1 + 18 * 0 + 0 * \frac{3}{9}\right) = \frac{1}{8}\left(24 * 2 + 18 * 0 + 0 * \frac{6}{9}\right) = 6$$

if $(a,b,c) = (1,0,3)$ or $(2,0,6)$ and is

$$\frac{1}{8}\left(24 * 1 + 18 * \frac{4}{3} + 0 * \frac{3}{9}\right) = 6$$

if $(a,b,c) = (1,4,3)$. □

Lemma 4.33. *The dimension of* $(\mathrm{Sym}^6 V)^{I_\infty}$ *is* 2 *if* $D_0 = G_{108}$ *and* 1 *if* $D_0 = G_{216}$.

Proof. Let $D_0 = \rho(I_\infty)$ be either G_{108} or G_{216}, which is a normal subgroup of $G = G_{216}$ or $G = G_{648}$ respectively. Let $\bar{a} \in G/D_0$ be the image of an element $a \in G$, then

$$\mathrm{Tr}(\bar{a} \mid (\mathrm{Sym}^k V)^{I_\infty}) = \frac{1}{\#D_0} \sum_{g \in a \cdot D_0} \mathrm{Tr}(g \mid V). \tag{4.34}$$

In particular, if we let $a = 1$, we get

$$\sum \dim(\mathrm{Sym}^k V)^{I_\infty} x^k = \frac{1}{\#D_0} \sum_{g \in D_0} \frac{-1}{P_{g,V}(x)} \tag{4.35}$$

where $P_{g,V}(x) = \det(x \cdot g - 1 \mid V)$ is the characteristic polynomial of g. This can be easily computed by Sagemath [The22], see Appendix A.4. Therefore, we get

$$\begin{cases} P(x) = -\dfrac{1 - x^3 + x^6 + x^{12} - x^{15} + x^{18}}{(-1+x^3)^3(1+x^3)^2(1+x^6)} & \text{if } D_0 = G_{108}; \\ \tilde{P}(x) = -\dfrac{1 - x^3 + x^9 - x^{15} + x^{18}}{(-1+x^3)^3(1+x^3)^2(1+x^6)} & \text{if } D_0 = G_{216}. \end{cases} \tag{4.36}$$

In particular, their coefficients of t^6 is 2 or 1 respectively. □

By Appendix A.1.1, we have

$$\mathrm{Tr}(\mathrm{Frob} \mid \mathrm{H}^1_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^6 \mathrm{Kl}_3)) = -820.$$

Combine Proposition 4.3 and [Kat88, Thm. 7.0.7], we deduce that

$$\mathrm{Tr}(\mathrm{Frob} \mid (\mathrm{Sym}^6 \mathrm{Kl}_3)^{I_0}) = 1 + p^2 + p^4 + p^6 = 820.$$

Using again the long exact sequence (1.7), we conclude that

$$\mathrm{Tr}(\mathrm{Frob} \mid \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^6\mathrm{Kl}_3)) = -\mathrm{Tr}(\mathrm{Frob} \mid (\mathrm{Sym}^6\mathrm{Kl}_3)^{I_\infty}) \tag{4.37}$$

and

$$\dim \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^6\mathrm{Kl}_3) = 2 - \dim(\mathrm{Sym}^6\mathrm{Kl}_3)^{I_\infty}.$$

If $D_0 = G_{216}$, then both $\dim(\mathrm{Sym}^6\mathrm{Kl}_3)^{I_\infty}$ and the middle cohomology are 1-dimensional. However, by (4.37), since $(\mathrm{Sym}^3\mathrm{Kl}_3)^{I_\infty}$ is pure of weight 12 and $\mathrm{H}^1_{\acute{e}t,mid}(\mathbb{G}_{m,\bar{\mathbb{F}}_3}, \mathrm{Sym}^6\mathrm{Kl}_3)$ is pure of weight 13, we get a contradiction.

In conclusion, the only possibility is $D_0 = G_{108}$. And its ramification filtration is given in terms of the triple $(1, 0, 3)$ in the proof of Lemma 4.32. □

The dimension of the mid-cohomology

Proposition 4.38. *When $p = 3$, the Swan conductor of the action of I_∞ on $(\mathrm{Sym}^k\mathrm{Kl}_3)\,|_{\eta_\infty}$ is given by*

$$\mathrm{Swan}_\infty(\mathrm{Sym}^k\mathrm{Kl}_3) = \begin{cases} \frac{1}{3}\binom{k+2}{2} & 3 \nmid k, \\ \frac{1}{4}\left(\binom{k+2}{2} - \frac{d(k,3,3)+2}{3}\right) & 3 \mid k. \end{cases}$$

Proof. If $3 \nmid k$, then there is no fixed vector of $\mathrm{Sym}^k\mathrm{Kl}_3\,|_{\eta_\infty}$ under the action of the group $<\omega I_3>$. So the Swan conductor will be

$$\sum_{i=1}^{\infty} \frac{\dim \mathrm{Sym}^k V - 0}{[D_0 : D_i]} = \frac{\dim \mathrm{Sym}^k V}{3} \cdot \sum \frac{3}{[D_0 : D_i]} = \frac{1}{3} \cdot \binom{k+2}{2}.$$

If $3 \mid k$, it is similar to the case that $k = 6$. In this case $D_4 =< \omega I_3 >$ acts trivially. The dimension of $\mathrm{Sym}^k V^{D_1}$ is computed in terms of invariant vectors under the action of S and T. We again let $\{v_i\}_{i=0,1,2}$ be the canonical basis of V and $f_i = v_0 + \omega^i v_1 + \omega^{2i} v_2$ for $i = 0, 1, 2$. If $Sf = Tf = f$, the vector f will be contained in the span of the set $\{\sum_{i=0}^{2} f^{\sigma^I} \mid I_0 \equiv I_1 \equiv I_2 \mod 3\}$. The dimension of the invariants of S and T is exactly the number $\frac{d(k,3,3)-2}{3} + 1 = \frac{d(k,3,3)+2}{3}$. In conclusion, the Swan conductor is

$$\sum_{i=1}^{\infty} \frac{\dim \mathrm{Sym}^k V - \dim \mathrm{Sym}^k V^{D_i}}{[D_0 : D_i]} = \frac{1}{4}\left(\binom{k+2}{2} - \frac{d(k,3,3)+2}{3}\right).$$

□

Proposition 4.39. *The invariance of inertia is*

$$(\mathrm{Sym}^k V)^{I_\infty} = \bar{\mathbb{Q}}_\ell(-k)^{\oplus \tilde{p}_k} \oplus \mathscr{L}_\theta(-k)^{\oplus p_k - \tilde{p}_k},$$

where θ is an unramified character which sends Frobenius to -1, and p_k and q_k are the k-th coefficients of the generating series $P(x)$ and $\tilde{P}(x)$ from (4.36). In particular, the dimension of $(\mathrm{Sym}^k V)^{I_\infty}$ is p_k.

Proof. Let ϕ be a lifting of the image of Frob_∞ in GL_3 and $\phi_1 = \frac{1}{3}\phi$ in SL_3. Since ϕ normalizes $D_0 = G_{108}$, it is in the normalizer of G_{108} in SL_3, i.e. G_{216}. By a direct computation, we know that G_{216}/D_1 is the quarternion group Q_8 and D_0/D_1 is a cyclic group. Notice that $\bar{\phi_1}^{-1} g \bar{\phi_1} = g^3$ for $g \in Q_8$, which implies that $\phi_1 \notin D_0$. In (4.34) we let $a = \phi_1$. Then we get

$$Q(x) := \sum_{s=0}^{\infty} \mathrm{Tr}(\phi_1 \mid (\mathrm{Sym}^s V)^{I_\infty}) x^s = \frac{1}{108} \sum_{g \in \phi_1 D_0} \frac{-1}{P_{g,V}(x)},$$

where $P_{g,V}(x)$ is again the characteristic polynomial. Since $\phi_1 \notin G_{108}$ and $G_{216} = G_{108} \cup \phi_1 G_{108}$, the series $Q(x)$ is nothing but

$$2\tilde{P}(x) - P(x) = \frac{-1 + x^3 - x^6}{(-1 + x^3)(1 + x^6)}.$$

Let p_k and q_k be the k-th coefficient of $P(x)$ and $\tilde{P}(x)$ respectively.

Notice that $\phi_1^2 \in G_{108}$, because $[G_{216} : G_{108}] = 2$. So the eigenvalues of ϕ_1 acting on $(\mathrm{Sym}^k V)^{I_\infty}$ are ± 1. Assume that the dimensions of eigenspaces of 1 and -1 are λ_1 and λ_{-1} respectively. Then $\lambda_1 + \lambda_{-1} = \dim(\mathrm{Sym}^k V)^{I_\infty}$ and $\lambda_1 - \lambda_{-1} = 2\tilde{p}_k - p_k$. Therefore, we get the desired decomposition Proposition 4.39. □

Corollary 4.40. *When $p = 3$ the dimension of the moments will be*

$$\dim \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3) = \begin{cases} \frac{1}{3}\binom{k+2}{2} - \lfloor \frac{k+2}{2} \rfloor & 3 \nmid k; \\ \frac{1}{4}\left(\binom{k+1}{2} - \frac{d(k,3,3)+2}{3}\right) - \lfloor \frac{k+2}{2} \rfloor - p_k & 3 \mid k. \end{cases}$$

4.5 The ε-factors of Kloosterman sheaves

Recall that the *ε-factor* of an ℓ-adic sheaf $\mathscr{F}$ on a smooth projective variety X over a finite field k is the ℓ-adic number

$$\varepsilon(X, \mathscr{F}) := \prod_{i=0}^{2} \det(-\mathrm{Frob}_k \mid \mathrm{H}^i_{\acute{e}t}(X_{\bar{k}}, \mathscr{F}))^{(-1)^{i+1}}.$$

The ε-factors of $\mathrm{Sym}^k \mathrm{Kl}_2$ are computed by Fu and Wan in [FW10]. In this section, we will prove the following result.

Theorem 4.41 (Theorem E)**.** *Let $\mathscr{F} = \mathrm{Kl}^\lambda_{n+1}$ be sheaves from Theorem A.*

(1). If $n|\lambda| + 1$ is even, then

$$\varepsilon\left(\mathbb{P}^1_{\mathbb{F}_p}, j_*\mathscr{F}\right) = p^{\frac{nk+1}{2} \dim \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\left(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathscr{F}\right)}.$$

(2). If $\mathscr{F} = \mathrm{Sym}^3 \mathrm{Kl}_4$, and $p \notin \{2, 3, 5\}$, then

$$\varepsilon\left(\mathbb{P}^1_{\mathbb{F}_p}, j_*\mathscr{F}\right) = \left(\frac{p}{15}\right) p^{10}.$$

Proof of Theorem 4.41(1). By Corollary 2.16, the middle de Rham cohomology $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\left(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1}\right)$ is a symplectic representation of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. So the determinant of Frob_p is a power of p. Taking into account the dimension and the weight of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\left(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1}\right)$, we conclude that

$$\begin{aligned} \varepsilon\left(\mathbb{P}^1_{\mathbb{F}_p}, j_*\mathrm{Kl}^\lambda_{n+1}\right) &= \det\left(-\mathrm{Frob}_p, \mathrm{H}^1_{\acute{e}t}\left(\mathbb{P}^1_{\mathbb{F}_p}, j_*\mathrm{Kl}^\lambda_{n+1}\right)\right) \\ &= \det\left(\mathrm{Frob}_p, \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\left(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1}\right)\right) = p^{\frac{n|\lambda|+1}{2} \dim \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\left(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1}\right)}. \end{aligned}$$

□

To prove the second part, we use Laumon's product formula, see Section 1.4.3.

4.5.1 Local ε-factors at 0

We compute the local ε-factors of $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ at 0 here.

Proposition 4.42. *We have*

$$\varepsilon\big(\mathbb{P}^1_{(0)}, j_*\mathrm{Sym}^k\mathrm{Kl}_{n+1}\mid_{\mathbb{P}^1_{(0)}}, \mathrm{d}t\mid_{\mathbb{P}^1_{(0)}}\big) = (-1)^{\binom{n+k}{k}+\sum_{i=0}^{\lfloor\frac{nk}{2}\rfloor} m_k(i)} \cdot p^{nk\binom{n+k}{k}/2-\sum_{i=0}^{\lfloor\frac{nk}{2}\rfloor} i\cdot m_k(i)},$$

where $m_k(i)$ are numbers generated by the series in (4.4).

Proof. By abuse of notation, we let $i\colon s_0 \to \mathbb{P}^1_{\mathbb{F}_p,(0)}$ and $j\colon \eta_0 \to \mathbb{P}^1_{\mathbb{F}_p,(0)}$ be the inclusions. By (1.11) and (1.13), we have

$$\varepsilon\big(\mathbb{P}^1_{(0)}, i_*i^*\bar{\mathbb{Q}}_\ell(i), \mathrm{d}t\big) = \det(-\mathrm{Frob}_0 \mid \bar{\mathbb{Q}}_\ell(i))^{-1} = -p^{-i} \tag{4.43}$$

and

$$\varepsilon\big(\mathbb{P}^1_{(0)}, \bar{\mathbb{Q}}_\ell(i), \mathrm{d}t\big) = 1.$$

By the short exact sequence

$$0 \to j_!j^*\bar{\mathbb{Q}}_\ell \to \bar{\mathbb{Q}}_\ell \to i_*i^*\bar{\mathbb{Q}}_\ell \to 0$$

and (1.10), we have

$$\varepsilon\big(\mathbb{P}^1_{(0)}, j_!j^*\bar{\mathbb{Q}}_\ell(i), \mathrm{d}t\big) = -p^i. \tag{4.44}$$

Then use (1.10), (4.43), and (4.44) several times we have

$$\begin{aligned}
&\varepsilon\big(\mathbb{P}^1_{(0)}, j_*\mathrm{Sym}^k\mathrm{Kl}_{n+1}\mid_{\mathbb{P}^1_{(0)}}, \mathrm{d}t\mid_{\mathbb{P}^1_{(0)}}\big)\\
=&\varepsilon\big(\mathbb{P}^1_{(x)}, j_!j^*\mathrm{Sym}^k\mathrm{Kl}_{n+1}\mid_{\mathbb{P}^1_{(0)}}, \mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{(0)}, i_*i^*(j_*\mathrm{Sym}^k\mathrm{Kl}_{n+1}), \mathrm{d}t\big)\\
=&\varepsilon\big(\mathbb{P}^1_{(0)}, j_!V_k, \mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{(0)}, i_*(\mathrm{Sym}^k\mathrm{Kl}^{I_0}_{n+1}), \mathrm{d}t\big)\\
=&\prod_{\underline{I}}\varepsilon\big(\mathbb{P}^1_{(0)}, j_!V_k^{\underline{I}}/V_k^{\underline{I},+}, \mathrm{d}t\big)\cdot\det\big(-\mathrm{Frob}_0 \mid \oplus_{i=0}^{\lfloor\frac{nk}{2}\rfloor}\bar{\mathbb{Q}}_\ell(-i)^{\oplus m_k(i)}\big)^{-1}\\
=&\prod_{\underline{I}}(-p^{nk-m_{\underline{I}}})\cdot\prod_{0\le i\le nk/2}(-p^{-i})^{m_k(i)}\\
=&(-1)^{\binom{n+k-1}{k-1}+\sum m_k(u)}\cdot p^{\sum_{\underline{I}}(nk-m_{\underline{I}})-\sum_{i=0}^{\lfloor\frac{nk}{2}\rfloor} i\cdot m_k(i)}.
\end{aligned}$$

Notice that $\sum_{\underline{I}}(nk-m_I) = \sum_{\underline{I}} m_I$, and $\sum_{\underline{I}}(nk-m_I)+\sum_{\underline{I}} m_I = \sum_{\underline{I}} nk = nk\binom{n+k}{k}$. The number $\sum_{\underline{I}} m_{\underline{I}}$ is $nk\binom{n+k}{k}/2$. □

4.5.2 Local ε-factors at ∞

Now we compute the local ε-factor of $\mathrm{Sym}^3\mathrm{Kl}_4$ at ∞.

Lemma 4.45. *Let $\mathbb{F}_q$ be a finite extension of $\mathbb{F}_p$. We denote by χ_q the unique nontrivial quadratic multiplicative character of $\mathbb{F}_q^\times$.*

(1). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)}, \mathscr{L}_\theta, \mathrm{d}t^m\big) = \theta(\mathrm{Frob}_q)^{m+1}q^{-m-1}$ *for any characters θ of* $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$,

(2). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)}, j_*\mathscr{L}_{\chi_q}, \mathrm{d}t\big) = -g(\chi_q,\psi_q)\cdot q^{-2}$,

(3). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)}, j_*\mathscr{L}_{\psi_q(a\cdot t)}, \mathrm{d}t\big) = q^{-1}$ *for $a\in\mathbb{F}_q^\times$*,

(4). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)}, j_*\mathscr{L}_{\psi_q(a\cdot t)}, \mathrm{d}t^4\big) = q^{-4}$ *for $a\in\mathbb{F}_q^\times$*,

(5). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(a\cdot t)},\mathrm{d}t\big)=q^{-1}\chi_q(a)$ *for* $a\in\mathbb{F}_q^\times$,

(6). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(a\cdot t)},\mathrm{d}t^4\big)=\chi_q(4a)q^{-4}$ *for* $a\in\mathbb{F}_q^\times$.

Proof. **(1).** By (1.15), we have

$$\begin{aligned}\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\bar{\mathbb{Q}}_\ell\otimes\mathscr{L}_\theta,\mathrm{d}t^m\big)=&\det(\mathrm{Frob}_\infty,\mathscr{L}_\theta)^{a\left(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\bar{\mathbb{Q}}_\ell,\mathrm{d}t^m\right)}\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\bar{\mathbb{Q}}_\ell,\mathrm{d}t^m\big)^{\mathrm{rk}\,\mathscr{L}_\theta}\\ =&\theta(\mathrm{Frob}_q)^{m+1}q^{-(m+1)},\end{aligned}$$

where $a\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\bar{\mathbb{Q}}_\ell,\mathrm{d}t^m\big)=-m-1$.

(2). Here we simply denote by χ the character induced by $j_*\mathscr{L}_{\chi_q}$. We can verify that, by [Ser13, Chap. XIV Prop. 8],

$$\chi((t^{-1})^m(a_0+a_1t^{-1}+\ldots))=\chi_q(-a_0^{-1}).$$

Then the conductor $a(\chi_q)$ is 1 and $v_0(\mathrm{d}t)=-2$. We have

$$\begin{aligned}\varepsilon(\chi,\psi_\omega)&=\int_{(t^{-1})\mathbb{F}_q[[t^{-1}]]^\times}\chi^{-1}(z)\Psi_{\mathrm{d}t}(z)\mathrm{d}z\\ &=\int_{\mathbb{F}_q[[t^{-1}]]^\times}\chi^{-1}(z)\Psi_{\mathrm{d}t}(t^{-1}z)\mathrm{d}(t^{-1}z)\\ &=q^{-1}\sum_{\alpha\in\mathbb{F}_q^\times}\chi_q^{-1}(-\alpha)\Psi_{\mathrm{d}t}(t^{-1}\alpha)\int_{1+t^{-1}\mathbb{F}_q[[t^{-1}]]}\mathrm{d}z\\ &=q^{-2}\sum_{\alpha}\chi_q\big(-\alpha^{-1}\big)\psi_q(-\alpha)=-g(\chi_q,\psi_q)\cdot q^{-2}.\end{aligned}$$

(3). Here we also denote by χ the character induced by $j_*\mathscr{L}_{\psi_k(a\cdot t)}$. By [Ser13, Chap. XIV Prop. 15] we can verify that

$$\chi(z)=\psi_{\mathbb{F}_q}\left(a\cdot\mathrm{Res}_\infty\left(\frac{1}{t^{-1}}\frac{\mathrm{d}z}{z}\right)\right).$$

Then the conductor $a(\chi)$ is 2 and $v_\infty(\mathrm{d}t)=-2$. We have

$$\begin{aligned}\varepsilon(\chi,\psi_\omega)&=\int_{\mathbb{F}_q[[t^{-1}]]^\times}\chi^{-1}(z)\Psi_{\mathrm{d}t}(z)\mathrm{d}z\\ &=\sum_{\alpha\in\mathbb{F}_q^\times,\beta\in\mathbb{F}_q}\chi^{-1}(\alpha+t^{-1}\beta)\Psi_{\mathrm{d}t}(\alpha+t^{-1}\beta)\int_{1+t^{-2}\mathbb{F}_q[[t^{-1}]]}\mathrm{d}z\\ &=q^{-2}\sum_{\alpha,\beta}\psi_{\mathbb{F}_q}\left(-\frac{a\beta}{\alpha}\right)\psi_{\mathbb{F}_q}(-\beta)\\ &=q^{-2}\sum_{\alpha,\beta}\psi_{\mathbb{F}_q}\left(-\beta\left(\frac{a}{\alpha}+1\right)\right)=q^{-1}.\end{aligned}$$

(4). Since $\mathrm{d}(t^4)=(4t^3)\mathrm{d}t$, by (1.14) and the description of the character χ associated with $j_*\mathscr{L}_{\psi(4at)}$ in (3), we have

$$\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},j_*\mathscr{L}_{\psi_q(a\cdot t)},\mathrm{d}t^4\big)=q^{-3}\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},j_*\mathscr{L}_{\psi_q(a\cdot t)},\mathrm{d}t\big)=q^{-4}.$$

(5). Let χ be the character induced by $\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(a\cdot t)}$, then combine the two propositions we used in (2) and (3), we have

$$\chi((t^{-1})^m(a_0+a_1t^{-1}+\ldots))=\chi_q(-a_0^{-1})\cdot\psi_q\Big(-\frac{a\cdot a_1}{a_0}\Big).$$

Then the conductor $a(\chi)$ is 2 and $v_\infty(\mathrm{d}t) = -2$. We have

$$\begin{aligned}\epsilon(\chi,\psi_\omega) &= \int_{\mathbb{F}_q[[t^{-1}]]^\times} \chi^{-1}(z)\Psi_{\mathrm{d}t}(z)\mathrm{d}z \\ &= \sum_{\alpha\in\mathbb{F}_q^\times,\beta\in\mathbb{F}_q} \chi^{-1}(\alpha+t^{-1}\beta)\Psi_{\mathrm{d}t}(\alpha+t^{-1}\beta)\int_{1+t^{-2}\mathbb{F}_q[[t^{-1}]]}\mathrm{d}z \\ &= q^{-2}\sum_{\alpha,\beta}\chi_q(-\alpha^{-1})\psi_{\mathbb{F}_q}\left(-\frac{a\beta}{\alpha}\right)\psi_{\mathbb{F}_q}(-\beta) \\ &= q^{-2}\sum_{\alpha,\beta}\chi_q(-\alpha^{-1})\psi_{\mathbb{F}_q}\left(-\beta\left(\frac{a}{\alpha}+1\right)\right) = \chi_q(a^{-1})q^{-1}.\end{aligned}$$

(6). Similar to (4).

□

Lemma 4.46. *Let $j\colon \mathbb{G}_m \to \mathbb{P}^1$ be the inclusion. Let k be a finite extension of $\mathbb{F}_q$, and $\pi_k\colon \eta_\infty\otimes k \to \eta_\infty$.*

(1). We have the decomposition

$$\pi_{k*}\bar{\mathbb{Q}}_\ell = \bigoplus_{\theta\colon \mathrm{Gal}(k/\mathbb{F}_q)\to\bar{\mathbb{Q}}_\ell^\times}\mathscr{L}_\theta,$$

(2). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\pi_{k*}\bar{\mathbb{Q}}_\ell,\mathrm{d}t\big) = \#k^{-2}$,

(3). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\pi_{k*}\bar{\mathbb{Q}}_\ell,\mathrm{d}t^{n+1}\big) = \#k^{-(n+2)}$,

(4). We have $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*\bar{\mathbb{Q}}_\ell,\mathrm{d}t\big) = -\left(\frac{-2}{p}\right)^{[\mathbb{F}_q:\mathbb{F}_p]}\cdot g(\chi,\psi_p)^{3[\mathbb{F}_q:\mathbb{F}_p]}\cdot q^{-8}$.

Proof. Assertions **(1)** can be proved by the definition of Induction of representation of finite cyclic groups and Galois theory.
(2). Use (1) and Lemma 4.45(1), we have

$$\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\pi_{k*}\bar{\mathbb{Q}}_\ell,\mathrm{d}t\big) = \prod_\theta \varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_\theta,\mathrm{d}t\big) = \prod_\theta q^{-2}\theta(\mathrm{Frob}_q)^2 = \#k^{-2},$$

where we used the fact that $(\prod\theta)^2 = 1$.
(3). Similar to the proof of (2).
(4). Use (2), (1.12), and Lemma 4.45(6), we have

$$\begin{aligned}&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*\bar{\mathbb{Q}}_\ell,\mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\psi_q(4t)}\otimes\mathscr{L}_{\chi_q}),\mathrm{d}t\big)^{-1} \\ =&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\bar{\mathbb{Q}}_\ell,\mathrm{d}t^4\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\psi_q(4t)}\otimes\mathscr{L}_{\chi_q},\mathrm{d}t^4\big)^{-1} \\ =&q^{-5}\cdot q^4 = q^{-1}.\end{aligned}$$

Apply Laumon's product formula in Theorem 1.9 to Kl_4, we have

$$1 = q^4\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(0)},j_*\mathrm{Kl}_4,\mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},j_*\mathrm{Kl}_4,\mathrm{d}t\big).$$

On the one hand, by Proposition 4.42, we have $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(0)},j_*\mathrm{Kl}_4,\mathrm{d}t\big) = -q^6$. On the other hand, by Lemma 4.16 and (1.15), we have

$$\begin{aligned}&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\psi(4t)}\otimes\mathscr{L}_{\chi_q})\otimes\mathscr{L}_{\theta_g\cdot\theta_{0,3}}(-1),\mathrm{d}t\big) \\ =&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\psi(4t)}\otimes\mathscr{L}_{\chi_q}),\mathrm{d}t\big)^1\cdot(\theta_g\cdot\theta_{0,3}(\mathrm{Frob}_q)\cdot q)^{-3} \\ =&q^{-3}\cdot\theta_{0,3}(\mathrm{Frob}_q)\cdot g(\chi,\psi_p)^{-3[\mathbb{F}_q:\mathbb{F}_p]}\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\psi(4t)}\otimes\mathscr{L}_{\chi_q}),\mathrm{d}t\big).\end{aligned}$$

Combining the above identities, we deduce that

$$\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*\big(\mathscr{L}_{\psi(4t)}\otimes\mathscr{L}_{\chi_q}\big),\mathrm{d}t\big)=-q^{-7}\cdot\theta_{0,3}(\mathrm{Frob}_q)\cdot g(\chi,\psi_p)^{3[\mathbb{F}_q:\mathbb{F}_p]}.$$

By the definition of $\theta_{0,3}$ in Section 4.1, it sends Frob_p to $(\frac{-2}{p})$. At last, by (1.12) and Lemma 4.45(6), we have

$$\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},[4]_*\bar{\mathbb{Q}}_\ell,\mathrm{d}t\big)=-\Big(\frac{-2}{p}\Big)^{[\mathbb{F}_q:\mathbb{F}_p]}\cdot g(\chi,\psi_p)^{3[\mathbb{F}_q:\mathbb{F}_p]}\cdot q^{-8}.$$

□

Remark 4.47. For simplicity, we can let

$$\delta_{q,3}=-\Big(\frac{-2}{p}\Big)^{[\mathbb{F}_q:\mathbb{F}_p]}\cdot\Big(g(\chi,\psi_p)/p^{1/2}\Big)^{3[\mathbb{F}_q:\mathbb{F}_p]}.$$

Lemma 4.48. *Notations are as the above. Let θ be a character of the $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)$ and $q\in\mathbb{F}_q^\times$. Let π be the map $\mathbb{P}^1_{\mathbb{F}_q,(\infty)}\to\mathbb{P}^1_{\mathbb{F}_p,(\infty)}$. We have*

(1). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\chi_q}\otimes\mathscr{L}_\theta,\mathrm{d}t\big)=-\theta^{-1}(\mathrm{Frob}_q)\cdot g(\chi_q,\psi_q)\cdot q^{-2}$,

(2). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta,\mathrm{d}t^4\big)=\theta(\mathrm{Frob}_q)^{-3}\cdot q^{-4}$,

(3). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta,\mathrm{d}t\big)=\theta(\mathrm{Frob}_q)^{-3}\cdot\delta_{q,3}\cdot q^{-\frac{11}{2}}$,

(4). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},\pi_*[4]_*(\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta,\mathrm{d}t)\big)=\theta(\mathrm{Frob}_q)^{-3}\delta_{q,3}\cdot q^{-\frac{11}{2}}$,

(5). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta,\mathrm{d}t^4\big)=\theta(\mathrm{Frob}_q)^{-3}\cdot\chi_q(a)\cdot q^{-4}$,

(6). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)=\theta(\mathrm{Frob}_q)^{-3}\cdot\chi_q(a)\cdot\delta_{q,3}\cdot q^{-\frac{11}{2}}$,

(7). $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},\pi_*[4]_*(\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4C_{\underline{I}}t)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)=\theta(\mathrm{Frob}_q)^{-3}\cdot\chi_q(a)\cdot\delta_{q,3}\cdot q^{-\frac{11}{2}}$.

Proof. **(1).** By (1.15) and Lemma 4.45(2), we have

$$\begin{aligned}\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\chi_q}\otimes\mathscr{L}_\theta,\mathrm{d}t\big)=&\det(\mathrm{Frob}_q,\mathscr{L}_\theta)^{-1}\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\chi_q},\mathrm{d}t\big)^1\\=&-\theta^{-1}(\mathrm{Frob}_q)\cdot(g(\chi_q,\psi_q))\cdot q^{-2}.\end{aligned}$$

(2). By (1.15) and Lemma 4.45(4), we have

$$\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta,\mathrm{d}t^4\big)=\theta(\mathrm{Frob}_q)^{-3}\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\psi_q(4at)},\mathrm{d}t^4\big)=\theta(\mathrm{Frob}_q)^{-3}\cdot q^{-4}.$$

(3). By (1.12), Lemma 4.46(4) and (2), we have

$$\begin{aligned}&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)\\=&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*\bar{\mathbb{Q}}_\ell,\mathrm{d}t\big)^{-1}\cdot\delta_{q,3}\cdot q^{-\frac{13}{2}}\\=&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta,\mathrm{d}t^4\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\bar{\mathbb{Q}}_\ell,\mathrm{d}t^4\big)^{-1}\cdot\delta_{q,3}\cdot q^{-\frac{13}{2}}\\=&\theta(\mathrm{Frob}_q)^{-3}\cdot q^{-4}\cdot q^5\cdot\delta_{q,3}\cdot q^{-\frac{13}{2}}=\theta(\mathrm{Frob}_q)^{-3}\cdot\delta_{q,3}\cdot q^{-\frac{11}{2}}\end{aligned}$$

(4). By (1.12), Lemma 4.46(4) and (3), we have

$$\begin{aligned}&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},\pi_*[4]_*(\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)\\=&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},\pi_*[4]_*(\mathscr{L}_{\psi_q(4at)})\otimes\mathscr{L}_\theta,\mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},\pi_*\bar{\mathbb{Q}}_\ell^4,\mathrm{d}t\big)^{-1}\cdot q^{-8}\\=&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta,\mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\bar{\mathbb{Q}}_\ell^4,\mathrm{d}t\big)^{-1}\cdot q^{-8}\\=&\theta(\mathrm{Frob}_q)^{-3}\delta_{q,3}\cdot q^{-\frac{11}{2}}\cdot q^8\cdot q^{-8}=\theta(\mathrm{Frob}_q)^{-3}\delta_{q,3}\cdot q^{-\frac{11}{2}}.\end{aligned}$$

(5). By (1.15) and Lemma 4.45(6), we have

$$\begin{aligned}&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_{\chi_q}\otimes\mathscr{L}_\theta,\mathrm{d}t^4\big)\\=&\theta(\mathrm{Frob}_q)^{-3}\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_{\chi_q},\mathrm{d}t^4\big)\\=&\theta(\mathrm{Frob}_q)^{-3}\cdot\chi_q(a)\cdot q^{-4}.\end{aligned}$$

(6). By (1.12), Lemma 4.46(4) and (5), we have

$$\begin{aligned}&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)\\=&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*\bar{\mathbb{Q}}_\ell,\mathrm{d}t\big)^{-1}\cdot\delta_{q,3}\cdot q^{-\frac{13}{2}}\\=&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta,\mathrm{d}t^4\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},\bar{\mathbb{Q}}_\ell,\mathrm{d}t^4\big)^{-1}\cdot\delta_{q,3}\cdot q^{-\frac{13}{2}}\\=&\theta(\mathrm{Frob}_q)^{-3}\cdot\chi_q(a)\cdot q^{-4}\cdot q^5\cdot\delta_{q,3}\cdot q^{-\frac{13}{2}}\\=&\theta(\mathrm{Frob}_q)^{-3}\cdot\chi_q(a)\cdot\delta_{q,3}\cdot q^{-\frac{11}{2}}.\end{aligned}$$

(7). By (1.12), Lemma 4.46(3) and (6), we have

$$\begin{aligned}&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},\pi_*[4]_*(\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)\\=&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},\pi_*[4]_*(\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},\pi_*\bar{\mathbb{Q}}^4_\ell,\mathrm{d}t\big)^{-1}\cdot q^{-8}\\=&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)\cdot q^8\cdot q^{-8}\\=&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*(\mathscr{L}_{\chi_q}\otimes\mathscr{L}_{\psi_q(4at)}\otimes\mathscr{L}_\theta),\mathrm{d}t\big)\cdot\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_q,(\infty)},[4]_*\bar{\mathbb{Q}}^4_\ell,\mathrm{d}t\big)^{-1}\cdot q^{-8}\\=&\theta(\mathrm{Frob}_q)^{-3}\cdot\chi_q(a)\cdot\delta_{q,3}\cdot q^{-\frac{11}{2}}\cdot q^8\cdot q^{-8}\\=&\theta(\mathrm{Frob}_q)^{-3}\cdot\chi_q(a)\cdot\delta_{q,3}\cdot q^{-\frac{11}{2}}.\end{aligned}$$

□

Proposition 4.49. *If $p\notin\{2,3,5\}\cup\{\ell\}$, we have*

$$\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},j_*\mathrm{Sym}^3\mathrm{Kl}_4,\mathrm{d}t\big)=-p^{-95}\Big(\frac{-15}{p}\Big).$$

Proof. **Assume first that $(\frac{-1}{p})=-1$.** In this case the 4-th primitive root of unity is not contained in $\mathbb{F}_p$. Let $\pi\colon\mathbb{P}^1_{\mathbb{F}_p(\zeta_4),(\infty)}\to\mathbb{P}^1_{\mathbb{F}_p,(\infty)}$ be the morphism induced from the inclusion $\mathbb{F}_p\to\mathbb{F}_p(\zeta_4)$. By Proposition 4.23, the local structure at infinity of $\mathrm{Sym}^3\mathrm{Kl}_4(3)$ is the direct sum of

$$[4]_*\big(\mathscr{L}_{\psi_p(12t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\eta_1}\big)\oplus[4]_*\big(\mathscr{L}_{\psi_p(-4t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\eta_2}\big)$$

and

$$[4]_*\big(\mathscr{L}_{\psi_p(-4t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\eta_3}\big)\oplus\pi_*[4]_*\big(\mathscr{L}_{\psi_{p^2}((8+4\zeta_4)t)}\otimes\mathscr{L}_{\chi_{p^2}}\otimes\mathscr{L}_{\theta_g}\big),$$

where θ_g,η_1,η_2 and η_3 send Frob_p to $-g(\chi_p,\psi_p)$, $(\frac{2}{p})$, $(\frac{2}{p})$ and $(\frac{-2}{p})$ respectively.

By Lemma 4.48, the local ε-factors of the above direct summands are

- $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},[4]_*\big(\mathscr{L}_{\psi_p(12t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\theta_{\eta_1}}\big),\mathrm{d}t\big)=-\big(\frac{-3}{p}\big)\cdot p^{-7}$;
- $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},[4]_*\big(\mathscr{L}_{\psi_p(-4t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\eta_2}\big),\mathrm{d}t\big)=-p^{-7}$;
- $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},[4]_*\big(\mathscr{L}_{\psi_p(-4t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g\cdot\eta_3}\big),\mathrm{d}t\big)=-\big(\frac{-1}{p}\big)p^{-7}$;
- $\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)},\pi_*[4]_*\big(\mathscr{L}_{\psi_{p^2}((8+4\zeta_4)t)}\otimes\mathscr{L}_{\chi_{p^2}}\otimes\mathscr{L}_{\theta_g}\big),\mathrm{d}t\big)=-\big(\frac{5}{p}\big)p^{-14}$.

Then, the local ε-factor $\varepsilon(\mathbb{P}^1_{\mathbb{F}_p,(\infty)}, j_*\mathrm{Sym}^3\mathrm{Kl}_4(4), \mathrm{d}t)$ is

$$\begin{aligned}&\Big(-\Big(\frac{-3}{p}\Big)\cdot p^{-7}\Big)\cdot(-p^{-7})\cdot\Big(-\Big(\frac{-1}{p}\Big)p^{-7}\Big)\cdot\Big(-\Big(\frac{5}{p}\Big)p^{-14}\Big)\\ =&p^{-35}\cdot\Big(\frac{15}{p}\Big)=-p^{-35}\cdot\Big(\frac{-15}{p}\Big).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)}, j_*\mathrm{Sym}^3\mathrm{Kl}_4, \mathrm{d}t\big)=\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)}, j_*\mathrm{Sym}^3\mathrm{Kl}_4(3)(-3), \mathrm{d}t\big)\\ =&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p,(\infty)}, j_*\mathrm{Sym}^3\mathrm{Kl}_4(3), \mathrm{d}t\big)\cdot(p^4)^{-15}=-p^{-95}\Big(\frac{-15}{p}\Big).\end{aligned}$$

Assume that $(\frac{-1}{p})=1$**.** In this case the 4-th primitive root of unity is in $\mathbb{F}_p$. The only difference is that we have now 5 direct summands. We replace

$$\pi_*[4]_*\big(\mathscr{L}_{\psi_p((8+4\zeta_4)t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g}\big)$$

by

$$[4]_*\big(\mathscr{L}_{\psi_p((8+4\zeta_4)t)}\otimes\mathscr{L}_{\chi}\otimes\mathscr{L}_{\theta_g}\big)\oplus[4]_*\big(\mathscr{L}_{\psi((8-4\zeta_4)t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g}\big).$$

Again we use Lemma 4.48 to get

- $[4]_*\big(\mathscr{L}_{\psi_p((8+4\zeta_4)t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g}\big)=-\big(\frac{-2(2+\zeta_4)}{p}\big)p^{-7}$;
- $[4]_*\big(\mathscr{L}_{\psi_p((8-4\zeta_4)t)}\otimes\mathscr{L}_{\chi_p}\otimes\mathscr{L}_{\theta_g}\big)=-\big(\frac{-2(2-\zeta_4)}{p}\big)p^{-7}$.

Therefore, the local ε-factor $\varepsilon(\mathbb{P}^1_{\mathbb{F}_p,(\infty)}, j_*\mathrm{Sym}^3\mathrm{Kl}_4(4), \mathrm{d}t)$ is

$$\begin{aligned}&\Big(-\Big(\frac{-3}{p}\Big)\cdot p^{-7}\Big)\cdot(-p^{-7})\cdot\Big(-\Big(\frac{-1}{p}\Big)p^{-7}\Big)\\ &\Big(-\Big(\frac{-2(2+\zeta_4)}{p}\Big)p^{-7}\Big)\cdot\Big(-\Big(\frac{-2(2-\zeta_4)}{p}\Big)p^{-7}\Big)\\ =&-p^{-35}\cdot\Big(\frac{15}{p}\Big)=-p^{-35}\cdot\Big(\frac{-15}{p}\Big).\end{aligned}$$

The rest is similar to the first case. □

4.5.3 Proof of Theorem 4.41.

Proof of 2. By Laumon's product formula 1.9, property (8) in Section 1.4.4, Proposition 4.42 and Proposition 4.49, we have

$$\begin{aligned}&\varepsilon\big(\mathbb{P}^1_{\mathbb{F}_p}, j_*\mathrm{Sym}^3\mathrm{Kl}_4\big)\\ =&p^{20}\cdot\varepsilon\big(\mathbb{P}^1_{(0)}, j_*\mathrm{Sym}^3\mathrm{Kl}_4\mid_{\mathbb{P}^1_{(0)}}, \mathrm{d}t\mid_{\mathbb{P}^1_{(0)}}\big)\cdot\varepsilon\big(\mathbb{P}^1_{(\infty)}, j_*\mathrm{Sym}^3\mathrm{Kl}_4\mid_{\mathbb{P}^1_{(\infty)}}, \mathrm{d}t\mid_{\mathbb{P}^1_{(\infty)}}\big)\\ =&p^{20}\cdot(-1)^{23}\cdot p^{85}\cdot\Big(-p^{-95}\Big(\frac{-15}{p}\Big)\Big)\\ =&p^{10}\Big(\frac{-15}{p}\Big)=p^{10}\Big(\frac{p}{15}\Big),\end{aligned}$$

where we used the quadratic reciprocity in the last identity. □

Chapter 5

L-functions of Kloosterman sheaves

The purpose of this chapter is to prove Theorem A and Theorem B.

5.1 Galois representations attached to Kloosterman sheaves

In this section, we prove Theorem 5.4, which implies Theorem C. Moreover, we describe the Galois representations attached to $\mathrm{Sym}^k \mathrm{Kl}_3$ in detail in Theorem 5.12.

5.1.1 A compactification of $\mathcal{K}$

Let k be an integer and p a prime number not dividing $n+1$. The Laurent polynomial $g_{n+1}^{\boxplus k} = \sum_{i=1}^{k} \left(\sum_{j=1}^{n} y_{i,j} + \frac{1}{\prod_j y_{i,j}} \right)$ on the torus $\mathbb{G}_{m,\mathbb{Q}}^{nk}$ defines a hypersurface $\mathcal{K}$. We choose a toric compactification X_{tor} of $\mathbb{G}_m^{nk}$ as the one in Section 3.3.5 now.

We begin with the pair $\left(\mathbb{G}^{nk}, g_{n+1}^{\boxplus k}\right)$. Let $M = \oplus_{i,j} \mathbb{Z} y_{i,j}$ be the lattice of monomials on $\mathbb{G}_m^{nk}$ and $N = \oplus_{i,j} \mathbb{Z} e_{i,j}$ be the dual lattice. We consider the toric compactification X of $\mathbb{G}_m^{nk}$ attached to the simplicial fan F in $N_{\mathbb{R}}$ generated by the rays

$$\mathbb{R}_{\geq 0} \cdot \textstyle\sum_{i,j} \epsilon_{i,j} e_{i,j}$$

where $\epsilon_{i,j} \in \{0, \pm 1\}$ and $(\epsilon_{i,j})_{i,j} \neq 0$. Each simplicial cone of maximal dimension nk in F provides an affine chart of X, which is isomorphic to $\mathbb{A}^{nk}$. On each chart, the function $g_{n+1}^{\boxplus k}$ has the same structure. For example, we can consider the maximal cone generated by

$$\gamma_{i_0,j_0} := \sum_{1 \leq i \leq i_0 - 1, 1 \leq j \leq n} e_{i,j} + \sum_{1 \leq j \leq j_0} e_{i_0,j}$$

for $1 \leq i_0 \leq k$ and $1 \leq j_0 \leq n$, where the affine ring associated with the dual cone is the polynomial ring $\mathbb{Q}[u_{i,j}]$ such that $u_{i,j} = \begin{cases} y_{i,j}/y_{i,j+1} & 1 \leq j < n \\ y_{i,j}/y_{i+1,1} & i < k, j = n \\ y_{k,n} & i = k, j = n. \end{cases}$

In this chart, we can rewrite $g_{n+1}^{\boxplus k}$ as $g_1 / \left(\prod_{1 \leq j \leq n} u_{1,j}^j \cdot \prod_{2 \leq i \leq k, j} u_{i,j}^n \right)$, where

$$g_1 = 1 + \sum_{e=1}^{k-1} \prod_{1 \leq j \leq n} u_{1,j}^j \cdot \prod_{\substack{2 \leq i \leq e \\ 1 \leq j \leq n}} u_{i,j}^n \cdot \prod_{1 \leq j \leq n} u_{e+1,j}^{n-j} + \prod_{1 \leq j \leq n} u_{1,j}^j \cdot \prod_{2 \leq i \leq k, j} u_{i,j}^n \cdot h$$

for a polynomial h. The toric variety X provides compactification of $(\mathbb{G}^{nk}, g_{n+1}^{\boxplus k})$, where the closure of the zero locus of $g_{n+1}^{\boxplus k}$, and $X \backslash \mathbb{G}_m^{nk}$ form a strict normal crossing divisor.

We take the zariski closure of the hypersurface $Z(g_{n+1}^{\boxplus k})$ inside X, denoted by $\bar{\mathcal{K}}$. One can check that

$$Z(g_1)\cap Z(u_{1,s})=\varnothing,\ Z(g_1)\cap Z(u_{r,s})=V\bigg(1+\sum_{e=1}^{r-1}\prod_j u_{1,j}^j\cdot\prod_{2\leq i\leq e,j}u_{i,j}^n\cdot\prod_j u_{e+1,j}^{n-j}\bigg)$$

and

$$Z(\partial g_1/\partial u_{1,1})\cap Z(u_{r,s})=V\bigg(\sum_{e=1}^{r-1}\prod_j u_{1,j}^j\cdot\prod_{2\leq i\leq e,j}u_{i,j}^n\cdot\prod_j u_{e+1,j}^{n-j}/u_{1,s}\bigg)$$

for $1\leq s\leq n$ and $2\leq r\leq k$. It follows that for $1\leq s\leq n$ and $1\leq r\leq k$, we have $Z(g_1)\cap Z(\partial g_1/\partial u_{1,1})\cap Z(u_{r,s})=\varnothing$. We deduce that $\bar{\mathcal{K}}$ is smooth along the divisor $Z(\prod u_{i,j})$. Moreover, one can check that $Z(\prod u_{i,j})\cap\bar{\mathcal{K}}$ satisfies the strict normal crossing property.

As for $\mathcal{K}=\bar{\mathcal{K}}\cap\mathbb{G}_m^{nk}$, one can check that $\mathcal{K}$ is smooth if $\gcd(k,n+1)=1$ and has isolated singularities inside $\mathbb{G}_m^{nk}$ if $\gcd(n+1,k)>1$. In the latter case, the singular locus Σ_0 of $\mathcal{K}$ has only finitely many $\bar{\mathbb{Q}}$-points or $\bar{\mathbb{F}}_p$-points, which are all ordinary quadratic. We do blow-ups of $\mathbb{G}_m^{nk}$ along the singular locus Σ_0 and denote by $\mathcal{K}'$ the strict transform of $\mathcal{K}$. For convenience, we also put $\mathcal{K}'=\mathcal{K}$ in the case $\gcd(k,n+1)=1$. We denote by $\bar{\mathcal{K}}'$ the closure of $\mathcal{K}'$ in X.

Lemma 5.1. *Let $\mathbb{F}$ be either $\bar{\mathbb{Q}}$ or $\bar{\mathbb{F}}_p$. Assume that $\gcd(k,n+1)>1$, $nk\geq 3$ and $p\nmid n+1$. Then we have*

$$\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}'_{\mathbb{F}})=\begin{cases}\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}_{\mathbb{F}})\oplus\mathbb{Q}_\ell(\frac{-nk+1}{2})^{\oplus\#\Sigma_0(\mathbb{F})} & 2\nmid nk,\\ \mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}_{\mathbb{F}}) & 2\mid nk.\end{cases}$$

Proof. Let T be the preimage of Σ_0 along the blow-up morphism $\bar{\mathcal{K}}'\to\bar{\mathcal{K}}$, which is a disjoint union of quadrics. Then consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc}\mathrm{H}_{\acute{e}t}^{nk-2}(T_{\mathbb{F}})&\longrightarrow&\mathrm{H}_{\acute{e}t,c}^{nk-1}((\mathcal{K}'\backslash T)_{\mathbb{F}})&\xrightarrow{\alpha}&\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}'_{\mathbb{F}})&\xrightarrow{\gamma}&\mathrm{H}_{\acute{e}t}^{nk-1}(T_{\mathbb{F}})\\ &&\downarrow\simeq&&\downarrow\beta&&\\ \mathrm{H}_{\acute{e}t}^{nk-2}((\Sigma_0)_{\mathbb{F}})&\longrightarrow&\mathrm{H}_{\acute{e}t,c}^{nk-1}((\mathcal{K}\backslash\Sigma_0)_{\mathbb{F}})&\longrightarrow&\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}_{\mathbb{F}})&\longrightarrow&\mathrm{H}_{\acute{e}t}^{nk-1}((\Sigma_0)_{\mathbb{F}}).\end{array}\tag{5.2}$$

By the assumption that $nk\geq 3$, the cohomology $\mathrm{H}_{\acute{e}t}^{nk-2}((\Sigma_0)_{\mathbb{F}})$ and $\mathrm{H}_{\acute{e}t}^{nk-1}((\Sigma_0)_{\mathbb{F}})$ are 0. In particular, γ will be surjective if we extend the diagram by one more column on the right.

If nk is odd, we have $\mathrm{H}_{\acute{e}t}^{nk-1}(T_{\mathbb{F}})=\mathbb{Q}_\ell(\frac{nk-1}{2})^{\oplus\#\Sigma_0(\mathbb{F})}$ since T is disjoint union of quadrics. Moreover, we have a short exact sequence

$$0\to\mathrm{H}_{\acute{e}t,c}^{nk-1}((\mathcal{K}'\backslash T)_{\mathbb{F}})\xrightarrow{\alpha}\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}'_{\mathbb{F}})\xrightarrow{\gamma}\mathrm{H}_{\acute{e}t}^{nk-1}(T_{\mathbb{F}})\to 0.$$

Since $\mathrm{H}_{\acute{e}t,c}^{nk-1}((\mathcal{K}'\backslash T)_{\mathbb{F}})\simeq\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}_{\mathbb{F}})$, the morphism β becomes a retraction of α. Hence, we have a direct sum decomposition

$$\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}'_{\mathbb{F}})=\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}_{\mathbb{F}})\oplus\mathbb{Q}_\ell(\tfrac{nk-1}{2})^{\oplus\#\Sigma_0(\mathbb{F})}.$$

If nk is even, the cohomology $\mathrm{H}_{\acute{e}t}^{nk-1}(T_{\mathbb{F}})=0$, because T is disjoint union of quadrics. From this we conclude that $\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}'_{\mathbb{F}})=\mathrm{H}_{\acute{e}t,c}^{nk-1}(\mathcal{K}_{\mathbb{F}})$. □

5.1.2 The ℓ-adic case in general

Let λ be a sequence of integers and G_λ be the finite group from Section 2.1. Let $\Sigma'(p)=\Sigma'(|\lambda|,n+1,p)$ be the singular set of $\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p}$. If $p\nmid n+1$, the singular points in $\Sigma'(p)$ are ordinary quadratic in the sense of [DK73, XII 1.1].

Assuming $p \nmid n+1$. We let $n|\lambda| = 2m+1$ (resp. $n|\lambda| = 2m+2$) and apply the Picard-Lefschetz formula [DK73, XV 3.4] to $\bar{\mathcal{K}}'_{\mathbb{Z}_p} \to \mathrm{Spec}(\mathbb{Z}_p)$. For each $x \in \Sigma'(p)$, there is a vanishing cycle class $\delta_x \in \mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}_p})(m)$, well-defined up to a sign. These vanishing cycle classes are orthogonal to each other and $<\delta_x, \delta_x>=(-1)^m 2$ (resp. $<\delta_x, \delta_x>=0$). We fix a place of $\bar{\mathbb{Q}}$ over p, and denote by I_p the corresponding inertia group. To each element $\sigma \in I_p$, the action on $\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}})$ is given by

$$\sigma(v) = \begin{cases} v + (-1)^m \sum_{x\in\Sigma'(p)} \frac{\epsilon(\sigma)-1}{2}(v,\delta_x)\delta_x & 2 \nmid n|\lambda| \\ v - (-1)^m \sum_{x\in\Sigma'(p)} \epsilon(\sigma)(v,\delta_x)\delta_x & 2 \mid n|\lambda| \end{cases} \tag{5.3}$$

where ϵ is the character $I_p \twoheadrightarrow \{\pm 1\}$ of order 2 if $n|\lambda|$ odd, and is the fundamental tame character $I_p \to \varprojlim_n \mu_{\ell^n}(\bar{\mathbb{Q}}_\ell)$ if $n|\lambda|$ is even. Moreover, we have an exact sequence

$$0 \to \mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p}) \to \mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}}) \xrightarrow{\gamma} \sum_{x\in\Sigma'(p)} \mathbb{Q}_\ell(m),$$

where γ is the sum of the intersections with the vanishing cycle classes δ_x.

Theorem 5.4. *Let $p \neq \ell$ be two different primes. Then there exists a finite set of primes $S = S(\lambda, n+1)$, such that*

(1). $(\mathrm{M}^\lambda_{n+1})_\ell$ is unramified at primes $p \notin S$, and $(\mathrm{M}^\lambda_{n+1})_\ell[\zeta_p]$ is isomorphic to $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}^\lambda_{n+1})$ as $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations, which is in particular unramified.

(2). If $n|\lambda|$ is odd, $p \nmid 2(n+1)$ and $p \in S$, then $(\mathrm{M}^\lambda_{n+1})_\ell$ is tamely ramified.

(3). If $n|\lambda|$ is even, $p \nmid (n+1)$ and $p \in S$, then $(\mathrm{M}^\lambda_{n+1})_\ell$ is tamely ramified.

Proof. Let S be a finite set of primes containing all primes p such that $p \mid n+1$ or the variety $\mathcal{K}'$ has bad reductions at p. We show that S satisfies our need.

For simplicity, we denote by $G = G_\lambda \times \mu_{n+1}$ and by $\chi = \chi_\lambda$ the groups and characters from Section 2.1, and we omit the coefficient $\mathbb{Q}_\ell$ in the cohomology. Set $\bar{\mathcal{K}}^{(0)} = \bar{\mathcal{K}}'$, and let $\bar{\mathcal{K}}^{(i)}$ be the disjoint union of all i-fold intersections of distinct irreducible components of $\bar{\mathcal{K}}' \backslash \mathcal{K}'$ for $i \geq 1$. Let $\mathbb{F}$ be either $\bar{\mathbb{Q}}$ or $\bar{\mathbb{F}}_p$. Consider the spectral sequence

$$(E_1^{p,q})_{\mathbb{F}} = \mathrm{H}^q_{\acute{e}t}(\bar{\mathcal{K}}^{(p)}_{\mathbb{F}}) \Rightarrow \mathrm{H}^{p+q}_{\acute{e}t,c}(\mathcal{K}^{(p)}_{\mathbb{F}}).$$

For the case $\mathbb{F} = \bar{\mathbb{Q}}$, since all the $\bar{\mathcal{K}}^{(i)}$ are proper smooth, morphisms in the E_2-page are 0 for the reason of weights. Therefore, the spectral sequence is degenerated at the E_2-page. It follows from the spectral sequence that

$$\begin{aligned} \mathrm{gr}^W_{n|\lambda|-1}\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{Q}}}) = (E_\infty^{0,n|\lambda|-1})_{\bar{\mathbb{Q}}} &= \ker(\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}}) \to \mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}^{(1)}_{\bar{\mathbb{Q}}})) \\ &= \mathrm{im}(\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{Q}}}) \xrightarrow{\alpha} \mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}})), \end{aligned}$$

where the second map is the quotient $\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{Q}}}) \to E_2^{0,n|\lambda|-1}$ induced from the abutment on $\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{Q}}})$. Notice that the above spectral sequence is equivariant with respect to the action of G. So using the isomorphism in (5.1) we conclude that

$$\mathrm{im}(\alpha)^{G,\chi} \simeq \begin{cases} (\mathrm{M}^\lambda_{n+1})_\ell(1) \oplus (\mathbb{Q}_\ell(m)^{\oplus \#\Sigma_0(\mathbb{Q})})^{G,\chi} & 2 \nmid n|\lambda|, \\ (\mathrm{M}^\lambda_{n+1})_\ell(1) & 2 \mid n|\lambda|. \end{cases} \tag{5.5}$$

For the case in characteristic p, i.e. $\mathbb{F}=\bar{\mathbb{F}}_p$, similar to the characteristic 0 case, we conclude that

$$\mathrm{gr}^W_{n|\lambda|-1}\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{Q}}})=\mathrm{gr}^W_{n|\lambda|-1}(E_\infty^{0,n|\lambda|-1})_{\bar{\mathbb{F}}_p}=\mathrm{gr}^W_{n|\lambda|-1}\mathrm{im}\Big(\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{F}}_p})\xrightarrow{\beta}\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p})\Big),$$

which is equivariant under the action of G. Recall that by Theorem 2.17, we have

$$\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_m,\mathrm{Kl}^\lambda_{n+1})\simeq\mathrm{gr}^W_{n|\lambda|+1}\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,\mathrm{c}}(\mathcal{K}_{\bar{\mathbb{F}}_p},\mathbb{Q}_\ell(\zeta))(-1)^{G,\chi}.$$

Therefore, $\mathrm{gr}^W_{n|\lambda|+1}\mathrm{im}(\beta)^{G,\chi}(-1)[\zeta_p]$ is isomorphic to

$$\begin{cases}\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Kl}^\lambda_{n+1})\oplus(\mathbb{Q}_\ell[\zeta_p](m)^{\oplus\#\Sigma_0(\bar{\mathbb{F}}_p)})^{G,\chi} & 2\nmid n|\lambda|,\\ \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Kl}^\lambda_{n+1}) & 2\mid n|\lambda|.\end{cases}\tag{5.6}$$

Assume from now that $p\nmid n+1$. Consider the G-equivariant commutative diagram with exact rows

$$\begin{array}{ccccccc}&&\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{F}}_p})&\xrightarrow{\iota_p}&\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{Q}}})&&\\ &&\downarrow\beta&&\downarrow\alpha&&\\ 0&\longrightarrow&\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p})&\xrightarrow{\iota}&\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}})&\xrightarrow{\gamma}&\oplus_{x\in\Sigma'(p)}\mathbb{Q}_\ell(-m),\end{array}\tag{5.7}$$

where the middle row is given by the Picard-Lefschetz formula. Notice that each class δ_x is a generator of $\mathrm{H}^{n|\lambda|-1}_{\{x\}}(\bar{\mathcal{K}}',\mathrm{R}\Psi(m))$, with support $\{x\}$. So $\Delta=\oplus\bar{\mathbb{Q}}_\ell(-m)\delta_x$ are contained in $\mathrm{im}(\alpha)$. If we take the (G,χ)-isotypic component on the second row, we have an exact sequence

$$0\to\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p})^{G,\chi}\xrightarrow{\iota}\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}})^{G,\chi}\xrightarrow{\gamma}\oplus_{x\in\Sigma}\left(\mathbb{Q}_\ell(-m)\right)^{G,\chi}.\tag{5.8}$$

By a diagram chasing argument, we get an inclusion $\mathrm{gr}^W_{n|\lambda|+1}\mathrm{im}(\beta)^{(G,\chi)}\hookrightarrow\mathrm{im}(\alpha)^{G,\chi}$ from (5.8). When $\mathcal{K}'$ has good reduction at p, the variety $\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p}$ is smooth proper and the morphisms ι and ι_p in (5.7) are isomorphisms. As a consequence of (5.5) and (5.6), we have an isomorphism

$$(\mathrm{M}^\lambda_{n+1})_\ell[\zeta_p]\simeq\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Kl}^\lambda_{n+1})$$

of unramified $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations. So according to our choice of S, the first statement is verified.

If $p\in S$, for example when $\Delta^{G,\chi}$ is nonzero, the Galois representation $(\mathrm{M}^\lambda_{n+1})_\ell$ is indeed ramified at p. By Picard-Lefschetz formula, $(\mathrm{M}^\lambda_{n+1})_\ell$ is tamely ramified if $(p+1)\cdot n|\lambda|$ even, and is possibly wildly ramified if $p=2$ and $n|\lambda|$ odd[1]. In conclusion, we proved the second and the third statement. □

Remark 5.9. In the above, the set S is not the smallest one, because we omit the case that $p\mid n+1$. When p divides $n+1$, the singular points of $\bar{\mathcal{K}}_{\bar{\mathbb{F}}_p}$ are isolated but not ordinary quadratic. So the Picard-Lefschetz formula is not valid in this case. However, the vanishing cycles with respect to $\bar{\mathcal{K}}'_{\bar{\mathbb{Z}}_p}\to\mathrm{Spec}(\bar{\mathbb{Z}}_p)$ are still 0 if $i\neq n|\lambda|-1$ [Ill03, Cor. 2.10]. By the long exact sequence associated with vanishing cycles [DK73, XIII (1.4.2.2)], the cospecialization morphism

$$\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}_{\bar{\mathbb{F}}_p})\to\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}_{\bar{\mathbb{Q}}})$$

is an injection. Hence, the diagram

$$\begin{array}{ccccc}&&\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{F}}_p})&\longrightarrow&\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{Q}}})\\ &&\downarrow&&\downarrow\\ 0&\longrightarrow&\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}_{\bar{\mathbb{F}}_p})&\longrightarrow&\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}_{\bar{\mathbb{Q}}})\end{array}$$

[1] This is because the character $\epsilon\colon I_p\to\{\pm1\}$ has order 2.

induces an injection

$$\mathrm{im}\Big(\mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}_{\bar{\mathbb{F}}_p}) \xrightarrow{\beta} \mathrm{H}^{n|\lambda|-1}_{\acute{e}t}(\bar{\mathcal{K}}_{\bar{\mathbb{Q}}})\Big)^{G,\chi}(-1) \to \mathrm{gr}^W_{n|\lambda|+1}(\mathrm{H}^{n|\lambda|-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{Q}}})^{G,\chi}(-1)) = (\mathrm{M}^{\lambda}_{n+1})_\ell. \tag{5.10}$$

As long as (5.10) is an isomorphism, i.e. the dimension of the source and the target are the same, we can conclude that $(\mathrm{M}^{\lambda}_{n+1})_\ell$ is unramified at p.

The morphism (5.10) is an isomorphism for $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ when $\gcd(p, n+1) = 1$ when $n \leq 2$, see [YV15, Cor. 4.3.5] and Corollary 4.40.

Inspired by these cases, we conjecture that the morphism (5.10) is an isomorphism for any n and λ when $N(V_\lambda) + E(V_\lambda)$ is invertible. Here $N(V_\lambda)$ and $E(\lambda)$ are matrices in Remark 2.7.

To finish this section, we give the dimension formula in the case of Symmetric powers of Kloosterman sheaves as a Corollary of Theorem 5.4.

Corollary 5.11. *The dimension of* $(\mathrm{M}^k_{n+1})_\ell$ *is*

$$\frac{1}{n+1}\left(\binom{k+n}{n} - d(k, n+1)\right) - \sum_{u=0}^{\lfloor \frac{nk}{2} \rfloor} m_k(u) - \delta(k, n+1)$$

where the numbers $d(k, n+1)$ *are defined in Section 4.1, the numbers* $m_k(u)$ *are defined in* (4.4)*, and*

$$\delta(k, n+1) = \begin{cases} a(k, n+1) & 2 \mid n, \\ 0 & 2 \nmid nk, \\ b(k, n+1) & 2 \nmid n \text{ and } 2 \mid k. \end{cases}$$

Proof. For p large enough, the numbers $d(k, n+1, p), a(k, n+1, p)$ and $b(k, n+1, p)$ are constants and are equal to $d(k, n+1), a(k, n+1)$ and $b(k, n+1)$ respectively. Moreover, the sequence (5.8) induces the isomorphism $\mathrm{im}(\beta)^{G,\chi} \simeq \mathrm{im}(\alpha)^{G,\chi}$. So the dimension of $(\mathrm{M}^k_{n+1})_\ell$ coincides with the dimension of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1})$ in Proposition 4.27. □

5.1.3 The ℓ-adic case for $\mathrm{Sym}^k\mathrm{Kl}_3$

Let ζ_3 be either a third primitive root of unity in $\bar{\mathbb{F}}_p$ or $\bar{\mathbb{Q}}_\ell$. Recall that each singular point x of $\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p}$ is of the form $x = (x_{i,j}) = (\zeta_3^{a_i})_{i,j}$ for some $a_i \in \{0, 1, 2\}$. The action of $S_k \times \mu_3$ on $\Sigma'(p) = \Sigma'(k, 3, p)$ is given by

$$(\sigma, \zeta_3^a) \cdot (x_{i,j}) = (\zeta_3^a \cdot x_{\sigma(i),j})$$

and there is also an action of the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ given by $\mathrm{Frob}_p(x_{i,j}) = (x^p_{i,j})$.

One can identify the S_k-orbits in $\Sigma'(p)$ with the set of multi-indices

$$\{\underline{I} \in \mathbb{N}^3 \mid |\underline{I}| = k,\ C_{\underline{I}} = 0\} \backslash \{(k/3, k/3, k/3)\},$$

by sending $x = (\zeta_3^{a_i})_{i,j}$ to $\underline{I}$ such that $I_j = \#\{i \mid a_i = j\}$. On multi-indices, the actions of μ_3 and $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ are given by $\zeta_3 \cdot (I_0, I_1, I_2) = (I_2, I_0, I_1)$ and $\mathrm{Frob}_p(I_0, I_1, I_2) = (I_0, I_2, I_1)$.

Theorem 5.12. *Let* $k \geq 2$ *be an integer, and* $p \neq \ell$ *be two different prime numbers. We take* $S = S(k, 3)$ *to be the set of primes* $\{p \neq 3 \mid d(k, 3, p) \neq d(k, 3)\}$ *if* $3 \nmid k$*, and* $\{p \neq 3 \mid d(k, 3, p) \neq d(k, 3)\} \cup \{3\}$ *if* $3 \mid k$*.*

(1). If $p \notin S$*, the Galois representation* $(\mathrm{M}^k_3)_\ell$ *is unramifed at* p*, and we have an isomorphism*

$$(\mathrm{M}^k_3)_\ell[\zeta_p] \simeq \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k\mathrm{Kl}_{n+1}).$$

(2). The conductor N is $\prod_p p^{c_p}$, where

$$c_p=\begin{cases}\frac{d(k,3,p)-d(k,3)}{3} & p\neq 2,3,\\ \frac{d(k,3,p)-d(k,3)}{3}-(1-k+2\lfloor\frac{k}{2}\rfloor) & p=2,\\ 0 & 3\nmid k \text{ and } p=3,\\ \text{an undetermined integer} & 3\mid k \text{ and } p=3.\end{cases}$$

(3). If $p\neq 3$, the inertia group $I_p\subset \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts unipotently on $(\mathrm{M}_3^k)_\ell$ such that $(\sigma-1)^2=0$ for any $\sigma\in I_p$. Let N be the nilpotent part of the monodromy operator. With respect to the intersection pairing, the image $U=N((\mathrm{M}_3^k)_\ell)$ is totally isotropic. The orthogonal complement of U is $(\mathrm{M}_3^k)_\ell^{I_p}$, and the induced map $\sigma-1\colon (\mathrm{M}_3^k)_\ell\mapsto(\mathrm{M}_3^k)_\ell/U$ is zero. At last, we have an isomorphism of $\mathrm{Gal}(\mathbb{F}_p/\mathbb{F}_p)$-modules

$$(\mathrm{M}_3^k)_\ell^{I_p}[\zeta_p]=\mathrm{H}^1_{\acute{e}t,\mathrm{c}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Sym}^k\mathrm{Kl}_3)/E[\zeta_3],$$

where E is mixed of weight $\leq 2k$, whose dimensions of graded quotients are given by

$$\mathrm{gr}_w^W E=\begin{cases}(1+2\lfloor\frac{k}{2}\rfloor-k)+(1+3\lfloor\frac{k^2}{3}\rfloor-k^2) & w=2k,\\ 1 & 4\leq w<2k \text{ and } 4\mid w,\\ 2 & w=2\\ 0 & \text{else.}\end{cases}$$

Proof. We take the notation from those in the proof of Theorem 5.4. If $p=3$, by Remark 5.9, the Galois representation $(\mathrm{M}_{n+1})_\ell$ is unramified at p when $3\nmid k$. If $p\notin S$ and $p\neq 3$, by Theorem 5.4.(1), the Galois representation $(\mathrm{M}_{n+1})_\ell$ is unramified at p. So part (1) is justified.

Assume that $p\neq 3$. In this case, all singularities of $\bar{\mathcal{K}}_{\bar{\mathbb{F}}_p}$ are ordinary quadratic. Consider again the spectral sequence used in Theorem 5.4

$$(E_1^{p,q})_{\mathbb{F}}=\mathrm{H}^q_{\acute{e}t}(\bar{\mathcal{K}}^{(p)}_{\mathbb{F}})\Rightarrow \mathrm{H}^{p+q}_{\acute{e}t,c}(\mathcal{K}^{(p)}_{\mathbb{F}}).$$

Since $\bar{\mathcal{K}}^{(i)}$ are smooth proper over both $\bar{\mathbb{Q}}$ and $\mathbb{F}_p$ if $i\geq 1$, we have the isomorphisms $\mathrm{H}^a_{\acute{e}t}(\bar{\mathcal{K}}^{(i)}_{\mathbb{F}_p})\simeq\mathrm{H}^a_{\acute{e}t}(\bar{\mathcal{K}}^{(i)}_{\bar{\mathbb{Q}}})$ for $i\geq 1$ and any $a\in\mathbb{Z}$. By the Picard-Lefschetz formula, we have isomorphisms $\mathrm{H}^a_{\acute{e}t}(\bar{\mathcal{K}}^{(i)}_{\mathbb{F}_p})\simeq\mathrm{H}^a_{\acute{e}t}(\bar{\mathcal{K}}^{(i)}_{\bar{\mathbb{Q}}})$ for $0\leq a\leq 2k-2$. So we have $(E_2^{i,2k-1-i})_{\bar{\mathbb{Q}}}\simeq(E_2^{i,2k-i-1})_{\mathbb{F}_p}$ and

$$(E_2^{i,2k-i-1})_{\mathbb{F}_p}=(E_\infty^{i,2k-i-1})_{\mathbb{F}_p}=(E_\infty^{i,2k-1-i})_{\bar{\mathbb{Q}}} \tag{5.13}$$

for $i\geq 1$.

Let $\Delta=\sum_{x\in\Sigma}\mathbb{Q}_\ell(-m)\delta_x$ be the subspace of $\mathrm{H}^{2k-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}})$ generated by vanishing cycle classes. We have a diagram

$$\begin{array}{ccccccc} & & \mathrm{im}(\beta)^{G,\chi} & & & & \\ & & \downarrow{\scriptstyle i_1} & & & & \\ 0 & \longrightarrow & \mathrm{im}(\alpha)^{G,\chi}\cap\mathrm{H}^{2k-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p}) & \hookrightarrow & \mathrm{im}(\alpha)^{G,\chi} & \longrightarrow & C \\ & & & & & & \downarrow{\scriptstyle i_2} \\ & & & & & & (\oplus_{x\in\Sigma'(p)}\bar{\mathbb{Q}}_\ell(m))^{G,\chi} \end{array} \tag{5.14}$$

where C is the image of the map γ.

Lemma 5.15. *In the diagram (5.14), the vertical map i_1 is an isomorphism. If $p = 2$ and k is even, the cokernel of the vertical map i_2 is one dimensional. Otherwise, the map i_2 is an isomorphism.*

Let us first assume this lemma. By the Picard-Lefschetz formula, the action of $\sigma \in I_p$ acting on a cohomology class $v \in \mathrm{H}^{2k-1}_{\acute{e}t}(\bar{K}_{\bar{\mathbb{Q}}})$ is given by

$$\sigma(v) = v - (-1)^{m+1} t_\ell(\sigma) \sum_{x \in \Sigma} < v, \delta_x > \delta_x,$$

which implies that $(\sigma - 1)^2 = 0$. It follows that the Galois representation $(\mathrm{M}_3^k)_\ell$ is tamely ramified at p, and the Swan conductor is 0 at p. In particular, the invariants of the inertia group I_p acting on $(\mathrm{M}_3^k)_\ell$ is $\mathrm{im}(\beta)^{G,\chi}(-1)$. So the exponent of p of the Artin conductor of $(\mathrm{M}_3^k)_\ell$ is $\dim C$. So we conclude the second part.

Finally, by (5.6), (5.13) and (5.14), we know that

$$(\mathrm{M}_3^k)_\ell^{I_p}[\zeta_p] = \mathrm{im}(\beta)^{G,\chi}(-1)[\zeta_p] = (E_\infty^{0,2k-1})_{\mathbb{F}_p}(-1) \simeq \mathrm{H}^1_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3)/E,$$

where $E = W_{\leq 2k}\mathrm{H}^1_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3)$. The structure of E can be made more explicit. Recall that in Theorem 3.33, we computed the Hodge numbers of $\mathrm{H}^1_{\mathrm{dR},c}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3)$. By a similar argument to that of (2.21), we have the exact sequence

$$0 \to \mathbb{Q} \to \mathrm{H}^1_{\mathrm{dR},c}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3) \to \mathrm{H}^{2k-1}_{\mathrm{dR},c}(\mathcal{K}_{\mathbb{C}})^{G,\chi}(-1) \to \mathbb{Q}(-1) \to 0,$$

from which we deduce the weight filtration of $\mathrm{H}^{2k-1}_{\acute{e}t,c}(\mathcal{K}_{\mathbb{Q}})(-1)$. We conclude that

$$\mathrm{gr}^W_w W_{\leq 2k}\mathrm{H}^{2k-1}_{\acute{e}t,c}(\mathcal{K}_{\mathbb{Q}})^{G,\chi}(-1) = \begin{cases} (1 + 2\lfloor \frac{k}{2} \rfloor - k) + (1 + 3\lfloor \frac{k^2}{3} \rfloor - k^2) & w = 2k, \\ 1 & 4 \leq w < 2k \text{ and } 4 \mid w, \\ 2 & w = 2 \\ 0 & \text{else} \end{cases}$$

and

$$\dim W_{w \leq 2k}\mathrm{H}^{2k-1}_{\acute{e}t,c}(\mathcal{K}_{\mathbb{Q}})^{G,\chi}(-1) = \sum_{i=1}^{2k-1} \dim(E_\infty^{i,2k-1-i})^{G,\chi}_{\bar{\mathbb{Q}}} = 1 + \left\lfloor \frac{k}{2} \right\rfloor + \left(1 + 3\left\lfloor \frac{4k^2}{3} \right\rfloor - 4k^2\right). \quad (5.16)$$

Because E is the successive extensions of $(E_\infty^{i,2k-i-1})_{\mathbb{F}_p}(-1)$, and $W_{\leq 2k}\mathrm{H}^{2k-1}_{\acute{e}t,c}(\mathcal{K}_{\mathbb{Q}})^{G,\chi}(-1)$ is the successive extensions of $(E_\infty^{i,2k-i-1})_{\bar{\mathbb{Q}}}(-1)$ for $i \geq 1$. By (5.13), $\dim \mathrm{gr}^W_w E = \dim \mathrm{gr}^W_w \mathrm{H}^{2k-1}_{\acute{e}t,c}(\mathcal{K}_{\mathbb{Q}})^{G,\chi}(-1)$. So part (3) is verified.

Proof of Lemma 5.15. By (5.5) and (5.11), we have

$$\begin{aligned} \dim \mathrm{im}(\alpha)^{G,\chi} &= \dim \mathrm{H}^{2k-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{Q}}}) - \sum_{i=1}^{2k-1} \dim(E_\infty^{i,2k-1-i})^{G,\chi}_{\bar{\mathbb{Q}}} \\ &= \frac{\binom{k+2}{2} - d(k,3)}{3} - 1 - \left\lfloor \frac{k}{2} \right\rfloor - \left(1 + 3\left\lfloor \frac{4k^2}{3} \right\rfloor - 4k^2\right), \end{aligned}$$

where the last term $\left(1 + 3\lfloor \frac{4k^2}{3} \rfloor - 4k^2\right)$ is in fact $d(k,3)$. As for the dimension of $\mathrm{im}(\beta)^{G,\chi}$, by the property of spectral sequences, (5.13) and (5.16), we know that

$$\begin{aligned} \dim \mathrm{im}(\beta)^{G,\chi} &= \dim \mathrm{H}^{2k-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{F}}_p}) - \sum_{i=1}^{2k-1} \dim(E_\infty^{i,2k-1-i})^{G,\chi}_{\bar{\mathbb{F}}_p} \\ &= \dim \mathrm{H}^1_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3) - 1 - \left\lfloor \frac{k}{2} \right\rfloor - d(k,3). \end{aligned}$$

Moreover, by the Grothendieck-Ogg-Shafarevich formula, we have

$$\dim \mathrm{H}^1_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3) = \frac{\binom{k+2}{2} - d(k,3,p)}{3} + \dim \mathrm{H}^2_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3).$$

By a direct computation, the dimension of $\Delta^{G,\chi}$ is $\frac{d(k,3,p)-d(k,3)}{3}$, the number of classes of singular points of $\mathcal{K}_{\mathbb{F}_p}$. Combine the formula in Proposition 4.27, we get the identity

$$\dim \mathrm{im}(\alpha)^{G,\chi} = \dim \mathrm{im}(\beta)^{G,\chi} + \dim \Delta^{G,\chi} - \dim \mathrm{H}^2_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3).$$

If $p \neq 2$ or k is odd, we have $\dim \mathrm{H}^2_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3) = \dim \mathrm{H}^0_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3) = 0$. So

$$\dim \mathrm{im}(\beta)^{G,\chi} + \dim C = \dim \mathrm{im}(\alpha)^{G,\chi} = \dim \mathrm{im}(\beta)^{G,\chi} + \dim \Delta^{G,\chi}.$$

It follows that $\dim C = \dim \Delta^{G,\chi}$. So the two vertical maps i_1 and i_2 are isomorphisms.

If $p = 2$ and k is even, we have $\dim \mathrm{H}^2_{\acute{e}t,c}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3) = \dim \mathrm{H}^0_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_3) = 1$. Consider the rest of the diagram (5.7), i.e.,

$$\begin{array}{ccccccccc} & & & & & & \mathrm{H}^{2k}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{F}}_p}) & \longrightarrow & \mathrm{H}^{2k}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{Q}}}) \\ & & & & & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \alpha'} \\ \mathrm{H}^{2k-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}}) & \xrightarrow{\gamma} & \oplus_{x\in\Sigma'(p)} \bar{\mathbb{Q}}_\ell(m) & \xrightarrow{\kappa} & \mathrm{H}^{2k}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p}) & \longrightarrow & \mathrm{H}^{2k}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}}) & \longrightarrow & 0 \end{array}$$

where the two vertical maps are the surjective edge map from the abutment $\mathrm{H}^{2k}_{\acute{e}t,c}(\mathcal{K}')$ to $E^{0,2k}$. By the same argument for the cohomology of degree $2k-1$, we have $\mathrm{im}(\alpha') = \mathrm{gr}^W_{2k}\mathrm{H}^{2k}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{Q}}})$, and by (2.21) an exact sequence

$$\mathrm{H}^2_{\acute{e}t,c}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3) \to \mathrm{H}^{2k}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{F}}_p})^{G,\chi}(-1)[\zeta_p] \to \mathbb{Q}_\ell(\zeta_p)(-2)^{G,\chi}, \tag{5.17}$$

Assume temporarily that p is outside the set S, then $\mathrm{im}(\beta') = \mathrm{im}(\alpha')$. Since $\mathrm{H}^2_{\acute{e}t,c}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3) = 0$ and $\mathrm{im}(\beta')$ is pure of weight $2k$, we have $\mathrm{im}(\beta')^{G,\chi} = 0$. This forces $\mathrm{im}(\alpha')^{G,\chi} = 0$, which does not depend on the choice of p.

By the property of the spectral sequence above, we have

$$\mathrm{gr}^W_{2k+2}\mathrm{H}^2_{\acute{e}t,c}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}_3) \simeq \mathrm{gr}^W_{2k+2}\mathrm{H}^{2k}_{\acute{e}t,c}(\mathcal{K}'_{\bar{\mathbb{F}}_p})^{G,\chi}(-1) = \mathrm{im}(\beta)^{G,\chi}(-1).$$

By (5.17), the image of β' is nonzero, because $\mathrm{gr}^W_{2k}\mathrm{H}^{2k}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{F}}_p})^{G,\chi} \neq 0$. Since $\mathrm{im}(\beta')^{G,\chi}$ is mapped to $\mathrm{im}(\alpha')^{G,\chi} = 0$, the image of κ is nonzero. Therefore, in (5.14), the cokernel of i_2 has dimension at least 1. By the inequality

$$\dim \mathrm{im}(\beta)^{G,\chi} + \dim C = \dim \mathrm{im}(\alpha)^{G,\chi} \geq \dim \mathrm{im}(\beta)^{G,\chi} + \dim \Delta^{G,\chi} - 1,$$

we know that $\dim C = \dim \Delta^{G,\chi} - 1$. So the vertical map i_1 is an isomorphism, and the cokernal of i_2 is 1-dimensional. □

□

5.1.4 The p-adic case

We study the p-adic Galois representations $(\mathrm{M}^\lambda_{n+1})_p$ in this section.

Proposition 5.18. *The p-adic representation $(\mathrm{M}^\lambda_{n+1})_p$ is de Rham. Let S a finite set of primes containing all primes p such that $p \mid n+1$ or $\bar{\mathcal{K}}'$ has bad reduction at p. If $p \notin S$, the representation $(\mathrm{M}^\lambda_{n+1})_p$ is crystalline and there is an isomorphism of Frobenius modules*

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_m/K, \mathrm{Kl}^\lambda_{n+1}) \simeq ((\mathrm{M}^\lambda_{n+1})_p \otimes \mathbf{B}_{crys})^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} \otimes K.$$

Proof. As in Section 5.1.1, we let $\mathcal{K}'$ be $\mathcal{K}$ if $\gcd(n+1,k)=1$ and the blow up of $\mathcal{K}$ along singular locus otherwise. By [Bei13, §3.3(i) and §3.4], since the p-adic representation $\mathrm{H}^{nk-1}_{\acute{e}t}(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_p)$ comes from a proper smooth variety, it is de Rham. Then we conclude the first assertion by the fact that the subquotient of a de Rham representation remains de Rham.

Now assume that $\gcd(p, n+1)=1$ and $\mathcal{K}'$ has good reduction at p. Then by the p-adic comparison theorem, the representation $\mathrm{H}^{nk-1}_{\acute{e}t}\big(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}_p}\big)$ is crystalline. Therefore, as a subquotient of $\mathrm{H}^{nk-1}_{\acute{e}t}\big(\bar{\mathcal{K}}'_{\bar{\mathbb{Q}}_p}\big)$, the representation $(\mathrm{M}^\lambda_{n+1})_p$ remains crystalline.

Recall that we have an isomorphism

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_m/K, \mathrm{Kl}^\lambda_{n+1}) \simeq \mathrm{gr}^W_{n|\lambda|-1}\mathrm{H}^{n|\lambda|-1}_{\mathrm{rig,c}}(\mathcal{K}/K)(-1)^{G_\lambda \times \mu_{n+1}, \chi_\lambda}[\varpi]$$

from Section 2.3.2. We have results similar to those in Lemma 5.1, by simply replacing étale cohomology by rigid cohomology everywhere. Consider the spectral sequence [Stu07, Prop. 8.2.17 and 8.2.18(ii)]

$$E_1^{i,j} = \mathrm{H}^j_{\mathrm{rig}}\big(\bar{\mathcal{K}}'^{(i)}_{\mathbb{F}_p}/\mathbb{Q}_p\big) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{rig,c}}(\mathcal{K}'_{\mathbb{F}_p}/\mathbb{Q}_p),$$

and we denote by

$$\alpha\colon \mathrm{H}^{n|\lambda|-1}\big(\bar{\mathcal{K}}'_{\bar{\mathbb{F}}_p}/\mathbb{Q}_p\big) \to \mathrm{H}^{n|\lambda|-1}(\bar{\mathcal{K}}'^{(1)}_{\bar{\mathbb{F}}_p}/\mathbb{Q}_p)$$

the differential from $E_1^{0,n|\lambda|-1}$ to $E_1^{1,n|\lambda|-1}$. Since the varieties $\bar{\mathcal{K}}'^{(i)}$ are smooth proper for all $i \geq 1$, the only contribution of weight $n|\lambda|-1$ to the abutment of the spectral sequence comes from the kernel of α. So

$$\mathrm{gr}^W_{n|\lambda|-1}\mathrm{H}^{n|\lambda|-1}_{\mathrm{rig,c}}(\mathcal{K}'_{\bar{\mathbb{F}}_p}/\mathbb{Q}_p) \simeq \mathrm{gr}^W_{n|\lambda|-1} \ker \alpha. \tag{5.19}$$

Then use the analog of (5.5), (5.19) and the p-adic comparison theorem, we get the isomorphism of Frobenius modules

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_m/K, \mathrm{Kl}^\lambda_{n+1}) \simeq ((\mathrm{M}^\lambda_{n+1})_p \otimes \mathbf{B}_{crys})^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} \otimes K.$$

□

5.2 The functional equations of L-functions attached to $\mathrm{Sym}^k\mathrm{Kl}_3$

For a prime p not in the finite set $S = S(k,3)$ from Theorem 5.12, we put

$$L(k,3,p;T) := \det(1 - \mathrm{Frob}_p T \mid (\mathrm{M}^k_3)_\ell)^{-1}$$

The *partial L-function* is defined by the Euler product

$$L^S(k,3;s) := \prod_{p \notin S} L(k,3,p;p^{-s}).$$

By considering the weight of Frobenius, we can show that it converges absolutely on the domain $\{s \mid \Re(s) > 1 + \frac{2k+1}{2}\}$. The aim of this section is to prove Theorem A.1, i.e., the function $L(k,3,p;T)$ extends to a meromorphic function on $\mathbb{C}$, and satisfies a functional equation. Although we do not give a precise formula of the functional equation, we can try to use similar computations as those in [FSY22, §5] to get as much information as possible.

Proof of Theorem A.1. Let S be the set of bad primes in Proposition 5.18. We first consider the L-function of the family of semisimplifications of ℓ-adic Galois representations $\mathscr{R} = \{(\mathrm{M}_3^k)_\ell^{ss}\}$, i.e., by replacing $(\mathrm{M}_3^k)_\ell$ with $(\mathrm{M}_3^k)_\ell^{ss}$ in the definition of local factors $L(k, 3, p; T)$. The goal is to prove the potential automorphy of $\mathscr{R}$.

We can verify that the family $\mathscr{R}$ is weakly compatible. In fact, it suffices to show that the conditions (1), (2), and (3) in Definition 1.20 hold for $\{(\mathrm{M}_3^k)_\ell\}$. The condition (1) is guaranteed by Theorem 5.12, and the condition (2) also holds because of Proposition 5.18. As for the condition (3), we choose an embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$ and use the p-adic comparison theorem. More precisely, we have

$$\begin{aligned}((\mathrm{M}_3^k)_p \otimes \mathbf{B}_{\mathrm{dR}})^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} \otimes \mathbb{C} &= (\mathrm{gr}^W_{2k+1}\mathrm{H}^{2k-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p)^{G,\chi}(-1) \otimes \mathbf{B}_{\mathrm{dR}})^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} \otimes \mathbb{C} \\ &= \mathrm{gr}^W_{2k+1}\mathrm{H}^{2k-1}_{\acute{e}t,c}(\mathcal{K}_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p)^{G,\chi}(-1) \otimes \mathbb{C} \simeq (\mathrm{M}_3^k)_{\mathrm{dR}}.\end{aligned}$$

To apply Theorem 1.21 to $\mathscr{R}$, we need to check the conditions in Theorem 1.21. The purity is satisfied because the Galois representations $(\mathrm{M}_3^k)_\ell$ (as well as their semisimplifications) are pure of weight $2k+1$. The regularity is also satisfied under the condition $k \leq 9$, because by Theorem D and the comparison isomorphism above, we know that the multiplicities of Hodge-Tate numbers of $(\mathrm{M}_3^k)_\ell$ (as well as their semisimplifications) are 0 or 1. So it suffices to check the odd essential self-duality.

By the perfect pairing in Proposition 2.3, the representations $(\mathrm{M}_3^k)_\ell$ factor through $\mathrm{GSP}((\mathrm{M}_3^k)_\ell)$ or $\mathrm{GSO}((\mathrm{M}_3^k)_\ell)$ with similitude character χ_{cyc}^{2k+1}. Choose a generator of $\bar{\mathbb{Q}}_\ell(-2k-1)$, and regard the perfect paring as a compatible nondegenerate bilinear form on the module $(\mathrm{M}_3^k)_\ell$ on the group ring of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ with the involution $g \mapsto \chi_{cyc}^{-2k-1}(g)g^{-1}$. By [Ser18, Thm. 4.2.1], the semi-simplification also factors through GSP or GSO with the same character. So we proved the odd essential self-duality.

It follows from Theorem 1.21 that $\mathscr{R}$ is potential automophic, and the partial L-function $L^S(\mathscr{R}, s)$ can be extended to a meromorphic function on $\mathbb{C}$, and satisfies the expected functional equation.

By Theorem 5.12 and the Lemma 5.20 below, the local factors of the semisimplifications of $(\mathrm{M}_3^k)_\ell$ coincide with those of $(\mathrm{M}_3^k)_\ell$ for each $p \notin S$. So $L^S(\mathscr{R}, s) = L^S(k, 3; s)$. By Corollary 1.22 the partial L-function $L^S(k, 3; s)$ can be completed to

$$\Lambda_k(s) = L_\infty(\mathscr{R}, s) \cdot L\left(\mathscr{R}, s\right),$$

which extends meromorphically to the whole complex plane, and satisfies a functional equation

$$\Lambda_k(s) = \varepsilon(\mathscr{R}, s)\Lambda_k(2k+2-s).$$

Lemma 5.20 ([FSY22, Lem. 5.40])**.** *Let p and ℓ be distinct prime numbers, and ρ: $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ an ℓ-adic representation. Suppose that there exists an increasing sequence*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_c = V$$

of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-stable subspaces such that the WD-representation associated with the induced representation $\bar{\rho}$ on $\bar{V} = \oplus_{i=1}^c V_i/V_{i-1}$ is pure. Then the WD-representation associated with ρ is also pure, and ρ and $\bar{\rho}$ have the same L-functions and ε-factors.

□

Remark 5.21. Yun constructed motives attached to Kloosterman sheaves for reductive groups [YV15]. When $G = \mathrm{SL}_{n+1}$, and for primes p outside a finite set of primes, the trace of Frob_p of the ℓ-adic realizations of these motives equals to the trace of Frob_p on M_{n+1}^λ. By Chebotarev density theorem, the two families of Galois representations have the same semi-simplification. Hence, they have the same Hodge-Tate numbers. So to prove Theorem A, we can also use the L-functions of motives constructed by Yun.

5.3 Conjectures of Evans type

In this section, we apply the computations of moments of Kloosterman sums to get some explicit information of the modular forms associated with Kloosterman sheaves in Theorem B, using an argument similar to that in [YV15, §4 & Appx.] with the help of the database LMFDB [LMF22].

5.3.1 $\mathrm{Sym}^4\mathrm{Kl}_3$

The motive M_3^4 from definition 2.1 is defined over $\mathbb{Q}$, pure of weight 9 and equipped with a skew-symmetric perfect pairing in Proposition 2.3. It is of dimension 2 and the Hodge numbers $h^{p,9-p}$ of its de Rham realization are 1 if $p = 3$ or 6 and 0 otherwise by Theorem D. We want to show that the compatible system of Galois representations $\{(\mathrm{M}_3^4)_\ell(6)\}$ is modular.

Theorem 5.22. *There exists a (unique) normalized cusp Hecke eigenform f in $S_4(\Gamma_0(14))$, such that for all primes $p \notin \{2, 7\}$, the Fourier coefficient $a_p(f)$ satisfies*

$$a_f(p) = -\frac{1}{p^3}(m_3^4(p) + 1 + p^2 + p^4),$$

where $m_3^4(p)$ is the symmetric power moments of $\mathrm{Sym}^4\mathrm{Kl}_3$. In particular, the label of this modular form in the database LMFDB is 14.4.a.b.

Proof. We know that the motive M_3^4 is of dimension 2, pure of weight 9, whose Hodge numbers are given by $h^{3,6} = h^{6,3} = 1$. Thus, the Hodge-Tate weight of $(\mathrm{M}_3^4)_\ell(6)$ is $(0, 3)$, with multiplicity 1 by the p-adic comparison theorem.

By Theorem 4.41, we have

$$\chi_{cyc}^6(p) \cdot \varepsilon\big(\mathbb{G}_{m,\mathbb{F}_p}, \mathrm{Sym}^4\mathrm{Kl}_3\big) = p^3$$

for $p \neq 2, 7$. By Chebotarev density theorem, the determinant $\det\big((\mathrm{M}_3^4)_\ell(6)\big)$ is χ_{cyc}^3. Since $\chi_{cyc,\ell}(c) = -1$, the representations $(\mathrm{M}_3^4)_\ell(6)$ are odd. So $\{(\mathrm{M}_3^4)_\ell(6)\}$ is modular by Theorem 1.17. By the exact sequence (1.7) and Proposition 4.3 and Proposition 4.26, we deduce that

$$\mathrm{Tr}(\mathrm{Frob}_p \mid (\mathrm{M}_3^4)_\ell) = -(m_3^4(p) + 1 + p + p^2).$$

It follows that for any $p \notin S \cup \{\ell\}$,

$$\begin{aligned} a_f(p) &= \mathrm{Tr}(\mathrm{Frob}_p^{-1} \mid (\mathrm{M}_3^4)_\ell(6)) \\ &= \det\big((\mathrm{M}_3^4)_\ell(6)\big)^{-1} \cdot \mathrm{Tr}(\mathrm{Frob}_p \mid (\mathrm{M}_3^4)_\ell(6)) \\ &= -\frac{1}{p^3}(m_3^4(p) + 1 + p + p^2). \end{aligned}$$

It suffices to determine the modular form. According to Theorem 5.12, the ℓ-adic representation $(\mathrm{M}_3^4)_\ell$ is unramified at p if and only if there $\dim \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^4\mathrm{Kl}_3\big) = 2$, and the representation is at most tamely ramified at every prime. If $p = 3$, the representation $(\mathrm{M}_3^4)_\ell$ is unramified by Remark 5.9. If $p \neq 3$, then $(\mathrm{M}_3^4)_\ell$ is unramified if $d(4, 3, p) = 0$, i.e., $p \neq 2, 7$. Moreover, by Theorem 5.12, the dimensions of $(\mathrm{M}_3^4)_\ell^{I_p}$ are 1 for $p = 2$ and 7. Therefore, the conductor of the compatible family $\{(\mathrm{M}_3^4)_\ell\}_\ell$ is 14.

In conclusion, the weight and the level of the corresponding modular form is $k = 4$, $N_f = 14$. By computing the Fourier coefficient $a_f(3)$ in Appendix A.1.1, we know that this f is labeled 14.4.a.b in the database LMFDB. □

5.3.2 $\mathrm{Sym}^3\mathrm{Kl}_4$

The motive M_4^3 is defined over $\mathbb{Q}$, pure of weight 10 and equipped with a symmetric perfect pairing in Proposition 2.3. It is of dimension 2 and the Hodge numbers $h^{p,10-p}$ of its de Rham realization are 1 if $p = 4$ or 6 and 0 otherwise by Proposition 3.57. We want to show that the compatible family of Galois representations $\{(\mathrm{M}_4^3)_\ell(6)\}$ is modular.

Theorem 5.23. *There exists a (unique) normalized cusp Hecke eigenform f in $S_3\big(\Gamma_0(15), \left(\frac{\cdot}{15}\right)\big)$ with complex multiplication, such that for all primes $p \notin \{2, 5\}$, the Fourier coefficient $a_f(p)$ satisfies*

$$a_f(p) = -\left(\frac{p}{15}\right)\frac{1}{p^4}(m_4^3(p) + 1 + p^2 + p^3). \tag{5.24}$$

Here $m_4^3(p)$ is the symmetric power moments of $\mathrm{Sym}^3\mathrm{Kl}_4$. In particular, the label of the corresponding modular form is 15.3.d.a in the database LMFDB.

Proof. By the p-adic comparison theorem and our computation on the Hodge numbers of the motive M_4^3, we know that the Hodge-Tate weight of $\{(\mathrm{M}_4^3)_\ell(6)\}$ is $(0, 2)$. Notice that these Galois representations are orthogonal, because we have a symmetric perfect paring on the motive M_4^3 in Proposition 2.3. By [Liv95, 1.4(2)], the associated Galois representation $\{(\mathrm{M}_4^3)_\ell(6)\}$ corresponds to a normalized cusp Hecke eigenform $f = q + \sum_{n=2}^{\infty} a_n q^n \in S_3(N_f, \varepsilon_f)$ of complex multiplication for some characters $\varepsilon_f \colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$. Now it suffices to determine the information of the modular form.

By the computation of ε-factors in Proposition 4.41, the values of the determinants of the Galois representations $(\mathrm{M}_4^3)_\ell$ at Frob_p are

$$\chi_{cyc}^{12}(p) \cdot \varepsilon\big(\mathbb{G}_{m,\mathbb{F}_p}, \mathrm{Sym}^3\mathrm{Kl}_4\big) = \left(\frac{p}{15}\right) p^{-2}$$

if $p > 5$. Hence, the determinants of the representations $(\mathrm{M}_4^3)_\ell(6)$ are $\left(\frac{\cdot}{15}\right)\chi_{cyc}^{-2}$ by the Chebotarev density theorem. It follows that the Nebentypus character ε_f of the modular form is $\left(\frac{\cdot}{15}\right)$.

Moreover, for any $p \notin \{2, 3, 5\} \cup \{\ell\}$ we deduce that

$$\begin{aligned} a_f(p) &= \mathrm{Tr}(\mathrm{Frob}_p^{-1} \mid (\mathrm{M}_4^3)_\ell(6)) \\ &= \det\big((\mathrm{M}_3^4)_\ell(6)\big)^{-1} \cdot \mathrm{Tr}(\mathrm{Frob}_p \mid (\mathrm{M}_4^3)_\ell(6)) \\ &= -\left(\frac{p}{15}\right)\frac{1}{p^4}(m_4^3(p) + 1 + p^2 + p^3). \end{aligned}$$

By Proposition 4.27, we can compute the dimension of middle ℓ-adic cohomology at $p \neq 2$. We know that $(\mathrm{M}_4^3)_\ell$ is unramified if $p \neq 2, 3, 5$, and tamely ramified if $p = 3, 5$. Moreover, $(\mathrm{M}_4^3)_\ell^{I_p} \simeq \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_4\big)$ has dimension 1. Then Theorem 5.4 ensures that the conductor N of $\{(\mathrm{M}_4^3)_\ell\}$ is of the form $2^s \cdot 15$ for some $s \in \mathbb{Z}_{\geq 0}$.

Lemma 5.25. *The representations $(\mathrm{M}_4^3)_\ell$ are unramified at $p = 2$. In particular, the level N_f of the modular form is 15.*

Proof. At $p = 2$, the Swan conductor of $\mathrm{Sym}^3\mathrm{Kl}_4$ is at most 5. Since the monodromy group of Kl_4 is Sp_4 and the symmetric power of standard representation of SP_4 remains irreducible, the 0-th cohomology $\mathrm{H}^0_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_4)$ is 0. By (1.7) and Grothendieck-Ogg-Shafarevich formula, we know that

$$\dim \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_4\big) = \mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_4) - 3 - \dim(\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty}.$$

Hence, we have $3 \leq \mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_4) \leq 5$.

By Appendix A.2.1, the trace of Frobenius at $p = 2$ on $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_4\big)$ is

$$-(m_4^3(p) + 1 + p^2 + p^3 + \mathrm{Tr}(\mathrm{Frob}_p \mid (\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty})) = -16 - \mathrm{Tr}(\mathrm{Frob}_p \mid (\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty}).$$

- If $\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_4) = 5$, the sheaf $\mathrm{Sym}^3\mathrm{Kl}_4$ only has one slope ($=1/4$) at ∞, which implies that $(\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty} = 0$. So the dimension of the middle ℓ-adic cohomology is 2. By Remark 5.9, the representation $(\mathrm{M}_4^3)_\ell$ is unramified at 2.
- If $\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_4) = 4$, then $\dim(\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty} \leq 1$.
  - Assume that $\dim(\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty} = 1$, the middle ℓ-adic cohomology is 0. The trace of Frob_2 on $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_4\big)$ is 0. So we have
  $$0 = -16 - \mathrm{Tr}(\mathrm{Frob}_p \mid (\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty}).$$
  This is impossible because $(\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty}$ is pure of weight 9 and of dimension 1.
  - Assume that $\dim(\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty} = 0$, the middle ℓ-adic cohomology is 1. The trace of Frob_2 on $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_4\big)$ is -16. Since $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_4\big)$ is pure of weight 10 and of dimension 1, this is impossible.
- If $\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_4) = 3$, then $\dim(\mathrm{Sym}^3\mathrm{Kl}_4)^{I_\infty} = 0$. So the dimension of the middle ℓ-adic cohomology is 0. However, the trace of Frob_2 on $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_4\big)$ is at the same time 0 and -16, which is absurd.

In conclusion, we have $\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_4) = 5$ and the representation $(\mathrm{M}_4^3)_\ell$ is unramified at 2. As a consequence, the conductor N is $2^0 \cdot 15 = 15$, which implies that the level of the modular form is 15. □

In conclusion, the modular form that we want is of weight $k = 3$, level 15, and with nebentypus $\varepsilon_f = (\frac{\cdot}{15})$. At last, we search the modular form in the database LMFDB, using the level, weight, character, and the trace $a_f(2) = -1$ of $(\mathrm{M}_4^3)_\ell(6)$ in section A.1.1. There is only one modular form left, labeled 15.3.*d.b* in LMFDB. □

5.3.3 $\mathrm{Sym}^4\mathrm{Kl}_4$

The motive M_4^4 is defined over $\mathbb{Q}$, pure of weight 13 and equipped with an antisymmetric perfect self-pairing. It is of dimension 2.

Theorem 5.26. *There exists a (unique) normalized cusp Hecke eigenform f in $S_6(\Gamma_0(10))$, such that for all primes $p \notin \{2, 5\}$, the Fourier coefficient $a_f(p)$ satisfies*

$$a_f(p) = -\frac{1}{p^4}(m_4^4(p) + 1 + p^2 + p^3 + p^4 + 2p^6),$$

where $m_4^4(p)$ is the symmetric power moments of $\mathrm{Sym}^4\mathrm{Kl}_4$. In particular, the label of the corresponding modular form is 10.6.a.a. in the database LMFDB.

Proof. There exists an integer $h \in \{0, 1, \ldots, 6\}$ such that the Hodge numbers of M_4^4 are

$$h^{p,13-p} = \begin{cases} 1 & p = h, 13-h \\ 0 & \text{else} \end{cases}$$

by the Hodge symmetry. The Hodge-Tate numbers of $(\mathrm{M}_4^4)_\ell(13-h)$ are $(0, 13-2h)$. The determinant of the Galois representations $(\mathrm{M}_4^4)_\ell(13-h)$ is an odd character χ_{cyc}^{13-2h} by Theorem 4.41 and Chebotarev density theorem. We use Theorem 1.17 to show the existence of the modular form.

Let S be the set of bad reductions of $(\mathrm{M}_4^4)_\ell(13-h)$. It follows that for any $p \notin S \cup \{\ell\}$,

$$\begin{aligned} a_f(p) &= \mathrm{Tr}(\mathrm{Frob}_p^{-1} \mid (\mathrm{M}_4^4)_\ell(13-h)) \\ &= \det(\mathrm{M}_4^4)_\ell(13-h))^{-1} \cdot \mathrm{Tr}(\mathrm{Frob}_p \mid (\mathrm{M}_4^4)_\ell(13-h)) \\ &= -\frac{1}{p^h}(m_4^4(p) + 1 + p^2 + p^3 + p^4 + 2p^6). \end{aligned}$$

One can compute that the dimension of the middle ℓ-adic cohomology of $\mathrm{Sym}^4\mathrm{Kl}_4$ is 1 when $p=5$ by 5.11. By Theorem 5.4, the representation $(\mathrm{M}_5^3)_\ell$ is tamely ramified at $p=5$, and possibly wildly ramified at $p=2$. So the conductor of $\{(\mathrm{M}_5^3)_\ell\}_\ell$ is $N=2^s\cdot 5$ for some $0\le s\le 8$.

At last, we can compute the Fourier coefficients $a_f(3)=-26\cdot 3^{4-h}$ and $a_f(7)=-22\cdot 7^{4-h}$ by results in Appendix A.2.2. We try $0\le h\le 6$ and $0\le s\le 8$ one by one.If $(s,h)=(8,0),(8,1),(8,2)$ $(8,3)$, $(8,4)$, $(7,0)$, $(7,1)$, $(7,2)$, $(7,3)$, $(6,0)$ or $(6,1)$, we compute the space of cuspidal new modular symbols over the finite field $\mathbb{F}_p$. We find that for some primes p, the numbers $a_f(p)$ are not roots of the characteristic polynomials of the Hecke operators T_p, see the table in Appendix A.2.2. Otherwise, we can use LMFDB. We find two modular forms left of weight 6 and having the prescribed Fourier coefficients. By the information of the level, there is only one left with label 10.6.*a.a.* in LMFDB, because the other one is of level 400. □

Remark 5.27. We deduce from the above proof that the nonzero Hodge numbers of the de Rham realization of M_4^4 are $h^{4,9}=h^{9,4}=1$. Although we did not compute the Hodge numbers directly, they can still be calculated by an argument similar to that of M_3^{3k} in section 3.3.5.

5.3.4 $\mathrm{Sym}^3\mathrm{Kl}_5$

The motive M_5^3 is defined over $\mathbb{Q}$, pure of weight 13 and equipped with an antisymmetric perfect pairing. It is of dimension 2. By Proposition 3.58, the Hodge numbers $h^{p,13-p}$ of its de Rham realization are 1 if $p=5$ or 8 and 0 otherwise. We want to show that the compatible family of Galois representations $\{(\mathrm{M}_5^3)_\ell\}_\ell(8)$ is modular.

Theorem 5.28. *There exists a (unique) normalized cusp Hecke eigenform f in $S_4(\Gamma_0(33))$, such that for all primes $p\notin\{3,11\}$, the Fourier coefficient $a_f(p)$ satisfies*

$$a_f(p)=-\frac{1}{p^5}(m_5^3(p)+1+p^2+p^3+p^4+p^6),$$

where $m_5^3(p)$ is the symmetric power moments of $\mathrm{Sym}^5\mathrm{Kl}_3$. In particular, the label of the corresponding modular form is 33.4.a.b in the database LMFDB.

Proof. Consider the Galois representations $(\mathrm{M}_5^3)_\ell(5)$. Their determinants are the odd characters χ_{cyc}^{-3} by Theorem 4.41 and the Chebotarev density theorem. We use Theorem 1.17 to show the existence of the modular form. Let S be the set of bad reductions of M_5^3. It follows that for any $p\notin S\cup\{\ell\}$ we have

$$\begin{aligned}a_f(p)&=\mathrm{Tr}(\mathrm{Frob}_p^{-1}\mid(\mathrm{M}_4^4)_\ell(9))\\&=\det\big((\mathrm{M}_4^4)_\ell(9)\big)^{-1}\cdot\mathrm{Tr}(\mathrm{Frob}_p\mid(\mathrm{M}_4^4)_\ell(9))\\&=-\frac{1}{p^4}(m_4^4(p)+1+p^2+p^3+p^4+p^6).\end{aligned}$$

One can compute that the set of primes of bad reduction of M_5^2 is a subset of $\{3,5,11\}$ by Theorem 5.4, which implies that the conductor of $\{(\mathrm{M}_5^3)_\ell(5)\}$ is $3^s\cdot 5^t\cdot 11^e$ for some $0\le s,e\le 2$ and $0\le t$.

Lemma 5.29. *The representations $(\mathrm{M}_5^3)_\ell$ are unramified at $p=5$.*

Proof. At $p=5$, the Swan conductor of $\mathrm{Sym}^3\mathrm{Kl}_5$ is at most 7. Since the monodromy group of Kl_5 is SL_5 and the symmetric power of standard representation of SL_5 remains irreducible, the 0-th cohomology $\mathrm{H}^0_{\acute{e}t}(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Sym}^3\mathrm{Kl}_5)$ is 0. By (1.7) and Grothendieck-Ogg-Shafarevich formula, we know that

$$\dim\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Sym}^3\mathrm{Kl}_5\big)=\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_5)-5-\dim(\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty}.$$

Hence, we have $5\le\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_5)\le 7$. By Appendix A.3, the trace of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p},\mathrm{Sym}^3\mathrm{Kl}_5\big)$ at prime $p=5$ is

$$-(m_5^3(p)+1+p^2+p^3+p^4+p^6+\mathrm{Tr}(\mathrm{Frob}_p\mid(\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty}))=-4\cdot 5^5-\mathrm{Tr}(\mathrm{Frob}_p\mid(\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty}).$$

- If $\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_5) = 7$, the sheaf $\mathrm{Sym}^3\mathrm{Kl}_5$ only has one slope ($=1/5$) at ∞, which implies that $\dim(\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty} = 0$. So the dimension of the middle ℓ-adic cohomology is 2. By Remark 5.9, the representation $(\mathrm{M}_5^3)_\ell$ is unramified at 5.

- If $\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_5) = 6$, then $\dim(\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty} \leq 1$.

  1. Assume that $\dim(\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty} = 1$, the middle ℓ-adic cohomology is 0. The trace of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_5\big)$ at prime $p = 5$ is 0. So we have
  $$0 = -4 \cdot 5^5 - \mathrm{Tr}(\mathrm{Frob}_p \mid (\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty}).$$
  Since $(\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty}$ is pure of weight 12 and of dimension 1, this is impossible.
  2. Assume that $\dim(\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty} = 0$, the middle ℓ-adic cohomology is 1. The trace of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_5\big)$ at prime $p = 5$ is $-4 \cdot 5^5$. Since $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_5\big)$ is pure of weight 13 and of dimension 1, this is impossible.

- If $\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_5) = 5$, then $\dim(\mathrm{Sym}^3\mathrm{Kl}_5)^{I_\infty} = 0$. So the dimension of the middle ℓ-adic cohomology of $\mathrm{Sym}^3\mathrm{Kl}_5$ is 0. The trace of $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_5\big)$ at prime $p = 5$ is at the same time 0 and $-4 \cdot 5^5$, which is absurd.

In conclusion, we have $\mathrm{Sw}(\mathrm{Sym}^3\mathrm{Kl}_5) = 7$ and the representation $(\mathrm{M}_5^3)_\ell$ is unramified at 5. □

So the modular form is of weight 4, of level $N_f = 3^s \cdot 11^e \leq 1089$, and of nebentypus the trivial character. Moreover, we compute the Fourier coefficients $a_2 = -1$ and $a_5 = -4$ in Appendix A.3. By this information, there is only one modular form left, of weight 3 and of level $N = 33$, whose label is 33.4.a.b in LMFDB. □

5.3.5 $\mathrm{Kl}_3^{(2,1)}$

The motive $\mathrm{M}_3^{(2,1)}$ is defined over $\mathbb{Q}$, pure of weight 9 and equipped with a skew-symmetric perfect pairing in Proposition 2.3. It is of dimension 2 and the Hodge numbers $h^{p,9-p}$ of its de Rham realization is 1 if $p = 4$ or 5 and is 0 otherwise. We want to show that the compatible family of Galois representations $\{\big(\mathrm{M}_2^{(2,1)}\big)_\ell(5)\}$ is modular.

Theorem 5.30. *There exists a (unique) normalized cusp Hecke eigenform $f \in S_2(\Gamma_0(14))$, such that for all primes $p \notin \{2, 3, 7\}$, the Fourier coefficient a_p satisfies*

$$a_f(p) = -\frac{1}{p^4}\Big(m_3^{(2,1)}(p) + p + p^2 + p^3\Big),$$

where $m_3^{(2,1)}(p)$ is the moments of the sheaf $\mathrm{Kl}_3^{(2,1)}$. In particular, this modular form is labeled 14.2.a.a *in the database LMFDB.*

Proof. The sheaf $\mathrm{Kl}_3^{(2,1)}$ is tamely ramified at 0 and wildly ramified at ∞. By Grothendieck-Ogg-Shafarevich formula, the dimension of the ℓ-adic cohomology equals to the Swan conductor at ∞. Similar to Proposition 4.38, the Swan conductor at ∞ is 5, because $\mathrm{Kl}_3^{(2,1)} \subset \mathrm{Kl}_3^{\otimes 4}$ and ζ_3 acts on $(\mathrm{Kl}_3^{\otimes 4})_{\eta_\infty}$ freely. Combining Proposition 4.25, we conclude that

$$\mathrm{Sw}_\infty\big(\mathrm{Kl}_3^{(2,1)}\big) = \begin{cases} 5 & p \neq 2, 7 \\ 4 & p = 2 \\ 3 & p = 7 \end{cases} \quad \text{and} \quad \dim\big(\mathrm{Kl}_3^{(2,1)}\big)_{\bar{\eta}_\infty}^{I_\infty} = \begin{cases} 5 & p \neq 2, 7 \\ 4 & p = 2 \\ 3 & p = 7 \end{cases}$$

by a direct computation. By the exact sequence (1.7) and Proposition 4.2, we have

$$\dim \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_3^{(2,1)}\big) = \begin{cases} 2 & p \neq 2,7 \\ 1 & p = 2,7 \end{cases}$$

and

$$\mathrm{Tr}\big(\mathrm{Frob}_p, \big(\mathrm{M}_3^{(2,1)}\big)_\ell(4)\big) = -p^{-4}(m_3^{(2,1)}(p) + p + p^2 + p^3).$$

By Theorem 5.4, the set of bad primes S is a subset of $\{2,3,7\}$ and the conductor of $\{\big(\mathrm{M}_3^{(2,1)}\big)_\ell\}_\ell$ is 14.

By Proposition 3.59, the Hodge numbers $h^{p,9-p}$ of its de Rham realization is 1 if $p = 4$ or 5 and is 0 otherwise. By Theorem 4.41 and the Chebotarev density theorem, the determinant of $\big(\mathrm{M}_3^{(2,1)}\big)_\ell(5)$ is χ_{cyc}^{-1}, which is an odd character. Then use Theorem 1.17 to show the existence of the modular form. It follows that for any $p \notin S \cup \{\ell\}$,

$$\begin{aligned} a_f(p) &= \mathrm{Tr}\big(\mathrm{Frob}_p^{-1} \mid \big(\mathrm{M}_3^{(2,1)}\big)_\ell(5)\big) \\ &= \det\big(\big(\mathrm{M}_3^{(2,1)}\big)_\ell(5)\big)^{-1} \cdot \mathrm{Tr}\big(\mathrm{Frob}_p \mid \big((\mathrm{M}_3^{(2,1)})_\ell(5)\big)\big) \\ &= -\frac{1}{p^4}\big(m_3^{(2,1)}(p) + p + p^2 + p^3\big). \end{aligned}$$

At last, by computations of Fourier coefficients $a_f(p)$ in Appendix A.1.2 for $p \leq 23$, we can determine the modular form in the database LMFDB. □

5.3.6 $\mathrm{Kl}_3^{(2,2)}$

The motive $\mathrm{M}_3^{(2,2)}$ is defined over $\mathbb{Q}$, pure of weight 13 and equipped with an antisymmetric perfect pairing in Proposition 2.3. We want to show the following:

Theorem 5.31. *There exists a (unique) normalized cusp Hecke eigenform $f = q + \sum_{n\geq 2}^{\infty} a_n q^n \in S_4(\Gamma_0(6))$, such that for all primes $p \notin \{2,3\}$, the Fourier coefficient $a_f(p)$ satisfies*

$$a_f(p) = -\frac{1}{p^5}\Big(m_3^{(2,2)}(p) + p^2 + p^3 + 2p^4 + 2p^6\Big),$$

where $m_3^{(2,2)}(p)$ is the moments of the sheaf $\mathrm{Kl}_3^{(2,2)}$. In particular, this modular form is labeled 6.4.a.a in the database LMFDB, the same as the modular form corresponding to $\mathrm{Sym}^6\mathrm{Kl}_2$.

Proof. The sheaf $\mathrm{Kl}_3^{(2,2)}$ is tamely ramified at 0 and wildly ramified at ∞. By Proposition 4.2, Proposition 4.25, and the long exact sequence (1.7), we know that

- The dimension of the ℓ-adic cohomology equals to the Swan conductor at ∞, which is 8 if $p \neq 2,3$.
- $\dim\big(\mathrm{Kl}_3^{(2,2)}\big)^{I_0}_{\bar{\eta}_0} = 5$ and $\dim\big(\mathrm{Kl}_3^{(2,2)}\big)^{I_\infty}_{\bar{\eta}_\infty} = \begin{cases} 1 & p \neq 2,3 \\ 2 & p = 2 \end{cases}$.
- $\dim \mathrm{H}^1_{\acute{e}t,\mathrm{mid}}\big(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_3^{(2,2)}\big) = \begin{cases} 2 & p \neq 2,3 \\ 1 & p = 2 \end{cases}$.
- $\mathrm{Tr}\big(\mathrm{Frob}_p, \big(\mathrm{M}_3^{(2,2)}\big)_\ell(5)\big) = -p^{-5}(m_3^{(2,2)}(p) + p^2 + p^3 + 2p^4 + 2p^6)$ if $p \neq 2,3$.

By Theorem 5.4, the set of bad primes S is a subset of $\{2,3\}$, and $\dim\big(\mathrm{M}_3^{(2,2)}\big)_\ell = 2$. Because $\big(\mathrm{M}_3^{(2,2)}\big)_\ell$ is at most tamely ramified at 2 and $\dim\big(\mathrm{M}_3^{(2,2)}\big)_\ell^{I_2} \geq 1$, the conductor of $\{\big(\mathrm{M}_3^{(2,2)}\big)_\ell\}_\ell$ is of the form $N = 2^s \cdot 3$ for some $0 \leq s \leq 8$.

There exists an integer $h \in \{0, 1, \dots, 6\}$ such that the Hodge numbers $h^{p,13-p}$ are $\begin{cases} 1 & p = h, 13-h \\ 0 & \text{else} \end{cases}$ by the Hodge symmetry. The Hodge-Tate numbers of $\big(\mathrm{M}_3^{(2,2)}\big)_\ell(13-h)$ are $(0, 13-2h)$.

By Theorem 4.41, we know that

$$\det\big(\mathrm{Frob}_p, \big(\mathrm{M}_3^{(2,2)}\big)_\ell\big) = p^{13}$$

for $p \notin \{2, 3\}$. It follows from the Chebotarev density theorem that $\det\big(\mathrm{M}_3^{(2,2)}\big)_\ell = \chi_{cyc}^{-13}$. Thus, the determinant of $\big(\mathrm{M}_3^{(2,2)}\big)_\ell(13-h)$ is χ_{cyc}^{13-2h}, which is an odd character.

By Theorem 1.17, there exists a modular form of weight $14-2h$ and of level $2^s \cdot 3$ such that $\big(\mathrm{M}_3^{(2,2)}\big)_\ell(13-h) \simeq \rho_{f,\ell}$. It follows that for any $p \notin S \cup \{\ell\}$,

$$\begin{aligned} a_f(p) &= \mathrm{Tr}(\mathrm{Frob}_p^{-1} \mid ((\mathrm{M}_3^{(2,2)})_\ell(13-h)) \\ &= \det\big(\big(\mathrm{M}_3^{(2,2)}\big)_\ell(13-h)\big)^{-1} \cdot \mathrm{Tr}\big(\mathrm{Frob}_p \mid \big(\mathrm{M}_3^{(2,2)}\big)_\ell(13-h)\big) \\ &= -\frac{1}{p^h}\big(m_3^{(2,2)}(p) + p^2 + p^3 + 2p^4 + 2p^6\big). \end{aligned}$$

To get the modular form, we compute the traces of Frob_p for $3 < p \le 13$ by the computations in Appendix A.1.2. We try $0 \le h \le 6$ and $0 \le s \le 8$ one by one. If $(s,h) \neq (8,0), (8,1), (8,2)$ or $(7,0)$, we can use LMFDB, and the only modular form left is of weight 4 and of level 6, where $(s,h) = (1,5)$ in this case. If $(s,h) = (8,0), (8,1), (8,2)$ or $(7,0)$, we compute the space of cuspidal new modular symbols over the finite field $\mathbb{F}_p$.We find that for some primes p, the numbers $a_f(p)$ are not roots of the characteristic polynomials of the Hecke operators T_p, see the table in Appendix A.1.2. □

Remark 5.32. The nonzero Hodge numbers of the de Rham realization of $\mathrm{M}_3^{(2,2)}$ are $h^{5,8} = h^{8,5} = 1$. This one cannot be calculated by our methods in Chapter 3, because the nilpotent part of the local monodromy of the connection $\mathrm{Kl}_3^{(2,2)}$ at 0 is not direct sum of Jordan blocks of different sizes (we have two blocks of size 4).

5.3.7 A conjecture

One interesting corollary of Theorem 5.31 is that for $p \nmid 6$, the moments of the sheaves $\mathrm{Sym}^6\mathrm{Kl}_2$ and $\mathrm{Kl}_3^{(2,2)}$ are the same, because they are both equal to the Fourier coefficients of the modular form of label 6.4.*a*.*a*. As a direct consequence, we have the identity

$$m_3^{(2,2)}(p) - p^3 m_2^6(p) = -2p^6 - 2p^4 - p^2.$$

In fact, we have isomorphisms of ℓ-adic Galois representations $(\mathrm{M}_2^6)_\ell(-3) \simeq \big(\mathrm{M}_3^{(2,2)}\big)_\ell$, which suggests us the following conjecture.

Conjecture 5.33. *The two motives* $\mathrm{M}_2^6(-3)$ *and* $\mathrm{M}_3^{(2,2)}$ *are isomorphic.*

Appendix A

Sagemath code

In our proof, we used several results computed by the software Sagemath [The22]. This appendix includes the Sagemath codes and explains the algorithms. We fix an embedding $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$ and identify ℓ-adic numbers with their images in $\mathbb{C}$ via ι.

A.1 Computations of $m_3^k(p)$, $m_3^{(2,1)}(p)$ and $m_3^{(2,2)}(p)$

A.1.1 $m_3^k(p)$

For a prime number p, after Deligne [Del77, Somme. Trig.], we know that for each $a \in \mathbb{F}_p^\times$, there exist 3 algebraic numbers α_a, β_a and γ_a, of absolute value p, such that

$$s_1(a) = \alpha_a + \beta_a + \gamma_a = \mathrm{Kl}_3(a; p)$$

and

$$s_3(a) = \alpha_a \cdot \beta_a \cdot \gamma_a = p^3.$$

Then the degree two elementary symmetric polynomials is

$$\begin{aligned} s_2(a) := \alpha_a\beta_a + \beta_a\gamma_a + \gamma_a\alpha_a &= p^3(\alpha_a^{-1} + \beta_a^{-1} + \gamma_a^{-1}) \\ &= p(\bar{\alpha}_a + \bar{\beta}_a + \bar{\gamma}_a) = p \cdot \overline{\mathrm{Kl}_3(a; q)}. \end{aligned}$$

The k-th symmetric power moments of Kl_3 are integers of the form

$$m_3^k(p) := \sum_{a \in \mathbb{F}_p^\times} \sum_{i+j+k=k} \alpha_a^i \beta_a^j \gamma_a^k,$$

which can be computed using the value of elementary symmetric polynomials. For example, the 3-rd, 4-th and 6-th symmetric power moments can be computed by

$$m_3^3(p) = \sum_a (s_1(a)^3 - 2s_1(a)s_2(a) + p^3),$$

$$m_3^4(p) = \sum_a (s_1(a)^4 - 3s_1(a)^2 s_2(a) + s_2(a)^2 + 2p^3 s_1(a)),$$

and

$$m_3^6(p) = \sum_a (s_1(a)^6 - 5s_1(a)^4 s_2 + 6s_1(a)^2 s_2(a)^2 - s_2(a)^3 + 4p^3 s_1(a)^3 - 6p^3 s_1(a)s_2(a) + p^6.$$

respectively. At last, let

$$a_3^4(p) = -\frac{1}{p^3}(m_3^4(p) + 1 + p^2 + p^4)$$

be the trace of the middle cohomology $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^4\mathrm{Kl}_3)$.

Part of the numerical results are the following:

	Primes	
	3	5
$m_3^3(p)$	-10	
$a_3^4(p)$	-2	-12
$m_3^6(p)$	-820	

A.1.2 $m_3^{(2,1)}(p)$ and $m_3^{(2,2)}(p)$

Since the representation $V_{2,1}$ is identified with the kernel of

$$\mathrm{Sym}^2\mathrm{Std} \otimes \wedge^2\mathrm{Std} \to \mathrm{Std},$$

we have an induced short exact sequence

$$0 \to \mathrm{Kl}_3^{(2,1)} \to \mathrm{Sym}^2\mathrm{Kl}_3 \otimes \wedge^2\mathrm{Kl}_3 \to \mathrm{Kl}_3(-3) \to 0.$$

The moment of $\mathrm{Kl}_{\mathrm{SL}_3}^{V_{2,1}}$ is therefore the difference of the moment of $\mathrm{Sym}^2\mathrm{Kl}_3 \otimes \wedge^2\mathrm{Kl}_3$ and the moment of $\mathrm{Kl}_3(-3)$. So

$$m_3^{(2,1)}(p) = \sum_a (s_1(a)^2 s_2(a) - s_2^2(a) - p^3 s_1(a)).$$

Let $a_3^{(2,1)}(p)$ be the traces of the middle cohomology $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_3^{(2,1)})$. Then

$$a_3^{(2,1)}(p) = -p^{-4}(m_3^{(2,1)}(p) + p + p^2 + p^3).$$

Similarly, we have an induced short exact sequence

$$0 \to \mathrm{Kl}_3^{(2,2)} \to \mathrm{Sym}^2\mathrm{Kl}_3 \otimes \mathrm{Sym}^2\left(\wedge^2\mathrm{Kl}_3\right) \to \mathrm{Kl}_3 \otimes \wedge^2\mathrm{Kl}_3(-3) \to 0.$$

Therefore, the moment of $\mathrm{Kl}_3^{(2,2)}$ is

$$m_3^{(2,2)}(p) = \sum_a \left((s_1(a)^2 - s_2(a))(s_2(a)^2 - p^3 s_1(a)) - p^3 s_1(a) \cdot s_2(a)\right).$$

Let $a_3^{(2,2)}(p)$ be the traces of the middle cohomology $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Kl}_3^{(2,3)})$. Then

$$a_3^{(2,2)}(p) = -p^{-5}(m_3^{(2,2)}(p) + p^2 + p^3 + 2p^4 + 2p^6)$$

Part of the numerical results are the following:

	Primes						
	5	7	11	13	17	19	23
$a_3^{(2,1)}$	0		0	-4	6	2	0
$a_3^{(2,2)}$	6	-16	12	38	-126	20	168

If $N \cdot k^2 \geq 40000$, the database LMFDB is not enough for us. So we follow the appendix in [YV15] to compute the space of cuspidal new modular symbols over some finite fields $\mathbb{F}_\ell$. Then we check if the prescribed traces are roots of the characteristic polynomials of T_p. Here are some numerical results.

Level N	weight k	Prime p	Finite field $\mathbb{F}_\ell$	$T_p(a_f(p))$
$2^8 \cdot 3$	14	17	$\mathbb{F}_{11}$	1
$2^8 \cdot 3$	12	5	$\mathbb{F}_{13}$	9
$2^8 \cdot 3$	10	7	$\mathbb{F}_{13}$	4
$2^7 \cdot 3$	14	17	$\mathbb{F}_{13}$	9

A.1.3 Sagemath code

```
p=3; #choose a prime p

import math;
#define Kl3
def Kl(a):
    import sage.rings.all as rings
    sum = 0;
    for x in range(1,p):
        for y in range(1,p):
            sum = sum + exp (2 * pi * I/p * (x + y + (a * x * y).inverse_mod(p)));
    return sum

def s2(a):
    v = conjugate(CC(Kl(a))) * p;
    return v

# moment of Sym^3Kl_3
mom3=0;
for a in range(1,p):
    a = ZZ(a);
    mom3 = mom3 + CC(Kl(a))^3 - 2 * CC(Kl(a)) * CC(s2(a)) + CC(p)^3;
a3= -(mom3 + 1 + p^2)
print(a3)

# moment of Sym^4Kl_3
mom4 = 0;
for a in range(1,p):
    a = ZZ(a);
    mom4 = mom4 + CC(Kl(a))^4 - 3 * CC(Kl(a))^2 * CC(s2(a)) + CC(s2(a))^2 + 2 * CC(Kl(a)) *
        CC(p)^3;
a4 = -p^(-3) * (mom4 + 1 + p^2 + p^4)
print(a4)

# moment of Sym^6Kl_3
mom6=0;
for a in range(1,p):
    a = ZZ(a);
    mom6 = mom6 + CC(Kl(a))^6 + 4 * CC(Kl(a))^3 * p^3 + p^6 - 5 * CC(Kl(a))^4 * CC(s2(a)) -
        6 * CC(Kl(a)) * p^3 * CC(s2(a)) + 6 * CC(Kl(a))^2 * CC(s2(a))^2 - CC(s2(a))^3;
print(mom6)

# moment of Kl_{SL_3}^{V_{2,1}}
mom21 = 0;
for a in range(1,p):
    a = ZZ(a);
    mom21 = mom21 +(CC(Kl(a))^2 * CC(s2(a)) - CC(s2(a))^2 - p^3 * CC(Kl(a)));
a21 = -p^(-4) * (mom21 + p + p^2 + p^3)
print(a21)

# moment of Kl_{SL_3}^{V_{2,2}}
mom22=0;
for a in range(1,p):
    a = ZZ(a);
    mom22 = mom22 + (CC(Kl(a))^2 - CC(s2(a))) * (CC(s2(a))^2 - p^3 * CC(Kl(a)))- p^3 *
        CC(Kl(a)) * CC(s2(a)) ;
```

```
a22 = - p^(-5) * (mom22 + p^2 + p^3 + 2 * p^4 + 2 * p^6)
print(a22)

# when N*k^2 is too big
M = ModularSymbols(Gamma0(N),k,sign=1,base_ring=GF(13))
Hp = M.new_subspace().cuspidal_subspace().hecke_polynomial(p)
print (Hp(a22))
```

A.2 Computation of $m_4^3(p)$ and $m_4^4(p)$

A.2.1 $m_4^3(2)$

Here we only compute the 3-rd symmetric power moment at $p = 2$. By Sagemath [The22], we know that $\mathrm{Kl}_4(1;2) = 1$ and $\mathrm{Kl}_4(1;4) = 11$. Let $\alpha_1, \dots, \alpha_5$ be the eigenvalues of Frob_2 acting on $(\mathrm{Kl}_4)_{\bar{1}}$ and let $s_1, \dots, s_4$ be the elementary symmetric polynomials on α_i. By the definition of Kl_4, we have

$$s_1 = \sum \alpha_i = -\mathrm{Kl}_4(1;2) = -1$$

and

$$s_1^2 - 2s_2 = \sum \alpha_i^2 = -\mathrm{Kl}_4(1;4) = -11.$$

Therefore, $s_1 = -1$ and $s_2 = 6$. Furthermore, since $\det \mathrm{Kl}_4 = \bar{\mathbb{Q}}_\ell(-6)$, we have

$$s_4 = \prod \alpha_i = p^6.$$

From this, and notice that $\alpha_i \cdot \bar{\alpha} = p^3$, we have

$$s_3 = p^3 \bar{s}_1 - 8.$$

Then the moments can be computed by

$$m_4^3(2) = \sum_{i,j,k} \alpha_i \alpha_j \alpha_k = s_1^3 - 2s_1 s_2 + s_3 = 3.$$

It follows that

$$a_4^3(2) = -\frac{1}{p^4}(m_4^3(p) + 1 + p^2 + p^3) = -1.$$

A.2.2 $m_4^4(p)$

Let $\alpha_1(a), \dots, \alpha_4(a)$ be the eigenvalues of Frob_p acting on $(\mathrm{Kl}_4)_{\bar{a}}$ for $a \in \mathbb{F}_p^\times$. We also denote by $s_1(a), \dots, s_4(a)$ the elementary symmetric polynomials on $\alpha_i(a)$. By the definition of Kl_4, we have

$$s_1(a) = \sum \alpha_i(a) = -\mathrm{Kl}_4(a;p) \text{ and } s_1(a)^2 - 2s_2(a) = \sum \alpha_i(a)^2 = -\mathrm{Kl}_4(a;p^2).$$

Furthermore, since $\det \mathrm{Kl}_4 = \bar{\mathbb{Q}}_\ell(-6)$, we have

$$s_4(a) = \prod \alpha_i = p^6.$$

From this, and notice that $\alpha_i(a) \cdot \bar{\alpha}_i(a) = p^3$, we have

$$s_3(a) = p^3 \overline{s_2(a)}.$$

Then the moments can be computed as

$$\begin{aligned} m_4^4(p) &= \sum_{a\in\mathbb{F}_p^\times} \sum_{i\le j\le k\le r} \alpha_i(a)\alpha_j(a)\alpha_k(a)\alpha_r(a) \\ &= \sum_a (s_1(a)^4 - 3s_1(a)^2 s_2(a) + s_2(a)^2 + 2p^3 s_1(a)\overline{s_1(a)} - p^6). \end{aligned}$$

At last the traces of the middle cohomology $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^4\mathrm{Kl}_4)$ are

$$a_4^4(p) = -\frac{1}{p^4}(m_4^4(p) + 1 + p^2 + p^3 + p^4 + 2p^6).$$

Some numerical results are listed below.

	Primes		
	2	3	7
$a_4^3(p)$	-1		
$a_4^4(p)$		-26	-22

If $N \cdot k^2 \ge 40000$, the database LMFDB is not enough for us. So we follow the appendix in [YV15] to compute the space of cuspidal new modular symbols over some finite fields $\mathbb{F}_\ell$. Then we check if the prescribed traces are roots of the characteristic polynomials of T_p. Here are some numerical results.

Level N	weight k	Prime p	Finite field $\mathbb{F}_\ell$	$T_p(a_f(p))$
$2^8 \cdot 5$	14	7	$\mathbb{F}_{11}$	3
$2^8 \cdot 5$	12	3	$\mathbb{F}_{17}$	15
$2^8 \cdot 5$	10	7	$\mathbb{F}_{11}$	3
$2^8 \cdot 5$	8	7	$\mathbb{F}_{11}$	5
$2^8 \cdot 5$	6	7	$\mathbb{F}_{11}$	4
$2^7 \cdot 5$	14	3	$\mathbb{F}_{17}$	8
$2^7 \cdot 5$	12	7	$\mathbb{F}_{17}$	8
$2^7 \cdot 5$	10	3	$\mathbb{F}_{11}$	5
$2^7 \cdot 5$	8	7	$\mathbb{F}_{11}$	3
$2^6 \cdot 5$	14	3	$\mathbb{F}_{17}$	3
$2^6 \cdot 5$	12	7	$\mathbb{F}_{13}$	7

A.2.3 Sagemath code

```
#choose a prime
p=7;

import math

#compute Kl_4(a;p), as a list Klp
k = GF(p,'r')
Klp = [-1];
for a in range(1,p):
    klp = 0;
    for x in k:
        for y in k:
            for z in k:
```

```
                if x*y*z!=0:
                    u = x + y + z + a/(x * y * z);
                    u = u.lift();
                    klp = klp + exp(2 * pi * I/p * u);
                    klp = n(klp);
    Klp.append(klp);

#compute Kl_4(a;p^2), as a list Klp2
K = GF(p^2,'r')
Klp2 = [-1];
for a in range(1,p):
    klp2 = 0;
    for x in K:
        for y in K:
            for z in K:
                if x*y*z!=0:
                    u = x + y + z + a/(x * y * z);
                    u = (u.trace());
                    u = u.lift();
                    klp2 = klp2 + exp(2 * pi * I/p * u);
                    klp2 = n(klp2);
    Klp2.append(klp2)

#compute s_1(a) and s_2(a) as lists S1 and S2
S1 = [1];
S2 = [1];
for a in range(1,p):
    s1 = - Klp[a];
    s2 = - Klp2[a];
    s2 = (s1^2 - s2)/2;
    S1.append(s1);
    S2.append(s2);

#compute moments

#compute moments m
mom3=0
for a in range(1,p):
    mom3 = mom3 + S1[a]^3 - 2 * S1[a] * S2[a] + p^3 * conjugate(S1[a]);

mom4=0;
for a in range(1,p):
    mom = mom + S1[a]^4 - 3 * S1[a]^2 * S2[a] + S2[a]^2 + 2 * p^3 * S1[a] *
        conjugate(S1[a]) - p^6;
print("mom_4^3(p) =",mom3,"and mom_4^3(p) =",mom4)

#compute trace a
a3 = -(mom+1 + p^2 + p^3)/p^4;
a4 = -(mom+1 + p^2 + p^3 + p^4 + 2 * p^6)/p^4;
print("a_4^3(p) =",a3, "and a_4^4(p) =",a4 )

# when N*k^2 is too big
M = ModularSymbols(Gamma0(N),k,sign=1,base_ring=GF(13))
Hp = M.new_subspace().cuspidal_subspace().hecke_polynomial(p)
print (Hp(a44))
```

A.3 Computation of $m_5^3(p)$

Let $\alpha_1(a), \dots, \alpha_5(a)$ be the eigenvalues of Frob_p acting on $(\mathrm{Kl}_5)_{\bar{a}}$ for $a \in \mathbb{F}_p^\times$. We also denote by $s_1(a), \dots, s_5(a)$ the elementary symmetric polynomials on $\alpha_i(a)$. By the definition of Kl_5, we have

$$s_1(a) = \sum \alpha_i(a) = \mathrm{Kl}_5(a;p) \text{ and } s_1(a)^2 - 2s_2(a) = \sum \alpha_i(a)^2 = \mathrm{Kl}_5(a;p^2).$$

Furthermore, since $\det \mathrm{Kl}_5 = \bar{\mathbb{Q}}_\ell(-10)$, we have

$$s_5(a) = \prod \alpha_i = p^{10}.$$

From this, and notice that $\alpha_i(a) \cdot \bar{\alpha}_i(a) = p^4$, we have

$$s_3(a) = p^2\overline{s_2(a)} \text{ and } s_4 = p^6\overline{s_1(a)}.$$

Then the moments can be computed as

$$m_5^3(p) = \sum_{a \in \mathbb{F}_p^\times} \sum_{i \le j \le k} \alpha_i(a)\alpha_j(a)\alpha_k(a) = \sum_a s_1(a)^3 - 2s_1(a)s_2(a) + 3s_3(a).$$

At last the traces of middle cohomology $\mathrm{H}^1_{\acute{e}t,\mathrm{mid}}(\mathbb{G}_{m,\bar{\mathbb{F}}_p}, \mathrm{Sym}^3\mathrm{Kl}_5)$ are

$$a_5^3(p) = -\frac{1}{p^5}(m_5^3(p) + 1 + p^2 + p^3 + p^4 + p^6).$$

The values at $p = 2, 5$ are listed below.

	Primes	
	2	5
$m_5^3(p)$	-61	3901
$a_5^3(p)$	-1	-4

A.3.1 Sagemath code

```
#choose a prime
p=2;

#compute Kl_5(a;p) and Kl_5(a;p^2), then print s_1(a) and s_2(a)
import math


#compute Kl_5(a;p) as the list Klp
k = GF(p,'r')
Klp = [];
for a in range(1,p):
    klp = 0;
    for x in k:
        for y in k:
            for z in k:
                for w in k:
                    if x*y*z*w!=0:
                        u = x + y + z + w + a/(x * y * z * w);
                        u = (u.trace());
                        u = u.lift();
```

```
                    klp = klp + exp(2 * pi * I/p * u);
                    klp = n(klp);
    Klp.append(klp);

#compute Kl_5(a;p^2) as the list Klp2
k = GF(p^2,'r')
Klp2 = [];
for a in range(1,p):
    klp2 = 0;
    for x in k:
        for y in k:
            for z in k:
                for w in k:
                    if x*y*z*w!=0:
                        u = x + y + z + w + a/(x * y * z * w);
                        u = (u.trace());
                        u = u.lift();
                        klp2 = klp2 + exp(2 * pi * I/p * u);
                        klp2 = n(klp2);
    Klp2.append(klp2)


#compute s_1(a) and s_2(a) as lists S1 and S2
S1 = [];
S2 = [];
for a in range(1,p):
    b = a-1;
    s1 = Klp[b];
    s2 = Klp2[b];
    s2 = (s1^2 - s2)/2;
    S1.append(s1);
    S2.append(s2);

#compute moments m
mom=0;
for a in range(1,p):
    b = a - 1;
    mom = S1[b]^3 - 2 * S1[b]*S2[b] + p^2 * conjugate(S2[b]);
print("mom_5^3(p) =",mom)

#compute trace a
a = -(mom + 1 + p^2 + p^3 + p^4 + p^6)/p^5;
print("a_5^3(p) =",a)
```

A.4 The computation of $P(x)$ and $\tilde{P}(s)$

```
# define the groups of orders 108 and 216
K = CyclotomicField(9);
a = K.gen();
w = a^6;
e = a^4;
l = w - w^2;
s = matrix(K, [[1,0,0],[0,w,0],[0,0,w^2]])
t = matrix(K, [[0,1,0],[0,0,1],[1,0,0]])
```

```
u = matrix(K,[[e,0,0],[0,e,0],[0,0,e*w]])
v = matrix(K,[[1/l,1/l,1/l],[1/l,w/l,w^2/l],[1/l,w^2/l,w/l]])
G = MatrixGroup(s,t,u,v);
G216 = G.subgroup([s,t,u * v * u.inverse(),v]);
G108 = MatrixGroup(s,t,v);
sum108=0;

#compute P(x) and tildeP(x)
for a in G108:
    A = (a).matrix();
    sum108 = sum108 - 1/A.charpoly();
print("P(x) =",sum108);
sum216=0;

for a in G216:
    A = (a).matrix();
    sum216 = sum216 - 1/A.charpoly();
print("tildeP(x) =",sum216)
```

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ECOLE
DOCTORALE
DE MATHEMATIQUES
HADAMARD

Titre : Fonctions L des faisceaux de Kloosterman

Mots clés : Fonctions L, Faisceaux de Kloosterman, Motifs, Théorie de Hodge, Represéntations galoisiennes

Résumé : Les faisceaux de Kloosterman classiques sont des systèmes locaux ℓ-adiques sur $\mathbb{G}_{m,\mathbb{F}_p}$ dont les traces de Frobenius sont des sommes de Kloosterman. Ce sont des cas particuliers des faisceaux de Kloosterman pour les groupes réductifs introduits par Heinloth, Ngô, et Yun. Cette thèse vise à généraliser le travail de Fresán, Sabbah et Yu sur les fonctions L des faisceaux de Kloosterman pour SL_2. On montre que les fonctions L de plusieurs faisceaux de Kloosterman pour SL_{n+1} admettent un prolongement méromorphe au plan complexe et satisfont à une équations fonctionnelle. En particulier, certaines fonctions L proviennent de formes modulaires, comme des conjectures de type Evans prédissent.
On construit d'abord des motifs sur $\mathbb{Q}$ tels que leurs réalisations ℓ-adiques ont les mêmes fonctions L que les faisceaux de Kloosterman pour SL_{n+1}. On calcule ensuite les nombres de Hodge des réalisations de Rham de ces motifs en calculant la filtration de Hodge irrégulière. Tant que l'on trouve un motif tel que les nombres de Hodge de sa réalisation de Rham soient soit zéro soit un, on peut appliquer un théorème de Patrikis et Taylor pour montrer l'automorphie potentielle des représentations galoisiennes. Contrairement au cas des faisceaux de Kloosterman pour SL_2, les nombres de Hodge peuvent être supérieurs à un.
Lorsque les motifs des faisceaux de Kloosterman sont de dimension deux, on peut utiliser la conjecture de modularité de Serre pour montrer que les fonctions L des faisceaux de Kloosterman coïncident avec des fonctions L de formes modulaires.

Title : L-functions of Kloosterman sheaves

Keywords : L-functions, Kloosterman sheaves, Motives, Hodge Theory, Galois representations

Abstract : The classical Kloosterman sheaves are ℓ-adic local systems over $\mathbb{G}_{m,\mathbb{F}_p}$ whose traces of Frobenius are Kloosterman sums. They are special cases of Kloosterman sheaves for reductive groups introduced by Heinloth, Ngô, and Yun. This thesis aims to generalize the work of Fresán, Sabbah, and Yu on the L-functions of Kloosterman sheaves for SL_2. We show that the L-functions of several Kloosterman sheaves for SL_{n+1} can be extended meromorphically to the complex plane and satisfy functional equations. In particular, some L-functions arise from modular forms, as predicted by conjectures of Evans type.
We first construct motives over $\mathbb{Q}$ such that their ℓ-adic realizations have the same L-functions as Kloosterman sheaves for SL_{n+1}. Then we compute the Hodge numbers of the de Rham realizations of these motives by computing the irregular Hodge filtration. As long as we find a motive such that the Hodge numbers of its de Rham realization are either zero or one, we can apply a theorem of Patrikis and Taylor to show the potential automorphy of Galois representations. Contrary to the case of Kloosterman sheaves for SL_2, the Hodge numbers can be bigger than one.
When the corresponding motives of Kloosterman sheaves have dimension 2, we can use Serre's modularity conjecture to show that the L-functions of Kloosterman sheaves coincide with the L-functions of modular forms.

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